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1 Functions and Models

1.1 Four Ways to Represent a Function

In exercises requiring estimations or approximations, your answers may vary slightly from the answers given here.

1. (a) The point $(-1, -2)$ is on the graph of f, so $f(-1) = -2$.

(b) When $x = 2$, y is about 2.8, so $f(2) \approx 2.8$.

(c) $f(x) = 2$ is equivalent to $y = 2$. When $y = 2$, we have $x = -3$ and $x = 1$.

(d) Reasonable estimates for x when $y = 0$ are $x = -2.5$ and $x = 0.3$.

(e) The domain of f consists of all x-values on the graph of f. For this function, the domain is $-3 \le x \le 3$. The range of f consists of all y-values on the graph of f. For this function, the range is $-2 \le y \le 3$.

(f) As x increases from -1 to 3, y increases from -2 to 3. Thus, f is increasing on the interval $[-1, 3]$.

2. (a) The point $(-4, -2)$ is on the graph of f, so $f(-4) = -2$. The point $(3, 4)$ is on the graph of g, so $g(3) = 4$.

(b) We are looking for the values of x for which the y-values are equal. The y-values for f and g are equal at the points $(-2, 1)$ and $(2, 2)$, so the desired values of x are -2 and 2.

(c) $f(x) = -1$ is equivalent to $y = -1$. When $y = -1$, we have $x = -3$ and $x = 4$.

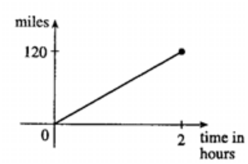
(d) As x increases from 0 to 4, y decreases from 3 to -1. Thus, f is decreasing on the interval $[0, 4]$.

(e) The domain of f consists of all x-values on the graph of f. For this function, the domain is $-4 \le x \le 4$. The range of f consists of all y-values on the graph of f. For this function, the range is $-2 \le y \le 3$.

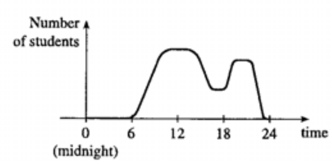
(f) The domain is $[-4, 3]$ and the range is $[0.5, 4]$.

3. From Figure 1 in the text, the lowest point occurs at about $(t, a) = (12, -85)$. The highest point occurs at about $(17, 115)$. Thus, the range of the vertical ground acceleration is $-85 \le a \le 115$. In Figure 11, the range of the north-south acceleration is approximately $-325 \le a \le 485$. In Figure 12, the range of the east-west acceleration is approximately $-210 \le a \le 200$.

4. *Example 1:* A car is driven at 60 mi/h for 2 hours. The distance d traveled by the car is a function of the time t. The domain of the function is $\{t \mid 0 \le t \le 2\}$, where t is measured in hours. The range of the function is $\{d \mid 0 \le d \le 120\}$, where d is measured in miles.

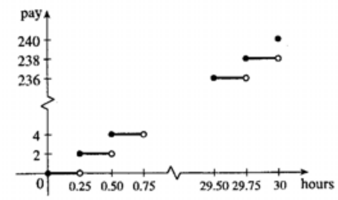


Example 2: At a certain university, the number of students N on campus at any time on a particular day is a function of the time t after midnight. The domain of the function is $\{t \mid 0 \le t \le 24\}$, where t is measured in hours. The range of the function is $\{N \mid 0 \le N \le k\}$, where N is an integer and k is the largest number of students on campus at once.



[continued]

Example 3: A certain employee is paid \$8.00 per hour and works a maximum of 30 hours per week. The number of hours worked is rounded down to the nearest quarter of an hour. This employee's gross weekly pay P is a function of the number of hours worked h. The domain of the function is $\{0, 0.25, 0.5, \ldots, 29.75, 30\}$ and the range of the function is $\{0, 2.00, 4.00, \ldots, 238.00, 240.00\}$.



5. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-3, 2]$ and the range is $[-2, 2]$.

6. No, the curve is not the graph of a function because a vertical line intersects the curve more than once and hence, the curve fails the Vertical Line Test.

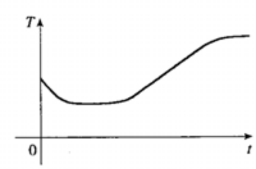
7. No, the curve is not the graph of a function since for $x = -1$ there are infinitely many points on the curve.

8. Yes, the curve is the graph of a function with domain $[-3, 2]$ and range $\{-2\} \cup (0, 3]$.

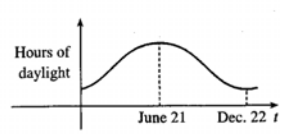
9. The person's weight increased to about 160 pounds at age 20 and stayed fairly steady for 10 years. The person's weight dropped to about 120 pounds for the next 5 years, then increased rapidly to about 170 pounds. The next 30 years saw a gradual increase to 190 pounds. Possible reasons for the drop in weight at 30 years of age: diet, exercise, health problems.

10. The salesman travels away from home from 8 to 9 A.M. and is then stationary until 10:00. The salesman travels farther away from 10 until noon. There is no change in his distance from home until 1:00, at which time the distance from home decreases until 3:00. Then the distance starts increasing again, reaching the maximum distance away from home at 5:00. There is no change from 5 until 6, and then the distance decreases rapidly until 7:00 P.M., at which time the salesman reaches home.

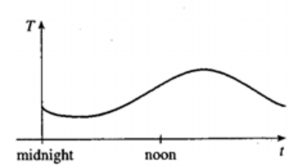
11. The water will cool down almost to freezing as the ice melts. Then, when the ice has melted, the water will slowly warm up to room temperature.



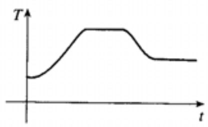
12. The summer solstice (the longest day of the year) is around June 21, and the winter solstice (the shortest day) is around December 22.



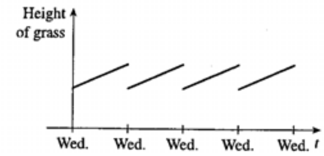
13. Of course, this graph depends strongly on the geographical location!



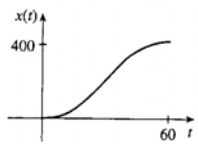
14. The temperature of the pie would increase rapidly, level off to oven temperature, decrease rapidly, and then level off to room temperature.



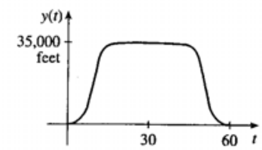
15.



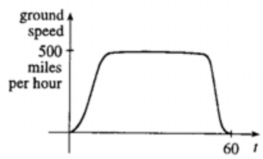
16. (a)



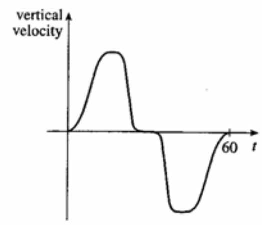
(b)



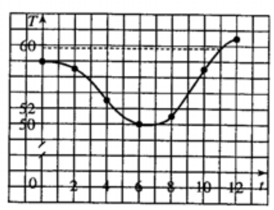
(c)



(d)

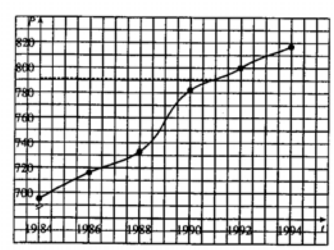


17. (a)



(b) $T(11) \approx 59°F$

18. (a)



(b) $P(1991) \approx 791{,}000$ people

19. $f(x) = 2x^2 + 3x - 4$, so $f(0) = 2(0)^2 + 3(0) - 4 = -4$,

$f(2) = 2(2)^2 + 3(2) - 4 = 10$, $f(\sqrt{2}) = 2(\sqrt{2})^2 + 3(\sqrt{2}) - 4 = 3\sqrt{2}$,

$f(1+\sqrt{2}) = 2(1+\sqrt{2})^2 + 3(1+\sqrt{2}) - 4 = 2\left(1+2+2\sqrt{2}\right) + 3 + 3\sqrt{2} - 4 = 5 + 7\sqrt{2}$,

$f(-x) = 2(-x)^2 + 3(-x) - 4 = 2x^2 - 3x - 4$,

$f(x+1) = 2(x+1)^2 + 3(x+1) - 4 = 2\left(x^2 + 2x + 1\right) + 3x + 3 - 4 = 2x^2 + 7x + 1$,

$2f(x) = 2\left(2x^2 + 3x - 4\right) = 4x^2 + 6x - 8$, and

$f(2x) = 2(2x)^2 + 3(2x) - 4 = 2\left(4x^2\right) + 6x - 4 = 8x^2 + 6x - 4$.

20. A spherical balloon with radius $r+1$ has volume $V(r+1)=\frac{4}{3}\pi(r+1)^3=\frac{4}{3}\pi\left(r^3+3r^2+3r+1\right)$. We wish to find the amount of air needed to inflate the balloon from a radius of r to $r+1$. Hence, we need to find the difference $V(r+1)-V(r)=\frac{4}{3}\pi\left(r^3+3r^2+3r+1\right)-\frac{4}{3}\pi r^3=\frac{4}{3}\pi\left(3r^2+3r+1\right)$.

21. $f(x)=x-x^2$, so $f(2+h)=2+h-(2+h)^2=2+h-4-4h-h^2=-\left(h^2+3h+2\right)$,
$f(x+h)=x+h-(x+h)^2=x+h-x^2-2xh-h^2$, and
$\dfrac{f(x+h)-f(x)}{h}=\dfrac{x+h-x^2-2xh-h^2-x+x^2}{h}=\dfrac{h-2xh-h^2}{h}=1-2x-h.$

22. $f(x)=\dfrac{x}{x+1}$, so $f(2+h)=\dfrac{2+h}{2+h+1}=\dfrac{2+h}{3+h}$, $f(x+h)=\dfrac{x+h}{x+h+1}$, and

$$\frac{f(x+h)-f(x)}{h}=\frac{\dfrac{x+h}{x+h+1}-\dfrac{x}{x+1}}{h}=\frac{(x+h)(x+1)-x(x+h+1)}{h(x+h+1)(x+1)}=\frac{1}{(x+h+1)(x+1)}.$$

23. $f(x)=\dfrac{x+2}{x^2-1}$ is defined for all x except when $x^2-1=0 \iff x=1$ or $x=-1$, so the domain is $\{x \mid x\neq\pm1\}$.

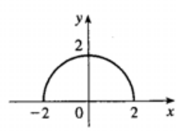
24. $f(x)=x^4/\left(x^2+x-6\right)$ is defined for all x except when $0=x^2+x-6=(x+3)(x-2) \iff x=-3$ or 2, so the domain is $\{x \mid x\neq-3,2\}$.

25. $g(x)=\sqrt[4]{x^2-6x}$ is defined when $0\le x^2-6x=x(x-6) \iff x\ge6$ or $x\le0$, so the domain is $(-\infty,0]\cup[6,\infty)$.

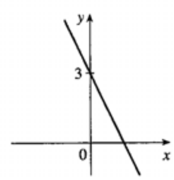
26. $h(x)=\sqrt[4]{7-3x}$ is defined when $7-3x\ge0$ or $x\le\frac{7}{3}$, so the domain is $\left(-\infty,\frac{7}{3}\right]$.

27. $f(t)=\sqrt[3]{t-1}$ is defined for every t, since every real number has a cube root. The domain is the set of all real numbers, $\mathbb{R}$.

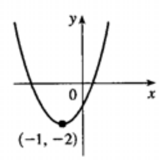
28. $h(x)=\sqrt{4-x^2}$. Now $y=\sqrt{4-x^2} \Rightarrow y^2=4-x^2 \iff x^2+y^2=4$, so the graph is the top half of a circle of radius 2 with center at the origin. The domain is $\left\{x \mid 4-x^2\ge0\right\}=[-2,2]$. From the graph, the range is $0\le y\le2$, or $[0,2]$.



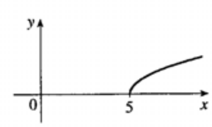
29. $f(x)=3-2x$. Domain is $\mathbb{R}$.



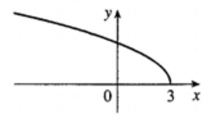
30. $f(x)=x^2+2x-1=\left(x^2+2x+1\right)-2=(x+1)^2-2$, so the graph is a parabola with vertex at $(-1,-2)$. The domain is $\mathbb{R}$.



31. $g(x)=\sqrt{x-5}$ is defined when $x-5\ge0$ or $x\ge5$, so the domain is $[5,\infty)$. Since $y=\sqrt{x-5} \Rightarrow y^2=x-5 \Rightarrow x=y^2+5$, we see that g is the top half of a parabola.



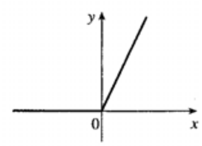
32. $g(x) = \sqrt{6 - 2x}$. The domain is $\{x \mid 6 - 2x \geq 0\} = (-\infty, 3]$.



33. $G(x) = |x| + x$. Since $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ we have

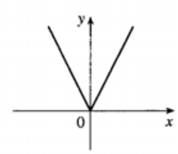
$$G(x) = \begin{cases} x + x & \text{if } x \geq 0 \\ -x + x & \text{if } x < 0 \end{cases} = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Domain is $\mathbb{R}$. Note that the negative x-axis is part of the graph of G.



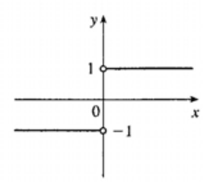
34. $H(x) = |2x| = \left\{ \begin{array}{ll} 2x & \text{if } 2x \geq 0 \\ -2x & \text{if } 2x < 0 \end{array} \right\} = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$

Domain is $\mathbb{R}$.

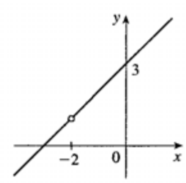


35. $f(x) = \dfrac{x}{|x|} = \left\{ \begin{array}{ll} x/x & \text{if } x > 0 \\ x/(-x) & \text{if } x < 0 \end{array} \right\} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

Note that we did not use $x \geq 0$, because $x \neq 0$. Hence, the domain of f is $\{x \mid x \neq 0\}$.

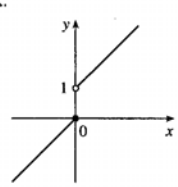


36. $f(x) = \dfrac{x^2 + 5x + 6}{x + 2} = \dfrac{(x + 3)(x + 2)}{x + 2}$, so for $x \neq -2$, $f(x) = x + 3$. Domain is $\{x \mid x \neq -2\}$. The hole in the graph can be found using the simplified function, $h(x) = x + 3$. $h(-2) = 1$ indicates that the hole has coordinates $(-2, 1)$.



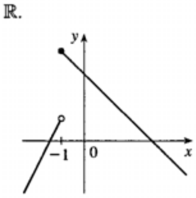
37. $f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x + 1 & \text{if } x > 0 \end{cases}$

Domain is $\mathbb{R}$.



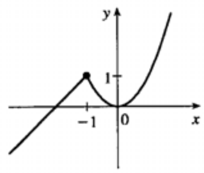
38. $f(x) = \begin{cases} 2x + 3 & \text{if } x < -1 \\ 3 - x & \text{if } x \geq -1 \end{cases}$

Domain is $\mathbb{R}$.



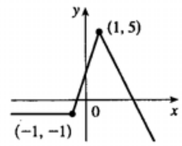
39. $f(x) = \begin{cases} x+2 & \text{if } x \le -1 \\ x^2 & \text{if } x > -1 \end{cases}$

Domain is $\mathbb{R}$.



40. $f(x) = \begin{cases} -1 & \text{if } x \le -1 \\ 3x+2 & \text{if } -1 < x < 1 \\ 7-2x & \text{if } x \ge 1 \end{cases}$

Domain is $\mathbb{R}$.



41. Recall that the slope m of a line between the two points (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and an equation of the line connecting those two points is $y - y_1 = m(x - x_1)$. The slope of this line segment is $\dfrac{-6-1}{4-(-2)} = -\dfrac{7}{6}$, so an equation is $y - 1 = -\frac{7}{6}(x+2)$. The function is $f(x) = -\frac{7}{6}x - \frac{4}{3}$, $-2 \le x \le 4$.

42. The slope of this line segment is $\dfrac{3-(-2)}{6-(-3)} = \dfrac{5}{9}$, so an equation is $y + 2 = \frac{5}{9}(x+3)$. The function is $f(x) = \frac{5}{9}x - \frac{1}{3}$, $-3 \le x \le 6$.

43. We need to solve the given equation for y. $x + (y-1)^2 = 0 \Rightarrow (y-1)^2 = -x \Rightarrow y - 1 = \pm\sqrt{-x} \Rightarrow y = 1 \pm \sqrt{-x}$. The expression with the positive radical represents the top half of the parabola, and the one with the negative radical represents the bottom half. Hence, we want $f(x) = 1 - \sqrt{-x}$, $x \le 0$.

44. $(x-1)^2 + y^2 = 1 \Rightarrow y = \pm\sqrt{1-(x-1)^2} = \pm\sqrt{2x - x^2}$. The top half is given by the function $f(x) = \sqrt{2x - x^2}$, $0 \le x \le 2$.

45. For $-1 \le x \le 2$, the graph is the line with slope 1 and y-intercept 1, that is, the line $y = x + 1$. For $2 < x \le 4$, the graph is the line with slope $-\frac{3}{2}$ and x-intercept 4, so $y = -\frac{3}{2}(x-4) = -\frac{3}{2}x + 6$. So the function is

$$f(x) = \begin{cases} x+1 & \text{if } -1 \le x \le 2 \\ -\frac{3}{2}x + 6 & \text{if } 2 < x \le 4 \end{cases}$$

46. For $x \le 0$, the graph is the line $y = 2$. For $0 < x \le 1$, the graph is the line with slope -2 and y-intercept 2, that is, the line $y = -2x + 2$. For $x > 1$, the graph is the line with slope 1 and x-intercept 1, that is, the line $y = 1(x-1) = x - 1$. So the function is $f(x) = \begin{cases} 2 & \text{if } x \le 0 \\ -2x+2 & \text{if } 0 < x \le 1 \\ x-1 & \text{if } 1 < x \end{cases}$

47. Let the length and width of the rectangle be L and W. Then the perimeter is $2L + 2W = 20$ and the area is $A = LW$. Solving the first equation for W in terms of L gives $W = \dfrac{20 - 2L}{2} = 10 - L$. Thus, $A(L) = L(10 - L) = 10L - L^2$. Since lengths are positive, the domain of A is $0 < L < 10$. If we further restrict L to be larger than W, then $5 < L < 10$ would be the domain.

48. Let the length and width of the rectangle be L and W. Then the area is $LW = 16$, so that $W = 16/L$. The perimeter is $P = 2L + 2W$, so $P(L) = 2L + 2(16/L) = 2L + 32/L$, and the domain of P is $L > 0$, since lengths must be positive quantities.

49. Let the length of a side of the equilateral triangle be x. Then by the Pythagorean Theorem, the height y of the triangle satisfies $y^2 + \left(\frac{1}{2}x\right)^2 = x^2$, so that $y = \frac{\sqrt{3}}{2}x$. Using the formula for the area A of a triangle, $A = \frac{1}{2}$ (base) (height), we obtain $A(x) = \frac{1}{2}(x)\left(\frac{\sqrt{3}}{2}x\right) = \frac{\sqrt{3}}{4}x^2$, with domain $x > 0$.

50. Let the volume of the cube be V and the length of an edge be L. Then $V = L^3$ so $L = \sqrt[3]{V}$, and the surface area is $S(V) = 6\left(\sqrt[3]{V}\right)^2 = 6V^{2/3}$, with domain $V > 0$.

51. Let each side of the base of the box have length x, and let the height of the box be h. Since the volume is 2, we know that $2 = hx^2$, so that $h = 2/x^2$, and the surface area is $S = x^2 + 4xh$. Thus, $S(x) = x^2 + 4x\left(2/x^2\right) = x^2 + 8/x$, with domain $x > 0$.

52. The area of the window is $A = xh + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = xh + \frac{\pi x^2}{8}$, where h is the height of the rectangular portion of the window. The perimeter is $P = 2h + x + \frac{1}{2}\pi x = 30 \Leftrightarrow 2h = 30 - x - \frac{1}{2}\pi x \Leftrightarrow h = \frac{1}{4}(60 - 2x - \pi x)$. Thus,

$$A(x) = x\frac{60 - 2x - \pi x}{4} + \frac{\pi x^2}{8} = 15x - \frac{1}{2}x^2 - \frac{\pi}{4}x^2 + \frac{\pi}{8}x^2 = 15x - \frac{4}{8}x^2 - \frac{\pi}{8}x^2 = 15x - x^2\left(\frac{\pi + 4}{8}\right)$$

Since the lengths x and h must be positive quantities, we have $x > 0$ and $h > 0$. For $h > 0$, we have $2h > 0 \Leftrightarrow 30 - x - \frac{1}{2}\pi x > 0 \Leftrightarrow 60 > 2x + \pi x \Leftrightarrow x < \frac{60}{2 + \pi}$. Hence, the domain of A is $0 < x < \frac{60}{2 + \pi}$.

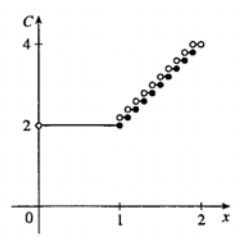
53. The height of the box is x and the length and width are $L = 20 - 2x$, $W = 12 - 2x$. Then $V = LWx$ and so

$$\begin{aligned} V(x) &= (20 - 2x)(12 - 2x)(x) = 4(10 - x)(6 - x)(x) = 4x\left(60 - 16x + x^2\right) \\ &= 4x^3 - 64x^2 + 240x \end{aligned}$$

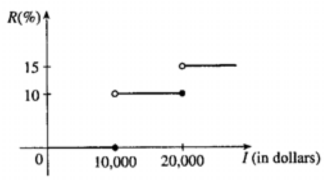
The sides L, W, and x must be positive. Thus, $L > 0 \Leftrightarrow 20 - 2x > 0 \Leftrightarrow x < 10$; $w > 0 \Leftrightarrow 12 - 2x > 0 \Leftrightarrow x < 6$; and $x > 0$. Combining these restrictions gives us the domain $0 < x < 6$.

54.

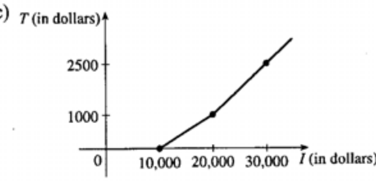
$$C(x) = \begin{cases} \$2.00 & \text{if } 0.0 < x \le 1.0 \\ 2.20 & \text{if } 1.0 < x \le 1.1 \\ 2.40 & \text{if } 1.1 < x \le 1.2 \\ 2.60 & \text{if } 1.2 < x \le 1.3 \\ 2.80 & \text{if } 1.3 < x \le 1.4 \\ 3.00 & \text{if } 1.4 < x \le 1.5 \\ 3.20 & \text{if } 1.5 < x \le 1.6 \\ 3.40 & \text{if } 1.6 < x \le 1.7 \\ 3.60 & \text{if } 1.7 < x \le 1.8 \\ 3.80 & \text{if } 1.8 < x \le 1.9 \\ 4.00 & \text{if } 1.9 < x < 2.0 \end{cases}$$



55. (a)



(c)



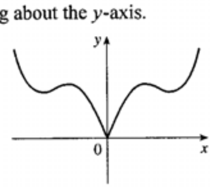
(b) On \$14,000, tax is assessed on \$4000, and $10\%\,(\$4000) = \$400$.
On \$26,000, tax is assessed on \$16,000, and $10\%\,(\$10{,}000) + 15\%\,(\$6000) = \$1000 + \$900 = \$1900$.

56. One example is the amount paid for cable or telephone system repair in the home, usually measured to the nearest quarter hour. Another example is the amount paid by a student in tuition fees, if the fees vary according to the number of credits for which the student has registered.

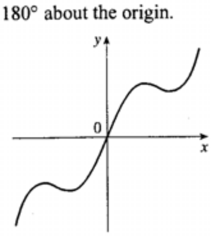
57. (a) Because an even function is symmetric with respect to the y-axis, and the point $(5, 3)$ is on the graph of this even function, the point $(-5, 3)$ must also be on its graph.

(b) Because an odd function is symmetric with respect to the origin, and the point $(5, 3)$ is on the graph of this odd function, the point $(-5, -3)$ must also be on its graph.

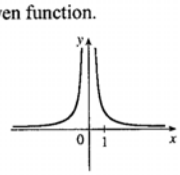
58. (a) If f is even, we get the rest of the graph by reflecting about the y-axis.



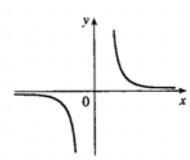
(b) If f is odd, we get the rest of the graph by rotating 180° about the origin.



59. $f(-x) = (-x)^{-2} = \dfrac{1}{(-x)^2} = \dfrac{1}{x^2}$
$= x^{-2} = f(x)$
so f is an even function.

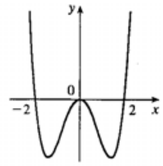


60. $f(-x) = (-x)^{-3} = \dfrac{1}{(-x)^3} = \dfrac{1}{-x^3}$
$= -\dfrac{1}{x^3} = -\left(x^{-3}\right) = -f(x)$
so f is odd.

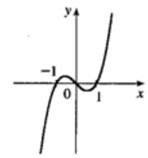


61. $f(-x) = (-x)^2 + (-x) = x^2 - x$. Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.

62. $f(-x) = (-x)^4 - 4(-x)^2$
$= x^4 - 4x^2 = f(x)$
so f is even.



63. $f(-x) = (-x)^3 - (-x) = -x^3 + x$
$= -(x^3 - x) = -f(x)$
so f is odd.



64. $f(-x) = 3(-x)^3 + 2(-x)^2 + 1 = -3x^3 + 2x^2 + 1$. Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.

1.2 Mathematical Models

1. (a) $f(x) = \sqrt[5]{x}$ is a root function.

(b) $g(x) = \sqrt{1 - x^2}$ is an algebraic function because it is a root of a polynomial.

(c) $h(x) = x^9 + x^4$ is a polynomial of degree 9.

(d) $r(x) = \dfrac{x^2 + 1}{x^3 + x}$ is a rational function because it is a ratio of polynomials.

(e) $s(x) = \tan 2x$ is a trigonometric function.

(f) $t(x) = \log_{10} x$ is a logarithmic function.

2. (a) $y = (x - 6)/(x + 6)$ is a rational function because it is a ratio of polynomials.

(b) $y = x + x^2/\sqrt{x - 1}$ is an algebraic function because it involves polynomials and roots of polynomials.

(c) $y = 10^x$ is an exponential function (notice that x is the *exponent*).

(d) $y = x^{10}$ is a power function (notice that x is the *base*).

(e) $y = 2t^6 + t^4 - \pi$ is a polynomial of degree 6.

(f) $y = \cos\theta + \sin\theta$ is a trigonometric function.

3. We notice from the figure that g and h are even functions (symmetric with respect to the y-axis) and that f is an odd function (symmetric with respect to the origin). So (b) $[y = x^5]$ must be f. Since g is flatter than h near the origin, we must have (c) $[y = x^8]$ matched with g and (a) $[y = x^2]$ matched with h.

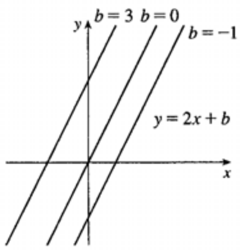
4. (a) The graph of $y = 3x$ is a line (choice G).

(b) $y = 3^x$ is an exponential function (choice f).

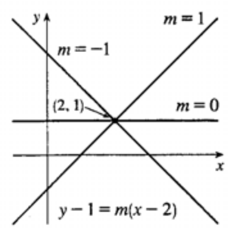
(c) $y = x^3$ is an odd polynomial function or power function (choice F).

(d) $y = \sqrt[3]{x} = x^{1/3}$ is a root function (choice g).

5. (a) An equation for the family of linear functions with slope 2 is $y = f(x) = 2x + b$, where b is the y-intercept.

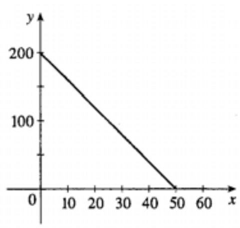


(b) $f(2) = 1$ means that the point $(2, 1)$ is on the graph of f. We can use the point-slope form of a line to obtain an equation for the family of linear functions through the point $(2, 1)$. $y - 1 = m(x - 2)$, which is equivalent to $y = mx + (1 - 2m)$ in slope-intercept form.



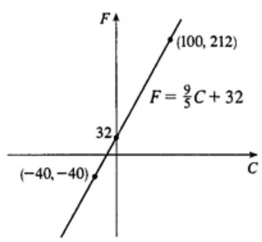
(c) The slope m must equal 2, so the equation in part (b), $y = mx + (1 - 2m)$, becomes $y = 2x - 3$. It is the *only* function that belongs to both families.

6. (a)



(b) The slope of -4 means that for each increase of 1 dollar for a rental space, the number of spaces rented *decreases* by 4. The y-intercept of 200 is the number of spaces that would be occupied if there were no charge for each space. The x-intercept of 50 is the smallest rental fee that results in no spaces rented.

7. (a)

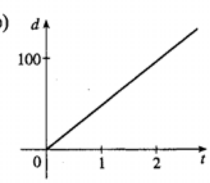


(b) The slope of $\frac{9}{5}$ means that F increases $\frac{9}{5}$ degrees for each increase of 1°C. (Equivalently, F increases by 9 when C increases by 5 and F decreases by 9 when C decreases by 5.) The F-intercept of 32 is the Fahrenheit temperature corresponding to a Celsius temperature of 0.

8. (a) Let d = distance traveled (in miles) and t = time elapsed (in hours). At $t = 0$, $d = 0$ and at $t = 50$ minutes $= 50 \cdot \frac{1}{60} = \frac{5}{6}$ h, $d = 40$. Thus we have two points: $(0, 0)$ and $\left(\frac{5}{6}, 40\right)$, so

$$m = \frac{40 - 0}{\frac{5}{6} - 0} = 48 \text{ and so } d = 48t.$$

(b)



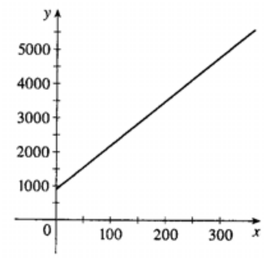
(c) The slope is 48 and represents the car's speed in mi/h.

9. (a) Using N in place of x and T in place of y, we find the slope to be $\dfrac{T_2 - T_1}{N_2 - N_1} = \dfrac{80 - 70}{173 - 113} = \dfrac{10}{60} = \dfrac{1}{6}$. So a linear equation is $T - 80 = \frac{1}{6}(N - 173) \Leftrightarrow T - 80 = \frac{1}{6}N - \frac{173}{6} \Leftrightarrow T = \frac{1}{6}N + \frac{307}{6}$ $[\frac{307}{6} = 51.1\overline{6}]$.

(b) The slope of $\frac{1}{6}$ means that the temperature in Fahrenheit degrees increases one-sixth as rapidly as the number of cricket chirps per minute. Said differently, each increase of 6 cricket chirps per minute corresponds to an increase of 1°F.

(c) When $N = 150$, the temperature is given approximately by $T = \frac{1}{6}(150) + \frac{307}{6} = 76.1\overline{6}°\text{F} \approx 76°\text{F}$.

10. (a) Let x denote the number of chairs produced in one day and y the associated cost. Using the points (100, 2200) and (300, 4800) we get the slope $\frac{4800-2200}{300-100} = \frac{2600}{200} = 13$. So $y - 2200 = 13(x - 100) \Leftrightarrow y = 13x + 900$.



(b) The slope of the line in part (a) is 13 and it represents the cost of producing each additional chair.

(c) The y-intercept is 900 and it represents the fixed daily costs of operating the factory.

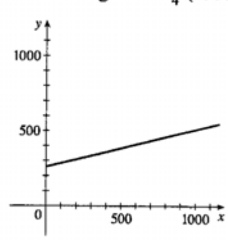
11. (a) We are given $\dfrac{\text{change in pressure}}{10 \text{ feet change in depth}} = \dfrac{4.34}{10} = 0.434$. Using P for pressure and d for depth with the point $(d, P) = (0, 15)$, we have $P - 15 = 0.434(d - 0) \Leftrightarrow P = 0.434d + 15$.

(b) When $P = 100$, then $100 = 0.434d + 15 \Leftrightarrow 0.434d = 85 \Leftrightarrow d \approx 195.85$ feet. Thus, the pressure is 100 lb/in^2 at a depth of approximately 196 feet.

12. (a) Using d in place of x and C in place of y, we find the slope to be $\dfrac{C_2 - C_1}{d_2 - d_1} = \dfrac{460 - 380}{800 - 480} = \dfrac{80}{320} = \dfrac{1}{4}$. So a linear equation is $C - 460 = \frac{1}{4}(d - 800) \Leftrightarrow C - 460 = \frac{1}{4}d - 200 \Leftrightarrow C = \frac{1}{4}d + 260$.

(b) Letting $d = 1500$ we get $C = \frac{1}{4}(1500) + 260 = 635$. The cost of driving 1500 miles is \$635.

(c)



The slope of the line represents the cost per mile, \$0.25.

(d) The y-intercept represents the fixed cost, \$260.

(e) Because you have fixed monthly costs such as insurance and car payments, as well as costs that increase as you drive, such as gasoline, oil, and tires, and the cost of these for each additional mile driven is a constant.

13. (a) The data appear to be periodic and a sine or cosine function would make the best model. A model of the form $f(x) = a\cos(bx) + c$ seems appropriate.

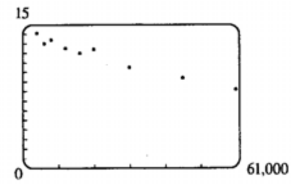
(b) The data appear to be decreasing in a linear fashion. A model of the form $f(x) = mx + b$ seems appropriate.

14. (a) The data appear to be increasing exponentially. A model of the form $f(x) = a \cdot b^x$ or $f(x) = a \cdot b^x + c$ seems appropriate.

(b) The data appear to be decreasing similarly to the values of the reciprocal function. A model of the form $f(x) = a/x$ seems appropriate.

Some values are given to many decimal places. These are the results given by several computer algebra systems — rounding is left to the reader.

15. (a)

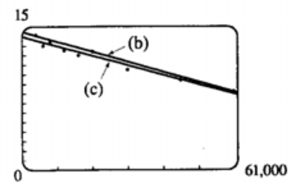


A linear model does seem appropriate.

(b) Using the points (4000, 14.1) and (60,000, 8.2), we obtain

$y - 14.1 = \dfrac{8.2 - 14.1}{60{,}000 - 4000}(x - 4000)$ or, equivalently,

$y \approx -0.000105357x + 14.521429.$



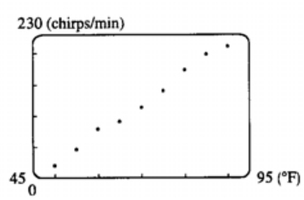
(c) Using a computing device, we obtain the least squares regression line $y = -0.0000997855x + 13.950764$.

(d) When $x = 25{,}000$, $y \approx 11.456$; or about 11.5 per 100 population.

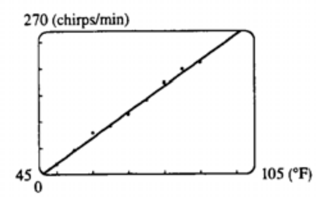
(e) When $x = 80{,}000$, $y \approx 5.968$; or about a 6% chance.

(f) When $x = 200{,}000$, y is negative, so the model does not apply.

16. (a)



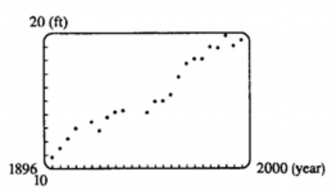
(b)



Using a computing device, we obtain the least squares regression line $y = 4.85\overline{6}x - 220.9\overline{6}$.

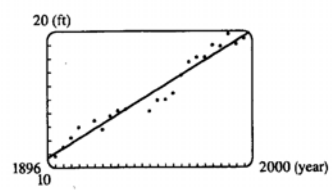
(c) When $x = 100°$F, $y = 264.7 \approx 265$ chirps/min.

17. (a)



A linear model does seem appropriate.

(b)

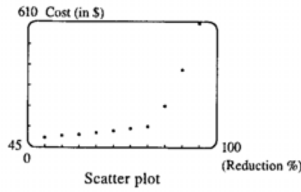


Using a computing device, we obtain the least squares regression line $y = -158.2403249x + 0.089119747$, where x is the year and y is the height in feet.

(c) When $x = 2000$, $y \approx 20.00$ ft.

(d) When $x = 2100$, $y \approx 28.91$ ft. This would be an increase of 9.49 ft from 1996 to 2100. Even though there was an increase of 8.59 ft from 1900 to 1996, it is unlikely that a similar increase will occur over the next 100 years.

18. By looking at the scatter plot of the data, we rule out the linear and logarithmic models.



Scatter plot

We try various models:

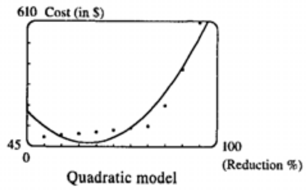
Quadratic: $y = 0.49\overline{6}x^2 - 62.28\overline{93}x + 1970.6\overline{39}$

Cubic: $y = 0.0201243201x^3 - 3.88037296x^2 + 247.6754468x - 5163.935198$

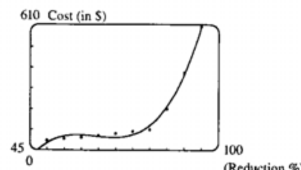
Quartic: $y = 0.0002951049x^4 - 0.0654560995x^3 + 5.27525641x^2 - 180.2266511x + 2203.210956$

Exponential: $y = 2.41422994\,(1.054516914)^x$

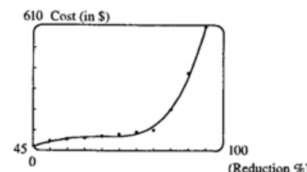
Power: $y = 0.000022854971x^{3.616078251}$



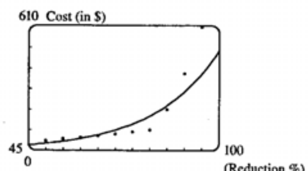
Quadratic model



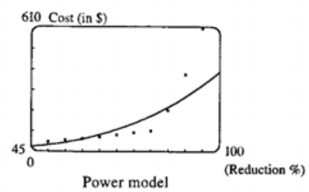
Cubic model



Quartic model



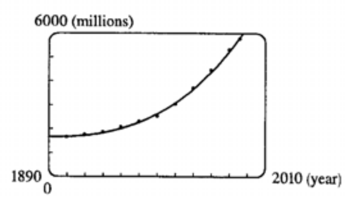
Exponential model



Power model

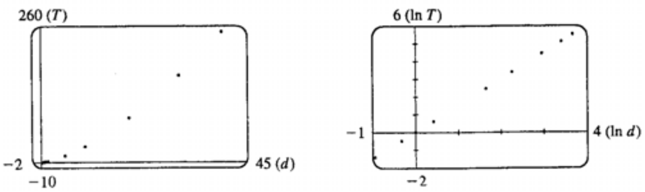
After examining the graphs of these models, we see that the cubic and quartic models are clearly the best.

19.



Using a computing device, we obtain the cubic function $y = ax^3 + bx^2 + cx + d$ with $a = 0.00232567051876$, $b = -13.064877957628$, $c = 24{,}463.10846422$, and $d = -15{,}265{,}793.872507$. When $x = 1925$, $y \approx 1922$ (millions).

20. (a)



The graph of T vs. d appears to be that of a power function and the graph of $\ln T$ vs. $\ln d$ appears to be linear, so a power model seems reasonable.

(b) $T = 1.000396048d^{1.499661718}$

(c) The power model in part (b) is approximately $T = d^{1.5}$. Squaring both sides gives us $T^2 = d^3$, so the model matches Kepler's Third Law, $T^2 = kd^3$.

1.3 New Functions from Old Functions

1. (a) If the graph of f is shifted 3 units upward, its equation becomes $y = f(x) + 3$.

(b) If the graph of f is shifted 3 units downward, its equation becomes $y = f(x) - 3$.

(c) If the graph of f is shifted 3 units to the right, its equation becomes $y = f(x - 3)$.

(d) If the graph of f is shifted 3 units to the left, its equation becomes $y = f(x + 3)$.

(e) If the graph of f is reflected about the x-axis, its equation becomes $y = -f(x)$.

(f) If the graph of f is reflected about the y-axis, its equation becomes $y = f(-x)$.

(g) If the graph of f is stretched vertically by a factor of 3, its equation becomes $y = 3f(x)$.

(h) If the graph of f is shrunk vertically by a factor of 3, its equation becomes $y = \frac{1}{3}f(x)$.

2. (a) To obtain the graph of $y = 5f(x)$ from the graph of $y = f(x)$, stretch the graph vertically by a factor of 5.

(b) To obtain the graph of $y = f(x - 5)$ from the graph of $y = f(x)$, shift the graph 5 units to the right.

(c) To obtain the graph of $y = -f(x)$ from the graph of $y = f(x)$, reflect the graph about the x-axis.

(d) To obtain the graph of $y = -5f(x)$ from the graph of $y = f(x)$, stretch the graph vertically by a factor of 5 and reflect it about the x-axis.

(e) To obtain the graph of $y = f(5x)$ from the graph of $y = f(x)$, shrink the graph horizontally by a factor of 5.

(f) To obtain the graph of $y = 5f(x) - 3$ from the graph of $y = f(x)$, stretch the graph vertically by a factor of 5 and shift it 3 units downward.

3. (a) (graph 3) The graph of f is shifted 4 units to the right and has equation $y = f(x - 4)$.

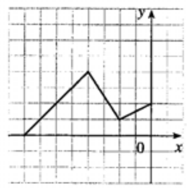
(b) (graph 1) The graph of f is shifted 3 units upward and has equation $y = f(x) + 3$.

(c) (graph 4) The graph of f is shrunk vertically by a factor of 3 and has equation $y = \frac{1}{3}f(x)$.

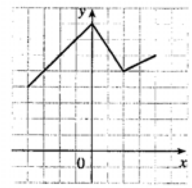
(d) (graph 5) The graph of f is shifted 4 units to the left and reflected about the x-axis. Its equation is $y = -f(x + 4)$.

(e) (graph 2) The graph of f is shifted 6 units to the left and stretched vertically by a factor of 2. Its equation is $y = 2f(x + 6)$.

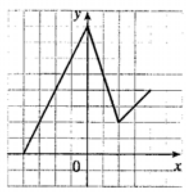
4. (a) To graph $y = f(x+4)$ we shift the graph of f 4 units to the left.



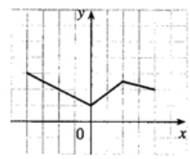
(b) To graph $y = f(x) + 4$ we shift the graph of f 4 units upward.



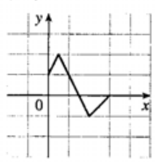
(c) To graph $y = 2f(x)$ we stretch the graph of f vertically by a factor of 2.



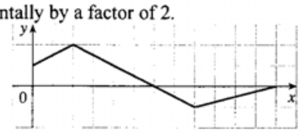
(d) To graph $y = -\frac{1}{2}f(x) + 3$, we shrink the graph of f vertically by a factor of 2, then reflect the resulting graph about the x-axis, then shift the resulting graph 3 units upward.



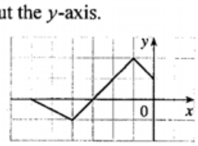
5. (a) To graph $y = f(2x)$ we shrink the graph of f horizontally by a factor of 2.



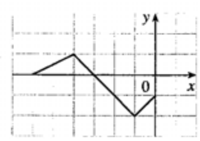
(b) To graph $y = f\left(\frac{1}{2}x\right)$ we stretch the graph of f horizontally by a factor of 2.



(c) To graph $y = f(-x)$ we reflect the graph of f about the y-axis.



(d) To graph $y = -f(-x)$ we reflect the graph of f about the y-axis, then about the x-axis.



6. The graph of $y = f(x) = \sqrt{3x - x^2}$ has been shifted 2 units to the right and stretched vertically by a factor of 2. Thus, a function describing the graph is

$$y = 2f(x-2) = 2\sqrt{3(x-2) - (x-2)^2} = 2\sqrt{3x - 6 - (x^2 - 4x + 4)} = 2\sqrt{-x^2 + 7x - 10}$$

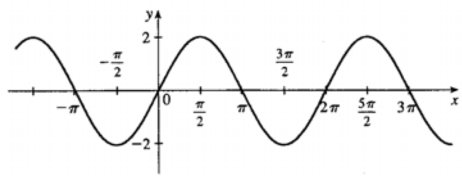
7. The graph of $y = f(x) = \sqrt{3x - x^2}$ has been shifted 4 units to the left, reflected about the x-axis, and shifted downward 1 unit. Thus, a function describing the graph is

$$y = \underbrace{-1\cdot}_{\substack{\text{reflect}\\ \text{about}\\ x\text{-axis}}} \; \underbrace{f(x+4)}_{\substack{\text{shift}\\ \text{4 units}\\ \text{left}}} \; \underbrace{-1}_{\substack{\text{shift}\\ \text{1 unit}\\ \text{down}}}$$

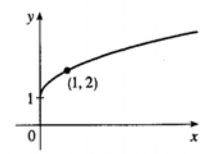
This function can be written as

$$\begin{aligned} y &= -f(x+4) - 1 = -\sqrt{3(x+4) - (x+4)^2} - 1 \\ &= -\sqrt{3x + 12 - (x^2 + 8x + 16)} - 1 = -\sqrt{-x^2 - 5x - 4} - 1 \end{aligned}$$

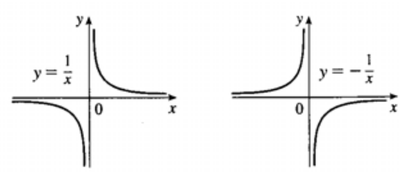
8. (a) The graph of $y = 2\sin x$ can be obtained from the graph of $y = \sin x$ by stretching it vertically by a factor of 2.



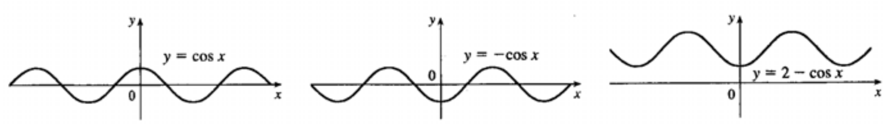
(b) The graph of $y = 1 + \sqrt{x}$ can be obtained from the graph of $y = \sqrt{x}$ by shifting it upward 1 unit.



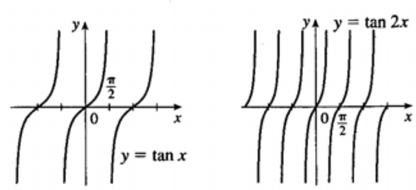
9. $y = -1/x$: Start with the graph of $y = 1/x$ and reflect about the x-axis.



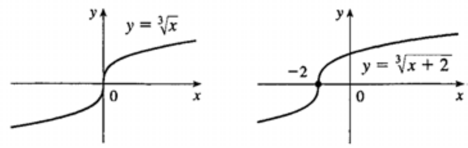
10. $y = 2 - \cos x$: Start with the graph of $y = \cos x$, reflect about the x-axis, and then shift 2 units upward.



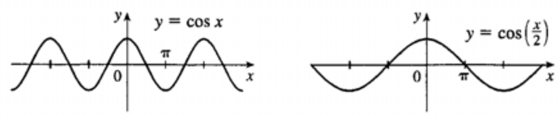
11. $y = \tan 2x$: Start with the graph of $y = \tan x$ and compress horizontally by a factor of 2.



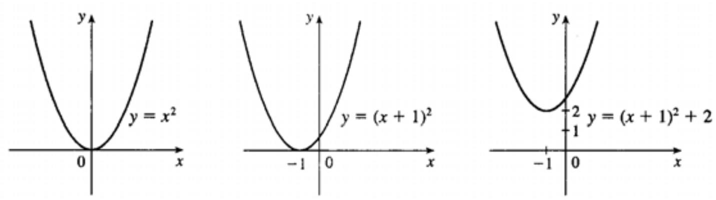
12. $y = \sqrt[3]{x+2}$: Start with the graph of $y = \sqrt[3]{x}$ and shift 2 units to the left.



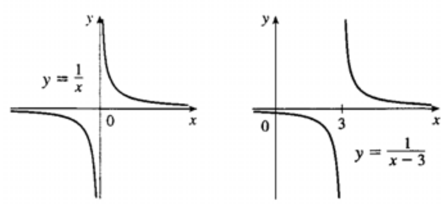
13. $y = \cos(x/2)$: Start with the graph of $y = \cos x$ and stretch horizontally by a factor of 2.



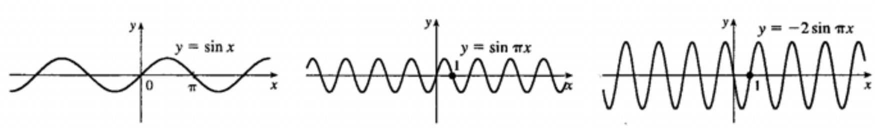
14. $y = x^2 + 2x + 3 = (x^2 + 2x + 1) + 2 = (x+1)^2 + 2$: Start with the graph of $y = x^2$, shift 1 unit left, and then shift 2 units upward.



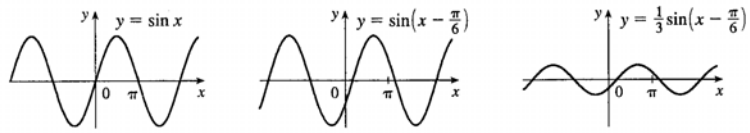
15. $y = \dfrac{1}{x-3}$: Start with the graph of $y = 1/x$ and shift 3 units to the right.



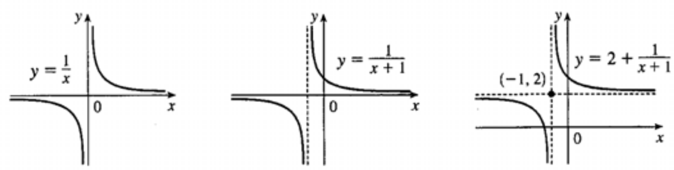
16. $y = -2\sin \pi x$: Start with the graph of $y = \sin x$, compress horizontally by a factor of π, stretch vertically by a factor of 2, and then reflect about the x-axis.



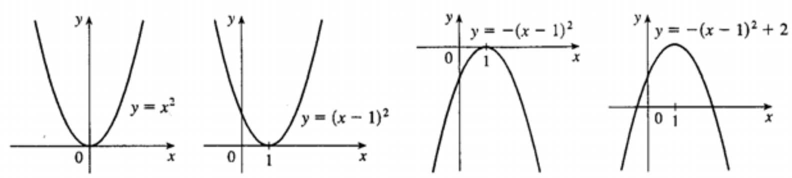
17. $y = \frac{1}{3}\sin\left(x - \frac{\pi}{6}\right)$: Start with the graph of $y = \sin x$, shift $\frac{\pi}{6}$ units to the right, and then compress vertically by a factor of 3.



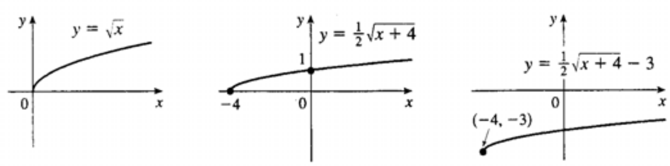
18. $y = 2 + \dfrac{1}{x+1}$: Start with the graph of $y = 1/x$, shift 1 unit left, and then shift 2 units upward.



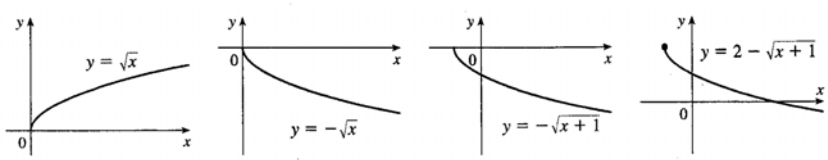
19. $y = 1 + 2x - x^2 = -x^2 + 2x + 1 = -\left(x^2 - 2x + 1\right) + 1 + 1 = -(x-1)^2 + 2$: Start with the graph of $y = x^2$, shift 1 unit right, reflect about the x-axis, and then shift 2 units upward.



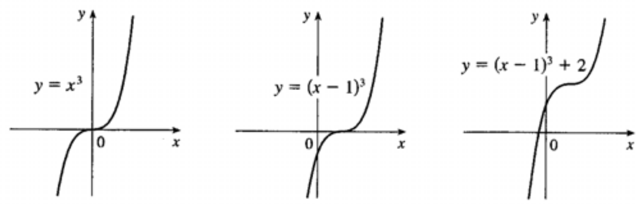
20. $y = \frac{1}{2}\sqrt{x+4} - 3$: Start with the graph of $y = \sqrt{x}$, shift 4 units to the left and compress vertically by a factor of 2, and then shift 3 units downward.



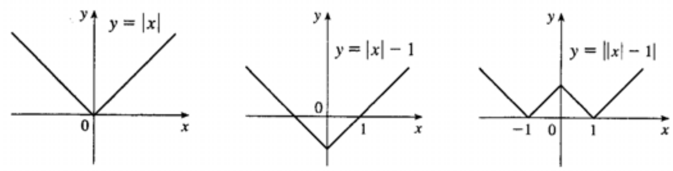
21. $y = 2 - \sqrt{x+1}$: Start with the graph of $y = \sqrt{x}$, reflect about the x-axis, shift 1 unit to the left, and then shift 2 units upward.



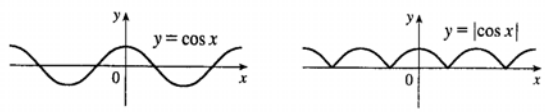
22. $y = (x-1)^3 + 2$: Start with the graph of $y = x^3$, shift 1 unit to the right, and then shift 2 units upward.



23. $y = ||x| - 1|$: Start with the graph of $y = |x|$, shift 1 unit downward, and then reflect the part of the graph from $x = -1$ to $x = 1$ about the x-axis.



24. $y = |\cos x|$: Start with the graph of $y = \cos x$ and reflect the parts of the graph that lie below the x-axis about the x-axis.

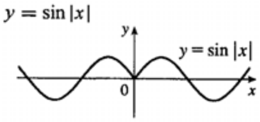


25. This is just like the solution to Example 4 except the amplitude of the curve is $14 - 12 = 2$. So the function is $L(t) = 12 + 2\sin\left[\frac{2\pi}{365}(t - 80)\right]$. March 31 is the 90th day of the year, so the model gives $L(90) \approx 12.34$ h. The daylight time (5:51 A.M. to 6:18 P.M.) is 12 hours and 27 minutes, or 12.45 h. The model value differs from the actual value by $\frac{12.45 - 12.34}{12.45} \approx 0.009$, less than 1%.

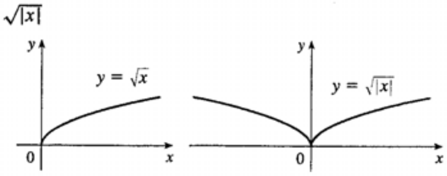
26. Using a sine function to model the brightness of Delta Cephei as a function of time, we take its period to be 5.4 days, its amplitude to be 0.35 (on the scale of magnitude), and its average magnitude to be 4.0. If we take $t = 0$ at a time of average brightness, then the magnitude (brightness) as a function of time t in days can be modeled by the formula $M(t) = 4.0 + 0.35\sin\left(\frac{2\pi}{5.4}t\right)$.

27. (a) To obtain $y = f(|x|)$, the portion of $y = f(x)$ to the right of the y-axis is reflected about the y-axis.

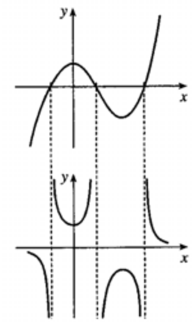
(b) $y = \sin|x|$



(c) $y = \sqrt{|x|}$

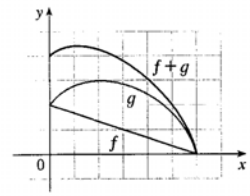


28. The most important features of the given graph are the x-intercepts and the maximum and minimum points. The graph of $y = 1/f(x)$ has vertical asymptotes at the x-values where there are x-intercepts on the graph of $y = f(x)$. The maximum of 1 on the graph of $y = f(x)$ corresponds to a minimum of $1/1 = 1$ on $y = 1/f(x)$. Similarly, the minimum on the graph of $y = f(x)$ corresponds to a maximum on the graph of $y = 1/f(x)$.



29. Assuming that successive horizontal and vertical gridlines are a unit apart, we can make a table of approximate values as follows.

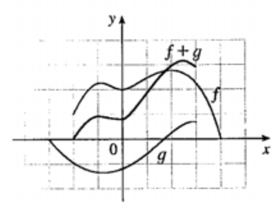
x	0	1	2	3	4	5	6
$f(x)$	2	1.7	1.3	1.0	0.7	0.3	0
$g(x)$	2	2.7	3	2.8	2.4	1.7	0
$f(x)+g(x)$	4	4.4	4.3	3.8	3.1	2.0	0



Connecting the points $(x, f(x)+g(x))$ with a smooth curve gives an approximation to the graph of $f+g$. Extra points can be plotted between those listed above if necessary.

30. First note that the domain of $f+g$ is the intersection of the domains of f and g; that is, $f+g$ is only defined where both f and g are defined. Taking the horizontal and vertical units of length to be the distances between successive vertical and horizontal gridlines, we can make a table of approximate values as follows:

x	-2	-1	0	1	2	2.5	3
$f(x)$	-1	2.2	2.0	2.4	2.7	2.7	2.3
$g(x)$	1	-1.3	-1.2	-0.6	0.3	0.5	0.7
$f(x)+g(x)$	0	0.9	0.8	1.8	3.0	3.2	3.0



Extra values of x (like the value 2.5 in the table above) can be added as needed.

31. $f(x) = x^3 + 2x^2$; $g(x) = 3x^2 - 1$. $D = \mathbb{R}$ for both f and g.

$(f+g)(x) = x^3 + 2x^2 + 3x^2 - 1 = x^3 + 5x^2 - 1$, $D = \mathbb{R}$.

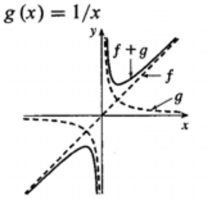
$(f-g)(x) = x^3 + 2x^2 - (3x^2 - 1) = x^3 - x^2 + 1$, $D = \mathbb{R}$.

$(fg)(x) = (x^3 + 2x^2)(3x^2 - 1) = 3x^5 + 6x^4 - x^3 - 2x^2$, $D = \mathbb{R}$.

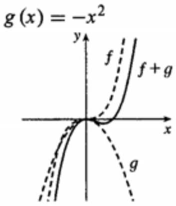
$\left(\dfrac{f}{g}\right)(x) = \dfrac{x^3 + 2x^2}{3x^2 - 1}$, $D = \left\{x \mid x \neq \pm\frac{1}{\sqrt{3}}\right\}$ since $3x^2 - 1 \neq 0$.

32. $f(x) = \sqrt{1+x}$, $D = [-1, \infty)$; $g(x) = \sqrt{1-x}$, $D = (-\infty, 1]$.
$(f+g)(x) = \sqrt{1+x} + \sqrt{1-x}$, $D = (-\infty, 1] \cap [-1, \infty) = [-1, 1]$.
$(f-g)(x) = \sqrt{1+x} - \sqrt{1-x}$, $D = [-1, 1]$.
$(fg)(x) = \sqrt{1+x} \cdot \sqrt{1-x} = \sqrt{1-x^2}$, $D = [-1, 1]$.
$\left(\dfrac{f}{g}\right)(x) = \dfrac{\sqrt{1+x}}{\sqrt{1-x}}$, $D = [-1, 1)$. We must exclude $x = 1$ since it would make $\dfrac{f}{g}$ undefined.

33. $f(x) = x$, $g(x) = 1/x$



34. $f(x) = x^3$, $g(x) = -x^2$



35. $f(x) = 2x^2 - x$; $g(x) = 3x + 2$. $D = \mathbb{R}$ for both f and g, and hence for their composites.
$(f \circ g)(x) = f(g(x)) = f(3x+2) = 2(3x+2)^2 - (3x+2) = 18x^2 + 21x + 6$.
$(g \circ f)(x) = g(f(x)) = g(2x^2 - x) = 3(2x^2 - x) + 2 = 6x^2 - 3x + 2$.
$(f \circ f)(x) = f(f(x)) = f(2x^2 - x) = 2(2x^2 - x)^2 - (2x^2 - x) = 8x^4 - 8x^3 + x$.
$(g \circ g)(x) = g(g(x)) = g(3x+2) = 3(3x+2) + 2 = 9x + 8$.

36. $f(x) = \sqrt{x-1}$, $D = [1, \infty)$; $g(x) = x^2$, $D = \mathbb{R}$.
$(f \circ g)(x) = f(g(x)) = f(x^2) = \sqrt{x^2 - 1}$,
$D = \{x \in \mathbb{R} \mid g(x) \in [1, \infty)\} = (-\infty, -1] \cup [1, \infty)$.
$(g \circ f)(x) = g(f(x)) = g(\sqrt{x-1}) = (\sqrt{x-1})^2 = x - 1$, $D = [1, \infty)$.
$(f \circ f)(x) = f(f(x)) = f(\sqrt{x-1}) = \sqrt{\sqrt{x-1} - 1}$,
$D = \{x \in [1, \infty) \mid \sqrt{x-1} \geq 1\} = [2, \infty)$.
$(g \circ g)(x) = g(g(x)) = g(x^2) = (x^2)^2 = x^4$, $D = \mathbb{R}$.

37. $f(x) = 1/x$, $D = \{x \mid x \neq 0\}$; $g(x) = x^3 + 2x$, $D = \mathbb{R}$.
$(f \circ g)(x) = f(g(x)) = f(x^3 + 2x) = 1/(x^3 + 2x)$, $D = \{x \mid x^3 + 2x \neq 0\} = \{x \mid x \neq 0\}$.
$(g \circ f)(x) = g(f(x)) = g(1/x) = 1/x^3 + 2/x$, $D = \{x \mid x \neq 0\}$.
$(f \circ f)(x) = f(f(x)) = f(1/x) = \dfrac{1}{1/x} = x$, $D = \{x \mid x \neq 0\}$.
$(g \circ g)(x) = g(g(x)) = g(x^3 + 2x) = (x^3 + 2x)^3 + 2(x^3 + 2x) = x^9 + 6x^7 + 12x^5 + 10x^3 + 4x$, $D = \mathbb{R}$.

38. $f(x) = \dfrac{1}{x-1}$, $D = \{x \mid x \neq 1\}$; $g(x) = \dfrac{x-1}{x+1}$, $D = \{x \mid x \neq -1\}$.
$(f \circ g)(x) = f\left(\dfrac{x-1}{x+1}\right) = \left(\dfrac{x-1}{x+1} - 1\right)^{-1} = \left(\dfrac{-2}{x+1}\right)^{-1} = \dfrac{-x-1}{2}$, $D = \{x \mid x \neq -1\}$.
$(g \circ f)(x) = g\left(\dfrac{1}{x-1}\right) = \dfrac{1/(x-1) - 1}{1/(x-1) + 1} = \dfrac{2-x}{x}$, $D = \{x \mid x \neq 0, 1\}$.
$(f \circ f)(x) = f\left(\dfrac{1}{x-1}\right) = \dfrac{1}{1/(x-1) - 1} = \dfrac{x-1}{2-x}$, $D = \{x \mid x \neq 1, 2\}$.
$(g \circ g)(x) = g\left(\dfrac{x-1}{x+1}\right) = \dfrac{(x-1)/(x+1) - 1}{(x-1)/(x+1) + 1} = -\dfrac{1}{x}$, $D = \{x \mid x \neq 0, -1\}$.

39. $f(x) = \sin x$, $D = \mathbb{R}$; $g(x) = 1 - \sqrt{x}$, $D = [0, \infty)$.

$(f \circ g)(x) = f(g(x)) = f(1 - \sqrt{x}) = \sin(1 - \sqrt{x})$, $D = [0, \infty)$.

$(g \circ f)(x) = g(f(x)) = g(\sin x) = 1 - \sqrt{\sin x}$. For $\sqrt{\sin x}$ to be defined, we must have $\sin x \geq 0 \iff x \in [0, \pi], [2\pi, 3\pi], [-2\pi, -\pi], [4\pi, 5\pi], [-4\pi, -3\pi], \ldots$, so $D = \{x \mid x \in [2n\pi, \pi + 2n\pi]$, where n is an integer$\}$.

$(f \circ f)(x) = f(f(x)) = f(\sin x) = \sin(\sin x)$, $D = \mathbb{R}$.

$(g \circ g)(x) = g(g(x)) = g(1 - \sqrt{x}) = 1 - \sqrt{1 - \sqrt{x}}$, $D = \{x \geq 0 \mid 1 - \sqrt{x} \geq 0\} = [0, 1]$.

40. $f(x) = \sqrt{x^2 - 1}$, $D = (-\infty, -1] \cup [1, \infty)$; $g(x) = \sqrt{1 - x}$, $D = (-\infty, 1]$.

$(f \circ g)(x) = f(g(x)) = f(\sqrt{1 - x}) = \sqrt{(\sqrt{1 - x})^2 - 1} = \sqrt{-x}$. To find the domain of $(f \circ g)(x)$, we must find the values of x that are in the domain of g such that $g(x)$ is in the domain of f. In symbols, we have $D = \{x \in (-\infty, 1] \mid \sqrt{1 - x} \in (-\infty, -1] \cup [1, \infty)\}$. First, we concentrate on the requirement that $\sqrt{1 - x} \in (-\infty, -1] \cup [1, \infty)$. Because $\sqrt{1 - x} \geq 0$, $\sqrt{1 - x}$ is not in $(-\infty, -1]$. If $\sqrt{1 - x}$ is in $[1, \infty)$, then we must have $\sqrt{1 - x} \geq 1 \Rightarrow 1 - x \geq 1 \Rightarrow x \leq 0$. Combining the restrictions $x \leq 0$ and $x \in (-\infty, 1]$, we obtain $D = (-\infty, 0]$.

$(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x^2 - 1}\right) = \sqrt{1 - \sqrt{x^2 - 1}}$,

$D = \left\{x \in (-\infty, -1] \cup [1, \infty) \mid \sqrt{x^2 - 1} \in (-\infty, 1]\right\}$. Now $\sqrt{x^2 - 1} \leq 1 \Rightarrow x^2 - 1 \leq 1 \Rightarrow x^2 \leq 2 \Rightarrow$ $|x| \leq \sqrt{2} \Rightarrow -\sqrt{2} \leq x \leq \sqrt{2}$. Combining this restriction with $x \in (-\infty, -1] \cup [1, \infty)$, we obtain $D = \left[-\sqrt{2}, -1\right] \cup \left[1, \sqrt{2}\right]$.

$(f \circ f)(x) = f(f(x)) = f\left(\sqrt{x^2 - 1}\right) = \sqrt{\left(\sqrt{x^2 - 1}\right)^2 - 1} = \sqrt{x^2 - 2}$,

$D = \left\{x \in (-\infty, -1] \cup [1, \infty) \mid \sqrt{x^2 - 1} \in (-\infty, -1] \cup [1, \infty)\right\}$. Now $\sqrt{x^2 - 1} \geq 1 \Rightarrow x^2 - 1 \geq 1 \Rightarrow$ $x^2 \geq 2 \Rightarrow |x| \geq \sqrt{2} \Rightarrow x \geq \sqrt{2}$ or $x \leq -\sqrt{2}$. Combining this restriction with $x \in (-\infty, -1] \cup [1, \infty)$, we obtain $D = (-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$.

$(g \circ g)(x) = g(g(x)) = g(\sqrt{1 - x}) = \sqrt{1 - \sqrt{1 - x}}$, $D = \{x \in (-\infty, 1] \mid \sqrt{1 - x} \in (-\infty, 1]\}$. Now $\sqrt{1 - x} \leq 1 \Rightarrow 1 - x \leq 1 \Rightarrow x \geq 0$. Combining this restriction with $x \in (-\infty, 1]$, we obtain $D = [0, 1]$.

41. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x - 1)) = f(\sqrt{x - 1}) = \sqrt{x - 1} - 1$

42. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^2 + 2)) = f\left((x^2 + 2)^3\right) = 1 \big/ (x^2 + 2)^3$

43. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt{x})) = f(\sqrt{x} - 5) = (\sqrt{x} - 5)^4 + 1$

44. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt[3]{x})) = f\left(\dfrac{\sqrt[3]{x}}{\sqrt[3]{x} - 1}\right) = \sqrt{\dfrac{\sqrt[3]{x}}{\sqrt[3]{x} - 1}}$

45. Let $g(x) = x - 9$ and $f(x) = x^5$. Then $(f \circ g)(x) = (x - 9)^5 = F(x)$.

46. Let $g(x) = \sqrt{x}$ and $f(x) = \sin x$. Then $(f \circ g)(x) = \sin\sqrt{x} = F(x)$.

47. Let $g(x) = x^2$ and $f(x) = \dfrac{x}{x + 4}$. Then $(f \circ g)(x) = \dfrac{x^2}{x^2 + 4} = G(x)$.

48. Let $g(x) = x + 3$ and $f(x) = 1/x$. Then $(f \circ g)(x) = 1/(x + 3) = G(x)$.

49. Let $g(t) = \cos t$ and $f(t) = \sqrt{t}$. Then $(f \circ g)(t) = \sqrt{\cos t} = u(t)$.

50. Let $g(t) = \pi t$ and $f(t) = \tan t$. Then $(f \circ g)(t) = \tan \pi t = u(t)$.

51. Let $h(x) = x^2$, $g(x) = 3^x$, and $f(x) = 1 - x$. Then $(f \circ g \circ h)(x) = 1 - 3^{x^2} = H(x)$.

52. Let $h(x) = \sqrt{x}$, $g(x) = x - 1$, and $f(x) = \sqrt[3]{x}$. Then $(f \circ g \circ h)(x) = \sqrt[3]{\sqrt{x} - 1} = H(x)$.

53. Let $h(x) = \sqrt{x}$, $g(x) = \sec x$, and $f(x) = x^4$. Then $(f \circ g \circ h)(x) = \left(\sec \sqrt{x}\right)^4 = \sec^4\left(\sqrt{x}\right) = H(x)$.

54. (a) $f(g(1)) = f(6) = 5$ (b) $g(f(1)) = g(3) = 2$

(c) $f(f(1)) = f(3) = 4$ (d) $g(g(1)) = g(6) = 3$

(e) $(g \circ f)(3) = g(f(3)) = g(4) = 1$ (f) $(f \circ g)(6) = f(g(6)) = f(3) = 4$

55. (a) $g(2) = 5$, because the point $(2, 5)$ is on the graph of g. Thus, $f(g(2)) = f(5) = 4$, because the point $(5, 4)$ is on the graph of f.

(b) $g(f(0)) = g(0) = 3$

(c) $(f \circ g)(0) = f(g(0)) = f(3) = 0$

(d) $(g \circ f)(6) = g(f(6)) = g(6)$. This value is not defined, because there is no point on the graph of g that has x-coordinate 6.

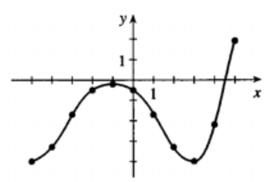
(e) $(g \circ g)(-2) = g(g(-2)) = g(1) = 4$

(f) $(f \circ f)(4) = f(f(4)) = f(2) = -2$

56. To find a particular value of $f(g(x))$, say for $x = 0$, we note from the graph that $g(0) \approx 2.8$ and $f(2.8) \approx -0.5$. Thus, $f(g(0)) \approx f(2.8) \approx -0.5$. The other values listed in the table were obtained in a similar fashion.

x	$g(x)$	$f(g(x))$
-5	-0.2	-4
-4	1.2	-3.3
-3	2.2	-1.7
-2	2.8	-0.5
-1	3	-0.2

x	$g(x)$	$f(g(x))$
0	2.8	-0.5
1	2.2	-1.7
2	1.2	-3.3
3	-0.2	-4
4	-1.9	-2.2
5	-4.1	1.9



57. (a) Using the relationship *distance* $=$ *rate* $\cdot$ *time* with the radius r as the distance, we have $r(t) = 60t$.

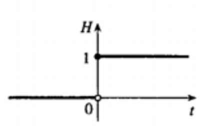
(b) $A = \pi r^2 \;\Rightarrow\; (A \circ r)(t) = A(r(t)) = \pi(60t)^2 = 3600\pi t^2$. This formula gives us the extent of the rippled area (in cm^2) at any time t.

58. (a) $d = rt \;\Rightarrow\; d = 350t$

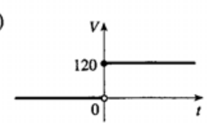
(b) There is a Pythagorean relationship involving the legs with lengths d and 1 and the hypotenuse with length s: $d^2 + 1^2 = s^2$. Thus, $s(d) = \sqrt{d^2 + 1}$.

(c) $(s \circ d)(t) = s(d(t)) = s(350t) = \sqrt{(350t)^2 + 1}$

59. (a)

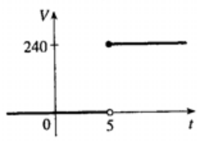


(b)



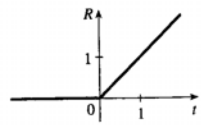
$V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 120 & \text{if } t \geq 0 \end{cases}$ so $V(t) = 120H(t)$.

(c)



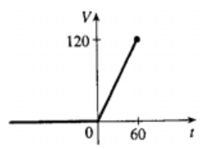
Starting with the formula in part (b), we replace 120 with 240 to reflect the different voltage. Also, because we are starting 5 units to the right of $t = 0$, we replace t with $t - 5$. Thus, the formula is $V(t) = 240H(t - 5)$.

60. (a) $R(t) = tH(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases}$



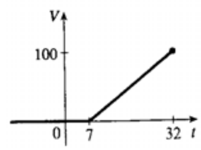
(b) $V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 2t & \text{if } 0 \leq t \leq 60 \end{cases}$

so $V(t) = 2tH(t)$, $t \leq 60$.



(c) $V(t) = \begin{cases} 0 & \text{if } t < 7 \\ 4(t-7) & \text{if } 7 \leq t \leq 32 \end{cases}$

so $V(t) = 4(t-7)H(t-7)$, $t \leq 32$.



61. (a) By examining the variable terms in g and h, we deduce that we must square g to get the terms $4x^2$ and $4x$ in h. If we let $f(x) = x^2 + c$, then $(f \circ g)(x) = f(g(x)) = f(2x+1) = (2x+1)^2 + c = 4x^2 + 4x + (1 + c)$. Since $h(x) = 4x^2 + 4x + 7$, we must have $1 + c = 7$. So $c = 6$ and $g(x) = x^2 + 6$.

(b) We need a function g so that $f(g(x)) = 3(g(x)) + 5 = h(x) = 3x^2 + 3x + 2 = 3(x^2 + x) + 2 = 3(x^2 + x - 1) + 5$. So we see that $g(x) = x^2 + x - 1$.

62. We need a function g so that $g(f(x)) = g(x+4) = h(x) = 4x - 1 = 4(x+4) - 17$. So we see that the function g must be $g(x) = 4x - 17$.

63. We need to examine $h(-x)$.

$$h(-x) = (f \circ g)(-x) = f(g(-x)) = f(g(x)) \quad [\text{because } g \text{ is even}] \quad = h(x)$$

Because $h(-x) = h(x)$, h is an even function.

64. $h(-x) = f(g(-x)) = f(-g(x))$. At this point, we can't simplify the expression, so we might try to find a counterexample to show that h is not an odd function. Let $g(x) = x$, an odd function, and $f(x) = x^2 + x$. Then $h(x) = x^2 + x$, which is neither even nor odd.

Now suppose f is an odd function. Then $f(-g(x)) = -f(g(x)) = -h(x)$. Hence, $h(-x) = -h(x)$, and so h is odd if both f and g are odd.

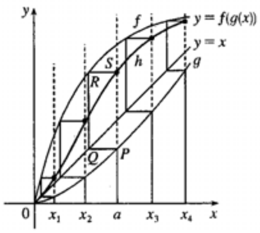
Now suppose f is an even function. Then $f(-g(x)) = f(g(x)) = h(x)$. Hence, $h(-x) = h(x)$, and so h is even if g is odd and f is even.

65. (a) $P = (a, g(a))$ and $Q = (g(a), g(a))$ because Q has the same y-value as P and it is on the line $y = x$.

(b) The x-value of Q is $g(a)$; this is also the x-value of R. The y-value of R is therefore $f(x\text{-value})$, that is, $f(g(a))$. Hence, $R = (g(a), f(g(a)))$.

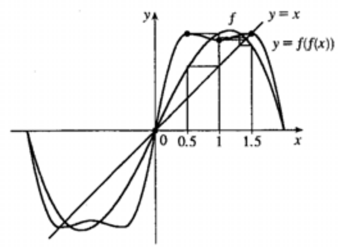
(c) The coordinates of S are $(a, f(g(a)))$ or, equivalently, $(a, h(a))$.

(d)



66. We only need to plot points for the first quadrant since we can see that f is an odd function, and by Exercise 64, we then know that $f \circ f$ is an odd function, and hence, symmetric with respect to the origin.

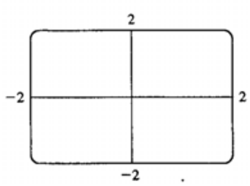
x	0	0.5	1	1.5	2
$f(x)$	0	1	1.5	1.4	0
$f(f(x))$	0	1.5	1.4	1.5	0



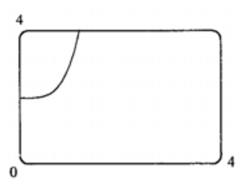
1.4 Graphing Calculators and Computers

1. $f(x) = x^4 + 2$

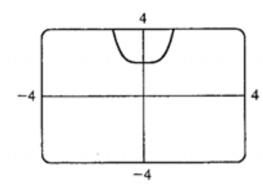
(a) $[-2, 2]$ by $[-2, 2]$



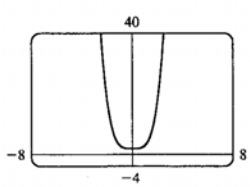
(b) $[0, 4]$ by $[0, 4]$



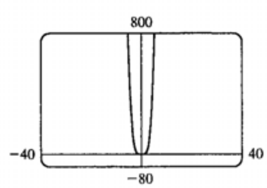
(c) $[-4, 4]$ by $[-4, 4]$



(d) $[-8, 8]$ by $[-4, 40]$



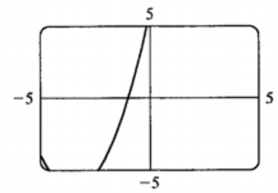
(e) $[-40, 40]$ by $[-80, 800]$



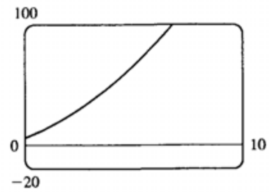
The most appropriate graph is produced in viewing rectangle (d).

2. $f(x) = x^2 + 7x + 6$

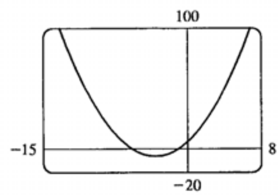
(a) $[-5, 5]$ by $[-5, 5]$



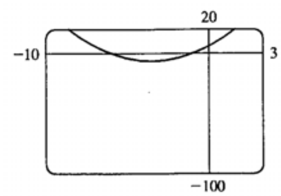
(b) $[0, 10]$ by $[-20, 100]$



(c) $[-15, 8]$ by $[-20, 100]$



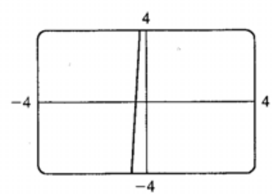
(d) $[-10, 3]$ by $[-100, 20]$



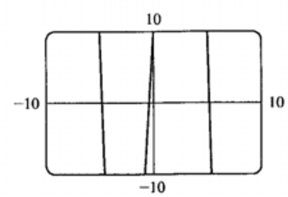
The most appropriate graph is produced in viewing rectangle (c).

3. $f(x) = 10 + 25x - x^3$

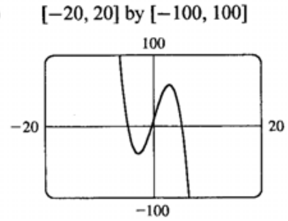
(a) $[-4, 4]$ by $[-4, 4]$



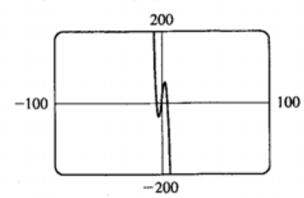
(b) $[-10, 10]$ by $[-10, 10]$



(c) $[-20, 20]$ by $[-100, 100]$



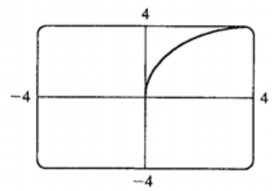
(d) $[-100, 100]$ by $[-200, 200]$



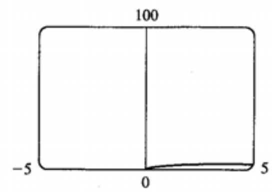
The most appropriate graph is produced in viewing rectangle (c) because the maximum and minimum points are fairly easy to see and estimate.

4. $f(x) = \sqrt{8x - x^2}$

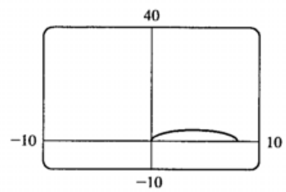
(a) $[-4, 4]$ by $[-4, 4]$



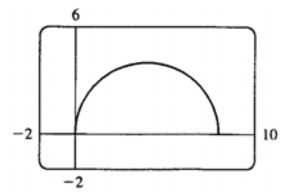
(b) $[-5, 5]$ by $[0, 100]$



(c) $[-10, 10]$ by $[-10, 40]$

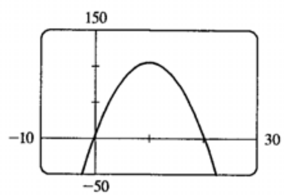


(d) $[-2, 10]$ by $[-2, 6]$

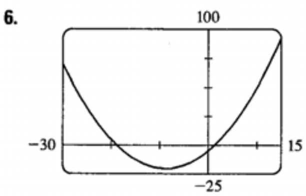


The most appropriate graph is produced in viewing rectangle (d).

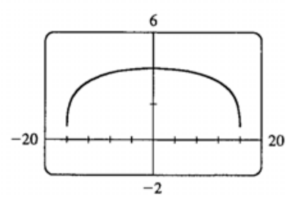
5. Since the graph of $f(x) = 5 + 20x - x^2$ is a parabola opening downward, an appropriate viewing rectangle should include the maximum point.



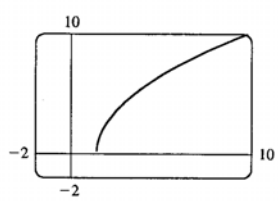
6.



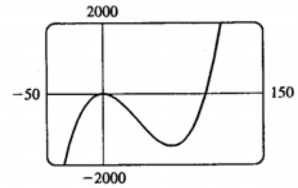
7. $f(x) = \sqrt[4]{256 - x^2}$. To find an appropriate viewing rectangle, we calculate f's domain and range: $256 - x^2 \geq 0 \quad \Leftrightarrow \quad x^2 \leq 256 \quad \Leftrightarrow \quad |x| \leq 16 \quad \Leftrightarrow \quad -16 \leq x \leq 16$, so the domain is $[-16, 16]$. Also, $0 \leq \sqrt[4]{256 - x^2} \leq \sqrt[4]{256} = 4$, so the range is $[0, 4]$. Thus, we choose the viewing rectangle to be $[-20, 20]$ by $[-2, 6]$.



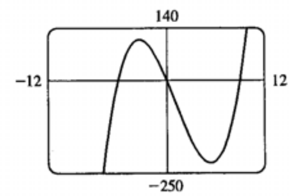
8. $f(x) = \sqrt{12x - 17}$



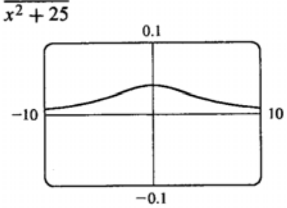
9. $f(x) = 0.01x^3 - x^2 + 5$. Graphing f in a standard viewing rectangle, $[-10, 10]$ by $[-10, 10]$, shows us what appears to be a parabola. But since this is a cubic polynomial, we know that a larger viewing rectangle will reveal a minimum point as well as the maximum point. After some trial and error, we choose the viewing rectangle $[-50, 150]$ by $[-2000, 2000]$.



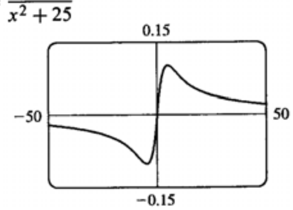
10. $f(x) = x(x+6)(x-9)$



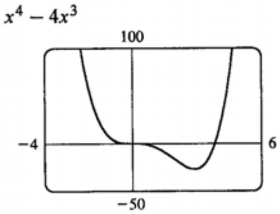
11. $y = \dfrac{1}{x^2 + 25}$



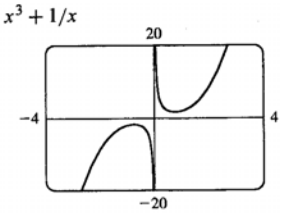
12. $y = \dfrac{x}{x^2 + 25}$



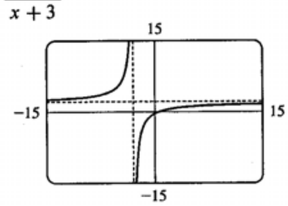
13. $y = x^4 - 4x^3$



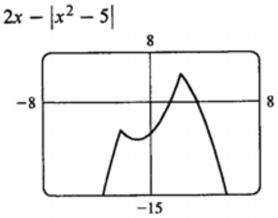
14. $y = x^3 + 1/x$



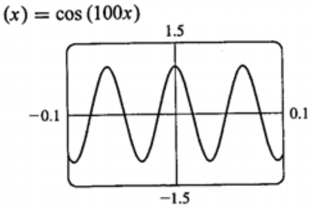
15. $y = \dfrac{2x - 1}{x + 3}$



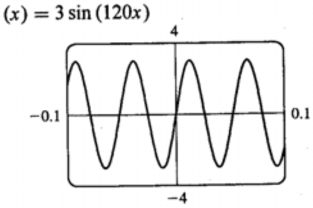
16. $y = 2x - |x^2 - 5|$



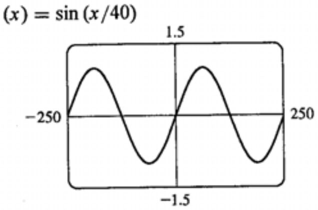
17. $f(x) = \cos(100x)$



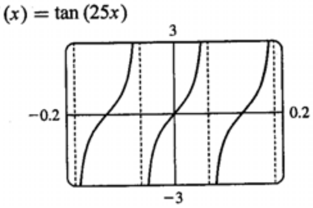
18. $f(x) = 3\sin(120x)$



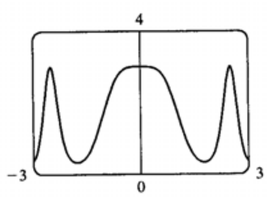
19. $f(x) = \sin(x/40)$

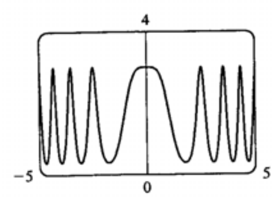


20. $f(x) = \tan(25x)$

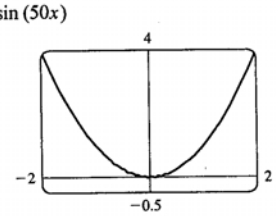


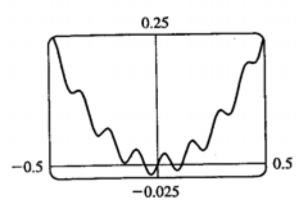
21. $y = 3^{\cos(x^2)}$



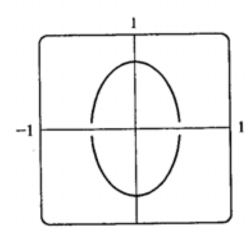


22. $y = x^2 + 0.02\sin(50x)$

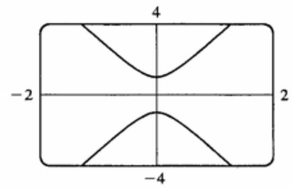




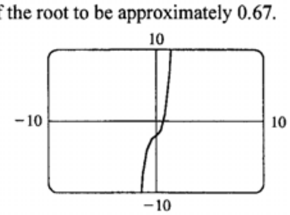
23. We must solve the given equation for y to obtain equations for the upper and lower halves of the ellipse. $4x^2 + 2y^2 = 1$ $\Leftrightarrow$ $2y^2 = 1 - 4x^2$ $\Leftrightarrow$ $y^2 = \dfrac{1-4x^2}{2}$ $\Leftrightarrow$ $y = \pm\sqrt{\dfrac{1-4x^2}{2}}$

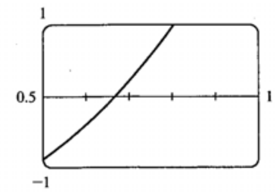


24. $y = \pm\sqrt{1 + 9x^2}$

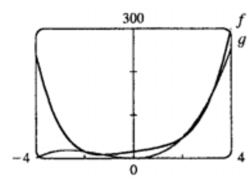


25. Graphing $f(x) = 3x^3 + x^2 + x - 2$ in a standard viewing rectangle, $[-10, 10]$ by $[-10, 10]$, reveals one real root between 0 and 1. The second figure shows a close-up of this region. By using a root finder or by zooming in, we find the value of the root to be approximately 0.67.

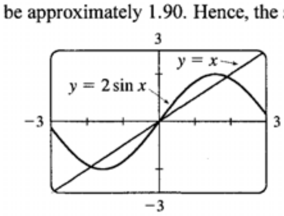


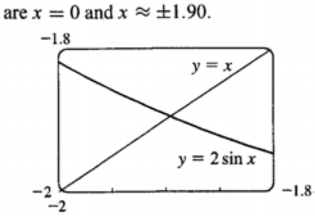


26. Graphing both $f(x) = x^4 + 8x + 16$ and $g(x) = 2x^3 + 8x^2$, it appears that there are four points of intersection (see the figure). We can now use an intersection finder or zoom in on the regions of interest to find the solutions $x \approx -2, -1.24, 2$, and 3.24.

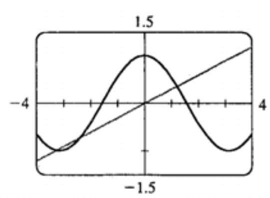


27. From the graph of $f(x) = 2\sin x$ and $g(x) = x$, we see that there are three points of intersection. The intersection point $(0, 0)$ is obvious and due to the symmetry of the graphs (both functions are odd), we only need to find one of the other two points of intersection. Using an intersection finder or zooming in, we find the x-value of the intersection to be approximately 1.90. Hence, the solutions are $x = 0$ and $x \approx \pm 1.90$.





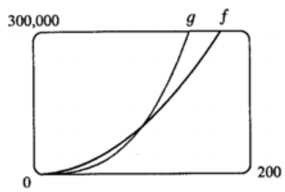
28. (a)



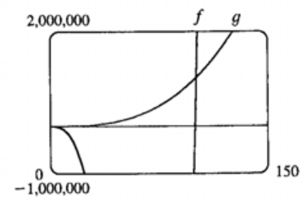
The x-coordinates of the three points of intersection are $x \approx -3.29, -2.36$ and 1.20.

(b) Using trial and error, we find that $m \approx 0.3365$. Note that m could also be negative.

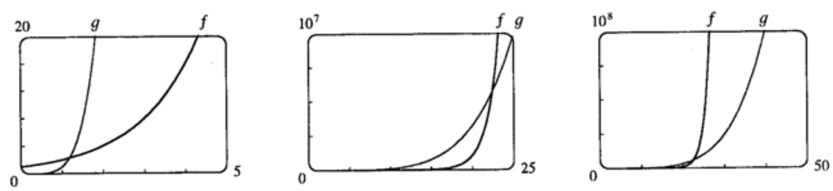
29. $g(x) = x^3/10$ is larger than $f(x) = 10x^2$ whenever $x > 100$.



30. $f(x) = x^4 - 100x^3$ is larger than $g(x) = x^3$ whenever $x > 101$.



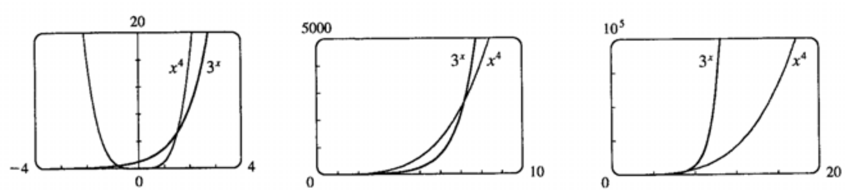
31. (a) (i) $[0, 5]$ by $[0, 20]$ (ii) $[0, 25]$ by $[0, 10^7]$ (iii) $[0, 50]$ by $[0, 10^8]$



As x gets large, $f(x) = 2^x$ grows more rapidly than $g(x) = x^5$.

