Space feddback control

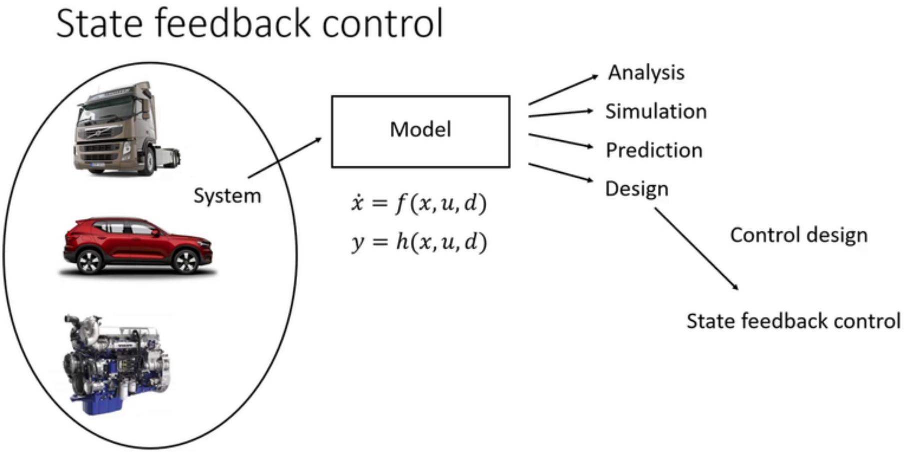
Dr. Ing. Rodrigo Gonzalez

rodralez@ingenieria.uncu.edu.ar

Control y Sistemas

Facultad de Ingeniería,
Universidad Nacional de Cuyo

State feedback control



State feedback control

Consider a linear time-invariant state-space model given by:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

where $x(t) \in \mathbb{R}^n$ is the state (vector), $u(t) \in \mathbb{R}^p$ is the input or control signal and $y(t) \in \mathbb{R}^q$ is the output signal. (For SISO case, $p = 1, q = 1$)

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The system poles are given by the eigenvalues of the system matrix $A \in \mathbb{R}^{n \times n}$.

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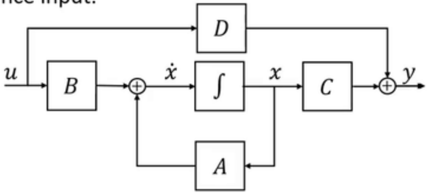

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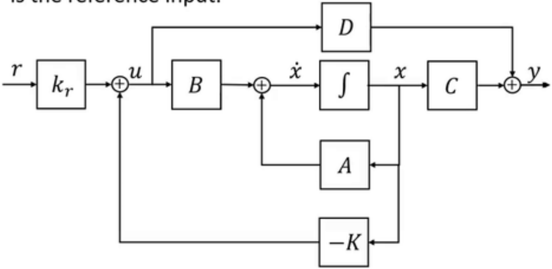

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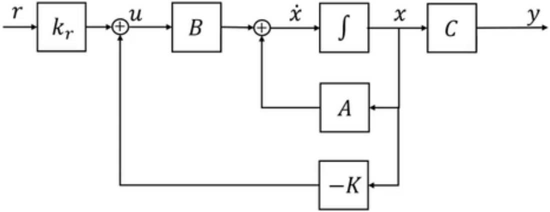

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Using the state feedback controller the closed loop dynamics becomes:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B(-Kx(t) + k_r r(t)) \\ &= (A - BK)x(t) + Bk_r r(t)\end{aligned}$$

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SISO case: *n parameters in K and n eigenvalues in A, so it might be possible!*

Reference tracking

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$$k_r = -(C(A - BK)^{-1}B)^{-1} \quad \text{or} \quad k_r = -1/C(A - BK)^{-1}B$$

Integral action

Using the steady-state feedback gain, k_{r}, can achieve zero steady-state error, but it does depend on the model parameters, as

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The new state-space model becomes:

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Given the new state-space model, we design a controller in the usual fashion and the resulting controller becomes:

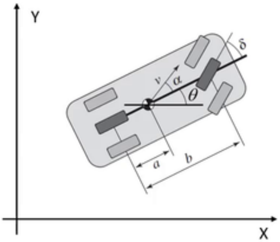
$$u(t) = -Kx(t) - K_I z(t) + k_r r(t)$$

Example 1 - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & v_0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}av_0/b\\ v_0/b\end{bmatrix}u$$
$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]\,u$$

where x_1 is the lateral position Y , x_2 is the heading orientation θ and u is the steering angle δ.



