

Space feedback control

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Control y Sistemas

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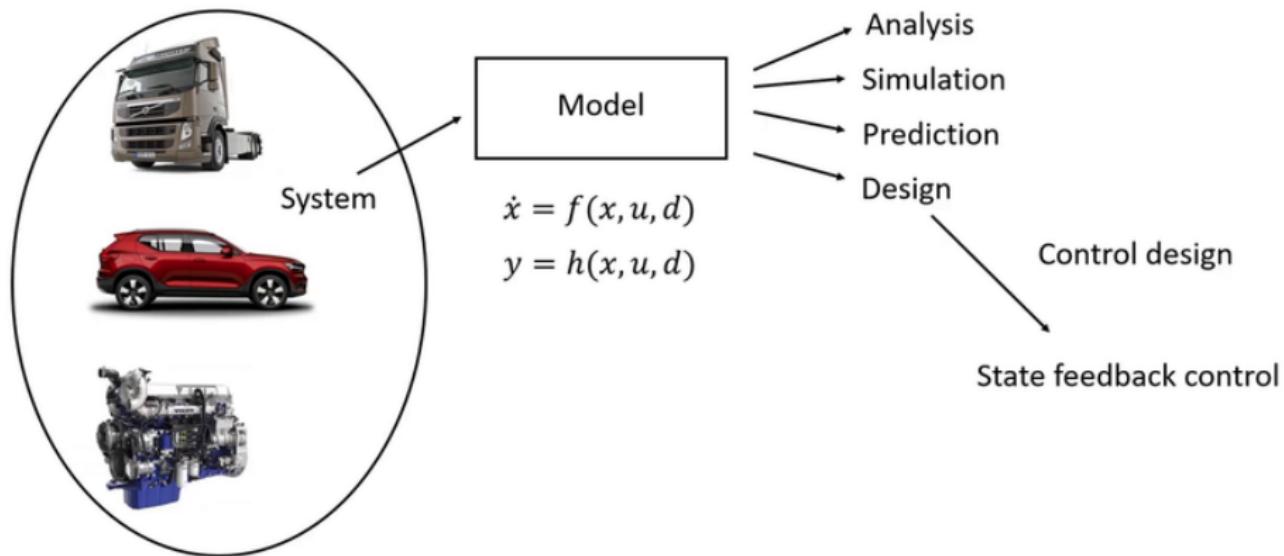
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State feedback control



State feedback control

Consider a linear time-invariant state-space model given by:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

where $x(t) \in \mathbb{R}^n$ is the state (vector), $u(t) \in \mathbb{R}^p$ is the input or control signal and $y(t) \in \mathbb{R}^q$ is the output signal. (For SISO case, $p = 1, q = 1$)

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The system poles are given by the eigenvalues of the system matrix $A \in \mathbb{R}^{n \times n}$.

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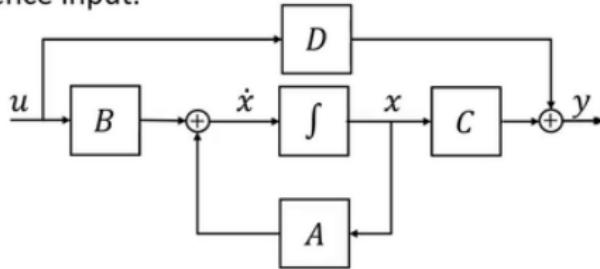
where $K \in \mathbb{R}^{p \times n}$ is the feedback gain, $k_r \in \mathbb{R}^{p \times r}$ is the steady-state reference gain and $r(t) \in \mathbb{R}^r$ is the reference input.

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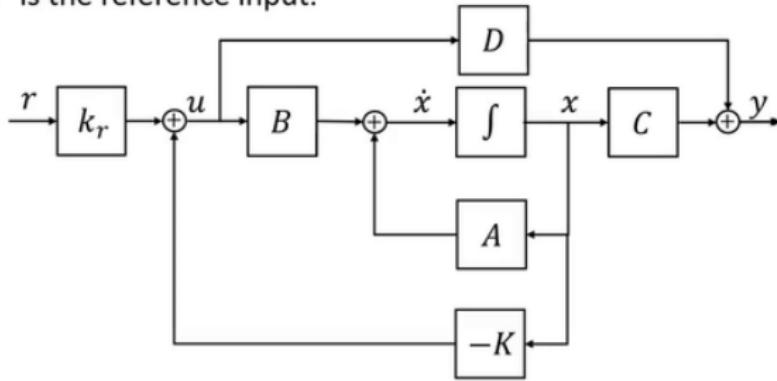


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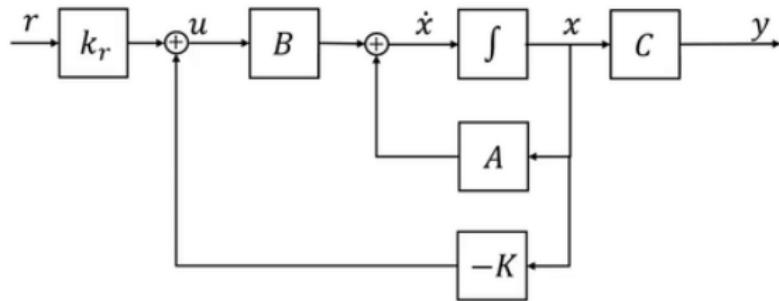


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SISO case: n parameters in K and n eigenvalues in A , so it might be possible!