(b) From the graphs in part (a), it appears that the two solutions are $x \approx 1.2$ and 22.4.

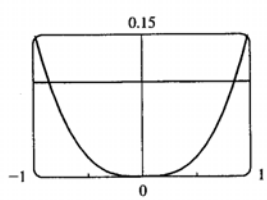
32. (a) (i) $[-4, 4]$ by $[0, 20]$ (ii) $[0, 10]$ by $[0, 5000]$ (iii) $[0, 20]$ by $[0, 10^5]$



As x gets large, $f(x) = 3^x$ grows more rapidly than $g(x) = x^4$.

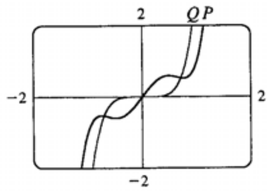
(b) From the graphs in (a), it appears that the three solutions are $x \approx -0.80$, 1.52 and 7.17.

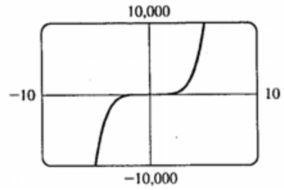
33.



We see from the graphs of $y = |\sin x - x|$ and $y = 0.1$ that there are two solutions to the equation $|\sin x - x| = 0.1$: $x \approx -0.85$ and $x \approx 0.85$. The condition $|\sin x - x| < 0.1$ holds for any x lying between these two values.

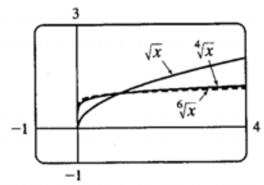
34.



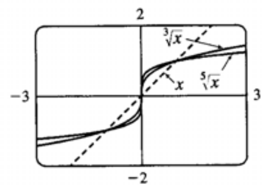


$P(x) = 3x^5 - 5x^3 + 2x$, $Q(x) = 3x^5$. These graphs are significantly different only in the region close to the origin. The larger a viewing rectangle one chooses, the more similar the two graphs look.

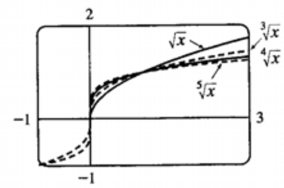
35. (a) The root functions $y = \sqrt{x}$, $y = \sqrt[4]{x}$ and $y = \sqrt[6]{x}$



(b) The root functions $y = x$, $y = \sqrt[3]{x}$ and $y = \sqrt[5]{x}$

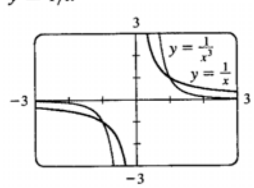


(c) The root functions $y = \sqrt{x}$, $y = \sqrt[3]{x}$, $y = \sqrt[4]{x}$ and $y = \sqrt[5]{x}$

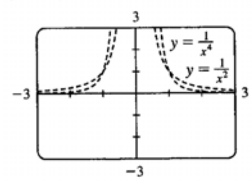


(d)
- For any n, the nth root of 0 is 0 and the nth root of 1 is 1; that is, all nth root functions pass through the points $(0, 0)$ and $(1, 1)$.
- For odd n, the domain of the nth root function is $\mathbb{R}$, while for even n, it is $\{x \in \mathbb{R} \mid x \geq 0\}$.
- Graphs of even root functions look similar to that of $\sqrt{x}$, while those of odd root functions resemble that of $\sqrt[3]{x}$.
- As n increases, the graph of $\sqrt[n]{x}$ becomes steeper near 0 and flatter for $x > 1$.

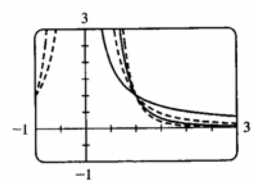
36. (a) The functions $y = 1/x$ and $y = 1/x^3$



(b) The functions $y = 1/x^2$ and $y = 1/x^4$

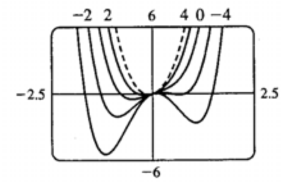


(c) The functions $y = 1/x$, $y = 1/x^2$, $y = 1/x^3$ and $y = 1/x^4$

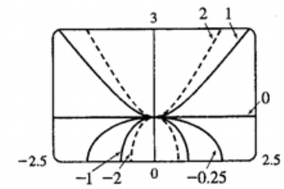


(d)
- The graphs of all functions of the form $y = 1/x^n$ pass through the point $(1, 1)$.
- If n is even, the graph of the function is entirely above the x-axis. The graphs of $1/x^n$ for n even are similar to one another.
- If n is odd, the function is positive for positive x and negative for negative x. The graphs of $1/x^n$ for n odd are similar to one another.
- As n increases, the graphs of $1/x^n$ approach 0 faster as $x \to \infty$.

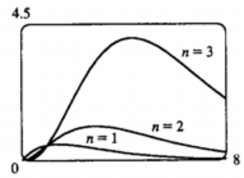
37. $f(x) = x^4 + cx^2 + x$. If $c < 0$, there are three humps: two minimum points and a maximum point. These humps get flatter as c increases, until at $c = 0$ two of the humps disappear and there is only one minimum point. This single hump then moves to the right and approaches the origin as c increases.

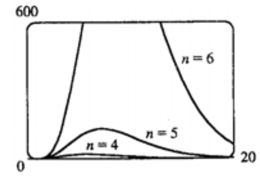


38. $f(x) = \sqrt{1 + cx^2}$. If $c < 0$, the function is only defined on $\left[-1/\sqrt{-c}\,, 1/\sqrt{-c}\,\right]$, and its graph is the top half of an ellipse. If $c = 0$, the graph is the line $y = 1$. If $c > 0$, the graph is the top half of a hyperbola. As c approaches 0, these curves become flatter and approach the line $y = 1$.

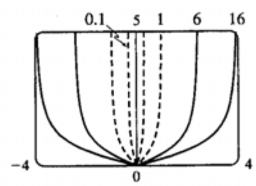


39. $y = x^n 2^{-x}$. As n increases, the maximum of the function moves further from the origin, and gets larger. Note, however, that regardless of n, the function approaches 0 as $x \to \infty$.



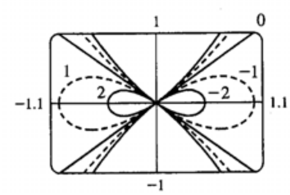


40. $y = \dfrac{|x|}{\sqrt{c - x^2}}$. The "bullet" becomes broader as c increases.



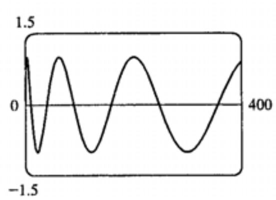
41. $y^2 = cx^3 + x^2$

If $c < 0$, the loop is to the right of the origin, and if c is positive, it is to the left. In both cases, the closer c is to 0, the larger the loop is. (In the limiting case, $c = 0$, the loop is "infinite", that is, it doesn't close.) Also, the larger $|c|$ is, the steeper the slope is on the loopless side of the origin.



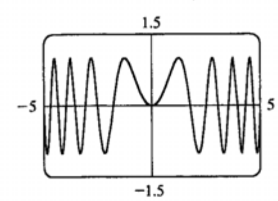
42. (a) $y = \sin\left(\sqrt{x}\right)$

This function is not periodic; it oscillates less frequently as x increases.



(b) $y = \sin\left(x^2\right)$

This function oscillates more frequently as $|x|$ increases. Note also that this function is even, whereas $\sin x$ is odd.



1 Review

CONCEPT CHECK

1. (a) A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B. The set A is called the **domain** of the function. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

(b) If f is a function with domain A, then its **graph** is the set of ordered pairs $\{(x, f(x)) \mid x \in A\}$.

(c) Use the Vertical Line Test on page 17.

2. The four ways to represent a function are: verbally, numerically, visually, and algebraically. An example of each is given below.
Verbally: An assignment of students to chairs in a classroom (a description in words)
Numerically: A tax table that assigns an amount of tax to an income (a table of values)
Visually: A graphical history of the Dow Jones average (a graph)
Algebraically: A relationship between distance, rate, and time: $d = rt$ (an explicit formula)

3. (a) An **even function** f satisfies $f(-x) = f(x)$ for every number x in its domain. It is symmetric with respect to the y-axis.

(b) An **odd function** g satisfies $g(-x) = -g(x)$ for every number x in its domain. It is symmetric with respect to the origin.

4. A function f is called **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

5. A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon.

6. (a) Linear function: $f(x) = 2x + 1$, $f(x) = ax + b$

(b) Power function: $f(x) = x^2$, $f(x) = x^a$

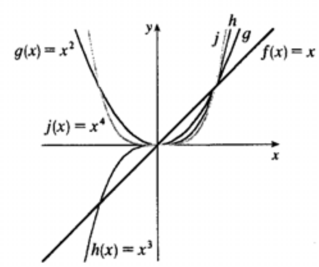
(c) Exponential function: $f(x) = 2^x$, $f(x) = a^x$

(d) Quadratic function: $f(x) = x^2 + x + 1$, $f(x) = ax^2 + bx + c$

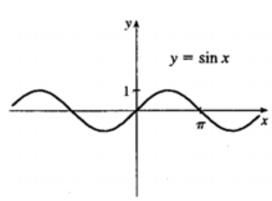
(e) Polynomial of degree 5: $f(x) = x^5 + 2$

(f) Rational function: $f(x) = \dfrac{x}{x+2}$, $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials

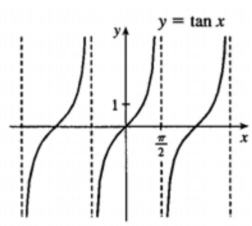
7.



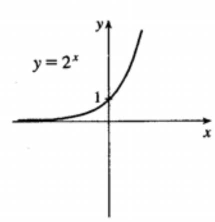
8. (a)



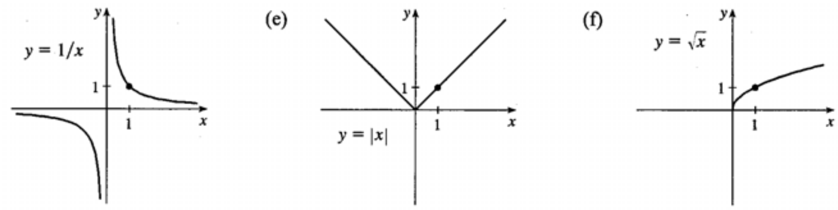
(b)



(c)



(d) (e) (f)



9. (a) The domain of $f + g$ is the intersection of the domain of f and the domain of g; that is, $A \cap B$.

(b) The domain of fg is also $A \cap B$.

(c) The domain of f/g must exclude values of x that make g equal to 0; that is, $\{x \in A \cap B \mid g(x) \neq 0\}$.

10. Given two functions f and g, the **composite** function $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

11. (a) If the graph of f is shifted 2 units upward, its equation becomes $y = f(x) + 2$.

(b) If the graph of f is shifted 2 units downward, its equation becomes $y = f(x) - 2$.

(c) If the graph of f is shifted 2 units to the right, its equation becomes $y = f(x - 2)$.

(d) If the graph of f is shifted 2 units to the left, its equation becomes $y = f(x + 2)$.

(e) If the graph of f is reflected about the x-axis, its equation becomes $y = -f(x)$.

(f) If the graph of f is reflected about the y-axis, its equation becomes $y = f(-x)$.

(g) If the graph of f is stretched vertically by a factor of 2, its equation becomes $y = 2f(x)$.

(h) If the graph of f is shrunk vertically by a factor of 2, its equation becomes $y = \frac{1}{2}f(x)$.

(i) If the graph of f is stretched horizontally by a factor of 2, its equation becomes $y = f\left(\frac{1}{2}x\right)$.

(j) If the graph of f is shrunk horizontally by a factor of 2, its equation becomes $y = f(2x)$.

TRUE-FALSE QUIZ

1. False. Let $f(x) = x^2$, $s = -1$, and $t = 1$. Then $f(s + t) = (-1 + 1)^2 = 0^2 = 0$, but $f(s) + f(t) = (-1)^2 + 1^2 = 2 \neq 0 = f(s + t)$.

2. False. Let $f(x) = x^2$. Then $f(-2) = 4 = f(2)$, but $-2 \neq 2$.

3. False. Let $f(x) = x^2$. Then $f(3x) = (3x)^2 = 9x^2$ and $3f(x) = 3x^2$. So $f(3x) \neq 3f(x)$.

4. True. If $x_1 < x_2$ and f is a decreasing function, then the y-values get smaller as we move from left to right. Thus, $f(x_1) > f(x_2)$.

5. True. See the Vertical Line Test.

6. False. Let $f(x) = x^2$ and $g(x) = 2x$. Then $(f \circ g)(x) = f(g(x)) = f(2x) = (2x)^2 = 4x^2$ and $(g \circ f)(x) = g(f(x)) = g(x^2) = 2x^2$. So $f \circ g \neq g \circ f$.

EXERCISES

1. (a) When $x = 2$, $y \approx 2.7$. Thus, $f(2) \approx 2.7$.

   (b) $f(x) = 3 \Rightarrow x \approx 2.3, 5.6$

   (c) The domain of f is $-6 \le x \le 6$, or $[-6, 6]$.

   (d) The range of f is $-4 \le y \le 4$, or $[-4, 4]$.

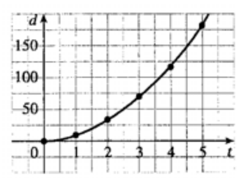
   (e) f is increasing on $(-4, 4)$.

   (f) f is odd since its graph is symmetric with respect to the origin.

2. (a) This curve *is not* the graph of a function of x since it *fails* the Vertical Line Test.

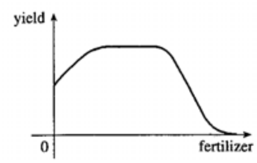
   (b) This curve *is* the graph of a function of x since it *passes* the Vertical Line Test. The domain is $[-3, 3]$ and the range is $[-2, 3]$.

3. (a)



   (b) From the graph, we see that the distance is slightly less than 150 feet.

4.



There will be some yield with no fertilizer, increasing yields with increasing fertilizer use, a leveling-off of yields at some point, and disaster with too much fertilizer use.

5. $f(x) = \sqrt{4 - 3x^2}$. Domain: $4 - 3x^2 \ge 0 \Rightarrow 3x^2 \le 4 \Rightarrow x^2 \le \frac{4}{3} \Rightarrow |x| \le \frac{2}{\sqrt{3}}$. Range: $y \ge 0$ and $y \le \sqrt{4} \Rightarrow 0 \le y \le 2$.

6. $g(x) = \dfrac{1}{x+1}$. Domain: $x + 1 \ne 0 \Rightarrow x \ne -1$. Range: all reals except 0 ($y = 0$ is the horizontal asymptote for g.)

7. $y = 1 + \sin x$. Domain: $\mathbb{R}$. Range: $-1 \le \sin x \le 1 \Rightarrow 0 \le 1 + \sin x \le 2 \Rightarrow 0 \le y \le 2$.

8. $y = \tan 2x$. Domain: $2x \ne \frac{\pi}{2} + \pi n \Rightarrow x \ne \frac{\pi}{4} + \frac{\pi}{2}n$. Range: the tangent function takes on all real values, so the range is $\mathbb{R}$.

9. (a) To obtain the graph of $y = f(x) + 8$, we shift the graph of $y = f(x)$ up 8 units.

   (b) To obtain the graph of $y = f(x + 8)$, we shift the graph of $y = f(x)$ left 8 units.

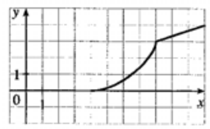
   (c) To obtain the graph of $y = 1 + 2f(x)$, we stretch the graph of $y = f(x)$ vertically by a factor of 2, and then shift the resulting graph 1 unit upward.

   (d) To obtain the graph of $y = f(x - 2) - 2$, we shift the graph of $y = f(x)$ right 2 units, and then shift the resulting graph 2 units downward.

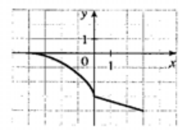
   (e) To obtain the graph of $y = -f(x)$, we reflect the graph of $y = f(x)$ about the x-axis.

   (f) To obtain the graph of $y = 3 - f(x)$, we reflect the graph of $y = f(x)$ about the x-axis, and then shift the resulting graph 3 units upward.

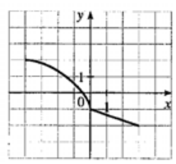
10. (a) To obtain the graph of $y = f(x-8)$, we shift the graph of $y = f(x)$ right 8 units.



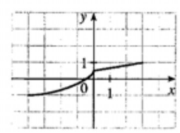
(b) To obtain the graph of $y = -f(x)$, we reflect the graph of $y = f(x)$ about the x-axis.



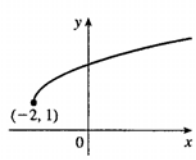
(c) To obtain the graph of $y = 2 - f(x)$, we reflect the graph of $y = f(x)$ about the x-axis, and then shift the resulting graph 2 units upward.



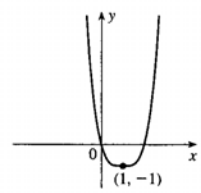
(d) To obtain the graph of $y = \frac{1}{2} f(x) - 1$, we shrink the graph of $y = f(x)$ by a factor of 2, and then shift the resulting graph 1 unit downward.



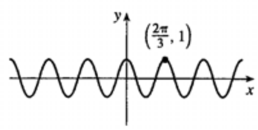
11. To sketch the graph of $y = 1 + \sqrt{x+2}$, we shift the graph of $y = \sqrt{x}$ left 2 units and up 1 unit.



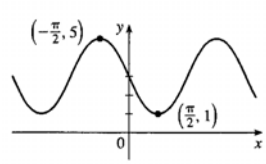
12. To sketch the graph of $y = (x-1)^4 - 1$, we shift the graph of $y = x^4$ right 1 unit and down 1 unit.



13. To sketch the graph of $y = \cos 3x$, we compress the graph of $y = \cos x$ horizontally by a factor of 3.



14. To sketch the graph of $y = 3 - 2\sin x$, we stretch the graph of $y = \sin x$ vertically by a factor of 2, reflect the resulting graph about the x-axis, and then shift that graph 3 units up.



15. (a) The terms of f are a mixture of odd and even powers of x, so f is neither even nor odd.

(b) The terms of f are all odd powers of x, so f is odd.

(c) $f(-x) = \cos\left((-x)^2\right) = \cos\left(x^2\right) = f(x)$, so f is even.

(d) $f(-x) = 1 + \sin(-x) = 1 - \sin x$. Now $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, so f is neither even nor odd.

16. For the line segment, the slope is $\dfrac{0-2}{-1+2} = -2$, and an equation is $y - 0 = -2(x+1)$ or, equivalently, $y = -2x - 2$. The circle has equation $x^2 + y^2 = 1$; the top half has equation $y = \sqrt{1-x^2}$ (we have solved for positive y.) Thus, $f(x) = \begin{cases} -2x - 2 & \text{if } -2 \leq x \leq -1 \\ \sqrt{1-x^2} & \text{if } -1 < x \leq 1 \end{cases}$

17. $f(x) = \sqrt{x}$, $D = [0, \infty)$; $g(x) = \sin x$, $D = \mathbb{R}$.

$(f \circ g)(x) = f(g(x)) = f(\sin x) = \sqrt{\sin x}$. For $\sqrt{\sin x}$ to be defined, we must have $\sin x \geq 0 \quad \Leftrightarrow \quad x \in [0, \pi]$, $[2\pi, 3\pi]$, $[-2\pi, -\pi]$, $[4\pi, 5\pi]$, $[-4\pi, -3\pi]$, …, so $D = \{x \mid x \in [2n\pi, \pi + 2n\pi]$, where n is an integer$\}$.

$(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x}\right) = \sin\sqrt{x}$. x must be greater than or equal to 0 for $\sqrt{x}$ to be defined, so $D = [0, \infty)$.

$(f \circ f)(x) = f(f(x)) = f\left(\sqrt{x}\right) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$. $D = [0, \infty)$.

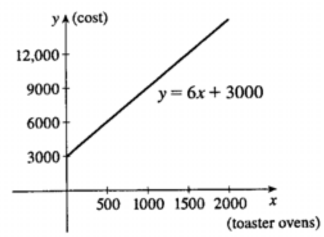
$(g \circ g)(x) = g(g(x)) = g(\sin x) = \sin(\sin x)$. $D = \mathbb{R}$.

18. Let $h(x) = x + \sqrt{x}$, $g(x) = \sqrt{x}$, and $f(x) = 1/x$. Then $(f \circ g \circ h)(x) = \dfrac{1}{\sqrt{x + \sqrt{x}}} = F(x)$.

19. The graphs of $f(x) = \sin^n x$, where n is a positive integer, all have domain $\mathbb{R}$. For odd n, the range is $[-1, 1]$ and for even n, the range is $[0, 1]$. For odd n, the functions are odd and symmetric with respect to the origin. For even n, the functions are even and symmetric with respect to the y-axis. As n becomes large, the graphs become less rounded and more "spiky".

20. (a) Let x denote the number of toaster ovens produced in one week and y the associated cost. Using the points (1000, 9000) and (1500, 12,000), we get an equation of a line: $y - 9000 = \dfrac{12{,}000 - 9000}{1500 - 1000}(x - 1000) \quad \Rightarrow$

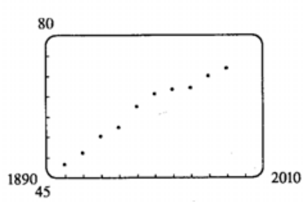
$y = 6(x - 1000) + 9000 \quad \Rightarrow \quad y = 6x + 3000$.



(b) The slope of 6 means that each additional toaster oven produced adds \$6 to the weekly production cost.

(c) The y-intercept of 3000 represents the overhead cost — the cost incurred without producing anything.

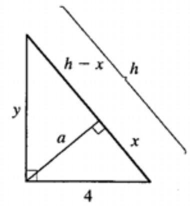
21.



Many models appear to be plausible. Your choice depends on whether you think medical advances will keep increasing life expectancy, or if there is bound to be a natural leveling-off of life expectancy. A linear model, $y = 0.263x - 450.034$, gives us an estimate of 76.0 years for the year 2000.

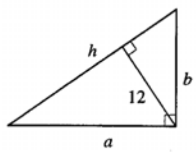
Principles of Problem Solving

1.



By using the area formula for a triangle, $\frac{1}{2}$ (base) (height), in two ways, we see that $\frac{1}{2}(4)(y) = \frac{1}{2}(h)(a)$, so $a = \dfrac{4y}{h}$. Since $4^2 + y^2 = h^2$, $y = \sqrt{h^2 - 16}$, and $a = \dfrac{4\sqrt{h^2 - 16}}{h}$.

2.



Refer to Example 1, where we obtained $h = \dfrac{P^2 - 100}{2P}$. The 100 came from 4 times the area of the triangle. In this case, the area of the triangle is $\frac{1}{2}(h)(12) = 6h$. Thus, $h = \dfrac{P^2 - 4(6h)}{2P} \;\Rightarrow\; 2Ph = P^2 - 24h \;\Rightarrow$ $2Ph + 24h = P^2 \;\Rightarrow\; h(2P + 24) = P^2 \;\Rightarrow\; h = \dfrac{P^2}{2P + 24}$.

3. $|2x - 1| = \begin{cases} 1 - 2x & \text{if } x < \frac{1}{2} \\ 2x - 1 & \text{if } x \geq \frac{1}{2} \end{cases}$ and $|x + 5| = \begin{cases} -x - 5 & \text{if } x < -5 \\ x + 5 & \text{if } x \geq -5 \end{cases}$

Therefore, we consider the three cases $x < -5$, $-5 \leq x < \frac{1}{2}$, and $x \geq \frac{1}{2}$.

If $x < -5$, we must have $1 - 2x - (-x - 5) = 3 \;\Leftrightarrow\; x = 3$, which is false, since we are considering $x < -5$.

If $-5 \leq x < \frac{1}{2}$, we must have $1 - 2x - (x + 5) = 3 \;\Leftrightarrow\; x = -\frac{7}{3}$.

If $x \geq \frac{1}{2}$, we must have $2x - 1 - (x + 5) = 3 \;\Leftrightarrow\; x = 9$.

So the two solutions of the equation are $x = -\frac{7}{3}$ and $x = 9$.

4. $|x - 1| = \begin{cases} 1 - x & \text{if } x < 1 \\ x - 1 & \text{if } x \geq 1 \end{cases}$ and $|x - 3| = \begin{cases} 3 - x & \text{if } x < 3 \\ x - 3 & \text{if } x \geq 3 \end{cases}$

Therefore, we consider the three cases $x < 1$, $1 \leq x < 3$, and $x \geq 3$.

If $x < 1$, we must have $1 - x - (3 - x) \geq 5 \;\Leftrightarrow\; 0 \geq 7$, which is false.

If $1 \leq x < 3$, we must have $x - 1 - (3 - x) \geq 5 \;\Leftrightarrow\; x \geq \frac{9}{2}$, which is false because $x < 3$.

If $x \geq 3$, we must have $x - 1 - (x - 3) \geq 5 \;\Leftrightarrow\; 2 \geq 5$, which is false.

All three cases lead to falsehoods, so the inequality has no solution.

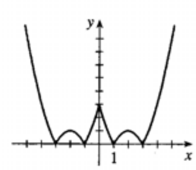
5. $f(x) = \left|x^2 - 4|x| + 3\right|$. If $x \geq 0$, then $f(x) = \left|x^2 - 4x + 3\right| = |(x - 1)(x - 3)|$.

Case (i): If $0 < x \leq 1$, then $f(x) = x^2 - 4x + 3$.

Case (ii): If $1 < x \leq 3$, then $f(x) = -\left(x^2 - 4x + 3\right) = -x^2 + 4x - 3$.

Case (iii): If $x > 3$, then $f(x) = x^2 - 4x + 3$.

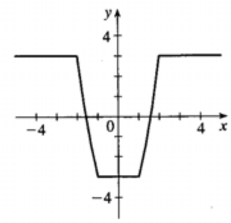
This enables us to sketch the graph for $x \geq 0$. Then we use the fact that f is an even function to reflect this part of the graph about the y-axis to obtain the entire graph. Or, we could consider also the cases $x < -3$, $-3 \leq x < -1$, and $-1 \leq x < 0$.



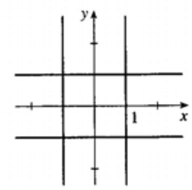
6. $g(x) = |x^2 - 1| - |x^2 - 4|$.

$$|x^2 - 1| = \begin{cases} x^2 - 1 & \text{if } |x| \ge 1 \\ 1 - x^2 & \text{if } |x| < 1 \end{cases} \quad \text{and } |x^2 - 4| = \begin{cases} x^2 - 4 & \text{if } |x| \ge 2 \\ 4 - x^2 & \text{if } |x| < 2 \end{cases}$$

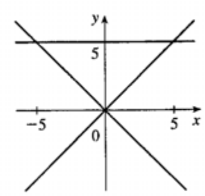
So for $0 \le |x| < 1$, $g(x) = 1 - x^2 - (4 - x^2) = -3$, for $1 \le |x| < 2$, $g(x) = x^2 - 1 - (4 - x^2) = 2x^2 - 5$, and for $|x| \ge 2$, $g(x) = x^2 - 1 - (x^2 - 4) = 3$.



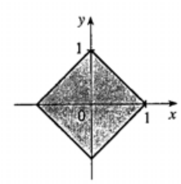
7. $|x| + |y| = 1 + |xy| \iff |xy| - |x| - |y| + 1 = 0 \iff |x|\,|y| - |x| - |y| + 1 = 0 \iff (|x| - 1)(|y| - 1) = 0 \iff x = \pm 1$ or $y = \pm 1$.



8. $x^2 y - y^3 - 5x^2 + 5y^2 = 0 \iff x^2(y - 5) - y^2(y - 5) = 0 \iff (x^2 - y^2)(y - 5) = 0 \iff x = \pm y$ or $y = 5$



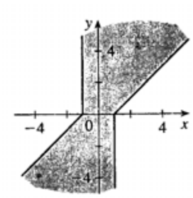
9. $|x| + |y| \le 1$. The boundary of the region has equation $|x| + |y| = 1$. In quadrants I, II, III, and IV, this becomes the lines $x + y = 1$, $-x + y = 1$, $-x - y = 1$, and $x - y = 1$ respectively.



10. $|x - y| + |x| - |y| \le 2$

Case (i):	$x > y > 0$	$\iff$	$x - y + x - y \le 2$	$\iff$	$x - y \le 1$	$\iff$	$y \ge x - 1$
Case (ii):	$y > x > 0$	$\iff$	$y - x + x - y \le 2$	$\iff$	$0 \le 2$ (true)		
Case (iii):	$x > 0$ and $y < 0$	$\iff$	$x - y + x + y \le 2$	$\iff$	$2x \le 2$	$\iff$	$x \le 1$
Case (iv):	$x < 0$ and $y > 0$	$\iff$	$y - x - x - y \le 2$	$\iff$	$-2x \le 2$	$\iff$	$x \ge -1$
Case (v):	$y < x < 0$	$\iff$	$x - y - x + y \le 2$	$\iff$	$0 \le 2$ (true)		
Case (vi):	$x < y < 0$	$\iff$	$y - x - x + y \le 2$	$\iff$	$y - x \le 1$	$\iff$	$y \le x + 1$

Note: Instead of considering cases (iv), (v), and (vi), we could have noted that the region is unchanged if x and y are replaced by $-x$ and $-y$, so the region is symmetric about the origin. Therefore, we need only draw cases (i), (ii), and (iii), and rotate through 180° about the origin.



11. Let d be the distance traveled on each half of the trip. Let t_1 and t_2 be the times taken for the first and second halves of the trip.

For the first half of the trip we have $t_1 = d/30$ and for the second half we have $t_2 = d/60$. Thus, the average speed for the entire trip is $\dfrac{\text{total distance}}{\text{total time}} = \dfrac{2d}{t_1 + t_2} = \dfrac{2d}{\dfrac{d}{30} + \dfrac{d}{60}} \cdot \dfrac{60}{60} = \dfrac{120d}{2d + d} = \dfrac{120d}{3d} = 40$. The average speed for the entire trip is 40 mi/h.

12. Let $f = \sin$, $g = x$, and $h = x$. Then the left-hand side of the equation is $f \circ (g + h) = \sin(x + x) = \sin 2x = 2 \sin x \cos x$; and the right-hand side is $f \circ g + f \circ h = \sin x + \sin x = 2 \sin x$. The two sides are not equal, so the given statement is false.

13. Let S_n be the statement that $7^n - 1$ is divisible by 6.

- S_1 is true because $7^1 - 1 = 6$ is divisible by 6.
- Assume S_k is true, that is, $7^k - 1$ is divisible by 6. In other words, $7^k - 1 = 6m$ for some positive integer m. Then $7^{k+1} - 1 = 7^k \cdot 7 - 1 = (6m + 1) \cdot 7 - 1 = 42m + 6 = 6(7m + 1)$, which is divisible by 6, so S_{k+1} is true.
- Therefore, by mathematical induction, $7^n - 1$ is divisible by 6 for every positive integer n.

14. Let S_n be the statement that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

- S_1 is true because $[2(1) - 1] = 1 = 1^2$.
- Assume S_k is true, that is, $1 + 3 + 5 + \cdots + (2k - 1) = k^2$. Then

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k - 1) + [(2k + 1) - 1] &= 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) \\ &= k^2 + (2k + 1) = (k + 1)^2 \end{aligned}$$

  which shows that S_{k+1} is true.
- Therefore, by mathematical induction, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for every positive integer n.

15. $f_0(x) = x^2$ and $f_{n+1}(x) = f_0(f_n(x))$ for $n = 0, 1, 2, \ldots$.

$f_1(x) = f_0(f_0(x)) = f_0(x^2) = (x^2)^2 = x^4$, $f_2(x) = f_0(f_1(x)) = f_0(x^4) = (x^4)^2 = x^8$, $f_3(x) = f_0(f_2(x)) = f_0(x^8) = (x^8)^2 = x^{16}, \ldots$. Thus, a general formula is $f_n(x) = x^{2^{n+1}}$.

16. (a) $f_0(x) = 1/(2 - x)$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \ldots$.

$$f_1(x) = f_0\left(\frac{1}{2 - x}\right) = \frac{1}{2 - \dfrac{1}{2 - x}} = \frac{2 - x}{2(2 - x) - 1} = \frac{2 - x}{3 - 2x},$$

$$f_2(x) = f_0\left(\frac{2 - x}{3 - 2x}\right) = \frac{1}{2 - \dfrac{2 - x}{3 - 2x}} = \frac{3 - 2x}{2(3 - 2x) - (2 - x)} = \frac{3 - 2x}{4 - 3x},$$

$$f_3(x) = f_0\left(\frac{3 - 2x}{4 - 3x}\right) = \frac{1}{2 - \dfrac{3 - 2x}{4 - 3x}} = \frac{4 - 3x}{2(4 - 3x) - (3 - 2x)} = \frac{4 - 3x}{5 - 4x}, \ldots$$

Thus, we conjecture that the general formula is $f_n(x) = \dfrac{n + 1 - nx}{n + 2 - (n + 1)x}$.

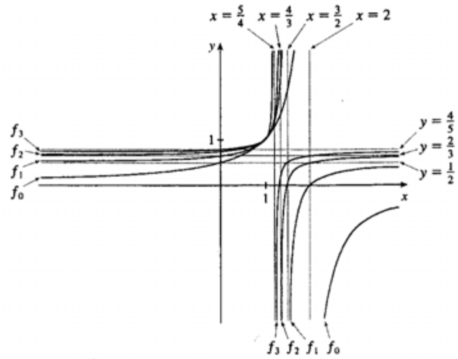
To prove this, we use the Principle of Mathematical Induction. We have already verified that f_n is true for $n = 1$. Assume that the formula is true for $n = k$; that is, $f_k(x) = \dfrac{k+1-kx}{k+2-(k+1)x}$. Then

$$f_{k+1}(x) = (f_0 \circ f_k)(x) = f_0(f_k(x)) = f_0\left(\frac{k+1-kx}{k+2-(k+1)x}\right) = \frac{1}{2 - \dfrac{k+1-kx}{k+2-(k+1)x}}$$

$$= \frac{k+2-(k+1)x}{2[k+2-(k+1)x]-(k+1-kx)} = \frac{k+2-(k+1)x}{k+3-(k+2)x}$$

This shows that the formula for f_n is true for $n = k + 1$. Therefore, by mathematical induction, the formula is true for all positive integers n.

(b)



From the graph, we can make several observations:

- The values at $x = a$ keep increasing as k increases.
- The vertical asymptote gets closer to $x = 1$ as k increases.
- The horizontal asymptote gets closer to $y = 1$ as k increases.
- The x-intercept for f_{k+1} is the value of the vertical asymptote for f_k.
- The y-intercept for f_k is the value of the horizontal asymptote for f_{k+1}.

2 Limits and Rates of Change

2.1 The Tangent and Velocity Problems

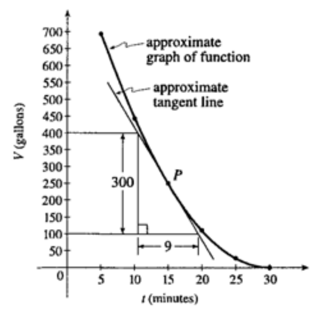
1. (a) Using $P\,(15, 250)$, we construct the following table:

t	Q	slope $= m_{PQ}$
5	$(5, 694)$	$\frac{694-250}{5-15} = -\frac{444}{10} = -44.4$
10	$(10, 444)$	$\frac{444-250}{10-15} = -\frac{194}{5} = -38.8$
20	$(20, 111)$	$\frac{111-250}{20-15} = -\frac{139}{5} = -27.8$
25	$(25, 28)$	$\frac{28-250}{25-15} = -\frac{222}{10} = -22.2$
30	$(30, 0)$	$\frac{0-250}{30-15} = -\frac{250}{15} = -16.\overline{6}$

(b) Using the values of t that correspond to the points closest to P ($t = 10$ and $t = 20$), we have

$$\frac{-38.8 + (-27.8)}{2} = -33.3.$$

(c) From the graph, we can estimate the slope of the tangent line at P to be $\frac{-300}{9} = -33.\overline{3}$.



2. (a) Slope $= \frac{2948 - 2530}{42 - 36} = \frac{418}{6} \approx 69.67$

(b) Slope $= \frac{2948 - 2661}{42 - 38} = \frac{287}{4} = 71.75$

(c) Slope $= \frac{2948 - 2806}{42 - 40} = \frac{142}{2} = 71$

(d) Slope $= \frac{3080 - 2948}{44 - 42} = \frac{132}{2} = 66$

From the data, we see that the patient's heart rate is decreasing from 71 to 66 heartbeats/minute after 42 minutes. After being stable for a while, the patient's heart rate is dropping.

3. For the curve $y = \sqrt{x}$ and the point $P\,(4, 2)$:

(a)

	x	Q	m_{PQ}
(i)	5	$(5, 2.236068)$	0.236068
(ii)	4.5	$(4.5, 2.121320)$	0.242641
(iii)	4.1	$(4.1, 2.024846)$	0.248457
(iv)	4.01	$(4.01, 2.002498)$	0.249844
(v)	4.001	$(4.001, 2.000250)$	0.249984
(vi)	3	$(3, 1.732051)$	0.267949
(vii)	3.5	$(3.5, 1.870829)$	0.258343
(viii)	3.9	$(3.9, 1.974842)$	0.251582
(ix)	3.99	$(3.99, 1.997498)$	0.250156
(x)	3.999	$(3.999, 1.999750)$	0.250016

(b) The slope appears to be $\frac{1}{4}$.

(c) $y - 2 = \frac{1}{4}(x - 4)$ or $y = \frac{1}{4}x + 1$.

4. For the curve $y = 1/x$ and the point $P\,(0.5, 2)$:

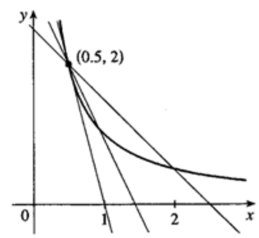
(a)

	x	Q	m_{PQ}
(i)	2	$(2, 0.5)$	-1
(ii)	1	$(1, 1)$	-2
(iii)	0.9	$(0.9, 1.111111)$	-2.222222
(iv)	0.8	$(0.8, 1.25)$	-2.5
(v)	0.7	$(0.7, 1.428571)$	-2.857143
(vi)	0.6	$(0.6, 1.666667)$	-3.333333
(vii)	0.55	$(0.55, 1.818182)$	-3.636364
(viii)	0.51	$(0.51, 1.960784)$	-3.921569
(ix)	0.45	$(0.45, 2.222222)$	-4.444444
(x)	0.49	$(0.49, 2.040816)$	-4.081633

(b) The slope appears to be -4.

(c) $y - 2 = -4\,(x - 0.5)$ or $y = -4x + 4$

(d)



5. (a) At $t = 2$, $y = 40\,(2) - 16\,(2)^2 = 16$. The average velocity between times 2 and $2 + h$ is

$$\frac{40\,(2+h) - 16\,(2+h)^2 - 16}{h} = \frac{-24h - 16h^2}{h} = -24 - 16h, \text{ if } h \neq 0.$$

(i) $h = 0.5$, -32 ft/s (ii) $h = 0.1$, -25.6 ft/s

(iii) $h = 0.05$, -24.8 ft/s (iv) $h = 0.01$, -24.16 ft/s

(b) The instantaneous velocity when $t = 2$ is -24 ft/s.

6. The average velocity between t and $t + h$ seconds is

$$\frac{58\,(t+h) - 0.83\,(t+h)^2 - \left(58t - 0.83t^2\right)}{h} = \frac{58h - 1.66th - 0.83h^2}{h} = 58 - 1.66t - 0.83h \text{ if } h \neq 0$$

(a) Here $t = 1$, so the average velocity is $58 - 1.66 - 0.83h = 56.34 - 0.83h$.

(i) $[1, 2]$: $h = 1$, 55.51 m/s (ii) $[1, 1.5]$: $h = 0.5$, 55.925 m/s

(iii) $[1, 1.1]$: $h = 0.1$, 56.257 m/s (iv) $[1, 1.01]$: $h = 0.01$, 56.3317 m/s

(v) $[1, 1.001]$: $h = 0.001$, 56.33917 m/s

(b) The instantaneous velocity after 1 second is 56.34 m/s.

7. Average velocity between times 1 and $1 + h$ is

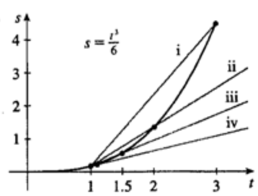
$$\frac{s\,(1+h) - s\,(1)}{h} = \frac{(1+h)^3/6 - 1/6}{h} = \frac{h^3 + 3h^2 + 3h}{6h} = \frac{h^2 + 3h + 3}{6} \text{ if } h \neq 0$$

(a) (i) $[1, 3]$: $h = 2$, $\frac{13}{6}$ ft/s (ii) $[1, 2]$: $h = 1$, $\frac{7}{6}$ ft/s

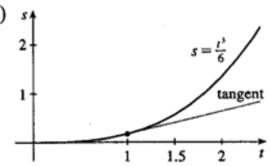
(iii) $[1, 1.5]$: $h = 0.5$, $\frac{19}{24}$ ft/s (iv) $[1, 1.1]$: $h = 0.1$, $\frac{331}{600}$ ft/s

(b) As h approaches 0, the velocity approaches $\frac{1}{2}$ ft/s.

(c)



(d)



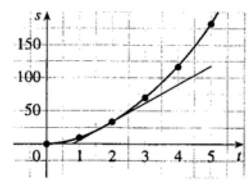
8. Average velocity between times $t = 2$ and $t = 2 + h$ is given by $\dfrac{s(2+h) - s(2)}{h}$

(a) (i) $h = 3 \Rightarrow v_{\text{av}} = \dfrac{s(5) - s(2)}{5-2} = \dfrac{178 - 32}{3} = \dfrac{146}{3} \approx 48.7$ ft/s

(ii) $h = 2 \Rightarrow v_{\text{av}} = \dfrac{s(4) - s(2)}{4-2} = \dfrac{119 - 32}{2} = \dfrac{87}{2} = 43.5$ ft/s

(iii) $h = 1 \Rightarrow v_{\text{av}} = \dfrac{s(3) - s(2)}{3-2} = \dfrac{70 - 32}{1} = 38$ ft/s

(b) Using the points $(0.8, 0)$ and $(5, 118)$ from the approximate tangent line, the instantaneous velocity at $t = 2$ is about $\frac{118-0}{5-0.8} \approx 28$ ft/s.



9. For the curve $y = \sin(10\pi/x)$ and the point $P(1, 0)$:

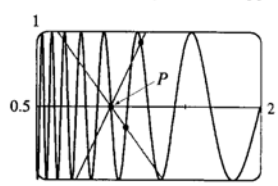
(a)

x	Q	m_{PQ}
2	(2, 0)	0
1.5	(1.5, 0.8660)	1.7321
1.4	(1.4, −0.4339)	−1.0847
1.3	(1.3, −0.8230)	−2.7433
1.2	(1.2, 0.8660)	4.3301
1.1	(1.1, −0.2817)	−2.8173

x	Q	m_{PQ}
0.5	(0.5, 0)	0
0.6	(0.6, 0.8660)	−2.1651
0.7	(0.7, 0.7818)	−2.6061
0.8	(0.8, 1)	−5
0.9	(0.9, −0.3420)	3.4202
0.99	(0.99, 0.3120)	−31.2033

As x approaches 1, the slopes do not appear to be approaching any particular value.

(b)



We see that problems with estimation are caused by the frequent oscillations of the graph. The tangent is so steep at P that we need to take x-values much closer to 1 in order to get accurate estimates of its slope.

(c) If we choose $x = 1.001$, then the point Q is $(1.001, -0.0314)$ and $m_{PQ} \approx -31.3794$. If $x = 0.999$, then Q is $(0.999, 0.0314)$ and $m_{PQ} = -31.4422$. Averaging these two slopes gives us the estimate -31.4108.

2.2 The Limit of a Function

1. As x approaches 2, $f(x)$ approaches 5. [Or, the values of $f(x)$ can be made as close to 5 as we like by taking x sufficiently close to 2 (but $x \neq 2$).] Yes, the graph could have a hole at $(2, 5)$ and be defined such that $f(2) = 3$.

2. As x approaches 1 from the left, $f(x)$ approaches 3; and as x approaches 1 from the right, $f(x)$ approaches 7. No, the limit does not exist because the left- and right-hand limits are different.

3. (a) $\lim_{x\to -3} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to -3 (but not equal to -3).

(b) $\lim_{x\to 4^+} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to 4 through values larger than 4.

4. (a) $\lim_{x\to 0} f(x) = 3$ (b) $\lim_{x\to 3^-} f(x) = 4$ (c) $\lim_{x\to 3^+} f(x) = 2$

(d) $\lim_{x\to 3} f(x)$ does not exist because the limits in part (b) and part (c) are not equal.

(e) $f(3) = 3$

5. (a) $\lim_{x\to 1} f(x) = 3$ (b) $\lim_{x\to 3^-} f(x) = 2$ (c) $\lim_{x\to 3^+} f(x) = -2$

(d) $\lim_{x\to 3} f(x)$ doesn't exist because the limits in part (b) and part (c) are not equal.

(e) $f(3) = 1$ (f) $\lim_{x\to -2^-} f(x) = -1$ (g) $\lim_{x\to -2^+} f(x) = -1$

(h) $\lim_{x\to -2} f(x) = -1$ (i) $f(-2) = -3$

6. (a) $\lim_{x\to -2^-} g(x) = -1$ (b) $\lim_{x\to -2^+} g(x) = 1$ (c) $\lim_{x\to -2} g(x)$ doesn't exist

(d) $g(-2) = 1$ (e) $\lim_{x\to 2^-} g(x) = 1$ (f) $\lim_{x\to 2^+} g(x) = 2$

(g) $\lim_{x\to 2} g(x)$ doesn't exist (h) $g(2) = 2$ (i) $\lim_{x\to 4^+} g(x)$ doesn't exist

(j) $\lim_{x\to 4^-} g(x) = 2$ (k) $g(0)$ doesn't exist (l) $\lim_{x\to 0} g(x) = 0$

7. (a) $\lim_{x\to 3} f(x) = 2$ (b) $\lim_{x\to 1} f(x) = -1$ (c) $\lim_{x\to -3} f(x) = 1$

(d) $\lim_{x\to 2^-} f(x) = 1$ (e) $\lim_{x\to 2^+} f(x) = 2$

(f) $\lim_{x\to 2} f(x)$ doesn't exist because the limits in part (d) and part (e) are not equal.

8. (a) $\lim_{x\to -6} g(x) = 0$ (b) $\lim_{x\to 0^-} g(x) = \infty$ (c) $\lim_{x\to 0^+} g(x) = -\infty$

(d) $\lim_{x\to 4} g(x) = -\infty$ (e) The equations of the vertical asymptotes: $x = -5$, $x = 0$, $x = 4$

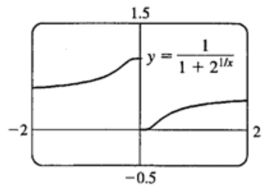
9. (a) $\lim_{x\to 3} f(x) = \infty$ (b) $\lim_{x\to 7} f(x) = -\infty$ (c) $\lim_{x\to -4} f(x) = -\infty$

(d) $\lim_{x\to -9^-} f(x) = \infty$ (e) $\lim_{x\to -9^+} f(x) = -\infty$

(f) The equations of the vertical asymptotes: $x = -9$, $x = -4$, $x = 3$, $x = 7$

10. $\lim_{t\to 12^-} f(t) = 150$ mg and $\lim_{t\to 12^+} f(t) = 300$ mg. These limits show that there is an abrupt change in the amount of drug in the patient's bloodstream at $t = 12$ h. The left-hand limit represents the amount of the drug just before the fourth injection. The right-hand limit represents the amount of the drug just after the fourth injection.

11.

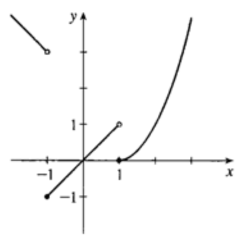


(a) $\lim\limits_{x\to 0^-} f(x) = 1$

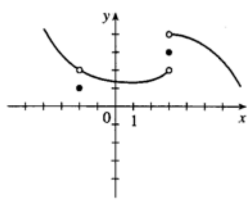
(b) $\lim\limits_{x\to 0^+} f(x) = 0$

(c) $\lim\limits_{x\to 0} f(x) = 0$ does not exist because the limits in part (a) and part (b) are not equal.

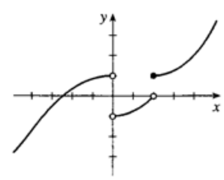
12. $\lim\limits_{x\to a} f(x)$ exists for all a except $a = \pm 1$.



13.



14.



15. For $g(x) = \dfrac{x-1}{x^3-1}$:

x	$g(x)$
0.2	0.806452
0.4	0.641026
0.6	0.510204
0.8	0.409836
0.9	0.369004
0.99	0.336689

x	$g(x)$
1.8	0.165563
1.6	0.193798
1.4	0.229358
1.2	0.274725
1.1	0.302115
1.01	0.330022

It appears that $\lim\limits_{x\to 1} \dfrac{x-1}{x^3-1} = 0.\overline{3} = \frac{1}{3}$.

16. For $g(x) = \dfrac{1-x^2}{x^2+3x-10}$:

x	$g(x)$
3	−1
2.1	−4.8028
2.01	−43.368
2.001	−429.08
2.0001	−4286.2
2.00001	−42858

It appears that $\lim\limits_{x\to 2^+} \dfrac{1-x^2}{x^2+3x-10} = -\infty$.

17. For $F(x) = \dfrac{(1/\sqrt{x}) - \frac{1}{5}}{x - 25}$:

x	$F(x)$
26	-0.003884
25.5	-0.003941
25.1	-0.003988
25.05	-0.003994
25.01	-0.003999

x	$F(x)$
24	-0.004124
24.5	-0.004061
24.9	-0.004012
24.95	-0.004006
24.99	-0.004001

It appears that $\lim\limits_{x\to 25} F(x) = -0.004$.

18. For $F(t) = \dfrac{\sqrt[3]{t} - 1}{\sqrt{t} - 1}$:

t	$F(t)$
1.5	0.643905
1.2	0.656488
1.1	0.661358
1.01	0.666114
1.001	0.666611

It appears that $\lim\limits_{t\to 1} \dfrac{\sqrt[3]{t} - 1}{\sqrt{t} - 1} = 0.\overline{6} = \frac{2}{3}$.

19. For $f(x) = \dfrac{1 - \cos x}{x^2}$:

x	$f(x)$
1	0.459698
0.5	0.489670
0.4	0.493369
0.3	0.496261
0.2	0.498336
0.1	0.499583
0.05	0.499896
0.01	0.499996

It appears that $\lim\limits_{x\to 0} \dfrac{1 - \cos x}{x^2} = 0.5$.

20. For $g(x) = \dfrac{\cos x - 1}{\sin x}$:

x	$g(x)$
1	-0.546302
0.5	-0.255342
0.4	-0.202710
0.3	-0.151135
0.2	-0.100335
0.1	-0.050042
0.05	-0.025005
0.01	-0.005000

It appears that $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{\sin x} = 0$.

21. $\lim\limits_{x\to 5^+} \dfrac{6}{x - 5} = \infty$ since $(x - 5) \to 0$ as $x \to 5^+$ and $\dfrac{6}{x - 5} > 0$ for $x > 5$.

22. $\lim\limits_{x\to 5^-} \dfrac{6}{x - 5} = -\infty$ since $(x - 5) \to 0$ as $x \to 5^-$ and $\dfrac{6}{x - 5} < 0$ for $x < 5$.

23. $\lim\limits_{x\to 3} \dfrac{1}{(x - 3)^8} = \infty$ since $(x - 3) \to 0$ as $x \to 3$ and $\dfrac{1}{(x - 3)^8} > 0$.

24. $\lim\limits_{x\to 0} \dfrac{x - 1}{x^2(x + 2)} = -\infty$ since $x^2 \to 0$ as $x \to 0$ and $\dfrac{x - 1}{x^2(x + 2)} < 0$ for $0 < x < 1$ and for $-2 < x < 0$.

25. $\lim\limits_{x\to -2^+} \dfrac{x - 1}{x^2(x + 2)} = -\infty$ since $(x + 2) \to 0$ as $x \to 2^+$ and $\dfrac{x - 1}{x^2(x + 2)} < 0$ for $-2 < x < 0$.

26. $\lim\limits_{x\to \pi^-} \csc x = \lim\limits_{x\to \pi^-} (1/\sin x) = \infty$ since $\sin x \to 0$ as $x \to \pi^-$ and $\sin x > 0$ for $0 < x < \pi$.

27. $\lim\limits_{x\to (-\pi/2)^-} \sec x = \lim\limits_{x\to (-\pi/2)^-} (1/\cos x) = -\infty$ since $\cos x \to 0$ as $x \to (-\pi/2)^-$ and $\cos x < 0$ for $-\pi < x < -\pi/2$.

28. $\lim\limits_{x\to 1^+} \dfrac{x + 1}{x \sin \pi x} = -\infty$ since $\dfrac{x + 1}{x} \to 2$ as $x \to 1^+$ and $\sin \pi x \to 0$ through negative values as $x \to 1^+$.

29. (a)

x	$f(x)$
0.5	−1.14
0.9	−3.69
0.99	−33.7
0.999	−333.7
0.9999	−3333.7
0.99999	−33,333.7

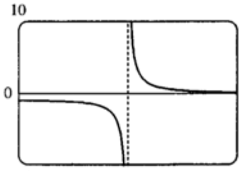
x	$f(x)$
1.5	0.42
1.1	3.02
1.01	33.0
1.001	333.0
1.0001	3333.0
1.00001	33,333.3

From these calculations, it seems that $\lim_{x \to 1^-} f(x) = -\infty$ and $\lim_{x \to 1^+} f(x) = \infty$.

(b) If x is slightly smaller than 1, then $x^3 - 1$ will be a negative number close to 0, and the reciprocal of $x^3 - 1$, that is, $f(x)$, will be a negative number with large absolute value. So $\lim_{x \to 1^-} f(x) = -\infty$.

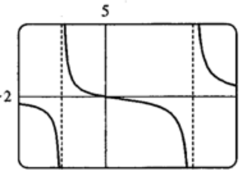
If x is slightly larger than 1, then $x^3 - 1$ will be a small positive number, and its reciprocal, $f(x)$, will be a large positive number. So $\lim_{x \to 1^+} f(x) = \infty$.

(c) It appears from the graph of f that $\lim_{x \to 1^-} f(x) = -\infty$ and $\lim_{x \to 1^+} f(x) = \infty$.



30. (a) $y = \dfrac{x}{x^2 - x - 2} = \dfrac{x}{(x-2)(x+1)}$. Therefore, as $x \to -1^+$ or $x \to 2^+$, the denominator approaches 0, and $y > 0$ for $x < -1$ and for $x > 2$, so $\lim_{x \to -1^+} y = \lim_{x \to 2^+} y = \infty$. Also, as $x \to -1^-$ or $x \to 2^-$, the denominator approaches 0 and $y < 0$ for $-1 < x < 2$, so $\lim_{x \to -1^-} y = \lim_{x \to 2^-} y = -\infty$.

(b)



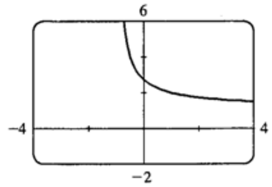
31. (a) Let $h(x) = (1+x)^{1/x}$.

x	$h(x)$
-0.001	2.71964
-0.0001	2.71842
-0.00001	2.71830
-0.000001	2.71828
0.000001	2.71828
0.00001	2.71827
0.0001	2.71815
0.001	2.71692

It appears that $\lim\limits_{x\to 0}(1+x)^{1/x} \approx 2.71828$, which is approximately e.

In Section 7.4 we'll see that the value of the limit is exactly e.

(b)

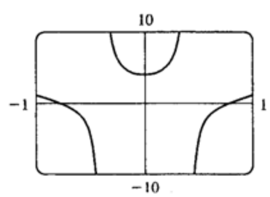


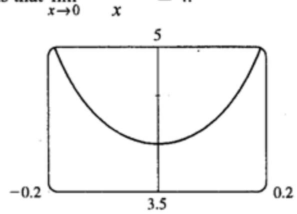
32. For the curve $y = 2^x$ and the points $P(0,1)$ and $Q(x, 2^x)$:

x	Q	m_{PQ}
0.1	(0.1, 1.0717735)	0.71773
0.01	(0.01, 1.0069556)	0.69556
0.001	(0.001, 1.0006934)	0.69339
0.0001	(0.0001, 1.0000693)	0.69317

The slope appears to be about 0.693.

33. (a) From the following graphs, it seems that $\lim\limits_{x\to 0}\dfrac{\tan(4x)}{x} = 4$.

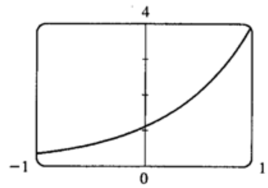


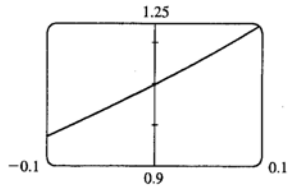


(b)

x	$f(x)$
± 0.1	4.227932
± 0.01	4.002135
± 0.001	4.000021
± 0.0001	4.000000

34. (a) From the following graphs, it seems that $\lim_{x \to 0} \frac{6^x - 2^x}{x} \approx 1.10$.





(b)

x	$f(x)$
-0.01	1.085052
-0.001	1.097248
-0.0001	1.098476
0.0001	1.098749
0.001	1.099978
0.01	1.112353

35. For $f(x) = x^2 - (2^x/1000)$:

(a)

x	$f(x)$
1	0.998000
0.8	0.638259
0.6	0.358484
0.4	0.158680
0.2	0.038851
0.1	0.008928
0.05	0.001465

It appears that $\lim_{x \to 0} f(x) = 0$.

(b)

x	$f(x)$
0.04	0.000572
0.02	-0.000614
0.01	-0.000907
0.005	-0.000978
0.003	-0.000993
0.001	-0.001000

It appears that $\lim_{x \to 0} f(x) = -0.001$.

36. $h(x) = \frac{\tan x - x}{x^3}$

(a)

x	$h(x)$
1.0	0.55740773
0.5	0.37041992
0.1	0.33467209
0.05	0.33366700
0.01	0.33334667
0.005	0.33333667

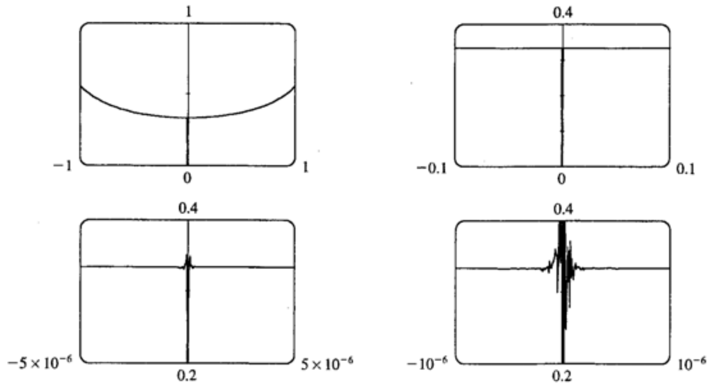
(b) It seems that $\lim_{x \to 0} h(x) = \frac{1}{3}$.

(c)

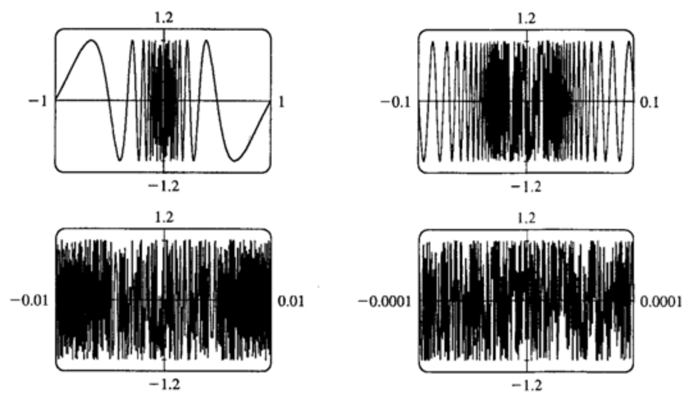
x	$h(x)$
0.001	0.33333350
0.0005	0.33333344
0.0001	0.33333000
0.00005	0.33333600
0.00001	0.33300000
0.000001	0.00000000

Here the values will vary from one calculator to another. Every calculator will eventually give *false values*.

(d) As in part (c), when we take a small enough viewing rectangle we get incorrect output.

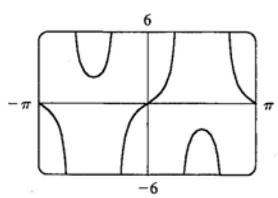


37. No matter how many times we zoom in towards the origin, the graphs appear to consist of almost-vertical lines. This indicates more and more frequent oscillations as $x \to 0$.



38. $\lim\limits_{v\to c^-} m = \lim\limits_{v\to c^-} \dfrac{m_0}{\sqrt{1-v^2/c^2}}$. As $v \to c^-$, $\sqrt{1-v^2/c^2} \to 0^+$, and $m \to \infty$.

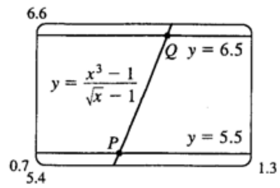
39.



There appear to be vertical asymptotes at $x \approx \pm 0.90$ and $x \approx \pm 2.24$. To find the exact equations of these asymptotes, we note that the graph of the tangent function has vertical asymptotes at $x = \frac{\pi}{2} + \pi n$. Thus, we must have $2\sin x = \frac{\pi}{2} + \pi n$, or equivalently, $\sin x = \frac{\pi}{4} + \frac{\pi}{2}n$. Since $-1 \le \sin x \le 1$, we must have $\sin x = \pm\frac{\pi}{4}$ and so $x = \pm\sin^{-1}\frac{\pi}{4}$ (corresponding to $x \approx \pm 0.90$). Just as 150° is the reference angle for 30°, $\pi - \sin^{-1}\frac{\pi}{4}$ is the reference angle for $\sin^{-1}\frac{\pi}{4}$. So $x = \pm\left(\pi - \sin^{-1}\frac{\pi}{4}\right)$ are also equations of the vertical asymptotes (corresponding to $x \approx \pm 2.24$).

40. (a)

x	y
0.99	5.92531
0.999	5.99250
0.9999	5.99925
1.01	6.07531
1.001	6.00750
1.0001	6.00075



From the table and the graph, we guess that the limit is 6.

(b) We need to have $5.5 < \dfrac{x^3 - 1}{\sqrt{x} - 1} < 6.5$. From the graph we obtain the approximate points of intersection $P\,(0.9313853, 5.5)$ and $Q\,(1.0649004, 6.5)$. Now $1 - 0.9313853 \approx 0.0686$ and $1.0649004 - 1 \approx 0.0649$, so by requiring that x be within 0.0649 of 1, we ensure that y is within 0.5 of 6.

2.3 Calculating Limits Using the Limit Laws

1. (a) $\lim\limits_{x\to a}[f(x) + h(x)] = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} h(x) = -3 + 8 = 5$

(b) $\lim\limits_{x\to a}[f(x)]^2 = \left[\lim\limits_{x\to a} f(x)\right]^2 = (-3)^2 = 9$

(c) $\lim\limits_{x\to a}\sqrt[3]{h(x)} = \sqrt[3]{\lim\limits_{x\to a} h(x)} = \sqrt[3]{8} = 2$

(d) $\lim\limits_{x\to a}\dfrac{1}{f(x)} = \dfrac{1}{\lim\limits_{x\to a} f(x)} = \dfrac{1}{-3} = -\dfrac{1}{3}$

(e) $\lim\limits_{x\to a}\dfrac{f(x)}{h(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} h(x)} = \dfrac{-3}{8} = -\dfrac{3}{8}$

(f) $\lim\limits_{x\to a}\dfrac{g(x)}{f(x)} = \dfrac{\lim\limits_{x\to a} g(x)}{\lim\limits_{x\to a} f(x)} = \dfrac{0}{-3} = 0$

(g) The limit does not exist, since $\lim\limits_{x\to a} g(x) = 0$ but $\lim\limits_{x\to a} f(x) \neq 0$.

(h) $\lim\limits_{x\to a}\dfrac{2f(x)}{h(x) - f(x)} = \dfrac{2\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} h(x) - \lim\limits_{x\to a} f(x)} = \dfrac{2(-3)}{8 - (-3)} = -\dfrac{6}{11}$

2. (a) $\lim\limits_{x\to 2}[f(x) + g(x)] = \lim\limits_{x\to 2} f(x) + \lim\limits_{x\to 2} g(x) = 2 + 0 = 2$

(b) $\lim\limits_{x\to 1} g(x)$ does not exist since its left- and right-hand limits are not equal, so the given limit does not exist.

(c) $\lim\limits_{x\to 0}[f(x)\,g(x)] = \lim\limits_{x\to 0} f(x) \cdot \lim\limits_{x\to 0} g(x) = 0 \cdot 1.3 = 0$

(d) Since $\lim\limits_{x\to -1} g(x) = 0$ and g is in the denominator, the given limit does not exist.

(e) $\lim\limits_{x\to 2} x^3 f(x) = \left[\lim\limits_{x\to 2} x^3\right]\left[\lim\limits_{x\to 2} f(x)\right] = 2^3 \cdot 2 = 16$

(f) $\lim\limits_{x\to 1}\sqrt{3 + f(x)} = \sqrt{3 + \lim\limits_{x\to 1} f(x)} = \sqrt{3 + 1} = 2$

3.
$$\begin{aligned}
\lim_{x\to 4}\left(5x^2 - 2x + 3\right) &= \lim_{x\to 4} 5x^2 - \lim_{x\to 4} 2x + \lim_{x\to 4} 3 && \text{(Limit Laws 2 \& 1)}\\
&= 5\lim_{x\to 4} x^2 - 2\lim_{x\to 4} x + 3 && \text{(3 \& 7)}\\
&= 5\,(4)^2 - 2\,(4) + 3 = 75 && \text{(9 \& 8)}
\end{aligned}$$

4. $\lim_{x\to3}\left(x^3+2\right)\left(x^2-5x\right)=\lim_{x\to3}\left(x^3+2\right)\lim_{x\to3}\left(x^2-5x\right)$ (Limit Law 4)

$=\left(\lim_{x\to3}x^3+\lim_{x\to3}2\right)\left(\lim_{x\to3}x^2-5\lim_{x\to3}x\right)$ (1, 2 & 3)

$=\left(3^3+2\right)\left(3^2-5\cdot3\right)$ (9, 7 & 8)

$=29\,(-6)=-174$

5. $\lim_{x\to-1}\dfrac{x-2}{x^2+4x-3}=\dfrac{\lim_{x\to-1}(x-2)}{\lim_{x\to-1}\left(x^2+4x-3\right)}$ (5)

$=\dfrac{\lim_{x\to-1}x-\lim_{x\to-1}2}{\lim_{x\to-1}x^2+4\lim_{x\to-1}x-\lim_{x\to-1}3}$ (2, 1 & 3)

$=\dfrac{(-1)-2}{(-1)^2+4\,(-1)-3}=\dfrac{1}{2}$ (8, 7 & 9)

6. $\lim_{x\to1}\left(\dfrac{x^4+x^2-6}{x^4+2x+3}\right)^2=\left[\dfrac{\lim_{x\to1}\left(x^4+x^2-6\right)}{\lim_{x\to1}\left(x^4+2x+3\right)}\right]^2$ (6 & 5)

$=\left(\dfrac{\lim_{x\to1}x^4+\lim_{x\to1}x^2-\lim_{x\to1}6}{\lim_{x\to1}x^4+2\lim_{x\to1}x+\lim_{x\to1}3}\right)^2$ (1, 2 & 3)

$=\left(\dfrac{1^4+1^2-6}{1^4+2\cdot1+3}\right)^2$ (9, 7 & 8)

$=\left(\frac{-4}{6}\right)^2=\left(-\frac{2}{3}\right)^2=\frac{4}{9}$

7. $\lim_{t\to-2}(t+1)^9\left(t^2-1\right)=\lim_{t\to-2}(t+1)^9\lim_{t\to-2}\left(t^2-1\right)$ (4)

$=\left[\lim_{t\to-2}(t+1)\right]^9\lim_{t\to-2}\left(t^2-1\right)$ (6)

$=\left[\lim_{t\to-2}t+\lim_{t\to-2}1\right]^9\left[\lim_{t\to-2}t^2-\lim_{t\to-2}1\right]$ (1 & 2)

$=\left[(-2)+1\right]^9\left[(-2)^2-1\right]=-3$ (8, 7 & 9)

8. $\lim_{u\to-2}\sqrt{u^4+3u+6}=\sqrt{\lim_{u\to-2}\left(u^4+3u+6\right)}$ (11)

$=\sqrt{\lim_{u\to-2}u^4+3\lim_{u\to-2}u+\lim_{u\to-2}6}$ (1, 2 & 3)

$=\sqrt{(-2)^4+3\,(-2)+6}$ (9, 8 & 7)

$=\sqrt{16-6+6}=\sqrt{16}=4$

9. $\lim_{x\to4^-}\sqrt{16-x^2}=\sqrt{\lim_{x\to4^-}\left(16-x^2\right)}$ (11)

$=\sqrt{\lim_{x\to4^-}16-\lim_{x\to4^-}x^2}$ (2)

$=\sqrt{16-(4)^2}=0$ (7 & 9)

10. (a) The left-hand side of the equation is not defined for $x = 2$, but the right-hand side is.

(b) Since the equation holds for all $x \neq 2$, it follows that both sides of the equation approach the same limit as $x \to 2$, just as in Example 3. Remember that in finding $\lim_{x\to a} f(x)$, we never consider $x = a$.

11. $\lim_{x\to -3} \dfrac{x^2 - x + 12}{x + 3}$ does not exist since $x + 3 \to 0$ but $x^2 - x + 12 \to 24$ as $x \to -3$.

12. $\lim_{x\to -3} \dfrac{x^2 - x - 12}{x + 3} = \lim_{x\to -3} \dfrac{(x+3)(x-4)}{x+3} = \lim_{x\to -3} (x - 4) = -3 - 4 = -7$

13. $\lim_{x\to -2} \dfrac{x+2}{x^2 - x - 6} = \lim_{x\to -2} \dfrac{x+2}{(x-3)(x+2)} = \lim_{x\to -2} \dfrac{1}{x-3} = -\dfrac{1}{5}$

14. $\lim_{x\to 1} \dfrac{x^2 + x - 2}{x^2 - 3x + 2} = \lim_{x\to 1} \dfrac{(x+2)(x-1)}{(x-2)(x-1)} = \lim_{x\to 1} \dfrac{x+2}{x-2} = \dfrac{1+2}{1-2} = -3$

15. $\lim_{h\to 0} \dfrac{(h-5)^2 - 25}{h} = \lim_{h\to 0} \dfrac{(h^2 - 10h + 25) - 25}{h} = \lim_{h\to 0} \dfrac{h^2 - 10h}{h} = \lim_{h\to 0} (h - 10) = -10$

16. $\lim_{x\to 1} \dfrac{x^3 - 1}{x^2 - 1} = \lim_{x\to 1} \dfrac{(x-1)(x^2 + x + 1)}{(x-1)(x+1)} = \lim_{x\to 1} \dfrac{x^2 + x + 1}{x+1} = \dfrac{1^2 + 1 + 1}{1+1} = \dfrac{3}{2}$

17. $\lim_{h\to 0} \dfrac{(1+h)^4 - 1}{h} = \lim_{h\to 0} \dfrac{(1 + 4h + 6h^2 + 4h^3 + h^4) - 1}{h} = \lim_{h\to 0} \dfrac{4h + 6h^2 + 4h^3 + h^4}{h}$

$= \lim_{h\to 0} (4 + 6h + 4h^2 + h^3) = 4$

18. $\lim_{h\to 0} \dfrac{(2+h)^3 - 8}{h} = \lim_{h\to 0} \dfrac{(8 + 12h + 6h^2 + h^3) - 8}{h} = \lim_{h\to 0} \dfrac{12h + 6h^2 + h^3}{h} = \lim_{h\to 0} (12 + 6h + h^2) = 12$

19. $\lim_{t\to 9} \dfrac{9 - t}{3 - \sqrt{t}} = \lim_{t\to 9} \dfrac{(3 + \sqrt{t})(3 - \sqrt{t})}{3 - \sqrt{t}} = \lim_{t\to 9} (3 + \sqrt{t}) = 3 + \sqrt{9} = 6$

20. $\lim_{t\to 2} \dfrac{t^2 + t - 6}{t^2 - 4} = \lim_{t\to 2} \dfrac{(t+3)(t-2)}{(t+2)(t-2)} = \lim_{t\to 2} \dfrac{t+3}{t+2} = \dfrac{5}{4}$

21. $\lim_{t\to 0} \dfrac{\sqrt{2-t} - \sqrt{2}}{t} = \lim_{t\to 0} \dfrac{\sqrt{2-t} - \sqrt{2}}{t} \cdot \dfrac{\sqrt{2-t} + \sqrt{2}}{\sqrt{2-t} + \sqrt{2}} = \lim_{t\to 0} \dfrac{-t}{t\left(\sqrt{2-t} + \sqrt{2}\right)} = \lim_{t\to 0} \dfrac{-1}{\sqrt{2-t} + \sqrt{2}}$

$= -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4}$

22. $\lim_{x\to 2} \dfrac{x^4 - 16}{x - 2} = \lim_{x\to 2} \dfrac{(x+2)(x-2)(x^2+4)}{x-2} = \lim_{x\to 2} (x+2)(x^2+4) = \lim_{x\to 2} (x+2) \lim_{x\to 2} (x^2+4)$

$= (2+2)(2^2+4) = 32$

23. $\lim_{x\to 9} \dfrac{x^2 - 81}{\sqrt{x} - 3} = \lim_{x\to 9} \dfrac{(x-9)(x+9)}{\sqrt{x} - 3} = \lim_{x\to 9} \dfrac{(\sqrt{x} - 3)(\sqrt{x} + 3)(x+9)}{\sqrt{x} - 3}$

$= \lim_{x\to 9} (\sqrt{x} + 3)(x+9) = \lim_{x\to 9} (\sqrt{x} + 3) \lim_{x\to 9} (x+9) = \left(\sqrt{9} + 3\right)(9+9) = 108$

24. $\lim_{x\to 1} \left(\dfrac{1}{x-1} - \dfrac{2}{x^2-1}\right) = \lim_{x\to 1} \dfrac{(x+1) - 2}{(x-1)(x+1)} = \lim_{x\to 1} \dfrac{x-1}{(x-1)(x+1)} = \lim_{x\to 1} \dfrac{1}{x+1} = \dfrac{1}{2}$

25. $\lim_{t\to 0} \left(\dfrac{1}{t\sqrt{1+t}} - \dfrac{1}{t}\right) = \lim_{t\to 0} \dfrac{1 - \sqrt{1+t}}{t\sqrt{1+t}} = \lim_{t\to 0} \dfrac{(1 - \sqrt{1+t})(1 + \sqrt{1+t})}{t\sqrt{t+1}(1 + \sqrt{1+t})} = \lim_{t\to 0} \dfrac{-t}{t\sqrt{1+t}(1 + \sqrt{1+t})}$

$= \lim_{t\to 0} \dfrac{-1}{\sqrt{1+t}(1 + \sqrt{1+t})} = \dfrac{-1}{\sqrt{1+0}(1 + \sqrt{1+0})} = -\dfrac{1}{2}$

26. $\displaystyle\lim_{h\to 0}\frac{(3+h)^{-1}-3^{-1}}{h}=\lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h}=\lim_{h\to 0}\frac{3-(3+h)}{h\,(3+h)\,3}=\lim_{h\to 0}\frac{-h}{h\,(3+h)\,3}$

$\displaystyle=\lim_{h\to 0}\left[-\frac{1}{3\,(3+h)}\right]=-\frac{1}{\lim_{h\to 0}[3\,(3+h)]}=-\frac{1}{3\,(3+0)}=-\frac{1}{9}$

27. $\displaystyle\lim_{x\to 2}\frac{1/x-\frac{1}{2}}{x-2}=\lim_{x\to 2}\frac{2-x}{2x\,(x-2)}=\lim_{x\to 2}\frac{-1}{2x}=-\frac{1}{4}$

28. $\displaystyle\lim_{x\to 1}\frac{\sqrt{x}-x^2}{1-\sqrt{x}}=\lim_{x\to 1}\frac{\sqrt{x}\,(1-x^{3/2})}{1-\sqrt{x}}=\lim_{x\to 1}\frac{\sqrt{x}\,(1-\sqrt{x})\,(1+\sqrt{x}+x)}{1-\sqrt{x}}$ (difference of cubes)

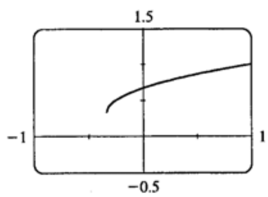
$\displaystyle=\lim_{x\to 1}\left[\sqrt{x}\,(1+\sqrt{x}+x)\right]=\lim_{x\to 1}[1\,(1+1+1)]=3$

Another Method: We "add and subtract" 1 in the numerator, and then split up the fraction:

$\displaystyle\lim_{x\to 1}\frac{\sqrt{x}-x^2}{1-\sqrt{x}}=\lim_{x\to 1}\frac{(\sqrt{x}-1)+(1-x^2)}{1-\sqrt{x}}=\lim_{x\to 1}\left[-1+\frac{(1-x)\,(1+x)}{1-\sqrt{x}}\right]$

$\displaystyle=\lim_{x\to 1}\left[-1+\frac{(1-\sqrt{x})\,(1+\sqrt{x})\,(1+x)}{1-\sqrt{x}}\right]=-1+\left(1+\sqrt{1}\right)(1+1)=3$

29. (a)



$\displaystyle\lim_{x\to 0}\frac{x}{\sqrt{1+3x}-1}\approx\frac{2}{3}$

(b)

x	$f(x)$
−0.001	0.6661663
−0.0001	0.6666167
−0.00001	0.6666617
−0.000001	0.6666662
0.000001	0.6666672
0.00001	0.6666717
0.0001	0.6667167
0.001	0.6671663

The limit appears to be $\frac{2}{3}$.

(c) $\displaystyle\lim_{x\to 0}\left(\frac{x}{\sqrt{1+3x}-1}\cdot\frac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1}\right)=\lim_{x\to 0}\frac{x\,(\sqrt{1+3x}+1)}{(1+3x)-1}=\lim_{x\to 0}\frac{x\,(\sqrt{1+3x}+1)}{3x}$

$\displaystyle=\tfrac{1}{3}\lim_{x\to 0}\left(\sqrt{1+3x}+1\right)$ (Limit Law 3)

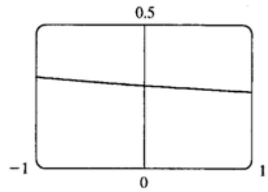
$\displaystyle=\tfrac{1}{3}\left[\sqrt{\lim_{x\to 0}(1+3x)}+\lim_{x\to 0}1\right]$ (1 & 11)

$\displaystyle=\tfrac{1}{3}\left(\sqrt{\lim_{x\to 0}1+3\lim_{x\to 0}x}+1\right)$ (1, 3 & 7)

$\displaystyle=\tfrac{1}{3}\left(\sqrt{1+3\cdot 0}+1\right)$ (7 & 8)

$=\frac{2}{3}$

30. (a)



$$\lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3}}{x}\approx 0.29$$

(b)

x	$f(x)$
-0.001	0.2886992
-0.0001	0.2886775
-0.00001	0.2886754
-0.000001	0.2886752
0.000001	0.2886751
0.00001	0.2886749
0.0001	0.2886727
0.001	0.2886511

The limit appears to be approximately 0.2887.

(c) $$\lim_{x\to 0}\left(\frac{\sqrt{3+x}-\sqrt{3}}{x}\cdot\frac{\sqrt{3+x}+\sqrt{3}}{\sqrt{3+x}+\sqrt{3}}\right)=\lim_{x\to 0}\frac{(3+x)-3}{x\left(\sqrt{3+x}+\sqrt{3}\right)}=\lim_{x\to 0}\frac{1}{\sqrt{3+x}+\sqrt{3}}$$

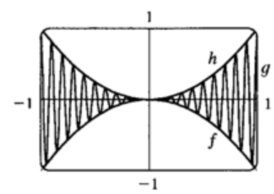
$$=\frac{\lim_{x\to 0}1}{\lim_{x\to 0}\sqrt{3+x}+\lim_{x\to 0}\sqrt{3}}\qquad\text{(Limit Laws 5 \& 1)}$$

$$=\frac{1}{\sqrt{\lim_{x\to 0}(3+x)}+\sqrt{3}}\qquad\text{(7 \& 11)}$$

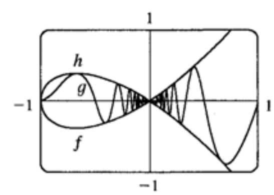
$$=\frac{1}{\sqrt{3+0}+\sqrt{3}}\qquad\text{(1, 7 \& 8)}$$

$$=\frac{1}{2\sqrt{3}}$$

31. Let $f(x)=-x^2$, $g(x)=x^2\cos 20\pi x$ and $h(x)=x^2$. Then $-1\le\cos 20\pi x\le 1\ \Rightarrow\ -x^2\le x^2\cos 20\pi x\le x^2\ \Rightarrow$ $f(x)\le g(x)\le h(x)$. So since $\lim_{x\to 0}f(x)=\lim_{x\to 0}h(x)=0$, by the Squeeze Theorem we have $\lim_{x\to 0}g(x)=0$.



32. Let $f(x)=-\sqrt{x^3+x^2}$, $g(x)=\sqrt{x^3+x^2}\sin(\pi/x)$, and $h(x)=\sqrt{x^3+x^2}$. Then $-1\le\sin(\pi/x)\le 1\ \Rightarrow$ $-\sqrt{x^3+x^2}\le\sqrt{x^3+x^2}\sin(\pi/x)\le\sqrt{x^3+x^2}\ \Rightarrow$ $f(x)\le g(x)\le h(x)$. So since $\lim_{x\to 0}f(x)=\lim_{x\to 0}h(x)=0$, by the Squeeze Theorem we have $\lim_{x\to 0}g(x)=0$.



33. $1\le f(x)\le x^2+2x+2$ for all x. Now $\lim_{x\to -1}1=1$ and $\lim_{x\to -1}\left(x^2+2x+2\right)=\lim_{x\to -1}x^2+2\lim_{x\to -1}x+\lim_{x\to -1}2=(-1)^2+2(-1)+2=1$. Therefore, by the Squeeze Theorem, $\lim_{x\to -1}f(x)=1$.

34. $3x \le f(x) \le x^3 + 2$ for $0 \le x \le 2$. Now $\lim_{x\to 1} 3x = 3$ and $\lim_{x\to 1}(x^3+2) = \lim_{x\to 1} x^3 + \lim_{x\to 1} 2 = 1^3 + 2 = 3$. Therefore, by the Squeeze Theorem, $\lim_{x\to 1} f(x) = 3$.

35. $-1 \le \cos(2/x) \le 1 \quad\Rightarrow\quad -x^4 \le x^4\cos(2/x) \le x^4$. Since $\lim_{x\to 0}(-x^4) = 0$ and $\lim_{x\to 0} x^4 = 0$, we have $\lim_{x\to 0}\left[x^4\cos(2/x)\right] = 0$ by the Squeeze Theorem.

36. $-1 \le \sin(2\pi/x) \le 1 \quad\Rightarrow\quad 0 \le \sin^2(2\pi/x) \le 1 \quad\Rightarrow\quad 1 \le 1 + \sin^2(2\pi/x) \le 2 \quad\Rightarrow$ $\sqrt{x} \le \sqrt{x}\left[1 + \sin^2(2\pi/x)\right] \le 2\sqrt{x}$. Since $\lim_{x\to 0^+}\sqrt{x} = 0$ and $\lim_{x\to 0^+} 2\sqrt{x} = 0$, we have $\lim_{x\to 0^+}\left[\sqrt{x}\left(1 + \sin^2(2\pi/x)\right)\right] = 0$ by the Squeeze Theorem.

37. If $x > -4$, then $|x+4| = x+4$, so $\lim_{x\to -4^+}|x+4| = \lim_{x\to -4^+}(x+4) = -4+4 = 0$.
If $x < -4$, then $|x+4| = -(x+4)$, so $\lim_{x\to -4^-}|x+4| = \lim_{x\to -4^-} -(x+4) = -(-4+4) = 0$.
Since the right and left limits are equal, $\lim_{x\to -4}|x+4| = 0$.

38. If $x < -4$, then $|x+4| = -(x+4)$, so $\lim_{x\to -4^-}\dfrac{|x+4|}{x+4} = \lim_{x\to -4^-}\dfrac{-(x+4)}{x+4} = \lim_{x\to -4^-}(-1) = -1$.

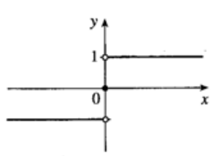
39. If $x > 2$, then $|x-2| = x-2$, so $\lim_{x\to 2^+}\dfrac{|x-2|}{x-2} = \lim_{x\to 2^+}\dfrac{x-2}{x-2} = \lim_{x\to 2^+} 1 = 1$. If $x < 2$, then $|x-2| = -(x-2)$, so $\lim_{x\to 2^-}\dfrac{|x-2|}{x-2} = \lim_{x\to 2^-}\dfrac{-(x-2)}{x-2} = \lim_{x\to 2^-} -1 = -1$. The right and left limits are different, so $\lim_{x\to 2}\dfrac{|x-2|}{x-2}$ does not exist.

40. If $x > \frac{3}{2}$, then $|2x-3| = 2x-3$, so
$\lim_{x\to 1.5^+}\dfrac{2x^2-3x}{|2x-3|} = \lim_{x\to 1.5^+}\dfrac{2x^2-3x}{2x-3} = \lim_{x\to 1.5^+}\dfrac{x(2x-3)}{2x-3} = \lim_{x\to 1.5^+} x = 1.5$. If $x < \frac{3}{2}$, then $|2x-3| = 3-2x$, so $\lim_{x\to 1.5^-}\dfrac{2x^2-3x}{|2x-3|} = \lim_{x\to 1.5^-}\dfrac{2x^2-3x}{-(2x-3)} = \lim_{x\to 1.5^-}\dfrac{x(2x-3)}{-(2x-3)} = \lim_{x\to 1.5^-} -x = -1.5$. The right and left limits are different, so $\lim_{x\to 1.5}\dfrac{2x^2-3x}{|2x-3|}$ does not exist.

41. Since $|x| = -x$ for $x < 0$, we have $\lim_{x\to 0^-}\left(\dfrac{1}{x} - \dfrac{1}{|x|}\right) = \lim_{x\to 0^-}\left(\dfrac{1}{x} - \dfrac{1}{-x}\right) = \lim_{x\to 0^-}\dfrac{2}{x}$, which does not exist since the denominator approaches 0 and the numerator does not.

42. Since $|x| = x$ for $x > 0$, we have $\lim_{x\to 0^+}\left(\dfrac{1}{x} - \dfrac{1}{|x|}\right) = \lim_{x\to 0^+}\left(\dfrac{1}{x} - \dfrac{1}{x}\right) = \lim_{x\to 0^+} 0 = 0$.

43. (a)



(b) (i) Since $\operatorname{sgn} x = 1$ for $x > 0$, $\lim_{x\to 0^+}\operatorname{sgn} x = \lim_{x\to 0^+} 1 = 1$.

(ii) Since $\operatorname{sgn} x = -1$ for $x < 0$, $\lim_{x\to 0^-}\operatorname{sgn} x = \lim_{x\to 0^-} -1 = -1$.

(iii) Since $\lim_{x\to 0^-}\operatorname{sgn} x \ne \lim_{x\to 0^+}\operatorname{sgn} x$, $\lim_{x\to 0}\operatorname{sgn} x$ does not exist.

(iv) Since $|\operatorname{sgn} x| = 1$ for $x \ne 0$, $\lim_{x\to 0}|\operatorname{sgn} x| = \lim_{x\to 0} 1 = 1$.

44. (a) $\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^-} (x^2 - 2x + 2)$

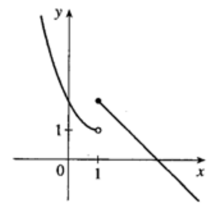
$= \lim\limits_{x\to1^-} x^2 - 2\lim\limits_{x\to1^-} x + \lim\limits_{x\to1^-} 2$

$= 1^2 - 2 + 2 = 1$

$\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+} (3 - x) = \lim\limits_{x\to1^+} 3 - \lim\limits_{x\to1^+} x = 3 - 1 = 2$

(b) $\lim\limits_{x\to1} f(x)$ does not exist because $\lim\limits_{x\to1^-} f(x) \neq \lim\limits_{x\to1^+} f(x)$.

(c)

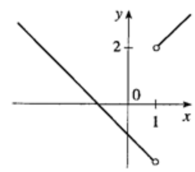


45. (a) (i) $\lim\limits_{x\to1^+} \dfrac{x^2-1}{|x-1|} = \lim\limits_{x\to1^+} \dfrac{x^2-1}{x-1} = \lim\limits_{x\to1^+} (x+1) = 2$

(ii) $\lim\limits_{x\to1^-} \dfrac{x^2-1}{|x-1|} = \lim\limits_{x\to1^-} \dfrac{x^2-1}{-(x-1)} = \lim\limits_{x\to1^-} -(x+1) = -2$

(b) No, $\lim\limits_{x\to1} F(x)$ does not exist since $\lim\limits_{x\to1^+} F(x) \neq \lim\limits_{x\to1^-} F(x)$.

(c)



46. (a) (i) $\lim\limits_{x\to0^+} h(x) = \lim\limits_{x\to0^+} x^2 = 0^2 = 0$

(ii) $\lim\limits_{x\to0^-} h(x) = \lim\limits_{x\to0^-} x = 0$, so $\lim\limits_{x\to0} h(x) = 0$.

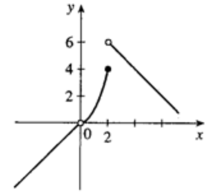
(iii) $\lim\limits_{x\to1} h(x) = \lim\limits_{x\to1} x^2 = 1^2 = 1$

(iv) $\lim\limits_{x\to2^-} h(x) = \lim\limits_{x\to2^-} x^2 = 2^2 = 4$

(v) $\lim\limits_{x\to2^+} h(x) = \lim\limits_{x\to2^+} (8 - x) = 8 - 2 = 6$

(vi) Since $\lim\limits_{x\to2^-} h(x) \neq \lim\limits_{x\to2^+} h(x)$, $\lim\limits_{x\to2} h(x)$ does not exist.

(b)



47. (a) (i) $[\![x]\!] = -2$ for $-2 \le x < -1$, so $\lim\limits_{x\to-2^+} [\![x]\!] = \lim\limits_{x\to-2^+} (-2) = -2$

(ii) $[\![x]\!] = -3$ for $-3 \le x < -2$, so $\lim\limits_{x\to-2^-} [\![x]\!] = \lim\limits_{x\to-2^-} (-3) = -3$. The right and left limits are different, so $\lim\limits_{x\to-2} [\![x]\!]$ does not exist.

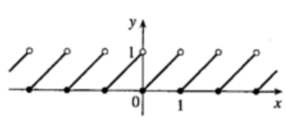
(iii) $[\![x]\!] = -3$ for $-3 \le x < -2$, so $\lim\limits_{x\to-2.4} [\![x]\!] = \lim\limits_{x\to-2.4} (-3) = -3$.

(b) (i) $[\![x]\!] = n - 1$ for $n - 1 \le x < n$, so $\lim\limits_{x\to n^-} [\![x]\!] = \lim\limits_{x\to n^-} (n-1) = n - 1$.

(ii) $[\![x]\!] = n$ for $n \le x < n + 1$, so $\lim\limits_{x\to n^+} [\![x]\!] = \lim\limits_{x\to n^+} n = n$.

(c) $\lim\limits_{x\to a} [\![x]\!]$ exists $\Leftrightarrow$ a is not an integer.

48. (a)



(b) (i) $\lim\limits_{x\to n^-} f(x) = \lim\limits_{x\to n^-} (x - [\![x]\!]) = \lim\limits_{x\to n^-} [x - (n-1)] = n - (n-1) = 1$

(ii) $\lim\limits_{x\to n^+} f(x) = \lim\limits_{x\to n^+} (x - [\![x]\!]) = \lim\limits_{x\to n^+} (x - n) = n - n = 0$

(c) $\lim\limits_{x\to a} f(x)$ exists $\Leftrightarrow$ a is not an integer.

