Aula PL #05

17 de outubro de 2023 17:03

(1)

$\alpha = swap \cdot (id \times swap)$

$A \xrightarrow{id} A, \quad B \times C \xrightarrow{swap} C \times B$

$A \times (B \times C) \xrightarrow{id \times swap} A \times (C \times B)$

$D \times E \xrightarrow{swap} E \times D$

$\{ A \times (C \times B) = D \times E ?$

$D = A; \quad E = C \times B$

$$\begin{tikzcd}
A \times (C \times B) \arrow[d, "swap"'] & A \times (B \times C) \arrow[l, "id \times swap"'] \arrow[d, "swap"] \\
(C' \times B') \times A' & (B' \times C') \times A' \arrow[l, "swap \times id"]
\end{tikzcd}$$

$$\begin{tikzcd}
A \times (B \times C) \arrow[r, "\alpha"] \arrow[d, "h \times (g \times f)"'] & (C \times B) \times A \arrow[d, "(f \times g) \times h"] \\
A' \times (B' \times C') \arrow[r, "\alpha"] & (C' \times B') \times A'
\end{tikzcd}$$

$\boxed{(f \times g) \times h \cdot \alpha = \alpha \cdot h \times (g \times f)}$

(2) $join = [id, id]$
$dup = \langle id, id \rangle$

$\alpha = dup \cdot join$

$join : A + B \longrightarrow C$

$dup : A \longrightarrow A \times A$ ou A^2

$$\begin{tikzcd}
A + A \arrow[r, "join"] \arrow[d, "(f+f)"'] & A \arrow[r, "dup"] & C \times C \arrow[d, "(h \times h)"] \\
A' + A' \arrow[r, "join"'] & C' \arrow[r, "dup"'] & C' \times C'
\end{tikzcd}$$

$\Leftarrow\Rightarrow \alpha$

$\Leftarrow\Rightarrow \alpha$

$\cancel{\alpha \cdot (f+f) = (h+h) \cdot \alpha}$

$$\begin{tikzcd}
& C & \\
A \arrow[ur, "f"] \arrow[r, "i_1"'] & A + B \arrow[u, "{[f, g]}"] & B \arrow[l, "i_2"] \arrow[ul, "g"']
\end{tikzcd}$$

$$\begin{tikzcd}
A + A \arrow[r, "join"] \arrow[d, "(f+f)"'] & A \arrow[r, "dup"] & A \times A \arrow[d, "(f \times f)"] \\
A' + A' \arrow[r, "join"'] & A' \arrow[r, "dup"'] & A' \times A'
\end{tikzcd}$$

$$\begin{tikzcd}
B & B \times C \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C \\
& A \arrow[ul, "f"] \arrow[u, "\langle f, g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$(f \times f) \cdot \alpha = (f + f) \cdot \alpha$

(3) $iso = \langle ! + !, join \rangle$

$! : A \longrightarrow 1$

$(id \times f) \cdot iso = iso \cdot (f + f)$

$$\begin{tikzcd}
1 + 1 \arrow[r] & (1+1) \times A & A \arrow[l, "i_2"'] \\
& A + A \arrow[ul] \arrow[u, "{\langle !+!, join \rangle}"] \arrow[ur, "{[id, id]}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
A + A \arrow[r, "iso"] \arrow[d, "f + f"'] & (1+1) \times A \arrow[d, "(id + id) \times f"] \\
A' + A' \arrow[r, "iso"] & (1+1) \times A'
\end{tikzcd}$$

$(id + id) \times f \cdot \alpha = \alpha \cdot (f + f)$

$\Leftrightarrow (id \times f) \cdot \langle ! + !, [id, id] \rangle = \langle ! + !, [id, id] \rangle \cdot (f + f)$

$\Leftrightarrow \langle id \cdot (! + !), f \cdot [id, id] \rangle = \langle (! + !) \cdot (f + f), [id, id] \cdot (f + f) \rangle$

$\Leftrightarrow \begin{cases} ! + ! = (! + !) \cdot (f + f) \\ f \cdot [id, id] = [id, id] \cdot (f + f) \end{cases}$

$\Leftrightarrow \begin{cases} ! + ! = (! \cdot f) + (! \cdot f) = ! + ! \\ [f, f] = [f, f] \end{cases} \Leftrightarrow TRUE$

(4) $[\nabla \circ |i_1| = id$

$\Leftrightarrow \begin{cases} !+! = (!\circ f)+(!\circ f) = !+! \\ [f,f] = [f,f] \end{cases} \Leftrightarrow \text{TRUE}$

$iso = \langle \circ\circ\circ \rangle$ Lei da troca

$\Leftrightarrow iso = [\langle i_1\circ !, i_2 \cdot ! \rangle, \langle i_1 \circ id, i_2 \cdot id \rangle]$

