# LINEAR INSTABILITY OF THE PRANDTL EQUATIONS VIA HYPERGEOMETRIC FUNCTIONS AND THE HARMONIC OSCILLATOR

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ABSTRACT. We establish a deep connection between the Prandtl equations linearised around a quadratic shear flow, confluent hypergeometric functions of the first kind, and the Schrödinger operator.

Our first result concerns an ODE and a spectral condition derived in [10], associated with unstable quasieigenmodes of the Prandtl equations. We entirely determine the space of solutions in terms of Kummer's functions. By classifying their asymptotic behaviour, we verify that the spectral condition has a unique, explicitly determined pair of eigenvalue and eigenfunction, the latter being expressible as a combination of elementary functions.

Secondly, we prove that any quasi-eigenmode solution of the linearised Prandtl equations around a quadratic shear flow can be explicitly determined from algebraic eigenfunctions of the Schrödinger operator with quadratic potential.

We show finally that the obtained analytical formulation of the velocity align with previous numerical simulations in the literature.

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#### 1. Introduction

This paper has two main objectives, aiming to establish a deep analytical connection between a family of solutions to the Prandtl equations linearised around a parabolic shear layer, hypergeometric functions, and algebraic eigenfunctions of the harmonic oscillator.

The first objective is to address a criterion for an ordinary differential equation (cf. Criterion 1) introduced in [10], which asymptotically characterizes instabilities near points of loss of monotonicity of the flow. Our first result states that all solutions of this particular ODE can be explicitly expressed in terms of Kummer's hypergeometric functions (cf. Theorem 1.1 and Corollary 1.2).

The second objective is to further develop these relations directly to the following linearised Prandtl system:

$$\begin{cases} \partial_t u + U_{\rm sh} \partial_x u + v U_{\rm sh}' - \partial_y^2 u = 0, & (t, x, y) \in (0, T) \times \mathbb{T} \times \mathbb{R}_+, \\ \partial_x u + \partial_y v = 0, & (t, x, y) \in (0, T) \times \mathbb{T} \times \mathbb{R}_+, \end{cases}$$
(1)

where (u, v) = (u(t, x, y), v(t, x, y)) is the velocity field, while the shear flow  $U_{\rm sh} = U_{\rm sh}(y)$ , with  $y \in \mathbb{R}_+$ , is considered as a parabolic shear layer characterized by

$$U_{\rm sh}(y) = \alpha + \beta (y - a)^2$$
, for given  $a \ge 0$ ,  $\alpha \in \mathbb{R}$  and  $\beta < 0$ . (2)

We determine explicitly all solutions of (1) that extend to complex values and can be written in the form

$$u(t, x, y) = u_k(y) \exp\left(ikx + \sigma t \sqrt{|k|}\right), \qquad v(t, x, y) = v_k(y) \exp\left(ikx + \sigma t \sqrt{|k|}\right), \tag{3}$$

for a general frequency  $k \in \mathbb{Z} \setminus \{0\}$  and an arbitrary  $\sigma \in \mathbb{C}$ . Our second result (cf. Section 1.2 and Theorem 1.4) states that the functions  $u_k, v_k : \mathbb{R}_+ \to \mathbb{C}$  can be written in terms of hypergeometric functions and vice versa suitable algebraic eigenfunctions of the Schrödinger operator. Additionally, we identify those that satisfy the no-slip boundary conditions at y = 0:

$$u_{|y=0} = v_{|y=0} = 0, (t, x) \in (0, T) \times \mathbb{T}.$$
 (4)

The paper is structured as follows: we state our main results in Section 1.1 and Section 1.2, and we prove them in Section 3 and Section 4, respectively. In Remark 1.5, we introduce a method to explicitly compute solutions of the form (3) in terms of eigenfunctions of the Schrödinger operator, which we further develop through examples in Section 6. Finally, in Section 7, we apply this method to compute unstable (quasi-)modes of the Prandtl equations, expressing with analytical precision (cf. (67)), the so-called "shear-layer corrector" of [10].

#### 1.1. The criterion of unstable modes by Kummer's functions.

Instabilities in boundary layers often arise from flow separations, a phenomenon in which the velocity field loses monotonicity and reverses direction. The distinction between monotonic and non-monotonic behaviour is particularly evident in the analysis of the Prandtl equations, where well-posedness in Sobolev spaces is locally ensured by monotonicity [1, 3, 19, 20], while significantly stronger regularities are required for non-monotonic profiles (analyticity and Gevrey classes [4, 5, 7, 11, 12, 15, 16, 18]).

For non-monotonic flows, unstable modes emerge already in the linearised equations (1) around shear flows  $U_{\rm sh}(y)$  exhibiting a critical point  $U'_{\rm sh}(a)=0$  which is non-degenerate  $(U''_{\rm sh}(a)\neq 0)$ . Under these conditions, the linear equations (1) are ill-posed in Sobolev spaces [10], with solutions that exhibit rapid norm inflation in time (cf. [14] for the nonlinear case and [13] for initial profiles for which no solution exists). In [10], in particular, unstable modes in the tangential frequencies are built relying on a perturbative argument of the following spectral condition for a reduced ODE (cf. (1.7) in [10]):

**Criterion 1.** There exist  $\tau \in \mathbb{C}$  with  $\operatorname{Im} \tau < 0$  and a smooth solution  $W : \mathbb{R} \to \mathbb{C}$  of

$$(\tau - z^2)^2 W'(z) + i \frac{d^3}{dz^3} \left[ (\tau - z^2) W(z) \right] = 0,$$
 (5)

 $such\ that\ \lim_{z\to -\infty} W(z) = 0\ and\ \lim_{z\to +\infty} W(z) = 1.$ 

In essence, the variable  $z \in \mathbb{R}$  is a rescaled version of the normal coordinate  $y \in \mathbb{R}_+$  from (1), given by  $z = \sqrt[4]{|k||\beta|}(y-a)$ . It is centered at the critical point y=a and describes regions that become increasingly localised around a as the tangential frequencies  $k \to +\infty$  (cf. also Section 2 and (26) for a detailed discussion within our parabolic shear flow (2)). The function  $(\tau - z^2)W(z)$  approximates  $v_k(y)$  near y=a (up to a multiplicative constant), while  $\sigma \sim i\tau$  in (3), at least asymptotically as  $k \to +\infty$ .

Gérard-Varet and Dormy [10] implicitly determined a couple  $(W, \tau)$  satisfying Criterion 1 through an auxiliary eigenvalue problem. Based on this result, they constructed solutions of the linearised Prandtl's system that, for large tangential frequencies  $k \gg 1$ , grow as  $k^m e^{i\tau\sqrt{k}t}$ , for a positive m > 0, over short k-dependent time scales (at least up to  $t \sim \ln(k)/\sqrt{k}$ , as shown in [6]), leading to rapid norm inflation in Sobolev spaces. Criterion 1 has moreover played a major role in additional instabilities in boundary layers (cf. for instance Section 4 in [13], Lemma 1.2 in [8] and (2.11) in [17]).

Our first result simplifies the statement of Criterion 1 by showing that, for any  $\tau \in \mathbb{C}$ , the ODE (5) can be solved explicitly in terms of hypergeometric functions. Moreover, we establish that the function W and the parameter  $\tau$  in Criterion 1 are uniquely determined and can be expressed in terms of elementary functions, such as the Gauss error function (cf. Corollary 1.2).

Hypergeometric functions play a crucial role in fluid dynamics and turbulence studies. For instance, Bertolotti [2] established a connection between cross-flow vortices and attachment-line instabilities by representing the velocity and pressure fields in swept Hiemenz flows using Kummer and Tricomi functions. More recently, Salin and Talon [22] analysed core-annular flows, identifying unstable solutions for the axisymmetric linearized equations of each fluid using Bessel and confluent hypergeometric functions. See also [9] for applications in Rayleigh's stability equation.

To the best of our knowledge, however, the use of hypergeometric functions in solving Prandtl-type equations has not yet been analytically established. Our first result states that the derivative X(z) = W'(z) of any solution W of equation (5), without prescribed boundary conditions, can be expressed using the Kummer's confluent function  $\mathcal{M}$ :

$$\mathcal{M}(\mathbf{a}, \mathbf{c}, \zeta) = \sum_{n=0}^{\infty} \frac{(\mathbf{a})_n}{(\mathbf{c})_n} \frac{\zeta^n}{n!}, \quad \text{for any } \zeta \in \mathbb{C},$$
 (6)

where  $a \in \mathbb{C}$ ,  $c \in \mathbb{C} \setminus \{0, -1, -2, ...\}$  and  $(\cdot)_n$  stands for the Pochhammer symbol. For any  $\tau \in \mathbb{C}$ , we set  $\mathcal{A}_{\tau} = \{z \in \mathbb{C} \text{ such that } z^2 = \tau\}$  and we determine in terms of  $\mathcal{M}$  all solutions X(z) = W'(z) of the following complex ordinary differential equation:

$$i(\tau - z^2)X''(z) - 6izX'(z) + ((\tau - z^2)^2 - 6i)X(z) = 0, \qquad z \in \mathbb{C} \setminus \mathcal{A}_{\tau}.$$
 (7)

**Theorem 1.1.** Let  $\tau \in \mathbb{C}$  be an arbitrary complex number and define the constants  $\mathbf{a}_{\tau}, \mathbf{b}_{\tau}, \mathbf{c}_{\tau}, \mathbf{d}_{\tau} \in \mathbb{C}$  as

$$\mathbf{a}_{\tau} := -\frac{1}{4} \Big( 1 + \tau e^{\frac{7\pi}{4} \mathrm{i}} \Big), \quad \mathbf{b}_{\tau} := \frac{1}{4} \Big( 3 - \tau e^{\frac{7\pi}{4} \mathrm{i}} \Big), \quad \mathbf{c}_{\tau} := \frac{1}{4} \Big( 1 - \tau e^{\frac{7\pi}{4} \mathrm{i}} \Big), \quad \mathbf{d}_{\tau} := \frac{1}{4} \Big( 5 - \tau e^{\frac{7\pi}{4} \mathrm{i}} \Big).$$

Every solution  $X: \mathbb{C} \setminus A_{\tau} \to \mathbb{C}$  of (7) can be written explicitly as  $X(z) = c_1 X_{\tau,1}(z) + c_2 X_{\tau,2}(z)$ , where  $c_1, c_2 \in \mathbb{C}$  are arbitrary constants, and the functions  $X_{\tau,1}$  and  $X_{\tau,2}$  are meromorphic and defined by

$$X_{\tau,1}(z) = \frac{\exp\left(\frac{1}{2}e^{\frac{3\pi}{4}i}z^{2}\right)}{(\tau - z^{2})^{2}} \left[\tau \mathcal{M}\left(\mathbf{a}_{\tau}, \frac{1}{2}, e^{\frac{7\pi}{4}i}z^{2}\right) - 4\mathbf{a}_{\tau}z^{2} \mathcal{M}\left(\mathbf{b}_{\tau}, \frac{3}{2}, e^{\frac{7\pi}{4}i}z^{2}\right)\right],$$

$$X_{\tau,2}(z) = z \frac{\exp\left(\frac{1}{2}e^{\frac{3\pi}{4}i}z^{2}\right)}{(\tau - z^{2})^{2}} \left[\mathcal{M}\left(\mathbf{c}_{\tau}, \frac{3}{2}, e^{\frac{7\pi}{4}i}z^{2}\right) + e^{\frac{7\pi}{4}i}z^{2} \mathcal{M}\left(\mathbf{d}_{\tau}, \frac{5}{2}, e^{\frac{7\pi}{4}i}z^{2}\right)\right].$$
(8)

The proof of Theorem 1.1 is given in Section 3, however some remarks on its implications are already in order. There is no restriction on the choice of  $\tau \in \mathbb{C}$  and the functions  $X_{\tau,1}$  and  $X_{\tau,2}$  are even and odd, respectively, ensuring their linear independence. Moreover, the prefactor  $\exp(\frac{1}{2}e^{\frac{3\pi}{4}i}z^2)$  decays exponentially for real  $z \in \mathbb{R}$ . Meanwhile, the Kummer's functions in (8) have a well-known asymptotic expansion at  $z \to \pm \infty$  (cf. Chapter 7.10 in [21]), typically of exponential growth, except when they collapse to polynomials. This corresponds mainly to the discrete values of  $\mathbf{a}_{\tau} \mathbf{b}_{\tau} \in -\mathbb{N}_0$  (if  $c_1 \neq 0$ ) and  $\mathbf{c}_{\tau}, \mathbf{d}_{\tau} \in -\mathbb{N}_0$  (if  $c_2 \neq 0$ ).

These remarks highlight the key impact of Theorem 1.1, namely it determines a pair  $(\tau, W)$  that satisfies all conditions of Criterion 1. Setting  $\tau = e^{\frac{5\pi}{4}i} = -(1+i)/\sqrt{2}$  yields  $\mathbf{a}_{\tau} = 0$ ,  $\mathbf{b}_{\tau} = 1$ ,  $\mathbf{c}_{\tau} = 1/2$ , and  $\mathbf{d}_{\tau} = 3/2$ . Consequently, the first Kummer function in  $X_{1,\tau}$  simplifies to  $\mathcal{M}(\mathbf{a}_{\tau}, 1/2, e^{\frac{7\pi}{4}i}z^2) = 1$ , while  $\mathbf{a}_{\tau}\mathcal{M}(\mathbf{b}_{\tau}, 3/2, e^{\frac{7\pi}{4}i}z^2)$  vanishes due to  $\mathbf{a}_{\tau} = 0$ . Choosing  $c_1 \in \mathbb{C} \setminus \{0\}$  and  $c_2 = 0$ , the corresponding solution X exhibits exponential decay as follows:

$$\tau = e^{\frac{5\pi}{4}i}, X(z) = c_1 e^{\frac{5\pi}{4}i} \frac{\exp\left(\frac{1}{2}e^{\frac{3\pi}{4}i}z^2\right)}{\left(e^{\frac{5\pi}{4}i} - z^2\right)^2} = c_1 e^{\frac{5\pi}{4}i} \frac{\exp\left(-\frac{1}{2}e^{-\frac{\pi i}{4}}z^2\right)}{\left(e^{\frac{\pi}{4}i} + z^2\right)^2} = c_1 e^{\frac{3\pi}{4}i} \frac{\exp\left(-\frac{\eta(z)^2}{2}\right)}{\left(1 + \eta(z)^2\right)^2}, (9)$$

with  $\eta(z) = e^{-\frac{\pi i}{8}}z$ . Since Im  $\tau = -1/\sqrt{2} < 0$ , Criterion 1 is governed by a primitive function W of X in (9), with a suitable  $c_1 \neq 0$  that ensures the correct limits as  $z \to \pm \infty$ . The function W is stated in the following corollary.

Corollary 1.2. There exists a unique  $\tau \in \mathbb{C}$  with  $\operatorname{Im} \tau < 0$  and a unique function W = W(z) satisfying the conditions of Criterion 1, given by

$$\tau = e^{\frac{5\pi}{4}i} = -\frac{1+i}{\sqrt{2}}, \qquad W(z) = \frac{1}{2} \left( 1 + \operatorname{erf}\left(\frac{\eta(z)}{\sqrt{2}}\right) + \sqrt{\frac{2}{\pi}} \frac{\eta(z)}{1+\eta(z)^2} e^{-\frac{\eta(z)^2}{2}} \right) \quad \text{with} \quad \eta(z) = e^{-\frac{i\pi}{8}}z,$$

where  $\operatorname{erf}(\zeta) = \frac{2}{\sqrt{\pi}} \int_0^{\zeta} e^{-\omega^2} d\omega$  is the Gauss error function in  $\zeta \in \mathbb{C}$ .

In this paper, we integrate only entire functions with the definite integrals understood as any path integrals. The existence part of Corollary 1.2 follows from setting  $c_1 = e^{\frac{\pi i}{8}} \sqrt{2/\pi} \neq 0$  in (9), along with the following identity and limits of the error function (cf. the asymptotic expansion 7.12.1 in [25]):

$$\frac{d}{d\eta} \left( \frac{1}{2} \operatorname{erf} \left( \frac{\eta}{\sqrt{2}} \right) + \frac{1}{\sqrt{2\pi}} \frac{\eta e^{-\frac{\eta^2}{2}}}{1 + \eta^2} \right) = \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{\eta^2}{2}}}{(1 + \eta^2)^2}, \quad \lim_{\substack{\zeta \to \infty_{\mathbb{C}} \\ |\operatorname{arg}(\zeta)| < \frac{\pi}{4} - \delta}} \operatorname{erf}(\zeta) = 1, \quad \lim_{\substack{\zeta \to \infty_{\mathbb{C}} \\ |\operatorname{arg}(-\zeta)| < \frac{\pi}{4} - \delta}} \operatorname{erf}(\zeta) = -1,$$

for any  $\delta > 0$ . The uniqueness in Corollary 1.2, however, requires a detailed analysis of the asymptotic expansions of the solutions (8) for general  $\tau \in \mathbb{C}$ , with  $\operatorname{Im} \tau < 0$ . We briefly postpone its explanation to Corollary 1.7.

Remark 1.3. Using a shooting method, Gérard-Varet and Dormy numerically approximated  $\tau \approx -0.706-0.706i$  (cf. Section 5.1 in [10]). This value is close to  $\tau = e^{\frac{5\pi}{4}i} \approx -0.707107-0.707107i$ , as established in Corollary 1.2. They also constructed their instability mechanism using the so-called "shear-layer" velocity, defined in terms of W. Since W is now explicit, we can formulate this shear layer in terms of elementary functions. Details are provided in Section 7, and the corresponding plot (cf. Figure 3 at the end of this paper) matches their numerical simulation (cf. Figure 3, page 607 in [10]).

As we describe in the next section, Theorem 1.1 takes a more natural form under the parabolic shear flow  $U_{\rm sh}$  introduced in (2). In this setting, equation (5) exactly corresponds to the modes of the Prandtl system (1) at any positive frequency  $k \in \mathbb{N}$ , not just asymptotically (cf. Section 2). This perspective provides a direct method for constructing solutions of the Prandtl system (1) in terms of Kummer's functions.

## 1.2. Explicit solutions of the Prandtl equations via harmonic oscillator.

From Theorem 1.1 and (8), we observe that there exists a sequence of  $\tau \in \mathbb{C}$  that generates solutions X decaying exponentially as  $z \to \pm \infty$ :

$$\tau = e^{\frac{\pi \mathrm{i}}{4}}(2n-1), \quad n \in \mathbb{N}_0 \setminus \{1\} \Rightarrow \begin{cases} X = X_{1,\tau} \text{ with } \mathbf{a}_\tau = -\frac{n}{2} \text{ and } \mathbf{b}_\tau = 1 - \frac{n}{2} & \text{if } n \text{ is even,} \\ X = X_{2,\tau} \text{ with } \mathbf{c}_\tau = \frac{1-n}{2} \text{ and } \mathbf{d}_\tau = \frac{3-n}{2} & \text{if } n \text{ is odd.} \end{cases}$$

The case n=0 was described in (9). Since  $\tau$  plays a role similar to an eigenvalue, this behaviour bears a resemblance to the bound state spectrum of the Schrödinger operator and the quantization of energy levels. Our second result aims to demonstrate that this correlation is indeed not accidental.