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$$k_r = -(C(A - BK)^{-1}B)^{-1} \quad \text{or} \quad k_r = -1/C(A - BK)^{-1}B$$

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Given the new state-space model, we design a controller in the usual fashion and the resulting controller becomes:

$$u(t) = -Kx(t) - K_I z(t) + k_r r(t)$$

Example 1

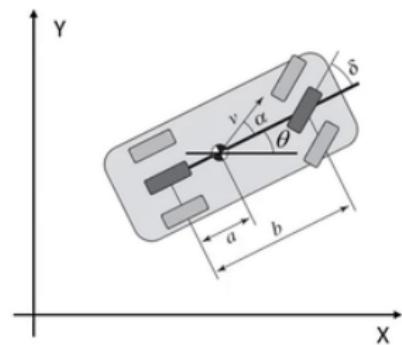
Example 1 - Vehicle steering (Ex 7.4)

Consider the following system:

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where x_1 is the lateral position Y , x_2 is the heading orientation θ and u is the steering angle δ .



Vehicle data: $v_0 = 12 \text{ m/s}$

$a = 2 \text{ m}$

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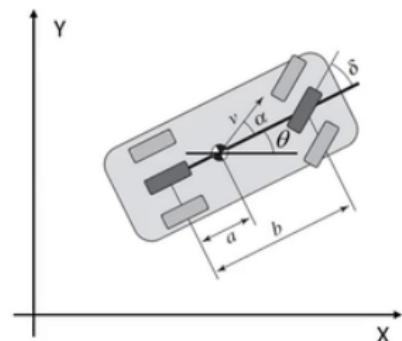
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The idea is to design a controller that ***stabilizes*** the dynamics and ***tracks*** a given lateral position of the vehicle.



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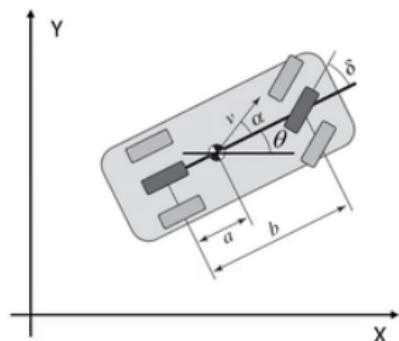
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Specification: Desired characteristic polynomial:

$$p_{des}(\lambda) = \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2$$



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The closed loop system has the characteristic polynomial

$$\det(\lambda I - A + BK) = \dots = \lambda^2 + \frac{v_0}{b}(ak_1 + k_2)\lambda + \frac{k_1 v_0^2}{b}$$

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Matching with desired characteristic polynomial gives:

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 \equiv \lambda^2 + \frac{v_0}{b}(ak_1 + k_2)\lambda + \frac{k_1 v_0^2}{b}$$

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The steady-state gain can be determined:

$$k_r = -1/C(A - BK)^{-1}B = \dots = k_1 = \frac{b\omega_n^2}{v_0^2}$$

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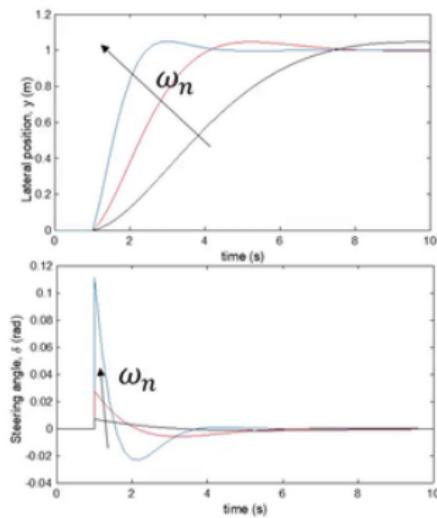
Inserting these control design parameters into the feedback controller gives:

$$u = -k_1x_1 - k_2x_2 + k_r r = -\frac{b\omega_n^2}{v_0^2}x_1 - \left(\frac{2\zeta\omega_n b}{v_0} - \frac{ab\omega_n^2}{v_0^2}\right)x_2 + \frac{b\omega_n^2}{v_0^2}r$$

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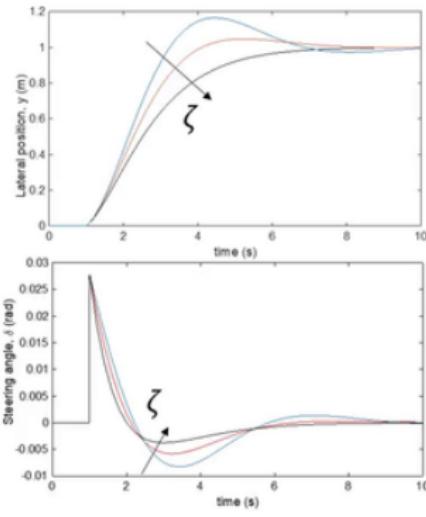
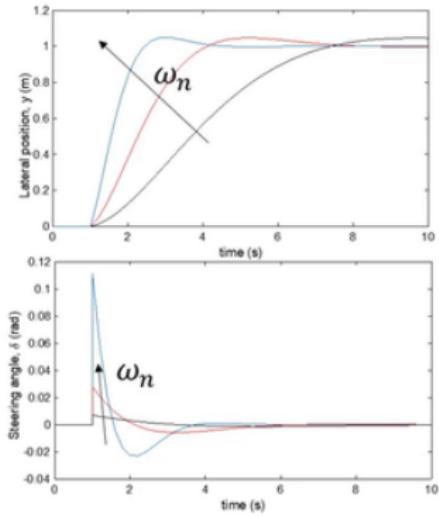
Simulations with different values of ζ and ω_n :



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Simulations with different values of ζ and ω_n :



Example 2

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Consider the following system (similar to Example 1 with one difference):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
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Again, the idea is to design a controller that **stabilizes** the dynamics and **tracks** a given reference signal.