Vehicle data: $v_0 = 12\ m/s$
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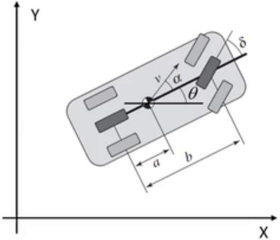

The idea is to design a controller that ***stabilizes*** the dynamics and ***tracks*** a given lateral position of the vehicle.

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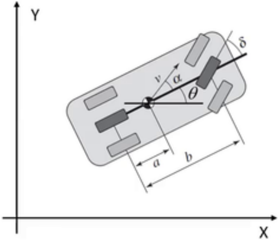

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The closed loop system has the characteristic polynomial

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Matching with desired characteristic polynomial gives:

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 \equiv \lambda^2 + \frac{v_0}{b}(ak_1 + k_2)\lambda + \frac{k_1 v_0^2}{b}$$

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The steady-state gain can be determined:

$$k_r = -1/C(A-BK)^{-1}B = \cdots = k_1 = \frac{b\omega_n^2}{v_0^2}$$

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Inserting these control design parameters into the feedback controller gives:

$$u = -k_1x_1 - k_2x_2 + k_rr = -\frac{b\omega_n^2}{v_0^2}x_1 - \left(\frac{2\zeta\omega_nb}{v_0} - \frac{ab\omega_n^2}{v_0^2}\right)x_2 + \frac{b\omega_n^2}{v_0^2}r$$

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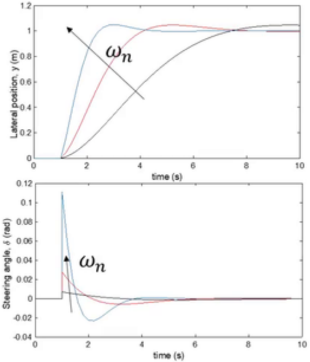
Simulations with different values of ζ and ω_n:



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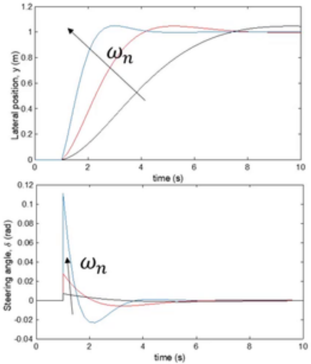

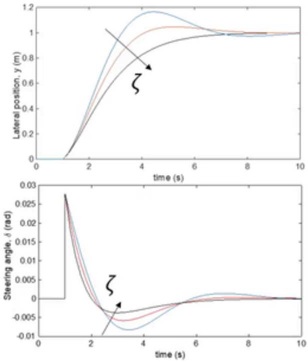

Example 2

Consider the following system (similar to Example 1 with one difference):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

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Again, the idea is to design a controller that ***stabilizes*** the dynamics and ***tracks*** a given reference signal.

Specification: Desired characteristic polynomial:

$$p_{des}(\lambda) = \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2$$

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$$u = -Kx + k_r r = -k_1 x_1 - k_2 x_2 + k_r r$$

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The closed loop system dynamics becomes

$$\dot{x} = (A - BK)x + Bk_r r = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} k_r \\ 0 \end{bmatrix} r$$

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The closed loop system has the characteristic polynomial

$$\det(\lambda I - A + BK) = \det \begin{pmatrix} \lambda + k_1 & k_2 - 1 \\ 0 & \lambda \end{pmatrix} = \lambda^2 + k_1 \lambda = \lambda(\lambda + k_1)$$

