State estimation

Dr. Ing. Rodrigo Gonzalez

rodralez@ingenieria.uncu.edu.ar

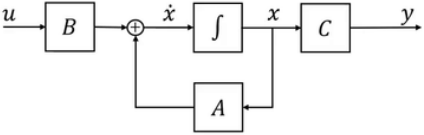
Control y Sistemas

Facultad de Ingeniería,
Universidad Nacional de Cuyo

Summary

State feedback control

Idea with state feedback control design: Modify the eigenvalues of the system by using the input, $u = -Kx + k_r r$.



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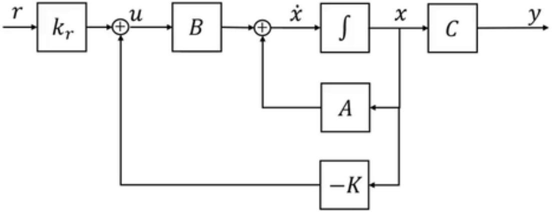
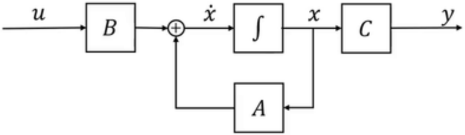


Problem: Requires full access to the state vector, $u = -Kx$

State estimation

Idea of state estimation: Develop an observer of the dynamic system that provides an estimate, $\hat{x}$, of the system's states.



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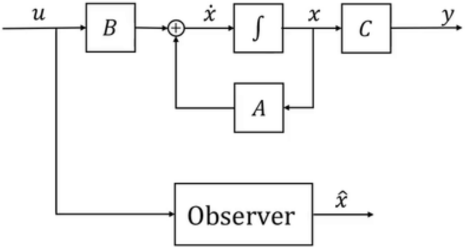

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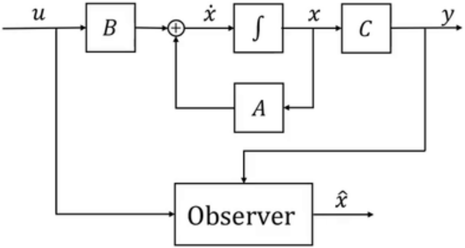
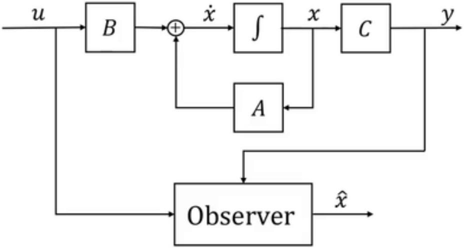


State estimation

Goal of state estimation: $\tilde{x} = x - \hat{x} = 0$



State estimation – Open loop estimator

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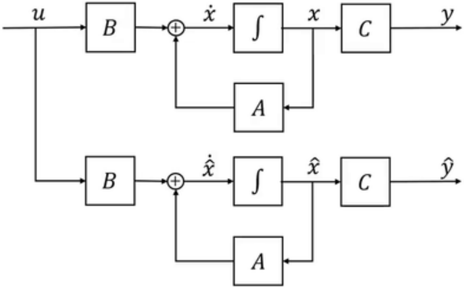
Idea: Use a ***copy*** of the model description in the observer: $\dot{\hat{x}} = A\hat{x} + Bu$



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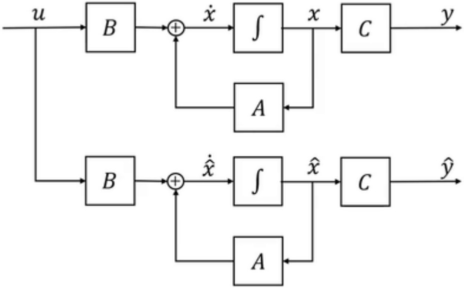

Analyze the error dynamics:

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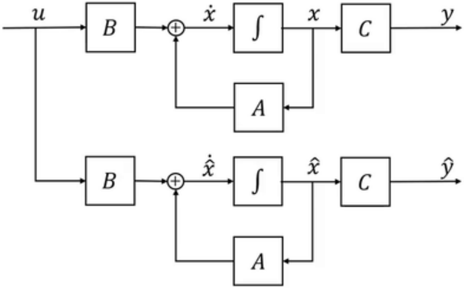

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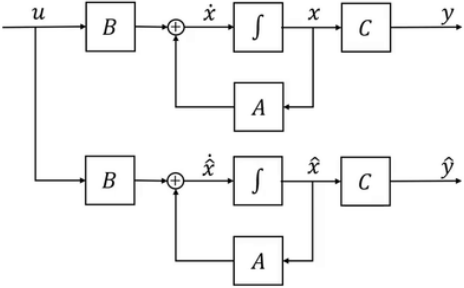

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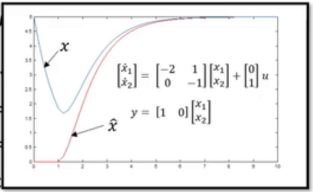

