AI: DD2380 Adversarial search

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Introduction to Games

- Some problems are multi-agent
  - Cooperative vs competitive interaction
- Search problems in these environments are called games
- In competitive environments also adversarial search

Why study games?

- Precise rules
- Limited number of actions
- They are hard problems!
- Example: Chess
  - Average branch factor of 35
  - Typically 50 moves for each player
  - $\rightarrow 35^{100} \approx 10^{154}$ nodes in search tree

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- (10^{80} atoms in the observable universe)

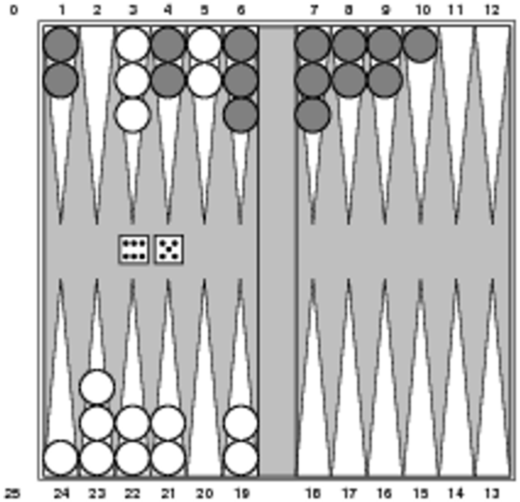
Types of games

	Deterministic	Chance
Perfect Information	Chess, Checkers go, othello	Backgammon monopoly
Imperfect Information	battleships	bridge,poker scrabble, paper-scissor-...

Perfect information = fully observable

Chance games: Backgammon

First game where a computer defeated a human world champion.

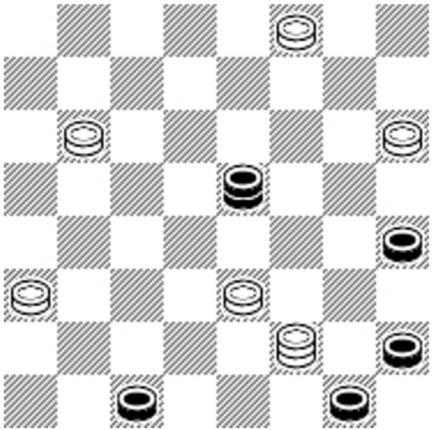


Deterministic games: Chess

- DeepBlue defeated human world champion Gary Kasparov in a six game match 1997.
- DeepBlue searched 200 million positions/s.
- Up to depth 40 in search.

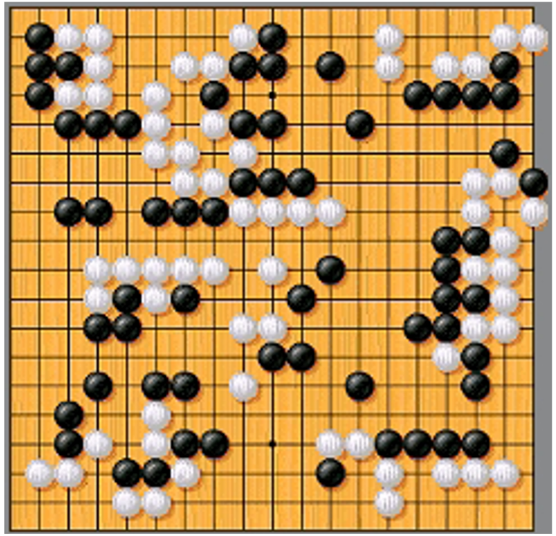
Deterministic games: Checkers

- “Solved” in 2007
- 18 years of computing on a cluster with 15+ machines
  → endgame table (<=10 pieces)
- → 39 trillion entries
- search+end game+computer → success



Deterministic games: Go

- Enormous branch factor (b>300)
- AlphaGo – a bit different principle



Today

Turn-taking, two-player, **zero-sum** games

Gain of winner same as loss of loser

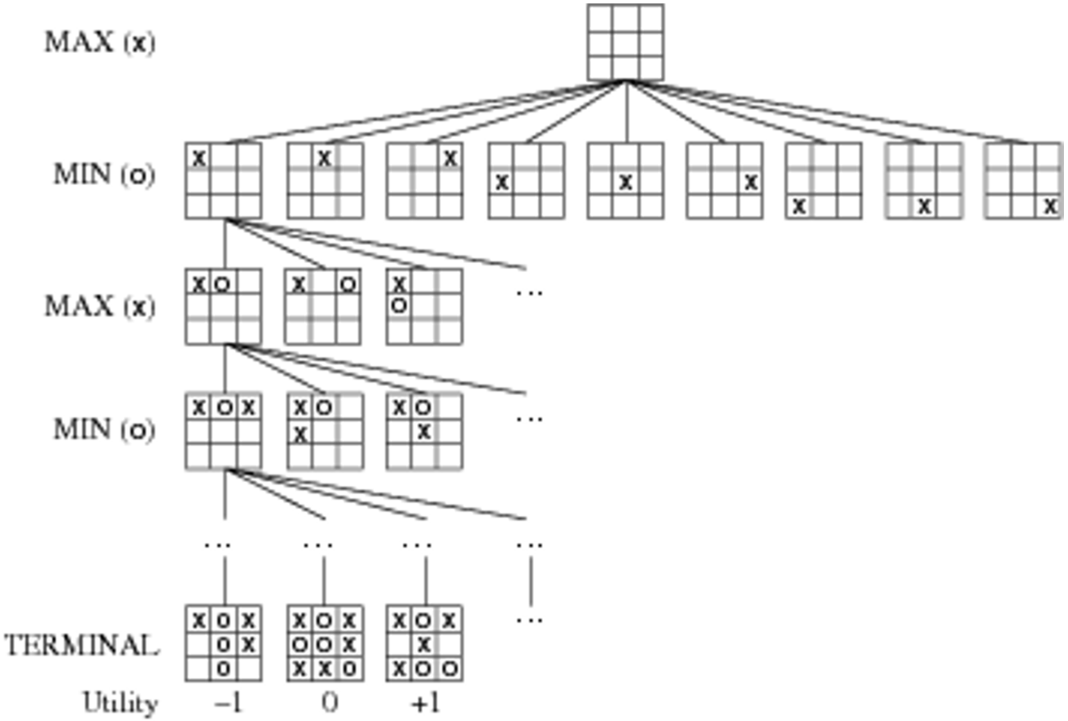
Formalizing the problem: MAX and MIN

- Two players MAX and MIN
- An initial state (the start situation)
- A successor function (move, state)
- A terminal state - when is the game over?
- A utility (objective, payoff) function gives value for terminal state

  typically (+1,-1,0)

- The progression can be modeled as a (game-) tree

Game tree
(2-player, deterministic, turn-based)

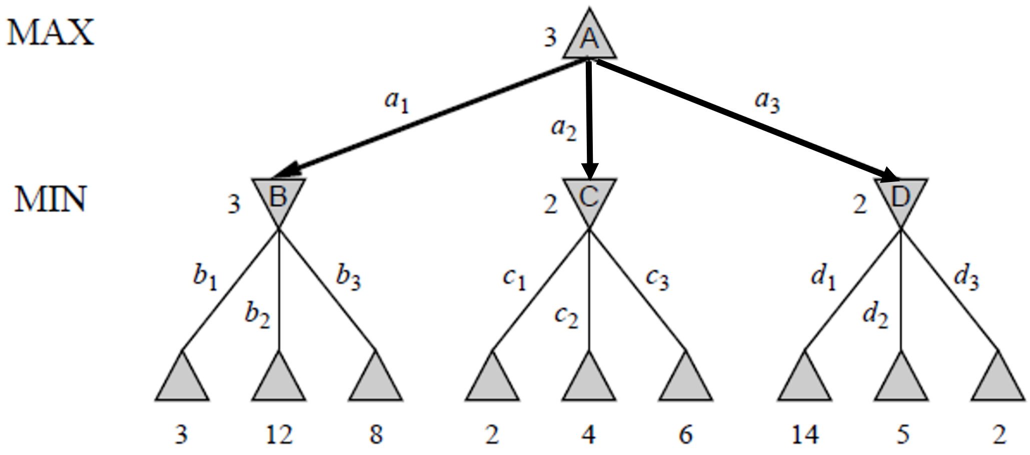


Optimal strategies

- In “normal search”: reach the goal configuration
- In adversarial search: not enough, MAX needs to reach the goal configuration regardless of what MIN does
- Optimal strategy: at least as good as any other strategy when playing against an infallible MIN

Q

- Consider a 2-ply (half-move) game tree
  - MAX has actions a_1,a_2,a_3 and MIN has actions $b_1,b_2,b_3,$ $c_1,c_2,c_3,$ d_1,d_2,d_3



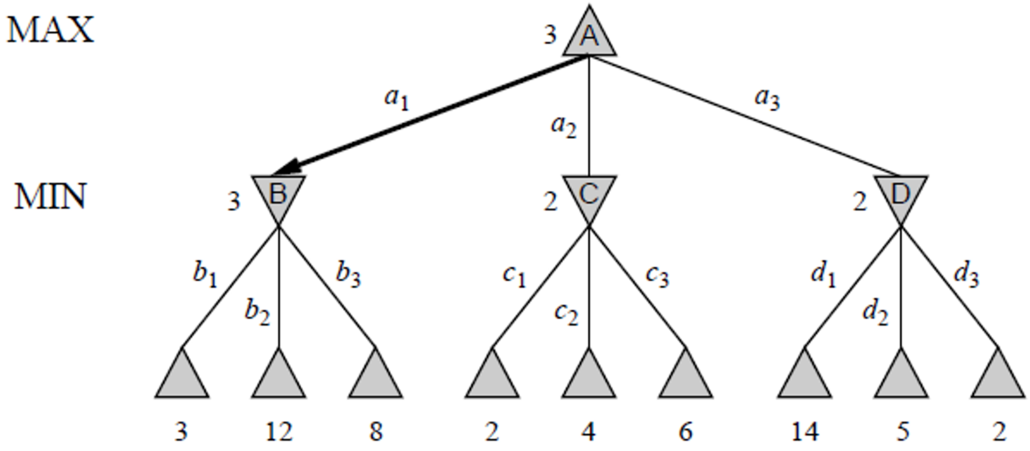
- What is the best strategy for MAX?