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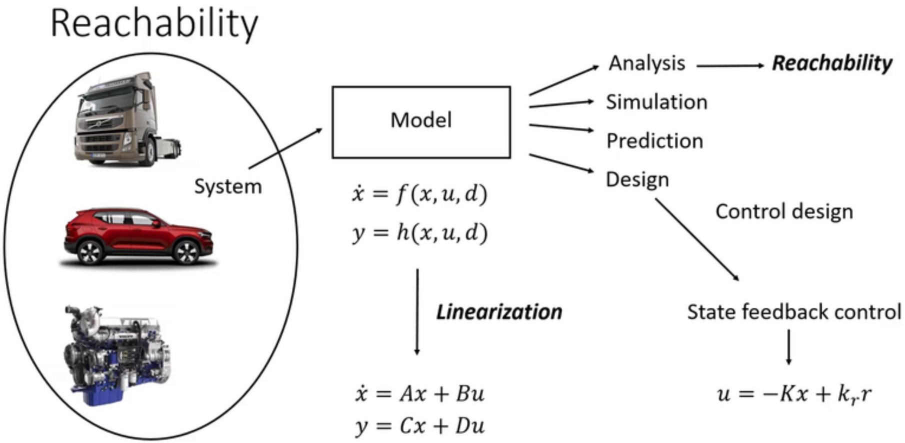
While matching with the desired characteristic polynomial

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 \equiv \lambda^2 + k_1\lambda$$

we notice that it is not possible to make them identical to each other.

It is not possible to control all the eigenvalues (shape the dynamics). We say that the system is not ***controllable*** or not ***reachable***. Furthermore, one eigenvalue is always 0, which means that the closed loop system is unstable.


Reachability
System
Model
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$y = h(x,u,d)$
Analysis
Reachability
Simulation
Prediction
Design
Control design
Linearization
State feedback control
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Revisit - Example

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State feedback control: $u = -Kx + k_r r = -k_1 x_1 - k_2 x_2 + k_r r$

The closed loop system dynamics becomes

$$\dot{x} = (A - BK)x + Bk_r r = \begin{bmatrix}-k_1 & 1 - k_2\\ 0 & 0\end{bmatrix}x + \begin{bmatrix}k_r\\ 0\end{bmatrix}r$$

Uncontrollable

and the characteristic polynomial is given as

Controllable

$$\det(\lambda I - A + BK) = \det\begin{pmatrix}\lambda + k_1 & k_2 - 1\\ 0 & \lambda\end{pmatrix} = \lambda^2 + k_1\lambda = \lambda(\lambda + k_1)$$

Reachability

Definition (Reachability): A linear system is ***reachable*** if for any $x_0, x_f \in \mathbb{R}^n$ there exists a $T > 0$ and $u\colon [0,T] \to \mathbb{R}$ such that if $x(0) = x_0$ then the corresponding solution satisfies $x(T) = x_f$.

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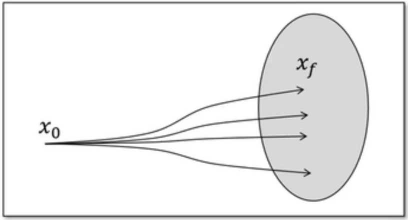

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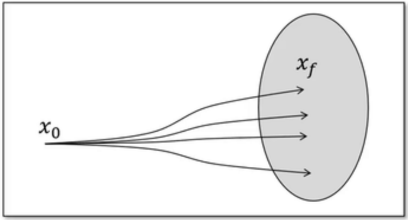
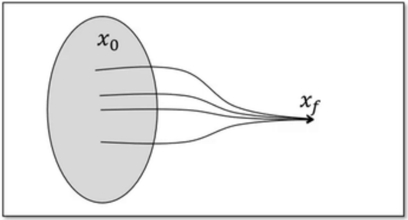


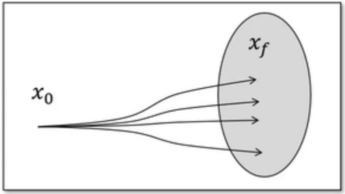


Sometimes the definition of ***controllable*** and ***controllability*** is used, and that is similar.

Linear system representations

$$
\begin{tikzcd}
a\left(\frac{d}{dt}\right)y(t) = b\left(\frac{d}{dt}\right)u(t) \arrow[r, "\mathcal{L}"] & a(s)Y(s) = b(s)U(s) \arrow[d, "G(s) = \frac{b(s)}{a(s)}"] \\
y(t) = \int_0^t g(t-\tau)\,u(\tau)d\tau & Y(s) = G(s)U(s) \arrow[l, "\mathcal{L}^{-1}"'] \\
\begin{gathered} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{gathered} \arrow[u, "g(t) = Ce^{At}B + D\delta(t)"'] \arrow[r, "\mathcal{L}"] & \begin{gathered} sX(s) - x(0) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{gathered} \arrow[u, "G(s) = C(sI-A)^{-1}B + D"']
\end{tikzcd}
$$

Reachability



To see that an arbitrary point can be reached, we can use the convolution equation.