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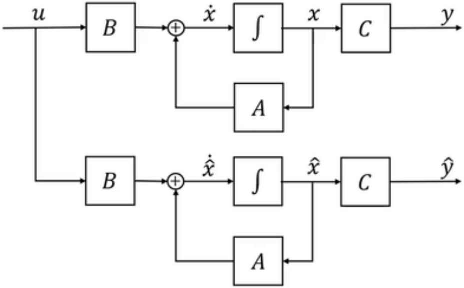

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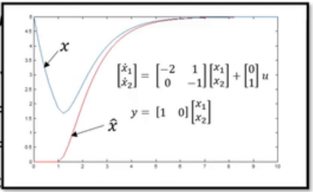

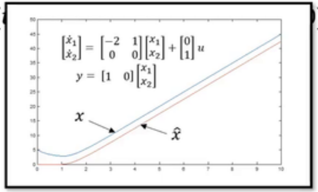

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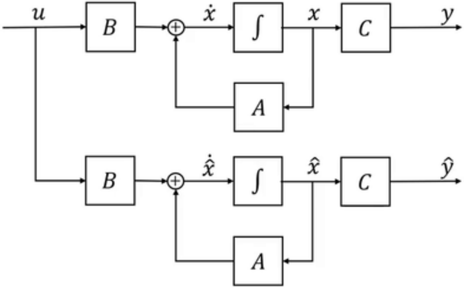

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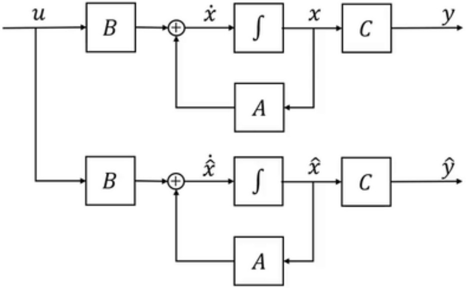

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Open loop estimation does not seem to be a good idea!

State estimation – Closed loop estimator

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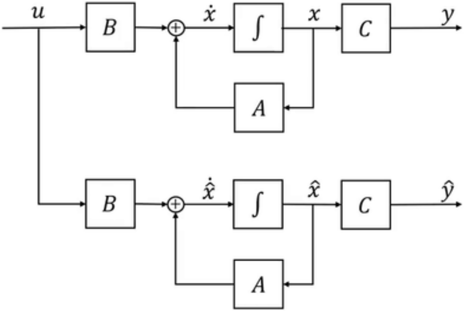
Idea: Use feedback from y in the observer to improve the estimates.



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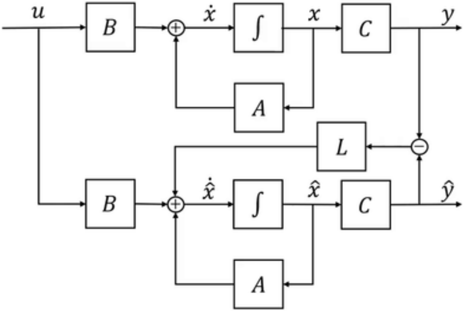

Compare the estimated output with measured:

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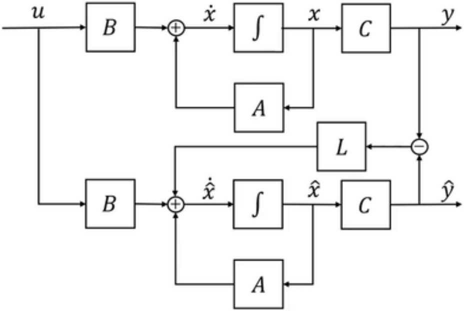

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Feed back the error to the open loop estimator via a feedback gain L:

$$\dot{\hat{x}} = A\hat{x} + Bu + L\tilde{y}$$

$$\hat{y} = C\hat{x}$$

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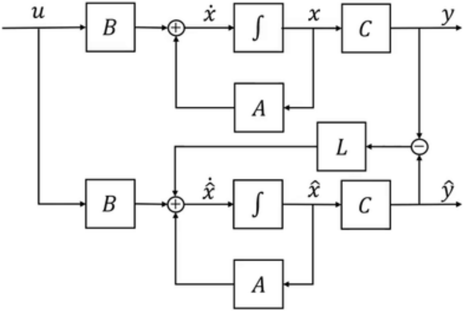

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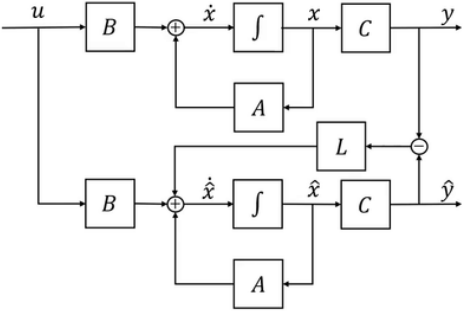