49. The graph of $f(x) = [\![x]\!] + [\![-x]\!]$ is the same as the graph of $g(x) = -1$ with holes at each integer, since $f(a) = 0$ for any integer a. Thus, $\lim\limits_{x\to 2^-} f(x) = -1$ and $\lim\limits_{x\to 2^+} f(x) = -1$, so $\lim\limits_{x\to 2} f(x) = -1$.
$f(2) = [\![2]\!] + [\![-2]\!] = 2 + (-2) = 0$.

50. $\lim\limits_{v\to c^-}\left(L_0\sqrt{1-\dfrac{v^2}{c^2}}\right) = L_0\sqrt{1-1} = 0$. As the velocity approaches the speed of light, the length approaches 0. A left-hand limit is necessary since L is not defined for $v > c$.

51. Since $p(x)$ is a polynomial, $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. Thus, by the Limit Laws,

$$\begin{aligned}\lim_{x\to a} p(x) &= \lim_{x\to a}\left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right)\\ &= a_0 + a_1\lim_{x\to a} x + a_2\lim_{x\to a} x^2 + \cdots + a_n\lim_{x\to a} x^n\\ &= a_0 + a_1a + a_2a^2 + \cdots + a_na^n = p(a)\end{aligned}$$

Thus, for any polynomial p, $\lim\limits_{x\to a} p(x) = p(a)$.

52. Let $r(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are any polynomials, and suppose that $q(a) \neq 0$. Thus,

$$\lim_{x\to a} r(x) = \lim_{x\to a}\frac{p(x)}{q(x)} = \frac{\lim\limits_{x\to a} p(x)}{\lim\limits_{x\to a} q(x)} \quad \text{(Limit Law 5)} \quad = \frac{p(a)}{q(a)} \quad \text{(Exercise 51)} \quad = r(a).$$

53. Observe that $0 \le f(x) \le x^2$ for all x, and $\lim\limits_{x\to 0} 0 = 0 = \lim\limits_{x\to 0} x^2$. So, by the Squeeze Theorem, $\lim\limits_{x\to 0} f(x) = 0$.

54. Let $f(x) = [\![x]\!]$ and $g(x) = -[\![x]\!]$. Then $\lim\limits_{x\to 3} f(x)$ and $\lim\limits_{x\to 3} g(x)$ do not exist (Example 10) but $\lim\limits_{x\to 3}[f(x) + g(x)] = \lim\limits_{x\to 3}([\![x]\!] - [\![x]\!]) = \lim\limits_{x\to 3} 0 = 0$.

55. Let $f(x) = H(x)$ and $g(x) = 1 - H(x)$, where H is the Heaviside function defined in Exercise 1.3.59. Thus, either f or g is 0 for any value of x. Then $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to 0} g(x)$ do not exist, but $\lim\limits_{x\to 0}[f(x)\,g(x)] = \lim\limits_{x\to 0} 0 = 0$.

56.
$$\begin{aligned}\lim_{x\to 2}\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} &= \lim_{x\to 2}\left(\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}\cdot\frac{\sqrt{6-x}+2}{\sqrt{6-x}+2}\cdot\frac{\sqrt{3-x}+1}{\sqrt{3-x}+1}\right)\\ &= \lim_{x\to 2}\left[\frac{\left(\sqrt{6-x}\right)^2-2^2}{\left(\sqrt{3-x}\right)^2-1^2}\cdot\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}\right] = \lim_{x\to 2}\left(\frac{6-x-4}{3-x-1}\cdot\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}\right)\\ &= \lim_{x\to 2}\frac{(2-x)\left(\sqrt{3-x}+1\right)}{(2-x)\left(\sqrt{6-x}+2\right)} = \lim_{x\to 2}\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} = \frac{1}{2}\end{aligned}$$

57. Since the denominator approaches 0 as $x \to -2$, the limit will exist only if the numerator also approaches 0 as $x \to -2$. In order for this to happen, we need $\lim\limits_{x\to -2}\left(3x^2 + ax + a + 3\right) = 0 \iff$
$3(-2)^2 + a(-2) + a + 3 = 0 \iff 12 - 2a + a + 3 = 0 \iff a = 15$. With $a = 15$, the limit becomes
$\lim\limits_{x\to -2}\dfrac{3x^2+15x+18}{x^2+x-2} = \lim\limits_{x\to -2}\dfrac{3(x+2)(x+3)}{(x-1)(x+2)} = \dfrac{3(-2+3)}{-2-1} = -1.$

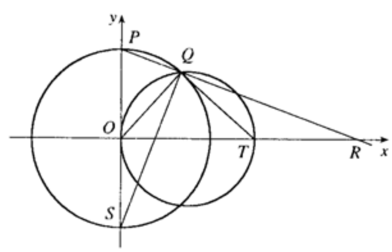
58. *Solution 1:* First, we find the coordinates of P and Q as functions of r. Then we can find the equation of the line determined by these two points, and thus find the x-intercept (the point R), and take the limit as $r \to 0$.
The coordinates of P are $(0, r)$. The point Q is the point of intersection of the two circles $x^2 + y^2 = r^2$ and $(x-1)^2 + y^2 = 1$. Eliminating y from these equations, we get $r^2 - x^2 = 1 - (x-1)^2 \quad \Leftrightarrow \quad r^2 = 1 + 2x - 1$ $\Leftrightarrow \quad x = \frac{1}{2}r^2$. Substituting back into the equation of the shrinking circle to find the y-coordinate, we get $\left(\frac{1}{2}r^2\right)^2 + y^2 = r^2 \quad \Leftrightarrow \quad y^2 = r^2\left(1 - \frac{1}{4}r^2\right) \quad \Leftrightarrow \quad y = r\sqrt{1 - \frac{1}{4}r^2}$ (the positive y-value). So the coordinates of Q are $\left(\frac{1}{2}r^2, r\sqrt{1 - \frac{1}{4}r^2}\right)$. The equation of the line joining P and Q is thus

$y - r = \dfrac{r\sqrt{1 - \frac{1}{4}r^2} - r}{\frac{1}{2}r^2 - 0}(x - 0)$. We set $y = 0$ in order to find the x-intercept, and get

$x = -r\dfrac{\frac{1}{2}r^2}{r\left(\sqrt{1 - \frac{1}{4}r^2} - 1\right)} = \dfrac{-\frac{1}{2}r^2\left(\sqrt{1 - \frac{1}{4}r^2} + 1\right)}{1 - \frac{1}{4}r^2 - 1} = 2\left(\sqrt{1 - \frac{1}{4}r^2} + 1\right)$. Now we take the limit as $r \to 0^+$:

$\lim\limits_{r \to 0^+} x = \lim\limits_{r \to 0^+} 2\left(\sqrt{1 - \frac{1}{4}r^2} + 1\right) = \lim\limits_{r \to 0^+} 2\left(\sqrt{1} + 1\right) = 4$. So the limiting position of R is the point $(4, 0)$.

Solution 2: We add a few lines to the diagram, as shown. Note that $\angle PQS = 90°$ (subtended by diameter PS). So $\angle SQR = 90° = \angle OQT$ (subtended by diameter OT). It follows that $\angle OQS = \angle TQR$. Also $\angle PSQ = 90° - \angle SPQ = \angle ORP$. Since $\triangle QOS$ is isosceles, so is $\triangle QTR$, implying that $QT = TR$. As the circle shrinks, the point Q plainly approaches the origin, so the point R must approach a point twice as far from the origin as T, that is, the point $(4, 0)$, as above.



2.4 The Precise Definition of a Limit

1. (a) To have $5x + 3$ within a distance of 0.1 of 13, we must have $12.9 \le 5x + 3 \le 13.1 \quad \Rightarrow \quad 9.9 \le 5x \le 10.1$ $\Rightarrow \quad 1.98 \le x \le 2.02$. Thus, x must be within 0.02 units of 2 so that $5x + 3$ is within 0.1 of 13.

(b) Use 0.01 in place of 0.1 in part (a) to obtain 0.002.

2. (a) To have $6x - 1$ within a distance of 0.01 of 29, we must have $28.99 \le 6x - 1 \le 29.01 \quad \Rightarrow$ $29.99 \le 6x \le 30.01 \quad \Rightarrow \quad 4.998\overline{3} \le x \le 5.001\overline{6}$. Thus, x must be within $0.001\overline{6}$ units of 5 so that $6x - 1$ is within 0.01 of 29.

(b) As in part (a) with 0.001 in place of 0.01, we obtain $0.0001\overline{6}$.

(c) As in part (a) with 0.0001 in place of 0.01, we obtain $0.00001\overline{6}$.

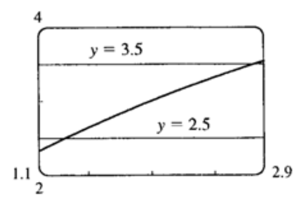
3. On the left side, we need $|x-2| < \left|\frac{10}{7}-2\right| = \frac{4}{7}$. On the right side, we need $|x-2| < \left|\frac{10}{3}-2\right| = \frac{4}{3}$. For both of these conditions to be satisfied at once, we need the more restrictive of the two to hold, that is, $|x-2| < \frac{4}{7}$. So we can choose $\delta = \frac{4}{7}$, or any smaller positive number.

4. On the left side, we need $|x-5| < |4-5| = 1$. On the right side, we need $|x-5| < |5.7-5| = 0.7$. For both conditions to be satisfied at once, we need the more restrictive condition to hold; that is, $|x-5| < 0.7$. So we can choose $\delta = 0.7$, or any smaller positive number.

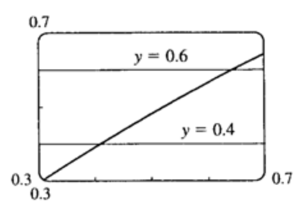
5. The leftmost question mark is the solution of $\sqrt{x} = 1.6$ and the rightmost, $\sqrt{x} = 2.4$. So the values are $1.6^2 = 2.56$ and $2.4^2 = 5.76$. On the left side, we need $|x-4| < |2.56-4| = 1.44$. On the right side, we need $|x-4| < |5.76-4| = 1.76$. To satisfy both conditions, we need the more restrictive condition to hold — namely, $|x-4| < 1.44$. Thus, we can choose $\delta = 1.44$, or any smaller positive number.

6. The left-hand question mark is the positive solution of $x^2 = \frac{1}{2}$, that is, $x = \frac{1}{\sqrt{2}}$, and the right-hand question mark is the positive solution of $x^2 = \frac{3}{2}$, that is, $x = \sqrt{\frac{3}{2}}$. On the left side, we need $|x-1| < \left|\frac{1}{\sqrt{2}}-1\right| \approx 0.293$. On the right side, we need $|x-1| < \left|\sqrt{\frac{3}{2}}-1\right| \approx 0.224$ (rounding down to be safe). The more restrictive of these two conditions must apply, so we choose $\delta = 0.224$ (or any smaller positive number).

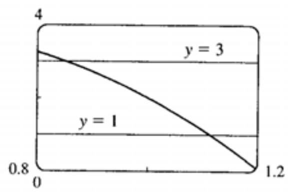
7. $\left|\sqrt{4x+1}-3\right| < 0.5 \quad \Leftrightarrow \quad 2.5 < \sqrt{4x+1} < 3.5$. We plot the three parts of this inequality on the same screen and identify the x-coordinates of the points of intersection using the cursor. It appears that the inequality holds for $1.32 \le x \le 2.81$. Since $|2-1.32| = 0.68$ and $|2-2.81| = 0.81$, we choose $0 < \delta \le \min\{0.68, 0.81\} = 0.68$.

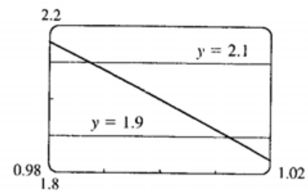


8. $\left|\sin x - \frac{1}{2}\right| < 0.1 \quad \Leftrightarrow \quad 0.4 < \sin x < 0.6$. From the graph, we see that for this inequality to hold, we need $0.42 \le x \le 0.64$. So since $|0.5-0.42| = 0.08$ and $|0.5-0.64| = 0.14$, we choose $0 < \delta \le \min\{0.08, 0.14\} = 0.08$.

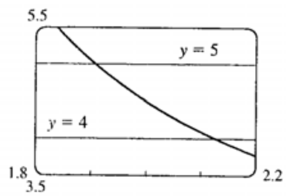


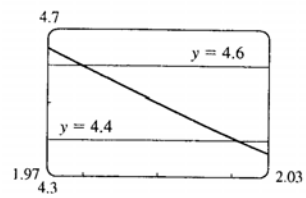
9. For $\varepsilon = 1$, the definition of a limit requires that we find δ such that $\left|(4 + x - 3x^3) - 2\right| < 1 \iff 1 < 4 + x - 3x^3 < 3$ whenever $|x - 1| < \delta$. If we plot the graphs of $y = 1$, $y = 4 + x - 3x^3$ and $y = 3$ on the same screen, we see that we need $0.86 \leq x \leq 1.11$. So since $|1 - 0.86| = 0.14$ and $|1 - 1.11| = 0.11$, we choose $\delta = 0.11$ (or any smaller positive number). For $\varepsilon = 0.1$, we must find δ such that $\left|(4 + x - 3x^3) - 2\right| < 0.1 \iff 1.9 < 4 + x - 3x^3 < 2.1$ whenever $|x - 1| < \delta$. From the graph, we see that we need $0.988 \leq x \leq 1.012$. So since $|1 - 0.988| = 0.012$ and $|1 - 1.012| = 0.012$, we must choose $\delta = 0.012$ (or any smaller positive number) for the inequality to hold.



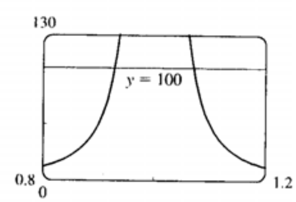


10. For $\varepsilon = 0.5$, we need $1.91 \leq x \leq 2.125$. So since $|2 - 1.91| = 0.09$ and $|2 - 2.125| = 0.125$, we can take $0 < \delta \leq 0.09$. For $\varepsilon = 0.1$, we need $1.980 \leq 2.021$. So since $|2 - 1.980| = 0.02$ and $|2 - 2.021| = 0.021$, we can take $\delta = 0.02$ (or any smaller positive number).

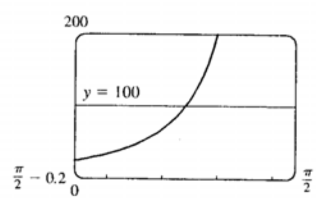


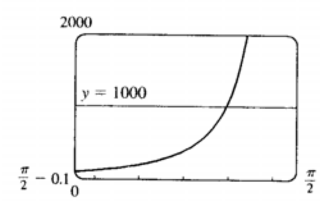


11. From the graph, we see that $\dfrac{x}{(x^2 + 1)(x - 1)^2} > 100$ whenever $0.93 \leq x \leq 1.07$. So since $|1 - 0.93| = 0.07$ and $|1 - 1.07| = 0.07$, we can take $\delta = 0.07$ (or any smaller positive number).



12. For $M = 100$, we need $1.48 \leq x \leq \frac{\pi}{2} \approx 1.5708$, so since $\left|\frac{\pi}{2} - 1.48\right| \approx 0.09$ we choose $0 < \delta \leq 0.09$. For $M = 1000$, we need $1.54 \leq x \leq \frac{\pi}{2}$, so since $\left|\frac{\pi}{2} - 1.54\right| \approx 0.03$, we choose $\delta \leq 0.03$.



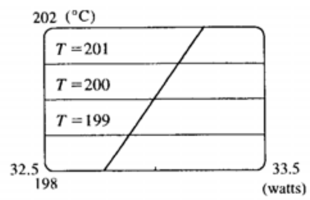


13. (a) $A = \pi r^2$ and $A = 1000\text{ cm}^2 \Rightarrow \pi r^2 = 1000 \Rightarrow r^2 = \frac{1000}{\pi} \Rightarrow r = \sqrt{\frac{1000}{\pi}}\ (r > 0) \approx 17.8412$ cm.

(b) $|A - 1000| \leq 5 \Rightarrow 1000 - 5 \leq \pi r^2 \leq 1000 + 5 \Rightarrow \sqrt{\frac{995}{\pi}} \leq r \leq \sqrt{\frac{1005}{\pi}} \Rightarrow$ $17.7966 \leq r \leq 17.8858$. $\sqrt{\frac{1000}{\pi}} - \sqrt{\frac{995}{\pi}} \approx 0.04466$ and $\sqrt{\frac{1005}{\pi}} - \sqrt{\frac{1000}{\pi}} \approx 0.04455$. So if the machinist gets the radius within 0.0445 cm of 17.8412, the area will be within 5 cm^2 of 1000.

(c) x is the radius, $f(x)$ is the area, a is the target radius given in part (a), L is the target area (1000), ε is the tolerance in the area (5), and δ is the tolerance in the radius given in part (b).

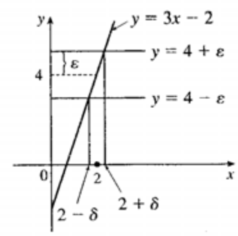
14. (a) $T = 0.1w^2 + 2.155w + 20$ and $T = 200 \Rightarrow$ $0.1w^2 + 2.155w + 20 = 200 \Rightarrow$ (by the quadratic formula or from the graph) $w \approx 33.0$ watts $(w > 0)$



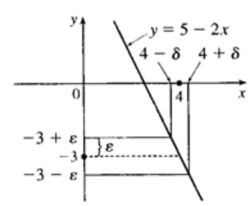
(b) From the graph, $199 \leq T \leq 201 \Rightarrow$ $32.89 < w < 33.11$.

(c) x is the input power, $f(x)$ is the temperature, a is the target input power given in part (a), L is the target temperature (200), ε is the tolerance in the temperature (1), and δ is the tolerance in the power input in watts indicated in part (b) (0.11 watts).

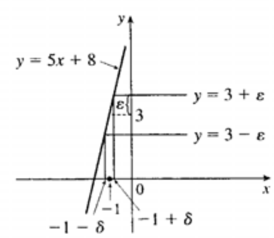
15. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $|x - 2| < \delta$, then $|(3x - 2) - 4| < \varepsilon \Leftrightarrow |3x - 6| < \varepsilon \Leftrightarrow 3|x - 2| < \varepsilon \Leftrightarrow$ $|x - 2| < \varepsilon/3$. So if we choose $\delta = \varepsilon/3$, then $|x - 2| < \delta \Rightarrow$ $|(3x - 2) - 4| < \varepsilon$. Thus, $\lim\limits_{x\to 2}(3x - 2) = 4$ by the definition of a limit.



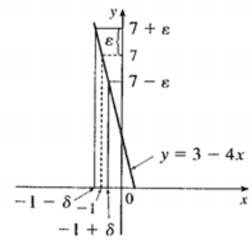
16. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $|x - 4| < \delta$, then $|(5 - 2x) - (-3)| < \varepsilon \Leftrightarrow |-2x + 8| < \varepsilon \Leftrightarrow 2|x - 4| < \varepsilon$ $\Leftrightarrow |x - 4| < \varepsilon/2$. So choose $\delta = \varepsilon/2$. Then $|x - 4| < \delta \Rightarrow$ $|(5 - 2x) - (-3)| < \varepsilon$. Thus, $\lim\limits_{x\to 4}(5 - 2x) = -3$ by the definition of a limit.



17. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $|x - (-1)| < \delta$, then $|(5x + 8) - 3| < \varepsilon \Leftrightarrow |5x + 5| < \varepsilon \Leftrightarrow 5|x + 1| < \varepsilon \Leftrightarrow$ $|x - (-1)| < \varepsilon/5$. So if we choose $\delta = \varepsilon/5$, then $|x - (-1)| < \delta$ $\Rightarrow |(5x + 8) - 3| < \varepsilon$. Thus, $\lim\limits_{x\to -1}(5x + 8) = 3$ by the definition of a limit.



18. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $|x-(-1)| < \delta$, then $|(3-4x)-7| < \varepsilon \Leftrightarrow |-4x-4| < \varepsilon \Leftrightarrow 4|x+1| < \varepsilon \Leftrightarrow$ $|x-(-1)| < \varepsilon/4$. So choose $\delta = \varepsilon/4$. Then $|x-(-1)| < \delta \Rightarrow$ $|(3-4x)-7| < \varepsilon$. Thus, $\lim\limits_{x\to -1}(3-4x) = 7$ by the definition of a limit.



19. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-3| < \delta$, then $\left|\dfrac{x}{5}-\dfrac{3}{5}\right| < \varepsilon \Leftrightarrow \frac{1}{5}|x-3| < \varepsilon \Leftrightarrow$ $|x-3| < 5\varepsilon$. So choose $\delta = 5\varepsilon$. Then $0 < |x-3| < \delta \Rightarrow |x-3| < 5\varepsilon \Rightarrow \dfrac{|x-3|}{5} < \varepsilon \Rightarrow$ $\left|\dfrac{x}{5}-\dfrac{3}{5}\right| < \varepsilon$. By the definition of a limit, $\lim\limits_{x\to 3}\dfrac{x}{5} = \dfrac{3}{5}$.

20. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-6| < \delta$, then $\left|\left(\frac{x}{4}+3\right)-\frac{9}{2}\right| < \varepsilon \Leftrightarrow \left|\frac{x}{4}-\frac{3}{2}\right| < \varepsilon \Leftrightarrow$ $\frac{1}{4}|x-6| < \varepsilon \Leftrightarrow |x-6| < 4\varepsilon$. So choose $\delta = 4\varepsilon$. Then $0 < |x-6| < \delta \Rightarrow |x-6| < 4\varepsilon \Rightarrow$ $\dfrac{|x-6|}{4} < \varepsilon \Rightarrow \left|\frac{x}{4}-\frac{6}{4}\right| < \varepsilon \Rightarrow \left|\left(\frac{x}{4}+3\right)-\frac{9}{2}\right| < \varepsilon$. By the definition of a limit, $\lim\limits_{x\to 6}\left(\frac{x}{4}+3\right) = \frac{9}{2}$.

21. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $|x-(-5)| < \delta$ then $\left|\left(4-\frac{3}{5}x\right)-7\right| < \varepsilon \Leftrightarrow \frac{3}{5}|x+5| < \varepsilon \Leftrightarrow$ $|x-(-5)| < \frac{5}{3}\varepsilon$. So take $\delta = \frac{5}{3}\varepsilon$. Then $|x-(-5)| < \delta \Rightarrow \left|\left(4-\frac{3}{5}x\right)-7\right| < \varepsilon$. Thus, $\lim\limits_{x\to -5}\left(4-\frac{3}{5}x\right) = 7$ by the definition of a limit.

22. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-3| < \delta$, then $\left|\dfrac{x^2+x-12}{x-3}-7\right| < \varepsilon$. Notice that if $0 < |x-3|$, then $x \neq 3$, so $\dfrac{x^2+x-12}{x-3} = \dfrac{(x+4)(x-3)}{x-3} = x+4$. Thus, when $0 < |x-3|$, we have $\left|\dfrac{x^2+x-12}{x-3}-7\right| < \varepsilon \Leftrightarrow |(x+4)-7| < \varepsilon \Leftrightarrow |x-3| < \varepsilon$. We take $\delta = \varepsilon$ and see that $0 < |x-3| < \delta$ $\Rightarrow \left|\dfrac{x^2+x-12}{x-3}-7\right| < \varepsilon$. By the definition of a limit, $\lim\limits_{x\to 3}\dfrac{x^2+x-12}{x-3} = 7$.

23. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $|x-a| < \delta$ then $|x-a| < \varepsilon$. So $\delta = \varepsilon$ will work.

24. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $|x-a| < \delta$ then $|c-c| < \varepsilon$. But $|c-c| = 0$, so this will be true no matter what δ we pick.

25. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $|x| < \delta$ then $\left|x^2-0\right| < \varepsilon \Leftrightarrow x^2 < \varepsilon \Leftrightarrow |x| < \sqrt{\varepsilon}$. Take $\delta = \sqrt{\varepsilon}$. Then $|x-0| < \delta \Rightarrow \left|x^2-0\right| < \varepsilon$. Thus, $\lim\limits_{x\to 0}x^2 = 0$ by the definition of a limit.

26. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $|x| < \delta$ then $\left|x^3-0\right| < \varepsilon \Leftrightarrow |x|^3 < \varepsilon \Leftrightarrow |x| < \sqrt[3]{\varepsilon}$. Take $\delta = \sqrt[3]{\varepsilon}$. Then $|x-0| < \delta \Rightarrow \left|x^3-0\right| < \delta^3 = \varepsilon$. Thus, $\lim\limits_{x\to 0}x^3 = 0$ by the definition of a limit.

27. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $|x-0| < \delta$ then $||x|-0| < \varepsilon$. But $||x|| = |x|$. So this is true if we pick $\delta = \varepsilon$.

28. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $9 - \delta \le x < 9$, then $\left|\sqrt[4]{9-x} - 0\right| < \varepsilon \quad\Leftrightarrow\quad \sqrt[4]{9-x} < \varepsilon \quad\Leftrightarrow$ $9 - x < \varepsilon^4 \quad\Leftrightarrow\quad 9 - \varepsilon^4 < x < 9$. So take $\delta = \varepsilon^4$. Then $9 - \delta \le x < 9 \quad\Rightarrow\quad \left|\sqrt[4]{9-x} - 0\right| < \varepsilon$. Thus, $\lim\limits_{x\to 9^-} \sqrt[4]{9-x} = 0$ by the definition of a limit.

29. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $|x-2| < \delta$, then $\left|(x^2 - 4x + 5) - 1\right| < \varepsilon \quad\Leftrightarrow\quad \left|x^2 - 4x + 4\right| < \varepsilon \quad\Leftrightarrow$ $\left|(x-2)^2\right| < \varepsilon$. So take $\delta = \sqrt{\varepsilon}$. Then $|x-2| < \delta \quad\Leftrightarrow\quad |x-2| < \sqrt{\varepsilon} \quad\Leftrightarrow\quad \left|(x-2)^2\right| < \varepsilon$. So $\lim\limits_{x\to 2}(x^2 - 4x + 5) = 1$ by the definition of a limit.

30. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $|x-3| < \delta$, then $\left|(x^2 + x - 4) - 8\right| < \varepsilon \quad\Leftrightarrow\quad \left|x^2 + x - 12\right| < \varepsilon \quad\Leftrightarrow$ $|(x-3)(x+4)| < \varepsilon$. Notice that if $|x-3| < 1$, then $-1 < x - 3 < 1 \quad\Rightarrow\quad 6 < x + 4 < 8 \quad\Rightarrow\quad |x+4| < 8$. So take $\delta = \min\{1, \varepsilon/8\}$. Then $|x-3| < \delta \quad\Leftrightarrow\quad |(x-3)(x+4)| \le |8(x-3)| = 8 \cdot |x-3| < 8\delta \le \varepsilon$. So $\lim\limits_{x\to 3}(x^2 + x - 4) = 8$ by the definition of a limit.

31. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $|x - (-2)| < \delta$ then $\left|(x^2 - 1) - 3\right| < \varepsilon$ or upon simplifying we need $\left|x^2 - 4\right| < \varepsilon$ whenever $|x+2| < \delta$. Notice that if $|x+2| < 1$, then $-1 < x + 2 < 1 \quad\Rightarrow\quad -5 < x - 2 < -3$ $\Rightarrow\quad |x-2| < 5$. So take $\delta = \min\{\varepsilon/5, 1\}$. Then $|x-2| < 5$ and $|x+2| < \varepsilon/5$, so $\left|(x^2 - 1) - 3\right| = |(x+2)(x-2)| = |x+2|\,|x-2| < (\varepsilon/5)(5) = \varepsilon$. Therefore, by the definition of a limit, $\lim\limits_{x\to -2}(x^2 - 1) = 3$.

32. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $|x-2| < \delta$, then $\left|x^3 - 8\right| < \varepsilon$. Now $\left|x^3 - 8\right| = \left|(x-2)(x^2 + 2x + 4)\right|$. If $|x-2| < 1$, that is, $1 < x < 3$, then $x^2 + 2x + 4 < 3^2 + 2(3) + 4 = 19$ and so $\left|x^3 - 8\right| = |x-2|\,(x^2 + 2x + 4) < 19\,|x-2|$. So if we take $\delta = \min\left\{1, \frac{\varepsilon}{19}\right\}$, then $|x-2| < \delta \quad\Rightarrow$ $\left|x^3 - 8\right| = |x-2|\,(x^2 + 2x + 4) < \frac{\varepsilon}{19} \cdot 19 = \varepsilon$. So by the definition of a limit, $\lim\limits_{x\to 2} x^3 = 8$.

33. Given $\varepsilon > 0$, we let $\delta = \min\left\{2, \frac{\varepsilon}{8}\right\}$. If $0 < |x-3| < \delta$ then $|x-3| < 2 \quad\Rightarrow\quad 1 < x < 5 \quad\Rightarrow\quad |x+3| < 8$. Also $|x-3| < \frac{\varepsilon}{8}$, so $\left|x^2 - 9\right| = |x+3|\,|x-3| < 8 \cdot \frac{\varepsilon}{8} = \varepsilon$. Thus, $\lim\limits_{x\to 3} x^2 = 9$.

34. *1. Guessing a value for δ* Let $\varepsilon > 0$ be given. We have to find a number $\delta > 0$ such that $\left|\dfrac{1}{x} - \dfrac{1}{2}\right| < \varepsilon$ whenever $0 < |x-2| < \delta$. But $\left|\dfrac{1}{x} - \dfrac{1}{2}\right| = \left|\dfrac{2-x}{2x}\right| = \dfrac{|x-2|}{|2x|} < \varepsilon$. We find a positive constant C such that $\dfrac{1}{|2x|} < C \quad\Rightarrow$ $\dfrac{|x-2|}{|2x|} < C\,|x-2|$ and we can make $C\,|x-2| < \varepsilon$ by taking $|x-2| < \dfrac{\varepsilon}{C} = \delta$. We restrict x to lie in the interval $|x-2| < 1 \quad\Rightarrow\quad 1 < x < 3$ so $1 > \dfrac{1}{x} > \dfrac{1}{3} \quad\Rightarrow\quad \dfrac{1}{6} < \dfrac{1}{2x} < \dfrac{1}{2} \quad\Rightarrow\quad \dfrac{1}{|2x|} < \dfrac{1}{2}$. So $C = \dfrac{1}{2}$ is suitable. Thus, we should choose $\delta = \min\{1, 2\varepsilon\}$.

2. Showing that δ works Given $\varepsilon > 0$ we let $\delta = \min\{1, 2\varepsilon\}$. If $0 < |x-2| < \delta$, then $|x-2| < 1 \quad\Rightarrow$ $1 < x < 3 \quad\Rightarrow\quad \dfrac{1}{|2x|} < \dfrac{1}{2}$ (as in part 1). Also $|x-2| < 2\varepsilon$, so $\left|\dfrac{1}{x} - \dfrac{1}{2}\right| = \dfrac{|x-2|}{|2x|} < \dfrac{1}{2} \cdot 2\varepsilon = \varepsilon$. This shows that $\lim\limits_{x\to 2}(1/x) = \frac{1}{2}$.

35. *1. Guessing a value for δ* Given $\varepsilon > 0$, we must find $\delta > 0$ such that $\left|\sqrt{x} - \sqrt{a}\right| < \varepsilon$ whenever $0 < |x-a| < \delta$. But $\left|\sqrt{x} - \sqrt{a}\right| = \dfrac{|x-a|}{\sqrt{x}+\sqrt{a}} < \varepsilon$ (from the hint). Now if we can find a positive constant C such that $\sqrt{x}+\sqrt{a} > C$ then $\dfrac{|x-a|}{\sqrt{x}+\sqrt{a}} < \dfrac{|x-a|}{C} < \varepsilon$, and we take $|x-a| < C\varepsilon$. We can find this number by restricting x to lie in some interval centered at a. If $|x-a| < \frac{1}{2}a$, then $\frac{1}{2}a < x < \frac{3}{2}a \;\Rightarrow$ $\sqrt{x}+\sqrt{a} > \sqrt{\frac{1}{2}a}+\sqrt{a}$, and so $C = \sqrt{\frac{1}{2}a}+\sqrt{a}$ is a suitable choice for the constant. So $|x-a| < \left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon$. This suggests that we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon\right\}$.

2. Showing that δ works Given $\varepsilon > 0$, we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon\right\}$. If $0 < |x-a| < \delta$, then $|x-a| < \frac{1}{2}a \;\Rightarrow\; \sqrt{x}+\sqrt{a} > \sqrt{\frac{1}{2}a}+\sqrt{a}$ (as in part 1). Also $|x-a| < \left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon$, so $\left|\sqrt{x}+\sqrt{a}\right| = \dfrac{|x-a|}{\sqrt{x}+\sqrt{a}} < \dfrac{\left(\sqrt{a/2}+\sqrt{a}\right)\varepsilon}{\left(\sqrt{a/2}+\sqrt{a}\right)} = \varepsilon$. Therefore, $\lim\limits_{x\to a}\sqrt{x} = \sqrt{a}$ by the definition of a limit.

36. Suppose that $\lim\limits_{t\to 0} H(t) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |t| < \delta \;\Rightarrow\; |H(t) - L| < \frac{1}{2} \;\Leftrightarrow$ $L - \frac{1}{2} < H(t) < L + \frac{1}{2}$. For $0 < t < \delta$, $H(t) = 1$, so $1 < L + \frac{1}{2} \;\Rightarrow\; L > \frac{1}{2}$. For $-\delta < t < 0$, $H(t) = 0$, so $L - \frac{1}{2} < 0 \;\Rightarrow\; L < \frac{1}{2}$. This contradicts $L > \frac{1}{2}$. Therefore, $\lim\limits_{t\to 0} H(t)$ does not exist.

37. Suppose that $\lim\limits_{x\to 0} f(x) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |x| < \delta \;\Rightarrow\; |f(x) - L| < \frac{1}{2}$. Take any rational number r with $0 < |r| < \delta$. Then $f(r) = 0$, so $|0 - L| < \frac{1}{2}$, so $L \le |L| < \frac{1}{2}$. Now take any irrational number s with $0 < |s| < \delta$. Then $f(s) = 1$, so $|1 - L| < \frac{1}{2}$. Hence, $1 - L < \frac{1}{2}$, so $L > \frac{1}{2}$. This contradicts $L < \frac{1}{2}$, so $\lim\limits_{x\to 0} f(x)$ does not exist.

38. First suppose that $\lim\limits_{x\to a} f(x) = L$. Then, given $\varepsilon > 0$ there exists $\delta > 0$ so that $0 < |x-a| < \delta \;\Rightarrow$ $|f(x) - L| < \varepsilon$. Then $a - \delta < x < a \;\Rightarrow\; 0 < |x-a| < \delta$ so $|f(x) - L| < \varepsilon$. Thus, $\lim\limits_{x\to a^-} f(x) = L$. Also $a < x < a + \delta \;\Rightarrow\; 0 < |x-a| < \delta$ so $|f(x) - L| < \varepsilon$. Hence, $\lim\limits_{x\to a^+} f(x) = L$.

Now suppose $\lim\limits_{x\to a^-} f(x) = L = \lim\limits_{x\to a^+} f(x)$. Let $\varepsilon > 0$ be given. Since $\lim\limits_{x\to a^-} f(x) = L$, there exists $\delta_1 > 0$ so that $a - \delta_1 < x < a \;\Rightarrow\; |f(x) - L| < \varepsilon$. Since $\lim\limits_{x\to a^+} f(x) = L$, there exists $\delta_2 > 0$ so that $a < x < a + \delta_2$ $\Rightarrow\; |f(x) - L| < \varepsilon$. Let δ be the smaller of δ_1 and δ_2. Then $0 < |x-a| < \delta \;\Rightarrow\; a - \delta_1 < x < a$ or $a < x < a + \delta_2$ so $|f(x) - L| < \varepsilon$. Hence, $\lim\limits_{x\to a} f(x) = L$. So we have proved that $\lim\limits_{x\to a} f(x) = L \;\Leftrightarrow$ $\lim\limits_{x\to a^-} f(x) = L = \lim\limits_{x\to a^+} f(x)$.

39. $\dfrac{1}{(x+3)^4} > 10{,}000 \;\Leftrightarrow\; (x+3)^4 < \dfrac{1}{10{,}000} \;\Leftrightarrow\; |x-(-3)| = |x+3| < \dfrac{1}{10}$

40. Given $M > 0$, we need $\delta > 0$ such that $|x+3| < \delta \;\Rightarrow\; 1/(x+3)^4 > M$. Now $\dfrac{1}{(x+3)^4} > M \;\Leftrightarrow$ $(x+3)^4 < \dfrac{1}{M} \;\Leftrightarrow\; |x+3| < \dfrac{1}{\sqrt[4]{M}}$. So take $\delta = \dfrac{1}{\sqrt[4]{M}}$. Then $0 < |x+3| < \delta = \dfrac{1}{\sqrt[4]{M}} \;\Rightarrow\; \dfrac{1}{(x+3)^4} > M$, so $\lim\limits_{x\to -3} \dfrac{1}{(x+3)^4} = \infty$.

41. Let $N < 0$ be given. Then, for $x < -1$, we have $\dfrac{5}{(x+1)^3} < N \iff \dfrac{5}{N} < (x+1)^3 \iff \sqrt[3]{\dfrac{5}{N}} < x+1$. Let $\delta = -\sqrt[3]{\dfrac{5}{N}}$. Then $-1-\delta < x < -1 \Rightarrow \sqrt[3]{\dfrac{5}{N}} < x+1 < 0 \Rightarrow \dfrac{5}{(x+1)^3} < N$, so $\displaystyle\lim_{x\to -1^-} \frac{5}{(x+1)^3} = -\infty$.

42. (a) Let M be given. Since $\lim_{x\to a} f(x) = \infty$, there exists $\delta_1 > 0$ such that $0 < |x-a| < \delta_1 \Rightarrow f(x) > M + 1 - c$. Since $\lim_{x\to a} g(x) = c$, there exists $\delta_2 > 0$ such that $0 < |x-a| < \delta_2 \Rightarrow |g(x) - c| < 1 \Rightarrow g(x) > c - 1$. Let δ be the smaller of δ_1 and δ_2. Then $0 < |x-a| < \delta \Rightarrow f(x) + g(x) > (M+1-c) + (c-1) = M$. Thus, $\lim_{x\to a} [f(x) + g(x)] = \infty$.

(b) Let $M > 0$ be given. Since $\lim_{x\to a} g(x) = c > 0$, there exists $\delta_1 > 0$ such that $0 < |x-a| < \delta_1 \Rightarrow |g(x) - c| < c/2 \Rightarrow g(x) > c/2$. Since $\lim_{x\to a} f(x) = \infty$, there exists $\delta_2 > 0$ such that $0 < |x-a| < \delta_2 \Rightarrow f(x) > 2M/c$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x-a| < \delta \Rightarrow f(x)\, g(x) > \dfrac{2M}{c}\dfrac{c}{2} = M$, so $\lim_{x\to a} f(x)\, g(x) = \infty$.

(c) Let $N < 0$ be given. Since $\lim_{x\to a} g(x) = c < 0$, there exists $\delta_1 > 0$ such that $0 < |x-a| < \delta_1 \Rightarrow |g(x) - c| < -c/2 \Rightarrow g(x) < c/2$. Since $\lim_{x\to a} f(x) = \infty$, there exists $\delta_2 > 0$ such that $0 < |x-a| < \delta_2 \Rightarrow f(x) > 2N/c$. (Note that $c < 0$ and $N < 0 \Rightarrow 2N/c > 0$.) Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x-a| < \delta \Rightarrow f(x) > 2N/c \Rightarrow f(x)\, g(x) < \dfrac{2N}{c} \cdot \dfrac{c}{2} = N$, so $\lim_{x\to a} f(x)\, g(x) = -\infty$.

2.5 Continuity

1. From Equation 1, $\lim_{x\to 4} f(x) = f(4)$.

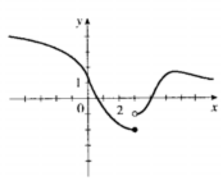
2. The graph of f doesn't have any holes, jumps, or vertical asymptotes.

3. (a) The following are the numbers at which f is discontinuous and the type of discontinuity at that number: -5 (jump), -3 (infinite), -1 (undefined), 3 (removable), 5 (infinite), 8 (jump), 10 (undefined).

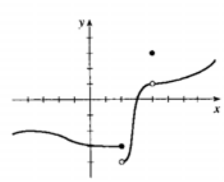
(b) f is continuous from the left at -5 and -3, and continuous from the right at 8. It is continuous from neither side at -1, 3, 5, and 10.

4. g is continuous on $[-6, -5]$, $(-5, -3)$, $(-3, -2]$, $(-2, 1)$, $(1, 3)$, $[3, 5]$, and $(5, 7]$.

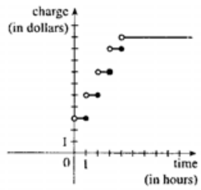
5.



6.



7. (a)



(b) There are discontinuities at $t = 1, 2, 3$, and 4. A person parking in the lot would want to keep in mind that the charge will jump at the beginning of each hour.

8. (a) Continuous; at the location in question, the temperature changes smoothly as time passes, without any instantaneous jumps from one temperature to another.

(b) Continuous; the temperature at a specific time changes smoothly as the distance due west from New York City increases, without any instantaneous jumps.

(c) Discontinuous; as the distance due west from New York City increases, the altitude above sea level may jump from one height to another without going through all of the intermediate values — at a cliff, for example.

(d) Discontinuous; as the distance traveled increases, the cost of the ride jumps in small increments.

(e) Discontinuous; when the lights are switched on (or off), the current suddenly changes between 0 and some nonzero value, without passing through all of the intermediate values. This is debatable, though, depending on your definition of current.

9. Since f and g are continuous functions,

$$\begin{aligned}\lim_{x\to 3}[2f(x)-g(x)] &= 2\lim_{x\to 3} f(x) - \lim_{x\to 3} g(x) \quad \text{(by Limit Laws 2 \& 3)}\\ &= 2f(3) - g(3) \quad \text{(by continuity of } f \text{ and } g \text{ at } x=3)\\ &= 2\cdot 5 - g(3) = 10 - g(3)\end{aligned}$$

Since it is given that $\lim_{x\to 3}[2f(x)-g(x)] = 4$, we have $10 - g(3) = 4$, or $g(3) = 6$.

10. $\lim_{x\to 4} f(x) = \lim_{x\to 4}\left(x^2+\sqrt{7-x}\right) = \lim_{x\to 4} x^2 + \sqrt{\lim_{x\to 4} 7 - \lim_{x\to 4} x} = 4^2 + \sqrt{7-4} = 16+\sqrt{3} = f(4)$. By the definition of continuity, f is continuous at $a = 4$.

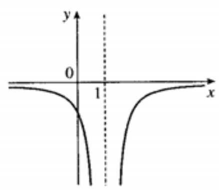
11. $\lim_{x\to -1} f(x) = \lim_{x\to -1}\left(x+2x^3\right)^4 = \left(\lim_{x\to -1} x + 2\lim_{x\to -1} x^3\right)^4 = \left[-1+2(-1)^3\right]^4 = (-3)^4 = 81 = f(-1)$. By the definition of continuity, f is continuous at $a = -1$.

12. $\lim_{x\to 4} g(x) = \lim_{x\to 4}\dfrac{x+1}{2x^2-1} = \dfrac{\lim_{x\to 4} x + \lim_{x\to 4} 1}{2\lim_{x\to 4} x^2 - \lim_{x\to 4} 1} = \dfrac{4+1}{2(4)^2-1} = \dfrac{5}{31} = g(4)$. So g is continuous at 4.

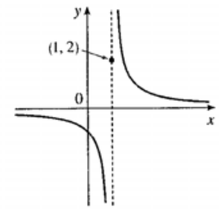
13. For $-4 < a < 4$ we have $\lim_{x\to a} f(x) = \lim_{x\to a} x\sqrt{16-x^2} = \lim_{x\to a} x\sqrt{\lim_{x\to a} 16 - \lim_{x\to a} x^2} = a\sqrt{16-a^2} = f(a)$, so f is continuous on $(-4, 4)$. Similarly, we get $\lim_{x\to 4^-} f(x) = 0 = f(4)$ and $\lim_{x\to -4^+} f(x) = 0 = f(-4)$, so f is continuous from the left at 4 and from the right at -4. Thus, f is continuous on $[-4, 4]$.

14. For $a < 3$, $\lim_{x\to a} F(x) = \lim_{x\to a}\dfrac{x+1}{x-3} = \dfrac{\lim_{x\to a} x + \lim_{x\to a} 1}{\lim_{x\to a} x - \lim_{x\to a} 3} = \dfrac{a+1}{a-3} = F(a)$, so F is continuous on $(-\infty, 3)$.

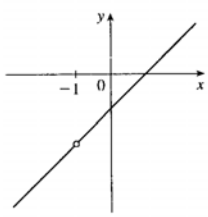
15. $f(x) = -\dfrac{1}{(x-1)^2}$ is discontinuous at 1 since $f(1)$ is not defined.



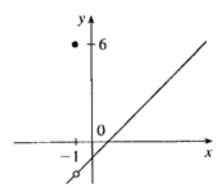
16. $f(x) = \begin{cases} 1/(x-1) & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$ is discontinuous at 1 because $\lim\limits_{x \to 1} f(x)$ does not exist.



17. $f(x) = \dfrac{x^2 - 1}{x + 1}$ is discontinuous at -1 because $f(-1)$ is not defined.



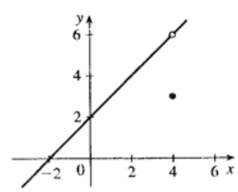
18. Since $f(x) = \dfrac{x^2 - 1}{x + 1}$ for $x \neq -1$, we have $\lim\limits_{x \to -1} f(x) = \lim\limits_{x \to -1} \dfrac{x^2 - 1}{x + 1} = \lim\limits_{x \to -1} (x - 1) = -2$. But $f(-1) = 6$, so $\lim\limits_{x \to -1} f(x) \neq f(-1)$. Therefore, f is discontinuous at -1.



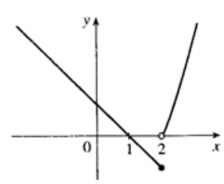
19. Since $f(x) = \dfrac{x^2 - 2x - 8}{x - 4}$ if $x \neq 4$,

$$\lim_{x \to 4} f(x) = \lim_{x \to 4} \frac{x^2 - 2x - 8}{x - 4} = \lim_{x \to 4} \frac{(x-4)(x+2)}{x - 4}$$
$$= \lim_{x \to 4} (x + 2) = 4 + 2 = 6.$$

But $f(4) = 3$, so $\lim\limits_{x \to 4} f(x) \neq f(4)$. Therefore, f is discontinuous at 4.



20. $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (1 - x) = 1 - 2 = -1$ and $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} \left(x^2 - 2x\right) = (2)^2 - 2(2) = 0$. Since $\lim\limits_{x \to 2^-} f(x) \neq \lim\limits_{x \to 2^+} f(x)$, $\lim\limits_{x \to 2} f(x)$ does not exist and therefore f is discontinuous at 2 [by Note 2 after Definition 1].



21. $F(x) = \dfrac{x}{x^2 + 5x + 6}$ is a rational function. So by Theorem 5 (or Theorem 7), F is continuous at every number in its domain, $\left\{x \mid x^2 + 5x + 6 \neq 0\right\} = \{x \mid x \neq -2, -3\}$ or $(-\infty, -3) \cup (-3, -2) \cup (-2, \infty)$.

22. $G(t) = 25 - t^2$ is a polynomial, so it is continuous (Theorem 5). $F(x) = \sqrt{x}$ is continuous by Theorem 7. So, by Theorem 9, $F(G(t)) = \sqrt{25 - t^2}$ is continuous on its domain, which is $\{t \mid 25 - t^2 \geq 0\} = \{t \mid |t| \leq 5\} = [-5, 5]$. Also, $2t$ is continuous on $\mathbb{R}$, so by Theorem 4 #1, $f(t) = 2t + \sqrt{25 - t^2}$ is continuous on its domain, which is $[-5, 5]$.

23. $g(x) = x - 1$ and $G(x) = x^2 - 2$ are both polynomials, so by Theorem 5 they are continuous. Also $f(x) = \sqrt[5]{x}$ is continuous by Theorem 6, so $f(g(x)) = \sqrt[5]{x - 1}$ is continuous on $\mathbb{R}$ by Theorem 8. Thus, the product $h(x) = \sqrt[5]{x - 1}\,(x^2 - 2)$ is continuous on $\mathbb{R}$ by Theorem 4 #4.

24. By Theorem 7, the trigonometric function $\sin x$ and the polynomial function $x + 1$ are continuous on $\mathbb{R}$. By Theorem 4, $h(x) = \dfrac{\sin x}{x + 1}$ is continuous on its domain, $\{x \mid x \neq -1\}$.

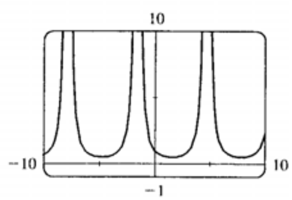
25. By Theorem 5, the polynomial $1 - x^2$ is continuous on $(-\infty, \infty)$. By Theorem 7, cos is continuous on its domain, $\mathbb{R}$. By Theorem 9, $\cos(1 - x^2)$ is continuous on its domain, which is $\mathbb{R}$.

26. By Theorem 5, the polynomial $2x$ is continuous on $(-\infty, \infty)$. By Theorem 7, tan is continuous at every number in its domain; that is, $\{x \mid x \neq \frac{\pi}{2} + \pi n\}$. By Theorem 9, $\tan 2x$ is continuous on its domain, which is $\{x \mid 2x \neq \frac{\pi}{2} + \pi n\} = \{x \mid x \neq \frac{\pi}{4} + \frac{\pi}{2} n\}$ (the odd multiples of $\frac{\pi}{4}$).

27. By Theorem 7, the root function $\sqrt{x}$ and the trigonometric function $\sin x$ are continuous on their domains, $[0, \infty)$ and $(-\infty, \infty)$, respectively. Thus, the product $F(x) = \sqrt{x}\sin x$ is continuous on the intersection of those domains, $[0, \infty)$, by Theorem 4 #4.

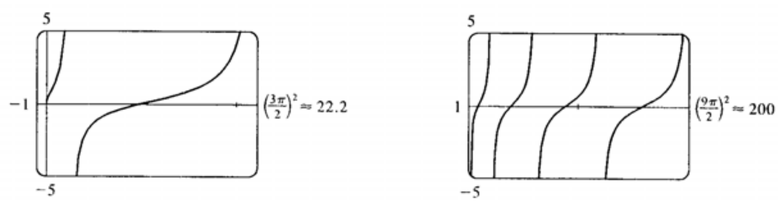
28. The sine and cosine functions are continuous everywhere by Theorem 7, so $F(x) = \sin(\cos(\sin x))$, which is the composite of sine, cosine, and (once again) sine, is continuous everywhere by Theorem 9.

29.



$y = \dfrac{1}{1 + \sin x}$ is undefined and hence discontinuous when $1 + \sin x = 0 \iff \sin x = -1 \iff x = -\frac{\pi}{2} + 2\pi n$, n an integer.

30.



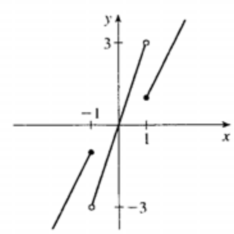
The function $y = f(x) = \tan\sqrt{x}$ is continuous throughout its domain because it is the composite of a trigonometric function and a root function. The square root function has domain $[0, \infty)$ and the tangent function has domain $\{x \mid x \neq \frac{\pi}{2} + \pi n\}$. So f is discontinuous when $x < 0$ and when $\sqrt{x} = \frac{\pi}{2} + \pi n \Rightarrow x = \left(\frac{\pi}{2} + \pi n\right)^2$, where n is a nonnegative integer. Note that as x increases, the distance between discontinuities increases.

31. Because we are dealing with root functions, $5+\sqrt{x}$ is continuous on $[0,\infty)$, $\sqrt{x+5}$ is continuous on $[-5,\infty)$, so the quotient $f(x)=\dfrac{5+\sqrt{x}}{\sqrt{5+x}}$ is continuous on $[0,\infty)$. Since f is continuous at $x=4$, $\lim\limits_{x\to 4} f(x)=f(4)=\frac{7}{3}$.

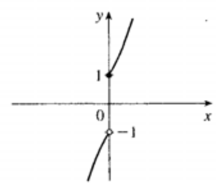
32. Because x is continuous on $\mathbb{R}$, $\sin x$ is continuous on $\mathbb{R}$, and $x+\sin x$ is continuous on $\mathbb{R}$, the composite function $f(x)=\sin(x+\sin x)$ is continuous on $\mathbb{R}$, so $\lim\limits_{x\to\pi} f(x)=f(\pi)=\sin(\pi+\sin\pi)=\sin\pi=0$.

33. f is continuous on $(-\infty,3)$ and $(3,\infty)$ since on each of these intervals it is a polynomial. Also $\lim\limits_{x\to 3^+} f(x)=\lim\limits_{x\to 3^+}(5-x)=2$ and $\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^-}(x-1)=2$, so $\lim\limits_{x\to 3} f(x)=2$. Since $f(3)=5-3=2$, f is also continuous at 3. Thus, f is continuous on $(-\infty,\infty)$.

34. f is continuous on $(-\infty,-1)$, $(-1,1)$ and $(1,\infty)$ since on each of these intervals it is a polynomial. Now $\lim\limits_{x\to -1^-} f(x)=\lim\limits_{x\to -1^-}(2x+1)=-1$ and $\lim\limits_{x\to -1^+} f(x)=\lim\limits_{x\to -1^+} 3x=-3$, so f is discontinuous at -1. Since $f(-1)=-1$, f is continuous from the left at -1. Also $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-} 3x=3$ and $\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(2x-1)=1$, so f is discontinuous at 1. Since $f(1)=1$, f is continuous from the right at 1.



35. f is continuous on $(-\infty,0)$ and $(0,\infty)$ since on each of these intervals it is a polynomial. Now $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}(x-1)^3=-1$ and $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}(x+1)^3=1$. Thus, $\lim\limits_{x\to 0} f(x)$ does not exist, so f is discontinuous at 0. Since $f(0)=1$, f is continuous from the right at 0.



36. By Theorem 5, each piece of F is continuous on its domain. We need to check for continuity at $r=R$. $\lim\limits_{r\to R^-} F(r)=\lim\limits_{r\to R^-}\dfrac{GMr}{R^3}=\dfrac{GM}{R^2}$ and $\lim\limits_{r\to R^+} F(r)=\lim\limits_{r\to R^+}\dfrac{GM}{r^2}=\dfrac{GM}{R^2}$, so $\lim\limits_{r\to R} F(r)=\dfrac{GM}{R^2}$. Since $F(R)=\dfrac{GM}{R^2}$, F is continuous at R. Therefore, F is a continuous function of r.

37. f is continuous on $(-\infty,3)$ and $(3,\infty)$. Now $\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^-}(cx+1)=3c+1$ and $\lim\limits_{x\to 3^+} f(x)=\lim\limits_{x\to 3^+}(cx^2-1)=9c-1$. So f is continuous $\Leftrightarrow\ 3c+1=9c-1\ \Leftrightarrow\ 6c=2\ \Leftrightarrow\ c=\frac{1}{3}$. Thus, for f to be continuous on $(-\infty,\infty)$, $c=\frac{1}{3}$.

38. The functions x^2-c^2 and $cx+20$, considered on the intervals $(-\infty,4)$ and $[4,\infty)$ respectively, are continuous for any value of c. So the only possible discontinuity is at $x=4$. For the function to be continuous at $x=4$, the left-hand and right-hand limits must be the same. Now $\lim\limits_{x\to 4^-} g(x)=\lim\limits_{x\to 4^-}\left(x^2-c^2\right)=16-c^2$ and $\lim\limits_{x\to 4^-}\left(x^2-c^2\right)=g(4)=4c+20$. Thus, $4^2-c^2=4c+20\ \Leftrightarrow\ c^2+4c+4=0\ \Leftrightarrow\ c=-2$.

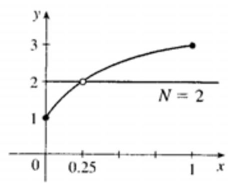
39. (a) $f(x)=\dfrac{x^2-2x-8}{x+2}=\dfrac{(x-4)(x+2)}{x+2}$ has a removable discontinuity at -2 because $g(x)=x-4$ is continuous on $\mathbb{R}$ and $f(x)=g(x)$ for $x\neq -2$. [The discontinuity is removed by defining $f(-2)=-6$.]

(b) $f(x) = \dfrac{x-7}{|x-7|} \Rightarrow \lim\limits_{x\to 7^-} f(x) = -1$ and $\lim\limits_{x\to 7^+} f(x) = 1$. Thus, $\lim\limits_{x\to 7} f(x)$ does not exist, so the discontinuity is not removable. (It is a jump discontinuity.)

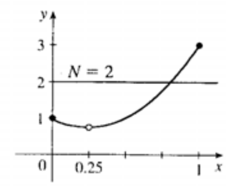
(c) $f(x) = \dfrac{x^3+64}{x+4} = \dfrac{(x+4)(x^2-4x+16)}{x+4}$ has a removable discontinuity at -4 because $g(x) = x^2 - 4x + 16$ is continuous on $\mathbb{R}$ and $f(x) = g(x)$ for $x \neq -4$. [The discontinuity is removed by defining $f(-4) = 48$.]

(d) $f(x) = \dfrac{3-\sqrt{x}}{9-x} = \dfrac{3-\sqrt{x}}{(3-\sqrt{x})(3+\sqrt{x})}$ has a removable discontinuity at 9 because $g(x) = 1/(3+\sqrt{x})$ is continuous on $\mathbb{R}$ and $f(x) = g(x)$ for $x \neq 9$. [The discontinuity is removed by defining $f(9) = \frac{1}{6}$.]

40.



f does not satisfy the conclusion of the Intermediate Value Theorem.



f does satisfy the conclusion of the Intermediate Value Theorem.

41. $f(x) = x^3 - x^2 + x$ is continuous on the interval $[2, 3]$, $f(2) = 6$, and $f(3) = 21$. Since $6 < 10 < 21$, there is a number c in $(2, 3)$ such that $f(c) = 10$ by the Intermediate Value Theorem.

42. $f(x) = x^2$ is continuous on the interval $[1, 2]$, $f(1) = 1$, and $f(2) = 4$. Since $1 < 2 < 4$, there is a number c in $(1, 2)$ such that $f(c) = c^2 = 2$ by the Intermediate Value Theorem.

43. $f(x) = x^3 - 3x + 1$ is continuous on the interval $[0, 1]$, $f(0) = 1$, and $f(1) = -1$. Since $-1 < 0 < 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^3 - 3x + 1 = 0$ in the interval $(0, 1)$.

44. $f(x) = x^2 - \sqrt{x+1}$ is continuous on the interval $[1, 2]$, $f(1) = 1 - \sqrt{2}$, and $f(2) = 4 - \sqrt{3}$. Since $1 - \sqrt{2} < 0 < 4 - \sqrt{3}$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^2 - \sqrt{x+1} = 0$, or $x^2 = \sqrt{x+1}$, in the interval $(1, 2)$.

45. $f(x) = \cos x - x$ is continuous on the interval $[0, 1]$, $f(0) = 1$, and $f(1) = \cos 1 - 1 \approx -0.46$. Since $-0.46 < 0 < 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos x - x = 0$, or $\cos x = x$, in the interval $(0, 1)$.

46. $f(x) = \tan x - 2x$ is continuous on the interval $[0, 1.4]$, $f(1) = \tan 1 - 2 \approx -0.44$, and $f(1.4) = \tan 1.4 - 2.8 \approx 3.00$. Since $-0.44 < 0 < 3.00$, there is a number c in $(0, 1.4)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\tan x - 2x = 0$, or $\tan x = 2x$, in the interval $(0, 1.4)$.

47. (a) $f(x) = \sin x - 2 + x$ is continuous on $[0, 2]$, $f(0) = -2$, and $f(2) = \sin 2 \approx 0.91$. Since $-2 < 0 < 0.91$, there is a number c in $(0, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sin x - 2 + x = 0$, or $\sin x = 2 - x$, in the interval $(0, 2)$.

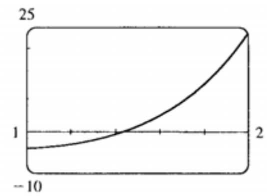
(b) $f(1.10) \approx -0.009$ and $f(1.11) \approx 0.006$, so there is a root between 1.10 and 1.11.

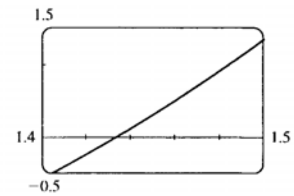
48. (a) $f(x) = x^5 - x^2 + 2x + 3$ is continuous on $[-1, 0]$, $f(-1) = -1$, and $f(0) = 3$. Since $-1 < 0 < 3$, there is a number c in $(-1, 0)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^5 - x^2 + 2x + 3 = 0$ in the interval $(-1, 0)$.

(b) $f(-0.88) \approx -0.062$ and $f(-0.87) \approx 0.0047$, so there is a root between -0.88 and -0.87.

49. (a) Let $f(x) = x^5 - x^2 - 4$. Then $f(1) = 1^5 - 1^2 - 4 = -4 < 0$ and $f(2) = 2^5 - 2^2 - 4 = 24 > 0$. So by the Intermediate Value Theorem, there is a number c in $(1, 2)$ such that $c^5 - c^2 - 4 = 0$.

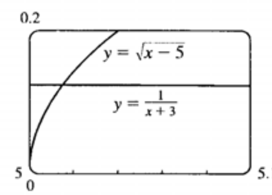
(b) We can see from the graphs that, correct to three decimal places, the root is $x \approx 1.434$.





50. (a) Let $f(x) = \sqrt{x-5} - \dfrac{1}{x+3}$. Then $f(5) = -\frac{1}{8} < 0$ and $f(6) = \frac{8}{9} > 0$, and f is continuous on $[5, \infty)$. So by the Intermediate Value Theorem, there is a number c in $(5, 6)$ such that $f(c) = 0$. This implies that $\dfrac{1}{c+3} = \sqrt{c-5}$.

(b) Using the intersect feature of the graphing device, we find that the root of the equation is $x = 5.016$, correct to three decimal places.



51. $(\Rightarrow)$ If f is continuous at a, then by Theorem 7 with $g(h) = a + h$, we have

$$\lim_{h \to 0} f(a+h) = f\left(\lim_{h \to 0}(a+h)\right) = f(a).$$

$(\Leftarrow)$ Let $\varepsilon > 0$. Since $\lim\limits_{h \to 0} f(a+h) = f(a)$, there exists $\delta > 0$ such that $|h| < \delta \Rightarrow$ $|f(a+h) - f(a)| < \varepsilon$. So if $|x - a| < \delta$, then $|f(x) - f(a)| = |f(a + (x - a)) - f(a)| < \varepsilon$. Thus, $\lim\limits_{x \to a} f(x) = f(a)$ and so f is continuous at a.

52.
$$\begin{aligned}
\lim_{h \to 0} \sin(a+h) &= \lim_{h \to 0} (\sin a \cos h + \cos a \sin h) \\
&= \lim_{h \to 0} (\sin a \cos h) + \lim_{h \to 0} (\cos a \sin h) \\
&= \left(\lim_{h \to 0} \sin a\right)\left(\lim_{h \to 0} \cos h\right) + \left(\lim_{h \to 0} \cos a\right)\left(\lim_{h \to 0} \sin h\right) \\
&= (\sin a)(1) + (\cos a)(0) = \sin a
\end{aligned}$$

53. As in the previous exercise, we must show that $\lim\limits_{h\to 0}\cos(a+h)=\cos a$ to prove that the cosine function is continuous.

$$\begin{aligned}\lim_{h\to 0}\cos(a+h) &= \lim_{h\to 0}(\cos a\cos h-\sin a\sin h)\\ &= \lim_{h\to 0}(\cos a\cos h)-\lim_{h\to 0}(\sin a\sin h)\\ &= \left(\lim_{h\to 0}\cos a\right)\left(\lim_{h\to 0}\cos h\right)-\left(\lim_{h\to 0}\sin a\right)\left(\lim_{h\to 0}\sin h\right)\\ &= (\cos a)(1)-(\sin a)(0)=\cos a\end{aligned}$$

54. (a) Since f is continuous at a, $\lim\limits_{x\to a}f(x)=f(a)$. Thus, using the Constant Multiple Law of Limits, we have $\lim\limits_{x\to a}(cf)(x)=\lim\limits_{x\to a}cf(x)=c\lim\limits_{x\to a}f(x)=cf(a)=(cf)(a)$. Therefore, cf is continuous at a.

(b) Since f and g are continuous at a, $\lim\limits_{x\to a}f(x)=f(a)$ and $\lim\limits_{x\to a}g(x)=g(a)$. Since $g(a)\neq 0$, we can use the Quotient Law of limits: $\lim\limits_{x\to a}\left(\dfrac{f}{g}\right)(x)=\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}g(x)}=\dfrac{f(a)}{g(a)}=\left(\dfrac{f}{g}\right)(a)$. Thus, $\dfrac{f}{g}$ is continuous at a.

55. $f(x)=\begin{cases}0 & \text{if } x \text{ is rational}\\ 1 & \text{if } x \text{ is irrational}\end{cases}$ is continuous nowhere. For, given any number a and any $\delta>0$, the interval $(a-\delta,a+\delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $f(a)=0$ or 1, there are infinitely many numbers x with $|x-a|<\delta$ and $|f(x)-f(a)|=1$. Thus, $\lim\limits_{x\to a}f(x)\neq f(a)$. [In fact, $\lim\limits_{x\to a}f(x)$ does not even exist.]

56. $g(x)=\begin{cases}0 & \text{if } x \text{ is rational}\\ x & \text{if } x \text{ is irrational}\end{cases}$ is continuous at 0. To see why, note that $-|x|\le g(x)\le |x|$, so by the Squeeze Theorem $\lim\limits_{x\to 0}g(x)=0=g(0)$. But g is continuous nowhere else. For if $a\neq 0$ and $\delta>0$, the interval $(a-\delta,a+\delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $g(a)=0$ or a, there are infinitely many numbers x with $|x-a|<\delta$ and $|g(x)-g(a)|>|a|/2$. Thus, $\lim\limits_{x\to a}g(x)\neq g(a)$.

57. If there is such a number, it satisfies the equation $x^3+1=x \Leftrightarrow x^3-x+1=0$. Let the LHS of this equation be called $f(x)$. Now $f(-2)=-5<0$, and $f(-1)=1>0$. Note also that $f(x)$ is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number c between -2 and -1 such that $f(c)=0$, so that $c=c^3+1$.

58. (a) $\lim\limits_{x\to 0^+}F(x)=0$ and $\lim\limits_{x\to 0^-}F(x)=0$, so $\lim\limits_{x\to 0}F(x)=0$, which is $F(0)$, and hence F is continuous at $x=a$ if $a=0$. For $a>0$, $\lim\limits_{x\to a}F(x)=\lim\limits_{x\to a}x=a=F(a)$. For $a<0$, $\lim\limits_{x\to a}F(x)=\lim\limits_{x\to a}(-x)=-a=F(a)$. Thus, F is continuous at $x=a$; that is, continuous everywhere.

(b) Assume that f is continuous on the interval I. Then for $a\in I$, $\lim\limits_{x\to a}|f(x)|=\left|\lim\limits_{x\to a}f(x)\right|=|f(a)|$ by Theorem 7. (If a is an endpoint of I, use the appropriate one-sided limit.) So $|f|$ is continuous on I.

(c) No, the converse is false. For example, the function $f(x)=\begin{cases}1 & \text{if } x\ge 0\\ -1 & \text{if } x<0\end{cases}$ is not continuous at $x=0$, but $|f(x)|=1$ is continuous on $\mathbb{R}$.

59. Define $u(t)$ to be the monk's distance from the monastery, as a function of time, on the first day, and define $d(t)$ to be his distance from the monastery, as a function of time, on the second day. Let D be the distance from the monastery to the top of the mountain. From the given information we know that $u(0) = 0$, $u(12) = D$, $d(0) = D$ and $d(12) = 0$. Now consider the function $u - d$, which is clearly continuous. We calculate that $(u - d)(0) = -D$ and $(u - d)(12) = D$. So by the Intermediate Value Theorem, there must be some time t_0 between 0 and 12 such that $(u - d)(t_0) = 0 \Leftrightarrow u(t_0) = d(t_0)$. So at time t_0 after 7:00 A.M., the monk will be at the same place on both days.

2.6 Tangents, Velocities, and Other Rates of Change

1. (a) This is just the slope of the line through two points: $m_{PQ} = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x) - f(3)}{x - 3}$.

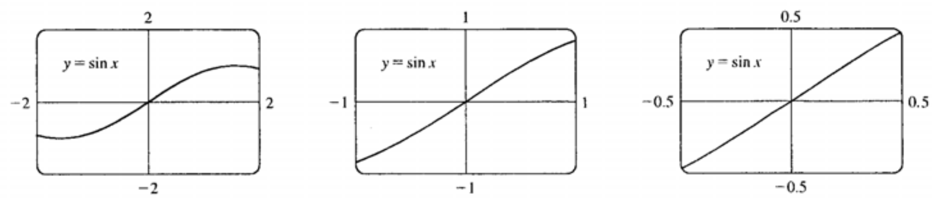
(b) This is the limit of the slope of the secant line PQ as Q approaches P: $m = \lim\limits_{x \to 3} \dfrac{f(x) - f(3)}{x - 3}$.

2. (a) Average velocity $= \dfrac{\Delta s}{\Delta t} = \dfrac{f(a+h) - f(a)}{(a+h) - a} = \dfrac{f(a+h) - f(a)}{h}$

(b) Instantaneous velocity $= \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

3. The slope at D is the largest positive slope, followed by the positive slope at E. The slope at C is zero. The slope at B is steeper than at A (both are negative). In decreasing order, we have the slopes at: D, E, C, A, B.

4. The curve looks more like a line as the viewing rectangle gets smaller.



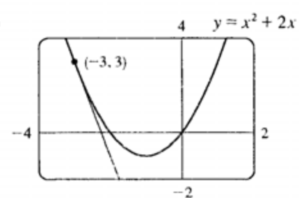
5. (a) (i) $m = \lim\limits_{x \to -3} \dfrac{f(x) - f(-3)}{x - (-3)} = \lim\limits_{x \to -3} \dfrac{(x^2 + 2x) - (3)}{x - (-3)} = \lim\limits_{x \to -3} \dfrac{(x+3)(x-1)}{x+3} = \lim\limits_{x \to -3} (x - 1)$
$= -4$

(ii) $m = \lim\limits_{h \to 0} \dfrac{f(-3+h) - f(-3)}{h} = \lim\limits_{h \to 0} \dfrac{\left[(-3+h)^2 + 2(-3+h)\right] - (3)}{h}$

$= \lim\limits_{h \to 0} \dfrac{9 - 6h + h^2 - 6 + 2h - 3}{h} = \lim\limits_{h \to 0} \dfrac{h(h-4)}{h} = \lim\limits_{h \to 0} (h - 4) = -4$

(b) The equation of the tangent line is $y - 3 = -4(x + 3)$ or $y = -4x - 9$.

(c)



6. (a) (i) $m = \lim_{x\to -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x\to -1} \frac{x^3 - (-1)}{x+1} = \lim_{x\to -1} \frac{(x+1)(x^2 - x + 1)}{x+1}$

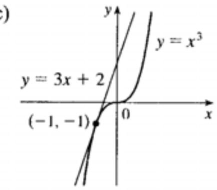
$= \lim_{x\to -1} (x^2 - x + 1) = 3$

(ii) $m = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{(-1+h)^3 - (-1)}{h} = \lim_{h\to 0} \frac{h^3 - 3h^2 + 3h - 1 + 1}{h}$

$= \lim_{h\to 0} (h^2 - 3h + 3) = 3$

(b) $y - (-1) = 3[x - (-1)] \;\Rightarrow\; y = 3x + 2$

(c)



7. Using (1),

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to -2} \frac{(1 - 2x - 3x^2) - (-7)}{x - (-2)} = \lim_{x\to -2} \frac{-3x^2 - 2x + 8}{x+2} = \lim_{x\to -2} \frac{(-3x+4)(x+2)}{x+2}$$

$$= \lim_{x\to -2} (-3x + 4) = 10$$

Thus, an equation of the tangent is $y + 7 = 10(x+2)$ or $y = 10x + 13$.
Alternate Solution: Using (2),

$$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h\to 0} \frac{\left[1 - 2(-2+h) - 3(-2+h)^2\right] - (-7)}{h}$$

$$= \lim_{h\to 0} \frac{(-3h^2 + 10h - 7) + 7}{h} = \lim_{h\to 0} \frac{h(-3h+10)}{h} = \lim_{h\to 0} (-3h + 10) = 10$$

8. Using (1), $m = \lim_{x\to 1} \frac{1/\sqrt{x} - 1}{x - 1} = \lim_{x\to 1} \frac{-(\sqrt{x} - 1)}{\sqrt{x}(\sqrt{x} - 1)(\sqrt{x} + 1)} = \lim_{x\to 1} \frac{-1}{\sqrt{x}(\sqrt{x}+1)} = -\frac{1}{2}$. Thus, an equation of the tangent line is $y - 1 = -\frac{1}{2}(x - 1)$ or $y = -\frac{1}{2}x + \frac{3}{2}$.

9. Using (1), $m = \lim_{x\to -2} \frac{1/x^2 - \frac{1}{4}}{x - (-2)} = \lim_{x\to -2} \frac{4 - x^2}{4x^2(x+2)} = \lim_{x\to -2} \frac{(2-x)(2+x)}{4x^2(x+2)} = \lim_{x\to -2} \frac{2-x}{4x^2} = \frac{1}{4}$.
Thus, an equation of the tangent line is $y - \frac{1}{4} = \frac{1}{4}(x+2) \;\Rightarrow\; y = \frac{1}{4}x + \frac{3}{4}$.

10. Using (1), $m = \lim_{x\to 0} \frac{x/(1-x) - 0}{x - 0} = \lim_{x\to 0} \frac{x}{x(1-x)} = \lim_{x\to 0} \frac{1}{1-x} = 1$.
Thus, an equation of the tangent line is $y - 0 = 1(x - 0) \;\Rightarrow\; y = x$.

11. (a) $m = \lim_{x\to a} \frac{2/(x+3) - 2/(a+3)}{x - a} = \lim_{x\to a} \frac{2(a-x)}{(x-a)(x+3)(a+3)} = \lim_{x\to a} \frac{-2}{(x+3)(a+3)} = \frac{-2}{(a+3)^2}$

(b) (i) $a = -1 \;\Rightarrow\; m = \frac{-2}{(-1+3)^2} = -\frac{1}{2}$ (ii) $a = 0 \;\Rightarrow\; m = \frac{-2}{(0+3)^2} = -\frac{2}{9}$

(iii) $a = 1 \;\Rightarrow\; m = \frac{-2}{(1+3)^2} = -\frac{1}{8}$

12. (a) Using (1),

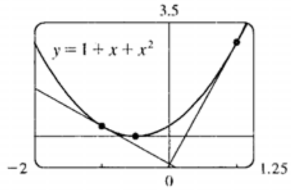
$$m = \lim_{x\to a} \frac{(1+x+x^2)-(1+a+a^2)}{x-a} = \lim_{x\to a} \frac{x+x^2-a-a^2}{x-a} = \lim_{x\to a} \frac{x-a+(x-a)(x+a)}{x-a}$$
$$= \lim_{x\to a} \frac{(x-a)(1+x+a)}{x-a} = \lim_{x\to a} (1+x+a) = 1+2a$$

(b) (i) $x=-1 \Rightarrow m = 1+2(-1) = -1$

(ii) $x = -\frac{1}{2} \Rightarrow m = 1+2\left(-\frac{1}{2}\right) = 0$

(iii) $x = 1 \Rightarrow m = 1+2(1) = 3$

(c)

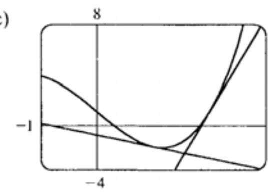


13. (a) Using (1),

$$m = \lim_{x\to a} \frac{(x^3-4x+1)-(a^3-4a+1)}{x-a} = \lim_{x\to a} \frac{(x^3-a^3)-4(x-a)}{x-a}$$
$$= \lim_{x\to a} \frac{(x-a)(x^2+ax+a^2)-4(x-a)}{x-a} = \lim_{x\to a} \left(x^2+ax+a^2-4\right) = 3a^2-4$$

(b) At $(1,-2)$: $m = 3(1)^2-4=-1$, so an equation of the tangent line is $y-(-2) = -1(x-1) \Leftrightarrow y = -x-1$. At $(2,1)$: $m = 3(2)^2-4 = 8$, so an equation of the tangent line is $y-1 = 8(x-2) \Leftrightarrow y = 8x-15$.

(c)

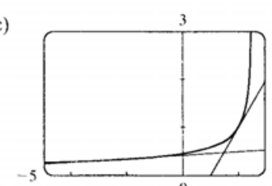


14. (a) Using (1),

$$m = \lim_{x\to a} \frac{\dfrac{1}{\sqrt{5-2x}} - \dfrac{1}{\sqrt{5-2a}}}{x-a} = \lim_{x\to a} \frac{\sqrt{5-2a}-\sqrt{5-2x}}{(x-a)\sqrt{5-2x}\sqrt{5-2a}}$$
$$= \lim_{x\to a} \frac{2(x-a)}{(x-a)\sqrt{(5-2x)(5-2a)}\left(\sqrt{5-2a}+\sqrt{5-2x}\right)}$$
$$= \lim_{x\to a} \frac{2}{(x-a)\sqrt{(5-2x)(5-2a)}\left(\sqrt{5-2a}+\sqrt{5-2x}\right)} = \frac{2}{2(5-2a)^{3/2}}$$
$$= (5-2a)^{-3/2}$$

(b) At $(2,1)$: $m = [5-2(2)]^{-3/2} = 1 \Leftrightarrow y-1 = 1(x-2) \Leftrightarrow y = x-1$. At $\left(-2, \frac{1}{3}\right)$: $m = [5-2(-2)]^{-3/2} = \frac{1}{27} \Leftrightarrow y - \frac{1}{3} = \frac{1}{27}[x-(-2)] \Leftrightarrow y = \frac{1}{27}x + \frac{11}{27}$.

(c)



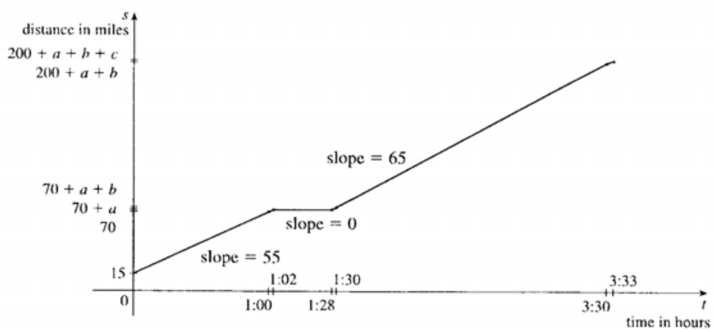
15. (a) Since the slope of the tangent at $s = 0$ is 0, the car's initial velocity was 0.

(b) The slope of the tangent is greater at C than at B, so the car was going faster at C.

(c) Near A, the tangent lines are becoming steeper as x increases, so the velocity was increasing, so the car was speeding up. Near B, the tangent lines are becoming less steep, so the car was slowing down. The steepest tangent near C is the one at C, so at C the car had just finished speeding up, and was about to start slowing down.

(d) Between D and E, the slope of the tangent is 0, so the car did not move during that time.

16.



Let a denote the distance traveled from 1:00 to 1:02, b from 1:28 to 1:30, and c from 3:30 to 3:33, where all the times are relative to $t = 0$ at the beginning of the trip.

17. Let $s(t) = 40t - 16t^2$.

$$v(2) = \lim_{t\to 2} \frac{s(t) - s(2)}{t - 2} = \lim_{t\to 2} \frac{(40t - 16t^2) - 16}{t - 2} = \lim_{t\to 2} \frac{-8(t-2)(2t-1)}{t - 2}$$
$$= -8 \lim_{t\to 2} (2t - 1) = -24$$

Thus, the instantaneous velocity when $t = 2$ is -24 ft/s.

18. (a) $$v(1) = \lim_{h\to 0} \frac{H(1+h) - H(1)}{h}$$
$$= \lim_{h\to 0} \frac{(58 + 58h - 0.83 - 1.66h - 0.83h^2) - 57.17}{h} = \lim_{h\to 0} (56.34 - 0.83h) = 56.34 \text{ m/s}$$

(b) $$v(a) = \lim_{h\to 0} \frac{H(a+h) - H(a)}{h}$$
$$= \lim_{h\to 0} \frac{(58a + 58h - 0.83a^2 - 1.66ah - 0.83h^2) - (58a - 0.83a^2)}{h}$$
$$= \lim_{h\to 0} (58 - 1.66a - 0.83h) = 58 - 1.66a \text{ m/s}$$

(c) The arrow strikes the moon when the height is 0, that is, $58t - 0.83t^2 = 0 \Leftrightarrow t(58 - 0.83t) = 0 \Leftrightarrow t = \frac{58}{0.83} \approx 69.9$ s (since t can't be 0).

(d) Using the time from part (c), $v\left(\frac{58}{0.83}\right) = 58 - 1.66\left(\frac{58}{0.83}\right) = -58$ m/s. Thus, the arrow will have a velocity of -58 m/s.

19. $v(a) = \lim_{h\to 0} \frac{s(a+h)-s(a)}{h} = \lim_{h\to 0} \frac{4(a+h)^3 + 6(a+h) + 2 - (4a^3+6a+2)}{h}$

$= \lim_{h\to 0} \frac{4a^3 + 12a^2h + 12ah^2 + 4h^3 + 6a + 6h + 2 - 4a^3 - 6a - 2}{h}$

$= \lim_{h\to 0} \frac{12a^2h + 12ah^2 + 4h^3 + 6h}{h} = \lim_{h\to 0} (12a^2 + 12ah + 4h^2 + 6)$

$= (12a^2 + 6)$ m/s

So $v(1) = 12(1)^2 + 6 = 18$ m/s, $v(2) = 12(2)^2 + 6 = 54$ m/s, and $v(3) = 12(3)^2 + 6 = 114$ m/s.

20. (a) The average velocity between times t and $t+h$ is

$$\frac{s(t+h)-s(t)}{(t+h)-t} = \frac{(t+h)^2 - 8(t+h) + 18 - (t^2 - 8t + 18)}{h}$$

$$= \frac{t^2 + 2th + h^2 - 8t - 8h + 18 - t^2 + 8t - 18}{h} = \frac{2th + h^2 - 8h}{h}$$

$$= (2t + h - 8) \text{ m/s}$$

(i) $[3, 4]$: $t = 3$, $h = 4 - 3 = 1$, so the average velocity is $2(3) + 1 - 8 = -1$ m/s.

(ii) $[3.5, 4]$: $t = 3.5$, $h = 0.5$, so the average velocity is $2(3.5) + 0.5 - 8 = -0.5$ m/s.

(iii) $[4, 5]$: $t = 4$, $h = 1$, so the average velocity is $2(4) + 1 - 8 = 1$ m/s.

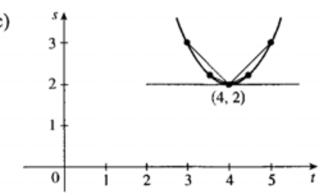
(iv) $[4, 4.5]$: $t = 4$, $h = 0.5$, so the average velocity is $2(4) + 0.5 - 8 = 0.5$ m/s.

(b) $v(t) = \lim_{h\to 0} \frac{s(t+h)-s(t)}{h}$

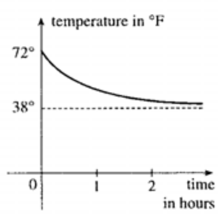
$= \lim_{h\to 0} (2t + h - 8) = 2t - 8$,

so $v(4) = 0$.

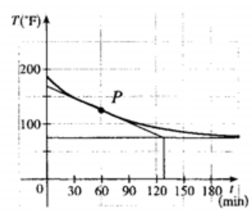
(c)



21. The sketch shows the graph for a room temperature of 72° and a refrigerator temperature of 38°. The initial rate of change is greater in magnitude than the rate of change after an hour.



22. The slope of the tangent (that is, the rate of change of temperature with respect to time) at $t = 1$ h seems to be about $\frac{75-168}{132-0} \approx -0.7°\text{F/min}$.

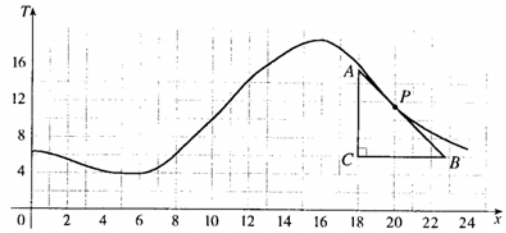


23. (a) (i) $[8, 11]$: $\dfrac{7.9 - 11.5}{3} = -1.2°/\text{h}$

(ii) $[8, 10]$: $\dfrac{9.0 - 11.5}{2} = -1.25°/\text{h}$

(iii) $[8, 9]$: $\dfrac{10.2 - 11.5}{1} = -1.3°/\text{h}$

(b) In the figure, we estimate A to be $(18, 15.5)$ and B as $(23, 6)$. So the slope is $\dfrac{6 - 15.5}{23 - 18} = -1.9°/\text{h}$ at 8:00 P.M.



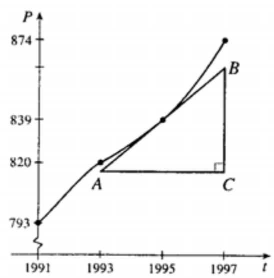
24. (a) (i) $\dfrac{P(1995) - P(1991)}{1995 - 1991} = \dfrac{839 - 793}{4} = \dfrac{46}{4} = 11.5$ thousand people per year

(ii) $\dfrac{839 - 820}{1995 - 1993} = \dfrac{19}{2} = 9.5$ thousand people per year

(iii) $\dfrac{874 - 839}{1997 - 1995} = \dfrac{35}{2} = 17.5$ thousand people per year

(b) Using the values from (a)(ii) and (a)(iii), we have $\dfrac{9.5 + 17.5}{2} = 13.5$ thousand people per year.

(c)



Estimating A as $(1993, 816)$ and B as $(1997, 862)$, the slope at 1995 is $\dfrac{862 - 816}{1997 - 1993} = 11.5$ thousand people per year.

25. (a) (i) $\dfrac{\Delta C}{\Delta x} = \dfrac{C(105) - C(100)}{105 - 100} = \dfrac{6601.25 - 6500}{5} = 20.25/\text{unit}.$

(ii) $\dfrac{\Delta C}{\Delta x} = \dfrac{C(101) - C(100)}{101 - 100} = \dfrac{6520.05 - 6500}{1} = 20.05/\text{unit}.$

(b) $$\frac{C(100 + h) - C(100)}{h} = \frac{\left[5000 + 10(100 + h) + 0.05(100 + h)^2\right] - 6500}{h} = \frac{20h + 0.05h^2}{h}$$
$$= 20 + 0.05h,\ h \neq 0$$

So the instantaneous rate of change is $\displaystyle\lim_{h \to 0} \frac{C(100 + h) - C(100)}{h} = \lim_{h \to 0} (20 + 0.05h) = \$20/\text{unit}.$

26. $\Delta V = V(t+h) - V(t) = 100{,}000\left(1 - \dfrac{t+h}{60}\right)^2 - 100{,}000\left(1 - \dfrac{t}{60}\right)^2$

$= 100{,}000\left[\left(1 - \dfrac{t+h}{30} + \dfrac{(t+h)^2}{3600}\right) - \left(1 - \dfrac{t}{30} + \dfrac{t^2}{3600}\right)\right] = 100{,}000\left(-\dfrac{h}{30} + \dfrac{2th}{3600} + \dfrac{h^2}{3600}\right)$

$= \frac{100{,}000}{3600}h(-120 + 2t + h) = \frac{250}{9}h(-120 + 2t + h)$

Dividing ΔV by h and then letting $h \to 0$, we see that the instantaneous rate of change is $\frac{500}{9}(t - 60)$ gal/min.

t	Flow rate (gal/min)	Water remaining $V(t)$ (gal)
0	$-3333.\overline{3}$	$100,000$
10	$-2777.\overline{7}$	$69,444.\overline{4}$
20	$-2222.\overline{2}$	$44,444.\overline{4}$
30	$-1666.\overline{6}$	$25,000$
40	$-1111.\overline{1}$	$11,111.\overline{1}$
50	$-555.\overline{5}$	$2,777.\overline{7}$
60	0	0

The magnitude of the flow rate is greatest at the beginning and gradually decreases to 0.

2 Review

CONCEPT CHECK

1. (a) $\lim_{x\to a} f(x) = L$: See Definition 2.2.1 and Figures 1 and 2 in Section 2.2.

(b) $\lim_{x\to a^+} f(x) = L$: See the paragraph after Definition 2.2.2 and Figure 9(b) in Section 2.2.

(c) $\lim_{x\to a^-} f(x) = L$: See Definition 2.2.2 and Figure 9(a) in Section 2.2.

(d) $\lim_{x\to a} f(x) = \infty$: See Definition 2.2.4 and Figure 12 in Section 2.2.

(e) $\lim_{x\to a} f(x) = -\infty$: See Definition 2.2.5 and Figure 13 in Section 2.2.

2. See Definition 2.2.6 and Figures 12–14 in Section 2.2.

3. (a)–(g) See the statements of Limit Laws 1–6 and 11 in Section 2.3.

4. See Theorem 3 in Section 2.3.

5. (a) A function f is continuous at a number a if $f(x)$ gets closer to $f(a)$ as x gets close to a, that is, $\lim_{x\to a} f(x) = f(a)$.

(b) A function f is continuous on the interval $(-\infty, \infty)$ if f is continuous at every real number a. The graph of such a function has no breaks and every vertical line crosses it.

6. See Theorem 2.5.10.

7. See Definition 2.6.1.

8. See the paragraph containing Formula 3 in Section 2.6.

9. (a) The average rate of change of y with respect to x over the interval $[x_1, x_2]$ is $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$.

(b) The instantaneous rate of change of y with respect to x at $x = x_1$ is $\lim_{x_2\to x_1} \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$.

TRUE-FALSE QUIZ

1. False. Limit Law 2 applies only if the individual limits exist (these don't.)

2. False. Limit Law 5 cannot be applied if the limit of the denominator is 0 (it is.)

3. True. Limit Law 5 applies.

4. True. The limit doesn't exist since $f(x)/g(x)$ doesn't approach any real number as x approaches 5. (The denominator approaches 0 and the numerator doesn't.)

5. False. Consider $\lim\limits_{x\to 5}\dfrac{x^2-5x}{x-5}$ or $\lim\limits_{x\to 5}\dfrac{\sin(x-5)}{x-5}$. By Example 3 in Section 2.2, we know that the latter limit exists (and it is equal to 1).

6. False. Consider $\lim\limits_{x\to 6}[f(x)\,g(x)] = \lim\limits_{x\to 6}\left[(x-6)\dfrac{1}{x-6}\right]$. It exists (its value is 1) but $f(6)=0$ and $g(6)$ does not exist, so $f(6)\,g(6)\neq 1$.

7. True. A polynomial is continuous everywhere, so $\lim\limits_{x\to b} p(x)$ exists and is equal to $p(b)$.

8. False. Consider $\lim\limits_{x\to 0}[f(x)-g(x)] = \lim\limits_{x\to 0}\left(\dfrac{1}{x^2}-\dfrac{1}{x^4}\right)$. This limit is $-\infty$ (not 0), but each of the individual functions approaches ∞.

9. False. Consider $f(x)=\begin{cases} 1/(x-1) & \text{if } x\neq 1 \\ 2 & \text{if } x=1 \end{cases}$

10. False. The function f must be *continuous* in order to use the Intermediate Value Theorem. For example, let

$f(x)=\begin{cases} 1 & \text{if } 0\le x<3 \\ -1 & \text{if } x=3 \end{cases}$ There is no number $c\in[0,3]$ with $f(c)=0$.

11. True. Use Theorem 2.5.8 with $a=2$, $b=5$, and $g(x)=4x^2-11$. Note that $f(4)=3$ is not needed.

12. True. Use the Intermediate Value Theorem with $a=-1$, $b=1$, $N=\pi$, since $3<\pi<4$.

13. True, by the definition of a limit with $\varepsilon=1$.

14. False. For example, let $f(x)=\begin{cases} x^2+1 & \text{if } x\neq 0 \\ 2 & \text{if } x=0 \end{cases}$ Then $f(x)>1$ for all x, but

$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\left(x^2+1\right)=1$.

