Review of Basic Statistics

Topics

- *What is a random variable?*
- *What can we tell about its distribution: density, moments*
- *How do we estimate the moments of stock returns?*
- *What is a regression?*
- *How do we test a hypothesis?*

I. *Probability Concepts*

A. Random Variables

1. Uncertainty

a. We know exactly what are the various things that *can* happen, but we don't know what *will* happen.

b. In other words, investors know exactly which “states of the world” (also called “events”) can occur, and exactly what will happen in each of these states.

c. However, they don't know what state will actually occur (they will learn that afterwards).

d. At the point in time when they have to make decisions, they only know what the probabilities of each of the states are.

2. Suppose that there are n possible states of the world at some future date

Example: state is technological state of IBM a year from today.

3. The probability that state i will materialize is p_i, i=1,...,n.

a. Example: 3 states, p_i = probability that IBM develops technology i

- i=1: no new technology, $p_1 = 1/2$
- i=2: improvements of current technology, $p_2 = 1/4$
- i=3: big technological leap, $p_3 = 1/4$

b. The probabilities satisfy the conditions:

- $0 \le p_i \le 1$ for all i

- $\sum_{i=1}^{n} p_i = p_1 + p_2 + \ldots p_n = 1$

- In the example: 1/2 + 1/4 + 1/4 = 1.

c. An event that is certain has probability $p_i = 1$, an impossible event probability $p_i = 0$.

4. A *random variable* X is a quantity that assumes a given value X_i in each event i.

Example: X = return from holding one share of IBM stock between now and a year from today

5. X will take the value x_i when event i occurs, and this happens with probability: $\text{Prob}(X{=}x_i) = p_i$.

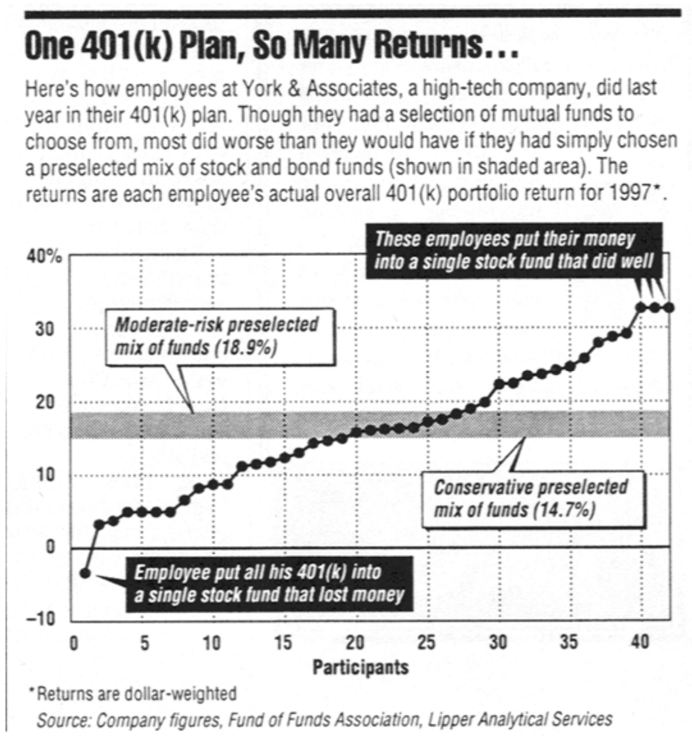
Example:

- if IBM makes no technological discovery (event i=1), its return over the coming year will be $x_1 = -20\%$

- in event i=2, IBM makes modest improvements to its current technology and the return is $x_2 = +5\%$

- in event i=3, IBM discovers a new valuable technology and the return is $x_3 = +25\%$.

6. Randomness means that different outcomes are possible



(WSJ)

B. Moments

1. The following *moments* are used to describe the distribution of the random variable X

2. Expectation or mean

$$M_1 = E[X] = \sum_{i=1}^{n} p_i x_i$$

   a. Characterizes the average value of X

   b. In the example:

$$\begin{aligned} M_1 &= 0.5*(-20\%)+0.25*(+5\%)+0.25*(+25\%) \\ &= -2.5\% \end{aligned}$$

3. Variance

$$M_2 = V[X] = E\left[\left(X - E[X]\right)^2\right]$$
$$= \sum_{i=1}^{n} p_i \left(x_i - E[X]\right)^2$$

a. Characterizes the dispersion of X around its average value

b. In the example:

$$M_2 = 0.5*(-20\%-(-2.5\%))^2 + 0.25*(+5\%-(-2.5\%))^2$$
$$+0.25*(+25\%-(-2.5\%))^2$$
$$= 0.035625$$

c. Note that when X is measured in %, M_2 is measured in "$\%^2$". Not very helpful unit...

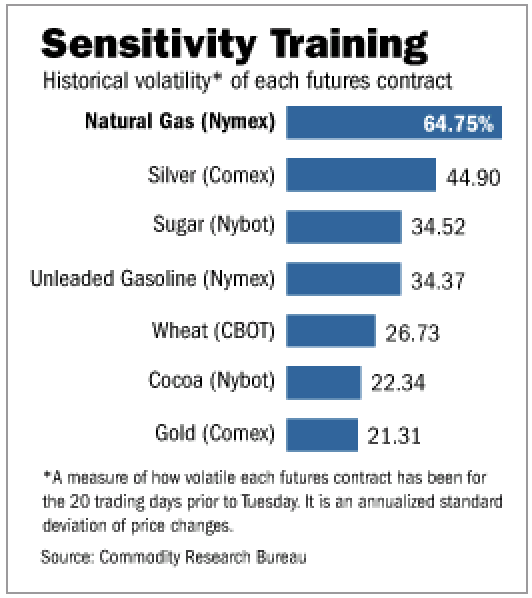
4. Standard deviation

$$\sigma(X) = \sqrt{V[X]}$$

a. Like the variance, measures the dispersion of X around its mean value, except is measured in the same units as X

b. In the example: $\sigma(X) = \sqrt{0.035625} = 0.1887 = 18.87\%$

c. $\sigma(X)$ is often called the *volatility* of the returns of asset X



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5. Skewness

a. Third moment:

$$M_3 = E\left[\left(X - E[X]\right)^3\right] = \sum_{i=1}^{n} p_i \left(x_i - E[X]\right)^3$$

b. Skewness:

$$Sk[X] = M_3 / \left(M_2\right)^{3/2}$$

c. Measures whether the dispersion arises because of many small positive deviations from the mean and a few large negative deviations or the opposite

d. For example, characterizes whether stock market crashes are more likely than rallies (left tail vs. right tail) = negative skewness observed in stock returns

6. Kurtosis

a. Fourth moment:

$$M_4 = E\left[\left(X - E[X]\right)^4\right] = \sum_{i=1}^{n} p_i \left(x_i - E[X]\right)^4$$

b. Kurtosis:

$$\text{Kurt}[X] = M_4 \big/ \left(M_2\right)^2$$

c. Measures whether the distribution is more peaked around the mean, and has fatter tails.

d. "Excess" kurtosis = Kurt[X] - 3 (because 3 is the kurtosis of a Gaussian variable)

7. Correlation

a. If Y is another random variable taking value y_i in event i, then the link between the distributions of X and Y can be characterized by:

b. Covariance

$$\begin{aligned} \text{cov}(X,Y) &= E\left[\left(X - E[X]\right)\left(Y - E[Y]\right)\right] \\ &= E[XY] - E[X]\,E[Y] \\ &= \sum_{i=1}^{n} p_i x_i y_i - \left(\sum_{i=1}^{n} p_i x_i\right)\left(\sum_{i=1}^{n} p_i y_i\right) \end{aligned}$$

• characterizes how close the distributions of X and Y are to each other

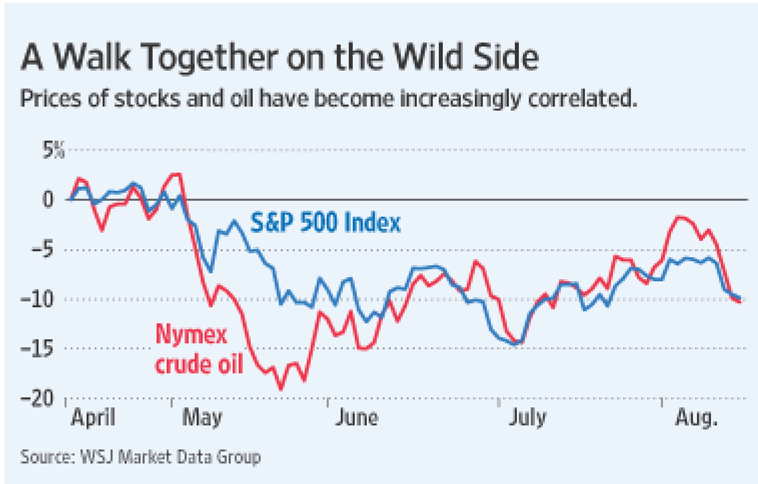
c. Correlation coefficient

$$\rho(X,Y) = \text{cor}(X,Y) = \frac{\text{cov}(X,Y)}{\sqrt{V[X]\,V[Y]}}$$

(satisfies $-1 \le \rho(X,Y) \le +1$)

d. X and Y are *uncorrelated* when cor(X,Y)=0. They are said to be correlated otherwise:

• positively correlated when $\mathrm{cor}(X, Y) > 0$



(WSJ 8/16/10)

• negatively correlated when $\mathrm{cor}(X, Y) < 0$

e. Beware: correlation does not imply causality!

We hear this all the time when we search for significant reasons behind the meanderings of the dollar, stocks and other assets. A quick synopsis of a typical interview:

Why are stocks up? "It's dollar weakness!" answers the Wall Street strategist.

The next day: "Why is the dollar down?" we ask. "Because stocks are up!" comes the answer.

(WSJ 11/12/09)

8. It's important to realize that all these numbers are not immutable; they change through time

a. For example, the USD and S&P500 were usually positively correlated; since 2008, however, they have become strongly negatively correlated

DECEMBER 8, 2009, 1:53 PM ET

Dollar/Stocks Inverse Relationship

(WSJ 12/8/09)

b. That is, since 2008, when the U.S. dollar went up, stocks fell, and vice versa.

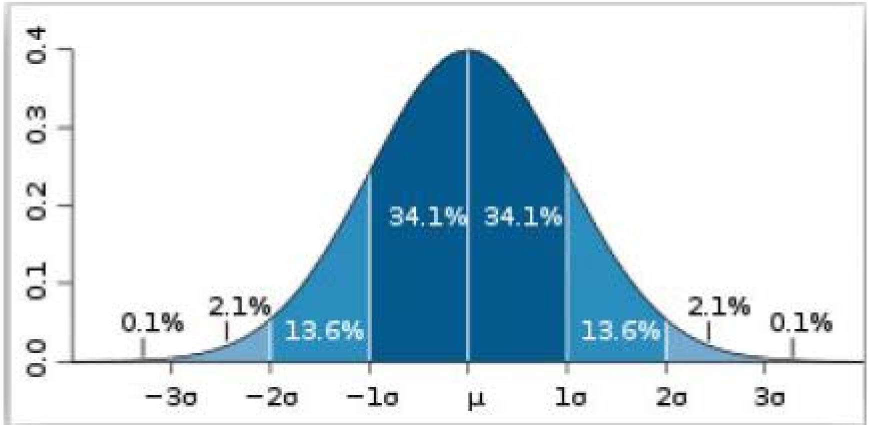
C. *Properties of Moments*

1. Let a and b denote constants (such as numbers of shares), X and Y are random variables (such as stock returns)

- $E[aX+b]=a\,E[X]+b$
- $E[aX+bY]=a\,E[X]+b\,E[Y]$
- $V[aX+b]=a^2\,V[X]$
- $V[aX+bY]=a^2V[X]+b^2V[Y]+2ab\,\mathrm{cov}(X,Y)$

2. We will use these relationships extensively in portfolio theory, the CAPM, etc.

D. *The Normal or Gaussian Distribution*



(Source: Wikipedia)

1. Properties

a. The distribution is entirely characterized by its first two moments. Write $X \sim N(\mu,\sigma^2)$.

b. $E[X] = \mu$ and $V[X] = \sigma^2$, Skewness=0, Kurtosis=3

c. $\dfrac{X-\mu}{\sigma} \sim N(0,1)$ (standard normal).

2. Cumulative distribution function of a standard normal N(0,1)

$$\Phi(x) \text{ or } N(x), \text{ for } -\infty < x < +\infty$$

(see tables of the standard normal distribution)

a. If $Z \sim N(0,1)$, then $\Phi(x)$ is the probability that Z be less than x, that is: $\Phi(x) = P(Z \le x)$.

b. $\Phi(x) = P(Z \le x)$ is given by the area under the density function up to x.

3. Useful for confidence intervals, assuming that the returns are Normal

   a. Example: Suppose that the monthly return R on IBM stock follows $R \sim N\left(\mu, \sigma^2\right)$ with μ=1.5% and σ=2.7%.

   b. A 95% *confidence interval* for R is an interval centered on μ that will contain the actual value of the monthly IBM return 95% of the times:

      • Construct $Z = \dfrac{R-\mu}{\sigma} \sim N(0,1)$. Let α = 0.05 = 5%. We want to find a number z such that:

      $$P(-z \le Z \le z) = 1-\alpha$$

      • See tables of the normal distribution to find z = 1.96.

      • Then:

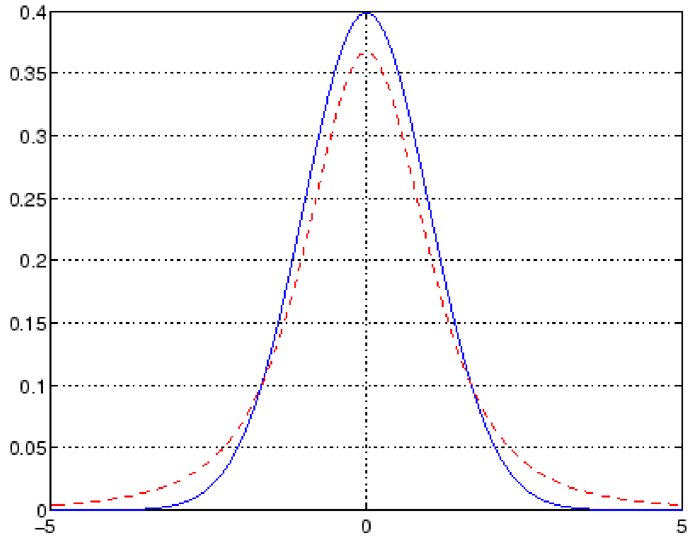
      $$\begin{aligned} 95\% = 1-\alpha &= P(-1.96 \le Z \le 1.96) \\ &= P\left(-1.96 \le \frac{R-\mu}{\sigma} \le 1.96\right) \\ &= P(\mu - 1.96\sigma \le R \le \mu + 1.96\sigma) \end{aligned}$$

      so the probability that R be between μ - 1.96σ = -3.79% and μ + 1.96σ = 6.79%, is 95%.

      • The 95% confidence interval is therefore:

      [-3.79% , 6.79%]

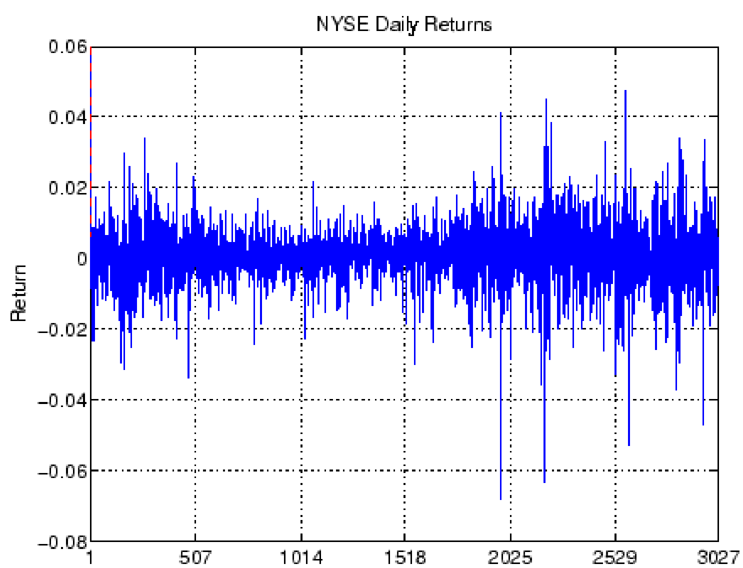
4. The Normal distribution can be a poor representation of asset returns, because it does not have fat tails



(Source: Matlab)

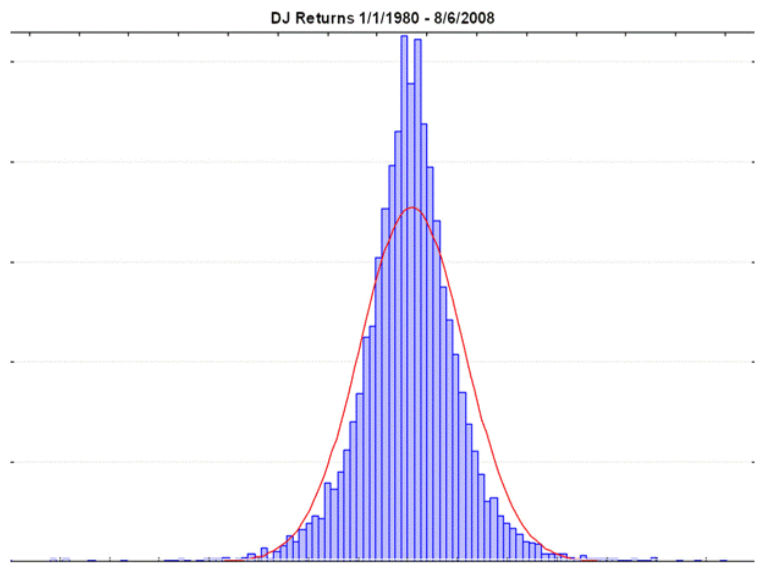
a. For example, under a Normal distribution with realistic values of μ and σ for asset returns a daily return greater than 7% (in absolute value) should happen once every 300,000 years.

b. But there were 48 such days for the DJIA during the 20th century.



(Source: Matlab)

c. Fat tails are an essential empirical feature of asset returns



(Source: Optionetics)

5. The Normal distribution plays a central role for two reasons

   a. Mathematical tractability, not realism!

   b. And because of the Central Limit Theorem, which shows that under some assumptions, large averages of random variables tend to be Normally distributed, no matter what the original distribution of these random variables are, as long as they are not too heavily dependent.

II. *Sample Statistics*

A. Sample Mean and Variance

1. Example

a. Suppose that we want to estimate the mean μ and variance σ^2 of the monthly return on IBM.

b. We have data on IBM stock returns for the past T = 24 months: $R_1,\ldots,R_T$.

i. Assumed to be *identically distributed* (but not necessarily Gaussian) with common mean μ and variance σ^2.

ii. Meaning: each random variable R_t, the return during period t, has the same distribution with mean μ and variance σ^2.

2. To estimate the mean and variance of IBM returns, we can use the average of the past returns.

a. $\hat{\mu} = \bar{R} = \frac{1}{T}\sum_{t=1}^{T} R_t$: *sample mean*

b. $\hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}\left(R_t - \bar{R}\right)^2$: *sample variance*.

c. To estimate the standard deviation σ, use $\hat{\sigma} = \sqrt{\hat{\sigma}^2}$, the *sample standard deviation*.

d. These estimators are *unbiased*, meaning that they are "right" on average:

$$E\left[\hat{\mu}\right] = \mu \text{ and } E\left[\hat{\sigma}^2\right] = \sigma^2$$

e. For this method to be accurate, the mean and variance of the stock returns must be relatively stable over time.

B. Sample Correlation

1. Suppose that we also have data on the Apple stock return, S_1, ..., S_T and want to estimate the correlation between the movements of IBM and Apple stock prices.

2. Compute first the sample covariance:

$$\hat{c} = \frac{1}{T-1}\sum_{t=1}^{T}\left(R_t - \bar{R}\right)\left(S_t - \bar{S}\right)$$

3. Then the sample correlation coefficient is:

$$\hat{\rho} = \frac{\hat{c}}{\sqrt{\hat{\sigma}^2_{IBM}}\sqrt{\hat{\sigma}^2_{APL}}}.$$

III. Regression Analysis

A. Model Specification

1. Example: IBM stock return R is assumed to follow $R_t = \alpha + \beta R_{Mt} + \varepsilon_t$ at every period t=1,...,T. R_M is the return on the entire market.

2. This model says that at each period t, the return on IBM R_t can be explained (or predicted) by the linear term $\alpha + \beta R_{Mt}$.

   a. The *disturbance* ε_t is random (and not observable), and represents random departures from the simple relationship.

   b. α (the intercept) and β (the slope) are unknown but constant *coefficients* or *parameters* to be estimated from the historical data $R_1,...,R_T$ and $R_{M1},...,R_{MT}$.

   c. The IBM return R is the *dependent variable*, the market return R_M is the *explanatory variable*, or *regressor*.

B. Estimation of the Parameters

1. For the model to be well-specified, the disturbance, or noise, ε must be uncorrelated with the regressor, and have zero mean.

2. $\hat{\beta}$ and $\hat{\alpha}$ obtained by a regression are called ordinary least squares estimates (OLS) of β and α respectively.

IV. *Hypothesis Testing*

A. Null Hypothesis and Test Statistics

1. Example: Suppose that we want to test the *null hypothesis* that the mean μ of the monthly IBM stock return, $R \sim N(\mu, \sigma^2)$, is equal to μ_0=1.5%

   a. Null hypothesis is $H_0 : \mu = \mu_0$ vs. alternative $H_1 : \mu \neq \mu_0$.

   b. We have data on IBM stock returns for the past T months: $R_1,\ldots,R_T$ as before.

   c. Both μ and σ are unknown.

   d. So they must first be estimated.

2. Recall from above that we know how to compute:

$$\hat{\mu} = \overline{R} = \frac{1}{T}\sum_{t=1}^{T} R_t \text{ and } \hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}\left(R_t - \overline{R}\right)^2.$$

3. If the null hypothesis is true then the *test statistics* $\sqrt{T}\left(\hat{\mu} - \mu_0\right)/\hat{\sigma}$ is distributed as a Student-t distribution with T-1 degrees of freedom:

$$\frac{\sqrt{T}\left(\hat{\mu} - \mu_0\right)}{\hat{\sigma}} \sim t$$

a. Why $\sqrt{T}$ there? Because the estimated standard deviation of $\hat{\mu}$ is $se(\hat{\mu}) = \hat{\sigma}/\sqrt{T}$

b. To form a t-statistic we divide the estimate by its standard deviation:

$$t = \hat{\mu}/se(\hat{\mu}) = \hat{\mu}/\left(\hat{\sigma}/\sqrt{T}\right) = \sqrt{T}\hat{\mu}/\hat{\sigma}.$$

c. This would be the t-stat for testing H_0: $\mu=0$.

d. To test H_0: $\mu=\mu_0$, note that μ_0 is a constant, and so $se(\hat{\mu} - \mu_0) = se(\hat{\mu})$. Hence the t-stat in this case is:

$$\sqrt{T}(\hat{\mu} - \mu_0)/\hat{\sigma}$$

B. Testing the Hypothesis

1. Intuition

a. If the null is true, $\hat{\mu}$ (which is close to μ) should be close to μ_0.

b. If it actually is, then the null will be "accepted" (= not rejected) whereas if $\hat{\mu}$ is far from μ_0 then the hypothesis is rejected.

2. Statistical meaning of "close" and "far":

a. At the 95% level, provided that T is large, if either

$$\sqrt{T}(\hat{\mu} - \mu_0)/\hat{\sigma} > 1.96 \text{ or } \sqrt{T}(\hat{\mu} - \mu_0)/\hat{\sigma} < -1.96$$

then $\hat{\mu}$ and μ_0 are "far apart" and the hypothesis is rejected.

b. That is, 1.96 is such that:

i. 2.5% of the mass is in [-∞,-1.96]

ii. 95% of the mass is in [-1.96,+1.96]

iii. 2.5% of the mass is in [+1.96,+ ∞]

c. In practice, we often use 2 instead of 1.96.

3. One-sided Tests

   a. If the hypothesis to be tested is instead $H_0 : \mu \geq \mu_0$, then we will only reject at the 95% level (for large T) if the t-stat is below the critical value -1.64:

      i. 5% of the mass is in [-∞,-1.64]

      ii. 95% of the mass is in [-1.64,+ ∞]

   b. To test $H_0 : \mu \leq \mu_0$: reject if the t-stat is above +1.64.

4. We can also test hypotheses regarding the coefficients of a regression model (α and β), using their standard errors.

   a. Easy: in addition to the estimated coefficients, your software will print out the estimated s.e. of the coefficients.

   b. For instance, to test H_0: β=1 at the 5% level, see whether $\left(\hat{\beta}-1\right)/\text{se}\left(\hat{\beta}\right)$ falls in [-1.96,+1.96].

   c. Or your software will report the coefficient's t-statistics = $\hat{\beta}/\text{se}\left(\hat{\beta}\right)$, from which $\left(\hat{\beta}-1\right)/\text{se}\left(\hat{\beta}\right)$ can be immediately computed.

   d. Same applies to hypotheses about the coefficient α.

   e. We can also test joint hypotheses about α and β (for instance H_0: α=0 and β=1) using F-statistics. See your statistics course.

