Lista de exercícios 01 - PCC116

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1 Item 1.5.7 - Prove os seguintes sequentes usando dedução natural. Tente demonstrá-los sem utilizar a regra RAA.

(a)

$$\{(P \wedge Q) \wedge R,\ S \wedge T\} \vdash Q \wedge S$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(P \wedge Q) \wedge R}\{_{ID}\}}{P \wedge Q}\{\wedge_{EE}\}}{Q}\{\wedge_{ED}\} \qquad \dfrac{\dfrac{}{S \wedge T}\{_{ID}\}}{S}\{\wedge_{EE}\}}{Q \wedge S}\{\wedge_{I}\}$$

(b)

$$\{(P \wedge Q) \wedge R\} \vdash (P \wedge R) \vee Z$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(P \wedge Q) \wedge R}\{_{ID}\}}{P \wedge Q}\{\wedge_{EE}\}}{P}\{\wedge_{EE}\} \qquad \dfrac{\dfrac{}{(P \wedge Q) \wedge R}\{_{ID}\}}{R}\{\wedge_{ED}\}}{P \wedge R^{1}}\{\wedge_{I}\}}{(P \wedge R) \vee Z}\{\vee_{IE}\}^{1}$$

(c)

$$\{P,\ P \to (P \to Q)\} \vdash Q$$

$$\dfrac{\dfrac{\dfrac{}{P \to (P \to Q)}\{_{ID}\} \qquad \dfrac{}{P}\{_{ID}\}}{P \to Q}\{\to_{E}\} \qquad \dfrac{}{P}\{_{ID}\}}{Q}\{\to_{E}\}$$

(d)

$$\vdash (P \wedge Q) \to P$$

$$\dfrac{\dfrac{\dfrac{}{(P \wedge Q)^{1}}\{_{ID}\}}{P}\{\wedge_{EE}\}}{(P \wedge Q) \to P}\{\to_{I}\}^{1}$$

(e)

$$\{P\} \vdash Q \to (P \wedge Q)$$

$$\dfrac{\dfrac{\dfrac{}{P}\{_{ID}\} \qquad \dfrac{}{Q^1}\{_{ID}\}}{P \wedge Q}\{\wedge_I\}}{Q \to (P \wedge Q)}\{\to_I\}^1$$

(f)

$$\{P\} \vdash (P \to Q) \to Q$$

$$\dfrac{\dfrac{\dfrac{}{P \to Q^1}\{_{ID}\} \qquad \dfrac{}{P}\{_{ID}\}}{Q}\{\to_E\}}{(P \to Q) \to Q}\{\to_I\}^1$$

(g)

$$\vdash (P \wedge Q) \to (P \vee Q)$$

$$\dfrac{\dfrac{\dfrac{}{P \wedge Q^1}\{_{ID}\}}{P \vee Q}\{\vee_{IE}\} \qquad \dfrac{\dfrac{}{P \wedge Q^1}\{_{ID}\}}{P \vee Q}\{\vee_{ID}\}}{(P \wedge Q) \to (P \vee Q)}\{\to_I\}^1$$

(h)

$$\{Q \to (P \to R),\ \neg R,\ Q\} \vdash \neg P$$

$$\dfrac{\dfrac{\dfrac{}{Q \to (P \to R)}\{_{ID}\} \qquad \dfrac{}{Q}\{_{ID}\}}{P \to R}\{\to_E\} \qquad \dfrac{}{\neg R}\{_{ID}\}}{\neg P}\{\to_E\}$$

(i)

$$\{P\} \vdash Q \to (P \wedge Q)$$

$$\dfrac{\dfrac{P \qquad Q^1}{P \wedge Q}\{\wedge_1\}}{Q \to (P \wedge Q)}\{\to_I\}^1$$

(j)

$$\{(P \to R) \wedge (Q \to R),\ P \wedge Q\} \vdash Q \wedge R$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(P \to R) \wedge (Q \to R)}\{_{ID}\}}{Q \to R}\{\wedge_{ED}\} \qquad \dfrac{\dfrac{}{P \wedge Q}\{_{ID}\}}{Q}\{\wedge_{ED}\}}{R}\{\to_E\} \qquad \dfrac{\dfrac{}{P \wedge Q}\{_{ID}\}}{Q}\{\wedge_{ED}\}}{Q \wedge R}\{\wedge_I\}$$

