

$$\alpha = \text{swap} \circ (\text{id} \times \text{swap}) \quad (1)$$

$$A \xrightarrow{\text{id}} A, B \times C \xrightarrow{\text{swap}} C \times B$$

$$A \times (B \times C) \xrightarrow{\text{id} \times \text{swap}} A \times (C \times B)$$

$$D \times E \xrightarrow{\text{swap}} E \times D$$

$$\{A \times (C \times B) = D \times E\}$$

$$D = A; E = C \times B$$

$$A \times (C \times B) \xleftarrow{\text{id} \times \text{swap}} A \times (B \times C)$$

$$\begin{array}{ccc} \downarrow \text{swap} & & \downarrow \text{swap} \\ (C \times B) \times A' & \xleftarrow{\text{swap} \times \text{id}} & (B \times C) \times A' \end{array}$$

$$\begin{array}{ccc} A \times (B \times C) & \xrightarrow{\alpha} & (C \times B) \times A \\ \downarrow h \times (g \times f) & & \downarrow (f \times g) \times h \\ A' \times (B' \times C') & \xrightarrow{\alpha} & (C' \times B') \times A' \end{array}$$

$$(f \times g) \times h \circ \alpha = \alpha \circ h \times (g \times f)$$

$$\textcircled{2} \text{ join} = [\text{id}, \text{id}]$$

$$\text{dup} = \langle \text{id}, \text{id} \rangle$$

$$\alpha = \text{dup} \circ \text{join}$$

$$\text{join} : A + B \longrightarrow C$$

$$\text{dup} : A \longrightarrow A \times A \text{ ou } A^2$$

$$\begin{array}{ccc} A + A & \xrightarrow{\text{join}} & A \\ \downarrow (f+g) & \searrow \text{dup} & \downarrow \text{dup} \\ A + A' & \xrightarrow{\text{join}} & C' \\ \downarrow (f+g) & \searrow \text{dup} & \downarrow \text{dup} \\ A' + A' & \xrightarrow{\text{join}} & C' \end{array}$$

$$\begin{array}{ccc} & C & \\ & \uparrow [\text{id}, \text{id}] & \\ A & \xrightarrow{i_1} A + B & \xleftarrow{i_2} B \end{array}$$

$$\alpha \circ (f+g) = (h+h) \circ \alpha$$

$$A + A \xrightarrow{\text{join}} A \xrightarrow{\text{dup}} A \times A \quad B \xleftarrow{\pi_1} B \times C \xrightarrow{\pi_2} C$$

$$\begin{array}{ccc} (f+g) & & (f+g) \\ \downarrow & & \downarrow \\ A + A' & \xrightarrow{\text{join}} & A \xrightarrow{\text{dup}} A' \times A' \\ (f \times g) \circ \alpha & = & (f+g) \circ \alpha \end{array}$$

$$\textcircled{3} \text{ iso} = \langle !+, \text{join} \rangle$$

$$! : A \rightarrow 1$$

$$(\text{id} \times f) \circ \text{iso} = \text{iso} \circ (f+g)$$

$$A + A \xrightarrow{\text{iso}} (1+1) \times A$$

$$\begin{array}{ccc} \downarrow f & & \downarrow (f+g) \\ A' + A' & \xrightarrow{\text{iso}} & (1+1) \times A' \end{array}$$

$$(\text{id} + \text{id}) \times f \circ \alpha = \alpha \circ (f+g)$$

$$\begin{array}{ccc} 1+1 & \xrightarrow{\quad} & (1+1) \times A \xleftarrow{i_2} A \\ & \nwarrow \langle !+, \text{join} \rangle & \uparrow [\text{id}, \text{id}] \\ & A + A & \end{array}$$

$$\Rightarrow (\text{id} \times f) \circ \langle !+, [\text{id}, \text{id}] \rangle = \langle !+, [\text{id}, \text{id}] \rangle \circ (f+g)$$

$$\Rightarrow \langle \text{id} \circ (!+), f \circ [\text{id}, \text{id}] \rangle = \langle (!+) \circ (f+g), [\text{id}, \text{id}] \circ (f+g) \rangle$$

$$\Rightarrow \begin{cases} !+ = (!+) \circ (f+g) \\ f \circ [\text{id}, \text{id}] = [\text{id}, \text{id}] \circ (f+g) \end{cases}$$

$$\Rightarrow \begin{cases} !+ = (! \circ f) + (! \circ g) = !+ \\ [f, g] = [f, f] \quad \Rightarrow \text{TRUE} \end{cases}$$