Risk and Return

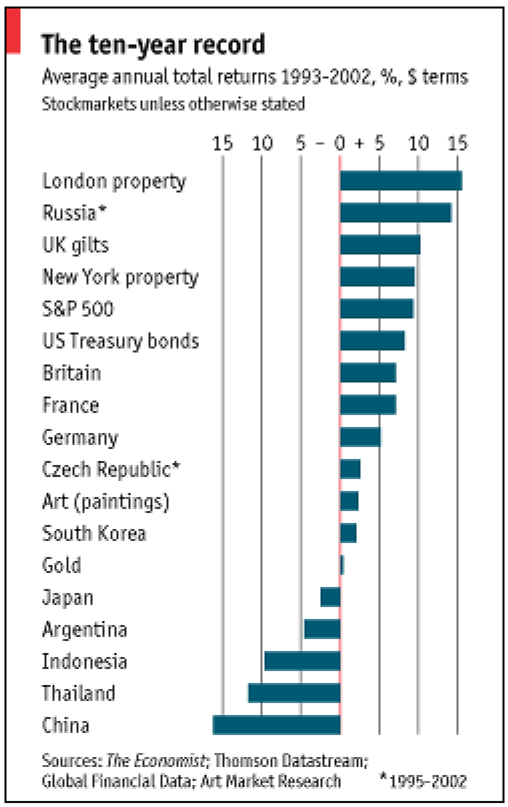
Topics

- *Defining returns*
- *The historical record*
- *Compounding*
- *The present value formula*
- *Dividend growth models*
- *Discounted cash flows*
- *Defining risk*
- *Risk aversion*
- *Risk-return trade-offs*

V. ***Return***

A. Measuring Returns

1. To make investment decisions, we need to be able to *compare* the returns of all sorts of assets:



(The Economist)

2. So it is critical that we agree on common standards for measuring asset returns: where is the return coming from? is it nominal or real? in which currency? is it annualized? compounded? how often? etc.

3. Components of the return on an asset

a. Income yield

• The issuer of the asset makes cash payment to the holder of the security (d_{t+1} = income = total cash received between date t and t+1).

• The percentage d_{t+1}/p_t gives the yield.

b. Capital gain or loss

= the difference between the price at which the security is traded at the beginning of the period and the end of period price = $p_{t+1} - p_t$

c. Total return r_{t+1} between t and t+1 = income + capital gain or loss as a % of initial investment, that is:

$$r_{t+1} = \frac{d_{t+1} + (p_{t+1} - p_t)}{p_t}$$

4. What is known, when?

a. In general, all the variables with subscript t+1 are not known until (the future) date t+1

b. In other words, as of date t (now), they are *random* variables.

5. Examples

a. A 3-month T-Bill provides no income (d=0), the return comes exclusively in the form of price appreciation.

  i. Interesting feature: if held until maturity, i.e., t+1 = 3 months from now, then p_{t+1} is not random.

  ii. Not true if t+1 is anytime before the maturity date of the bill.

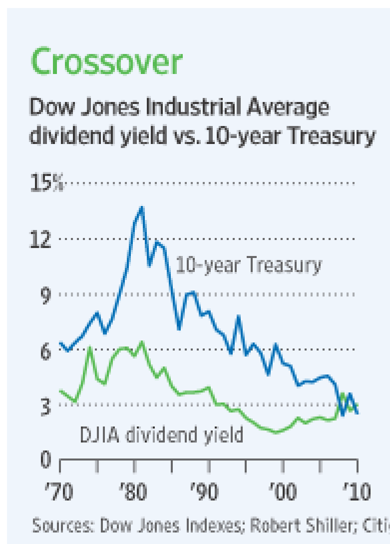
b. A 30-year T-Bond provides both income (semi-annual coupon payments) and some capital gain or loss:

i. If the investor holds the bond for the entire 30 years, the bond is sold by the investor in year 30 for its face value, but it was bought in year 0 for close to that amount (that is how the coupon rate is determined).

ii. When the bond is traded in the secondary market between years 0 and 30, its price is generally different from its face value (typically higher if the coupon rate is higher than the current interest rate, and vice-versa).

c. A stock provides income through d_{t+1} = dividends received between t and t+1, as well as capital gains or losses.

Playing the Yield Game on Stocks



(WSJ 8/30/10)

d. Even in a market that is flat for a long period (= no capital appreciation), dividends have typically allowed the S&P500 to beat T-Bonds

e. Same for mutual funds

Total return is a popular concept with mutual funds. By law, funds have to distribute most of the gains that they realize each year, often leading to hefty capital gains and income distributions. Many investors use these dividends to buy additional fund shares.

When a fund makes one of these distributions, it directly reduces the fund's share price. Result: The long-run trend in a fund's shares may bear no relationship to the fund's performance. Instead, to get a handle on a fund's performance, you need to look at the fund's total return, which includes reinvested dividends.

(WSJ)

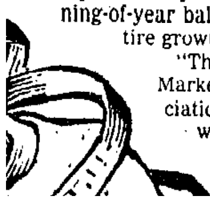
6. Return on a portfolio

a. Let the amount invested at t to form the portfolio be W_t and the value of the portfolio at t+1 be W_{t+1}

b. Let the income received during the period be d_{t+1}

Instead of looking at total return, investors often just compare their year-end portfolio value with the beginning-of-year balance and then presume that the entire growth is due to investment gains.

"That's a common mistake," says John Markese, president of the American Association of Individual Investors. What's wrong? This method ignores any additional investments or withdrawals made during the year.



(WSJ)

c. Then the rate of return of the investment in the portfolio is:

$$r_{t+1} = \frac{d_{t+1} + \left(W_{t+1} - W_t\right)}{W_t}$$

B. Realized vs. Expected Returns

1. Realized Return = actual ex-post return earned on a security during the period.

a. Generally, can only be determined at the end of the period, because d_{t+1} and/or p_{t+1} are not known when the period begins (only p_t is).

b. But may be known ahead of time in some cases (e.g., T-Bill). In such a case, the return is sure or certain (there is no risk).

2. Expected Return = the ex-ante return expected to be earned during the period

   a. It is subject to uncertainty because the income yield and/or capital gain components are uncertain.

   b. Investors base their investment decisions partly on expected returns, but expected returns may not materialize. At the end of period t, the realized return for period t can be computed. It may well be different from the return for the period that was expected at the beginning of period t.

C. Nominal vs. Real Returns, Dollar Returns vs. Foreign Returns

1. Real Return = Nominal Return - Inflation Rate

   a. Comparing nominal returns of assets over the long run can cause problems if the inflation rate is not the same (e.g., US vs. UK)

   b. As long as the inflation rate is the same (e.g., US assets), we can work exclusively with nominal returns.

2. When comparing foreign assets, compare their returns in a common currency (e.g., US$).

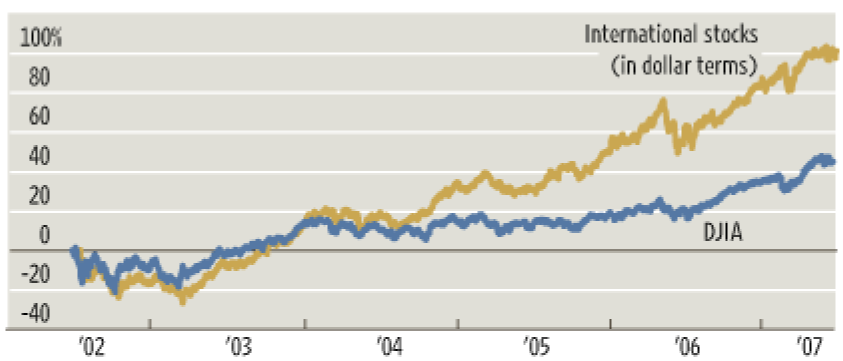
Gains in European Stocks Swept Away by the Sinking Euro

Americans who have invested in European stock markets this year have been dealt a cruel blow by the euro, which seems to be on a relentless decline. But European stocks rose so sharply at the end of last year that American investors still came out well ahead. For example, a 39.1 percent surge in the DAX was tempered to a 20.3 percent rise in dollar terms. The 51.1 percent moonshot by the French index in euros became 30.7 percent in dollars. And in Italy, the 22.3 percent rise in euro terms was translated into a 5.7 percent gain.

3. And the reverse in the past few years

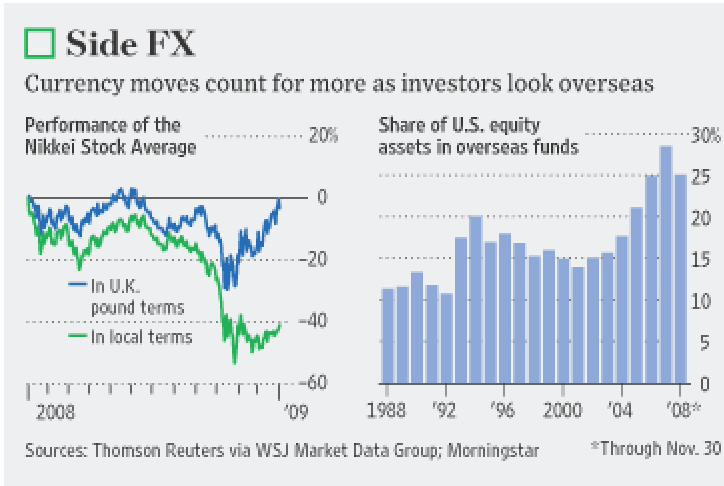
Rewards of Going Abroad

Over the past five years, U.S. investors have done far better buying foreign stocks (as measured by the MSCI EAFE Index) than large U.S. stocks (as measured by the Dow Jones Industrial Average).



Sources: Reuters, WSJ Market Data Group

4. Especially in 2008-10 for foreign investors in Japan due to the rise of the yen



(WSJ 1/5/09)

5. In terms of dollar returns, we can compare the nominal returns of every asset, including foreign assets

   a. The currency expected depreciation takes into account the inflation rate differential

   b. So dollar returns already adjust for inflation differentials, and we can again use nominal returns.

(The Economist 8/12/10)

D. Annualized Returns and Compounding Frequencies

1. Returns over Holding Periods

a. Definition: You earn a rate of return r_1=5% during year 2007, if starting with $1 on 1/1/07 you end up with 1+r'=$1.05 on 12/31/07.

b. Similarly, you earn a rate of return r_2=4% during year 2008, if starting with $1 on 1/1/08 you end up with 1+r'=$1.04 on 12/31/08.

c. Your $1 capital will grow to $\$\left(1+r_1\right)\left(1+r_2\right)$ after two years:

$$\begin{aligned}\left(1+r'\right) &= \left(1+r_1\right)\left(1+r_2\right) \\ &= \underbrace{1}_{\substack{\text{capital}\\ \text{preserved}}} + \underbrace{\left\{r_1+r_2\right\}}_{\substack{\text{simple}\\ \text{interest}}} + \underbrace{\left\{r_1.r_2\right\}}_{\substack{\text{compound}\\ \text{interest}}}\end{aligned}$$

2. Annualized rate of return r

a. r is the rate of return that you would have had to receive between 1/1/07 and 12/31/07 and between 1/1/08 and 12/31/08 to end up with the same amount of money as you had when receiving r' between 1/1/07 and 12/31/08.

b. That is: $(1+r)^2 = (1+r')$.

c. Notation: r' is non-annualized, r is annualized.

■ **10 Plus 10 is 21**

Imagine you invest $100, which earns 10% this year and 10% next. How much have you made? If you answered 21%, go to the head of the class.

Here's how the math works. This year's 10% gain turns your $100 into $110. Next year, you also earn 10%, but you start the year with $110. Result? You earn $11, boosting your wealth to $121.

Thus, your portfolio has earned a *cumulative* 21% return over two years, but the *annualized* return is just 10%. The fact that 21% is more than double 10% can be attributed to the effect of investment compounding, the way that you earn money each year not only on your original investment, but also on earnings from prior years that you've reinvested.

(WSJ)

3. Compounding Frequency

a. Different business situations typically require different lengths of time as a natural unit of time:

   i. Money market accounts and credit card companies keep track of interest daily.

   ii. A mortgage banker or home borrower works on a monthly cycle.

   iii. US Treasuries typically make coupon payments every six months.

   iv. Eurobonds make annual payments.

b. Regardless of the unit of choice, it is customary to quote interest rates on an *annualized* basis.

   i. A loan has an annualized interest rate of 12% compounded monthly if you have to pay 12%/12 = 1% of the borrowed amount every month.

   ii. A loan has an annualized interest rate of 12% compounded semi-annually if you have to pay 12%/2 = 6% of the borrowed amount every six months.

   iii. These two loans are different. You are better off receiving 1% every month than 6% every six months, because every $1 you lent would grow to

$$\left(1+0.01\right)^{12}=1.1268$$

instead of

$$\left(1+0.06\right)^{2}=1.1236\,.$$

VI. Prices and Expected Returns

A. The Present Value Formula

1. The rate of return on a stock is computed as:

$$r_{t+1} = \frac{d_{t+1} + (p_{t+1} - p_t)}{p_t}$$

2. Its expected return is:

$$E[r_{t+1}] = E\left[\frac{d_{t+1} + (p_{t+1} - p_t)}{p_t}\right] = \frac{E[d_{t+1}] + (E[p_{t+1}] - p_t)}{p_t}$$

3. Therefore:

$$p_t = \frac{E[d_{t+1}] + E[p_{t+1}]}{1 + E[r_{t+1}]}$$

a. Let $E[r] = E[r_{t+1}]$ be the expected return of the stock (for simplicity assume it's the same at every future date).

b. Imagine that you by the stock today (date t) for p_t, and will resell it one period later (t+1).

c. The price today (p_t) is the sum of the expected dividend to be received between t and t+1 ($E[d_{t+1}]$), and the expected resale value of the stock at t+1 ($E[p_{t+1}]$), discounted at rate r.

4. What if you buy the asset at t+1, and sell it at t+2:

$$p_t = \frac{E[d_{t+1}] + E[p_{t+1}]}{1 + E[r]}, \text{ but also } p_{t+1} = \frac{E[d_{t+2}] + E[p_{t+2}]}{1 + E[r]}.$$

a. Substitute back to obtain:

$$p_t = \frac{E[d_{t+1}]}{1+E[r]} + \frac{E[d_{t+2}]}{(1+E[r])^2} + \frac{E[p_{t+2}]}{(1+E[r])^2}$$

5. And if you continue iterating:

$$p_t = \frac{E[d_{t+1}]}{1+E[r]} + \frac{E[d_{t+2}]}{(1+E[r])^2} + \ldots + \frac{E[d_{t+T}]}{(1+E[r])^T} + \frac{E[p_{t+T}]}{(1+E[r])^T}$$

   a. So that the price is the discounted sum of the expected value of the dividends, discounted at the stock's expected rate of return (E[r])

   b. Plus the terminal expected value of the stock, $E[p_{t+T}]$.

6. Be careful here: we need to discount expected dividends, not earnings.

   a. It is sensible to include dividends *including repurchases of stock*

      i. Stock buybacks reduce the number of shares outstanding, hence increasing dividends per share.

      ii. Just like p above is the price per share, d is the dividend *per share*.

   b. How about a high tech company that has never paid a dividend (largely for tax reasons)?

      i. What makes it worth a lot is the expectation that stockholders will at some future date either receive large dividends or have their stock bought back by the company at high prices

      ii. Or everything hinges upon a very high resale value (high $E[p_{t+T}]$).

      iii. MSFT started paying dividends in 2005, but many others still do not.

   c. The bottom line is that a stock is worth the discounted value of the cash the investor expect to actually receive

      i. Not what he hypothetically might have received (such as earnings, free cash flow, or whatever other measure you select).

      ii. Earnings affect the formula only indirectly, through the expectation of higher future dividends and/or a higher resale value.

d. Finally, be careful if you take this formula to the macro level.

i. Today's S&P500 is *not* a claim on the aggregate earnings of the U.S. corporate sector into the indefinite future.

ii. They are a claim on the earnings of today's corporations into the indefinite future.

iii. U.S. companies will most likely be earning a lot of money 50 years from now; but much of that money will be earned by companies that do not now exist.

iv. On the other hand, one can expect S&P to adjust the composition of its index so that it will reflect that.

B. Application to Valuation

1. Dividend growth

a. The PV relationship is sometimes called the "dividend discount model" but there is no model so far (i.e., no assumptions were needed to get the PV formula). This equation is an identity.

b. To get a "model" we need to make assumptions on the growth of the dividends.

2. Constant growth rate g of dividends: the Gordon model

$$E\left[d_{t+1}\right]=(1+g)d_t,\ E\left[d_{t+2}\right]=(1+g)^2 d_t,\ E\left[d_{t+T}\right]=(1+g)^T d_t$$

a. Therefore:

$$\begin{aligned}
p_t &= \frac{(1+g)d_t}{1+E[r]}+\frac{(1+g)^2 d_t}{\left(1+E[r]\right)^2}+\ldots+\frac{(1+g)^T d_t}{\left(1+E[r]\right)^T}+\frac{E\left[p_{t+T}\right]}{\left(1+E[r]\right)^T} \\
&= d_t\left\{x+x^2+\ldots+x^T\right\}+\frac{E\left[p_{t+T}\right]}{\left(1+E[r]\right)^T} \\
&= d_t\frac{x\left(1-x^T\right)}{1-x}+\frac{E\left[p_{t+T}\right]}{\left(1+E[r]\right)^T}
\end{aligned}$$

where $x = \dfrac{1+g}{1+E[r]}$.

b. If now we let T go to infinity (assuming a constant g to hold to infinity is a big assumption!):

$$\frac{E[p_{t+T}]}{(1+E[r])^T} \xrightarrow[T\to\infty]{} 0$$

and $x^T \xrightarrow[T\to\infty]{} 0$ because g < E[r] and so 0 < x < 1

3. Intrinsic stock value

a. The model predicts that the stock should be worth:

$$p_t = d_t \frac{x}{1-x} = d_t \frac{\dfrac{1+g}{1+E[r]}}{1-\dfrac{1+g}{1+E[r]}} = d_t \frac{1+g}{E[r]-g}.$$

b. $p_t = d_t \dfrac{1+g}{E[r]-g}$ is the stock's intrinsic value

c. For example, if current dividends are d_t = \$0.75 per share, g = 5% and E[r] = 7%, then

$$p_t = 0.75 \times \frac{1+0.05}{0.07-0.05} = \$39.375$$

d. This number should be used for a "sanity" check only

i. It is unlikely to be a good approximation to what the company is worth, because of all the (unrealistic) assumptions that went into it.

ii. But if the market price is 10 times this number, then…

4. The discounted cash flow formula:

a. We can equivalently compute E[r]:

$$\underbrace{E[r]}_{\substack{\text{expected}\\\text{total}\\\text{return}}} = \frac{E[d_{t+1}] + (E[p_{t+1}] - p_t)}{p_t} = \underbrace{\frac{d_t(1+g)}{p_t}}_{\substack{\text{expected}\\\text{dividend}\\\text{yield}}} + \underbrace{g}_{\substack{\text{expected}\\\text{capital}\\\text{gain}\\\text{(also =dividend}\\\text{growth rate)}}}$$

b. For example, if g=5%, current dividends are d_t = \$0.75 per share, and the market price of the stock is p_t = \$39.375 then

$$E[r] = \frac{d_t(1+g)}{p_t} + g = \frac{0.75 \times (1+0.05)}{39.375} + 0.05 = 7\%$$

c. This is the first example of a result that tells us how a value could be obtained for E[r]. Others will follow…

5. P/E Ratio

a. In this model, the firm pays dividend d_t = (1-b) e_t, i.e., the firm retains a constant fraction b of earnings and pays back the rest (1-b) as dividends

b. The fraction 1-b is called the dividend *payout* ratio and b is the *plowback* ratio.

c. For example, if b = 33%, the price earnings ratio is

$$\frac{p_t}{e_t} = \frac{p_t}{d_t}(1-b) = \frac{(1-b)(1+g)}{E[r]-g}$$

d. In the example, the P/E ratio is $\frac{p_t}{e_t} = \frac{(1-0.33)(1+0.05)}{0.07-0.05} = 33.5$.

C. Why do Prices Change?

1. From the PV formula:

$$p_t = \frac{E[d_{t+1}]}{1+E[r]} + \frac{E[d_{t+2}]}{(1+E[r])^2} + \ldots + \frac{E[d_{t+T}]}{(1+E[r])^T} + \frac{E[p_{t+T}]}{(1+E[r])^T}$$

2. p_t will change only if one or both of the following happen:

   a. Expected future dividends ($E[d_{t+1}]$, $E[d_{t+2}]$, etc.) and/or future resale value ($E[p_{t+T}]$) change

   b. Expected returns on the stock (E[r]) change

3. Example: Possible explanations for bubbles

   a. Expect rise in future dividends (as a result of the expected rise in corporate earnings)

   b. Future expected returns on stocks fall

      • But why?

      • The expected return is

$$E[r] = r_f + \underbrace{\{E[r] - r_f\}}_{\text{risk premium}}$$

      • So the riskfree rate falls and/or investors require a lower risk premium.

         • The riskfree rate is

$$r_f = r_f^{\text{nominal}} = r_f^{\text{real}} + E[\text{inflation}]$$

            • The real rate r_f^{real} may fall because baby boomers are putting their savings in the stock market, so the supply of capital outstrips demand, and the real rate adjusts to balance the market

• Or expectations of future inflation go down

• The risk premium may fall because:

• Investors become less risk-averse

• Or they think that stocks have become less risky, so they don't need to be compensated as much for holding stocks

c. These are all explanations assuming that investors are fully rational. But there are alternative explanations if investors can be irrational:

i. It's a "bubble", "markets are exuberant", etc.

ii. People are extrapolating reports of rise of corporate earnings too far into the future

iii. People have short memories, they forget that stocks can go down as well as up. They can't help overweighing the performance of stocks in the recent past.

D. Stocks, Bonds and Inflation Reports

1. If the inflation number (CPI or PPI) just released is higher than expected, bond prices fall

a. Because the expected return on bonds (just like any other asset) is higher

$$E[r] = r_f^{real} + E[\text{inflation}] + \{E[r] - r_f\}$$

b. And future cashflows ($E[d_{t+1}]$, $E[d_{t+2}]$, etc.) are unchanged

c. So:

$$p_t = \frac{E[d_{t+1}]}{1+E[r]} + \frac{E[d_{t+2}]}{(1+E[r])^2} + \frac{E[d_{t+3}]}{(1+E[r])^3} + \dots$$

goes down.

2. What happens to stocks?

   a. Just like bonds, E[r] goes up.

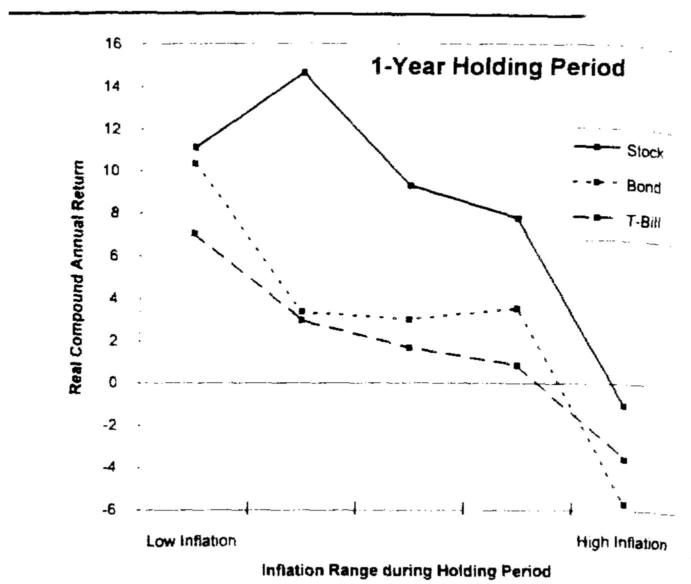
   b. However expected dividends also go up in nominal terms

   c. Over the long run the two effects cancel each other, but in the short run the first effect may dominate and stock prices may fall.

   d. Because of expected dividends go up, stocks offer some protection against inflation.

      i. Bonds do not

      ii. Unless they are inflation-indexed (introduced in the US in 1997, very popular in the UK). The amount to be paid at maturity, i.e., the bond's last cash flow, goes up with inflation.



