Topic:

Demonstration of the Black-Scholes formula

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August 5, 2024

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Introduction

The Black-Scholes formula is a fundamental pillar in quantitative finance. particularly for an IT-Quant, who plays a crucial role in the development and implementation of option pricing models, this skill is indispensable. Understanding and demonstrating this formula not only allows for the theoretical calculation of European option prices but also provides insight into the underlying dynamics of financial markets. Mastering the demonstration of the Black-Scholes formula is essential for developing robust trading algorithms, optimizing investment strategies, and accurately assessing financial risks. Here is a detailed demonstration of the Black-Scholes formula for a call option.

I- Assumptions of the Black-Scholes Model.

Statement

This model is based on the following assumptions:

1- The price of the underlying asset follows a geometric Brownian motion dynamic.

2- The risk-free interest rate is constant and known.

3- The volatility of the underlying asset is constant and known.

4- No dividends are paid by the underlying asset.

5- The market is efficient (i.e., asset prices reflect all available information at any given time) and liquid.

6- Options can only be exercised at the expiration date (European options).

I- Assumptions of the Black-Scholes Model.

Result

From the first assumption, it follows that the price of the underlying asset S_t is a process that follows the differential equation:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t^{\mathbb{P}}$$

where μ is the expected rate of return, σ is the volatility, and W_t is a standard Brownian motion under the real-world probability measure $\mathbb{P}$

Note: Investors are generally risk-averse, meaning they prefer safe investments over risky ones, given the same return. However, to properly evaluate the prices of options and other financial derivatives, it is necessary to simplify this risk aversion to make the calculations more tractable.

II- Transformation of the differential equation followed by S_t in the risk-neutral framework.

Girsanov's Theorem

It allows us to change the probability measure under which a stochastic process is observed. More precisely, it transforms the derivative of a Brownian motion when transitioning from the real probability measure $\mathbb{P}$ to a risk-neutral probability measure $\mathbb{Q}$ under which has a derivative equal to the risk-free rate r.

Summary:

- Original Measure $\mathbb{P}$: The process W_t is a Brownian motion.
- New Measure $\mathbb{Q}$: The process $\tilde{W}_t = W_t + \int_0^t \theta_s \, ds$ is a Brownian motion.

II- Transformation of the differential equation followed by S_t in the risk-neutral framework.

Application of Girsanov's Theorem

Under the new measure $\mathbb{Q}$, the process $W_t^{\mathbb{Q}}$ is defined by:
$W_t^{\mathbb{Q}} = W_t^{\mathbb{P}} + \int_0^t \theta\, ds$ where $\theta = \frac{\mu - r}{\sigma}$ is the market price of risk.
$=> dW_t^{\mathbb{P}} = dW_t^{\mathbb{Q}} - \theta\, dt$
$=> dS_t = \mu S_t\, dt + \sigma S_t (dW_t^{\mathbb{Q}} - \theta\, dt)$
$=> dS_t = \mu S_t\, dt + \sigma S_t (dW_t^{\mathbb{Q}} - (\frac{\mu - r}{\sigma})\, dt)$
$=> dS_t = \mu S_t\, dt + \sigma S_t\, dW_t^{\mathbb{Q}} - (\mu - r) S_t\, dt)$
Thus $dS_t = r S_t\, dt + \sigma S_t\, dW_t^{\mathbb{Q}}$

Note: under the new risk-neutral measure $\mathbb{Q}$, the process S_t has a derivative equal to the risk-free rate r.

III-Formulation of the option price as an expectation.

Expectation

The price of a call option can be expressed as the expected discounted payoff at maturity under the risk-neutral measure $\mathbb{Q}$.
$C = \mathbb{E}[e^{-rT}(S_T - K) \cdot \mathbb{1}_{S_T > K}]$ ($\cdot \mathbb{1}_{S_T > K}$ is the indicator function which equals 1 if S_T>K and 0 else.)
=> $C = e^{-rT}\mathbb{E}[S_T \cdot \mathbb{1}_{S_T > K}] - Ke^{-rT}\mathbb{E}[\cdot \mathbb{1}_{S_T > K}]$
=> $C = e^{-rT}\mathbb{E}[S_T \cdot \mathbb{1}_{S_T > K}] - Ke^{-rT}\mathbb{P}(\cdot \mathbb{1}_{S_T > K})$
Thus $C = e^{-rT}\mathbb{E}[S_T \cdot \mathbb{1}_{S_T > K}] - Ke^{-rT}\mathbb{P}(S_T > K)$
These last two terms can be computed explicitly using the expression for the underlying asset price that we will establish subsequently.
Consider a function f(t,S(t)) where S(t) follows a given diffusion process:
$dS_t = rS_t\, dt + \sigma S_t\, dW_t^{\mathbb{Q}}$