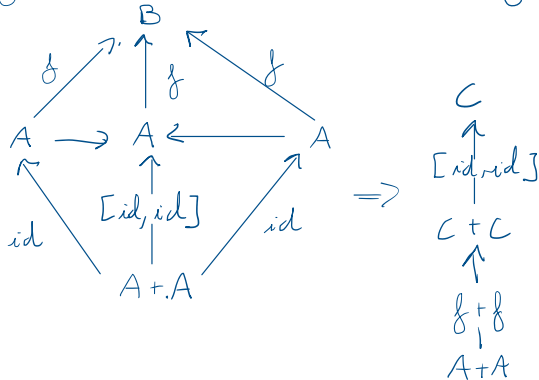
$$\textcircled{4} [\nabla \circ !, \text{id}] = \text{id}$$

$$\textcircled{4} \begin{cases} \nabla \circ i_1 = id \\ \nabla \circ i_2 = id \end{cases}$$

$$f \circ \nabla = \nabla \circ (f + f)$$

$$[id, id] = \nabla$$

$$f \circ [id, id] = [id, id] \circ (f + f)$$



$$\Rightarrow \begin{cases} !+! = (! \circ f) + (! \circ f) = !+! \\ [f, f] = [f, f] \end{cases} \quad \Rightarrow \text{TRUE}$$

$$i_{\text{iso}} = \langle \dots \rangle \text{ Lei du troc}$$

$$\Rightarrow i_{\text{iso}} = [\langle i_1 \circ !, i_2 \circ ! \rangle, \langle i_1 \circ id, i_2 \circ id \rangle]$$

$$Universal +$$

$$[f \circ id, f \circ id] = [id \circ f, id \circ f]$$

$$\textcircled{5} (f + h) \circ \alpha = \alpha \circ (f + g \times h)$$

$$\begin{array}{ccc} A + (B \times C) & \xrightarrow{\alpha} & A + B \\ \downarrow f + (g \times h) & & \downarrow f + h \\ A' + (B' \times C') & \xrightarrow{\alpha} & A' + B' \end{array}$$

$$\begin{array}{c} A + B \\ \uparrow \\ \alpha = (id + \pi_1) \\ \downarrow \\ A + (B \times C) \end{array}$$

$$\textcircled{6} \begin{aligned} \alpha \circ g = h &\equiv g = \alpha^\circ \circ h \\ g \circ \alpha = h &\equiv g = h \circ \alpha^\circ \end{aligned}$$

$$\underbrace{h \cdot (g \times id + g \times \alpha)}_g = \underbrace{k \cdot undistr}_{h \alpha^\circ}$$

$$\underbrace{h \cdot distr \cdot (g \times (id + \alpha))}_h = k$$

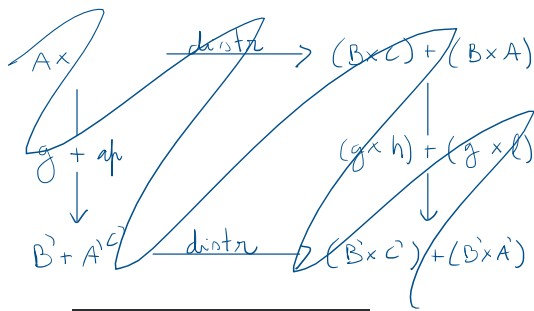
$$h \circ \alpha = k$$

$$\Rightarrow h \circ \alpha \circ \alpha^\circ = k \circ \alpha^\circ$$

$$\Rightarrow h = k \circ \alpha^\circ$$

```
distl :: (Either c a, b) -> Either (c, b) (a, b)
distl = uncurry (either (curry i1) (curry i2))
```

```
distr :: (b, Either c a) -> Either (b, c) (b, a)
distr = (swap -|- swap) . distl . swap
```