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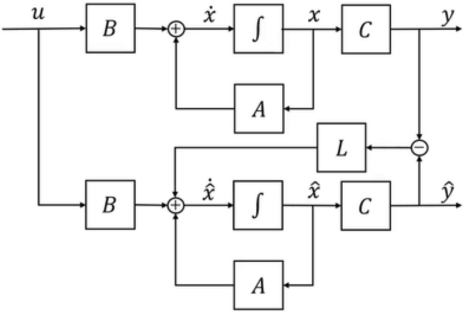

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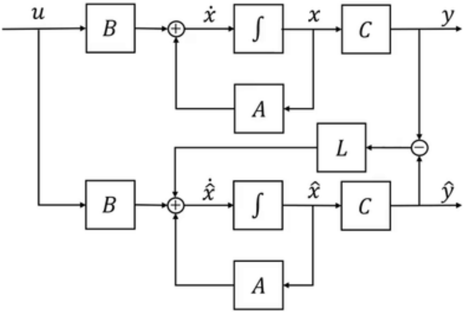

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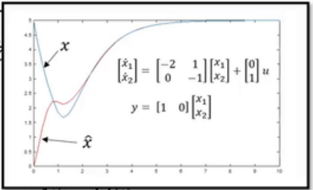

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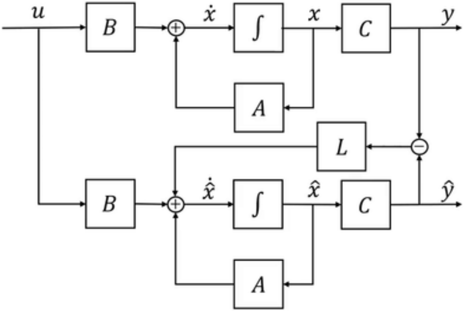

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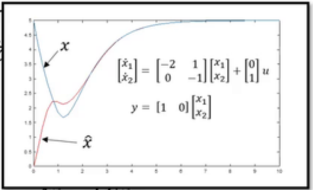

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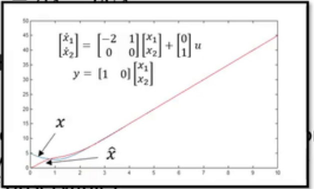

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The way to choose estimator gain is similar to that used for control design. Using the closed loop estimator, the error dynamics becomes:

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Use pole placement with a desired characteristic polynomial to choose the estimator gain, L.

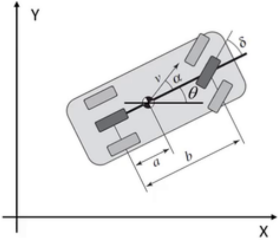
Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 12\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}6\\ 3\end{bmatrix} u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]\, u$$

where x_1 is the lateral position Y , x_2 is the heading orientation θ and u is the steering angle δ.



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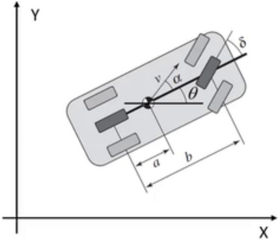

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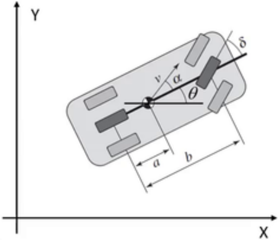

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Specification: Desired characteristic polynomial:

$$p_{des}(\lambda) = (\lambda + 6)(\lambda + 4) = \lambda^2 + 10\lambda + 24$$

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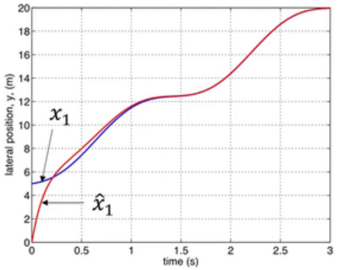
Revisit Example - Vehicle steering (Ex 7.4)

Simulations using a sinusiodal input, with $x(0)$ = (5, 0.1) and $\hat{x}(0)$ = (0, 0):



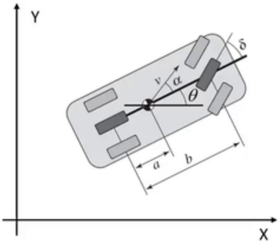
Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 12\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}6\\ 3\end{bmatrix} u$$

$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]\, u$$

where x_1 is the lateral position Y , x_2 is the heading orientation θ and u is the steering angle δ.



Design a state estimator to estimate the vehicle's states, from measurement of the heading angle.

