Aula PL #02

26 de setembro de 2023 18:02

1. $\langle h, k \rangle \cdot f = \langle h \cdot f, k \cdot f \rangle$

$$\begin{tikzcd}
C & C \times D \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & D \\
& B \arrow[ul, "h"] \arrow[u, "\langle h,k \rangle"] \arrow[ur, "k"'] & \\
& A \arrow[u, "f"] \arrow[uul, bend left, "\langle h \cdot f ,"] \arrow[uur, bend right, "k \cdot f \rangle"'] &
\end{tikzcd}$$

Aplicar Universal-×

$$\underbrace{\langle h, k \rangle \cdot f}_{k} = \overbrace{\langle h \cdot f, k \cdot f \rangle}^{\text{SPLIT}}$$

$$\begin{cases} \pi_1 \cdot (\langle h, k \rangle \cdot f) = h \cdot f \\ \pi_2 \cdot (\langle h, k \rangle \cdot f) = k \cdot f \end{cases}$$

$$\Leftrightarrow \begin{cases} (\pi_1 \cdot \langle h, k \rangle) \cdot f = h \cdot f \\ (\pi_2 \cdot \langle h, k \rangle) \cdot f = k \cdot f \end{cases}$$

$$\Leftrightarrow \begin{cases} h \cdot f = h \cdot f \\ k \cdot f = k \cdot f \end{cases}$$

2. Uma das operações essenciais em processamento da informação é a sua duplicação:

$$x \to \begin{cases} x \\ x \end{cases} \qquad dup\ x = (x, x)$$

$$dup \cdot f = \langle f, f \rangle$$

$$\text{Fusão-}\times : \langle g, h \rangle \cdot f = \langle g \cdot h, h \cdot f \rangle$$

$$(x, x) \cdot f = \langle f, f \rangle$$

$$B \leftarrow B \times B \rightarrow B \qquad \begin{array}{c} B \times B \\ \uparrow f \\ A \times A \end{array}$$

$$
\begin{tikzcd}
B & B\times B \arrow[l] \arrow[r] & B \\
A \arrow[u, "f"] & A\times A \arrow[l] \arrow[r] \arrow[u, "f\times f"] & A \arrow[u, "f"'] \\
& A \arrow[ul, "id"] \arrow[u, "\langle id, id\rangle"] \arrow[ur, "id"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
{} \\
A\times A \arrow[u, "f"] \\
A \arrow[u, "dup"]
\end{tikzcd}
$$

$$\langle id, id\rangle \cdot f$$
$$= \langle id \cdot f, id \cdot f\rangle$$

$$dup \cdot f = \langle f, f\rangle$$
$$\Leftrightarrow \forall x \mid (dup \cdot f)\, x = \langle f, f\rangle\, x \qquad (72$$
$$\Leftrightarrow \forall x \mid dup\, (f x) = (f x, f x) \qquad (73$$
$$\Leftrightarrow \forall x \mid (f x, f x) = (f x, f x)$$

$$dup \cdot f = \langle f, f\rangle$$
$$\Leftrightarrow \langle id, id\rangle \cdot f = \langle f, f\rangle$$
$$\Leftrightarrow \langle id \cdot f, id \cdot f\rangle = \langle f \cdot f\rangle$$
$$\Leftrightarrow \langle f, f\rangle = \langle f, f\rangle$$
$$\Leftrightarrow \text{TRUE}$$

3. $assocl = \langle id \times \pi_1, \pi_2 \cdot \pi_2\rangle$

$$
\begin{tikzcd}
(A\times B)\times C \arrow[rr, bend left, "assocr"] \arrow[rr, phantom, "\cong"] & & A\times(B\times C) \arrow[ll, bend left, "assocl"]
\end{tikzcd}
$$

$$
\begin{array}{ll}
& \mathsf{assocl} \cdot \mathsf{assocr} = id \\
\equiv & \{ \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad \} \\
& \begin{cases} (id \times \pi_1) \cdot \mathsf{assocr} = \pi_1 \\ \pi_2 \cdot \pi_2 \cdot \mathsf{assocr} = \pi_2 \end{cases} \\
\equiv & \{ \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad \} \\
& \begin{cases} \begin{cases} \pi_1 \cdot \mathsf{assocr} = \pi_1 \cdot \pi_1 \\ \pi_1 \cdot \pi_2 \cdot \mathsf{assocr} = \pi_2 \cdot \pi_1 \end{cases} \\ \pi_2 \cdot \pi_2 \cdot \mathsf{assocr} = \pi_2 \end{cases} \\
\equiv & \{ \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad \} \\
& \begin{cases} \pi_1 \cdot \mathsf{assocr} = \pi_1 \cdot \pi_1 \\ \begin{cases} \pi_1 \cdot \pi_2 \cdot \mathsf{assocr} = \pi_2 \cdot \pi_1 \\ \pi_2 \cdot \pi_2 \cdot \mathsf{assocr} = \pi_2 \end{cases} \end{cases} \\
\equiv & \{ \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad \} \\
& \begin{cases} \pi_1 \cdot \mathsf{assocr} = \pi_1 \cdot \pi_1 \\ \pi_2 \cdot \mathsf{assocr} = \langle \pi_2 \cdot \pi_1, \pi_2 \rangle \end{cases} \\
\equiv & \{ \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad \} \\
& \mathsf{assocr} = \langle \pi_1 \cdot \pi_1, \pi_2 \times id \rangle
\end{array}
$$