Assume that the system starts from zero, the state of a linear system is given by:

$$x(t) = \int_0^t e^{A(t-\tau)} B\, u(\tau) d\tau = \int_0^t e^{A\tau} B\, u(t-\tau) d\tau$$

From linear theory it can be shown that

$$e^{A\tau} = I\alpha_0(\tau) + A\alpha_1(\tau) + \cdots + A^{n-1}\alpha_{n-1}(\tau)$$

where $\alpha_i(t)$ are scalar functions, so that

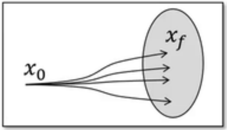
$$x(t) = B \int_0^t \alpha_0(\tau)\, u(t-\tau) d\tau + AB \int_0^t \alpha_1(\tau)\, u(t-\tau) d\tau + \cdots + A^{n-1} B \int_0^t \alpha_{n-1}(\tau)\, u(t-\tau) d\tau$$

Reachability

By writing it in "vector form", we get:

$$x(t) = \underbrace{[B \quad AB \quad \dots \quad A^{n-1}B]}_{W_r} \begin{bmatrix} \int_0^t \alpha_0(\tau)\, u(t-\tau)d\tau \\ \int_0^t \alpha_1(\tau)\, u(t-\tau)d\tau \\ \vdots \\ \int_0^t \alpha_{n-1}(\tau)\, u(t-\tau)d\tau \end{bmatrix}$$

Reachability



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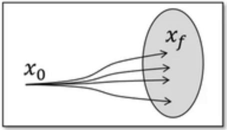
To reach an arbitrary point is state-space, we require that W_r is nonsingular. The matrix W_r is called the ***reachability matrix***.

Reachability



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Theorem (Reachability rank condition): *A linear system is reachable if and only if the reachability matrix is invertable (has full rank).*

Revisit - Example

Return to our example, with the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}u$$

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Reachability matrix:

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Note: A square matrix M ($n \times n$) has full rank n iff the det(M)≠0

Compute the determinant:

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Compute the determinant:

$$\det(W_r) = \begin{vmatrix}1 & 0\\0 & 0\end{vmatrix} = 1\cdot 0 - 0\cdot 0 = 0$$

The system is not reachable!

Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & v_0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}av_0/b\\ v_0/b\end{bmatrix}u$$

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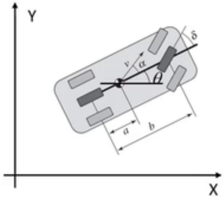

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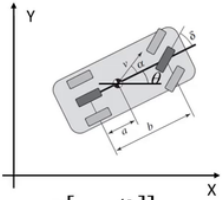

Reachability matrix:

$$W_r = [B \quad AB] = \left[\begin{matrix}av_0/b\\ v_0/b\end{matrix} \quad \begin{bmatrix}0 & v_0\\ 0 & 0\end{bmatrix}\begin{bmatrix}av_0/b\\ v_0/b\end{bmatrix}\right]$$

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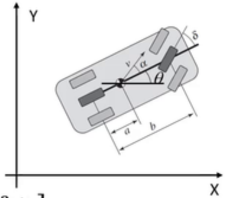

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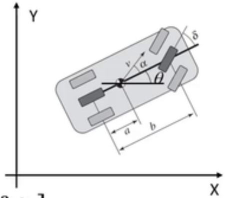

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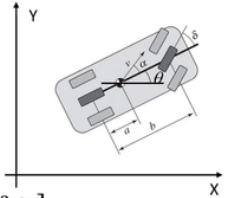

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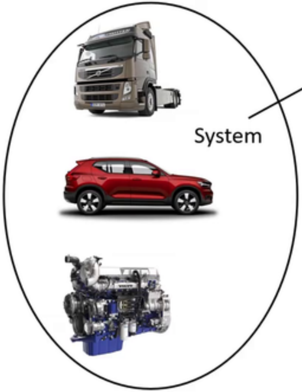
The system is reachable, as long as $v_0 \neq 0$.