(k)

$$\{P \to Q \to R,\ P \to Q\} \vdash P \to R$$

$$\dfrac{\dfrac{\dfrac{}{P \to Q \to R}\{_{ID}\} \quad \dfrac{}{P^1}\{_{ID}\}}{Q \to R}\{\to_E\}^1 \qquad \dfrac{\dfrac{}{P \to Q)}\{_{ID}\} \quad \dfrac{}{P^1}\{_{ID}\}}{Q}\{\to_E\}}{P \to R}\{\to_E\}^1$$

(l)

$$\{P \to Q,\ R \to S\} \vdash (P \lor R) \to (Q \lor S)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \to Q}\{_{ID}\}}{Q \lor S}\{\lor_{IE}\} \qquad \dfrac{\dfrac{}{R \to S}\{_{ID}\}}{Q \lor S}\{\lor_{ID}\}}{Q \lor S}\{\lor_E\}}{(P \lor R) \to (Q \lor S)}\{\to_I\}$$

(m)

$$\{Q \to R\} \vdash (P \to Q) \to (P \to R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{Q \to R}\{_{ID}\} \qquad \dfrac{\dfrac{}{P}\{_{ID}\} \quad \dfrac{}{P \to Q^1}\{_{ID}\}}{Q}\{\to_E\}}{R}\{\to_E\}}{P \to R}\{\to_I\}}{(P \to Q) \to (P \to R)}\{\to_I\}^1$$

(n)

$$\{(P \land Q) \lor (P \land R)\} \vdash P \land (Q \lor R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(P \land Q) \lor (P \land R)}\{_{ID}\}}{P \land Q}\{\land_{IE}\}}{P}\{\land_{EE}\} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{(P \land Q) \lor (P \land R)}\{_{ID}\}}{P \land Q}\{\land_{EE}\}}{Q}\{\land_{EE}\}}{Q \lor R}\{\lor_{IE}\}}{P \land (Q \lor R)}\{\land_I\}$$

(o)

$$\{P \to (Q \land R)\} \vdash (P \to Q) \land (P \to R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \to (Q \land R)}\{_{ID}\} \quad \dfrac{}{P^1}\{_{ID}\}}{Q \land R}\{\to_E\}}{Q}\{\land_{EE}\}}{P \to Q}\{\to_I\} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{P \to (Q \land R)}\{_{ID}\} \quad \dfrac{}{P^1}\{_{ID}\}}{Q \land R}\{\to_E\}^1}{R}\{\land_{ED}\}}{P \to R}\{\to_I\}}{(P \to Q) \land (P \to R)}\{\land_I\}$$

(p)

$$\{(P \to Q) \land (P \to R)\} \vdash P \to (Q \land R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(P\to Q)\wedge(P\to R)}\{ID\}}{P\to Q}\{\wedge EE\} \quad \dfrac{}{P_1}\{ID\}}{Q}\{\to E\} \quad \dfrac{\dfrac{\dfrac{}{(P\to Q)\wedge(P\to R)}\{ID\}}{P\to R}\{\wedge ED\} \quad \dfrac{}{P_1}\{ID\}}{R}\{\to E\}}{Q\wedge R}\{\wedge I\}}{P\to(Q\wedge R)}\{\to I\}^1$$

(q)

$$\{P\to Q\}\vdash((P\wedge Q)\to P)\wedge(P\to(P\wedge Q))$$

$$\dfrac{\dfrac{\dfrac{}{P^1}\{ID\}}{(P\wedge Q)\to P}\{\to I\}^1 \quad \dfrac{\dfrac{\dfrac{}{P^1}\{ID\} \quad \dfrac{\dfrac{}{P\to Q}\{ID\}}{Q}\{\to E\}}{P\wedge Q}\{\wedge I\}}{P\to(P\wedge Q)}\{\to I\}}{((P\wedge Q)\to P)\wedge(P\to(P\wedge Q))}\{\wedge I\}$$

(r)

$$\{P\to(Q\vee R),\ Q\to S,\ R\to S\}\vdash P\to S$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P}\{ID\} \quad \dfrac{}{P\to(Q\vee R)}\{ID\}}{Q\vee R}\{\to E\} \quad Q\to S \quad R\to S}{S}\{\vee E\}}{P\to S}\{\to I\}$$

(s)

$$\vdash\neg P\to P\to P\to Q$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg P^1}\{ID\} \quad \dfrac{}{P^2}\{ID\}}{\bot}\{\to E\}}{Q}\{CTR\}}{P\to Q}\{\to I\}}{P\to(P\to Q)}\{\to I\}^2}{\neg P\to(P\to(P\to Q))}\{\to I\}^1$$