E. Stocks, Bonds and Unemployment Reports

1. If the unemployment rate just released is higher than expected (bad economic news) bond prices rise:

Bond Market Rises Sharply On Jobs Data

Fall in Industrial Hiring Eases Inflation Worries

(NYT)

a. Because the expected return on bonds is lower

$$E[r] = r_f^{real} + E[\text{inflation}] + \{E[r] - r_f\}$$

• People will want to borrow less in a declining economy, which pushes down r_f^{real}

• The Fed may lower interest rates to prevent a recession

• Expected inflation may go down

• On the other hand, people may become more risk averse, thereby demanding a higher risk premium $E[r] - r_f$, which has the opposite effect

b. And future cash flows ($E[d_{t+1}]$, $E[d_{t+2}]$, etc.) are unchanged.

c. Therefore bond prices will typically rise on bad economic news.

d. If the economic news are good, the opposite typically happens:

Bond Prices Post Steep Losses on Strong Economic News

CREDIT MARKETS | JUNE 3, 2010, 4:22 P.M. ET

Treasury Prices Lower Ahead of Jobs Report

NEW YORK—Prices of Treasury securities fell Thursday, led by long-dated maturities, as many investors bet on a strong key U.S. jobs report,

2. What happens to stocks when good economic news are released?

   a. Just like bonds, E[r] goes up, which pushes prices lower.

   b. However expected dividends also go up in nominal terms, because of an expanding economy:

      i. As long as the good unemployment numbers are not accompanied by signs of wage pressure

      ii. Which can lead to both inflation and lower corporate earnings

Rising Rates May Diminish Earnings Impact

   c. Over the long run the two effects cancel each other, but in the short run the first effect may dominate and stock prices may fall:

Jobs Data Spark 115-Point Plunge in Industrials

F. Stocks, Bonds and Risk Aversion

Stocks Fall
As Risk Aversion Picks Up

THE WALL STREET JOURNAL. ONLINE 10/22/07

1. An increase in risk aversion means that the required risk premium of all assets go up

$$E[r] = r_f + \underbrace{\{E[r] - r_f\}}_{\text{risk premium}}$$

2. Consequently, the prices of all assets go down.

G. How to Get Numbers?

1. What is the “right” price p_t for a share of IBM?

$$p_t = \frac{E[d_{t+1}]}{1+E[r]} + \frac{E[d_{t+2}]}{(1+E[r])^2} + \frac{E[d_{t+3}]}{(1+E[r])^3} + \ldots$$

   a. Need to forecast future dividends

   b. Need to forecast future expected returns

$$E[r] = r_f + \underbrace{\{E[r] - r_f\}}_{\text{risk premium}}$$

$$= r_f^{\text{real}} + E[\text{inflation}] + \underbrace{\{E[r] - r_f\}}_{\text{risk premium}}$$

      i. Need a number for the real interest rate

      ii. Need inflation forecasts

      iii. Most importantly, need a number for IBM’s risk premium

2. So, what is the appropriate risk premium to use for IBM?

    a. That is, how much compensation (risk premium) should we demand in return for facing IBM's risk?

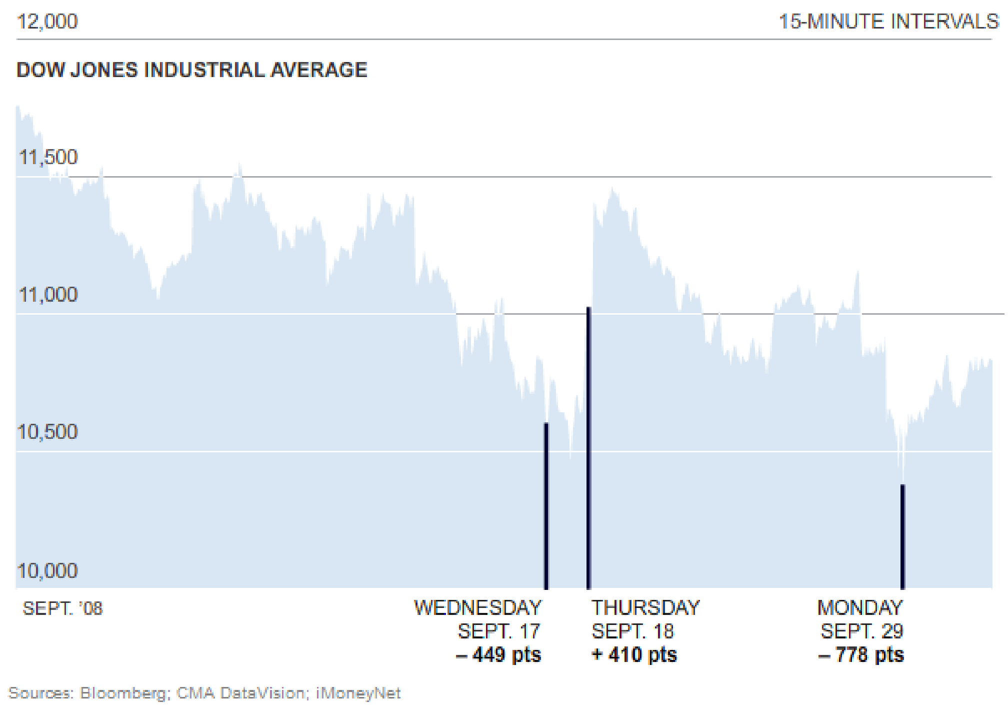
        2% per year? 5%? 10%?

    b. Discounted cash flows give us one possible answer.

    c. CAPM (and more sophisticated risk factor models) will also do it for us.

VII. Risk: Variability in Returns

One Sure Thing for Investors? Volatility



(NYT 10/15/08)

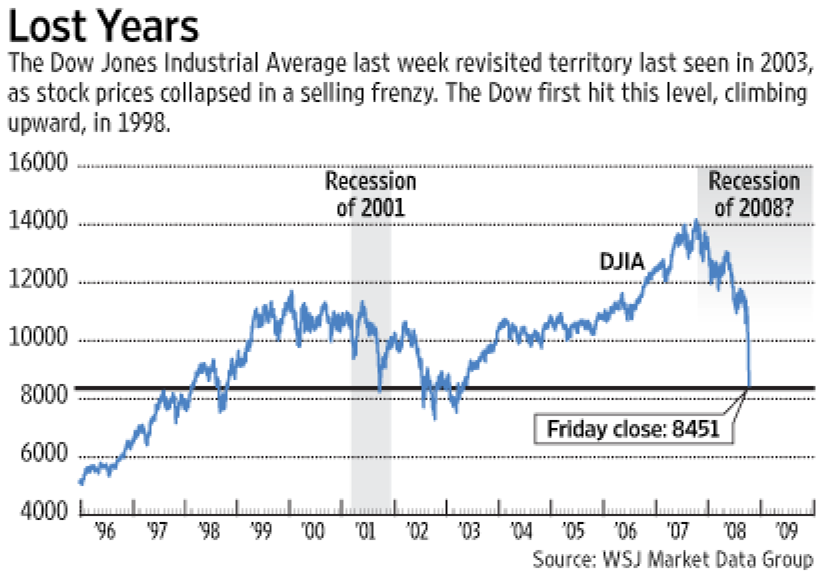
A. Sources of Risk in Financial Assets

1. Interest rate risk

a. Bond prices *always* move with interest rates in opposite direction

b. And stock prices *tend* to move with interest rates in opposite direction

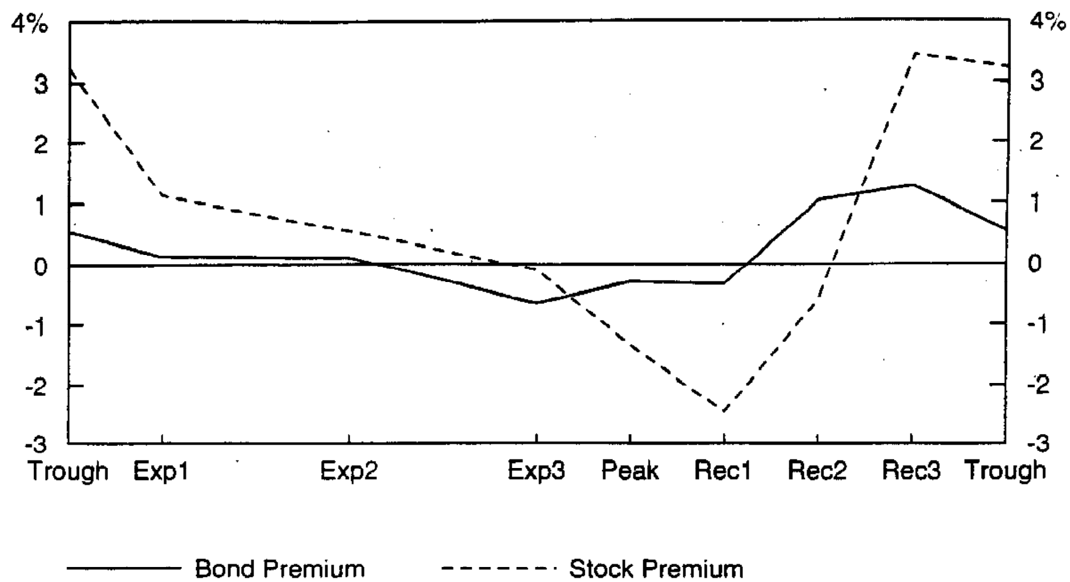
2. Business cycle / systemic risk



(WSJ 10/12/08)

a. Each of the past six recessions has been accompanied by a bear market, with the Standard & Poor's 500-stock index losing, on average, 31%.

b. There is a consistent pattern: expected returns are negative as we head into a recession but high towards the end of a recession, as we head out of it.



(Salomon Brothers)

[An asset risk premium = its expected return minus the riskfree rate,

ex post: replace expected return by realized return]

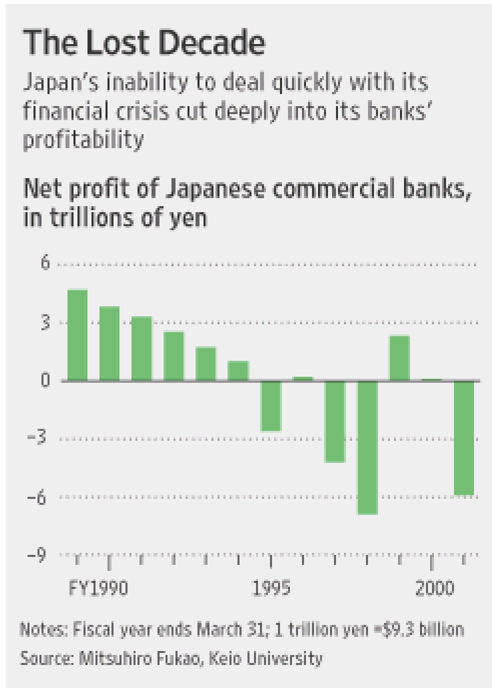
c. Not all stocks react the same way to the business cycle.

d. Cyclical stocks (e.g., airline and hotel stocks) are highly dependent on the state of the economy. When things slow down, their earnings tend to fall rapidly, along with their stock prices. and vice versa as the economy recovers.

e. Defensive industries, such as food, drugs and energy, are typically less affected by changes in the business cycle because consumption of such goods is more stable.

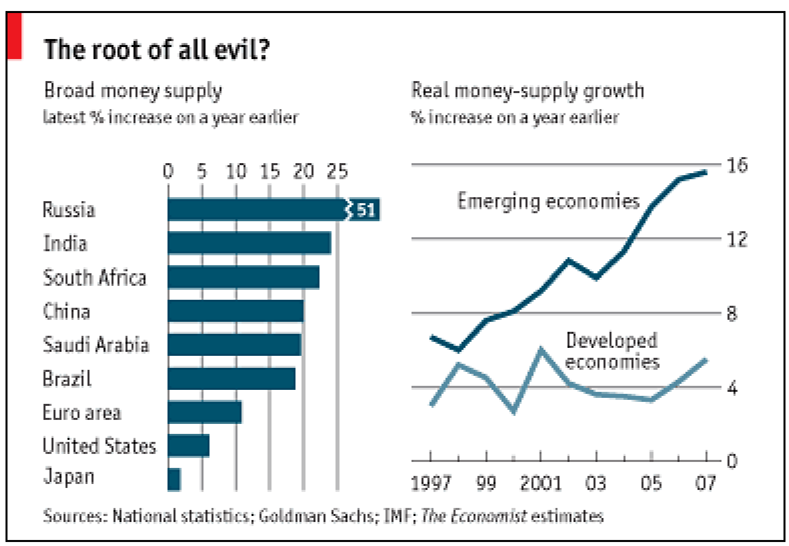
3. Macroeconomic Risk

a. For instance, Japan's stock index (Nikkei) is now below 10,000, after having been at 40,000 in the late 1980s…



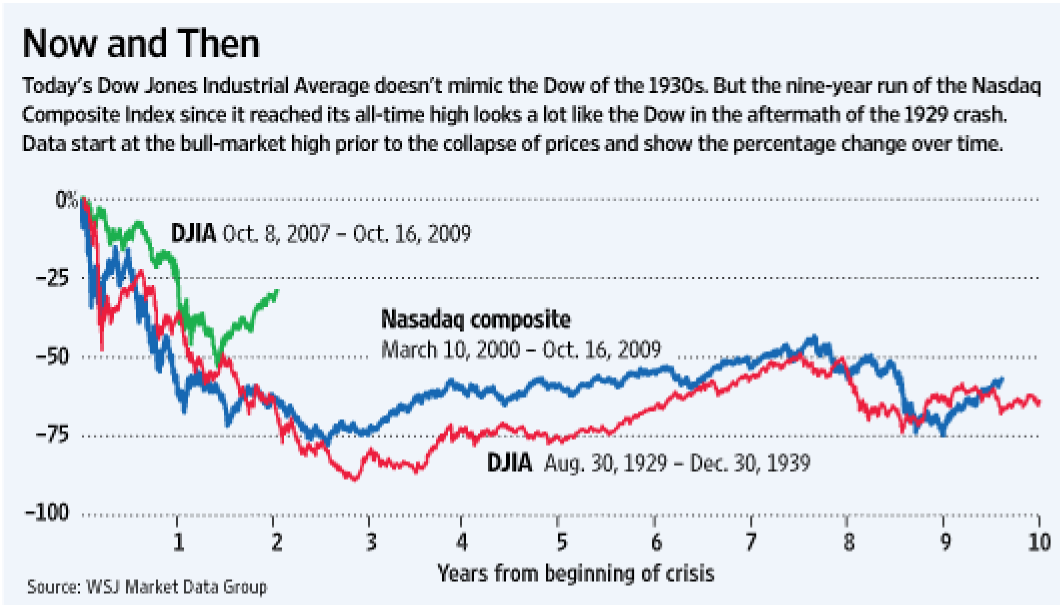
(WSJ 9/15/08)

b. Irresponsible monetary and/or fiscal policy can wreak havoc on the markets…



(The Economist 8/9/07)

4. History often repeats itself...



(WSJ 10/18/09)

5. Political risks

a. Election cycle

b. Country risk, includes things such as

i. Expropriation of assets

ii. Restrictions on foreign exchange

iii. Political instability

Country	S&P	Moody's
Australia	AA	Aa2
Austria	AAA	Aaa
Belgium	AA+	Aa1
Canada	AA+	Aa2
Denmark	AA+	Aa1
Finland	AA–	Aa2
France	AAA	Aaa
Germany	AAA	Aaa
Ireland	AA	Aa2
Italy	AA	A1
Japan	AAA	Aaa
Netherlands	AAA	Aaa
New Zealand	AA	Aa2
Norway	AAA	Aa1
Portugal	AA–	A1
Spain	AA	Aa2
Sweden	AA+	Aa3
Switzerland	AAA	Aaa
United Kingdom	AAA	Aaa
United States	AAA	Aaa

6. Industry risk

(The Economist 8/16/07)

7. Exchange rate risk

(The Economist 8/12/10)

8. Liquidity risk

a. This is the risk that you may not be able to sell an asset at a reasonable price because no buyer is available.

b. For example, mortgage lenders are cutting risky loans after a jump in subprime-mortgage defaults. Banks have started calling in collateral from hedge funds that invest in exotic securities linked to subprime loans.

With Many Holding Same Hands, Quant Funds Find Exiting Costly

August 13, 2007

THE WALL STREET JOURNAL.

9. Demographic trends

   a. Peak spending age is 46.5

   b. Strong correlation, at least historically between the proportion of the population around that age and the performance of the stock market.

B. Measuring Risk

1. We will measure risk using the standard deviation of the (annualized) returns $\sigma = S[r]$.

   a. Often use 24 to 36 months of past data to estimate σ.

   b. S[r] is also called the *volatility* of a given market or asset:

Those Turbulent U.S. Markets?

As wild as the U.S. stock and bond markets may feel lately, they traditionally have been much more volatile. And their gyrations pale compared with those of major foreign markets.

	1993	FIVE YEARS	10 YEARS	20 YEARS
Standard & Poor's 500-stock index	6.69%	14.80%	17.98%	18.11%
Long-Term Treasury Bonds	8.38	8.52	11.73	12.49
British stocks*	15.12	23.24	26.24	32.70
Japanese stocks*	40.13	29.66	33.34	27.78
Hong Kong stocks*	72.70	38.58	41.82	45.47

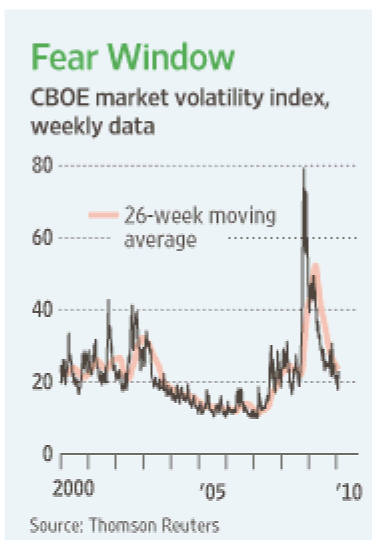
Note: All figures are based on total returns, including capital appreciation and investment income, with volatility measured by annualized monthly standard deviation, a statistical yardstick showing the degree to which returns tend to vary from the mean.

*Data for Japan and Hong Kong are based on Morgan Stanley Capital International indexes, with returns denominated in U.S. dollars.

Source: Ibbotson Associates

(WSJ)

c. "Implied volatility" is a measure of the volatility implicit in the market prices of options, and the S&P500's volatility index, or VIX, is often used as a measure of the broader market "fear level"



(WSJ 1/23/10)

C. Other Possible Concepts of "Risk"

Investors Rethink Risk

a. The range from low to high price reached during the day (called the range)

b. The number of days in a given period where the daily price change has been greater than some threshold, for example 2%:

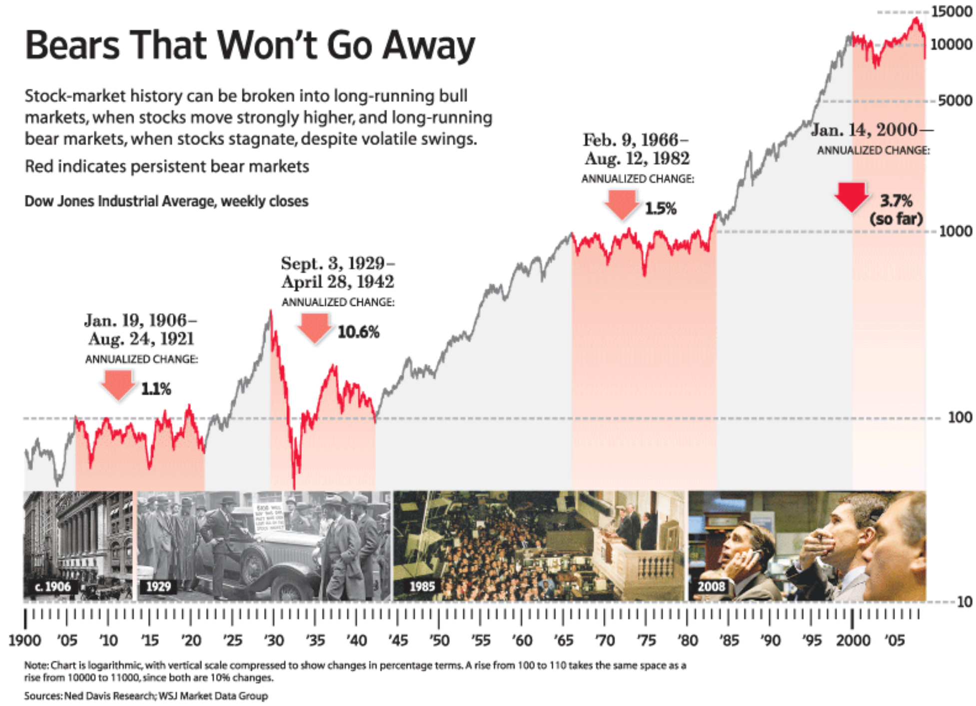
c. Some measure of maximum loss that has been recorded in the past n years.

d. The probability that a loss greater than a certain threshold occurs

   i. For example the probability of losing more than 25% of the value of the portfolio over the next three months

   ii. If we think that we have a 95% chance of not losing more than 20% of the portfolio, that 20% is often called the "Value-at-Risk" or VaR

e. The time it would take to recover from a bear market (estimated from past episodes)



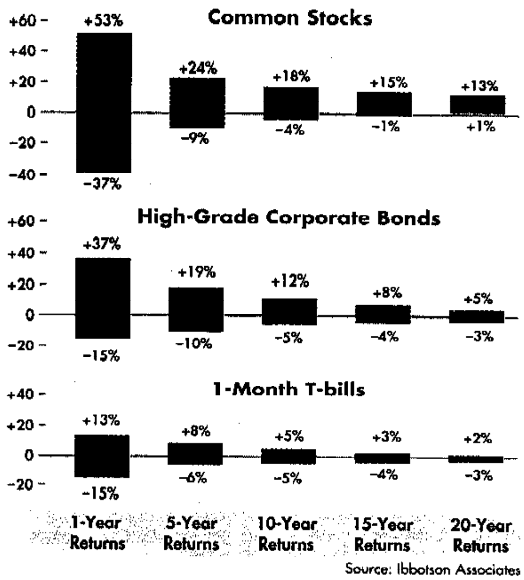
(WSJ)

f. Measures that assess how large and frequent the *losses* are when they occur, i.e., penalize deviations from the mean only on the downside (vs. standard deviation, which penalizes deviations equally on the downside and upside)

g. While these concepts all make sense, we will mostly use the standard deviation of the returns as our measure of risk.

D. *Volatility and Holding Periods*

1. The volatility of average returns cancels out over longer holding periods



a. As the horizon grows, average returns are more likely to end up close to their expected value.

b. In other words, the average returns measured over a period of 20 years are much less volatile than average returns measured over one year.

c. Thus the range of average returns shrinks towards the expected return as the horizon grows.

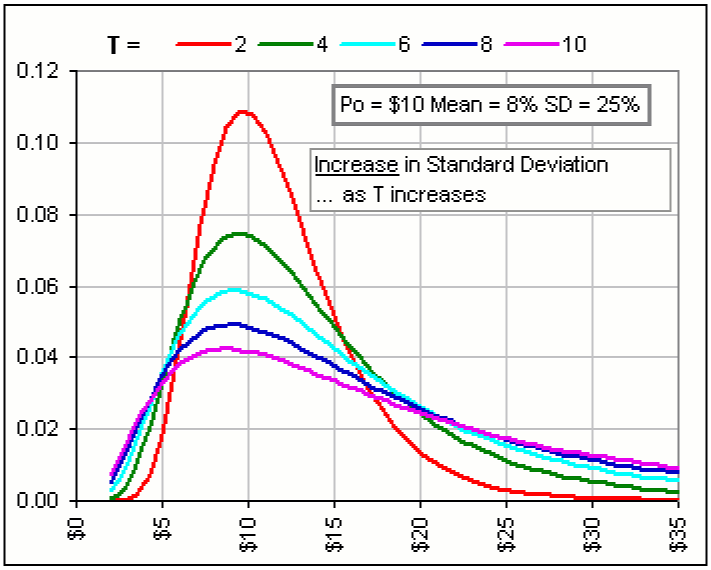
d. But this argument, based on the "safety of the average return" is a fallacy

   i. Investors care about the total rate of return on their investments over the horizon they invested in;

   ii. And not the average rate of return over that horizon.

   iii. In other words, small differences in the average rate of return over long periods make a big difference to your bottom line. So, not surprisingly, the average tends to be subject to less deviations (or volatility).

iv. But the standard deviation of the total return increases with the horizon



2. Probability that one asset class outperforms another, as the length of the horizon increases

Holding Period Comparisons Since 1802 (Percentage of Periods When Returns on One Asset Exceed Another Asset)

Holding Period	Stocks Outperform Bonds	Stocks Outperform T-Bills	Bonds Outperform T-Bills
1 Year	60.21	61.26	49.74
2 Years	64.74	64.74	53.16
5 Years	69.52	72.73	51.87
10 Years	79.67	79.12	52.75
20 Years	91.28	94.19	51.74
30 Years	99.38	96.91	46.91

a. As seen above, stocks provide a higher rate of return than bonds, but at the cost of higher volatility.

b. High expected returns assets (such as stocks) are more likely to win than low expected returns assets (such as bonds) over long horizons.

c. But over short periods of time, bonds may significantly outperform stocks (2002, when the stock market dropped while interest rates went down, and of course 2008).

VIII. Risk Aversion

A. Risk Premium

1. Risk arises because more than one outcome is possible, otherwise S[r]=0

a. Make an investment in stocks of \$1,000 today, in one year your portfolio may be worth \$1,300 (return r_1=30%) with probability p=.6 or \$900 (return r_2=-10%) with probability $1-p$=.4

b. The expected return of the portfolio is:

$$\begin{aligned} E[r] &= p r_1 + (1-p) r_2 \\ &= 0.6x(0.3) + 0.4x(-0.1) \\ &= 0.14 = 14\% \end{aligned}$$

c. The standard deviation of the portfolio return is:

$$\begin{aligned} S[r] &= \sqrt{V[r]} \\ &= \sqrt{p(r_1 - E[r])^2 + (1-p)(r_2 - E[r])^2} \\ &= \sqrt{0.6x(0.3-0.14)^2 + 0.4x(-0.1-0.14)^2} \\ &= 0.1960 = 19.6\% \end{aligned}$$

2. Now consider a T-Bill investment:

a. Invest \$1,000 today

b. Receive for sure \$1,050 in one year (interest rate = 5%), yielding a sure return of r_f = 5%.

3. Clearly the risky investment yields a larger expected return than the sure investment.

   a. The difference 14% - 5% = 9% is called the *risk premium*. of the risky investment.

   b. It compensates ex-ante the investor for making a risky investment, i.e., facing 19.6% in volatility.