EXERCISES

1. (a) (i) $\lim\limits_{x\to 2^+} f(x)=3$ (ii) $\lim\limits_{x\to -3^+} f(x)=0$

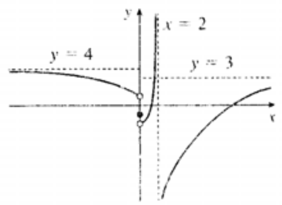
(iii) $\lim\limits_{x\to -3} f(x)$ does not exist since the left and right limits are not equal. (The left limit is -2.)

(iv) $\lim\limits_{x\to 4} f(x)=2$ (v) $\lim\limits_{x\to 0} f(x)=\infty$ (vi) $\lim\limits_{x\to 2^-} f(x)=-\infty$

(b) The vertical asymptotes are $x=0$ and $x=2$.

(c) f is discontinuous at $x=-3$, 0, 2, and 4. The discontinuities are jump, infinite, infinite, and removable, respectively.

2.



3. $\lim\limits_{x\to 0} \tan\left(x^2\right) = \tan 0 = 0$ because the tangent function is continuous at $x = 0$.

4. $\lim\limits_{t\to 4} \dfrac{t-4}{t^2-3t-4} = \lim\limits_{t\to 4} \dfrac{t-4}{(t-4)(t+1)} = \lim\limits_{t\to 4} \dfrac{1}{t+1} = \dfrac{1}{4+1} = \dfrac{1}{5}$

5. $\lim\limits_{h\to 0} \dfrac{(1+h)^2-1}{h} = \lim\limits_{h\to 0} \dfrac{1+2h+h^2-1}{h} = \lim\limits_{h\to 0} \dfrac{2h+h^2}{h} = \lim\limits_{h\to 0} (2+h) = 2$

6. $\lim\limits_{h\to 0} \dfrac{(1+h)^{-2}-1}{h} = \lim\limits_{h\to 0} \dfrac{1-(1+h)^2}{h(1+h)^2} = \lim\limits_{h\to 0} \dfrac{-2h-h^2}{h(1+h)^2} = \lim\limits_{h\to 0} \dfrac{-2-h}{(1+h)^2} = \dfrac{-2-0}{(1+0)^2} = -2$

7. $\lim\limits_{x\to -1} \dfrac{x^2-x-2}{x^2+3x-2} = \dfrac{\lim\limits_{x\to -1}\left(x^2-x-2\right)}{\lim\limits_{x\to -1}\left(x^2+3x-2\right)} = \dfrac{(-1)^2-(-1)-2}{(-1)^2+3(-1)-2} = \dfrac{0}{-4} = 0$

8. $\lim\limits_{x\to -1} \dfrac{x^2-x-2}{x^2+3x+2} = \lim\limits_{x\to -1} \dfrac{(x+1)(x-2)}{(x+1)(x+2)} = \lim\limits_{x\to -1} \dfrac{x-2}{x+2} = \dfrac{-1-2}{-1+2} = -3$

9. $\lim\limits_{t\to 6} \dfrac{17}{(t-6)^2} = \infty$ since $(t-6)^2 \to 0$ and $\dfrac{17}{(t-6)^2} > 0$.

10. $\lim\limits_{x\to -6^+} \dfrac{x}{x+6} = -\infty$ since $x+6 \to 0$ as $x \to -6^+$ and $\dfrac{x}{x+6} < 0$ for $-6 < x < 0$.

11. $\lim\limits_{s\to 16} \dfrac{4-\sqrt{s}}{s-16} = \lim\limits_{s\to 16} \dfrac{4-\sqrt{s}}{\left(\sqrt{s}+4\right)\left(\sqrt{s}-4\right)} = \lim\limits_{s\to 16} \dfrac{-1}{\sqrt{s}+4} = \dfrac{-1}{\sqrt{16}+4} = -\dfrac{1}{8}$

12. $\lim\limits_{v\to 2} \dfrac{v^2+2v-8}{v^4-16} = \lim\limits_{v\to 2} \dfrac{(v+4)(v-2)}{(v+2)(v-2)\left(v^2+4\right)} = \lim\limits_{v\to 2} \dfrac{v+4}{(v+2)\left(v^2+4\right)} = \dfrac{2+4}{(2+2)\left(2^2+4\right)} = \dfrac{3}{16}$

13. $\lim\limits_{x\to 8^-} \dfrac{|x-8|}{x-8} = \lim\limits_{x\to 8^-} \dfrac{-(x-8)}{x-8} = \lim\limits_{x\to 8^-} (-1) = -1$

14. $\lim\limits_{x\to 9^+} \left(\sqrt{x-9} + [\![x+1]\!]\right) = \lim\limits_{x\to 9^+} \sqrt{x-9} + \lim\limits_{x\to 9^+} [\![x+1]\!] = \sqrt{9-9} + 10 = 10$

15. $\lim\limits_{x\to 0} \dfrac{1-\sqrt{1-x^2}}{x} \cdot \dfrac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}} = \lim\limits_{x\to 0} \dfrac{1-\left(1-x^2\right)}{x\left(1+\sqrt{1-x^2}\right)} = \lim\limits_{x\to 0} \dfrac{x^2}{x\left(1+\sqrt{1-x^2}\right)} = \lim\limits_{x\to 0} \dfrac{x}{1+\sqrt{1-x^2}} = 0$

16. $\lim\limits_{x\to 2} \dfrac{\sqrt{x+2}-\sqrt{2x}}{x(x-2)} \cdot \dfrac{\sqrt{x+2}+\sqrt{2x}}{\sqrt{x+2}+\sqrt{2x}} = \lim\limits_{x\to 2} \dfrac{-(x-2)}{x(x-2)\left(\sqrt{x+2}+\sqrt{2x}\right)} = \lim\limits_{x\to 2} \dfrac{-1}{x\left(\sqrt{x+2}+\sqrt{2x}\right)} = -\dfrac{1}{8}$

17. Since $2x-1 \le f(x) \le x^2$ for $0 < x < 3$ and $\lim\limits_{x\to 1} (2x-1) = 1 = \lim\limits_{x\to 1} x^2$, we have $\lim\limits_{x\to 1} f(x) = 1$ by the Squeeze Theorem.

18. Let $f(x) = -x^2$, $g(x) = x^2 \cos(1/x^2)$ and $h(x) = x^2$. Then since $|\cos(1/x^2)| \leq 1$ for $x \neq 0$, we have $f(x) \leq g(x) \leq h(x)$ for $x \neq 0$, and so $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0 \Rightarrow \lim_{x\to 0} g(x) = 0$ by the Squeeze Theorem.

19. Given $\varepsilon > 0$, we need $\delta > 0$ so that if $|x-5| < \delta$ then $|(7x-27)-8| < \varepsilon \Leftrightarrow |7x-35| < \varepsilon \Leftrightarrow$ $|x-5| < \varepsilon/7$. So take $\delta = \varepsilon/7$. Then $|x-5| < \delta \Rightarrow |(7x-27)-8| < \varepsilon$. Thus, $\lim_{x\to 5}(7x-27) = 8$ by the definition of a limit.

20. Given $\varepsilon > 0$ we must find $\delta > 0$ so that if $|x-0| < \delta$, then $\left|\sqrt[3]{x}-0\right| < \varepsilon$. Now $\left|\sqrt[3]{x}-0\right| = \left|\sqrt[3]{x}\right| < \varepsilon \Rightarrow$ $|x| = \left|\sqrt[3]{x}\right|^3 < \varepsilon^3$. So take $\delta = \varepsilon^3$; then $|x-0| = |x| < \varepsilon^3 \Rightarrow \left|\sqrt[3]{x}-0\right| = \left|\sqrt[3]{x}\right| = \sqrt[3]{|x|} < \sqrt[3]{\varepsilon^3} = \varepsilon$. Therefore, by the definition of a limit, $\lim_{x\to 0}\sqrt[3]{x} = 0$.

21. Given $\varepsilon > 0$, we need $\delta > 0$ so that if $|x-2| < \delta$ then $\left|x^2-3x-(-2)\right| < \varepsilon$. First, note that if $|x-2| < 1$, then $-1 < x-2 < 1$, so $0 < x-1 < 2 \Rightarrow |x-1| < 2$. Now let $\delta = \min\{\varepsilon/2, 1\}$. Then $|x-2| < \delta \Rightarrow$ $\left|x^2-3x-(-2)\right| = |(x-2)(x-1)| = |x-2||x-1| < (\varepsilon/2)(2) = \varepsilon$.
Thus, $\lim_{x\to 2}\left(x^2-3x\right) = -2$ by the definition of a limit.

22. Given $M > 0$, we need $\delta > 0$ such that if $0 < x-4 < \delta$ then $2/\sqrt{x-4} > M$. This is true $\Leftrightarrow \sqrt{x-4} < 2/M$ $\Leftrightarrow x-4 < 4/M^2$. So if we choose $\delta = 4/M^2$, then $0 < x-4 < \delta \Rightarrow 2/\sqrt{x-4} > M$. So by the definition of a limit, $\lim_{x\to 4^+}\left(2/\sqrt{x-4}\right) = \infty$.

23. (a) $f(x) = \sqrt{-x}$ if $x < 0$, $f(x) = 3-x$ if $0 \leq x < 3$, $f(x) = (x-3)^2$ if $x > 3$.

(i) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(3-x) = 3$

(ii) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\sqrt{-x} = 0$

(iii) Because of (i) and (ii), $\lim_{x\to 0} f(x)$ does not exist.

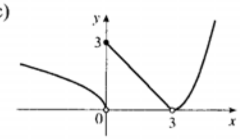
(iv) $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(3-x) = 0$

(v) $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(x-3)^2 = 0$

(vi) Because of (iv) and (v), $\lim_{x\to 3} f(x) = 0$.

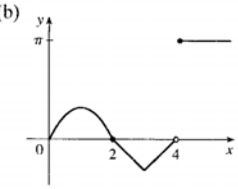
(b) f is discontinuous at 0 since $\lim_{x\to 0} f(x)$ does not exist. f is discontinuous at 3 since $f(3)$ does not exist.

(c)



24. (a) $g(x) = 2x - x^2$ if $0 \leq x \leq 2$, $g(x) = 2-x$ if $2 < x \leq 3$, $g(x) = x-4$ if $3 < x < 4$, $g(x) = \pi$ if $x \geq 4$. Therefore, $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-}\left(2x-x^2\right) = 0$ and $\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+}(2-x) = 0$. Thus, $\lim_{x\to 2} g(x) = 0 = g(2)$, so g is continuous at 2. $\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-}(2-x) = -1$ and $\lim_{x\to 3^+} g(x) = \lim_{x\to 3^+}(x-4) = -1$. Thus, $\lim_{x\to 3} g(x) = -1 = g(3)$, so g is continuous at 3. $\lim_{x\to 4^-} g(x) = \lim_{x\to 4^-}(x-4) = 0$ and $\lim_{x\to 4^+} g(x) = \lim_{x\to 4^+}\pi = \pi$. Thus, $\lim_{x\to 4} g(x)$ does not exist, so g is discontinuous at 4. But $\lim_{x\to 4^+} g(x) = \pi = g(4)$, so g is continuous from the right at 4.

(b)



25. x^3 is continuous on $\mathbb{R}$ since it is a polynomial and $\cos x$ is also continuous on $\mathbb{R}$, so the product $x^3 \cos x$ is continuous on $\mathbb{R}$. The root function $\sqrt[4]{x}$ is continuous on its domain, $[0, \infty)$, and so the sum, $h(x) = \sqrt[4]{x} + x^3 \cos x$, is continuous on its domain, $[0, \infty)$.

26. $x^2 - 9$ is continuous on $\mathbb{R}$ since it is a polynomial and $\sqrt{x}$ is continuous on $[0, \infty)$, so the composition $\sqrt{x^2 - 9}$ is continuous on $\{x \mid x^2 - 9 \geq 0\} = (-\infty, -3] \cup [3, \infty)$. Note that $x^2 - 2 \neq 0$ on this set and so the quotient function $g(x) = \dfrac{\sqrt{x^2 - 9}}{x^2 - 2}$ is continuous on its domain, $(-\infty, -3] \cup [3, \infty)$.

27. $f(x) = 2x^3 + x^2 + 2$ is a polynomial, so it is continuous on $[-2, -1]$ and $f(-2) = -10 < 0 < 1 = f(-1)$. So by the Intermediate Value Theorem there is a number c in $(-2, -1)$ such that $f(c) = 0$, that is, the equation $2x^3 + x^2 + 2 = 0$ has a root in $(-2, -1)$.

28. Let $f(x) = 2\sin x - 3 + 2x$. Now f is continuous on $[0, 1]$ and $f(0) = -3 < 0$ and $f(1) = 2\sin 1 - 1 \approx 0.68 > 0$. So by the Intermediate Value Theorem there is a number c in $(0, 1)$ such that $f(c) = 0$, that is, the equation $2\sin x = 3 - 2x$ has a root in $(0, 1)$.

29. (a) The slope of the tangent line at $(2, 1)$ is

$$\begin{aligned}\lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} &= \lim_{x\to 2} \frac{9 - 2x^2 - 1}{x - 2} = \lim_{x\to 2} \frac{8 - 2x^2}{x - 2} = \lim_{x\to 2} \frac{-2(x^2 - 4)}{x - 2} = \lim_{x\to 2} \frac{-2(x - 2)(x + 2)}{x - 2} \\ &= \lim_{x\to 2} -2(x + 2) = -8\end{aligned}$$

(b) An equation of this tangent line is $y - 1 = -8(x - 2)$ or $y = -8x + 17$.

30. For a general point with x-coordinate a, we have

$$\begin{aligned} m &= \lim_{x\to a} \frac{2/(1 - 3x) - 2/(1 - 3a)}{x - a} = \lim_{x\to a} \frac{2(1 - 3a) - 2(1 - 3x)}{(1 - 3a)(1 - 3x)(x - a)} \\ &= \lim_{x\to a} \frac{6(x - a)}{(1 - 3a)(1 - 3x)(x - a)} = \lim_{x\to a} \frac{6}{(1 - 3a)(1 - 3x)} = \frac{6}{(1 - 3a)^2}\end{aligned}$$

For $a = 0$, $m = 6$ and $f(0) = 2$, so an equation of the tangent line is $y - 2 = 6(x - 0)$ or $y = 6x + 2$. For $a = -1$, $m = \frac{3}{8}$ and $f(-1) = \frac{1}{2}$, so an equation of the tangent line is $y - \frac{1}{2} = \frac{3}{8}(x + 1)$ or $y = \frac{3}{8}x + \frac{7}{8}$.

31. (a) $s = 1 + 2t + t^2/4$. The average velocity over the time interval $[1, 1 + h]$ is

$\dfrac{s(1 + h) - s(1)}{h} = \dfrac{1 + 2(1 + h) + (1 + h)^2/4 - 13/4}{h} = \dfrac{10h + h^2}{4h} = \dfrac{10 + h}{4}$. So for the following intervals the average velocities are:

(i) $[1, 3]$: $(10 + 2)/4 = 3$ m/s

(ii) $[1, 2]$: $(10 + 1)/4 = 2.75$ m/s

(iii) $[1, 1.5]$: $(10 + 0.5)/4 = 2.625$ m/s

(iv) $[1, 1.1]$: $(10 + 0.1)/4 = 2.525$ m/s

(b) When $t = 1$ the velocity is $\displaystyle\lim_{h\to 0} \frac{s(1 + h) - s(1)}{h} = \lim_{h\to 0} \frac{10 + h}{4} = 2.5$ m/s.

32. (a) When V increases from 200 in^3 to 250 in^3, we have $\Delta V = 250 - 200 = 50\ \text{in}^3$, and since $P = 800/V$,

$\Delta P = P(250) - P(200) = \dfrac{800}{250} - \dfrac{800}{200} = 3.2 - 4 = -0.8\ \text{lb/in}^2$. So the average rate of change is

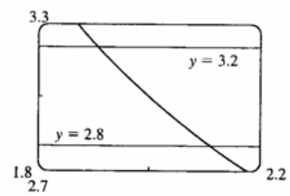
$\dfrac{\Delta P}{\Delta V} = \dfrac{-0.8}{50} = -0.016\ \dfrac{\text{lb/in}^2}{\text{in}^3}$.

(b) Since $V = 800/P$, the instantaneous rate of change of V with respect to P is

$$\lim_{h\to 0}\frac{\Delta V}{\Delta P} = \lim_{h\to 0}\frac{V(P+h)-V(P)}{h} = \lim_{h\to 0}\frac{800/(P+h)-800/P}{h}$$
$$= \lim_{h\to 0}\frac{800[P-(P+h)]}{h(P+h)P} = \lim_{h\to 0}\frac{-800}{(P+h)P} = -\frac{800}{P^2}$$

which is inversely proportional to the square of P.

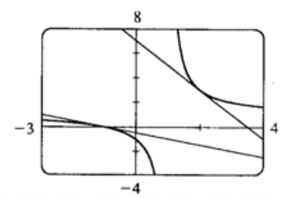
33. The inequality $\left|\dfrac{x+1}{x-1}-3\right| < 0.2$ is equivalent to the double inequality $2.8 < \dfrac{x+1}{x-1} < 3.2$. Graphing the functions $y = 2.8$, $y = |(x+1)/(x-1)|$ and $y = 3.2$ on the interval $[1.9, 2.15]$, we see that the inequality holds whenever $1.91 < x < 2.11$ (approximately). So since $|2-1.91| = 0.09$ and $|2-2.15| = 0.15$, any positive $\delta \le 0.09$ will do.



34. The slope of the tangent to $y = \dfrac{x+1}{x-1}$ is

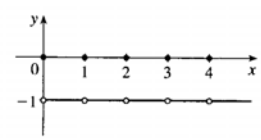
$$\lim_{h\to 0}\frac{\dfrac{(x+h)+1}{(x+h)-1}-\dfrac{x+1}{x-1}}{h} = \lim_{h\to 0}\frac{(x-1)(x+h+1)-(x+1)(x+h-1)}{h(x-1)(x+h-1)}$$
$$= \lim_{h\to 0}\frac{-2h}{h(x-1)(x+h-1)} = -\frac{2}{(x-1)^2}$$

So at $(2,3)$, $m = -\dfrac{2}{(2-1)^2} = -2 \;\Rightarrow\; y-3 = -2(x-2) \;\Rightarrow$ $y = -2x+7$. At $(-1,0)$, $m = -\dfrac{2}{(-1-1)^2} = -\dfrac{1}{2} \;\Rightarrow$ $y = -\frac{1}{2}(x+1) \;\Rightarrow\; y = -\frac{1}{2}x - \frac{1}{2}$.



35. $|f(x)| \le g(x) \;\Leftrightarrow\; -g(x) \le f(x) \le g(x)$ and $\lim_{x\to a} g(x) = 0 = \lim_{x\to a} -g(x)$. Thus, by the Squeeze Theorem, $\lim_{x\to a} f(x) = 0$.

36. (a) Note that f is an even function since $f(x) = f(-x)$. Now for any integer n, $[\![n]\!] + [\![-n]\!] = n - n = 0$, and for any real number k which is not an integer, $[\![k]\!] + [\![-k]\!] = [\![k]\!] + (-[\![k]\!] - 1) = -1$. So $\lim_{x\to a} f(x)$ exists (and is equal to -1) for all values of a.



(b) f is discontinuous at all integers.

Problems Plus

1. Let $t = \sqrt[6]{x}$, so $x = t^6$. Then $t \to 1$ as $x \to 1$, so

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt{x}-1} = \lim_{t\to 1}\frac{t^2-1}{t^3-1} = \lim_{t\to 1}\frac{(t-1)(t+1)}{(t-1)(t^2+t+1)} = \lim_{t\to 1}\frac{t+1}{t^2+t+1} = \frac{1+1}{1^2+1+1} = \frac{2}{3}.$$

Another Method: Multiply both numerator and denominator by $(\sqrt{x}+1)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)$.

2. First rationalize the numerator: $\lim\limits_{x\to 0}\dfrac{\sqrt{ax+b}-2}{x}\cdot\dfrac{\sqrt{ax+b}+2}{\sqrt{ax+b}+2} = \lim\limits_{x\to 0}\dfrac{ax+b-4}{x\left(\sqrt{ax+b}+2\right)}$. Now since the denominator approaches 0 as $x \to 0$, the limit will exist only if the numerator also approaches 0 as $x \to 0$. So we require that $a(0)+b-4=0 \Rightarrow b=4$. So the equation becomes $\lim\limits_{x\to 0}\dfrac{a}{\sqrt{ax+4}+2} = 1 \Rightarrow \dfrac{a}{\sqrt{4}+2} = 1$ $\Rightarrow a = 4$. Therefore, $a = b = 4$.

3. For $-\frac{1}{2} < x < \frac{1}{2}$, we have $2x-1<0$ and $2x+1>0$, so $|2x-1| = -(2x-1)$ and $|2x+1| = 2x+1$. Therefore, $\lim\limits_{x\to 0}\dfrac{|2x-1|-|2x+1|}{x} = \lim\limits_{x\to 0}\dfrac{-(2x-1)-(2x+1)}{x} = \lim\limits_{x\to 0}\dfrac{-4x}{x} = \lim\limits_{x\to 0}(-4) = -4$.

4. Let R be the midpoint of OP, so the coordinates of R are $\left(\frac{1}{2}x, \frac{1}{2}x^2\right)$ since the coordinates of P are (x, x^2). Let $Q = (0, a)$. Since the slope $m_{OP} = \dfrac{x^2}{x} = x$, $m_{QR} = -\dfrac{1}{x}$ (negative reciprocal). But $m_{QR} = \dfrac{\frac{1}{2}x^2 - a}{\frac{1}{2}x - 0} = \dfrac{x^2-2a}{x}$, so we conclude that $-1 = x^2 - 2a \Rightarrow 2a = x^2+1 \Rightarrow a = \frac{1}{2}x^2 + \frac{1}{2}$. As $x \to 0$, $a \to \frac{1}{2}$, and the limiting position of Q is $\left(0, \frac{1}{2}\right)$.

5. Since $[\![x]\!] \le x < [\![x]\!] + 1$, we have $1 \le \dfrac{x}{[\![x]\!]} \le 1 + \dfrac{1}{[\![x]\!]}$ for $x > 0$. As $x \to \infty$, $[\![x]\!] \to \infty$, so $\dfrac{1}{[\![x]\!]} \to 0$ and $1 + \dfrac{1}{[\![x]\!]} \to 1$. Thus, $\lim\limits_{x\to\infty}\dfrac{x}{[\![x]\!]} = 1$ by the Squeeze Theorem.

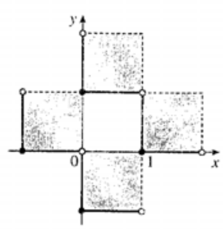
6. (a) $[\![x]\!]^2 + [\![y]\!]^2 = 1$. Since $[\![x]\!]^2$ and $[\![y]\!]^2$ are positive integers or 0, there are only 4 cases:

Case (i): $[\![x]\!] = 1$, $[\![y]\!] = 0 \Rightarrow 1 \le x < 2$ and $0 \le y < 1$

Case (ii): $[\![x]\!] = -1$, $[\![y]\!] = 0 \Rightarrow -1 \le x < 0$ and $0 \le y < 1$

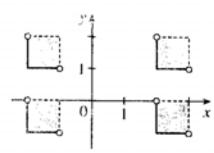
Case (iii): $[\![x]\!] = 0$, $[\![y]\!] = 1 \Rightarrow 0 \le x < 1$ and $1 \le y < 2$

Case (iv): $[\![x]\!] = 0$, $[\![y]\!] = -1 \Rightarrow 0 \le x < 1$ and $-1 \le y < 0$

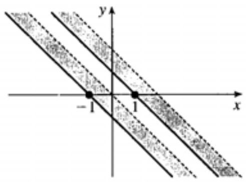


(b) $[\![x]\!]^2 - [\![y]\!]^2 = 3$. The only integral solution of $n^2 - m^2 = 3$ is $n = \pm 2$ and $m = \pm 1$. So the graph is

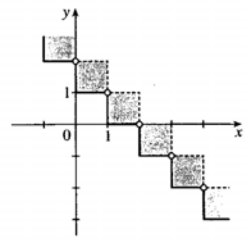
$$\{(x,y) \mid [\![x]\!] = \pm 2,\ [\![y]\!] = \pm 1\} = \left\{(x,y)\ \middle|\ \begin{array}{l} 2 \le x \le 3 \text{ or } -2 \le x < 1, \\ 1 \le y < 2 \text{ or } -1 \le y < 0 \end{array}\right\}.$$



(c) $[\![x+y]\!]^2 = 1 \;\Rightarrow\; [\![x+y]\!] = \pm 1$
$\Rightarrow\; 1 \le x+y < 2$ or
$-1 \le x+y < 0$

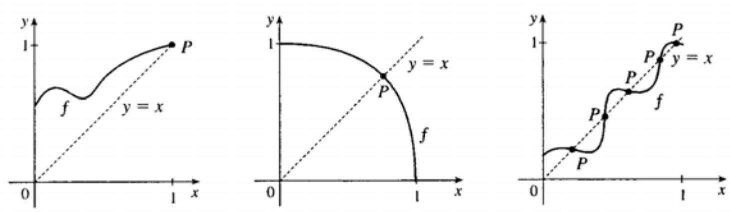


(d) For $n \le x < n+1$, $[\![x]\!] = n$. Then $[\![x]\!] + [\![y]\!] = 1 \;\Rightarrow$
$[\![y]\!] = 1 - n \;\Rightarrow\; 1 - n \le y < 2 - n$. Choosing integer values for n produces the graph.



7. f is continuous on $(-\infty, a)$ and (a, ∞). To make f continuous on $\mathbb{R}$, we must have continuity at a. Thus,
$\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) \;\Rightarrow\; \lim\limits_{x\to a^+} x^2 = \lim\limits_{x\to a^-} (x+1) \;\Rightarrow\; a^2 = a + 1 \;\Rightarrow\; a = \left(1 \pm \sqrt{5}\right)\big/ 2 \approx 1.618$
or -0.618.

8. (a) Here are a few possibilities:



(b) The "obstacle" is the line $x = y$ (see diagram). Any intersection of the graph of f with the line $y = x$ constitutes a fixed point, and if the graph of the function does not cross the line somewhere in $(0, 1)$, then it must either start at $(0, 0)$ (in which case 0 is a fixed point) or finish at $(1, 1)$ (in which case 1 is a fixed point).

(c) Consider the function $F(x) = f(x) - x$, where f is any continuous function with domain $[0, 1]$ and range in $[0, 1]$. We shall prove that f has a fixed point.
Now if $f(0) = 0$ then we are done: f has a fixed point (the number 0), which is what we are trying to prove. So assume $f(0) \ne 0$. For the same reason we can assume that $f(1) \ne 1$. Then $F(0) = f(0) > 0$ and $F(1) = f(1) - 1 < 0$. So by the Intermediate Value Theorem, there exists some number c in the interval $(0, 1)$ such that $F(c) = f(c) - c = 0$. So $f(c) = c$, and therefore f has a fixed point.

9.
$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}\left(\tfrac{1}{2}[f(x)+g(x)] + \tfrac{1}{2}[f(x)-g(x)]\right)\\ &= \tfrac{1}{2}\lim_{x\to a}[f(x)+g(x)] + \tfrac{1}{2}\lim_{x\to a}[f(x)-g(x)]\\ &= \tfrac{1}{2}\cdot 2 + \tfrac{1}{2}\cdot 1 = \tfrac{3}{2}, \text{ and}\end{aligned}$$
$\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a}([f(x)+g(x)] - f(x)) = \lim\limits_{x\to a}[f(x)+g(x)] - \lim\limits_{x\to a} f(x) = 2 - \tfrac{3}{2} = \tfrac{1}{2}$.
So $\lim\limits_{x\to a}[f(x)g(x)] = \left[\lim\limits_{x\to a} f(x)\right]\left[\lim\limits_{x\to a} g(x)\right] = \tfrac{3}{2}\cdot\tfrac{1}{2} = \tfrac{3}{4}$.

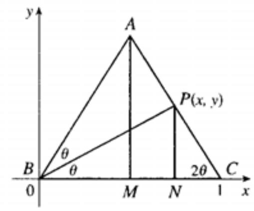
Another Solution: Since $\lim_{x\to a}[f(x)+g(x)]$ and $\lim_{x\to a}[f(x)-g(x)]$ exist, we must have

$\lim_{x\to a}[f(x)+g(x)]^2 = \left(\lim_{x\to a}[f(x)+g(x)]\right)^2$ and $\lim_{x\to a}[f(x)-g(x)]^2 = \left(\lim_{x\to a}[f(x)-g(x)]\right)^2$, so

$$\lim_{x\to a}[f(x)\,g(x)] = \lim_{x\to a}\tfrac{1}{4}\left([f(x)+g(x)]^2 - [f(x)-g(x)]^2\right) \text{ (because all of the } f^2 \text{ and } g^2 \text{ cancel)}$$

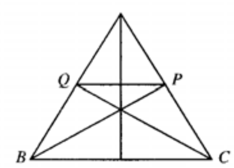
$$= \tfrac{1}{4}\left(\lim_{x\to a}[f(x)+g(x)]^2 - \lim_{x\to a}[f(x)-g(x)]^2\right) = \tfrac{1}{4}\left(2^2-1^2\right) = \tfrac{3}{4}.$$

10. (a) *Solution 1:* We introduce a coordinate system and drop a perpendicular from P, as shown. We see from $\angle NCP$ that $\tan 2\theta = \dfrac{y}{1-x}$, and from $\angle NBP$ that $\tan\theta = y/x$. Using the double-angle formula for tangents, we get $\dfrac{y}{1-x} = \tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{2(y/x)}{1-(y/x)^2}$. After a bit of simplification, this becomes $\dfrac{1}{1-x} = \dfrac{2x}{x^2-y^2} \Leftrightarrow$

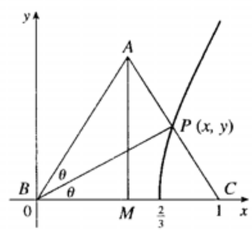


$y^2 = x(3x-2)$. As the altitude AM decreases in length, the point P will approach the x-axis, that is, $y \to 0$, so the limiting location of P must be one of the roots of the equation $x(3x-2)=0$. Obviously it is not $x=0$ (the point P can never be to the left of the altitude AM, which it would have to be in order to approach 0) so it must be $3x-2=0$, that is, $x=\frac{2}{3}$.

Solution 2: We add a few lines to the original diagram, as shown. Now note that $\angle BPQ = \angle PBC$ (alternate angles; $QP \parallel BC$ by symmetry) and similarly $\angle CQP = \angle QCB$. So ΔBPQ and ΔCQP are isosceles, and the line segments BQ, QP and PC are all of equal length. As $|AM| \Rightarrow 0$, P and Q approach points on the base, and the point P is seen to approach a position two-thirds of the way between B and C, as above.



(b) The equation $y^2 = x(3x-2)$ calculated in part (a) is the equation of the curve traced out by P. Now as $|AM| \to \infty$, $2\theta \to \frac{\pi}{2}$, $\theta \to \frac{\pi}{4}$, $x \to 1$, and since $\tan\theta = y/x$, $y \to 1$. Thus, P only traces out the part of the curve with $0 \le y < 1$.



11. (a) Consider $G(x) = T(x+180°) - T(x)$. Fix any number a. If $G(a)=0$, we are done: Temperature at a = Temperature at $a+180°$. If $G(a)>0$, then $G(a+180°) = T(a+360°) - T(a+180°) = T(a) - T(a+180°) = -G(a) < 0$. Also, G is continuous since temperature varies continuously. So, by the Intermediate Value Theorem, G has a zero on the interval $[a, a+180°]$. If $G(a)<0$, then a similar argument applies.

(b) Yes. The same argument applies.

(c) The same argument applies for quantities that vary continuously, such as barometric pressure. But one could argue that altitude above sea level is sometimes discontinuous, so the result might not always hold for that quantity.

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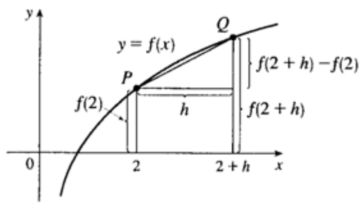
3 Derivatives

3.1 Derivatives

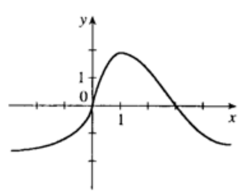
1. 

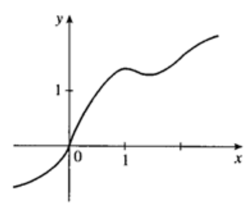
The line from P to Q is the line that has slope $\dfrac{f(2+h)-f(2)}{h}$.

2. As h decreases, the line PQ becomes steeper, so its slope increases. So $0 < \dfrac{f(4)-f(2)}{4-2} < \dfrac{f(3)-f(2)}{3-2} < \lim\limits_{x\to 2}\dfrac{f(x)-f(2)}{x-2}$. Thus, $0 < \frac{1}{2}[f(4)-f(2)] < f(3)-f(2) < f'(2)$.

3. $g'(0)$ is the only negative value. The slope at $x = 4$ is smaller than the slope at $x = 2$ and both are smaller than the slope at $x = -2$. Thus, $g'(0) < 0 < g'(4) < g'(2) < g'(-2)$.

4. Since $(4, 3)$ is on $y = f(x)$, $f(4) = 3$. The slope of the tangent line between $(0, 2)$ and $(4, 3)$ is $\frac{1}{4}$, so $f'(4) = \frac{1}{4}$.

5. 

6. 

7. $f'(2) = \lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h} = \lim\limits_{h\to 0}\dfrac{\left[3(2+h)^2 - 5(2+h)\right] - \left[3(2)^2 - 5(2)\right]}{h}$

$= \lim\limits_{h\to 0}\dfrac{\left(12 + 12h + 3h^2 - 10 - 5h\right) - (2)}{h} = \lim\limits_{h\to 0}\dfrac{3h^2 + 7h}{h} = \lim\limits_{h\to 0}(3h + 7) = 7$

So an equation of the tangent line at $(2, 2)$ is $y - 2 = 7(x - 2)$ or $y = 7x - 12$.

8. $g'(0) = \lim\limits_{h\to 0}\dfrac{g(0+h)-g(0)}{h} = \lim\limits_{h\to 0}\dfrac{\left[1-(0+h)^3\right] - \left(1 - 0^3\right)}{h} = \lim\limits_{h\to 0}\dfrac{\left(1-h^3\right)-1}{h}$

$= \lim\limits_{h\to 0}\left(-h^2\right) = 0$. So an equation of the tangent line is $y - 1 = 0(x - 0)$ or $y = 1$.

9. (a) $F'(1) = \lim\limits_{x \to 1} \dfrac{F(x) - F(1)}{x - 1} = \lim\limits_{x \to 1} \dfrac{\left(x^3 - 5x + 1\right) - (-3)}{x - 1} = \lim\limits_{x \to 1} \dfrac{x^3 - 5x + 4}{x - 1}$

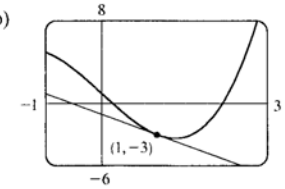
$= \lim\limits_{x \to 1} \dfrac{(x - 1)\left(x^2 + x - 4\right)}{x - 1} = \lim\limits_{x \to 1} \left(x^2 + x - 4\right) = -2$

So an equation of the tangent line at $(1, -3)$ is

$y - (-3) = -2(x - 1) \quad \Leftrightarrow \quad y = -2x - 1.$

Note: Instead of using Equation 3 to compute $F'(1)$, we could have used Equation 1.

(b)



10. (a) $G'(a) = \lim\limits_{h \to 0} \dfrac{G(a+h) - G(a)}{h} = \lim\limits_{h \to 0} \dfrac{\dfrac{a+h}{1 + 2(a+h)} - \dfrac{a}{1 + 2a}}{h}$

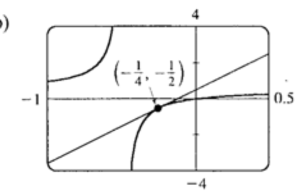
$= \lim\limits_{h \to 0} \dfrac{a + 2a^2 + h + 2ah - a - 2a^2 - 2ah}{h(1 + 2a + 2h)(1 + 2a)} = \lim\limits_{h \to 0} \dfrac{1}{(1 + 2a + 2h)(1 + 2a)} = (1 + 2a)^{-2}$

So the slope of the tangent at the point $\left(-\frac{1}{4}, -\frac{1}{2}\right)$ is

$m = \left[1 + 2\left(-\frac{1}{4}\right)\right]^{-2} = 4$, and thus an equation is

$y + \frac{1}{2} = 4\left(x + \frac{1}{4}\right)$ or $y = 4x + \frac{1}{2}$.

(b)



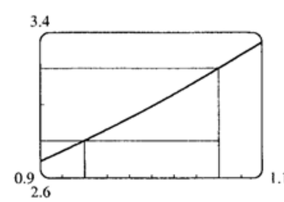
11. (a) $f'(1) = \lim\limits_{h \to 0} \dfrac{f(1+h) - f(1)}{h} = \lim\limits_{h \to 0} \dfrac{3^{1+h} - 3^1}{h}$.

So let $F(h) = \dfrac{3^{1+h} - 3}{h}$. We calculate:

h	$F(h)$
0.1	3.484
0.01	3.314
0.001	3.298
0.0001	3.296
−0.1	3.121
−0.01	3.278
−0.001	3.294
−0.0001	3.296

We estimate that $f'(1) \approx 3.296$.

(b)



From the graph, we estimate that the slope of the tangent is about

$\dfrac{3.2 - 2.8}{1.06 - 0.94} = \dfrac{0.4}{0.12} \approx 3.3.$

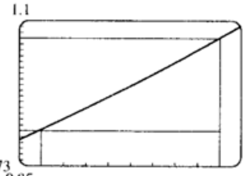
12. (a) $g'\left(\frac{\pi}{4}\right) = \lim\limits_{h\to 0} \dfrac{g\left(\frac{\pi}{4}+h\right) - g\left(\frac{\pi}{4}\right)}{h} = \lim\limits_{h\to 0} \dfrac{\tan\left(\frac{\pi}{4}+h\right) - \tan\left(\frac{\pi}{4}\right)}{h}$.

So let $G(h) = \dfrac{\tan\left(\frac{\pi}{4}+h\right) - 1}{h}$. We calculate:

h	$G(h)$
0.1	2.2305
0.01	2.0203
0.001	2.0020
0.0001	2.0002
−0.1	1.8237
−0.01	1.9803
−0.001	1.9980
−0.0001	1.9998

We estimate that $g'\left(\frac{\pi}{4}\right) = 2$.

(b)



From the graph, we estimate that the slope of the tangent is about $\dfrac{1.07 - 0.91}{0.82 - 0.74} = \dfrac{0.16}{0.08} = 2$.

13. $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h} = \lim\limits_{h\to 0} \dfrac{\left[1 + (a+h) - 2(a+h)^2\right] - \left(1 + a - 2a^2\right)}{h}$

$= \lim\limits_{h\to 0} \dfrac{h - 4ah - 2h^2}{h} = \lim\limits_{h\to 0} (1 - 4a - 2h) = 1 - 4a$

14. $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h} = \lim\limits_{h\to 0} \dfrac{\left[(a+h)^3 + 3(a+h)\right] - \left(a^3 + 3a\right)}{h}$

$= \lim\limits_{h\to 0} \dfrac{3a^2h + 3ah^2 + h^3 + 3h}{h} = \lim\limits_{h\to 0} \left(3a^2 + 3ah + h^2 + 3\right) = 3a^2 + 3$

15. $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{a+h}{2(a+h)-1} - \dfrac{a}{2a-1}}{h}$

$= \lim\limits_{h\to 0} \dfrac{(a+h)(2a-1) - a(2a+2h-1)}{h(2a+2h-1)(2a-1)} = \lim\limits_{h\to 0} \dfrac{-h}{h(2a+2h-1)(2a-1)}$

$= \lim\limits_{h\to 0} \dfrac{-1}{(2a+2h-1)(2a-1)} = -\dfrac{1}{(2a-1)^2}$

16. $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{a+h}{(a+h)^2-1} - \dfrac{a}{a^2-1}}{h}$

$= \lim\limits_{h\to 0} \dfrac{(a+h)\left(a^2-1\right) - a\left(a^2+2ah+h^2-1\right)}{h\left(a^2-1\right)\left(a^2+2ah+h^2-1\right)} = \lim\limits_{h\to 0} \dfrac{h\left(-a^2-1-ah\right)}{h\left(a^2-1\right)\left(a^2+2ah+h^2-1\right)}$

$= \lim\limits_{h\to 0} \dfrac{-a^2-1-ah}{\left(a^2-1\right)\left(a^2+2ah+h^2-1\right)} = \dfrac{-a^2-1}{\left(a^2-1\right)\left(a^2-1\right)} = -\dfrac{a^2+1}{\left(a^2-1\right)^2}$

17. $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{2}{\sqrt{3-(a+h)}} - \dfrac{2}{\sqrt{3-a}}}{h} = \lim\limits_{h\to 0} \dfrac{2\left(\sqrt{3-a} - \sqrt{3-a-h}\right)}{h\sqrt{3-a-h}\sqrt{3-a}}$

$= \lim\limits_{h\to 0} \dfrac{2\left(\sqrt{3-a} - \sqrt{3-a-h}\right)}{h\sqrt{3-a-h}\sqrt{3-a}} \cdot \dfrac{\sqrt{3-a} + \sqrt{3-a-h}}{\sqrt{3-a} + \sqrt{3-a-h}}$

$= \lim\limits_{h\to 0} \dfrac{2\,[3-a-(3-a-h)]}{h\sqrt{3-a-h}\sqrt{3-a}\left(\sqrt{3-a} + \sqrt{3-a-h}\right)}$

$= \lim\limits_{h\to 0} \dfrac{2}{\sqrt{3-a-h}\sqrt{3-a}\left(\sqrt{3-a} + \sqrt{3-a-h}\right)}$

$= \dfrac{2}{\sqrt{3-a}\sqrt{3-a}\left(2\sqrt{3-a}\right)} = \dfrac{1}{(3-a)^{3/2}}$

18. $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt{3(a+h)+1} - \sqrt{3a+1}}{h}$

$= \lim\limits_{h\to 0} \dfrac{\left(\sqrt{3a+3h+1} - \sqrt{3a+1}\right)\left(\sqrt{3a+3h+1} + \sqrt{3a+1}\right)}{h\left(\sqrt{3a+3h+1} + \sqrt{3a+1}\right)}$

$= \lim\limits_{h\to 0} \dfrac{(3a+3h+1) - (3a+1)}{h\left(\sqrt{3a+3h+1} + \sqrt{3a+1}\right)}$

$= \lim\limits_{h\to 0} \dfrac{3}{\sqrt{3a+3h+1} + \sqrt{3a+1}} = \dfrac{3}{2\sqrt{3a+1}}$

Note that the answers to Exercises 19–24 are not unique.

19. By Equation 1, $\lim\limits_{h\to 0} \dfrac{\sqrt{1+h} - 1}{h} = f'(1)$, where $f(x) = \sqrt{x}$.

Or: $f'(0)$, where $f(x) = \sqrt{1+x}$

20. By Equation 1, $\lim\limits_{h\to 0} \dfrac{(2+h)^3 - 8}{h} = f'(2)$, where $f(x) = x^3$.

21. By Equation 3, $\lim\limits_{x\to 1} \dfrac{x^9 - 1}{x - 1} = f'(1)$, where $f(x) = x^9$.

22. By Equation 3, $\lim\limits_{x\to 3\pi} \dfrac{\cos x + 1}{x - 3\pi} = f'(3\pi)$, where $f(x) = \cos x$.

23. By Equation 1, $\lim\limits_{t\to 0} \dfrac{\sin\left(\frac{\pi}{2} + t\right) - 1}{t} = f'\left(\frac{\pi}{2}\right)$, where $f(x) = \sin x$.

24. By Equation 3, $\lim\limits_{x\to 0} \dfrac{3^x - 1}{x} = f'(0)$, where $f(x) = 3^x$.

25. $v(2) = f'(2) = \lim\limits_{h\to 0} \dfrac{f(2+h) - f(2)}{h} = \lim\limits_{h\to 0} \dfrac{\left[(2+h)^2 - 6(2+h) - 5\right] - \left[2^2 - 6(2) - 5\right]}{h}$

$= \lim\limits_{h\to 0} \dfrac{(4 + 4h + h^2 - 12 - 6h - 5) - (-13)}{h} = \lim\limits_{h\to 0} \dfrac{h^2 - 2h}{h} = \lim\limits_{h\to 0} (h-2) = -2$ m/s

26. $v(2) = f'(2) = \lim\limits_{h\to 0} \dfrac{f(2+h) - f(2)}{h} = \lim\limits_{h\to 0} \dfrac{\left[2(2+h)^3 - (2+h) + 1\right] - \left[2(2)^3 - 2 + 1\right]}{h}$

$= \lim\limits_{h\to 0} \dfrac{\left(2h^3 + 12h^2 + 24h + 16 - 2 - h + 1\right) - 15}{h} = \lim\limits_{h\to 0} \dfrac{2h^3 + 12h^2 + 23h}{h}$

$= \lim\limits_{h\to 0} \left(2h^2 + 12h + 23\right) = 23$ m/s

27. (a) $f'(x)$ is the rate of change of the production cost with respect to the number of ounces of gold produced. Its units are dollars per ounce.

(b) After 800 ounces of gold have been produced, the rate at which the production cost is increasing is \$17/ounce. So the cost of producing the 800th (or 801st) ounce is about \$17.

(c) In the short term, the values of $f'(x)$ will decrease because more efficient use is made of start-up costs as x increases. But eventually $f'(x)$ might increase due to large-scale operations.

28. (a) $f'(5)$ is the rate of growth of the bacteria population when $t = 5$ hours. Its units are bacteria per hour.

(b) With unlimited space and nutrients, f' should increase as t increases; so $f'(5) < f'(10)$. If the supply of nutrients is limited, the growth rate slows down at some point in time, and the opposite may be true.

29. (a) $f'(v)$ is the rate at which the fuel consumption is changing with respect to the speed. Its units are $(\text{gal/h})/(\text{mi/h})$.

(b) The fuel consumption is decreasing by $0.05\ (\text{gal/h})/(\text{mi/h})$ as the car's speed reaches 20 mi/h. So if you increase your speed to 21 mi/h, you could expect to decrease your fuel consumption by about $0.05\ (\text{gal/h})/(\text{mi/h})$.

30. (a) $f'(8)$ is the rate of change of the quantity of coffee sold with respect to the price per pound when the price is \$8 per pound. The units for $f'(8)$ are pounds/(dollars/pound).

(b) $f'(8)$ is negative since the quantity of coffee sold will decrease as the price charged for it increases. People are generally less willing to buy a product when its price increases.

31. $C'(1980)$ is the rate of change of U.S. cash per capita in circulation with respect to time. To estimate the value of $C'(1980)$, we will average the difference quotients obtained using the years 1970 and 1990.
Let $A = \dfrac{C(1970) - C(1980)}{1970 - 1980} = \dfrac{265 - 571}{-10} = 30.6$ and $B = \dfrac{C(1990) - C(1980)}{1990 - 1980} = \dfrac{1063 - 571}{10} = 49.2$.
Then $C'(1980) = \displaystyle\lim_{t \to 1980} \frac{C(t) - C(1980)}{t - 1980} \approx \frac{A + B}{2} = 39.9$ dollars per year.

32. **For 1910:** We will average the difference quotients obtained using the years 1900 and 1920.
Let $A = \dfrac{E(1900) - E(1910)}{1900 - 1910} = \dfrac{48.3 - 51.1}{-10} = 0.28$ and
$B = \dfrac{E(1920) - E(1910)}{1920 - 1910} = \dfrac{55.2 - 51.1}{10} = 0.41$. Then
$E'(1910) = \displaystyle\lim_{t \to 1910} \frac{E(t) - E(1910)}{t - 1910} \approx \frac{A + B}{2} = 0.345$. This means that life expectancy at birth was increasing at about 0.345 years/year in 1910.
For 1950: Using data for 1940 and 1960 in a similar fashion, we obtain $E'(1950) \approx [0.31 + 0.10]/2 = 0.205$. So life expectancy at birth was increasing at about 0.205 years/year in 1950.

33. Since $f(x) = x \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have
$f'(0) = \displaystyle\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{h \sin(1/h) - 0}{h} = \lim_{h \to 0} \sin(1/h)$. This limit does not exist since $\sin(1/h)$ takes the values -1 and 1 on any interval containing 0. (Compare with Example 4 in Section 2.2.)

34. Since $f(x) = x^2 \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

$f'(0) = \lim_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \dfrac{h^2 \sin(1/h) - 0}{h} = \lim_{h\to 0} h\sin(1/h)$. Since $-1 \leq \sin\dfrac{1}{h} \leq 1$, we have

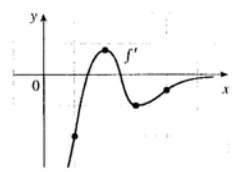
$-|h| \leq h\sin\dfrac{1}{h} \leq |h|$. Because $\lim_{h\to 0}(-|h|) = 0$ and $\lim_{h\to 0}|h| = 0$, we know that $\lim_{h\to 0}\left(h\sin\dfrac{1}{h}\right) = 0$ by the Squeeze Theorem. Thus, $f'(0) = 0$.

3.2 The Derivative as a Function

1. *Note:* Your answers may vary depending on your estimates. From the graph of f, it appears that

(a) $f'(1) \approx -2$ (b) $f'(2) \approx 0.8$

(c) $f'(3) \approx -1$ (d) $f'(4) \approx -0.5$

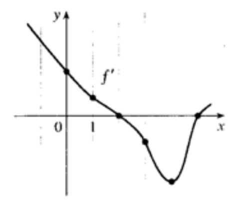


2. From the graph of f, it appears that

(a) $f'(0) \approx 1.8$ (b) $f'(1) \approx 0.8$

(c) $f'(2) \approx 0$ (d) $f'(3) \approx -1$

(e) $f'(4) \approx -2.5$ (f) $f'(5) \approx 0$



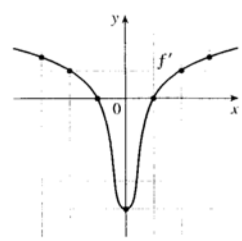
3. It appears that f is an odd function, so f' will be an even function — that is, $f'(-a) = f'(a)$.

(a) $f'(-3) \approx 1.5$ (b) $f'(-2) \approx 1$

(c) $f'(-1) \approx 0$ (d) $f'(0) \approx -4$

(e) $f'(1) \approx 0$ (f) $f'(2) \approx 1$

(g) $f'(3) \approx 1.5$



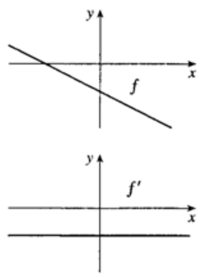
4. (a)$'$ = II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0, then positive, then 0, then negative again. The actual function values in graph II follow the same pattern.

(b)$'$ = IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.

(c)$'$ = I, since the slopes of the tangents to graph (c) are negative for $x < 0$ and positive for $x > 0$, as are the function values of graph I.

(d)$'$ = III, since from left to right, the slopes of the tangents to graph (d) are positive, then 0, then negative, then 0, then positive, then 0, then negative again, and the function values in graph III follow the same pattern.

5.



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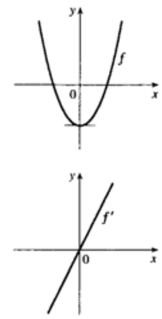

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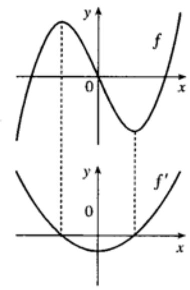

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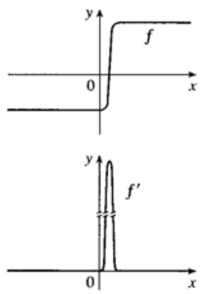

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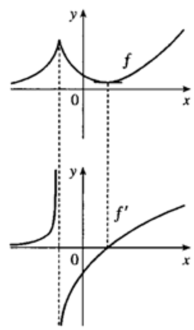

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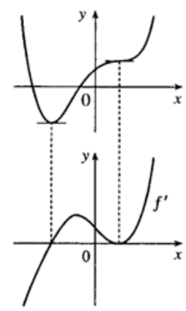

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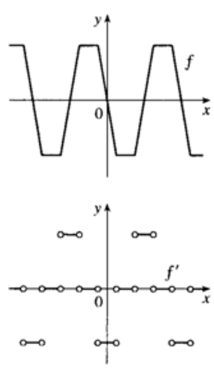

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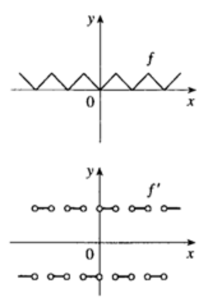
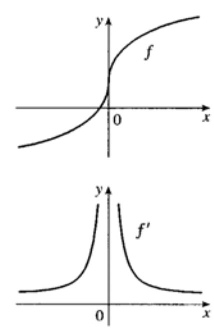
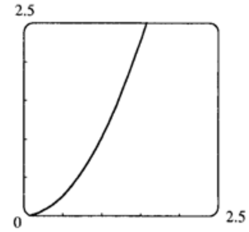


13.



14. See Figure 1 in Section 3.5.

15. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) = 1$, $f'(1) = 2$, and $f'(2) = 4$.



(b) By symmetry, $f'(-x) = -f'(x)$. So $f'\left(-\frac{1}{2}\right) = -1$, $f'(-1) = -2$, and $f'(-2) = -4$.

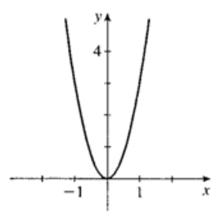
(c) It appears that $f'(x)$ is twice the value of x, so we guess that $f'(x) = 2x$.

(d) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{(x+h)^2 - x^2}{h} = \lim\limits_{h\to 0} \dfrac{\left(x^2 + 2hx + h^2\right) - x^2}{h}$

$$= \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0} (2x+h) = 2x$$

16. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) \approx 0.75$, $f'(1) \approx 3$, $f'(2) \approx 12$, and $f'(3) \approx 27$.

(c)



(b) By symmetry, $f'(-x) = f'(x)$. So $f'\left(-\frac{1}{2}\right) \approx 0.75$, $f'(-1) \approx 3$, $f'(-2) \approx 12$, and $f'(-3) \approx 27$.

(d) Since $f'(0) = 0$, it appears that f' may have the form $f'(x) = ax^2$. Using $f'(1) = 3$, we have $a = 3$, so $f'(x) = 3x^2$.

(e) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{(x+h)^3 - x^3}{h} = \lim\limits_{h\to 0} \dfrac{\left(x^3 + 3x^2h + 3xh^2 + h^3\right) - x^3}{h}$

$$= \lim_{h\to 0} \frac{h\left(3x^2 + 3xh + h^2\right)}{h} = \lim_{h\to 0} \left(3x^2 + 3xh + h^2\right) = 3x^2$$

17. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{[5(x+h)+3] - (5x+3)}{h} = \lim\limits_{h\to 0} \dfrac{5h}{h} = \lim\limits_{h\to 0} 5 = 5$

Domain of f = domain of $f' = \mathbb{R}$.

18. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\left[5 - 4(x+h) + 3(x+h)^2\right] - \left[5 - 4x + 3x^2\right]}{h}$

$$= \lim_{h\to 0} \frac{\left[5 - 4x - 4h + 3x^2 + 6xh + 3h^2\right] - \left[5 - 4x + 3x^2\right]}{h}$$

$$= \lim_{h\to 0} \frac{-4h + 6xh + 3h^2}{h} = \lim_{h\to 0} (-4 + 6x + 3h) = -4 + 6x$$

Domain of f = domain of $f' = \mathbb{R}$.

19. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\left[(x+h)^3 - (x+h)^2 + 2(x+h)\right] - \left(x^3 - x^2 + 2x\right)}{h}$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 2xh - h^2 + 2h}{h} = \lim_{h\to 0} \left(3x^2 + 3xh + h^2 - 2x - h + 2\right)$$

$$= 3x^2 - 2x + 2$$

Domain of f = domain of $f' = \mathbb{R}$.

20. $f'(x) = \lim\limits_{h\to0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to0} \dfrac{\left(x+h+\sqrt{x+h}\right)-\left(x+\sqrt{x}\right)}{h}$

$= \lim\limits_{h\to0}\left(\dfrac{h}{h} + \dfrac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$

$= \lim\limits_{h\to0}\left[1 + \dfrac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\right] = \lim\limits_{h\to0}\left(1+\dfrac{1}{\sqrt{x+h}+\sqrt{x}}\right)$

$= 1 + \dfrac{1}{\sqrt{x}+\sqrt{x}} = 1 + \dfrac{1}{2\sqrt{x}}$

Domain of $f = [0,\infty)$, domain of $f' = (0,\infty)$.

21. $g'(x) = \lim\limits_{h\to0} \dfrac{g(x+h)-g(x)}{h} = \lim\limits_{h\to0} \dfrac{\sqrt{1+2(x+h)}-\sqrt{1+2x}}{h}\left[\dfrac{\sqrt{1+2(x+h)}+\sqrt{1+2x}}{\sqrt{1+2(x+h)}+\sqrt{1+2x}}\right]$

$= \lim\limits_{h\to0} \dfrac{(1+2x+2h)-(1+2x)}{h\left[\sqrt{1+2(x+h)}+\sqrt{1+2x}\right]} = \lim\limits_{h\to0}\dfrac{2}{\sqrt{1+2(x+h)}+\sqrt{1+2x}} = \dfrac{1}{\sqrt{1+2x}}$

Domain of $g = \left[-\frac{1}{2},\infty\right)$, domain of $g' = \left(-\frac{1}{2},\infty\right)$.

22. $f'(x) = \lim\limits_{h\to0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to0} \dfrac{\dfrac{x+h+1}{x+h-1}-\dfrac{x+1}{x-1}}{h}$

$= \lim\limits_{h\to0}\dfrac{(x+h+1)(x-1)-(x+1)(x+h-1)}{h(x+h-1)(x-1)} = \lim\limits_{h\to0}\dfrac{-2h}{h(x+h-1)(x-1)}$

$= \lim\limits_{h\to0}\dfrac{-2}{(x+h-1)(x-1)} = \dfrac{-2}{(x-1)^2}$

Domain of f = domain of $f' = \{x \mid x \neq 1\}$.

23. $G'(x) = \lim\limits_{h\to0} \dfrac{G(x+h)-G(x)}{h} = \lim\limits_{h\to0} \dfrac{\dfrac{4-3(x+h)}{2+(x+h)}-\dfrac{4-3x}{2+x}}{h}$

$= \lim\limits_{h\to0}\dfrac{(4-3x-3h)(2+x)-(4-3x)(2+x+h)}{h(2+x+h)(2+x)} = \lim\limits_{h\to0}\dfrac{-10h}{h(2+x+h)(2+x)}$

$= \lim\limits_{h\to0}\dfrac{-10}{(2+x+h)(2+x)} = \dfrac{-10}{(2+x)^2}$

Domain of G = domain of $G' = \{x \mid x \neq -2\}$.

24. $g'(x) = \lim\limits_{h\to0} \dfrac{g(x+h)-g(x)}{h} = \lim\limits_{h\to0} \dfrac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h} = \lim\limits_{h\to0}\dfrac{x^2-(x+h)^2}{h(x+h)^2x^2}$

$= \lim\limits_{h\to0}\dfrac{x^2-\left(x^2+2xh+h^2\right)}{h} = \lim\limits_{h\to0}\dfrac{-2xh-h^2}{h(x+h)^2x^2} = \lim\limits_{h\to0}\dfrac{-2x-h}{(x+h)^2x^2} = \dfrac{-2x}{x^4}$

$= -2x^{-3}$

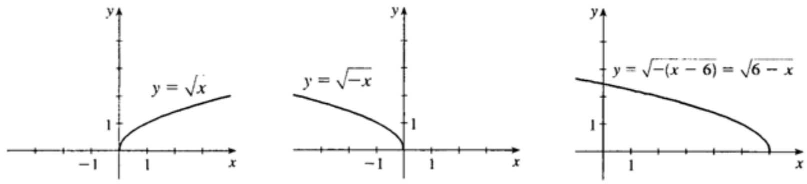
Domain of g = domain of $g' = \{x \mid x \neq 0\}$.

25. $f'(x) = \lim\limits_{h\to0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to0}\dfrac{(x+h)^4-x^4}{h} = \lim\limits_{h\to0}\dfrac{4x^3h+6x^2h^2+4xh^3+h^4}{h}$

$= \lim\limits_{h\to0}\left(4x^3+6x^2h+4xh^2+h^3\right) = 4x^3$

Domain of f = domain of $f' = \mathbb{R}$.

26. (a)

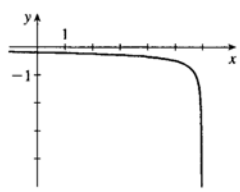


(b) Note that the third graph in part (a) has small negative values for its slope, f'; but as $x \to 6^-$, $f' \to -\infty$.

(c) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{6-(x+h)} - \sqrt{6-x}}{h}\left[\frac{\sqrt{6-(x+h)} + \sqrt{6-x}}{\sqrt{6-(x+h)} + \sqrt{6-x}}\right]$$
$$= \lim_{h\to 0} \frac{[6-(x+h)] - (6-x)}{h\left[\sqrt{6-(x+h)} + \sqrt{6-x}\right]} = \lim_{h\to 0} \frac{-h}{h\left(\sqrt{6-x-h} + \sqrt{6-x}\right)}$$
$$= \lim_{h\to 0} \frac{-1}{\sqrt{6-x-h} + \sqrt{6-x}} = \frac{-1}{2\sqrt{6-x}}$$

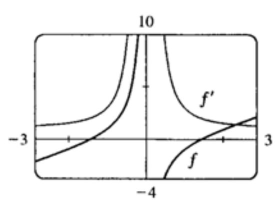
Domain of $f = (-\infty, 6]$, domain of $f' = (-\infty, 6)$.

(d)



27. (a) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left[x + h - \left(\frac{2}{x+h}\right)\right] - \left[x - \left(\frac{2}{x}\right)\right]}{h}$$
$$= \lim_{h\to 0} \left[\frac{h + \frac{2}{x} - \frac{2}{(x+h)}}{h}\right] = \lim_{h\to 0} \left[1 + \frac{2(x+h) - 2x}{(h)(x)(x+h)}\right] = \lim_{h\to 0} \left[1 + \frac{2}{x(x+h)}\right]$$
$$= 1 + 2x^{-2}$$

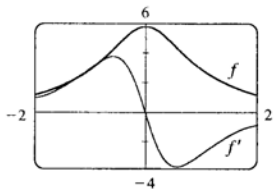
(b) Notice that when f has steep tangent lines, $f'(x)$ is very large. When f is flatter, $f'(x)$ is smaller.



28. (a) $f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h\to 0} \frac{\frac{6}{1+(t+h)^2} - \frac{6}{1+t^2}}{h} = \lim_{h\to 0} \frac{6 + 6t^2 - 6 - 6(t+h)^2}{h\left[1+(t+h)^2\right](1+t^2)}$

$= \lim_{h\to 0} \frac{-12th - 6h^2}{h\left[1+(t+h)^2\right](1+t^2)} = \lim_{h\to 0} \frac{-12t - 6h}{\left[1+(t+h)^2\right](1+t^2)} = \frac{-12t}{(1+t^2)^2}$

(b) Notice that f has a horizontal tangent when $t = 0$. This corresponds to $f'(0) = 0$. f' is positive when f is increasing and negative when f is decreasing.



29. (a) $U'(t)$ is the rate at which the unemployment rate is changing with respect to time. Its units are percent per year.

(b) To find $U'(t)$, we use $\lim_{h\to 0} \frac{U(t+h) - U(t)}{h} \approx \frac{U(t+h) - U(t)}{h}$ for small values of h.

For 1988: $U'(1988) = \frac{U(1989) - U(1988)}{1989 - 1988} = \frac{5.3 - 5.5}{1} = -0.20$

For 1989: We estimate $U'(1989)$ both using $h = -1$ and using $h = 1$, and then average the two results to obtain a final estimate.

$h = -1 \;\Rightarrow\; U'(1989) \approx \frac{U(1988) - U(1989)}{1988 - 1989} = \frac{5.5 - 5.3}{-1} = -0.20;$

$h = 1 \;\Rightarrow\; U'(1989) \approx \frac{U(1990) - U(1989)}{1990 - 1989} = \frac{5.6 - 5.3}{1} = 0.30.$

So we estimate that $U'(1989) \approx \frac{1}{2}(-0.20 + 0.30) = 0.05$.

t	1988	1989	1990	1991	1992	1993	1994	1995	1996	1997
$U'(t)$	−0.20	0.05	0.75	0.95	0.05	−0.70	−0.65	−0.35	−0.35	−0.50

30. (a) $S'(t)$ is the rate at which the smoking rate is changing with respect to time. Its units are percent per year.

(b) To find $S'(t)$, we use $\lim_{h\to 0} \frac{S(t+h) - S(t)}{h} \approx \frac{S(t+h) - S(t)}{h}$ for small values of h.

For 1978: $S'(1978) \approx \frac{S(1980) - S(1978)}{1980 - 1978} = \frac{21.4 - 27.5}{2} = -3.05$

For 1980: We estimate $S'(1980)$ both using $h = -2$ and using $h = 2$, and then average the two results to obtain a final estimate.

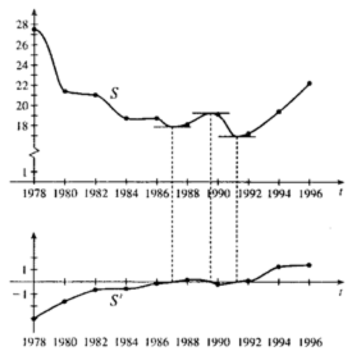
$h = -2 \;\Rightarrow\; S'(1980) = \frac{S(1978) - S(1980)}{1978 - 1980} = \frac{27.5 - 21.4}{-2} = -3.05$

$h = 2 \;\Rightarrow\; S'(1980) = \frac{S(1982) - S(1980)}{1982 - 1980} = \frac{21.0 - 21.4}{2} = -0.20$

So we estimate that $S'(1980) \approx \frac{1}{2}(-3.05 - 0.20) = -1.625$.

t	1978	1980	1982	1984	1986	1988	1990	1992	1994	1996
$S'(t)$	−3.05	−1.625	−0.675	−0.575	−0.15	0.10	−0.225	0.075	1.25	1.40

(c)



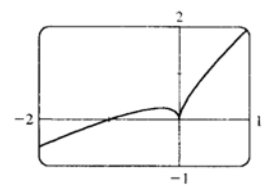
(d) We could get more accurate values for $S'(t)$ by obtaining data for the odd-numbered years.

31. f is not differentiable at $x = -1$ or at $x = 11$ because the graph has vertical tangents at those points; at $x = 4$, because there is a discontinuity there; and at $x = 8$, because the graph has a corner there.

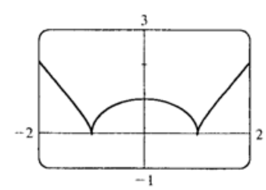
32. (a) g is discontinuous at $x = -2$ (a removable discontinuity), at $x = 0$ (g is not defined there), and at $x = 5$ (a jump discontinuity).

(b) g is not differentiable at the above points (by Theorem 4), and also at $x = -1$ (corner), at $x = 2$ (vertical tangent), and at $x = 4$ (vertical tangent).

33. As we zoom in toward $(-1, 0)$, the curve appears more and more like a straight line, so f is differentiable at $x = -1$. But no matter how much we zoom in toward the origin, the curve doesn't straighten out — we can't eliminate the sharp point (a cusp). So f is not differentiable at $x = 0$.



34. As we zoom in toward $(0, 1)$, the curve appears more and more like a straight line, so f is differentiable at $x = 0$. But no matter how much we zoom in toward $(1, 0)$ or $(-1, 0)$, the curve doesn't straighten out — we can't eliminate the sharp point (a cusp). So f is not differentiable at $x = \pm 1$.



35. (a) $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{x - a} = \lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{\left(x^{1/3} - a^{1/3}\right)\left(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}\right)}$$
$$= \lim_{x \to a} \frac{1}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} = \lim_{x \to a} \frac{1}{3x^{2/3}} = \frac{1}{3a^{2/3}}$$

(b) $f'(0) = \lim\limits_{h \to 0} \dfrac{f(0 + h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{\sqrt[3]{h} - 0}{h} = \lim\limits_{h \to 0} \dfrac{1}{h^{2/3}}$. This limit does not exist, and therefore $f'(0)$ does not exist.

(c) $\lim\limits_{x \to 0} \left|f'(x)\right| = \lim\limits_{x \to 0} \dfrac{1}{3x^{2/3}} = \infty$ and f is continuous at $x = 0$ (root function), so f has a vertical tangent at $x = 0$.

36. (a) $g'(0) = \lim\limits_{x\to 0} \dfrac{g(x) - g(0)}{x - 0} = \lim\limits_{x\to 0} \dfrac{x^{2/3} - 0}{x} = \lim\limits_{x\to 0} \dfrac{1}{x^{1/3}}$, which does not exist.

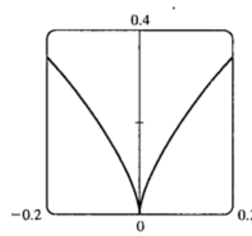
(b) $g'(a) = \lim\limits_{x\to a} \dfrac{g(x) - g(a)}{x - a} = \lim\limits_{x\to a} \dfrac{x^{2/3} - a^{2/3}}{x - a} = \lim\limits_{x\to a} \dfrac{\left(x^{1/3} - a^{1/3}\right)\left(x^{1/3} + a^{1/3}\right)}{\left(x^{1/3} - a^{1/3}\right)\left(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}\right)}$

$= \lim\limits_{x\to a} \dfrac{x^{1/3} + a^{1/3}}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} = \dfrac{2a^{1/3}}{3a^{2/3}} = \dfrac{2}{3a^{1/3}}$

(c) $g(x) = x^{2/3}$ is continuous at $x = 0$ and

$\lim\limits_{x\to 0} |g'(x)| = \lim\limits_{x\to 0} \dfrac{2}{3|x|^{1/3}} = \infty$. This shows that g has a vertical tangent line at $x = 0$.

(d)

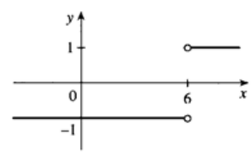


37. $f(x) = |x - 6| = \begin{cases} 6 - x & \text{if } x < 6 \\ x - 6 & \text{if } x \ge 6 \end{cases}$

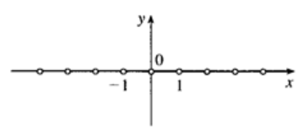
$\lim\limits_{x\to 6^+} \dfrac{f(x) - f(6)}{x - 6} = \lim\limits_{x\to 6^+} \dfrac{|x - 6| - 0}{x - 6} = \lim\limits_{x\to 6^+} \dfrac{x - 6}{x - 6} = \lim\limits_{x\to 6^+} 1 = 1.$

But $\lim\limits_{x\to 6^-} \dfrac{f(x) - f(6)}{x - 6} = \lim\limits_{x\to 6^-} \dfrac{|x - 6| - 0}{x - 6} = \lim\limits_{x\to 6^-} \dfrac{6 - x}{x - 6} = \lim\limits_{x\to 6^-} (-1) = -1.$

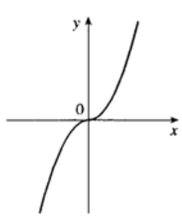
So $f'(6) = \lim\limits_{x\to 6} \dfrac{f(x) - f(6)}{x - 6}$ does not exist. However, $f'(x) = \begin{cases} -1 & \text{if } x < 6 \\ 1 & \text{if } x > 6 \end{cases}$



38. $f(x) = [\![x]\!]$ is not continuous at any integer n, so f is not differentiable at n by Theorem 4. If a is not an integer, then f is constant on an open interval containing a, so $f'(a) = 0$. Thus, $f'(x) = 0$, x not an integer.



39. (a) $f(x) = x|x| = \begin{cases} x^2 & \text{if } x \ge 0 \\ -x^2 & \text{if } x < 0 \end{cases}$



(b) Since $f(x) = x^2$ for $x \ge 0$, we have $f'(x) = 2x$ for $x > 0$. [See Exercise 3.2.15(d).] Similarly, since $f(x) = -x^2$ for $x < 0$, we have $f'(x) = -2x$ for $x < 0$. At $x = 0$, we have

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x|x|}{x} = \lim_{x\to 0} |x| = 0$$

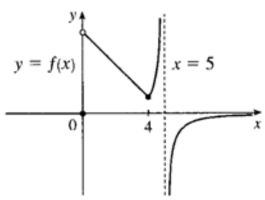
So f is differentiable at 0. Thus, f is differentiable for all x.

(c) From part (b), we have $f'(x) = \left\{ \begin{array}{ll} 2x & \text{if } x \ge 0 \\ -2x & \text{if } x < 0 \end{array} \right\} = 2|x|.$

40. (a) $f'_{-}(4) = \lim\limits_{h\to 0^-} \dfrac{f(4+h)-f(4)}{h} = \lim\limits_{h\to 0^-} \dfrac{5-(4+h)-1}{h} = \lim\limits_{h\to 0^-} \dfrac{-h}{h} = -1$ and

$$f'_{+}(4) = \lim_{h\to 0^+} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0^+} \frac{\dfrac{1}{5-(4+h)}-1}{h} = \lim_{h\to 0^+} \frac{1-(1-h)}{h(1-h)} = \lim_{h\to 0^+} \frac{1}{1-h} = 1.$$

(b)



(c) $f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 5-x & \text{if } 0 < x < 4 \\ 1/(5-x) & \text{if } x \ge 4 \end{cases}$

These expressions show that f is continuous on the intervals $(-\infty, 0)$, $(0, 4)$, $(4, 5)$ and $(5, \infty)$. Since $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} (5-x) = 5 \ne 0 = \lim\limits_{x\to 0^-} f(x)$, $\lim\limits_{x\to 0} f(x)$ does not exist, so f is discontinuous (and therefore not differentiable) at 0.

At 4 we have $\lim\limits_{x\to 4^-} f(x) = \lim\limits_{x\to 4^-} (5-x) = 1$ and $\lim\limits_{x\to 4^+} f(x) = \lim\limits_{x\to 4^+} \dfrac{1}{5-x} = 1$, so $\lim\limits_{x\to 4} f(x) = 1 = f(4)$ and f is continuous at 4. Since $f(5)$ is not defined, f is discontinuous at 5.

(d) From (a), f is not differentiable at 4 since $f'_{-}(4) \ne f'_{+}(4)$, and from (c), f is not differentiable at 0 or 5.

41. (a) If f is even, then

$$\begin{aligned} f'(-x) &= \lim_{h\to 0} \frac{f(-x+h)-f(-x)}{h} = \lim_{h\to 0} \frac{f(x-h)-f(x)}{h} \\ &= -\lim_{h\to 0} \frac{f(x-h)-f(x)}{-h} \quad [\text{let } \Delta x = -h] \quad = -\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = -f'(x) \end{aligned}$$

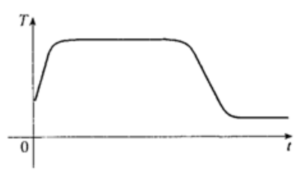
Therefore, f' is odd.

(b) If f is odd, then

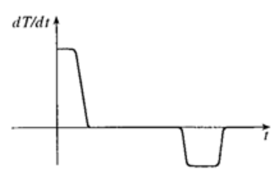
$$\begin{aligned} f'(-x) &= \lim_{h\to 0} \frac{f(-x+h)-f(-x)}{h} = \lim_{h\to 0} \frac{-f(x-h)+f(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x-h)-f(x)}{-h} \quad [\text{let } \Delta x = -h] \quad = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x) \end{aligned}$$

Therefore, f' is even.

42. (a)

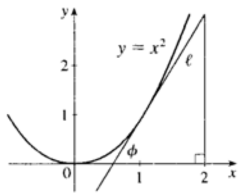


(c)



(b) The initial temperature of the water is close to room temperature because of the water that was in the pipes. When the water from the hot water tank starts coming out, dT/dt is large and positive as T increases to the temperature of the water in the tank. In the next phase, $dT/dt = 0$ as the water comes out at a constant, high temperature. After some time, dT/dt becomes small and negative as the contents of the hot water tank are exhausted. Finally, when the hot water has run out, dT/dt is once again 0 as the water maintains its (cold) temperature.

43.



From the diagram, we see that the slope of the tangent is equal to $\tan\phi$, and also that $0 < \phi < \frac{\pi}{2}$. We know (see Exercise 15) that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. So the slope of the tangent to the curve at the point $(1, 1)$ is 2. So ϕ is the angle between 0 and $\frac{\pi}{2}$ whose tangent is 2, that is, $\phi = \tan^{-1} 2 \approx 63°$.

3.3 Differentiation Formulas

1. $f(x) = 5x - 1 \;\Rightarrow\; f'(x) = 5 - 0 = 5$

2. $F(x) = -4x^{10} \;\Rightarrow\; F'(x) = -4\left(10x^{10-1}\right) = -40x^9$

3. $f(x) = x^2 + 3x - 4 \;\Rightarrow\; f'(x) = 2x^{2-1} + 3 - 0 = 2x + 3$

4. $g(x) = 5x^8 - 2x^5 + 6 \;\Rightarrow\; g'(x) = 5\left(8x^{8-1}\right) - 2\left(5x^{5-1}\right) + 0 = 40x^7 - 10x^4$

5. $V(r) = \frac{4}{3}\pi r^3 \;\Rightarrow\; V'(r) = \frac{4}{3}\pi\left(3r^2\right) = 4\pi r^2$

6. $s(t) = t^3 - 3t^2 + 12t \;\Rightarrow\; s'(t) = 3t^{3-1} - 3\left(2t^{2-1}\right) + 12 = 3t^2 - 6t + 12$

7. $F(x) = (16x)^3 = 4096x^3 \;\Rightarrow\; F'(x) = 4096\left(3x^2\right) = 12{,}288x^2$

8. $H(s) = (s/2)^5 = s^5/2^5 = \frac{1}{32}s^5 \;\Rightarrow\; H'(s) = \frac{1}{32}\left(5s^{5-1}\right) = \frac{5}{32}s^4$

9. $Y(t) = 6t^{-9} \;\Rightarrow\; Y'(t) = 6(-9)t^{-10} = -54t^{-10}$

10. $R(t) = 5t^{-3/5} \;\Rightarrow\; R'(t) = 5\left[-\frac{3}{5}t^{(-3/5)-1}\right] = -3t^{-8/5}$

11. $y = 4\pi^2 \;\Rightarrow\; y' = 0$ since $4\pi^2$ is a constant.

12. $R(x) = \dfrac{\sqrt{10}}{x^7} = \sqrt{10}x^{-7} \;\Rightarrow\; R'(x) = -7\sqrt{10}x^{-8} = -\dfrac{7\sqrt{10}}{x^8}$

13. $g(x) = x^2 + \dfrac{1}{x^2} = x^2 + x^{-2} \;\Rightarrow\; g'(x) = 2x + (-2)x^{-3} = 2x - \dfrac{2}{x^3}$

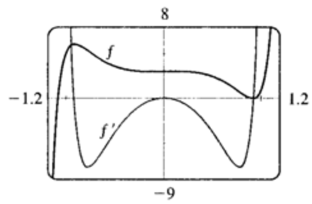
14. $f(t) = \sqrt{t} - \dfrac{1}{\sqrt{t}} = t^{1/2} - t^{-1/2} \;\Rightarrow\; f'(t) = \frac{1}{2}t^{-1/2} - \left(-\frac{1}{2}t^{-3/2}\right) = \dfrac{1}{2\sqrt{t}} + \dfrac{1}{2t\sqrt{t}}$

15. $y = \sqrt{5x} = \sqrt{5}x^{1/2} \;\Rightarrow\; y' = \sqrt{5}\left(\frac{1}{2}\right)x^{-1/2} = \dfrac{\sqrt{5}}{2\sqrt{x}}$

16. $y = x^{4/3} - x^{2/3} \;\Rightarrow\; y' = \frac{4}{3}x^{1/3} - \frac{2}{3}x^{-1/3}$

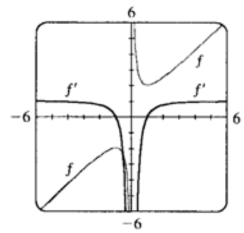
17. Product Rule: $y = \left(x^2 + 1\right)\left(x^3 + 1\right) \;\Rightarrow$
$y' = \left(x^2 + 1\right)\left(3x^2\right) + \left(x^3 + 1\right)(2x) = 3x^4 + 3x^2 + 2x^4 + 2x = 5x^4 + 3x^2 + 2x.$
Multiplying first: $y = \left(x^2 + 1\right)\left(x^3 + 1\right) = x^5 + x^3 + x^2 + 1 \;\Rightarrow\; y' = 5x^4 + 3x^2 + 2x$ (equivalent)

43. $f(x) = 3x^{15} - 5x^3 + 3 \quad \Rightarrow$
$f'(x) = 3 \cdot 15x^{14} - 5 \cdot 3x^2 = 45x^{14} - 15x^2$



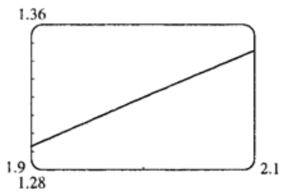
Note that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

44. $f(x) = x + 1/x = x + x^{-1} \quad \Rightarrow$
$f'(x) = 1 - x^{-2} = 1 - 1/x^2$



Note that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

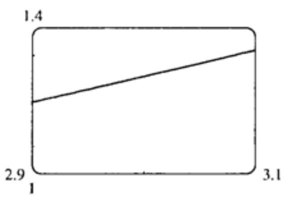
45. (a)



The endpoints of f in this graph are about $(1.9, 1.2927)$ and $(2.1, 1.3455)$. An estimate of $f'(2)$ is $\frac{1.3455 - 1.2927}{2.1 - 1.9} = \frac{0.0528}{0.2} = 0.264$.

(b) $f(x) = x^{2/5} \quad \Rightarrow \quad f'(x) = \frac{2}{5}x^{-3/5} = 2 / (5x^{3/5})$.
$f'(2) = 2 / (5 \cdot 2^{3/5}) \approx 0.263902$.

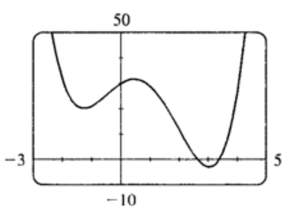
46. (a)



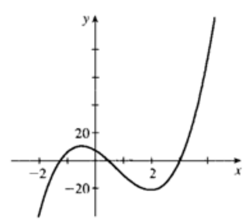
The endpoints of f in this graph are about $(2.9, 1.19706)$ and $(3.1, 1.33932)$. An estimate of $f'(3)$ is
$$\frac{1.33932 - 1.19706}{3.1 - 2.9} = \frac{0.14226}{0.2} = 0.7113.$$

(b) $f(x) = x - \sqrt{x} \quad \Rightarrow \quad f'(x) = 1 - \frac{1}{2}x^{-1/2} = 1 - \frac{1}{2\sqrt{x}}$.
$f'(3) = 1 - \frac{1}{2\sqrt{3}} \approx 0.7113$.

47. (a)



(b)



From the graph in part (a), it appears that f' is zero at $x_1 \approx -1.25$, $x_2 \approx 0.5$, and $x_3 \approx 3$. The slopes are negative (so f' is negative) on $(-\infty, x_1)$ and (x_2, x_3). The slopes are positive (so f' is positive) on (x_1, x_2) and (x_3, ∞).

32. $y = \dfrac{4t+5}{2-3t} \quad\Rightarrow\quad y' = \dfrac{(2-3t)(4)-(4t+5)(-3)}{(2-3t)^2} = \dfrac{23}{(2-3t)^2}$

33. $y = x + \sqrt[5]{x^2} = x + x^{2/5} \quad\Rightarrow\quad y' = 1 + \frac{2}{5}x^{-3/5} = 1 + \dfrac{2}{5\sqrt[5]{x^3}}$

34. $u = \sqrt[3]{t^2} + 2\sqrt{t^3} = t^{2/3} + 2t^{3/2} \quad\Rightarrow\quad u' = \frac{2}{3}t^{-1/3} + 2\left(\frac{3}{2}\right)t^{1/2} = \dfrac{2}{3\sqrt[3]{t}} + 3\sqrt{t}$

35. $v = x\sqrt{x} + \dfrac{1}{x^2\sqrt{x}} = x^{3/2} + x^{-5/2} \quad\Rightarrow\quad v' = \frac{3}{2}x^{1/2} - \frac{5}{2}x^{-7/2} = \frac{3}{2}\sqrt{x} - \dfrac{5}{2x^3\sqrt{x}}$

36. $v = \dfrac{6}{\sqrt[3]{t^5}} = 6t^{-5/3} \quad\Rightarrow\quad v' = 6\left(-\frac{5}{3}\right)t^{-8/3} = -\dfrac{10}{\sqrt[3]{t^8}}$

37. $f(x) = \dfrac{x}{x+c/x} \quad\Rightarrow\quad f'(x) = \dfrac{(x+c/x)(1) - x\left(1-c/x^2\right)}{(x+c/x)^2} = \dfrac{x+c/x-x+c/x}{\left(\dfrac{x^2+c}{x}\right)^2} \cdot \dfrac{x^2}{x^2} = \dfrac{2cx}{\left(x^2+c\right)^2}$

38. $f(x) = \dfrac{ax+b}{cx+d} \quad\Rightarrow\quad f'(x) = \dfrac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2} = \dfrac{acx+ad-acx-bc}{(cx+d)^2} = \dfrac{ad-bc}{(cx+d)^2}$

39. $f(x) = \dfrac{x^5}{x^3-2} \quad\Rightarrow\quad f'(x) = \dfrac{\left(x^3-2\right)\left(5x^4\right) - x^5\left(3x^2\right)}{\left(x^3-2\right)^2} = \dfrac{2x^4\left(x^3-5\right)}{\left(x^3-2\right)^2}$

40. $s = \sqrt{t}\left(t^3 - \sqrt{t} + 1\right) = t^{7/2} - t + t^{1/2} \quad\Rightarrow\quad s' = \frac{7}{2}t^{5/2} - 1 + \dfrac{1}{2\sqrt{t}}.$

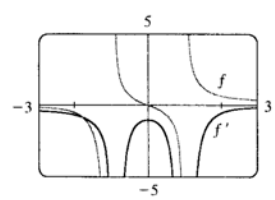
Another Method: Use the Product Rule.

41. $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 \quad\Rightarrow$
$P'(x) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1$

42. $f(x) = \dfrac{x}{x^2-1} \quad\Rightarrow$

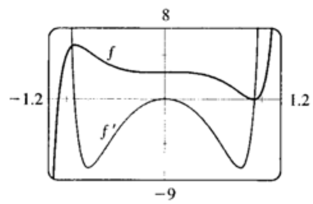
$f'(x) = \dfrac{\left(x^2-1\right)1 - x(2x)}{\left(x^2-1\right)^2} = \dfrac{-x^2-1}{\left(x^2-1\right)^2} = -\dfrac{x^2+1}{\left(x^2-1\right)^2}$

Notice that the slopes of all tangents to f are negative and $f'(x) < 0$ always.



43. $f(x) = 3x^{15} - 5x^3 + 3 \quad \Rightarrow$

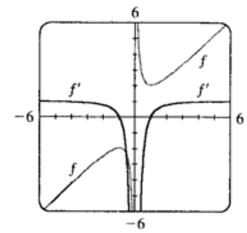
$f'(x) = 3 \cdot 15x^{14} - 5 \cdot 3x^2 = 45x^{14} - 15x^2$



Note that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

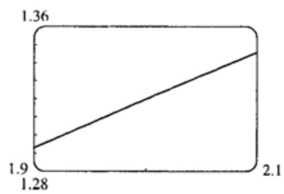
44. $f(x) = x + 1/x = x + x^{-1} \quad \Rightarrow$

$f'(x) = 1 - x^{-2} = 1 - 1/x^2$



Note that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

45. (a)

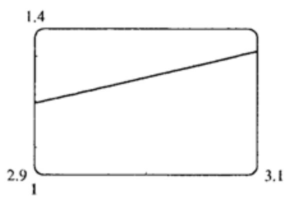


The endpoints of f in this graph are about $(1.9, 1.2927)$ and $(2.1, 1.3455)$. An estimate of $f'(2)$ is $\frac{1.3455 - 1.2927}{2.1 - 1.9} = \frac{0.0528}{0.2} = 0.264$.

(b) $f(x) = x^{2/5} \quad \Rightarrow \quad f'(x) = \frac{2}{5}x^{-3/5} = 2\big/\left(5x^{3/5}\right)$.

$f'(2) = 2\big/\left(5 \cdot 2^{3/5}\right) \approx 0.263902$.

46. (a)



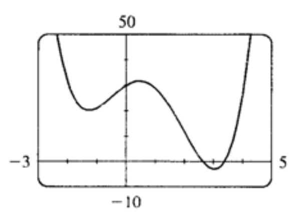
The endpoints of f in this graph are about $(2.9, 1.19706)$ and $(3.1, 1.33932)$. An estimate of $f'(3)$ is

$$\frac{1.33932 - 1.19706}{3.1 - 2.9} = \frac{0.14226}{0.2} = 0.7113.$$

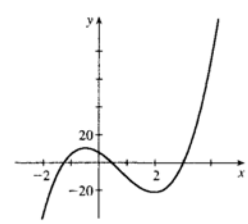
(b) $f(x) = x - \sqrt{x} \quad \Rightarrow \quad f'(x) = 1 - \frac{1}{2}x^{-1/2} = 1 - \dfrac{1}{2\sqrt{x}}$.

$f'(3) = 1 - \frac{1}{2\sqrt{3}} \approx 0.7113$.

47. (a)

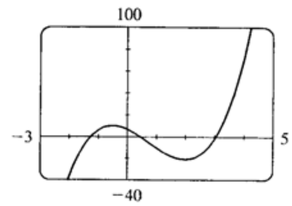


(b)

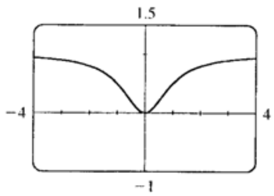


From the graph in part (a), it appears that f' is zero at $x_1 \approx -1.25$, $x_2 \approx 0.5$, and $x_3 \approx 3$. The slopes are negative (so f' is negative) on $(-\infty, x_1)$ and (x_2, x_3). The slopes are positive (so f' is positive) on (x_1, x_2) and (x_3, ∞).

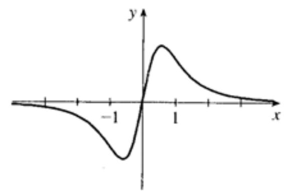
(c) $f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30 \quad \Rightarrow$
$f'(x) = 4x^3 - 9x^2 - 12x + 7$



48. (a)



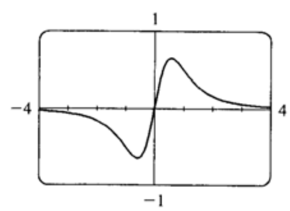
(b)



From the graph in part (a), it appears that g' is zero at $x = 0$. The slopes are negative (so g' is negative) on $(-\infty, 0)$. The slopes are positive (so g' is positive) on $(0, \infty)$.

(c) $g(x) = \dfrac{x^2}{x^2+1} \quad \Rightarrow$

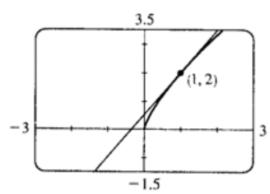
$$g'(x) = \frac{(x^2+1)(2x) - x^2(2x)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}$$



49. $y = \dfrac{2x}{x+1} \quad \Rightarrow \quad y' = \dfrac{(x+1)(2) - (2x)(1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$. At $(1, 1)$, $y' = \frac{1}{2}$, and an equation of the tangent line is $y - 1 = \frac{1}{2}(x - 1)$, or $y = \frac{1}{2}x + \frac{1}{2}$.

50. $y = \dfrac{\sqrt{x}}{x+1} \quad \Rightarrow \quad y' = \dfrac{(x+1)\left(\dfrac{1}{2\sqrt{x}}\right) - \sqrt{x}\,(1)}{(x+1)^2} = \dfrac{(x+1)-(2x)}{2\sqrt{x}\,(x+1)^2} = \dfrac{1-x}{2\sqrt{x}\,(x+1)^2}$. At $(4, 0.4)$, $y' = -\frac{3}{100} = -0.03$, and an equation of the tangent line is $y - 0.4 = -0.03(x - 4)$, or $y = -0.03x + 0.52$.