Universal-+

(4) $\begin{cases} \nabla \circ \boxed{i_1} = id \\ \nabla \circ \boxed{i_2} = id \end{cases}$

$f \circ \nabla = \nabla \cdot (f+f)$

$[id, id] = \nabla$

$f \cdot [id, id] = [id, id] \circ (f+f)$

$$\begin{tikzcd} & B & \\ A \arrow[ur, "f"] \arrow[r] & A \arrow[u, "f"] & A \arrow[ul, "f"'] \arrow[l] \\ & A+A \arrow[ul, "id"] \arrow[u, "{[id,id]}"] \arrow[ur, "id"'] & \end{tikzcd} \Rightarrow \begin{tikzcd} C \\ C+C \arrow[u, "{[id,id]}"] \\ A+A \arrow[u, "f+f"] \end{tikzcd}$$

$[f \cdot id, f \cdot id] = [id \circ f, id \circ f]$

(5) $(f+h)\circ\alpha = \alpha \circ (f+g\times h)$

$$\begin{tikzcd} A+(B\times C) \arrow[r, "\alpha"] \arrow[d, "f+(g\times h)"'] & A+B \arrow[d, "f+h"] \\ A'+(B'\times C') \arrow[r, "\alpha"] & A'+B' \end{tikzcd}$$

$$\begin{tikzcd} A+B \\ A+(B\times C) \arrow[u, "\alpha = (id+\pi_1)"] \end{tikzcd}$$

(6) $\alpha \cdot g = h \equiv g = \alpha^\circ \cdot h$

$g \cdot \alpha = h \equiv g = h \cdot \alpha^\circ$

$\underbrace{h \cdot (g \times id + g \times \alpha)}_{g} = \underbrace{k}_{h} \cdot \underbrace{\mathsf{undistr}}_{\alpha^\circ}$

$h \circ \alpha = K$

$\Leftrightarrow h \circ \alpha \cdot \alpha^\circ = K \cdot \alpha^\circ$

$\Leftrightarrow h = K \cdot \alpha^\circ$

$h \cdot \mathsf{distr} \cdot (g \times (id + \alpha)) = \underbrace{k}_{h}$

```
distl :: (Either c a, b) -> Either (c, b) (a, b)
distl = uncurry (either (curry i1)(curry i2))

distr :: (b, Either c a) -> Either (b, c) (b, a)
distr = (swap -|- swap) . distl . swap
```

$$\begin{tikzcd} A \times \arrow[r, "distr"] \arrow[d, "g + ap"'] & (B \times C) + (B \times A) \arrow[d, "(g \times h) + (g \times f)"] \\ B' + A'^{C} \arrow[r, "distr"] & (B' \times C') + (B' \times A') \end{tikzcd}$$

$h \cdot \mathsf{distr} \cdot (g \times (id + \alpha)) = k$

$$\begin{tikzcd} A \times (B + C) \arrow[r, "distr"] \arrow[d, "f \times (g + h)"'] & (A \times B) + (A \times C) \arrow[d, "(f \times g) + (f \times h)"] \\ A' \times (B' + C') \arrow[r, "distr"] & (A' \times B') + (A' \times C') \end{tikzcd}$$

$$\underbrace{h \circ ((g \times id) + (g \times \alpha))}_{g} \circ \underbrace{distr}_{\alpha} = \underbrace{k}_{h}$$

$$\underbrace{h \cdot (g \times id + g \times \alpha)}_{g} = \underbrace{k}_{h} \cdot \underbrace{\mathsf{undistr}}_{\alpha^{o}}$$

$$h \circ ((g \times id) + (g \times \alpha)) \circ distr = k$$

$$= h \circ ((g \times id) + (g \times \alpha)) \circ distr \circ undistr = k \circ undistr$$

$$= h \circ ((g \times id) + (g \times \alpha)) = k \circ undistr$$

(7) Propriedade de cancelamento da exponenciação

$$ap \circ (\overline{f} \times id) = f$$

$\Leftrightarrow$

$$curry\ f\ a\ b = f(a, b)$$

$$(ap \circ (\overline{f} \times id))(a, b) = f(a, b)$$

$$\Leftrightarrow\ ap\ (\overline{f}\ a,\ id\ b) = f(a, b)$$

$$\Leftrightarrow\ ap\ (\overline{f}\ a,\ b)$$

$$\begin{tikzcd} C^{B} \times B \arrow[r, "ap"] & C \\ \arrow[u, "\overline{f} \times id"] \arrow[ur] & \end{tikzcd}$$