Summary - state feedback control



Model

$$\dot{x} = f(x, u, d)$$
$$y = h(x, u, d)$$

Analysis → ***Reachability***

Simulation

Prediction

Design

Control design

Tuning

Linearization

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

State feedback control

$$u = -Kx + k_r r$$

Pole placement

So far we have learnt how a state feedback looks like and when it is possible to design a state feedback controller to stabilize a system:

$$\dot{x} = Ax + Bu$$
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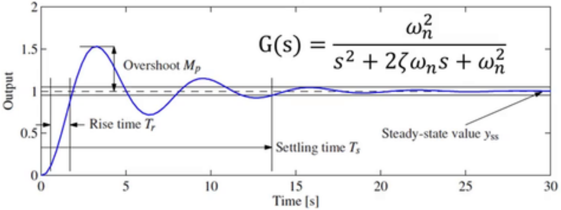

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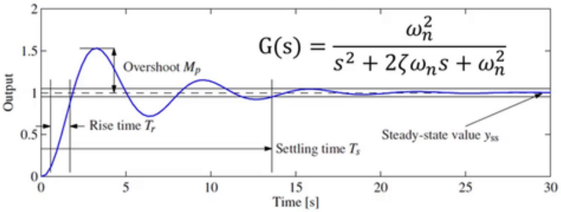
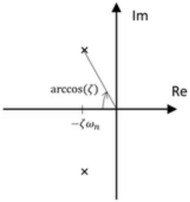
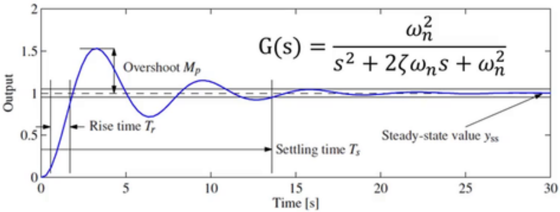
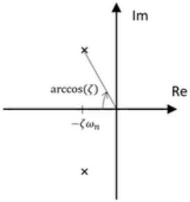




Specifications and pole placement

Property	Value	$\zeta = 0.5$	$\zeta = 1/\sqrt{2}$	$\zeta = 1$
Rise time	$T_r \approx 1/\omega_n \cdot e^{arccos\zeta/\tan(arccos\zeta)}$	$1.8/\omega_n$	$2.2/\omega_n$	$2.7/\omega_n$
Overshoot	$M_p \approx e^{-\pi\zeta\sqrt{1-\zeta^2}}$	16%	4%	0%
Settling time (2%)	$T_s \approx 4/\zeta\omega_n$	$8.0/\omega_n$	$5.9/\omega_n$	$4.0/\omega_n$





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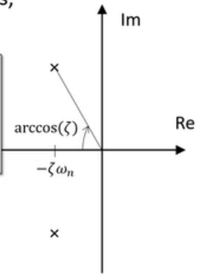

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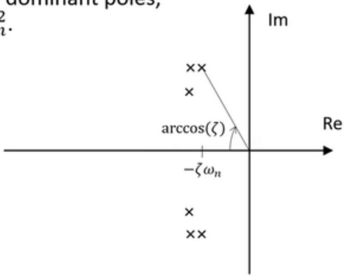

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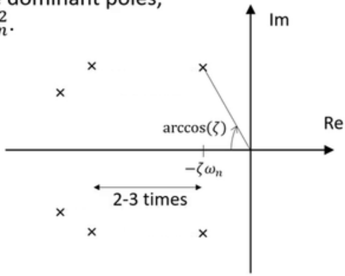

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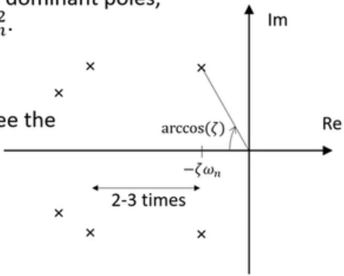

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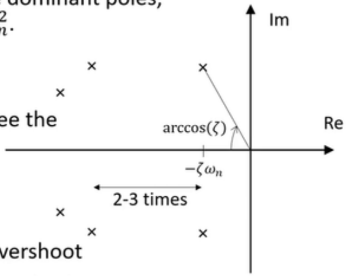

Usually you end up with some zeros as well:

- Zeros in the left half plane give additional overshoot
- Zeros in the right half plane give a negative undershoot

Pole placement (Ackermann's formula)

Pole placement is performed by matching the desired characteristic polynomial with the closed loop system's characteristic polynomial.

