

# State estimation

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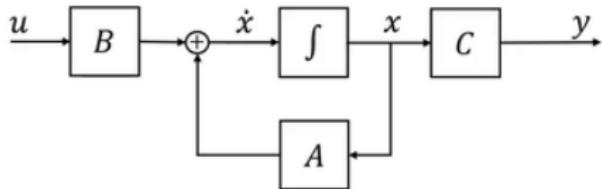
1 Observers

2 Observability

3 Kalman filtering

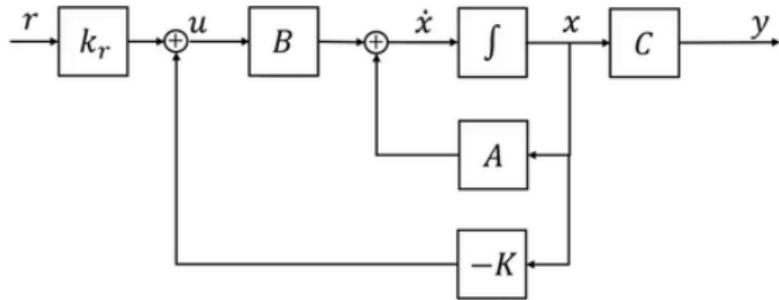
## State feedback control

**Idea with state feedback control design:** Modify the eigenvalues of the system by using the input,  $u = -Kx + k_r r$ .



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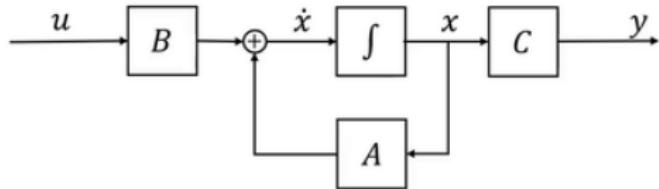
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**Problem:** Requires full access to the state vector,  $u = -Kx$

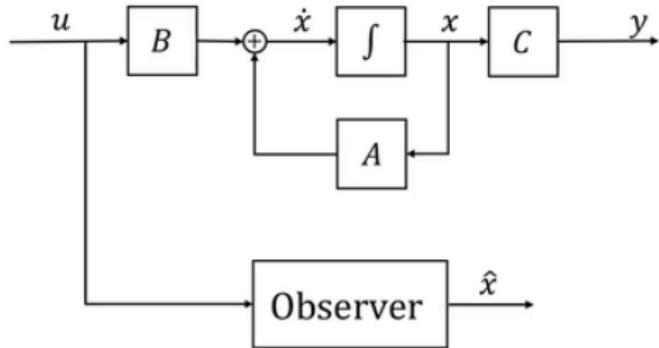
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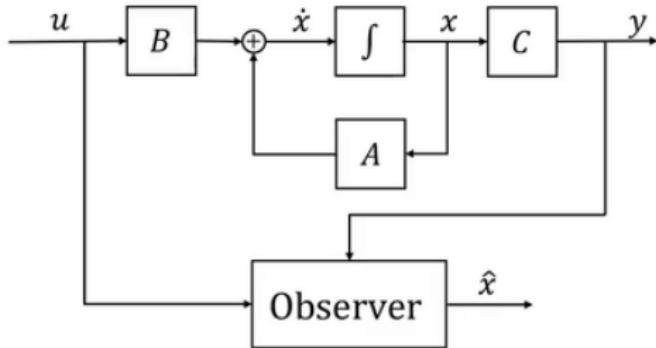
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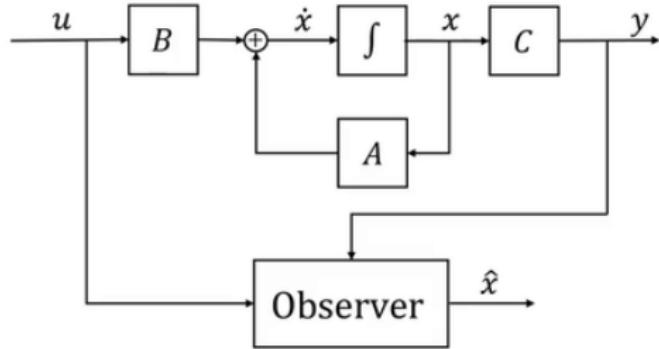
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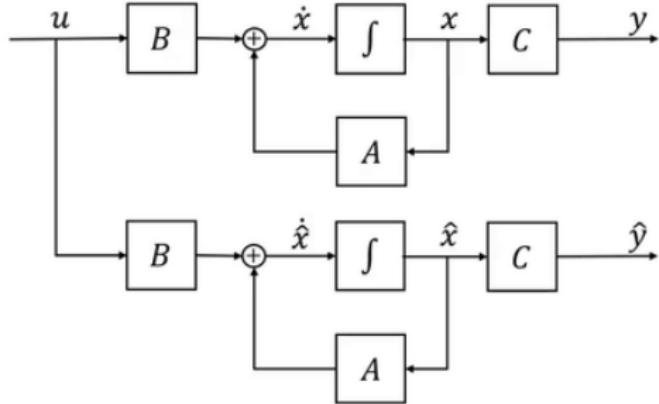
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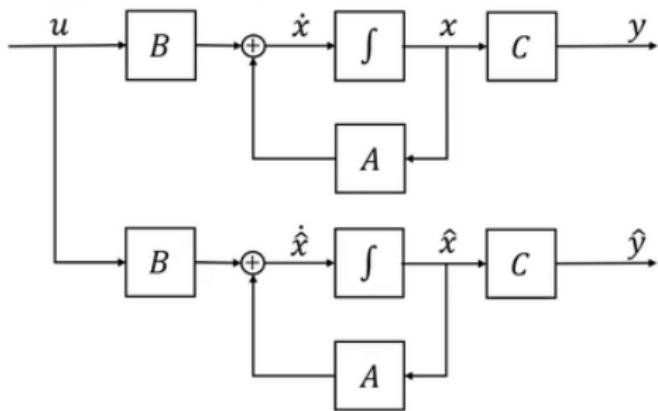
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*Realistic?*

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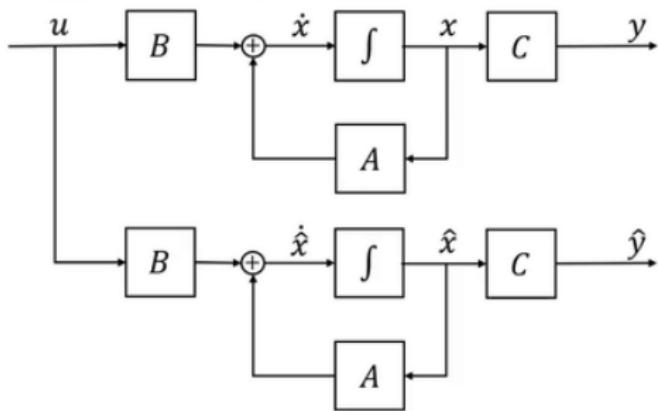
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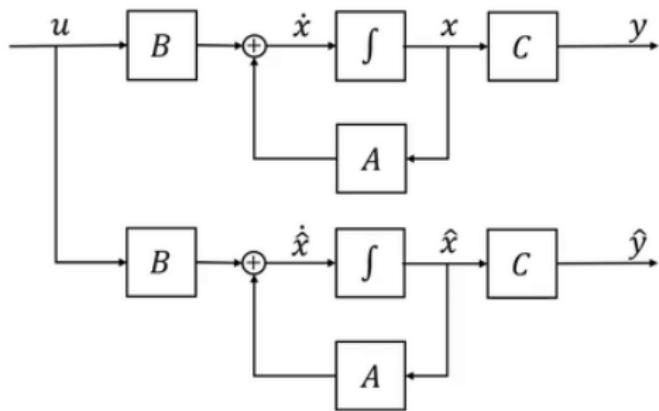
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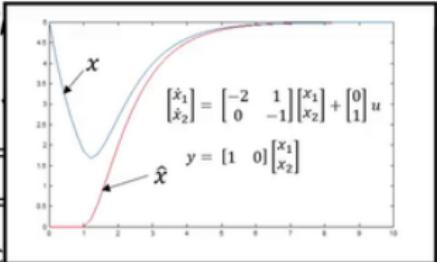