As further intuition, we draw inspiration from the derivation of the Orr-Sommerfeld equation (e.g. [23]) and recent studies on the well-posedness of the Prandtl equations in Gevrey classes [7]. Defining  $\pm = \operatorname{sgn}(k)$  and introducing the following variable and parameter equivalences (cf. Section 2 for all details):

$$y \in \mathbb{R}_{+} \quad \leftrightarrow \quad z = \sqrt[4]{|\beta||k|}(y - a) \in \mathbb{R} \qquad \leftrightarrow \quad \eta = e^{\mp \frac{\pi}{8}i}z \in \mathbb{C}$$

$$\sigma \in \mathbb{C} \qquad \leftrightarrow \quad \tau = -\frac{i\sigma}{\sqrt{|\beta|}} \pm \alpha \sqrt{\frac{|k|}{|\beta|}} \in \mathbb{C} \quad \leftrightarrow \quad \mu = -i\tau e^{\pm \frac{\pi i}{4}} \in \mathbb{C},$$

the Prandtl eigenproblem, when expressed in the rotated variable  $\eta \in \mathbb{C}$ , takes the form (cf. (29)):

$$-\Upsilon'''(\eta) + (\eta^2 - \mu)\Upsilon'(\eta) - 2\eta\Upsilon(\eta) = 0 \qquad \eta \in \mathbb{C}, \tag{10}$$

where  $\Upsilon(\eta)$  corresponds to  $v_k(y)$ , up to a multiplicative constant. If the last minus sign were positive, the resulting equation would correspond to the derivative of the Schrödinger equation with a quadratic potential  $V(\eta) = \eta^2$ . This observation motivates the introduction of the operator

$$\mathcal{B}_{\mu} = -\frac{d^2}{d\eta^2} + V(\eta) - \mu \quad \Rightarrow \quad \frac{d}{d\eta} \Big( \mathcal{B}_{\mu} \Upsilon(\eta) \Big) = 4\eta \Upsilon(\eta)$$

and heuristically a new quantity  $\Phi = \Phi(\eta)$  defined by  $\mathcal{B}_{\mu}\Phi = 2\Upsilon$ . In resemblance to [7] and [23], a notable simplification arises when considering the commutator of the derivative and the operator  $\mathcal{B}^2_{\mu}$ :

$$\frac{d}{d\eta}\mathcal{B}_{\mu}^{2} = \mathcal{B}_{\mu}^{2}\frac{d}{d\eta} - 4\frac{d}{d\eta} + 4\eta\mathcal{B}_{\mu} \quad \Rightarrow \quad \left[\frac{d}{d\eta}, \mathcal{B}_{\mu}^{2}\right]\Phi = 4\eta\mathcal{B}_{\mu}\Phi - 4\Phi'.$$

Setting  $\psi(\eta) = \Phi'(\eta)$ , we therefore obtain that the  $\Upsilon$ -equation reduces to the following identity

$$\mathcal{B}_{\mu}^{2}\psi - 4\psi = 0. \tag{11}$$

Since  $\mathcal{B}_{\mu} = \mathcal{B}_{\mu-2} - 2$  and  $\mathcal{B}_{\mu} = \mathcal{B}_{\mu+2} + 2$ , one has that  $\mathcal{B}_{\mu}^2 = (\mathcal{B}_{\mu-2} - 2)(\mathcal{B}_{\mu+2} + 2) = \mathcal{B}_{\mu-2}\mathcal{B}_{\mu+2} + 4$ , hence we can factorise the latter equation into

$$\mathcal{B}_{\mu-2}\mathcal{B}_{\mu+2}\psi=0.$$

Both operators  $\mathcal{B}_{\mu+2}$  and  $\mathcal{B}_{\mu-2}$  commute and leave us investigating one of the two kernels, for instance

$$\mathcal{B}_{\mu+2}\psi = -\psi'' + \eta^2\psi - (\mu+2)\psi = 0,$$

which is indeed the Schrödinger equation. Moreover, any solution  $\psi$ ,  $\Upsilon$  (and thus  $v_k$ ) is then determined by the integral (c.f. Proposition 4.1)

$$\Upsilon(\eta) = \frac{1}{2} \mathcal{B}_{\mu} \left[ \int_{\eta_*}^{\eta} \psi(\xi) d\xi \right] 
= \int_{\eta_*}^{\eta} \left( 1 + \frac{\eta^2 - \xi^2}{2} \right) \psi(\xi) d\xi - \frac{\psi'(\eta_*)}{2},$$
(12)

for a constant  $\eta_* \in \mathbb{C}$  that will set the boundary conditions (cf. Theorem 1.4). At this point, we want to clarify that the shift  $\mu \mapsto \mu + 2$  of the associated "energy" spectrum is creating instability. While all (functional analytic) eigenvalues of the quantum harmonic oscillator are positive odd numbers, the smallest value here is  $\mu = 1 - 2 = -1$  leading to a "negative energy" and, in consequence, to an unstable eigenmode.

We aim to make the latter Ansatz explicit employing Kummer's functions. To this end we first recall some result of the harmonic oscillator. For a given constant  $\mu \in \mathbb{C}$ , the general solution  $\psi$  of the equation

$$-\psi''(\eta) + \eta^2 \psi(\eta) = (\mu + 2)\psi(\eta), \qquad \eta \in \mathbb{C}, \tag{13}$$

is given by  $\psi = c_1 \psi_{\mu,1} + c_2 \psi_{\mu,2}$  for arbitrary constants  $c_1, c_2 \in \mathbb{C}$  and functions  $\psi_{\mu,1}$  and  $\psi_{\mu,2}$  defined in terms of the following Kummer's functions (although a classical result, we provide a short proof in Lemma A.1):

$$\psi_{\mu,1}(\eta) := \mathcal{M}\left(-\frac{1+\mu}{4}, \frac{1}{2}, \eta^2\right) \exp\left(-\frac{\eta^2}{2}\right),$$

$$\psi_{\mu,2}(\eta) := \eta \,\mathcal{M}\left(\frac{1-\mu}{4}, \frac{3}{2}, \eta^2\right) \exp\left(-\frac{\eta^2}{2}\right).$$
(14)

For a given  $\eta_* \in \mathbb{C}$ , we define the function  $\Upsilon_{\mu,\eta_*,1} : \mathbb{C} \to \mathbb{C}$  through the operator (12) applied to  $\psi = \psi_{\mu,1}$ :

$$\Upsilon_{\mu,\eta_*,1}(\eta) := \int_{\eta_*}^{\eta} \left( 1 + \frac{\eta^2 - \xi^2}{2} \right) \psi_{\mu,1}(\xi) d\xi - \frac{\psi'_{\mu,1}(\eta_*)}{2}. \tag{15}$$

Similarly, we define  $\Upsilon_{\mu,\eta_*,2}:\mathbb{C}\to\mathbb{C}$  considering two separate cases  $\mu\in\mathbb{C}\setminus\{1\}$  and  $\mu=1$  (cf. also Remark 1.6):

$$\Upsilon_{\mu,\eta_*,2}(\eta) := \begin{cases}
\int_{\eta_*}^{\eta} \left(1 + \frac{\eta^2 - \xi^2}{2}\right) \psi_{\mu,2}(\xi) d\xi - \frac{\psi'_{\mu,2}(\eta_*)}{2} & \text{if } \mu \neq 1, \\
\int_{\eta_*}^{\eta} \left(1 + \frac{\eta^2 - \xi^2}{2}\right) g(\xi) d\xi - \frac{g'(\eta_*)}{2} & \text{with } g(\xi) = e^{\frac{\xi^2}{2}} \operatorname{erf}(\xi) & \text{if } \mu = 1.
\end{cases}$$
(16)

Our second result states that the normal component  $v_k$  of the velocity field in (3) is a linear combination of  $\Upsilon_{\mu,\eta_*,1}$ ,  $\Upsilon_{\mu,\eta_*,2}$  and a quadratic function, provided the variable  $\eta \in \mathbb{C}$  is appropriately related to  $y \in \mathbb{R}_+$  and the constant  $\mu \in \mathbb{C}$  is linked to  $\sigma \in \mathbb{C}$  in (3).

**Theorem 1.4.** Let  $U_{\rm sh}(y) = \alpha + \beta(y-a)^2$  be as in (2) and consider  $k \in \mathbb{Z} \setminus \{0\}$  and  $\sigma \in \mathbb{C}$ . Denote  $\pm = {\rm sgn}(k)$ ,  $\mp = -{\rm sgn}(k)$ , and define the constant  $\mu = \mu(k, \sigma, \alpha, \beta) \in \mathbb{C}$  and  $\eta : \mathbb{R}_+ \to \mathbb{C}$  as

$$\mu := -\frac{\sigma}{\sqrt{|\beta|}} e^{\pm \frac{\pi i}{4}} + \frac{\alpha \sqrt{|k|}}{\sqrt{|\beta|}} e^{\mp \frac{\pi i}{4}} \in \mathbb{C}, \qquad \eta(y) := e^{\mp \frac{\pi}{8} i} \sqrt[4]{|\beta| |k|} (y - a) \in \mathbb{C}.$$
 (17)

Then  $u(t,x,y) = \phi'_k(y)e^{ikx+\sigma\sqrt{|k|}t}$  and  $v(t,x,y) = -ik\phi_k(y)e^{ikx+\sigma\sqrt{|k|}t}$  generate a smooth solution of Equation (1) (without boundary conditions) if and only if  $\phi_k$  satisfies

$$\phi_k(y) = c_0 \left( \mu - \eta(y)^2 \right) + c_1 \left( \Upsilon_{\mu, \eta_*, 1} \circ \eta \right) (y) + c_2 \left( \Upsilon_{\mu, \eta_*, 2} \circ \eta \right) (y) \qquad \forall y \in \mathbb{R}_+, \tag{18}$$

where  $\eta_*, c_0, c_1, c_2 \in \mathbb{C}$  are general constants and  $\Upsilon_{\mu,\eta_*,1}$  and  $\Upsilon_{\mu,\eta_*,2}$  are set in (15) and (16), respectively. Additionally, the no-slip boundary conditions  $\phi_k(0) = \phi_k'(0) = 0$  are satisfied if  $\eta_* = -e^{\mp \frac{\pi}{8}i} a \sqrt[4]{|\beta||k|}$  and the triple  $(c_0, c_1, c_2) \in \mathbb{C}^3$  is a solution of the linear system

$$\begin{pmatrix} \mu - \eta_*^2 & -\frac{1}{2}\psi'_{\mu,1}(\eta_*) & -\frac{1}{2}\psi'_{\mu,2}(\eta_*) \\ -2\eta_* & \psi_{\mu,1}(\eta_*) & \psi_{\mu,2}(\eta_*) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{if } \mu \neq 1,$$
 (19)

while, in the case  $\mu = 1$ ,  $(c_0, c_1 c_2) \in \mathbb{C}^3$  satisfies (19) with  $\psi_{\mu,2}(\eta)$  replaced by  $g(\eta) = e^{\frac{\eta^2}{2}} \operatorname{erf}(\eta)$ .

**Remark 1.5.** Theorem 1.4 provides a systematic approach to computing all solutions of System (1) in the form (3), as well as selecting those that satisfy the no-slip boundary conditions (4) at y = 0:

- (i) Choose a general  $\sigma \in \mathbb{C}$  and  $k \in \mathbb{Z} \setminus \{0\}$  in (3), and compute the constant  $\mu \in \mathbb{C}$  as defined in (17).
- (ii) Determine the two algebraic eigenfunctions  $\psi_{\mu,1}$  and  $\psi_{\mu,2}$  of the Schrödinger operator in (14). These correspond to bound state eigenfunctions if  $\mu = 2n 1$  for some  $n \in \mathbb{N}_0 \setminus \{1\}$  (where Kummer's functions reduce to polynomials, simplifying the next steps).
- (iii) Set  $\eta_* = -e^{\mp \frac{\pi}{8}i} a \sqrt[4]{|\beta||k|}$  to enforce the no-slip boundary conditions. Otherwise, choose any  $\eta_* \in \mathbb{C}$ ; calculations are simplified when setting  $\eta_* = 0$ .
- (iv) Compute the functions  $\Upsilon_{\mu,\eta_*,1}$  and  $\Upsilon_{\mu,\eta_*,2}$  explicitly, as defined in (15) and (16), and define u and v according to the stream function  $\phi_k$  in (18).
- (v) For solutions satisfying the no-slip boundary conditions, select  $c_0$ ,  $c_1$ , and  $c_2$  as given in (19); otherwise, these constants can be chosen freely.

Examples illustrating this method are provided in Section 6.

Remark 1.6. The case  $\mu=1$  in Theorem 1.4 is not particularly relevant for identifying unstable modes, as it forces  $\sigma$  to have negative real part leading to stable modes. However, we can employ a separate analysis because  $\psi_{1,2}(\eta) = \eta e^{-\frac{\eta^2}{2}}$  in (14), and the corresponding function  $\Upsilon(\eta)$  from (12) is then proportional to  $1-\eta^2 = \mu-\eta^2$ . To obtain a linearly independent function  $\Upsilon_{\mu,\eta_*,2}$  in (16), we observe that any function in the kernel of either  $\mathcal{B}_{\mu+2}$  or  $\mathcal{B}_{\mu-2}$  can be chosen. For  $\mu \neq 1$ , we select  $\mathcal{B}_{\mu+2}$ , while for  $\mu=1$ , it is more convenient to consider  $\mathcal{B}_{\mu-2}=\mathcal{B}_{-1}$ . The function  $g(\eta)=e^{\frac{\eta^2}{2}}\operatorname{erf}(\eta)$  lies in the kernel of  $\mathcal{B}_{-1}$  and is linearly independent of both  $\psi_{1,1}(\eta)=e^{\frac{\eta^2}{2}}-\sqrt{\pi}\eta\operatorname{erf}(\eta)$  and  $\psi_{1,2}(\eta)=\eta e^{-\frac{\eta^2}{2}}$ .

We conclude our analysis with a result that transfers well-known asymptotics of Kummer's functions, as defined in (14), to the functions  $\Upsilon_{\mu,\eta_*,1}$  and  $\Upsilon_{\mu,\eta_*,2}$ . In particular, we state the following corollary.

Corollary 1.7. Let  $\mu \in \mathbb{C}$  be such that  $\mu \neq 2n-1$  for any  $n \in \mathbb{N}_0$ , and let the constants  $\eta_*, c_0, c_1, c_2 \in \mathbb{C}$  be arbitrary with  $(c_1, c_2) \neq (0, 0)$ . Then, the function

$$\Upsilon(\eta) = c_0(\mu - \eta^2) + c_1 \Upsilon_{\mu, \eta_*, 1}(\eta) + c_2 \Upsilon_{\mu, \eta_*, 2}(\eta)$$
(20)

satisfies the following asymptotics:

$$\lim_{\begin{subarray}{c} \eta \to \infty, \\ |\arg(\eta)| < \frac{\pi}{4} \end{subarray}} \left| \frac{\Upsilon(\eta)}{\mu - \eta^2} \right| = +\infty \qquad or \qquad \lim_{\begin{subarray}{c} \eta \to \infty, \\ |\arg(-\eta)| < \frac{\pi}{4} \end{subarray}} \left| \frac{\Upsilon(\eta)}{\mu - \eta^2} \right| = +\infty.$$

In case of  $\mu = -1$ , it holds

$$\lim_{\begin{subarray}{c} \eta \to \infty, \\ |\arg(\eta)| < \frac{\pi}{4} \end{subarray}} \left| \frac{\Upsilon(\eta)}{\mu - \eta^2} \right| \neq 0 \qquad or \qquad \lim_{\begin{subarray}{c} \eta \to \infty, \\ |\arg(-\eta)| < \frac{\pi}{4} \end{subarray}} \left| \frac{\Upsilon(\eta)}{\mu - \eta^2} \right| \neq 0.$$

This result ultimately establishes the uniqueness of the function W stated in Corollary 1.2. In Section 2, indeed, we express any pair  $(\tau,W)$  satisfying the ODE (5) as  $(\tau-z^2)W(z)=\Upsilon(\eta(z))$ , with  $\eta(z)=e^{-\frac{\pi}{8}\mathrm{i}}z$ ,  $\mu=e^{-\frac{\pi}{4}\mathrm{i}}\tau$ , and  $\Upsilon$  satisfying the Prandtl eigenproblem (10) (cf. Remark 2.2). Due to Corollary 1.7 and the limits  $\lim_{z\to -\infty}W(z)=0$  and  $\lim_{z\to +\infty}W(z)=1$ , it follows that  $\tau=e^{\frac{\pi}{4}\mathrm{i}}(2n-1)$  for some  $n\in\mathbb{N}_0$ . Furthermore, since  $\mathrm{Im}\,\tau$  must be negative, the only remaining possibility is n=0, leading to  $\tau=e^{\frac{5\pi}{4}\mathrm{i}}$ .

#### 2. Equivalent forms of the linearised Prandtl's System

In this section, we begin our analysis with a review of equivalent forms of the equations (1). We briefly revise two transformations: the first rescales  $y \in \mathbb{R}_+$  into a variable  $z \in \mathbb{R}$  as a spatial scale in k around y = a (cf. (22)); the second rotates  $z \in \mathbb{R}$  to a complex variable  $\eta \in \mathbb{C}$  (cf. (28)), yielding eventually to a third-order linear ODE, a Prandtl eigenproblem, where most coefficients are real (cf. (29)). The novelty of our work does not lie in the equations themselves (some of them already introduced in [10]) but in their explicit resolution (starting from Section 3). Additionally, since our approach is not asymptotic, we also track the corresponding boundary conditions in terms of both z and  $\eta$ . For clarity, we summarize all these change of variables, new functions and parameters in Table 1 (we recall that  $\pm = \operatorname{sgn}(k)$ ).

Variables		
$y \in \mathbb{R}_+$	$z = \sqrt[4]{ \beta  k }(y-a) \in \mathbb{R}$	$\eta = e^{\mp \frac{\pi}{8}i} z \in \mathbb{R} \cdot e^{\mp \frac{\pi}{8}i}$
y = 0	$z_* = -a\sqrt[4]{ \beta  k }$	$\eta_* = -ae^{\mp \frac{\pi}{8}i} \sqrt[4]{ \beta  k }$
Parameters		
$\sigma\in\mathbb{C}$	$\tau = -\frac{\mathrm{i}\sigma}{\sqrt{ \beta }} \pm \alpha \sqrt{\frac{ k }{ \beta }} \in \mathbb{C}$	$\mu = \tau e^{-\frac{\pi}{2}i \pm \frac{\pi}{4}i} \in \mathbb{C}$
Functions		
$u_k(y) = \phi'_k(y)$ $v_k(y) = -ik\phi_k(y)$	$F_{\pm}(z) = \phi_k \left( a + \frac{1}{\sqrt[4]{ \beta  k }} z \right)$ $= (\tau \mp z^2) W_{\pm}(z)$	$\Upsilon(\eta) = F_{\pm}(e^{\pm \frac{\pi}{8}i}\eta)$
	$X_{\pm}(z) = W'_{\pm}(z)$	$Y(\eta) = X_{\pm}(e^{\pm \frac{\pi}{8}i}\eta)$

Table 1. Summary of considered variables, parameters and functions

We begin with formally considering a solution  $(u, v) = (u(t, x, y), v(t, x, y)) \in \mathbb{C}^2$  of System (1) of the form

$$u(t, x, y) = u_k(y)e^{\mathrm{i}kx + \sigma\sqrt{|k|}t}, \qquad v(t, x, y) = v_k(y)e^{\mathrm{i}kx + \sigma\sqrt{|k|}t},$$

for general  $k \in \mathbb{Z} \setminus \{0\}$  and  $\sigma \in \mathbb{C}$ . From the incompressibility condition, we formally look for a stream function  $\phi_k : \mathbb{R}_+ \to \mathbb{C}$  with sufficient regularity (eventually in  $\mathcal{C}^{\infty}(\mathbb{R}_+)$ ), such that  $u_k(y) = \phi'_k(y)$  and  $v_k(y) := -\mathrm{i}k\phi_k(y)$  at any  $y \in \mathbb{R}_+$ . In particular, thanks to (1),  $\phi_k$  satisfies the following equation and boundary conditions:

$$\begin{cases} i\left(-i\sigma\sqrt{|k|} + k\alpha + k\beta(y-a)^2\right)\phi_k'(y) - 2ik\beta(y-a)\phi_k(y) - \phi_k'''(y) = 0, & y \in \mathbb{R}_+, \\ \phi_k(0) = \phi_k'(0) = 0. \end{cases}$$
(21)

where we recall that  $\alpha \in \mathbb{R}$ ,  $\beta < 0$  and  $a \ge 0$  are the parameters of  $U_{\rm sh}(y) = \alpha + \beta(y - a)^2$ .