The minimax algorithm

- Assumes that the agents are rational, self-interested
- The strategy of best achievable payoff against best play by opponent
- $$\text{MINIMAX}(s) = \begin{cases} \text{UTILITY}(s) & \text{if TERMINAL-TEST}(s) \\ \max_{a \in Actions(s)} \text{MINIMAX}(\text{RESULT}(s, a)) & \text{if PLAYER}(s) = \text{MAX} \\ \min_{a \in Actions(s)} \text{MINIMAX}(\text{RESULT}(s, a)) & \text{if PLAYER}(s) = \text{MIN} \end{cases}$$

The best strategy

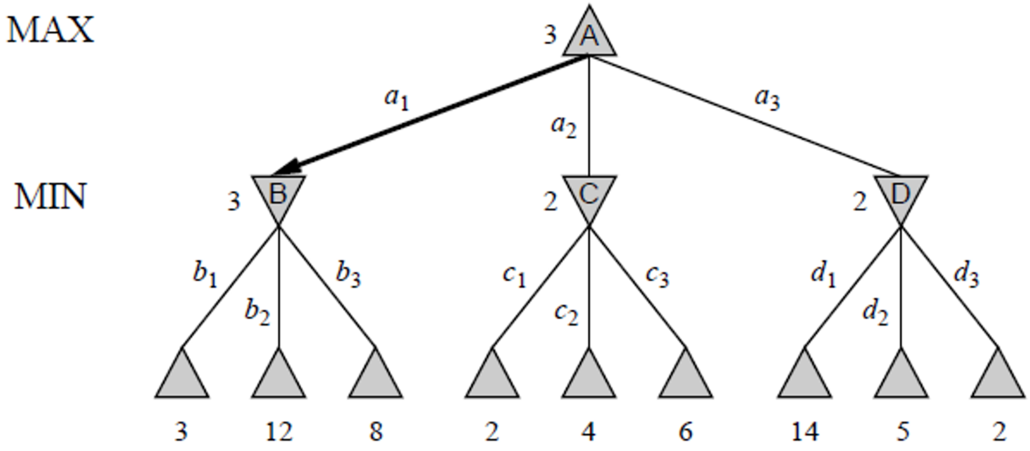
- If MAX picks a_1 MIN will pick $b_1 \rightarrow 3$, if MAX picks a_2 MIN will pick $c_1 \rightarrow 2$, a_3 will give $d_3 \rightarrow 2$



Best to pick a_1 as it gives utility 3

The best strategy

- Q: What if MIN is not infallible, is it still the best strategy?



Recursion

- Make sure you know how to work with recursive functions

Minimax algorithm

function MINIMAX-DECISION(*state*) **returns** *an action*
  return $\arg\max_{a \in \text{ACTIONS}(s)}$ MIN-VALUE(RESULT(*state*, *a*))

---

function MAX-VALUE(*state*) **returns** *a utility value*
  if TERMINAL-TEST(*state*) **then return** UTILITY(*state*)
  $v \leftarrow -\infty$
  for each *a* **in** ACTIONS(*state*) **do**
    $v \leftarrow$ MAX(v, MIN-VALUE(RESULT(s, a)))
  return v

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function MIN-VALUE(*state*) **returns** *a utility value*
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Properties of minimax

- Complexity $O(b^m)$

Optimal strategies

- In “normal search”: reach the goal configuration
- In a game (adversarial search): not enough, MAX needs to reach the goal configuration regardless of what MIN does
- Optimal strategy: at least as good as any other strategy when playing against an infallible MIN
- **Analysis of the full game tree can be demanding or impossible**

Repeated states

- Transposition causes repeated states (among others)
  - Many different moves result in same configuration
  - Example: chess
- Store evaluation in transposition (hash) table

Handling resource limitations

- Do not use real utility but a **heuristic** evaluation function that estimates the value of a particular strategy/position
  - Should order the terminal states the same as the utility function
  - Should be fast to compute
  - Should reflect chances to win in every state
    - > *Chess: number of white pawns, black pawns, white queens, black queens, etc.*

Handling resource limitations

- Cuttoff test instead of terminal test
  - Limited depth search
  - Iterative deepening

α-β pruning

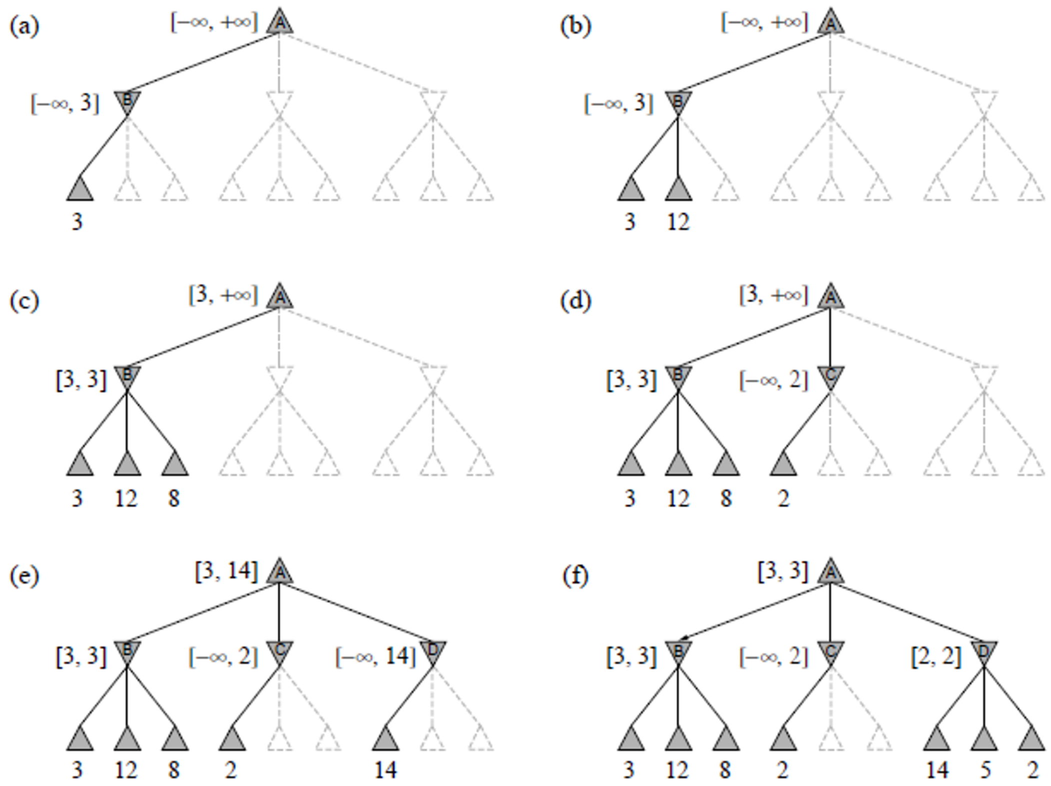
- Do we really need to analyze all nodes?
- Stop investigate unrealistic plays as early as possible, i.e. prune the tree
- Cuts down computations, but still gives the same Minimax result

α-β pruning

- α: the value of the best choice for MAX
- β: the value of the best choice for MIN

- found along the path till now

Re-analyze the 2-ply game tree



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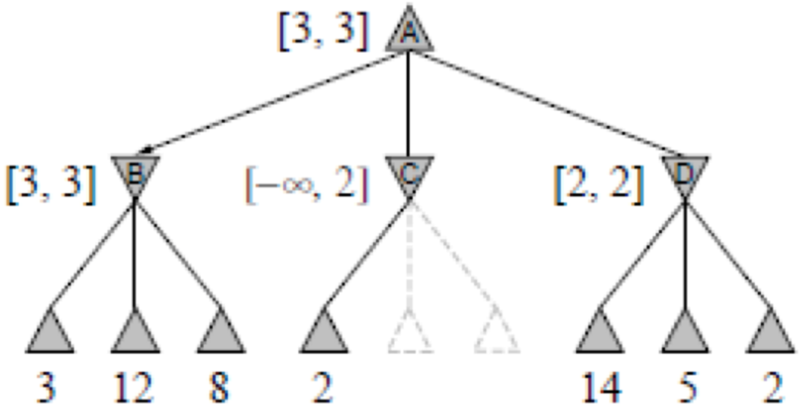

$$
\begin{aligned}
\text{MINIMAX}(root) &= \max(\min(3,12,8), \min(2,x,y), \min(14,5,2)) \\
&= \max(3, \min(2,x,y), 2) \\
&= \max(3, z, 2) \qquad \text{where } z = \min(2,x,y) \leq 2 \\
&= 3.
\end{aligned}
$$

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function MAX-VALUE(state) returns a utility value
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α-β pruning algorithm