$$A \times B$$

$$\text{curry } f\, a\, b = f(a,b)$$
$$\Leftrightarrow \bar{f}\, a\, b = f(a,b)$$
$$\Leftrightarrow ap(\bar{f}\, a, b) = f(a,b)$$
$$\Leftrightarrow ap(\bar{f}\, a, id\, b) = f(a,b)$$
$$\Leftrightarrow ap((\bar{f} \times id)(a,b)) = f(a,b)$$
$$\Leftrightarrow (ap \circ (\bar{f} \times id))\cancel{(a,b)} = f\cancel{(a,b)}$$
$$\Leftrightarrow ap \circ (\bar{f} \times id) = f$$

```
ghci> f (a,b) = a + b
ghci> curry f 1 2 == f (1,2)
True
```

```
ghci> ap (curry f 1, 2)
3
```

```
ghci> testFunction = ap . (curry f >< id)
ghci> testFunction (1,2) == f (1,2)
True
```

```haskell
ap_1 :: (a -> b, a) -> b
ap_1 = uncurry ($)

ap_2 :: (a -> b, a) -> b
ap_2 = uncurry (\f x -> f x)

ap_3 :: (a -> b, a) -> b
ap_3 = uncurry id

ap_4 :: (a -> b, a) -> b
ap_4 (f, a) = f a
```

Mistérios da vida

8. $\text{curry} \circ \text{uncurry } g\, a\, b = \text{uncurry } g\, (a,b)$

$$\text{uncurry } g\, (a,b) = g\, a\, b$$
$$\hat{g}\, (a,b) = g\, a\, b$$
$$\Leftrightarrow \hat{g}\, (a,b) = ap(g\, a, b)$$
$$\Leftrightarrow \hat{g}\, (a,b) = ap((g \times id)(a,b))$$
$$\Leftrightarrow \hat{g}\, \cancel{(a,b)} = (ap\, (g \times id))\cancel{(a,b)}$$
$$\Leftrightarrow \hat{g} = ap \circ (g \times id)$$

```
ghci> :{
ghci| g :: Num a -> a -> a -> a
ghci| g a b = a + b
ghci| :}
```

```
ghci> g 1 2
3
ghci> uncurry g (1,2)
3
```

```
ghci> test = ap . (g >< id)
ghci> uncurry g (1,2) == test (1,2)
True
```

(9) 1 t. Point a = Point {x :: a, y :: a, z :: a}

```
True
```

(9) data Point a = Point { x :: a, y :: a, z :: a }

Point :: a -> a -> a -> Point

$$\begin{tikzcd}
\text{Point } A \arrow[rr, bend left, "out = \langle\langle x,y\rangle, z\rangle"] & & (A\times A)\times A \arrow[ll, bend left, "in = \dots"]
\end{tikzcd}$$

$$\begin{tikzcd}
A\times A \arrow[r, dash] & (A\times A)\times A & & A \arrow[ll] \\
 & \text{Point } A \arrow[ul, "\langle x,y\rangle"] \arrow[u, "\langle\langle x,y\rangle, z\rangle"] \arrow[urr, "z"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\text{Point } A \arrow[rr, "out = \langle\langle x,y\rangle, z\rangle"] & & (A\times A)\times A \arrow[ll, bend left, "in ="]
\end{tikzcd}$$

$$\begin{tikzcd}
C^B & & C \\
A \arrow[u, "\overline{g}"] & \Leftrightarrow & A\times B \arrow[u, "g"']
\end{tikzcd}$$

$$\begin{tikzcd}
C^{A\times B} \arrow[rr, bend left, "curry"] & & (C^B)^A \arrow[ll, bend left, "uncurry"]
\end{tikzcd}$$

$$\begin{tikzcd}
\text{Point}^{A^A} & (\text{Point } A)^A \times A \arrow[rr, "ap"] & & \text{Point } A \\
A \arrow[u, "\overline{\text{Point}}"] & ((A\times A)\times A) \arrow[u, "\overline{\text{Point}} \times id"'] \arrow[urr, "ap \circ uncurry\ \text{Point}"'] & &
\end{tikzcd}$$

!

```
ghci> test1 = uncurry Point
ghci> :t test1
test1 :: (b, b) -> b -> Point b
```

```haskell
data Point a = Point { x :: a, y :: a, z :: a } deriving (
-- :t Point
-- Point :: a -> a -> a -> Point a

out :: Point a -> ((a, a), a)
out (Point x y z) = ((x, y), z)

out2 :: Point a -> ((a, a), a)
out2 = split (split x y) z

in' :: ((a, a), a) -> Point a
in' ((x, y), z)= Point x y z

in'' :: ((a, a), a) -> Point a
in'' = ap . (uncurry Point >< id)
```