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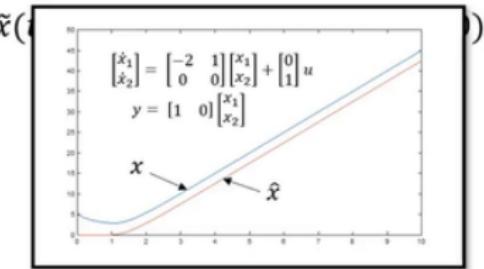
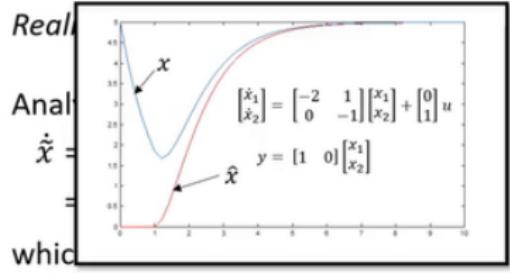
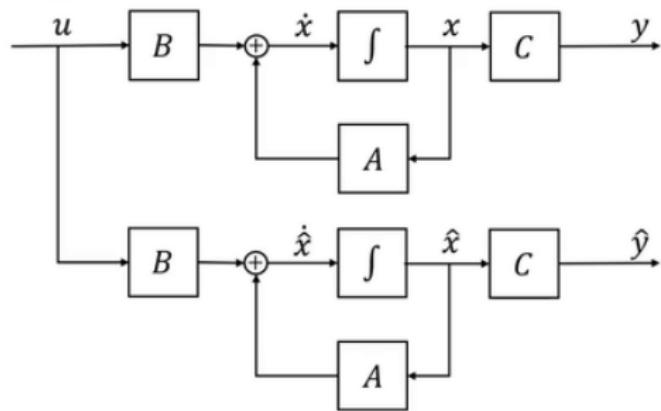


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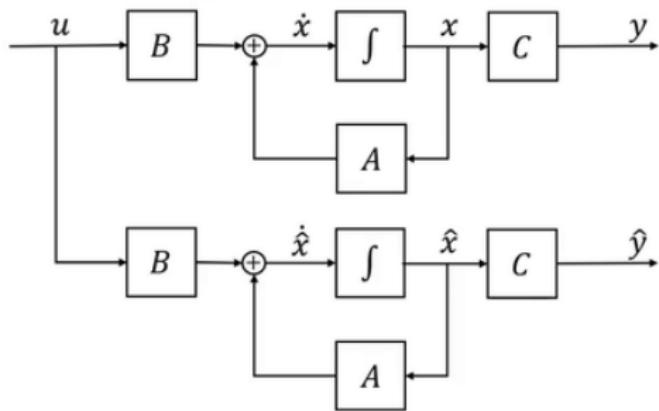
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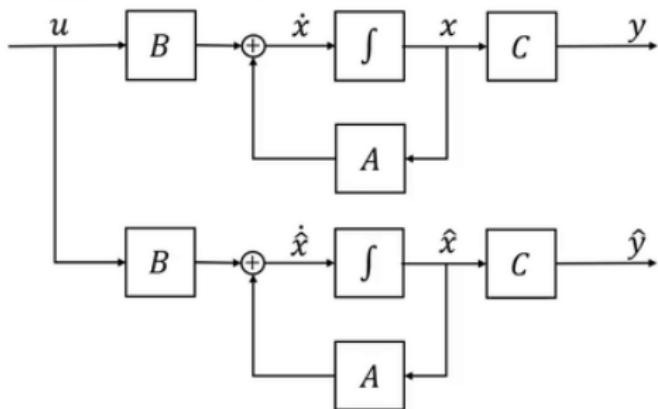
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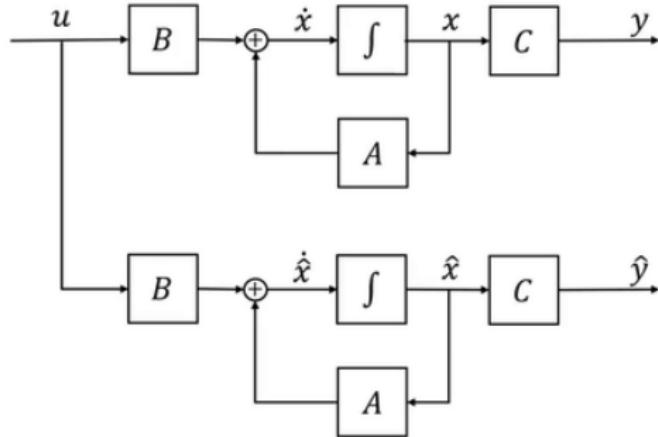
*Open loop estimation does not seem to be a good idea!*