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function ALPHA-BETA-SEARCH(state) returns an action
   v ← MAX-VALUE(state, −∞, +∞)
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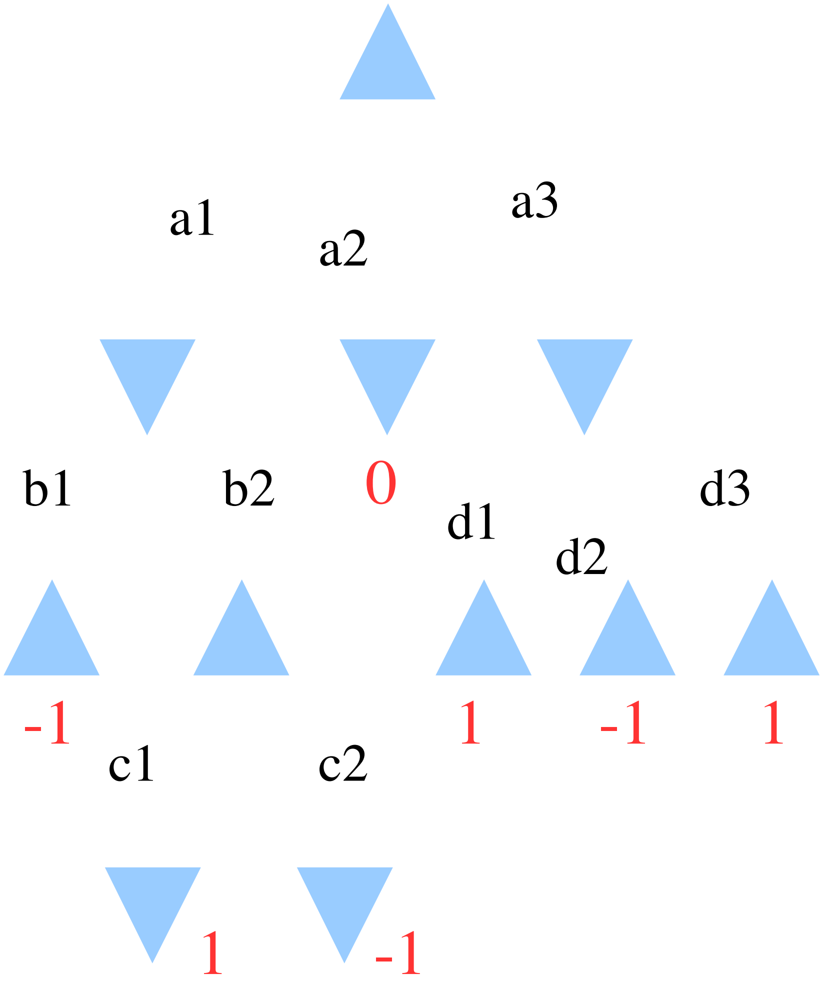

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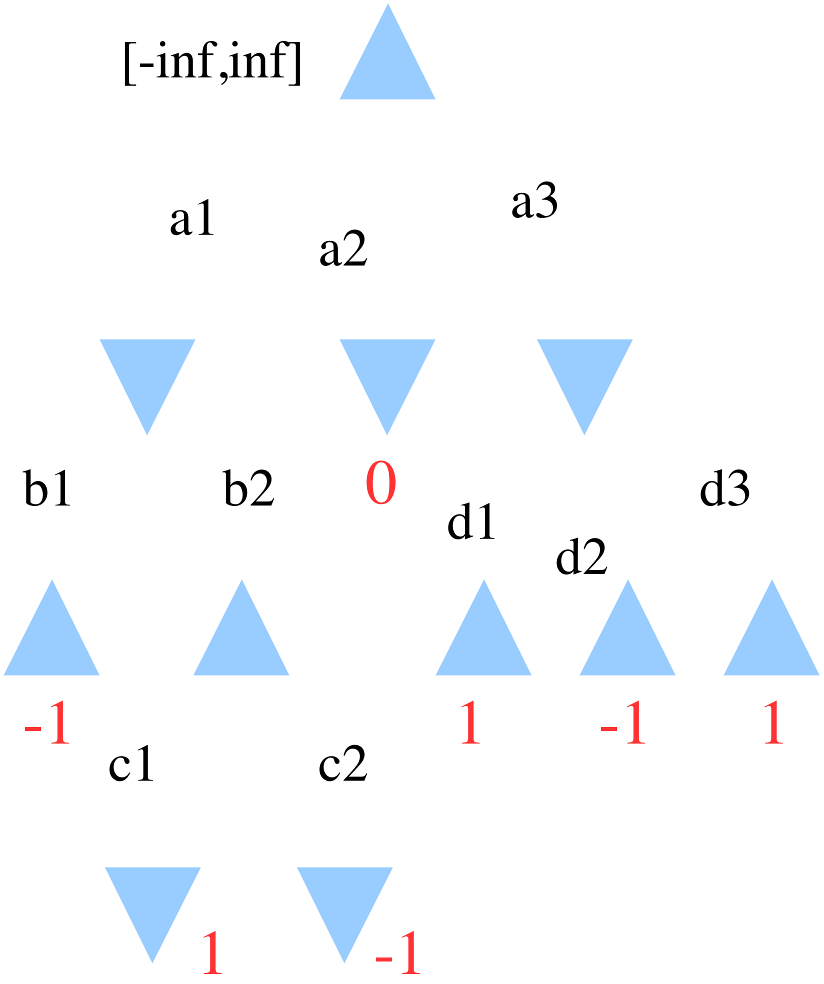

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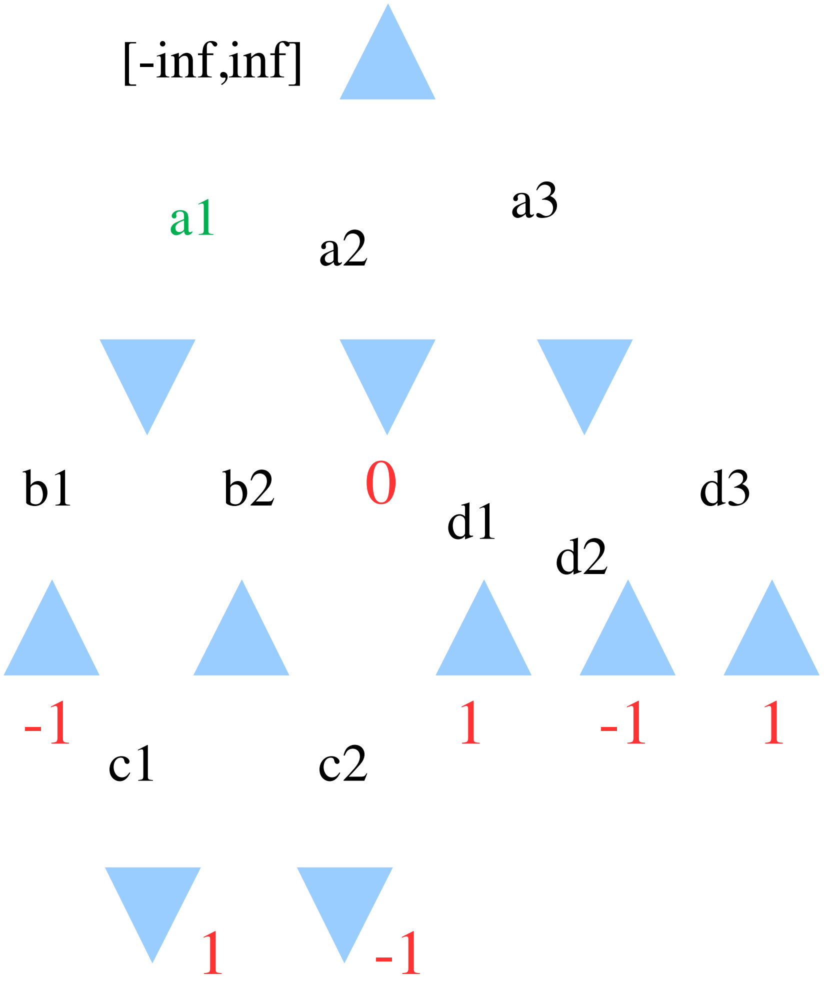

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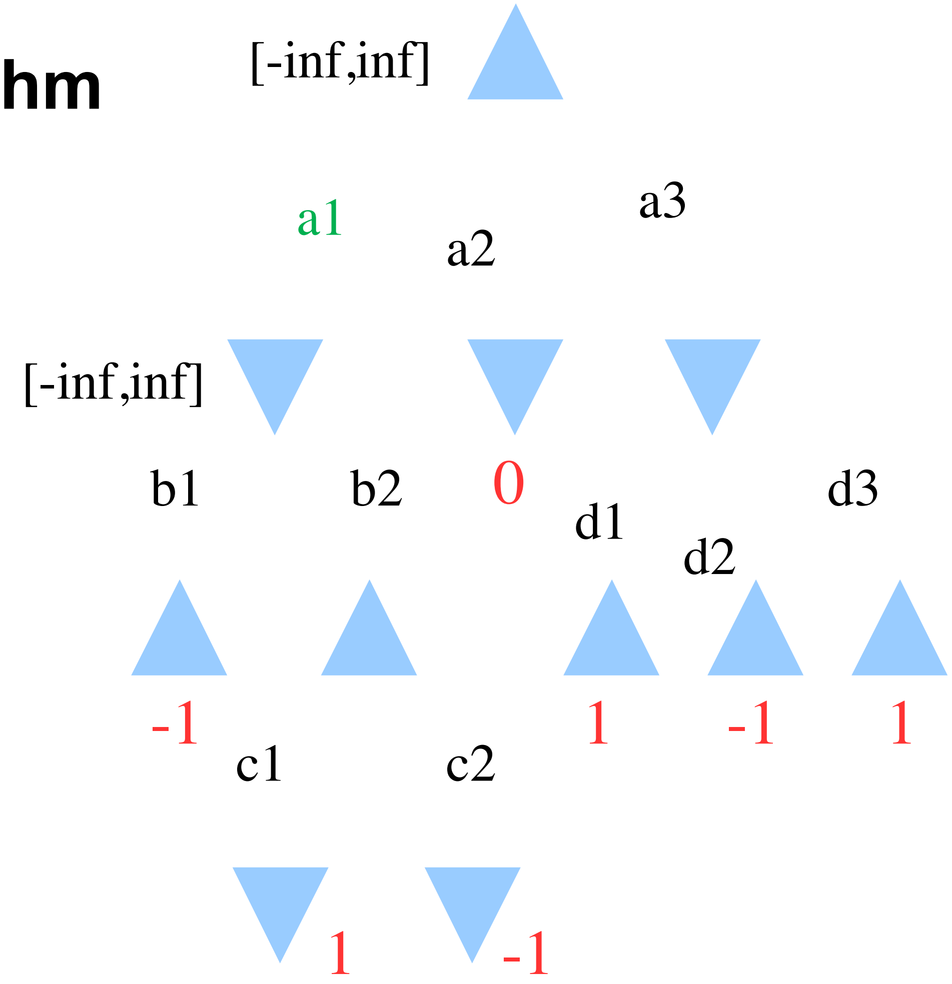