4. Why do investors require compensation for making risky investments compared to sure ones?

   a. Because they are *risk-averse*. A risk-averse investor penalizes the expected rate of return of a risky investment to account for the risk involved

   b. A *risk-lover* investor would do the opposite, preferring investments that have more risk for a given amount of expected return

   c. A *risk-neutral* investor would be indifferent to risk, looking only at the expected return, and paying no attention to risk.

5. By how much should the expected rate of return of a risky investment be penalized by a risk-averse investor?

   a. Suppose that the standard deviation of the stock investment is fixed. For a very low expected return (say 5%, exactly the same as the sure return of the T-Bill), a risk-averse investor will prefer the T-Bill.

      i. When facing two investment prospects that yield the same expected rate of return (this is called a *fair bet)*, a risk-averse individual will choose the investment with the least amount of risk.

      ii. This is risk-aversion.

   b. If the expected return on the stock portfolio is very high, say 25%, the investor might decide that the risk is worth taking, and invest in stocks instead of the 5%-yielding T-Bills.

c. Somewhere in the middle, there is a level of expected return on the stock portfolio that makes the investor indifferent between the two investments.

   i. This level of expected return for the risky investment is called the *certainty equivalent*.

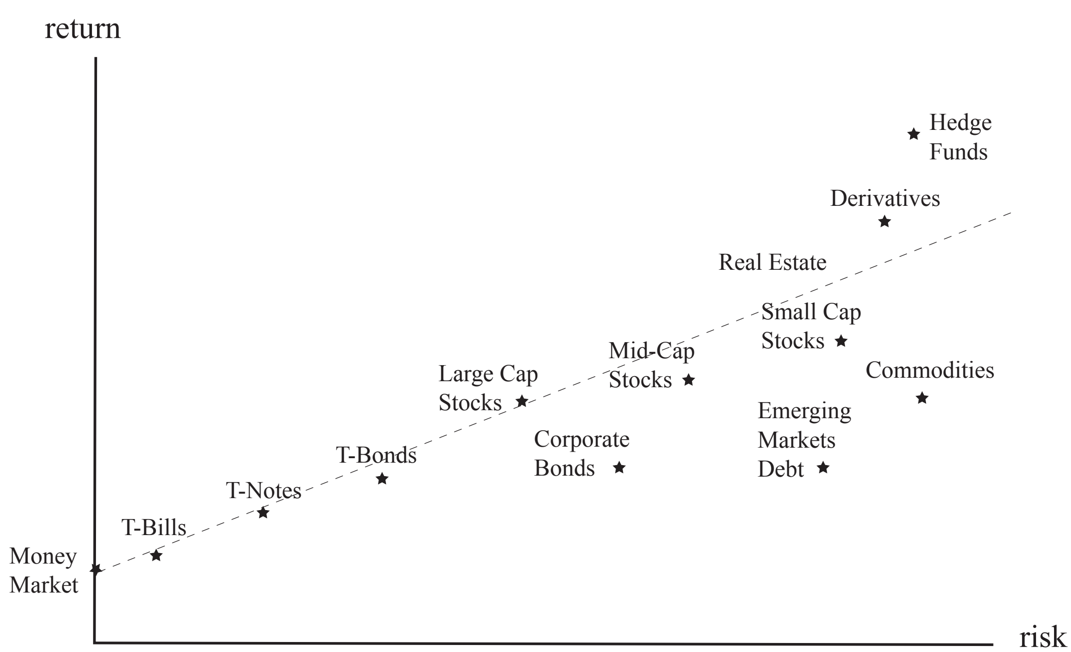
   ii. It makes the risk-averse investor indifferent between the risky and the riskfree investments.

B. Risk-Return Trade-Offs

1. The trade-off is formalized through the investor's *utility function*.

2. Each investment is characterized by the pair (E,S) where E is the expected return of the investment and S its standard deviation

3. So an asset is a certain point on a graph where the x-axis represents S and the y-axis represents E



a. The utility function U assigns a value U(E,S) to each investment.

b. Everybody likes expected profits, so U is increasing in E

c. Risk-averse investors also dislike risk, so their U would be decreasing in S, for example:

$$U(E,S) = E - \gamma S^2$$

i. γ is the coefficient of risk-aversion, assumed to be a constant parameter

ii. $\gamma>0$ means risk aversion.

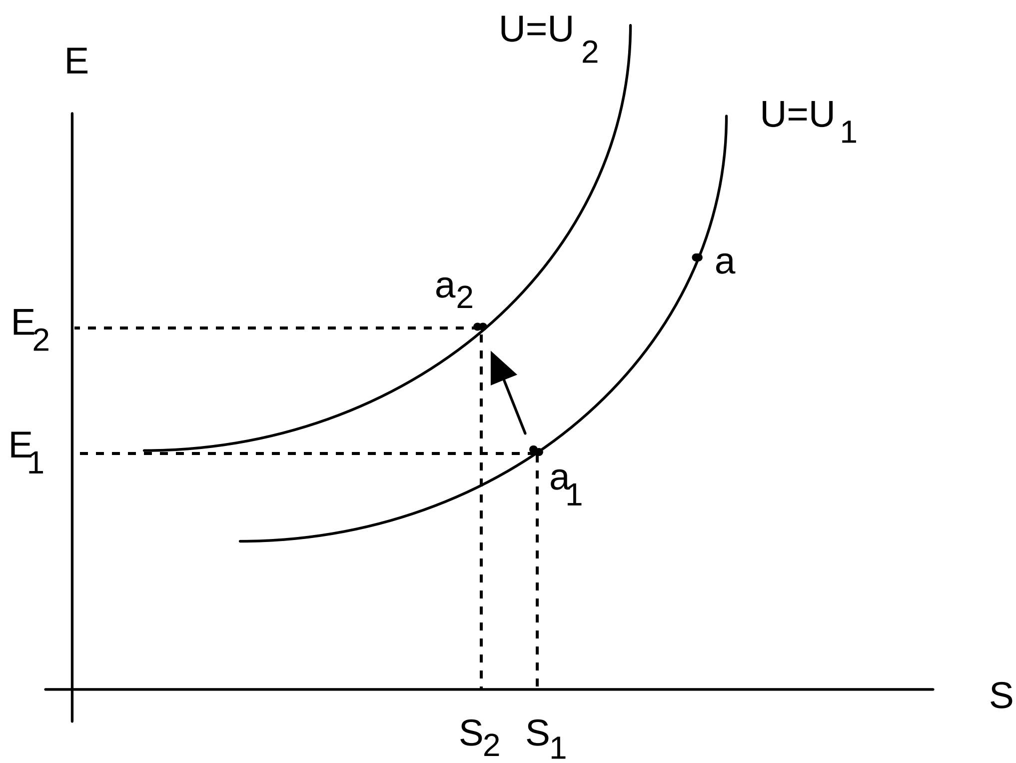
iii. High $\gamma>0$ corresponds to high risk-aversion; low $\alpha>0$ to low risk-aversion

iv. $\gamma=0$ corresponds to risk neutrality

v. $\gamma<0$ is the case of a risk-lover investor

d. *Indifference curve*: the investor is indifferent between portfolios a and a_1.

i. For a risk-averse investor, indifference curves look like:

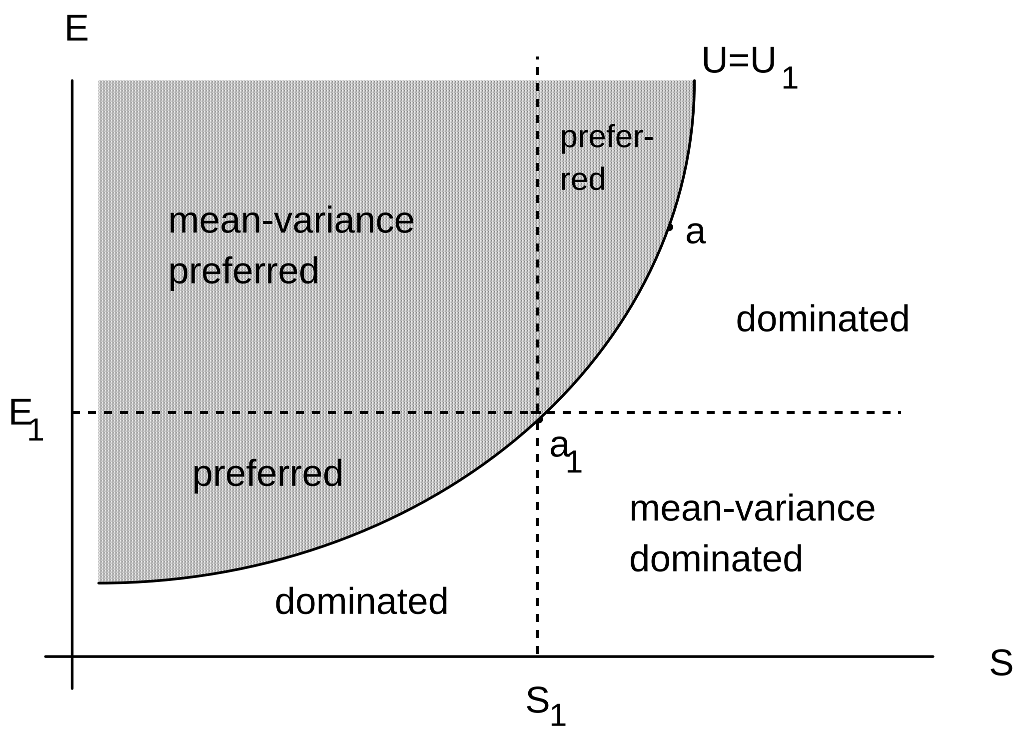


ii. Asset a is riskier, but provides higher expected return than a_1. And does it in just the right amount, given the investor's γ, for the investor to be indifferent between a and a_1.

iii. The investor is indifferent between assets a and a_1, but prefers a_2 to either one of them.

iv. In fact, the investor prefers any asset on the "higher utility" curve $U=U_2$ to any asset on the "lower utility" curve $U=U_1$.

v. Mean-variance domination occurs when an asset is preferred on both counts:



e. In practice, investment companies try to assess their clients indifference curves by having them answer questions like:

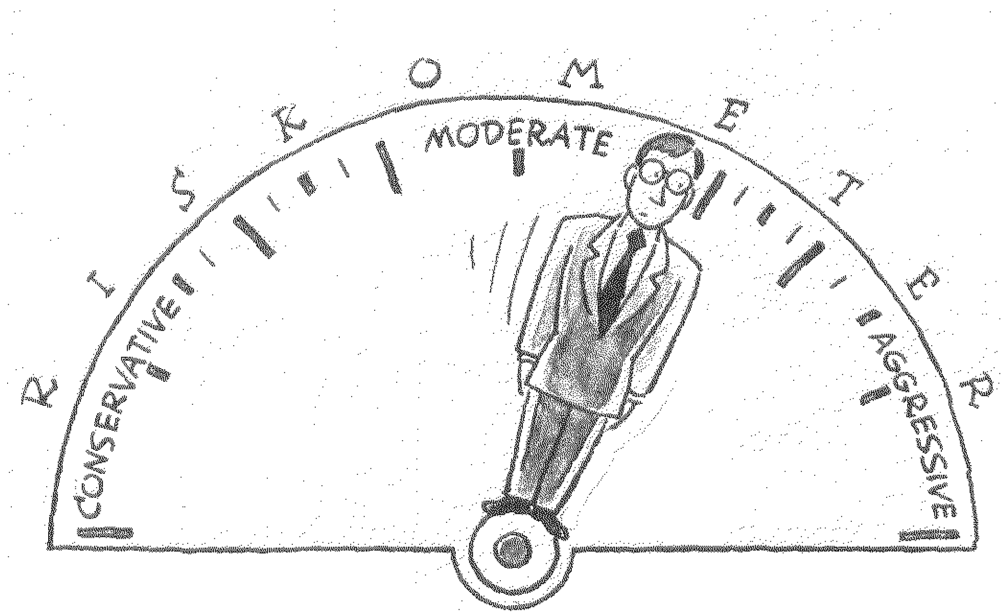
Fidelity Investments®

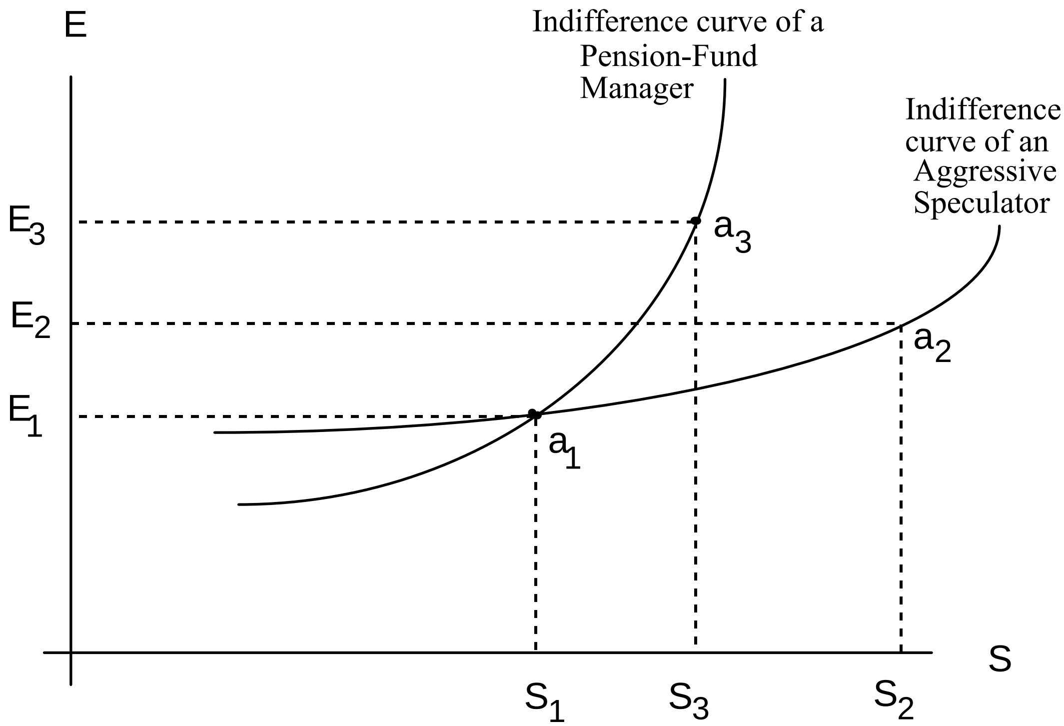
FundsNetwork®
Investment Planner

10. If you could increase your chances of improving your returns by taking more risk, would you:

	Points
Be willing to take *a lot more risk* with *all* your money?	16
Be willing to take *a lot more risk* with *some* of your money?	12
Be willing to take *a little more risk* with *all* your money?	10
Be willing to take *a little more risk* with *some* of your money?	5
Be unlikely to take much more risk?	2

4. Varying the level of risk-aversion





a. The pension-fund manager, with high risk-aversion, requires a large increase in expected return (from E_1 to E_3) to be compensated for the small increase in risk (from S_1 to S_3), and be indifferent between the two portfolios a_1 and a_3.

b. The speculator, with low risk-aversion, requires a much lower increase in expected return (from E_1 to E_2) to be compensated for the large increase in risk (from S_1 to S_2), and be indifferent between portfolios a_1 and a_2.

c. Therefore:

   i. Steeper indifference curves correspond to a more risk averse investor

   ii. Flatter indifference curves correspond to a less risk averse investor

   iii. Completely flat (= horizontal) indifference curves correspond to a risk neutral investor: this is someone who cares only about E and ignores S

   iv. Vertical indifference curves correspond to an infinitely risk averse investor: this is someone who cares only about S and ignores E

5. Figuring out the right value of γ to use is tricky

   a. Investment investors often use psychological questionnaires along the lines of the example above.

   b. Or they set γ based on objective criteria such as age, wealth, time-to-withdrawal, investment objective, etc.

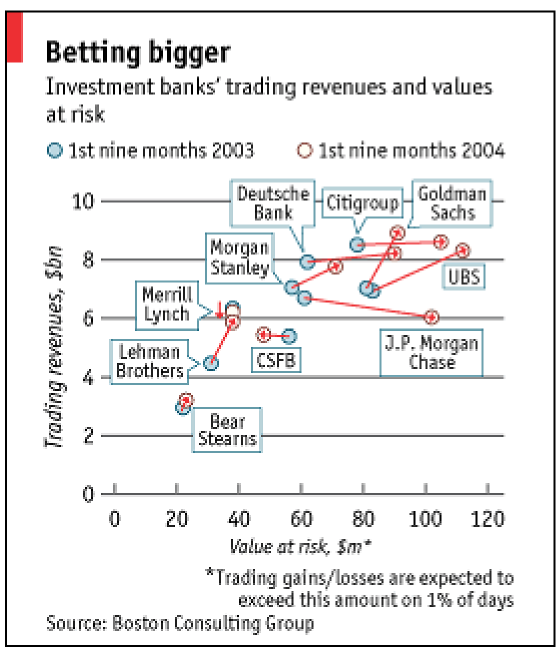
   c. Most of us aren't honest with ourselves about how much investment risk we can really handle, but we find that out only when things turn sour.

   d. Even worse, we tend to change our minds at market tops and bottoms, making the wrong choices at precisely the wrong moments.

6. The same type of risk-return tradeoff occurs with alternative risk and return measures

   a. In the example below, "risk" is measured by VaR, or value-at-risk

   b. And "return" by trading revenues



(The Economist 1/3/05)

Portfolio Theory

Topics

- *How to create an optimal portfolio:*
  - *If you can only invest in stocks and the money market*
  - *If you can only invest in stocks and bonds*
  - *If you can invest in a number of risky assets, but no money market*
  - *If you can invest in anything you want*
- *The importance of correlation*
- *Efficient frontier*
- *Capital market line*

I. *One Risky Asset and the Risk-Free Asset*

A. Portfolio Expected Return and Risk

1. Call the risky asset "STOCKS." Form a portfolio by investing in STOCKS (risky) and a Money-Market account (risk-free).

   a. Initial wealth W_0=\$100,000.

   b. Suppose that you invest a share ω=60% of your wealth in STOCKS, and a share 1-ω=40% in MM.

   c. The \$ amounts invested in STOCKS and the MM are respectively ωW_0=\$60,000 and $(1-\omega)W_0$=\$40,000.

   d. Today's STOCKS trade for S_0=\$100, and the price of one share of MM is M_0=\$1.

   e. With the \$ amounts allocated to each asset, we can buy $s=\omega W_0/S_0$=600 shares of STOCKS and $m=(1-\omega)W_0/M_0$=40,000 shares of MM.

2. The cost of forming the portfolio is: sS_0+mM_0. All your wealth is invested so $W_0=sS_0+mM_0$.

   a. A year from today, STOCKS will trade for S_1, and the MM will have a price M_1=\$1, and will have paid a dividend rate of r_f=5%.

   b. M_1 is known today (risk-*free* asset) and so is r_f

   c. S_1 however is unknown today: S_1 is a random variable (*risky* asset)

d. In a year, the portfolio will be worth $W_1 = sS_1 + mM_1$ which is unknown today since S_1 is unknown.

3. Rate of return on the portfolio (assume that STOCKS pay no dividends):

$$
\begin{aligned}
r &= \frac{r_f\, mM_0 + \left(W_1 - W_0\right)}{W_0} \\
&= \frac{r_f mM_0 + \left\{sS_1 + mM_1\right\} - \left\{sS_0 + mM_0\right\}}{W_0} \\
&= \frac{r_f mM_0 + s\left\{S_1 - S_0\right\} + m\underbrace{\left\{M_1 - M_0\right\}}_{=0}}{W_0}
\end{aligned}
$$

so that

$$
\begin{aligned}
r &= \frac{r_f mM_0 + s\left\{S_1 - S_0\right\}}{W_0} \\
&= \frac{r_f \dfrac{\left(1-\omega\right)W_0}{M_0} M_0 + \dfrac{\omega W_0}{S_0}\left\{S_1 - S_0\right\}}{W_0} \\
&= \left(1-\omega\right) r_f + \omega \underbrace{\frac{\left\{S_1 - S_0\right\}}{S_0}}_{= r_s}
\end{aligned}
$$

4. Thus the portfolio's rate of return is $r = \omega r_s + \left(1-\omega\right) r_f$

a. where r_s is the rate of return on STOCKS, and r_f is the rate of return on the MM account (risk-free rate of return).

b. Again, r_s is random so r is also a random variable. The *expected rate of return* on the portfolio is:

$$
\begin{aligned}
E[r] &= E\left[\omega r_s + (1-\omega) r_f\right] \\
&= \omega E[r_s] + (1-\omega) r_f
\end{aligned}
$$

(recall that $E[aX+b] = a\,E[X]+b$)

c. Interpretation:

$$E[r] = r_f + \omega \underbrace{\{E[r_s] - r_f\}}_{\substack{\text{STOCKS} \\ \text{risk premium}}}$$

5. Risk of the Portfolio

a. The variance of the portfolio is:

$$V[r] = V\left[\omega r_s + (1-\omega) r_f\right] = \omega^2\, V[r_s]$$

(recall that $V[aX+b] = a^2\, V[X]$)

b. b. So its standard deviation (our measure of risk) is:

$$S[r] = \sqrt{V[r]} = \sqrt{\omega^2\, V[r_s]} = \omega \sqrt{V[r_s]} = \omega S[r_s]$$

B. Investment Opportunities: The Capital Allocation Line

1. Investment Opportunity Set

a. It's the set of feasible combinations of risk and expected return of the portfolios corresponding to all the possible investment allocations

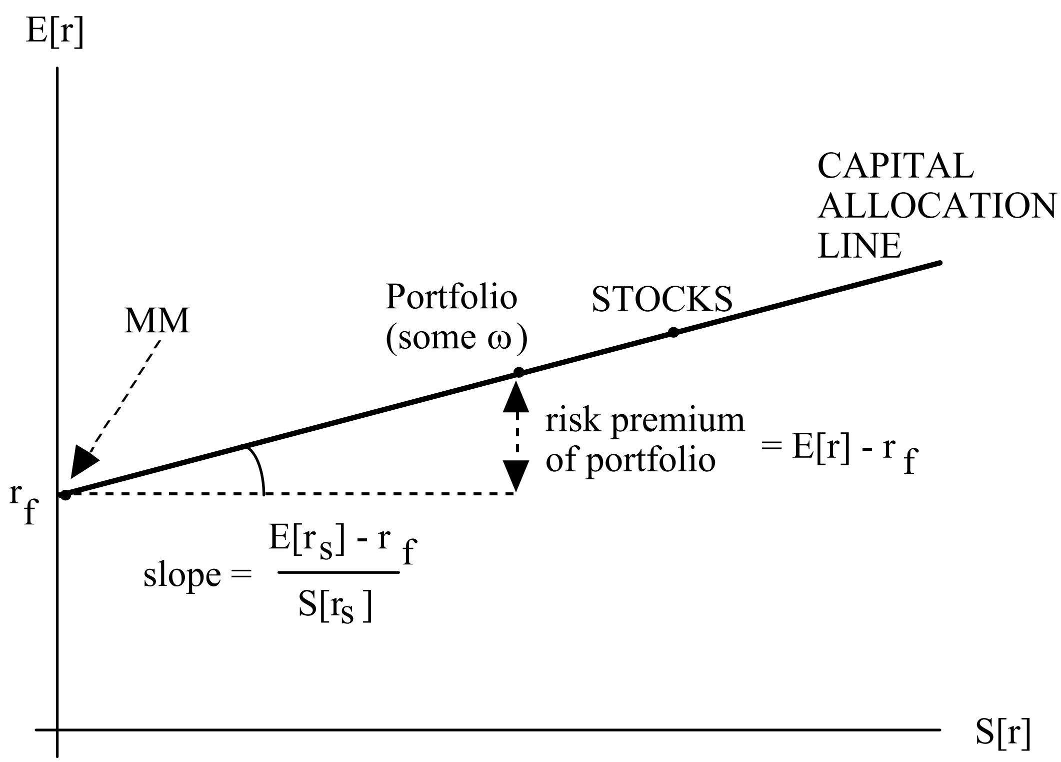
b. That is, it's the set of $(E[r], S[r])$ obtained by varying ω.

$$\begin{cases} E[r] = r_f + \omega\{E[r_s] - r_f\} \\ S[r] = \omega S[r_s] \end{cases}$$

$$\Rightarrow E[r] = r_f + \frac{\{E[r_s] - r_f\}}{S[r_s]} S[r]$$

c. This is the equation for a line when E[r] is the y-axis and S[r] is on the x-axis, since it's of the form y = intercept + slope * x.