Vehicle data: $v_0 = 12\ m/s$
$a = 2\ m$
$b = 4\ m$

Specification: Desired characteristic polynomial:

$$p_{des}(\lambda) = (\lambda + 6)(\lambda + 4) = \lambda^2 + 10\lambda + 24$$

Revisit Example - Vehicle steering (Ex 7.4)

State estimator:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

The closed loop estimator dynamics becomes

$$\dot{\tilde{x}} = (A - LC)\tilde{x} = \begin{bmatrix} 0 & 12 - l_1 \\ 0 & -l_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

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The closed loop system has the characteristic polynomial

$$\det(\lambda I - A + LC) = \begin{vmatrix} \lambda & -12 + l_1 \\ 0 & \lambda + l_2 \end{vmatrix} = \lambda(\lambda + l_2)$$

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Matching with desired characteristic polynomial gives:

$$\lambda^2 + 10\lambda + 24 \equiv \lambda^2 + l_2\lambda$$

It is not possible shape the error dynamics. We say that the system is not ***observable***.

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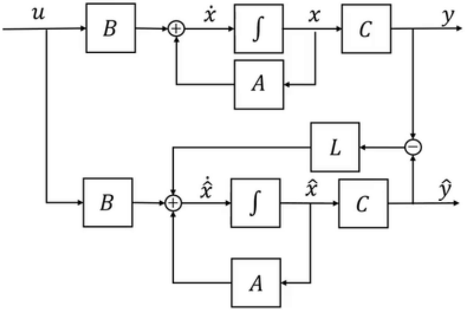
Control using estimated states

Use the estimated states for feedback, $u = -K\hat{x} + k_r r$.



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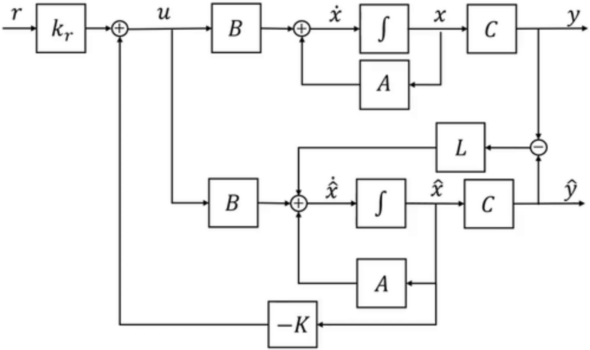

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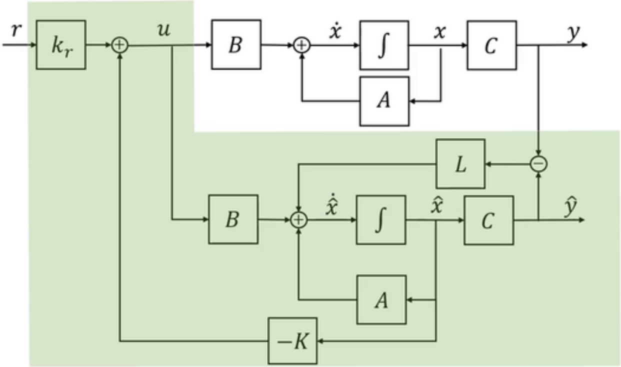

Control using estimated states

Given the system, $\dot{x} = Ax + Bu$, $y = Cx$, the controller, $u = -K\hat{x} + k_r r$, and the state estimator, $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$, the closed loop system can be written as:

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - CL \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} Bk_r \\ 0 \end{bmatrix} r$$

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Rule of thumb: Make the estimator poles 4-5 times faster then the "feedback" poles.

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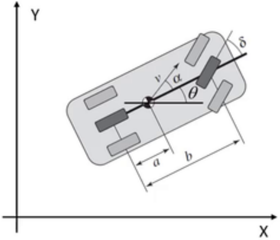

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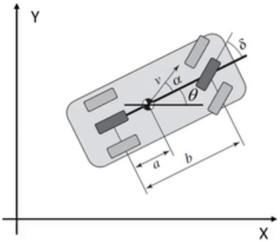

State feedback control (poles in -1 (double pole)):

$$u = -0.0278x_1 - 0.6111x_2 + 0.0278r$$

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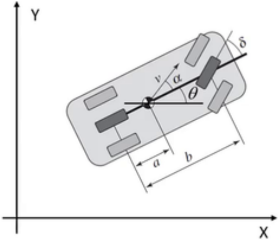

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State estimator (poles in -4 and -6):