## 2.1. Rescaling around the critical point.

We next introduce the variable z=z(y) and recast  $\phi_k$  by a function  $F_{\pm}=F_{\pm}(z)$  to reduce most of the k-dependence in (21):

$$z = \sqrt[4]{|\beta||k|}(y-a) \in \left[-a\sqrt[4]{|\beta||k|}, +\infty\right[ \text{ and } F_{\pm}(z) := \phi_k\left(a + \frac{1}{\sqrt[4]{|\beta||k|}}z\right). \tag{22}$$

The above identities are equivalent to  $y = a + z/\sqrt[4]{|\beta||k|}$  and  $\phi_k(y) = F_{\pm}(\sqrt[4]{|\beta||k|}(y-a))$ , for any  $y \in \mathbb{R}_+$ . Thus, we can rewrite System (21) in terms of  $F_{\pm}$  and z:

$$i\left(-i\sigma\sqrt{|k|} + \alpha k + \frac{\beta k}{\sqrt{|\beta||k|}}z^2\right)\sqrt[4]{|\beta||k|}F'_{\pm}(z) - 2ik\beta\frac{z}{\sqrt[4]{|\beta||k|}}F_{\pm}(z) - (|\beta||k|)^{\frac{3}{4}}F'''_{\pm}(z) = 0,$$

with boundary conditions  $F_{\pm}\left(-a\sqrt[4]{|\beta||k|}\right) = F'_{\pm}\left(-a\sqrt[4]{|\beta||k|}\right) = 0$ . We divide the equation by  $(|\beta||k|)^{\frac{3}{4}} > 0$ 

$$i\left(\frac{-i\sigma\sqrt{|k|} + \alpha k}{\sqrt{|\beta||k|}} + \frac{\beta k}{|\beta||k|}z^2\right)F'_{\pm}(z) - 2i\frac{\beta k}{|\beta||k|}zF_{\pm}(z) - F'''_{\pm}(z) = 0.$$

We recall that  $\beta < 0 \Rightarrow \beta/|\beta| = -1$  and we set both the constant  $\tau \in \mathbb{C}$  and the starting point  $z_* \in \mathbb{R}$  by

$$\tau := -\frac{\mathrm{i}\sigma}{\sqrt{|\beta|}} \pm \alpha \sqrt{\frac{|k|}{|\beta|}} \in \mathbb{C}, \qquad z_* := -a\sqrt[4]{|\beta||k|} \in \mathbb{R}. \tag{23}$$

The function  $F_{\pm}$  is therefore a solution of the following third-order linear ordinary differential equation

$$(\tau \mp z^2)F'_{\pm}(z) \pm 2zF_{\pm}(z) + iF'''_{\pm}(z) = 0 \qquad z \in [z_*, \infty[,$$
(24)

where  $\pm$  and  $\mp$  represent  $\operatorname{sgn}(k)$  and  $-\operatorname{sgn}(k)$ , respectively. Moreover, the boundary conditions in (21) are reduced to  $F_{\pm}(z_*) = F'_{\pm}(z_*) = 0$ . We shall remark that if  $\tau = 0$  and  $z_* = 0$  in (24), the function  $F_{\pm}$  is up to a multiplicative constant the trivial solution  $F_{\pm}(z) = z^2$ . The focus of the next steps shall be on different cases. The notation  $F_{\pm}$  indicates that the equation (24) primarily depends on  $\pm = \operatorname{sgn}(k)$ , although this is a slight abuse of notation, as both  $z_*$  and  $\tau$  in (23) depend on  $k \in \mathbb{Z} \setminus \{0\}$ . Additionally, while we focus on real  $z \in [z_*, \infty[$ , equation (24) can also be addressed in the complex plane  $z \in \mathbb{C}$ .

Notably, the function  $z \in [z_*, \infty[ \mapsto (\tau \mp z^2) \in \mathbb{C}]$  satisfies the equation in (24) (though not the boundary conditions). Thus we can apply a reduction of order by introducing the ansatz

$$F_{\pm}(z) = (\tau \mp z^2)W_{\pm}(z),$$
 (25)

for a suitable function  $W_{\pm} = W_{\pm}(z)$  satisfying

$$(\tau \mp z^2)^2 W'_{\pm}(z) + i \frac{d^3}{dz^3} \Big( (\tau \mp z^2) W_{\pm}(z) \Big) = 0 \qquad z \in [z_*, \infty[,$$
 (26)

The boundary conditions are  $(\tau \mp z_*^2)^2 W_{\pm}(z_*) = 0$  and  $(\tau \mp z_*^2)^2 W'_{\pm}(z_*) \mp 2z_* W_{\pm}(z_*) = 0$ , which always implies  $W_{\pm}(z_*) = 0$  for  $(\tau, z_*) \neq (0, 0)$  (as mentioned above, for  $(\tau, z_*) = (0, 0)$ ,  $W_{\pm}$  has to be chosen constant). If  $\tau \mp z_*^2 \neq 0$ , these reduce to  $W(z_*) = W'(z_*) = 0$ .

Remark 2.1. For positive frequencies k > 0, the ODE in (26) coincides with the one introduced in Criterion 1, with  $W = W_+$ . Moreover, recalling that  $z_* = -a \sqrt[4]{|\beta| |k|}$ , if a > 0 and  $\tau \neq 0$ , the function W shall vanish at  $z = -\infty$  in the asymptotic limit  $k \to +\infty$ , as stated in Criterion 1.

The reduction of order is then complete by denoting  $X_{\pm}(z) := W'_{\pm}(z)$  for  $z \in [z_*, \infty[$  and formally writing the second-order ODE

$$i(\tau \mp z^2)X''_{\pm}(z) \mp 6izX'_{\pm}(z) + ((\tau \mp z^2)^2 \mp 6i)X_{\pm}(z) = 0 \qquad z \in [z_*, \infty[.$$
 (27)

If  $\tau \mp z_*^2 \neq 0$ , the boundary condition is moreover  $X_{\pm}(z_*) = 0$ , while if  $\tau \mp z_*^2 = 0$  (but  $(\tau, z_*) \neq (0, 0)$ ) any solution  $X_{\pm}$  will yield the boundary conditions of  $F_{\pm}$ . While our focus is on real  $z \in [z_*, \infty[$ ,  $X_{\pm}$  can be extended meromorphically in  $\mathbb C$  with singularities only at solutions of  $\tau \mp z^2 = 0$ .

When  $\pm = +$  (i.e. for positive frequencies), equation (27) coincides with equation (7) stated in the introduction. Our analysis requires a further change of variable to eliminate the imaginary parts of most coefficients in both (24) and (27) as well as the dependence of the equations on  $\pm = \operatorname{sgn}(k)$ .

## 2.2. A further reduction through a rotation.

We introduce the new variable  $\eta \in \mathbb{C}$  together with the function  $\Upsilon : \mathbb{C} \to \mathbb{C}$  by means of the relations:

$$\eta := e^{\mp \frac{\pi}{8}i} z = e^{\mp \frac{\pi}{8}i} \sqrt[4]{|\beta||k|} (y - a) \in \mathbb{C}, \qquad \Upsilon(\eta) := F_{\pm} \left( e^{\pm \frac{\pi}{8}i} \eta \right) = \phi_k \left( a + \frac{e^{\pm \frac{\pi}{8}i}}{\sqrt[4]{|\beta||k|}} \eta \right). \tag{28}$$

Once again, while our main interest is in  $\eta \in \mathbb{R} \cdot e^{\mp \frac{\pi}{8}i}$ , the function  $F_{\pm}$  entirely extends to  $\mathbb{C}$ , allowing us to seek *a-priori* a function  $\Upsilon$  defined in  $\mathbb{C}$ . By setting the parameter and initial point

$$\mu = -\mathrm{i}\,e^{\pm\frac{\pi}{4}\mathrm{i}}\tau = -\frac{e^{\pm\frac{\pi}{4}\mathrm{i}}}{\sqrt{|\beta|}}\left(\sigma \pm \mathrm{i}\alpha\sqrt{|k|}\right) \in \mathbb{C}, \qquad \eta_* = e^{\mp\frac{\pi}{8}\mathrm{i}}z_* = -ae^{\mp\frac{\pi}{8}\mathrm{i}}\sqrt[4]{|\beta||k|} \in \mathbb{R} \cdot e^{\mp\frac{\pi}{8}\mathrm{i}},$$

we obtain from System (24) that  $\Upsilon(\eta) = F_{\pm}(e^{\pm \frac{\pi}{8}i}\eta)$  must satisfy the third-order linear ODE

$$\mathrm{i} \big(\tau \mp \big(e^{\pm \frac{\pi}{8}\mathrm{i}}\eta\big)^2\big)e^{\mp \frac{\pi}{8}\mathrm{i}}\Upsilon'(\eta) \pm 2\mathrm{i} e^{\pm \frac{\pi}{8}\mathrm{i}}\eta\Upsilon(\eta) - e^{\mp \frac{3\pi}{8}\mathrm{i}}\Upsilon'''(\eta) = 0.$$

Multiplying the last equation by  $-e^{\pm \frac{3\pi}{8}i}$  leads to

$$\left(\left(-i\tau e^{\pm\frac{\pi}{4}i}\right)\pm ie^{\pm\frac{\pi}{2}i}\eta^2\right)\Upsilon'(\eta)\mp 2ie^{\pm\frac{\pi}{2}i}\eta\Upsilon(\eta)+\Upsilon'''(\eta)=0,$$

which eventually corresponds to

$$(\mu - \eta^2)\Upsilon'(\eta) + 2\eta\Upsilon(\eta) + \Upsilon'''(\eta) = 0 \qquad \eta \in \mathbb{C}, \tag{29}$$

as it was stated in (10). The accompanied boundary conditions are  $\Upsilon(\eta_*) = \Upsilon'(\eta_*) = 0$ . Section 4 is devoted to determine the general solution of (29) (as well as the ones satisfying the boundary conditions). From the exact form of  $\Upsilon$  we can therefore determine the exact form of  $F_{\pm}$  and thus also of  $\phi_k$  using (28).

**Remark 2.2.** Using (25) and (28), the function  $\Upsilon$  satisfies (29) if and only if there exists  $W_{\pm}(z)$  satisfying (26) such that  $\Upsilon(e^{\mp \frac{\pi i}{8}}z) = (\tau \mp z^2)W_{\pm}(z)$ . By Corollary 1.7, the asymptotics of  $\Upsilon$  determine those of  $W_{\pm}$  as  $z \to \pm \infty$ .

Finally, we introduce a last function  $Y = Y(\eta)$ , which plays the homologous role of  $X_{\pm}$  with respect to the variable  $\eta$ . For any  $\eta \in \mathbb{C}$  with  $\eta^2 \neq \mu$ , we set  $Y(\eta) := X_{\pm}(e^{\pm \frac{\pi}{8}i}\eta)$  and from (27), it satisfies the equation

$$(\mu - \eta^2)Y''(\eta) - 6\eta Y'(\eta) + ((\mu - \eta^2)^2 - 6)Y(\eta) = 0, \qquad \eta \in \mathbb{C}.$$
(30)

The next section is devoted to determine explicitly all solutions of (30).

#### 3. Proof of Theorem 1.1

The following section is devoted to the proof of Theorem 1.1 and to determine the general solution X = X(z) of Equation (7). We rather address the equivalent form in  $Y = Y(\eta) = X\left(e^{\frac{\pi}{8}i}\eta\right)$ , that satisfies equation (30) with  $\mu = \tau e^{-\frac{\pi}{4}i}$ .

**Proposition 3.1.** Let  $\mu \in \mathbb{C}$  be an arbitrary complex number, and define  $\mathcal{A}_{\mu} = \{ \eta \in \mathbb{C} \mid \eta^2 = \mu \}$ . All solutions  $Y : \mathbb{C} \setminus \mathcal{A}_{\mu} \to \mathbb{C}$  of the complex ordinary differential equation

$$(\mu - \eta^2)Y''(\eta) - 6\eta Y'(\eta) + ((\mu - \eta^2)^2 - 6)Y(\eta) = 0, \qquad \eta \in \mathbb{C} \setminus \mathcal{A}_{\mu}, \tag{31}$$

can explicitly be written as

$$Y(\eta) = c_1 Y_{\mu,1}(\eta) + c_1 Y_{\mu,2}(\eta), \qquad \eta \in \mathbb{C} \setminus \mathcal{A}_{\mu}$$

where  $c_1$  and  $c_2$  are arbitrary complex numbers, while the functions  $Y_{\mu,1}$  and  $Y_{\mu,2}$  are given by

$$Y_{\mu,1}(\eta) := \frac{\exp\left(-\frac{\eta^2}{2}\right)}{(\mu - \eta^2)^2} R_{\mu,1}(\eta) \quad \text{with} \quad R_{\mu,1}(\eta) := \sum_{n=0}^{\infty} \frac{\mu - 2n}{n!} \frac{\left(-\frac{\mu+1}{4}\right)_n}{\left(\frac{1}{2}\right)_n} \eta^{2n},$$

$$Y_{\mu,2}(\eta) := \frac{\exp\left(-\frac{\eta^2}{2}\right)}{(\mu - \eta^2)^2} R_{\mu,2}(\eta) \quad \text{with} \quad R_{\mu,2}(\eta) := \eta + \frac{1}{4} \sum_{n=1}^{\infty} \frac{2n + 1 - \mu}{n!} \frac{\left(\frac{5 - \mu}{4}\right)_{n-1}}{\left(\frac{3}{2}\right)_n} \eta^{2n+1}.$$

$$(32)$$

**Remark 3.2.** The power series  $R_{\mu,1}$  and  $R_{\mu,2}$ , as defined in (32), are even and odd, respectively. Both have an infinite radius of convergence, which implies that they define entire functions in  $\mathbb{C}$ . The functions  $Y_{\mu,1}$  and  $Y_{\mu,2}$  are therefore meromorphic across  $\mathbb{C}$ , exhibiting singularities where  $\eta^2 = \mu$  and having at each singular point residue equal to zero. Consequently, both functions possess primitive functions that are meromorphic over the same domain.

Before proceeding with the proof of Proposition 3.1, we present a corollary (whose proof is at the end of this section), that expresses the solutions  $Y_{\mu,1}$  and  $Y_{\mu,2}$  in (32) in a compact form depending on the Kummer's hypergeometric function  $\mathcal{M}$ .

Corollary 3.3. Under the conditions of Proposition 3.1, the functions  $Y_{\mu,1}$  and  $Y_{\mu,2}$  defined in (32) satisfy

$$Y_{\mu,1}(\eta) := \frac{\exp\left(-\frac{\eta^2}{2}\right)}{(\mu - \eta^2)^2} \left[ \mu \mathcal{M}\left(-\frac{\mu + 1}{4}, \frac{1}{2}, \eta^2\right) + (\mu + 1)\eta^2 \mathcal{M}\left(\frac{3 - \mu}{4}, \frac{3}{2}, \eta^2\right) \right],$$

$$Y_{\mu,2}(\eta) := \eta \frac{\exp\left(-\frac{\eta^2}{2}\right)}{(\mu - \eta^2)^2} \left[ \mathcal{M}\left(\frac{1 - \mu}{4}, \frac{3}{2}, \eta^2\right) + \frac{\eta^2}{3} \mathcal{M}\left(\frac{5 - \mu}{4}, \frac{5}{2}, \eta^2\right) \right],$$

where  $\mathcal{M}$  is the Kummer's function introduced in (6).

Theorem 1.1 follows from Corollary 3.3 by setting  $\mu = \tau e^{-\frac{\pi i}{4}}$  and  $\eta = e^{-\frac{\pi i}{8}}z$ .

Proof of Proposition 3.1. We carry out the proof by three major steps:

(i) We reformulate the ordinary differential equation in (31) in terms of a new function  $R: \mathbb{C} \setminus \mathcal{A}_{\mu} \to \mathbb{C}$ , setting

$$Y(\eta) = \frac{e^{-\frac{\eta^2}{2}}}{(\mu - \eta^2)^2} R(\eta). \tag{33}$$

We show that this transformation recasts Equation (31) into the following ODE for R:

$$(\mu - \eta^2)R''(\eta) - 2\eta(\mu - 1 - \eta^2)R'(\eta) + (\mu + 1)(\mu - 2 - \eta^2)R(\eta) = 0 \qquad \eta \in \mathbb{C} \setminus \mathcal{A}_{\mu}.$$
 (34)

This is stated in Lemma 3.4.

(ii) We seek two independent non-trivial solutions R of (34) making use of a power series approach. To this end, we consider two general sequences  $(a_n)_{n\in\mathbb{N}_0}\subseteq\mathbb{C}$  and  $(b_n)_{n\in\mathbb{N}_0}\subseteq\mathbb{C}$ , assuming momentary that both

$$R_{\mu,1}(\eta) := \sum_{n=0}^{\infty} a_n \eta^{2n}, \qquad R_{\mu,2}(\eta) := \sum_{n=0}^{\infty} b_n \eta^{2n+1}.$$
 (35)

have positive radii of convergence. We then identify the recursive conditions on these sequences ensuring that  $R_{\mu,1}$  and  $R_{\mu,2}$  are solutions of (34) within their domain of definition. These are eventually:

$$\begin{cases}
2\mu a_1 + (\mu^2 - \mu - 2)a_0 = 0, \\
2\mu(2n^2 + 3n + 1)a_{n+1} - (4n^2 + (4\mu - 6)n - \mu^2 + \mu + 2)a_n + (4n - 5 - \mu)a_{n-1} = 0 \quad \forall n \in \mathbb{N},
\end{cases}$$
(36)

and

$$\begin{cases}
6\mu b_1 + (\mu^2 - 3\mu)b_0 = 0, \\
2\mu(2n^2 + 5n + 3)b_{n+1} - (4n^2 + (4\mu - 2)n - \mu^2 + 3\mu)b_n + (4n - 3 - \mu)b_{n-1} = 0
\end{cases} \quad \forall n \in \mathbb{N}.$$
(37)

This is stated in Lemma 3.5

(iii) We demonstrate that both (36) and (37) are met by the sequences written in (32), namely

$$a_n = \frac{\mu - 2n}{n!} \frac{\left(-\frac{\mu + 1}{4}\right)_n}{\left(\frac{1}{2}\right)_n} \quad \forall n \in \mathbb{N}_0 \quad \text{and} \quad b_n = \begin{cases} 1 & \text{if } n = 0, \\ \frac{1}{4} \frac{2n + 1 - \mu}{n!} \frac{\left(\frac{5 - \mu}{4}\right)_{n - 1}}{\left(\frac{3}{2}\right)_n} & \text{if } n \in \mathbb{N}. \end{cases}$$
(38)

The conclusion of the proof follows from the fact that the series  $R_{1,\mu}$  and  $R_{2,\mu}$ , generated by these sequences, are entire with infinite radii of convergence, making them solutions of (34), in fact, in all  $\mathbb{C}$ .