2. The investment opportunity set is a line in the space $(E[r], S[r])$, called the *Capital Allocation Line* (CAL).



C. *Optimal Asset Allocation*

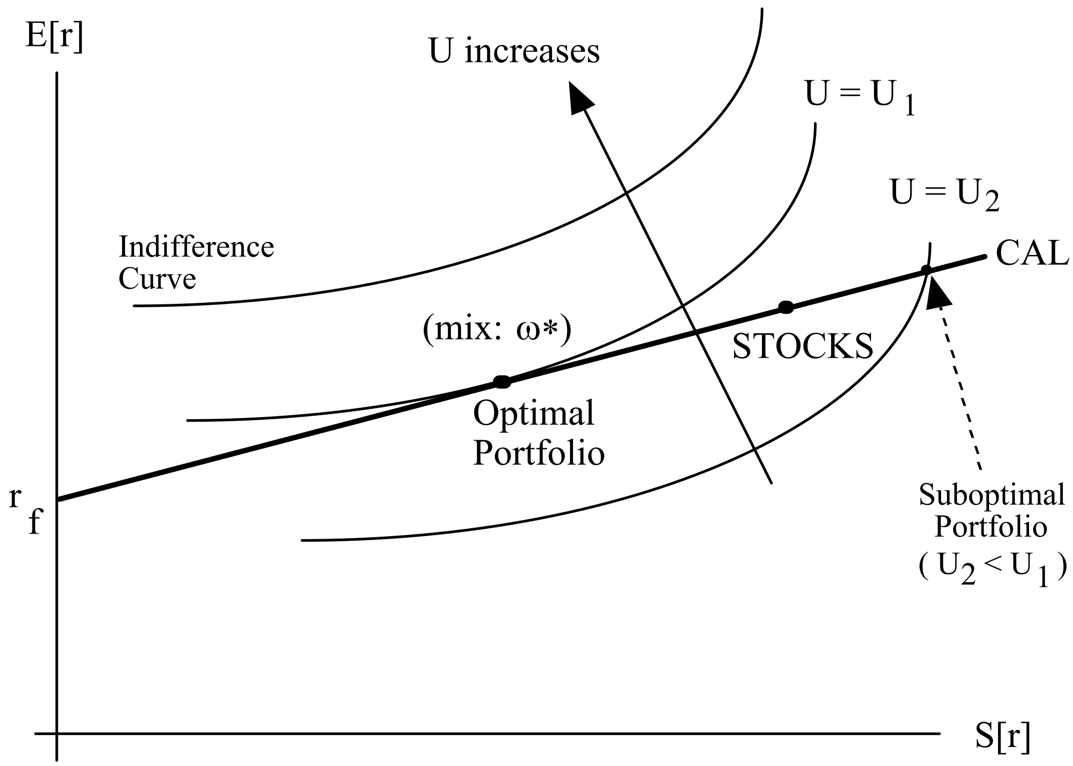
1. Given the set of feasible allocations (the CAL) the investor would choose his/her portfolio (that is ω) by maximizing the utility function:

$$U(E[r], S[r]) = E[r] - \alpha S[r]^2$$

a. If $\alpha > 0$, the investor is risk averse (the case on the graph below)

b. If $\alpha = 0$: risk-neutral

c. If $\alpha < 0$: risk-lover



2. Only the portfolios on the CAL are feasible.

   a. The optimal portfolio is the feasible portfolio that generates the highest possible level of utility.

   b. The optimal portfolio is characterized by the tangency of the CAL to the indifference curve.

3. Portfolios to the right of STOCKS on the CAL are *leveraged*.

   a. Any such portfolio is formed by borrowing the MM and investing more than 100% of your wealth in STOCKS (so $\omega > 1$ and $1 - \omega < 0$).

b. These portfolios are chosen by investors with very low risk aversion.

II. *Two Risky Assets*

A. Portfolio Expected Return and Risk

1. Form a portfolio by investing in STOCKS and BONDS (both risky)

a. Initial wealth W_0.

b. Invest a share ω of your wealth in STOCKS, and a share 1-ω in BONDS.

c. The \$ amounts invested in STOCKS and BONDS are respectively ωW_0 and $(1-\omega)W_0$.

d. Today's STOCKS and BONDS stock prices are S_0 and B_0 respectively.

e. With the \$ amounts allocated to each asset, we can buy $s = \omega W_0 / S_0$ shares of STOCKS and $b = (1-\omega) W_0 / B_0$ shares of BONDS.

2. The cost of forming the portfolio is:

$$W_0 = sS_0 + bB_0 .$$

a. A year from today, STOCKS will trade for S_1, and BONDS for B_1 (both random).

b. In a year, the portfolio will therefore be worth $W_1 = sS_1 + bB_1$ (random).

3. Rate of Return of the Portfolio

a. Assume that STOCKS pay no dividend and BONDS pay no coupon; so they're sold at a discount):

$$r = \frac{W_1 - W_0}{W_0} = \omega \underbrace{\frac{\{S_1 - S_0\}}{S_0}}_{= r_s} + (1-\omega) \underbrace{\frac{\{B_1 - B_0\}}{B_0}}_{= r_b}$$

b. Therefore again:

$$r = \omega r_s + (1-\omega) r_b$$

c. The difference is that now both r_s and r_b are random.

d. The expected rate of return on the portfolio is:

$$E[r] = E[\omega r_s + (1-\omega) r_b] = \omega E[r_s] + (1-\omega) E[r_b]$$

(recall that $E[aX + bY] = a\,E[X] + b\,E[Y]$)

e. Risk of the Portfolio

$$\begin{aligned} V[r] &= V[\omega r_s + (1-\omega) r_b] \\ &= \omega^2 V[r_s] + (1-\omega)^2 V[r_b] + 2\omega(1-\omega)\operatorname{cov}(r_s, r_b) \end{aligned}$$

(recall $V[aX + bY] = a^2 V[X] + b^2 V[Y] + 2ab\operatorname{cov}(X,Y)$)

and $S[r] = \sqrt{V[r]}$.

B. Investment Opportunities

1. Investment Opportunity Set

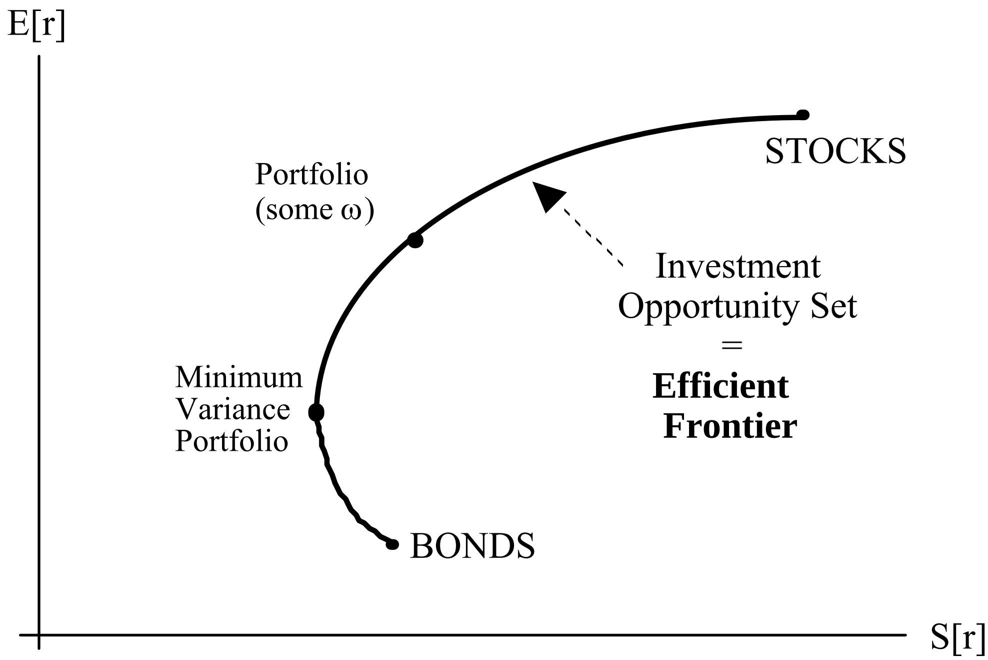
a. This is the set of feasible combinations of risk and expected return of the portfolios corresponding to all the possible investment allocations

b. It can be obtained by varying ω in $(E[r], S[r])$

c. Recall that:

$$\begin{cases} E[r] = \omega E[r_s] + (1-\omega) E[r_b] \\ S[r] = \sqrt{\omega^2 V[r_s] + (1-\omega)^2 V[r_b] + 2\omega(1-\omega)\operatorname{cov}(r_s, r_b)} \end{cases}$$

d. The set in the space $(E[r], S[r])$ is not a line when $-1 < \operatorname{cor}(r_s, r_b) < +1$. Instead it has the following shape:



e. The minimum-variance portfolio (obtained by finding ω to minimize S[r]) has a variance smaller than that of either STOCKS or BONDS: this is the effect of *diversification*.

Sleep at Night but at a Price

Smoothing the Bumps—At a Price

How different mixes of stocks and bonds have fared, Jan. 1945 - Dec. 1997

PORTFOLIO	COMPOUND ANN. TOT. RETURN	WORST YEAR'S RETURN	—NO. OF YEARS WITH LOSS GREATER THAN— 5%	10%	15%	20%
100% Stocks	12.9%	-26.5%	8	4	1	1
80% Stocks/20% Bonds	11.6	-20.5	8	2	1	1
60% Stocks/40% Bonds	10.3	-14.3	3	1	0	0
40% Stocks/60% Bonds	8.8	-7.9	1	0	0	0
20% Stocks/80% Bonds	7.3	-3.9	0	0	0	0
100% Bonds	5.8	-5.1	1	0	0	0

Note: Stocks = Standard & Poor's 500 Index Bonds = Ibbotson Intermediate Bond Index

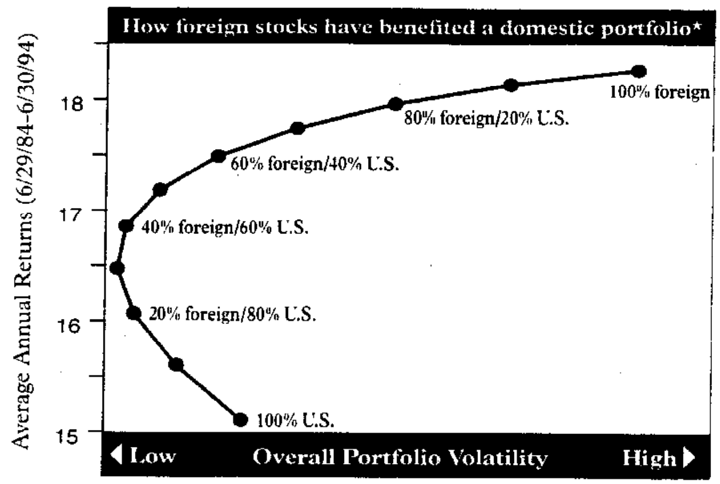
Source: Ibbotson Associates

f. The investment opportunity set that can be constructed with the risky assets is called the *efficient frontier.*

2. Example:

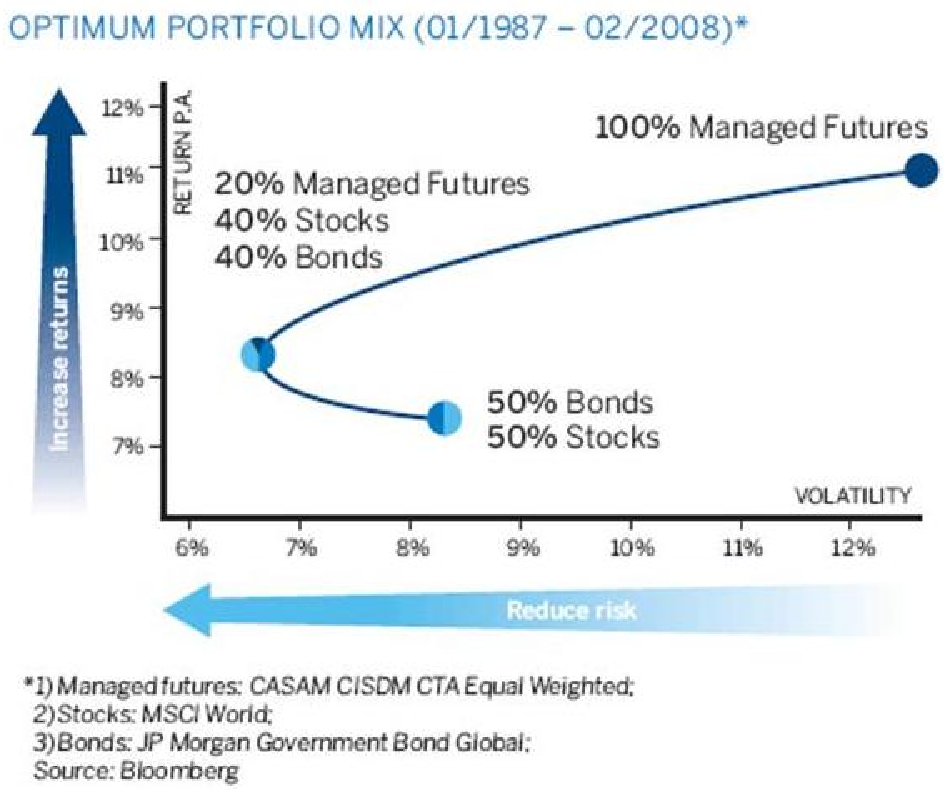
If you're a U.S. investor concerned with limiting volatility and seeking higher long-term returns, there are good reasons for adding a foreign component to your portfolio.

As the chart below shows, over the past 10 years, adding 20% to 40% of foreign stocks to a domestic portfolio actually reduced overall volatility and enhanced overall return.*



Dreyfus

3. Another example

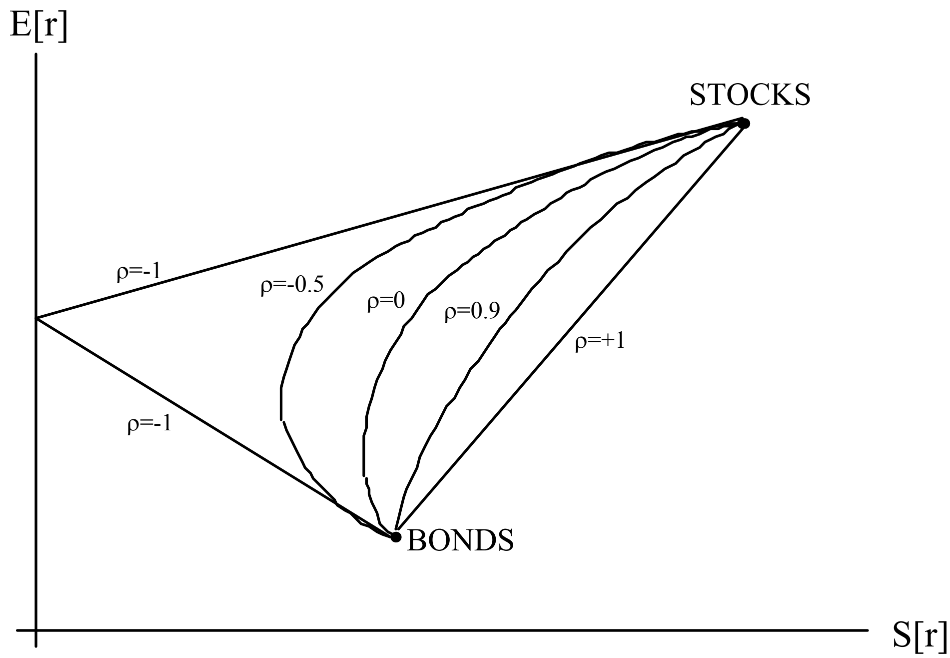


(Source: CME Group)

4. The effect of changing the correlation between the two assets

a. When the assets are perfectly correlated: if $\text{cor}(r_s, r_b) = -1$ the set consists of two lines, and of one line when $\text{cor}(r_s, r_b) = +1$.

b. This graph gives the shape of the efficient frontier for various values of $\rho = \mathrm{cor}(r_s, r_b)$:



c. So the more negative the correlation, the better

If Wall St. Zigs, Find Markets That Zag

d. Conversely, increases in correlation are bad

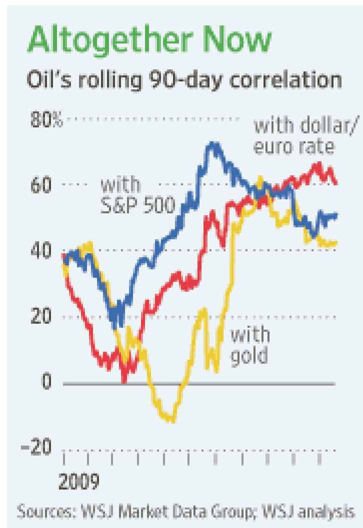
ABREAST OF THE MARKET | AUGUST 16, 2010

Oil Gets a New Dance Partner: Stocks

The Former Opposites Attract, as Prices for Both Are Proxies for Economic Outlook and Risk

HEARD ON THE STREET | NOVEMBER 20, 2009

Oil's Unfriendly Correlation



(WSJ 11/20/09)

e. Indeed, increases in correlation reduce the diversifying power of the combination of the two assets

FUND TRACK | DECEMBER 24, 2009

Stocks, Commodities Are Unlikely Buddies

Recent Moves Thwart Diversity Seekers

As stocks retreated and recovered in the past 15 months, commodities investments moved in step with the U.S. market.

That wasn't supposed to happen.

Investing in commodities long has been pushed as a useful way to cushion portfolio risk. "We haven't seen the independence [in commodities returns] that you'd hope for in a diversified portfolio," says Jay Feuerstein, chief executive of Chicago commodity-trading adviser 2100 Xenon Group.

Last year, commodities such as oil, corn, copper and grain crumbled as the stock market tanked. This year, they have climbed in tandem with stocks. And the performance of commodities mutual funds has striking parallels to the broad U.S. stock market.

f. Actual correlations between returns on different assets

Asset allocation: Searching for opposites

A well-constructed bulletproof portfolio should include assets whose returns do not rise or fall in step with one another. To see whether two types of investments are closely entwined or move in opposite directions, find the point at which they intersect. Example: Returns of small-company stock are only slightly correlated to those of corporate bonds. Assets whose prices tend to move in opposite directions are described as "opposite."

Correlation in returns

	Corp. Bonds	Common Stocks	Small Stocks	Internatl Stocks	Internatl Bonds	Equity Reits
Treasury bills	Very low	Opposite	Opposite	Opposite	Opposite	Opposite
Corporate bonds	—	Mod. Low	Low	Low	Mod. Low	Moderate
Common stocks	Mod. Low	—	High	Moderate	Low	High
Small stocks	Low	High	—	Moderate	Very low	High
Internatl stocks	Low	Moderate	Moderate	—	Mod. High	Mod. High
Internatl bonds	Mod. Low	Low	Very low	Mod. High	—	Mod. Low

Adapted from *Asset Allocation: Balancing Financial Risk*, by Roger Gibson (Business One Irwin)

(Kiplinger)

5. However, even if two markets exhibit low correlations on average, the protection may be tenuous when it is most needed, i.e. during market declines

    a. For instance when monetary policy becomes tighter in the US, as has been the case periodically

Developed countries	*Correlation to U.S. (last 5 years)*	*Correlation when U.S. interest rates rose (10 years)*
Austria	0.15	0.42
Italy	0.07	0.40
Japan	0.20	0.36
Emerging countries		
China	0.26	0.32
Colombia	0.04	0.04
Greece	0.24	0.21
Poland	0.28	0.46
Taiwan	0.07	0.45
Turkey	–0.15	0.40

Source: Smith Barney

(NYT)

Wall Street Wobbles, and the World Shakes

(WSJ)

b. Or during the bear market in the Fall of 2008:

Country-by-Country Derby

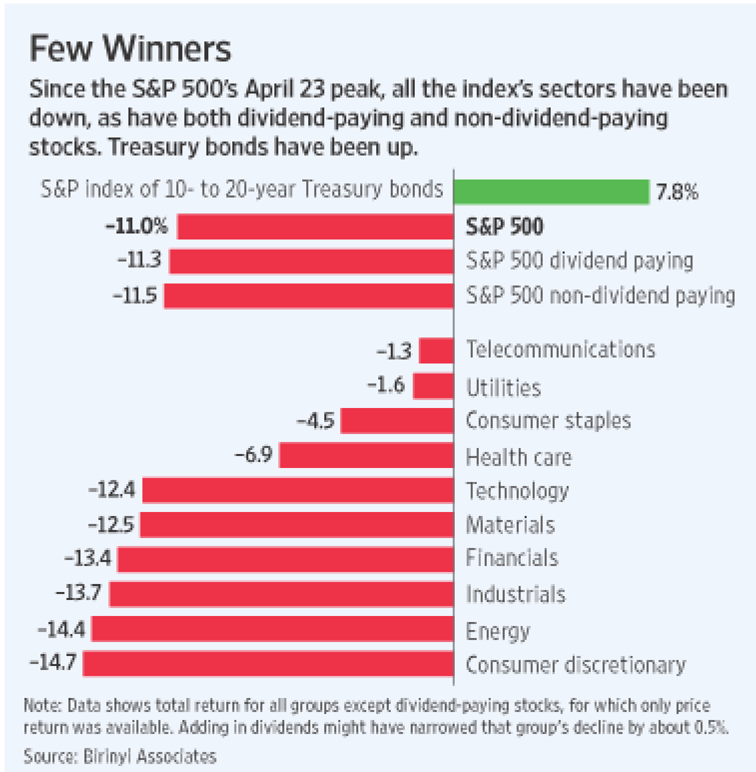
Third-quarter stock-market performance, based on the Dow Jones Global Indexes in U.S.-dollar and local-currency terms ranked by U.S.-dollar performance

Country	U.S. dollar	Local currency	Country	U.S. dollar	Local currency
Turkey	**−0.5%**	3.4%	Italy	**−23.2%**	−13.8%
Philippines	**−1.0**	3.8	Thailand	**−23.7**	−22.7
Peru	**−6.2**	−6.1	Sweden	**−24.1**	−12.1
U.S.	**−9.1**	−9.1	Czech Republic	**−24.6**	−13.3
Colombia	**−10.9**	2.0	Greece	**−24.9**	−15.7
Jordan	**−11.6**	−11.5	South Korea	**−25.0**	−13.4
Chile	**−13.1**	−9.1	Singapore	**−25.7**	−21.8
Switzerland	**−13.1**	−4.3	Netherlands	**−25.8**	−16.8
India	**−15.4**	−7.6	Hong Kong	**−26.6**	−26.9
Israel	**−15.9**	−13.0	Malta	**−27.0**	−18.1
Slovakia	**−16.1**	−5.7	Finland	**−27.5**	−18.7
Sri Lanka	**−16.6**	−16.2	Denmark	**−27.8**	−19.0
New Zealand	**−16.6**	−4.9	Australia	**−28.1**	−12.5
South Africa	**−16.8**	−12.0	Taiwan	**−28.8**	−24.5
Bahrain	**−18.5**	−18.5	Egypt	**−29.3**	−27.5
Japan	**−18.5**	−18.4	Slovenia	**−29.4**	−20.8
Mexico	**−18.7**	−13.3	Indonesia	**−29.6**	−28.0
Spain	**−18.9**	−9.1	Estonia	**−32.7**	−24.5
Kuwait	**−19.1**	−18.5	Cyprus	**−34.4**	−26.4
France	**−19.2**	−9.4	Lithuania	**−35.8**	−28.0
Malaysia	**−19.6**	−15.3	Pakistan	**−37.6**	−28.7
Poland	**−19.7**	−8.9	Brazil	**−38.1**	−25.0
Portugal	**−19.7**	−9.9	Austria	**−39.0**	−31.6
Argentina	**−19.9**	−17.1	Ireland	**−40.2**	−32.9
Belgium	**−20.5**	−10.8	Norway	**−41.1**	−31.6
Morocco	**−20.7**	−12.8	Iceland	**−41.8**	−22.6
Hungary	**−20.7**	−8.5	Latvia	**−41.9**	−34.4
Germany	**−21.0**	−11.4	Romania	**−43.4**	−34.7
U.K.	**−22.0**	−12.9	Bulgaria	**−43.8**	−37.2
China	**−22.8**	−23.2	Russia	**−47.8**	−42.8
Canada	**−23.1**	−19.4			

Source: Dow Jones Indexes

(WSJ 10/1/2008)

c. On the other hand, in a "risk-on/risk-off" scenario, as seems to be the case since the crisis began in 2008, when all segments of the US stock market decline together (positive correlation), US government bonds tend to do well (negative correlation with US stocks) as a "flight-to-quality" takes place



(WSJ 7/12/2010)

d. Compare the two correlation matrices

i. Estimates of correlation between asset classes over the last 20 years

		TIP	AGG	IGE	GSG	VNQ	RWX	EEM	EFA	VB	VV	VO	VGK	VPL
Inflation-protected Treasuries	TIP													
US Bonds	AGG	0.75												
Natural Resources (Oil)	IGE	0.13	-0.19											
Commodities Index	GSG	0.67	0.15	0.73										
US Real Estate	VNQ	-0.63	-0.53	0.16	-0.29									
International Real Estate	RWX	-0.77	-0.41	0.07	-0.58	0.67								
Emerging Markets	EEM	-0.60	-0.34	0.39	-0.28	0.52	0.89							
Europe, Australasia, Far East	EFA	-0.71	-0.38	0.23	-0.44	0.59	0.96	0.97						
US Small Cap Stocks	VB	-0.87	-0.70	0.12	-0.47	0.83	0.77	0.67	0.73					
US Large Cap Stocks	VV	-0.86	-0.51	0.10	-0.58	0.71	0.94	0.87	0.94	0.89				
US Mid Cap Stocks	VO	-0.82	-0.59	0.30	-0.40	0.75	0.90	0.88	0.91	0.92	0.97			
European Stocks	VGK	-0.70	-0.38	0.22	-0.44	0.58	0.95	0.96	1.00	0.73	0.93	0.90		
Pacific Rim Stocks	VPL	-0.74	-0.47	0.30	-0.37	0.63	0.94	0.96	0.98	0.78	0.93	0.94	0.97	
NASDAQ	^IXIC	-0.89	-0.63	0.05	-0.56	0.68	0.83	0.76	0.83	0.94	0.94	0.92	0.83	0.85

ii. Compare to estimates over August, September and October 2008

		TIP	AGG	IGE	GSG	VNQ	RWX	EEM	EFA	VB	VV	VO	VGK	VPL
Inflation-protected Treasuries	TIP													
US Bonds	AGG	0.49												
Natural Resources (Oil)	IGE	-0.05	0.22											
Commodities Index	GSG	0.02	0.08	0.70										
US Real Estate	VNQ	-0.34	-0.17	0.63	0.29									
International Real Estate	RWX	0.09	0.21	0.78	0.56	0.64								
Emerging Markets	EEM	-0.20	0.04	0.86	0.53	0.83	0.82							
Europe, Australasia, Far East	EFA	-0.12	0.20	0.88	0.59	0.78	0.88	0.94						
US Small Cap Stocks	VB	-0.16	0.06	0.81	0.45	0.91	0.82	0.89	0.89					
US Large Cap Stocks	VV	-0.16	0.15	0.87	0.48	0.87	0.83	0.94	0.96	0.95				
US Mid Cap Stocks	VO	-0.16	0.12	0.89	0.51	0.87	0.85	0.93	0.94	0.97	0.98			
European Stocks	VGK	-0.09	0.17	0.89	0.62	0.77	0.89	0.93	0.99	0.90	0.94	0.94		
Pacific Rim Stocks	VPL	-0.07	0.20	0.88	0.55	0.77	0.88	0.93	0.98	0.88	0.94	0.93	0.96	
NASDAQ	^IXIC	-0.25	0.10	0.82	0.48	0.88	0.81	0.93	0.95	0.95	0.97	0.97	0.93	0.92

(Source: www.assetcorrelation.com)

e. Consequently, in down markets, in the short term, diversification doesn't work well.