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From earlier example (vehicle steering) we have seen:

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 \equiv \lambda^2 + \frac{v_0}{b}(ak_1 + k_2)\lambda + \frac{k_1 v_0^2}{b}$$

$$k_1 = \frac{b\omega_n^2}{v_0^2} \qquad k_2 = \frac{2\zeta\omega_n b}{v_0} - \frac{ab\omega_n^2}{v_0^2}$$

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For low order systems it is ok, but for larger systems this is boring work.

Ackermann's formula offers us a method to do this in one computational step.

Pole placement (Ackermann's formula)

Consider a system $\dot{x} = Ax + Bu$ with the characteristic polynomial

$$a(s) = s^n + a_1 s^{n-1} + \cdots a_{n-1} s + a_n.$$

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Pole placement (Ackermann's formula)

The feedback gain is given by

$$K = [p_1 - a_1 \quad p_2 - a_2 \quad \dots \quad p_n - a_n]\widetilde{W}_r W_r^{-1}$$

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where W_r is the reachability matrix

$$W_r = [B \quad AB \quad \dots \quad A^{n-1}B]$$

and

$$\widetilde{W}_r = \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 1 & a_1 & \cdots & a_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_1 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}^{-1}$$

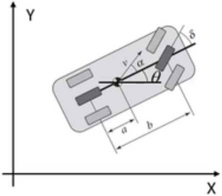
This is called ***Ackermann's formula***.

Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & v_0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}av_0/b\\ v_0/b\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]\,u$$



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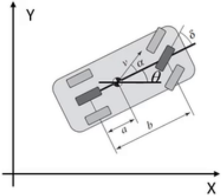
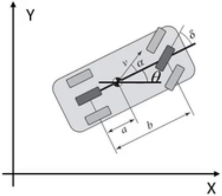


Determine the characteristic polynomial for the system:

$$\det(sI - A) = \begin{vmatrix}s & -v_0\\ 0 & s\end{vmatrix} = s^2 = s^2 + 0s + 0$$

Revisit Example - Vehicle steering (Ex 7.4)



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Desired characteristic polynomial for the closed loop system:

$$p(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$$

Revisit Example - Vehicle steering (Ex 7.4)

The feedback gain is given by

$$K = [p_1 - a_1 \quad p_2 - a_2]\widetilde{W}_r W_r^{-1} = [2\zeta\omega_n \quad \omega_n^2]\widetilde{W}_r W_r^{-1}$$

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$$A^T S + SA - SBQ_u^{-1} B^T S + Q_x = 0$$

This equation is called the ***algebraic Riccati equation***.

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The tuning of the LQR is to choose the weighting matrices Q_x and Q_u. To guarantee that a solution exists, the system must be ***reachable*** and that $\boldsymbol{Q_x} \succcurlyeq \mathbf{0}$ and $\boldsymbol{Q_u} \succ \mathbf{0}$.

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3. Diagnonal weighting.

$$Q_x = \begin{bmatrix} q_1 & & 0 \\ & \ddots & \\ 0 & & q_n \end{bmatrix} \quad Q_u = \begin{bmatrix} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_p \end{bmatrix}$$

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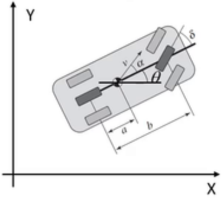
4. Trial and error

Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & v_0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} av_0/b \\ v_0/b \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\, u$$



Vehicle data: $v_0 = 12\ m/s$
$a = 2\ m$
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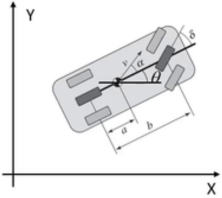
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 12\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}6\\ 3\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]\,u$$



Place the poles so that the closed loop system optimizes the cost function:

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Revisit Example - Vehicle steering (Ex 7.4)

For the case when

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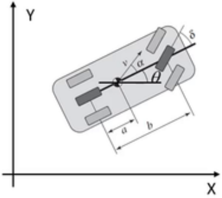