Given the technicality of Part (i) and Part (ii), we postpone the related proofs to Lemma 3.4 and Lemma 3.5 below, respectively. We focus the next steps on proving Part (iii), assuming momentarily that (i) and (ii) hold true. The sequence  $(a_n)_{n\in\mathbb{N}_0}$  in (38) satisfies the first identity of (36), since

$$2\mu a_1 + (\mu^2 - \mu - 2)a_0 = 2\mu(\mu - 2)\frac{-\frac{\mu + 1}{4}}{\frac{1}{2}} + (\mu^2 - \mu - 2)\mu = -\mu(\mu - 2)(\mu + 1) + (\mu^2 - \mu - 2)\mu = 0.$$

Similarly, the sequence  $(b_n)_{n\in\mathbb{N}_0}$  in (38) satisfies the first identity of (37), since

$$6\mu b_1 + (\mu^2 - 3\mu)b_0 = 6\mu \frac{1}{4}(3 - \mu)\frac{1}{\frac{3}{2}} + (\mu^2 - 3\mu) = \mu(3 - \mu) + \mu(\mu - 3) = 0.$$

It remains therefore to show that the recursive formula in (36) and (37) are satisfied. We note that the first term in the second equation of (36) can be written as

$$2\mu(2n^{2}+3n+1)a_{n+1} = 2\mu(n+1)(2n+1)\frac{\mu-2(n+1)}{(n+1)!}\frac{4n-1-\mu}{2(2n+1)}\frac{\left(-\frac{\mu+1}{4}\right)_{n}}{\left(\frac{1}{2}\right)_{n}}$$

$$= \mu(-2n+\mu-2)(4n-1-\mu)\frac{1}{n!}\frac{\left(-\frac{\mu+1}{4}\right)_{n}}{\left(\frac{1}{2}\right)_{n}}$$

$$= \left(-8\mu n^{2} + (6\mu^{2}-6\mu)n - \mu^{3} + \mu^{2} + 2\mu\right)\frac{1}{n!}\frac{\left(-\frac{\mu+1}{4}\right)_{n}}{\left(\frac{1}{2}\right)_{n}},$$
(39)

for any  $n \in \mathbb{N}$ . Additionally, for the second term of the recursive formula in (36), we have that

$$-(4n^{2} + (4\mu - 6)n - \mu^{2} + \mu + 2)a_{n} = -(4n^{2} + (4\mu - 6)n - \mu^{2} + \mu + 2)(\mu - 2n)\frac{1}{n!}\frac{\left(-\frac{\mu+1}{4}\right)_{n}}{\left(\frac{1}{2}\right)_{n}}$$

$$= \left(8n^{3} + (4\mu - 12)n^{2} + (4 + 8\mu - 6\mu^{2})n + \mu^{3} - \mu^{2} - 2\mu\right)\frac{1}{n!}\frac{\left(-\frac{\mu+1}{4}\right)_{n}}{\left(\frac{1}{2}\right)_{n}}.$$

$$(40)$$

Finally, the last term in the recursive formula of (36) can be expressed as

$$(4n - 5 - \mu)a_{n-1} = (2n - 1)\frac{4(n - 1) - (\mu + 1)}{2n - 1}a_{n-1} = (2n - 1)\frac{4}{2}\frac{-\frac{\mu + 1}{4} + n - 1}{\frac{1}{2} + n - 1}a_{n-1}$$

$$= (2n - 1)2\frac{-\frac{\mu + 1}{4} + n - 1}{\frac{1}{2} + n - 1}\frac{\mu - 2n + 2}{(n - 1)!}\frac{\left(-\frac{\mu + 1}{4}\right)_{n-1}}{\left(\frac{1}{2}\right)_{n-1}} = 2n(2n - 1)(\mu - 2n + 2)\frac{1}{n!}\frac{\left(-\frac{\mu + 1}{4}\right)_{n}}{\left(\frac{1}{2}\right)_{n}}$$

$$= \left(-8n^{3} + (4\mu + 12)n^{2} - (4 + 2\mu)n\right)\frac{1}{n!}\frac{\left(-\frac{\mu + 1}{4}\right)_{n}}{\left(\frac{1}{2}\right)_{n}}.$$

$$(41)$$

The total sum of the right-hand-sides in the identities (39), (40) and (41) is identically null. This implies that the sequence  $(a_n)_{n\in\mathbb{N}_0}$  satisfies the second equation in (36).

A similar argument holds true also for the sequence  $(b_n)_{n\in\mathbb{N}_0}$  in (38) and the second equation in (37). We indeed remark that for n=1

$$2\mu(2n^{2} + 5n + 3)b_{n+1} - \left(4n^{2} + (4\mu - 2)n - \mu^{2} + 3\mu\right)b_{n} + (4n - 3 - \mu)b_{n-1}$$

$$= 20 \mu b_{2} - (2 + 7\mu - \mu^{2})b_{1} + (1 - \mu)b_{0} = 20\mu \frac{(5 - \mu)^{2}}{120} - (2 + 7\mu - \mu^{2})\frac{3 - \mu}{6} + (1 - \mu)$$

$$= \frac{\mu^{3} - 10\mu^{2} + 25\mu}{6} - \frac{6 + 19\mu - 10\mu^{2} + \mu^{3}}{6} + 1 - \mu = 0.$$

Moreover, with  $n \ge 2$ , the first term  $2\mu(2n^2 + 5n + 3)b_{n+1}$  in the second equation of (37) is

$$2\mu(2n^{2} + 5n + 3)b_{n+1} = 2\mu(2n + 3)(n + 1)\frac{1}{4}\frac{2n + 3 - \mu}{(n + 1)!}\frac{\left(\frac{5 - \mu}{4}\right)_{n}}{\left(\frac{3}{2}\right)_{n+1}}$$

$$= \frac{\mu}{2}(2n + 3)(2n + 3 - \mu)\frac{1}{n!}\frac{\left(\frac{5 - \mu}{4} + n - 1\right)\left(\frac{5 - \mu}{4}\right)_{n-1}}{\left(\frac{3}{2} + n\right)\left(\frac{3}{2}\right)_{n}}$$

$$= \frac{\mu}{2}(2n + 3)(2n + 3 - \mu)\frac{1}{n!}\frac{1}{2}\frac{4n + 1 - \mu}{2n + 3}\frac{\left(\frac{5 - \mu}{4}\right)_{n-1}}{\left(\frac{3}{2}\right)_{n}}$$

$$= \frac{\mu}{4}(2n + 3 - \mu)(4n + 1 - \mu)\frac{1}{n!}\frac{\left(\frac{5 - \mu}{4}\right)_{n-1}}{\left(\frac{3}{2}\right)_{n}}$$

$$= \frac{1}{4}\left(8\mu n^{2} + (14\mu - 6\mu^{2})n + \mu^{3} - 4\mu^{2} + 3\mu\right)\frac{1}{n!}\frac{\left(\frac{5 - \mu}{4}\right)_{n-1}}{\left(\frac{3}{2}\right)_{n}},$$
(42)

while the second term in the recursive formula of (37) is

$$-(4n^{2} + (4\mu - 2)n - \mu^{2} + 3\mu)b_{n} = -(4n^{2} + (4\mu - 2)n - \mu^{2} + 3\mu)\frac{1}{4}\frac{2n + 1 - \mu}{n!}\frac{\left(\frac{5-\mu}{4}\right)_{n-1}}{\left(\frac{3}{2}\right)_{n}}$$

$$= \frac{1}{4}\left(-8n^{3} - 4\mu n^{2} + (6\mu^{2} - 12\mu + 2)n - \mu^{3} + 4\mu^{2} - 3\mu\right)\frac{1}{n!}\frac{\left(\frac{5-\mu}{4}\right)_{n-1}}{\left(\frac{3}{2}\right)_{n}}.$$
(43)

Finally, the last term in the recursive formula of (37) can be expressed as

$$(4n - 3 - \mu)b_{n-1} = (2n + 1)\frac{4(n - 2) + (5 - \mu)}{2n + 1}b_{n-1} = (2n + 1)\frac{4}{2}\frac{\frac{5 - \mu}{4} + n - 2}{\frac{3}{2} + n - 1}b_{n-1}$$

$$= (2n + 1)2\frac{\frac{5 - \mu}{4} + n - 2}{\frac{3}{2} + n - 1}\frac{1}{4}\frac{2n - 1 - \mu}{(n - 1)!}\frac{\left(\frac{5 - \mu}{4}\right)_{n-2}}{\left(\frac{3}{2}\right)_{n-1}} = \frac{1}{2}(2n + 1)(2n - 1 - \mu)\frac{n}{n!}\frac{\left(\frac{5 - \mu}{4}\right)_{n-1}}{\left(\frac{3}{2}\right)_n}$$

$$= \frac{1}{4}\left(8n^3 - 4\mu n^2 + (-2 - 2\mu)n\right)\frac{1}{n!}\frac{\left(\frac{5 - \mu}{4}\right)_n}{\left(\frac{3}{2}\right)_n}.$$
(44)

The total sum of the right-hand-sides in the identities (42), (43) and (44) is identically null. This implies that the sequence  $(b_n)_{n\in\mathbb{N}_0}$  satisfies the recursive equation in (37). This concludes the proof of Proposition 3.1.  $\square$ 

The following lemma addresses Part (i) of Proposition 3.1, reformulating the ODE (31) for Y as the equivalent ODE (34) in terms of the function R in (33).

**Lemma 3.4.** Let  $\mu \in \mathbb{C}$  be arbitrary and let  $\mathcal{A}_{\mu} = \{ \eta \in \mathbb{C} \mid \eta^2 = \mu \}$ . A function  $Y : \mathbb{C} \setminus A_{\mu} \to \mathbb{C}$  is a solution of

$$(\mu - \eta^2)Y''(\eta) - 6\eta Y'(\eta) + ((\mu - \eta^2)^2 - 6)Y(\eta) = 0, \qquad \eta \in \mathbb{C} \setminus \mathcal{A}_{\mu}, \tag{45}$$

if and only if there exists a function  $R: \mathbb{C} \setminus A_{\mu} \to \mathbb{C}$  that satisfies both (33) and the ODE

$$(\mu - \eta^2)R''(\eta) - 2\eta(\mu - 1 - \eta^2)R'(\eta) + (\mu + 1)(\mu - 2 - \eta^2)R(\eta) = 0 \qquad \eta \in \mathbb{C} \setminus \mathcal{A}_{\mu}. \tag{46}$$

*Proof.* The proof follows a straightforward approach: the equation (45) is reformulated so that every term involving Y aligns with the form given by R in (33). We first observe that (45) can be written as

$$\frac{1}{(\mu - \eta^2)^2} \frac{d}{d\eta} \Big( (\mu - \eta^2)^3 Y'(\eta) \Big) + \Big( (\mu - \eta^2)^2 - 6 \Big) Y(\eta) = 0, \qquad \eta \in \mathbb{C} \setminus A_{\mu}.$$

Using the product rule, we hence collect outside the derivative of  $Y'(\eta)$ :

$$\frac{1}{(\mu - \eta^2)^2} \frac{d^2}{d\eta^2} \Big( (\mu - \eta^2)^3 Y(\eta) \Big) + \frac{1}{(\mu - \eta^2)^2} \frac{d}{d\eta} \Big( 6\eta(\mu - \eta^2)^2 Y(\eta) \Big) + \Big( (\mu - \eta^2)^2 - 6 \Big) Y(\eta) = 0.$$

Hence, developing the derivative in the second term while retaining the product  $(\mu - \eta^2)^2 Y(\eta)$  together, we get

$$\frac{1}{(\mu - \eta^2)^2} \frac{d^2}{d\eta^2} \Big( (\mu - \eta^2)^3 Y(\eta) \Big) + \frac{6\eta}{(\mu - \eta^2)^2} \frac{d}{d\eta} \Big( (\mu - \eta^2)^2 Y(\eta) \Big) + (\mu - \eta^2)^2 Y(\eta) = 0. \tag{47}$$

Next, we handle the first term with the second derivatives and we rewrite it to highlight the dependence on the function  $(\mu - \eta^2)^2 Y(\eta)$ :

$$\begin{split} \frac{d^2}{d\eta^2} \Big( (\mu - \eta^2)^3 Y(\eta) \Big) &= \frac{d^2}{d\eta^2} \Big( (\mu - \eta^2) (\mu - \eta^2)^2 Y(\eta) \Big) \\ &= (\mu - \eta^2) \frac{d^2}{d\eta^2} \Big( (\mu - \eta^2)^2 Y(\eta) \Big) - 4\eta \frac{d}{d\eta} \Big( (\mu - \eta^2)^2 Y(\eta) \Big) - 2(\mu - \eta^2)^2 Y(\eta). \end{split}$$

Plugging the last relation into (47) yields therefore to the following identity:

$$\frac{1}{\mu - \eta^2} \frac{d^2}{d\eta^2} \Big( (\mu - \eta^2)^2 Y(\eta) \Big) + \frac{2\eta}{(\mu - \eta^2)^2} \frac{d}{d\eta} \Big( (\mu - \eta^2)^2 Y(\eta) \Big) + \Big( (\mu - \eta^2)^2 - 2 \Big) Y(\eta) = 0.$$

We next set  $R(\eta) = e^{\frac{\eta^2}{2}}(\mu - \eta^2)Y(\eta)$  and multiply the resulting equation by  $(\mu - \eta^2)^2$ . We thus obtain

$$(\mu - \eta^2) \frac{d^2}{d\eta^2} \left( e^{-\frac{\eta^2}{2}} R(\eta) \right) + 2\eta \frac{d}{d\eta} \left( e^{-\frac{\eta^2}{2}} R(\eta) \right) + \left( (\mu - \eta^2)^2 - 2 \right) e^{-\frac{\eta^2}{2}} R(\eta) = 0. \tag{48}$$

Hence, thanks to the identities

$$\frac{d}{d\eta} \Big( e^{-\frac{\eta^2}{2}} R(\eta) \Big) = e^{-\frac{\eta^2}{2}} \Big( R'(\eta) - \eta R(\eta) \Big), \quad \frac{d^2}{d\eta^2} \Big( e^{-\frac{\eta^2}{2}} R(\eta) \Big) = e^{-\frac{\eta^2}{2}} \Big( R''(\eta) - 2\eta R'(\eta) + (\eta^2 - 1) R(\eta) \Big),$$

we gather that the last equation in (48) corresponds to

$$(\mu - \eta^2)R''(\eta) + \left((\mu - \eta^2)(-2\eta) + 2\eta\right)R'(\eta) + \left((\mu - \eta^2)(\eta^2 - 1) + 2\eta(-\eta) + (\mu - \eta^2)^2 - 2\right)R(\eta) = 0.$$

Finally, since  $(\mu - \eta^2)(\eta^2 - 1) + (\mu - \eta^2)^2 = (\mu - \eta^2)(\mu - 1) = (\mu - \eta^2)(\mu + 1) - 2\mu + 2\eta^2$ , we obtain

$$(\mu - \eta^2)R''(\eta) - 2\eta(\mu - 1 - \eta^2)R'(\eta) + ((\mu - \eta^2)(\mu + 1) - 2\mu - 2)R(\eta) = 0,$$

which, together with  $(\mu - \eta^2)(\mu - \eta^2)(\mu + 1) - 2\mu - 2 = (\mu + 1)(\mu - 2 - \eta^2)$ , correspond to the ODE in (46).

To complete the proof of Proposition 3.1, it remains to establish Part (ii), concerning the recursive conditions required for the power series  $R_{\mu,1}$  and  $R_{\mu,2}$  in (35) to satisfy (at least locally) the ODE in (34).

**Lemma 3.5.** Let  $R_{\mu,1}(\eta) = \sum_{n=0}^{\infty} a_n \eta^{2n}$  and  $R_{\mu,2}(\eta) = \sum_{n=0}^{\infty} b_n \eta^{2n+1}$  be complex power series with nonzero radii of convergence. Then  $R_{\mu,1}$  and  $R_{\mu,2}$  are independent local solutions of the ODE (34) if and only if the sequences  $(a_n)_{n\in\mathbb{N}_0}$  and  $(b_n)_{n\in\mathbb{N}_0}$  satisfy the recursive conditions in (36) and (37).

*Proof.* We begin by expanding each component of the ODE (34). When applied to the power series  $R = R_{\mu,1}$ , the first term in (34) corresponds to

$$(\mu - \eta^2) R_{\mu,1}^{"}(\eta) = (\mu - \eta^2) \sum_{n=1}^{\infty} 2n(2n-1)a_n \eta^{2(n-1)} = \mu \sum_{k=0}^{\infty} 2(k+1)(2k+1)a_{k+1}\eta^{2k} +$$

$$- \sum_{n=1}^{\infty} 2n(2n-1)a_n \eta^{2n} = 2\mu a_1 + \sum_{n=1}^{\infty} \left(2\mu(n+1)(2n+1)a_{n+1} - 2n(2n-1)a_n\right) \eta^{2n}.$$

$$(49)$$

Next, we expand the second term  $-2\eta(\mu-1-\eta^2)R'(\eta)$  in (34). when replacing R with  $R_{\mu,1}$ :

$$-2\eta(\mu - 1 - \eta^{2})R'_{\mu,1}(\eta) = -2\eta(\mu - 1 - \eta^{2})\sum_{n=1}^{\infty} 2n \, a_{n}\eta^{2n-1}$$

$$= -2(\mu - 1)\sum_{n=1}^{\infty} 2n \, a_{n}\eta^{2n} + 2\sum_{k=2}^{\infty} 2(k-1) \, a_{k-1}\eta^{2k} = \sum_{n=1}^{\infty} \left(-4(\mu - 1)n \, a_{n} + 4\underbrace{(n-1)a_{n-1}}_{=0 \text{ if } n-1}\right)\eta^{2n}.$$
(50)

Finally, the last term  $(\mu + 1)(\mu - 2 - \eta^2)R(\eta)$  in Equation (34) with  $R = R_{\mu,1}$  can be expressed as

$$(\mu+1)(\mu-2-\eta^2)R_{\mu,1}(\eta) = (\mu+1)(\mu-2-\eta^2)\sum_{n=0}^{\infty} a_n\eta^{2n} = (\mu+1)(\mu-2)\sum_{n=0}^{\infty} a_n\eta^{2n} + (\mu+1)\sum_{k=1}^{\infty} a_{k-1}\eta^{2k} = (\mu+1)(\mu-2)a_0 + \sum_{n=1}^{\infty} ((\mu+1)(\mu-2)a_n - (\mu+1)a_{n-1})\eta^{2n}.$$
(51)

Summing the expressions in (49), (50) and (51), and observing that the following identities hold true

$$-2n(2n-1)a_n - 4(\mu-1)na_n + (\mu+1)(\mu-2)a_n = (-4n^2 + (-4\mu+6)n + \mu^2 - \mu - 2)a_n,$$
  
$$4(n-1)a_{n-1} - (\mu+1)a_{n-1} = (4n-5-\mu)a_{n-1},$$

we obtain that the power series  $R = R_{\mu,1}$  is a local solution of Equation (34) if and only if the following power series is identically null:

$$2\mu a_1 + (\mu + 1)(\mu - 2)a_0 + \sum_{n=1}^{\infty} \left\{ 2\mu(n+1)(2n+1)a_{n+1} - \left[4n^2 + (4\mu - 6)n - \mu^2 + \mu + 2\right]a_n + (4n - 5 - \mu)a_{n-1} \right\} \eta^{2n} = 0.$$

Since  $(\mu+1)(\mu-2) = \mu^2 - \mu - 2$  and  $2\mu(n+1)(2n+1) = 2\mu(2n^2+3n+1)$ , this holds if and only if the sequence  $(a_n)_{n\in\mathbb{N}}$  satisfies the recursive relations specified in (36).