(WSJ 10/13/08)

i. In the long run, however, asset classes don't stay highly correlated forever.

ii. Thus, investors are best served by sticking to their long-term asset allocation plan, even if it doesn't seem to be working in the short run.

iii. But undoubtedly the empirical increase in correlations in times of crises raises legitimate questions over the usefulness of the theory, at least in extreme circumstances.

f. Between 1976 and 1999, correlations between American and European markets were low, ranging from 0.24 for Italy to 0.5 for Britain. Since 2000, however, correlations have been as high as 0.9.

(The Economist)

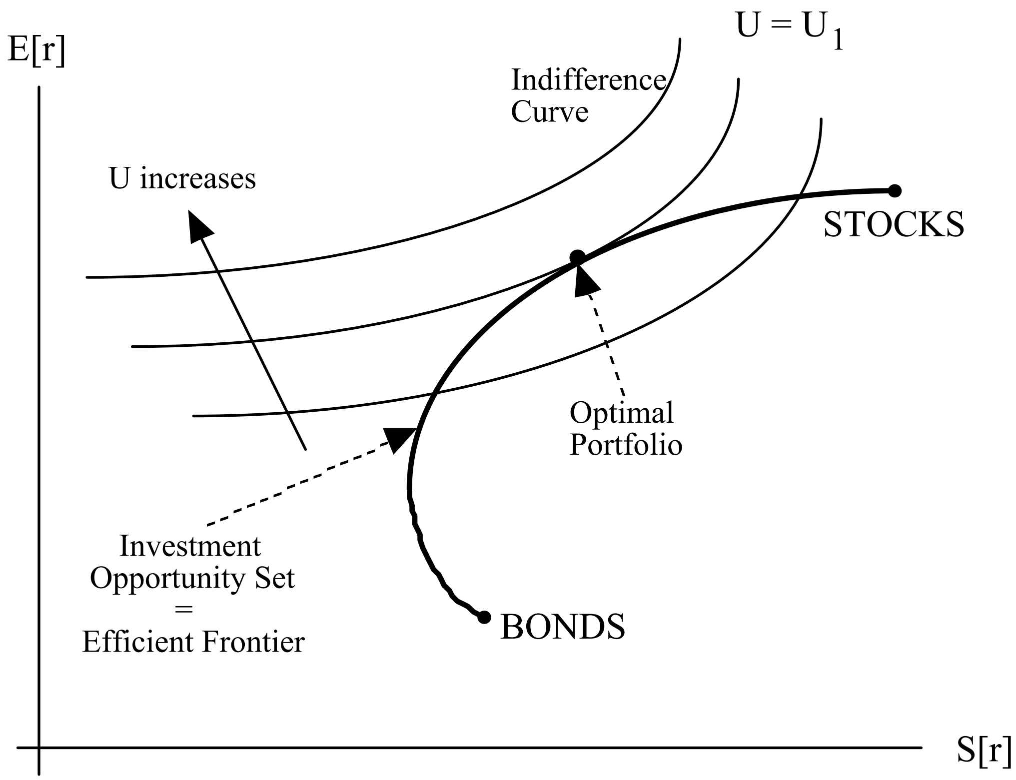
g. There is typically a link between the level of integration of two economies, or their business cycle, and the level of correlation between their respective stock returns.

(The Economist)

C. *Optimal Asset Allocation*

1. Given what precedes, we want to find the *feasible* portfolio that maximizes the utility function.

2. Being feasible means that the portfolio must be on the efficient frontier linking BONDS to STOCKS.



III. One Risk-Free Asset, Two Risky Assets

A. Portfolio Expected Return and Risk

1. Consider a portfolio P that invests in STOCKS, BONDS (both risky) and MM (risk-free).

   a. Invest a share ω_s of your initial wealth W_0 in STOCKS, ω_b in BONDS and ω_m=1-ω_s-ω_b in MM.

   b. For instance, $P = \begin{cases} 40\% \text{ in S} \\ 40\% \text{ in B} \\ 20\% \text{ in MM} \end{cases}$.

c. Now consider the portfolio $R = \begin{cases} 50\% \text{ in S} \\ 50\% \text{ in B} \\ 0\% \text{ in MM} \end{cases}$ that invests only in the risky assets.

2. P can be viewed as investing in R and MM:

$$P = \begin{cases} 40\% \text{ in S} \\ 40\% \text{ in B} \\ 20\% \text{ in MM} \end{cases} = \begin{cases} 80\% \text{ in R} \\ 20\% \text{ in MM} \end{cases}$$

3. R invests only in STOCKS and BONDS

   a. Therefore it must be on the efficient frontier linking S and B.

   b. That follows from what we learned above in the two risky assets case.

4. Now consider the line from MM to R.

   a. Because P invests in MM and R, it must be on that line

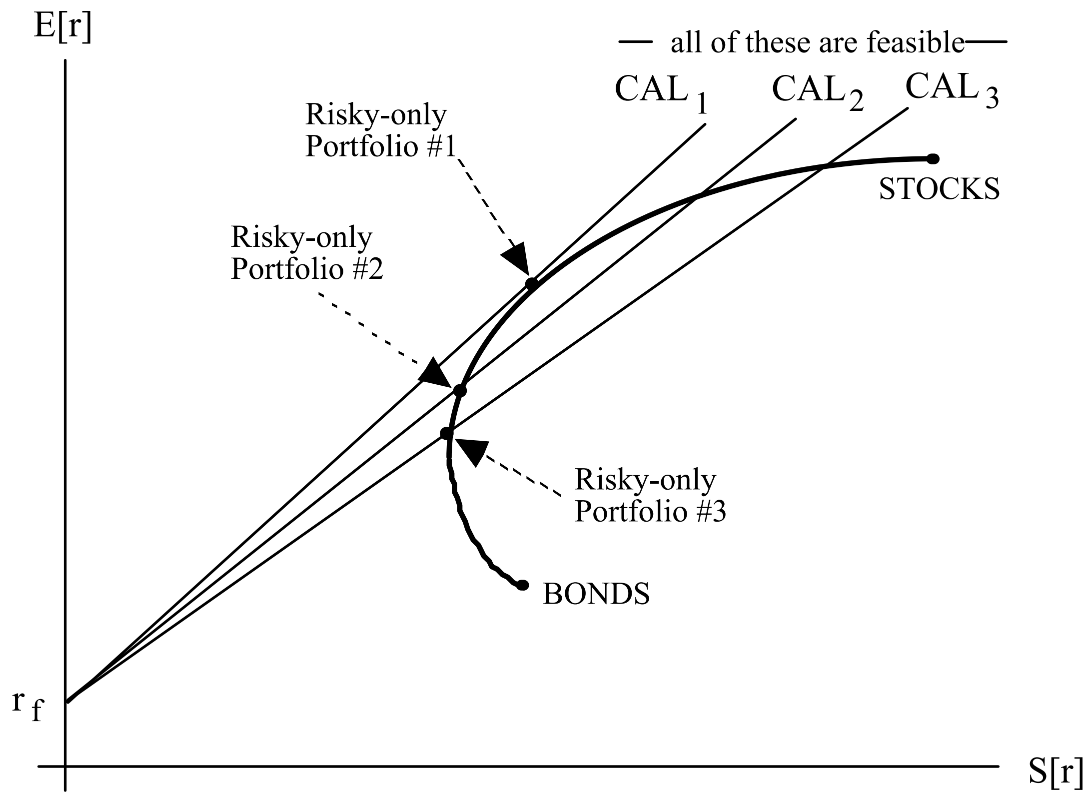
   b. That follows from what we learned above in the one riskfree, one risky asset case.

B. Investment Opportunities

1. All possible risky-only combinations of S and B are feasible, each giving rise to a new portfolio R.

   a. Then each combination of MM and R is feasible.

   b. For *each* risky-only portfolio formed with some combination of STOCKS and BONDS shares, the investment opportunity set corresponding to combinations of *that* risky-only portfolio with the MM is a Capital Allocation Line:



2. Each of these CALs is feasible. The opportunity set is the *collection* of these CALs.

   a. All the CALs start on the left at the MM risk-free rate r_f.

   b. To be feasible, a CAL must start at r_f on the y-axis and intersect somewhere the efficient frontier from BONDS to STOCKS.

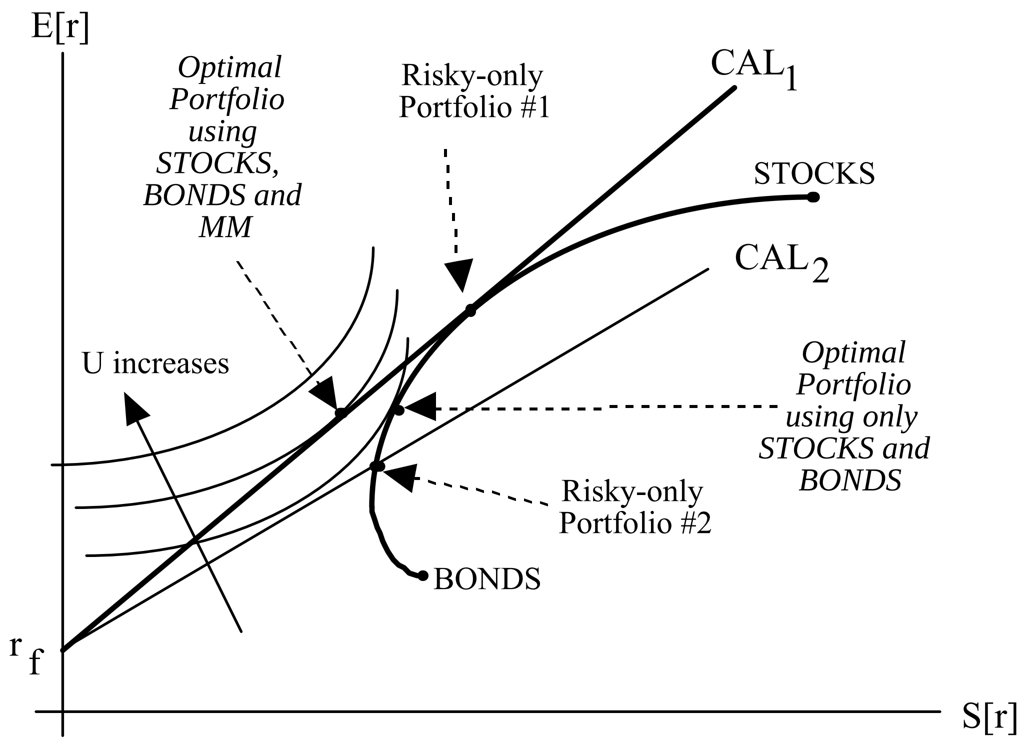
   c. No investor will ever want to be on CAL_2 if (s)he can be on CAL_1 instead: for each portfolio on CAL_2, there exists one directly above it on CAL_1 and it is feasible.

   d. By pushing the reasoning further, investors will all want to be on the one CAL with the highest possible slope. This is CAL_1.

   e. Hence the only CAL worth considering is the one that goes from r_f on the y-axis to the tangency point with the efficient frontier, corresponding to CAL_1 in the graph.

C. Optimal Asset Allocation

1. As always, find the *feasible* portfolio that maximizes utility:



2. Among all the possible CALs, the optimal portfolio is on CAL_1 which is tangent to the set of feasible portfolios using only STOCKS and BONDS.

   a. This is also the steepest feasible CAL.

   b. The optimal portfolio using STOCKS, BONDS and the MM is equivalent to the optimal portfolio using the risky-only portfolio #1 and the MM.

   c. The optimal portfolio using STOCKS, BONDS and the MM achieves a higher level of utility than the optimal portfolio using only STOCKS and BONDS and no MM (because increasing your available options cannot make your worse-off).

3. If you consider now an investor with a different degree of risk aversion, that investor will choose a different portfolio, but still on the same CAL, the steepest one.

   a. Consequently, the two investors have the same ratio of STOCKS to BONDS in their portfolio, given by the same Risky Only Portfolio #1.

   b. They differ in their respective allocation to MM and Risky Only Portfolio #1.

4. In practice:

Who Has the Best Blend?

Performance of asset-allocation blends recommended by 12 brokerage houses in periods ended Dec. 31, 1993. Houses are ranked by 12-month performance. Also shown is the mix each house now recommends. Figures do not include transaction costs.

	PERFORMANCE			RECOMMENDED BLEND		
BROKERAGE HOUSE	THREE MONTHS	ONE YEAR	FIVE YEARS	STOCKS	BONDS	CASH
PaineWebber	1.7%	**10.5%**	79.3%	68%	30%	2%
Kidder Peabody	1.5	**10.4**	84.9	70	30	0
Goldman Sachs	1.6	**9.9**	83.5	70	25	5
Lehman Brothers	1.7	**9.8**	87.2	50	35	15
A.G. Edwards	1.5	**9.8**	75.4	65	30	5
Dean Witter	1.4	**9.7**	78.7	60	25	15
Prudential	2.3	**9.7**	92.1	85	15	0
Smith Barney	1.2	**9.4**	82.9	55	30	15
Merrill Lynch	1.4	**9.4**	80.9	60	25	15

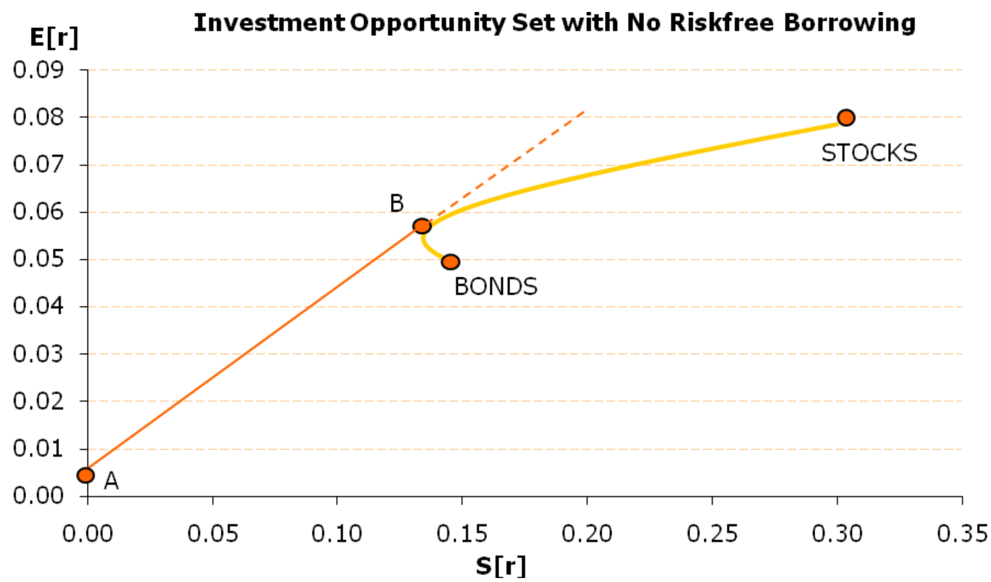
(WSJ)

   a. According to the theory, these brokers should only recommend a mix of STOCKS and BONDS, since this is investor-independent, leaving the mix of [Risky Only] and MM for the client to select (as a function of the client's risk aversion).

   b. So "Recommended Blend" should not contain a "Cash" column.

c. Why does the mix of STOCKS and BONDS vary across brokers? That's because they use different assumptions for the five inputs of the problem $E[r_s]$, $E[r_b]$, $S[r_s]$, $S[r_b]$ and $cor(r_s, r_b)$.

5. Without Borrowing



a. The previous discussion assumes that you can have a negative portfolio weight on MM, that is, borrow at the riskfree rate.

b. What happens to the investment opportunity set if you cannot borrow at all at the riskfree rate?

c. The relevant part of the feasible investment set consists of the line with the highest possible slope going from MM (point A in the figure) to the efficient frontier of the two risky assets: in the figure, that is the CAL going from A to the tangency point B, and ending at B.

d. Points on the CAL between A and B involve lending at the riskfree rate so they are feasible.

e. If you are not allowed to borrow at the riskfree rate, then the line ends at B because points past B on that CAL would involve borrowing at the rate indicated by A, which is 0.5% on the plot.

The portion of the CAL past B (dashed on the figure) is therefore not part of the investment opportunity set.

f. But the portion of the efficient frontier from B to asset 2 is also part of the investment opportunity set

    i. Since the CAL past B is no longer in it, this portion of the efficient frontier becomes the most North-West part of the investment opportunity set, relevant for an investor who has low risk-aversion.

    ii. Such an investor would choose to invest only in the two risky assets since she can't leverage up her portfolio by borrowing at the riskfree rate.

6. With Different Borrowing and Lending Rates

a. Suppose now that you can borrow, but at a different (= higher) rate than the one at which you lend.

b. The CAL from A to B is still feasible since it involves lending at the lower of the two riskfree rate, say 0.5% on the plot. As in the previous case, the dashed portion past B is not feasible.

c. Allowing borrowing expands the feasible investment set in the following way:

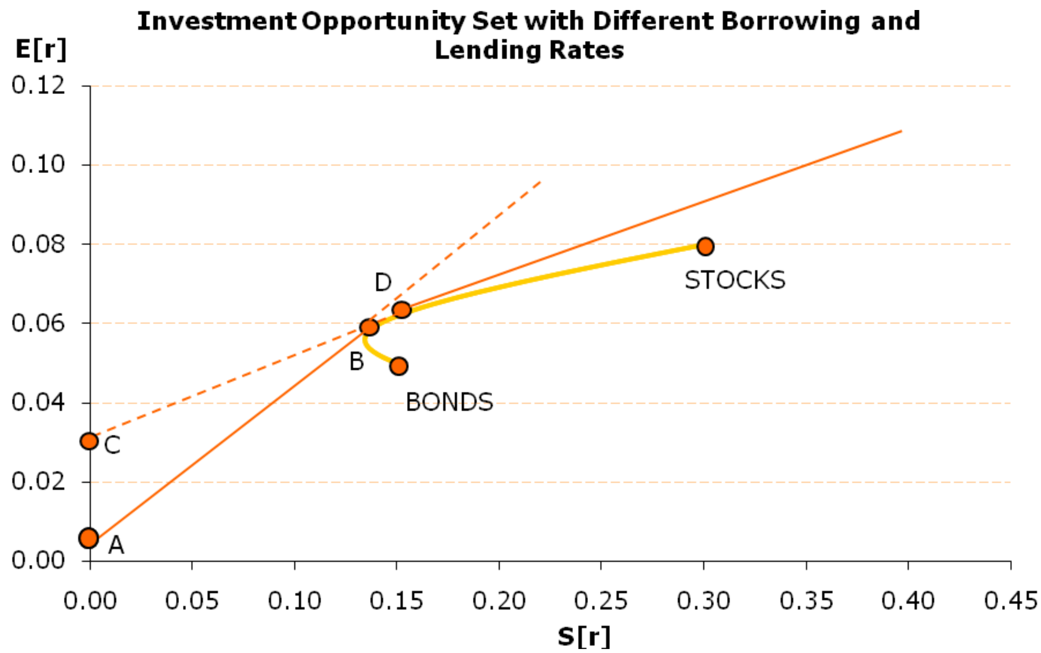
    i. Plot the CAL tangent to the efficient frontier that starts at C, the higher borrowing rate, say 3%.

    ii. Call the tangency point D.

    iii. Points that are between C and D on that CAL (dashed on the figure) are NOT feasible since they involve lending at 3% (which you cannot do, you can only lend at 0.5%).

    iv. However, points that are past D on that CAL are feasible since they involve borrowing at 3%.

d. The investment opportunity set consists of A to B (first CAL), B to D (on the efficient frontier), and the second CAL starting at D



IV. *Many Risky Assets and the Risk-Free Asset*

A. Efficient Frontier

1. Recall from the two-risky-asset-case that with two risky assets one can form a minimum-variance portfolio

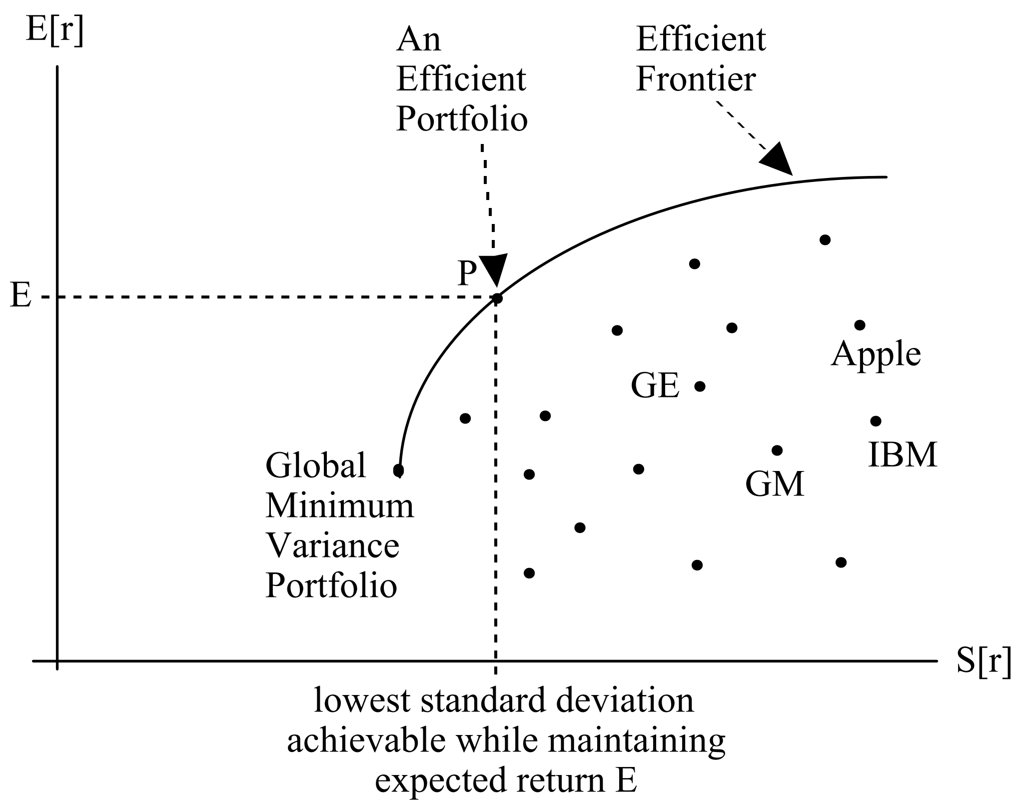
   a. By choosing the shares invested in each of the two assets in order the minimize the resulting portfolio's variance

   b. Without regard to the portfolio's expected rate of return.