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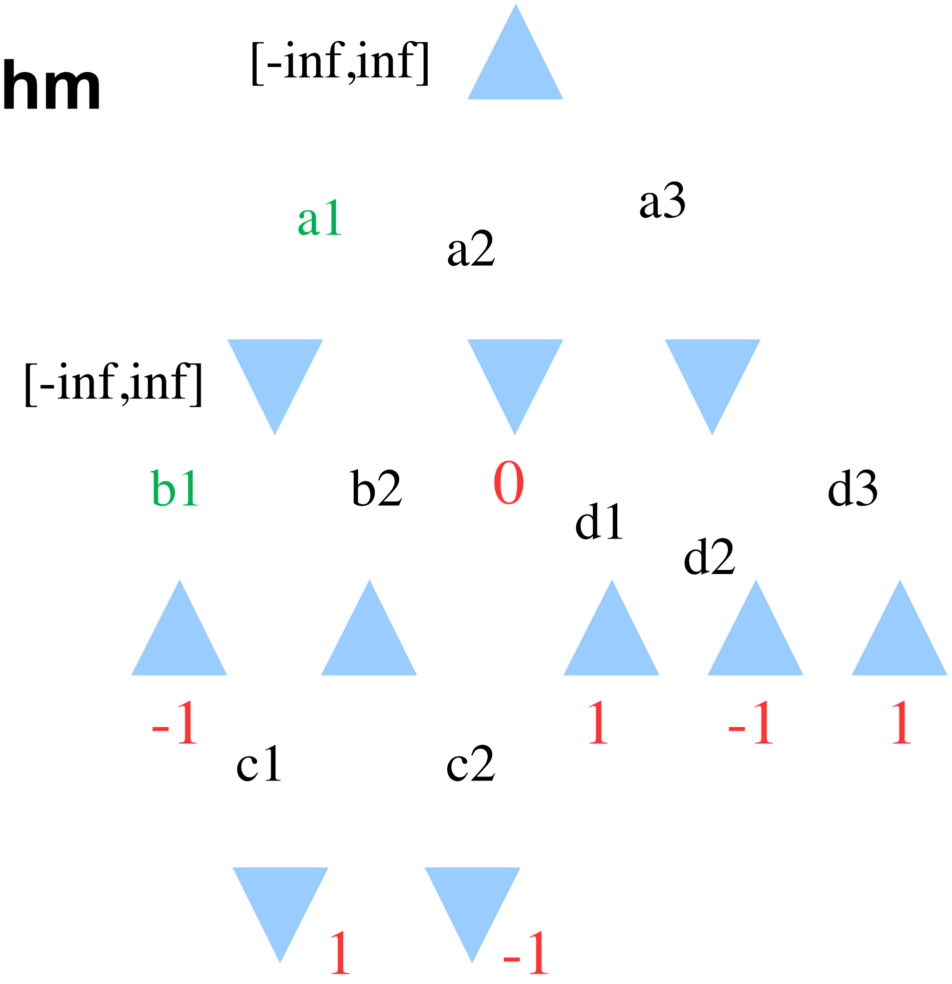

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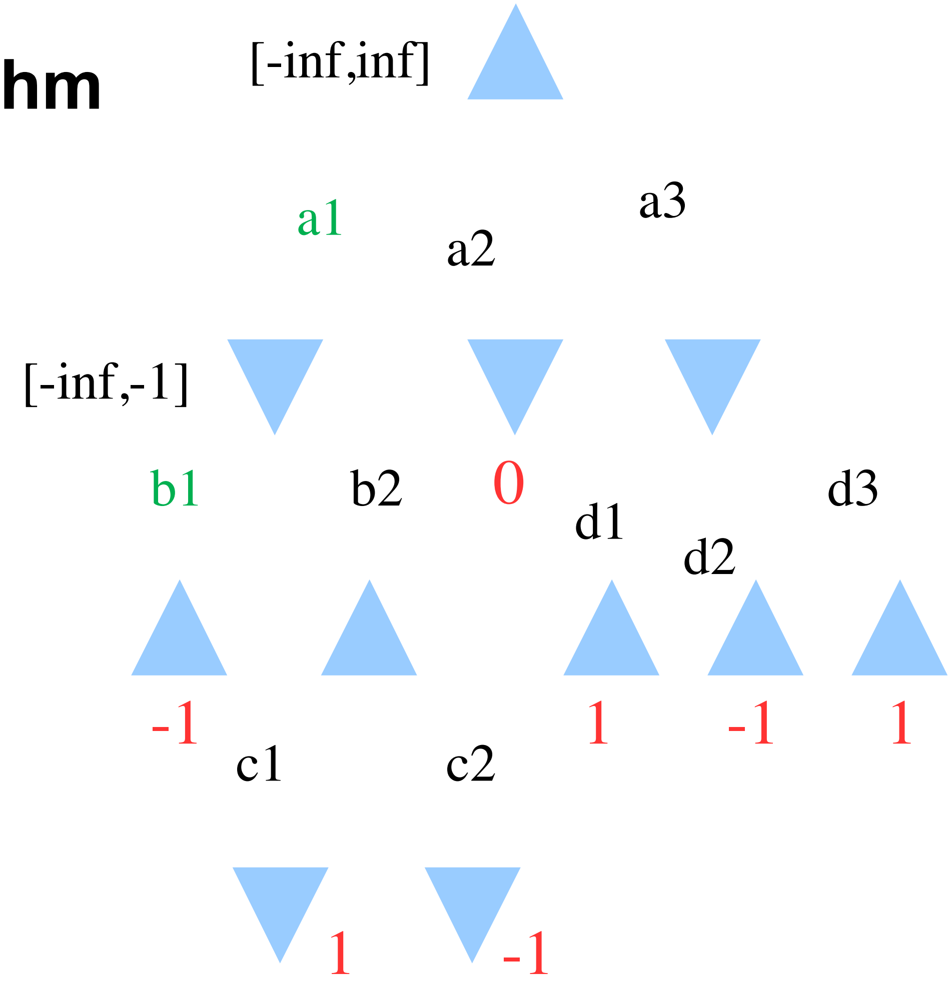

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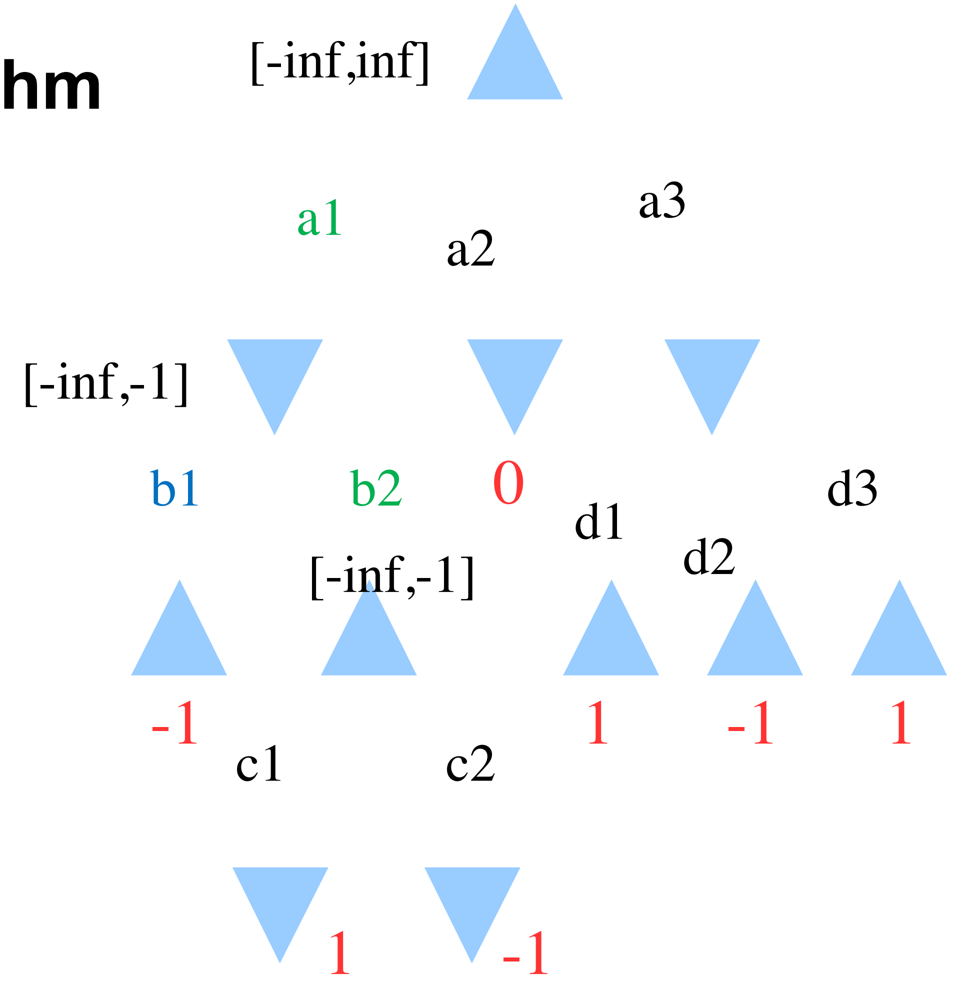