We consider next the power series  $R_{\mu,2}$ . Replacing  $R = R_{\mu,2}$  in the first term  $(\mu - \eta^2)R''(\eta)$  of (34), we obtain

$$(\mu - \eta^2) R_{\mu,2}^{"}(\eta) = (\mu - \eta^2) \sum_{n=1}^{\infty} (2n+1) 2n \, b_n \eta^{2n-1} = \mu \sum_{k=0}^{\infty} (2k+3) (2k+2) b_{k+1} \eta^{2k+1} +$$

$$- \sum_{n=1}^{\infty} (2n+1) 2n \, b_n \eta^{2n+1} = 6\mu \, b_1 \eta + \sum_{n=1}^{\infty} \left( 2\mu (2n+3) (n+1) b_{n+1} - 2n (2n+1) b_n \right) \eta^{2n+1}.$$
(52)

Analogously, setting  $R = R_{2,\mu}$  into the second term  $-2\eta(\mu - 1 - \eta^2)R'(\eta)$  of (34), we get

$$-2\eta(\mu - 1 - \eta^{2})R'_{\mu,2}(\eta) = -2\eta(\mu - 1 - \eta^{2})\sum_{n=0}^{\infty} (2n+1)b_{n}\eta^{2n} = -2(\mu - 1)\sum_{n=0}^{\infty} (2n+1)b_{n}\eta^{2n+1} + 2\sum_{k=1}^{\infty} (2k-1)b_{k-1}\eta^{2k+1} = -2(\mu - 1)b_{0}\eta + \sum_{n=1}^{\infty} \left(-2(\mu - 1)(2n+1)b_{n} + 2(2n-1)b_{n-1}\right)\eta^{2n+1}.$$
(53)

Finally, the last term  $(\mu + 1)(\mu - 2 - \eta^2)R(\eta)$  in Equation (34) with  $R = R_{\mu,2}$  corresponds to

$$(\mu+1)(\mu-2-\eta^2)R_{\mu,2}(\eta) = (\mu+1)(\mu-2-\eta^2)\sum_{n=0}^{\infty}b_n\eta^{2n+1} = (\mu+1)(\mu-2)\sum_{n=0}^{\infty}b_n\eta^{2n+1} + (\mu+1)\sum_{k=1}^{\infty}b_{k-1}\eta^{2k+1} = (\mu^2-\mu-2)b_0\eta + \sum_{n=1}^{\infty}\left((\mu^2-\mu-2)b_n - (\mu+1)b_{n-1}\right)\eta^{2n+1}.$$
(54)

Summing the expressions in (52), (53) and (54), and observing that the following identities hold true

$$-2n(2n+1)b_n - 2(\mu-1)(2n+1)b_n + (\mu^2 - \mu - 2)b_n = (-4n^2 + (-4\mu + 2)n + \mu^2 - 3\mu)b_n,$$
  
$$2(2n-1)b_{n-1} - (\mu+1)b_{n-1} = (4n-3-\mu)b_{n-1},$$

we deduce that the power series  $R_{\mu,2}$  is a local solution of Equation (34) if and only if

$$(6\mu b_1 + (\mu^2 - 3\mu)b_0)\eta + \sum_{n=1}^{\infty} \left\{ 2\mu(2n+3)(n+1)b_{n+1} - \left[ 4n^2 + (4\mu - 2)n - \mu^2 + 3\mu \right]b_n + (4n-3-\mu)b_{n-1} \right\}\eta^{2n+1} = 0.$$

Since  $2\mu(2n+3)(n+1)=2\mu(2n^2+5n+3)$ , this holds if and only if the sequence  $(b_n)_{n\in\mathbb{N}}$  satisfies the recursive relations specified in (37).

We conclude this section with the proof of Corollary 3.3.

*Proof of Corollary 3.3.* It follows from the definition (6) of the Kummer's function  $\mathcal{M}$  and the following relation on its derivative:

$$\mathcal{M}(a,c,\zeta) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{\zeta^n}{n!}, \qquad \frac{d}{d\zeta} M(a,c,\zeta) = \frac{a}{c} M(1+a,1+c,\zeta), \qquad \text{for all } \zeta \in \mathbb{C}.$$
 (55)

Indeed, we can expand the power series  $R_{u,1}$  defined in (32) as follows:

$$R_{\mu,1}(\eta) = \mu \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\left(-\frac{\mu+1}{4}\right)_n}{\left(\frac{1}{2}\right)_n} \eta^{2n} - \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\left(-\frac{\mu+1}{4}\right)_n}{\left(\frac{1}{2}\right)_n} \eta \frac{d}{d\eta} (\eta^{2n})$$
$$= \mu \mathcal{M}\left(-\frac{\mu+1}{4}, \frac{1}{2}, \eta^2\right) - \eta \frac{d}{d\eta} \left[\mathcal{M}\left(-\frac{\mu+1}{4}, \frac{1}{2}, \eta^2\right)\right]$$

Hence, applying the relation on the derivative of the Kummer's function as in (55), we obtain that

$$R_{\mu,1}(\eta) = \mu \mathcal{M}\left(-\frac{\mu+1}{4}, \frac{1}{2}, \eta^2\right) - 2\eta^2 \frac{-\frac{\mu+1}{4}}{\frac{1}{2}} \mathcal{M}\left(1 - \frac{\mu+1}{4}, 1 + \frac{1}{2}, \eta^2\right)$$
$$= \mu \mathcal{M}\left(-\frac{\mu+1}{4}, \frac{1}{2}, \eta^2\right) + (\mu+1)\eta^2 \mathcal{M}\left(\frac{3-\mu}{4}, \frac{3}{2}, \eta^2\right).$$

Similarly, we can develop the power series  $R_{\mu,2}$  defined in (32) as follows:

$$\begin{split} R_{\mu,2}(\eta) &= \eta + \frac{1}{4} \sum_{n=1}^{\infty} \frac{2n+1-\mu}{n!} \frac{\left(\frac{5-\mu}{4}\right)_{n-1}}{\left(\frac{3}{2}\right)_n} \eta^{2n+1} \\ &= \eta + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{1-\mu}{4} \frac{\left(1+\frac{1-\mu}{4}\right)_{n-1}}{\left(\frac{3}{2}\right)_n} \eta^{2n+1} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \frac{\left(\frac{5-\mu}{4}\right)_{n-1}}{\left(\frac{5}{2}-1\right)_n} \eta^{2n+1} \\ &= \eta + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\left(\frac{1-\mu}{4}\right)_n}{\left(\frac{3}{2}\right)_n} \eta^{2n+1} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \frac{\left(\frac{5-\mu}{4}\right)_{n-1}}{\frac{3}{2} \cdot \left(\frac{5}{2}\right)_{n-1}} \eta^{2n+1} \end{split}$$

Applying a substitution to the second series with k = n - 1, we further obtain that

$$R_{\mu,2}(\eta) = \eta \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\left(\frac{1-\mu}{4}\right)_n}{\left(\frac{3}{2}\right)_n} \eta^{2n} + \frac{\eta^3}{3} \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\left(\frac{5-\mu}{4}\right)_k}{\left(\frac{5}{2}\right)_k} \eta^{2k}$$
$$= \eta \mathcal{M}\left(\frac{1-\mu}{4}, \frac{3}{2}, \eta^2\right) + \frac{\eta^3}{3} \mathcal{M}\left(\frac{5-\mu}{4}, \frac{5}{2}, \eta^2\right).$$

This concludes the proof of Corollary 3.3.

#### 4. Proof of Theorem 1.4

The following section is devoted to the proof of Theorem 1.4, highlighting the deep connection between the Prandtl eigenvalue problem and the harmonic oscillator. We briefly recall from Section 2 that the variable  $\eta \in \mathbb{C}$  and the constant  $\mu \in \mathbb{C}$  are defined as

$$\eta = e^{\mp \frac{\pi}{8} \mathrm{i} \sqrt[4]{|\beta| |k|}} (y - a) \in \mathbb{C}, \qquad \mu = -\left(\frac{\sigma}{|\beta|} \pm \mathrm{i} \alpha \sqrt{\frac{|k|}{|\beta|}}\right) \exp\left(\pm \frac{\pi}{4} \mathrm{i}\right) \in \mathbb{C}.$$

In terms of  $\eta \in \mathbb{C}$ , the two functions  $u(t,x,y) = \phi_k'(y)e^{ikx+\sigma\sqrt{|k|}t}$  and  $v(t,x,y) = -ik\phi_k(y)e^{ikx+\sigma\sqrt{|k|}t}$  satisfy the Prandtl equations (1) (not necessarily with boundary conditions) if and only if  $\Upsilon(\eta) := \phi_k(a + e^{\pm\frac{\pi i}{8}}\eta/\sqrt[4]{\beta||k|})$  is a solution of the following Prandtl eigenproblem (cf. (29) in Section 2):

$$-\Upsilon'''(\eta) + \eta^2 \Upsilon'(\eta) - 2\eta \Upsilon(\eta) = \mu \Upsilon'(\eta), \qquad \eta \in \mathbb{C}. \tag{56}$$

Recalling that the definitions of  $\psi_{\mu,1}$ ,  $\psi_{\mu,2}$ ,  $\Upsilon_{\mu,\eta_*,1}$ , and  $\Upsilon_{\mu,\eta_*,2}$  are given in (14), (15), and (16), the first part of Theorem 1.4, in the absence of boundary conditions, follows from determining the general solution of (56). In the following proposition, we established in particular three linearly independent solutions.

**Proposition 4.1.** Let  $\mu \in \mathbb{C}$  and  $\eta_* \in \mathbb{C}$  be two arbitrary constants. Then all solutions  $\Upsilon : \mathbb{C} \to \mathbb{C}$  of (56) are entire and given by

$$\Upsilon(\eta) = c_0(\eta^2 - \mu) + c_1 \Upsilon_{\mu, \eta_*, 1}(\eta) + c_2 \Upsilon_{\mu, \eta_*, 2}(\eta)$$

for some constants  $c_0, c_1, c_2 \in \mathbb{C}$ . Here, the functions  $\Upsilon_{\mu,i} : \mathbb{C} \to \mathbb{C}$  are given by

$$\Upsilon_{\mu,i}(\eta) := \Upsilon_{\mu,\eta_*,i}(\eta) = \begin{cases} \int_{\eta_*}^{\eta} \left(1 + \frac{\eta^2 - \xi^2}{2}\right) \psi_{\mu,i}(\xi) d\xi - \frac{\psi'_{\mu,i}(\eta_*)}{2}, & \text{if } (\mu,i) \neq (1,2), \\ \int_{\eta_*}^{\eta} \left(1 + \frac{\eta^2 - \xi^2}{2}\right) g(\xi) d\xi - \frac{g'(\xi)}{2} & \text{with } g(\xi) = e^{\frac{\xi^2}{2}} \operatorname{erf}(\xi), & \text{if } (\mu,i) = (1,2), \end{cases}$$

for i = 1, 2.

Proof of Proposition 4.1. First, one easily checks that  $\Upsilon(\eta) = \mu - \eta^2$  solves (56), which leaves the remaining two linearly independent solutions to be found. Next, we show that any linear combination  $\Upsilon := c_1 \Upsilon_{\mu,\eta_*,1} + c_2 \Upsilon_{\mu,\eta_*,2}$  satisfies (56), where we recall that for any i = 1, 2

$$\begin{cases}
\Upsilon_{\mu,\eta_{*},i}(\eta) := \frac{1}{2}\mathcal{B}_{\mu} \int_{\eta_{*}}^{\eta} \psi_{\mu,i}(\xi) d\xi & \text{if } (\mu,i) \neq (1,2), \\
\Upsilon_{1,\eta_{*},2}(\eta) := \frac{1}{2}\mathcal{B}_{\mu} \int_{\eta_{*}}^{\eta} e^{\frac{\xi^{2}}{2}} \operatorname{erf}(\xi) d\xi & \text{otherwise,} 
\end{cases}$$
(57)

with  $\mathcal{B}_{\mu} := -\frac{d^2}{d\eta^2} + \eta^2 - \mu$  the Schrödinger operator. Recall that  $\psi_{\mu,1}$  is even and  $\psi_{\mu,2}$  is odd. Because of (57), we distinguish two cases: Part(i) with  $\mu \neq 1$  and Part(ii) with  $\mu = 1$ .

Part (i): We address first the case  $\mu \neq 1$  and we observe that  $\Upsilon_{\mu,\eta_*,1}$  is always the sum of an odd function and a multiple of  $\eta \mapsto \eta^2 - \mu$  and  $\Upsilon_{\mu,\eta_*,2}$  is always even. The linear combination  $\Upsilon = c_1 \Upsilon_{\mu,\eta_*,1} + c_2 \Upsilon_{\mu,\eta_*,2}$  satisfies

$$\begin{split} -\Upsilon''' + (\eta^2 - \mu)\Upsilon' - 2\eta\Upsilon &= \mathcal{B}_{\mu}\Upsilon' - 2\eta\Upsilon = \mathcal{B}_{\mu}\frac{d}{d\eta}\left(\frac{1}{2}\mathcal{B}_{\mu}\int_{\eta_*}^{\eta}\psi d\xi\right) - \eta\mathcal{B}_{\mu}\int_{\eta_*}^{\eta}\psi d\xi \\ &= \frac{1}{2}\mathcal{B}_{\mu}^2\frac{d}{d\eta}\left(\int_{\eta_*}^{\eta}\psi d\xi\right) + \frac{1}{2}\mathcal{B}_{\mu}\left[\frac{d}{d\eta},\mathcal{B}_{\mu}\right]\left(\int_{\eta_*}^{\eta}\psi d\xi\right) - \eta\mathcal{B}_{\mu}\int_{\eta_*}^{\eta}\psi d\xi \\ &= \frac{1}{2}\mathcal{B}_{\mu}^2\psi + \frac{1}{2}\mathcal{B}_{\mu}\left(2\eta\int_{\eta_*}^{\eta}\psi d\xi\right) - \eta\mathcal{B}_{\mu}\int_{\eta_*}^{\eta}\psi d\xi \\ &= \frac{1}{2}\mathcal{B}_{\mu}^2\psi + \eta\mathcal{B}_{\mu}\int_{\eta_*}^{\eta}\psi d\xi + [\mathcal{B}_{\mu},\eta]\int_{\eta_*}^{\eta}\psi d\xi - \eta\mathcal{B}_{\mu}\int_{\eta_*}^{\eta}\psi d\xi \\ &= \frac{1}{2}\mathcal{B}_{\mu}^2\psi - 2\frac{d}{d\eta}\int_{\eta_*}^{\eta}\psi d\xi = \frac{1}{2}\mathcal{B}_{\mu}^2\psi - 2\psi, \end{split}$$

with  $\psi = c_1 \psi_{\mu,1} + c_2 \psi_{\mu,2}$ . With the help of the latter identities and  $-\psi'' + \eta^2 \psi = (\mu + 2)\psi$ , we compute

$$-\Upsilon''' + (\eta^2 - \mu)\Upsilon' - 2\eta\Upsilon = \frac{1}{2} \left( -\frac{d^2}{d\eta^2} + \eta^2 - \mu \right)^2 \psi - 2\psi$$
$$= \left( -\frac{d^2}{d\eta^2} + \eta^2 - \mu \right) \psi - 2\psi = 2\psi - 2\psi = 0.$$

Hence we have shown that any  $\Upsilon = c_1 \Upsilon_{\mu,\eta_*,1} + c_2 \Upsilon_{\mu,\eta_*,2}$  solves indeed (56). Next, we show that  $\Upsilon_{\mu,\eta_*,1}$  and  $\Upsilon_{\mu,\eta_*,2}$  are actually linearly independent (in particular non-vanishing). First, we see that  $\Upsilon_{\mu,\eta_*,1} \neq \Upsilon_{\mu,\eta_*,2}$  by a comparison of parities. Now if one of these solutions vanished, we would have

$$\left(-\frac{d^2}{d\eta^2} + \eta^2 - \mu\right) \int_{\eta_*}^{\eta} \psi_{\mu,i}(\xi) d\xi = 0.$$

In this case, the integral expression must be an eigenfunction of the Schrödinger operator

$$\int_{\eta_*}^{\eta} \psi_{\mu,1} d\xi = \tilde{c}_1 \psi_{\mu-2,1} + \tilde{c}_2 \psi_{\mu-2,2},$$
$$\int_{\eta}^{\eta} \psi_{\mu,2} d\xi = \hat{c}_1 \psi_{\mu-2,1} + \hat{c}_2 \psi_{\mu-2,2},$$

for some  $\tilde{c}_1$ ,  $\tilde{c}_2$ ,  $\hat{c}_1$ ,  $\hat{c}_2 \in \mathbb{C}$ . Thus, the parities show that  $\tilde{c}_1 = \bar{c}_2 = 0$  since a non-vanishing constant function is never an algebraic eigenfunction of the Schrödinger operator. Next, we distinguish two cases: If  $\mu = -1$ , we have (up to multiplicative constants)

$$\psi_{\mu,1}(\eta) = e^{-\eta^2/2}, \qquad \psi_{\mu,2}(\eta) = e^{-\eta^2/2} \int_{\eta_*}^{\eta} e^{\xi^2} d\xi,$$

$$\psi_{\mu-2,2} = e^{\eta^2/2} \int_{\eta_*}^{\eta} e^{-\xi^2} d\xi, \qquad \psi_{\mu-2,1}(\eta) = e^{\eta^2/2}.$$

Taking a derivative, one easily checks that the above equations are never satisfied for any choice  $\tilde{c}_2$ ,  $\bar{c}_1 \in \mathbb{C}$ . If  $\mu \in \mathbb{C} \setminus \{1, -1\}$ , taking a derivative of the equations and using the up-operator  $\mathcal{A}_{\uparrow} := (\eta - \frac{d}{d\eta})$  and  $A_{\uparrow}\psi_{\mu-2} = \psi_{\mu} \neq 0$  which holds for any  $\mu \in \mathbb{C} \setminus \{1, -1\}$ , we can rewrite this equation equivalently by

$$(\tilde{c}+1)\psi'_{\mu-2} = \eta\psi_{\mu-2}, \quad \psi_{\mu-2}(\eta_*) = 0.$$

The unique solution is, however,  $\psi_{\mu-2} \equiv 0$  which implies  $\psi_{\mu} \equiv 0$ . Hence, we have found that  $\Upsilon_{\mu,\eta_*,1}$  and  $\Upsilon_{\mu,\eta_*,2}$  are two non-vanishing solutions, one odd plus a quadratic function and the second one even. In sum, they are two linearly independent solutions of (56).

In order to conclude the linear independence of all three solutions (the third one being  $\eta \mapsto \mu - \eta^2$ ) we need to show that the resulting even solution satisfies

$$\mathcal{B}_{\mu} \int_{\eta_{*}}^{\eta} \psi_{\mu,2} d\xi \neq c(\eta^{2} - \mu) \quad \Leftrightarrow \quad \mathcal{B}_{\mu} \left[ \int_{\eta_{*}}^{\eta} \psi_{\mu,2} d\xi - c \right] \neq 0$$

for any  $c \in \mathbb{C}$ . Similar to above, if this was not the case, we would have  $\int_{\eta_*}^{\eta} \psi_{\mu,2} d\xi - c = \tilde{c}\psi_{\mu-2,1}$  and equivalently

$$(\tilde{c}+1)\psi'_{\mu-2,1} = \eta\psi_{\mu-2,1}, \quad \psi_{\mu-2,1}(\eta_*) = -c.$$

Here the unique solution is  $\tilde{c} = -2$ ,  $\mu = 1$  and  $\psi_{-1,1}(\eta) = -ce^{-(\eta^2 - \eta_*^2)/2}$ . Thus, for  $\mu \neq 1$ , the three found solutions  $(\mu - \eta^2)$ ,  $\Upsilon_{\mu,\eta_*,1}$  and  $\Upsilon_{\mu,\eta_*,2}$  are linearly independent.

Part (ii): to conclude the proof of Proposition 4.1, we need to consider the case of  $\mu = 1$ . Here, we have a more direct approach, since the functions  $\Upsilon_{1,\eta_*,1}$  and  $\Upsilon_{1,\eta_*,2}$  in (57) can be determined explicitly:

$$\Upsilon_{1,\eta_*,1} = \frac{1}{2} \mathcal{B}_{\mu} \int_{\eta_*}^{\eta} \left( e^{\frac{\xi^2}{2}} - \sqrt{\pi} \xi \operatorname{erfi}(\xi) \right) d\xi, \qquad \Upsilon_{1,\eta_*,2} := \frac{1}{2} \mathcal{B}_{\mu} \int_{\eta_*}^{\eta} e^{\frac{\xi^2}{2}} \operatorname{erf}(\xi) d\xi$$

The fact that  $\mu - \eta^2$ ,  $\Upsilon_{1,\eta_*,1}$  and  $\Upsilon_{1,\eta_*,2}$  are linearly independent follows from a similar Ansatz as in Part (i). It remains to show that  $\Upsilon_{1,\eta_*,2}$  is a solution for (56) with  $\mu = 1$ . We remark that  $g(\eta) := e^{\frac{\eta^2}{2}} \operatorname{erf}(\eta)$  satisfies  $-g'' + \eta^2 g = -g \Rightarrow \mathcal{B}_1 g = -2g$ . Hence, dropping the indexes in  $\Upsilon = \Upsilon_{1,\eta_*,2}$ , we gather that

$$-\Upsilon''' + \eta^2 \Upsilon' - 2\eta \Upsilon - \Upsilon' = \frac{1}{2} \mathcal{B}_1^2 g - 2g = 2g - 2g = 0.$$

This concludes the proof of Proposition 4.1.