(t)

$$\{P\wedge Q\to R,\ R\to S,\ Q\wedge\neg S\}\vdash\neg P$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{Q\wedge\neg S}\{ID\}}{\neg S}\{\wedge ED\} \quad \dfrac{\dfrac{}{R\to S}\{ID\} \quad \dfrac{\dfrac{}{P\wedge Q\to R}\{ID\} \quad \dfrac{\dfrac{}{P}\{ID\} \quad \dfrac{\dfrac{}{Q\wedge\neg S}\{ID\}}{Q}\{\wedge EE\}}{P\wedge Q}\{\wedge I\}}{R}\{\to E\}}{S}\{\to E\}}{\bot}\{\to E\}}{\neg P}\{CTR\}^1$$

(u)

$$\{(P \to Q) \to R,\ S \to \neg P,\ T,\ \neg S \land T \to Q\} \vdash R$$

$$\dfrac{\dfrac{}{(P \to Q) \to R}\{ID\} \qquad \dfrac{\dfrac{\dfrac{}{\neg S \land T \to Q}\{ID\} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\neg S^1}\{ID\}}{\neg P \to \neg S}\{\to I\} \qquad \dfrac{\dfrac{}{S \to \neg P}\{ID\}}{\neg P}\{\to E\}}{\neg S}\{\to E\} \qquad \dfrac{}{T}\{ID\}}{\neg S \land T}\{\land I\}^1}{Q}\{\to E\}}{P \to Q}\{\to I\}}{R}\{\to E\}$$

(v)

$$\{(S \to P) \lor (T \to Q)\} \vdash (S \to Q) \lor (T \to P)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(S \to P) \lor (T \to Q)}\{ID\}}{S \to P}\{\lor EE\}}{P}\{\to E\}}{T \to P}\{\to I\}}{(S \to Q) \lor (T \to P)}\{\lor ID\}$$

(w)

$$\{\neg(P \to Q)\} \vdash Q \to P$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(P \to Q)}\{ID\} \qquad \dfrac{\dfrac{}{Q^1}\{ID\}}{(P \to Q)}\{\to_I\}}{\bot}}{P}\{CTR\}^1}{Q \to P}\{\to_I\}^1$$

2 Item 2.9 - Prove os seguintes sequentes usando dedução natural:

(a)

$$\{\forall x.(P(x) \to Q(x))\} \vdash (\forall x.\neg Q(x)) \to (\forall x.\neg P(x))$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x.(P(x) \to Q(x))}\{ID\} \qquad \dfrac{\dfrac{\dfrac{}{\forall x.\neg Q(x)}\{ID\}}{\neg Q(a)}\{\forall E\} \qquad \dfrac{}{P(a)}\{ID\}}{Q(a)}\{\to E\}}{\bot}\{\neg E\}}{\neg P(a)}\{\neg I\}}{\forall x.\neg P(x)}\{\forall I\}}{(\forall x.\neg Q(x)) \to (\forall x.\neg P(x))}\{\to I\}$$

(b)

$$\{\forall x.(P(x) \to \neg Q(x))\} \vdash \neg(\exists x.(P(x) \land Q(x)))$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\exists x.(P(x) \land Q(x))}\{_{ID}\}}{P(t) \land Q(t)}[\text{hipótese}]}{P(t)}\{\land E\} \qquad \dfrac{\dfrac{}{\forall x.(P(x) \to \neg Q(x))}\{_{ID}\}}{P(t) \to \neg Q(t)}\{\forall E\}}{\neg Q(t)}\{\to E\} \qquad \dfrac{\dfrac{}{P(t) \land Q(t)}\{_{ID}\}}{Q(t)}\{\land E\}}{\bot}}{\neg \exists x.(P(x) \land Q(x))}\{\neg I\}$$

(c)

$$\{\forall x.(A(x) \to (B(x) \lor C(x))),\ \forall x.\neg B(x)\} \vdash (\forall x.A(x)) \to (\forall x.C(x))$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x.A(x)}[\text{hipótese}]}{A(t)}\{\forall E\} \qquad \dfrac{\dfrac{}{\forall x.(A(x) \to (B(x) \lor C(x)))}\{_{ID}\}}{A(t) \to (B(t) \lor C(t))}\{\forall E\}}{B(t) \lor C(t)}\{\to E\} \qquad \dfrac{\dfrac{}{\forall x.\neg B(x)}\{_{ID}\}}{\neg B(t)}\{\forall E\}}{C(t)}\{\lor E\}}{\forall x.C(x)}\{\forall I\}}{(\forall x.A(x)) \to (\forall x.C(x))}\{\to I\}$$

