

Lista de exercícios 01 - PCC116

RODRIGO SENNA DA SILVA

Maio 2025

1 Item 1.5.7 - Prove os seguintes sequentes usando dedução natural. Tente demonstrá-los sem utilizar a regra RAA.

(a)

$$\{(P \wedge Q) \wedge R, S \wedge T\} \vdash Q \wedge S$$

$$\frac{\frac{\frac{\overline{(P \wedge Q) \wedge R} \{ID\}}{P \wedge Q} \{\wedge EE\}}{Q} \{\wedge ED\} \quad \frac{\frac{\overline{S \wedge T} \{ID\}}{S} \{\wedge EE\}}{Q \wedge S} \{\wedge I\}}$$

(b)

$$\{(P \wedge Q) \wedge R\} \vdash (P \wedge R) \vee Z$$

$$\frac{\frac{\frac{\overline{(P \wedge Q) \wedge R} \{ID\}}{P \wedge Q} \{\wedge EE\}}{P} \{\wedge EE\} \quad \frac{\frac{\overline{(P \wedge Q) \wedge R} \{ID\}}{R} \{\wedge ED\}}{P \wedge R^1} \{\wedge I\}}{(P \wedge R) \vee Z} \{\vee IE\}^1$$

(c)

$$\{P, P \rightarrow (P \rightarrow Q)\} \vdash Q$$

$$\frac{\frac{\overline{P \rightarrow (P \rightarrow Q)} \{ID\}}{P \rightarrow Q} \{\rightarrow E\} \quad \frac{\overline{P} \{ID\}}{Q} \{\rightarrow E\}}{Q} \{\rightarrow E\}$$

(d)

$$\vdash (P \wedge Q) \rightarrow P$$

$$\frac{\frac{\overline{(P \wedge Q)^1} \{ID\}}{P} \{\wedge EE\}}{(P \wedge Q) \rightarrow P} \{\rightarrow I\}^1$$

(e)

$$\{P\} \vdash Q \rightarrow (P \wedge Q)$$

$$\frac{\frac{\overline{P} \{ID\} \quad \overline{Q^1} \{ID\}}{P \wedge Q} \{\wedge_I\}}{Q \rightarrow (P \wedge Q)} \{\rightarrow_I\}^1$$

(f)

$$\{P\} \vdash (P \rightarrow Q) \rightarrow Q$$

$$\frac{\frac{\overline{P \rightarrow Q^1} \{ID\} \quad \overline{P} \{ID\}}{Q} \{\rightarrow_E\}}{(P \rightarrow Q) \rightarrow Q} \{\rightarrow_I\}^1$$

(g)

$$\vdash (P \wedge Q) \rightarrow (P \vee Q)$$

$$\frac{\frac{\overline{P \wedge Q^1} \{ID\}}{P \vee Q} \{\vee_{IE}\} \quad \frac{\overline{P \wedge Q^1} \{ID\}}{P \vee Q} \{\vee_{ID}\}}{(P \wedge Q) \rightarrow (P \vee Q)} \{\rightarrow_I\}^1$$

(h)

$$\{Q \rightarrow (P \rightarrow R), \neg R, Q\} \vdash \neg P$$

$$\frac{\frac{\overline{Q \rightarrow (P \rightarrow R)} \{ID\} \quad \overline{Q} \{ID\}}{P \rightarrow R} \{\rightarrow_E\} \quad \overline{\neg R} \{ID\}}{\neg P} \{\rightarrow_E\}$$

(i)

$$\{P\} \vdash Q \rightarrow (P \wedge Q)$$

$$\frac{\frac{P \quad Q^1}{P \wedge Q} \{\wedge_1\}}{Q \rightarrow (P \wedge Q)} \{\rightarrow_I\}^1$$

(j)

$$\{(P \rightarrow R) \wedge (Q \rightarrow R), P \wedge Q\} \vdash Q \wedge R$$

$$\frac{\frac{\overline{(P \rightarrow R) \wedge (Q \rightarrow R)} \{ID\}}{Q \rightarrow R} \{\wedge_{ED}\} \quad \frac{\overline{P \wedge Q} \{ID\}}{Q} \{\wedge_{ED}\} \quad \frac{\overline{P \wedge Q} \{ID\}}{Q} \{\wedge_{ED}\}}{\frac{R \quad Q}{Q \wedge R} \{\wedge_I\}}$$