It is obvious that the constants  $c_0, c_1$  and  $c_2$  above can always be chosen in such a way that  $\Upsilon$  satisfies the boundary conditions of the Prandtl equation  $\Upsilon(\eta_*) = \Upsilon'(\eta_*) = 0$  for an arbitrary  $\eta_* \in \mathbb{C}$ .

**Proposition 4.2.** Under the conditions of Proposition 4.1, the boundary conditions  $\Upsilon(\eta_*) = \Upsilon'(\eta_*)$  are satisfied if the triple  $(c_0, c_1, c_2) \in \mathbb{C}^3$  is a solution of the linear system

$$\begin{pmatrix} \mu - \eta_*^2 & -\frac{1}{2}\psi'_{\mu,1}(\eta_*) & -\frac{1}{2}\psi'_{\mu,2}(\eta_*) \\ -2\eta_* & \psi_{\mu,1}(\eta_*) & \psi_{\mu,2}(\eta_*) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad if \ \mu \neq 1,$$
 (58)

while, in the case  $\mu = 1$ ,  $(c_0, c_1, c_2) \in \mathbb{C}^3$  satisfies (58) with  $\psi_{\mu,2}(\eta)$  replaced by  $g(\eta) = e^{\frac{\eta^2}{2}} \operatorname{erf}(\eta)$ .

*Proof.* By Proposition 4.1, the condition  $\Upsilon_{\mu}(\eta_*) = 0$  immediately implies the first row of the linear system whereas  $\Upsilon'(\eta_*) = 0$  leads to the second one.

## 5. Proof of Corollary 1.7: The Behaviour at infinity

An important aspect of  $\Upsilon$  in Proposition 4.1 is its behavior at infinity, which determines the asymptotics of a general solution  $(\tau, W)$  to (5). From Table 1, for positive frequencies  $k \in \mathbb{N}$ , we recall the relations  $\eta = e^{-\frac{\pi}{8}i}z$ ,  $\tau = \mu e^{\frac{\pi}{4}i}$ , and  $\Upsilon(\eta) = (\tau - z^2)W(z)$ . Thus,  $(\mu, \Upsilon)$  solves (56) if and only if  $(\tau, W)$  solves (5).

We have established that for  $\tau=e^{\frac{5\pi i}{4}}$  (i.e.,  $\mu=-1$ ), there exists an explicit solution W satisfying Criterion 1 (cf. Corollary 1.2 and (63) for  $\Upsilon$ ), with the boundary conditions  $\lim_{z\to -\infty}W(z)=0$  and  $\lim_{z\to +\infty}W(z)=1$ , namely

$$\lim_{\eta \to \infty_{\mathbb{C}}, \arg(-\eta) = -\frac{\pi}{8}i} \Upsilon(\eta) = 0 \qquad \lim_{\eta \to \infty_{\mathbb{C}}, \arg(\eta) = -\frac{\pi}{8}i} \Upsilon(\eta) = 1.$$

The question now is whether another pair  $(\tau, W)$  with  $\operatorname{Im} \tau < 0$  (i.e.  $\operatorname{Re} \mu < \operatorname{Im} \mu$ ) satisfies the same criterion. The answer is negative.

The key idea is that if  $\mu \neq 2n-1$  for any  $n \in \mathbb{N}_0 \setminus \{1\}$  (i.e.,  $\tau \neq (2n-1)e^{\frac{\pi}{4}i}$ ), then any  $\Upsilon$  from Proposition 4.1 with  $\eta_* = 0$  and

$$\Upsilon(\eta) = c_0(\mu - \eta^2) + c_1 \Upsilon_{\mu,0,1}(\eta) + c_2 \Upsilon_{\mu,0,2}(\eta) \quad \text{with} \quad (c_1, c_2) \neq (0, 0), \tag{59}$$

exhibits exponential growth in at least one of the sectors  $|\arg(\eta)| < \pi/4$  or  $|\arg(-\eta)| < \pi/4$ . Note that if  $(c_1, c_2) = (0, 0)$ , then  $\Upsilon$  is purely quadratic, and W(z) is constant, meaning it cannot satisfy Criterion 1. Corollary 1.2 thus requires  $\mu = -1$ . In this case,  $\Upsilon_{-1,0,2}$  grows exponentially in both sectors  $|\arg(\eta)| < \pi/4$  and  $|\arg(-\eta)| < \pi/4$ , while  $\Upsilon_{-1,0,1}$  exhibits quadratic growth on one side and exponential decay on the other (cf. Lemma 5.9). The boundary conditions of W(z) in Criterion 1 then impose  $c_2 = 0$ , leaving only two degrees of freedom in  $c_0$  and  $c_1$ , ensuring the uniqueness of W.

In what follows, we formalize the above heuristics. For simplicity, we exclude the case  $\mu = 1$  as it does not satisfy Re  $\mu < \text{Im } \mu$ . Recall that for any  $\mu \in \mathbb{C} \setminus \{1\}$ , the functions  $\Upsilon_{\mu,0,i}(\eta)$  in (59) for i = 1,2 are defined as

$$\Upsilon_{\mu,0,i}(\eta) = \int_0^{\eta} \left( 1 + \frac{\eta^2 - \xi^2}{2} \right) \psi_{i,\mu}(\xi) d\xi \quad \text{with} \quad \begin{cases} \psi_{\mu,1}(\eta) = \mathcal{M}\left( -\frac{1+\mu}{4}, \frac{1}{2}, \eta^2 \right) e^{-\frac{\eta^2}{2}}, \\ \psi_{\mu,2}(\eta) = \eta \, \mathcal{M}\left( \frac{1-\mu}{4}, \frac{3}{2}, \eta^2 \right) e^{-\frac{\eta^2}{2}}. \end{cases}$$
(60)

The asymptotics of  $\Upsilon_{\mu,0,i}$  are therefore related to the asymptotics of Kummer's Hypergeometric functions, of which we repeatedly make use throughout this section:

**Proposition 5.1.** For every  $a, c \in \mathbb{C}$  with  $a, c - a \neq -2n$  for any  $n \in \mathbb{N}_0$ , the following asymptotic expansion holds true:

$$\mathcal{M}\left(a,c,\zeta\right) \quad \sim \quad \frac{\Gamma(c)}{\Gamma(a)} e^{\zeta} \zeta^{a-c} \left(1 + (1-a)(c-a)\frac{1}{\zeta} + \mathcal{O}\left(\frac{1}{\zeta^2}\right)\right)$$

as  $\zeta \to \infty_{\mathbb{C}}$  with  $|\arg(\zeta)| < \pi/2 - \delta$  for any  $\delta > 0$ .

*Proof.* This is a classical result (see, e.g., Section 10.1 in [21]), noting that we have included only the leading term determined by the branch  $|\arg(\zeta)| < \pi/2 - \delta$ . Additionally, because of this branch, the asymptotics is single valued.

Ultimately, we are only interested in  $\zeta=\eta^2=e^{\pm\frac{\pi}{4}\mathrm{i}}z^2$  as  $z\to\pm\infty$  in  $\mathbb R$  and we recall that for given functions  $h,\phi:\mathbb R\to\mathbb C$ , the Poincaré expansion  $h(z)\sim\phi(z)$  as  $z\to\pm\infty$  means

$$h(z) - \phi(z) = o(\phi(z))$$
 as  $z \to \pm \infty$ .

By setting  $\zeta = \eta^2$  in Proposition 5.1, we remark that, for the most values of  $\mu$ ,  $\psi_{\mu,1}$  and  $\psi_{\mu,2}$  behave asymptotically as  $e^{\eta^2/2}$ . The main technical challenge is extending this behaviour to the integrals of  $\Upsilon_{\mu,0,i}$ . In essence, we seek to avoid pathological examples such as

$$f(z) = \sin(\exp(z^2))$$
  $\Rightarrow$   $f'(z) = 2ze^{z^2}\cos(\exp(z^2))$ 

showing that the primitive of an asymptotically exponentially growing function does not need to be unbounded (on the real line  $z \in \mathbb{R}$ ). However, the idea is that the oscillations of the hypergeometric functions are not sufficiently large in order to compensate the exponential growth of the derivative.

Since in (60) we have  $(a,c) = (-(1+\mu)/4,2)$  or  $(a,c) = ((1-\mu)/4,3/2)$ , Proposition 5.1 does not apply to  $\mu \in 2\mathbb{Z} - 1$ , of which only  $\mu \in -2\mathbb{N} - 1$  is relevant to our analysis. Hence we work two separated cases: first  $\mu \notin 2\mathbb{Z} - 1$  (Lemma 5.3) and afterwards proceed by a semi-explicit representation in case  $\mu$  is odd and negative (Lemma 5.5 and Lemma 5.7). In both we need a simple auxiliary lemma:

**Lemma 5.2.** Let  $\gamma \in \mathbb{C}$ ,  $b \in \mathbb{C} \setminus \{0\}$  with  $\arg(b) \in ]-\pi/4, \pi/4[$  and  $\arg(b^2) \in ]-\pi/2, \pi, 2[$ . Then, for  $z \in \mathbb{R}$  the following asymptotics hold true:

$$\int_{\operatorname{sgn}(z)}^{z} \exp\left(b^{2} \frac{\tilde{z}^{2}}{2}\right) (b^{2} \tilde{z}^{2})^{\gamma} d\tilde{z} \qquad \sim \quad \frac{1}{b} \exp\left(b^{2} \frac{z^{2}}{2}\right) \frac{(b^{2} z^{2})^{\gamma}}{bz}, \qquad as \ z \to \pm \infty,$$

$$\int_{\operatorname{sgn}(z)}^{z} \exp\left(b^{2} \frac{\tilde{z}^{2}}{2}\right) (b^{2} \tilde{z}^{2})^{\gamma} (b\tilde{z}) d\tilde{z} \qquad \sim \quad \frac{1}{b} \exp\left(b^{2} \frac{z^{2}}{2}\right) (b^{2} z^{2})^{\gamma}, \qquad as \ z \to \pm \infty.$$

We start the integrals from  $\tilde{z} = \operatorname{sgn}(z) = \pm 1$  to avoid the possible singularity at  $\tilde{z} = 0$ , though, in reality the considered functions in the next lemmas are entire, making this a mere technicality. We note moreover that  $\operatorname{arg}(b^2) \in ]-\pi/2, \pi/2[$  is set, thus  $(b^2z^2)^{\gamma}$  is single-valued as  $(b^2z^2)^{\gamma} = (|b|^2z^2)^{\gamma}e^{i\operatorname{arg}(b^2)}$ . While the lemma extends to other values of b (with possible multi-valued asymptotics), this is beyond our scope.

*Proof.* The result follows from applying integration by parts twice. We carry it out only for the first integral, as the second follows an identical procedure. We have

$$\begin{split} \int_{\mathrm{sgn}(z)}^z \exp\left(b^2 \frac{\tilde{z}^2}{2}\right) (b^2 \tilde{z}^2)^{\gamma} d\tilde{z} &= \int_{\mathrm{sgn}(z)}^z \frac{d}{d\tilde{z}} \left(\exp\left(b^2 \frac{\tilde{z}^2}{2}\right)\right) \frac{1}{b} \frac{(b^2 \tilde{z}^2)^{\gamma}}{b\tilde{z}} d\tilde{z} \\ &= \frac{1}{b} \exp\left(b^2 \frac{\tilde{z}^2}{2}\right) \frac{(b^2 z^2)^{\gamma}}{bz} \bigg|_{\mathrm{sgn}(z)}^z + (2\gamma - 1) \int_{\mathrm{sgn}(z)}^z \exp\left(b^2 \frac{\tilde{z}^2}{2}\right) (b^2 \tilde{z}^2)^{\gamma - 2} d\tilde{z} \\ &= \frac{1}{b} \exp\left(b^2 \frac{z^2}{2}\right) \frac{(b^2 z^2)^{\gamma}}{bz} - \frac{1}{b} \exp\left(\frac{b^2}{2}\right) \frac{(b^2)^{\gamma}}{b} + (2\gamma - 1) \int_{\mathrm{sgn}(z)}^z \exp\left(b^2 \frac{\tilde{z}^2}{2}\right) (b^2 \tilde{z}^2)^{\gamma - 2} d\tilde{z}. \end{split}$$

The first term on the right-hand side of the last identity is as in the claimed asymptotics. Additionally, the remaining terms are of order  $o\left(\exp\left(b^2z^2/2\right)(b^2z^2)^{\gamma}/(bz)\right)$ : repeating the same integration by parts with  $\gamma$  replaced by  $\gamma - 2$ , the last integral develops as

$$\int_{\text{sgn}(z)}^{z} \exp\left(b^{2} \frac{\tilde{z}^{2}}{2}\right) (b^{2} \tilde{z}^{2})^{\gamma - 2} d\tilde{z} 
= \frac{1}{b} \exp\left(b^{2} \frac{z^{2}}{2}\right) \frac{(b^{2} z^{2})^{\gamma - 2}}{bz} - \frac{1}{b} \exp\left(\frac{b^{2}}{2}\right) \frac{b^{\gamma - 2}}{b} + (2\gamma - 3) \int_{\text{sgn}(z)}^{z} \exp\left(b^{2} \frac{\tilde{z}^{2}}{2}\right) (b^{2} \tilde{z}^{2})^{\gamma - 4} d\tilde{z}.$$

The first term is  $o\left(\exp\left(b^2z^2/2\right)(b^2z^2)^{\gamma}/(bz)\right)$  as  $z\to\pm\infty$ . For the remaining term, we apply the mean value theorem: we know that  $\mathbb{R}\ni z\mapsto |e^{b^2z^2/2}(b^2z^2)^{\gamma-4}|=|(b^2)^{\gamma-4}|e^{z^2/(2\sqrt{2})}(z^2)^{\mathrm{Re}\gamma-4}$  is convex for sufficiently large

|z|, saying  $|z| > |\bar{z}| \gg 1$  with  $\operatorname{sgn}(\bar{z}) = \operatorname{sgn}(z)$ . Hence we can estimate the integral using the mean value theorem in the interval  $[\bar{z}, z]$  or  $[z, \bar{z}]$ :

$$\left| \int_{\operatorname{sgn}(z)}^{z} \exp\left(b^{2} \frac{\tilde{z}^{2}}{2}\right) (b^{2} \tilde{z}^{2})^{\gamma - 4} d\tilde{z} \right| \leq \int_{\operatorname{sgn}(z)}^{\bar{z}} \left| \exp\left(b^{2} \frac{\tilde{z}^{2}}{2}\right) (b^{2} \tilde{z}^{2})^{\gamma - 4} \right| d\tilde{z} + |z - \bar{z}| \left| \exp\left(b^{2} \frac{z^{2}}{2}\right) (b^{2} z^{2})^{\gamma - 4} \right|.$$

The last term is  $o\left(\exp\left(b^2z^2/2\right)(b^2z^2)^{\gamma-1}\right)$ , this concludes the proof of the lemma.

## 5.1. Asymptotic expansions for $\mu \notin 2\mathbb{Z} - 1$ .

We apply the result to the integral of the hypergeometric functions in the case  $\mu \notin 2\mathbb{Z} - 1$ , where the asymptotic expansions of Proposition 5.1 holds true.

**Lemma 5.3.** Let  $b = e^{\pm i\frac{\pi}{8}}$  and set  $\arg(b) := \pm \frac{\pi}{8}$ ,  $\arg(b^2) := \pm \frac{\pi}{4}$ . Consider  $\mu \notin 2\mathbb{Z} - 1$  and  $m \in \{0, 1\}$ . Then, the following asymptotics hold true as  $z \to \pm \infty$ :

$$\psi_{\mu,1}(bz) \sim \mathcal{C}_{\mu,1} \exp\left(b^2 \frac{z^2}{2}\right) \left(b^2 z^2\right)^{-\frac{\mu+3}{4}}, \qquad \int_0^z (b^2 \tilde{z}^2)^m \psi_{\mu,1}(b\tilde{z}) d\tilde{z} \sim \frac{\mathcal{C}_{\mu,1}}{b} \exp\left(b^2 \frac{z^2}{2}\right) \frac{\left(b^2 z^2\right)^{m-\frac{\mu+3}{4}}}{bz}$$

$$\psi_{\mu,2}(bz) \sim \mathcal{C}_{\mu,2} \exp\left(b^2 \frac{z^2}{2}\right) \left(b^2 z^2\right)^{-\frac{\mu+5}{4}} (bz), \qquad \int_0^z (b^2 \tilde{z}^2)^m \psi_{\mu,2}(b\tilde{z}) d\tilde{z} \sim \frac{\mathcal{C}_{\mu,2}}{b} \exp\left(b^2 \frac{z^2}{2}\right) \left(b^2 z^2\right)^{m-\frac{\mu+5}{4}}$$

$$with \ \mathcal{C}_{\mu,1} := \frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{\mu+1}{2})} \ and \ \mathcal{C}_{\mu,2} := \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1-\mu}{2})}.$$

*Proof.* The asymptotics of  $\psi_{\mu,1}$  and  $\psi_{\mu,2}$  are a direct application of Proposition 5.1 together with (60) and  $\zeta = \eta^2 = b^2 z^2$  (thus  $\arg(\zeta) = \pm \pi/4 \in ]-\pi/2 - \delta, \pi/2 + \delta[$ ). We now evaluate the integrals, considering only  $\psi_{\mu,1}$ , as  $\psi_{\mu,2}$  follows by an identical procedure. We define

$$f_1(z) := \mathcal{C}_{\mu,1} \exp\left(b^2 \frac{z^2}{2}\right) \left(b^2 z^2\right)^{m - \frac{\mu + 3}{4}},$$

Thanks to Lemma 5.2 with  $b = e^{\pm \frac{\pi}{8}i}$  and  $\gamma = -(\mu + 3)/4$ , we gather that

$$\int_{\operatorname{sgn}(z)}^{z} f_1(\tilde{z}) d\tilde{z} \sim \mathcal{C}_{\mu,1} \exp\left(b^2 \frac{z^2}{2}\right) \frac{\left(b^2 z^2\right)^{m - \frac{\mu_1 + \nu}{4}}}{bz} =: h_1(z).$$

From  $(b^2z^2)^m\psi_{\mu,1} \sim f_1$  we deduce that  $(b^2z^2)^m\psi_{\mu,1} - f_1$  is  $o(f_1(z))$  as  $z \to \pm \infty$ . In particular, for any  $\varepsilon > 0$  there exists  $z_{\varepsilon} \gg 1$  such that

$$|(b^2z^2)^m\psi_{u,1}(z)-f_1(z)|<\varepsilon|f_1(z)|,$$
 for any  $|z|>z_{\varepsilon}$ .

It follows the following identities

$$\left| \int_{0}^{z} (b^{2}z^{2})^{m} \psi_{\mu,1}(\tilde{z}) d\tilde{z} - h_{1}(z) \right| \leq \left| \int_{0}^{z} (b^{2}z^{2})^{m} \psi_{\mu,1}(\tilde{z}) d\tilde{z} - \int_{\mathrm{sgn}(z)}^{z} f_{1}(\tilde{z}) d\tilde{z} \right| + o(h_{1}(z))$$

$$\leq \varepsilon \left| \int_{\mathrm{sgn}(z)z_{\varepsilon}}^{z} |f_{1}(\tilde{z})| d\tilde{z} \right| + o(h_{1}(z))$$

$$\leq \varepsilon |\mathcal{C}_{\mu,1}| \left| (b^{2})^{-\frac{\mu+3}{4}} \right| \int_{z_{\varepsilon}}^{|z|} e^{\frac{\tilde{z}^{2}}{2\sqrt{2}}} (\tilde{z}^{2})^{m - \frac{\mathrm{Re}(\mu) + 3}{4}} d\tilde{z} + o(h_{1}(z)).$$

Using Lemma 5.2 with  $b = 1/\sqrt{2\sqrt{2}}$ ,  $\arg(b) = 0$  and  $\arg(b^2) = 0$  implies that following asymptotics:

$$\int_{z_{\varepsilon}}^{|z|} e^{\frac{\tilde{z}^2}{2\sqrt{2}}} \left(\frac{\tilde{z}^2}{2\sqrt{2}}\right)^{m - \frac{\operatorname{Re}(\mu) + 3}{4}} d\tilde{z} \sim \sqrt{2\sqrt{2}} e^{\frac{z^2}{2\sqrt{2}}} \left(\frac{z^2}{2\sqrt{2}}\right)^{m - \frac{\operatorname{Re}(\mu) + 3}{4} - 1} = \mathcal{O}\left(h_1(z)\right).$$

We thus obtain that

$$\left| \int_0^z (b^2 z^2)^m \psi_{\mu,1}(\tilde{z}) d\tilde{z} - h_1(z) \right| = \varepsilon \mathcal{O}\left(h_1(z)\right) + o\left(h_1(z)\right).$$

The claim follows from the arbitrariness of  $\varepsilon > 0$ .