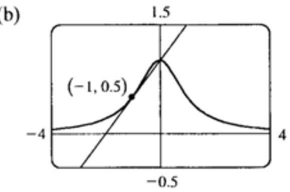
51. $y = f(x) = x + \sqrt{x} \quad \Rightarrow \quad f'(x) = 1 + \frac{1}{2}x^{-1/2}$. So the slope of the tangent line at $(1, 2)$ is $f'(1) = 1 + \frac{1}{2}(1) = \frac{3}{2}$ and its equation is $y - 2 = \frac{3}{2}(x - 1)$ or $y = \frac{3}{2}x + \frac{1}{2}$.



52. $y = x\sqrt{x} = x^{3/2} \quad \Rightarrow \quad y' = \frac{3}{2}x^{1/2}$. At $(1, 1)$, $y' = \frac{3}{2}$, and an equation of the tangent line is $y - 1 = \frac{3}{2}(x - 1)$, or $y = \frac{3}{2}x - \frac{1}{2}$.

53. (a) $y = f(x) = \dfrac{1}{1+x^2} \Rightarrow f'(x) = \dfrac{-2x}{\left(1+x^2\right)^2}$. So the slope of the tangent line at the point $\left(-1, \frac{1}{2}\right)$ is $f'(-1) = \frac{2}{2^2} = \frac{1}{2}$ and its equation is $y - \frac{1}{2} = \frac{1}{2}(x+1)$ or $y = \frac{1}{2}x + 1$.

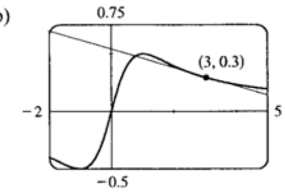
(b)



54. (a) $y = f(x) = \dfrac{x}{1+x^2} \Rightarrow$

$$f'(x) = \frac{\left(1+x^2\right)1 - x(2x)}{\left(1+x^2\right)^2} = \frac{1-x^2}{\left(1+x^2\right)^2}.$$ So the slope of the tangent line at the point $(3, 0.3)$ is $f'(3) = \frac{-8}{100}$ and its equation is $y - 0.3 = -0.08(x-3)$ or $y = -0.08x + 0.54$.

(b)



55. (a) $(fg)'(5) = f(5)g'(5) + g(5)f'(5) = (1)(2) + (-3)(6) = 2 - 18 = -16$

(b) $\left(\dfrac{f}{g}\right)'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \dfrac{(-3)(6) - (1)(2)}{(-3)^2} = -\dfrac{20}{9}$

(c) $\left(\dfrac{g}{f}\right)'(5) = \dfrac{f(5)g'(5) - g(5)f'(5)}{[f(5)]^2} = \dfrac{(1)(2) - (-3)(6)}{(1)^2} = 20$

56. (a) $(f+g)'(3) = f'(3) + g'(3) = -6 + 5 = -1$

(b) $(fg)'(3) = f(3)g'(3) + g(3)f'(3) = (4)(5) + (2)(-6) = 20 - 12 = 8$

(c) $\left(\dfrac{f}{g}\right)'(3) = \dfrac{g(3)f'(3) - f(3)g'(3)}{[g(3)]^2} = \dfrac{(2)(-6) - (4)(5)}{(2)^2} = \dfrac{-32}{4} = -8$

(d) $$\left(\frac{f}{f-g}\right)'(3) = \frac{[f(3) - g(3)]f'(3) - f(3)\left[f'(3) - g'(3)\right]}{[f(3) - g(3)]^2} = \frac{(4-2)(-6) - 4(-6-5)}{(4-2)^2} = \frac{-12+44}{2^2} = 8$$

57. $f(x) = \sqrt{x}\,g(x) \Rightarrow f'(x) = \sqrt{x}\,g'(x) + g(x)\cdot\frac{1}{2}x^{-1/2}$, so $f'(4) = \sqrt{4}\,g'(4) + g(4)\cdot\frac{1}{2\sqrt{4}} = 2\cdot 7 + 8\cdot\frac{1}{4} = 16$.

58. $$\frac{d}{dx}\left[\frac{h(x)}{x}\right] = \frac{xh'(x) - h(x)\cdot 1}{x^2} \Rightarrow \frac{d}{dx}\left[\frac{h(x)}{x}\right]_{x=2} = \frac{2h'(2) - h(2)}{2^2} = \frac{2(-3) - (4)}{4} = \frac{-10}{4} = -2.5$$

59. (a) $u(x) = f(x)g(x)$, so $u'(1) = f(1)g'(1) + g(1)f'(1) = 2\cdot(-1) + 1\cdot 2 = 0$

(b) $v(x) = f(x)/g(x)$, so $v'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \dfrac{2\left(-\frac{1}{3}\right) - 3\cdot\frac{2}{3}}{2^2} = -\dfrac{2}{3}$

60. (a) $y = x^2 f(x) \Rightarrow y' = x^2 f'(x) + f(x)(2x)$

(b) $y = \dfrac{f(x)}{x^2} \Rightarrow y' = \dfrac{x^2 f'(x) - f(x)(2x)}{\left(x^2\right)^2} = \dfrac{xf'(x) - 2f(x)}{x^3}$

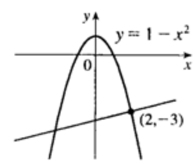
(c) $y = \dfrac{x^2}{f(x)} \Rightarrow y' = \dfrac{f(x)(2x) - x^2 f'(x)}{[f(x)]^2}$

(d) $y = \dfrac{1 + xf(x)}{\sqrt{x}} \quad \Rightarrow$

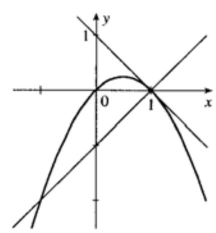
$$y' = \frac{\sqrt{x}\left[xf'(x) + f(x)\right] - \left[1 + xf(x)\right]\dfrac{1}{2\sqrt{x}}}{\left(\sqrt{x}\right)^2}$$

$$= \frac{x^{3/2}f'(x) + x^{1/2}f(x) - \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2}f(x)}{x} \cdot \frac{2x^{1/2}}{2x^{1/2}} = \frac{xf(x) + 2x^2 f'(x) - 1}{2x^{3/2}}$$

61. $y = f(x) = 1 - x^2 \quad \Rightarrow \quad f'(x) = -2x$, so the tangent line at $(2, -3)$ has slope $f'(2) = -4$. The normal line has slope $-\frac{1}{-4} = \frac{1}{4}$ and equation $y + 3 = \frac{1}{4}(x - 2)$ or $y = \frac{1}{4}x - \frac{7}{2}$.

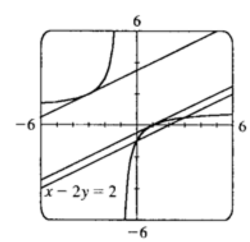


62. $y = f(x) = x - x^2 \quad \Rightarrow f'(x) = 1 - 2x$. So $f'(1) = -1$, and the slope of the normal line is the negative reciprocal of that of the tangent line, that is, $-1/(-1) = 1$. So the equation of the normal line at $(1, 0)$ is $y - 0 = 1(x - 1)$ $\Leftrightarrow \quad y = x - 1$. Substituting this into the equation of the parabola, we obtain $x - 1 = x - x^2 \quad \Leftrightarrow \quad x = \pm 1$. The solution $x = -1$ is the one we require. Substituting $x = -1$ into the equation of the parabola to find the y-coordinate, we have $y = -2$. So the point of intersection is $(-1, -2)$, as shown in the sketch.



63. $y = x^3 - x^2 - x + 1$ has a horizontal tangent when $y' = 3x^2 - 2x - 1 = 0$. $y' = (3x + 1)(x - 1) = 0 \quad \Leftrightarrow$ $x = 1$ or $-\frac{1}{3}$. Therefore, the points are $(1, 0)$ and $\left(-\frac{1}{3}, \frac{32}{27}\right)$.

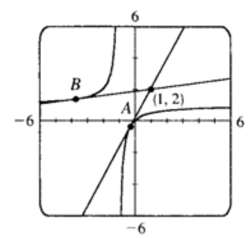
64. $y = \dfrac{x - 1}{x + 1} \quad \Rightarrow \quad y' = \dfrac{(x + 1)(1) - (x - 1)(1)}{(x + 1)^2} = \dfrac{2}{(x + 1)^2}$. If the tangent intersects the curve when $x = a$, then its slope is $2/(a + 1)^2$. But if the tangent is parallel to $x - 2y = 2$, that is, $y = \frac{1}{2}x - 1$, then its slope is $\frac{1}{2}$. Thus, $\dfrac{2}{(a + 1)^2} = \dfrac{1}{2} \quad \Rightarrow$ $(a + 1)^2 = 4 \quad \Rightarrow \quad a + 1 = \pm 2 \quad \Rightarrow \quad a = 1$ or -3. When $a = 1$, $y = 0$ and the equation of the tangent is $y - 0 = \frac{1}{2}(x - 1)$ or $y = \frac{1}{2}x - \frac{1}{2}$. When $a = -3$, $y = 2$ and the equation of the tangent is $y - 2 = \frac{1}{2}(x + 3)$ or $y = \frac{1}{2}x + \frac{7}{2}$.



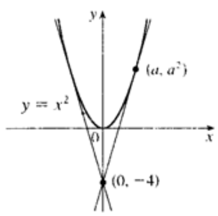
65. If $y = f(x) = \dfrac{x}{x+1}$ then $f'(x) = \dfrac{(x+1)(1) - x(1)}{(x+1)^2} = \dfrac{1}{(x+1)^2}$. When $x = a$, the equation of the tangent line is $y - \dfrac{a}{a+1} = \dfrac{1}{(a+1)^2}(x - a)$. This line passes through $(1, 2)$ when $2 - \dfrac{a}{a+1} = \dfrac{1}{(a+1)^2}(1 - a) \Leftrightarrow$ $2(a+1)^2 = a(a+1) + (1-a) = a^2 + 1 \Leftrightarrow a^2 + 4a + 1 = 0$. The quadratic formula gives the roots of this equation as $-2 \pm \sqrt{3}$, so there are two such tangent lines, which touch the curve at

$A\left(-2+\sqrt{3}, \frac{1-\sqrt{3}}{2}\right) \approx (-0.27, -0.37)$ and

$B\left(-2-\sqrt{3}, \frac{1+\sqrt{3}}{2}\right) \approx (-3.73, 1.37)$.



66.



Let (a, a^2) be a point on the parabola at which the tangent line passes through the point $(0, -4)$. The tangent line has slope $2a$ and equation $y - (-4) = 2a(x - 0) \Rightarrow y = 2ax - 4$. Since (a, a^2) also lies on the line, $a^2 = 2a(a) - 4$, or $a^2 = 4$. So $a = \pm 2$ and the points are $(2, 4)$ and $(-2, 4)$.

67. $y = 6x^3 + 5x - 3 \Rightarrow m = y' = 18x^2 + 5$, but $x^2 \geq 0$ for all x, so $m \geq 5$ for all x.

68. If $y = x^2 + x$, then $y' = 2x + 1$. If the point at which a tangent meets the parabola is $(a, a^2 + a)$, then the slope of the tangent is $2a + 1$. But since it passes through $(2, -3)$, the slope must also be $\dfrac{a^2 + a + 3}{a - 2}$. Therefore, $2a + 1 = \dfrac{a^2 + a + 3}{a - 2}$. Solving this equation for a we get $a^2 + a + 3 = 2a^2 - 3a - 2 \Leftrightarrow$ $a^2 - 4a - 5 = (a-5)(a+1) = 0 \Leftrightarrow a = 5$ or -1. If $a = -1$, the point is $(-1, 0)$ and the slope is -1, so the equation is $y - 0 = (-1)(x + 1)$ or $y = -x - 1$. If $a = 5$, the point is $(5, 30)$ and the slope is 11, so the equation is $y - 30 = 11(x - 5)$ or $y = 11x - 25$.

We will sometimes use the form $f'g + fg'$ rather than the form $fg' + gf'$ for the Product Rule.

69. (a) $(fgh)' = [(fg)h]' = (fg)'h + (fg)h' = (f'g + fg')h + (fg)h' = f'gh + fg'h + fgh'$

(b) $y = \sqrt{x}\,(x^4 + x + 1)(2x - 3)$. Using part (a), we have

$$\begin{aligned} y' &= \frac{1}{2\sqrt{x}}\left(x^4 + x + 1\right)(2x - 3) + \sqrt{x}\left(4x^3 + 1\right)(2x - 3) + \sqrt{x}\left(x^4 + x + 1\right)(2) \\ &= \left(x^4 + x + 1\right)\frac{2x - 3}{2\sqrt{x}} + \sqrt{x}\left[\left(4x^3 + 1\right)(2x - 3) + 2\left(x^4 + x + 1\right)\right] \end{aligned}$$

70. (a) Putting $f = g = h$ in Exercise 69, we have

$$\frac{d}{dx}[f(x)]^3 = (fff)' = f'ff + ff'f + fff' = 3fff' = 3[f(x)]^2 f'(x).$$

(b) $y = \left(x^4 + 3x^3 + 17x + 82\right)^3 \Rightarrow y' = 3\left(x^4 + 3x^3 + 17x + 82\right)^2\left(4x^3 + 9x^2 + 17\right)$

71. $y = f(x) = ax^3 + bx^2 + cx + d \Rightarrow f'(x) = 3ax^2 + 2bx + c$. $f(-2) = 6 \Rightarrow$ $-8a + 4b - 2c + d = 6$ **(1)**. $f(2) = 0 \Rightarrow 8a + 4b + 2c + d = 0$ **(2)**. Since there are horizontal tangents at $(-2, 6)$ and $(2, 0)$, $f'(\pm 2) = 0$. $f'(-2) = 0 \Rightarrow 12a - 4b + c = 0$ **(3)** and $f'(2) = 0 \Rightarrow$ $12a + 4b + c = 0$ **(4)**. Subtracting equation **(3)** from **(4)** gives $8b = 0 \Rightarrow b = 0$. Adding **(1)** and **(2)** gives $8b + 2d = 6$, so $d = 3$ since $b = 0$. From **(3)** we have $c = -12a$, so **(2)** becomes $8a + 4(0) + 2(-12a) + 3 = 0$ $\Rightarrow 3 = 16a \Rightarrow a = \frac{3}{16}$. Now $c = -12a = -12\left(\frac{3}{16}\right) = -\frac{9}{4}$ and the desired cubic function is $y = \frac{3}{16}x^3 - \frac{9}{4}x + 3$.

72. (a) $s(0) = 100{,}000$ subscribers and $n(0) = 1.2$ phone lines per subscriber. $s'(0) = 1000$ subscribers/month and $n'(0) = 0.01$ phone line per subscriber/month.

(b) The total number of lines is given by $L(t) = s(t)\,n(t)$. To find $L'(0)$, we first find $L'(t)$ using the Product Rule. $L'(t) = s(t)\,n'(t) + n(t)\,s'(t) \Rightarrow$ $L'(0) = s(0)\,n'(0) + n(0)\,s'(0) = 100{,}000\,(0.01) + 1.2\,(1000) = 2200$ phone lines/month.

73. Let $P(t)$ be the population and let $A(t)$ be the average annual income at time t, where t is measured in years and $t = 0$ corresponds to July 1993. Then the total personal income is given by $T(t) = P(t)\,A(t)$. We wish to find $T'(0)$. $T'(t) = P(t)\,A'(t) + A(t)\,P'(t)$. The term $P(t)\,A'(t)$ represents the portion of the rate of change of total income due to the existing population's increasing income. The term $A(t)\,P'(t)$ represents the portion of the rate of change of total income due to the increasing population.
$T'(0) = P(0)\,A'(0) + A(0)\,P'(0) \approx (3{,}354{,}000)\,(1900) + (21{,}107)\,(45{,}000) = 7{,}322{,}415{,}000$. So the total personal income was rising at a rate of about \$7.322 billion per year.

74. (a) $f(20) = 10{,}000$ means that when the price of the fabric is \$20/yard, 10,000 yards will be sold.
$f'(20) = -350$ means that as the price of the fabric increases past \$20/yard, the amount of fabric which will be sold is decreasing at a rate of 350 yards per (dollar per yard).

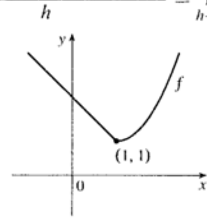
(b) $R'(p) = pf'(p) + f(p) \cdot 1 \Rightarrow R'(20) = 20f'(20) + f(20) \cdot 1 = 20(-350) + 10{,}000 = 3000$. This means that as the price of the fabric increases past \$20/yard, the total revenue is increasing at \$3000/ (\$/yard). Note that the Product Rule indicates that we will lose \$7000/ (\$/yard) due to selling less fabric, but that that loss is more than made up for by the additional revenue due to the increase in price.

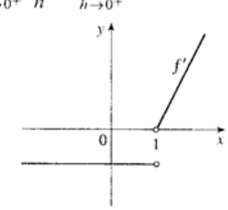
75. $f(x) = 2 - x$ if $x \le 1$ and $f(x) = x^2 - 2x + 2$ if $x > 1$. Now we compute the right- and left-hand derivatives defined in Exercise 3.2.40:

$$f'_-(1) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{2 - (1+h) - 1}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1 \text{ and}$$

$$f'_+(1) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{(1+h)^2 - 2(1+h) + 2 - 1}{h} = \lim_{h \to 0^+} \frac{h^2}{h} = \lim_{h \to 0^+} h = 0.$$

Thus, $f'(1)$ does not exist since $f'_-(1) \ne f'_+(1)$, so f is not differentiable at 1. But $f'(x) = -1$ for $x < 1$ and $f'(x) = 2x - 2$ if $x > 1$.





76. $g(x) = \begin{cases} 1-2x & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x \le 1 \\ x & \text{if } x > 1 \end{cases}$

$\lim\limits_{h\to 0^-} \dfrac{g(-1+h)-g(-1)}{h} = \lim\limits_{h\to 0^-} \dfrac{[-1-2(-1+h)]-1}{h} = \lim\limits_{h\to 0^-} \dfrac{-2h}{h} = \lim\limits_{h\to 0^-} (-2) = -2$ and

$\lim\limits_{h\to 0^+} \dfrac{g(-1+h)-g(-1)}{h} = \lim\limits_{h\to 0^+} \dfrac{(-1+h)^2-1}{h} = \lim\limits_{h\to 0^+} \dfrac{-2h+h^2}{h} = \lim\limits_{h\to 0^+} (-2+h) = -2,$

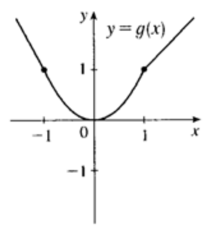
so g is differentiable at -1 and $g'(-1) = -2$.

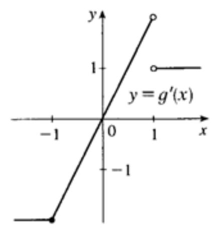
$\lim\limits_{h\to 0^-} \dfrac{g(1+h)-g(1)}{h} = \lim\limits_{h\to 0^-} \dfrac{(1+h)^2-1}{h} = \lim\limits_{h\to 0^-} \dfrac{2h+h^2}{h} = \lim\limits_{h\to 0^-} (2+h) = 2$ and

$\lim\limits_{h\to 0^+} \dfrac{g(1+h)-g(1)}{h} = \lim\limits_{h\to 0^+} \dfrac{(1+h)-1}{h} = \lim\limits_{h\to 0^+} \dfrac{h}{h} = \lim\limits_{h\to 0^+} 1 = 1$, so $g'(1)$ does not exist.

Thus, g is differentiable except when $x = 1$, and

$g'(x) = \begin{cases} -2 & \text{if } x < -1 \\ 2x & \text{if } -1 \le x < 1 \\ 1 & \text{if } x > 1 \end{cases}$





77. (a) Note that $x^2 - 9 < 0$ for $x^2 < 9 \;\Leftrightarrow\; |x| < 3 \;\Leftrightarrow\; -3 < x < 3$. So

$$f(x) = \begin{cases} x^2-9 & \text{if } x \le -3 \\ -x^2+9 & \text{if } -3 < x < 3 \\ x^2-9 & \text{if } x \ge 3 \end{cases} \quad\Rightarrow\quad f'(x) = \begin{cases} 2x & \text{if } x < -3 \\ -2x & \text{if } -3 < x < 3 \\ 2x & \text{if } x > 3 \end{cases}$$

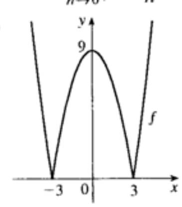
To show that $f'(3)$ does not exist we investigate $\lim\limits_{h\to 0} \dfrac{f(3+h)-f(3)}{h}$ by computing the left- and right-hand derivatives defined in Exercise 3.2.40.

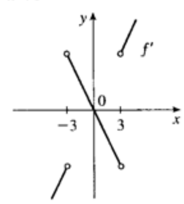
$f'_-(3) = \lim\limits_{h\to 0^-} \dfrac{f(3+h)-f(3)}{h} = \lim\limits_{h\to 0^-} \dfrac{\left(-(3+h)^2+9\right)-0}{h} = \lim\limits_{h\to 0^-} (-6+h) = -6$ and

$f'_+(3) = \lim\limits_{h\to 0^+} \dfrac{f(3+h)-f(3)}{h} = \lim\limits_{h\to 0^+} \dfrac{\left[(3+h)^2+9\right]-0}{h} = \lim\limits_{h\to 0^+} \dfrac{6h+h^2}{h} = \lim\limits_{h\to 0^+} (6+h) = 6.$

Since the left and right limits are different, $\lim\limits_{h\to 0} \dfrac{f(3+h)-f(3)}{h}$ does not exist, that is, $f'(3)$ does not exist. Similarly, $f'(-3)$ does not exist. Therefore, f is not differentiable at 3 or at -3.

(b)





78. If $x \geq 1$, then $h(x) = |x-1| + |x+2| = x - 1 + x + 2 = 2x + 1$.
If $-2 < x < 1$, then $h(x) = -(x-1) + x + 2 = 3$.
If $x \leq -2$, then $h(x) = -(x-1) - (x+2) = -2x - 1$. Therefore,

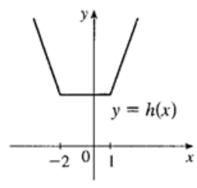
$$h(x) = \begin{cases} -2x - 1 & \text{if } x \leq -2 \\ 3 & \text{if } -2 < x < 1 \\ 2x + 1 & \text{if } x \geq 1 \end{cases} \quad \Rightarrow \quad h'(x) = \begin{cases} -2 & \text{if } x < -2 \\ 0 & \text{if } -2 < x < 1 \\ 2 & \text{if } x > 1 \end{cases}$$

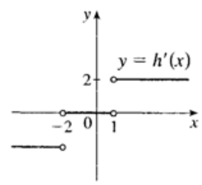
To see that $h'(1) = \lim\limits_{x \to 1} \dfrac{h(x) - h(1)}{x - 1}$ does not exist, observe that

$\lim\limits_{x \to 1^-} \dfrac{h(x) - h(1)}{x - 1} = \lim\limits_{x \to 1^-} \dfrac{3 - 3}{3 - 1} = 0$ but

$\lim\limits_{x \to 1^+} \dfrac{h(x) - h(1)}{x - 1} = \lim\limits_{x \to 1^+} \dfrac{2x - 2}{x - 1} = 2$.

Similarly, $h'(-2)$ does not exist.





79. $y = f(x) = ax^2 \quad \Rightarrow \quad f'(x) = 2ax$. So the slope of the tangent to the parabola at $x = 2$ is $m = 2a(2) = 4a$. The slope of the given line is seen to be -2, so we must have $4a = -2 \quad \Leftrightarrow \quad a = -\frac{1}{2}$. So the point in question has y-coordinate $-\frac{1}{2} \cdot 2^2 = -2$. Now we simply require that the given line, whose equation is $2x + y = b$, pass through the point $(2, -2)$: $2(2) + (-2) = b \quad \Leftrightarrow \quad b = 2$. So we must have $a = -\frac{1}{2}$ and $b = 2$.

80. f is clearly differentiable for $x < 2$ and for $x > 2$. For $x < 2$, $f'(x) = 2x$, so $f'_-(2) = 4$. For $x > 2$, $f'(x) = m$, so $f'_+(2) = m$. For f to be differentiable at $x = 2$, we need $4 = f'_-(2) = f'_+(2) = m$. So $f(x) = 4x + b$. We must also have continuity at $x = 2$, so $4 = f(2) = \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (4x + b) = 8 + b$. Hence, $b = -4$.

81. $F = f/g \quad \Rightarrow \quad f = Fg \quad \Rightarrow \quad f' = F'g + Fg' \quad \Rightarrow \quad F' = \dfrac{f' - Fg'}{g} = \dfrac{f' - (f/g)g'}{g} = \dfrac{f'g - fg'}{g^2}$

82. (a) $xy = c \quad \Rightarrow \quad y = \dfrac{c}{x}$. Let $P = \left(a, \dfrac{c}{a}\right)$. The slope of the tangent line at $x = a$ is $y'(a) = -\dfrac{c}{a^2}$. Its equation is $y - \dfrac{c}{a} = -\dfrac{c}{a^2}(x - a)$ or $y = -\dfrac{c}{a^2}x + \dfrac{2c}{a}$, so its y-intercept is $\dfrac{2c}{a}$. Setting $y = 0$ gives $x = 2a$, so the x-intercept is $2a$. The midpoint of the line segment joining $\left(0, \dfrac{2c}{a}\right)$ and $(2a, 0)$ is $\left(a, \dfrac{c}{a}\right) = P$.

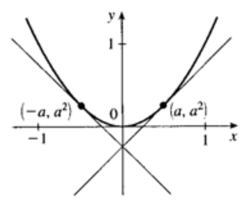
(b) We know the x- and y-intercepts of the tangent line from part (a), so the area of the triangle bounded by the axes and the tangent is $\frac{1}{2}xy = \frac{1}{2}(2a)(2c/a) = 2c$, a constant.

83. *Solution 1:* Let $f(x) = x^{1000}$. Then, by the definition of derivative,

$f'(1) = \lim\limits_{x \to 1} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1} \dfrac{x^{1000} - 1}{x - 1}$. But this is just the limit we want to find, and we know (from the Power Rule) that $f'(x) = 1000x^{999}$, so $f'(1) = 1000(1)^{999} = 1000$. So $\lim\limits_{x \to 1} \dfrac{x^{1000} - 1}{x - 1} = 1000$.

Solution 2: Note that $(x^{1000} - 1) = (x - 1)(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)$. So

$$\begin{aligned}\lim_{x \to 1} \frac{x^{1000} - 1}{x - 1} &= \lim_{x \to 1} \frac{(x - 1)(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)}{x - 1} \\ &= \lim_{x \to 1} \left(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1\right) = \underbrace{1 + 1 + 1 + \cdots + 1 + 1 + 1}_{1000 \text{ ones}} \\ &= 1000, \text{ as above.}\end{aligned}$$

84. 

In order for the two tangents to intersect on the y-axis, the points of tangency must be at equal distances from the y-axis, since the parabola $y = x^2$ is symmetric about the y-axis. Say the points of tangency are (a, a^2) and $(-a, a^2)$, for some $a > 0$. Then since the derivative of $y = x^2$ is $dy/dx = 2x$, the left-hand tangent has slope $-2a$ and equation $y - a^2 = -2a(x + a)$, or $y = -2ax - a^2$, and similarly the right-hand tangent line has equation $y = 2ax - a^2$. So the two lines intersect at $(0, -a^2)$. Now if the lines are perpendicular, then the product of their slopes is -1, so $(-2a)(2a) = -1 \quad \Leftrightarrow \quad a^2 = \frac{1}{4} \quad \Leftrightarrow \quad a = \frac{1}{2}$. So the lines intersect at $\left(0, -\frac{1}{4}\right)$.

3.4 Rates of Change in the Natural and Social Sciences

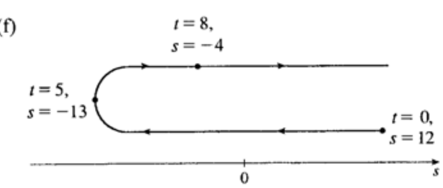
1. (a) $s = f(t) = t^2 - 10t + 12 \quad \Rightarrow \quad v(t) = f'(t) = (2t - 10)\text{ft/s}$

(b) $v(3) = 2(3) - 10 = -4$ ft/s

(c) The particle is at rest when $v(t) = 0 \quad \Leftrightarrow \quad 2t - 10 = 0 \quad \Leftrightarrow \quad t = 5$ s.

(d) The particle is moving in the positive direction when $v(t) > 0 \quad \Leftrightarrow \quad 2t - 10 > 0 \quad \Leftrightarrow \quad 2t > 10 \quad \Leftrightarrow$ $t > 5$.

(e) Since the particle is moving in the positive direction and in the negative direction, we need to calculate the distance traveled in the intervals [0, 5] and [5, 8] separately. $|f(5) - f(0)| = |-13 - 12| = 25$ ft and $|f(8) - f(5)| = |-4 - (-13)| = 9$ ft. The total distance traveled during the first 8 s is $25 + 9 = 34$ ft.

(f) 

2. (a) $s = f(t) = t^3 - 9t^2 + 15t + 10 \quad \Rightarrow \quad v(t) = f'(t) = 3t^2 - 18t + 15 = 3(t-1)(t-5)$ ft/s

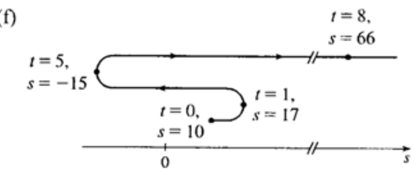
(b) $v(3) = 3(2)(-2) = -12$ ft/s

(c) $v(t) = 0 \quad \Leftrightarrow \quad t = 1$ s or 5 s

(d) $v(t) > 0 \quad \Leftrightarrow \quad 0 \le t < 1$ or $t > 5$

(e) $|f(1) - f(0)| = |17 - 10| = 7$,
$|f(5) - f(1)| = |-15 - 17| = 32$, and
$|f(8) - f(5)| = |66 - (-15)| = 81$.
Total distance $= 7 + 32 + 81 = 120$ ft.

(f)



3. (a) $s = f(t) = t^3 - 12t^2 + 36t \quad \Rightarrow \quad v(t) = f'(t) = 3t^2 - 24t + 36$

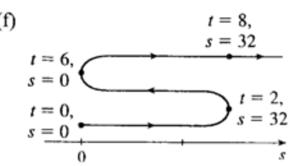
(b) $v(3) = 27 - 72 + 36 = -9$ ft/s

(c) The particle is at rest when $v(t) = 0$. $3t^2 - 24t + 36 = 0 \quad \Rightarrow \quad 3(t-2)(t-6) = 0 \quad \Rightarrow \quad t = 2, 6$.

(d) The particle is moving in the positive direction when $v(t) > 0$. $3(t-2)(t-6) > 0 \quad \Leftrightarrow \quad 0 \le t < 2$ or $t > 6$.

(e) Since the particle is moving in the positive direction and in the negative direction, we need to calculate the distance traveled in the intervals [0, 2], [2, 6], and [6, 8] separately.
$|f(2) - f(0)| = |32 - 0| = 32$. $|f(6) - f(2)| = |0 - 32| = 32$.
$|f(8) - f(6)| = |32 - 0| = 32$. The total distance is
$32 + 32 + 32 = 96$ ft.

(f)



4. (a) $s = f(t) = t^4 - 4t + 1 \quad \Rightarrow \quad v(t) = f'(t) = 4t^3 - 4$

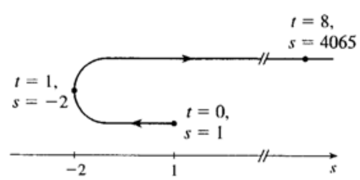
(b) $v(3) = 4(3)^3 - 4 = 104$ ft/s

(c) It is at rest when $v(t) = 4(t^3 - 1) = 4(t-1)(t^2 + t + 1) = 0 \quad \Leftrightarrow \quad t = 1$.

(d) It moves in the positive direction when $4(t^3 - 1) > 0 \quad \Leftrightarrow \quad t > 1$.

(e) Distance in positive direction $= |f(8) - f(1)| = |4065 - (-2)| = 4067$ ft
Distance in negative direction $= |f(1) - f(0)| = |-2 - 1| = 3$ ft
Total distance traveled $= 4067 + 3 = 4070$ ft

(f)



5. (a) $s = \dfrac{t}{t^2+1} \quad \Rightarrow \quad v(t) = s'(t) = \dfrac{(t^2+1)(1) - t(2t)}{(t^2+1)^2} = \dfrac{1-t^2}{(t^2+1)^2}$

(b) $v(3) = \dfrac{1-(3)^2}{(3^2+1)^2} = -\dfrac{2}{25}$ ft/s

(c) It is at rest when $v = 0 \quad \Leftrightarrow \quad 1 - t^2 = 0 \quad \Leftrightarrow \quad t = 1$.

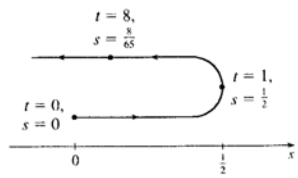
(d) It moves in the positive direction when $v > 0 \quad \Leftrightarrow \quad 1 - t^2 > 0 \quad \Leftrightarrow \quad t^2 < 1 \quad \Leftrightarrow \quad 0 \le t < 1$.

(e) Distance in positive direction $= |s(1) - s(0)| = \left|\frac{1}{2} - 0\right| = \frac{1}{2}$ ft

Distance in negative direction $= |s(8) - s(1)| = \left|\frac{8}{65} - \frac{1}{2}\right| = \frac{49}{130}$ ft

Total distance traveled $= \frac{1}{2} + \frac{49}{130} = \frac{57}{65}$ ft

(f)



6. (a) $s = \sqrt{t}\left(3t^2 - 35t + 90\right) = 3t^{5/2} - 35t^{3/2} + 90t^{1/2} \quad \Rightarrow$

$v(t) = s'(t) = \frac{15}{2}t^{3/2} - \frac{105}{2}t^{1/2} + 45t^{-1/2} = \frac{15}{2}t^{-1/2}\left(t^2 - 7t + 6\right) = \dfrac{15}{2\sqrt{t}}(t-1)(t-6)$

(b) $v(3) = \frac{15}{2\sqrt{3}}(2)(-3) = -15\sqrt{3}$ ft/s

(c) It is at rest when $v = 0 \quad \Leftrightarrow \quad t = 1$ s or 6 s.

(d) It moves in the positive direction when $v > 0 \quad \Leftrightarrow \quad (t-1)(t-6) > 0 \quad \Leftrightarrow \quad 0 \le t < 1$ or $t > 6$.

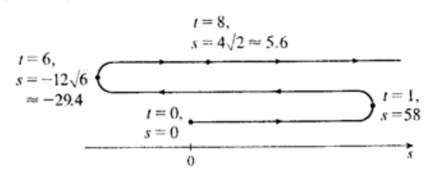
(e) Distance in positive direction $= |s(1) - s(0)| + |s(8) - s(6)| = |58 - 0| + \left|4\sqrt{2} - \left(-12\sqrt{6}\right)\right|$

$= 58 + 4\sqrt{2} + 12\sqrt{6} \approx 93.05$ ft

Distance in negative direction $= |s(6) - s(1)| = \left|-12\sqrt{6} - 58\right| = 58 + 12\sqrt{6} \approx 87.39$ ft

Total distance traveled $= 58 + 4\sqrt{2} + 12\sqrt{6} + 58 + 12\sqrt{6} = 116 + 4\sqrt{2} + 24\sqrt{6} \approx 180.44$ ft

(f)



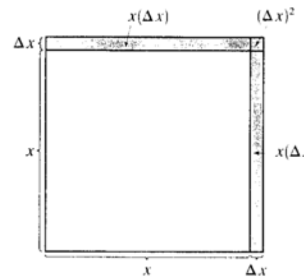
7. $s(t) = t^3 - 4.5t^2 - 7t \quad \Rightarrow \quad v(t) = s'(t) = 3t^2 - 9t - 7 = 5 \quad \Leftrightarrow \quad 3t^2 - 9t - 12 = 0 \quad \Leftrightarrow$
$3(t-4)(t+1) = 0 \quad \Leftrightarrow \quad t = 4$ or -1. Since $t \ge 0$, the particle reaches a velocity of 5 m/s at $t = 4$ s.

8. (a) At maximum height the velocity of the ball is 0 ft/s. $v(t) = s'(t) = 80 - 32t = 0 \quad \Leftrightarrow \quad 32t = 80 \quad \Leftrightarrow$
$t = \frac{5}{2}$. So the maximum height is $s\left(\frac{5}{2}\right) = 80\left(\frac{5}{2}\right) - 16\left(\frac{5}{2}\right)^2 = 200 - 100 = 100$ ft.

(b) $s(t) = 80t - 16t^2 = 96 \quad \Leftrightarrow \quad 16t^2 - 80t + 96 = 0 \quad \Leftrightarrow \quad 16(t-3)(t-2) = 0$. So the ball has a height of 96 ft on the way up at $t = 2$ and on the way down at $t = 3$. At these times the velocities are $v(2) = 80 - 32(2) = 16$ ft/s and $v(3) = 80 - 32(3) = -16$ ft/s respectively.

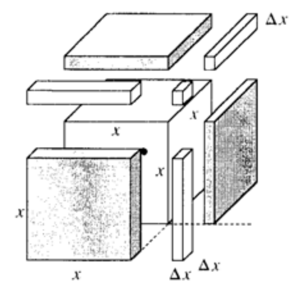
9. (a) $A(x) = x^2 \Rightarrow A'(x) = 2x$. $A'(15) = 30\ \text{mm}^2/\text{mm}$ is the rate at which the area is increasing with respect to the side length as x reaches 15 mm.

(b) The perimeter is $P(x) = 4x$, so $A'(x) = 2x = \frac{1}{2}(4x) = \frac{1}{2}P(x)$. The figure suggests that if Δx is small, then the change in the area of the square is approximately half of its perimeter (2 of the 4 sides) times Δx. From the figure, $\Delta A = 2x(\Delta x) + (\Delta x)^2$. If Δx is small, then $\Delta A \approx 2x(\Delta x)$ and so $\Delta A/\Delta x \approx 2x$.



10. (a) $V(x) = x^3 \Rightarrow \dfrac{dV}{dx} = 3x^2$. $\left.\dfrac{dV}{dx}\right|_{x=3} = 3(3)^2 = 27\ \text{mm}^3/\text{mm}$ is the rate at which the volume is increasing as x increases past 15 mm.

(b) The surface area is $S(x) = 6x^2$, so $V'(x) = 3x^2 = \frac{1}{2}(6x^2) = \frac{1}{2}S(x)$. The figure suggests that if Δx is small, then the change in the volume of the cube is approximately half of its surface area (the area of 3 of the 6 faces) times Δx. From the figure, $\Delta V = 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3$. If Δx is small, then $\Delta V \approx 3x^2(\Delta x)$ and so $\Delta V/\Delta x \approx 3x^2$.



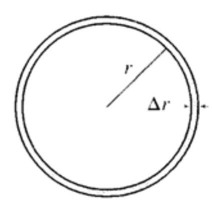
11. (a) $A(r) = \pi r^2$, so the average rate of change is:

(i) $\dfrac{A(3) - A(2)}{3 - 2} = \dfrac{9\pi - 4\pi}{1} = 5\pi$ (ii) $\dfrac{A(2.5) - A(2)}{2.5 - 2} = \dfrac{6.25\pi - 4\pi}{0.5} = 4.5\pi$

(iii) $\dfrac{A(2.1) - A(2)}{2.1 - 2} = \dfrac{4.41\pi - 4\pi}{0.1} = 4.1\pi$

(b) $A'(r) = 2\pi r$, so $A'(2) = 4\pi$.

(c) The circumference is $C(r) = 2\pi r = A'(r)$. The figure suggests that if Δr is small, then the change in the area of the circle (a ring around the outside) is approximately equal to its circumference times Δr. Straightening out this ring gives us a shape that is approximately rectangular with length $2\pi r$ and width Δr, so $\Delta A \approx 2\pi r(\Delta r)$. Algebraically, $\Delta A = A(r + \Delta r) - A(r) = \pi(r + \Delta r)^2 - \pi r^2 = 2\pi r(\Delta r) + \pi(\Delta r)^2$. So we see that if Δr is small, then $\Delta A \approx 2\pi r(\Delta r)$ and therefore, $\Delta A/\Delta r \approx 2\pi r$.



12. After t seconds the radius is $r = 60t$, so the area is $A(t) = \pi(60t)^2 = 3600\pi t^2 \Rightarrow A'(t) = 7200\pi t \Rightarrow$

(a) $A'(1) = 7200\pi\ \text{cm}^2/\text{s}$ (b) $A'(3) = 21{,}600\pi\ \text{cm}^2/\text{s}$ (c) $A'(5) = 36{,}000\pi\ \text{cm}^2/\text{s}$

As time goes by, the area grows at an increasing rate. In fact, the rate of change is linear with respect to time.

13. $S(r) = 4\pi r^2 \Rightarrow S'(r) = 8\pi r \Rightarrow$

(a) $S'(1) = 8\pi\ \text{ft}^2/\text{ft}$ (b) $S'(2) = 16\pi\ \text{ft}^2/\text{ft}$ (c) $S'(3) = 24\pi\ \text{ft}^2/\text{ft}$

As the radius increases, the surface area grows at an increasing rate. In fact, the rate of change is linear with respect to the radius.

14. (a) $V(r) = \frac{4}{3}\pi r^3 \quad \Leftrightarrow \quad$ the average rate of change is

(i) $\dfrac{V(8) - V(5)}{8-5} = \dfrac{\frac{4}{3}\pi(512) - \frac{4}{3}\pi(125)}{3} = 172\pi \ \mu\text{m}^3/\mu\text{m}$

(ii) $\dfrac{V(6) - V(5)}{6-5} = \dfrac{\frac{4}{3}\pi(216) - \frac{4}{3}\pi(125)}{1} = 121.\overline{3}\pi \ \mu\text{m}^3/\mu\text{m}$

(iii) $\dfrac{V(5.1) - V(5)}{5.1-5} = \dfrac{\frac{4}{3}\pi(5.1)^3 - \frac{4}{3}\pi(5)^3}{0.1} = 102.01\overline{3}\pi \ \mu\text{m}^3/\mu\text{m}$

(b) $V'(r) = 4\pi r^2$, so $V'(5) = 100\pi \ \mu\text{m}^3/\mu\text{m}$.

(c) $V(r) = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad V'(r) = 4\pi r^2 = S(r)$. By analogy with Exercise 11(c), we can say that the change in the volume of the spherical shell, ΔV, is approximately equal to its thickness, Δr, times the surface area of the inner sphere. Thus, $\Delta V \approx 4\pi r^2 (\Delta r)$ and so $\Delta V/\Delta r \approx 4\pi r^2$.

15. $f(x) = 3x^2$, so the linear density at x is $\rho(x) = f'(x) = 6x$.

(a) $\rho(1) = 6$ kg/m (b) $\rho(2) = 12$ kg/m (c) $\rho(3) = 18$ kg/m

Since ρ is an increasing function, the density will be the highest at the right end of the rod and lowest at the left end.

16. $V(t) = 5000(1 - t/40)^2 = 5000\left(1 - \frac{1}{20}t + \frac{1}{1600}t^2\right) \quad \Rightarrow \quad V'(t) = 5000\left(-\frac{1}{20} + \frac{1}{800}t\right) = -250\left(1 - \frac{1}{40}t\right)$

(a) $V'(5) = -250\left(1 - \frac{5}{40}\right) = -218.75$ gal/min (b) $V'(10) = -250\left(1 - \frac{10}{40}\right) = -187.5$ gal/min

(c) $V'(20) = -250\left(1 - \frac{20}{40}\right) = -125$ gal/min (d) $V'(40) = -250\left(1 - \frac{40}{40}\right) = 0$ gal/min

The water is flowing out the fastest at the beginning — when $t = 0$, $V'(t) = -250$ gal/min. The water is flowing out the slowest at the end — when $t = 40$, $V'(t) = 0$. As the tank empties, the water flows out more slowly.

17. $Q(t) = t^3 - 2t^2 + 6t + 2$, so the current is $Q'(t) = 3t^2 - 4t + 6$.

(a) $Q'(0.5) = 3(0.5)^2 - 4(0.5) + 6 = 4.75$ A

(b) $Q'(1) = 3(1)^2 - 4(1) + 6 = 5$ A

The current is lowest when Q' has a minimum. $Q''(t) = 6t - 4 < 0$ when $t < \frac{2}{3}$. So the current decreases when $t < \frac{2}{3}$ and increases when $t > \frac{2}{3}$. Thus, the current is lowest at $\frac{2}{3}$ s.

18. (a) $F = \dfrac{GmM}{r^2} = (GmM)r^{-2} \quad \Rightarrow \quad \dfrac{dF}{dr} = -2(GmM)r^{-3} = -\dfrac{2GmM}{r^3}$, which is the rate of change of the force with respect to the distance between the bodies. The minus sign indicates that as the distance r between the bodies increases, the magnitude of the force F exerted by the body of mass m on the body of mass M is decreasing.

(b) Given $F'(20{,}000) = -2$, find $F'(10{,}000)$. $-2 = -\dfrac{2GmM}{20{,}000^3} \quad \Rightarrow \quad GmM = 20{,}000^3$.

$$F'(10{,}000) = -\frac{2\left(20{,}000^3\right)}{10{,}000^3} = -2 \cdot 2^3 = -16 \text{ N/km}$$

19. (a) $PV = C \quad \Rightarrow \quad V = \dfrac{C}{P} \quad \Rightarrow \quad \dfrac{dV}{dP} = -\dfrac{C}{P^2}$

(b) From the formula for dV/dP in part (a), we see that as P increases, the absolute value of dV/dP decreases. Thus, the volume is decreasing more rapidly at the beginning.

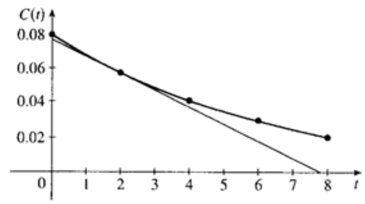
(c) $\beta = -\dfrac{1}{V}\dfrac{dV}{dP} = -\dfrac{1}{V}\left(-\dfrac{C}{P^2}\right) = \dfrac{C}{(PV)P} = \dfrac{C}{CP} = \dfrac{1}{P}$

20. (a) (i) $\dfrac{C(6)-C(2)}{6-2}=\dfrac{0.0295-0.0570}{4}$
$=-0.006875\ (\text{moles/L})/\text{min}$

(ii) $\dfrac{C(4)-C(2)}{4-2}=\dfrac{0.0408-0.0570}{2}$
$=-0.0081\ (\text{moles/L})/\text{min}$

(iii) $\dfrac{C(2)-C(0)}{2-0}=\dfrac{0.0570-0.0800}{2}$
$=-0.0115\ (\text{moles/L})/\text{min}$

(b) Slope $=\dfrac{\Delta C}{\Delta t}\approx-\dfrac{0.077}{7.8}$
$\approx-0.01\ (\text{moles/L})/\text{min}$



21. (a) **1920:** $m_1=\frac{1860-1750}{1920-1910}=\frac{110}{10}=11$, $m_2=\frac{2070-1860}{1930-1920}=\frac{210}{10}=21$,
$(m_1+m_2)/2=(11+21)/2=16$ million/year

1980: $m_1=\frac{4450-3700}{1980-1970}=\frac{750}{10}=75$, $m_2=\frac{5300-4450}{1990-1980}=\frac{850}{10}=85$,
$(m_1+m_2)/2=(75+85)/2=80$ million/year

(b) $P(t)=at^3+bt^2+ct+d$ where $a=2325.67$, $b=-1.306488\times10^7$, $c=2.44631\times10^{10}$, and $d=-1.52658\times10^{13}$.

(c) $P(t)=at^3+bt^2+ct+d \quad\Rightarrow\quad P'(t)=3at^2+2bt+c$

(d) $P'(1920)=3\,(2325.67)\,(1920)^2+2\left(-1.306448\times10^7\right)(1920)+2.44631\times10^{10}$
$=14{,}010{,}464/\text{year}$ [smaller than the answer in part (a), but close to it]

$P'(1980)=78{,}845{,}204/\text{year}$ (smaller, but close)

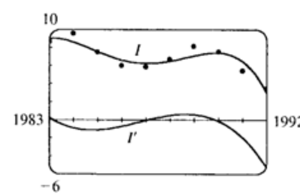
(e) $P'(1985)=86{,}515{,}627.25/\text{year}$, so the rate of growth in 1985 was about 86.5 million/year.

22. (a) $I(t)=at^4+bt^3+ct^2+dt+e$ for $1983\le t\le1992$, where $a=-0.0145512821$, $b=115.636927$, $c=-344{,}605.8704$, $d=456{,}421{,}256$, and $e=-2.266939\times10^{11}$.

(b) Using the values in part (a), $I'(t)=4at^3+3bt^2+2ct+d$.

(c) $I'(1988)\approx0.49559$ and $I'(1991)\approx-1.95946$, so the interest rate was increasing at about $\frac{1}{2}$ percent per year in 1988 and decreasing at about 2 percent per year in 1991.

(d)



23. (a) $[\mathrm{C}]=\dfrac{a^2kt}{akt+1} \quad\Rightarrow$

rate of reaction $=\dfrac{d[\mathrm{C}]}{dt}=\dfrac{(akt+1)\left(a^2k\right)-\left(a^2kt\right)(ak)}{(akt+1)^2}=\dfrac{a^2k\,(akt+1-akt)}{(akt+1)^2}=\dfrac{a^2k}{(akt+1)^2}$

(b) If $x=[\mathrm{C}]$, then $a-x=a-\dfrac{a^2kt}{akt+1}=\dfrac{a^2kt+a-a^2kt}{akt+1}=\dfrac{a}{akt+1}$.

So $k\,(a-x)^2=k\left(\dfrac{a}{akt+1}\right)^2=\dfrac{a^2k}{(akt+1)^2}=\dfrac{d[\mathrm{C}]}{dt}$ [from part (a)] $=\dfrac{dx}{dt}$.

24. $\dfrac{1}{p} = \dfrac{1}{f} - \dfrac{1}{q} \;\Leftrightarrow\; \dfrac{1}{p} = \dfrac{q-f}{fq} \;\Leftrightarrow\; p = \dfrac{fq}{q-f}$. So $\dfrac{dp}{dq} = \dfrac{f(q-f)-fq}{(q-f)^2} = -\dfrac{f^2}{(q-f)^2}$.

25. (a) Using $v = \dfrac{P}{4\eta l}\left(R^2 - r^2\right)$ with $R = 0.01$, $l = 3$, $P = 3000$, and $\eta = 0.027$, we have v as a function of r:

$v(r) = \dfrac{3000}{4(0.027)3}\left(0.01^2 - r^2\right)$. $v(0) = 0.\overline{925}$ cm/s, $v(0.005) = 0.69\overline{4}$ cm/s, $v(0.01) = 0$.

(b) $v(r) = \dfrac{P}{4\eta l}\left(R^2 - r^2\right) \;\Rightarrow\; v'(r) = \dfrac{P}{4\eta l}(-2r) = -\dfrac{Pr}{2\eta l}$. When $l = 3$, $P = 3000$, and $\eta = 0.027$, we have

$v'(r) = -\dfrac{3000r}{2(0.027)3}$. $v'(0) = 0$, $v'(0.005) = -92.\overline{592}$ (cm/s)/cm, and $v'(0.01) = -185.\overline{185}$ (cm/s)/cm.

(c) The velocity is greatest where $r = 0$ (at the center) and the velocity is changing most where $r = R = 0.01$ cm (at the edge).

26. (a) (i) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{1}{2}\sqrt{\dfrac{T}{\rho}}\right)L^{-1} \;\Rightarrow\; \dfrac{df}{dL} = -\left(\dfrac{1}{2}\sqrt{\dfrac{T}{\rho}}\right)L^{-2} = -\dfrac{1}{2L^2}\sqrt{\dfrac{T}{\rho}}$

(ii) $f = \left(\dfrac{1}{2L\sqrt{\rho}}\right)T^{1/2} \;\Rightarrow\; \dfrac{df}{dT} = \dfrac{1}{2}\left(\dfrac{1}{2L\sqrt{\rho}}\right)T^{-1/2} = \dfrac{1}{4L\sqrt{T\rho}}$

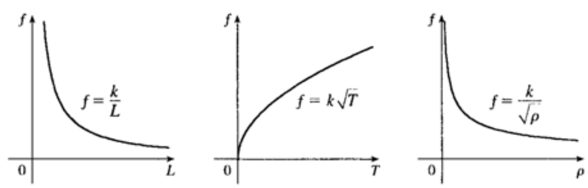
(iii) $f = \left(\dfrac{\sqrt{T}}{2L}\right)\rho^{-1/2} \;\Rightarrow\; \dfrac{df}{d\rho} = -\dfrac{1}{2}\left(\dfrac{\sqrt{T}}{2L}\right)\rho^{-3/2} = -\dfrac{\sqrt{T}}{4L\rho^{3/2}}$

(b) *Note:* Illustrating tangent lines on the generic figures may help to explain the results.

(i) $\dfrac{df}{dL} < 0$ and L is decreasing $\Rightarrow$ f is increasing $\Rightarrow$ higher note

(ii) $\dfrac{df}{dT} > 0$ and T is increasing $\Rightarrow$ f is increasing $\Rightarrow$ higher note

(iii) $\dfrac{df}{d\rho} < 0$ and ρ is increasing $\Rightarrow$ f is decreasing $\Rightarrow$ lower note



27. (a) $C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3 \;\Rightarrow\; C'(x) = 3 + 0.02x + 0.0006x^2$

(b) $C'(100) = 3 + 0.02(100) + 0.0006(10{,}000) = 3 + 2 + 6 = \$11$/yard. $C'(100)$ is the rate at which costs are increasing as the 100th yard is produced. It predicts the cost of the 101st yard.

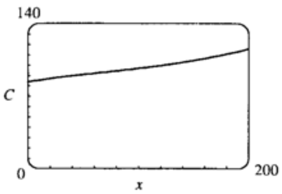
(c) The cost of manufacturing the 101st yard is

$$\begin{aligned} C(101) - C(100) &= (2000 + 303 + 102.01 + 206.0602) - (2000 + 300 + 100 + 200) \\ &= 11.0702 \approx \$11.07/\text{yard} \end{aligned}$$

28. (a) $C(x) = 84 + 0.16x - 0.0006x^2 + 0.000003x^3 \;\Rightarrow\; C'(x) = 0.16 - 0.0012x + 0.000009x^2 \;\Rightarrow$ $C'(100) = 0.13$. This is the rate at which costs are increasing as the 100th item is produced.

(b) $C(101) - C(100) = 97.13030299 - 97 \approx 0.13$.

(c)



From the graph, we can estimate the x-coordinate of the point of inflection to be between 60 and 80.

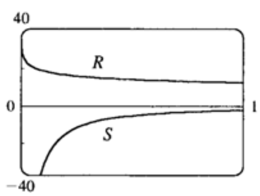
(d) $C''(x) = -0.0012 + 0.000018x = 0 \Rightarrow$ $x = 66\frac{2}{3}$ and $C''(x)$ changes from negative to positive at this value of x. This is where the marginal cost changes from decreasing to increasing and so has its minimum value.

29. (a) $A(x) = \dfrac{p(x)}{x} \Rightarrow A'(x) = \dfrac{xp'(x) - p(x) \cdot 1}{x^2}$. $A'(x) > 0 \Rightarrow A(x)$ is increasing; that is, the average productivity increases as the size of the workforce increases.

(b) Suppose $p'(x) > A(x)$. Then $p'(x) > \dfrac{p(x)}{x} \Rightarrow xp'(x) > p(x) \Rightarrow xp'(x) - p(x) > 0 \Rightarrow$ $\dfrac{xp'(x) - p(x)}{x^2} > 0 \Rightarrow A'(x) > 0$.

30. (a)
$$S = \frac{dR}{dx} = \frac{(1+4x^{0.4})(9.6x^{-0.6}) - (40+24x^{0.4})(1.6x^{-0.6})}{(1+4x^{0.4})^2}$$
$$= \frac{9.6x^{-0.6} + 38.4x^{-0.2} - 64x^{-0.6} - 38.4x^{-0.2}}{(1+4x^{0.4})^2} = -\frac{54.4x^{-0.6}}{(1+4x^{0.4})^2}$$

(b)



At low levels of brightness, R is quite large $[R(0) = 40]$ and is quickly decreasing, that is, S is negative with large absolute value. This is to be expected: at low levels of brightness, the eye is more sensitive to slight changes than it is at higher levels of brightness.

31. $PV = nRT \Rightarrow T = \dfrac{PV}{nR} = \dfrac{PV}{(10)(0.0821)} = \dfrac{1}{0.821}(PV)$.

$$\frac{dT}{dt} = \frac{1}{0.821}\left[P(t)V'(t) + V(t)P'(t)\right] = \frac{1}{0.821}[(8)(-0.15) + (10)(0.10)] \approx -0.2436 \text{ K/min}$$

32. (a) If $dP/dt = 0$, the population is stable (it is constant.)

(b) $\dfrac{dP}{dt} = 0 \Rightarrow \beta P = r_0\left(1 - \dfrac{P}{P_c}\right)P \Rightarrow \dfrac{\beta}{r_0} = 1 - \dfrac{P}{P_c} \Rightarrow \dfrac{P}{P_c} = 1 - \dfrac{\beta}{r_0} \Rightarrow P = P_c\left(1 - \dfrac{\beta}{r_0}\right)$.

If $P_c = 10{,}000$, $r_0 = 0.05$, and $\beta = 0.04$, then $P = 10{,}000\left(1 - \frac{4}{5}\right) = 2000$.

(c) If $\beta = 0.05$, then $P = 0$. There is no stable population.

33. (a) $\dfrac{dC}{dt} = 0$ and $\dfrac{dW}{dt} = 0$.

(b) The caribou go extinct $\Leftrightarrow C = 0$.

(c) We have **(1)** $0.05C - 0.001CW = 0$ and **(2)** $-0.05W + 0.0001CW = 0$. Adding 10 times **(2)** to **(1)** gives us $0.05C - 0.5W = 0 \Rightarrow C = 10W$. Substituting $C = 10W$ into **(1)** results in $W = 0$ or 50 and hence, $C = 0$ or 500. The pairs are $(0, 0)$ and $(500, 50)$. So it is possible for the two species to live in harmony.

3.5 Derivatives of Trigonometric Functions

1. $f(x) = x - 3\sin x \quad \Rightarrow \quad f'(x) = 1 - 3\cos x$

2. $f(x) = x\sin x \quad \Rightarrow \quad f'(x) = x \cdot \cos x + (\sin x) \cdot 1 = x\cos x + \sin x$

3. $y = \sin x + \cos x \quad \Rightarrow \quad dy/dx = \cos x - \sin x$

4. $y = \cos x - 2\tan x \quad \Rightarrow \quad dy/dx = -\sin x - 2\sec^2 x$

5. $g(t) = t^3\cos t \quad \Rightarrow \quad g'(t) = t^3(-\sin t) + (\cos t) \cdot 3t^2 = 3t^2\cos t - t^3\sin t$

6. $g(t) = 4\sec t + \tan t \quad \Rightarrow \quad g'(t) = 4\sec t\tan t + \sec^2 t$

7. $h(\theta) = \theta\csc\theta - \cot\theta \quad \Rightarrow \quad h'(\theta) = \theta(-\csc\theta\cot\theta) + (\csc\theta) \cdot 1 - \left(-\csc^2\theta\right) = \csc\theta - \theta\csc\theta\cot\theta + \csc^2\theta$

8. $h(\theta) = \sqrt{\theta}\cot\theta \quad \Rightarrow \quad h'(\theta) = \sqrt{\theta}\left(-\csc^2\theta\right) + (\cot\theta)\left(\frac{1}{2}\theta^{-1/2}\right) = \dfrac{1}{2\sqrt{\theta}}\cot\theta - \sqrt{\theta}\csc^2\theta$

9. $y = \dfrac{\tan x}{x} \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{x\sec^2 x - \tan x}{x^2}$

10. $y = \dfrac{\sin x}{1 + \cos x} \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{(1 + \cos x)\cos x - \sin x(-\sin x)}{(1 + \cos x)^2} = \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2} = \frac{\cos x + 1}{(1 + \cos x)^2} = \frac{1}{1 + \cos x}$$

11. $y = \dfrac{x}{\sin x + \cos x} \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{(\sin x + \cos x) - x(\cos x - \sin x)}{(\sin x + \cos x)^2} = \frac{(1 + x)\sin x + (1 - x)\cos x}{\sin^2 x + \cos^2 x + 2\sin x\cos x} = \frac{(1 + x)\sin x + (1 - x)\cos x}{1 + \sin 2x}$$

12. $y = \dfrac{\tan x - 1}{\sec x} \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{\sec x\sec^2 x - (\tan x - 1)\sec x\tan x}{\sec^2 x} = \dfrac{\sec x\left(\sec^2 x - \tan^2 x + \tan x\right)}{\sec^2 x} = \dfrac{1 + \tan x}{\sec x}$

Another Method: Simplify y first: $y = \sin x - \cos x \quad \Rightarrow \quad y' = \cos x + \sin x$.

13. $y = \dfrac{\sin x}{x^2} \quad \Rightarrow \quad y' = \dfrac{x^2\cos x - (\sin x)(2x)}{\left(x^2\right)^2} = \dfrac{x(x\cos x - 2\sin x)}{x^4} = \dfrac{x\cos x - 2\sin x}{x^3}$

14. $y = \tan\theta(\sin\theta + \cos\theta) \quad \Rightarrow$

$y' = \tan\theta(\cos\theta - \sin\theta) + (\sin\theta + \cos\theta)\sec^2\theta = \sin\theta - \sin\theta\tan\theta + \sin\theta\sec^2\theta + \sec\theta$

15. $y = \csc x\cot x \quad \Rightarrow \quad dy/dx = (-\csc x\cot x)\cot x + \csc x\left(-\csc^2 x\right) = -\csc x\left(\cot^2 x + \csc^2 x\right)$

16. Recall that if $y = fgh$, then $y' = f'gh + fg'h + fgh'$. $y = x\sin x\cos x \quad \Rightarrow$

$$\frac{dy}{dx} = \sin x\cos x + x\cos x\cos x + x\sin x(-\sin x) = \sin x\cos x + x\cos^2 x - x\sin^2 x$$

17. $\dfrac{d}{dx}(\csc x) = \dfrac{d}{dx}\left(\dfrac{1}{\sin x}\right) = \dfrac{(\sin x)(0) - 1(\cos x)}{\sin^2 x} = \dfrac{-\cos x}{\sin^2 x} = -\dfrac{1}{\sin x} \cdot \dfrac{\cos x}{\sin x} = -\csc x\cot x$

18. $\dfrac{d}{dx}(\sec x) = \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right) = \dfrac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x} = \dfrac{\sin x}{\cos^2 x} = \dfrac{1}{\cos x} \cdot \dfrac{\sin x}{\cos x} = \sec x\tan x$

19. $\dfrac{d}{dx}(\cot x) = \dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right) = \dfrac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = -\dfrac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\dfrac{1}{\sin^2 x}$

$= -\csc^2 x$

20. $f(x) = \cos x \quad \Rightarrow$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h\to 0} \left(\cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h} \right) = \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h}$$

$$= (\cos x)(0) - (\sin x)(1) = -\sin x$$

21. $y = \tan x \quad \Rightarrow \quad y' = \sec^2 x \Rightarrow$ the slope of the tangent line at $\left(\frac{\pi}{4}, 1\right)$ is $\sec^2 \frac{\pi}{4} = \left(\sqrt{2}\right)^2 = 2$ and an equation is $y - 1 = 2\left(x - \frac{\pi}{4}\right)$ or $y = 2x + 1 - \frac{\pi}{2}$.

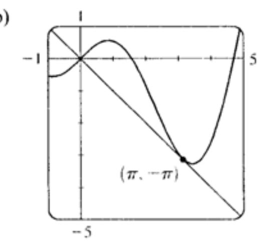
22. $y = 2\sin x \quad \Rightarrow \quad y' = 2\cos x \quad \Rightarrow$ the slope of the tangent line at $\left(\frac{\pi}{6}, 1\right)$ is $2\cos\frac{\pi}{6} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$ and an equation is $y - 1 = \sqrt{3}\left(x - \frac{\pi}{6}\right)$ or $y = \sqrt{3}x + 1 - \frac{\sqrt{3}\pi}{6}$.

23. $y = x + \cos x \quad \Rightarrow \quad y' = 1 - \sin x$. At $(0, 1)$, $y' = 1$, and an equation of the tangent line is $y - 1 = 1(x - 0)$, or $y = x + 1$.

24. $y = \dfrac{1}{\sin x + \cos x} \quad \Rightarrow \quad y' = -\dfrac{\cos x - \sin x}{(\sin x + \cos x)^2}$ (Reciprocal Rule). At $(0, 1)$, $y' = -\dfrac{1 - 0}{(0+1)^2} = -1$, and an equation of the tangent line is $y - 1 = -1(x - 0)$, or $y = -x + 1$.

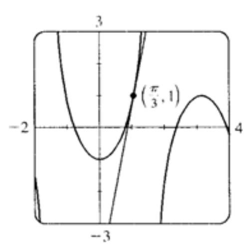
25. (a) $y = x\cos x \quad \Rightarrow \quad y' = x(-\sin x) + \cos x(1) = \cos x - x\sin x$. So the slope of the tangent at the point $(\pi, -\pi)$ is $\cos\pi - \pi\sin\pi = -1 - \pi(0) = -1$, and its equation is $y + \pi = -(x - \pi) \quad \Leftrightarrow \quad y = -x$.

(b)



26. (a) $y = \sec x - 2\cos x \quad \Rightarrow \quad y' = \sec x \tan x + 2\sin x \quad \Rightarrow$ the slope of the tangent line at $\left(\frac{\pi}{3}, 1\right)$ is $\sec\frac{\pi}{3}\tan\frac{\pi}{3} + 2\sin\frac{\pi}{3} = 2\sqrt{3} + 2 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3}$ and an equation is $y - 1 = 3\sqrt{3}\left(x - \frac{\pi}{3}\right)$ or $y = 3\sqrt{3}x + 1 - \pi\sqrt{3}$.

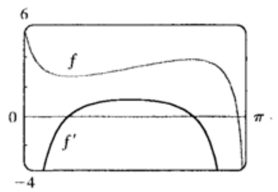
(b)



27. (a) $f(x) = 2x + \cot x \quad \Rightarrow$

$f'(x) = 2 - \csc^2 x$

(b)



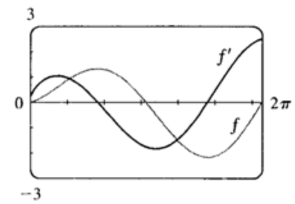
Notice that $f'(x) = 0$ when f has a horizontal tangent. Also, $f'(x)$ is large negative when the graph of f is steep.

28. (a) $f(x) = \sqrt{x}\sin x \quad \Rightarrow$

$$f'(x) = \sqrt{x}\cos x + (\sin x)\left(\tfrac{1}{2}x^{-1/2}\right)$$

$$= \sqrt{x}\cos x + \frac{\sin x}{2\sqrt{x}}$$

(b)



Notice that $f'(x) = 0$ when f has a horizontal tangent. f' is positive when f is increasing and f' is negative when f is decreasing.

29. $y = x + 2\sin x$ has a horizontal tangent when $y' = 1 + 2\cos x = 0 \quad \Leftrightarrow \quad \cos x = -\frac{1}{2} \quad \Leftrightarrow \quad x = \frac{2\pi}{3} + 2\pi n$ or $\frac{4\pi}{3} + 2\pi n$ or, equivalently, $(2n+1)\pi \pm \frac{\pi}{3}$, n an integer.

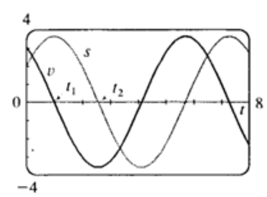
30. $$y = \frac{\cos x}{2 + \sin x} \quad \Rightarrow \quad y' = \frac{-\sin x\,(2 + \sin x) - \cos x\cos x}{(2 + \sin x)^2} = \frac{-2\sin x - \sin^2 x - \cos^2 x}{(2 + \sin x)^2} = \frac{-2\sin x - 1}{(2 + \sin x)^2} = 0$$
when $-2\sin x - 1 = 0 \quad \Leftrightarrow \quad \sin x = -\frac{1}{2} \quad \Leftrightarrow \quad x = \frac{11\pi}{6} + 2\pi n$ or $x = \frac{7\pi}{6} + 2\pi n$, n an integer. So $y = \frac{1}{\sqrt{3}}$ or $y = -\frac{1}{\sqrt{3}}$ and the points on the curve with horizontal tangents are: $\left(\frac{11\pi}{6} + 2\pi n, \frac{1}{\sqrt{3}}\right)$, $\left(\frac{7\pi}{6} + 2\pi n, -\frac{1}{\sqrt{3}}\right)$, n an integer.

31. (a) $x(t) = 8\sin t \quad \Rightarrow \quad v(t) = x'(t) = 8\cos t$

(b) $x\left(\frac{2\pi}{3}\right) = 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}$, $v\left(\frac{2\pi}{3}\right) = 8\left(-\frac{1}{2}\right) = -4$. Since $v\left(\frac{2\pi}{3}\right) < 0$, the particle is moving to the left.

32. (a) $s(t) = 2\cos t + 3\sin t \quad \Rightarrow \quad v(t) = -2\sin t + 3\cos t$

(b)

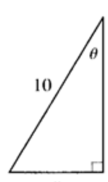


(c) $s = 0 \quad \Rightarrow \quad t_2 \approx 2.55$. So the mass passes through the equilibrium position for the first time when $t \approx 2.55$ s.

(d) $v = 0 \quad \Rightarrow \quad t_1 \approx 0.98$, $s(t_1) \approx 3.61$ cm. So the mass travels a maximum of about 3.6 cm (upward and downward) from its equilibrium position.

(e) The speed $|v|$ is greatest when $s = 0$; that is, when $t = t_2 + n\pi$, n a positive integer.

33.



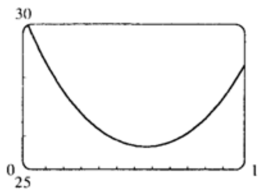
From the diagram we can see that $\sin\theta = x/10 \quad \Leftrightarrow \quad x = 10\sin\theta$. But we want to find the rate of change of x with respect to θ, that is, $dx/d\theta$. Taking the derivative of the above expression, $dx/d\theta = 10(\cos\theta)$. So when $\theta = \frac{\pi}{3}$,

$dx/d\theta = 10\cos\frac{\pi}{3} = 10\left(\frac{1}{2}\right) = 5$ ft/rad.

34. (a) $F = \dfrac{\mu W}{\mu \sin\theta + \cos\theta} \;\Rightarrow\; \dfrac{dF}{d\theta} = \dfrac{(\mu \sin\theta + \cos\theta)(0) - \mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2} = \dfrac{\mu W(\sin\theta - \mu\cos\theta)}{(\mu\sin\theta+\cos\theta)^2}$

(b) $\dfrac{dF}{d\theta} = 0 \;\Rightarrow\; \mu W(\sin\theta - \mu\cos\theta) = 0 \;\Rightarrow\; \sin\theta = \mu\cos\theta \;\Rightarrow\; \tan\theta = \mu \;\Rightarrow\; \theta = \tan^{-1}\mu$

(c)



From the graph, we see that $\dfrac{dF}{d\theta} = 0 \;\Rightarrow\; \theta \approx 0.54$. Checking this with part (b) and $\mu = 0.6$, we calculate $\theta = \tan^{-1} 0.6 \approx 0.54$. So the graph is consistent with part (b).

35. $\displaystyle\lim_{t\to 0}\frac{\sin 5t}{t} = \lim_{t\to 0}\frac{5\sin 5t}{5t} = 5\lim_{t\to 0}\frac{\sin 5t}{5t} = 5\cdot 1 = 5$

36. $\displaystyle\lim_{t\to 0}\frac{\sin 8t}{\sin 9t} = \lim_{t\to 0}\frac{8\left(\dfrac{\sin 8t}{8t}\right)}{9\left(\dfrac{\sin 9t}{9t}\right)} = \frac{8\lim\limits_{t\to 0}\dfrac{\sin 8t}{8t}}{9\lim\limits_{t\to 0}\dfrac{\sin 9t}{9t}} = \frac{8\cdot 1}{9\cdot 1} = \frac{8}{9}$

37. $\displaystyle\lim_{\theta\to 0}\frac{\sin(\cos\theta)}{\sec\theta} = \frac{\sin\left(\lim\limits_{\theta\to 0}\cos\theta\right)}{\lim\limits_{\theta\to 0}\sec\theta} = \frac{\sin 1}{1} = \sin 1$

38. $\displaystyle\lim_{\theta\to 0}\frac{\cos\theta - 1}{\sin\theta} = \lim_{\theta\to 0}\frac{\dfrac{\cos\theta-1}{\theta}}{\dfrac{\sin\theta}{\theta}} = \frac{\lim\limits_{\theta\to 0}\dfrac{\cos\theta-1}{\theta}}{\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}} = \frac{0}{1} = 0$

39. $\displaystyle\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta} = \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)\sin\theta = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\lim_{\theta\to 0}\sin\theta = 1\cdot 0 = 0$

40. $\displaystyle\lim_{x\to 0}\frac{\tan x}{4x} = \lim_{x\to 0}\frac{1}{4}\frac{\sin x}{x}\cdot\frac{1}{\cos x} = \frac{1}{4}\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{1}{\cos x} = \tfrac{1}{4}\cdot 1\cdot 1 = \tfrac{1}{4}$

41. $\displaystyle\lim_{x\to 0}\frac{\cot 2x}{\csc x} = \lim_{x\to 0}\frac{\cos 2x\sin x}{\sin 2x} = \lim_{x\to 0}\cos 2x\left[\frac{(\sin x)/x}{(\sin 2x)/x}\right] = \lim_{x\to 0}\cos 2x\left[\frac{\lim\limits_{x\to 0}[(\sin x)/x]}{2\lim\limits_{x\to 0}[(\sin 2x)/2x]}\right]$

$\displaystyle = 1\cdot\frac{1}{2\cdot 1} = \frac{1}{2}$

42. $\displaystyle\lim_{x\to\pi/4}\frac{\sin x - \cos x}{\cos 2x} = \lim_{x\to\pi/4}\frac{\sin x - \cos x}{\cos^2 x - \sin^2 x} = \lim_{x\to\pi/4}\frac{\sin x - \cos x}{(\cos x + \sin x)(\cos x - \sin x)}$

$\displaystyle = \lim_{x\to\pi/4}\frac{-1}{\cos x + \sin x} = \frac{-1}{\cos\frac{\pi}{4} + \sin\frac{\pi}{4}} = \frac{-1}{\sqrt{2}}$

43. Divide numerator and denominator by θ. ($\sin\theta$ also works.)

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta + \tan\theta} = \lim_{\theta\to 0}\frac{\dfrac{\sin\theta}{\theta}}{1 + \dfrac{\sin\theta}{\theta}\cdot\dfrac{1}{\cos\theta}} = \frac{\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}}{1 + \lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}\lim\limits_{\theta\to 0}\dfrac{1}{\cos\theta}} = \frac{1}{1 + 1\cdot 1} = \frac{1}{2}$$

44. $\displaystyle\lim_{x\to 1}\frac{\sin(x-1)}{x^2+x-2} = \lim_{x\to 1}\frac{\sin(x-1)}{(x+2)(x-1)} = \lim_{x\to 1}\frac{1}{x+2}\lim_{x\to 1}\frac{\sin(x-1)}{x-1} = \tfrac{1}{3}\cdot 1 = \tfrac{1}{3}$

45. (a) $\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} \Rightarrow \sec^2 x = \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$. So $\sec^2 x = \frac{1}{\cos^2 x}$.

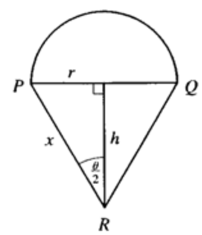
(b) $\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} \Rightarrow \sec x\tan x = \frac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x}$. So $\sec x\tan x = \frac{\sin x}{\cos^2 x}$.

(c) $\frac{d}{dx}(\sin x + \cos x) = \frac{d}{dx}\frac{1+\cot x}{\csc x} \Rightarrow$

$$\cos x - \sin x = \frac{\csc x(-\csc^2 x) - (1+\cot x)(-\csc x\cot x)}{\csc^2 x} = \frac{-\csc^2 x + \cot^2 x + \cot x}{\csc x}$$

So $\cos x - \sin x = \frac{\cot x - 1}{\csc x}$.

46. Let $|PR| = x$. Then we get the following formulas for r and h in terms of θ and x:



$\sin\frac{\theta}{2} = \frac{r}{x} \Rightarrow r = x\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2} = \frac{h}{x} \Rightarrow h = x\cos\frac{\theta}{2}$. Now

$A(\theta) = \frac{1}{2}\pi r^2$ and $B(\theta) = \frac{1}{2}(2r)h = rh$. So

$$\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} = \lim_{\theta\to 0^+}\frac{\frac{1}{2}\pi r^2}{rh} = \lim_{\theta\to 0^+}\frac{\frac{1}{2}\pi x\sin(\theta/2)}{x\cos(\theta/2)}$$
$$= \lim_{\theta\to 0^+}\tfrac{1}{2}\pi\tan(\theta/2) = 0.$$

47. By the definition of radian measure, $s = r\theta$, where r is the radius of the circle.

By drawing the bisector of the angle θ, we can see that $\sin\frac{\theta}{2} = \frac{d/2}{r} \Rightarrow d = 2r\sin\frac{\theta}{2}$.

So $\lim_{\theta\to 0^+}\frac{s}{d} = \lim_{\theta\to 0^+}\frac{r\theta}{2r\sin(\theta/2)} = \lim_{\theta\to 0^+}\frac{2\cdot(\theta/2)}{2\sin(\theta/2)} = \lim_{\theta\to 0}\frac{\theta/2}{\sin(\theta/2)} = 1$. [This is just the reciprocal of the limit $\lim_{x\to 0}\frac{\sin x}{x} = 1$ combined with the fact that as $\theta \to 0$, $\frac{\theta}{2} \to 0$ also.]

3.6 The Chain Rule

1. Let $u = g(x) = x^2 + 4x + 6$ and $y = f(u) = u^5$.
Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (5u^4)(2x+4) = 5(x^2+4x+6)^4(2x+4) = 10(x^2+4x+6)^4(x+2)$.

2. Let $u = g(x) = 3x$ and $y = f(u) = \tan u$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\sec^2 u)(3) = 3\sec^2 3x$.

3. Let $u = g(x) = \tan x$ and $y = f(u) = \cos u$.
Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (-\sin u)(\sec^2 x) = -\sin(\tan x)\sec^2 x$.

4. Let $u = g(x) = 1 + x^3$ and $y = f(u) = u^{1/3}$.
Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{3}u^{-2/3}(3x^2) = (1+x^3)^{-2/3}x^2 = \frac{x^2}{(1+x^3)^{2/3}}$.

5. Let $u = g(x) = \sin x$ and $y = f(u) = \sqrt{u}$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2}u^{-1/2}\cos x = \frac{\cos x}{2\sqrt{u}} = \frac{\cos x}{2\sqrt{\sin x}}$.

6. Let $u = g(x) = \sqrt{x}$ and $y = f(u) = \sin u$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\cos u)\left(\frac{1}{2}x^{-1/2}\right) = \frac{\cos u}{2\sqrt{x}} = \frac{\cos\sqrt{x}}{2\sqrt{x}}$.

7. $F(x) = (x^3+4x)^7 \Rightarrow F'(x) = 7(x^3+4x)^6(3x^2+4)$ [or $7x^6(x^2+4)^6(3x^2+4)$]

8. $F(x) = (x^2-x+1)^3 \Rightarrow F'(x) = 3(x^2-x+1)^2(2x-1)$

9. $g(x) = \sqrt{x^2 - 7x} = (x^2 - 7x)^{1/2} \quad \Rightarrow \quad g'(x) = \frac{1}{2}(x^2 - 7x)^{-1/2}(2x - 7) = \dfrac{2x - 7}{2\sqrt{x^2 - 7x}}$

10. $f(t) = \dfrac{1}{(t^2 - 2t - 5)^4} = (t^2 - 2t - 5)^{-4} \quad \Rightarrow$

$f'(t) = -4(t^2 - 2t - 5)^{-5}(2t - 2) = \dfrac{8(1 - t)}{(t^2 - 2t - 5)^5}$

11. $h(t) = (t - 1/t)^{3/2} \quad \Rightarrow \quad h'(t) = \frac{3}{2}(t - 1/t)^{1/2}(1 + 1/t^2)$

12. $f(t) = \sqrt[3]{1 + \tan t} = (1 + \tan t)^{1/3} \quad \Rightarrow \quad f'(t) = \frac{1}{3}(1 + \tan t)^{-2/3}\sec^2 t = \dfrac{\sec^2 t}{3\sqrt[3]{(1 + \tan t)^2}}$

13. $y = \cos(a^3 + x^3) \quad \Rightarrow \quad y' = -\sin(a^3 + x^3) \cdot 3x^2 = -3x^2 \sin(a^3 + x^3)$

14. $y = a^3 + \cos^3 x \quad \Rightarrow \quad y' = 3(\cos x)^2(-\sin x) = -3\sin x \cos^2 x$

15. $y = \cot(x/2) \quad \Rightarrow \quad y' = -\csc^2(x/2) \cdot \frac{1}{2} = -\frac{1}{2}\csc^2(x/2)$

16. $y = 4\sec 5x \quad \Rightarrow \quad y' = 4\sec 5x \tan 5x\,(5) = 20\sec 5x \tan 5x$

17. $G(x) = (3x - 2)^{10}(5x^2 - x + 1)^{12} \quad \Rightarrow$

$$\begin{aligned} G'(x) &= (3x - 2)^{10}(12)(5x^2 - x + 1)^{11}(10x - 1) + 10(3x - 2)^9(3)(5x^2 - x + 1)^{12} \\ &= 6(3x - 2)^9(5x^2 - x + 1)^{11}[2(3x - 2)(10x - 1) + 5(5x^2 - x + 1)] \\ &= 6(3x - 2)^9(5x^2 - x + 1)^{11}(85x^2 - 51x + 9) \end{aligned}$$

18. $g(t) = (6t^2 + 5)^3(t^3 - 7)^4 \quad \Rightarrow$

$$\begin{aligned} g'(t) &= (6t^2 + 5)^3(4)(t^3 - 7)^3(3t^2) + 3(6t^2 + 5)^2(12t)(t^3 - 7)^4 \\ &= 12t(6t^2 + 5)^2(t^3 - 7)^3[t(6t^2 + 5) + 3(t^3 - 7)] \\ &= 12t(6t^2 + 5)^2(t^3 - 7)^3(9t^3 + 5t - 21) \end{aligned}$$

19. $y = (2x - 5)^4(8x^2 - 5)^{-3} \quad \Rightarrow$

$$\begin{aligned} y' &= 4(2x - 5)^3(2)(8x^2 - 5)^{-3} + (2x - 5)^4(-3)(8x^2 - 5)^{-4}(16x) \\ &= 8(2x - 5)^3(8x^2 - 5)^{-3} - 48x(2x - 5)^4(8x^2 - 5)^{-4} \end{aligned}$$

[This simplifies to $8(2x - 5)^3(8x^2 - 5)^{-4}(-4x^2 + 30x - 5)$.]

20. $y = (x^2 + 1)(x^2 + 2)^{1/3} \quad \Rightarrow$

$y' = 2x(x^2 + 2)^{1/3} + (x^2 + 1)\left(\frac{1}{3}\right)(x^2 + 2)^{-2/3}(2x) = 2x(x^2 + 2)^{1/3}\left[1 + \dfrac{x^2 + 1}{3(x^2 + 2)}\right]$

21. $y = x^3 \cos nx \quad \Rightarrow \quad y' = x^3(-\sin nx)(n) + \cos nx\,(3x^2) = x^2(3\cos nx - nx\sin nx)$

22. $F(s) = \sqrt{s^3 + 1}\,(s^2 + 1)^4 = (s^3 + 1)^{1/2}(s^2 + 1)^4 \quad \Rightarrow$

$$\begin{aligned} F'(s) &= \tfrac{1}{2}\left(s^3 + 1\right)^{-1/2}\left(3s^2\right)\left(s^2 + 1\right)^4 + \left(s^3 + 1\right)^{1/2}(4)\left(s^2 + 1\right)^3(2s) \\ &= \frac{3s^2(s^2 + 1)^4}{2\sqrt{s^3 + 1}} + 8s\left(s^2 + 1\right)^3\sqrt{s^3 + 1} \end{aligned}$$

23. $F(y) = \left(\dfrac{y - 6}{y + 7}\right)^3 \Rightarrow F'(y) = 3\left(\dfrac{y - 6}{y + 7}\right)^2 \dfrac{(y + 7)(1) - (y - 6)(1)}{(y + 7)^2} = 3\left(\dfrac{y - 6}{y + 7}\right)^2 \dfrac{13}{(y + 7)^2} = \dfrac{39(y - 6)^2}{(y + 7)^4}$

24. $s(t) = \left(\dfrac{t^3+1}{t^3-1}\right)^{1/4} \Rightarrow$

$s'(t) = \dfrac{1}{4}\left(\dfrac{t^3+1}{t^3-1}\right)^{-3/4} \dfrac{3t^2(t^3-1)-(t^3+1)(3t^2)}{(t^3-1)^2} = \dfrac{1}{2}\left(\dfrac{t^3+1}{t^3-1}\right)^{-3/4} \dfrac{-3t^2}{(t^3-1)^2}$

25. $f(z) = (2z-1)^{-1/5} \Rightarrow f'(z) = -\frac{1}{5}(2z-1)^{-6/5}(2) = -\frac{2}{5}(2z-1)^{-6/5}$

26. $f(x) = \dfrac{x}{\sqrt{7-3x}} \Rightarrow$

$f'(x) = \dfrac{\sqrt{7-3x} - x\left(\frac{1}{2}\right)(7-3x)^{-1/2}(-3)}{7-3x} = \dfrac{1}{\sqrt{7-3x}} + \dfrac{3x}{2(7-3x)^{3/2}}$ or $\dfrac{14-3x}{2(7-3x)^{3/2}}$

27. $y = \tan(\cos x) \Rightarrow y' = \sec^2(\cos x)\cdot(-\sin x) = -\sin x\sec^2(\cos x)$

28. $y = \dfrac{\sin^2 x}{\cos x} \Rightarrow$

$y' = \dfrac{\cos x(2\sin x\cos x) - \sin^2 x(-\sin x)}{\cos^2 x} = \dfrac{\sin x(2\cos^2 x + \sin^2 x)}{\cos^2 x} = \dfrac{\sin x(1+\cos^2 x)}{\cos^2 x}$

$= \sin x(1+\sec^2 x)$

Another Method: $y = \tan x\sin x \Rightarrow y' = \sec^2 x\sin x + \tan x\cos x = \sec^2 x\sin x + \sin x$

29. $y = \sec^2 2x - \tan^2 2x \Rightarrow y' = 2\sec 2x(\sec 2x\tan 2x)(2) - 2\tan 2x\sec^2(2x)(2) = 0$

Easier method: $y = \sec^2 2x - \tan^2 2x = 1 \Rightarrow y' = 0$

30. $y = \sqrt{1+2\tan x} \Rightarrow y' = \frac{1}{2}(1+2\tan x)^{-1/2}2\sec^2 x = \dfrac{\sec^2 x}{\sqrt{1+2\tan x}}$

31. $y = \sin^3 x + \cos^3 x \Rightarrow y' = 3\sin^2 x\cos x + 3\cos^2 x(-\sin x) = 3\sin x\cos x(\sin x - \cos x)$

32. $y = \sin^2(\cos kx) \Rightarrow y' = 2\sin(\cos kx)\cos(\cos kx)(-\sin kx)(k) = -k\sin kx\sin(2\cos kx)$

33. $y = (1+\cos^2 x)^6 \Rightarrow y' = 6(1+\cos^2 x)^5 2\cos x(-\sin x) = -12\cos x\sin x(1+\cos^2 x)^5$

34. $y = x\sin\dfrac{1}{x} \Rightarrow y' = \sin\dfrac{1}{x} + x\cos\dfrac{1}{x}\left(-\dfrac{1}{x^2}\right) = \sin\dfrac{1}{x} - \dfrac{1}{x}\cos\dfrac{1}{x}$

35. $y = \dfrac{1+\sin 2x}{1-\sin 2x} \Rightarrow y' = \dfrac{(1-\sin 2x)(2\cos 2x) - (1+\sin 2x)(-2\cos 2x)}{(1-\sin 2x)^2} = \dfrac{4\cos 2x}{(1-\sin 2x)^2}$

36. $y = \tan(x^2) + \tan^2 x \Rightarrow y' = \sec^2(x^2)(2x) + 2\tan x\sec^2 x$

37. $y = \tan^2(x^3) \Rightarrow y' = 2\tan(x^3)\sec^2(x^3)(3x^2) = 6x^2\tan(x^3)\sec^2(x^3)$

38. $y = \sin(\sin(\sin x)) \Rightarrow y' = \cos(\sin(\sin x))\dfrac{d}{dx}(\sin(\sin x)) = \cos(\sin(\sin x))\cos(\sin x)\cos x$

39. $y = \sqrt{x+\sqrt{x}} \Rightarrow y' = \frac{1}{2}(x+\sqrt{x})^{-1/2}\left(1+\frac{1}{2}x^{-1/2}\right) = \dfrac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\dfrac{1}{2\sqrt{x}}\right)$

40. $y = \sqrt{x+\sqrt{x+\sqrt{x}}} \Rightarrow y' = \frac{1}{2}\left(x+\sqrt{x+\sqrt{x}}\right)^{-1/2}\left[1+\frac{1}{2}(x+\sqrt{x})^{-1/2}\left(1+\frac{1}{2}x^{-1/2}\right)\right]$

41. $y = \sin\left(\tan\sqrt{\sin x}\right) \Rightarrow y' = \cos\left(\tan\sqrt{\sin x}\right)\left(\sec^2\sqrt{\sin x}\right)\left(\dfrac{1}{2\sqrt{\sin x}}\right)(\cos x)$

42. $y = \sqrt{\cos(\sin^2 x)} \Rightarrow y' = \frac{1}{2}(\cos(\sin^2 x))^{-1/2}\left[-\sin(\sin^2 x)\right](2\sin x\cos x) = -\dfrac{\sin(\sin^2 x)\sin x\cos x}{\sqrt{\cos(\sin^2 x)}}$

43. $y = f(x) = \dfrac{8}{\sqrt{4+3x}} = 8(4+3x)^{-1/2} \Rightarrow f'(x) = 8\left(-\frac{1}{2}\right)(4+3x)^{-3/2}(3) = -12(4+3x)^{-3/2}$. The slope of the tangent at $(4,2)$ is $f'(4) = -\frac{12}{64} = -\frac{3}{16}$ and its equation is $y - 2 = -\frac{3}{16}(x-4)$ or $y = -\frac{3}{16}x + \frac{11}{4}$.

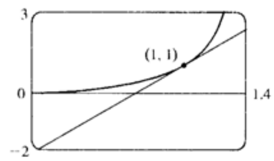
44. $y = f(x) = \sin x + \cos 2x \Rightarrow f'(x) = \cos x - 2\sin 2x$. The slope of the tangent at $\left(\frac{\pi}{6}, 1\right)$ is $f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - 2\left(\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{2}$ and its equation is $y - 1 = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right)$ or $\sqrt{3}x + 2y = 2 + \frac{\sqrt{3}}{6}\pi$.

45. $y = \sin(\sin x) \Rightarrow y' = \cos(\sin x)\cdot\cos x$. At $(\pi, 0)$, $y' = \cos(\sin\pi)\cdot\cos\pi = \cos(0)\cdot(-1) = 1(-1) = -1$, and an equation of the tangent line is $y - 0 = -1(x - \pi)$, or $y = -x + \pi$.

46. $y = \sqrt{5 + x^2} \Rightarrow y' = \frac{1}{2}\left(5 + x^2\right)^{-1/2}(2x) = x\Big/\sqrt{5 + x^2}$. At $(2, 3)$, $y' = \frac{2}{3}$, and an equation of the tangent line is $y - 3 = \frac{2}{3}(x - 2)$, or $y = \frac{2}{3}x + \frac{5}{3}$.

47. (a) $y = f(x) = \tan\left(\frac{\pi}{4}x^2\right) \Rightarrow f'(x) = \sec^2\left(\frac{\pi}{4}x^2\right)\left(2\cdot\frac{\pi}{4}x\right)$. The slope of the tangent at $(1, 1)$ is thus $f'(1) = \sec^2\frac{\pi}{4}\left(\frac{\pi}{2}\right) = 2\cdot\frac{\pi}{2} = \pi$, and its equation is $y - 1 = \pi(x - 1)$ or $y = \pi x - \pi + 1$.