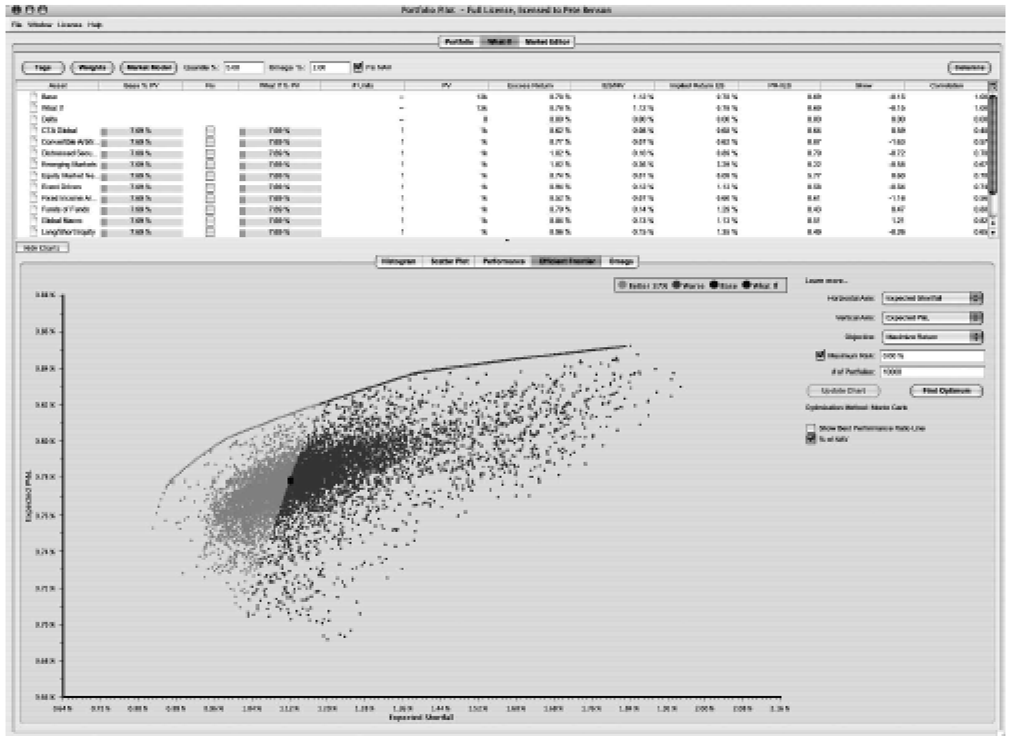
2. With many assets, we can find many different portfolios that will achieve a given expected rate of return.

   a. Among those, one will have the lowest variance. So for each expected rate of return, we can find a minimum-variance portfolio by choosing the weights on each of the individual assets appropriately.

   b. The collection of these portfolios is called the *efficient frontier*:

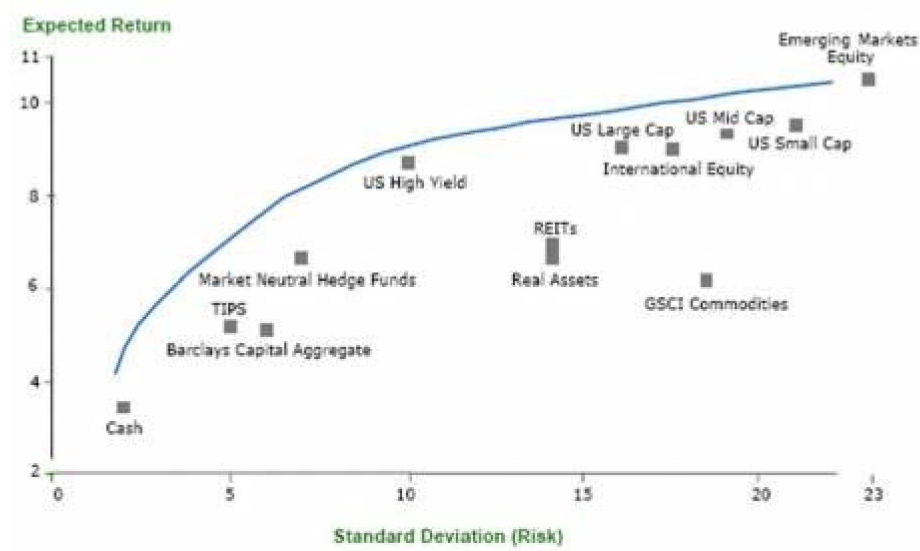


c. In practice, this can be implemented with thousands of assets



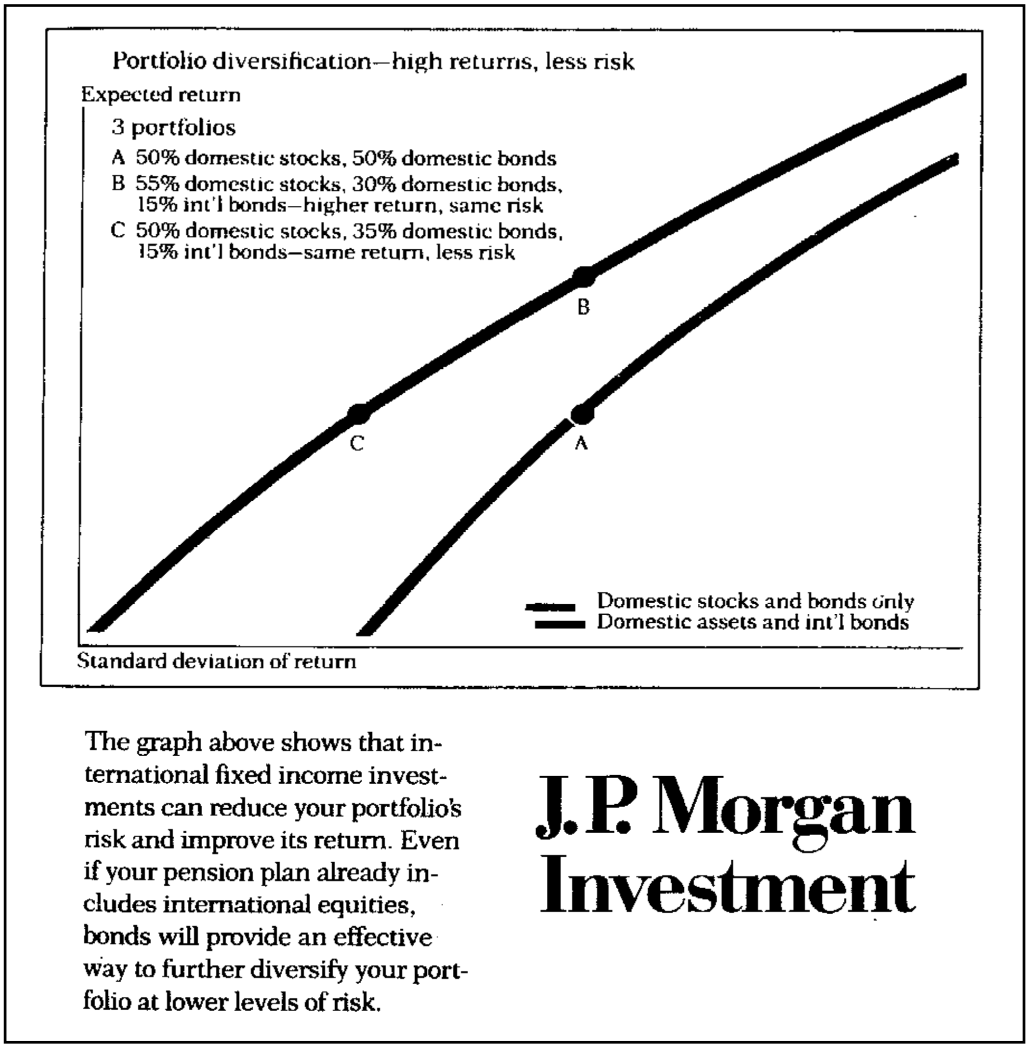
(Riskmetrics.com Portfolio Pilot)

d. Or asset classes

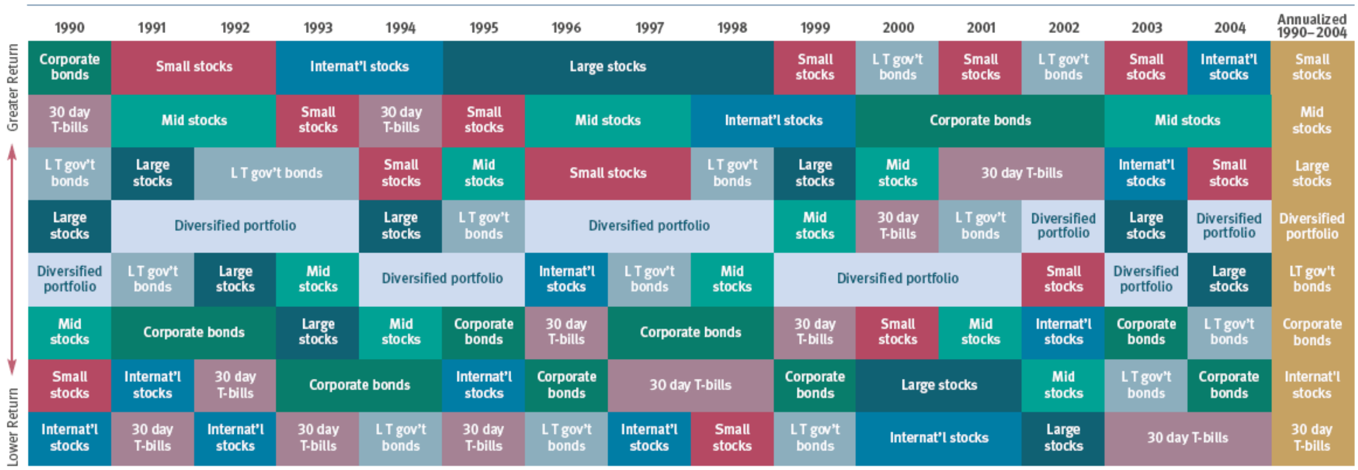


(Source: State Street Investors)

3. If more risky assets are added, the efficient frontier shifts to the North-West:



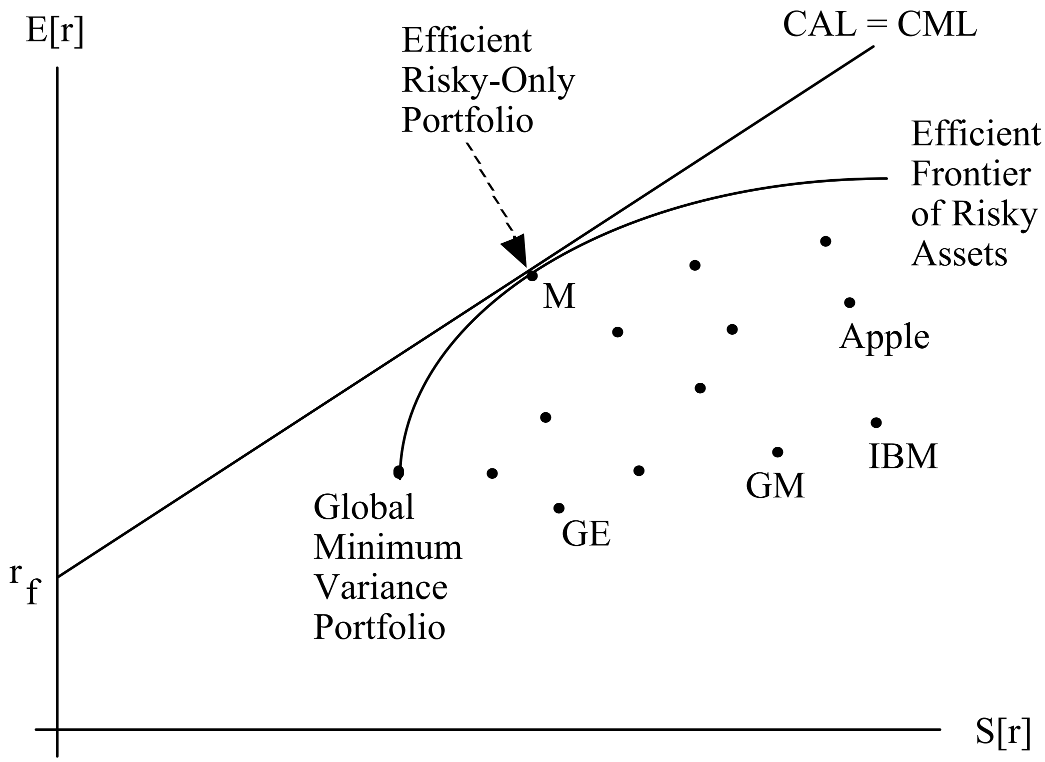
4. Diversification works to reduce the overall variance of the portfolio



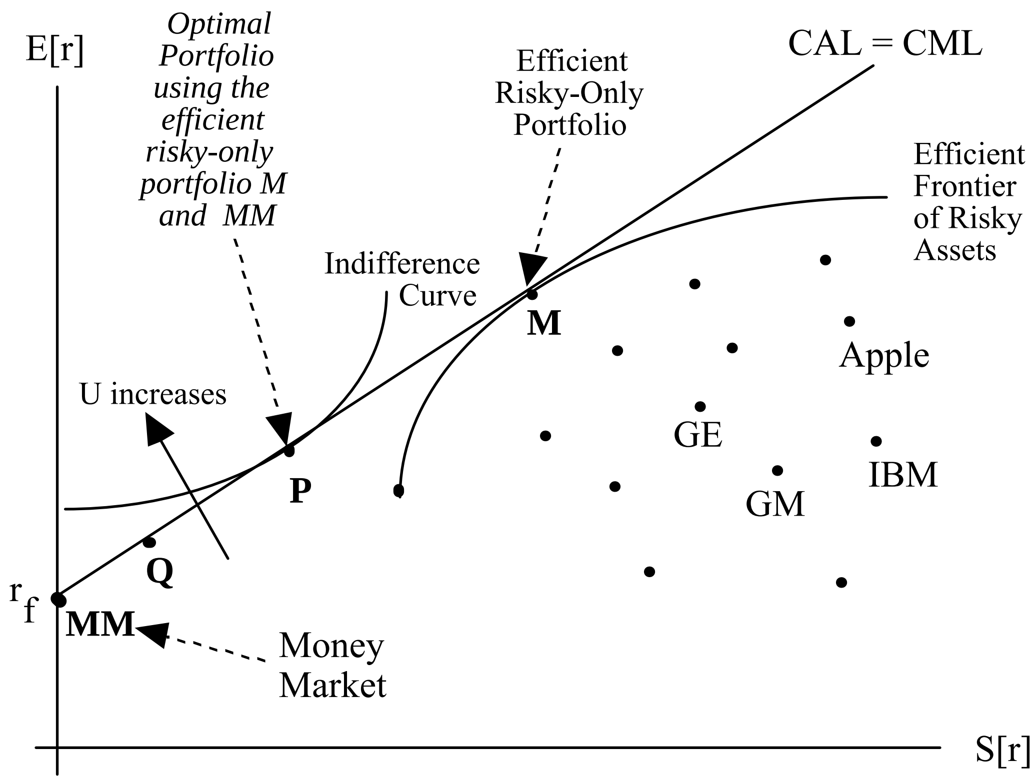
(Source: Pioneer Investments)

B. Optimal Portfolio

1. With a MM present, the only relevant part of the investment opportunity set becomes the CAL *tangent* to the efficient frontier:



2. Given his/her utility function, an investor will choose portfolio P:



3. So finding the optimal portfolio P involves two separate steps:

   a. First find the efficient risky-only portfolio, M, as the point of tangency between the CAL with the highest slope linking the MM to the efficient frontier. This first step is common to all investors.

   b. Then find P on the CML by maximizing utility. P will wary according to the investor's utility function (i.e., risk-aversion).

4. This result is Tobin's ***separation theorem*** of portfolio investing:

   a. Separation because the there are two separate steps:

i. First find the CML (that’s a purely technological step that is independent of preferences)

ii. Then, based on your preferences, find out where you want to be on the CML.

b. For example, a more risk-averse investor (i.e. with steepest indifference curves) would choose another portfolio on the CAL, Q between P and MM.

i. Compared to P, Q would involve investing a bigger share in MM and less in M.

ii. It is not optimal for investors to pick their own portfolio of risky assets: everyone should invest in the mix of risky assets determined by M.

5. The separation theorem is often called the “mutual funds principle”.

a. *Index funds* are mutual funds designed to replicate M.

b. According to the theory, investors should invest all their wealth in money market accounts and index funds;

c. And not pick out stocks and other risky assets individually: the right proportions to hold the risky assets are given by those of M.

V. Static Portfolio Optimization: Mathematical Characterization of the Optimal Solution

A. Computation of the Efficient Frontier

1. The problem we have just finished solving using graphical tools is called a “static” optimization because we have only one target date

a. Investment decisions are made at date 0

b. And results get realized at date 1

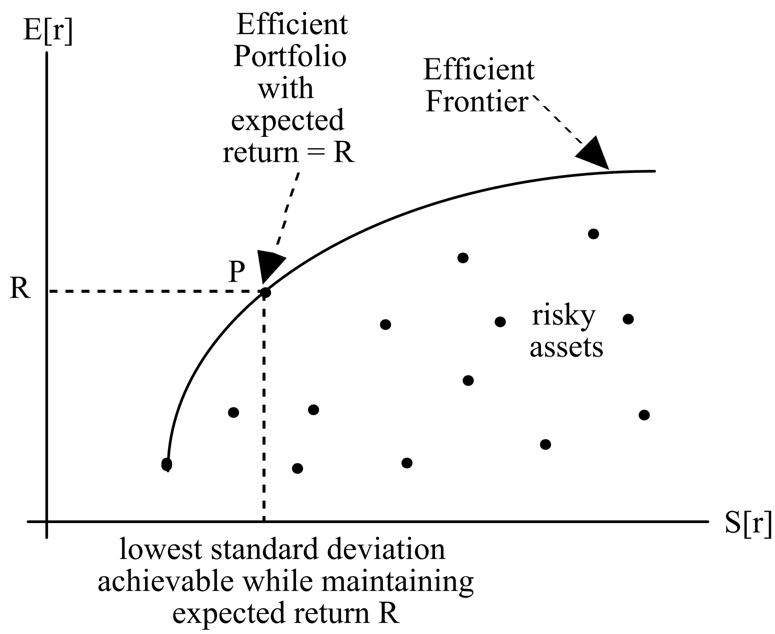
2. Here is now the formal mathematical derivation of the solution

   a. Given N assets, i=1,2,...,N, with expected returns $E[r_i] \equiv \bar{r}_i$, standard deviations $S[r_i] \equiv \sigma_i$ and correlation coefficients ρ_{ij} between assets i and j, the expected return and variance of a portfolio with weights ω_i are given by:

$$E[r] = \sum_{i=1}^{N} \omega_i \bar{r}_i$$

$$V[r] \equiv \sigma^2 = \sum_{i=1}^{N} \omega_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \omega_i \omega_j \rho_{ij} \sigma_i \sigma_j$$

   b. To compute the efficient frontier point-by-point, we fix the target level of expected return at a level R and look for the portfolio weights ω_i which will minimize the portfolio standard deviation (or equivalently variance).



3. That is, for each value of R we solve a minimization-under-constraints problem

$$\min\text{imize}_{\text{over }\omega_1,\ldots,\omega_n} \quad \sum_{i=1}^{N} \omega_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \omega_i \omega_j \rho_{ij} \sigma_i \sigma_j$$

$$\text{subject to:} \quad (1) \ \sum_{i=1}^{N} \omega_i = 1$$

$$(2) \ \sum_{i=1}^{N} \omega_i \bar{r}_i = R$$

4. This is called a quadratic programming problem under linear constraints

   a. Because the objective function contains terms like ω_i^2 and $\omega_i \omega_j$ while the constraints (1) and (2) contain terms like ω_i.

   b. By varying the value of the target R and re-solving the problem for each value of R, we can trace out the entire efficient frontier.

   c. Any short sale constraint on stock i would add a constraint on the sign of ω_i; for example, $\omega_i \geq 0$ if no short sales are allowed.

      i. To learn how to deal with them, look up "Kuhn-Tucker conditions" in your calculus calculus textbook.

      ii. I do not put any constraints on short sales here, to keep things simple.

5. This problem is easier to write down if we use matrix notation.

   a. So define the portfolio weight vector ω, the expected return vector of the assets $\bar{r}$, the vector of ones 1 and the variance-covariance matrix of the assets Σ as:

$$\omega \equiv \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix}, \quad \overline{r} \equiv \begin{bmatrix} \overline{r}_1 \\ \vdots \\ \overline{r}_n \end{bmatrix}, \quad \mathbf{1} \equiv \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\Sigma \equiv \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \rho_{12}\sigma_1\sigma_2 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \rho_{1n}\sigma_1\sigma_n & \cdots & \cdots & \sigma_n^2 \end{bmatrix}$$

b. Then we have that (the superscript ^{T} means that the vector is transposed):

$$E[r] = \sum_{i=1}^{N} \omega_i \overline{r}_i = \omega^T \overline{r}$$

$$V[r] = \sum_{i=1}^{N} \omega_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \omega_i \omega_j \rho_{ij} \sigma_i \sigma_j = \omega^T \Sigma \omega .$$

c. Therefore we find the efficient frontier by solving:

$$\begin{aligned} &\text{minimize}_{\text{over } \omega} \quad \omega^T \Sigma \omega \\ &\text{subject to:} \quad (1) \ \ \omega^T \mathbf{1} = 1 \\ &\qquad\qquad\quad\ \ (2) \ \ \omega^T \overline{r} = R \end{aligned}$$

where 1 denotes a vector of ones.

6. To find the solution, introduce the two Lagrange multipliers λ_1 and λ_2 each associated with one of the two constraints, and define the Lagrangian:

$$\ell(\omega, \lambda_1, \lambda_2) \equiv \omega^T \Sigma \omega - \lambda_1 \left(\omega^T \mathbf{1} - 1 \right) - \lambda_2 \left(\omega^T \overline{r} - R \right)$$

a. To learn about this technique, look in a calculus textbook under "constrained optimization;" the topic is also covered in most microeconomics textbooks.

b. The solution solves the system of equations:

$$\frac{\partial \ell}{\partial \omega} = 0,\ \frac{\partial \ell}{\partial \lambda_1} = 0,\ \frac{\partial \ell}{\partial \lambda_2} = 0,$$

c. that is:

$$\begin{cases} 2\Sigma\omega - \lambda_1 \mathbf{1} - \lambda_2 \overline{r} = \mathbf{0} \\ \omega^T \mathbf{1} - 1 = 0 \\ \omega^T \overline{r} - R = 0 \end{cases}$$

d. Premultiply the first equation by ω^T to get: $2\omega^T\Sigma\omega = \lambda_1\omega^T\mathbf{1} + \lambda_2\omega^T\overline{r} = \lambda_1 + \lambda_2 R$, which is two times the portfolio's variance.

e. We then determine the two Lagrange multipliers:

$$\begin{cases} \omega = \frac{1}{2}\left(\lambda_1\Sigma^{-1}\mathbf{1} + \lambda_2\Sigma^{-1}\overline{r}\right) \\ \omega^T\mathbf{1} = \frac{1}{2}\left(\lambda_1\mathbf{1}^T\Sigma^{-1}\mathbf{1} + \lambda_2\overline{r}^T\Sigma^{-1}\mathbf{1}\right) = 1 \\ \omega^T\overline{r} = \frac{1}{2}\left(\lambda_1\mathbf{1}^T\Sigma^{-1}\overline{r} + \lambda_2\overline{r}^T\Sigma^{-1}\overline{r}\right) = R \end{cases}$$

f. Here Σ^{-1} denote the inverse of the matrix Σ.

g. The last two equations (two linear equations in the two unknown λ_1 and λ_2) can easily be solved using Cramer's rule.

h. The solution is:

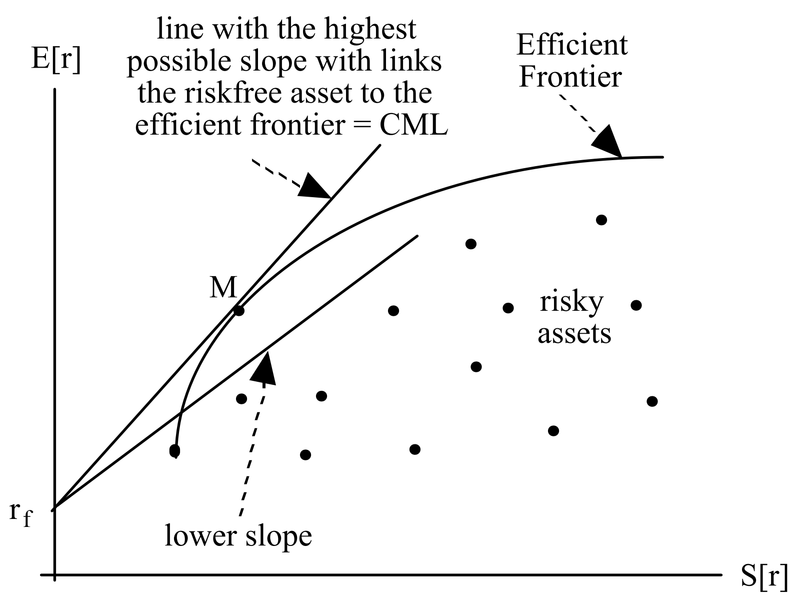
$$\begin{cases} \lambda_1 = 2\dfrac{\overline{r}^T\Sigma^{-1}\overline{r} - R\,\overline{r}^T\Sigma^{-1}\mathbf{1}}{\left(\mathbf{1}^T\Sigma^{-1}\mathbf{1}\right)\left(\overline{r}^T\Sigma^{-1}\overline{r}\right) - \left(\overline{r}^T\Sigma^{-1}\mathbf{1}\right)^2} \\ \lambda_2 = 2\dfrac{R\,\mathbf{1}^T\Sigma^{-1}\mathbf{1} - \overline{r}^T\Sigma^{-1}\mathbf{1}}{\left(\mathbf{1}^T\Sigma^{-1}\mathbf{1}\right)\left(\overline{r}^T\Sigma^{-1}\overline{r}\right) - \left(\overline{r}^T\Sigma^{-1}\mathbf{1}\right)^2} \end{cases}$$

i. Replace these values into the variance expression, then compute the standard deviation $S[r] = \sqrt{\omega^T \Sigma \omega} = \sqrt{(\lambda_1 + \lambda_2 R)/2}$.