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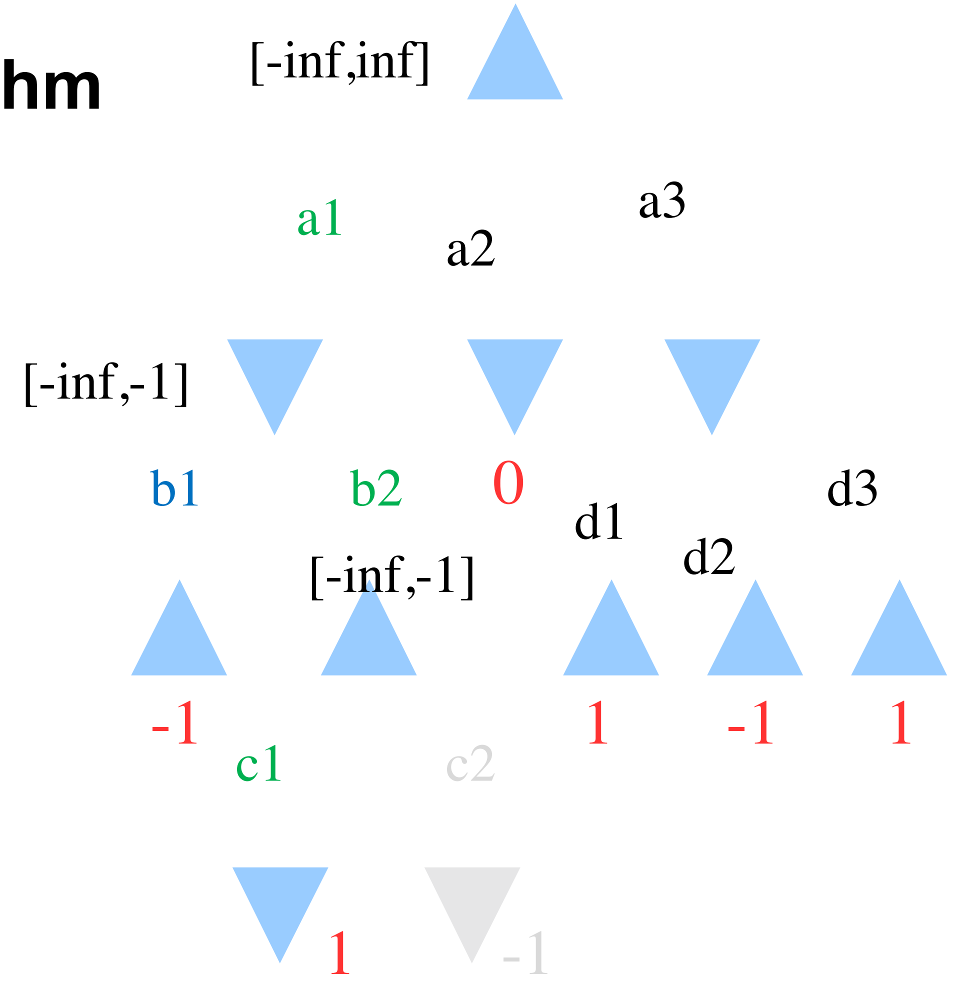

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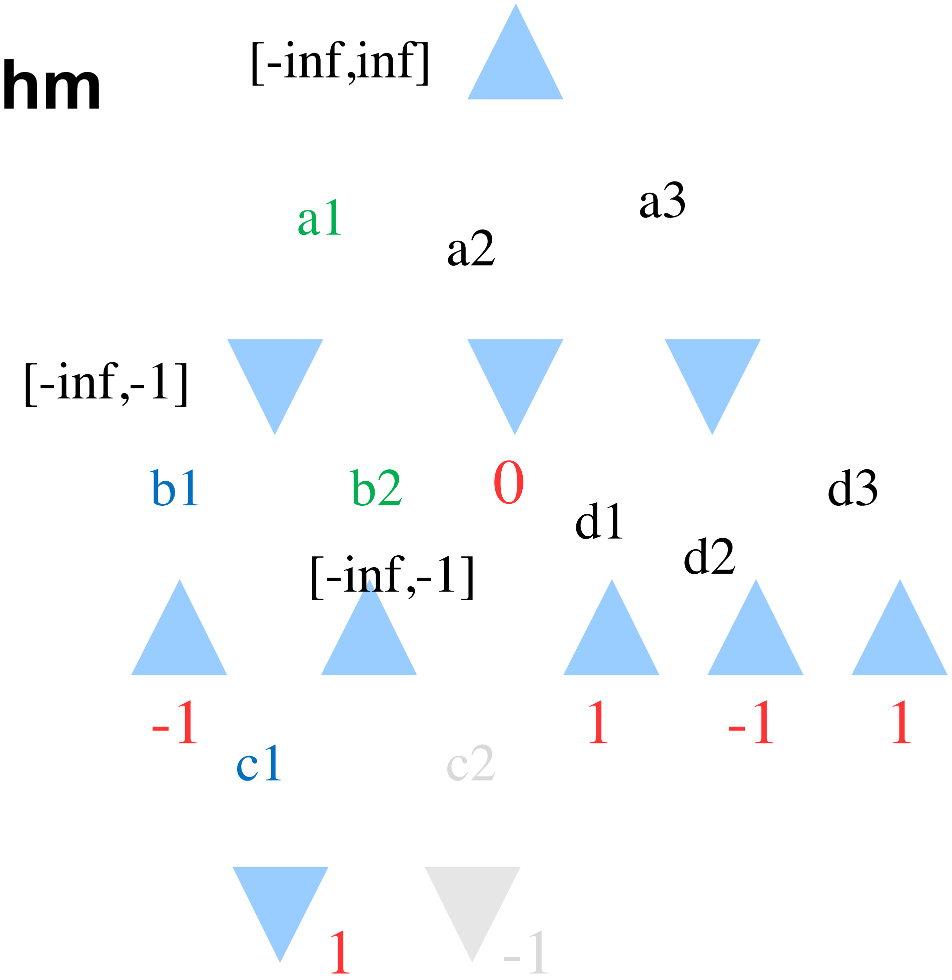

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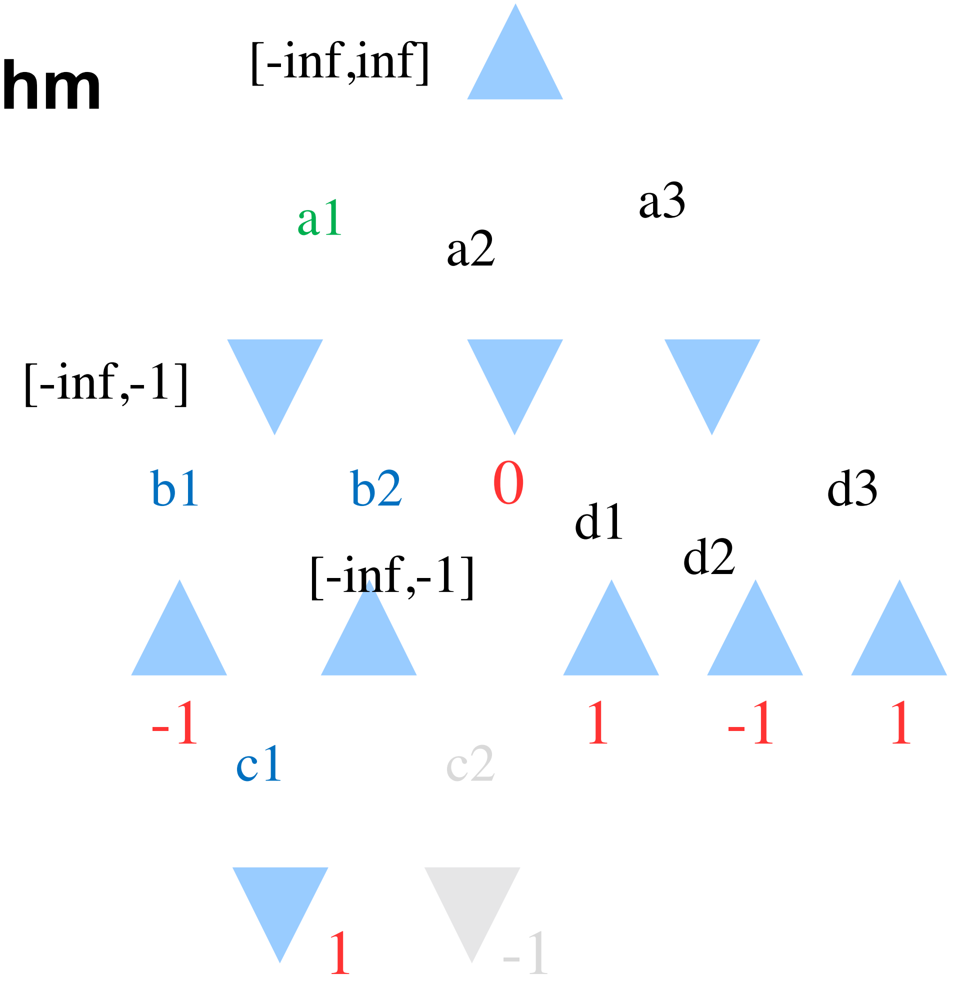