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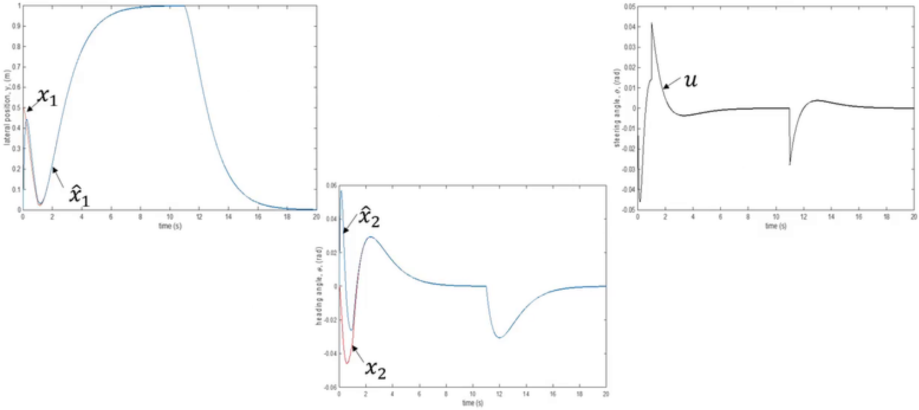
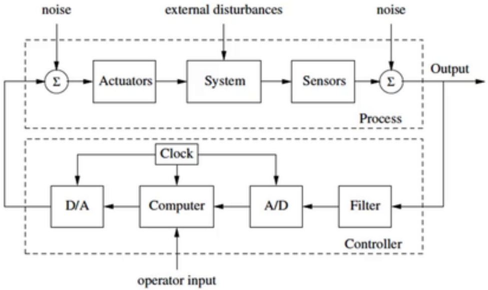
Revisit Example - Vehicle steering (Ex 7.4)



Implementation



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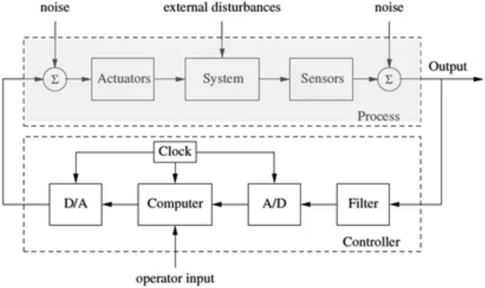

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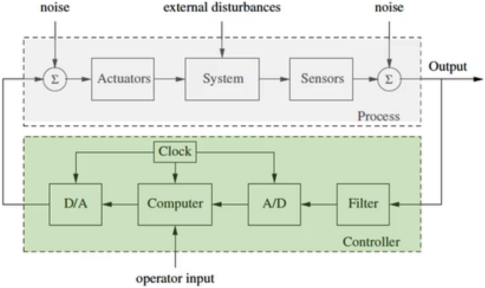

Implementation

Our controller consists of the state feedback controller,

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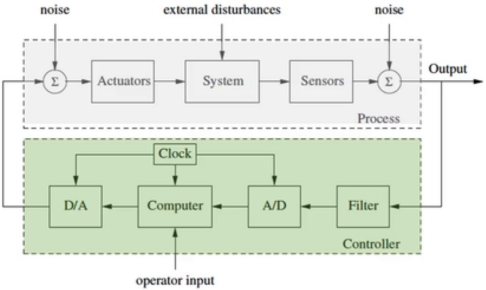

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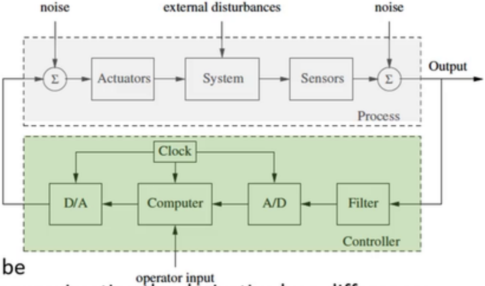

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We need to discretize the controller to be able to implement it in a computer, by approximating the derivative by a difference:

$$\dot{\hat{x}} \approx \frac{\hat{x}(t_{k+1}) - \hat{x}(t_k)}{t_{k+1} - t_k} = A\hat{x}(t_k) + Bu(t_k) + L(y(t_k) - C\hat{x}(t_k)),$$

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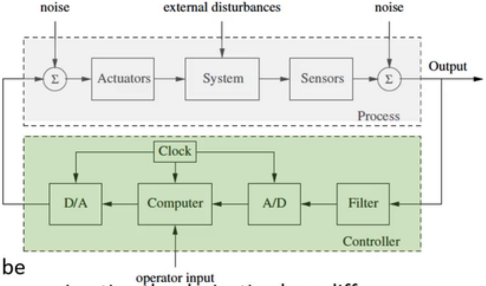

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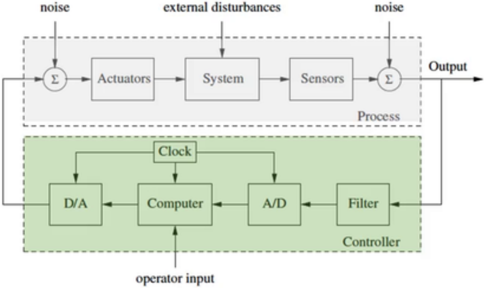
Rewriting it as a difference equation:

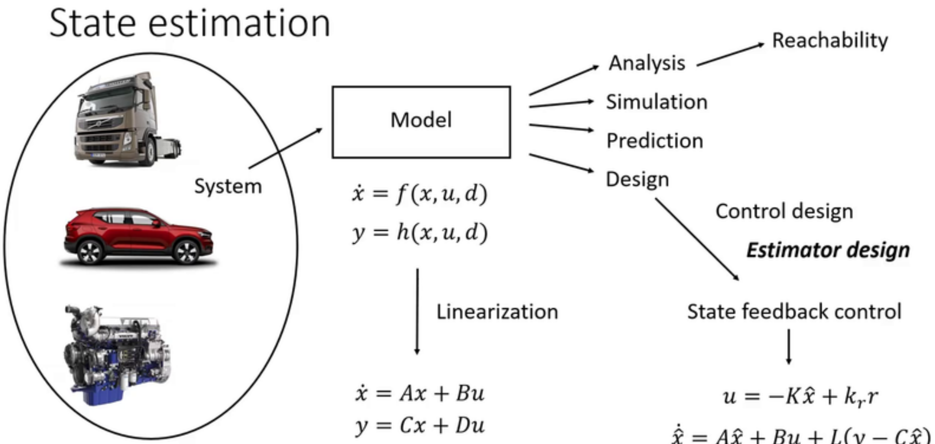
$$\hat{x}(t_{k+1}) = \hat{x}(t_k) + \underbrace{(t_{k+1} - t_k)}_{h\text{ - sampling time}}(A\hat{x}(t_k) + Bu(t_k) + L(y(t_k) - C\hat{x}(t_k))),$$

Implementation



In pseudocode:

```
% Control algorithm - main loop
r = adin(ch1)                          % read reference
y = adin(ch2)                          % get process output
u = -K*xhat + kr*r                     % compute control variable
daout(ch1, u)                          % set analog output
xhat = xhat + h*(A*x+B*u+L*(y-C*x))    % update state estimate
```


State estimation
System
Model
$\dot{x} = f(x, u, d)$
$y = h(x, u, d)$
Analysis
Reachability
Simulation
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Design
Control design
Estimator design
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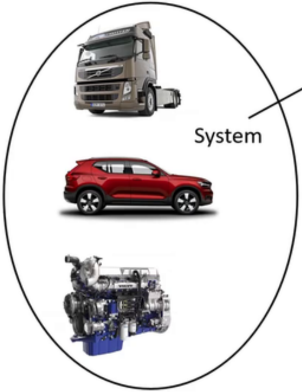
State estimation



Model

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Linearization

$$\dot{x} = Ax + Bu$$
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- Analysis
  - Reachability
  - **Observability**
- Simulation
- Prediction
- Design
  - Control design
  - ***Estimator design***

State feedback control

$$u = -K\hat{x} + k_r r$$
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Observability

Definition (Observability): A linear system is ***observable*** if for every $T > 0$ it is possible to determine the state of the system $x(T)$ through the measurements of $y(t)$ and $u(t)$ on the interval $[0, T]$.

Recall the solution to the differential equation:

$$y(t) = Ce^{At}x(0)$$

Since we know $u(t)$, we only needs to consider zero input case.

So, if $x(0)$ can be determined, then we can reconstruct $x(t)$ exactly.

Observability

Definition: A ***state*** $x^* \neq 0$ is said to be ***unobservable*** if the zero-input solution $y(t) = Ce^{At}x(0)$, with $x(0) = x^*$, is zero for all $t \geq 0$.

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So, if we can find a state x^* for which $Ce^{At}x(0) = 0$ for all $t \geq 0$, then the system is unobservable. For this to hold, all derivatives must be zero at $t = 0$.

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$$
\begin{aligned}
Ce^{At}x^*\Big|_{t=0} &= 0 \quad \Longrightarrow \quad Cx^*=0 \\
\frac{d}{dt}Ce^{At}x^*\Big|_{t=0} &= 0 \quad \Longrightarrow \quad CAe^{At}x^*\Big|_{t=0} = CAx^*=0 \\
\frac{d^2}{dt^2}Ce^{At}x^*\Big|_{t=0} &= 0 \quad \Longrightarrow \quad CA^2e^{At}x^*\Big|_{t=0} = CA^2x^*=0 \\
&\ \ \vdots \\
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$$\left.\begin{array}{lll}
\left. Ce^{At}x^* \right|_{t=0} = 0 & \Rightarrow & Cx^* = 0 \\
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\left. \frac{d^{n-1}}{dt^{n-1}} Ce^{At}x^* \right|_{t=0} = 0 & \Rightarrow & \left. CA^{n-1}e^{At}x^* \right|_{t=0} = CA^{n-1}x^* = 0
\end{array}\right\} \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{W_o} x^* = 0$$

Observability

Theorem (Observability rank condition): *A linear system is observable if and only if the observability matrix reachability* W_o

$$W_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full row rank.