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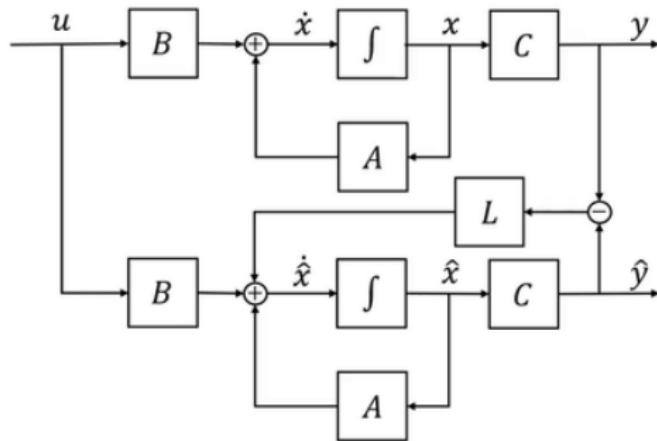
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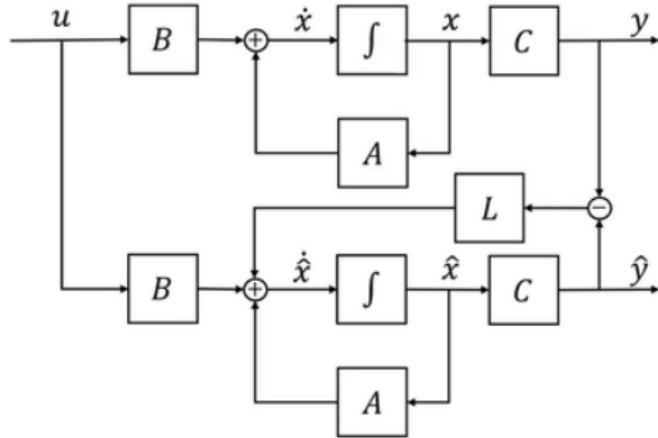
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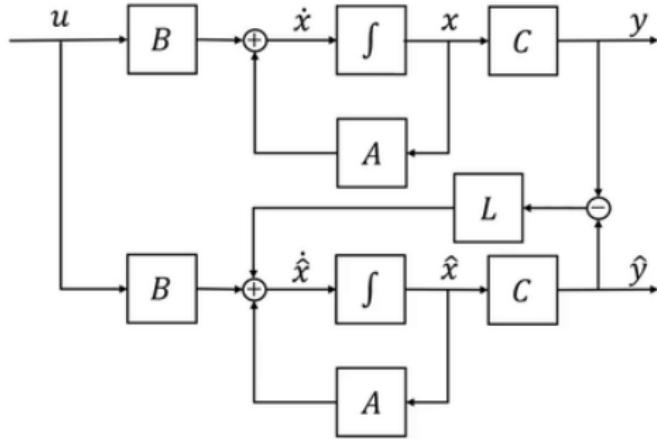
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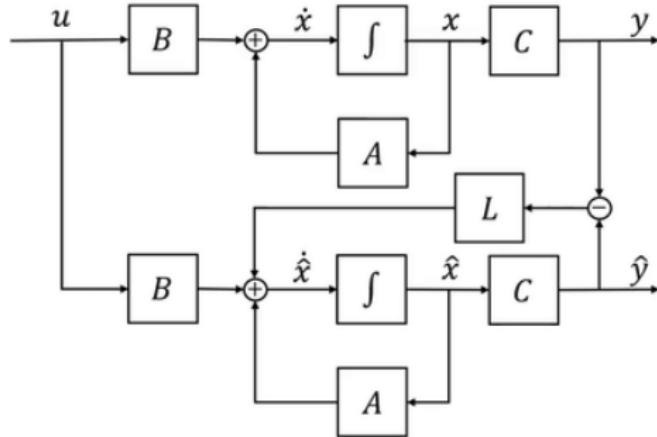
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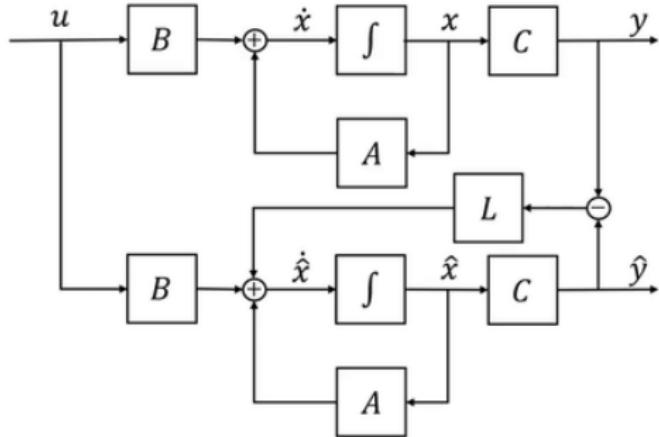
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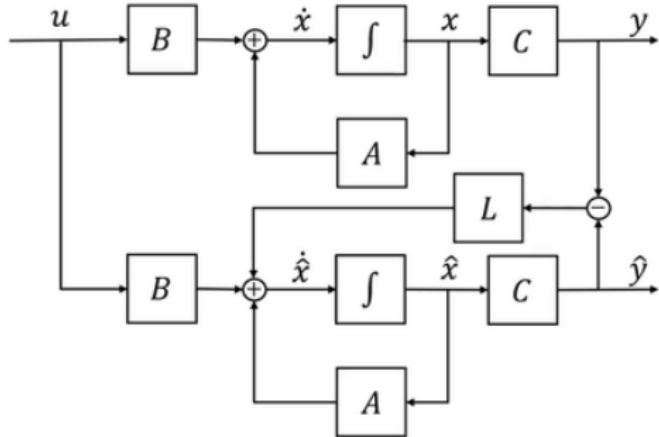
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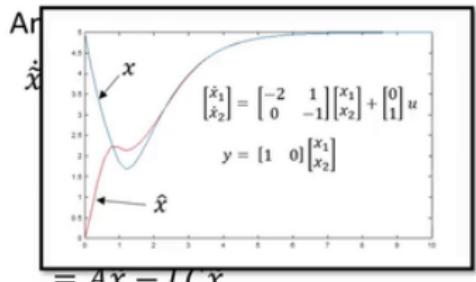
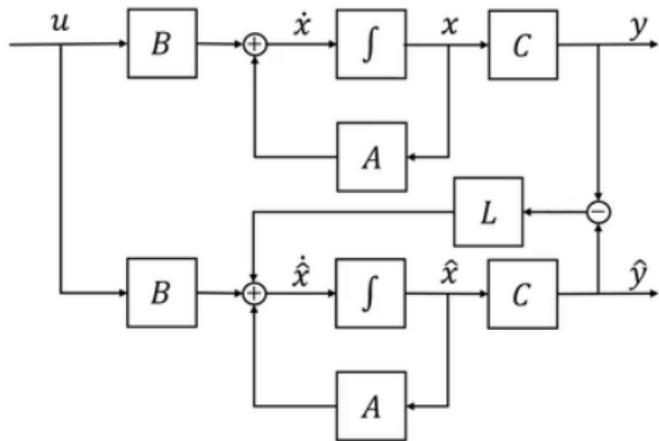
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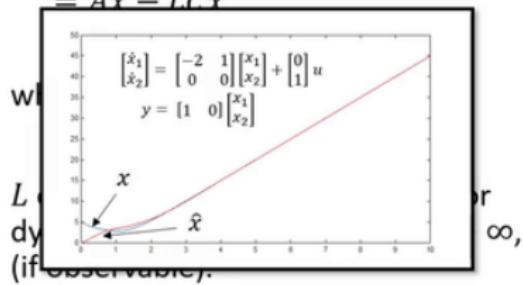
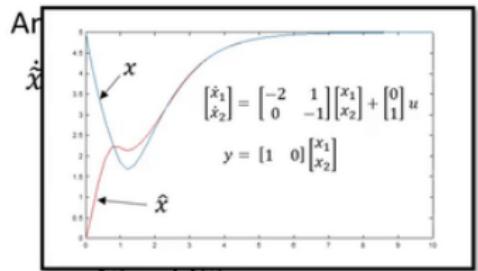
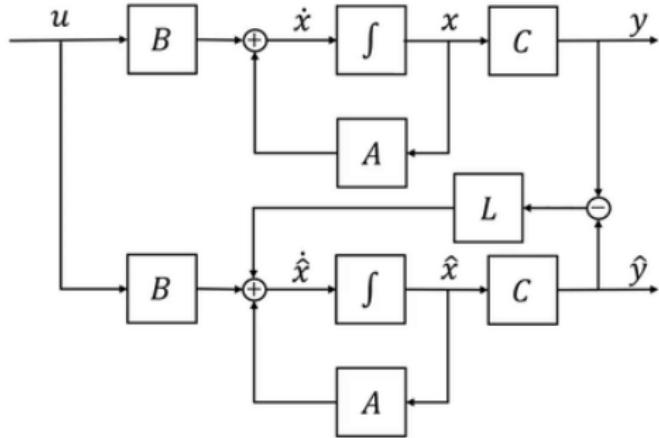
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Use pole placement with a desired characteristic polynomial to choose the estimator gain,  $L$ .

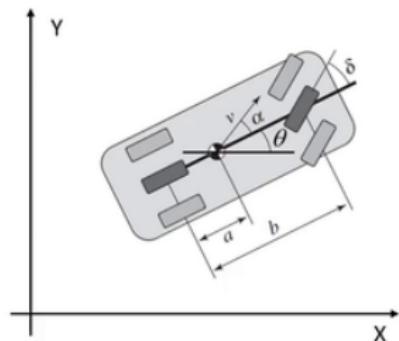
## Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0] u$$

where  $x_1$  is the lateral position  $Y$ ,  $x_2$  is the heading orientation  $\theta$  and  $u$  is the steering angle  $\delta$ .