(b)



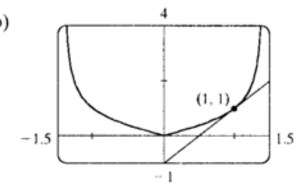
48. (a) For $x > 0$, $y = f(x) = \dfrac{x}{\sqrt{2 - x^2}} \Rightarrow$

$$f'(x) = \frac{\sqrt{2 - x^2}\,(1) - x\left(\frac{1}{2}\right)\left(2 - x^2\right)^{-1/2}(-2x)}{2 - x^2}\cdot\frac{\left(2 - x^2\right)^{1/2}}{\left(2 - x^2\right)^{1/2}}$$

$$= \frac{\left(2 - x^2\right) + x^2}{\left(2 - x^2\right)^{3/2}} = \frac{2}{\left(2 - x^2\right)^{3/2}}$$

So at $(1, 1)$, the slope of the tangent is $f'(1) = 2$ and its equation is $y - 1 = 2(x - 1)$ or $y = 2x - 1$.

(b)

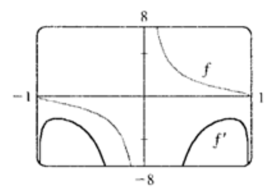


49. (a) $f(x) = \dfrac{\sqrt{1 - x^2}}{x} \Rightarrow$

$$f'(x) = \frac{x\cdot\frac{1}{2}\left(1 - x^2\right)^{-1/2}(-2x) - \sqrt{1 - x^2}}{x^2}\cdot\frac{\sqrt{1 - x^2}}{\sqrt{1 - x^2}}$$

$$= \frac{-x^2 - \left(1 - x^2\right)}{x^2\sqrt{1 - x^2}} = \frac{-1}{x^2\sqrt{1 - x^2}}$$

(b)



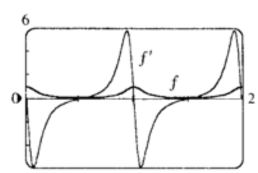
Notice that all tangents to the graph of f have negative slopes and $f'(x) < 0$ always.

50. (a) $f(x) = \dfrac{1}{\cos^2\pi x + 9\sin^2\pi x} = \dfrac{1}{1 + 8\sin^2\pi x} \Rightarrow$

$$f'(x) = -\left(1 + 8\sin^2\pi x\right)^{-2}(16\sin\pi x)(\cos\pi x)\,\pi$$

$$= \frac{-16\pi\sin\pi x\cos\pi x}{\left(1 + 8\sin^2\pi x\right)^2}$$

(b)



Notice that $f'(x) = 0$ when f has horizontal tangents.

51. For the tangent line to be horizontal $f'(x) = 0$. $f(x) = 2\sin x + \sin^2 x \Rightarrow$
$f'(x) = 2\cos x + 2\sin x \cos x = 0 \Leftrightarrow 2\cos x(1 + \sin x) = 0 \Leftrightarrow \cos x = 0$ or $\sin x = -1$, so
$x = \left(n + \frac{1}{2}\right)\pi$ or $\left(2n + \frac{3}{2}\right)\pi$ where n is any integer. So the points on the curve with a horizontal tangent are
$\left(\left(2n + \frac{1}{2}\right)\pi, 3\right)$ and $\left(\left(2n + \frac{3}{2}\right)\pi, -1\right)$ where n is any integer.

52. $f(x) = \sin 2x - 2\sin x \Rightarrow f'(x) = 2\cos 2x - 2\cos x = 4\cos^2 x - 2\cos x - 2$, and
$4\cos^2 x - 2\cos x - 2 = 0 \Leftrightarrow (\cos x - 1)(4\cos x + 2) = 0 \Leftrightarrow \cos x = 1$ or $\cos x = -\frac{1}{2}$. So $x = 2n\pi$ or
$(2n + 1)\pi \pm \frac{\pi}{3}$, n any integer.

53. $F(x) = f(g(x)) \Rightarrow$
$F'(x) = f'(g(x))g'(x)$, so $F'(3) = f'(g(3))g'(3) = f'(6)g'(3) = 7 \cdot 4 = 28$.

54. $w = u \circ v \Rightarrow w'(x) = u'(v(x))v'(x)$, so $w'(0) = u'(v(0))v'(0) = u'(2)v'(0) = 4 \cdot 5 = 20$.

55. (a) $h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x)) \cdot g'(x)$, so $h'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot 6 = 5 \cdot 6 = 30$.

(b) $H(x) = g(f(x)) \Rightarrow H'(x) = g'(f(x)) \cdot f'(x)$, so $H'(1) = g'(f(1)) \cdot f'(1) = g'(3) \cdot 4 = 9 \cdot 4 = 36$.

56. (a) $F(x) = f(f(x)) \Rightarrow F'(x) = f'(f(x)) \cdot f'(x)$, so $F'(2) = f'(f(2)) \cdot f'(2) = f'(1) \cdot 5 = 4 \cdot 5 = 20$.

(b) $G(x) = g(g(x)) \Rightarrow G'(x) = g'(g(x)) \cdot g'(x)$, so $G'(3) = g'(g(3)) \cdot g'(3) = g'(2) \cdot 9 = 7 \cdot 9 = 63$.

57. (a) $u(x) = f(g(x)) \Rightarrow u'(x) = f'(g(x))g'(x)$.
So $u'(1) = f'(g(1))g'(1) = f'(3)g'(1) = \left(-\frac{1}{4}\right)(-3) = \frac{3}{4}$.

(b) $v(x) = g(f(x)) \Rightarrow v'(x) = g'(f(x))f'(x)$. So $v'(1) = g'(f(1))f'(1) = g'(2)f'(1)$, which does not exist since $g'(2)$ does not exist.

(c) $w(x) = g(g(x)) \Rightarrow w'(x) = g'(g(x))g'(x)$.
So $w'(1) = g'(g(1))g'(1) = g'(3)g'(1) = \left(\frac{2}{3}\right)(-3) = -2$.

58. (a) $h(x) = f(f(x)) \Rightarrow h'(x) = f'(f(x))f'(x)$.
So $h'(2) = f'(f(2))f'(2) = f'(1)f'(2) \approx (-1)(-1) = 1$.

(b) $g(x) = f(x^2) \Rightarrow g'(x) = f'(x^2)(2x)$. So $g'(2) = f'(2^2)(2 \cdot 2) = 4f'(4) \approx 4(1.5) = 6$.

59. $h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x))g'(x)$. So $h'(0.5) = f'(g(0.5))g'(0.5) = f'(0.1)g'(0.5)$. We can estimate the derivatives by taking the average of two secant slopes.

For $f'(0.1)$: $m_1 = \dfrac{14.8 - 12.6}{0.1 - 0} = 22$, $m_2 = \dfrac{18.4 - 14.8}{0.2 - 0.1} = 36$. So $f'(0.1) \approx \dfrac{m_1 + m_2}{2} = 29$.

For $g'(0.5)$: $m_1 = \dfrac{0.10 - 0.17}{0.5 - 0.4} = -0.7$, $m_2 = \dfrac{0.05 - 0.10}{0.6 - 0.5} = -0.5$.

So $g'(0.5) \approx (m_1 + m_2)/2 = -0.6$. Hence, $h'(0.5) \approx (29)(-0.6) = -17.4$.

60. $g(x) = f(f(x)) \Rightarrow g'(x) = f'(f(x))f'(x)$. So $g'(1) = f'(f(1))f'(1) = f'(2)f'(1)$.

For $f'(2)$: $m_1 = \dfrac{3.1 - 2.4}{2.0 - 1.5} = 1.4$, $m_2 = \dfrac{4.4 - 3.1}{2.5 - 2.0} = 2.6$. So $f'(2) \approx \dfrac{m_1 + m_2}{2} = 2$.

For $f'(1)$: $m_1 = \dfrac{2.0 - 1.8}{1.0 - 0.5} = 0.4$, $m_2 = \dfrac{2.4 - 2.0}{1.5 - 1.0} = 0.8$. So $f'(1) \approx \dfrac{m_1 + m_2}{2} = 0.6$.

Hence, $g'(1) \approx (2)(0.6) = 1.2$.

61. (a) $F(x) = f(\cos x) \Rightarrow F'(x) = f'(\cos x)\dfrac{d}{dx}(\cos x) = -\sin x f'(\cos x)$

(b) $G(x) = \cos(f(x)) \Rightarrow G'(x) = -\sin(f(x))f'(x)$

62. (a) $F(x) = f(x^\alpha) \quad \Rightarrow \quad F'(x) = f'(x^\alpha)\dfrac{d}{dx}(x^\alpha) = f'(x^\alpha)\,\alpha x^{\alpha-1}$

(b) $G(x) = [f(x)]^\alpha \quad \Rightarrow \quad G'(x) = \alpha\,[f(x)]^{\alpha-1} f'(x)$

63. $s(t) = 10 + \frac{1}{4}\sin(10\pi t) \quad \Rightarrow$ the velocity after t seconds is
$v(t) = s'(t) = \frac{1}{4}\cos(10\pi t)(10\pi) = \frac{5\pi}{2}\cos(10\pi t)$ cm/s.

64. (a) $s = A\cos(\omega t + \delta) \quad \Rightarrow \quad$ velocity $= s' = -\omega A\sin(\omega t + \delta)$.

(b) If $A \neq 0$ and $\omega \neq 0$, then $s' = 0 \quad \Leftrightarrow \quad \sin(\omega t + \delta) = 0 \quad \Leftrightarrow \quad \omega t + \delta = n\pi \quad \Leftrightarrow \quad t = \dfrac{n\pi - \delta}{\omega}$, n an integer.

65. (a) $B(t) = 4.0 + 0.35\sin\dfrac{2\pi t}{5.4} \quad \Rightarrow \quad \dfrac{dB}{dt} = \left(0.35\cos\dfrac{2\pi t}{5.4}\right)\left(\dfrac{2\pi}{5.4}\right) = \dfrac{7\pi}{54}\cos\dfrac{2\pi t}{5.4}$

(b) At $t = 1$, $\dfrac{dB}{dt} = \dfrac{7\pi}{54}\cos\dfrac{2\pi}{5.4} \approx 0.16$.

66. $L(t) = 12 + 2.8\sin\left(\frac{2\pi}{365}(t - 80)\right) \quad \Rightarrow \quad L'(t) = 2.8\cos\left(\frac{2\pi}{365}(t - 80)\right)\left(\frac{2\pi}{365}\right)$.
On March 21, $t = 80$, and $L'(80) \approx 0.0482$ hours per day. On May 21, $t = 141$, and $L'(141) \approx 0.02398$, which is approximately one-half of $L'(80)$.

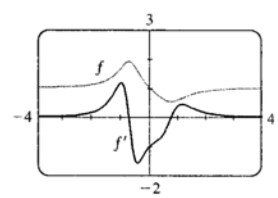
67. (a) Derive gives $g'(t) = \dfrac{45(t-2)^8}{(2t+1)^{10}}$ without simplifying. With either Maple or Mathematica, we first get
$g'(t) = 9\dfrac{(t-2)^8}{(2t+1)^9} - 18\dfrac{(t-2)^9}{(2t+1)^{10}}$, and the simplification command results in the above expression.

(b) Derive gives $y' = 2\left(x^3 - x + 1\right)^3 (2x+1)^4 \left(17x^3 + 6x^2 - 9x + 3\right)$ without simplifying.
With either Maple or Mathematica, we first get
$y' = 10(2x+1)^4\left(x^3 - x + 1\right)^4 + 4(2x+1)^5\left(x^3 - x + 1\right)^3\left(3x^2 - 1\right)$. If we use Mathematica's `Factor` or `Simplify`, or Maple's `factor`, we get the above expression, but Maple's `simplify` gives the polynomial expansion instead. For locating horizontal tangents, the factored form is the most helpful.

68. (a) $f(x) = \left(\dfrac{x^4 - x + 1}{x^4 + x + 1}\right)^{1/2}$. Derive gives $f'(x) = \dfrac{\left(3x^4 - 1\right)\sqrt{\dfrac{x^4 - x + 1}{x^4 + x + 1}}}{\left(x^4 + x + 1\right)\left(x^4 - x + 1\right)}$ whereas Maple and Mathematica give $f'(x) = \dfrac{3x^4 - 1}{\sqrt{\dfrac{x^4 - x + 1}{x^4 + x + 1}}\left(x^4 + x + 1\right)^2}$ after simplification.

(b) $f'(x) = 0 \quad \Leftrightarrow \quad 3x^4 - 1 = 0 \quad \Leftrightarrow \quad x = \pm\sqrt[4]{\frac{1}{3}} \approx \pm 0.7598$.

(c) $f'(x) = 0$ where f has horizontal tangents. f' has two maxima and one minimum where f has inflection points.



69. (a) If f is even, then $f(x) = f(-x)$. Using the Chain Rule to differentiate this equation, we get

$f'(x) = f'(-x)\dfrac{d}{dx}(-x) = -f'(-x)$. Thus, $f'(-x) = -f'(x)$, so f' is odd.

(b) If f is odd, then $f(x) = -f(-x)$. Differentiating this equation, we get $f'(x) = -f'(-x)(-1) = f'(-x)$, so f' is even.

70. $$\left[\frac{f(x)}{g(x)}\right]' = \left[f(x)[g(x)]^{-1}\right]' = f'(x)[g(x)]^{-1} + (-1)[g(x)]^{-2}g'(x)f(x)$$
$$= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

71. (a) $$\frac{d}{dx}\left(\sin^n x\cos nx\right) = n\sin^{n-1}x\cos x\cos nx + \sin^n x(-n\sin nx)$$
$$= n\sin^{n-1}x(\cos nx\cos x - \sin nx\sin x) = n\sin^{n-1}x\cos(nx + x)$$
$$= n\sin^{n-1}x\cos[(n+1)x]$$

(b) $$\frac{d}{dx}(\cos^n x\cos nx) = n\cos^{n-1}x(-\sin x)\cos nx + \cos^n x(-n\sin nx)$$
$$= -n\cos^{n-1}x(\cos nx\sin x + \sin nx\cos x) = -n\cos^{n-1}x\sin(nx + x)$$
$$= -n\cos^{n-1}x\sin[(n+1)x]$$

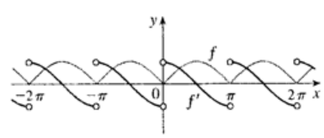
72. $80\dfrac{dy}{dx} = \dfrac{d}{dx}y^5 = 5y^4\dfrac{dy}{dx} \quad\Leftrightarrow\quad 80 = 5y^4$ (Note that $dy/dx \neq 0$ since the curve never has a horizontal tangent) $\Leftrightarrow\quad y^4 = 16 \quad\Leftrightarrow\quad y = 2$ (since $y > 0$ for all x)

73. Since $\theta^\circ = \left(\frac{\pi}{180}\right)\theta$ rad, we have $\dfrac{d}{d\theta}(\sin\theta^\circ) = \dfrac{d}{d\theta}\left(\sin\frac{\pi}{180}\theta\right) = \frac{\pi}{180}\cos\frac{\pi}{180}\theta = \frac{\pi}{180}\cos\theta^\circ$.

74. (a) $f(x) = |x| = \sqrt{x^2} \quad\Rightarrow\quad f'(x) = \frac{1}{2}\left(x^2\right)^{-1/2}(2x) = x/\sqrt{x^2} = x/|x|$

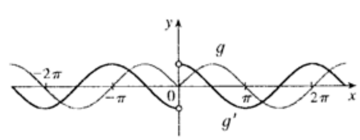
(b) $f(x) = |\sin x| = \sqrt{\sin^2 x} \quad\Rightarrow$

$$f'(x) = \tfrac{1}{2}\left(\sin^2 x\right)^{-1/2}2\sin x\cos x = \frac{\sin x}{|\sin x|}\cos x = \begin{cases} \cos x & \text{if } \sin x > 0 \\ -\cos x & \text{if } \sin x < 0 \end{cases}$$



f is not differentiable when $x = n\pi$, n an integer.

(c) $g(x) = \sin|x| = \sin\sqrt{x^2} \quad\Rightarrow\quad g'(x) = \cos|x|\cdot\dfrac{x}{|x|} = \dfrac{x}{|x|}\cos x = \begin{cases} \cos x & \text{if } x > 0 \\ -\cos x & \text{if } x < 0 \end{cases}$



g is not differentiable at 0.

3.7 Implicit Differentiation

1. (a) $\frac{d}{dx}\left(xy + 2x + 3x^2\right) = \frac{d}{dx}(4) \Rightarrow (x \cdot y' + y \cdot 1) + 2 + 6x = 0 \Rightarrow xy' = -y - 2 - 6x \Rightarrow$
$y' = \frac{-y - 2 - 6x}{x}$ or $y' = -6 - \frac{y+2}{x}$.

(b) $xy + 2x + 3x^2 = 4 \Rightarrow xy = 4 - 2x - 3x^2 \Rightarrow y = \frac{4 - 2x - 3x^2}{x} = \frac{4}{x} - 2 - 3x$, so $y' = -\frac{4}{x^2} - 3$.

(c) From part (a), $y' = \frac{-y - 2 - 6x}{x} = \frac{-(4/x - 2 - 3x) - 2 - 6x}{x} = \frac{-4/x - 3x}{x} = -\frac{4}{x^2} - 3$.

2. (a) $\frac{d}{dx}\left(4x^2 + 9y^2\right) = \frac{d}{dx}(36) \Rightarrow 8x + 18y \cdot y' = 0 \Rightarrow y' = -\frac{8x}{18y} = -\frac{4x}{9y}$

(b) $4x^2 + 9y^2 = 36 \Rightarrow 9y^2 = 36 - 4x^2 \Rightarrow y^2 = \frac{4}{9}\left(9 - x^2\right) \Rightarrow y = \pm\frac{2}{3}\sqrt{9 - x^2}$, so
$y' = \pm\frac{2}{3} \cdot \frac{1}{2}\left(9 - x^2\right)^{-1/2}(-2x) = \mp\frac{2x}{3\sqrt{9 - x^2}}$

(c) From part (a), $y' = -\frac{4x}{9y} = -\frac{4x}{9\left(\pm\frac{2}{3}\sqrt{9 - x^2}\right)} = \mp\frac{2x}{3\sqrt{9 - x^2}}$.

3. (a) $\frac{d}{dx}\left(\frac{1}{x} + \frac{1}{y}\right) = \frac{d}{dx}(1) \Rightarrow -\frac{1}{x^2} - \frac{1}{y^2}y' = 0 \Rightarrow -\frac{1}{y^2}y' = \frac{1}{x^2} \Rightarrow y' = -\frac{y^2}{x^2}$

(b) $\frac{1}{x} + \frac{1}{y} = 1 \Rightarrow \frac{1}{y} = 1 - \frac{1}{x} = \frac{x-1}{x} \Rightarrow y = \frac{x}{x-1}$, so $y' = \frac{(x-1)(1) - (x)(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$.

(c) $y' = -\frac{y^2}{x^2} = -\frac{[x/(x-1)]^2}{x^2} = -\frac{x^2}{x^2(x-1)^2} = -\frac{1}{(x-1)^2}$

4. (a) $\frac{d}{dx}\left(\sqrt{x} + \sqrt{y}\right) = \frac{d}{dx}(4) \Rightarrow \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}y' = 0 \Rightarrow y' = -\frac{\sqrt{y}}{\sqrt{x}}$

(b) $\sqrt{y} = 4 - \sqrt{x} \Rightarrow y = \left(4 - \sqrt{x}\right)^2 = 16 - 8\sqrt{x} + x \Rightarrow y' = -\frac{4}{\sqrt{x}} + 1$

(c) $y' = -\frac{\sqrt{y}}{\sqrt{x}} = -\frac{4 - \sqrt{x}}{\sqrt{x}} = -\frac{4}{\sqrt{x}} + 1$

5. $\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(1) \Rightarrow 2x + 2yy' = 0 \Rightarrow 2yy' = -2x \Rightarrow y' = -\frac{x}{y}$

6. $\frac{d}{dx}\left(x^2 - y^2\right) = \frac{d}{dx}(1) \Rightarrow 2x - 2yy' = 0 \Rightarrow 2x = 2yy' \Rightarrow y' = \frac{x}{y}$

7. $\frac{d}{dx}\left(x^3 + x^2y + 4y^2\right) = \frac{d}{dx}(6) \Rightarrow 3x^2 + \left(x^2y' + y \cdot 2x\right) + 8yy' = 0 \Rightarrow x^2y' + 8yy' = -3x^2 - 2xy \Rightarrow$
$\left(x^2 + 8y\right)y' = -3x^2 - 2xy \Rightarrow y' = -\frac{3x^2 + 2xy}{x^2 + 8y}$

8. $\dfrac{d}{dx}\left(x^2-2xy+y^3\right)=\dfrac{d}{dx}(c) \Rightarrow 2x-2\left(xy'+y\cdot 1\right)+3y^2y'=0 \Rightarrow 2x-2y=2xy'-3y^2y' \Rightarrow$
$2x-2y=y'\left(2x-3y^2\right) \Rightarrow y'=\dfrac{2x-2y}{2x-3y^2}$

9. $\dfrac{d}{dx}\left(x^2y+xy^2\right)=\dfrac{d}{dx}(3x) \Rightarrow \left(x^2y'+y\cdot 2x\right)+\left(x\cdot 2yy'+y^2\cdot 1\right)=3 \Rightarrow x^2y'+2xyy'=3-2xy-y^2$
$\Rightarrow y'\left(x^2+2xy\right)=3-2xy-y^2 \Rightarrow y'=\dfrac{3-2xy-y^2}{x^2+2xy}$

10. $\dfrac{d}{dx}\left(y^5+x^2y^3\right)=\dfrac{d}{dx}\left(1+x^4y\right) \Rightarrow 5y^4y'+x^2\cdot 3y^2y'+y^3\cdot 2x=0+x^4y'+y\cdot 4x^3 \Rightarrow$
$y'\left(5y^4+3x^2y^2-x^4\right)=4x^3y-2xy^3 \Rightarrow y'=\dfrac{4x^3y-2xy^3}{5y^4+3x^2y^2-x^4}$

11. $\dfrac{y}{x-y}=x^2+1 \Rightarrow 2x=\dfrac{(x-y)\,y'-y\left(1-y'\right)}{(x-y)^2}=\dfrac{xy'-y}{(x-y)^2} \Rightarrow y'=\dfrac{y}{x}+2\,(x-y)^2$

Another Method: Write the equation as $y=(x-y)\left(x^2+1\right)=x^3+x-yx^2-y$. Then $y'=\dfrac{3x^2+1-2xy}{x^2+2}$.

12. $\sqrt{x+y}+\sqrt{xy}=6 \Rightarrow \frac{1}{2}(x+y)^{-1/2}\left(1+y'\right)+\frac{1}{2}(xy)^{-1/2}\left(y+xy'\right)=0 \Rightarrow$
$(x+y)^{-1/2}+(x+y)^{-1/2}y'+(xy)^{-1/2}y+(xy)^{-1/2}xy'=0 \Rightarrow$
$y'=-\dfrac{(x+y)^{-1/2}+(xy)^{-1/2}y}{(x+y)^{-1/2}+(xy)^{-1/2}x}\cdot\dfrac{(x+y)^{1/2}(xy)^{1/2}}{(x+y)^{1/2}(xy)^{1/2}}=-\dfrac{\sqrt{xy}+y\sqrt{x+y}}{\sqrt{xy}+x\sqrt{x+y}}$

13. $\sqrt{xy}=1+x^2y \Rightarrow \frac{1}{2}(xy)^{-1/2}\left(xy'+y\cdot 1\right)=0+x^2y'+y\cdot 2x \Rightarrow \dfrac{x}{2\sqrt{xy}}y'+\dfrac{y}{2\sqrt{xy}}=x^2y'+2xy \Rightarrow$
$y'\left(\dfrac{x}{2\sqrt{xy}}-x^2\right)=2xy-\dfrac{y}{2\sqrt{xy}} \Rightarrow y'\left(\dfrac{x-2x^2\sqrt{xy}}{2\sqrt{xy}}\right)=\dfrac{4xy\sqrt{xy}-y}{2\sqrt{xy}} \Rightarrow y'=\dfrac{4xy\sqrt{xy}-y}{x-2x^2\sqrt{xy}}$

14. $\sqrt{1+x^2y^2}=2xy \Rightarrow \frac{1}{2}\left(1+x^2y^2\right)^{-1/2}\left(x^2\cdot 2yy'+y^2\cdot 2x\right)=2\left(xy'+y\cdot 1\right) \Rightarrow$
$\dfrac{2x^2y}{2\sqrt{1+x^2y^2}}y'+\dfrac{2xy^2}{2\sqrt{1+x^2y^2}}=2xy'+2y \Rightarrow y'\left(\dfrac{x^2y}{\sqrt{1+x^2y^2}}-2x\right)=2y-\dfrac{xy^2}{\sqrt{1+x^2y^2}}$
$\Rightarrow y'\left(\dfrac{x^2y-2x\sqrt{1+x^2y^2}}{\sqrt{1+x^2y^2}}\right)=\dfrac{2y\sqrt{1+x^2y^2}-xy^2}{\sqrt{1+x^2y^2}} \Rightarrow$
$y'=\dfrac{2y\sqrt{1+x^2y^2}-xy^2}{x^2y-2x\sqrt{1+x^2y^2}}=\dfrac{y\left(2\sqrt{1+x^2y^2}-xy\right)}{x\left(xy-2\sqrt{1+x^2y^2}\right)}=-\dfrac{y}{x}$

Another Method: Since $1+x^2y^2$ is positive, we can square both sides first and then differentiate implicitly.

15. $4\cos x\sin y=1 \Rightarrow 4\cos x\cos y\cdot y'+4\sin y\,(-\sin x)=0 \Rightarrow y'=\dfrac{4\sin x\sin y}{4\cos x\cos y}=\tan x\tan y$

16. $x\sin y+\cos 2y=\cos y \;\Rightarrow\; \sin y+(x\cos y)\,y'-(2\sin 2y)\,y'=(-\sin y)\,y' \;\Rightarrow$
$\sin y=(2\sin 2y)\,y'-(x\cos y)\,y'-(\sin y)\,y' \;\Rightarrow\; y'=\dfrac{\sin y}{2\sin 2y-x\cos y-\sin y}$

17. $\cos(x-y)=y\sin x \;\Rightarrow\; -\sin(x-y)\left(1-y'\right)=y'\sin x+y\cos x \;\Rightarrow\; y'=\dfrac{\sin(x-y)+y\cos x}{\sin(x-y)-\sin x}$

18. $x\cos y+y\cos x=1 \;\Rightarrow\; \cos y+x(-\sin y)\,y'+y'\cos x-y\sin x=0 \;\Rightarrow\; y'=\dfrac{y\sin x-\cos y}{\cos x-x\sin y}$

19. $xy=\cot(xy) \;\Rightarrow\; y+xy'=-\csc^2(xy)\left(y+xy'\right) \;\Rightarrow\; \left(y+xy'\right)\left[1+\csc^2(xy)\right]=0 \;\Rightarrow$
$y+xy'=0$ [since $1+\csc^2(xy)>0$] $\Rightarrow\; y'=-y/x$

20. $\sin x+\cos y=\sin x\cos y \;\Rightarrow\; \cos x-\sin y\cdot y'=\sin x\left(-\sin y\cdot y'\right)+\cos y\cos x \;\Rightarrow$
$(\sin x\sin y-\sin y)\,y'=\cos x\cos y-\cos x \;\Rightarrow\; y'=\dfrac{\cos x(\cos y-1)}{\sin y(\sin x-1)}$

21. $x\,[f(x)]^3+xf(x)=6 \;\Rightarrow\; [f(x)]^3+3x\,[f(x)]^2f'(x)+f(x)+xf'(x)=0 \;\Rightarrow$
$f'(x)=-\dfrac{[f(x)]^3+f(x)}{3x\,[f(x)]^2+x} \;\Rightarrow\; f'(3)=-\dfrac{(1)^3+1}{3(3)(1)^2+3}=-\dfrac{1}{6}$

22. $[g(x)]^2+12x=x^2g(x) \;\Rightarrow\; 2g(x)\,g'(x)+12=2xg(x)+x^2g'(x) \;\Leftrightarrow\; g'(x)=\dfrac{2xg(x)-12}{2g(x)-x^2} \;\Rightarrow$
$g'(4)=\dfrac{2(4)(12)-12}{2(12)-(4)^2}=\dfrac{21}{2}$

23. $y^4+x^2y^2+yx^4=y+1 \;\Rightarrow\; 4y^3+2x\dfrac{dx}{dy}y^2+2x^2y+x^4+4yx^3\dfrac{dx}{dy}=1 \;\Rightarrow$
$\dfrac{dx}{dy}=\dfrac{1-4y^3-2x^2y-x^4}{2xy^2+4yx^3}$

24. $\left(x^2+y^2\right)^2=ax^2y \;\Rightarrow\; 2\left(x^2+y^2\right)\left(2x\dfrac{dx}{dy}+2y\right)=2ayx\dfrac{dx}{dy}+ax^2 \;\Rightarrow\; \dfrac{dx}{dy}=\dfrac{ax^2-4y\left(x^2+y^2\right)}{4x\left(x^2+y^2\right)-2axy}$

25. $\dfrac{x^2}{16}-\dfrac{y^2}{9}=1 \;\Rightarrow\; \dfrac{x}{8}-\dfrac{2yy'}{9}=0 \;\Rightarrow\; y'=\dfrac{9x}{16y}$. When $x=-5$ and $y=\frac{9}{4}$ we have $y'=\dfrac{9(-5)}{16(9/4)}=-\dfrac{5}{4}$ so an equation of the tangent is $y-\frac{9}{4}=-\frac{5}{4}(x+5)$ or $y=-\frac{5}{4}x-4$.

26. $\dfrac{x^2}{9}+\dfrac{y^2}{36}=1 \;\Rightarrow\; \dfrac{2x}{9}+\dfrac{yy'}{18}=0 \;\Rightarrow\; y'=-\dfrac{4x}{y}$. When $x=-1$ and $y=4\sqrt{2}$ we have
$y'=-\dfrac{4(-1)}{4\sqrt{2}}=\dfrac{1}{\sqrt{2}}$ so an equation of the tangent line is $y-4\sqrt{2}=\frac{1}{\sqrt{2}}(x+1)$ or $y=\frac{1}{\sqrt{2}}(x+9)$.

27. $y^2=x^3(2-x)=2x^3-x^4 \;\Rightarrow\; 2yy'=6x^2-4x^3 \;\Rightarrow\; y'=\dfrac{3x^2-2x^3}{y}$. When $x=y=1$,
$y'=\dfrac{3(1)^2-2(1)^3}{1}=1$, so an equation of the tangent line is $y-1=1(x-1)$ or $y=x$.

28. $x^{2/3}+y^{2/3}=4 \Rightarrow \frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y'=0 \Rightarrow \frac{1}{\sqrt[3]{x}}+\frac{y'}{\sqrt[3]{y}}=0 \Rightarrow y'=-\frac{\sqrt[3]{y}}{\sqrt[3]{x}}$. When $x=-3\sqrt{3}$ and $y=1$, we have $y'=-\frac{1}{\left(-3\sqrt{3}\right)^{1/3}}=-\frac{\left(-3\sqrt{3}\right)^{2/3}}{-3\sqrt{3}}=\frac{3}{3\sqrt{3}}=\frac{1}{\sqrt{3}}$, so an equation of the tangent is $y-1=\frac{1}{\sqrt{3}}\left(x+3\sqrt{3}\right)$ or $y=\frac{1}{\sqrt{3}}x+4$.

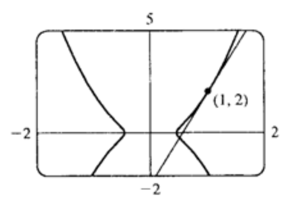
29. $2\left(x^2+y^2\right)^2=25\left(x^2-y^2\right) \Rightarrow 4\left(x^2+y^2\right)\left(2x+2yy'\right)=25\left(2x-2yy'\right) \Rightarrow$
$4yy'\left(x^2+y^2\right)+25yy'=25x-4x\left(x^2+y^2\right) \Rightarrow y'=\frac{25x-4x\left(x^2+y^2\right)}{25y+4y\left(x^2+y^2\right)}$. When $x=3$ and $y=1$,
$y'=\frac{75-120}{25+40}=-\frac{9}{13}$ so an equation of the tangent is $y-1=-\frac{9}{13}(x-3)$ or $y=-\frac{9}{13}x+\frac{40}{13}$.

30. $x^2y^2=(y+1)^2\left(4-y^2\right) \Rightarrow 2xy^2+2x^2yy'=2(y+1)y'\left(4-y^2\right)+(y+1)^2\left(-2yy'\right) \Rightarrow$
$y'=\frac{xy^2}{(y+1)\left(4-y^2\right)-y(y+1)^2-x^2y}=0$ when $x=0$. So an equation of the tangent line at $(0,-2)$ is
$y+2=0(x-0)$ or $y=-2$.

31. (a) $y^2=5x^4-x^2 \Rightarrow 2yy'=5\left(4x^3\right)-2x \Rightarrow y'=\frac{10x^3-x}{y}$.

So at the point $(1,2)$ we have $y'=\frac{10(1)^3-1}{2}=\frac{9}{2}$, and an equation of the tangent line is $y-2=\frac{9}{2}(x-1)$ or $y=\frac{9}{2}x-\frac{5}{2}$.

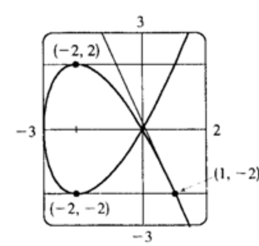
(b)



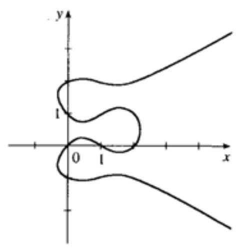
32. (a) $y^2=x^3+3x^2 \Rightarrow 2yy'=3x^2+3(2x) \Rightarrow y'=\frac{3x^2+6x}{2y}$. So at the point $(1,-2)$ we have
$y'=\frac{3(1)^2+6(1)}{2(-2)}=-\frac{9}{4}$, and an equation of the tangent is $y-(-2)=-\frac{9}{4}(x-1) \Leftrightarrow y=-\frac{9}{4}x+\frac{1}{4}$.

(b) The curve has a horizontal tangent where $y'=0 \Leftrightarrow 3x^2+6x=0$ $\Leftrightarrow 3x(x+2)=0 \Leftrightarrow x=0$ or $x=-2$. But note that at $x=0$, $y=0$ also, so the derivative does not exist. At $x=-2$, $y^2=(-2)^3+3(-2)^2=-8+12=4$, so $y=\pm 2$. So the two points at which the curve has a horizontal tangent are $(-2,-2)$ and $(-2,2)$.

(c)



33. (a)



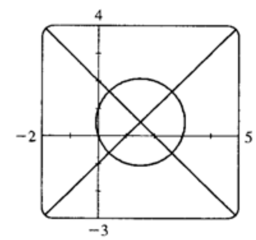
There are eight points with horizontal tangents: four at $x \approx 1.57735$ and four at $x \approx 0.42265$.

(b) $y' = \dfrac{3x^2 - 6x + 2}{2\left(2y^3 - 3y^2 - y + 1\right)} \Rightarrow y' = -1$ at $(0, 1)$ and $y' = \frac{1}{3}$ at $(0, 2)$.

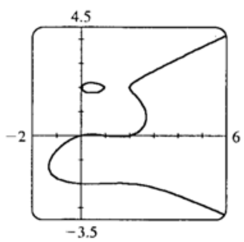
Equations of the tangent lines are $y = -x + 1$ and $y = \frac{1}{3}x + 2$.

(c) $y' = 0 \Rightarrow 3x^2 - 6x + 2 = 0 \Rightarrow x = 1 \pm \frac{1}{3}\sqrt{3}$

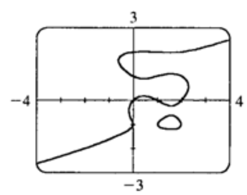
(d) By multiplying the right side of the equation by $x - 3$, we obtain the first graph.
By modifying the equation in other ways, we can generate the other graphs.



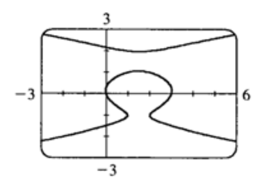
$y\left(y^2 - 1\right)(y - 2)$
$= x(x - 1)(x - 2)(x - 3)$



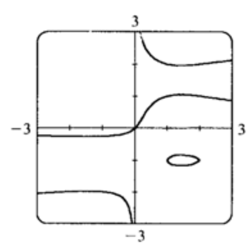
$y\left(y^2 - 4\right)(y - 2)$
$= x(x - 1)(x - 2)$



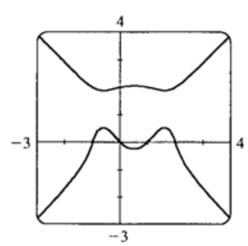
$y(y + 1)\left(y^2 - 1\right)(y - 2)$
$= x(x - 1)(x - 2)$



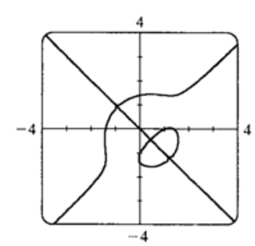
$(y + 1)\left(y^2 - 1\right)(y - 2)$
$= (x - 1)(x - 2)$



$x(y + 1)\left(y^2 - 1\right)(y - 2)$
$= y(x - 1)(x - 2)$

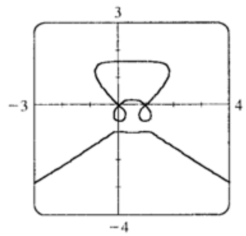


$y\left(y^2 + 1\right)(y - 2)$
$= x\left(x^2 - 1\right)(x - 2)$



$y(y + 1)\left(y^2 - 2\right)$
$= x(x - 1)\left(x^2 - 2\right)$

34. (a)



(b) There are 9 points with horizontal tangents: 3 at $x = 0$, 3 at $x = \frac{1}{2}$, and 3 at $x = 1$. The three horizontal tangents along the top of the wagon are hard to find, but by limiting the y-range of the graph (to $[1.6, 1.7]$, for example) they are distinguishable.

35. From Exercise 29, a tangent to the lemniscate will be horizontal $\Rightarrow$ $y' = 0$ $\Rightarrow$ $25x - 4x\left(x^2 + y^2\right) = 0$ $\Rightarrow$ $x^2 + y^2 = \frac{25}{4}$. (Note that $x = 0$ $\Rightarrow$ $y = 0$ and there is no horizontal tangent at the origin.) Putting this in the equation of the lemniscate, we get $x^2 - y^2 = \frac{25}{8}$. Solving these two equations we have $x^2 = \frac{75}{16}$ and $y^2 = \frac{25}{16}$, so the four points are $\left(\pm\frac{5\sqrt{3}}{4}, \pm\frac{5}{4}\right)$.

36. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $\Rightarrow$ $\dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0$ $\Rightarrow$ $y' = -\dfrac{b^2x}{a^2y}$ $\Rightarrow$ the equation of the tangent at (x_0, y_0) is $y - y_0 = \dfrac{-b^2x_0}{a^2y_0}(x - x_0)$. Multiplying both sides by $\dfrac{y_0}{b^2}$ gives $\dfrac{y_0y}{b^2} - \dfrac{y_0^2}{b^2} = -\dfrac{x_0x}{a^2} + \dfrac{x_0^2}{a^2}$. Since (x_0, y_0) lies on the ellipse, we have $\dfrac{x_0x}{a^2} + \dfrac{y_0y}{b^2} = \dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} = 1$.

37. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ $\Rightarrow$ $\dfrac{2x}{a^2} - \dfrac{2yy'}{b^2} = 0$ $\Rightarrow$ $y' = \dfrac{b^2x}{a^2y}$ $\Rightarrow$ the equation of the tangent at (x_0, y_0) is $y - y_0 = \dfrac{b^2x_0}{a^2y_0}(x - x_0)$. Multiplying both sides by $\dfrac{y_0}{b^2}$ gives $\dfrac{y_0y}{b^2} - \dfrac{y_0^2}{b^2} = \dfrac{x_0x}{a^2} - \dfrac{x_0^2}{a^2}$. Since (x_0, y_0) lies on the hyperbola, we have $\dfrac{x_0x}{a^2} - \dfrac{y_0y}{b^2} = \dfrac{x_0^2}{a^2} - \dfrac{y_0^2}{b^2} = 1$.

38. $\sqrt{x} + \sqrt{y} = \sqrt{c}$ $\Rightarrow$ $\dfrac{1}{2\sqrt{x}} + \dfrac{y'}{2\sqrt{y}} = 0$ $\Rightarrow$ $y' = -\dfrac{\sqrt{y}}{\sqrt{x}}$ $\Rightarrow$ the equation of the tangent line at (x_0, y_0) is $y - y_0 = -\dfrac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$. Now $x = 0$ $\Rightarrow$ $y = y_0 - \dfrac{\sqrt{y_0}}{\sqrt{x_0}}(-x_0) = y_0 + \sqrt{x_0}\sqrt{y_0}$, so the y-intercept is $y_0 + \sqrt{x_0}\sqrt{y_0}$. Also $y = 0$ $\Rightarrow$ $-y_0 = -\dfrac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$, so the x-intercept is $x_0 + \sqrt{x_0}\sqrt{y_0}$. The sum of the intercepts is $\left(y_0 + \sqrt{x_0}\sqrt{y_0}\right) + \left(x_0 + \sqrt{x_0}\sqrt{y_0}\right) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = \left(\sqrt{x_0} + \sqrt{y_0}\right)^2 = \left(\sqrt{c}\right)^2 = c$.

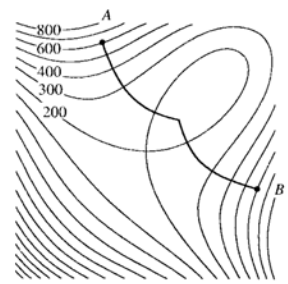
39. If the circle has radius r, its equation is $x^2 + y^2 = r^2$ $\Rightarrow$ $2x + 2yy' = 0$ $\Rightarrow$ $y' = -\dfrac{x}{y}$, so the slope of the tangent line at $P(x_0, y_0)$ is $-\dfrac{x_0}{y_0}$. The slope of OP is $\dfrac{y_0}{x_0} = \dfrac{-1}{-x_0/y_0}$, so the tangent is perpendicular to OP.

40. $y^q = x^p$ $\Rightarrow$ $qy^{q-1}y' = px^{p-1}$ $\Rightarrow$ $y' = \dfrac{px^{p-1}}{qy^{q-1}} = \dfrac{px^{p-1}y}{qy^q} = \dfrac{px^{p-1}x^{p/q}}{qx^p} = \dfrac{p}{q}x^{(p/q)-1}$

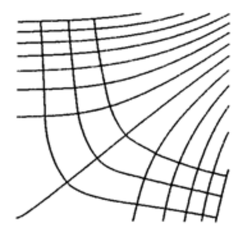
41. $2x^2 + y^2 = 3$ and $x = y^2$ intersect when $2x^2 + x - 3 = (2x + 3)(x - 1) = 0$ $\Leftrightarrow$ $x = -\frac{3}{2}$ or 1, but $-\frac{3}{2}$ is extraneous. $2x^2 + y^2 = 3$ $\Rightarrow$ $4x + 2yy' = 0$ $\Rightarrow$ $y' = -2x/y$, and $x = y^2$ $\Rightarrow$ $1 = 2yy'$ $\Rightarrow$ $y' = 1/(2y)$. At $(1, 1)$ the slopes are $m_1 = -2$ and $m_2 = \frac{1}{2}$, so the curves are orthogonal there. By symmetry they are also orthogonal at $(1, -1)$.

42. $x^2 - y^2 = 5$ and $4x^2 + 9y^2 = 72$ intersect when $4x^2 + 9\left(x^2 - 5\right) = 72 \quad \Leftrightarrow \quad 13x^2 = 117 \quad \Leftrightarrow \quad x = \pm 3$, so there are four points of intersection: $(\pm 3, \pm 2)$. $x^2 - y^2 = 5 \quad \Rightarrow \quad 2x - 2yy' = 0 \quad \Rightarrow \quad y' = x/y$ and $4x^2 + 9y^2 = 72 \quad \Rightarrow \quad 8x + 18yy' = 0 \quad \Leftrightarrow \quad y' = -4x/9y$. At $(3, 2)$ the slopes are $m_1 = \frac{3}{2}$ and $m_2 = -\frac{2}{3}$, so the curves are orthogonal there. By symmetry, they are also orthogonal at $(3, -2)$, $(-3, 2)$ and $(-3, -2)$.

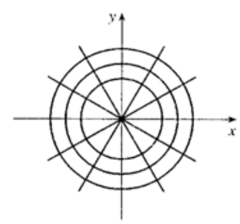
43.



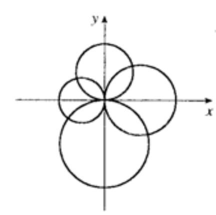
44. The orthogonal family represents the direction of the wind.



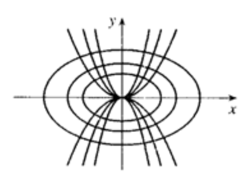
45. $x^2 + y^2 = r^2$ is a circle with center O and $ax + by = 0$ is a line through O. $x^2 + y^2 = r^2 \quad \Rightarrow \quad 2x + 2yy' = 0 \quad \Rightarrow \quad y' = -x/y$, so the slope of the tangent line at $P_0\,(x_0, y_0)$ is $-x_0/y_0$. The slope of the line OP is y_0/x_0, which is the negative reciprocal of $-x_0/y_0$. Hence, the curves are orthogonal.



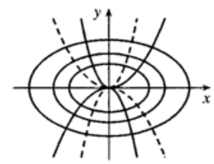
46. The circles $x^2 + y^2 = ax$ and $x^2 + y^2 = by$ intersect at the origin where the tangents are vertical and horizontal. If (x_0, y_0) is the other point of intersection, then $ax_0 = x_0^2 + y_0^2 = by_0$ (★). Now $x^2 + y^2 = ax \quad \Rightarrow \quad 2x + 2yy' = a \quad \Rightarrow$ $y' = \dfrac{a - 2x}{2y}$ and $x^2 + y^2 = by \;\Rightarrow\; 2x + 2yy' = by' \;\Rightarrow\; y' = \dfrac{2x}{b - 2y}$. Thus, the curves are orthogonal at $(x_0, y_0) \quad \Leftrightarrow \quad \dfrac{a - 2x_0}{2y_0} = -\dfrac{b - 2y_0}{2x_0} \quad \Leftrightarrow$ $2ax_0 - 4x_0^2 = 4y_0^2 - 2by_0 \quad \Leftrightarrow \quad 2ax_0 + 2by_0 = 4\left(x_0^2 + y_0^2\right)$, which is true by (★).



47. $y = cx^2 \quad \Rightarrow \quad y' = 2cx$ and $x^2 + 2y^2 = k \quad \Rightarrow \quad 2x + 4yy' = 0 \quad \Rightarrow$ $y' = -\dfrac{x}{2y} = -\dfrac{x}{2cx^2} = -\dfrac{1}{2cx}$, so the curves are orthogonal.



48. $y = ax^3 \Rightarrow y' = 3ax^2$ and $x^2 + 3y^2 = b \Rightarrow 2x + 6yy' = 0 \Rightarrow$
$y' = -\dfrac{x}{3y} = -\dfrac{x}{3ax^3} = -\dfrac{1}{3ax^2}$, so the curves are orthogonal.

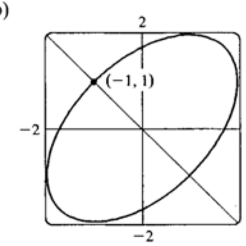


49. $y = 0 \Rightarrow x^2 - x(0) + 0^2 = 3 \Leftrightarrow x = \pm\sqrt{3}$. So the graph of the ellipse crosses the x-axis at the points $\left(\pm\sqrt{3}, 0\right)$. Using implicit differentiation to find y', we get $2x - xy' - y + 2yy' = 0 \Rightarrow y'(2y - x) = y - 2x$ $\Leftrightarrow y' = \dfrac{y - 2x}{2y - x}$. So $y'\left(\sqrt{3}, 0\right) = \dfrac{0 - 2\sqrt{3}}{2(0) - \sqrt{3}} = 2$, and $y'\left(-\sqrt{3}, 0\right) = \dfrac{0 + 2\sqrt{3}}{2(0) + \sqrt{3}} = 2 = y'\left(\sqrt{3}, 0\right)$. So the tangent lines at these points are parallel.

50. (a) We use implicit differentiation to find $y' = \dfrac{y - 2x}{2y - x}$ as in Exercise 49.

The slope of the tangent line at $(-1, 1)$ is $m = \dfrac{1 - 2(-1)}{2(1) - (-1)} = 1$, so the slope of the normal line is $-\dfrac{1}{m} = -1$, and its equation is $y - 1 = -(x + 1) \Leftrightarrow y = -x$. Substituting this into the equation of the ellipse, we get $x^2 - x(-x) + (-x)^2 = 3 \Rightarrow 3x^2 = 3 \Leftrightarrow x = \pm 1$. So the normal line must intersect the ellipse again at $x = 1$, and since the equation of the line is $y = -x$, the other point of intersection must be $(1, -1)$.

(b)



51. $x^2y^2 + xy = 2 \Rightarrow 2xy^2 + 2x^2yy' + y + xy' = 0 \Leftrightarrow y'\left(2x^2y + x\right) = -2xy^2 - y \Leftrightarrow y' = -\dfrac{2xy^2 + y}{2x^2y + x}$.
So $-\dfrac{2xy^2 + y}{2x^2y + x} = -1 \Leftrightarrow 2xy^2 + y = 2x^2y + x \Leftrightarrow y(2xy + 1) = x(2xy + 1) \Leftrightarrow$ $(2xy + 1)(y - x) = 0 \Leftrightarrow y = x$ or $xy = -\frac{1}{2}$. But $xy = -\frac{1}{2} \Rightarrow x^2y^2 + xy = \frac{1}{4} - \frac{1}{2} \neq 2$ so we must have $x = y$. Then $x^2y^2 + xy = 2 \Rightarrow x^4 + x^2 = 2 \Leftrightarrow x^4 + x^2 - 2 = 0 \Leftrightarrow \left(x^2 + 2\right)\left(x^2 - 1\right) = 0$. So $x^2 = -2$, which is impossible, or $x^2 = 1 \Leftrightarrow x = \pm 1$. So the points on the curve where the tangent line has a slope of -1 are $(-1, -1)$ and $(1, 1)$.

52. Using implicit differentiation, $2x + 8yy' = 0$ so $y' = -\dfrac{x}{4y}$. Let (a, b) be a point on $x^2 + 4y^2 = 36$ whose tangent line passes through $(12, 3)$. The tangent line is then $y - 3 = -\dfrac{a}{4b}(x - 12)$, so $b - 3 = -\dfrac{a}{4b}(a - 12)$. Multiplying both sides by $4b$ gives $4b^2 - 12b = -a^2 + 12a$, so $4b^2 + a^2 = 12(a + b)$. But $4b^2 + a^2 = 36$, so $36 = 12(a + b) \Rightarrow a + b = 3 \Rightarrow b = 3 - a$. Substituting into $x^2 + 4y^2 = 36$ gives $a^2 + 4(3 - a)^2 = 36$, so $a^2 + 36 - 24a + 4a^2 = 36$. Hence, $0 = 5a^2 - 24a = a(5a - 24)$, so $a = 0$ or $a = \frac{24}{5}$. Thus, if $a = 0$, $b = 3 - 0 = 3$ while if $a = \frac{24}{5}$, $b = 3 - \frac{24}{5} = -\frac{9}{5}$. So the two points are $(0, 3)$ and $\left(\frac{24}{5}, -\frac{9}{5}\right)$. A check shows that both points satisfy the necessary hypothesis.

53. $x^2 + 4y^2 = 5 \Rightarrow 2x + 4(2yy') = 0 \Rightarrow y' = -\dfrac{x}{4y}$. Now let h be the height of the lamp, and let (a, b) be the point of tangency of the line passing through the points $(3, h)$ and $(-5, 0)$. This line has slope $(h - 0) / [3 - (-5)] = \frac{1}{8}h$. But the slope of the tangent line through the point (a, b) can be expressed as $y' = -\dfrac{a}{4b}$, or as $\dfrac{b - 0}{a - (-5)} = \dfrac{b}{a + 5}$ [since the line passes through $(-5, 0)$ and (a, b)], so $-\dfrac{a}{4b} = \dfrac{b}{a + 5} \Leftrightarrow$ $4b^2 = -a^2 - 5a \Leftrightarrow a^2 + 4b^2 = -5a$. But $a^2 + 4b^2 = 5$, since (a, b) is on the ellipse, so $5 = -5a \Leftrightarrow$ $a = -1$. Then $4b^2 = -1 - 5(-1) = 4 \Rightarrow b = 1$, since the point is on the top half of the ellipse. So $\dfrac{h}{8} = \dfrac{b}{a + 5} = \dfrac{1}{-1 + 5} = \dfrac{1}{4} \Rightarrow h = 2$. So the lamp is located 2 units above the x-axis.

3.8 Higher Derivatives

1. $a = f$, $b = f'$, $c = f''$. We can see this because where a has a horizontal tangent, $b = 0$, and where b has a horizontal tangent, $c = 0$. We can immediately see that c can be neither f nor f', since at the points where c has a horizontal tangent, neither a nor b is equal to 0.

2. Where d has horizontal tangents, only c is 0, so $d' = c$. c has negative tangents for $x < 0$ and b is the only graph that is negative for $x < 0$, so $c' = b$. b has positive tangents on $\mathbb{R}$ (except at $x = 0$), and the only graph that is positive on the same domain is a, so $b' = a$. We conclude that $d = f$, $c = f'$, $b = f''$, and $a = f'''$.

3. We can immediately see that a is the graph of the acceleration function, since at the points where a has a horizontal tangent, neither c nor b is equal to 0. Next, we note that $a = 0$ at the point where b has a horizontal tangent, so b must be the graph of the velocity function, so that $b' = a$. We conclude that c is the graph of the position function.

4. a must be the jerk since none of the graphs are 0 at its high and low points. a is 0 where b has a maximum, so $b' = a$. b is 0 where c has a maximum, so $c' = b$. We conclude that d is the position function, c is the velocity, b is the acceleration, and a is the jerk.

5. $f(x) = x^5 + 6x^2 - 7x \Rightarrow f'(x) = 5x^4 + 12x - 7 \Rightarrow f''(x) = 20x^3 + 12$

6. $f(t) = t^8 - 7t^6 + 2t^4 \Rightarrow f'(t) = 8t^7 - 42t^5 + 8t^3 \Rightarrow f''(t) = 56t^6 - 210t^4 + 24t^2$

7. $y = \cos 2\theta \Rightarrow y' = -2\sin 2\theta \Rightarrow y'' = -4\cos 2\theta$

8. $y = \theta \sin\theta \Rightarrow y' = \theta\cos\theta + \sin\theta \Rightarrow y'' = \theta(-\sin\theta) + \cos\theta \cdot 1 + \cos\theta = 2\cos\theta - \theta\sin\theta$

9. $h(x) = \sqrt{x^2 + 1} \Rightarrow h'(x) = \dfrac{1}{2}(x^2 + 1)^{-1/2}(2x) = \dfrac{x}{\sqrt{x^2 + 1}} \Rightarrow$

$$h''(x) = \frac{\sqrt{x^2 + 1} - x\left(x/\sqrt{x^2 + 1}\right)}{x^2 + 1} = \frac{x^2 + 1 - x^2}{(x^2 + 1)^{3/2}} = \frac{1}{(x^2 + 1)^{3/2}}$$

10. $G(r) = \sqrt{r} + \sqrt[3]{r} \Rightarrow G'(r) = \frac{1}{2}r^{-1/2} + \frac{1}{3}r^{-2/3} \Rightarrow G''(r) = -\frac{1}{4}r^{-3/2} - \frac{2}{9}r^{-5/3}$

11. $F(s) = (3s + 5)^8 \Rightarrow F'(s) = 8(3s + 5)^7(3) = 24(3s + 5)^7 \Rightarrow$ $F''(s) = 168(3s + 5)^6(3) = 504(3s + 5)^6$

12. $g(u) = 1/\sqrt{1 - u} = (1 - u)^{-1/2} \Rightarrow g'(u) = -\frac{1}{2}(1 - u)^{-3/2}(-1) = \frac{1}{2}(1 - u)^{-3/2} \Rightarrow$ $g''(u) = -\frac{3}{4}(1 - u)^{-5/2}(-1) = \frac{3}{4}(1 - u)^{-5/2}$.

13. $y = \dfrac{x}{1 - x} \Rightarrow y' = \dfrac{1(1 - x) - x(-1)}{(1 - x)^2} = \dfrac{1}{(1 - x)^2} \Rightarrow y'' = -2(1 - x)^{-3}(-1) = \dfrac{2}{(1 - x)^3}$

14. $y = x^n \Rightarrow y' = nx^{n-1} \Rightarrow y'' = n(n-1)x^{n-2}$

15. $y = (1-x^2)^{3/4} \Rightarrow y' = \frac{3}{4}(1-x^2)^{-1/4}(-2x) = -\frac{3}{2}x(1-x^2)^{-1/4} \Rightarrow$

$$y'' = -\tfrac{3}{2}(1-x^2)^{-1/4} - \tfrac{3}{2}x\left(-\tfrac{1}{4}\right)(1-x^2)^{-5/4}(-2x) = -\tfrac{3}{2}(1-x^2)^{-1/4} - \tfrac{3}{4}x^2(1-x^2)^{-5/4}$$
$$= \tfrac{3}{4}(1-x^2)^{-5/4}(x^2-2)$$

16. $y = \dfrac{x^2}{x+1} \Rightarrow y' = \dfrac{(x+1)2x - x^2}{(x+1)^2} = \dfrac{x^2+2x}{(x+1)^2} \Rightarrow$

$$y'' = \frac{(x+1)^2(2x+2) - (x^2+2x)(2)(x+1)}{(x+1)^4} = \frac{2(x+1)\left[(x+1)^2 - (x^2+2x)\right]}{(x+1)^4} = \frac{2}{(x+1)^3}$$

17. $H(t) = \tan 3t \Rightarrow H'(t) = 3\sec^2 3t \Rightarrow$

$$H''(t) = 2\cdot 3\sec 3t\,\frac{d}{dt}(\sec 3t) = 6\sec 3t\,(3\sec 3t\tan 3t) = 18\sec^2 3t\tan 3t$$

18. $g(s) = s^2\cos s \Rightarrow g'(s) = 2s\cos s - s^2\sin s \Rightarrow$

$$g''(s) = 2\cos s - 2s\sin s - 2s\sin s - s^2\cos s = (2-s^2)\cos s - 4s\sin s$$

19. $g(\theta) = \theta\csc\theta \Rightarrow g'(\theta) = -\theta\csc\theta\cot\theta + \csc\theta \Rightarrow$

$$g''(\theta) = (-1)\csc\theta\cot\theta + (-\theta)(-\csc\theta\cot\theta)\cot\theta + (-\theta\csc\theta)\left(-\csc^2\theta\right) - \csc\theta\cot\theta$$
$$= \csc\theta\left(\theta\csc^2\theta + \theta\cot^2\theta - 2\cot\theta\right)$$

20. $h(x) = \dfrac{x+3}{x^2+2x} \Rightarrow$

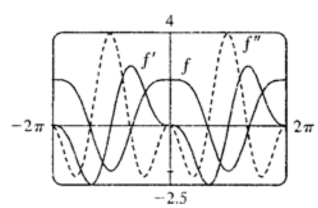
$$h'(x) = \frac{(x^2+2x)(1) - (x+3)(2x+2)}{(x^2+2x)^2} = \frac{(x^2+2x) - (2x^2+8x+6)}{(x^2+2x)^2} = -\frac{x^2+6x+6}{(x^2+2x)^2} \Rightarrow$$

$$h''(x) = -\frac{(x^2+2x)^2(2x+6) - (x^2+6x+6)\,2(x^2+2x)(2x+2)}{\left[(x^2+2x)^2\right]^2}$$
$$= -\frac{2(x^2+2x)\left[(x^2+2x)(x+3) - (x^2+6x+6)(2x+2)\right]}{(x^2+2x)^4}$$
$$= -\frac{2\left[(x^3+5x^2+6x) - (2x^3+14x^2+24x+12)\right]}{(x^2+2x)^3} = \frac{2(x^3+9x^2+18x+12)}{(x^2+2x)^3}$$

21. (a) $f(x) = 2\cos x + \sin^2 x \Rightarrow f'(x) = 2(-\sin x) + 2\sin x(\cos x) = \sin 2x - 2\sin x \Rightarrow$

$f''(x) = 2\cos 2x - 2\cos x = 2(\cos 2x - \cos x)$

(b)

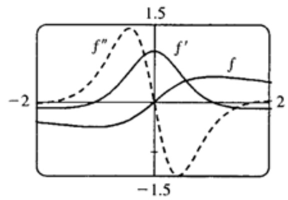


We can see that our answers are plausible, since f has horizontal tangents where $f'(x) = 0$, and f' has horizontal tangents where $f''(x) = 0$.

22. (a) $f(x) = \dfrac{x}{x^2+1} \quad \Rightarrow \quad f'(x) = \dfrac{(x^2+1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} \quad \Rightarrow$

$$f''(x) = \frac{(x^2+1)^2(-2x) - (1-x^2)(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{2x(2x^2-2-x^2-1)}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3}$$

(b)



We can see that our answers are plausible, since f has horizontal tangents where $f'(x) = 0$, and f' has horizontal tangents where $f''(x) = 0$.

23. $y = \sqrt{2x+3} = (2x+3)^{1/2} \quad \Rightarrow \quad y' = \frac{1}{2}(2x+3)^{-1/2} \cdot 2 = (2x+3)^{-1/2} \quad \Rightarrow$

$y'' = -\frac{1}{2}(2x+3)^{-3/2} \cdot 2 = -(2x+3)^{-3/2} \quad \Rightarrow \quad y''' = \frac{3}{2}(2x+3)^{-5/2} \cdot 2 = 3(2x+3)^{-5/2}$

24. $y = \dfrac{1-x}{1+x} \quad \Rightarrow \quad y' = \dfrac{(1+x)(-1) - (1-x)}{(1+x)^2} = \dfrac{-2}{(1+x)^2} = -2(1+x)^{-2} \quad \Rightarrow \quad y'' = 4(1+x)^{-3} \quad \Rightarrow$

$y''' = -12(1+x)^{-4}$

25. $f(x) = (2-3x)^{-1/2} \quad \Rightarrow \quad f(0) = 2^{-1/2} = \frac{1}{\sqrt{2}}$

$f'(x) = -\frac{1}{2}(2-3x)^{-3/2}(-3) = \frac{3}{2}(2-3x)^{-3/2} \quad \Rightarrow \quad f'(0) = \frac{3}{2}(2)^{-3/2} = \frac{3}{4\sqrt{2}}$

$f''(x) = -\frac{9}{4}(2-3x)^{-5/2}(-3) = \dfrac{27}{4}(2-3x)^{-5/2} \quad \Rightarrow \quad f''(0) = \frac{27}{4}(2)^{-5/2} = \frac{27}{16\sqrt{2}}$

$f'''(x) = \frac{405}{8}(2-3x)^{-7/2} \quad \Rightarrow \quad f'''(0) = \frac{405}{8}(2)^{-7/2} = \frac{405}{64\sqrt{2}}$

26. $g(t) = (2-t^2)^6 \quad \Rightarrow \quad g(0) = 2^6 = 64$

$g'(t) = 6(2-t^2)^5(-2t) = -12t(2-t^2)^5 \quad \Rightarrow \quad g'(0) = 0$

$g''(t) = -12(2-t^2)^5 + 120t^2(2-t^2)^4 \quad \Rightarrow \quad g''(0) = -12(2)^5 = -384$

$g'''(t) = 360t(2-t^2)^4 - 960t^3(2-t^2)^3 \quad \Rightarrow \quad g'''(0) = 0$

27. $f(\theta) = \cot\theta \quad \Rightarrow \quad f'(\theta) = -\csc^2\theta \quad \Rightarrow \quad f''(\theta) = -2\csc\theta(-\csc\theta\cot\theta) = 2\csc^2\theta\cot\theta \quad \Rightarrow$

$f'''(\theta) = 2(-2\csc^2\theta\cot\theta)\cot\theta + 2\csc^2\theta(-\csc^2\theta) = -2\csc^2\theta(2\cot^2\theta + \csc^2\theta) \quad \Rightarrow$

$f'''\left(\frac{\pi}{6}\right) = -2(2)^2\left[2\left(\sqrt{3}\right)^2 + (2)^2\right] = -80$

28. $g(x) = \sec x \quad \Rightarrow \quad g'(x) = \sec x \tan x \quad \Rightarrow$

$g''(x) = \sec x \sec^2 x + \tan x(\sec x \tan x) = \sec^3 x + \sec x \tan^2 x = \sec^3 x + \sec x(\sec^2 x - 1)$

$= 2\sec^3 x - \sec x \quad \Rightarrow$

$g'''(x) = 6\sec^2 x(\sec x \tan x) - \sec x \tan x = \sec x \tan x(6\sec^2 x - 1) \quad \Rightarrow$

$g'''\left(\frac{\pi}{4}\right) = \sqrt{2}(1)(6 \cdot 2 - 1) = 11\sqrt{2}$

29. $x^3 + y^3 = 1 \quad \Rightarrow \quad 3x^2 + 3y^2y' = 0 \quad \Rightarrow \quad y' = -\dfrac{x^2}{y^2} \quad \Rightarrow$

$y'' = -\dfrac{2xy^2 - 2x^2yy'}{y^4} = -\dfrac{2xy^2 - 2x^2y(-x^2/y^2)}{y^4} = -\dfrac{2xy^3 + 2x^4}{y^5} = -\dfrac{2x(y^3+x^3)}{y^5} = -\dfrac{2x}{y^5}$, since x and y must satisfy the original equation, $x^3 + y^3 = 1$.

30. $\sqrt{x}+\sqrt{y}=1 \quad\Rightarrow\quad \dfrac{1}{2\sqrt{x}}+\dfrac{y'}{2\sqrt{y}}=0 \quad\Rightarrow\quad y'=-\sqrt{y}/\sqrt{x} \quad\Rightarrow$

$$y''=-\frac{\sqrt{x}\left[1/\left(2\sqrt{y}\right)\right]y'-\sqrt{y}\left[1/\left(2\sqrt{x}\right)\right]}{x}=-\frac{\sqrt{x}\left(1/\sqrt{y}\right)\left(-\sqrt{y}/\sqrt{x}\right)-\sqrt{y}\left(1/\sqrt{x}\right)}{2x}=\frac{1+\sqrt{y}/\sqrt{x}}{2x}$$

$=\dfrac{\sqrt{x}+\sqrt{y}}{2x\sqrt{x}}=\dfrac{1}{2x\sqrt{x}}$ since x and y must satisfy the original equation, $\sqrt{x}+\sqrt{y}=1$.

31. $x^2+xy+y^2=1 \quad\Rightarrow\quad 2x+xy'+y+2yy'=0 \quad\Rightarrow\quad y'(x+2y)=-2x-y \quad\Rightarrow\quad y'=-\dfrac{2x+y}{x+2y} \quad\Rightarrow$

$$y''=-\frac{(x+2y)(2+y')-(2x+y)(1+2y')}{(x+2y)^2}=-\frac{(2x+xy'+4y+2yy')-(2x+4xy'+y+2yy')}{(x+2y)^2}$$

$$=-\frac{-3xy'+3y}{(x+2y)^2}=-\frac{-3x\left(-\dfrac{2x+y}{x+2y}\right)+3y}{(x+2y)^2}=-\frac{\dfrac{3x(2x+y)+3y(x+2y)}{x+2y}}{(x+2y)^2}$$

$$=-\frac{6x^2+3xy+3xy+6y^2}{(x+2y)^3}=-\frac{6\left(x^2+xy+y^2\right)}{(x+2y)^3}$$

$=-\dfrac{6}{(x+2y)^3}$, since x and y must satisfy the original equation, $x^2+xy+y^2=1$.

32. $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1 \quad\Rightarrow\quad \dfrac{2x}{a^2}-\dfrac{2yy'}{b^2}=0 \quad\Rightarrow\quad y'=\dfrac{b^2x}{a^2y} \quad\Rightarrow$

$y''=\dfrac{b^2}{a^2}\dfrac{y-xy'}{y^2}=\dfrac{b^2}{a^2}\dfrac{y-x\dfrac{b^2x}{a^2y}}{y^2}=\dfrac{b^2}{a^2}\dfrac{a^2y^2-b^2x^2}{a^2y^3}=\dfrac{b^4}{a^2y^3}\left(\dfrac{y^2}{b^2}-\dfrac{x^2}{a^2}\right)=-\dfrac{b^4}{a^2y^3}$ since x and y must satisfy the original equation.

33. $f(x)=x^n \quad\Rightarrow\quad f'(x)=nx^{n-1} \quad\Rightarrow\quad f''(x)=n(n-1)x^{n-2} \quad\Rightarrow\quad\cdots\quad\Rightarrow$
$f^{(n)}(x)=n(n-1)(n-2)\cdots 2\cdot 1x^{n-n}=n!$

34. $f(x)=(1-x)^{-2} \quad\Rightarrow\quad f'(x)=-2(1-x)^{-3}(-1)=2(1-x)^{-3} \quad\Rightarrow$
$f''(x)=2(-3)(1-x)^{-4}(-1)=2\cdot 3(1-x)^{-4} \quad\Rightarrow$
$f'''(x)=2\cdot 3(-4)(1-x)^{-5}(-1)=2\cdot 3\cdot 4(1-x)^{-5} \quad\Rightarrow\quad\cdots\quad\Rightarrow$
$f^{(n)}(x)=2\cdot 3\cdot 4\cdot\cdots\cdot n(n+1)(1-x)^{-(n+2)}=\dfrac{(n+1)!}{(1-x)^{n+2}}$

35. $f(x)=(1+x)^{-1} \quad\Rightarrow\quad f'(x)=-1(1+x)^{-2}$, $f''(x)=1\cdot 2(1+x)^{-3}$, $f^{(3)}(x)=-1\cdot 2\cdot 3(1+x)^{-4}$,
$f^{(4)}(x)=1\cdot 2\cdot 3\cdot 4(1+x)^{-5},\ldots, f^{(n)}(x)=(-1)^n n!\,(1+x)^{-(n+1)}$

36. $f(x)=\sqrt{x}=x^{1/2} \quad\Rightarrow\quad f'(x)=\frac{1}{2}x^{-1/2} \quad\Rightarrow\quad f''(x)=\frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2} \quad\Rightarrow$

$f'''(x)=\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2} \quad\Rightarrow\quad f^{(4)}(x)=\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-7/2}=-\dfrac{1\cdot 3\cdot 5}{2^4}x^{-7/2} \quad\Rightarrow$

$f^{(5)}(x)=\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)x^{-9/2}=\dfrac{1\cdot 3\cdot 5\cdot 7}{2^5}x^{-9/2} \quad\Rightarrow\quad\cdots\quad\Rightarrow$

$f^{(n)}(x)=\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(\frac{1}{2}-n+1\right)x^{-(2n-1)/2}=(-1)^{n-1}\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n}x^{-(2n-1)/2}$

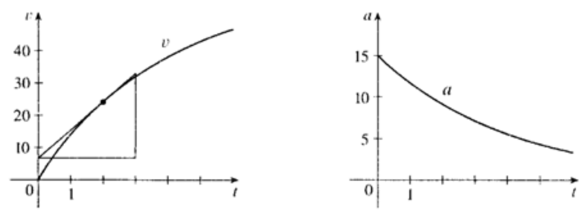
37. $f(x)=1/\left(3x^3\right)=\frac{1}{3}x^{-3} \quad\Rightarrow\quad f'(x)=\frac{1}{3}(-3)x^{-4} \quad\Rightarrow$
$f''(x)=\frac{1}{3}(-3)(-4)x^{-5} \quad\Rightarrow\quad f'''(x)=\frac{1}{3}(-3)(-4)(-5)x^{-6} \quad\Rightarrow\quad\cdots\quad\Rightarrow$
$f^{(n)}(x)=\frac{1}{3}(-3)(-4)\cdots[-(n+2)]x^{-(n+3)}=\dfrac{(-1)^n\cdot 3\cdot 4\cdot 5\cdot\cdots\cdot(n+2)}{3x^{n+3}}=\dfrac{(-1)^n(n+2)!}{6x^{n+3}}$

38. $D\sin x = \cos x \quad\Rightarrow\quad D^2\sin x = -\sin x \quad\Rightarrow\quad D^3\sin x = -\cos x \quad\Rightarrow\quad D^4\sin x = \sin x$
The derivatives of $\sin x$ occur in a cycle of four. Since $99 = 4\,(24) + 3$, we have $D^{99}\sin x = D^3\sin x = -\cos x$.

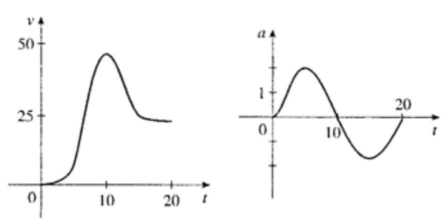
39. In general, $Df\,(2x) = 2f'\,(2x)$, $D^2 f\,(2x) = 4f''\,(2x)$, ..., $D^n f\,(2x) = 2^n f^{(n)}\,(2x)$. Specifically, since $f\,(x) = \cos x$ and $50 = 4\,(12) + 2$, we have $f^{(50)}\,(x) = f^{(2)}\,(x) = -\cos x$, so $D^{50}\cos 2x = -2^{50}\cos 2x$.

40. Let $f\,(x) = x\sin x$ and $h\,(x) = \sin x$, so $f\,(x) = xh\,(x)$. Then
$f'\,(x) = h\,(x) + xh'\,(x)$, $f''\,(x) = h'\,(x) + h'\,(x) + xh''\,(x) = 2h'\,(x) + xh''\,(x)$,
$f'''\,(x) = 2h''\,(x) + h''\,(x) + xh'''\,(x) = 3h''\,(x) + xh'''\,(x)$, ..., $f^{(n)}\,(x) = nh^{(n-1)}\,(x) + xh^{(n)}\,(x)$. Since $34 = 4\,(8) + 2$, we have $h^{(34)}\,(x) = h^{(2)}\,(x) = D^2\sin x = -\sin x$ and $h^{(35)}\,(x) = -\cos x$. Thus,
$D^{(35)}x\sin x = 35h^{(34)}\,(x) + xh^{(35)}\,(x) = -35\sin x - x\cos x$.

41. By measuring the slope of the graph of $s = f\,(t)$ at $t = 0, 1, 2, 3, 4$, and 5, and using the method of Example 1 in Section 3.2, we plot the graph of the velocity function $v = f'\,(t)$ in the first figure. The acceleration when $t = 2$ s is $a = f''\,(2)$, the slope of the tangent line to the graph of f' when $t = 2$. We estimate the slope of this tangent line to be $a\,(2) = f''\,(2) = v'\,(2) \approx \frac{27}{3} = 9\ \text{ft/s}^2$. Similar measurements enable us to graph the acceleration function in the second figure.



42. (a) Since we estimate the velocity to be a maximum at $t = 10$, the acceleration is 0 at $t = 10$.



(b) Drawing a tangent line at $t = 10$ on the graph of a, a appears to decrease by $10\ \text{ft/s}^2$ over a period of 20 s. So at $t = 10$ s, the jerk is approximately $-10/20 = -0.5\ \left(\text{ft/s}^2\right)/\text{s}$ or ft/s^3.

43. (a) $s = t^3 - 3t \quad\Rightarrow\quad v\,(t) = s'\,(t) = 3t^2 - 3 \quad\Rightarrow\quad a\,(t) = v'\,(t) = 6t$

(b) $a\,(1) = 6\,(1) = 6\ \text{m/s}^2$

(c) $v\,(t) = 3t^2 - 3 = 0$ when $t^2 = 1$, that is, $t = 1$ and $a\,(1) = 6\ \text{m/s}^2$.

44. (a) $s = t^2 - t + 1 \quad\Rightarrow\quad v\,(t) = s'\,(t) = 2t - 1 \quad\Rightarrow\quad a\,(t) = v'\,(t) = 2$

(b) $a\,(1) = 2\ \text{m/s}^2$

(c) $v\,(t) = 2t - 1 = 0$ when $t = \frac{1}{2}$ and $a\left(\frac{1}{2}\right) = 2\ \text{m/s}^2$.

45. (a) $s = \sin 2\pi t \;\Rightarrow\; v(t) = s'(t) = 2\pi \cos 2\pi t \;\Rightarrow\; a(t) = v'(t) = -4\pi^2 \sin 2\pi t$

(b) $a(1) = -4\pi^2 \sin 2\pi(1) = -4\pi^2(0) = 0 \text{ m/s}^2$

(c) $v(t) = 2\pi \cos 2\pi t = 0$ when $2\pi t$ is an odd multiple of $\frac{\pi}{2} \;\Leftrightarrow\; t$ is an odd multiple of $\frac{1}{4} \;\Leftrightarrow$ $t \in \left\{\pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{5}{4}, \ldots\right\}$. When $t \in \left\{\ldots, -\frac{7}{4}, -\frac{3}{4}, \frac{1}{4}, \frac{5}{4}, \ldots\right\}$, $\sin 2\pi t = 1$ and $a(t) = -4\pi^2 \text{ m/s}^2$. When $t \in \left\{\ldots, -\frac{5}{4}, -\frac{1}{4}, \frac{3}{4}, \frac{7}{4}, \ldots\right\}$, $\sin 2\pi t = -1$ and $a(t) = 4\pi^2 \text{ m/s}^2$.

46. (a) $s = 2t^3 - 7t^2 + 4t + 1 \;\Rightarrow\; v(t) = s'(t) = 6t^2 - 14t + 4 \;\Rightarrow\; a(t) = v'(t) = 12t - 14$

(b) $a(1) = 12 - 14 = -2 \text{ m/s}^2$

(c) $v(t) = 2(3t^2 - 7t + 2) = 2(3t-1)(t-2) = 0$ when $t = \frac{1}{3}$ or 2 and $a\left(\frac{1}{3}\right) = 12\left(\frac{1}{3}\right) - 14 = -10 \text{ m/s}^2$, $a(2) = 12(2) - 14 = 10 \text{ m/s}^2$.

47. (a) $s(t) = t^4 - 4t^3 + 2 \;\Rightarrow\; v(t) = s'(t) = 4t^3 - 12t^2 \;\Rightarrow\; a(t) = v'(t) = 12t^2 - 24t = 12t(t-2) = 0$ when $t = 0$ or 2.

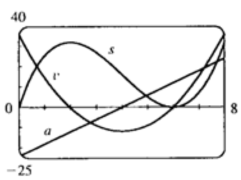
(b) $s(0) = 2$ m, $v(0) = 0$ m/s, $s(2) = -14$ m, $v(2) = -16$ m/s

48. (a) $s(t) = 2t^3 - 9t^2 \;\Rightarrow\; v(t) = s'(t) = 6t^2 - 18t \;\Rightarrow\; a(t) = v'(t) = 12t - 18 = 0$ when $t = 1.5$.

(b) $s(1.5) = -13.5$ m, $v(1.5) = -13.5$ m/s

49. (a) $s = f(t) = t^3 - 12t^2 + 36t \;\Rightarrow\; v(t) = s'(t) = 3t^2 - 24t + 36 \;\Rightarrow\; a(t) = v'(t) = 6t - 24$. $a(3) = -6$ (m/s)/s or m/s^2

(b)

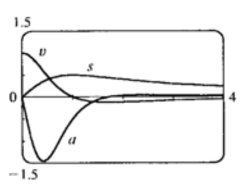


(c) First note that $v(t) = 0$ when $t = 2$ and when $t = 6$, and $a(t) = 0$ when $t = 4$. The particle is speeding up when v and a have the same sign. This occurs when $2 < t < 4$ and when $t > 6$. It is slowing down when v and a have opposite signs; that is, when $0 \le t < 2$ and $4 < t < 6$.

50. (a) $x(t) = \dfrac{t}{1+t^2} \;\Rightarrow\; v(t) = x'(t) = \dfrac{1-t^2}{(1+t^2)^2} \;\Rightarrow\; a(t) = v'(t) = \dfrac{2t(t^2-3)}{(1+t^2)^3}$.

$a(t) = 0 \;\Rightarrow\; 2t(t^2-3) = 0 \;\Rightarrow\; t = 0$ or $\sqrt{3}$

(b)



(c) The particle is speeding up when v and a have the same sign; that is, when $1 < t < \sqrt{3}$. The particle is slowing down when v and a have opposite signs; that is, when $0 < t < 1$ and when $t > \sqrt{3}$.

51. (a) $y(t) = A \sin \omega t \;\Rightarrow\; v(t) = y'(t) = A\omega \cos \omega t \;\Rightarrow\; a(t) = v'(t) = -A\omega^2 \sin \omega t$

(b) $a(t) = -A\omega^2 \sin \omega t = -\omega^2 y(t)$

(c) $|v(t)| = A\omega |\cos \omega t|$ is a maximum when $\cos \omega t = \pm 1 \;\Leftrightarrow\; \sin \omega t = 0 \;\Leftrightarrow\; a(t) = -A\omega^2 \sin^2 \omega t = 0$.

52. $a(t) = \dfrac{dv}{dt} = \dfrac{dv}{ds}\dfrac{ds}{dt} = v(t)\dfrac{dv}{ds}$. Then $\dfrac{dv}{dt}$ is the rate of change of the velocity with respect to time (in other words, the acceleration) whereas dv/ds is the rate of change of the velocity with respect to the displacement.

53. Let $P(x) = ax^2 + bx + c$. Then $P'(x) = 2ax + b$ and $P''(x) = 2a$.
$P''(2) = 2 \Rightarrow 2a = 2 \Rightarrow a = 1$. $P'(2) = 3 \Rightarrow 4a + b = 4 + b = 3 \Rightarrow b = -1$.
$P(2) = 5 \Rightarrow 2^2 - 2 + c = 5 \Rightarrow c = 3$. So $P(x) = x^2 - x + 3$.

54. Let $Q(x) = ax^3 + bx^2 + cx + d$. Then $Q'(x) = 3ax^2 + 2bx + c$, $Q''(x) = 6ax + 2b$ and $Q'''(x) = 6a$. Thus, $Q(1) = a + b + c + d = 1$, $Q'(1) = 3a + 2b + c = 3$, $Q''(1) = 6a + 2b = 6$ and $Q'''(1) = 6a = 12$. Solving these four equations in four unknowns a, b, c and d we get $a = 2$, $b = -3$, $c = 3$ and $d = -1$, so $Q(x) = 2x^3 - 3x^2 + 3x - 1$.

55. $y = A\sin x + B\cos x \Rightarrow y' = A\cos x - B\sin x \Rightarrow y'' = -A\sin x - B\cos x$. Substituting into $y'' + y' - 2y = \sin x$ gives us $(-3A - B)\sin x + (A - 3B)\cos x = 1\sin x$, so we must have $-3A - B = 1$ and $A - 3B = 0$. Solving for A and B, we add the first equation to three times the second to get $B = -\frac{1}{10}$ and $A = -\frac{3}{10}$.

56. $y = Ax^2 + Bx + C \Rightarrow y' = 2Ax + B \Rightarrow y'' = 2A$. We substitute these expressions into the equation $y'' + y' - 2y = x^2$ to get

$$(2A) + (2Ax + B) - 2\left(Ax^2 + Bx + C\right) = x^2$$
$$2A + 2Ax + B - 2Ax^2 - 2Bx - 2C = x^2$$
$$(-2A)x^2 + (2A - 2B)x + (2A + B - 2C) = (1)x^2 + (0)x + (0)$$

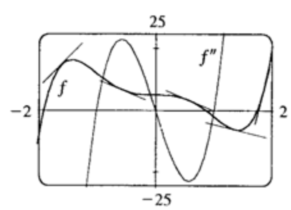
The coefficients of x^2 on each side must be equal, so $-2A = 1 \Rightarrow A = -\frac{1}{2}$. Similarly, $2A - 2B = 0 \Rightarrow A = B = -\frac{1}{2}$ and $2A + B - 2C = 0 \Rightarrow -1 - \frac{1}{2} - 2C = 0 \Rightarrow C = -\frac{3}{4}$.

57. $f(x) = xg(x^2) \Rightarrow f'(x) = g(x^2) + xg'(x^2)2x = g(x^2) + 2x^2g'(x^2) \Rightarrow$
$f''(x) = 2xg'(x^2) + 4xg'(x^2) + 4x^3g''(x^2) = 6xg'(x^2) + 4x^3g''(x^2)$

58. $f(x) = \dfrac{g(x)}{x} \Rightarrow f'(x) = \dfrac{xg'(x) - g(x)}{x^2} \Rightarrow$
$$f''(x) = \frac{x^2\left[g'(x) + xg''(x) - g'(x)\right] - 2x\left[xg'(x) - g(x)\right]}{x^4} = \frac{x^2g''(x) - 2xg'(x) + 2g(x)}{x^3}$$

59. $f(x) = g(\sqrt{x}) \Rightarrow f'(x) = \dfrac{g'(\sqrt{x})}{2\sqrt{x}} \Rightarrow f''(x) = \dfrac{\dfrac{g''(\sqrt{x})}{2\sqrt{x}} \cdot 2\sqrt{x} - \dfrac{g'(\sqrt{x})}{\sqrt{x}}}{4x} = \dfrac{\sqrt{x}g''(\sqrt{x}) - g'(\sqrt{x})}{4x\sqrt{x}}$

60.



$f(x) = 3x^5 - 10x^3 + 5 \Rightarrow f'(x) = 15x^4 - 30x^2 \Rightarrow$
$f''(x) = 60x^3 - 60x = 60x(x^2 - 1) = 60x(x + 1)(x - 1)$
So $f''(x) > 0$ when $-1 < x < 0$ or $x > 1$, and on these intervals the graph of f lies above its tangent lines; and $f''(x) < 0$ when $x < -1$ or $0 < x < 1$, and on these intervals the graph of f lies below its tangent lines.

61. (a) $f(x) = \dfrac{1}{x^2+x} \quad \Rightarrow \quad f'(x) = \dfrac{-(2x+1)}{(x^2+x)^2} \quad \Rightarrow$

$$f''(x) = \frac{(x^2+x)^2(-2) + (2x+1)(2)(x^2+x)(2x+1)}{(x^2+x)^4} = \frac{2(3x^2+3x+1)}{(x^2+x)^3} \quad \Rightarrow$$

$$f'''(x) = \frac{(x^2+x)^3(2)(6x+3) - 2(3x^2+3x+1)(3)(x^2+x)^2(2x+1)}{(x^2+x)^6}$$

$$= \frac{-6(4x^3+6x^2+4x+1)}{(x^2+x)^4} \quad \Rightarrow$$

$$f^{(4)}(x) = \frac{(x^2+x)^4(-6)(12x^2+12x+4) + 6(4x^3+6x^2+4x+1)(4)(x^2+x)^3(2x+1)}{(x^2+x)^8}$$

$$= \frac{24(5x^4+10x^3+10x^2+5x+1)}{(x^2+x)^5}$$

$f^{(5)}(x) = ?$

(b) $f(x) = \dfrac{1}{x(x+1)} = \dfrac{1}{x} - \dfrac{1}{x+1} \quad \Rightarrow \quad f'(x) = -x^{-2} + (x+1)^{-2} \quad \Rightarrow$

$f''(x) = 2x^{-3} - 2(x+1)^{-3} \quad \Rightarrow \quad f'''(x) = (-3)(2)x^{-4} + (3)(2)(x+1)^{-4} \quad \Rightarrow \quad \cdots \quad \Rightarrow$

$f^{(n)}(x) = (-1)^n n! \left[x^{-(n+1)} - (x+1)^{-(n+1)}\right]$

62. (a) Use the Product Rule repeatedly: $F = fg \quad \Rightarrow \quad F' = f'g + fg' \quad \Rightarrow$

$F'' = (f''g + f'g') + (f'g' + fg'') = f''g + 2f'g' + fg''$.

(b) $F''' = f'''g + f''g' + 2(f''g' + f'g'') + f'g'' + fg''' = f'''g + 3f''g' + 3f'g'' + fg''' \quad \Rightarrow$

$$F^{(4)} = f^{(4)}g + f'''g' + 3(f'''g' + f''g'') + 3(f''g'' + f'g''') + f'g''' + fg^{(4)}$$

$$= f^{(4)}g + 4f'''g' + 6f''g'' + 4f'g''' + fg^{(4)}$$

(c) By analogy with the Binomial Theorem, we make the guess:

$$F^{(n)} = f^{(n)}g + nf^{(n-1)}g' + \binom{n}{2}f^{(n-2)}g'' + \cdots + \binom{n}{k}f^{(n-k)}g^{(k)} + \cdots + nf'g^{(n-1)} + fg^{(n)}$$

where $\dbinom{n}{k} = \dfrac{n!}{k!\,(n-k)!} = \dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$.

63. The Chain Rule says that $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{du}\frac{du}{dx}\right) = \left[\frac{d}{dx}\left(\frac{dy}{du}\right)\right]\frac{du}{dx} + \frac{dy}{du}\frac{d}{dx}\left(\frac{du}{dx}\right) \quad \text{(Product Rule)}$$

$$= \left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\frac{du}{dx} + \frac{dy}{du}\frac{d^2u}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}$$

64. From Exercise 63, $\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{du^2}\left(\dfrac{du}{dx}\right)^2 + \dfrac{dy}{du}\dfrac{d^2u}{dx^2} \quad \Rightarrow$

$$\begin{aligned}
\frac{d^3y}{dx^3} &= \frac{d}{dx}\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2\right] + \frac{d}{dx}\left[\frac{dy}{du}\frac{d^2u}{dx^2}\right] \\
&= \left[\frac{d}{dx}\left(\frac{d^2y}{du^2}\right)\right]\left(\frac{du}{dx}\right)^2 + \left[\frac{d}{dx}\left(\frac{du}{dx}\right)^2\right]\frac{d^2y}{du^2} + \left[\frac{d}{dx}\left(\frac{dy}{du}\right)\right]\frac{d^2u}{dx^2} + \left[\frac{d}{dx}\left(\frac{d^2u}{dx^2}\right)\right]\frac{dy}{du} \\
&= \left[\frac{d}{du}\left(\frac{d^2y}{du^2}\right)\frac{du}{dx}\right]\left(\frac{du}{dx}\right)^2 + 2\frac{du}{dx}\frac{d^2u}{dx^2}\frac{d^2y}{du^2} + \left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\left(\frac{d^2u}{dx^2}\right) + \frac{d^3u}{dx^3}\frac{dy}{du} \\
&= \frac{d^3y}{du^3}\left(\frac{du}{dx}\right)^3 + 3\frac{du}{dx}\frac{d^2u}{dx^2}\frac{d^2y}{du^2} + \frac{dy}{du}\frac{d^3u}{dx^3}
\end{aligned}$$

Applied Project □ Where Should a Pilot Start Descent?

1. Condition (i) will hold if and only if all of the following four conditions hold:

(α) $P(0) = 0$

(β) $P'(0) = 0$ (for a smooth landing)

(γ) $P'(\ell) = 0$ (since the plane is cruising horizontally when it begins its descent)

(δ) $P(\ell) = h$.

First of all, condition α implies that $P(0) = d = 0$, so $P(x) = ax^3 + bx^2 + cx \quad \Rightarrow \quad P'(x) = 3ax^2 + 2bx + c$. But $P'(0) = c = 0$ by condition β. So $P'(\ell) = 3a\ell^2 + 2b\ell = \ell(3a\ell + 2b)$. Now by condition γ, $3a\ell + 2b = 0$ $\Rightarrow \quad a = -\dfrac{2b}{3\ell}$. Therefore, $P(x) = -\dfrac{2b}{3\ell}x^3 + bx^2$. Setting $P(\ell) = h$ for condition δ, we get

$P(\ell) = -\dfrac{2b}{3\ell}\ell^3 + b\ell^2 = h \quad \Rightarrow \quad -\frac{2}{3}b\ell^2 + b\ell^2 = h \quad \Rightarrow \quad b = \dfrac{3h}{\ell^2} \quad \Rightarrow \quad a = -\dfrac{2h}{\ell^3}$. So

$P(x) = -\dfrac{2h}{\ell^3}x^3 + \dfrac{3h}{\ell^2}x^2$.