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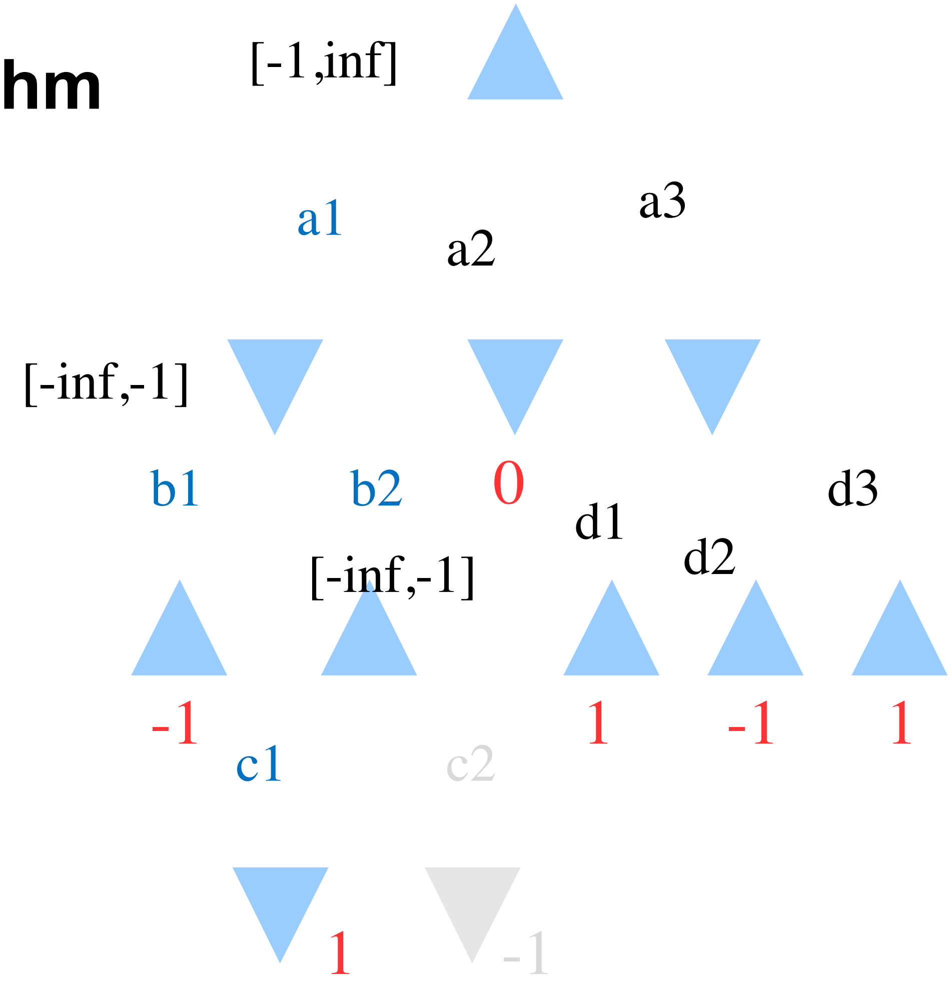