(d)

$$\{\exists x.(P(x) \land Q(x)), \forall x.(P(x) \to R(x))\} \vdash \exists x.(R(x) \land Q(x))$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x.(P(x) \to R(x))}\{_{ID}\}}{P(t) \land Q(t)}[\text{hipótese}]}{P(t)}\{\land E\} \qquad \dfrac{\dfrac{}{\forall x.(P(x) \to R(x))}\{_{ID}\}}{P(t) \to R(t)}\{\forall E\}}{R(t)}\{\to E\} \qquad \dfrac{\dfrac{}{P(t) \land Q(t)}\{_{ID}\}}{Q(t)}\{\land E\}}{R(t) \land Q(t)}\{\land I\}}{\exists x.(R(x) \land Q(x))}\{\exists I\}$$

(e)

$$\{\forall x.P(a,x,x),\ \forall x.\forall y.\forall z.(P(x,y,z) \to P(f(x),y,f(z)))\} \vdash P(f(a),a,f(a))$$

$$\dfrac{\dfrac{\forall x.P(a,x,x)}{P(a,a,a)}\{\forall E\} \qquad \dfrac{\dfrac{\dfrac{\forall x.\forall y.\forall z.(P(x,y,z) \to P(f(x),y,f(z)))}{\forall y.\forall z.(P(a,y,z) \to P(f(a),y,f(z)))}\{\forall E\}}{\forall z.(P(a,a,z) \to P(f(a),a,f(z)))}\{\forall E\}}{P(a,a,a) \to P(f(a),a,f(a))}\{\forall E\}}{P(f(a),a,f(a))}\{\to E\}$$

(f)

$$\{\forall x.(P(x) \to Q(x))\} \vdash \forall x.P(x) \to \forall x.Q(x)$$

$$\dfrac{\dfrac{\dfrac{}{\forall x.P(x)}\,[\text{hipótese}]}{P(t)}\,\{\forall E\} \qquad \dfrac{\dfrac{}{\forall x.(P(x) \to Q(x))}\,\{{\scriptstyle ID}\}}{P(t) \to Q(t)}\,\{\forall E\}}{\dfrac{\dfrac{Q(t)}{\forall x.Q(x)}\,\{\forall I\}}{\forall x.P(x) \to \forall x.Q(x)}\,\{\to I\}}\,\{\to E\}$$

(g)

$$\{\exists x.\neg P(x)\} \vdash \neg\forall x.P(x)$$

$$\dfrac{\dfrac{\dfrac{}{\exists x.\neg P(x)}\,\{{\scriptstyle ID}\}}{\neg P(t)} \qquad \dfrac{\dfrac{}{\forall x.P(x)}\,\{{\scriptstyle ID}\}}{P(t)}\,\{\forall E\}}{\dfrac{\perp}{\neg\forall x.P(x)}\,\{\neg I\}}$$

3 Item 5.2.1 - Prove os seguintes teoremas

(a). Para todo $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} 3^i = \frac{3^{n+1} - 1}{2}$$

Base da indução: Para $n = 0$, temos:

$$\sum_{i=0}^{0} 3^i = 3^0 = 1$$

e

$$\frac{3^{0+1} - 1}{2} = \frac{3 - 1}{2} = \frac{2}{2} = 1$$

Logo, a base da indução é válida.

Passo indutivo: Suponha que a fórmula seja válida para um certo $n \in \mathbb{N}$, ou seja,

$$\sum_{i=0}^{n} 3^i = \frac{3^{n+1} - 1}{2}$$

Vamos provar que:

$$\sum_{i=0}^{n+1} 3^i = \frac{3^{n+2} - 1}{2}$$

De fato:

$$\begin{aligned}
\sum_{i=0}^{n+1} 3^i &= \left(\sum_{i=0}^{n} 3^i\right) + 3^{n+1} \\
&= \frac{3^{n+1}-1}{2} + 3^{n+1} \\
&= \frac{3^{n+1}-1+2\cdot 3^{n+1}}{2} \\
&= \frac{3\cdot 3^{n+1}-1}{2} \\
&= \frac{3^{n+2}-1}{2}
\end{aligned}$$

Portanto, a fórmula vale para $n+1$. Pelo princípio da indução matemática, a identidade é válida para todo $n \in \mathbb{N}$.