Vehicle data:  $v_0 = 12 \text{ m/s}$

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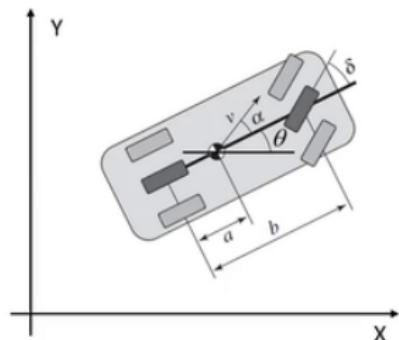
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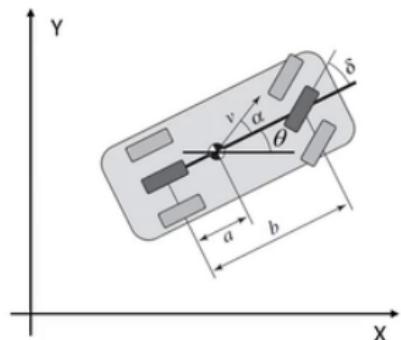
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**Specification:** Desired characteristic polynomial:

$$p_{des}(\lambda) = (\lambda + 6)(\lambda + 4) = \lambda^2 + 10\lambda + 24$$

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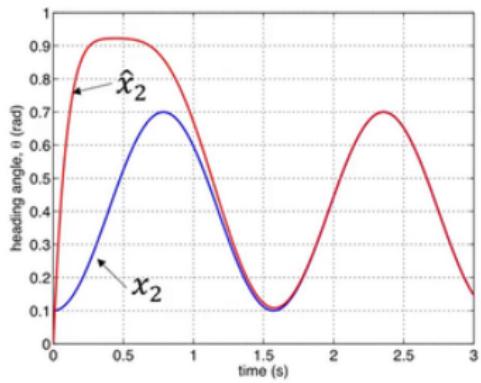
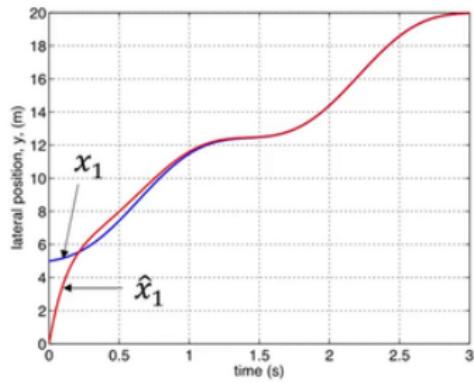
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Simulations using a sinusoidal input, with  $x(0) = (5, 0.1)$  and  $\hat{x}(0) = (0, 0)$ :

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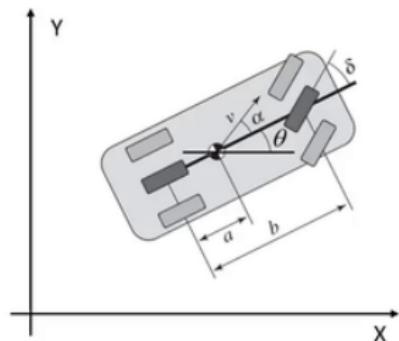
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where  $x_1$  is the lateral position  $Y$ ,  $x_2$  is the heading orientation  $\theta$  and  $u$  is the steering angle  $\delta$ .

Design a state estimator to estimate the vehicle's states, from measurement of the **heading angle**.



Vehicle data:  $v_0 = 12 \text{ m/s}$

$a = 2 \text{ m}$

$b = 4 \text{ m}$

**Specification:** Desired characteristic polynomial:

$$p_{des}(\lambda) = (\lambda + 6)(\lambda + 4) = \lambda^2 + 10\lambda + 24$$

## Revisit Example - Vehicle steering (Ex 7.4)

State estimator:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

The closed loop estimator dynamics becomes

$$\dot{\tilde{x}} = (A - LC)\tilde{x} = \begin{bmatrix} 0 & 12 - l_1 \\ 0 & -l_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

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$$\det(\lambda I - A + LC) = \begin{vmatrix} \lambda & -12 + l_1 \\ 0 & \lambda + l_2 \end{vmatrix} = \lambda(\lambda + l_2)$$

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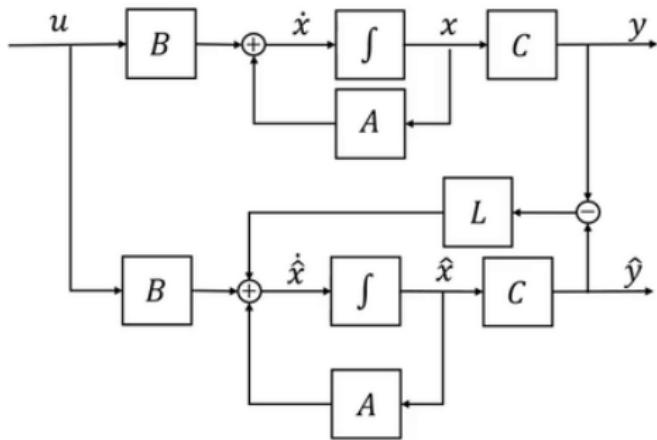
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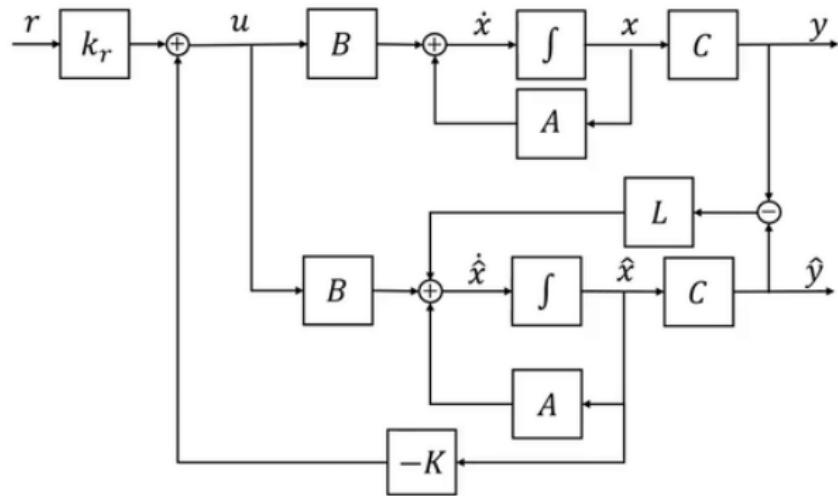
## Control using estimated states

Use the estimated states for feedback,  $u = -K\hat{x} + k_r r$ .



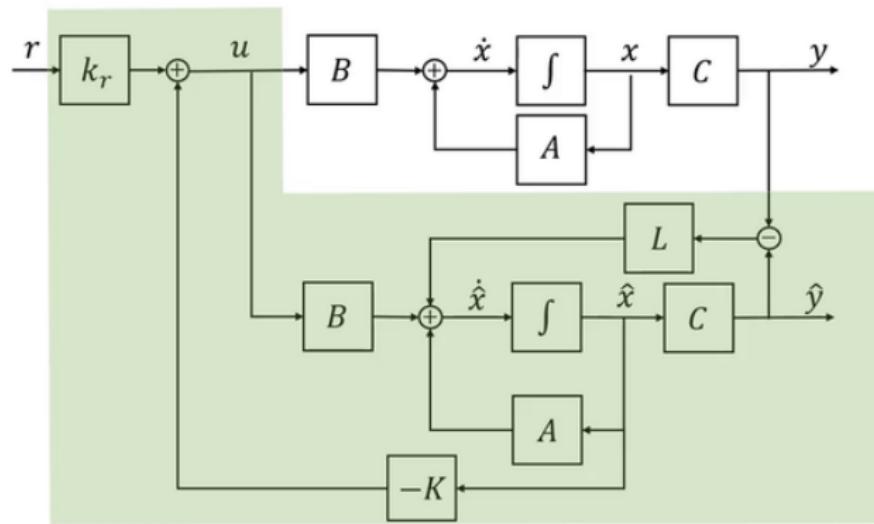
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Given the system,  $\dot{x} = Ax + Bu$ ,  $y = Cx$ , the controller,  $u = -K\hat{x} + k_r r$ , and the state estimator,  $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$ , the closed loop system can be written as:

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - CL \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} Bk_r \\ 0 \end{bmatrix} r$$

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**Rule of thumb:** Make the estimator poles 4-5 times faster than the "feedback" poles.