$$h \cdot \text{distr} \cdot (g \times (\text{id} + \alpha)) = k$$

$$\begin{array}{ccc} A \times (B+C) & \xrightarrow{\text{distl}} & (A \times B) + (A \times C) \\ g \times (g+h) & & (g \times g) + (g \times h) \\ \downarrow & & \downarrow \\ A' \times (B+C') & \xrightarrow{\text{distr}} & (A' \times B') + (A' \times C') \end{array}$$

$$\underbrace{h \cdot ((g \times \text{id}) + (g \times \alpha))}_{g} \cdot \underbrace{\text{distl}}_{\alpha} = \underbrace{k}_{h}$$

$$\underbrace{h \cdot (g \times \text{id} + g \times \alpha)}_g = \underbrace{k \cdot \text{undistr}}_{h \cdot \alpha^0}$$

$$\begin{aligned} h \cdot ((g \times \text{id}) + (g \times \alpha)) \cdot \text{distl} &= k \\ = h \cdot ((g \times \text{id}) + (g \times \alpha)) \cdot \text{distl} \cdot \text{undistr} &= k \cdot \text{undistr} \\ = h \cdot ((g \times \text{id}) + (g \times \alpha)) &= k \cdot \text{undistr} \end{aligned}$$

⑦ Propriedade de cancelamento da exponenciação

$$\text{ap} \circ (\bar{f} \times \text{id}) = f$$

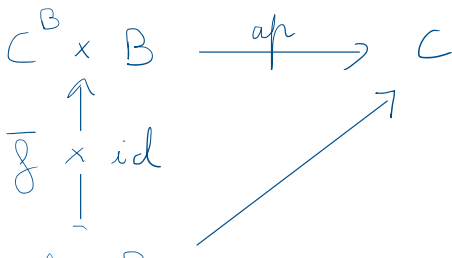
\Rightarrow

$$\text{curry } f \circ b = f(a, b)$$

$$(\text{ap} \circ (\bar{f} \times \text{id}))(a, b) = f(a, b)$$

$$\Rightarrow \text{ap} (\bar{f} \ a, \text{id } b) = f(a, b)$$

$$\Rightarrow \text{ap} (\bar{f} \ a, b)$$



$$\begin{array}{c} f : A \times B \rightarrow C \\ \text{---} \\ A \times B \end{array}$$

$$\text{curry } f \ a \ b = f \ (a, b)$$

$$\Rightarrow \bar{f} \ a \ b = f \ (a, b)$$

$$\Rightarrow \text{ap} (\bar{f} \ a, b) = f \ (a, b)$$

$$\Rightarrow \text{ap} (\bar{f} \ a, \text{id } b) = f \ (a, b)$$

$$\Rightarrow \text{ap} ((\bar{f} \times \text{id}) \ (a, b)) = f \ (a, b)$$

$$\Rightarrow (\text{ap} \circ (\bar{f} \times \text{id})) \ (a, b) = f \ (a, b)$$

$$\Rightarrow \text{ap} \circ (\bar{f} \times \text{id}) = f$$

```
ghci> f (a,b) = a + b
ghci> curry f 1 2 == f (1,2)
True
```

```
ghci> ap (curry f 1, 2)
3
```

```
ghci> testFunction = ap . (curry f << id)
ghci> testFunction (1,2) == f (1,2)
True
```

```
ap_1 :: (a -> b, a) -> b
ap_1 = uncurry ($)

ap_2 :: (a -> b, a) -> b
ap_2 = uncurry (\f x -> f x)

ap_3 :: (a -> b, a) -> b
ap_3 = uncurry id

ap_4 :: (a -> b, a) -> b
ap_4 (f, a) = f a
```

Mistérios
da vida

$$8. \quad \text{curry} \circ \text{uncurry } g \ a \ b = \text{uncurry } g \ (a, b)$$