We can transfer the above asymptotics directly to  $\Upsilon_{\mu,0,1}$  and  $\Upsilon_{\mu,0,2}$  using (60):

Corollary 5.4. Let  $b = e^{\pm i\frac{\pi}{8}}$  and set  $\arg(b) := \pm \frac{\pi}{8}$ ,  $\arg(b^2) := \pm \frac{\pi}{4}$ . Consider  $\mu \notin 2\mathbb{Z} - 1$ . Then, the following asymptotics hold true as  $z \to \pm \infty$ :

$$\Upsilon_{\mu,0,1}(bz) \sim C_{\mu,1} \exp\left(b^2 \frac{z^2}{2}\right) \frac{\left(b^2 z^2\right)^{-\frac{\mu+3}{4}}}{bz}, \qquad \Upsilon_{\mu,0,2}(bz) \sim C_{\mu,2} \exp\left(b^2 \frac{z^2}{2}\right) \left(b^2 z^2\right)^{-\frac{\mu+5}{4}},$$

$$with \ \mathcal{C}_{\mu,1} := \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{\mu+1}{4}\right)} \ and \ \mathcal{C}_{\mu,2} := \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1-\mu}{4}\right)}.$$

## 5.2. Asymptotic expansions for $\mu \in -2\mathbb{N} - 1$ .

We next address the values  $\mu \in -2\mathbb{N} - 1$  and we begin expressing  $\psi_{\mu,1}$  and  $\psi_{\mu,2}$  with a semi-explicit formula.

**Lemma 5.5.** Let  $\mu = -(2m+3) \in -2\mathbb{N} - 1$  with  $m \in \mathbb{N}_0$ . Define the down-operator  $\mathcal{A}_{\downarrow} := \zeta + \frac{d}{d\zeta}$ . Then, with indexes  $i = (3 + (-1)^{m+1})/2 \in \{1, 2\}$  and  $j = (3 + (-1)^m)/2 \in \{1, 2\}$ , one has

$$\psi_{\mu,i}(\eta) = \frac{\left\lceil \frac{m}{2} \right\rceil!}{\left(2 \left\lceil \frac{m}{2} \right\rceil\right)!} e^{-\eta^2/2} \frac{d^m}{d\eta^m} e^{\eta^2}, \qquad \qquad \psi_{\mu,j}(\eta) = \frac{1}{\left\lceil \frac{m-1}{2} \right\rceil!} \frac{1}{4^{\left\lceil \frac{m-1}{2} \right\rceil}} \mathcal{A}_{\downarrow}^m \left(e^{\eta^2/2} \int_0^{\eta} e^{-\xi^2} d\xi\right),$$

for any  $\eta \in \mathbb{C}$ , where  $\lceil \cdot \rceil$  is the ceiling function. In particular, it follows that there exist three polynomials  $p_m$ ,  $q_m$ ,  $\tilde{q} \in \mathbb{R}[\eta]$  of degree m such that  $p_m$ ,  $q_m$  have leading-order coefficient  $2^m \lceil \frac{m}{2} \rceil ! / \left(2 \lceil \frac{m}{2} \rceil\right) !$  and  $2^m / \left(\lceil \frac{m-1}{2} \rceil ! 4^{\lceil \frac{m-1}{2} \rceil}\right)$ , respectively, as well as

$$\psi_{\mu,i}(\eta) = p_m(\eta)e^{\eta^2/2},\tag{61}$$

$$\psi_{\mu,j}(\eta) = q_m(\eta)e^{\eta^2/2} \int_0^{\eta} e^{-\xi^2} d\xi + \tilde{q}(\eta)e^{-\eta^2/2}.$$
 (62)

*Proof.* We consider only the case  $m \in 2\mathbb{N}_0$ , as  $m \in 2\mathbb{N}_0 + 1$  can be treated similarly. If  $m \in 2\mathbb{N}_0$ , we have that i = 1 and j = 2 and the functions

$$h_m(\eta) := e^{-\eta^2/2} \frac{d^m}{d\eta^m} e^{\eta^2}, \qquad g_m(\eta) := \mathcal{A}_{\downarrow}^m \left( e^{\eta^2/2} \int_0^{\eta} e^{-\xi^2} d\xi \right),$$

are even and odd, respectively. The functions  $h_m$ ,  $g_m$  are solutions of the harmonic oscillator  $-\psi'' + \eta^2 \psi = (\mu + 2)\psi = -(2m + 1)\psi$  (cf. Lemma A.3). By showing that

$$h_m(0) = \frac{m!}{(\frac{m}{2})!}$$
  $g'_m(0) = (\frac{m}{2})!2^m$ ,

then the statement is obtained by uniqueness of the solutions of the harmonic oscillator, since  $\psi_{\mu,1}(0) = \psi'_{\mu,2}(0) = 1$ . First, we have that

$$h_m(0) = \frac{d^m}{d\eta^m} \Big|_{\eta=0} \left( \sum_{k=0}^{\infty} \frac{\eta^{2k}}{k!} \right) = \sum_{k=\frac{m}{2}}^{\infty} \frac{(2k)!}{(2k-m)!} \frac{\eta^{2k-m}}{k!} \Big|_{\eta=0} = \frac{m!}{\left(\frac{m}{2}\right)!}.$$

We show by induction the second identity: if m=0 then we have that  $g_0(\eta)=e^{\eta^2/2}\int_0^{\eta}e^{-\xi^2}d\xi$ , implying that  $g_0'(0)=1$ . Assume now that  $g_m'(0)=(m/2)!2^m$ , for a given  $m\in\mathbb{N}_0$ . Then we remark that

$$g_{m+2}(\eta) = \mathcal{A}_{\downarrow}^2 g_m(\eta) = (\eta^2 + 1) g_m(\eta) + 2\eta g'_m(\eta) + g''_m(\eta)$$
  
=  $(2\eta^2 + 2m + 2) g_m(\eta) + 2\eta g'_m(\eta)$ .

Applying one derivative, computing the result in  $\eta = 0$  and recalling that  $g_m$  is odd, we obtain

$$g'_{m+2}(0) = (2m+4)g'_m(0) = 2^2 \frac{m+2}{2} 2^m \left(\frac{m}{2}\right)! = 2^{m+2} \left(\frac{m+2}{2}\right)!.$$

This concludes the proof of the lemma.

**Remark 5.6.** Actually, the form of the polynomials  $p_m$  is well-known since it is given by the Hermite polynomials  $H_m$  via

$$p_m(\eta) = c_m H_m(i\eta)$$

up to differing conventions and a multiplicative factor  $c_m \in \mathbb{C} \setminus \{0\}$ . However, we only use the facts of Lemma 5.5 in the following argument.

Using the preceding qualitative representation of solutions, we are in the position to determine the asymptotic behaviour of  $\Upsilon_{\mu,0,1}$  and  $\Upsilon_{\mu,0,2}$  for  $\mu \in -2\mathbb{N} - 1$ . We consider the two solutions branches  $\psi_{\mu,i}$  and  $\psi_{\mu,j}$  of Lemma 5.5, separately. We begin with  $\psi_{\mu,i}$ .

**Lemma 5.7.** Let  $b = e^{\pm i\frac{\pi}{8}}$  and consider  $\mu = -(2m+3) \in -(2\mathbb{N}+1)$  with  $m \in \mathbb{N}_0$ . Setting the index  $i = (3+(-1)^{m+1})/2 \in \{1,2\}$ , then it holds the asymptotic expansion

$$\Upsilon_{\mu,0,i}(bz) \sim 2^m (m+3) \frac{\left\lceil \frac{m}{2} \right\rceil!}{\left(2 \left\lceil \frac{m}{2} \right\rceil\right)!} (bz)^{m-1} \exp\left(b^2 \frac{z^2}{2}\right)$$

as  $z \to \pm \infty$  in  $\mathbb{R}$ .

We note that  $(bz)^{m-1}$  is single-valued for any choice of arg(b).

*Proof.* We take  $\psi = \psi_{\mu,i}(\eta) = p_m(\eta)e^{\eta^2/2}$  as in Lemma 5.5 and we determine the asymptotic of  $\Upsilon = \Upsilon_{\mu,0,i}$  through the identity

$$\Upsilon(\eta) = \left(-\frac{d^2}{d\eta^2} + \eta^2 - \mu\right) \int_0^{\eta} \psi(\xi) d\xi = -\psi'(\eta) + \psi'(0) + (\eta^2 - \mu) \int_0^{\eta} \psi(\xi) d\xi.$$

 $\psi'(0)$  is constant, while  $-\psi'(\eta) = -\psi'_{\mu,i}(\eta)$  with  $\eta = bz$  satisfies the identity

$$-\psi'(bz) = -\left[p'_m(bz) + bz\,p_m(bz)\right] \exp\left(b^2\frac{z^2}{2}\right).$$

We have by Lemma 5.2 and  $\eta = bz$ 

$$(\eta^2 - \mu) \int_0^{\eta} \psi(\xi) d\xi = (bz^2 - \mu) \int_0^{bz} p_m(\xi) e^{\frac{\xi^2}{2}} d\xi \quad \sim \quad \frac{b^2 z^2 - \mu}{bz} p_m(bz) \exp\left(b^2 \frac{z^2}{2}\right).$$

In sum, we have the following asymptotics of  $\Upsilon = \Upsilon_{\mu,0,i}$ :

$$\Upsilon(bz) \sim -\left(p_m'(bz) + \mu \frac{p_m(bz)}{bz}\right) \exp\left(b^2 \frac{z^2}{2}\right), \text{ as } z \to \pm \infty \text{ in } \mathbb{R}.$$

The right-hand side is not identically null, since the equation  $\eta f'(\eta) = -\mu f(\eta) = (2m+3)f(\eta)$  is only solved by monomials  $f(\eta) = c\eta^{2m+3}$  but the polynomial  $p_m$  has degree m. Thanks to the leading order of  $p_m$  we finally obtain that

$$\Upsilon_{\mu,0,i}(bz) \sim -\frac{2^m \left\lceil \frac{m}{2} \right\rceil!}{\left(2 \left\lceil \frac{m}{2} \right\rceil\right)!} (m+\mu)(bz)^{m-1} \exp\left(b^2 \frac{z^2}{2}\right).$$

The assertion follows with  $m + \mu = -(m + 3)$ .

Next, we address the asymptotics of the second solution branch given by Lemma 5.5. Since the computations are slightly more extended, we give the result in the following:

**Lemma 5.8.** Let  $b = e^{\pm i\frac{\pi}{8}}$  and consider  $\mu = -(2m+3) \in -(2\mathbb{N}+1)$  with  $m \in \mathbb{N}_0$ . Then, with index  $j = (3+(-1)^m)/2 \in \{1,2\}$ , it holds the asymptotic expansion

$$\Upsilon_{\mu,0,j}(bz) \quad \sim \quad \pm \frac{1}{\left\lceil \frac{m-1}{2} \right\rceil!} \frac{1}{4^{\left\lceil \frac{m-1}{2} \right\rceil}} \frac{\sqrt{\pi}}{2} (m+3) 2^m (bz)^{m-1} \exp\left(b^2 \frac{z^2}{2}\right).$$

as  $z \to \pm \infty$ .

*Proof.* By Lemma 5.5, we know that  $\psi = \psi_{\mu,j}$  satisfies

$$\psi(\eta) = q_m(\eta)e^{\eta^2/2} \int_0^{\eta} e^{-\xi^2} d\xi + \tilde{q}(\eta)e^{-\eta^2/2}.$$

Similar to the above argument, we calculate the summands of  $\Upsilon(\eta) = -\psi'(\eta) + \psi'(0) + (\eta^2 - \mu) \int_0^{\eta} \psi(\xi) d\xi$ . But  $\psi'(0)$  is constant and for the derivative in  $\eta = bz$ , we obtain

$$-\psi'(bz) = -[q'_m(bz) + bzq_m(bz)] \exp\left(b^2 \frac{z^2}{2}\right) b \int_0^{bz} \exp\left(-\xi^2\right) d\xi + \hat{q}(bz) \exp\left(-b^2 \frac{z^2}{2}\right),$$

where  $\hat{q}$  is a polynomial. Note that the last summand on the right-hand side is exponentially decreasing as  $z \to \pm \infty$ , thus it has no relevant contribution on the asymptotics of  $\Upsilon$ .

Next, we address the leading term for the asymptotics of  $\int_0^{\eta} \psi(\xi) d\xi$  which is due to the contribution of  $q_m$ . We apply an integration by parts:

$$\begin{split} &\int_0^z q_m(b\tilde{z}) \exp\left(b^2 \frac{\tilde{z}^2}{2}\right) \left(\int_0^{b\tilde{z}} e^{-\xi^2} d\xi\right) b \cdot d\tilde{z} = \int_0^z \frac{d}{d\tilde{z}} \left(\int_0^{\tilde{z}} q_m(bs) \exp\left(b^2 \frac{s^2}{2}\right) ds\right) \left(\int_0^{b\tilde{z}} e^{-\xi^2} d\xi\right) b \cdot d\tilde{z} \\ &= \left\{ \left(\int_0^{\tilde{z}} q_m(bs) \exp\left(b^2 \frac{s^2}{2}\right) ds\right) \left(\int_0^{b\tilde{z}} e^{-\xi^2} d\xi\right) \right\} \Big|_{\tilde{z}=0}^{\tilde{z}=z} - \int_0^z \left(\int_0^{\tilde{z}} q_m(bs) \exp\left(b^2 \frac{s^2}{2}\right) ds\right) \cdot b \exp\left(-b^2 \tilde{z}^2\right) d\tilde{z} \\ &\sim \int_0^z q_m(bs) \exp\left(b^2 \frac{s^2}{2}\right) ds \cdot b \int_0^z \exp\left(-b^2 \tilde{z}^2\right) d\tilde{z} \sim \frac{q_m(bz)}{bz} \exp\left(b^2 \frac{z^2}{2}\right) \int_0^z \exp\left(-b^2 \tilde{z}^2\right) d\tilde{z}. \end{split}$$

Note that, passing the summand with negative sign in the second line is of lower order (asymptotically constant), while in the last relation we have used Lemma 5.2. In sum, it holds the following asymptotics

$$\Upsilon_{\mu,0,j}(bz) = -\psi'(bz) + b(b^2z^2 - \mu) \int_0^z \psi(b\tilde{z})d\tilde{z}$$

$$\sim -\left(q'_m(bz) + \mu \frac{q_m(bz)}{bz}\right) \exp\left(b^2 \frac{z^2}{2}\right) b \int_0^z \exp\left(-b^2 \tilde{z}^2\right) d\tilde{z}$$

$$\sim \mp \frac{1}{\left\lceil \frac{m-1}{2} \right\rceil!} \frac{1}{4^{\left\lceil \frac{m-1}{2} \right\rceil}} \frac{\pi}{2} (m+\mu) 2^m (bz)^{m-1} \exp\left(b^2 \frac{z^2}{2}\right).$$

The claim follows from  $m + \mu = -(m + 3)$ .

#### 5.3. Exact formulation for $\mu = -1$ .

As a consequence we can extend the assertion of Lemma 5.3 to any  $\mu$  lying on the negative real axis with the exception of the crucial value  $\mu = -1$ . However, we are able to give explicit solutions in this case:

**Lemma 5.9.** Let  $\eta \in \mathbb{C}$  and consider  $\mu = -1$ . Then it holds

$$\Upsilon_{-1,0,1}(\eta) = \frac{1}{2} \left( \eta e^{-\frac{\eta^2}{2}} + \frac{\sqrt{\pi}}{2} (1 + \eta^2) \operatorname{erf} \left( \frac{\eta}{\sqrt{2}} \right) \right)$$

$$\Upsilon_{-1,0,2}(\eta) = \frac{\sqrt{\pi}}{2} \left( 1 + \frac{\eta^2}{2} \right) \int_0^{\eta} \operatorname{erfi}(\xi) e^{-\frac{\xi^2}{2}} d\xi - \frac{\sqrt{\pi}}{4} \int_0^{\eta} \xi^2 \operatorname{erfi}(\xi) e^{-\frac{\xi^2}{2}} d\xi - \frac{1}{2}.$$

## 5.4. Proof of Corollary 1.7.

Finally, we revise the obtained asymptotics to address the statement of Corollary 1.7. By Proposition 4.1, we know the general solution of equation 10 is given by

$$\Upsilon(\eta) = c_0(\mu - \eta^2)(\eta) + c_1 \Upsilon_{\mu,0,1}(\eta) + c_2 \Upsilon_{\mu,0,2}(\eta).$$

Clearly, the first term  $\mu - \eta^2$  grows quadratically. In view of the previous lemmata, setting  $\eta = bz$  with  $b := e^{\pm \frac{\pi}{8}i}$  we hence need to exclude the possibility that the different summands cancel each other at both  $z \to \pm \infty$  for certain combinations of coefficients  $c_0, c_1, c_2$ . We proceed by investigating three cases:

Case  $\mu = -1$ : Referring to Lemma 5.9, the function  $\Upsilon_{-1,0,1}(bz)$  grows quadratically on both side of  $\mathbb{R}$  with odd parity. Regarding  $\Upsilon_{-1,0,2}(bz)$ , the asymptotic behaviour is exponential by the same lemmata. Hence for every combination of  $c_0$  and  $c_1$  we must have

$$\lim_{\begin{subarray}{c} \eta \to \infty, \\ |\arg(\eta)| < \frac{\pi}{4} \end{subarray}} \left| \frac{\Upsilon(\eta)}{\mu - \eta^2} \right| \neq 0 \qquad \text{or} \qquad \lim_{\begin{subarray}{c} \eta \to \infty, \\ |\arg(-\eta)| < \frac{\pi}{4} \end{subarray}} \left| \frac{\Upsilon(\eta)}{\mu - \eta^2} \right| \neq 0.$$

Case  $\mu \in -(2\mathbb{N}+1)$ :

The asymptotic behaviour of the summands of  $\Upsilon_{\mu,0,1}$  and  $\Upsilon_{\mu,0,1}$  are given in Lemma 5.7 and Lemma 5.8. In particular, these asymptotics are of exponential order but with differing parities for  $\Upsilon_{\mu,0,1}$  and  $\Upsilon_{\mu,0,2}$ . Hence for every combination  $(c_1, c_2) \neq (0, 0)$ , we have

$$\lim_{\substack{\eta \to \infty, \\ |\arg(\eta)| < \frac{\pi}{4}}} \left| \frac{\Upsilon(\eta)}{\mu - \eta^2} \right| = +\infty \qquad \text{or} \qquad \lim_{\substack{\eta \to \infty, \\ |\arg(\eta)| < \frac{\pi}{4}}} \left| \frac{\Upsilon(\eta)}{\mu - \eta^2} \right| = +\infty.$$

Case  $\mu \in \mathbb{C} \setminus (2\mathbb{Z} - 1)$ :

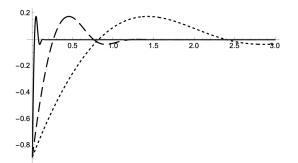
For all remaining  $\mu \in \mathbb{C} \setminus (2\mathbb{Z} - 1)$ , we realize that the asymptotics of Corollary 5.4 implies that  $\Upsilon_{\mu,0,1}$  and  $\Upsilon_{\mu,0,2}$  grow exponentially with different parities. Hence any linear combination needs to grow at least for  $z \to +\infty$  or  $z \to -\infty$  exponentially. This proves the claim.