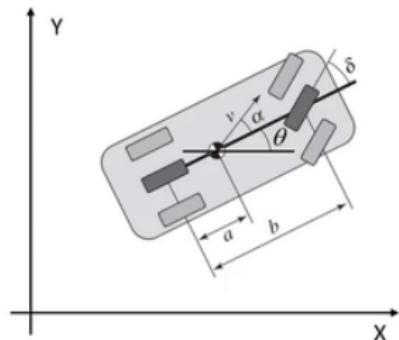
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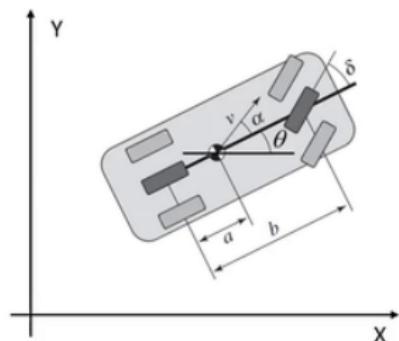
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State feedback control (poles in -1 (double pole)):

$$u = -0.0278x_1 - 0.6111x_2 + 0.0278r$$



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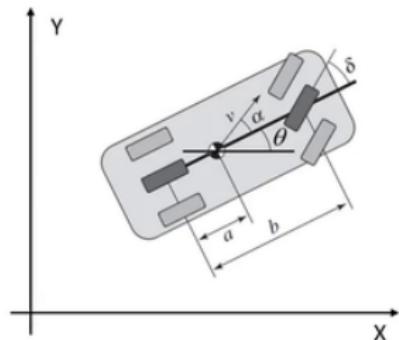
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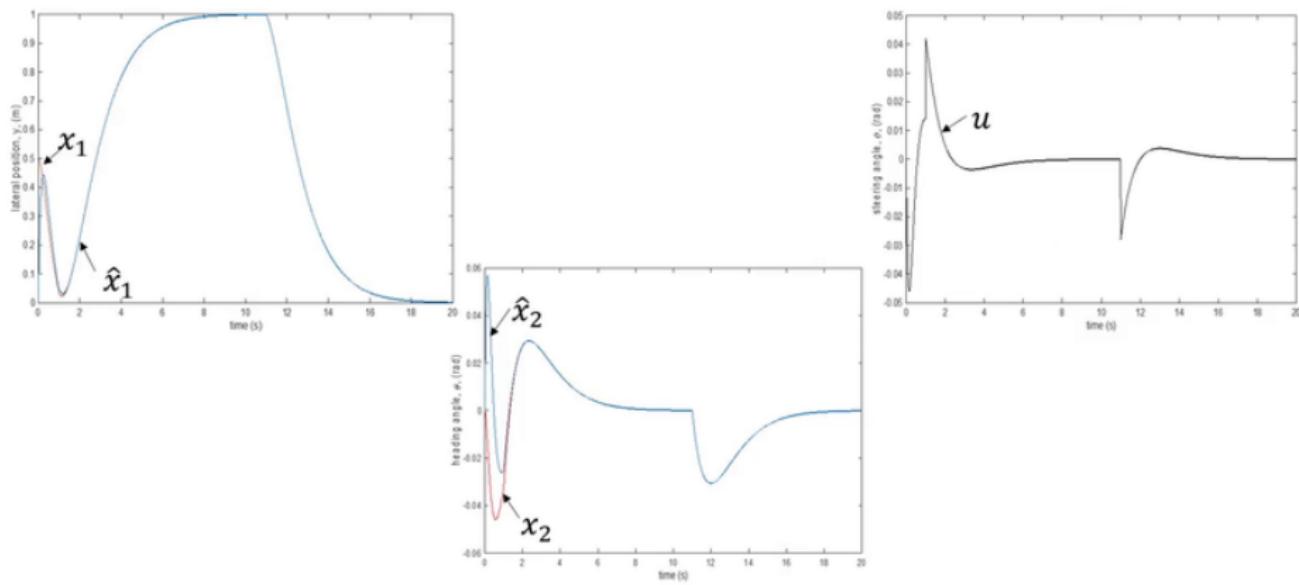


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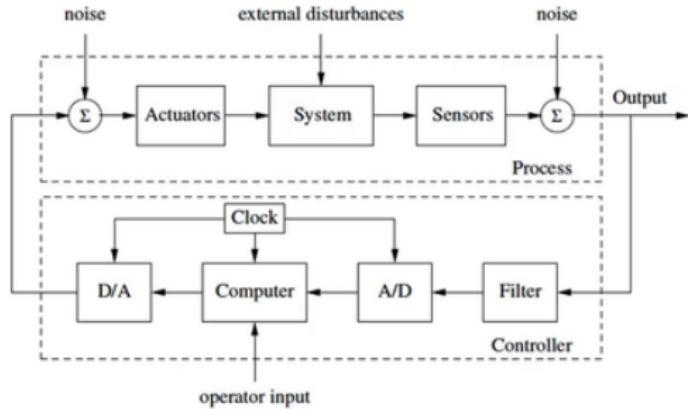
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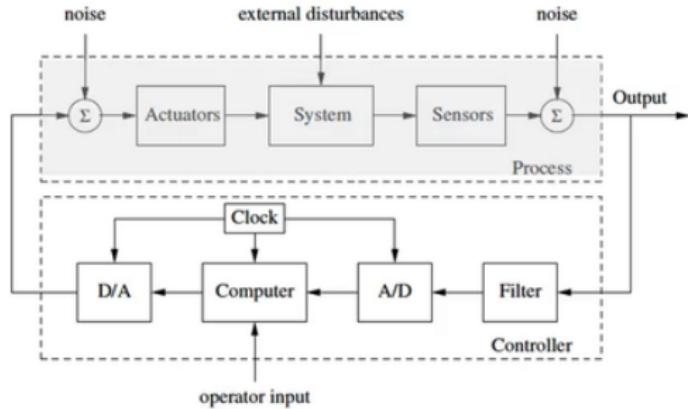
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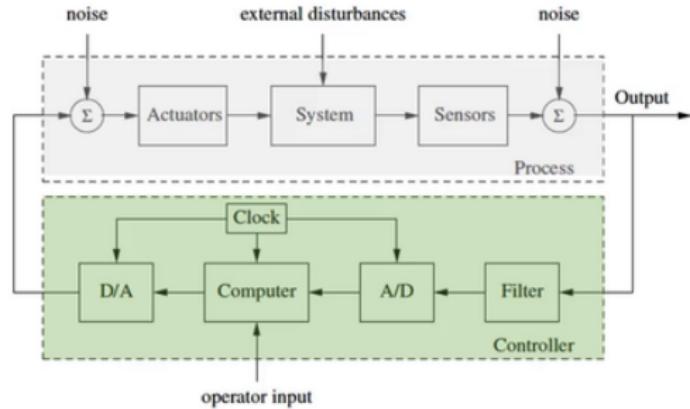
## Implementation



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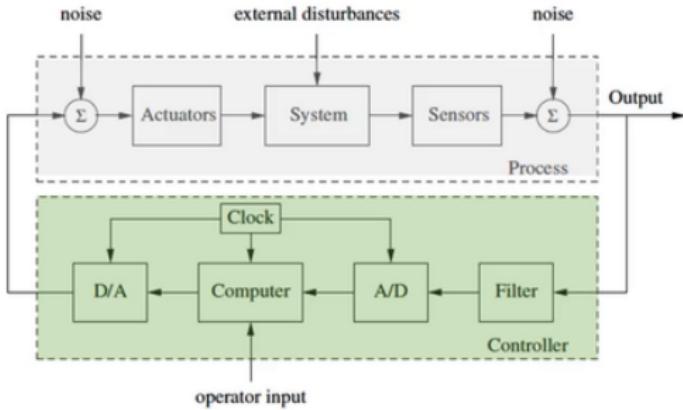
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Our controller consists of the state feedback controller,