$$\text{uncurry } g \ (a, b) = g \ a \ b$$

$$\hat{g} \ (a, b) = g \ a \ b$$

$$\Rightarrow \hat{g} \ (a, b) = \text{ap} (g \ a, b)$$

$$\Rightarrow \hat{g} \ (a, b) = \text{ap} ((g \times \text{id}) \ (a, b))$$

$$\Rightarrow \hat{g} \ (a, b) = (\text{ap} (g \times \text{id})) \ (a, b)$$

$$\Rightarrow \hat{g} = \text{ap} \circ (g \times \text{id})$$

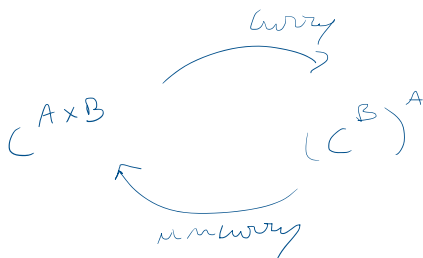
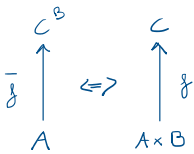
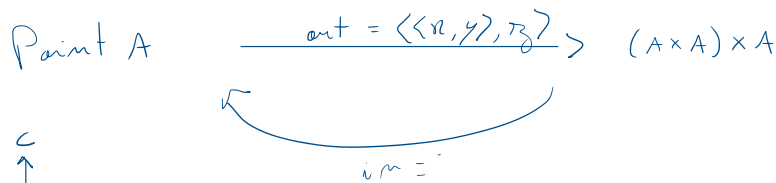
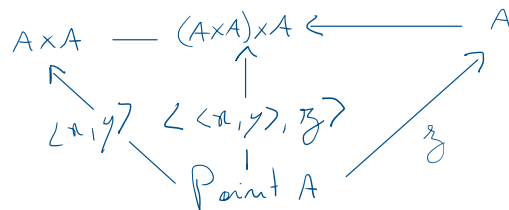
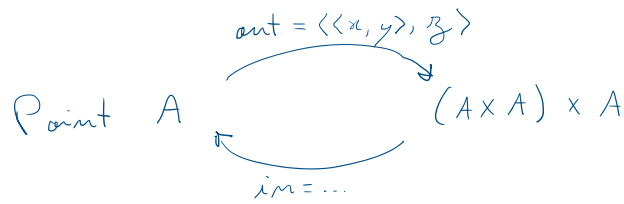
```
ghci> {
ghci| g :: Num a -> a -> a -> a
ghci| g a b = a + b
ghci| }
```

```
ghci> g 1 2
3
ghci> uncurry g (1,2)
3
```

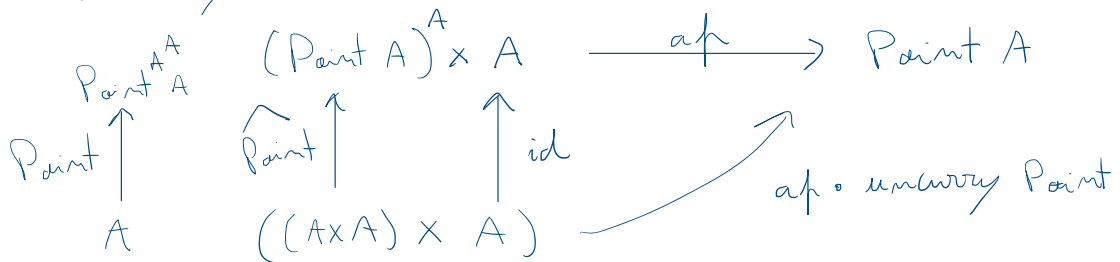
```
ghci> test = ap . (g << id)
ghci> uncurry g (1,2) == test (1,2)
True
```

$$(9) \quad \text{Point } a = \text{Point } \{ x :: a, y :: a, z :: a \}$$

⑨ $\text{data Point } a = \text{Point } \{ x :: a, y :: a, z :: a \}$
 $\text{Point} :: a \rightarrow a \rightarrow a \rightarrow \text{Point}$



```
ghci> test1 = uncurry Point
ghci> :t test1
test1 :: (b, b) -> b -> Point b
```



```
data Point a = Point { x :: a, y :: a, z :: a } deriving (Eq, Ord)
-- :t Point
-- Point :: a -> a -> a -> Point a

out :: Point a -> ((a, a), a)
out (Point x y z) = ((x, y), z)

out2 :: Point a -> ((a, a), a)
out2 = split (split x y) z

in' :: ((a, a), a) -> Point a
in' ((x, y), z) = Point x y z

in'' :: ((a, a), a) -> Point a
in'' = ap . (uncurry Point >> id)
```