## 6. Some examples of explicit solutions

In this section, we present two examples illustrating how to generate solutions to Equation (1), following the methodology introduced in Remark 1.5. To simplify the calculations, we focus on the specific case where  $U_{\rm sh}(y)=-y^2$ , corresponding to  $\alpha=0,\,\beta=-1,$  and a=0.

We consider positive frequencies k > 0 and, in both examples, set  $\sigma = e^{\frac{7\pi}{4}i} = (1-i)/\sqrt{2} \in \mathbb{C}$ , which implies  $\tau = -i\sigma = e^{\frac{5\pi}{4}i} = -(1+i)/\sqrt{2} \in \mathbb{C}$  and  $\mu = -\sigma e^{\frac{\pi}{4}i} = -1$ . Next, we compute the two solutions  $\psi_{\mu,1}$  and  $\psi_{\mu,2}$ , which we abbreviate as  $\psi_1$  and  $\psi_2$ :

$$\psi_1(\eta) = \mathcal{M}\left(0, \frac{1}{2}, \eta^2\right) e^{-\frac{\eta^2}{2}} = e^{-\frac{\eta^2}{2}},$$
  
$$\psi_2(\eta) = \eta \mathcal{M}\left(\frac{1}{2}, \frac{3}{2}, \eta^2\right) e^{-\frac{\eta^2}{2}} = \frac{\sqrt{\pi}}{2} \operatorname{erfi}(\eta) e^{-\frac{\eta^2}{2}},$$



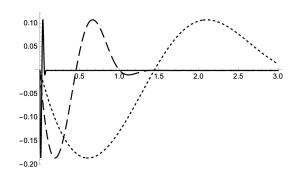


FIGURE 1. Plot of the stream function  $\phi_k = \phi_k(y)$  defined by (64) for k = 1 (dotted line),  $k = 10^2$  (dashed line) and  $k = 10^6$  (full line). With  $\sigma = (1 - i)/\sqrt{2}$ , thus this profile inflates in time as  $\exp(t\sqrt{k}/2\sqrt{2})$ . The no-slip boundary conditions  $\phi_k(0) = \phi'_k(0) = 0$  are not satisfied. Left and right correspond to the real and imaginary parts, respectively.

where  $\operatorname{erfi}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{\zeta^2} d\zeta$ . We remark that  $\eta_* = -a \sqrt[4]{|\beta| |k|}$  is null for any  $k \in \mathbb{N}$ , thus we can compute the functions  $\Upsilon_{\mu,\eta_*,1}$  and  $\Upsilon_{\mu,\eta_*,2}$  (abbreviated as  $\Upsilon_1$  and  $\Upsilon_2$ ) through the expressions

$$\Upsilon_1(\eta) = \int_0^{\eta} \left( 1 + \frac{\eta^2 - \xi^2}{2} \right) \psi_1(\xi) d\xi - \frac{\psi_1'(0)}{2} = \frac{1}{2} \left( \eta e^{-\frac{\eta^2}{2}} + \frac{\sqrt{\pi}}{2} (1 + \eta^2) \operatorname{erf}\left(\frac{\eta}{\sqrt{2}}\right) \right)$$
(63)

while for  $\Upsilon_2(\eta)$ , we obtain

$$\begin{split} \Upsilon_2(\eta) &= \int_0^{\eta} \left( 1 + \frac{\eta^2 - \xi^2}{2} \right) \psi_2(\xi) d\xi - \frac{\psi_2'(0)}{2} \\ &= \frac{\sqrt{\pi}}{2} \left( 1 + \frac{\eta^2}{2} \right) \int_0^{\eta} \text{erfi}(\xi) e^{-\frac{\xi^2}{2}} d\xi - \frac{\sqrt{\pi}}{4} \int_0^{\eta} \xi^2 \, \text{erfi}(\xi) e^{-\frac{\xi^2}{2}} d\xi - \frac{1}{2}. \end{split}$$

It is possible to further develop the last integral in terms of Owen's T function, but this lies beyond the scope of the present paper. Nevertheless, we note that  $\Upsilon_2$  is well defined, entire, and can be computationally evaluated. The general solution of the Prandtl eigenproblem (10) is therefore

$$\Upsilon(\eta) = c_0(-1 - \eta^2) + c_1 \Upsilon_1(\eta) + c_2 \Upsilon_2(\eta), \quad \text{with } c_0, c_1, c_2 \in \mathbb{C}.$$

If we do not enforce the no-slip boundary conditions at y=0, the constants can be chosen arbitrarily. For instance, setting  $c_0=-\frac{\sqrt{\pi}}{2}$ ,  $c_1=2$ , and  $c_2=0$ , the function  $\Upsilon(\eta)$  simplifies to

$$\Upsilon(\eta) = \eta e^{-\frac{\eta^2}{2}} + \frac{\sqrt{\pi}}{2} (1 + \eta^2) \left( \operatorname{erf} \left( \frac{\eta}{\sqrt{2}} \right) - 1 \right). \tag{64}$$

The corresponding stream function is given by  $\phi_k(y) = \Upsilon(e^{-\frac{\pi}{8}\mathrm{i}}\sqrt[4]{k}y)$ , leading to the velocity components  $u(t,x,y) = \phi_k'(y)e^{\mathrm{i}kx+t\sqrt{k}\frac{1-\mathrm{i}}{\sqrt{2}}}$  and  $v(t,x,y) = -\mathrm{i}k\phi_k(y)e^{\mathrm{i}kx+t\sqrt{k}\frac{1-\mathrm{i}}{\sqrt{2}}}$ . These satisfy Equations (1). However, we note that  $\phi_k(0) = -\frac{\sqrt{\pi}}{2} \neq 0$  and  $\phi_k'(0) = e^{-\frac{\pi}{8}\mathrm{i}}\sqrt[4]{k}(1+\frac{1}{\sqrt{2}}) \neq 0$ . The plot of  $\phi_k$  is provided in Figure 1. To pursue the no-slip boundary conditions  $\phi_k(0) = \phi_k'(0) = 0$ , we need to impose (19) and we first compute

To pursue the no-slip boundary conditions  $\phi_k(0) = \phi_k'(0) = 0$ , we need to impose (19) and we first compute  $\psi_1(0) = 1$ ,  $\psi_2(0) = 0$ ,  $\psi_1'(0) = 0$  and  $\psi_2'(0) = 1$ . The constants must satisfy  $2c_0 + c_2 = 0$  and  $c_1 = 0$ . Choosing  $c_0 = -1/2$  and  $c_2 = 1$  leads to

$$\Upsilon(\eta) = \frac{\eta^2}{2} + \frac{\sqrt{\pi}}{2} \left( 1 + \frac{\eta^2}{2} \right) \int_0^{\eta} \operatorname{erfi}(\xi) e^{-\frac{\xi^2}{2}} d\xi - \frac{\sqrt{\pi}}{4} \int_0^{\eta} \xi^2 \operatorname{erfi}(\xi) e^{-\frac{\xi^2}{2}} d\xi.$$
 (65)

The corresponding stream function  $\phi_k(y) = \Upsilon(e^{-\frac{\pi}{8}i}\sqrt[4]{k}y)$  is plotted in Figure 2. Unfortunately,  $\phi_k(y)$  grows and oscillate in space as  $\phi_k(y) \approx e^{\sqrt{k}\frac{1-i}{\sqrt{2}}\frac{y^2}{2}}$  making it not amenable for the ill-posedness of (1) in Gevrey-classes.

## 7. An explicit formulation of the shear-layer velocity

In this final section, we highlight how our Corollary 1.2 enables us to obtain, with analytical precision, the shear-layer velocity of Gérard-Varet and Dormy [10], which has been asymptotically associated with the instability of the Prandtl equations. The following formula can be applied to any shear flow  $U_{\rm sh}(y)$  that is sufficiently regular and satisfies  $U'_{\rm sh}(a)=0$  and  $U''_{\rm sh}(a)<0$  (not only the quadratic case of (2)).

For the sake of comparison, we adopt the same notation as in [10]. Setting  $\varepsilon = 1/k$ , the shear-layer velocity  $v_{\varepsilon}^{\rm sl}$  is defined as

$$v_{\varepsilon}^{\mathrm{sl}}(y) = \varepsilon^{\frac{1}{2}} V\left(\frac{y-a}{\varepsilon^{\frac{1}{4}}}\right) = \varepsilon^{\frac{1}{2}} V\left(z\right), \quad \text{with } z = \frac{y-a}{\varepsilon^{\frac{1}{4}}}.$$
 (66)

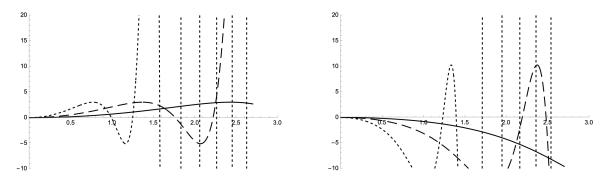


FIGURE 2. Plot of the stream function  $\phi_k = \phi_k(y)$  defined by (65) for k = 1 (full line), k = 10 (dashed line) and  $k = 10^2$  (dotted line). With  $\sigma = (1 - i)/\sqrt{2}$ , this profile inflates in time as  $\exp(t\sqrt{k}/\sqrt{2})$ . The no-slip boundary conditions  $\phi_k(0) = \phi_k'(0) = 0$  are satisfied but generates exponential growth and oscillations in y. Left and right correspond to the real and imaginary parts, respectively.

(cf. formula below (2.5) in [10]). Making use of the definition of V and  $\tilde{V}$  at the pages 596 and 597 of [10] (being careful that  $\tau$  here is  $\tilde{\tau}$  in [10]), one has that

$$\tilde{V}(z) := V(z) + \left(\frac{|U_{\rm sh}''(a)|^{\frac{1}{2}}}{\sqrt{2}}\tau + U_{\rm sh}''(a)\frac{z^2}{2}\right)H(z) = \left(\frac{|U_{\rm sh}''(a)|^{\frac{1}{2}}}{\sqrt{2}}\tau + U_{\rm sh}''(a)\frac{z^2}{2}\right)W\left(\sqrt[4]{\frac{|U_{\rm sh}''(a)|}{2}}z\right),$$

with H denoting the Heaviside function. On the other hand, Corollary 1.2 implies that  $\tau = e^{\frac{5\pi}{4}i}$  and that W can be expressed in terms of erf and other elementary functions. More precisely, our result implies that the function V is given by

$$V(z) = \left(\frac{|U_{\rm sh}''(a)|^{\frac{1}{2}}}{\sqrt{2}}\tau + U_{\rm sh}''(a)\frac{z^{2}}{2}\right) \left(W\left(\sqrt[4]{\frac{|U_{\rm sh}''(a)|}{2}}z\right) - H(z)\right)$$

$$= e^{\frac{5\pi}{4}i}\frac{|U_{\rm sh}''(a)|^{\frac{1}{2}}}{\sqrt{2}}\left(1 + f(z)^{2}\right)\left(\frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{f(z)}{\sqrt{2}}\right) + \frac{1}{\sqrt{2\pi}}\frac{f(z)}{1 + f(z)^{2}}e^{-\frac{f(z)^{2}}{2}} - H(z)\right),$$
(67)

where the function f(z) is defined as

$$f(z) := \frac{|U_{\mathrm{sh}}''(a)|^{\frac{1}{4}}}{\sqrt[4]{2}} \eta(z) = \frac{|U_{\mathrm{sh}}''(a)|^{\frac{1}{4}}}{\sqrt[4]{2}e^{\frac{\pi}{8}i}} z.$$

Expression (67) generates the "shear-layer" velocity (66) for any  $U_{\rm sh}(y)$  satisfying  $U'_{\rm sh}(a) = 0$  and  $U''_{\rm sh}(a) < 0$ , depending only on  $U''_{\rm sh}(a)$ .

We can plot the formula (67) using the example from Section 5.2 of [10], where  $U_{\rm sh}(y) = 2y \exp(-y^2)$ . The singular point is  $a = 1/\sqrt{2} > 0$ , with  $U_s'(1/\sqrt{2}) = 0$  and  $U_{\rm sh}''(1/\sqrt{2}) = -4\sqrt{2}/\sqrt{e}$ . The plot of the resulting function V is shown in Figure 3, which precisely matches the plot of the shear-layer correction  $v_{\rm in}^{\rm th}$  in [10] (cf. Figure 3, page 607).

# APPENDIX A. THE HARMONIC OSCILLATOR

For the sake of completeness, we give some facts on the solutions of the harmonic oscillator. First, it follows a representation of the algebraic solutions of the harmonic oscillator in terms of confluent hypergeometric functions:

**Lemma A.1.** Every solution  $\psi : \mathbb{C} \to \mathbb{C}$  of the differential equation

$$-\psi'' + \eta^2 \psi = (\mu + 2)\psi, \qquad \mu \in \mathbb{C}$$
(68)

is of the form

$$\psi(\eta) = e^{-\frac{\eta^2}{2}} \left[ \tilde{c}_1 \mathcal{M} \left( -\frac{1+\mu}{4}, \frac{1}{2}, \eta^2 \right) + \tilde{c}_2 \eta \mathcal{M} \left( \frac{1-\mu}{4}, \frac{3}{2}, \eta^2 \right) \right]$$

for arbitrary  $\tilde{c}_1, \tilde{c}_2 \in \mathbb{C}$ .

*Proof.* Following well-known reductions from quantum physics (e.g. [24]), we have that  $\phi(\eta) := e^{\frac{\eta^2}{2}} \psi(\eta)$  solves

$$\phi'' - 2n\phi' + (\mu + 1)\phi = 0.$$

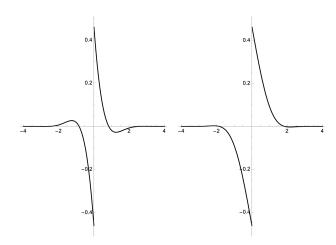


FIGURE 3. Plot of the "shear-layer" function V (67), explicitly defined in terms of the erf and exp functions in z. It coincides with the shear-layer correction in [10] for the shear layer  $U_{\rm sh}(y) = 2y \exp(-y^2)$ . Left and right correspond to the real and imaginary parts, respectively.

Setting  $w(z) := \mathcal{M}(a, c, z)$ , we have that zw''(z) + (b-z)w'(z) - aw(z) = 0 for any  $z \in \mathbb{C}$ . For  $a = -\frac{1+\mu}{4}$ ,  $c = \frac{1}{2}$  and  $z = \eta^2$ , we have

$$\begin{split} \frac{d^2}{d^2\eta} w(\eta^2) - 2\eta \frac{d}{d\eta} w(\eta^2) + (\mu + 1) w(\eta^2) \\ &= 4 \left[ \eta^2 w''(\eta^2) + \frac{1}{2} w'(\eta) - \eta^2 w'(\eta^2) - \left( -\frac{1+\mu}{4} \right) w(\eta^2) \right] \\ &= 0. \end{split}$$

Analogously, for  $a = \frac{1-\mu}{4}, c = \frac{3}{2}$  and  $\bar{w}(\eta) := \eta w(\eta^2)$ , it is

$$\begin{split} \frac{d^2}{d^2\eta} \bar{w}(\eta) &- 2\eta \frac{d}{d\eta} \bar{w}(\eta) + (\mu + 1)\bar{w}(\eta) \\ &= 4\eta^3 w''(\eta^2) + 6\eta w'(\eta^2) - 4\eta^3 w'(\eta^2) - 2\eta w(\eta^2) + (\mu + 1)\eta w(\eta^2) \\ &= 4\eta \left[ \eta^2 w''(\eta^2) + \left(\frac{3}{2} - \eta^2\right) w'(\eta^2) - \frac{1 - \mu}{4} w(\eta^2) \right] \\ &= 0. \end{split}$$

As both functions,  $\eta \mapsto w(\eta^2)$ ,  $\bar{w}(\eta)$ , are linearly independent, the assertion follows.

Another important tool consists of the up- and down-operators

$$\mathcal{A}_{\uparrow} := \eta - \frac{d}{d\eta}, \qquad \mathcal{A}_{\downarrow} := \eta + \frac{d}{d\eta}.$$

In our notation, a given solution  $\psi_{\mu}$  (68) transcends to another solution of the algebraic eigenvalue  $\mu \pm 2$  by applying the up- respectively down-operator. It holds:

**Lemma A.2.** If  $\psi = \psi_{\mu} : \mathbb{C} \to \mathbb{C}$  is a solution to (68), then

$$\psi_{\mu+2} := \mathcal{A}_{\uparrow} \psi_{\mu}, \qquad \psi_{\mu-2} := \mathcal{A}_{\downarrow} \psi_{\mu} \tag{69}$$

are solutions of (68) with  $\mu$  replaced by  $\mu + 2$  and  $\mu - 2$ , respectively.

*Proof.* We prove the first equality of (69) as the second one follows analogously. Recall that  $\mathcal{B}_{\mu} = -\frac{d^2}{d\eta^2} + \eta^2 - \mu$ . Then we compute the commutator

$$[\mathcal{B}_{\mu},\mathcal{A}_{\uparrow}] = \left[ -rac{d^2}{d\eta^2}, \eta 
ight] - \left[ \eta^2, rac{d}{d\eta} 
ight] = -2rac{d}{d\eta} + 2\eta = 2\mathcal{A}_{\uparrow},$$

for any  $\mu \in \mathbb{C}$ . With this at hand, we have

$$\mathcal{B}_{\mu+2}\mathcal{A}_{\uparrow}\psi_{\mu} = \mathcal{A}_{\uparrow}\mathcal{B}_{\mu+2}\psi_{\mu} + [\mathcal{B}_{\mu+2}, \mathcal{A}_{\uparrow}]\psi_{\mu} = 2\mathcal{A}_{\uparrow}\psi_{\mu}.$$

This proves the claim.

It should be pointed out that the operator  $\mathcal{A}_{\uparrow}$  and  $\mathcal{A}_{\downarrow}$  have one-dimensional kernels coinciding with solutions to (68), namely

$$\ker \mathcal{A}_{\uparrow} = \langle \eta \mapsto e^{-\eta^2/2} \rangle = \langle \psi_{-1,1} \rangle,$$
$$\ker \mathcal{A}_{\downarrow} = \langle \eta \mapsto e^{\eta^2/2} \rangle = \langle \psi_{-3,1} \rangle.$$

Hence, when constructing new solutions by those operators, one needs to ensure that these are non-vanishing. With this in mind, we give an auxiliary result concerning solutions to negative odd eigenvalues  $\mu$ .

**Lemma A.3.** Let  $\mu \in -(2\mathbb{N}+1)$  and  $m=-\frac{\mu+3}{2} \in \mathbb{N}_0$ . Then, with indexes  $i=(3+(-1)^{m+1})/2 \in \{1,2\}$  and  $j=(3+(-1)^m)/2 \in \{1,2\}$ , for every  $m \in \mathbb{N}$ , the two linearly independent solutions to (68) are given by

$$\psi_{\mu,i}(\eta) = \mathcal{A}_{\downarrow}^{m} e^{\eta^{2}/2} = e^{-\eta^{2}/2} \frac{d^{m}}{d\eta^{m}} e^{\eta^{2}}, \qquad \psi_{\mu,j}(\eta) = \mathcal{A}_{\downarrow}^{m} \left( e^{\eta^{2}/2} \int_{0}^{\eta} e^{-\xi^{2}} d\xi \right)$$
 (70)

up to a non-vanishing multiplicative factor.

*Proof.* In case m=0 which is  $\mu=-3$ , a simple computation shows

$$\psi_{-3,1} = e^{\eta^2/2}, \qquad \qquad \psi_{-3,2} = e^{\eta^2/2} \int_0^{\eta} e^{-\xi^2} d\xi.$$

Furthermore, it holds  $\mathcal{A}_{\downarrow}(e^{-\eta^2/2}f(\eta)) = e^{-\eta^2/2}f'(\eta)$ , which proves the second equality in (70). The remaining claims follow by induction over  $m \in \mathbb{N}_0$  using Lemma A.2 and realizing that the functions  $\psi_{\mu,i}$  and  $\psi_{\mu,j}$  do not lie in ker  $\mathcal{A}_{\downarrow}$ .

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