(k)

$$\{P \rightarrow Q \rightarrow R, P \rightarrow Q\} \vdash P \rightarrow R$$

$$\frac{\frac{\overline{P \rightarrow Q \rightarrow R} \{ID\}}{Q \rightarrow R} \quad \frac{\overline{P^1} \{ID\}}{\{\rightarrow_E\}^1} \quad \frac{\overline{P \rightarrow Q} \{ID\}}{Q} \quad \frac{\overline{P^1} \{ID\}}{\{\rightarrow_E\}}}{P \rightarrow R} \{\rightarrow_E\}^1$$

(l)

$$\{P \rightarrow Q, R \rightarrow S\} \vdash (P \vee R) \rightarrow (Q \vee S)$$

$$\frac{\frac{\overline{P \rightarrow Q} \{ID\}}{Q \vee S} \{\vee_{IE}\} \quad \frac{\overline{R \rightarrow S} \{ID\}}{Q \vee S} \{\vee_{ID}\}}{Q \vee S} \{\vee_E\} \quad \frac{}{(P \vee R) \rightarrow (Q \vee S)} \{\rightarrow_I\}$$

(m)

$$\{Q \rightarrow R\} \vdash (P \rightarrow Q) \rightarrow (P \rightarrow R)$$

$$\frac{\frac{\overline{Q \rightarrow R} \{ID\}}{Q} \quad \frac{\overline{P} \{ID\} \quad \overline{P \rightarrow Q^1} \{ID\}}{Q} \{\rightarrow_E\}}{\frac{R}{P \rightarrow R} \{\rightarrow_I\}} \{\rightarrow_I\}^1$$

(n)

$$\{(P \wedge Q) \vee (P \wedge R)\} \vdash P \wedge (Q \vee R)$$

$$\frac{\frac{\overline{(P \wedge Q) \vee (P \wedge R)} \{ID\}}{P \wedge Q} \{\wedge_{IE}\} \quad \frac{\overline{(P \wedge Q) \vee (P \wedge R)} \{ID\}}{P \wedge Q} \{\wedge_{EE}\}}{\frac{P \wedge Q}{P} \{\wedge_{EE}\} \quad \frac{Q}{Q \vee R} \{\vee_{IE}\}} \{\wedge_I\}$$

(o)

$$\{P \rightarrow (Q \wedge R)\} \vdash (P \rightarrow Q) \wedge (P \rightarrow R)$$

$$\frac{\frac{\overline{P \rightarrow (Q \wedge R)} \{ID\}}{Q \wedge R} \{\rightarrow_E\} \quad \frac{\overline{P^1} \{ID\}}{P \rightarrow (Q \wedge R)} \{\rightarrow_E\}^1 \quad \frac{\overline{P \rightarrow (Q \wedge R)} \{ID\}}{Q \wedge R} \{\rightarrow_E\}^1 \quad \frac{\overline{P^1} \{ID\}}{P \rightarrow (Q \wedge R)} \{\rightarrow_E\}^1}{\frac{Q \wedge R}{Q} \{\wedge_{EE}\} \quad \frac{Q \wedge R}{R} \{\wedge_{ED}\}} \{\wedge_I\}$$

(p)

$$\{(P \rightarrow Q) \wedge (P \rightarrow R)\} \vdash P \rightarrow (Q \wedge R)$$

$$\frac{\frac{\frac{\overline{(P \rightarrow Q) \wedge (P \rightarrow R)} \{ID\}}{P \rightarrow Q} \{\wedge EE\} \quad \frac{\overline{P_1} \{ID\}}{P_1} \{\rightarrow E\} \quad \frac{\frac{\frac{\overline{(P \rightarrow Q) \wedge (P \rightarrow R)} \{ID\}}{P \rightarrow R} \{\wedge ED\} \quad \frac{\overline{P_1} \{ID\}}{P_1} \{\rightarrow E\}}{R} \{\wedge I\}}{Q \wedge R} \{\rightarrow I\}^1}{P \rightarrow (Q \wedge R)} \{\rightarrow I\}^1$$

(q)

$$\{P \rightarrow Q\} \vdash ((P \wedge Q) \rightarrow P) \wedge (P \rightarrow (P \wedge Q))$$

$$\frac{\frac{\frac{\overline{P^1} \{ID\}}{(P \wedge Q) \rightarrow P} \{\rightarrow I\}^1 \quad \frac{\frac{\frac{\overline{P^1} \{ID\}}{P \wedge Q} \{\wedge I\} \quad \frac{\frac{\overline{P \rightarrow Q} \{ID\}}{Q} \{\rightarrow E\}}{P \rightarrow (P \wedge Q)} \{\rightarrow I\}}{((P \wedge Q) \rightarrow P) \wedge (P \rightarrow (P \wedge Q))} \{\wedge I\}}{((P \wedge Q) \rightarrow P) \wedge (P \rightarrow (P \wedge Q))} \{\wedge I\}$$