Observability

Example - Vehicle steering (Ex 7.4)

Return to our vehicle steering example, with lateral position as output signal:

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Observability matrix:

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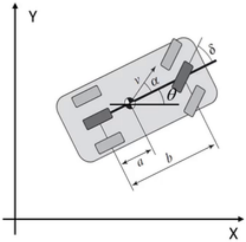

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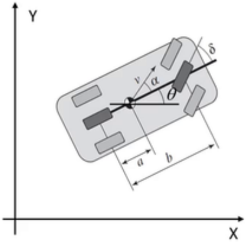

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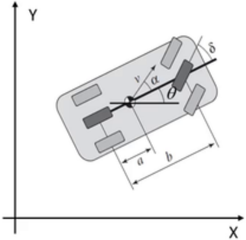

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Note: A square matrix M ($n \times n$) has full rank n iff the det(M)≠0

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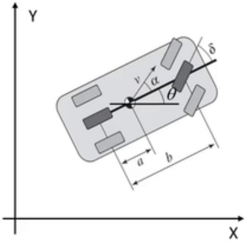

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Compute the determinant:

$$\det(W_o) = \begin{vmatrix}1 & 0\\ 0 & 12\end{vmatrix} = 1\cdot 12 - 0\cdot 0 = 12$$

The system is observable!

Observability

Example - Vehicle steering (Ex 7.4)

Return to our vehicle steering example, with heading angle as output signal:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\, u$$

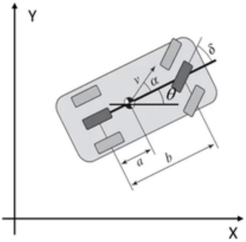
Observability matrix:

$$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Compute the determinant:

$$\det(W_o) = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 \cdot 0 - 1 \cdot 0 = 0$$

The system is not observable!



Observability

Definition: A ***state*** $x^* \neq 0$ is said to be ***unobservable*** if the zero-input solution $y(t) = Ce^{At}x(0)$, with $x(0) = x^*$, is zero for all $t \geq 0$.

So, look at our vehicle steering example with heading angle as output:

$$\begin{bmatrix} C \\ CA \end{bmatrix} x^* = 0 \quad \Rightarrow \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x^* = 0 \quad \Rightarrow \quad x^* = \begin{bmatrix} x_1(0) \neq 0 \\ 0 \end{bmatrix}$$

State estimation

Consider a linear time-invariant state-space model given by:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

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where x is the state (vector), u is the input or control signal and y is the output signal, v is the process disturbance and w is measurement noise. The disturbance v and the noise w are zero mean and Gaussian.

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$$\mathbb{E}(v(s)v^T(t)) = R_v\delta(t-s)$$

$$\mathbb{E}(w(s)w^T(t)) = R_w\delta(t-s)$$

where δ is the unit impulse function (dirac function).

State estimation

The state estimator (observer) is given as:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

The estimation error $\tilde{x} = x - \hat{x}$ can be computed as

$$\begin{aligned}\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} &= Ax + Bu + v - A\hat{x} - Bu - L(Cx + w - C\hat{x}) \\ &= (A - LC)\tilde{x} + v - Lw\end{aligned}$$

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The optimal observer minimizes $P_{\tilde{x}}$.

State estimation

The optimal observer gain, if the system is *observable*, is:

$$L = P_{\tilde{x}} C^T R_w^{-1}$$

where $P_{\tilde{x}} = P_{\tilde{x}}^T \geq 0$ is the solution to the Riccati equation:

$$0 = AP_{\tilde{x}} + P_{\tilde{x}} A^T + R_v - P_{\tilde{x}} C^T R_w^{-1} C P_{\tilde{x}}$$

The observer is called the ***Kalman-Bucy filter.***

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Similarities with LQR:

LQR ⟷ Kalman

$A \leftrightarrow A^T$ $\quad B \leftrightarrow C^T$

$S \leftrightarrow P$ $\quad K \leftrightarrow L^T$

$Q_x \leftrightarrow R_v$ $\quad Q_u \leftrightarrow R_w$

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- always stable.

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- always stable.
- *the optimal linear filter* for state estimation.
- R_v and R_w are regarded as the design parameters.

Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 12\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}6\\ 3\end{bmatrix}u$$

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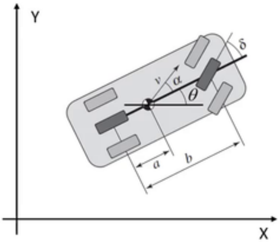
where x_1 is the lateral position Y , x_2 is the heading orientation θ and u is the steering angle δ.



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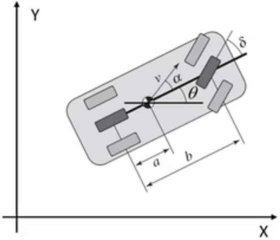

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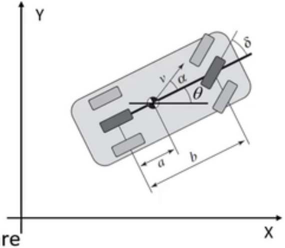

The process disturbance and the measurement noise are zero mean with covariance

$$R_v = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad R_w = \rho$$

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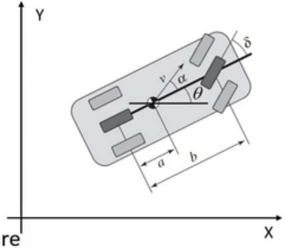

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Design a Kalman filter to estimate the vehicle's states, from measurement of the lateral position.