$$\mathrm{assocl} \cdot \mathrm{assocr} = id$$

$$\Leftrightarrow \underbrace{\langle id \times \pi_1, \pi_2 \cdot \pi_2 \rangle \cdot \mathrm{assocr}}_{\text{Aplicar Fusão-}\times\text{, para transformar num SPLIT}} = id$$

$$\Leftrightarrow \langle (id \times \pi_1) \cdot \mathrm{assocr}, (\pi_2 \cdot \pi_2) \cdot \mathrm{assocr} \rangle = id \qquad \{9\}$$

$$\Leftrightarrow \begin{cases} \mathrm{assocr} \cdot (id \times \pi_1) = \pi_1 \cdot id \\ \pi_2 \cdot \pi_2 \cdot \mathrm{assocr} = \pi_2 \cdot id \end{cases} \qquad \{6, 1 \times (2)\}$$

$$\text{Def-}\times : f \times g = \langle f \cdot \pi_1, g \cdot \pi_2 \rangle$$

$$\Leftrightarrow \Big\{ \underline{\langle id \cdot \pi_1, \pi_1 \cdot \pi_2 \rangle} \circ \mathrm{assocr} = \pi_1$$

$$\Leftrightarrow \Big\{ \underline{\langle id \cdot \pi_1 \cdot \mathrm{assocr}}, \pi_1 \cdot \pi_2 \cdot \mathrm{assocr} \rangle = \pi_1$$

$$\Leftrightarrow \begin{cases} \begin{cases} \pi_1 \cdot \pi_1 = \pi_1 \cdot \mathrm{assocr} \\ \pi_2 \cdot \pi_1 = \pi_1 \cdot \pi_2 \cdot \mathrm{assocr} \end{cases} \\ \pi_2 \cdot \pi_2 \cdot \mathrm{assocr} = \pi_2 \end{cases} \qquad (6) \qquad \{ \Leftrightarrow \text{CONJUNÇÃO}$$

$$\Leftrightarrow \begin{cases} \pi_1 \cdot \pi_1 = \pi_1 \cdot assocr \\ \begin{cases} \underbrace{\pi_1}_{\pi_{1,2}} \cdot \underbrace{\pi_2 \cdot assocr}_{k} = \underbrace{\pi_2 \cdot \pi_1}_{f} \\ \pi_2 \cdot \pi_2 \cdot assocr = \underbrace{\pi_2}_{g} \end{cases} \end{cases}$$

$$\Leftrightarrow \begin{cases} \underbrace{\pi_1}_{\pi_{1,2}} \cdot \underbrace{assocr}_{k} = \underbrace{\pi_1 \cdot \pi_1}_{f} \\ \pi_2 \cdot assocr = \underbrace{\langle \pi_2 \cdot \pi_1 , \pi_2 \rangle}_{g} \end{cases}$$

$$\Leftrightarrow \quad assocr = \langle \pi_1 \cdot \pi_1 , \langle \pi_2 \cdot \pi_1 , \pi_2 \rangle \rangle$$
$$= \langle \pi_1 \cdot \pi_1 , \langle \pi_2 \cdot \pi_1 , id \cdot \pi_2 \rangle \rangle$$
$$= \langle \pi_1 \cdot \pi_1 , \pi_2 \times id \rangle \quad \text{(10) Def-}\times$$

```
ghci> assocr ((a,b),c) = (a,(b,c))
ghci> assocr ((1,2),3)
(1,(2,3))
ghci>
```

$$assocr \; ((a,b),c) = \langle \pi_1 \cdot \pi_1 , \pi_2 \times id \rangle \, ((a,b),c)$$
$$= ((\pi_1 \cdot \pi_1)((a,b),c) , (\pi_2 \times id)((a,b),c))$$
$$= (\pi_1 (\pi_1 ((a,b),c)), (\pi_2 (a,b), id (c)))$$
$$= (a, (b,c))$$