(r)

$$\{P \rightarrow (Q \vee R), Q \rightarrow S, R \rightarrow S\} \vdash P \rightarrow S$$

$$\frac{\frac{\overline{P} \{ID\}}{Q \vee R} \{\rightarrow E\} \quad \frac{\frac{\overline{P \rightarrow (Q \vee R)} \{ID\}}{Q \rightarrow S} \{\rightarrow E\} \quad \frac{R \rightarrow S}{P \rightarrow S} \{\vee E\}}{P \rightarrow S} \{\rightarrow I\}$$

(s)

$$\vdash \neg P \rightarrow P \rightarrow P \rightarrow Q$$

$$\frac{\frac{\frac{\overline{\neg P^1} \{ID\}}{\perp} \{CTR\} \quad \frac{\overline{P^2} \{ID\}}{P \rightarrow Q} \{\rightarrow I\}^2}{\neg P \rightarrow (P \rightarrow (P \rightarrow Q))} \{\rightarrow I\}^1$$

(t)

$$\{P \wedge Q \rightarrow R, R \rightarrow S, Q \wedge \neg S\} \vdash \neg P$$

$$\frac{\frac{\frac{\overline{Q \wedge \neg S} \{ID\}}{\neg S} \{\wedge ED\} \quad \frac{\overline{R \rightarrow S} \{ID\}}{S} \{\rightarrow E\} \quad \frac{\frac{\frac{\overline{P \wedge Q \rightarrow R} \{ID\}}{R} \{\rightarrow E\} \quad \frac{\frac{\overline{P} \{ID\}}{P \wedge Q} \{\wedge I\} \quad \frac{\overline{Q \wedge \neg S} \{ID\}}{Q} \{\wedge EE\}}{R} \{\rightarrow E\}}{S} \{\rightarrow E\}}{\perp} \{CTR\}^1}{\neg P} \{CTR\}^1$$

(u) $\{(P \rightarrow Q) \rightarrow R, S \rightarrow \neg P, T, \neg S \wedge T \rightarrow Q\} \vdash R$

$$\{(S \rightarrow P) \vee (T \rightarrow Q)\} \vdash (S \rightarrow Q) \vee (T \rightarrow P)$$

$$\{\neg(P \rightarrow Q)\} \vdash Q \rightarrow P$$

$$\{\forall x.(P(x) \rightarrow Q(x))\} \vdash (\forall x.\neg Q(x)) \rightarrow (\forall x.\neg P(x))$$

5

(b)

$$\{\forall x.(P(x) \rightarrow \neg Q(x))\} \vdash \neg(\exists x.(P(x) \wedge Q(x)))$$

$$\frac{\frac{\frac{\overline{\exists x.(P(x) \wedge Q(x))} \{ID\}}{P(t) \wedge Q(t)} [\text{hipótese}]}{P(t)} \{\wedge E\} \quad \frac{\frac{\overline{\forall x.(P(x) \rightarrow \neg Q(x))} \{ID\}}{P(t) \rightarrow \neg Q(t)} \{\forall E\} \quad \frac{\overline{P(t) \wedge Q(t)} \{ID\}}{Q(t)} \{\wedge E\}}{\neg Q(t)} \{\rightarrow E\} \quad \frac{\perp}{\neg \exists x.(P(x) \wedge Q(x))} \{\neg I\}$$

(c)

$$\{\forall x.(A(x) \rightarrow (B(x) \vee C(x))), \forall x.\neg B(x)\} \vdash (\forall x.A(x)) \rightarrow (\forall x.C(x))$$

$$\frac{\frac{\frac{\overline{\forall x.(A(x) \rightarrow (B(x) \vee C(x)))} \{ID\}}{\forall x.A(x)} [\text{hipótese}]}{A(t)} \forall E \quad \frac{\frac{\overline{\forall x.(A(x) \rightarrow (B(x) \vee C(x)))} \{ID\}}{A(t) \rightarrow (B(t) \vee C(t))} \forall E \quad \frac{\frac{\overline{\forall x.\neg B(x)} \{ID\}}{\neg B(t)} \forall E \quad \frac{\overline{\forall x.\neg B(x)}}{B(t)} \rightarrow}{\frac{B(t) \vee C(t)}{C(t)} \perp E} \rightarrow E \quad \frac{C(t)}{\forall x.C(x)} \forall I \rightarrow I$$