Revisit Example - Vehicle steering (Ex 7.4)

State estimator:

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The optimal observer gain, if the system is *observable*, is:

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MATLAB:

```
[P,E,L]=care(A',C',[[1 0];[0 1]],1)
```

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$$\rho = 1 \quad \Longrightarrow \quad P_{\tilde{x}} = \begin{bmatrix} 5.0 & 1.0 \\ 1.0 & 0.4167 \end{bmatrix} \quad \Longrightarrow \quad L = \begin{bmatrix} 5.0 \\ 1.0 \end{bmatrix}$$

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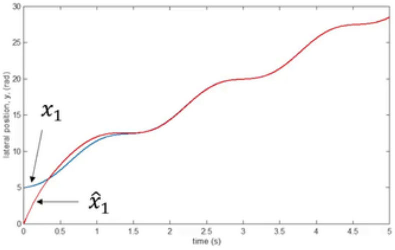
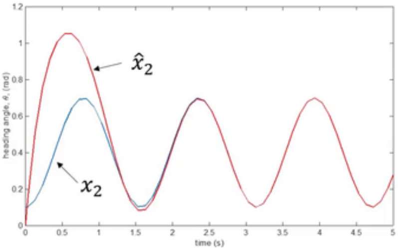
Simulations using a sinusiodal input, with $x(0)$ = (5, 0.1) and $\hat{x}(0)$ = (0, 0):





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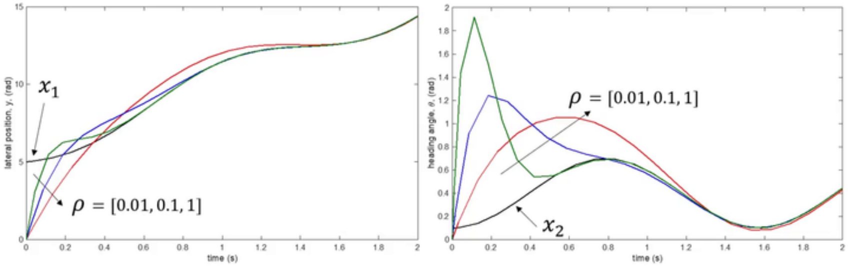

State estimation – discrete time case

Consider a linear time-invariant state-space model in discrete time given by:

$$x[k+1] = Ax[k] + Bu[k] + v[k]$$
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where x is the state (vector), u is the input or control signal and y is the output signal, v is the process disturbance and w is measurement noise. The disturbance v and the noise w are zero mean and Gaussian.

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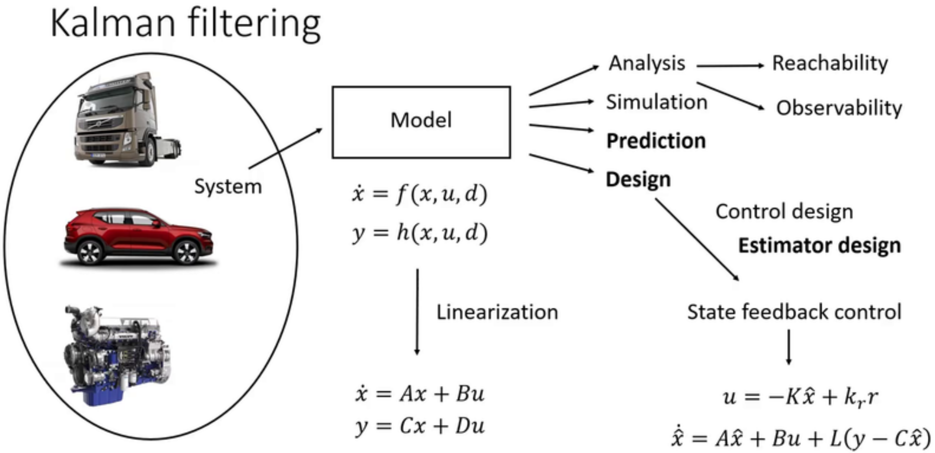
The state estimator (observer) is given as:

$$\hat{x}[k+1] = A\hat{x}[k] + Bu[k] + L[k](y[k] - C\hat{x}[k])$$

and the estimation error $\tilde{x}[k] = x[k] - \hat{x}[k]$ can be computed as:

$$\tilde{x}[k+1] = (A - L[k]C)\tilde{x}[k] + v[k] - Lw[k]$$

Kalman filtering



Bibliography

- Karl J. Astrom and Richard M. Murray *Feedback Systems*. Version v3.0i. Princeton University Press. September 2018. Chapter 8.