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The closed loop system dynamics becomes

$$\begin{aligned}\dot{x} &= (A - BK)x + Bk_r r = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} k_r \\ 0 \end{bmatrix} r \\ y &= Cx + Du = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\end{aligned}$$

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The closed loop system has the characteristic polynomial

$$\det(\lambda I - A + BK) = \det \begin{pmatrix} \lambda + k_1 & k_2 - 1 \\ 0 & \lambda \end{pmatrix} = \lambda^2 + k_1\lambda = \lambda(\lambda + k_1)$$

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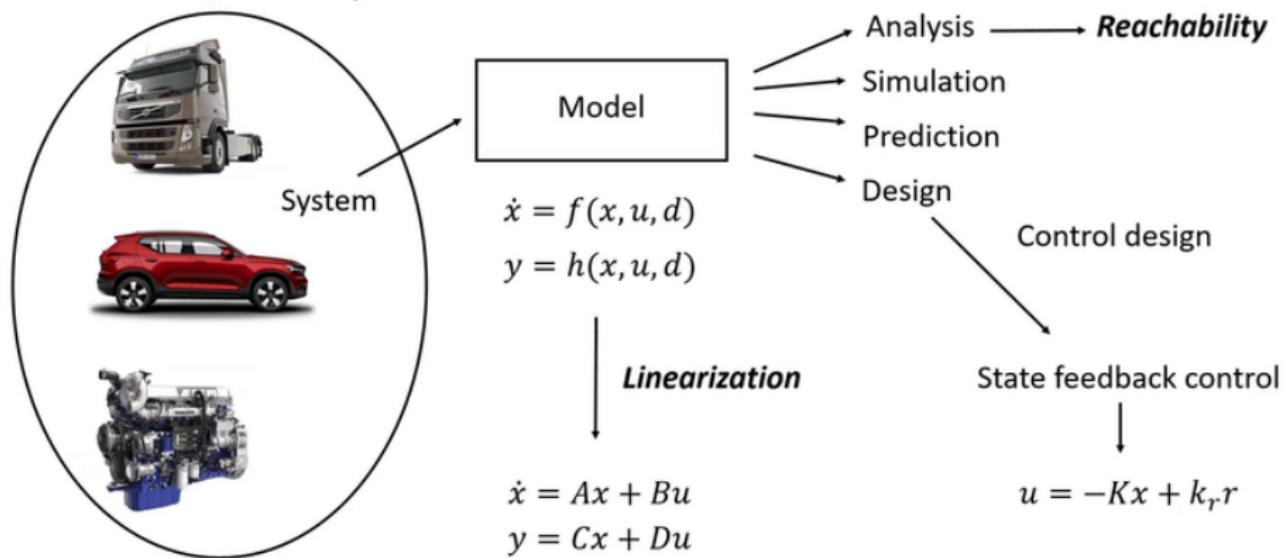
While matching with the desired characteristic polynomial

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 \equiv \lambda^2 + k_1\lambda$$

we notice that it is not possible to make them identical to each other.

It is not possible to control all the eigenvalues (shape the dynamics). We say that the system is not **controllable** or not **reachable**. Furthermore, one eigenvalue is always 0, which means that the closed loop system is unstable.

Reachability



Revisit - Example

Consider the following system:

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$$\dot{x} = (A - BK)x + Bk_rr = \begin{bmatrix} -k_1 & 1 - k_2 \\ 0 & 0 \end{bmatrix}x + \begin{bmatrix} k_r \\ 0 \end{bmatrix}r$$

and the characteristic polynomial is given as

$$\det(\lambda I - A + BK) = \det\begin{pmatrix} \lambda + k_1 & k_2 - 1 \\ 0 & \lambda \end{pmatrix} = \lambda^2 + k_1\lambda = \lambda(\lambda + k_1)$$

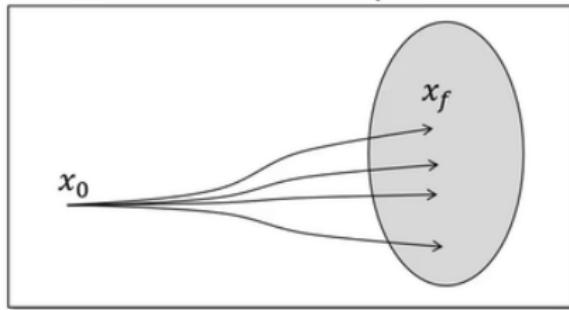
Uncontrollable
Controllable

Reachability

Definition (Reachability): A linear system is **reachable** if for any $x_0, x_f \in \mathbb{R}^n$ there exists a $T > 0$ and $u: [0, T] \rightarrow \mathbb{R}$ such that if $x(0) = x_0$ then the corresponding solution satisfies $x(T) = x_f$.

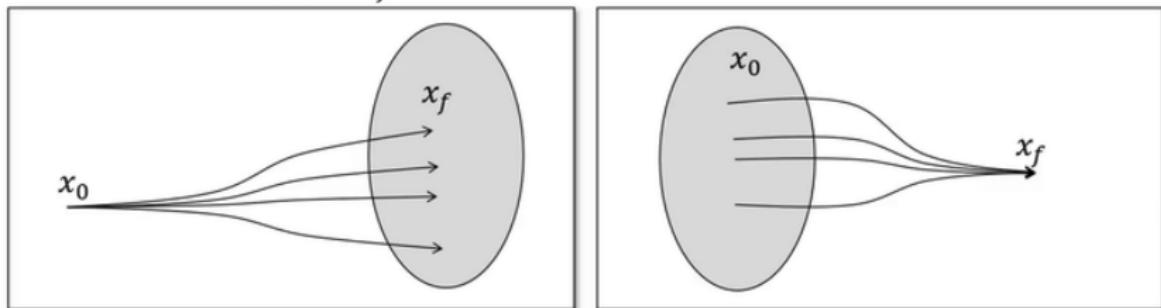
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Sometimes the definition of **controllable** and **controllability** is used, and that is similar.