By the same way put option $P = \mathbb{E}[e^{-rT}(K - S_T) \cdot \mathbb{1}_{K > S_T}] => P = Ke^{-rT}\mathbb{P}(K > S_T) - e^{rT}\mathbb{E}[S_T \cdot \mathbb{1}_{K > S_T}]$

Itô's Lemma

$$df(t,S(t)) = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial f}{\partial S}\sigma S dW(t)$$

Application

Consider f(t,S(t)) = ln(S(t)) and determine d(f(t,S(t)):

$$\frac{\partial f}{\partial t} = 0, \frac{\partial f}{\partial S} = \frac{1}{S}, \frac{\partial^2 f}{\partial S^2} = -\frac{1}{S^2}$$

$$=> d(\ln(S(t))) = \left(0 + \frac{1}{S(t)}(rS(t)) + \frac{1}{2}\left(-\frac{1}{S(t)^2}\right)\sigma^2 S(t)^2\right) dt + \frac{1}{S(t)}\sigma S(t) dW(t)$$

Thus

$$d(f(t,S(t)) = d(\ln(S(t))) = \left(r - \frac{1}{2}\sigma^2\right) dt + \sigma dW(t)$$

III-Formulation of the option price as an expectation.

finding S(t)

$$d(f(t,S(t)) = d(\ln(S(t))) = \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma dW(t)$$

By integration from 0 to t:

$$\int_0^t d(\ln(S(t)))dt = \int_0^t (\left(r - \frac{1}{2}\sigma^2\right)dt + \sigma dW(t))dt$$
$$=> ln(S_t) - ln(S_0) = (\left(r - \frac{1}{2}\sigma^2\right) + \sigma W_t) - (\left(r - \frac{1}{2}\sigma^2\right)(0) + \sigma W_0)$$
$$=> ln(S_t) = ln(S_0) + (\left(r - \frac{1}{2}\sigma^2\right) + \sigma W_t)$$

Thus $S_t = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right) + \sigma W_t}$

III-Formulation of the option price as an expectation.

finding $\mathbb{P}(S_T > K)$

Note that the random variable W_t follows a normal distribution $\mathscr{N}(0,t)$.

$$\mathbb{P}(S_T > K) = \mathbb{P}\left((r - \frac{\sigma^2}{2})T + \sigma W_T > \ln\left(\frac{K}{S_0}\right)\right)$$

$$= \mathbb{P}\left(\frac{1}{\sqrt{T}} W_T > \frac{\ln\left(\frac{K}{S_0}\right) - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

$$= \mathbb{P}(G > -d_2)$$

Where $G \sim \mathscr{N}(0,1)$ and $d_2 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right) T\right]$

Since G and -G follow the same distribution :

Thus $\mathbb{P}(S_T > K) = \mathbb{P}(-G > -d_2) = \mathbb{P}(G < d_2) = \mathscr{N}(d_2)$

III-Formulation of the option price as an expectation.

finding $\mathbb{E}[S_T \cdot \mathbb{1}_{S_T>K}]$

$$S_T = S_0 \exp\left((r - \frac{\sigma^2}{2})T + \sigma W_T\right)$$

$$\mathbb{E}[S_T \cdot \mathbb{1}_{S_T>K}] = \mathbb{E}\left[S_0 \exp\left((r - \frac{\sigma^2}{2})T + \sigma W_T\right) \cdot \mathbb{1}_{S_0 \exp\left((r-\frac{\sigma^2}{2})T+\sigma W_T\right)>K}\right]$$

$$\mathbb{E}[S_T \cdot \mathbb{1}_{S_T>K}] = \mathbb{E}\left[S_0 \exp\left((r - \frac{\sigma^2}{2})T + \sigma W_T\right) \cdot \mathbb{1}_{\frac{W_T}{\sqrt{T}}>\frac{1}{\sigma\sqrt{T}}\left(\ln\frac{K}{S_0}-(r-\frac{\sigma^2}{2})T\right)}\right]$$