(d)

$$\{\exists x.(P(x) \wedge Q(x)), \forall x.(P(x) \rightarrow R(x))\} \vdash \exists x.(R(x) \wedge Q(x))$$

$$\frac{\frac{\frac{\overline{\forall x.(P(x) \rightarrow R(x))} \{ID\}}{P(t) \wedge Q(t)} [\text{hipótese}]}{P(t)} \{\wedge E\} \quad \frac{\frac{\overline{\forall x.(P(x) \rightarrow R(x))} \{ID\}}{P(t) \rightarrow R(t)} \{\forall E\} \quad \frac{\overline{\forall x.(P(x) \rightarrow R(x))} \{ID\}}{P(t) \wedge Q(t)} [\text{hipótese}]}{Q(t)} \{\wedge E\}}{R(t)} \{\rightarrow E\} \quad \frac{R(t) \wedge Q(t)}{\exists x.(R(x) \wedge Q(x))} \{\exists I\}$$

(e)

$$\{\forall x.P(a, x, x), \forall x.\forall y.\forall z.(P(x, y, z) \rightarrow P(f(x), y, f(z)))\} \vdash P(f(a), a, f(a))$$

$$\frac{\frac{\overline{\forall x.P(a, x, x)} \{ID\}}{P(a, a, a)} \{\forall E\} \quad \frac{\frac{\frac{\overline{\forall x.\forall y.\forall z.(P(x, y, z) \rightarrow P(f(x), y, f(z)))} \{ID\}}{\forall y.\forall z.(P(a, y, z) \rightarrow P(f(a), y, f(z)))} \{\forall E\} \quad \frac{\overline{\forall z.(P(a, a, z) \rightarrow P(f(a), a, f(z)))} \{\forall E\}}{P(a, a, a) \rightarrow P(f(a), a, f(a))} \{\forall E\}}{P(f(a), a, f(a))} \{\rightarrow E\}$$

(f)

$$\{\forall x.(P(x) \rightarrow Q(x))\} \vdash \forall x.P(x) \rightarrow \forall x.Q(x)$$

$$\frac{\frac{\overline{\forall x.P(x)} \quad \{ID\}}{P(t)} \quad \{VE\} \quad \frac{\overline{\forall x.(P(x) \rightarrow Q(x))} \quad \{ID\}}{P(t) \rightarrow Q(t)} \quad \{VE\}}{\frac{Q(t)}{\forall x.Q(x)} \quad \{VI\}} \quad \{ \rightarrow E \}$$

$$\frac{}{\forall x.P(x) \rightarrow \forall x.Q(x)} \quad \{ \rightarrow I \}$$

(g)

$$\{\exists x.\neg P(x)\} \vdash \neg \forall x.P(x)$$

$$\frac{\frac{\overline{\exists x.\neg P(x)} \quad \{ID\}}{\frac{\perp}{\neg P(x)} \quad \{\neg I\}^1} \quad \frac{\overline{P(x)^1} \quad \{ID\}}{\exists x.P(x)} \quad \{VE\}}{\frac{\perp}{\neg \forall x.P(x)} \quad \{VI\}} \quad \perp$$

3 Item 5.2.1 - Prove os seguintes teoremas

(a). Para todo $n \in \mathbb{N}$,

$$\sum_{i=0}^n 3^i = \frac{3^{n+1} - 1}{2}$$

Base da indução: Para $n = 0$, temos:

$$\sum_{i=0}^0 3^i = 3^0 = 1$$

e

$$\frac{3^{0+1} - 1}{2} = \frac{3 - 1}{2} = \frac{2}{2} = 1$$

Logo, a base da indução é válida.

Passo indutivo: Suponha que a fórmula seja válida para um certo $n \in \mathbb{N}$, ou seja,

$$\sum_{i=0}^n 3^i = \frac{3^{n+1} - 1}{2}$$

Vamos provar que:

$$\sum_{i=0}^{n+1} 3^i = \frac{3^{n+2} - 1}{2}$$

De fato:

$$\begin{aligned}
 \sum_{i=0}^{n+1} 3^i &= \left(\sum_{i=0}^n 3^i \right) + 3^{n+1} \\
 &= \frac{3^{n+1} - 1}{2} + 3^{n+1} \\
 &= \frac{3^{n+1} - 1 + 2 \cdot 3^{n+1}}{2} \\
 &= \frac{3 \cdot 3^{n+1} - 1}{2} \\
 &= \frac{3^{n+2} - 1}{2}
 \end{aligned}$$

Portanto, a fórmula vale para $n + 1$. Pelo princípio da indução matemática, a identidade é válida para todo $n \in \mathbb{N}$.