Linear system representations

$$\begin{array}{ccc}
 a\left(\frac{d}{dt}\right)y(t) = b\left(\frac{d}{dt}\right)u(t) & \xrightarrow{\mathcal{L}} & a(s)Y(s) = b(s)U(s) \\
 & & \downarrow G(s) = \frac{b(s)}{a(s)} \\
 y(t) = \int_0^t g(t-\tau) u(\tau) d\tau & \xleftarrow{\mathcal{L}^{-1}} & Y(s) = G(s)U(s) \\
 & \uparrow g(t) = Ce^{At}B + D\delta(t) & \uparrow G(s) = C(sI - A)^{-1}B + D \\
 \dot{x}(t) = Ax(t) + Bu(t) & \xrightarrow{\mathcal{L}} & sX(s) - x(0) = AX(s) + BU(s) \\
 y(t) = Cx(t) + Du(t) & & \quad \quad \quad Y(s) = CX(s) + DU(s)
 \end{array}$$

Reachability

To see that an arbitrary point can be reached, we can use the convolution equation.

Assume that the system starts from zero, the state of a linear system is given by:

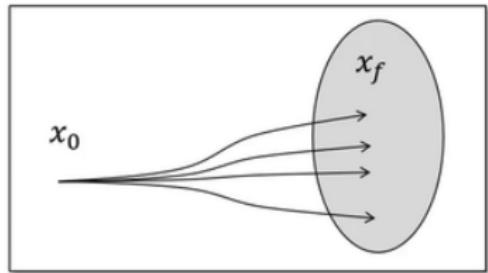
$$x(t) = \int_0^t e^{A(t-\tau)} B u(\tau) d\tau = \int_0^t e^{A\tau} B u(t-\tau) d\tau$$

From linear theory it can be shown that

$$e^{A\tau} = I\alpha_0(\tau) + A\alpha_1(\tau) + \cdots + A^{n-1}\alpha_{n-1}(\tau)$$

where $\alpha_i(t)$ are scalar functions, so that

$$\begin{aligned} x(t) = & B \int_0^t \alpha_0(\tau) u(t-\tau) d\tau + AB \int_0^t \alpha_1(\tau) u(t-\tau) d\tau + \cdots \\ & + A^{n-1}B \int_0^t \alpha_{n-1}(\tau) u(t-\tau) d\tau \end{aligned}$$



Reachability

By writing it in "vector form", we get:

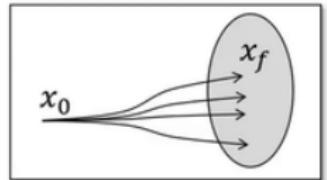
$$x(t) = [B \quad AB \quad \dots \quad A^{n-1}B] \underbrace{\left[\begin{array}{c} \int_0^t \alpha_0(\tau) u(t-\tau) d\tau \\ \int_0^t \alpha_1(\tau) u(t-\tau) d\tau \\ \vdots \\ \int_0^t \alpha_{n-1}(\tau) u(t-\tau) d\tau \end{array} \right]}$$

W_r

Reachability

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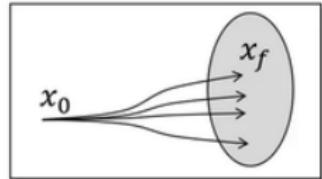


To reach an arbitrary point in state-space, we require that W_r is nonsingular. The matrix W_r is called the **reachability matrix**.

Reachability

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To reach an arbitrary point in state-space, we require that W_r is nonsingular. The matrix W_r is called the **reachability matrix**.

Theorem (Reachability rank condition): *A linear system is reachable if and only if the reachability matrix is invertable (has full rank).*

Revisit - Example

Return to our example, with the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0] u$$

Reachability matrix:

$$W_r = [B \quad AB] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

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Reachability matrix:

$$W_r = [B \ AB] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Note: A square matrix $M (n \times n)$ has full rank n iff $\det(M) \neq 0$

Compute the determinant:

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$$\det(W_r) = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 1 \cdot 0 - 0 \cdot 0 = 0$$

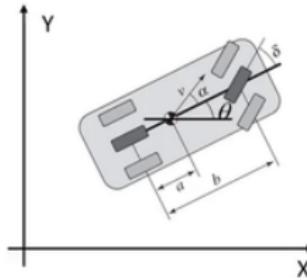
The system is not reachable!

Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & v_0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} av_0/b \\ v_0/b \end{bmatrix} u$$
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Revisit Example - Vehicle steering (Ex 7.4)

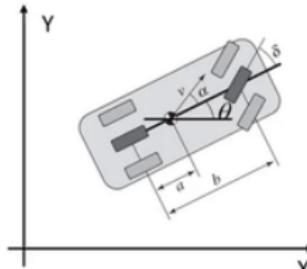
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Reachability matrix:

$$W_r = [B \quad AB] = \begin{bmatrix} av_0/b & [av_0/b \\ v_0/b] \\ v_0/b & [0 \quad 0] \begin{bmatrix} av_0/b \\ v_0/b \end{bmatrix} \end{bmatrix}$$



Revisit Example - Vehicle steering (Ex 7.4)

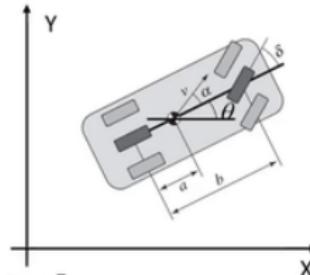
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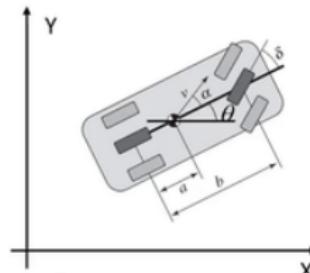
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$$\begin{aligned}\dot{\begin{bmatrix}x_1 \\ x_2\end{bmatrix}} &= \begin{bmatrix}0 & v_0 \\ 0 & 0\end{bmatrix} \begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}av_0/b \\ v_0/b\end{bmatrix} u \\ y &= [1 \quad 0] \begin{bmatrix}x_1 \\ x_2\end{bmatrix} + [0] u\end{aligned}$$