Define $G = \frac{W_T}{\sqrt{T}}$ and $-d_2 = \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{K}{S_0} - (r - \frac{\sigma^2}{2})T\right)$. Thus, $\mathbb{1}_{G>-d_2}$ corresponds to the indicator of the event that $G > -d_2$.

$$\mathbb{E}[S_T \mathbb{1}_{S_T>K}] = S_0 \mathbb{E}\left[e^{(r-\frac{\sigma^2}{2})T+\sigma G}\mathbb{1}_{G>-d2}\right]$$

III-Formulation of the option price as an expectation.

finding $\mathbb{E}[S_T \cdot 1\!\!1_{S_T > K}]$

G follows $\mathcal{N}(0,1)$ and its density is:

$\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{G^2}{2}\right)$.

$$\mathbb{E}\left[\exp\left((r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}G\right) \cdot 1\!\!1_{G > -d_2}\right]$$

$$= \int_{-d_2}^{\infty} \exp\left((r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}x\right) \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx.$$

$$= \exp\left((r - \frac{\sigma^2}{2})T\right) \int_{-d_2}^{\infty} \exp\left(\sigma\sqrt{T}x - \frac{x^2}{2}\right) \cdot \frac{1}{\sqrt{2\pi}} dx.$$

Thus $\mathbb{E}[S_T \cdot 1\!\!1_{S_T > K}] = S_0 \int_{-d_2}^{\infty} \exp\left(rT - \frac{x^2}{2} + \sigma\sqrt{T}x - \frac{\sigma^2}{2}T\right) \cdot \frac{dx}{\sqrt{2\pi}}$.

III-Formulation of the option price as an expectation.

Call option price

Then:

$$e^{-rT}\mathbb{E}[S_T \cdot 1\!\!1_{S_T>K}] = S_0 \int_{-d_2}^{\infty} \exp\left(-\frac{x^2}{2} + \sigma\sqrt{T}x - \frac{\sigma^2}{2}T\right) \cdot \frac{dx}{\sqrt{2\pi}}.$$

$$e^{-rT}\mathbb{E}[S_T 1\!\!1_{S_T>K}] = S_0 \int_{-d_2}^{+\infty} \mathrm{e}^{-\frac{1}{2}(x-\sigma\sqrt{T})^2} \frac{dx}{\sqrt{2\pi}}$$

$$e^{-rT}\mathbb{E}[S_T \cdot 1\!\!1_{S_T>K}] = S_0 \int_{-d_2-\sigma\sqrt{T}}^{+\infty} \mathrm{e}^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$e^{-rT}\mathbb{E}[S_T \cdot 1\!\!1_{S_T>K}] = S_0 \int_{-\infty}^{d_1} \mathrm{e}^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$e^{-rT}\mathbb{E}[S_T \cdot 1\!\!1_{S_T>K}] = S_0 N(d_1)$$

Finally, by combining the last two results, we obtain:

$$C = S_0 N(d_1) - K\mathrm{e}^{-rT} N(d_2)$$

By the same way put option $P = K\mathrm{e}^{-rT}N(-d_2) - S_0 N(-d_1)$

IV- Implementation in Python and C#.

1- **Continuous Black-Scholes:** The traditional analytical approach.
We will price by using this formula $C = S_0 N(d_1) - Ke^{-rT} N(d_2)$ demonstrated above.

2- **Numeric Monte Carlo Simulation:** A simulation-based method.
We will simulate many asset price paths by using this formula $S_t = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right) + \sigma W_t}$ found above; calculate Payoff for Each Path; discount Payoffs to Present Value and average the Discounted Payoffs.
All this will be done using asynchronous and parallel programming to bring forward performance, which is usually overlooked.

Conclusion

In conclusion, the demonstration of the Black-Scholes formula shows its crucial importance in quantitative finance. This formula not only facilitates the theoretical calculation of European option prices, it also improves our understanding of financial market dynamics and will enable us to understand all other Black-Scholes derivative models just as easily. By transforming the differential equations into the risk-neutral framework and solving them, we will apply numerical methods such as Monte Carlo simulations to obtain accurate option pricing. Mastering the demonstration of this formula is essential for developing efficient trading algorithms, optimizing investment strategies and accurately assessing financial risks, thus making a significant contribution to the field of financial engineering.

Thanks for your kind attention

See you soon.