(b). Para todo $n \in \mathbb{N}$,

$$\sum_{i=0}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

Base: Para $n = 0$,

$$\sum_{i=0}^0 i^2 = 0, \quad \text{e} \quad \frac{0(0+1)(2 \cdot 0 + 1)}{6} = 0$$

Passo indutivo: Suponha a fórmula válida para n ,

$$\sum_{i=0}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

Prove para $n + 1$:

$$\begin{aligned}
 \sum_{i=0}^{n+1} i^2 &= \sum_{i=0}^n i^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
 &= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} = \frac{(n+1)[n(2n+1) + 6(n+1)]}{6} \\
 &= \frac{(n+1)(2n^2 + 7n + 6)}{6} = \frac{(n+1)(n+2)(2n+3)}{6} \\
 &\Rightarrow \sum_{i=0}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6}
 \end{aligned}$$

Logo, a fórmula vale para todo $n \in \mathbb{N}$.

(c). Para todo $n \in \mathbb{N}$,

$$\sum_{i=1}^n (8i - 5) = 4n^2 - n$$

Base: Para $n = 1$,

$$8(1) - 5 = 3, \quad 4(1)^2 - 1 = 3$$

Passo indutivo: Suponha que

$$\sum_{i=1}^n (8i - 5) = 4n^2 - n$$

então

$$\begin{aligned} \sum_{i=1}^{n+1} (8i - 5) &= \sum_{i=1}^n (8i - 5) + (8(n+1) - 5) \\ &= 4n^2 - n + 8n + 3 = 4n^2 + 7n + 3 = 4(n+1)^2 - (n+1) \end{aligned}$$

(d). Para todo $n \geq 1$,

$$5 \mid (n^5 - n)$$

Base: Para $n = 1$, temos $1^5 - 1 = 0$, e $5 \mid 0$

Passo indutivo: Suponha $5 \mid n^5 - n$, ou seja, $n^5 - n = 5k$

Verificar para $n + 1$:

$$(n+1)^5 - (n+1) = n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1 - n - 1 = (n^5 - n) + 5(n^4 + 2n^3 + 2n^2 + n)$$

Como $5 \mid n^5 - n$ e o restante é múltiplo de 5, segue que $5 \mid (n+1)^5 - (n+1)$

(e). Para todo $n \geq 1$,

$$6 \mid (7^n - 1)$$

Provar que $6 \mid (7^n - 1)$ para todo $n \geq 1$

““latex **Base:** Para $n = 1$, $7^1 - 1 = 6 \Rightarrow 6 \mid 6$

Passo indutivo: Suponha $6 \mid (7^n - 1) \Rightarrow 7^n \equiv 1 \pmod{6}$

Multiplicando ambos os lados por 7:

$$7^{n+1} = 7 \cdot 7^n \equiv 7 \cdot 1 = 7 \equiv 1 \pmod{6} \Rightarrow 6 \mid (7^{n+1} - 1)$$

(f). Para todo $n \geq 1$,

$$6 \mid (n^3 + 5n)$$

Provar que $6 \mid (n^3 + 5n)$

“**Base:** Para $n = 1$, $1 + 5 = 6 \Rightarrow 6 \mid 6$

Passo indutivo: Suponha $6 \mid (n^3 + 5n)$

$$(n+1)^3 + 5(n+1) = n^3 + 3n^2 + 3n + 1 + 5n + 5 = (n^3 + 5n) + 3n^2 + 3n + 6$$

Como $6 \mid (n^3 + 5n)$ e os demais termos somam múltiplo de 6, temos $6 \mid (n+1)^3 + 5(n+1)$

(g). Para todo $n \in \mathbb{N}$,

$$2 \mid (n^2 + n)$$

Provar que $2 \mid (n^2 + n)$

“**Base:** Para $n = 0$, $0^2 + 0 = 0 \Rightarrow 2 \mid 0$

Passo indutivo: Suponha $2 \mid n^2 + n$

$$(n+1)^2 + (n+1) = n^2 + 2n + 1 + n + 1 = (n^2 + n) + 2n + 2$$

Como $2 \mid (n^2 + n)$ e $2n + 2$ é par, então $2 \mid (n+1)^2 + (n+1)$