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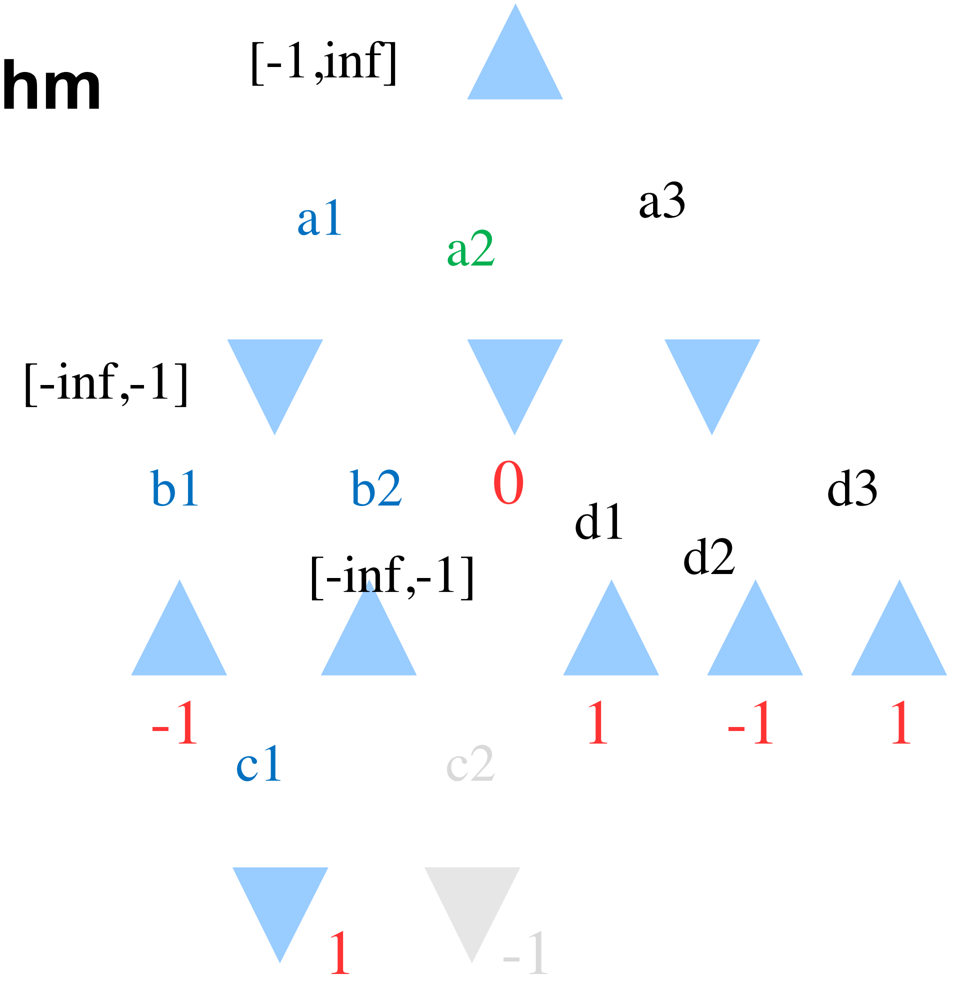
function MIN-VALUE(state, α, β) returns a utility value
  if TERMINAL-TEST(state) then return UTILITY(state)
  v ← +∞
  for each a in ACTIONS(state) do
    v ← MIN(v, MAX-VALUE(RESULT(s,a) , α, β))
    if v ≤ α then return v
    β ← MIN(β, v)
  return v
```



α-β pruning algorithm

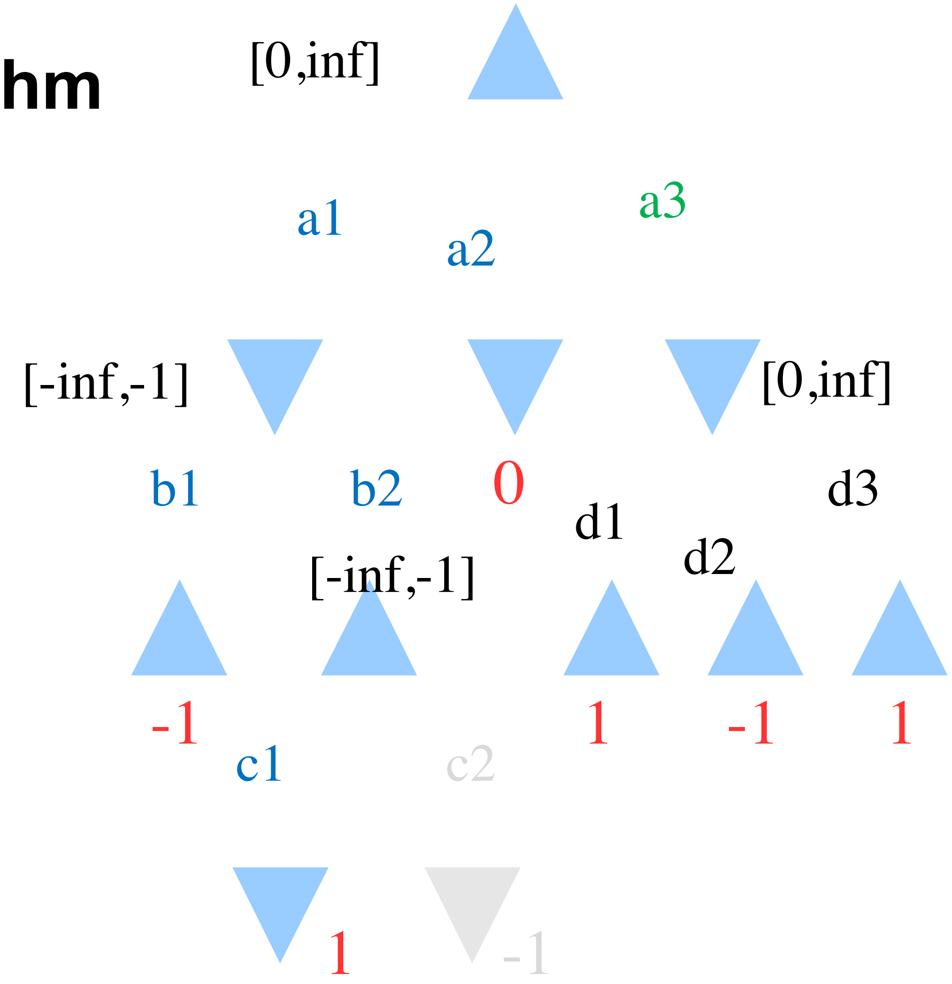
function ALPHA-BETA-SEARCH(*state*) **returns** an action
  $v \leftarrow$ MAX-VALUE(*state*, $-\infty, +\infty$)
  return the *action* in ACTIONS(*state*) with value v

---

function MAX-VALUE(*state*, α, β) **returns** *a utility value*
  if TERMINAL-TEST(*state*) **then return** UTILITY(*state*)
  $v \leftarrow -\infty$
  for each a **in** ACTIONS(*state*) **do**
    $v \leftarrow$ MAX(v, MIN-VALUE(RESULT(s,a), α, β))
    if $v \geq \beta$ **then return** v
    $\alpha \leftarrow$ MAX(α, v)
  return v

---

function MIN-VALUE(*state*, α, β) **returns** *a utility value*
  if TERMINAL-TEST(*state*) **then return** UTILITY(*state*)
  $v \leftarrow +\infty$
  for each a **in** ACTIONS(*state*) **do**
    $v \leftarrow$ MIN(v, MAX-VALUE(RESULT(s,a) , α, β))
    if $v \leq \alpha$ **then return** v
    $\beta \leftarrow$ MIN(β, v)
  return v



α-β pruning algorithm

function ALPHA-BETA-SEARCH($state$) **returns** an action
  $v \leftarrow$ MAX-VALUE($state, -\infty, +\infty$)
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function MAX-VALUE($state, \alpha, \beta$) **returns** *a utility value*
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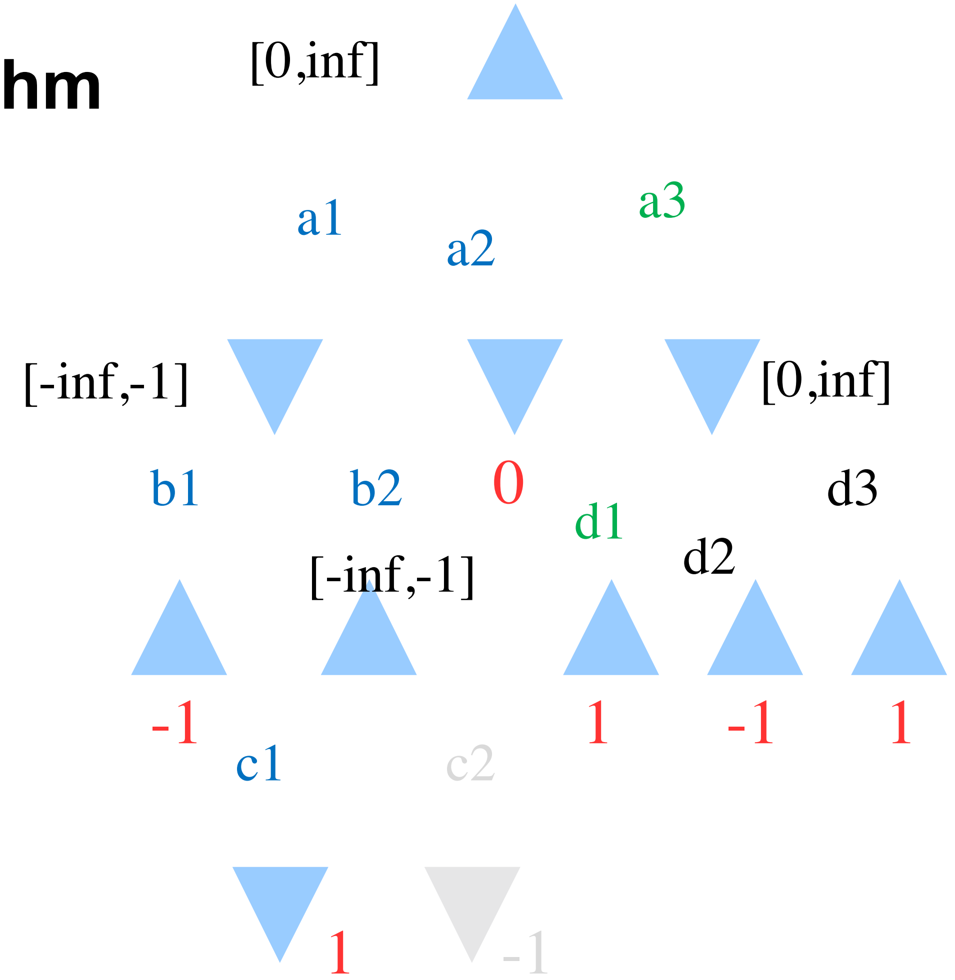

α-β pruning algorithm

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function ALPHA-BETA-SEARCH(state) returns an action
   v ← MAX-VALUE(state, −∞, +∞)
   return the action in ACTIONS(state) with value v

function MAX-VALUE(state, α, β) returns a utility value
   if TERMINAL-TEST(state) then return UTILITY(state)
   v ← −∞
   for each a in ACTIONS(state) do
      v ← MAX(v, MIN-VALUE(RESULT(s,a), α, β))
      if v ≥ β then return v
      α ← MAX(α, v)
   return v