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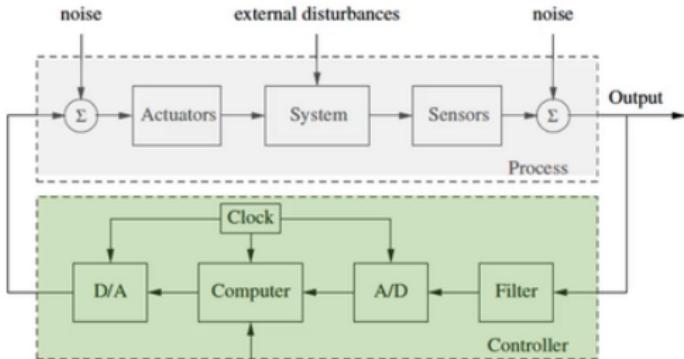
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We need to discretize the controller to be able to implement it in a computer, by approximating the derivative by a difference:

$$\dot{\hat{x}} \approx \frac{\hat{x}(t_{k+1}) - \hat{x}(t_k)}{t_{k+1} - t_k} = A\hat{x}(t_k) + Bu(t_k) + L(y(t_k) - C\hat{x}(t_k)),$$



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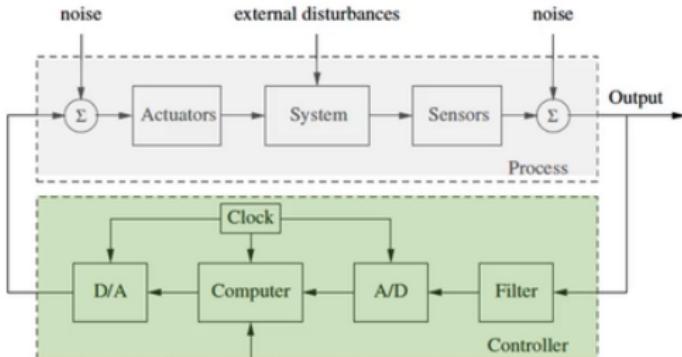
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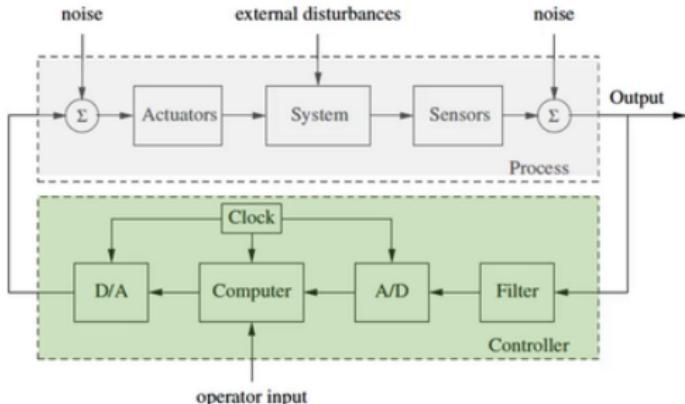
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Rewriting it as a difference equation:

$$\hat{x}(t_{k+1}) = \hat{x}(t_k) + \underbrace{(t_{k+1} - t_k)(A\hat{x}(t_k) + Bu(t_k) + L(y(t_k) - C\hat{x}(t_k)))}_{h \text{- sampling time}}$$



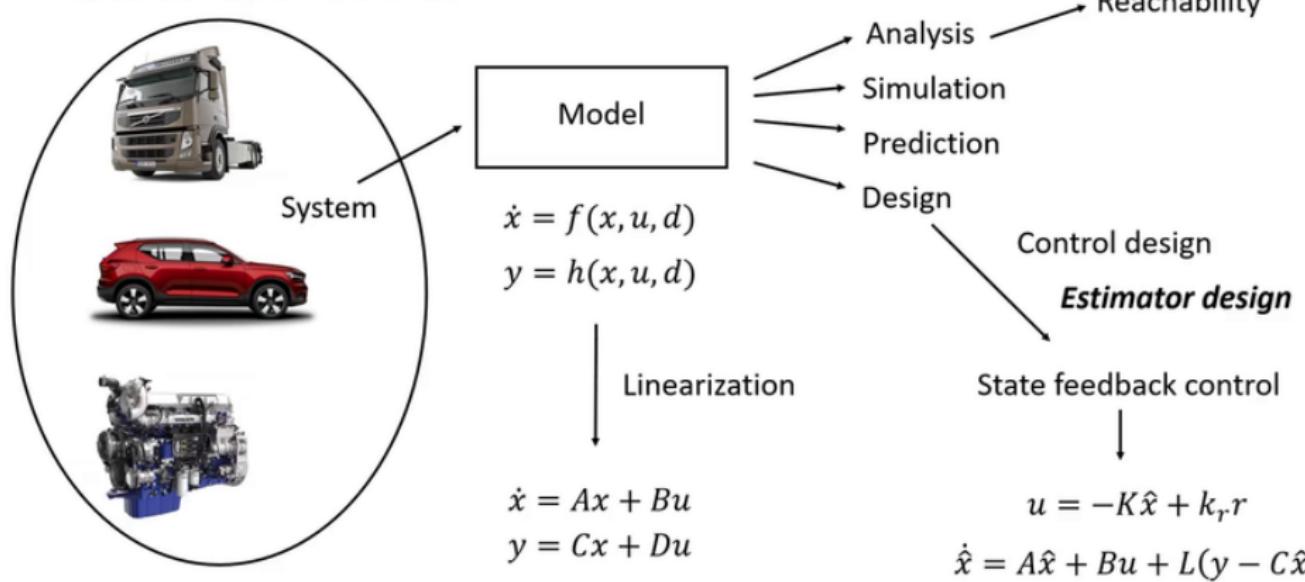
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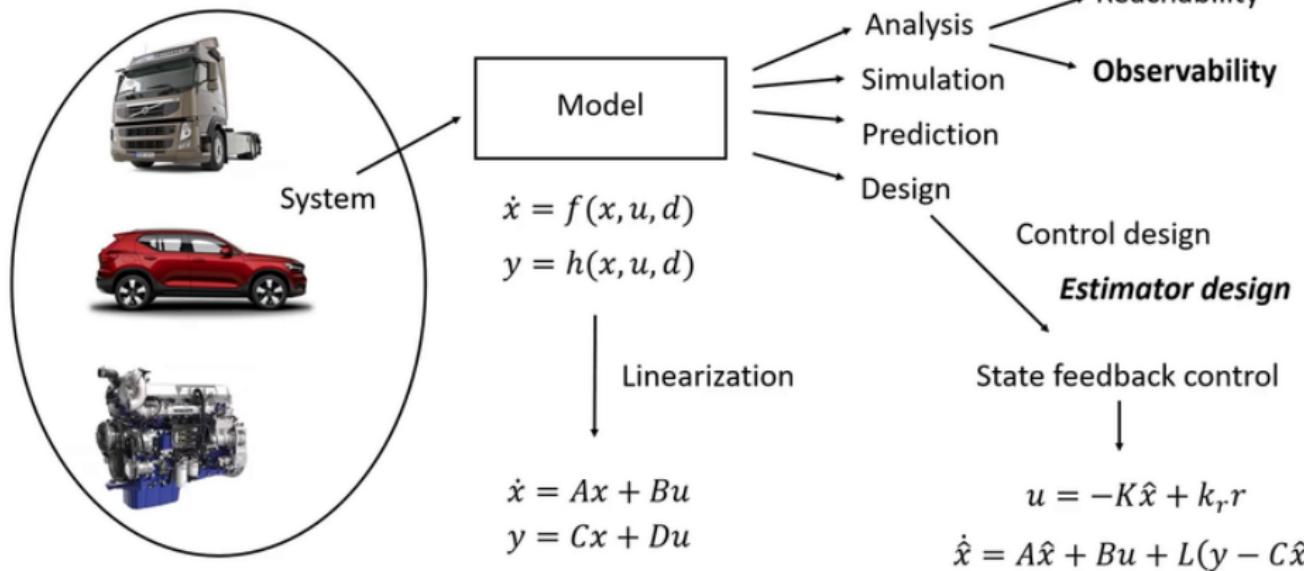
In pseudocode:

```
% Control algorithm - main loop
r = adin(ch1)                                % read reference
y = adin(ch2)                                % get process output
u = -K*xhat + kr*r                            % compute control variable
daout(ch1, u)                                 % set analog output
xhat = xhat + h*(A*x+B*u+L*(y-C*x))        % update state estimate
```

## State estimation



## State estimation



# Observability

**Definition** (Observability): A linear system is ***observable*** if for every  $T > 0$  it is possible to determine the state of the system  $x(T)$  through the measurements of  $y(t)$  and  $u(t)$  on the interval  $[0, T]$ .