4 $\quad xr \cdot \underbrace{\langle \langle f,g \rangle , h \rangle}_{id\,?} = \langle \langle f,h \rangle , g \rangle$

$$xr = \langle \pi_1 \times id , \pi_2 \cdot \pi_1 \rangle$$

$$\{ xr \cdot id = xr = \langle \langle f,h \rangle , g \rangle \} \; \times$$

$$\Leftrightarrow \begin{cases} \pi_1 \cdot \overbrace{xr}^{k?} = \langle f,h \rangle \\ \pi_2 \cdot xr = g \end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{cases} \pi_1 \cdot (\pi_1 \cdot xr) = f \\ \pi_2 \cdot (\pi_1 \cdot xr) = h \end{cases} \\ \pi_2 \cdot xr = g \end{cases}$$

$$\begin{cases} \pi_2 \circ (\pi_1 \cdot xr) = h \\ \pi_2 \circ xr = g \end{cases}$$

$$\Leftrightarrow \begin{cases} \pi_1 \cdot (\pi_1 \cdot xr) = f \\ \pi_1 \cdot (\pi_2 \circ xr) = h \\ \pi_2 \circ xr = g \end{cases}$$

$$\Leftrightarrow \begin{cases} \pi_1 \circ (\pi_1 \circ xr) = f \\ \pi_1 \circ g = h \\ \pi_2 \cdot xr = g \end{cases}$$

$$\Leftrightarrow$$

$\times$

$$\boxed{\langle\langle f, g\rangle, h\rangle = id}$$

$$\Leftrightarrow \begin{cases} \langle f, g\rangle = \pi_1 \cdot id \\ h = \pi_2 \cdot id \end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{cases} f = \pi_1 \cdot \pi_1 \\ g = \pi_2 \cdot \pi_1 \end{cases} \\ h = \pi_2 \cdot id = \pi_2 \end{cases}$$

$$xr \circ id = \langle\langle f, h\rangle, g\rangle$$

$$xr = \langle\langle \pi_1 \cdot \pi_1, \pi_2\rangle, \pi_2 \cdot \pi_1\rangle$$

$$= \langle\langle \pi_1 \cdot \pi_1, id \times \pi_2, \pi_2 \cdot \pi_1\rangle$$

$$= \langle \pi_1 \times id, \pi_2 \circ \pi_1\rangle$$

5

$$\text{const} :: a \to b \to a$$

$$\text{const}\ a\ b = a$$

$$\underline{(b, a)} = \langle \underline{b}, \underline{g}\rangle$$

$$\underline{(b, a)} = \text{const}\ (b, a)$$

const :: $a \to b \to a$
const a $b = a$

$$\cancel{\begin{cases} \pi_1 \circ (b, a) = b \\ \pi_2 \circ (b, a) = a \end{cases}}$$

$$b \to a \to b$$

$$\text{const}\ (b, a) = \langle \text{const}\ b, \text{const}\ a)$$

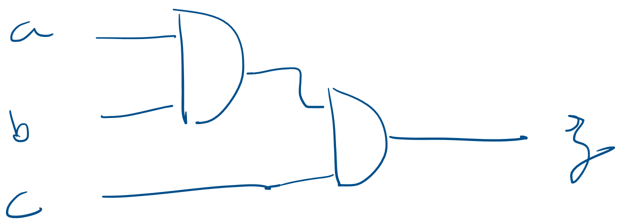
$$\begin{cases} \pi_1 \circ \text{const}\ (b, a) = \text{const}\ \underline{b} \\ \pi_2 \circ \text{const}\ (b, a) = \text{const}\ a \end{cases}$$

$$\begin{cases} \pi_1 \cdot const\,(b,a) = const\ b \\ \pi_2 \cdot const\,(b,a) = const\ a \end{cases}$$

$$\Leftrightarrow \begin{cases} \pi_1\ (const\,(b,a)) = const\ b \\ \pi_2\ (const\,(b,a)) = const\ a \end{cases}$$

$$\Leftrightarrow \begin{cases} const\ b = const\ b \\ const\ a = const\ a \end{cases}$$

⑥



$$f((a,b),c) = (a \wedge b) \oplus c$$

$\oplus \rightarrow$ exclusive or :

A	B	$A \oplus B$
False	False	False
False	True	True
True	False	True
True	True	False

$$(\mathbb{B} \times \mathbb{B}) \times \mathbb{B} \xrightarrow{f} \mathbb{B}$$

$f\ ((a,b),c) = (a \wedge b) \oplus c$

$\equiv$ { Operadores infixos para notação prefixa }

$f\ ((a,b),c) = \ \oplus\,(\wedge\,(a,b),c)$

$\equiv$ { Lei 74, Def-id }

$f\ ((a,b),c) = \ \oplus\,(\wedge\,(a,b), id\ c)$

$\equiv$ { Lei 78, Def-$\times$ }

$f\ ((a,b),c) = \ \oplus\,((\wedge \times id)\ ((a,b),c))$

$\equiv$ { Lei 73, Def-comp }

$f\ ((a,b),c) = (\oplus\ \cdot\ (\wedge \times id))\ ((a,b),c)$

$\equiv$ { Lei 72, Igualdade extensional }

$f = \oplus\ \cdot\ (\wedge \times id)$

$\cancel{A \times ((B \times B) \times B) \xrightarrow{\langle \pi_1, f \rangle} A \times B}$

$$\begin{tikzcd} (\mathbb{B}\times\mathbb{B}) & (\mathbb{B}\times\mathbb{B})\times\mathbb{B} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathbb{B} \\ & (\mathbb{B}+\mathbb{B})\times\mathbb{B} \arrow[ul, "\pi_1"] \arrow[u, "\langle \pi_1, f \rangle"] \arrow[ur] & \end{tikzcd}$$