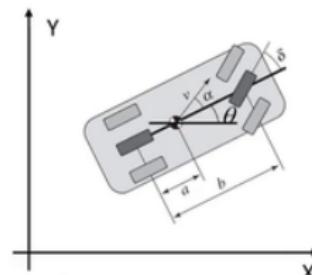
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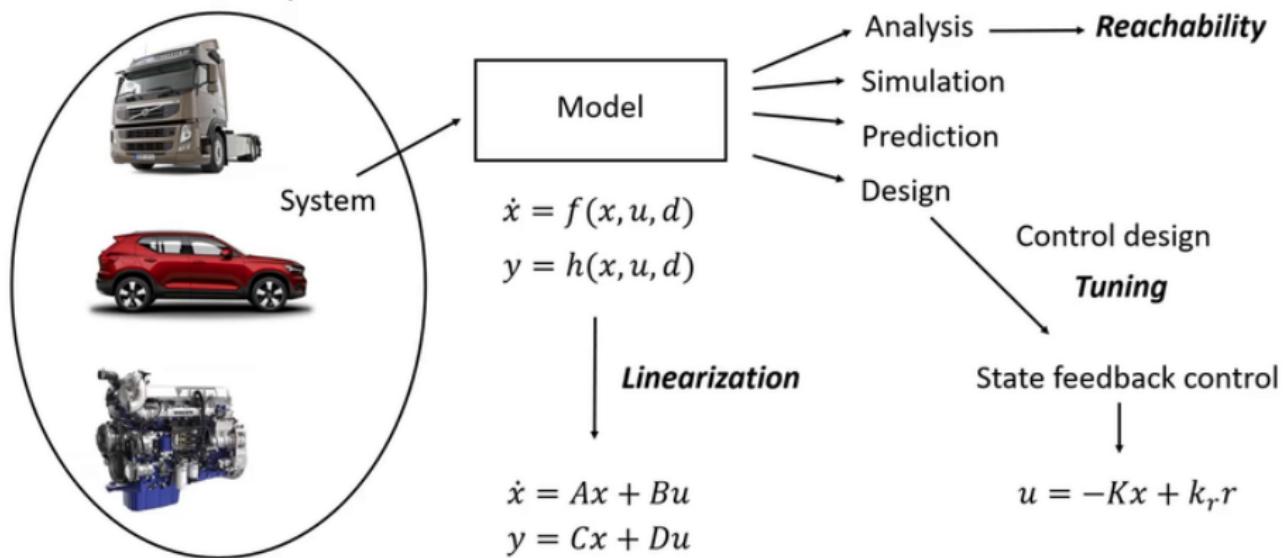
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The system is reachable, as long as $v_0 \neq 0$.



Summary - state feedback control



Pole placement

So far we have learnt how a state feedback looks like and when it is possible to design a state feedback controller to stabilize a system:

$$\dot{x} = Ax + Bu$$

$$u = -Kx + k_r r$$

$$\dot{x} = (A - BK)x + Bk_r r$$

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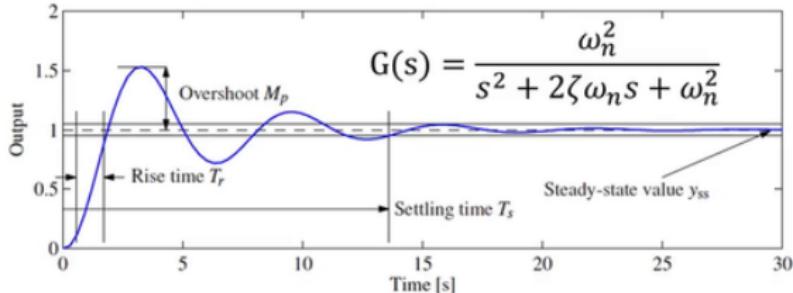
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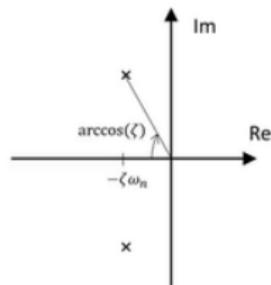
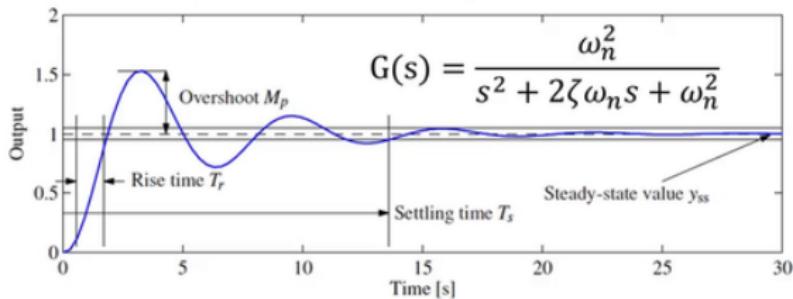
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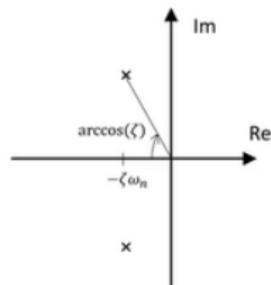
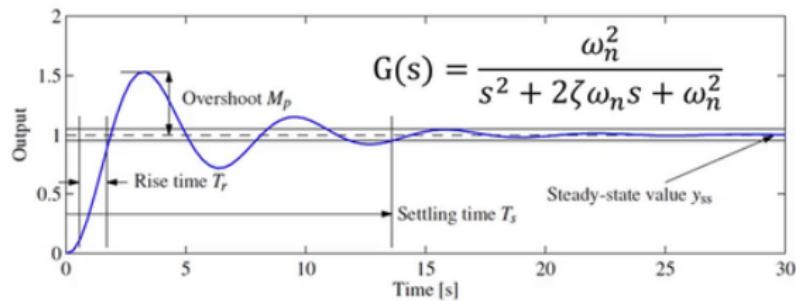
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Specifications and pole placement

Property	Value	$\zeta = 0.5$	$\zeta = 1/\sqrt{2}$	$\zeta = 1$
Rise time (inverse slope)	$T_r = e^{\varphi/\tan\varphi} / \omega_0$	$1.8/\omega_0$	$2.2/\omega_0$	$2.7/\omega_0$
Overshoot	$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$	16%	4%	0%
Settling time (2%)	$T_s \approx 4/\zeta\omega_0$	$8.0/\omega_0$	$5.6/\omega_0$	$4.0/\omega_0$



Pole placement

Where do we place the closed loop system's poles?