(b). Para todo $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Base: Para $n = 0$,

$$\sum_{i=0}^{0} i^2 = 0, \quad \text{e} \quad \frac{0(0+1)(2\cdot 0+1)}{6} = 0$$

Passo indutivo: Suponha a fórmula válida para n,

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Prove para $n+1$:

$$\sum_{i=0}^{n+1} i^2 = \sum_{i=0}^{n} i^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \frac{n(n+1)(2n+1)+6(n+1)^2}{6} = \frac{(n+1)\left[n(2n+1)+6(n+1)\right]}{6}$$

$$= \frac{(n+1)(2n^2+7n+6)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$\Rightarrow \sum_{i=0}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

Logo, a fórmula vale para todo $n \in \mathbb{N}$.

Para todo $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Base: Para $n = 0$,

$$\sum_{i=0}^{0} i^2 = 0, \quad \text{e} \quad \frac{0(0+1)(2 \cdot 0+1)}{6} = 0$$

Passo indutivo: Suponha a fórmula válida para n,

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Prove para $n+1$:

$$\sum_{i=0}^{n+1} i^2 = \sum_{i=0}^{n} i^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} = \frac{(n+1)\left[n(2n+1) + 6(n+1)\right]}{6}$$

$$= \frac{(n+1)(2n^2 + 7n + 6)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$\Rightarrow \sum_{i=0}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

Logo, a fórmula vale para todo $n \in \mathbb{N}$.

(c). Para todo $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} (8i - 5) = 4n^2 - n$$

Base: Para $n = 1$,

$$8(1) - 5 = 3, \quad 4(1)^2 - 1 = 3$$

Passo indutivo: Suponha que

$$\sum_{i=1}^{n} (8i - 5) = 4n^2 - n$$

então

$$\sum_{i=1}^{n+1}(8i-5) = \sum_{i=1}^{n}(8i-5) + (8(n+1)-5)$$

$$= 4n^2 - n + 8n + 3 = 4n^2 + 7n + 3 = 4(n+1)^2 - (n+1)$$

(d). Para todo $n \geq 1$,

$$5 \mid (n^5 - n)$$

Base: Para $n = 1$, temos $1^5 - 1 = 0$, e $5 \mid 0$
Passo indutivo: Suponha $5 \mid n^5 - n$, ou seja, $n^5 - n = 5k$
Verificar para $n + 1$:

$$(n+1)^5-(n+1) = n^5+5n^4+10n^3+10n^2+5n+1-n-1 = (n^5-n)+5(n^4+2n^3+2n^2+n)$$

Como $5 \mid n^5 - n$ e o restante é múltiplo de 5, segue que $5 \mid (n+1)^5 - (n+1)$

(e). Para todo $n \geq 1$,

$$6 \mid (7^n - 1)$$

Provar que $6 \mid (7^n - 1)$ para todo $n \geq 1$
"'latex **Base:** Para $n = 1$, $7^1 - 1 = 6 \Rightarrow 6 \mid 6$
Passo indutivo: Suponha $6 \mid (7^n - 1) \Rightarrow 7^n \equiv 1 \mod 6$
Multiplicando ambos os lados por 7:

$$7^{n+1} = 7 \cdot 7^n \equiv 7 \cdot 1 = 7 \equiv 1 \mod 6 \Rightarrow 6 \mid (7^{n+1} - 1)$$

(f). Para todo $n \geq 1$,

$$6 \mid (n^3 + 5n)$$

Provar que $6 \mid (n^3 + 5n)$
"'latex **Base:** Para $n = 1$, $1 + 5 = 6 \Rightarrow 6 \mid 6$
Passo indutivo: Suponha $6 \mid (n^3 + 5n)$

$$(n+1)^3 + 5(n+1) = n^3 + 3n^2 + 3n + 1 + 5n + 5 = (n^3 + 5n) + 3n^2 + 3n + 6$$

Como $6 \mid (n^3 + 5n)$ e os demais termos somam múltiplo de 6, temos $6 \mid (n+1)^3 + 5(n+1)$

(g). Para todo $n \in \mathbb{N}$,

$$2 \mid (n^2 + n)$$

Provar que $2 \mid (n^2 + n)$

"'latex **Base:** Para $n = 0$, $0^2 + 0 = 0 \Rightarrow 2 \mid 0$

Passo indutivo: Suponha $2 \mid n^2 + n$

$$(n+1)^2 + (n+1) = n^2 + 2n + 1 + n + 1 = (n^2 + n) + 2n + 2$$

Como $2 \mid (n^2 + n)$ e $2n + 2$ é par, então $2 \mid (n+1)^2 + (n+1)$