Recall the solution to the differential equation:

$$y(t) = Ce^{At}x(0)$$

Since we know  $u(t)$ , we only needs to consider zero input case.

So, if  $x(0)$  can be determined, then we can reconstruct  $x(t)$  exactly.

## Observability

**Definition:** A *state*  $x^* \neq 0$  is said to be *unobservable* if the zero-input solution  $y(t) = Ce^{At}x(0)$ , with  $x(0) = x^*$ , is zero for all  $t \geq 0$ .

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$$\begin{aligned} Ce^{At}x^* \Big|_{t=0} &= 0 \quad \Rightarrow \quad Cx^* = 0 \\ \frac{d}{dt} Ce^{At}x^* \Big|_{t=0} &= 0 \quad \Rightarrow \quad CAe^{At}x^* \Big|_{t=0} = CAx^* = 0 \\ \frac{d^2}{dt^2} Ce^{At}x^* \Big|_{t=0} &= 0 \quad \Rightarrow \quad CA^2e^{At}x^* \Big|_{t=0} = CA^2x^* = 0 \\ &\vdots \\ \frac{d^{n-1}}{dt^{n-1}} Ce^{At}x^* \Big|_{t=0} &= 0 \quad \Rightarrow \quad CA^{n-1}e^{At}x^* \Big|_{t=0} = CA^{n-1}x^* = 0 \end{aligned}$$

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$$\left. \begin{array}{l}
 Ce^{At}x^* \Big|_{t=0} = 0 \Rightarrow Cx^* = 0 \\
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$W_o$

## Observability

**Theorem** (Observability rank condition): A linear system is observable if and only if the observability matrix reachability  $W_o$

$$W_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full row rank.

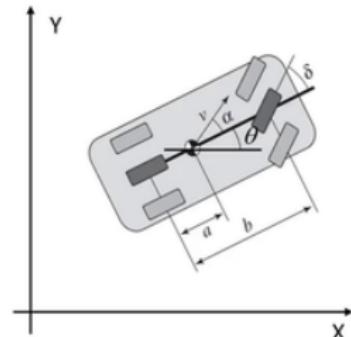
## Example - Vehicle steering (Ex 7.4)

Return to our vehicle steering example, with lateral position as output signal:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} u$$
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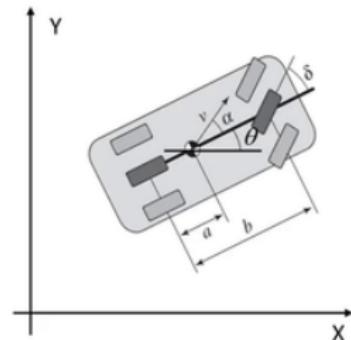
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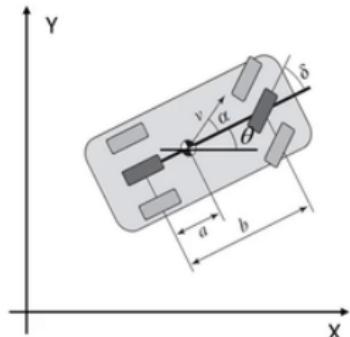
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Note: A square matrix  $M (n \times n)$  has full rank  $n$  iff  $\det(M) \neq 0$

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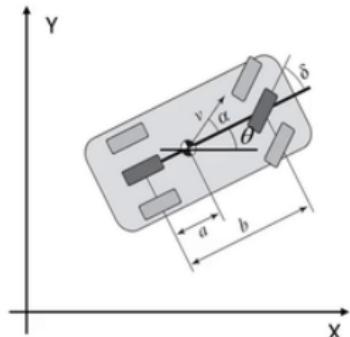
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Compute the determinant:

$$\det(W_o) = \begin{vmatrix} 1 & 0 \\ 0 & 12 \end{vmatrix} = 1 \cdot 12 - 0 \cdot 0 = 12$$

*The system is observable!*



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## Example - Vehicle steering (Ex 7.4)

Return to our vehicle steering example, with heading angle as output signal:

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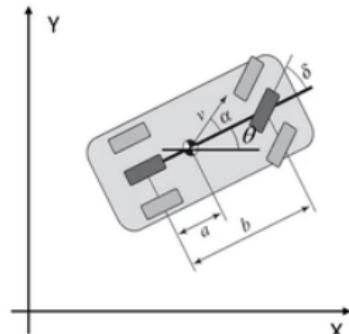
Observability matrix:

$$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Compute the determinant:

$$\det(W_o) = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 \cdot 0 - 1 \cdot 0 = 0$$

*The system is not observable!*



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So, look at our vehicle steering example with heading angle as output:

$$\begin{bmatrix} C \\ CA \end{bmatrix} x^* = 0 \quad \Rightarrow \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x^* = 0 \quad \Rightarrow \quad x^* = \begin{bmatrix} x_1(0) \neq 0 \\ 0 \end{bmatrix}$$

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Consider a linear time-invariant state-space model given by:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

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$$\mathbb{E}(v(s)v^T(t)) = R_v \delta(t - s)$$

$$\mathbb{E}(w(s)w^T(t)) = R_w \delta(t - s)$$

where  $\delta$  is the unit impulse function (dirac function).

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The state estimator (observer) is given as:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

The estimation error  $\tilde{x} = x - \hat{x}$  can be computed as

$$\begin{aligned}\dot{\tilde{x}} &= \dot{x} - \dot{\hat{x}} = Ax + Bu + v - A\hat{x} - Bu - L(Cx + w - C\hat{x}) \\ &= (A - LC)\tilde{x} + v - Lw\end{aligned}$$

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The optimal observer minimizes  $P_{\tilde{x}}$ .

## State estimation

The optimal observer gain, if the system is *observable*, is:

$$L = P_{\tilde{x}} C^T R_w^{-1}$$

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The observer is called the **Kalman-Bucy filter**.

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LQR  $\leftrightarrow$  Kalman

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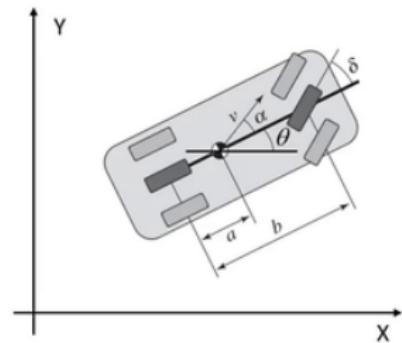
- always stable.
- *the optimal linear filter* for state estimation.
- $R_v$  and  $R_w$  are regarded as the design parameters.

## Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} u$$
$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where  $x_1$  is the lateral position  $Y$ ,  $x_2$  is the heading orientation  $\theta$  and  $u$  is the steering angle  $\delta$ .