Revisit Example - Vehicle steering (Ex 7.4)

For the case when

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The solution to the algebraic Ricatti equation is

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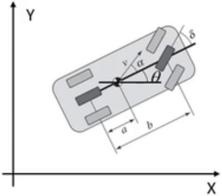

In MATLAB: The LQR problem can be solved using the `lqr` command

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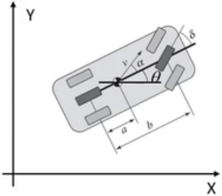
For the case when

$$Q_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad Q_u = 10$$

The solution to the algebraic Ricatti equation is

$$S = \begin{bmatrix} 0.292 & 0.470 \\ 0.470 & 2.754 \end{bmatrix}$$



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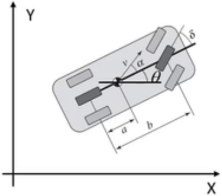
$$u = -Kx, \qquad K = Q_u^{-1} B^T S = \begin{bmatrix} 0.316 & 1.108 \end{bmatrix}$$



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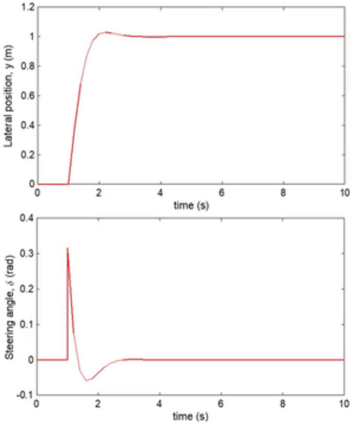

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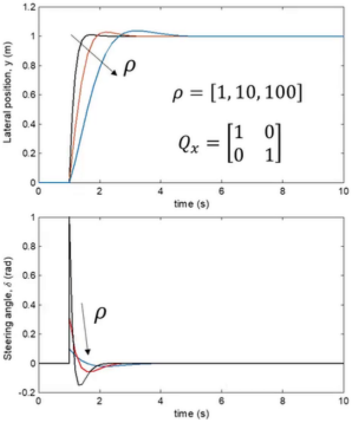

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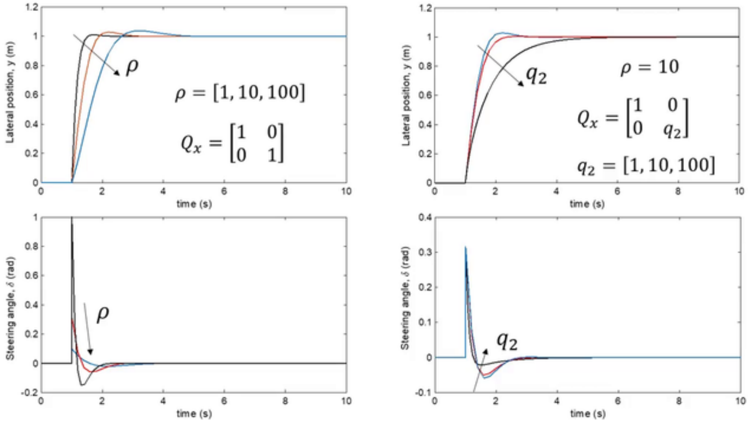

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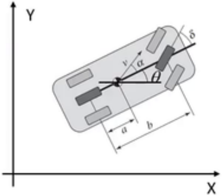

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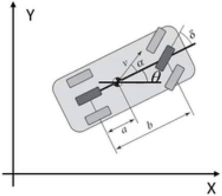
The closed loop system poles are

$$E = \begin{bmatrix} -2.6110 + 2.1371i \\ -2.6110 - 2.1371i \end{bmatrix}$$



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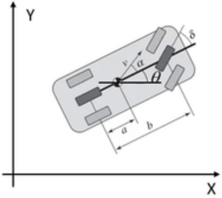

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Compared to the pole placement design, this corresponds to $\zeta = 0.77$ and $\omega_n = 3.44$.

In MATLAB: The LQR problem can be solved using the `lqr` command

Bibliography

- Karl J. Astrom and Richard M. Murray *Feedback Systems*. Version v3.0i. Princeton University Press. September 2018. Chapter 11.