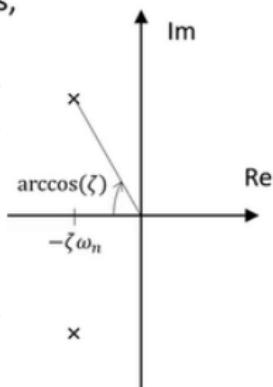
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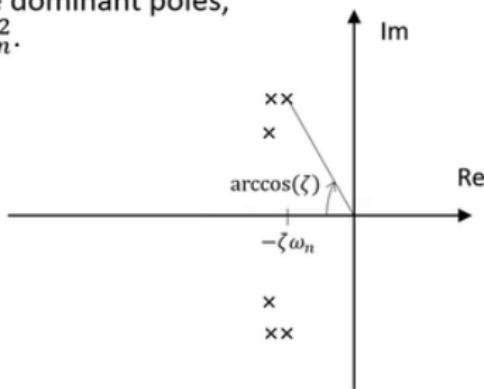


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- Place the rest of the poles so they become faster than the dominant poles.

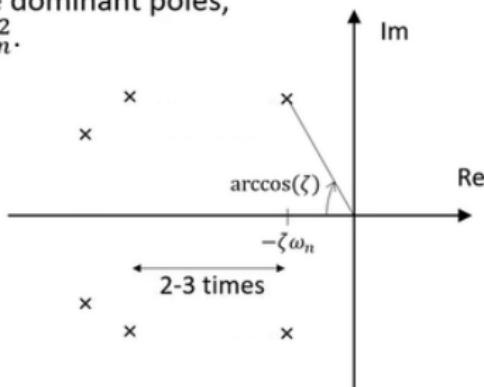


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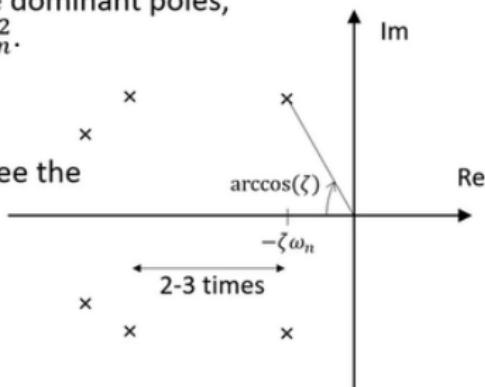
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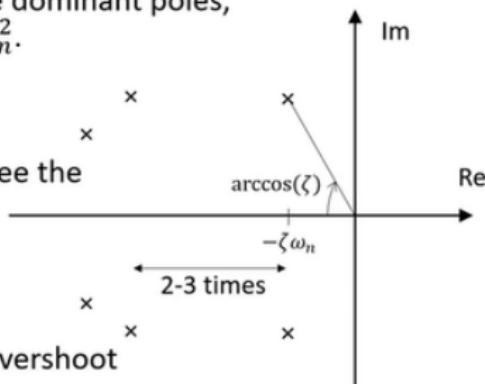
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Usually you end up with some zeros as well:

- Zeros in the left half plane give additional overshoot
- Zeros in the right half plane give a negative undershoot



Pole placement (Ackermann's formula)

Pole placement is performed by matching the desired characteristic polynomial with the closed loop system's characteristic polynomial.

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From earlier example (vehicle steering) we have seen:

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 \equiv \lambda^2 + \frac{v_0}{b}(ak_1 + k_2)\lambda + \frac{k_1 v_0^2}{b}$$
$$k_1 = \frac{b\omega_n^2}{v_0^2} \quad k_2 = \frac{2\zeta\omega_n b}{v_0} - \frac{ab\omega_n^2}{v_0^2}$$

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For low order systems it is ok, but for larger systems this is boring work.

Ackermann's formula offers us a method to do this in one computational step.

Pole placement (Ackermann's formula)

Consider a system $\dot{x} = Ax + Bu$ with the characteristic polynomial

$$a(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n.$$

Pole placement (Ackermann's formula)

Consider a system $\dot{x} = Ax + Bu$ with the characteristic polynomial

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If the system is reachable, then there exist a control law, $u = -Kx$, that gives a closed loop system with the characteristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \cdots p_{n-1} s + p_n.$$

Pole placement (Ackermann's formula)

The feedback gain is given by

$$K = [p_1 - a_1 \quad p_2 - a_2 \quad \dots \quad p_n - a_n] \tilde{W}_r W_r^{-1}$$

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$$K = [p_1 - a_1 \quad p_2 - a_2 \quad \dots \quad p_n - a_n] \tilde{W}_r W_r^{-1}$$

where W_r is the reachability matrix

$$W_r = [B \quad AB \quad \dots \quad A^{n-1}B]$$

and

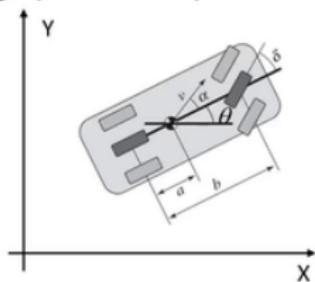
$$\tilde{W}_r = \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 1 & a_1 & \cdots & a_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}^{-1}$$

This is called **Ackermann's formula**.

Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

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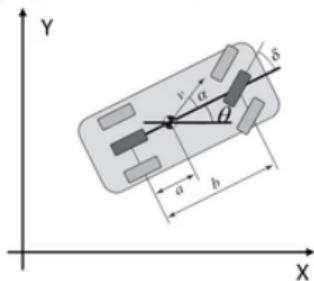
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Determine the characteristic polynomial for the system:

$$\det(sI - A) = \begin{vmatrix} s & -v_0 \\ 0 & s \end{vmatrix} = s^2 = s^2 + 0s + 0$$



Revisit Example - Vehicle steering (Ex 7.4)

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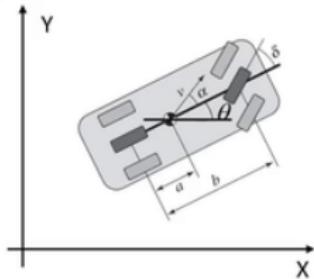
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Determine the characteristic polynomial for the system:

$$\det(sI - A) = \begin{vmatrix} s & -v_0 \\ 0 & s \end{vmatrix} = s^2 = s^2 + 0s + 0$$

Desired characteristic polynomial for the closed loop system:

$$p(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$$



Revisit Example - Vehicle steering (Ex 7.4)

The feedback gain is given by

$$K = [p_1 - a_1 \quad p_2 - a_2] \tilde{W}_r W_r^{-1} = [2\zeta\omega_n \quad \omega_n^2] \tilde{W}_r W_r^{-1}$$

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Linear Quadratic Regulator

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where S is a positive definite, symmetric matrix given by

$$A^T S + S A - S B Q_u^{-1} B^T S + Q_x = 0$$

This equation is called the **algebraic Riccati equation**.

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The tuning of the LQR is to choose the weighting matrices Q_x and Q_u . To guarantee that a solution exists, the system must be **reachable** and that $Q_x \geq \mathbf{0}$ and $Q_u > \mathbf{0}$.

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3. Diagonal weighting.

$$Q_x = \begin{bmatrix} q_1 & & 0 \\ 0 & \ddots & \\ & & q_n \end{bmatrix} \quad Q_u = \begin{bmatrix} \rho_1 & & 0 \\ 0 & \ddots & \\ & & \rho_p \end{bmatrix}$$

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$$\sum_{i=1}^n \alpha_i^2 = 1 \quad \sum_{i=1}^p \beta_i^2 = 1$$

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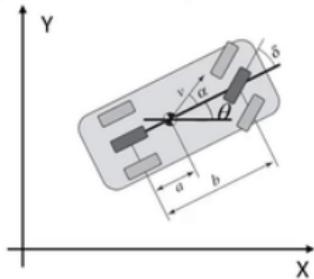
$$\sum_{i=1}^n \alpha_i^2 = 1 \quad \sum_{i=1}^p \beta_i^2 = 1$$

4. Trial and error

Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & v_0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} av_0/b \\ v_0/b \end{bmatrix} u$$
$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0] u$$



Vehicle data: $v_0 = 12 \text{ m/s}$
 $a = 2 \text{ m}$
 $b = 4 \text{ m}$

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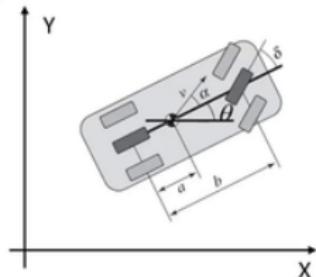
$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0] u$$

Place the poles so that the closed loop system optimizes the cost function:

$$J = \int_0^{\infty} (x^T Q_x x + u^T Q_u u) dt$$

where

$$Q_x = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} \quad Q_u = \rho$$

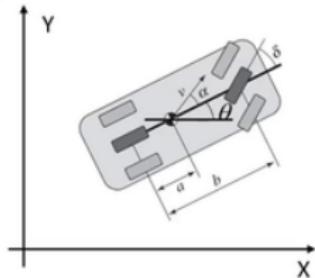


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Revisit Example - Vehicle steering (Ex 7.4)

For the case when

$$Q_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad Q_u = 10$$



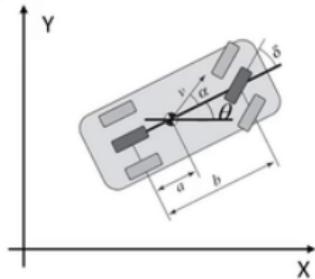
Revisit Example - Vehicle steering (Ex 7.4)

For the case when

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The solution to the algebraic Riccati equation is

$$A^T S + S A - S B Q_u^{-1} B^T S + Q_x = 0$$



In MATLAB: The LQR problem can be solved using the `lqr` command

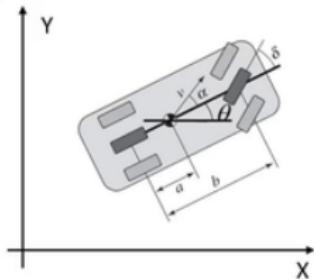
Revisit Example - Vehicle steering (Ex 7.4)

For the case when

$$Q_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad Q_u = 10$$

The solution to the algebraic Riccati equation is

$$S = \begin{bmatrix} 0.292 & 0.470 \\ 0.470 & 2.754 \end{bmatrix}$$



In MATLAB: The LQR problem can be solved using the `lqr` command

Revisit Example - Vehicle steering (Ex 7.4)