function MIN-VALUE(state, α, β) returns a utility value
   if TERMINAL-TEST(state) then return UTILITY(state)
   v ← +∞
   for each a in ACTIONS(state) do
      v ← MIN(v, MAX-VALUE(RESULT(s,a) , α, β))
      if v ≤ α then return v
      β ← MIN(β, v)
   return v
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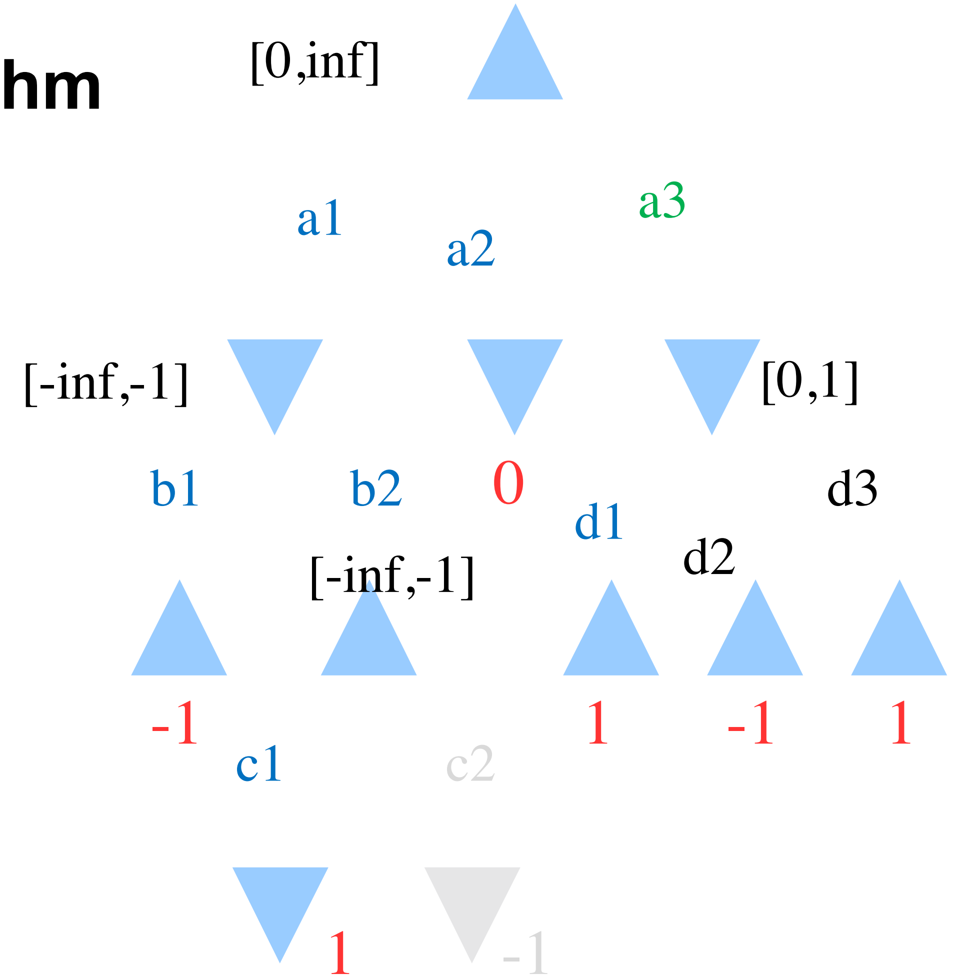

α-β pruning algorithm

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      if v ≥ β then return v
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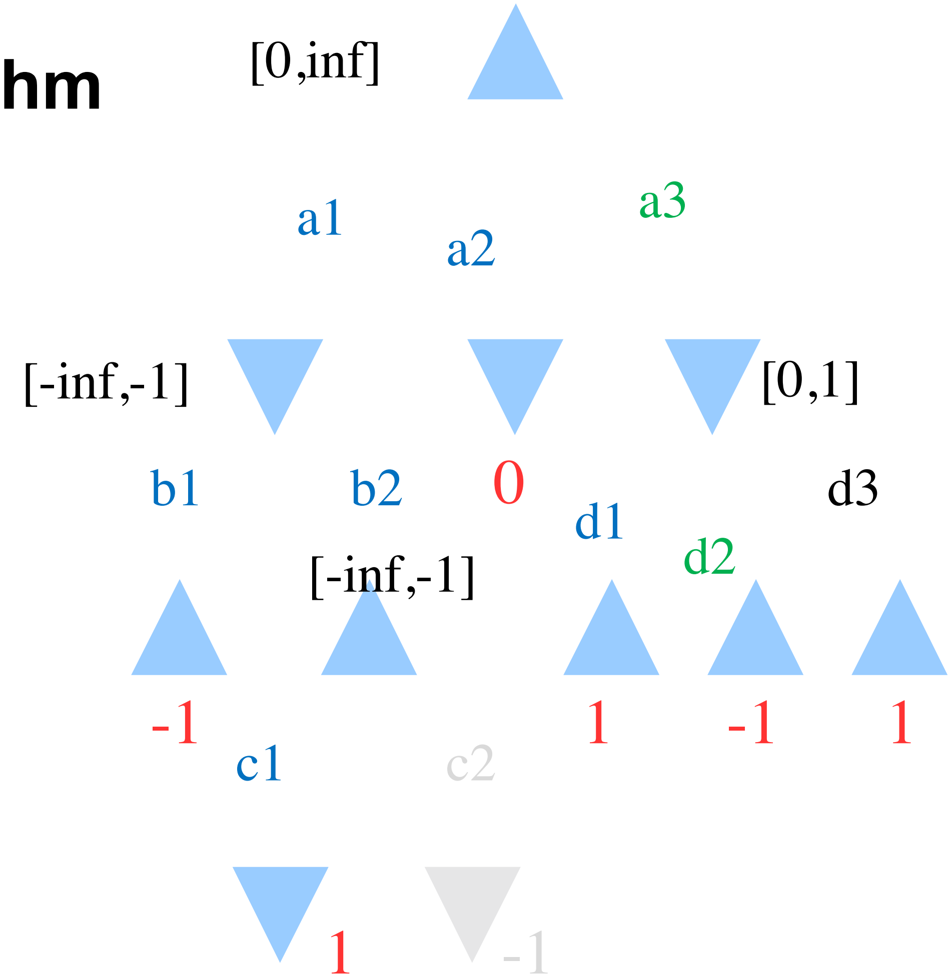

α-β pruning algorithm

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   v ← MAX-VALUE(state, −∞, +∞)
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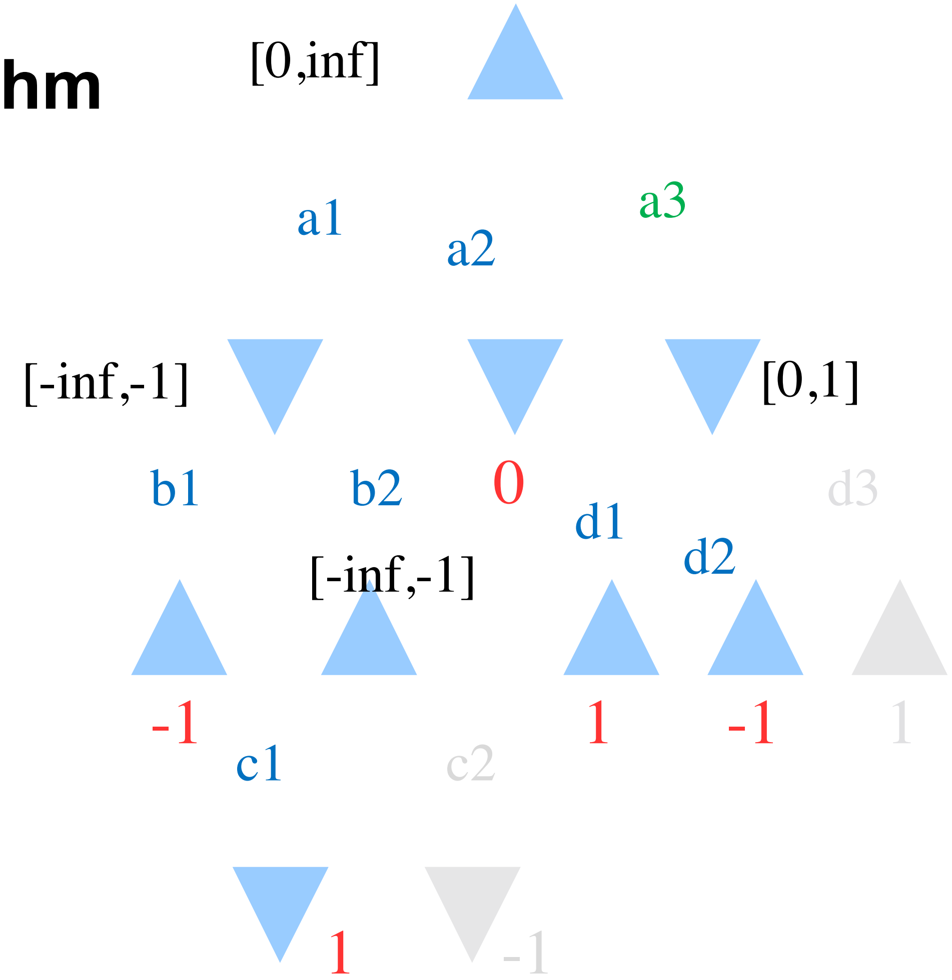
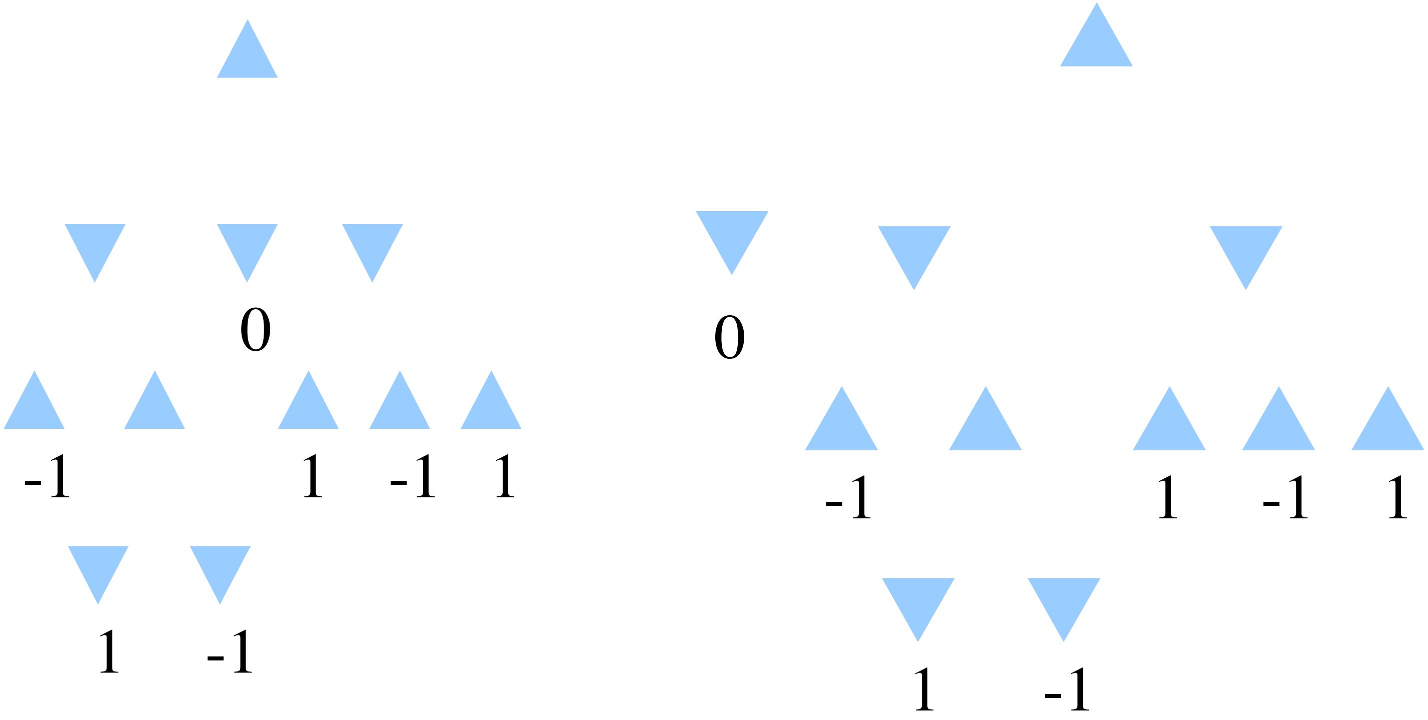


α-β pruning algorithm: order matters



Properties of α-β pruning

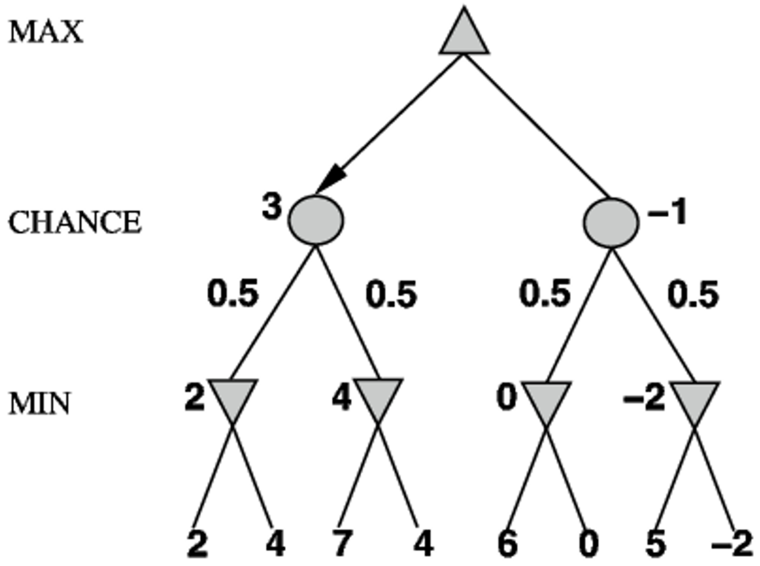
- Complexity - depends heavily on order of evaluation.
- With perfect ordering $O(b^{m/2})$
- → Can then look twice as far into the game as minimax in the same time

Reordering in α-β pruning

- Try likely the best first
- Chess: captures, threats, forward moves, backward moves
- Estimate through limited DFS
- Repeat killer moves

Non-deterministic games

- In nondeterministic games an element of chance is introduced
  - e.g. a coin-flip
- Expectiminimax



Games with imperfect information

- E.g., card games, where opponent's initial cards are unknown
- Typically one can compute a probability for each possible deal
- Idea
  - compute the minimax value of each action in each deal
  - choose the action with highest expected value over all deals

Summary adversarial search

- Competitive environment
- Four components: initial state, actions, terminal test, and utility function
- Perfect information allow use of minimax
  → pruning to optimize search
- Use of evaluation functions to allow depth limited search
- With an element of chance the expected value can be used similar to minimax (expectimax)
- Today computers are as good or better than humans at many different games