2. By condition (ii), $\dfrac{dx}{dt} = -v$ for all t, so $x(t) = \ell - vt$. Condition (iii) states that $\left|\dfrac{d^2y}{dt^2}\right| \le k$. By the Chain Rule, we have $\dfrac{dy}{dt} = -\dfrac{2h}{\ell^3}(3x^2)\dfrac{dx}{dt} + \dfrac{3h}{\ell^2}(2x)\dfrac{dx}{dt} = \dfrac{6hx^2v}{\ell^3} - \dfrac{6hxv}{\ell^2}$ (for $x \le \ell$) $\Rightarrow$

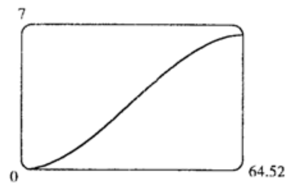
$\dfrac{d^2y}{dt^2} = \dfrac{6hv}{\ell^3}(2x)\dfrac{dx}{dt} - \dfrac{6hv}{\ell^2}\dfrac{dx}{dt} = -\dfrac{12hv^2}{\ell^3}x + \dfrac{6hv^2}{\ell^2}$. In particular, when $t = 0$, $x = \ell$ and so

$\left.\dfrac{d^2y}{dt^2}\right|_{t=0} = -\dfrac{12hv^2}{\ell^3}\ell + \dfrac{6hv^2}{\ell^2} = -\dfrac{6hv^2}{\ell^2}$. Thus, $\left|\dfrac{d^2y}{dt^2}\right|_{t=0} = \dfrac{6hv^2}{\ell^2} \le k$. (This condition also follows from taking $x = 0$.)

3. We substitute $k = 860\ \text{mi/h}^2$, $h = 35{,}000\ \text{ft} \times \dfrac{1\ \text{mi}}{5280\ \text{ft}}$, and $v = 300$ mi/h into the result of part (b):

$$\frac{6\left(35{,}000 \cdot \frac{1}{5280}\right)(300)^2}{\ell^2} \le 860 \quad \Leftrightarrow \quad \ell \ge 300\sqrt{6 \cdot \frac{35{,}000}{5280 \cdot 860}} \approx 64.5 \text{ miles.}$$

4. Substituting the values of h and ℓ in Problem 3 into $P(x) = -\frac{2h}{\ell^3}x^3 + \frac{3h}{\ell^2}x^2$ gives us $P(x) = ax^3 + bx^2$, where $a \approx 4.937 \times 10^{-5}$ and $b \approx 4.78 \times 10^{-3}$.



3.9 Related Rates

1. $V = x^3 \Rightarrow \frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = 3x^2\frac{dx}{dt}$

2. (a) $A = \pi r^2 \Rightarrow \frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt} = 2\pi r\frac{dr}{dt}$

(b) $\frac{dA}{dt} = 2\pi r\frac{dr}{dt} = 2\pi\,(30)\,(1) = 60\pi\ \text{m}^2/\text{s}$

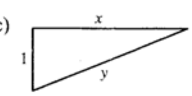
3. $y = x^3 + 2x \Rightarrow \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = (3x^2+2)\,(5) = 5\,(3x^2+2)$. When $x = 2$, $\frac{dy}{dt} = 5\,(14) = 70$.

4. $y = \sqrt{1+x^3} \Rightarrow \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{2}\left(1+x^3\right)^{-1/2}(3x^2)\frac{dx}{dt} = \frac{3x^2}{2\sqrt{1+x^3}}\frac{dx}{dt}$. With $\frac{dy}{dt} = 4$ when $x = 2$ and $y = 3$, we have $4 = \frac{3\,(4)}{2\,(3)}\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 2$ cm/s.

5. (a) Given: a plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. If we let t be time (in hours) and x be the horizontal distance traveled by the plane (in mi), then we are given that $dx/dt = 500$ mi/h.

(b) Unknown: the rate at which the distance from the plane to the station is increasing when it is 2 mi from the station. If we let y be the distance from the plane to the station, then we want to find dy/dt when $y = 2$ mi.

(c)



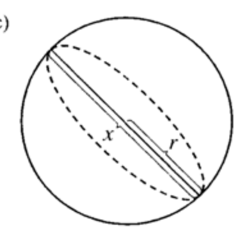
(d) By the Pythagorean Theorem, $y^2 = x^2 + 1 \Rightarrow 2y\,(dy/dt) = 2x\,(dx/dt)$.

(e) $\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt} = 500\frac{x}{y}$. When $y = 2$, $x = \sqrt{3}$, so $\frac{dy}{dt} = 500\left(\frac{\sqrt{3}}{2}\right) = 250\sqrt{3} \approx 433$ mi/h.

6. (a) Given: the rate of decrease of the surface area is 1 cm^2/min. If we let t be time (in minutes) and S be the surface area (in cm^2), then we are given that $dS/dt = -1$ cm^2/s.

(b) Unknown: the rate of decrease of the diameter when the diameter is 10 cm. If we let x be the diameter, then we want to find dx/dt when $x = 10$ cm.

(c)

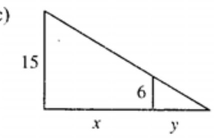


(d) If the radius is r and the diameter x, then $S = 4\pi r^2 = 4\pi\left(\frac{1}{2}x\right)^2 = \pi x^2 \Rightarrow dS/dx = 2\pi x\,(dx/dt)$.

(e) $-1 = \dfrac{dS}{dt} = 2\pi x \dfrac{dx}{dt} \Rightarrow \dfrac{dx}{dt} = -\dfrac{1}{2\pi x}$. When $x = 10$, $\dfrac{dx}{dt} = -\dfrac{1}{20\pi}$. So the rate of decrease is $\frac{1}{20\pi}$ cm/min.

7. (a) Given: a man 6 ft tall walks away from a street light mounted on a 15-ft-tall pole at a rate of 5 ft/s. If we let t be time (in s) and x be the distance from the pole to the man (in ft), then we are given that $dx/dt = 5$ ft/s.

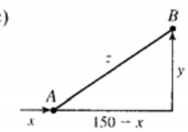
(b) Unknown: the rate at which the tip of his shadow is moving when he is 40 ft from the pole. If we let y be the distance from the man to the tip of his shadow (in ft), then we want to find $\dfrac{d}{dt}(x + y)$ when $x = 40$ ft.

(c)


(d) By similar triangles, $\dfrac{15}{6} = \dfrac{x+y}{y} \Rightarrow 15y = 6x + 6y \Rightarrow 9y = 6x \Rightarrow y = \frac{2}{3}x$.

(e) The tip of the shadow moves at a rate of $\dfrac{d}{dt}(x+y) = \dfrac{d}{dt}\left(x + \frac{2}{3}x\right) = \dfrac{5}{3}\dfrac{dx}{dt} = \frac{5}{3}(5) = \frac{25}{3}$ ft/s.

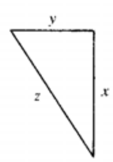
8. (a) Given: at noon, ship A is 150 km west of ship B; ship A is sailing east at 35 km/h, and ship B is sailing north at 25 km/h. If we let t be time (in hours), x be the distance traveled by ship A (in km), and y be the distance traveled by ship B (in km), then we are given that $dx/dt = 35$ km/h and $dy/dt = 25$ km/h.

(c)


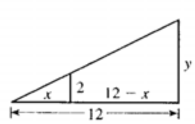
(b) Unknown: the rate at which the distance between the ships is changing at 4:00 P.M. If we let z be the distance between the ships, then we want to find dz/dt when $t = 4$ h.

(d) $z^2 = (150 - x)^2 + y^2 \Rightarrow 2z\dfrac{dz}{dt} = 2(150 - x)\left(-\dfrac{dx}{dt}\right) + 2y\dfrac{dy}{dt}$

(e) At 4:00 P.M., $x = 4(35) = 140$ and $y = 4(25) = 100 \Rightarrow z = \sqrt{(150 - 140)^2 + 100^2} = \sqrt{10{,}100}$. So
$\dfrac{dz}{dt} = \dfrac{1}{z}\left[(x - 150)\dfrac{dx}{dt} + y\dfrac{dy}{dt}\right] = \dfrac{-10(35) + 100(25)}{\sqrt{10{,}100}} = \dfrac{215}{\sqrt{101}} \approx 21.4$ km/h.

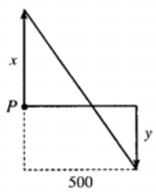
9.


We are given that $\dfrac{dx}{dt} = 60$ mi/h and $\dfrac{dy}{dt} = 25$ mi/h. $z^2 = x^2 + y^2 \Rightarrow$
$2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt}$. After 2 hours, $x = 2(60) = 120$ and $y = 2(25) = 50$
$\Rightarrow z = \sqrt{120^2 + 50^2} = 130$, so
$\dfrac{dz}{dt} = \dfrac{1}{z}\left(x\dfrac{dx}{dt} + y\dfrac{dy}{dt}\right) = \dfrac{120(60) + 50(25)}{130} = 65$ mi/h.

10.


We are given that $\dfrac{dx}{dt} = 1.6$ m/s. By similar triangles, $\dfrac{y}{12} = \dfrac{2}{x} \Rightarrow y = \dfrac{24}{x} \Rightarrow$
$\dfrac{dy}{dt} = -\dfrac{24}{x^2}\dfrac{dx}{dt} = -\dfrac{24}{x^2}(1.6)$. When $x = 8$, $\dfrac{dy}{dt} = -\dfrac{24(1.6)}{64} = -0.6$ m/s, so the shadow is decreasing at a rate of 0.6 m/s.

11.



We are given that $\dfrac{dx}{dt} = 4$ ft/s and $\dfrac{dy}{dt} = 5$ ft/s. $z^2 = (x+y)^2 + 500^2 \quad \Rightarrow$

$2z\dfrac{dz}{dt} = 2(x+y)\left(\dfrac{dx}{dt} + \dfrac{dy}{dt}\right)$. 15 minutes after the woman starts, we have

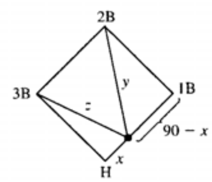
$x = (4 \text{ ft/s})(20 \text{ min})(60 \text{ s/min}) = 4800$ ft and $y = 5 \cdot 15 \cdot 60 = 4500 \quad \Rightarrow$

$z = \sqrt{(4800+4500)^2 + 500^2}$, so

$$\frac{dz}{dt} = \frac{x+y}{z}\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = \frac{4800+4500}{\sqrt{86{,}740{,}000}}(5+4) = \frac{837}{\sqrt{8674}} \approx 8.99 \text{ ft/s}.$$

12. We are given that $\dfrac{dx}{dt} = 24$ ft/s.

(a)



$y^2 = (90-x)^2 + 90^2 \quad \Rightarrow 2y\dfrac{dy}{dt} = 2(90-x)\left(-\dfrac{dx}{dt}\right)$.

When $x = 45$, $y = \sqrt{45^2 + 90^2} = 45\sqrt{5}$, so

$$\frac{dy}{dt} = \frac{90-x}{y}\left(-\frac{dx}{dt}\right) = \frac{45}{45\sqrt{5}}(-24) = -\frac{24}{\sqrt{5}},$$

so the distance from second base is decreasing at a rate of $\frac{24}{\sqrt{5}} \approx 10.7$ ft/s.

(b) Due to the symmetric nature of the problem in part (a), we expect to get the same answer — and we do.

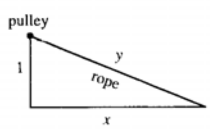
$z^2 = x^2 + 90^2 \Rightarrow 2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt}$. When $x = 45$, $z = 45\sqrt{5}$, so $\dfrac{dz}{dt} = \dfrac{45}{45\sqrt{5}}(24) = \dfrac{24}{\sqrt{5}} \approx 10.7$ ft/s.

13. $A = \frac{1}{2}bh$, where b is the base and h is the altitude. We are given that $\dfrac{dh}{dt} = 1$ cm/min and $\dfrac{dA}{dt} = 2$ cm^2/min.

Using the Product Rule, we have $\dfrac{dA}{dt} = \dfrac{1}{2}\left(b\dfrac{dh}{dt} + h\dfrac{db}{dt}\right)$. When $h = 10$ and $A = 100$, we have $b = 20$, so

$$2 = \frac{1}{2}\left(20 \cdot 1 + 10\frac{db}{dt}\right) \quad \Rightarrow \quad 4 = 20 + 10\frac{db}{dt} \quad \Rightarrow \quad \frac{db}{dt} = \frac{4-20}{10} = -1.6 \text{ cm/min}.$$

14.

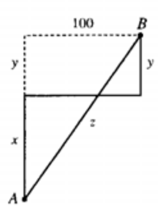


Given $\dfrac{dy}{dt} = -1$ m/s, find $\dfrac{dx}{dt}$ when $x = 8$ m. $y^2 = x^2 + 1 \quad \Rightarrow$

$2y\dfrac{dy}{dt} = 2x\dfrac{dx}{dt} \quad \Rightarrow \quad \dfrac{dx}{dt} = \dfrac{y}{x}\dfrac{dy}{dt} = -\dfrac{y}{x}$. When $x = 8$, $y = \sqrt{65}$, so

$\dfrac{dx}{dt} = -\dfrac{\sqrt{65}}{8}$. Thus, the boat approaches the dock at $\frac{\sqrt{65}}{8} \approx 1.01$ m/s.

15.



We are given that $\dfrac{dx}{dt} = 35$ km/h and $\dfrac{dy}{dt} = 25$ km/h. $z^2 = (x+y)^2 + 100^2$

$\Rightarrow \quad 2z\dfrac{dz}{dt} = 2(x+y)\left(\dfrac{dx}{dt} + \dfrac{dy}{dt}\right)$. At 4:00 P.M., $x = 140$ and $y = 100$

$\Rightarrow \quad z = 260$, so

$$\frac{dz}{dt} = \frac{x+y}{z}\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = \frac{140+100}{260}(35+25) = \frac{720}{13} \approx 55.4 \text{ km/h}.$$

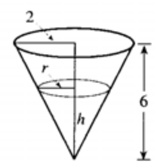
16. Let D denote the distance from the origin $(0, 0)$ to the point on the curve $y = \sqrt{x}$.

$D = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + (\sqrt{x})^2} = \sqrt{x^2 + x} \quad \Rightarrow$

$\dfrac{dD}{dt} = \frac{1}{2}\left(x^2 + x\right)^{-1/2}(2x+1)\dfrac{dx}{dt} = \dfrac{2x+1}{2\sqrt{x^2+x}}\dfrac{dx}{dt}$. With $\dfrac{dx}{dt} = 3$ when $x = 4$,

$\dfrac{dD}{dt} = \dfrac{9}{2\sqrt{20}}(3) = \dfrac{27}{4\sqrt{5}} \approx 3.02$ cm/s.

17.



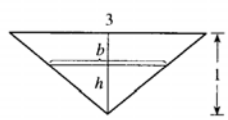
If $C =$ the rate at which water is pumped in, then $\dfrac{dV}{dt} = C - 10{,}000$, where $V = \frac{1}{3}\pi r^2 h$ is the volume at time t. By similar triangles, $\dfrac{r}{2} = \dfrac{h}{6} \quad \Rightarrow$

$r = \frac{1}{3}h \quad \Rightarrow \quad V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h = \frac{\pi}{27}h^3 \quad \Rightarrow \quad \dfrac{dV}{dt} = \frac{\pi}{9}h^2\dfrac{dh}{dt}$. When

$h = 200$, $\dfrac{dh}{dt} = 20$, so $C - 10{,}000 = \frac{\pi}{9}(200)^2(20) \quad \Rightarrow$

$C = 10{,}000 + \frac{800{,}000}{9}\pi \approx 289{,}253$ cm^3/min.

18.

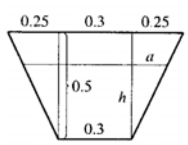


By similar triangles, $\dfrac{3}{1} = \dfrac{b}{h}$, so $b = 3h$. The trough has volume

$V = \frac{1}{2}bh(10) = 5(3h)h = 15h^2 \quad \Rightarrow \quad 12 = \dfrac{dV}{dt} = 30h\dfrac{dh}{dt} \quad \Rightarrow$

$\dfrac{dh}{dt} = \dfrac{2}{5h}$. When $h = \frac{1}{2}$, $\dfrac{dh}{dt} = \dfrac{2}{5 \cdot \frac{1}{2}} = \dfrac{4}{5}$ ft/min.

19.



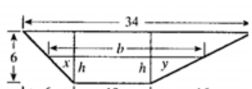
The figure is labeled in meters. The area A of a trapezoid is $\frac{1}{2}(\text{base}_1 + \text{base}_2)(\text{height})$, and the volume V of the trough is $10A$. Thus,

$V = \frac{1}{2}[0.3 + (0.3 + 2a)]h(10)$, where $\dfrac{a}{h} = \dfrac{0.25}{0.5} = \dfrac{1}{2}$ so $2a = h \quad \Rightarrow$

$V = 5(0.6 + h)h = 3h + 5h^2 \quad \Rightarrow \quad 0.2 = \dfrac{dV}{dt} = (3 + 10h)\dfrac{dh}{dt} \quad \Rightarrow$

$\dfrac{dh}{dt} = \dfrac{0.2}{3 + 10h}$. When $h = 0.3$, $\dfrac{dh}{dt} = \dfrac{0.2}{3 + 10(0.3)} = \dfrac{0.2}{6}$ m/min $= \dfrac{10}{3}$ cm/min.

20.



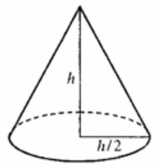
The figure is drawn without the top 3 feet.

$V = \frac{1}{2}(b + 12)h(20) = 10(b + 12)h$ and, from similar triangles, $x = h$ and

$\dfrac{y}{h} = \dfrac{16}{6} = \dfrac{8}{3}$, so $b = y + 12 + x = h + 12 + \dfrac{8h}{3} = 12 + \dfrac{11h}{3}$.

Thus, $V = 10\left(24 + \dfrac{11h}{3}\right)h = 240h + \dfrac{110h^2}{3}$ and so $0.8 = \dfrac{dV}{dt} = \left(240 + \dfrac{220}{3}h\right)\dfrac{dh}{dt}$. When $h = 5$,

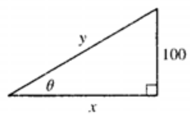
$\dfrac{dh}{dt} = \dfrac{0.8}{240 + 5(220/3)} = \dfrac{3}{2275} \approx 0.00132$ ft/min.

21. 

We are given that $\dfrac{dV}{dt} = 30\ \text{ft}^3/\text{min}$. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\dfrac{h}{2}\right)^2 h = \dfrac{\pi h^3}{12} \Rightarrow$

$30 = \dfrac{dV}{dt} = \dfrac{\pi h^2}{4}\dfrac{dh}{dt} \Rightarrow \dfrac{dh}{dt} = \dfrac{120}{\pi h^2}$. When $h = 10$ ft,

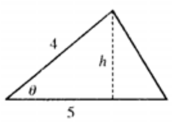
$\dfrac{dh}{dt} = \dfrac{120}{10^2\pi} = \dfrac{6}{5\pi} \approx 0.38$ ft/min.

22. 

We are given $dx/dt = 8$ ft/s. $\cot\theta = \dfrac{x}{100} \Rightarrow x = 100\cot\theta \Rightarrow$

$\dfrac{dx}{dt} = -100\csc^2\theta\dfrac{d\theta}{dt} \Rightarrow \dfrac{d\theta}{dt} = -\dfrac{\sin^2\theta}{100}\cdot 8$. When $y = 200$,

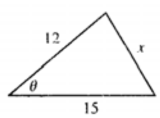
$\sin\theta = \dfrac{100}{200} = \dfrac{1}{2} \Rightarrow \dfrac{d\theta}{dt} = -\dfrac{(1/2)^2}{100}\cdot 8 = -\dfrac{1}{50}$ rad/s. The angle is decreasing at a rate of $\frac{1}{50}$ rad/s.

23. 

$A = \frac{1}{2}bh$, but $b = 5$ m and $h = 4\sin\theta$ so $A = 10\sin\theta$. We are given

$\dfrac{d\theta}{dt} = 0.06$ rad/s. $\dfrac{dA}{dt} = 10\cos\theta\dfrac{d\theta}{dt} = 0.6\cos\theta$. When $\theta = \frac{\pi}{3}$,

$\dfrac{dA}{dt} = 0.6\left(\cos\frac{\pi}{3}\right) = (0.6)\left(\frac{1}{2}\right) = 0.3\ \text{m}^2/\text{s}$.

24. 

We are given $d\theta/dt = 2^\circ/\text{min} = \frac{\pi}{90}$ rad/min. By the Law of Cosines,

$x^2 = 12^2 + 15^2 - 2(12)(15)\cos\theta = 369 - 360\cos\theta \Rightarrow$

$2x\dfrac{dx}{dt} = 360\sin\theta\dfrac{d\theta}{dt}$. When $\theta = 60^\circ$, $x = \sqrt{369 - 360\cos 60^\circ} = \sqrt{189}$, so

$\dfrac{dx}{dt} = \dfrac{360\sin 60^\circ}{2\left(3\sqrt{21}\right)}\dfrac{\pi}{90} = \dfrac{\pi\sqrt{3}}{3\sqrt{21}} = \dfrac{\sqrt{7}\pi}{21} \approx 0.396$ m/min.

25. Differentiating both sides of $PV = C$ with respect to t and using the Product Rule gives us $P\dfrac{dV}{dt} + V\dfrac{dP}{dt} = 0$ $\Rightarrow \dfrac{dV}{dt} = -\dfrac{V}{P}\dfrac{dP}{dt}$. When $V = 600$, $P = 150$ and $\dfrac{dP}{dt} = 20$, we have $\dfrac{dV}{dt} = -\dfrac{600}{150}(20) = -80$, so the volume is decreasing at a rate of 80 cm^3/min.

26. $PV^{1.4} = C \Rightarrow V^{1.4}\dfrac{dP}{dt} + 1.4PV^{0.4}\dfrac{dV}{dt} = 0 \Rightarrow \dfrac{dV}{dt} = -\dfrac{V}{1.4P}\dfrac{dP}{dt}$. When $V = 400$, $P = 80$ and $\dfrac{dP}{dt} = -10$, we have $\dfrac{dV}{dt} = -\dfrac{400}{1.4(80)}(-10) = \dfrac{250}{7}$, so the volume is increasing at a rate of $\frac{250}{7} \approx 36$ cm^3/min.

27. With $R_1 = 80$ and $R_2 = 100$, $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} = \dfrac{1}{80} + \dfrac{1}{100} = \dfrac{180}{8000} = \dfrac{9}{400}$, so $R = \dfrac{400}{9}$. Differentiating $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$ with respect to t, we have $-\dfrac{1}{R^2}\dfrac{dR}{dt} = -\dfrac{1}{R_1^2}\dfrac{dR_1}{dt} - \dfrac{1}{R_2^2}\dfrac{dR_2}{dt} \Rightarrow$

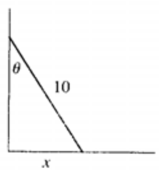
$\dfrac{dR}{dt} = R^2\left(\dfrac{1}{R_1^2}\dfrac{dR_1}{dt} + \dfrac{1}{R_2^2}\dfrac{dR_2}{dt}\right)$. When $R_1 = 80$ and $R_2 = 100$,

$\dfrac{dR}{dt} = \dfrac{400^2}{9^2}\left[\dfrac{1}{80^2}(0.3) + \dfrac{1}{100^2}(0.2)\right] = \dfrac{107}{810} \approx 0.132\ \Omega/\text{s}$.

28. We want to find $\frac{dB}{dt}$ when $L = 18$.

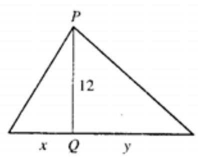
$$\frac{dB}{dt} = \frac{dB}{dW}\frac{dW}{dL}\frac{dL}{dt} = \left(0.007 \cdot \tfrac{2}{3}W^{-1/3}\right)\left(0.12 \cdot 2.53 \cdot L^{1.53}\right)\left(\frac{20-15}{10{,}000{,}000}\right)$$
$$= \left[0.007 \cdot \tfrac{2}{3}\left(0.12 \cdot 18^{2.53}\right)^{-1/3}\right]\left(0.12 \cdot 2.53 \cdot 18^{1.53}\right)\left(\frac{5}{10^7}\right) \approx 1.045 \times 10^{-8} \text{ g/yr}$$

29.



We are given that $\frac{dx}{dt} = 2$ ft/s. $x = 10\sin\theta \Rightarrow \frac{dx}{dt} = 10\cos\theta\frac{d\theta}{dt}$. When $\theta = \frac{\pi}{4}$, $\frac{d\theta}{dt} = \frac{2}{10\left(1/\sqrt{2}\right)} = \frac{\sqrt{2}}{5}$ rad/s.

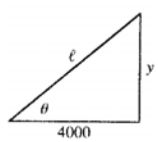
30.



Using Q for the origin, we are given $\frac{dx}{dt} = -2$ ft/s and need to find $\frac{dy}{dt}$ when $x = -5$. Using the Pythagorean Theorem twice, we have $\sqrt{x^2 + 12^2} + \sqrt{y^2 + 12^2} = 39$, the total length of the rope. Differentiating with respect to t, we get $\frac{x}{\sqrt{x^2+12^2}}\frac{dx}{dt} + \frac{y}{\sqrt{y^2+12^2}}\frac{dy}{dt} = 0$, so

$\frac{dy}{dt} = -\frac{x\sqrt{y^2+12^2}}{y\sqrt{x^2+12^2}}\frac{dx}{dt}$. Now when $x = -5$, $39 = \sqrt{(-5)^2 + 12^2} + \sqrt{y^2+12^2} = 13 + \sqrt{y^2+12^2} \Leftrightarrow$ $\sqrt{y^2+12^2} = 26$, and $y = \sqrt{26^2 - 12^2} = \sqrt{532}$. So when $x = -5$, $\frac{dy}{dt} = -\frac{(-5)(26)}{\sqrt{532}\,(13)}(-2) \approx -0.87$ ft/s. So cart B is moving towards Q at about 0.87 ft/s.

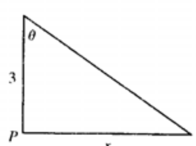
31. (a)



By the Pythagorean Theorem, $4000^2 + y^2 = \ell^2$. Differentiating with respect to t, we obtain $2y\frac{dy}{dt} = 2\ell\frac{d\ell}{dt}$. We know that $\frac{dy}{dt} = 600$, so when $y = 3000$ and $\ell = 5000$, $\frac{d\ell}{dt} = \frac{y\,(dy/dt)}{\ell} = \frac{3000\,(600)}{5000} = \frac{1800}{5} = 360$ ft/s.

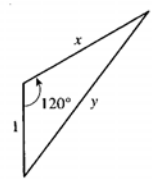
(b) Here $\tan\theta = y/4000$, so $\sec^2\theta\frac{d\theta}{dt} = \frac{1}{4000}\frac{dy}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{\cos^2\theta}{4000}\frac{dy}{dt}$. When $y = 3000$, $\frac{dy}{dt} = 600$, $\ell = 5000$ and $\cos\theta = \frac{4000}{\ell} = \frac{4000}{5000} = \frac{4}{5}$, so $\frac{d\theta}{dt} = \frac{(4/5)^2}{4000}(600) = 0.096$ rad/s.

32.



We are given that $\frac{d\theta}{dt} = 4\,(2\pi) = 8\pi$ rad/min. $x = 3\tan\theta \Rightarrow$ $\frac{dx}{dt} = 3\sec^2\theta\frac{d\theta}{dt}$. When $x = 1$, $\tan\theta = \frac{1}{3}$, so $\sec^2\theta = 1 + \left(\frac{1}{3}\right)^2 = \frac{10}{9}$ and $\frac{dx}{dt} = 3\left(\frac{10}{9}\right)(8\pi) = \frac{80\pi}{3} \approx 83.8$ km/min.

33.



We are given that $\frac{dx}{dt} = 300$ km/h. By the Law of Cosines,

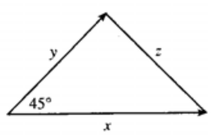
$y^2 = x^2 + 1 - 2(1)x\cos 120° = x^2 + 1 - 2x\left(-\frac{1}{2}\right) = x^2 + x + 1$, so

$2y\frac{dy}{dt} = 2x\frac{dx}{dt} + \frac{dx}{dt} \quad \Rightarrow \quad \frac{dy}{dt} = \frac{2x+1}{2y}\frac{dx}{dt}$. After 1 minute, $x = \frac{300}{60} = 5$

$\Rightarrow \; y = \sqrt{5^2 + 5 + 1} = \sqrt{31} \quad \Rightarrow$

$\frac{dy}{dt} = \frac{2(5)+1}{2\sqrt{31}}(300) = \frac{1650}{\sqrt{31}} \approx 296$ km/h.

34.



We are given that $\frac{dx}{dt} = 3$ mi/h and $\frac{dy}{dt} = 2$ mi/h. By the Law of Cosines,

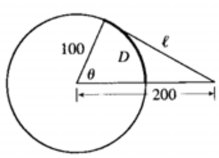
$z^2 = x^2 + y^2 - 2xy\cos 45° = x^2 + y^2 - \sqrt{2}xy \quad \Rightarrow$

$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} - \sqrt{2}x\frac{dy}{dt} - \sqrt{2}y\frac{dx}{dt}$. After 15 minutes, we have

$x = \frac{3}{4}$ and $y = \frac{1}{2} \quad \Rightarrow \quad z = \frac{\sqrt{13 - 6\sqrt{2}}}{4}$ and

$$\frac{dz}{dt} = \frac{2}{\sqrt{13 - 6\sqrt{2}}}\left[2\left(\tfrac{3}{4}\right)3 + 2\left(\tfrac{1}{2}\right)2 - \sqrt{2}\left(\tfrac{3}{4}\right)2 - \sqrt{2}\left(\tfrac{1}{2}\right)3\right] = \sqrt{13 - 6\sqrt{2}} \approx 2.125 \text{ mi/h}.$$

35.



Let the distance between the runner and the friend be ℓ. Then by the Law of Cosines,

$\ell^2 = 200^2 + 100^2 - 2\cdot 200\cdot 100\cdot\cos\theta = 50{,}000 - 40{,}000\cos\theta$ (★).

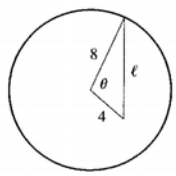
Differentiating implicitly with respect to t, we obtain

$2\ell\frac{d\ell}{dt} = -40{,}000(-\sin\theta)\frac{d\theta}{dt}$. Now if D is the distance run when the angle is θ radians, then by the formula for the length of an arc on a circle, $s = r\theta$, we have $D = 100\theta$, so $\theta = \frac{1}{100}D \quad \Rightarrow \quad \frac{d\theta}{dt} = \frac{1}{100}\frac{dD}{dt} = \frac{7}{100}$. To substitute into the expression for $\frac{d\ell}{dt}$, we must know $\sin\theta$ at the time when $\ell = 200$, which we find from (★): $200^2 = 50{,}000 - 40{,}000\cos\theta \quad \Leftrightarrow \quad \cos\theta = \frac{1}{4} \quad \Rightarrow$

$\sin\theta = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$. Substituting, we get $2(200)\frac{d\ell}{dt} = 40{,}000\frac{\sqrt{15}}{4}\left(\frac{7}{100}\right) \quad \Rightarrow$

$d\ell/dt = \frac{7\sqrt{15}}{4} \approx 6.78$ m/s. Whether the distance between them is increasing or decreasing depends on the direction in which the runner is running.

36.



The hour hand of a clock goes around once every 12 hours or, in radians per hour, $\frac{2\pi}{12} = \frac{\pi}{6}$ rad/h. The minute hand goes around once an hour, or at the rate of 2π rad/h. So the angle θ between them (measuring clockwise from the minute hand to the hour hand) is changing at the rate of $d\theta/dt = \frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$ rad/h. Now, to relate θ to ℓ, we use the Law of Cosines: $\ell^2 = 4^2 + 8^2 - 2 \cdot 4 \cdot 8 \cdot \cos\theta = 80 - 64\cos\theta$ (★).

Differentiating implicitly with respect to t, we get $2\ell\dfrac{d\ell}{dt} = -64\,(-\sin\theta)\,\dfrac{d\theta}{dt}$. At 1:00, the angle between the two hands is one-twelfth of the circle, that is, $\frac{2\pi}{12} = \frac{\pi}{6}$ radians. We use (★) to find ℓ at 1:00:

$\ell = \sqrt{80 - 64\cos\frac{\pi}{6}} = \sqrt{80 - 32\sqrt{3}}$. Substituting, we get $2\ell\dfrac{d\ell}{dt} = 64\sin\frac{\pi}{6}\left(-\frac{11\pi}{6}\right) \Rightarrow$

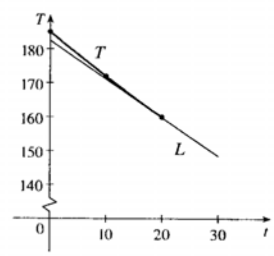
$\dfrac{d\ell}{dt} \approx \dfrac{64\left(\frac{1}{2}\right)\left(-\frac{11\pi}{6}\right)}{2\sqrt{80 - 32\sqrt{3}}} = -\dfrac{88\pi}{3\sqrt{80 - 32\sqrt{3}}} \approx -18.6$. So at 1:00, the distance between the tips of the hands is decreasing at a rate of 18.6 mm/h ≈ 0.005 mm/s.

3.10 Linear Approximations and Differentials

1. As in Example 1, $T(0) = 185$, $T(10) = 172$, $T(20) = 160$, and

$T'(20) \approx \dfrac{T(10) - T(20)}{10 - 20} = \dfrac{172 - 160}{-10} = -1.2°$ F/min.

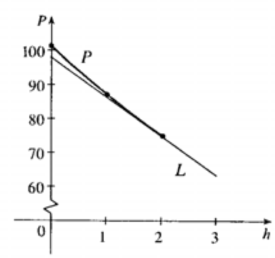
$T(30) \approx T(20) + T'(20)(30 - 20) \approx 160 - 1.2(10) = 148°$ F. We would expect the temperature of the turkey to get closer to 75° F as time increases. Since the temperature decreased 13° F in the first 10 minutes and 12° F in the second 10 minutes, we can assume that the slopes of the tangent line are increasing through negative values: $-1.3, -1.2, \ldots$. Hence, the tangent lines are under the curve and 148° F is an underestimate. From the figure, we estimate the slope of the tangent line at $t = 20$ to be $\frac{184 - 147}{0 - 30} = -\frac{37}{30}$. Then the linear approximation becomes $T(30) \approx T(20) + T'(20) \cdot 10 \approx 160 - \frac{37}{30}(10) = 147\frac{2}{3} \approx 147.7$.



2. $P'(2) \approx \dfrac{P(1) - P(2)}{1 - 2} = \dfrac{87.1 - 74.9}{-1} = -12.2$ kilopascals/km.

$P(3) \approx P(2) + P'(2)(3 - 2) \approx 74.9 - 12.2(1) = 62.7$ kPa. From the figure, we estimate the slope of the tangent line at $h = 2$ to be $\frac{98 - 63}{0 - 3} = -\frac{35}{3}$. Then the linear approximation becomes

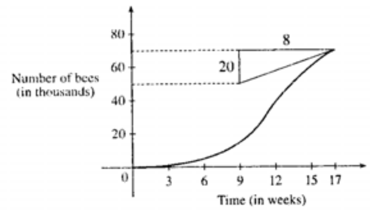
$P(3) \approx P(2) + P'(2) \cdot 1 \approx 74.9 - \frac{35}{3} \approx 63.23$ kPa.



3. If $C(t)$ represents the cash per capita in circulation in year t, then we estimate $C(2000)$ using a linear approximation based on the tangent line to the graph of C at $(1990, C(1990)) = (1990, 1063)$. $C'(1990) \approx \dfrac{C(1980) - C(1990)}{1980 - 1990} = \dfrac{571 - 1063}{-10} = 49.2\ \dfrac{\text{dollars}}{\text{year}}$.

$C(2000) \approx C(1990) + C'(1990)(2000 - 1990) \approx 1063 + 49.2(10) = 1555$. So our estimate of cash per capita in circulation in the year 2000 is \$1555. For the given data, C' is increasing, so the graph of C is concave upward and the tangent line approximations are below the curve, indicating that our prediction is an underestimate.

4. (a)



From the figure,

$P'(17) \approx \frac{20}{8} = 2.5$ thousand bees/week.

$P(18) \approx P(17) + P'(17)(18 - 17)$

$\approx 70 + 2.5(1) = 72.5$ or 72,500 bees.

$P(20) \approx P(17) + P'(17)(20 - 17)$

$\approx 70 + 2.5(3) = 77.5$ or 77,500 bees.

(b) Since the tangent line at $t = 17$ is above the graph, our predictions are overestimates.

(c) $P(18)$ is more accurate than $P(20)$ since it is closer to the given data.

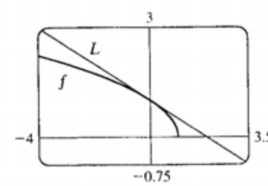
5. $f(x) = x^3 \Rightarrow f'(x) = 3x^2$ so $f(1) = 1$ and $f'(1) = 3$.
Thus, $L(x) = f(1) + f'(1)(x - 1) = 1 + 3(x - 1) = 3x - 2$.

6. $f(x) = 1/\sqrt{2 + x} = (2 + x)^{-1/2} \Rightarrow f'(x) = -\frac{1}{2}(2 + x)^{-3/2}$ so $f(0) = \frac{1}{\sqrt{2}}$ and $f'(0) = -1 \Big/ \left(4\sqrt{2}\right)$. So $L(x) = f(0) + f'(0)(x - 0) = \frac{1}{\sqrt{2}} - \frac{1}{4\sqrt{2}}(x - 0) = \frac{1}{\sqrt{2}}\left(1 - \frac{1}{4}x\right)$.

7. $f(x) = 1/x \Rightarrow f'(x) = -1/x^2$. So $f(4) = \frac{1}{4}$ and $f'(4) = -\frac{1}{16}$.
So $L(x) = f(4) + f'(4)(x - 4) = \frac{1}{4} + \left(-\frac{1}{16}\right)(x - 4) = \frac{1}{2} - \frac{1}{16}x$.

8. $f(x) = \sqrt[3]{x} = x^{1/3} \Rightarrow f'(x) = \frac{1}{3}x^{-2/3}$ so $f(-8) = -2$ and $f'(-8) = \frac{1}{12}$.
Thus, $L(x) = f(-8) + f'(-8)(x + 8) = -2 + \frac{1}{12}(x + 8) = \frac{1}{12}x - \frac{4}{3}$.

9. $f(x) = \sqrt{1 - x} \Rightarrow f'(x) = \dfrac{-1}{2\sqrt{1 - x}}$ so $f(0) = 1$ and $f'(0) = -\frac{1}{2}$.

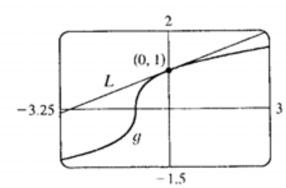


Therefore,

$$\sqrt{1 - x} = f(x) \approx f(0) + f'(0)(x - 0)$$
$$= 1 + \left(-\tfrac{1}{2}\right)(x - 0) = 1 - \tfrac{1}{2}x$$

So $\sqrt{0.9} = \sqrt{1 - 0.1} \approx 1 - \frac{1}{2}(0.1) = 0.95$ and
$\sqrt{0.99} = \sqrt{1 - 0.01} \approx 1 - \frac{1}{2}(0.01) = 0.995$.

10. $g(x) = \sqrt[3]{1 + x} = (1 + x)^{1/3} \Rightarrow g'(x) = \frac{1}{3}(1 + x)^{-2/3}$ so $g(0) = 1$ and $g'(0) = \frac{1}{3}$.

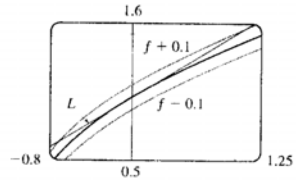


Therefore, $\sqrt[3]{1 + x} = g(x) \approx g(0) + g'(0)(x - 0) = 1 + \frac{1}{3}x$.
So $\sqrt[3]{0.95} = \sqrt[3]{1 + (-0.05)} \approx 1 + \frac{1}{3}(-0.05) = 0.98\overline{3}$, and
$\sqrt[3]{1.1} = \sqrt[3]{1 + 0.1} \approx 1 + \frac{1}{3}(0.1) = 1.0\overline{3}$.

11. $f(x) = \sqrt{1+x} \quad \Rightarrow \quad f'(x) = \dfrac{1}{2\sqrt{1+x}}$ so $f(0) = 1$ and $f'(0) = \frac{1}{2}$.

Thus, $f(x) \approx f(0) + f'(0)(x-0) = 1 + \frac{1}{2}(x-0) = 1 + \frac{1}{2}x$.

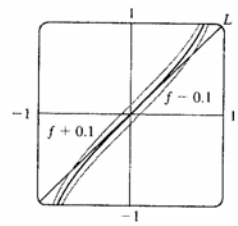
We need $\sqrt{1+x} - 0.1 < 1 + \frac{1}{2}x < \sqrt{1+x} + 0.1$. By zooming in or using an intersect feature, we see that this is true when $-0.69 < x < 1.09$.



12. $f(x) = \tan x \quad \Rightarrow \quad f'(x) = \sec^2 x$ so $f(0) = 0$ and $f'(0) = 1$.

Thus, $f(x) \approx f(0) + f'(0)(x-0) = 0 + 1(x-0) = x$.

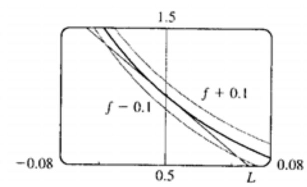
We need $\tan x - 0.1 < x < \tan x + 0.1$, which is true when $-0.63 < x < 0.63$.



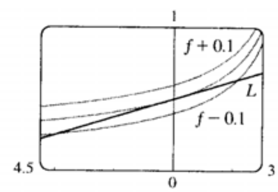
13. $f(x) = \dfrac{1}{(1+2x)^4} \quad \Rightarrow \quad f'(x) = \dfrac{-8}{(1+2x)^5}$ so $f(0) = 1$ and $f'(0) = -8$.

Thus, $f(x) \approx f(0) + f'(0)(x-0) = 1 + (-8)(x-0) = 1 - 8x$.

We need $1/(1+2x)^4 - 0.1 < 1 - 8x < 1/(1+2x)^4 + 0.1$, which is true when $-0.045 < x < 0.055$.



14. $f(x) = \dfrac{1}{\sqrt{4-x}} \quad \Rightarrow \quad f'(x) = \dfrac{1}{2(4-x)^{3/2}}$ so $f(0) = \frac{1}{2}$ and $f'(0) = \frac{1}{16}$. So $f(x) \approx \frac{1}{2} + \frac{1}{16}(x-0) = \frac{1}{2} + \frac{1}{16}x$. We need $\dfrac{1}{\sqrt{4-x}} - 0.1 < \frac{1}{2} + \frac{1}{16}x < \dfrac{1}{\sqrt{4-x}} + 0.1$, which is true when $-3.91 < x < 2.14$.



15. If $y = f(x)$, then the differential dy is equal to $f'(x)\,dx$. $y = x^4 + 5x \quad \Rightarrow \quad dy = \left(4x^3 + 5\right)dx$.

16. $y = \cos \pi x \quad \Rightarrow \quad dy = -\sin \pi x \cdot \pi\, dx = -\pi \sin \pi x\, dx$

17. $y = x^2 \tan x \quad \Rightarrow \quad dy = \left(x^2 \sec^2 x + \tan x \cdot 2x\right)dx = \left(x^2 \sec^2 x + 2x \tan x\right)dx$

18. $y = \sqrt{1+t^2} \quad \Rightarrow \quad dy = \frac{1}{2}\left(1+t^2\right)^{-1/2}(2t)\,dt = \dfrac{t}{\sqrt{1+t^2}}\,dt$

19. $y = \dfrac{u+1}{u-1} \quad \Rightarrow \quad dy = \dfrac{(u-1)(1) - (u+1)(1)}{(u-1)^2}\,du = \dfrac{-2}{(u-1)^2}\,du$

20. $y = (1+2r)^{-4} \quad \Rightarrow \quad dy = -4(1+2r)^{-5} \cdot 2\,dr = -8(1+2r)^{-5}\,dr$

21. (a) $y = x^2 + 2x \quad \Rightarrow \quad dy = (2x+2)\,dx$

(b) When $x = 3$ and $dx = \frac{1}{2}$, $dy = [2(3) + 2]\left(\frac{1}{2}\right) = 4$.

22. (a) $y = x^3 - 6x^2 + 5x - 7 \quad \Rightarrow \quad dy = \left(3x^2 - 12x + 5\right)dx$

(b) When $x = -2$ and $dx = 0.1$, $dy = (12 + 24 + 5)(0.1) = 4.1$.

23. (a) $y = (x^2+5)^3 \quad \Rightarrow \quad dy = 3(x^2+5)^2\, 2x\, dx = 6x(x^2+5)^2\, dx$

(b) When $x = 1$ and $dx = 0.05$, $dy = 6(1)(1^2+5)^2(0.05) = 10.8$.

24. (a) $y = \sqrt{1-x} \quad \Rightarrow \quad dy = \frac{1}{2}(1-x)^{-1/2}(-1)\,dx = -\dfrac{1}{2\sqrt{1-x}}dx$

(b) When $x = 0$ and $dx = 0.02$, $dy = -\frac{1}{2}(0.02) = -0.01$.

25. (a) $y = \cos x \quad \Rightarrow \quad dy = -\sin x\, dx$

(b) When $x = \frac{\pi}{6}$ and $dx = 0.05$, $dy = -\frac{1}{2}(0.05) = -0.025$.

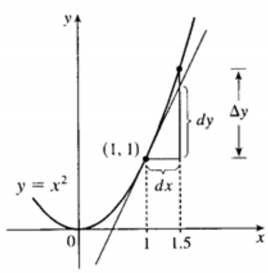
26. (a) $y = \sin x \quad \Rightarrow \quad dy = \cos x\, dx$

(b) When $x = \frac{\pi}{6}$ and $dx = -0.1$, $dy = \frac{\sqrt{3}}{2}(-0.1) = -\frac{\sqrt{3}}{20}$.

27. $y = x^2$, $x = 1$, $\Delta x = 0.5 \quad \Rightarrow$

$\Delta y = (1.5)^2 - 1^2 = 1.25$.

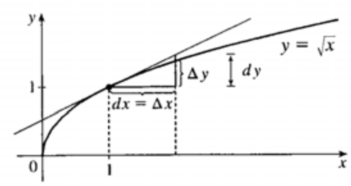
$dy = 2x\, dx = 2(1)(0.5) = 1$



28. $y = \sqrt{x}$, $x = 1$, $\Delta x = 1 \quad \Rightarrow$

$\Delta y = \sqrt{2} - \sqrt{1} = \sqrt{2} - 1 \approx 0.414$

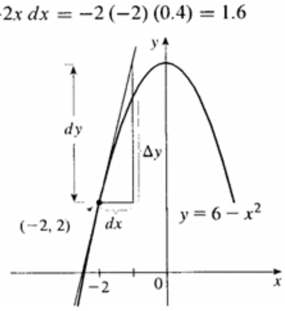
$dy = \dfrac{1}{2\sqrt{x}}dx = \frac{1}{2}(1) = 0.5$



29. $y = 6 - x^2$, $x = -2$, $\Delta x = 0.4 \quad \Rightarrow$

$\Delta y = \left(6 - (-1.6)^2\right) - \left(6 - (-2)^2\right) = 1.44$

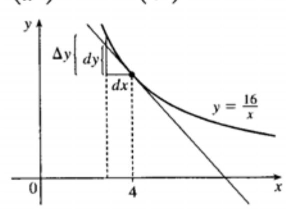
$dy = -2x\, dx = -2(-2)(0.4) = 1.6$



30. $y = \dfrac{16}{x}$, $x = 4$, $\Delta x = -1 \quad \Rightarrow$

$\Delta y = \frac{16}{3} - \frac{16}{4} = \frac{4}{3}$.

$dy = -\left(\dfrac{16}{x^2}\right)dx = -\left(\dfrac{16}{4^2}\right)(-1) = 1$



31. $y = f(x) = \sqrt{x} \quad \Rightarrow \quad dy = \dfrac{1}{2\sqrt{x}}\,dx$. When $x = 36$ and $dx = 0.1$, $dy = \frac{1}{2\sqrt{36}}(0.1) = \frac{1}{120}$, so

$\sqrt{36.1} = f(36.1) \approx f(36) + dy = \sqrt{36} + \frac{1}{120} \approx 6.0083$.

32. $y = f(x) = \sqrt[3]{x} + \sqrt[4]{x} \Rightarrow dy = \left(\frac{1}{3}x^{-2/3} + \frac{1}{4}x^{-3/4}\right)dx$. If $x = 1$ and $dx = 0.02$, then $dy = \left(\frac{1}{3} + \frac{1}{4}\right)(0.02) = \frac{7}{12}(0.02)$. Thus, $\sqrt[3]{1.02} + \sqrt[4]{1.02} = f(1.02) \approx f(1) + dy = 2 + \frac{7}{12}(0.02) \approx 2.0117$.

33. $y = f(x) = 1/x \Rightarrow dy = (-1/x^2)\,dx$. When $x = 10$ and $dx = 0.1$, $dy = \left(-\frac{1}{100}\right)(0.1) = -0.001$, so $\frac{1}{10.1} = f(10.1) \approx f(10) + dy = 0.1 - 0.001 = 0.099$.

34. $y = f(x) = x^6 \Rightarrow dy = 6x^5\,dx$. When $x = 2$ and $dx = -0.03$, $dy = 6(2)^5(-0.03) = -5.76$, so $(1.97)^6 = f(1.97) \approx f(2) + dy = 64 - 5.76 = 58.24$.

35. $y = f(x) = \sin x \Rightarrow dy = \cos x\,dx$. When $x = \frac{\pi}{3}$ and $dx = -\frac{\pi}{180}$, $dy = \cos\frac{\pi}{3}\left(-\frac{\pi}{180}\right) = -\frac{\pi}{360}$, so $\sin 59^\circ = f\left(\frac{59}{180}\pi\right) \approx f\left(\frac{\pi}{3}\right) + dy = \frac{\sqrt{3}}{2} - \frac{\pi}{360} \approx 0.857$.

36. $y = f(x) = \cos x \Rightarrow dy = -\sin x\,dx$. When $x = \frac{\pi}{6}$ and $dx = \frac{1.5}{180}\pi$, $dy = -\sin\frac{\pi}{6}\left(\frac{1.5}{180}\pi\right) = -\frac{1}{2}\left(\frac{\pi}{120}\right) = -\frac{\pi}{240}$, so $\cos 31.5^\circ = f\left(\frac{31.5}{180}\pi\right) \approx f\left(\frac{\pi}{6}\right) + dy = \frac{\sqrt{3}}{2} - \frac{\pi}{240} \approx 0.853$.

37. We can see from a graph of $y = \sec x$ that the tangent line approximation at $(0, 1)$ is the horizontal line $y = 1$. Since 0.08 is close to 0, approximating $\sec 0.08$ with 1 is reasonable.

38. If $y = x^6$, $y' = 6x^5$ and the tangent line approximation at $(1, 1)$ has slope 6. If the change in x is 0.01, the change in y on the tangent line is 0.06, and approximating $(1.01)^6$ with 1.06 is reasonable.

39. (a) If x is the edge length, then $V = x^3 \Rightarrow dV = 3x^2\,dx$. When $x = 30$ and $dx = 0.1$, $dV = 3(30)^2(0.1) = 270$, so the maximum error is about 270 cm^3.

(b) $S = 6x^2 \Rightarrow dS = 12x\,dx$. When $x = 30$ and $dx = 0.1$, $dS = 12(30)(0.1) = 36$, so the maximum error is about 36 cm^2.

40. (a) $A = \pi r^2 \Rightarrow dA = 2\pi r\,dr$. When $r = 24$ and $dr = 0.2$, $dA = 2\pi(24)(0.2) = 9.6\pi$, so the maximum error is about $9.6\pi \approx 30\text{ cm}^2$.

(b) Relative error $= \dfrac{\Delta A}{A} \approx \dfrac{dA}{A} = \dfrac{9.6\pi}{\pi(24)^2} = \dfrac{1}{60} \approx 0.0167$

41. (a) For a sphere of radius r, the circumference is $C = 2\pi r$ and the surface area is $S = 4\pi r^2$, so $r = C/(2\pi) \Rightarrow S = 4\pi(C/2\pi)^2 = C^2/\pi \Rightarrow dS = (2/\pi)\,C\,dC$. When $C = 84$ and $dC = 0.5$, $dS = \frac{2}{\pi}(84)(0.5) = \frac{84}{\pi}$, so the maximum error is about $\frac{84}{\pi} \approx 27\text{ cm}^2$. Relative error $\approx \dfrac{dS}{S} = \dfrac{84/\pi}{84^2/\pi} = \dfrac{1}{84} \approx 0.012$

(b) $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\dfrac{C}{2\pi}\right)^3 = \dfrac{C^3}{6\pi^2} \Rightarrow dV = \dfrac{1}{2\pi^2}C^2\,dC$. When $C = 84$ and $dC = 0.5$, $dV = \dfrac{1}{2\pi^2}(84)^2(0.5) = \dfrac{1764}{\pi^2}$, so the maximum error is about $\dfrac{1764}{\pi^2} \approx 179\text{ cm}^3$. The relative error is approximately $\dfrac{dV}{V} = \dfrac{1764/\pi^2}{(84)^3/(6\pi^2)} = \dfrac{1}{56} \approx 0.018$.

42. $V = \frac{2}{3}\pi r^3 \Rightarrow dV = 2\pi r^2\,dr$. When $r = 25$ and $dr = 0.0005$, $dV = 2\pi(25)^2(0.0005) = \frac{5\pi}{8}$, so the amount of paint is about $\frac{5\pi}{8} \approx 2\text{ m}^3$.

43. (a) $V = \pi r^2 h \Rightarrow \Delta V \approx dV = 2\pi rh\,dr = 2\pi rh\Delta r$

(b) $\Delta V = \pi(r + \Delta r)^2 h - \pi r^2 h$, so the error is $\Delta V - dv = \pi(r + \Delta r)^2 h - \pi r^2 h - 2\pi rh\Delta r = \pi(\Delta r)^2 h$

44. $F = kR^4 \;\Rightarrow\; dF = 4kR^3\,dR$. $\dfrac{dF}{F} = \dfrac{4kR^3\,dR}{kR^4} = 4\left(\dfrac{dR}{R}\right) \;\Rightarrow$ the relative change in F is about 4 times the relative change in R. So a 5% increase in the radius corresponds to a 20% increase in blood flow.

45. (a) $dc = \dfrac{dc}{dx}\,dx = 0\,dx = 0$

(b) $d\,(cu) = \dfrac{d}{dx}\,(cu)\,dx = c\dfrac{du}{dx}\,dx = c\,du$

(c) $d\,(u+v) = \dfrac{d}{dx}\,(u+v)\,dx = \left(\dfrac{du}{dx} + \dfrac{dv}{dx}\right)dx = \dfrac{du}{dx}\,dx + \dfrac{dv}{dx}\,dx = du + dv$

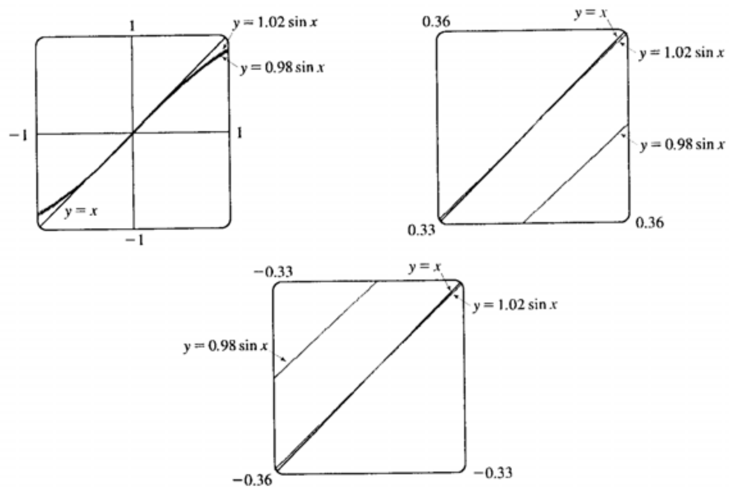
(d) $d\,(uv) = \dfrac{d}{dx}\,(uv)\,dx = \left(u\dfrac{dv}{dx} + v\dfrac{du}{dx}\right)dx = u\dfrac{dv}{dx}\,dx + v\dfrac{du}{dx}\,dx = u\,dv + v\,du$

(e) $d\left(\dfrac{u}{v}\right) = \dfrac{d}{dx}\left(\dfrac{u}{v}\right)dx = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}\,dx = \dfrac{v\dfrac{du}{dx}\,dx - u\dfrac{dv}{dx}\,dx}{v^2} = \dfrac{v\,du - u\,dv}{v^2}$

(f) $d\,(x^n) = \dfrac{d}{dx}\,(x^n)\,dx = nx^{n-1}\,dx$

46. (a) $f(x) = \sin x \;\Rightarrow\; f'(x) = \cos x$ so $f(0) = 0$ and $f'(0) = 1$.
Thus, $f(x) \approx f(0) + f'(0)(x-0) = 0 + 1\,(x-0) = x$.

(b)



We want to know the values of x for which $y = x$ approximates $y = \sin x$ with less than a 2% difference; that is, the values of x for which

$$\left|\frac{x - \sin x}{\sin x}\right| < 0.02 \quad\Leftrightarrow\quad \begin{cases} 0.98\sin x < x < 1.02\sin x & \text{where } \sin x > 0 \\ 1.02\sin x < x < 0.98\sin x & \text{where } \sin x < 0 \end{cases}$$

In the first figure, we see that the graphs are very close to each other near $x = 0$. Changing the viewing rectangle and using an intersect feature (see the second figure) we find that $y = x$ intersects $y = 1.02\sin x$ at $x \approx 0.344$. By symmetry, they also intersect at $x \approx -0.344$ (see the third figure.). Converting 0.344 radians to degrees, we get $0.344\left(\frac{180^\circ}{\pi}\right) \approx 19.7^\circ \approx 20^\circ$, which verifies the statement.

47. (a) The graph shows that $f'(1) = 2$, so $L(x) = f(1) + f'(1)(x-1) = 5 + 2(x-1) = 2x + 3$. $f(0.9) \approx L(0.9) = 4.8$ and $f(1.1) \approx L(1.1) = 5.2$.

(b) From the graph, we see that $f'(x)$ is positive and decreasing. This means that the slopes of the tangent lines are positive, but the tangents are becoming less steep. So the tangent lines lie *above* the curve. Thus, the estimates in part (a) are too large.

48. (a) $g'(x) = \sqrt{x^2+5} \;\Rightarrow\; g'(2) = \sqrt{9} = 3$. $g(1.95) \approx g(2) + g'(2)(1.95-2) = -4 + 3(-0.05) = -4.15$. $g(2.05) \approx g(2) + g'(2)(2.05-2) = -4 + 3(0.05) = -3.85$.

(b) The formula $g'(x) = \sqrt{x^2+5}$ shows that $g'(x)$ is positive and increasing. This means that the slopes of the tangent lines are positive and the tangents are getting steeper. So the tangent lines lie *below* the graph of g. Hence, the estimates in part (a) are too small.

Laboratory Project □ Taylor Polynomials

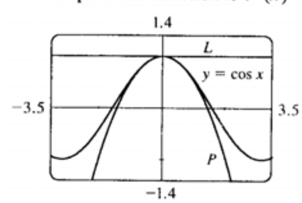
1. We first write the functions described in conditions (i), (ii), and (iii):

$$\begin{aligned} P(x) &= A + Bx + Cx^2 & f(x) &= \cos x \\ P'(x) &= B + 2Cx & f'(x) &= -\sin x \\ P''(x) &= 2C & f''(x) &= -\cos x \end{aligned}$$

So, taking $a = 0$, our three conditions become

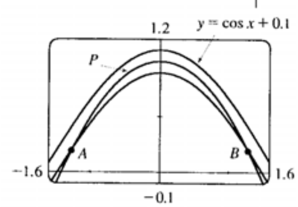
$$\begin{aligned} P(0) = f(0){:}\quad & A = \cos 0 = 1 \\ P'(0) = f'(0){:}\quad & B = -\sin 0 = 0 \\ P''(0) = f''(0){:}\quad & 2C = -\cos 0 = -1 \;\Rightarrow\; C = -\tfrac{1}{2} \end{aligned}$$

The desired quadratic function is $P(x) = 1 - \frac{1}{2}x^2$, so the quadratic approximation is $\cos x \approx 1 - \frac{1}{2}x^2$.



The figure shows a graph of the cosine function together with its linear approximation $L(x) = 1$ and quadratic approximation $P(x) = 1 - \frac{1}{2}x^2$ near 0. You can see that the quadratic approximation is much better than the linear one.

2. Accuracy to within 0.1 means that $\left|\cos x - \left(1 - \frac{1}{2}x^2\right)\right| < 0.1 \;\Leftrightarrow\; \cos x - 0.1 < 1 - \frac{1}{2}x^2 < \cos x + 0.1$.



From the figure we see that this is true between A and B. Zooming in or using an intersect feature, we find that the x-coordinates of B and A are about ± 1.26. Thus, the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ is accurate to within 0.1 when $-1.26 < x < 1.26$.

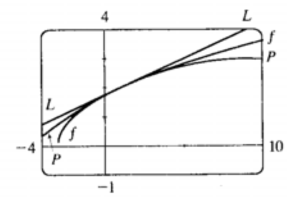
3. If $P(x) = A + B(x-a) + C(x-a)^2$, then $P'(x) = B + 2C(x-a)$ and $P''(x) = 2C$. Applying the conditions (i), (ii), and (iii), we get

$$\begin{aligned} P(a) = f(a)&: \quad A = f(a) \\ P'(a) = f'(a)&: \quad B = f'(a) \\ P''(a) = f''(a)&: \quad 2C = f''(a) \quad \Rightarrow \quad C = \tfrac{1}{2} f''(a) \end{aligned}$$

Thus, P can be written in the form $P(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2}(x-a)^2$.

4. From Example 2 in Section 3.10, we have $f(1) = 2$, $f'(1) = \frac{1}{4}$, and $f'(x) = \frac{1}{2}(x+3)^{-1/2}$. So $f''(x) = -\frac{1}{4}(x+3)^{-3/2} \Rightarrow$ $f''(1) = -\frac{1}{32}$. From Problem 3, the quadratic approximation $P(x)$ is

$$\begin{aligned} \sqrt{x+3} &\approx f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 \\ &= 2 + \tfrac{1}{4}(x-1) - \tfrac{1}{64}(x-1)^2 \end{aligned}$$



The figure shows the function $f(x) = \sqrt{x+3}$ together with its linear approximation $L(x) = \frac{1}{4}x + \frac{7}{4}$ and its quadratic approximation $P(x)$. You can see that $P(x)$ is a better approximation than $L(x)$ and this is borne out by the numerical values in the following chart.

	from $L(x)$	actual value	from $P(x)$
$\sqrt{3.98}$	1.9950	1.99499373...	1.99499375
$\sqrt{4.05}$	2.0125	2.01246118...	2.01246094
$\sqrt{4.2}$	2.0500	2.04939015...	2.04937500

5. $T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n$. If we put $x = a$ in this equation, then all terms after the first are 0 and we get $T_n(a) = c_0$. Now we differentiate $T_n(x)$ and obtain $T_n'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots + nc_n(x-a)^{n-1}$. Substituting $x = a$ gives $T_n'(a) = c_1$. Differentiating again, we have $T_n''(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4\left(x-a^2\right) + \cdots + (n-1)nc_n(x-a)^{n-2}$ and so $T_n''(a) = 2c_2$. Continuing in this manner, we get $T_n'''(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + \cdots + (n-2)(n-1)nc_n(x-a)^{n-3}$ and $T_n'''(a) = 2\cdot 3c_3$. By now we see the pattern. If we continue to differentiate and substitute $x = a$, we obtain $T_n^{(4)}(a) = 2\cdot 3\cdot 4c_4$ and in general, for any integer k between 1 and n,

$$T_n^{(k)}(a) = 2\cdot 3\cdot 4\cdot 5\cdot \cdots \cdot kc_k = k!\,c_k \quad \Rightarrow \quad c_k = \frac{T_n^{(k)}(a)}{k!}$$

Because we want T_n and f to have the same derivatives at a, we require that $c_k = \dfrac{f^{(k)}(a)}{k!}$ for $k = 1, 2, \ldots, n$.

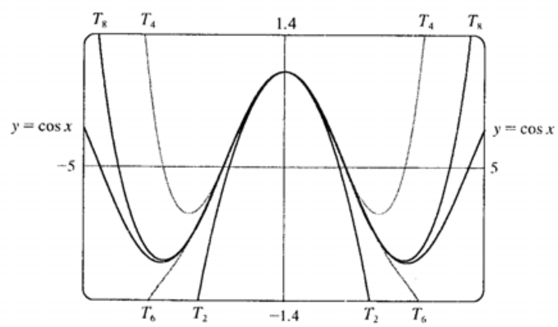
6. $T_n(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \cdots + \dfrac{f^{(n)}(a)}{n!}(x-a)^n$. To compute the coefficients in this equation we need to calculate the derivatives of f at 0:

$$\begin{aligned}
f(x) &= \cos x & f(0) &= \cos 0 = 1 \\
f'(x) &= -\sin x & f'(0) &= -\sin 0 = 0 \\
f''(x) &= -\cos x & f''(0) &= -1 \\
f'''(x) &= \sin x & f'''(0) &= 0 \\
f^{(4)}(x) &= \cos x & f^{(4)}(0) &= 1
\end{aligned}$$

We see that the derivatives repeat in a cycle of length 4, so $f^{(5)}(0) = 0$, $f^{(6)}(0) = -1$, $f^{(7)}(0) = 0$, and $f^{(8)}(0) = 1$. From the original expression for $T_n(x)$, with $n = 8$ and $a = 0$, we have

$$\begin{aligned}
T_8(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots + \frac{f^{(8)}(0)}{8!}(x-0)^8 \\
&= 1 + 0 \cdot x + \frac{-1}{2!}x^2 + 0 \cdot x^3 + \frac{1}{4!}x^4 + 0 \cdot x^5 + \frac{-1}{6!}x^6 + 0 \cdot x^7 + \frac{1}{8!}x^8 \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}
\end{aligned}$$

and the desired approximation is $\cos x \approx 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!}$. The Taylor polynomials T_2, T_4, and T_6 consist of the initial terms of T_8. Therefore, $T_2(x) = 1 - \dfrac{x^2}{2!}$, $T_4(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$, and $T_6(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!}$. We graph T_2, T_4, T_6, T_8, and f:



Notice that $T_2(x)$ is a good approximation to $\cos x$ near 0, $T_4(x)$ is a good approximation on a larger interval, $T_6(x)$ is a better approximation, and $T_8(x)$ is better still. Each successive Taylor polynomial is a good approximation on a larger interval than the previous one.

3 Review

CONCEPT CHECK

1. See Definition 3.1.2 and the subsequent discussions on the interpretations of the derivative as the slope of a tangent and as a rate of change.
2. (a) See Definition 3.2.3.
   (b) See Theorem 3.2.4 and the note following it.
3. (a) The Power Rule: If n is any real number, then $\frac{d}{dx}(x^n) = nx^{n-1}$. The derivative of a variable base raised to a constant power is the power times the base raised to the power minus one.
   (b) The Constant Multiple Rule: If c is a constant and f is a differentiable function, then $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$. The derivative of a constant times a function is the constant times the derivative of the function.
   (c) The Sum Rule: If f and g are both differentiable, then $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$. The derivative of a sum of functions is the sum of the derivatives.
   (d) The Difference Rule: If f and g are both differentiable, then $\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$. The derivative of a difference of functions is the difference of the derivatives.
   (e) The Product Rule: If f and g are both differentiable, then $\frac{d}{dx}[f(x)\,g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$. The derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.
   (f) The Quotient Rule: If f and g are both differentiable, then $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$.
   The derivative of a quotient of functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.
   (g) The Chain Rule: If f and g are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and F' is given by the product $F'(x) = f'(g(x))\,g'(x)$. The derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.
4. (a) $y = x^n \Rightarrow y' = nx^{n-1}$
   (b) $y = \sin x \Rightarrow y' = \cos x$
   (c) $y = \cos x \Rightarrow y' = -\sin x$
   (d) $y = \tan x \Rightarrow y' = \sec^2 x$
   (e) $y = \csc x \Rightarrow y' = -\csc x \cot x$
   (f) $y = \sec x \Rightarrow y' = \sec x \tan x$
   (g) $y = \cot x \Rightarrow y' = -\csc^2 x$
5. Implicit differentiation consists of differentiating both sides of an equation involving x and y with respect to x, and then solving the resulting equation for y'.
6. The second derivative of a function f is the rate of change of the first derivative f'. The third derivative is the derivative (rate of change) of the second derivative. If f is the position function of an object, f' is its velocity function, f'' is its acceleration function, and f''' is its jerk function.
7. (a) The linearization L of f at $x = a$ is $L(x) = f(a) + f'(a)(x-a)$.
   (b) If $y = f(x)$, then the differential dy is given by $dy = f'(x)\,dx$.

TRUE-FALSE QUIZ

1. False. See the warning after Theorem 3.2.4.

2. True. This is the Sum Rule.

3. False. See the warning before the Product Rule.

4. True. This is the Chain Rule.

5. True by the Chain Rule.

6. False. $\dfrac{d}{dx} f\left(\sqrt{x}\right) = \dfrac{f'\left(\sqrt{x}\right)}{2\sqrt{x}}$ by the Chain Rule.

7. False. $f(x) = \left|x^2 + x\right| = x^2 + x$ for $x \geq 0$ or $x \leq -1$ and $\left|x^2 + x\right| = -\left(x^2 + x\right)$ for $-1 < x < 0$. So $f'(x) = 2x + 1$ for $x > 0$ or $x < -1$ and $f'(x) = -(2x + 1)$ for $-1 < x < 0$. But $|2x + 1| = 2x + 1$ for $x \geq -\frac{1}{2}$ and $|2x + 1| = -2x - 1$ for $x < -\frac{1}{2}$.

8. True. $f'(r)$ exists $\Rightarrow$ f is differentiable at r $\Rightarrow$ f is continuous at r $\Rightarrow$ $\lim\limits_{x \to r} f(x) = f(r)$.

9. True. $g(x) = x^5 \Rightarrow g'(x) = 5x^4 \Rightarrow g'(2) = 5(2)^4 = 80$, and by the definition of the derivative, $\lim\limits_{x \to 2} \dfrac{g(x) - g(2)}{x - 2} = g'(2) = 80$.

10. False. $\dfrac{d^2y}{dx^2}$ is the second derivative while $\left(\dfrac{dy}{dx}\right)^2$ is the first derivative squared. For example, if $y = x$, then $\dfrac{d^2y}{dx^2} = 0$, but $\left(\dfrac{dy}{dx}\right)^2 = 1$.

11. False. A tangent to the parabola has slope $dy/dx = 2x$, so at $(-2, 4)$ the slope of the tangent is $2(-2) = -4$ and the equation is $y - 4 = -4(x + 2)$. [The equation $y - 4 = 2x(x + 2)$ is not even linear!]

12. True. $D\left(\tan^2 x\right) = 2 \tan x \sec^2 x$, and $D\left(\sec^2 x\right) = 2 \sec x (\sec x \tan x) = 2 \tan x \sec^2 x$. We can also show this by differentiating the identity $\tan^2 x + 1 = \sec^2 x$: we get $\dfrac{d}{dx}\left(\tan^2 x + 1\right) = \dfrac{d}{dx} \tan^2 x = \dfrac{d}{dx} \sec^2 x$.

EXERCISES

1. Estimating the slopes of the tangent lines at $x = 2, 3$, and 5, we obtain approximate values 0.4, 2, and 0.1. Since the graph is concave downward at $x = 5$, $f''(5)$ is negative. Arranging the numbers in increasing order, we have: $f''(5)$, 0, $f'(5)$, $f'(2)$, 1, $f'(3)$.

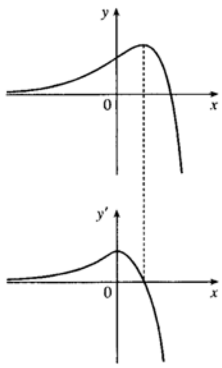
2. $2^6 = 64$, so $f(x) = x^6$ and $a = 2$.

3. (a) $f'(r)$ is the rate at which the total cost changes with respect to the interest rate. Its units are dollars/ (percent per year).

(b) The total cost of paying off the loan is increasing by \$1200/ (percent per year) as the interest rate reaches 10%. So if the interest rate goes up from 10% to 11%, the cost goes up approximately \$1200.

(c) As r increases, C increases. So $f'(r)$ will always be positive.

4.



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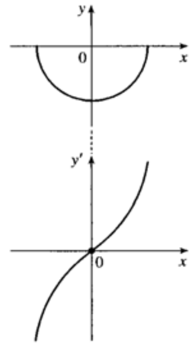
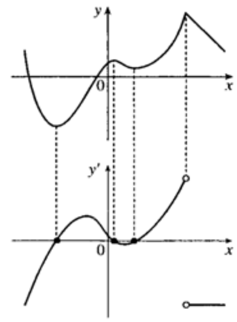


6.



7. The graph of a has tangent lines with positive slope for $x < 0$ and negative slope for $x > 0$, and the values of c fit this pattern, so c must be the graph of the derivative of the function for a. The graph of c has horizontal tangent lines to the left and right of the x-axis and b has zeros at these points. Hence, b is the graph of the derivative of the function for c. Therefore, a is the graph of f, c is the graph of f', and b is the graph of f''.

8. (a) Drawing slope triangles, we obtain the following estimates: $F'(1950) \approx \frac{1.1}{10} = 0.11$, $F'(1965) \approx \frac{-1.6}{10} = -0.16$, and $F'(1987) \approx \frac{0.2}{10} = 0.02$.

(b) The rate of change of the average number of children born to each woman was increasing by 0.11 in 1950, decreasing by 0.16 in 1965, and increasing by 0.02 in 1987.

(c) There are many possible reasons:

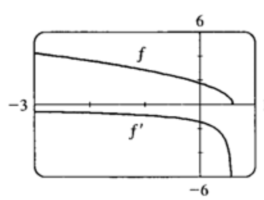
- In the baby boom era (post-WWII), there was optimism about the economy and family size was rising.
- In the baby bust era, there was less economic optimism, and it was considered less socially responsible to have a large family.
- In the baby boomlet era, there was increased economic optimism and a return to more conservative attitudes.

9. (a)
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{3-5(x+h)} - \sqrt{3-5x}}{h} \frac{\left(\sqrt{3-5(x+h)} + \sqrt{3-5x}\right)}{\left(\sqrt{3-5(x+h)} + \sqrt{3-5x}\right)}$$
$$= \lim_{h\to 0} \frac{[3-5(x+h)] - (3-5x)}{h\left(\sqrt{3-5(x+h)} + \sqrt{3-5x}\right)} = \lim_{h\to 0} \frac{-5}{\sqrt{3-5(x+h)} + \sqrt{3-5x}} = \frac{-5}{2\sqrt{3-5x}}$$

(b) Domain of f: $3 - 5x \geq 0 \;\Rightarrow\; 5x \leq 3 \;\Rightarrow\; x \in \left(-\infty, \frac{3}{5}\right]$

Domain of f': exclude $\frac{3}{5}$; $x \in \left(-\infty, \frac{3}{5}\right)$

(c) Our answer to part (a) is reasonable because $f'(x)$ is always negative and f is always decreasing.



10. $f(x) = \dfrac{4-x}{3+x} \quad \Rightarrow$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{4-(x+h)}{3+(x+h)} - \dfrac{4-x}{3+x}}{h}$$
$$= \lim_{h\to 0} \frac{(4-x-h)(3+x) - (4-x)(3+x+h)}{h(3+x+h)(3+x)} = \lim_{h\to 0} \frac{-7h}{h(3+x+h)(3+x)}$$
$$= \lim_{h\to 0} \frac{-7}{(3+x+h)(3+x)} = -\frac{7}{(3+x)^2}$$

11. $f(x) = x^3 + 5x + 4 \quad \Rightarrow$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 + 5(x+h) + 4 - \left(x^3 + 5x + 4\right)}{h}$$
$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 + 5h}{h} = \lim_{h\to 0} \left(3x^2 + 3xh + h^2 + 5\right) = 3x^2 + 5$$

12. $f(x) = x \sin x \quad \Rightarrow$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)\sin(x+h) - x\sin x}{h}$$
$$= \lim_{h\to 0} \frac{(x+h)(\sin x \cos h + \cos x \sin h) - x \sin x}{h}$$
$$= \lim_{h\to 0} \frac{x \sin x(\cos h - 1) + x\cos x \sin h + h(\sin x \cos h + \sin h \cos x)}{h}$$
$$= x\sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + x\cos x \lim_{h\to 0} \frac{\sin h}{h} + \sin x \lim_{h\to 0} \cos h + \cos x \lim_{h\to 0} \sin h$$
$$= x\sin x\,(0) + x\cos x\,(1) + \sin x\,(1) + \cos x\,(0) = x\cos x + \sin x$$

13. $y = (x+2)^8 (x+3)^6 \quad \Rightarrow$

$$y' = (x+2)^8\, 6(x+3)^5 + (x+3)^6\, 8(x+2)^7$$
$$= 2(x+2)^7 (x+3)^5 [3(x+2) + 4(x+3)] = 2(x+2)^7 (x+3)^5 (7x+18)$$

14. $y = \sqrt[3]{x} + 1/\sqrt[3]{x} = x^{1/3} + x^{-1/3} \quad \Rightarrow \quad y' = \frac{1}{3}x^{-2/3} - \frac{1}{3}x^{-4/3}$

15. $y = \dfrac{x}{\sqrt{9-4x}} \quad \Rightarrow \quad y' = \dfrac{\sqrt{9-4x} - x\left[-4/\left(2\sqrt{9-4x}\right)\right]}{9-4x} \cdot \dfrac{\sqrt{9-4x}}{\sqrt{9-4x}} = \dfrac{(9-4x) + 2x}{(9-4x)^{3/2}} = \dfrac{9-2x}{(9-4x)^{3/2}}$

16. $y = \left(x + 1/x^2\right)^{\sqrt{7}} \quad \Rightarrow \quad y' = \sqrt{7}\left(x + 1/x^2\right)^{\sqrt{7}-1}\left(1 - 2/x^3\right)$

17. $x^2y^3 + 3y^2 = x - 4y \quad \Rightarrow \quad 2xy^3 + 3x^2y^2y' + 6yy' = 1 - 4y' \quad \Rightarrow \quad y' = \dfrac{1 - 2xy^3}{3x^2y^2 + 6y + 4}$

18. $y = \left(1 - x^{-1}\right)^{-1} \quad \Rightarrow \quad y' = -\left(1 - x^{-1}\right)^{-2} x^{-2} = -(x-1)^{-2}$

19. $y = \sec 2\theta \quad \Rightarrow \quad y' = 2\sec 2\theta \tan 2\theta$

20. $y = -2/\sqrt[4]{t^3} = -2t^{-3/4} \quad \Rightarrow \quad y' = (-2)\left(-\frac{3}{4}\right)t^{-7/4} = \frac{3}{2}t^{-7/4}$

21. $y = \dfrac{x}{8-3x} \quad \Rightarrow \quad y' = \dfrac{(8-3x) - x(-3)}{(8-3x)^2} = \dfrac{8}{(8-3x)^2}$

22. $y\sqrt{x-1}+x\sqrt{y-1}=xy \quad\Rightarrow\quad y'\sqrt{x-1}+y\dfrac{1}{2\sqrt{x-1}}+\sqrt{y-1}+x\dfrac{1}{2\sqrt{x-1}}y'=y+xy' \quad\Rightarrow$

$$y'=\frac{y-\sqrt{y-1}-y\,/\left(2\sqrt{x-1}\right)}{\sqrt{x-1}-x+x\,/\left(2\sqrt{y-1}\right)}$$

23. $y=(x\tan x)^{1/5} \quad\Rightarrow\quad y'=\frac{1}{5}(x\tan x)^{-4/5}\left(\tan x+x\sec^2 x\right)$

24. $y=\sin(\cos x) \quad\Rightarrow\quad y'=\cos(\cos x)(-\sin x)=-\sin x\cos(\cos x)$

25. $x^2=y(y+1)=y^2+y \quad\Rightarrow\quad 2x=2yy'+y' \quad\Rightarrow\quad y'=2x/(2y+1)$

26. $y=\left(x+\sqrt{x}\right)^{-1/3} \quad\Rightarrow\quad y'=-\frac{1}{3}\left(x+\sqrt{x}\right)^{-4/3}\left(1+\dfrac{1}{2\sqrt{x}}\right)$

27. $y=\dfrac{(x-1)(x-4)}{(x-2)(x-3)}=\dfrac{x^2-5x+4}{x^2-5x+6} \quad\Rightarrow$

$$y'=\frac{\left(x^2-5x+6\right)(2x-5)-\left(x^2-5x+4\right)(2x-5)}{\left(x^2-5x+6\right)^2}=\frac{2(2x-5)}{(x-2)^2(x-3)^2}$$

28. $y=\sqrt{\sin\sqrt{x}} \quad\Rightarrow\quad y'=\frac{1}{2}\left(\sin\sqrt{x}\right)^{-1/2}\left(\cos\sqrt{x}\right)\left(\dfrac{1}{2\sqrt{x}}\right)=\dfrac{\cos\sqrt{x}}{4\sqrt{x}\sin\sqrt{x}}$

29. $y=\tan\sqrt{1-x} \quad\Rightarrow\quad y'=\left(\sec^2\sqrt{1-x}\right)\left(\dfrac{1}{2\sqrt{1-x}}\right)(-1)=-\dfrac{\sec^2\sqrt{1-x}}{2\sqrt{1-x}}$

30. Using the Reciprocal Rule, $g(x)=\dfrac{1}{f(x)} \quad\Rightarrow\quad g'(x)=-\dfrac{f'(x)}{[f(x)]^2}$, we have $y=\dfrac{1}{\sin(x-\sin x)} \quad\Rightarrow$

$$y'=-\frac{\cos(x-\sin x)(1-\cos x)}{\sin^2(x-\sin x)}$$

31. $y=\sin\left(\tan\sqrt{1+x^3}\right) \quad\Rightarrow\quad y'=\cos\left(\tan\sqrt{1+x^3}\right)\left(\sec^2\sqrt{1+x^3}\right)\left[3x^2\,/\left(2\sqrt{1+x^3}\right)\right]$

32. $y=\dfrac{(x+\lambda)^4}{x^4+\lambda^4} \quad\Rightarrow\quad y'=\dfrac{\left(x^4+\lambda^4\right)(4)(x+\lambda)^3-(x+\lambda)^4\left(4x^3\right)}{\left(x^4+\lambda^4\right)^2}=\dfrac{4(x+\lambda)^3\left(\lambda^4-\lambda x^3\right)}{\left(x^4+\lambda^4\right)^2}$

33. $y=\cot\left(3x^2+5\right) \quad\Rightarrow\quad y'=-\csc^2\left(3x^2+5\right)(6x)=-6x\csc^2\left(3x^2+5\right)$

34. $y=(\sin mx)/x \quad\Rightarrow\quad y'=(mx\cos mx-\sin mx)/x^2$

35. $y=\cos^2(\tan x) \quad\Rightarrow\quad y'=2\cos(\tan x)[-\sin(\tan x)]\sec^2 x=-\sin(2\tan x)\sec^2 x$

36. $x\tan y=y-1 \quad\Rightarrow\quad \tan y+\left(x\sec^2 y\right)y'=y' \quad\Rightarrow\quad y'=\dfrac{\tan y}{1-x\sec^2 y}$

37. $f(x)=(2x-1)^{-5} \quad\Rightarrow\quad f'(x)=-5(2x-1)^{-6}(2)=-10(2x-1)^{-6} \quad\Rightarrow$
$f''(x)=60(2x-1)^{-7}(2)=120(2x-1)^{-7}$. $f''(0)=120(-1)^{-7}=-120$

38. $g(t)=\csc 2t \quad\Rightarrow\quad g'(t)=-2\csc 2t\cot 2t \quad\Rightarrow$

$$\begin{aligned}g''(t)&=-2(-2\csc 2t\cot 2t)\cot 2t-2\csc 2t\left(-2\csc^2 2t\right)=4\csc 2t\left(\cot^2 2t+\csc^2 2t\right)\\&=8\csc^3 2t-4\csc 2t \quad\Rightarrow\end{aligned}$$

$g'''(t)=24\csc^2 2t\,(-2\csc 2t\cot 2t)-4(-2\csc 2t\cot 2t)=-48\csc^3 2t\cot 2t+8\csc 2t\cot 2t \quad\Rightarrow$

$g'''\left(-\frac{\pi}{8}\right)=-48\left(-\sqrt{2}\right)^3(-1)+8\left(-\sqrt{2}\right)(-1)=-88\sqrt{2}$

39. $x^6 + y^6 = 1 \;\Rightarrow\; 6x^5 + 6y^5y' = 0 \;\Rightarrow\; y' = -x^5/y^5 \;\Rightarrow$

$$y'' = -\frac{y^5\left(5x^4\right) - x^5\left(5y^4y'\right)}{\left(y^5\right)^2} = -\frac{5x^4y^4\left[y - x\left(-x^5/y^5\right)\right]}{y^{10}} = -\frac{5x^4\left[\left(y^6+x^6\right)/y^5\right]}{y^6} = -\frac{5x^4}{y^{11}}$$

40. $f(x) = (2-x)^{-1} \;\Rightarrow\; f'(x) = (2-x)^{-2} \;\Rightarrow\; f''(x) = 2(2-x)^{-3} \;\Rightarrow\; f'''(x) = 2\cdot 3(2-x)^{-4} \;\Rightarrow$
$f^{(4)}(x) = 2\cdot 3\cdot 4(2-x)^{-5}$. In general, $f^{(n)}(x) = 2\cdot 3\cdot 4\cdot\cdots\cdot n(2-x)^{-(n+1)} = \dfrac{n!}{(2-x)^{(n+1)}}$.

41. $\displaystyle\lim_{x\to 0}\frac{\sec x}{1-\sin x} = \frac{\sec 0}{1-\sin 0} = \frac{1}{1-0} = 1$

42. $\displaystyle\lim_{t\to 0}\frac{t^3}{\tan^3 2t} = \lim_{t\to 0}\frac{t^3\cos^3 2t}{\sin^3 2t} = \lim_{t\to 0}\cos^3 2t\cdot\frac{1}{8\dfrac{\sin^3 2t}{(2t)^3}} = \lim_{t\to 0}\frac{\cos^3 2t}{8\left(\lim\limits_{t\to 0}\dfrac{\sin 2t}{2t}\right)^3} = \frac{1}{8\cdot 1^3} = \frac{1}{8}$

43. $y = \dfrac{x}{x^2-2} \;\Rightarrow\; y' = \dfrac{\left(x^2-2\right) - x(2x)}{\left(x^2-2\right)^2} = \dfrac{-x^2-2}{\left(x^2-2\right)^2}$. When $x = 2$, $y' = \dfrac{-2^2-2}{\left(2^2-2\right)^2} = -\dfrac{3}{2}$, so an equation of the tangent line at $(2, 1)$ is $y - 1 = -\frac{3}{2}(x-2)$ or $y = -\frac{3}{2}x + 4$.

44. $\sqrt{x} + \sqrt{y} = 3 \;\Rightarrow\; \dfrac{1}{2\sqrt{x}} + \dfrac{1}{2\sqrt{y}}y' = 0 \;\Rightarrow\; y' = -\dfrac{\sqrt{y}}{\sqrt{x}}$. At $(4, 1)$, $y' = -\frac{1}{2}$, so an equation of the tangent line at $(4, 1)$ is $y - 1 = -\frac{1}{2}(x-4)$ or $y = -\frac{1}{2}x + 3$.

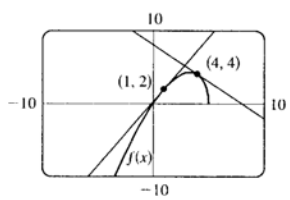
45. $y = \tan x \;\Rightarrow\; y' = \sec^2 x$. When $x = \frac{\pi}{3}$, $y' = 2^2 = 4$, so an equation of the tangent line at $\left(\frac{\pi}{3}, \sqrt{3}\right)$ is $y - \sqrt{3} = 4\left(x - \frac{\pi}{3}\right)$ or $y = 4x + \sqrt{3} - \frac{4}{3}\pi$.

46. $y = x\sqrt{1+x^2} \;\Rightarrow\; y' = \sqrt{1+x^2} + x^2/\sqrt{1+x^2}$. When $x = 1$, $y' = \sqrt{2} + \frac{1}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$, so an equation of the tangent line at $\left(1, \sqrt{2}\right)$ is $y - \sqrt{2} = \frac{3\sqrt{2}}{2}(x-1)$ or $y = \frac{3\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}$.

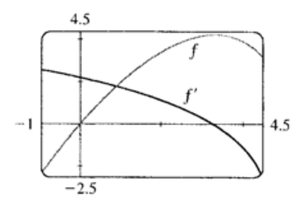
47. (a) $f(x) = x\sqrt{5-x} \;\Rightarrow\; f'(x) = x\left[\frac{1}{2}(5-x)^{-1/2}(-1)\right] + \sqrt{5-x} = \dfrac{-x}{2\sqrt{5-x}} + \dfrac{2(5-x)}{2\sqrt{5-x}} = \dfrac{10-3x}{2\sqrt{5-x}}$

(b) At $(1, 2)$: $f'(1) = \frac{7}{4}$. So an equation of the tangent is $y - 2 = \frac{7}{4}(x-1)$ or $y = \frac{7}{4}x + \frac{1}{4}$.
At $(4, 4)$: $f'(4) = -\frac{2}{2} = -1$. So an equation of the tangent is $y - 4 = -(x-4)$ or $y = -x + 8$.

(c)



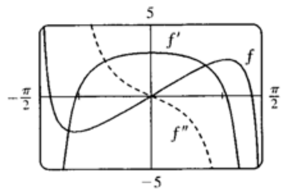
(d)



The graphs look reasonable, since f' is positive where f has tangents with positive slope, and f' is negative where f has tangents with negative slope.

48. (a) $f(x) = 4x - \tan x \quad \Rightarrow \quad f'(x) = 4 - \sec^2 x \quad \Rightarrow \quad f''(x) = -2\sec x\,(\sec x \tan x) = -2\sec^2 x \tan x$.

(b)



We can see that our answers are reasonable, since the graph of f' is 0 where f has a horizontal tangent, and the graph of f' is positive where f has tangents with positive slope and negative where f has tangents with negative slope. The same correspondence holds between the graphs of f' and f''.

49. $y = \sin x + \cos x \quad \Rightarrow \quad y' = \cos x - \sin x = 0 \quad \Leftrightarrow \quad \cos x = \sin x$ and $0 \le x \le 2\pi \quad \Leftrightarrow \quad x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$, so the points are $\left(\frac{\pi}{4}, \sqrt{2}\right)$ and $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$.

50. $x^2 + 2y^2 = 1 \quad \Rightarrow \quad 2x + 4yy' = 0 \quad \Rightarrow \quad y' = -x/(2y) = 1 \quad \Leftrightarrow \quad x = -2y$. Since the points lie on the ellipse, we have $(-2y)^2 + 2y^2 = 1 \quad \Rightarrow \quad 6y^2 = 1 \quad \Rightarrow \quad y = \pm\frac{1}{\sqrt{6}}$. The points are $\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ and $\left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$.

51. $f(x) = (x-a)(x-b)(x-c) \quad \Rightarrow \quad f'(x) = (x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)$. So

$$\frac{f'(x)}{f(x)} = \frac{(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)}{(x-a)(x-b)(x-c)} = \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}.$$

52. (a) $\cos 2x = \cos^2 x - \sin^2 x \quad \Rightarrow \quad -2\sin 2x = -2\cos x \sin x - 2\sin x \cos x \quad \Leftrightarrow \quad \sin 2x = 2\sin x \cos x$

(b) $\sin(x + a) = \sin x \cos a + \cos x \sin a \quad \Rightarrow \quad \cos(x + a) = \cos x \cos a - \sin x \sin a$.