j. All that remains to be done is to vary R in order to trace out the efficient frontier:

$$R \mapsto \sqrt{(\lambda_1 + \lambda_2 R)/2} = \sqrt{\left(\frac{\left(\mathbf{1}^T \Sigma^{-1} \mathbf{1}\right) R^2 - 2\left(\overline{r}^T \Sigma^{-1} \mathbf{1}\right) R + \left(\overline{r}^T \Sigma^{-1} \overline{r}\right)}{\left(\mathbf{1}^T \Sigma^{-1} \mathbf{1}\right)\left(\overline{r}^T \Sigma^{-1} \overline{r}\right) - \left(\overline{r}^T \Sigma^{-1} \mathbf{1}\right)^2} \right)}$$

B. Computation of the Capital Market Line



1. We must find, among all the CALs linking the riskfree rate to one of the portfolioP on the efficient frontier, the one with the highest slope.

   a. The slope of the CAL linking the riskfree asset to an efficient portfolio is given by $(E[r] - r_f)/S[r]$, which we want to maximize:

$$\begin{aligned}&\text{maximize}_{\text{over }\omega}\ \frac{\omega^T\bar{r}-r_f}{\sqrt{\omega^T\Sigma\omega}}\\&\text{subject to:}\quad \omega^T\mathbf{1}=1\end{aligned}$$

b. Rather than defining the Lagrange multiplier λ associated with the unique constraint, and the associated Lagrangian, let's be sneaky.

c. Note that, under the constraint, the problem can be replaced by:

$$\begin{aligned}&\text{maximize}_{\text{over }\omega}\ \frac{\omega^T\bar{r}-\omega^T\mathbf{1}r_f}{\sqrt{\omega^T\Sigma\omega}}\\&\text{subject to:}\quad \omega^T\mathbf{1}=1\end{aligned}$$

d. Now note that this problem is homogenous of degree zero, meaning that if ω maximizes $\left(\omega^T\bar{r}-\omega^T\mathbf{1}r_f\right)\Big/\sqrt{\omega^T\Sigma\omega}$ then for any constant a, aω also maximizes it.

e. Therefore we can simply solve the unconstrained problem:

$$\text{maximize}_{\text{over }\omega}\ \frac{\omega^T\bar{r}-\omega^T\mathbf{1}r_f}{\sqrt{\omega^T\Sigma\omega}}$$

f. Then find its solution and then multiply it by an appropriate constant a to insure that the final solution satisfies the constraint $\omega^T\mathbf{1}=1$.

g. So let's solve the unconstrained problem. We set the derivative w.r.t. ω of $\left(\omega^T\bar{r}-\omega^T\mathbf{1}r_f\right)\Big/\sqrt{\omega^T\Sigma\omega}$ to zero, i.e.:

$$\frac{\sqrt{\omega^T\Sigma\omega}\left(\bar{r}-\mathbf{1}r_f\right)-\left(\omega^T\bar{r}-\omega^T\mathbf{1}r_f\right)\dfrac{2\Sigma\omega}{2\sqrt{\omega^T\Sigma\omega}}}{\omega^T\Sigma\omega}=0$$

$$\left(\omega^T\Sigma\omega\right)\left(\bar{r}-\mathbf{1}r_f\right)-\left(\omega^T\bar{r}-\omega^T\mathbf{1}r_f\right)\Sigma\omega=0$$

$$\left(\omega^T\Sigma\omega\right)\Sigma^{-1}\left(\overline{r}-\mathbf{1}r_f\right)=\left(\omega^T\overline{r}-\omega^T\mathbf{1}r_f\right)\omega \text{ (premultiply by } \Sigma^{-1}\text{)}$$

h. Now premultiply by 1^T to get:

$$\left(\omega^T\Sigma\omega\right)\mathbf{1}^{\mathbf{T}}\Sigma^{-1}\left(\overline{r}-\mathbf{1}r_f\right)=\left(\omega^T\overline{r}-\omega^T\mathbf{1}r_f\right)\underbrace{\mathbf{1}^{\mathbf{T}}\omega}_{=1}.$$

i. Therefore:

$$\frac{\left(\omega^T\overline{r}-\omega^T\mathbf{1}r_f\right)}{\left(\omega^T\Sigma\omega\right)}=\mathbf{1}^{\mathbf{T}}\Sigma^{-1}\left(\overline{r}-\mathbf{1}r_f\right) \text{ i.e., } \omega=\frac{\Sigma^{-1}\left(\overline{r}-\mathbf{1}r_f\right)}{\mathbf{1}^{\mathbf{T}}\Sigma^{-1}\left(\overline{r}-\mathbf{1}r_f\right)}$$

j. Which is the final solution (weights of M) since it satisfies the constraint.

k. The slope of the CML is then given by:

$$\frac{\omega^T\overline{r}-\omega^T\mathbf{1}r_f}{\sqrt{\omega^T\Sigma\omega}}=\frac{\left(\overline{r}-\mathbf{1}r_f\right)^T\Sigma^{-1}\left(\overline{r}-\mathbf{1}r_f\right)}{\sqrt{\left(\overline{r}-\mathbf{1}r_f\right)^T\Sigma^{-1}\left(\overline{r}-\mathbf{1}r_f\right)}}$$

$$=\sqrt{\left(\overline{r}-\mathbf{1}r_f\right)^T\Sigma^{-1}\left(\overline{r}-\mathbf{1}r_f\right)}$$

l. The slope and the intercept (r_f) fully define the CML.

VI. *Practical Implications of Portfolio Theory for Asset Allocation Strategies*

A. Core Asset Allocation

1. Cover basic needs

a. Investments in cash, home, insurance, annuities to protect basic standard of living

b. Should be safe and hedged, with enough liquidity (a home is not!).

c. The primary objective is to limit downside risk, and so you should be prepared to accept below-market returns.

d. Seek to keep up with inflation

   i. But for the basket of goods that you actually consume: you want to preserve your standard of living

   ii. Not the standard CPI!

   iii. There is a role for inflation-indexed bonds.

2. Access to market risk through standard asset classes

a. Domestic equities and bonds

   i. These are large, liquid, and heavily researched markets

   ii. Objective is to achieve the risk-adjusted return provided by the market

b. Include US Treasury inflation-protected bonds (TIPS).

c. These markets are largely efficient, so should focus on:

   i. Diversification

   ii. Exposure to broad segments of the market (for equities, rather than the S&P500, look at the Wilshire 5000)

   iii. Tax efficiency: for example the Russell 2000, which consists of stocks ranked by market capitalization, as defined once a year, has a high turnover and generates tax bills.

iv. Minimizing costs

d. Means: indexing!

e. If you really insist upon active management in the standard asset classes:

    i. Seek managers with proven bottom-up fundamental research capabilities.

    ii. Seek managers with a **value bias**: those looking for out-of-favor securities that are cheap in relation to current fundamental measures such as book value, earnings, or cash flow.

    iii. Emphasize small-capitalization stocks, as they are less efficiently priced and offer greater opportunities to add value through active management.

3. International asset classes

a. Less efficient than the corresponding domestic markets, especially in emerging markets

b. Unfortunately less diversifying now than commonly assumed on the basis of long historical data series

c. Try to exploit compelling undervaluations in countries, sectors, and styles but this is easier said than done.

4. "Personal beta"

a. Individual investors should think more broadly about the correlation between asset classes and their non-investment income, instead of just the correlation between their different investments

b. Some professions (civil servants, tenured professors, etc.) have incomes that fluctuate very little with the asset markets.

c. But some others (investment bankers, financial advisors, etc.) have a high "personal beta".

d. Their financial investment strategies such as their retirement accounts should reflect the sensitivity of their personal income to the asset markets

i. Highly asset-sensitive individuals (high beta) should use their retirement accounts as a hedge (and be conservative there)

ii. Whereas low beta individuals can afford to take on more risk in their financial portfolio since their human capital has a lower beta with respect to financial assets.

B. Alternative Asset Classes

1. Access to other sources of risk

a. Concentrated stock and executive stock options, business partnerships, etc.

i. These are inherently undiversified and illiquid

ii. It is usually not a good idea to add further to them, especially using leverage (buying your own company stock on margin?!?)

> **Chesapeake Energy CEO Aubrey McClendon sells bulk of his stock to meet margin calls**
>
> Last month, Forbes showed McClendon at No. 134 on its list of the nation's 400 richest people, with a net worth of $3 billion. "I am very disappointed to have been required to sell substantially all of my shares of Chesapeake," McClendon said in a news release. "I have been the company's largest individual shareholder for the past three years and frequently purchased additional shares of stock on margin as an expression of my complete confidence in the value of the company's strategy and assets." (AP 10/10/08)

b. Hedge funds, private equity, commodities, real assets, derivatives, investment real estate, REITs, distressed debt, etc.

i. These asset classes have higher return and higher risk than the standard asset classes

ii. And benefit from potentially lower correlation with standard asset classes, although the reality of this lower correlation is debatable

iii. Private equity investments such as venture capital and leveraged buyout partnerships have some of the most attractive long-term risk-adjusted returns.

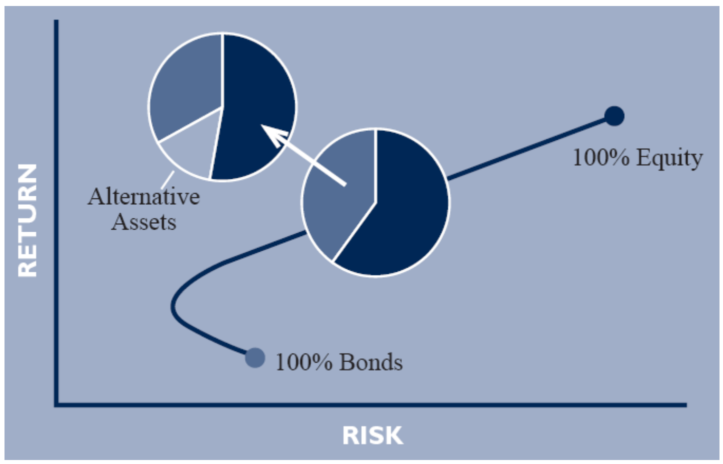
iv. Real estate and commodities (oil, gas, timber, etc.) provide diversification, are sensitive to inflation and as a result are a hedge against unanticipated inflation.

v. The illiquid nature of many of these assets combined with the expensive and time-consuming process of completing transactions creates a high cost of entry for casual investors – and opportunities for more sophisticated ones.

vi. If you are able to identify good managers (more on that later), active management makes some sense for alternative asset classes which are generally less efficient and accessible.

2. Objectives

a. Push the efficient frontier NW by adding asset classes



(Arrow Funds Ad)

b. Seek "absolute return" that is *less correlated* with the market risk of the standard asset classes.

c. The less correlated they are, the more the efficient frontier gets pushed in a good direction (NW).

d. Besides deep pockets, a long time horizon is needed to fully exploit these illiquid and less efficient markets.

3. Is the low correlation really there when you need it?

[The Morningstar 1000 Hedge Fund Index is a global index for hedge fund performance]

Third-Quarter 2008 Morningstar Hedge Fund Index Performance*	July-08	Aug-08	Sep-08	3Q 2008
Morningstar 1000 PR USD**	-2.88	-2.96	-7.87	-13.17
Morningstar HF of Funds PR USD	-2.96	-3.78	-7.68	-13.80
Hedge Fund Category Indexes				
Convertible Arbitrage PR USD	-1.57	-0.99	-12.39	-14.61
Corporate Actions PR USD	-3.60	-2.17	-10.60	-15.68
Debt Arbitrage PR USD	-0.31	-1.32	-4.39	-5.95
Developed Asia Equity	-3.00	-3.24	-8.41	-14.04
Distressed Securities PR USD	-1.53	-1.16	-6.21	-8.72
Emerging Market Equity PR USD	-4.55	-7.26	-13.07	-23.04
Equity Arbitrage PR USD	-1.90	-2.04	-4.60	-8.32
Europe Equity PR USD	-3.05	-3.37	-9.62	-15.33
Global Debt PR USD	-0.90	-2.38	-7.50	-10.51
Global Equity PR USD	-4.83	-4.13	-11.22	-19.00
Global Non Trend PR USD	-1.63	-1.76	-1.56	-4.88
Global Trend PR USD	-5.55	-5.20	-1.26	-11.59
Multistrategy PR USD	-2.79	-1.96	-9.84	-14.07
Short Equity PR USD	-0.20	-1.01	7.29	6.00
US Equity PR USD	-2.90	0.30	-10.25	-12.59
US Small-Cap Equity PR USD	-3.34	-2.58	-7.27	-12.67
Broad Hedge Fund Category Indexes				
Arbitrage PR USD	-1.27	-1.63	-6.15	-8.85
Equity PR USD	-3.81	-3.75	-10.56	-17.20
Event PR USD	-2.71	-1.77	-8.72	-12.77
Global Derivatives PR USD	-3.18	-3.06	-1.48	-7.54
Market Indexes				
MSCI World USD	-2.53	-1.60	-12.08	-15.67
S&P 500 TR	-0.84	1.45	-8.91	-8.37
Russell 2000 TR USD	3.70	3.61	-7.97	-1.11
MSCI Europe USD	-2.97	-4.20	-15.19	-21.16
MSCI AC Asia USD	-2.56	-5.41	-14.02	-20.75
MSCI EM USD	-4.16	-8.22	-17.71	-27.61
LB Global Agg TR USD	0.14	-1.65	-2.35	-3.83
LB U.S. Agg Bond TR USD	-0.08	0.95	-1.34	-0.49
LB Global High Yield TR USD	-1.14	-0.41	-8.26	-9.68
DJ AIG Commodity TR USD	-11.85	-7.28	-11.53	-27.70

4. And access to these asset classes comes at substantially higher cost.

    a. Many managers like having endowments and large institutional investors as clients (as a validation of their own quality)

    b. So they provide them with preferential terms, such as:

        i. Lower fees

        ii. Performance-related incentive fees that include downside provisions

        iii. Hurdle rates (a minimum rate of return that the fund must achieve before its general partners or managers receive an increased interest in the proceeds of the fund)

        iv. Clawback provisions (a clawback guarantees that limited partners, i.e., investors, will get their initial capital back before the general partners, i.e., the managers, get their share of the profits)

        v. Co-investment requirements (requiring managers to invest a significant portion of their net worth alongside you, to avoid the agency problem)

        vi. Access to timely information about the strategies followed by the managers, etc.

    c. Keep that in mind when you are looking at the big endowments' allocation to these asset classes and you attempt to mimic it

        i. The playing field is not level.

        ii. As a "regular" client, it is difficult to ensure that the interests of the managers are perfectly aligned with yours.

    d. Endowments have other advantages

        i. Extremely long horizons, measured in decades

        ii. Each endowment has a single client and can adjust its spending rule downwards if really needed

        iii. Unlike pension funds, they do not have to worry about matching assets with liabilities

        iv. This means endowments can tolerate lots of volatility, which in turn allows them to make, and stick to, contrarian bets: they can perhaps stay solvent longer than a segment of the market can stay irrational.

v. Size matters:

- For instance, Harvard Management has three professional lumberjacks on staff. They go and look at the trees.
- There are very few people who will show up at an auction in New Zealand to buy a $600 million piece of timber with cash (like they did).

vi. University endowments were among the first movers (in the late 80s) in this area:

- Yale Investments has been a pioneer.
- For example, Harvard Management was the first institutional client of the Kleiner Perkins fund.

e. Even with these advantages, many of these endowments suffered large losses in 2008-09, with declines between 20 and 30%.

5. Rebalance your portfolio

a. To reflect changes in expected returns and risk assessment

b. But also because rebalancing at regular intervals forces you to sell asset classes that have outperformed and purchase those that have underperformed.

Asset Class	June 2009 Actual	June 2009 Target
Absolute Return	24.3%	15.0%
Domestic Equity	7.5	7.5
Fixed Income	4.0	4.0
Foreign Equity	9.8	10.0
Private Equity	24.3	26.0
Real Assets	32.0	37.0
Cash	-1.9	0.5

(Source: Yale University 2009 Endowment Report)

c. It's a disciplined way to "sell high and buy low."

d. For example, rebalancing discipline would force an investor to go back into equities during the worst equity market falls (e.g., early 2009) when behavioral considerations make it unlikely that the investor would otherwise be willing to purchase stocks.

Yale Sticks With Investment Model

The endowment in fact increased its allocations to illiquid, or hard to sell, assets that caused some funds trouble when markets cratered, such as private equity and real estate, according to the report, which

(WSJ 3/19/10)

6. Want to have fun with your investments?

a. Open a separate “casino account” to implement your investment ideas.

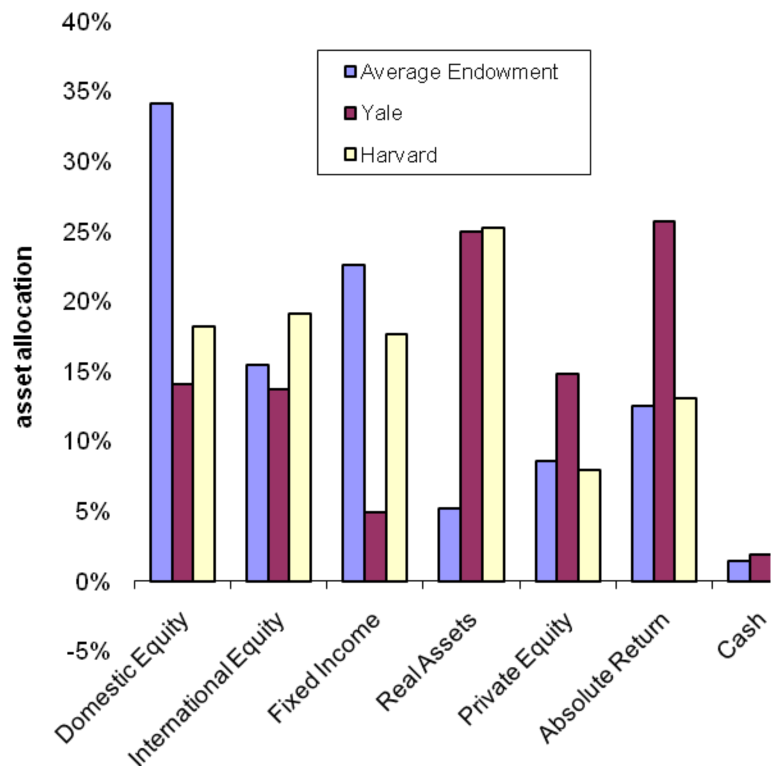
   i. Keep it out of your asset allocation process, view it as entertainment.

   ii. There is little point in attempting to time the market in your asset allocation. Leave that to your casino account!

b. Be wary of behavioral biases, so commit to investing only the amount in your casino account (no replenishing!)

C. Implementing Portfolio Theory

1. Start with best practices



a. Harvard and Yale are the two most successful endowments

i. On the face of it, they do things differently: Yale farms out most of its management, Harvard does most of it in-house; Harvard allocates significant amount to fixed income, Yale very little.

ii. But in fact they are quite similar: both are very patient investors, who stick with an asset allocation policy even if it doesn't work in the short term; they rebalance; they invest in illiquid asset classes, which is where bargains can be found; they are first movers in new asset classes; their asset allocation is driven by quantitative analysis, not by the seat of their pants.

iii. They were the pioneers in getting into real assets, hedge funds, etc., but this is a relatively recent development: in the 1980s, Harvard's endowment was invested in stocks, bonds and cash.

b. Of course, the endowments have specific advantages which make them less replicable by regular investors

i. Start by adjusting standard classes up and alternative classes down relative to the above to acknowledge the unique advantages of the endowments

ii. Do not set your expectations on the basis of how well the endowments do. Instead, think of the returns on the endowments as upper bounds to what you might achieve!

c. Insert constraints on portfolio weights in the optimizer

i. Want to avoid deviating too far from a sensible asset allocation, even if the assumed inputs take your portfolio optimizer there

ii. Because this may reflect unreasonable inputs (i.e., assumptions)

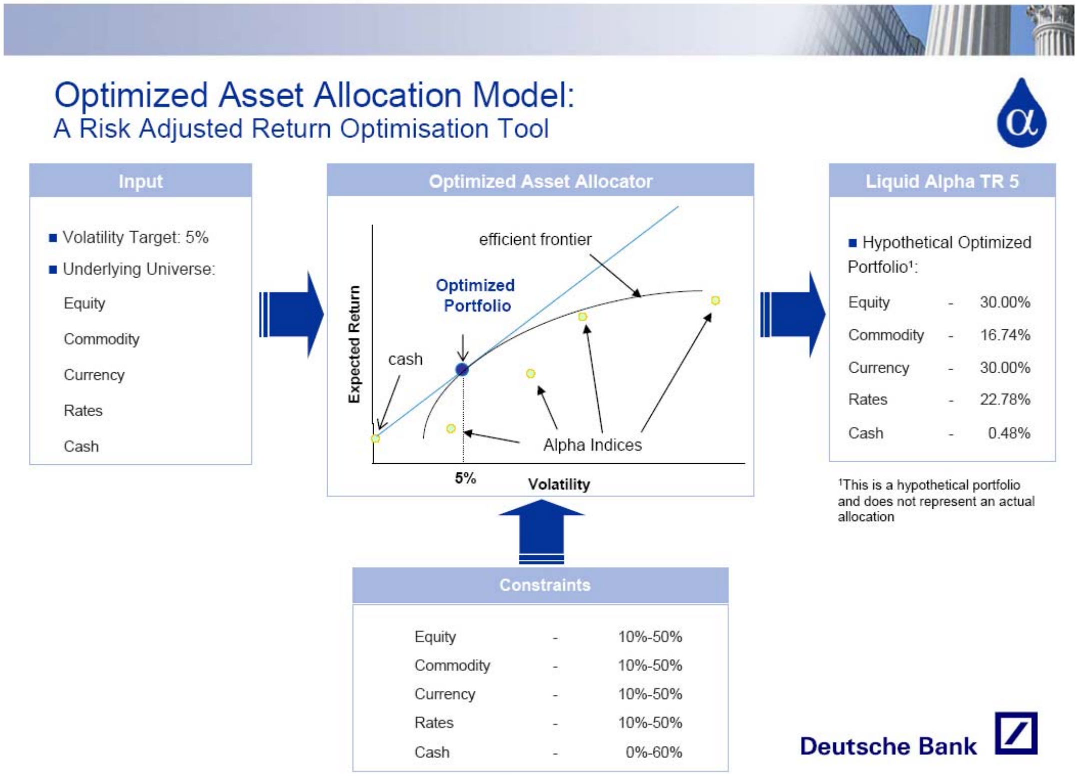
iii. Examples of reasonable constraints

- No shorting (each $\omega_i > 0$)
- Maintain sufficient diversification (for example each $\omega_i < 25\%$)

iv. Typical self-imposed :constraints

Policy Range Constraints	
Domestic Equity	10%–40%
International Equity	10%–35%
Emerging Markets	5%–15%
Marketable Alternatives	15%–25%
Non-Marketable Alternatives	5%–20%
Real Assets	5%–20%
Fixed Income	5%–35%

2. A typical example



3. Adjust your risk aversion to match your investment horizon

a. Different objectives have different horizons: college, retirement, philanthropy, passing wealth to the next generation, etc.

b. Longer horizons make you more able to stomach short-term volatility, afford the lack of liquidity and capture the higher returns that generally come with it.

c. You are more likely to get the average return of each asset class over long horizons than over shorter ones

   i. This is known as the law of large numbers

   ii. But you don't care about our average return over the last 20 years, you care about the total cumulated return you achieved

   iii. In other words, starting 20 years ago with $1, how much do I have now?

d. Keep in mind that diversification unfortunately does not work that well in times of crisis

Broad Equity Index Performance, Ranked by Year-to-Date Total Return						
		Through November 07, 2008				
		One	Year to	Annualized		
Asset Class	Representative Index	Month	Date	1-year	3-year	5-year
U.S. Small Cap Stocks	S&P SmallCap 600	-8.6%	-31.0%	-32.0%	-7.0%	1.5%
U.S. Value Stocks	Russell 3000 Value	-5.5%	-35.0%	-34.9%	-6.8%	1.0%
U.S. Large Cap Stocks	S&P 500	-5.3%	-35.4%	-35.5%	-6.8%	-0.6%
U.S. Mid Cap Stocks	S&P MidCap 400	-8.6%	-36.4%	-37.1%	-7.8%	0.5%
U.S. Growth Stocks	Russell 3000 Growth	-6.0%	-37.0%	-37.5%	-7.7%	-2.1%
U.S. Micro-Cap Stocks	Russell Micro Cap	-6.8%	-37.1%	-38.7%	-11.7%	-3.9%
Foreign Large Cap Stocks	MSCI EAFE	-6.9%	-43.4%	-45.5%	-5.5%	3.5%
Foreign Small Cap Stocks	MSCI EAFE Small Cap	-6.6%	-47.1%	-51.8%	-10.9%	2.2%
Emerging Markets Stocks	MSCI Emerging Markets	-6.6%	-53.6%	-55.6%	-1.7%	8.9%

4. Take all the asset classes together in a portfolio optimizer

a. Three groups: cash-like instruments, market classes, alternative classes

i. The objective of the first one is guarantee the investor's lifestyle

ii. The role of the second is to participate in the market in a cost-efficient manner

iii. The role of the third is to increase risk in order to grow one's wealth.

b. Need estimates of expected returns, standard deviations and correlations and throw them into an optimizer (e.g, spreadsheet-based)

c. For standard deviations and correlations, can use historical data on asset class returns (RiskMetrics, etc.)

d. For expected returns, there are three basic methods

i. Average performance of the different asset classes over the long run

ii. Fundamental analysis of the individual securities

iii. Quantitative methods: CAPM, APT; those are the methods we'll talk about now.