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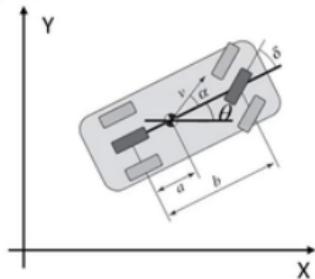
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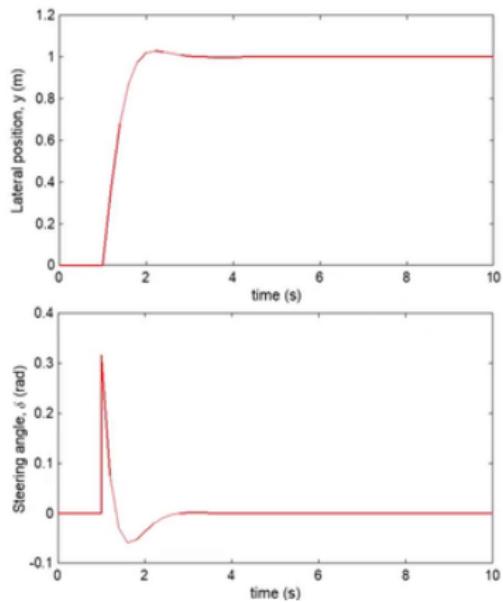
and the corresponding control law becomes

$$u = -Kx, \quad K = Q_u^{-1}B^T S = [0.316 \quad 1.108]$$

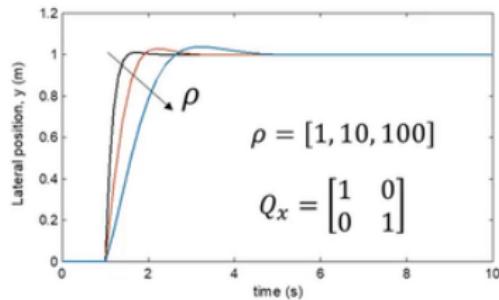


In MATLAB: The LQR problem can be solved using the `lqr` command

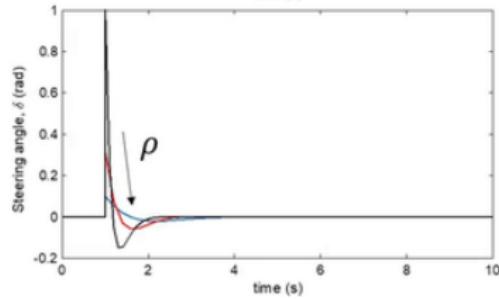
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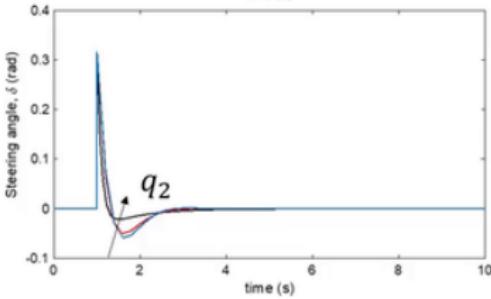
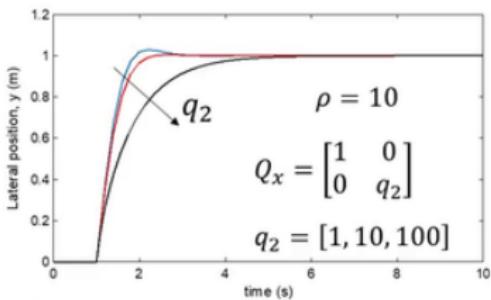
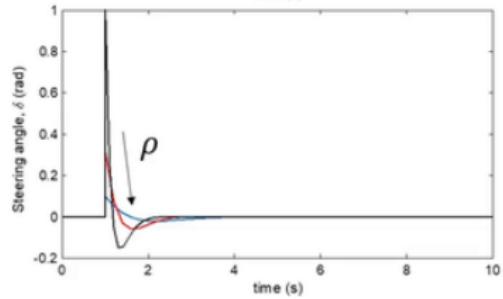
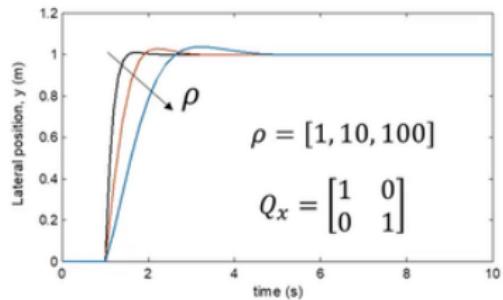
$$Q_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Tuning or Pole placement

Optimal control, example

Revisit Example - Vehicle steering (Ex 7.4)



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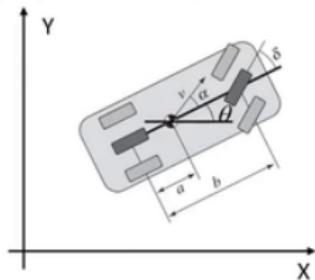
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In MATLAB: The LQR problem can be solved using the `lqr` command

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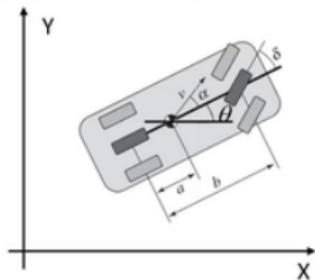
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The closed loop system poles are

$$E = \begin{bmatrix} -2.6110 + 2.1371i \\ -2.6110 - 2.1371i \end{bmatrix}$$



In MATLAB: The LQR problem can be solved using the `lqr` command

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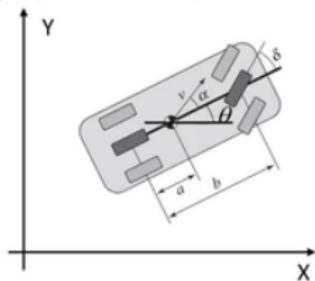
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Compared to the pole placement design, this corresponds to $\zeta = 0.77$ and $\omega_n = 3.44$.



In MATLAB: The LQR problem can be solved using the `lqr` command

Bibliography

- Karl J. Astrom and Richard M. Murray *Feedback Systems*. Version v3.0i. Princeton University Press. September 2018. Chapter 7.