Vehicle data:  $v_0 = 12 \text{ m/s}$

$a = 2 \text{ m}$

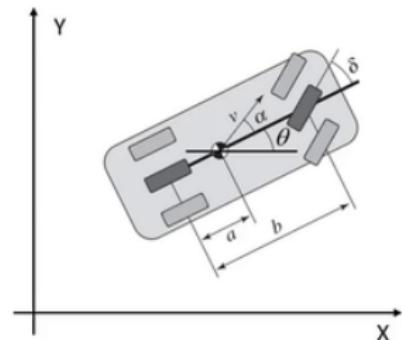
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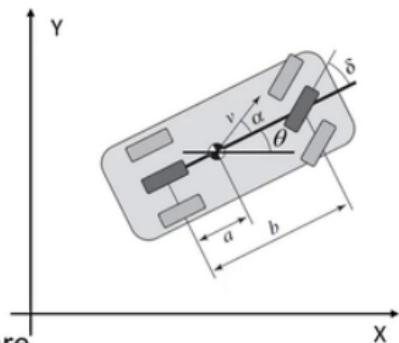
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The process disturbance and the measurement noise are zero mean with covariance

$$R_v = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$$

$$R_w = \rho$$



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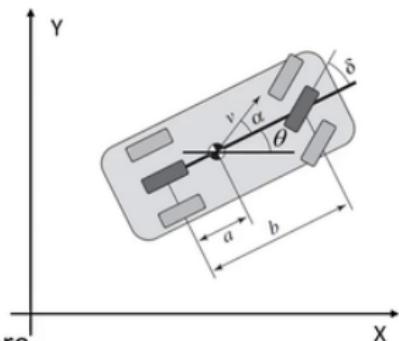
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$$R_v = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} \quad R_w = \rho$$

Design a Kalman filter to estimate the vehicle's states, from measurement of the lateral position.



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State estimator:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

The optimal observer gain, if the system is *observable*, is:

$$L = P_{\tilde{x}}C^TR_w^{-1}$$

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**MATLAB:**

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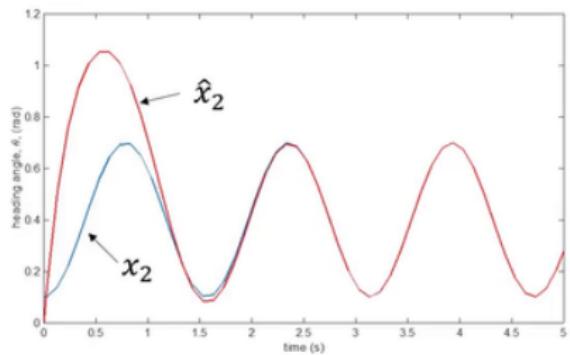
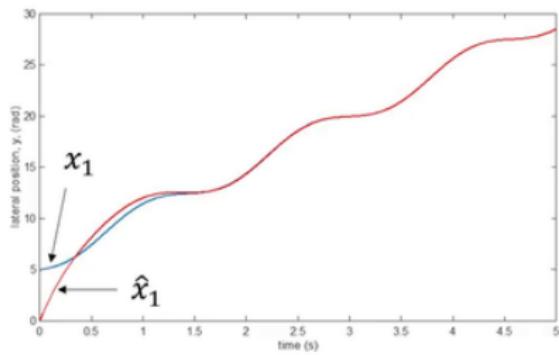
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$$\rho = 1 \implies P_{\tilde{x}} = \begin{bmatrix} 5.0 & 1.0 \\ 1.0 & 0.4167 \end{bmatrix} \implies L = \begin{bmatrix} 5.0 \\ 1.0 \end{bmatrix}$$

**MATLAB:**  
`[P,E,L]=care(A',C',[1 0];[0 1]),1)`

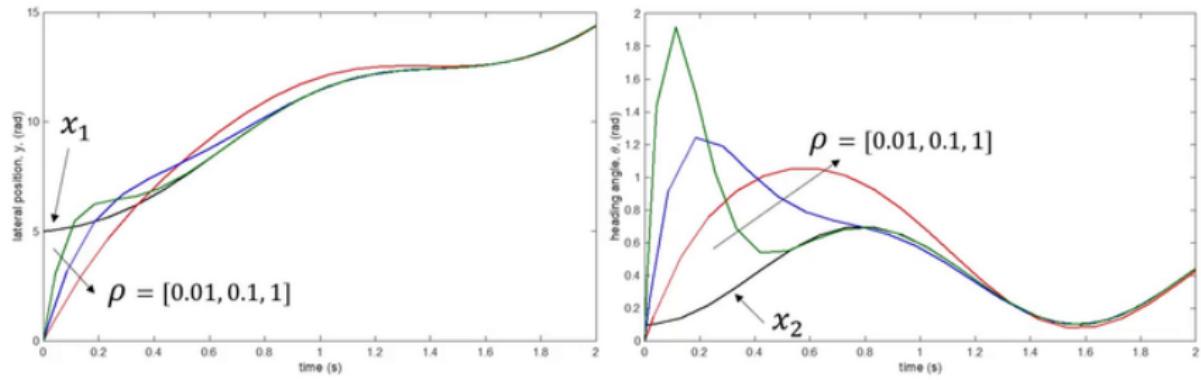
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Simulations using a sinusoidal input, with  $x(0) = (5, 0.1)$  and  $\hat{x}(0) = (0, 0)$ :



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## State estimation – discrete time case

Consider a linear time-invariant state-space model in discrete time given by:

$$\begin{aligned}x[k+1] &= Ax[k] + Bu[k] + v[k] \\y[k] &= Cx[k] + w[k]\end{aligned}$$

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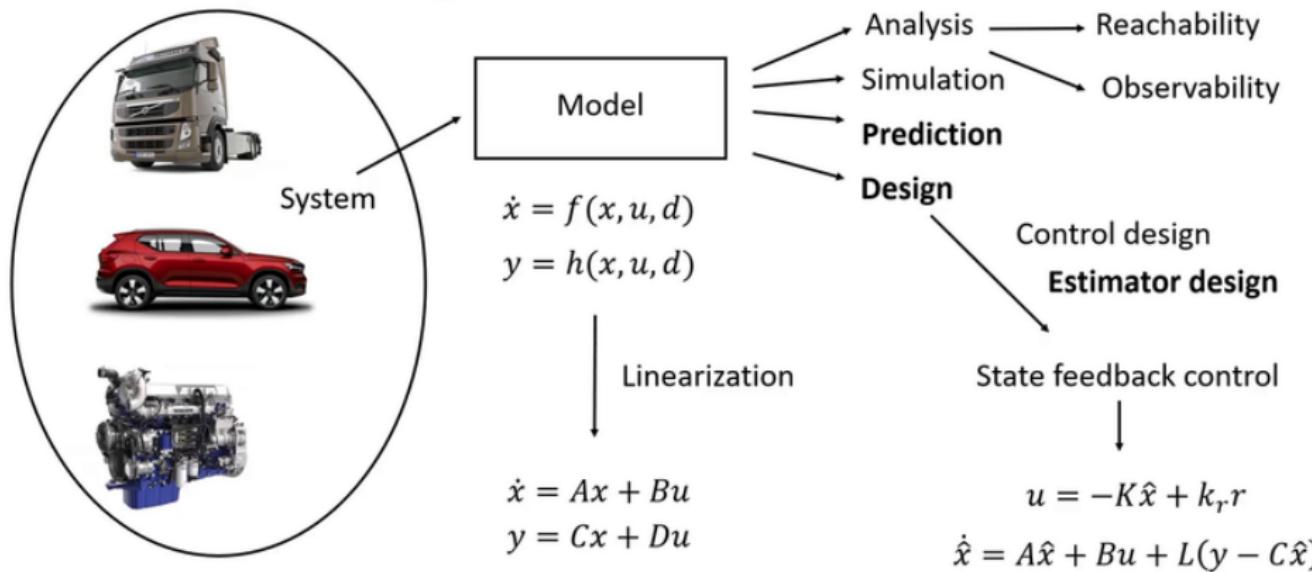
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## Kalman filtering



# Bibliography

- Karl J. Astrom and Richard M. Murray *Feedback Systems*. Version v3.0i. Princeton University Press. September 2018. Chapter 8.