53. (a) $h(x) = f(x)\,g(x) \quad \Rightarrow \quad h'(x) = f'(x)\,g(x) + f(x)\,g'(x) \quad \Rightarrow$
$h'(2) = f'(2)\,g(2) + f(2)\,g'(2) = (-2)(5) + (3)(4) = 2$

(b) $F(x) = f(g(x)) \quad \Rightarrow \quad F'(x) = f'(g(x))\,g'(x) \quad \Rightarrow$
$F'(2) = f'(g(2))\,g'(2) = f'(5)(4) = 11 \cdot 4 = 44$

54. (a) $P(x) = f(x)\,g(x) \quad \Rightarrow \quad P'(x) = f(x)\,g'(x) + g(x)\,f'(x) \quad \Rightarrow$
$P'(2) = f(2)\,g'(2) + g(2)\,f'(2) = (1)(2) + (4)(-1) = -2$

(b) $Q(x) = \dfrac{f(x)}{g(x)} \quad \Rightarrow \quad Q'(x) = \dfrac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2} \quad \Rightarrow$

$$Q'(2) = \frac{g(2)\,f'(2) - f(2)\,g'(2)}{[g(2)]^2} = \frac{(4)(-1) - (1)(2)}{4^2} = -\frac{3}{8}$$

(c) $C(x) = f(g(x)) \quad \Rightarrow \quad C'(x) = f'(g(x))\,g'(x) \quad \Rightarrow$
$C'(2) = f'(g(2))\,g'(2) = f'(4)\,g'(2) = (3)(2) = 6$

55. $f(x) = x^2 g(x) \quad \Rightarrow \quad f'(x) = 2xg(x) + x^2 g'(x) = x\left[2g(x) + xg'(x)\right]$

56. $f(x) = g\left(x^2\right) \quad \Rightarrow \quad f'(x) = g'\left(x^2\right)(2x) = 2xg'\left(x^2\right)$

57. $f(x) = [g(x)]^2 \quad \Rightarrow \quad f'(x) = 2g(x)\,g'(x)$

58. $f(x) = x^a g\left(x^b\right) \quad \Rightarrow \quad f'(x) = ax^{a-1} g\left(x^b\right) + x^a g'\left(x^b\right)\left(bx^{b-1}\right) = ax^{a-1} g\left(x^b\right) + bx^{a+b-1} g'\left(x^b\right)$

59. $f(x) = g(g(x)) \quad \Rightarrow \quad f'(x) = g'(g(x))\,g'(x)$

60. $f(x) = g\left(\tan\sqrt{x}\right) \quad \Rightarrow$

$$f'(x) = g'\left(\tan\sqrt{x}\right) \cdot \frac{d}{dx}\left(\tan\sqrt{x}\right) = g'\left(\tan\sqrt{x}\right) \cdot \sec^2\sqrt{x} \cdot \frac{d}{dx}\left(\sqrt{x}\right) = \frac{g'\left(\tan\sqrt{x}\right)\sec^2\sqrt{x}}{2\sqrt{x}}$$

61. $h(x) = \dfrac{f(x)g(x)}{f(x)+g(x)} \quad \Rightarrow$

$$h'(x) = \frac{[f(x)+g(x)]\left[f(x)g'(x)+g(x)f'(x)\right] - f(x)g(x)\left[f'(x)+g'(x)\right]}{[f(x)+g(x)]^2}$$

$$= \frac{[f(x)]^2 g'(x) + f(x)g(x)f'(x) + f(x)g(x)g'(x) + [g(x)]^2 f'(x) - f(x)g(x)f'(x) - f(x)g(x)g'(x)}{[f(x)+g(x)]^2}$$

$$= \frac{f'(x)[g(x)]^2 + g'(x)[f(x)]^2}{[f(x)+g(x)]^2}$$

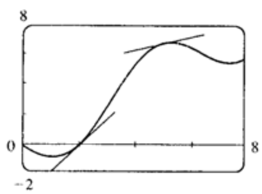
62. $h(x) = \sqrt{\dfrac{f(x)}{g(x)}} \quad \Rightarrow \quad h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{2\sqrt{f(x)/g(x)}\,[g(x)]^2} = \dfrac{f'(x)g(x) - f(x)g'(x)}{2[g(x)]^{3/2}\sqrt{f(x)}}$

63. Using the Chain Rule repeatedly, $h(x) = f(g(\sin 4x)) \quad \Rightarrow$

$$h'(x) = f'(g(\sin 4x)) \cdot \frac{d}{dx}(g(\sin 4x)) = f'(g(\sin 4x)) \cdot g'(\sin 4x) \cdot \frac{d}{dx}(\sin 4x)$$

$$= f'(g(\sin 4x))\, g'(\sin 4x)(\cos 4x)(4)$$

64. (a)



(b) The average rate of change is larger on [2, 3].

(c) The instantaneous rate of change (the slope of the tangent) is larger at $x = 2$.

(d) $f(x) = x - 2\sin x \quad \Rightarrow \quad f'(x) = 1 - 2\cos x$, so $f'(2) = 1 - 2\cos 2 \approx 1.8323$ and $f'(5) = 1 - 2\cos 5 \approx 0.4327$. So $f'(2) > f'(5)$, as predicted in part (c).

65. f is not differentiable: at $x = -4$ because f is not continuous, at $x = -1$ because f has a corner, at $x = 2$ because f is not continuous, and at $x = 5$ because f has a vertical tangent.

66. (a) $x = \sqrt{b^2 + c^2t^2} \quad \Rightarrow \quad v(t) = x' = \left[1/\left(2\sqrt{b^2+c^2t^2}\right)\right]2c^2t = c^2t/\sqrt{b^2+c^2t^2} \quad \Rightarrow$

$$a(t) = v'(t) = \frac{c^2\sqrt{b^2+c^2t^2} - c^2t\left(c^2t/\sqrt{b^2+c^2t^2}\right)}{b^2+c^2t^2} = \frac{b^2c^2}{\left(b^2+c^2t^2\right)^{3/2}}$$

(b) $v(t) > 0$ for $t > 0$, so the particle always moves in the positive direction.

67. (a) $y = t^3 - 12t + 3 \quad \Rightarrow \quad v(t) = y' = 3t^2 - 12 \quad \Rightarrow \quad a(t) = v'(t) = 6t$

(b) $v(t) = 3\left(t^2 - 4\right) > 0$ when $t > 2$, so it moves upward when $t > 2$ and downward when $0 \le t < 2$.

(c) Distance upward $= y(3) - y(2) = -6 - (-13) = 7$,
Distance downward $= y(0) - y(2) = 3 - (-13) = 16$. Total distance $= 7 + 16 = 23$

68. (a) $V = \frac{1}{3}\pi r^2 h \quad \Rightarrow \quad dV/dh = \frac{1}{3}\pi r^2$

(b) $dV/dr = \frac{2}{3}\pi rh$

69. The linear density ρ is the rate of change of mass m with respect to length x. $m = x\left(1+\sqrt{x}\right) = x + x^{3/2} \quad \Rightarrow$ $\rho = dm/dx = 1 + \frac{3}{2}\sqrt{x}$, so the linear density when $x = 4$ is $1 + \frac{3}{2}\sqrt{4} = 4$ kg/m.

70. (a) $C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3 \quad \Rightarrow \quad C'(x) = 2 - 0.04x + 0.00021x^2$

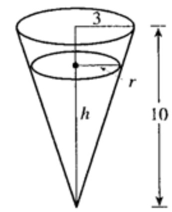
(b) $C'(100) = \$0.10/\text{unit}$. This value represents the rate at which costs are increasing as the hundredth unit is produced, and is the approximate cost of producing the 101st unit.

(c) $C(101) - C(100) = 990.10107 - 990 = \$0.10107$, slightly larger than $C'(100)$.

71. If x = edge length, then $V = x^3 \Rightarrow dV/dt = 3x^2 dx/dt = 10 \Rightarrow dx/dt = 10/(3x^2)$ and $S = 6x^2 \Rightarrow$ $dS/dt = (12x)\,dx/dt = 12x[10/(3x^2)] = 40/x$. When $x = 30$, $dS/dt = \frac{40}{30} = \frac{4}{3}$ cm^2/min.

72. Given $dV/dt = 2$, find dh/dt when $h = 5$. $V = \frac{1}{3}\pi r^2 h$ and, from similar triangles, $\frac{r}{h} = \frac{3}{10} \Rightarrow V = \frac{\pi}{3}\left(\frac{3h}{10}\right)^2 h = \frac{3\pi}{100}h^3$, so

$2 = \frac{dV}{dt} = \frac{9\pi}{100}h^2\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{200}{9\pi h^2} = \frac{200}{9\pi\,(5)^2} = \frac{8}{9\pi}$ cm/s when $h = 5$.

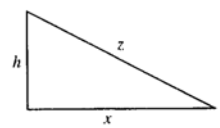


73. Given $dh/dt = 5$ and $dx/dt = 15$, find dz/dt. $z^2 = x^2 + h^2 \Rightarrow$

$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2h\frac{dh}{dt} \Rightarrow \frac{dz}{dt} = \frac{1}{z}(15x + 5h)$. When $t = 3$,

$h = 45 + 3\,(5) = 60$ and $x = 15\,(3) = 45 \Rightarrow z = 75$, so

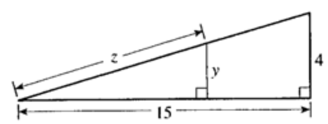
$\frac{dz}{dt} = \frac{1}{75}[15\,(45) + 5\,(60)] = 13$ ft/s.



74. We are given $dz/dt = 30$ ft/s. By similar triangles,

$\frac{y}{z} = \frac{4}{\sqrt{241}} \Rightarrow y = \frac{4}{\sqrt{241}}z$, so

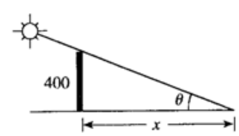
$\frac{dy}{dt} = \frac{4}{\sqrt{241}}\frac{dz}{dt} = \frac{120}{\sqrt{241}} \approx 7.7$ ft/s.



75. We are given $d\theta/dt = -0.25$ rad/h. $x = 400\cot\theta \Rightarrow$

$\frac{dx}{dt} = -400\csc^2\theta\frac{d\theta}{dt}$. When $\theta = \frac{\pi}{6}$,

$\frac{dx}{dt} = -400\,(2)^2\,(-0.25) = 400$ ft/h.

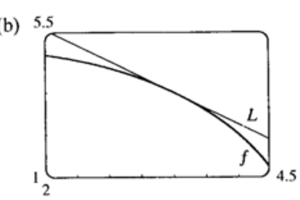


76. (a) $f(x) = \sqrt{25 - x^2} \Rightarrow$

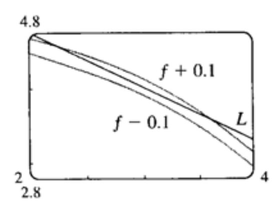
$f'(x) = \frac{-2x}{2\sqrt{25 - x^2}} = -x\left(25 - x^2\right)^{-1/2}$. So the linear approximation to $f(x)$ near 3 is

$f(x) \approx f(3) + f'(3)(x - 3) = 4 - \frac{3}{4}(x - 3)$.

(b)

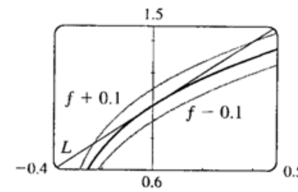


(c) For the required accuracy, we want $\sqrt{25 - x^2} - 0.1 < 4 - \frac{3}{4}(x - 3)$ and $4 - \frac{3}{4}(x - 3) < \sqrt{25 - x^2} + 0.1$. From the graph, it appears that these both hold for $2.24 < x < 3.66$.



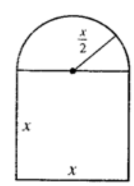
77. (a) $f(x) = \sqrt[3]{1+3x} = (1+3x)^{1/3} \Rightarrow f'(x) = (1+3x)^{-2/3}$ so $L(x) = f(0) + f'(0)(x-0) = 1^{1/3} + 1^{-2/3}x = 1 + x$. Thus, $\sqrt[3]{1+3x} \approx 1 + x \Rightarrow \sqrt[3]{1.03} = \sqrt[3]{1+3(0.01)} \approx 1 + (0.01) = 1.01$.

(b) The linear approximation is $\sqrt[3]{1+3x} \approx 1 + x$, so for the required accuracy we want $\sqrt[3]{1+3x} - 0.1 < 1 + x < \sqrt[3]{1+3x} + 0.1$. From the graph, it appears that this is true when $-0.23 < x < 0.40$.



78. $y = x^3 - 2x^2 + 1 \Rightarrow dy = (3x^2 - 4x)\,dx$. When $x = 2$ and $dx = 0.2$, $dy = [3(2)^2 - 4(2)](0.2) = 0.8$.

79. $A = x^2 + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = \left(1 + \frac{\pi}{8}\right)x^2 \Rightarrow dA = \left(2 + \frac{\pi}{4}\right)x\,dx$. When $x = 60$ and $dx = 0.1$, $dA = \left(2 + \frac{\pi}{4}\right)60\,(0.1) = 12 + \frac{3\pi}{2}$, so the maximum error is approximately $12 + \frac{3\pi}{2} \approx 16.7\text{ cm}^2$.



80. $\lim\limits_{x\to 1} \dfrac{x^{17} - 1}{x - 1} = \left[\dfrac{d}{dx}x^{17}\right]_{x=1} = 17(1)^{16} = 17$

81. $\lim\limits_{h\to 0} \dfrac{\sqrt[4]{16+h} - 2}{h} = \left[\dfrac{d}{dx}\sqrt[4]{x}\right]_{x=16} = \frac{1}{4}x^{-3/4}\Big|_{x=16} = \dfrac{1}{4\left(\sqrt[4]{16}\right)^3} = \dfrac{1}{32}$

82. $\lim\limits_{\theta\to\pi/3} \dfrac{\cos\theta - 0.5}{\theta - \pi/3} = \left[\dfrac{d}{d\theta}\cos\theta\right]_{\theta=\pi/3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

83.
$$\lim_{x\to 0} \frac{\sqrt{1+\tan x} - \sqrt{1+\sin x}}{x^3} = \lim_{x\to 0} \frac{\left(\sqrt{1+\tan x} - \sqrt{1+\sin x}\right)\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)}{x^3\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)}$$
$$= \lim_{x\to 0} \frac{(1+\tan x) - (1+\sin x)}{x^3\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)} = \lim_{x\to 0} \frac{\sin x\,(1/\cos x - 1)\cos x}{x^3\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)\cos x}$$
$$= \lim_{x\to 0} \frac{\sin x\,(1 - \cos x)(1 + \cos x)}{x^3\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)\cos x\,(1+\cos x)}$$
$$= \lim_{x\to 0} \frac{\sin x \cdot \sin^2 x}{x^3\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)\cos x\,(1+\cos x)}$$
$$= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^3 \lim_{x\to 0} \frac{1}{\left(\sqrt{1+\tan x} + \sqrt{1+\sin x}\right)\cos x\,(1+\cos x)}$$
$$= 1^3 \cdot \frac{1}{\left(\sqrt{1} + \sqrt{1}\right)\cdot 1 \cdot (1+1)} = \frac{1}{4}$$

84. Differentiating the first given equation implicitly with respect to x and using the Chain Rule, we obtain $f(g(x)) = x \quad\Rightarrow\quad f'(g(x))\,g'(x) = 1 \quad\Rightarrow\quad g'(x) = \dfrac{1}{f'(g(x))}$. Using the second given equation to expand the denominator of this expression gives $g'(x) = \dfrac{1}{1 + [f(g(x))]^2}$. But the first given equation states that $f(g(x)) = x$, so $g'(x) = \dfrac{1}{1 + x^2}$.

85. $\dfrac{d}{dx}[f(2x)] = x^2 \quad\Rightarrow\quad f'(2x)\cdot 2 = x^2 \quad\Rightarrow\quad f'(2x) = \frac{1}{2}x^2$. Let $t = 2x$. Then $f'(t) = \frac{1}{2}\left(\frac{1}{2}t\right)^2 = \frac{1}{8}t^2$, so $f'(x) = \frac{1}{8}x^2$.

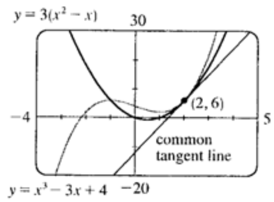
86. Let (b, c) be on the curve, that is, $b^{2/3} + c^{2/3} = a^{2/3}$. Now $x^{2/3} + y^{2/3} = a^{2/3} \quad\Rightarrow\quad \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{y^{1/3}}{x^{1/3}} = -\left(\dfrac{y}{x}\right)^{1/3}$, so at (b, c) the slope of the tangent line is $-(c/b)^{1/3}$ and an equation of the tangent line is $y - c = -(c/b)^{1/3}(x - b)$ or $y = -(c/b)^{1/3}x + \left(c + b^{2/3}c^{1/3}\right)$. Setting $y = 0$, we find that the x-intercept is $b^{1/3}c^{2/3} + b$ and setting $x = 0$ we find that the y-intercept is $c + b^{2/3}c^{1/3}$. So the length of the tangent line between these two points is

$$\begin{aligned}
\sqrt{\left[b^{1/3}\left(c^{2/3} + b^{2/3}\right)\right]^2 + \left[c^{1/3}\left(c^{2/3} + b^{2/3}\right)\right]^2} &= \sqrt{b^{2/3}\left(a^{2/3}\right)^2 + c^{2/3}\left(a^{2/3}\right)^2} \\
&= \sqrt{\left(b^{2/3} + c^{2/3}\right)a^{4/3}} = \sqrt{a^{2/3}a^{4/3}} \\
&= \sqrt{a^2} = a = \text{constant}
\end{aligned}$$

Problems Plus

1. Let a be the x-coordinate of Q. Since the derivative of $y = 1 - x^2$ is $y' = -2x$, the slope at Q is $-2a$. But since the triangle is equilateral, $\overline{AO}/\overline{OC} = \sqrt{3}/1$, so the slope at Q is $-\sqrt{3}$. Therefore, we must have that $-2a = -\sqrt{3}$ $\Rightarrow$ $a = \frac{\sqrt{3}}{2}$. Thus, the point Q has coordinates $\left(\frac{\sqrt{3}}{2}, 1 - \left(\frac{\sqrt{3}}{2}\right)^2\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$ and by symmetry, P has coordinates $\left(-\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$.

2. $y = x^3 - 3x + 4$ $\Rightarrow$ $y' = 3x^2 - 3$, and $y = 3\left(x^2 - x\right)$ $\Rightarrow$ $y' = 6x - 3$. The slopes of the tangents of the two curves are equal when $3x^2 - 3 = 6x - 3$, that is, when $x = 0$ or 2. At $x = 0$, both tangents have slope -3, but the curves do not intersect. At $x = 2$, both tangents have slope 9 and the curves intersect at (2, 6). So there is a common tangent line at (2, 6), $y = 9x - 12$.



3. (a) Put $x = 0$ and $y = 0$ in the equation: $f(0) = f(0 + 0) = f(0) + f(0) + 0^2 \cdot 0 + 0 \cdot 0^2 = 2f(0)$. Subtracting $f(0)$ from each side of this equation gives $f(0) = 0$.

(b) $$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\left[f(0) + f(h) + 0^2 h + 0h^2\right] - f(0)}{h}$$
$$= \lim_{h\to 0} \frac{f(h)}{h} = \lim_{x\to 0} \frac{f(x)}{x} = 1$$

(c) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left[f(x) + f(h) + x^2 h + xh^2\right] - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{f(h) + x^2 h + xh^2}{h} = \lim_{h\to 0} \left[\frac{f(h)}{h} + x^2 + xh\right] = 1 + x^2$$

4. We find the equation of the parabola by substituting the point $(-100, 100)$, at which the car is situated, into the general equation $y = ax^2$: $100 = a(-100)^2$ $\Rightarrow$ $a = \frac{1}{100}$. Now we find the equation of a tangent to the parabola at the point (x_0, y_0). We can show that $y' = a(2x) = \frac{1}{100}(2x) = \frac{1}{50}x$, so an equation of the tangent is $y - y_0 = \frac{1}{50}x_0(x - x_0)$. Since the point (x_0, y_0) is on the parabola, we must have $y_0 = \frac{1}{100}x_0^2$, so our equation of the tangent can be simplified to $y = \frac{1}{100}x_0^2 + \frac{1}{50}x_0(x - x_0)$. We want the statue to be located on the tangent line, so we substitute its coordinates (100, 50) into this equation: $50 = \frac{1}{100}x_0^2 + \frac{1}{50}x_0(100 - x_0)$ $\Rightarrow$ $x_0^2 - 200x_0 + 5000 = 0$ $\Rightarrow$ $x_0 = \frac{1}{2}\left[200 \pm \sqrt{200^2 - 4(5000)}\right]$ $\Rightarrow$ $x_0 = 100 \pm 50\sqrt{2}$. But $x_0 < 100$, so the car's headlights illuminate the statue when it is located at the point $\left(100 - 50\sqrt{2}, 150 - 100\sqrt{2}\right) \approx (29.3, 8.6)$, that is, about 29.3 m east and 8.6 m north of the origin.

5. We use mathematical induction. Let S_n be the statement that $\dfrac{d^n}{dx^n}\left(\sin^4 x+\cos^4 x\right)=4^{n-1}\cos\left(4x+n\pi/2\right)$. S_1 is true because

$$\begin{aligned}\frac{d}{dx}\left(\sin^4 x+\cos^4 x\right)&=4\sin^3 x\cos x-4\cos^3 x\sin x=4\sin x\cos x\left(\sin^2 x-\cos^2 x\right)\\&=-4\sin x\cos x\cos 2x=-2\sin 2x\cos 2x=-\sin 4x\\&=\cos\left(\tfrac{\pi}{2}-(-4x)\right)=\cos\left(\tfrac{\pi}{2}+4x\right)=4^{n-1}\cos\left(4x+n\tfrac{\pi}{2}\right)\text{ when }n=1\end{aligned}$$

Now assume S_k is true, that is, $\dfrac{d^k}{dx^k}\left(\sin^4 x+\cos^4 x\right)=4^{k-1}\cos\left(4x+k\frac{\pi}{2}\right)$. Then

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}\left(\sin^4 x+\cos^4 x\right)&=\frac{d}{dx}\left[\frac{d^k}{dx^k}\left(\sin^4 x+\cos^4 x\right)\right]=\frac{d}{dx}\left[4^{k-1}\cos\left(4x+k\tfrac{\pi}{2}\right)\right]\\&=-4^{k-1}\sin\left(4x+k\tfrac{\pi}{2}\right)\cdot\frac{d}{dx}\left(4x+k\tfrac{\pi}{2}\right)=-4^k\sin\left(4x+k\tfrac{\pi}{2}\right)\\&=4^k\sin\left(-4x-k\tfrac{\pi}{2}\right)=4^k\cos\left(\tfrac{\pi}{2}-\left(-4x-k\tfrac{\pi}{2}\right)\right)\\&=4^k\cos\left(4x+(k+1)\tfrac{\pi}{2}\right)\end{aligned}$$

which shows that S_{k+1} is true.

Therefore, $\dfrac{d_n}{dx_n}\left(\sin^4 x+\cos^4 x\right)=4^{n-1}\cos\left(4x+n\frac{\pi}{2}\right)$ for every positive integer n, by mathematical induction.

Another Proof: First write

$\sin^4 x+\cos^4 x=\left(\sin^2 x+\cos^2 x\right)^2-2\sin^2 x\cos^2 x=1-\frac{1}{2}\sin^2 2x=1-\frac{1}{4}(1-\cos 4x)=\frac{3}{4}+\frac{1}{4}\cos 4x.$

Then we have $\dfrac{d^n}{dx^n}\left(\sin^4 x+\cos^4 x\right)=\dfrac{d^n}{dx^n}\left(\frac{3}{4}+\frac{1}{4}\cos 4x\right)=\frac{1}{4}\cdot 4^n\cos\left(4x+n\frac{\pi}{2}\right)=4^{n-1}\cos\left(4x+n\frac{\pi}{2}\right).$

6. If we divide $1-x$ into x^n by long division, we find that $f(x)=\dfrac{x^n}{1-x}=-x^{n-1}-x^{n-2}-\cdots-x-1+\dfrac{1}{1-x}$. This can also be seen by multiplying the last expression by $1-x$ and canceling terms on the right-hand side. So we let $g(x)=1+x+x^2+\cdots+x^{n-1}$, so that $f(x)=\dfrac{1}{1-x}-g(x)\quad\Rightarrow\quad f^{(n)}(x)=\left(\dfrac{1}{1-x}\right)^{(n)}-g^{(n)}(x)$.

But g is a polynomial of degree $(n-1)$, so its nth derivative is 0, and therefore $f^{(n)}(x)=\left(\dfrac{1}{1-x}\right)^{(n)}$. Now

$\dfrac{d}{dx}(1-x)^{-1}=(-1)(1-x)^{-2}(-1)=(1-x)^{-2}$, $\dfrac{d^2}{dx^2}(1-x)^{-1}=(-2)(1-x)^{-3}(-1)=2(1-x)^{-3}$,

$\dfrac{d^3}{dx^3}(1-x)^{-1}=(-3)\cdot 2(1-x)^{-4}(-1)=3\cdot 2(1-x)^{-4}$, $\dfrac{d^4}{dx^4}(1-x)^{-1}=4\cdot 3\cdot 2(1-x)^{-5}$, and so on. So after n differentiations, we will have $f^{(n)}(x)=\left(\dfrac{1}{1-x}\right)^{(n)}=\dfrac{n!}{(1-x)^{n+1}}$.

7. We must find a value x_0 such that the normal lines to the parabola $y = x^2$ at $x = \pm x_0$ intersect at a point one unit from the points $(\pm x_0, x_0^2)$. The normals to $y = x^2$ at $x = \pm x_0$ have slopes $-\dfrac{1}{\pm 2x_0}$ and pass through $(\pm x_0, x_0^2)$ respectively, so the normals have the equations $y - x_0^2 = -\dfrac{1}{2x_0}(x - x_0)$ and $y - x_0^2 = \dfrac{1}{2x_0}(x + x_0)$. The common y-intercept is $x_0^2 + \frac{1}{2}$. We want to find the value of x_0 for which the distance from $\left(0, x_0^2 + \frac{1}{2}\right)$ to (x_0, x_0^2) equals 1. The square of the distance is $(x_0 - 0)^2 + \left[x_0^2 - \left(x_0^2 + \frac{1}{2}\right)\right]^2 = x_0^2 + \frac{1}{4} = 1 \iff x_0 = \pm\frac{\sqrt{3}}{2}$. For these values of x_0, the y-intercept is $x_0^2 + \frac{1}{2} = \frac{5}{4}$, so the center of the circle is at $\left(0, \frac{5}{4}\right)$.

Another Solution: Let the center of the circle be $(0, a)$. Then the equation of the circle is $x^2 + (y - a)^2 = 1$. Solving with the equation of the parabola, $y = x^2$, we get $x^2 + (x^2 - a)^2 = 1 \iff x^2 + x^4 - 2ax^2 + a^2 = 1$ $\iff x^4 + (1 - 2a)x^2 + a^2 - 1 = 0$. The parabola and the circle will be tangent to each other when this quadratic equation in x^2 has equal roots, that is, when the discriminant is 0. Thus, $(1 - 2a)^2 - 4(a^2 - 1) = 0 \iff$ $1 - 4a + 4a^2 - 4a^2 + 4 = 0 \iff 4a = 5$, so $a = \frac{5}{4}$. The center of the circle is $\left(0, \frac{5}{4}\right)$.

8.
$$\begin{aligned}
\lim_{x\to a} \frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}} &= \lim_{x\to a}\left[\frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}} \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}}\right] \\
&= \lim_{x\to a}\left[\frac{f(x) - f(a)}{x - a} \cdot \left(\sqrt{x} + \sqrt{a}\right)\right] = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x\to a}\left(\sqrt{x} + \sqrt{a}\right) \\
&= f'(a) \cdot \left(\sqrt{a} + \sqrt{a}\right) \\
&= 2\sqrt{a}\, f'(a)
\end{aligned}$$

9. We can assume without loss of generality that $\theta = 0$ at time $t = 0$, so that $\theta = 12\pi t$ rad. [The angular velocity of the wheel is 360 rpm $= 360 \cdot (2\pi \text{ rad}) / (60 \text{ s}) = 12\pi$ rad/s.] Then the position of A as a function of time is $A = (40\cos\theta, 40\sin\theta) = (40\cos 12\pi t, 40\sin 12\pi t)$, so $\sin\alpha = \dfrac{40\sin\theta}{120} = \dfrac{\sin\theta}{3} = \frac{1}{3}\sin 12\pi t$.

(a) Differentiating the expression for $\sin\alpha$, we get $\cos\alpha \cdot \dfrac{d\alpha}{dt} = \frac{1}{3} \cdot 12\pi \cdot \cos 12\pi t = 4\pi\cos\theta$. When $\theta = \frac{\pi}{3}$, we have $\sin\alpha = \frac{1}{3}\sin\theta = \frac{\sqrt{3}}{6}$, so $\cos\alpha = \sqrt{1 - \left(\frac{\sqrt{3}}{6}\right)^2} = \sqrt{\frac{11}{12}}$ and

$$\frac{d\alpha}{dt} = \frac{4\pi\cos\frac{\pi}{3}}{\cos\alpha} = \frac{2\pi}{\sqrt{11/12}} = \frac{4\pi\sqrt{3}}{\sqrt{11}} \approx 6.56 \text{ rad/s}.$$

(b) By the Law of Cosines, $|AP|^2 = |OA|^2 + |OP|^2 - 2|OA||OP|\cos\theta \Rightarrow$ $120^2 = 40^2 + |OP|^2 - 2 \cdot 40|OP|\cos\theta \Rightarrow |OP|^2 - (80\cos\theta)|OP| - 12{,}800 = 0 \Rightarrow$

$$\begin{aligned}
|OP| &= \tfrac{1}{2}\left(80\cos\theta \pm \sqrt{6400\cos^2\theta + 51{,}200}\right) = 40\cos\theta \pm 40\sqrt{\cos^2\theta + 8} \\
&= 40\left(\cos\theta + \sqrt{8 + \cos^2\theta}\right) \text{ cm (since } |OP| > 0)
\end{aligned}$$

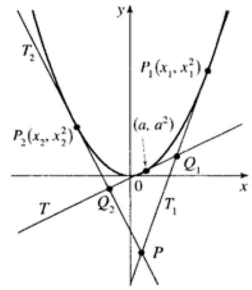
As a check, note that $|OP| = 160$ cm when $\theta = 0$ and $|OP| = 80\sqrt{2}$ cm when $\theta = \frac{\pi}{2}$.

(c) By part (b), the x-coordinate of P is given by $x = 40\left(\cos\theta + \sqrt{8 + \cos^2\theta}\right)$, so

$$\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} = 40\left(-\sin\theta - \frac{2\cos\theta\sin\theta}{2\sqrt{8 + \cos^2\theta}}\right) \cdot 12\pi = -480\pi\sin\theta\left(1 + \frac{\cos\theta}{\sqrt{8 + \cos^2\theta}}\right) \text{ cm/s}.$$

In particular, $dx/dt = 0$ cm/s when $\theta = 0$ and $dx/dt = -480\pi$ cm/s when $\theta = \frac{\pi}{2}$.

10. The equation of T_1 is $y - x_1^2 = 2x_1(x - x_1) = 2x_1 x - 2x_1^2$ or $y = 2x_1 x - x_1^2$. The equation of T_2 is $y = 2x_2 x - x_2^2$. Solving for the point of intersection, we get $2x(x_1 - x_2) = x_1^2 - x_2^2 \Rightarrow x = \frac{1}{2}(x_1 + x_2)$. Therefore, the coordinates of P are $\left(\frac{1}{2}(x_1 + x_2), x_1 x_2\right)$. So if the point of contact of T is (a, a^2), then Q_1 is $\left(\frac{1}{2}(a + x_1), a x_1\right)$ and Q_2 is $\left(\frac{1}{2}(a + x_2), a x_2\right)$.



Therefore, $|PQ_1|^2 = \frac{1}{4}(a - x_2)^2 + x_1^2(a - x_2)^2 = (a - x_2)^2\left(\frac{1}{4} + x_1^2\right)$ and $|PP_1|^2 = \frac{1}{4}(x_1 - x_2)^2 + x_1^2(x_1 - x_2)^2 = (x_1 - x_2)^2\left(\frac{1}{4} + x_1^2\right)$. So $\dfrac{|PQ_1|^2}{|PP_1|^2} = \dfrac{(a - x_2)^2}{(x_1 - x_2)^2}$, and similarly $\dfrac{|PQ_2|^2}{|PP_2|^2} = \dfrac{(x_1 - a)^2}{(x_1 - x_2)^2}$. Finally, $\dfrac{|PQ_1|}{|PP_1|} + \dfrac{|PQ_2|}{|PP_2|} = \dfrac{a - x_2}{x_1 - x_2} + \dfrac{x_1 - a}{x_1 - x_2} = 1$.

11. It seems from the figure that as P approaches the point $(0, 2)$ from the right, $x_T \to \infty$ and $y_T \to 2^+$. As P approaches the point $(3, 0)$ from the left, it appears that $x_T \to 3^+$ and $y_T \to \infty$. So we guess that $x_T \in (3, \infty)$ and $y_T \in (2, \infty)$. It is more difficult to estimate the range of values for x_N and y_N. We might perhaps guess that $x_N \in (0, 3)$, and $y_N \in (-\infty, 0)$ or $(-2, 0)$.

In order to actually solve the problem, we implicitly differentiate the equation of the ellipse to find the equation of the tangent line: $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1 \Rightarrow \dfrac{2x}{9} + \dfrac{2y}{4}y' = 0$, so $y' = -\dfrac{4}{9}\dfrac{x}{y}$. So at the point (x_0, y_0) on the ellipse, an equation of the tangent line is $y - y_0 = -\dfrac{4}{9}\dfrac{x_0}{y_0}(x - x_0)$ or $4x_0 x + 9y_0 y = 4x_0^2 + 9y_0^2$. This can be written as $\dfrac{x_0 x}{9} + \dfrac{y_0 y}{4} = \dfrac{x_0^2}{9} + \dfrac{y_0^2}{4} = 1$, because (x_0, y_0) lies on the ellipse. So an equation of the tangent line is $\dfrac{x_0 x}{9} + \dfrac{y_0 y}{4} = 1$.

Therefore, the x-intercept x_T for the tangent line is given by $\dfrac{x_0 x_T}{9} = 1 \Leftrightarrow x_T = \dfrac{9}{x_0}$, and the y-intercept y_T is given by $\dfrac{y_0 y_T}{4} = 1 \Leftrightarrow y_T = \dfrac{4}{y_0}$.

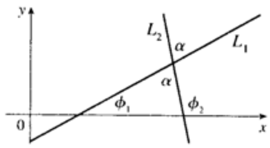
So as x_0 takes on all values in $(0, 3)$, x_T takes on all values in $(3, \infty)$, and as y_0 takes on all values in $(0, 2)$, y_T takes on all values in $(2, \infty)$.

At the point (x_0, y_0) on the ellipse, the slope of the normal line is $-\dfrac{1}{y'(x_0, y_0)} = \dfrac{9}{4}\dfrac{y_0}{x_0}$, and its equation is $y - y_0 = \dfrac{9}{4}\dfrac{y_0}{x_0}(x - x_0)$. So the x-intercept x_N for the normal line is given by $0 - y_0 = \dfrac{9}{4}\dfrac{y_0}{x_0}(x_N - x_0) \Rightarrow$ $x_N = -\dfrac{4x_0}{9} + x_0 = \dfrac{5x_0}{9}$, and the y-intercept y_N is given by $y_N - y_0 = \dfrac{9}{4}\dfrac{y_0}{x_0}(0 - x_0) \Rightarrow$ $y_N = -\dfrac{9y_0}{4} + y_0 = -\dfrac{5y_0}{4}$.

So as x_0 takes on all values in $(0, 3)$, x_N takes on all values in $\left(0, \frac{5}{3}\right)$, and as y_0 takes on all values in $(0, 2)$, y_N takes on all values in $\left(-\frac{5}{2}, 0\right)$.

12. $\displaystyle\lim_{x\to 0}\frac{\sin(3+x)^2-\sin 9}{x}=f'(3)$ where $f(x)=\sin x^2$. Now $f'(x)=\left(\cos x^2\right)(2x)$, so $f'(3)=6\cos 9$.

13. (a)



If the two lines L_1 and L_2 have slopes m_1 and m_2 and angles of inclination ϕ_1 and ϕ_2, then $m_1=\tan\phi_1$ and $m_2=\tan\phi_2$. The triangle in the figure shows that $\phi_1+\alpha+\left(180^\circ-\phi_2\right)=180^\circ$ and so $\alpha=\phi_2-\phi_1$. Therefore, using the identity for $\tan(x-y)$, we have $\tan\alpha=\tan\left(\phi_2-\phi_1\right)=\dfrac{\tan\phi_2-\tan\phi_1}{1+\tan\phi_2\tan\phi_1}$ and so $\tan\alpha=\dfrac{m_2-m_1}{1+m_1m_2}$.

(b) (i) The parabolas intersect when $x^2=(x-2)^2 \Rightarrow x=1$. If $y=x^2$, then $y'=2x$, so the slope of the tangent to $y=x^2$ at $(1,1)$ is $m_1=2(1)=2$. If $y=(x-2)^2$, then $y'=2(x-2)$, so the slope of the tangent to $y=(x-2)^2$ at $(1,1)$ is $m_2=2(1-2)=-2$. Therefore, $\tan\alpha=\dfrac{m_2-m_1}{1+m_1m_2}=\dfrac{-2-2}{1+2(-2)}=\frac{4}{3}$ and so $\alpha=\tan^{-1}\frac{4}{3}\approx 53^\circ$.

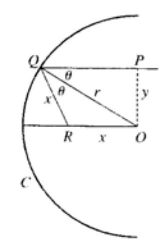
(ii) $x^2-y^2=3$ and $x^2-4x+y^2+3=0$ intersect when $x^2-4x+\left(x^2-3\right)+3=0 \Leftrightarrow 2x(x-2)=0 \Rightarrow x=0$ or 2, but 0 is extraneous. If $x=2$, then $y=\pm 1$. If $x^2-y^2=3$ then $2x-2yy'=0 \Rightarrow y'=x/y$ and $x^2-4x+y^2+3=0 \Rightarrow 2x-4+2yy'=0 \Rightarrow y'=\dfrac{2-x}{y}$. At $(2,1)$ the slopes are $m_1=2$ and $m_2=0$, so $\tan\alpha=\frac{0-2}{1+2\cdot 0}=-2 \Rightarrow \alpha\approx 117^\circ$. At $(2,-1)$ the slopes are $m_1=-2$ and $m_2=0$, so $\tan\alpha=\frac{0-(-2)}{1+(-2)(0)}=2 \Rightarrow \alpha\approx 63^\circ$.

14. $y^2=4px \Rightarrow 2yy'=4p \Rightarrow y'=2p/y \Rightarrow$ slope of tangent at $P\left(x_1,y_1\right)$ is $m_1=2p/y_1$. The slope of FP is $m_2=\dfrac{y_1}{x_1-p}$, so by the formula from Problem 13(a),

$$\begin{aligned}\tan\alpha&=\frac{y_1/\left(x_1-p\right)-2p/y_1}{1+\left(2p/y_1\right)\left[y_1/\left(x_1-p\right)\right]}\cdot\frac{y_1\left(x_1-p\right)}{y_1\left(x_1-p\right)}=\frac{y_1^2-2p\left(x_1-p\right)}{y_1\left(x_1-p\right)+2py_1}\\&=\frac{4px_1-2px_1+2p^2}{x_1y_1-py_1+2py_1}=\frac{2p\left(p+x_1\right)}{y_1\left(p+x_1\right)}=\frac{2p}{y_1}=\text{slope of tangent at }P=\tan\beta\end{aligned}$$

Since $0\le\alpha,\beta\le\frac{\pi}{2}$, this proves that $\alpha=\beta$.

15. Since $\angle ROQ=\angle OQP=\theta$, the triangle QOR is isosceles, so $|QR|=|RO|=x$. By the Law of Cosines, $x^2=x^2+r^2-2rx\cos\theta$. Hence, $2rx\cos\theta=r^2$, so $x=\dfrac{r^2}{2r\cos\theta}=\dfrac{r}{2\cos\theta}$. Note that as $y\to 0^+$, $\theta\to 0^+$ (since $\sin\theta=y/r$), and hence $x\to\dfrac{r}{2\cos 0}=\dfrac{r}{2}$. Thus, as P is taken closer and closer to the x-axis, the point R approaches the midpoint of the radius AO.



16. $$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f(x)-0}{g(x)-0}=\lim_{x\to 0}\frac{f(x)-f(0)}{g(x)-g(0)}=\lim_{x\to 0}\frac{\dfrac{f(x)-f(0)}{x-0}}{\dfrac{g(x)-g(0)}{x-0}}=\frac{\displaystyle\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}}{\displaystyle\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}}=\frac{f'(0)}{g'(0)}$$

17. $\displaystyle\lim_{x\to 0}\frac{\sin(a+2x)-2\sin(a+x)+\sin a}{x^2}$

$$=\lim_{x\to 0}\frac{\sin a\cos 2x+\cos a\sin 2x-2\sin a\cos x-2\cos a\sin x+\sin a}{x^2}$$

$$=\lim_{x\to 0}\frac{\sin a\,(\cos 2x-2\cos x+1)+\cos a\,(\sin 2x-2\sin x)}{x^2}$$

$$=\lim_{x\to 0}\frac{\sin a\,(2\cos^2 x-1-2\cos x+1)+\cos a\,(2\sin x\cos x-2\sin x)}{x^2}$$

$$=\lim_{x\to 0}\frac{\sin a\,(2\cos x)(\cos x-1)+\cos a\,(2\sin x)(\cos x-1)}{x^2}$$

$$=\lim_{x\to 0}\frac{2(\cos x-1)[\sin a\cos x+\cos a\sin x](\cos x+1)}{x^2(\cos x+1)}$$

$$=\lim_{x\to 0}\frac{-2\sin^2 x\,[\sin(a+x)]}{x^2(\cos x+1)}=-2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\frac{\sin(a+x)}{\cos x+1}=-2(1)^2\frac{\sin(a+0)}{\cos 0+1}=-\sin a$$

18. Suppose that $y=mx+c$ is a tangent line to the ellipse. Then it intersects the ellipse at only one point, so the discriminant of the equation $\dfrac{x^2}{a^2}+\dfrac{(mx+c)^2}{b^2}=1 \quad\Leftrightarrow\quad (b^2+a^2m^2)x^2+2mca^2x+a^2c^2-a^2b^2=0$ must be 0, that is,

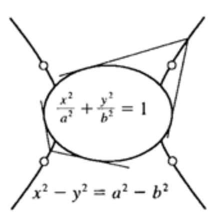
$$0=\left(2mca^2\right)^2-4\left(b^2+a^2m^2\right)\left(a^2c^2-a^2b^2\right)$$
$$=4a^4c^2m^2-4a^2b^2c^2+4a^2b^4-4a^4c^2m^2+4a^4b^2m^2=4a^2b^2\left(a^2m^2+b^2-c^2\right)$$

Therefore, $a^2m^2+b^2-c^2=0$.

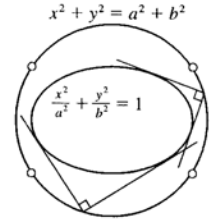
Now if a point (α,β) lies on the line $y=mx+c$, then $c=\beta-m\alpha$, so from above,

$0=a^2m^2+b^2-(\beta-m\alpha)^2=\left(a^2-\alpha^2\right)m^2+2\alpha\beta m+b^2-\beta^2 \quad\Leftrightarrow\quad m^2+\dfrac{2\alpha\beta}{a^2-\alpha^2}m+\dfrac{b^2-\beta^2}{a^2-\alpha^2}=0.$

(a) Suppose that the two tangent lines from the point (α,β) to the ellipse have slopes m and $\dfrac{1}{m}$. Then m and $\dfrac{1}{m}$ are roots of the equation $z^2+\dfrac{2\alpha\beta}{a^2-\alpha^2}z+\dfrac{b^2-\beta^2}{a^2-\alpha^2}=0$. This implies that $(z-m)\left(z-\dfrac{1}{m}\right)=0 \quad\Leftrightarrow$ $z^2-\left(m+\dfrac{1}{m}\right)z+m\left(\dfrac{1}{m}\right)=0$, so equating the constant terms in the two quadratic equations, we get $\dfrac{b^2-\beta^2}{a^2-\alpha^2}=m\left(\dfrac{1}{m}\right)=1$, and hence $b^2-\beta^2=a^2-\alpha^2$. So (α,β) lies on the hyperbola $x^2-y^2=a^2-b^2$.



(b) If the two tangent lines from the point (α, β) to the ellipse have slopes m and $-\frac{1}{m}$, then m and $-\frac{1}{m}$ are roots of the quadratic equation, and so $(z-m)\left(z+\frac{1}{m}\right)=0$, and equating the constant terms as in part (a), we get $\frac{b^2-\beta^2}{a^2-\alpha^2}=-1$, and hence $b^2-\beta^2=\alpha^2-a^2$. So the point (α, β) lies on the circle $x^2+y^2=a^2+b^2$.



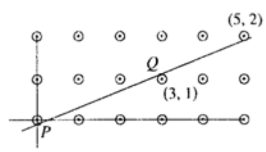
19. $y = x^4 - 2x^2 - x \quad \Rightarrow \quad y' = 4x^3 - 4x - 1$. The equation of the tangent line at $x = a$ is $y - (a^4 - 2a^2 - a) = (4a^3 - 4a - 1)(x - a)$ or $y = (4a^3 - 4a - 1)x + (-3a^4 + 2a^2)$ and similarly for $x = b$. So if at $x = a$ and $x = b$ we have the same tangent line, then $4a^3 - 4a - 1 = 4b^3 - 4b - 1$ and $-3a^4 + 2a^2 = -3b^4 + 2b^2$. The first equation gives $a^3 - b^3 = a - b \quad \Rightarrow \quad (a-b)(a^2 + ab + b^2) = (a - b)$. Assuming $a \neq b$, we have $1 = a^2 + ab + b^2$. The second equation gives $3(a^4 - b^4) = 2(a^2 - b^2) \quad \Rightarrow$ $3(a^2 - b^2)(a^2 + b^2) = 2(a^2 - b^2)$ which is true if $a = -b$. Substituting into $1 = a^2 + ab + b^2$ gives $1 = a^2 - a^2 + a^2 \quad \Rightarrow \quad a = \pm 1$ so that $a = 1$ and $b = -1$ or vice versa. Thus, the points $(1, -2)$ and $(-1, 0)$ have a common tangent line.

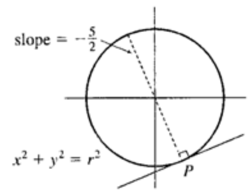
As long as there are only two such points, we are done. So we show that these are in fact the only two such points. Suppose that $a^2 - b^2 \neq 0$. Then $3(a^2 - b^2)(a^2 + b^2) = 2(a^2 - b^2)$ gives $3(a^2 + b^2) = 2$ or $a^2 + b^2 = \frac{2}{3}$. Thus, $ab = (a^2 + ab + b^2) - (a^2 + b^2) = 1 - \frac{2}{3} = \frac{1}{3}$, so $b = \frac{1}{3a}$. Hence, $a^2 + \frac{1}{9a^2} = \frac{2}{3}$, so $9a^4 + 1 = 6a^2 \quad \Rightarrow \quad 0 = 9a^4 - 6a^2 + 1 = (3a^2 - 1)^2$. So $3a^2 - 1 = 0 \quad \Rightarrow \quad a^2 = \frac{1}{3} \quad \Rightarrow$ $b^2 = \frac{1}{9a^2} = \frac{1}{3} = a^2$, contradicting our assumption that $a^2 \neq b^2$.

20. Suppose that the normal lines at the three points (a_1, a_1^2), (a_2, a_2^2), and (a_3, a_3^2) intersect at a common point. Now if one of the a_i is 0 (suppose $a_1 = 0$) then by symmetry $a_2 = -a_3$, so $a_1 + a_2 + a_3 = 0$. So we can assume that none of the a_i is 0.

The slope of the tangent line at (a_i, a_i^2) is $2a_i$, so the slope of the normal line is $-\frac{1}{2a_i}$ and its equation is $y - a_i^2 = -\frac{1}{2a_i}(x - a_i)$. We solve for the x-coordinate of the intersection of the normal lines from (a_1, a_1^2) and (a_2, a_2^2): $y = a_1^2 - \frac{1}{2a_1}(x - a_1) = a_2^2 - \frac{1}{2a_2}(x - a_2) \quad \Rightarrow \quad x\left(\frac{1}{2a_2} - \frac{1}{2a_1}\right) = a_2^2 - a_1^2 \quad \Rightarrow$ $x\left(\frac{a_1 - a_2}{2a_1 a_2}\right) = -(a_1 - a_2)(a_1 + a_2) \quad \Leftrightarrow \quad x = -2a_1 a_2 (a_1 + a_2)$ ($\bigstar$). Similarly, solving for the x-coordinate of the intersections of the normal lines from (a_1, a_1^2) and (a_3, a_3^2) gives $x = -2a_1 a_3 (a_1 + a_3)$ ($\dagger$). Equating ($\bigstar$) and ($\dagger$) gives $a_2(a_1 + a_2) = a_3(a_1 + a_3) \quad \Leftrightarrow \quad a_1(a_2 - a_3) = a_3^2 - a_2^2 = -(a_2 + a_3)(a_2 - a_3) \quad \Leftrightarrow$ $a_1 = -(a_2 + a_3) \quad \Leftrightarrow \quad a_1 + a_2 + a_3 = 0$.

21.



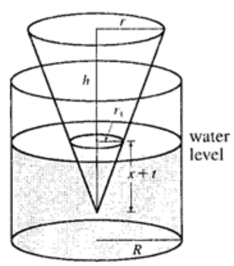


Because of the periodic nature of the lattice points, it suffices to consider the points in the 5×2 grid shown. We can see that the minimum value of r occurs when there is a line with slope $\frac{2}{5}$ which touches the circle centered at $(3, 1)$ and the circles centered at $(0, 0)$ and $(5, 2)$. To find P, the point at which the line is tangent to the circle at $(0, 0)$, we simultaneously solve $x^2 + y^2 = r^2$ and $y = -\frac{5}{2}x \Rightarrow x^2 + \frac{25}{4}x^2 = r^2 \Rightarrow x^2 = \frac{4}{29}r^2 \Rightarrow$ $x = \frac{2}{\sqrt{29}}r$, $y = -\frac{5}{\sqrt{29}}r$. To find Q, we either use symmetry or solve $(x-3)^2 + (y-1)^2 = r^2$ and $y - 1 = -\frac{5}{2}(x-3)$. As above, we get $x = 3 - \frac{2}{\sqrt{29}}r$, $y = 1 + \frac{5}{\sqrt{29}}r$. Now the slope of the line PQ is $\frac{2}{5}$, so

$$
\begin{aligned}
m_{PQ} = \frac{1 + \frac{5}{\sqrt{29}}r - \left(-\frac{5}{\sqrt{29}}r\right)}{3 - \frac{2}{\sqrt{29}}r - \frac{2}{\sqrt{29}}r} = \frac{1 + \frac{10}{\sqrt{29}}r}{3 - \frac{4}{\sqrt{29}}r} &= \frac{\sqrt{29} + 10r}{3\sqrt{29} - 4r} = \frac{2}{5} \\
\Rightarrow \qquad 5\sqrt{29} + 50r &= 6\sqrt{29} - 8r \\
\Leftrightarrow \qquad 58r &= \sqrt{29} \\
\Leftrightarrow \qquad r &= \tfrac{\sqrt{29}}{58}
\end{aligned}
$$

So the minimum value of r for which any line with slope $\frac{2}{5}$ intersects circles with radius r centered at the lattice points on the plane is $r = \frac{\sqrt{29}}{58} \approx 0.093$.

22.



Assume the axes of the cone and the cylinder are parallel. Let H denote the initial height of the water. When the cone has been dropping for t seconds, the water level has risen x meters, so the tip of the cone is $x + 1t$ meters below the water line. We want to find dx/dt when $x + t = h$ (when the cone is completely submerged). Using similar triangles, $\frac{r_1}{x+t} = \frac{r}{h} \Rightarrow r_1 = \frac{r}{h}(x+t)$.

volume of water and cone at time t	=	original volume of water	+	volume of submerged part of cone
$\pi R^2 (H + x)$	=	$\pi R^2 H$	+	$\frac{1}{3}\pi r_1^2 (x+t)$
$\pi R^2 H + \pi R^2 x$	=	$\pi R^2 H$	+	$\frac{1}{3}\pi \frac{r^2}{h^2}(x+t)^3$
$3h^2 R^2 x$	=	$r^2 (x+t)^3$		

Differentiating implicitly with respect to t gives us

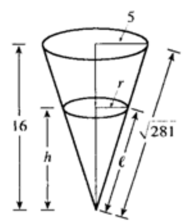
$$3h^2R^2\frac{dx}{dt} = r^2\left[3(x+t)^2\frac{dx}{dt} + 3(x+t)^2\frac{dt}{dt}\right] \quad\Rightarrow$$

$$\frac{dx}{dt} = \frac{r^2(x+t)^2}{h^2R^2 - r^2(x+t)^2} \quad\Rightarrow$$

$$\left.\frac{dx}{dt}\right|_{x+t=h} = \frac{r^2h^2}{h^2R^2 - r^2h^2} = \frac{r^2}{R^2 - r^2}$$

Thus, the water level is rising at a rate of $\dfrac{r^2}{R^2 - r^2}$ cm/s at the instant the cone is completely submerged.

23.



By similar triangles, $\dfrac{r}{5} = \dfrac{h}{16} \;\Rightarrow\; r = \dfrac{5h}{16}$. The volume of the cone is $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\dfrac{5h}{16}\right)^2 h = \dfrac{25\pi}{768}h^3$, so $\dfrac{dV}{dt} = \dfrac{25\pi}{256}h^2\dfrac{dh}{dt}$. Now the rate of change of the volume is also equal to the difference of what is being added ($2\text{ cm}^3/\text{min}$) and what is oozing out ($k\pi rl$, where πrl is the area of the cone and k is a proportionality constant). Thus, $\dfrac{dV}{dt} = 2 - k\pi rl$.

Equating the two expressions for $\dfrac{dV}{dt}$ and substituting $h = 10$, $\dfrac{dh}{dt} = -0.3$, $r = \dfrac{5(10)}{16} = \dfrac{25}{8}$, and $\dfrac{l}{\sqrt{281}} = \dfrac{10}{16}$ $\Leftrightarrow$ $l = \frac{5}{8}\sqrt{281}$, we get $\frac{25\pi}{256}(10)^2(-0.3) = 2 - k\pi\frac{25}{8}\cdot\frac{5}{8}\sqrt{281}$ $\Leftrightarrow$ $\dfrac{125k\pi\sqrt{281}}{64} = 2 + \dfrac{750\pi}{256}$. Solving for k gives us $k = \dfrac{256 + 375\pi}{250\pi\sqrt{281}}$. To maintain a certain height, the rate of oozing, $k\pi rl$, must equal the rate of the liquid being poured in, that is, $dV/dt = 0$. $k\pi rl = \dfrac{256 + 375\pi}{250\pi\sqrt{281}}\cdot\pi\cdot\dfrac{25}{8}\cdot\dfrac{5\sqrt{281}}{8} = \dfrac{256 + 375\pi}{128} \approx 11.204\text{ cm}^3/\text{min}$.

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4 Applications of Differentiation

4.1 Maximum and Minimum Values

1. A function f has an **absolute minimum** at $x = c$ if $f(c)$ is the smallest function value on the entire domain of f, whereas f has a **local minimum** at c if $f(c)$ is the smallest function value when x is near c.

2. (a) The Extreme Value Theorem

(b) See the Closed Interval Method.

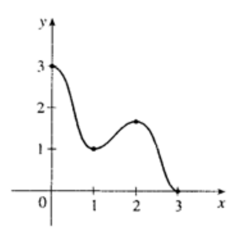
3. Absolute maximum at b; absolute minimum at d; local maxima at b, e; local minima at d, s; neither a maximum nor a minimum at a, c, r, and t.

4. Absolute maximum at e; absolute minimum at t; local maxima at c, e, s; local minima at b, c, d, r; neither a maximum nor a minimum at a.

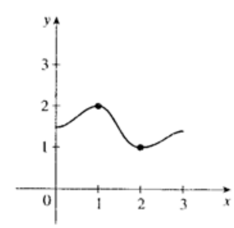
5. Absolute maximum value is $f(4) = 4$; absolute minimum value is $f(7) = 0$; local maximum values are $f(4) = 4$ and $f(6) = 3$; local minimum values are $f(2) = 1$ and $f(5) = 2$.

6. Absolute maximum value is $f(7) = 5$; absolute minimum value is $f(1) = 0$; local maximum values are $f(0) = 2$, $f(3) = 4$, and $f(5) = 3$; local minimum values are $f(1) = 0$, $f(4) = 2$, and $f(6) = 1$.

7.



8.



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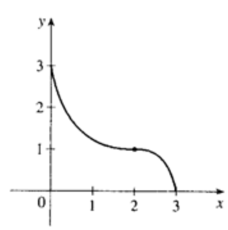

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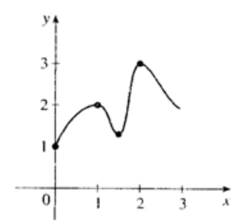
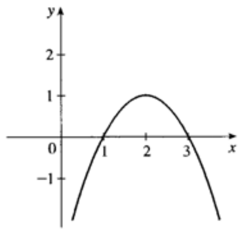
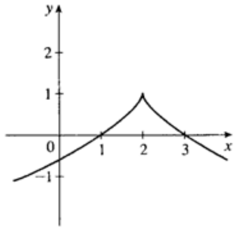


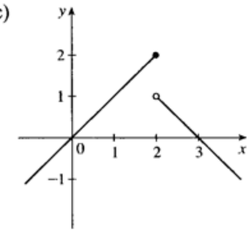
11. (a)



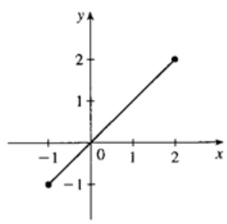
(b)



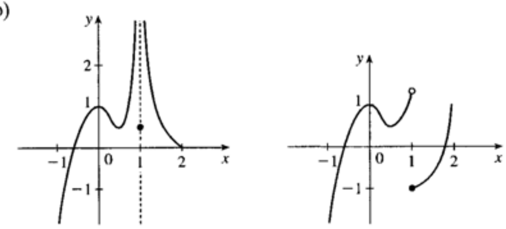
(c)



12. (a) Note that a local maximum cannot occur at an endpoint.

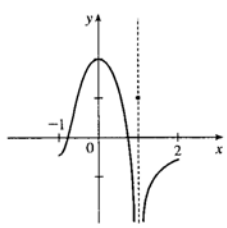


(b)

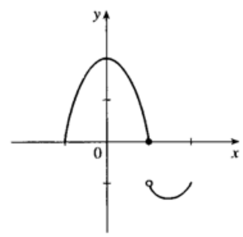


Note: By the Extreme Value Theorem, f must *not* be continuous.

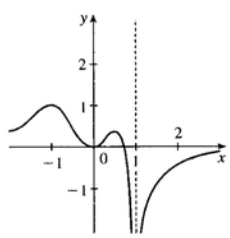
13. (a) *Note:* By the Extreme Value Theorem, f must *not* be continuous; because if it were, it would attain an absolute minimum.



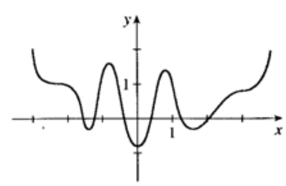
(b)



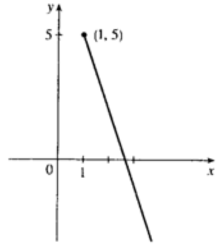
14. (a)



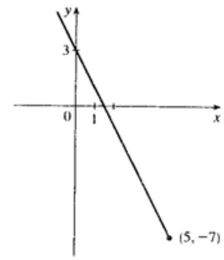
(b)



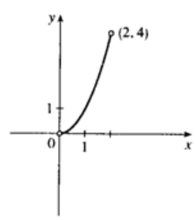
15. $f(x) = 8 - 3x$, $x \geq 1$. Absolute maximum $f(1) = 5$; no local maximum. No absolute or local minimum.



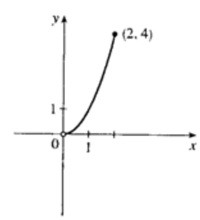
16. $f(x) = 3 - 2x$, $x \leq 5$. Absolute minimum $f(5) = -7$; no local minimum. No absolute or local maximum.



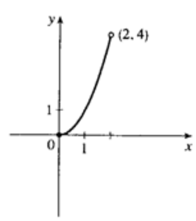
17. $f(x) = x^2$, $0 < x < 2$. No absolute or local maximum or minimum value.



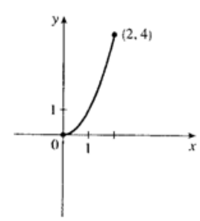
18. $f(x) = x^2$, $0 < x \leq 2$. Absolute maximum $f(2) = 4$; no local maximum. No absolute or local minimum.



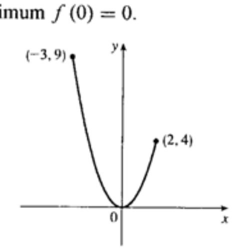
19. $f(x) = x^2$, $0 \leq x < 2$. Absolute minimum $f(0) = 0$; no local minimum. No absolute or local maximum.



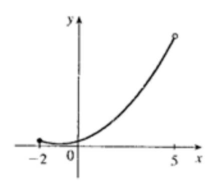
20. $f(x) = x^2$, $0 \leq x \leq 2$. Absolute maximum $f(2) = 4$. Absolute minimum $f(0) = 0$. No local maximum or minimum.



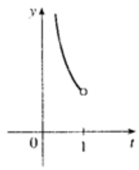
21. $f(x) = x^2$, $-3 \leq x \leq 2$. Absolute maximum $f(-3) = 9$. No local maximum. Absolute and local minimum $f(0) = 0$.



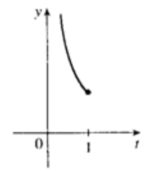
22. $f(x) = 1 + (x+1)^2$, $-2 \leq x < 5$. No absolute or local maximum. Absolute and local minimum $f(-1) = 1$.



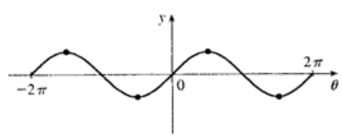
23. $f(t) = 1/t$, $0 < t < 1$. No maximum or minimum.



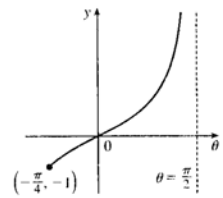
24. $f(t) = 1/t$, $0 < t \le 1$. Absolute minimum $f(1) = 1$; no local minimum. No local or absolute maximum.



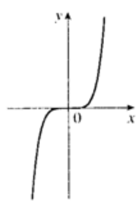
25. $f(\theta) = \sin\theta$, $-2\pi \le \theta \le 2\pi$. Absolute and local maxima $f\left(-\frac{3\pi}{2}\right) = f\left(\frac{\pi}{2}\right) = 1$. Absolute and local minima $f\left(-\frac{\pi}{2}\right) = f\left(\frac{3\pi}{2}\right) = -1$.



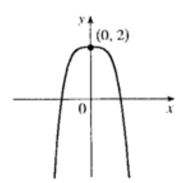
26. $f(\theta) = \tan\theta$, $-\frac{\pi}{4} \le \theta < \frac{\pi}{2}$. Absolute minimum $f\left(-\frac{\pi}{4}\right) = -1$; no local minimum. No absolute or local maximum.



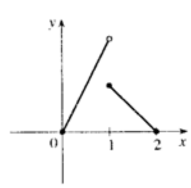
27. $f(x) = x^5$. No maximum or minimum.



28. $f(x) = 2 - x^4$. Local and absolute maximum $f(0) = 2$. No local or absolute minimum.

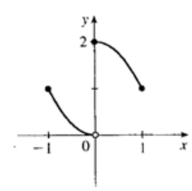


29. $f(x) = \begin{cases} 2x & \text{if } 0 \le x < 1 \\ 2 - x & \text{if } 1 \le x \le 2 \end{cases}$ Absolute minima $f(0) = f(2) = 0$; no local minimum. No absolute or local maximum.



30. $f(x) = \begin{cases} x^2 & \text{if } -1 \le x < 0 \\ 2 - x^2 & \text{if } 0 \le x \le 1 \end{cases}$

Absolute and local maximum $f(0) = 2$. No absolute or local minimum.



31. $f(x) = 5x^2 + 4x \Rightarrow f'(x) = 10x + 4$. $f'(x) = 0 \Rightarrow x = -\frac{2}{5}$, so $-\frac{2}{5}$ is the only critical number.

32. $f(x) = 5 + 6x - 2x^3 \Rightarrow f'(x) = 6 - 6x^2 = 6(1+x)(1-x)$. $f'(x) = 0 \Rightarrow x = \pm 1$, so ± 1 are the critical numbers.

33. $f(t) = 2t^3 + 3t^2 + 6t + 4 \Rightarrow f'(t) = 6t^2 + 6t + 6$. But $t^2 + t + 1 = 0$ has no real solution since $b^2 - 4ac = 1 - 4(1)(1) = -3 < 0$. No critical number.

34. $f(x) = 4x^3 - 9x^2 - 12x + 3 \Rightarrow f'(x) = 12x^2 - 18x - 12 = 6(2x^2 - 3x - 2) = 6(2x+1)(x-2)$.
$f'(x) = 0 \Rightarrow x = -\frac{1}{2}, 2$; so the critical numbers are $x = -\frac{1}{2}, 2$.

35. $s(t) = 2t^3 + 3t^2 - 6t + 4 \Rightarrow s'(t) = 6t^2 + 6t - 6 = 6(t^2 + t - 1)$. By the quadratic formula, the critical numbers are $t = \left(-1 \pm \sqrt{5}\right)/2$.

36. $s(t) = t^4 + 4t^3 + 2t^2 \Rightarrow s'(t) = 4t^3 + 12t^2 + 4t = 4t(t^2 + 3t + 1) = 0$ when $t = 0$ or $t^2 + 3t + 1 = 0$. By the quadratic formula, the critical numbers are $t = 0, \frac{-3 \pm \sqrt{5}}{2}$.

37. $f(r) = \dfrac{r}{r^2+1} \Rightarrow f'(r) = \dfrac{(r^2+1)1 - r(2r)}{(r^2+1)^2} = \dfrac{-r^2+1}{(r^2+1)^2} = 0 \Leftrightarrow r^2 = 1 \Leftrightarrow r = \pm 1$, so these are the critical numbers. Note that $f'(x)$ always exists since $r^2 + 1 \neq 0$.

38. $f(z) = \dfrac{z+1}{z^2+z+1} \Rightarrow f'(z) = \dfrac{1(z^2+z+1) - (z+1)(2z+1)}{(z^2+z+1)^2} = \dfrac{-z^2 - 2z}{(z^2+z+1)^2} = 0 \Leftrightarrow z(z+2) = 0$
$\Rightarrow z = 0, -2$ are the critical numbers. (Note that $z^2 + z + 1 \neq 0$ since the discriminant < 0.)

39. $g(x) = |2x+3| = \begin{cases} 2x+3 & \text{if } 2x+3 \geq 0 \\ -(2x+3) & \text{if } 2x+3 < 0 \end{cases} \Rightarrow g'(x) = \begin{cases} 2 & \text{if } x > -\frac{3}{2} \\ -2 & \text{if } x < -\frac{3}{2} \end{cases}$
$g'(x)$ is never 0, but $g'(x)$ does not exist for $x = -\frac{3}{2}$, so $-\frac{3}{2}$ is the only critical number.

40. $g(x) = x^{1/3} - x^{-2/3} \Rightarrow g'(x) = \frac{1}{3}x^{-2/3} + \frac{2}{3}x^{-5/3} = \frac{1}{3}x^{-5/3}(x+2) = \dfrac{x+2}{3x^{5/3}}$. $g'(-2) = 0$ and $g'(0)$ does not exist, but 0 is not in the domain of g, so the only critical number is -2.

41. $g(t) = 5t^{2/3} + t^{5/3} \Rightarrow g'(t) = \frac{10}{3}t^{-1/3} + \frac{5}{3}t^{2/3}$. $g'(0)$ does not exist, so $t = 0$ is a critical number.
$g'(t) = \frac{5}{3}t^{-1/3}(2+t) = 0 \Leftrightarrow t = -2$, so $t = -2$ is also a critical number.

42. $g(t) = \sqrt{t}(1-t) = t^{1/2} - t^{3/2} \Rightarrow g'(t) = \dfrac{1}{2\sqrt{t}} - \frac{3}{2}\sqrt{t}$. $g'(0)$ does not exist, so $t = 0$ is a critical number.
$0 = g'(t) = \dfrac{1-3t}{2\sqrt{t}} \Rightarrow t = \frac{1}{3}$, so $t = \frac{1}{3}$ is also a critical number.

43. $F(x) = x^{4/5}(x-4)^2 \Rightarrow$
$$F'(x) = \tfrac{4}{5}x^{-1/5}(x-4)^2 + 2x^{4/5}(x-4) = \tfrac{1}{5}x^{-1/5}(x-4)[4(x-4) + 10x]$$
$$= \frac{(x-4)(14x-16)}{5x^{1/5}} = \frac{2(x-4)(7x-8)}{5x^{1/5}} = 0 \text{ when } x = 4, \tfrac{8}{7}\text{; and } F'(0) \text{ does not exist.}$$
Critical numbers are $0, \frac{8}{7}, 4$.

44. $G(x) = \sqrt[3]{x^2 - x} \Rightarrow G'(x) = \frac{1}{3}(x^2 - x)^{-2/3}(2x - 1)$. $G'(x)$ does not exist when $x^2 - x = 0$ or $x = 0, 1$.
$G'(x) = 0 \Leftrightarrow 2x - 1 = 0 \Leftrightarrow x = \frac{1}{2}$. So the critical numbers are $x = 0, \frac{1}{2}, 1$.

45. $f(\theta) = \sin^2(2\theta) \Rightarrow f'(\theta) = 2\sin(2\theta)\cos(2\theta)(2) = 2(2\sin 2\theta\cos 2\theta) = 2[\sin(2\cdot 2\theta)] = 2\sin 4\theta = 0$ $\Leftrightarrow \sin 4\theta = 0 \Leftrightarrow 4\theta = n\pi$, n an integer. So $\theta = n\pi/4$ are the critical numbers.

46. $g(\theta) = \theta + \sin\theta \Rightarrow g'(\theta) = 1 + \cos\theta = 0 \Leftrightarrow \cos\theta = -1$. The critical numbers are $\theta = \pi + 2n\pi = (2n+1)\pi$, n an integer.

47. $f(x) = 3x^2 - 12x + 5$, $[0,3]$. $f'(x) = 6x - 12 = 0 \Leftrightarrow x = 2$. $f(0) = 5$, $f(2) = -7$, $f(3) = -4$. So $f(0) = 5$ is the absolute maximum and $f(2) = -7$ is the absolute minimum.

48. $f(x) = x^3 - 3x + 1$, $[0,3]$. $f'(x) = 3x^2 - 3 = 0 \Leftrightarrow x = \pm 1$, but -1 is not in $[0,3]$. $f(0) = 1$, $f(1) = -1$, $f(3) = 19$. So $f(3) = 19$ is the absolute maximum and $f(1) = -1$ is the absolute minimum.

49. $f(x) = 2x^3 + 3x^2 + 4$, $[-2,1]$. $f'(x) = 6x^2 + 6x = 6x(x+1) = 0 \Leftrightarrow x = -1, 0$. $f(-2) = 0$, $f(-1) = 5$, $f(0) = 4$, $f(1) = 9$. So $f(1) = 9$ is the absolute maximum and $f(-2) = 0$ is the absolute minimum.

50. $f(x) = 18x + 15x^2 - 4x^3$, $[-3,4]$. $f'(x) = 18 + 30x - 12x^2 = 6(3-x)(1+2x) = 0 \Leftrightarrow x = 3, -\frac{1}{2}$. $f(-3) = 189$, $f\left(-\frac{1}{2}\right) = -\frac{19}{4}$, $f(3) = 81$, $f(4) = 56$. So $f(-3) = 189$ is the absolute maximum and $f\left(-\frac{1}{2}\right) = -\frac{19}{4}$ is the absolute minimum.

51. $f(x) = x^4 - 4x^2 + 2$, $[-3,2]$. $f'(x) = 4x^3 - 8x = 4x\left(x^2 - 2\right) = 0 \Leftrightarrow x = 0, \pm\sqrt{2}$. $f(-3) = 47$, $f\left(-\sqrt{2}\right) = -2$, $f(0) = 2$, $f\left(\sqrt{2}\right) = -2$, $f(2) = 2$, so $f\left(\pm\sqrt{2}\right) = -2$ is the absolute minimum and $f(-3) = 47$ is the absolute maximum.

52. $f(x) = 3x^5 - 5x^3 - 1$, $[-2,2]$. $f'(x) = 15x^4 - 15x^2 = 15x^2(x+1)(x-1) = 0 \Leftrightarrow x = -1, 0, 1$. $f(-2) = -57$, $f(-1) = 1$, $f(0) = -1$, $f(1) = -3$, $f(2) = 55$. So $f(-2) = -57$ is the absolute minimum and $f(2) = 55$ is the absolute maximum.

53. $f(x) = x^2 + \dfrac{2}{x}$, $\left[\frac{1}{2}, 2\right]$. $f'(x) = 2x - \dfrac{2}{x^2} = 2\dfrac{x^3 - 1}{x^2} = 0 \Leftrightarrow x^3 - 1 = 0 \Leftrightarrow (x-1)\left(x^2 + x + 1\right) = 0$, but $x^2 + x + 1 \neq 0$, so $x = 1$. The denominator is 0 at $x = 0$, but not in the desired interval. $f\left(\frac{1}{2}\right) = \frac{17}{4}$, $f(1) = 3$, $f(2) = 5$. So $f(1) = 3$ is the absolute minimum and $f(2) = 5$ is the absolute maximum.

54. $f(x) = \sqrt{9 - x^2}$, $[-1,2]$. $f'(x) = -x\Big/\sqrt{9 - x^2} = 0 \Leftrightarrow x = 0$. $f(-1) = 2\sqrt{2}$, $f(0) = 3$, $f(2) = \sqrt{5}$. So $f(2) = \sqrt{5}$ is the absolute minimum and $f(0) = 3$ is the absolute maximum.

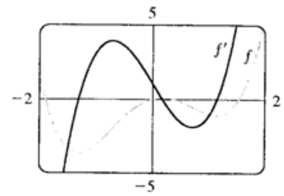
55. $f(x) = \dfrac{x}{x^2 + 1}$, $[0,2]$. $f'(x) = \dfrac{\left(x^2 + 1\right) - x(2x)}{\left(x^2 + 1\right)^2} = \dfrac{1 - x^2}{\left(x^2 + 1\right)^2} = 0 \Leftrightarrow x = \pm 1$, but -1 is not in $[0,2]$. $f(0) = 0$, $f(1) = \frac{1}{2}$, $f(2) = \frac{2}{5}$. So $f(1) = \frac{1}{2}$ is the absolute maximum and $f(0) = 0$ is the absolute minimum.

56. $f(x) = \dfrac{x}{x+1}$, $[1,2]$. $f'(x) = \dfrac{(x+1) - x}{(x+1)^2} = \dfrac{1}{(x+1)^2} \neq 0 \Rightarrow$ no critical number. $f(1) = \frac{1}{2}$ and $f(2) = \frac{2}{3}$, so $f(1) = \frac{1}{2}$ is the absolute minimum and $f(2) = \frac{2}{3}$ is the absolute maximum.

57. $f(x) = \sin x + \cos x$, $\left[0, \frac{\pi}{3}\right]$. $f'(x) = \cos x - \sin x = 0 \Leftrightarrow \sin x = \cos x \Rightarrow \dfrac{\sin x}{\cos x} = 1 \Rightarrow \tan x = 1$ $\Rightarrow x = \frac{\pi}{4}$. $f(0) = 1$, $f\left(\frac{\pi}{4}\right) = \sqrt{2} \approx 1.41$, $f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}+1}{2} \approx 1.37$. So $f(0) = 1$ is the absolute minimum and $f\left(\frac{\pi}{4}\right) = \sqrt{2}$ is the absolute maximum.

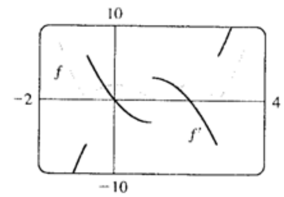
58. $f(x) = x - 2\cos x$, $[-\pi, \pi]$. $f'(x) = 1 + 2\sin x = 0 \Leftrightarrow \sin x = -\frac{1}{2} \Leftrightarrow x = -\frac{5\pi}{6}, -\frac{\pi}{6}$.
$f(-\pi) = 2 - \pi \approx -1.14$, $f\left(-\frac{5\pi}{6}\right) = \sqrt{3} - \frac{5\pi}{6} \approx -0.886$, $f\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} - \sqrt{3} \approx -2.26$,
$f(\pi) = \pi + 2 \approx 5.14$. So $f\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} - \sqrt{3}$ is the absolute minimum and $f(\pi) = \pi + 2$ is the absolute maximum.

59.



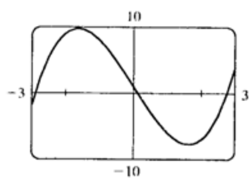
We see that $f'(x) = 0$ at about $x = -1.3, 0.2,$ and 1.1. Since f' exists everywhere, these are the only critical numbers.

60.



We see that $f'(x) = 0$ at about $x = 0.0$ and 2.0, and that $f'(x)$ does not exist at about $x = -0.7$, 1.0, and 2.7, so the critical numbers of f are about -0.7, 0.0, 1.0, 2.0, and 2.7.

61. (a)



From the graph, it appears that the absolute maximum value is about $f(-1.63) = 9.71$, and the absolute minimum value is about $f(1.63) = -7.71$. These values make sense because the graph is symmetric about the point $(0, 1)$. ($y = x^3 - 8x$ is symmetric about the origin.)

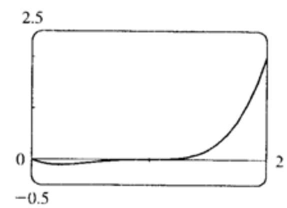
(b) $f(x) = x^3 - 8x + 1 \Rightarrow f'(x) = 3x^2 - 8$. So $f'(x) = 0 \Rightarrow x = \pm\sqrt{\frac{8}{3}}$.

$$f\left(\pm\sqrt{\tfrac{8}{3}}\right) = \left(\pm\sqrt{\tfrac{8}{3}}\right)^3 - 8\left(\pm\sqrt{\tfrac{8}{3}}\right) + 1 = \pm\tfrac{8}{3}\sqrt{\tfrac{8}{3}} \mp 8\sqrt{\tfrac{8}{3}} + 1$$

$$= -\tfrac{16}{3}\sqrt{\tfrac{8}{3}} + 1 = 1 - \tfrac{32\sqrt{6}}{9} \text{ (minimum) or } \tfrac{16}{3}\sqrt{\tfrac{8}{3}} + 1 = 1 + \tfrac{32\sqrt{6}}{9} \text{ (maximum)}$$

(From the graph, we see that the extreme values do not occur at the endpoints.)

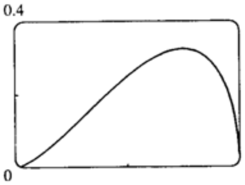
62. (a)



From the graph, it appears that the absolute maximum value is $f(2) = 2$, and that the absolute minimum value is about $f(0.25) = -0.11$.

(b) $f(x) = x^4 - 3x^3 + 3x^2 - x \Rightarrow f'(x) = 4x^3 - 9x^2 + 6x - 1 = (4x - 1)(x - 1)^2$. So $f'(x) = 0 \Rightarrow$ $x = \frac{1}{4}$ or $x = 1$. Now $f(1) = 1^4 - 3 \cdot 1^3 + 3 \cdot 1^2 - 1 = 0$ (not an extremum) and
$f\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^4 - 3\left(\frac{1}{4}\right)^3 + 3\left(\frac{1}{4}\right)^2 - \frac{1}{4} = -\frac{27}{256}$ (minimum). At the right endpoint we have
$f(2) = 2^4 - 3 \cdot 2^3 + 3 \cdot 2^2 - 2 = 2$ (maximum).

63. (a)



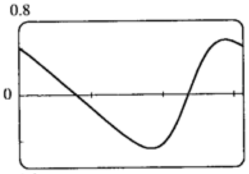
From the graph, it appears that the absolute maximum value is about $f(0.75) = 0.32$, and the absolute minimum value is $f(0) = f(1) = 0$, that is, at both endpoints.

(b) $f(x) = x\sqrt{x - x^2} \;\Rightarrow\; f'(x) = x \cdot \dfrac{1 - 2x}{2\sqrt{x - x^2}} + \sqrt{x - x^2} = \dfrac{(x - 2x^2) + (2x - 2x^2)}{2\sqrt{x - x^2}} = \dfrac{3x - 4x^2}{2\sqrt{x - x^2}}$.

So $f'(x) = 0 \;\Rightarrow\; 3x - 4x^2 = 0 \;\Rightarrow\; x = 0$ or $\frac{3}{4}$. $f(0) = f(1) = 0$ (minima), and

$f\left(\frac{3}{4}\right) = \frac{3}{4}\sqrt{\frac{3}{4} - \left(\frac{3}{4}\right)^2} = \frac{3\sqrt{3}}{16}$ (maximum).

64. (a)



From the graph, it appears that the absolute maximum value is about $f(5.76) = 0.58$, and the absolute minimum value is about $f(3.67) = -0.58$.

(b) $f(x) = \dfrac{\cos x}{2 + \sin x} \;\Rightarrow\; f'(x) = \dfrac{(2 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(2 + \sin x)^2} = \dfrac{-1 - 2\sin x}{(2 + \sin x)^2}$. So $f'(x) = 0 \;\Rightarrow$

$\sin x = -\frac{1}{2} \;\Rightarrow\; x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$. Now $f\left(\frac{7\pi}{6}\right) = \frac{-\sqrt{3}/2}{3/2} = -\frac{1}{\sqrt{3}}$ (minimum), and $f\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}/2}{3/2} = \frac{1}{\sqrt{3}}$ (maximum).

65. The density is defined as $\rho = \dfrac{\text{mass}}{\text{volume}} = \dfrac{1000}{V(T)}$ (in g/cm^3). But a critical point of ρ will also be a critical point of V $\left[\text{since } \dfrac{d\rho}{dT} = -1000V^{-2}\dfrac{dV}{dT} \text{ and } V \text{ is never } 0\right]$, and V is easier to differentiate than ρ. $V(T) = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3 \;\Rightarrow$

$V'(T) = -0.06426 + 0.0170086T - 0.0002037T^2$. Setting this equal to 0 and using the quadratic formula to find T, we get $T = \dfrac{-0.0170086 \pm \sqrt{0.0170086^2 - 4 \cdot 0.0002037 \cdot 0.06426}}{2(-0.0002037)} \approx 3.9665°$ or $79.5318°$. Since we are only interested in the region $0° \le T \le 30°$, we check the density ρ at the endpoints and at $3.9665°$:

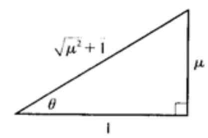
$\rho(0) \approx \dfrac{1000}{999.87} \approx 1.00013$; $\rho(30) \approx \dfrac{1000}{1003.7641} \approx 0.99625$; $\rho(3.9665) \approx \dfrac{1000}{999.7447} \approx 1.000255$. So water has its maximum density at about $3.9665°$C.

66. $F = \dfrac{\mu W}{\mu \sin\theta + \cos\theta} \;\Rightarrow\; \dfrac{dF}{d\theta} = \dfrac{(\mu\sin\theta + \cos\theta)(0) - \mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2} = \dfrac{-\mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2}$. So

$\dfrac{dF}{d\theta} = 0 \;\Rightarrow\; \mu\cos\theta - \sin\theta = 0 \;\Rightarrow\; \mu = \tan\theta$. Substituting $\tan\theta$ for μ in F gives us

$$F = \frac{(\tan\theta)\,W}{(\tan\theta)\sin\theta + \cos\theta} = \frac{W\tan\theta}{\dfrac{\sin^2\theta}{\cos\theta} + \cos\theta} = \frac{W\tan\theta\cos\theta}{\sin^2\theta + \cos^2\theta} = \frac{W\sin\theta}{1} = W\sin\theta.$$

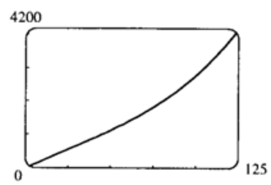
If $\tan\theta = \mu$, then $\sin\theta = \dfrac{\mu}{\sqrt{\mu^2+1}}$ (see the figure), so $F = \dfrac{\mu}{\sqrt{\mu^2+1}}W$. We compare this with the value of F at the endpoints: $F(0) = \mu W$ and $F\left(\frac{\pi}{2}\right) = W$.



Now because $\dfrac{\mu}{\sqrt{\mu^2+1}} \le 1$ and $\dfrac{\mu}{\sqrt{\mu^2+1}} \le \mu$, we have that $\dfrac{\mu}{\sqrt{\mu^2+1}}W$ is less than or equal to each of $F(0)$ and $F\left(\frac{\pi}{2}\right)$. Hence, $\dfrac{\mu}{\sqrt{\mu^2+1}}W$ is the absolute minimum value of $F(\theta)$, and it occurs when $\tan\theta = \mu$.

67. We apply the Closed Interval Method to the continuous function I on $[0, 10]$. Its derivative is $I'(t) = 0.00045225t^4 + 0.005752t^3 - 0.19683tss + 0.9196t - 0.6270$. Since I' exists for all t, the only critical numbers of I occur when $I'(t) = 0$. We use a root-finder on a computer algebra system (or a graphing device) to find that $I'(t) = 0$ when $t \approx -29.7186, 0.8231, 5.1309$, or 11.0459, but only the second and third roots lie in the interval $[0, 10]$. The values of I at these critical numbers are $I(0.8231) \approx 99.09$ and $I(5.1309) \approx 100.67$. The values of I at the endpoints of the interval are $I(0) = 99.33$ and $I(10) \approx 96.86$. Comparing these four numbers, we see that food was most expensive at $t \approx 5.1309$ (corresponding roughly to August, 1989) and cheapest at $t = 10$ (midyear 1994).

68. (a)



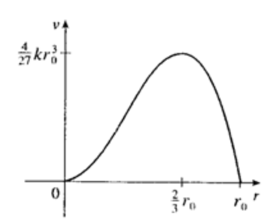
The equation of the graph in the figure is
$v(t) = 0.00146t^3 - 0.11553t^2$
$+24.98169t - 21.26872$

(b) $a(t) = v'(t) = 0.00438t^2 - 0.23106t + 24.98169$ $\Rightarrow$ $a'(t) = 0.00876t - 0.23106$. $a'(t) = 0$ $\Rightarrow$ $t_1 = \frac{0.23106}{0.00876} \approx 26.4$. $a(0) \approx 24.98$, $a(t_1) \approx 21.94$, and $a(126) \approx 64.60$. The maximum acceleration is about $64.6\ \text{ft/s}^2$ and the minimum acceleration is about $21.94\ \text{ft/s}^2$.

69. (a) $v(r) = k(r_0 - r)r^2 = kr_0r^2 - kr^3$ $\Rightarrow$ $v'(r) = 2kr_0r - 3kr^2$. $v'(r) = 0$ $\Rightarrow$ $kr(2r_0 - 3r) = 0$ $\Rightarrow r = 0$ or $\frac{2}{3}r_0$ (but 0 is not in the interval). Evaluating v at $\frac{1}{2}r_0$, $\frac{2}{3}r_0$, and r_0, we get $v\left(\frac{1}{2}r_0\right) = \frac{1}{8}kr_0^3$, $v\left(\frac{2}{3}r_0\right) = \frac{4}{27}kr_0^3$, and $v(r_0) = 0$. Since $\frac{4}{27} > \frac{1}{8}$, v attains its maximum value at $r = \frac{2}{3}r_0$. This supports the statement in the text.

(b) From part (a), the maximum value of v is $\frac{4}{27}kr_0^3$.

(c)



70. $g(x) = 2 + (x-5)^3$ $\Rightarrow$ $g'(x) = 3(x-5)^2$ $\Rightarrow$ $f'(5) = 0$, so 5 is a critical number. But $g(5) = 2$ and g takes on values > 2 and values < 2 in any open interval containing 5, so g does not have a local maximum or minimum at 5.

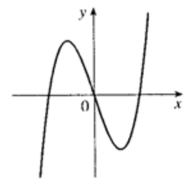
71. $f(x) = x^{101} + x^{51} + x + 1 \Rightarrow f'(x) = 101x^{100} + 51x^{50} + 1 \geq 1$ for all x, so $f'(x) = 0$ has no solution. Thus, $f(x)$ has no critical number, so $f(x)$ can have no local maximum or minimum.

72. Suppose that f has a minimum value at c, so $f(x) \geq f(c)$ for all x near c. Then $g(x) = -f(x) \leq -f(c)$ for all x near c, so $g(x)$ has a maximum value at c.

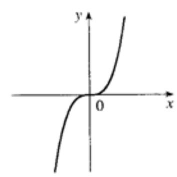
73. If f has a local minimum at c, then $g(x) = -f(x)$ has a local maximum at c, so $g'(c) = 0$ by the case of Fermat's Theorem proved in the text. Thus, $f'(c) = -g'(c) = 0$.

74. (a) $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$. So $f'(x) = 3ax^2 + 2bx + c$ is a quadratic and hence has either 2, 1, or 0 real roots, so $f(x)$ has either 2, 1 or 0 critical numbers.

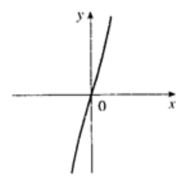
Case (i) (2 critical numbers): $f(x) = x^3 - 3x \Rightarrow f'(x) = 3x^2 - 3$, so $x = -1, 1$ are critical numbers.



Case (ii) (1 critical number): $f(x) = x^3 \Rightarrow f'(x) = 3x^2$, so $x = 0$ is the only critical number.



Case (iii) (no critical number): $f(x) = x^3 + 3x \Rightarrow f'(x) = 3x^2 + 3$, so there are no real roots.



(b) Since there are at most two critical numbers, it can have at most two local extreme values and by (i) this can occur. By (iii) it can have no local extreme value. However, if there is only one critical number, then there is no local extreme value.

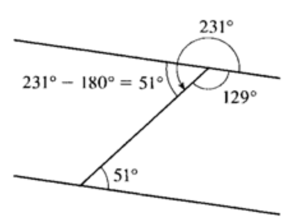
Applied Project □ The Calculus of Rainbows

1. From Snell's Law, we have $\sin\alpha = \frac{4}{3}\sin\beta \;\Rightarrow\; \dfrac{d}{d\alpha}(\sin\alpha) = \dfrac{4}{3}\dfrac{d}{d\alpha}(\sin\beta) \;\Rightarrow\; \cos\alpha = \dfrac{4}{3}\cos\beta\,\dfrac{d\beta}{d\alpha} \;\Rightarrow$ $\dfrac{d\beta}{d\alpha} = \dfrac{3\cos\alpha}{4\cos\beta}$. Now $D(\alpha) = \pi + 2\alpha - 4\beta \;\Rightarrow\; D'(\alpha) = 2 - 4\dfrac{d\beta}{d\alpha} = 2 - 3\dfrac{\cos\alpha}{\cos\beta}$. So $D'(\alpha) = 0 \;\Leftrightarrow$ $2\cos\beta = 3\cos\alpha$. Thus, $4\cos^2\beta = 9\cos^2\alpha \;\Rightarrow\; 4 - 4\sin^2\beta = 9 - 9\sin^2\alpha$. Since $3\sin\alpha = 4\sin\beta$, $\sin\beta = \frac{3}{4}\sin\alpha \;\Rightarrow\; 4 - 4\left(\frac{3}{4}\sin\alpha\right)^2 = 9 - 9\sin^2\alpha \;\Rightarrow\; \left(9 - \frac{9}{4}\right)\sin^2\alpha = 9 - 4 = 5 \;\Rightarrow\; \sin^2\alpha = \frac{20}{27} \;\Rightarrow$ $\sin\alpha = \sqrt{\frac{20}{27}}$. So $\alpha \approx 1.037$ radians or $59.4°$. We show that this α does give the minimum on $\left[0, \frac{\pi}{2}\right]$: When $\alpha = 0$, $\sin\alpha = \frac{4}{3}\sin\beta \;\Rightarrow\; \beta = 0$, or $D(0) = \pi \approx 3.14$. When $\alpha = \frac{\pi}{2}$, $1 = \sin\frac{\pi}{2} = \frac{4}{3}\sin\beta \;\Rightarrow\; \sin\beta = \frac{3}{4} \;\Rightarrow$ $\beta \approx 0.85$. So $D\left(\frac{\pi}{2}\right) \approx \pi + \pi - 4(0.85) \approx 2.88$. For $\alpha \approx 1.037$, $\sin\beta = \frac{3}{4}\sin\alpha = \frac{3}{4}\sqrt{\frac{20}{27}}$, so $\beta \approx 0.702 \;\Rightarrow$ $D(\alpha) \approx \pi + 2(1.036) - 4(0.702) \approx 2.41$. So the minimum occurs when $\alpha \approx 1.04$ radians or $59.4°$.

2. We repeat Problem 1 with k in place of $\frac{4}{3}$. So $\sin\alpha = k\sin\beta \;\Rightarrow\; \dfrac{d\beta}{d\alpha} = \dfrac{1}{k}\dfrac{\cos\alpha}{\cos\beta}$. Then $D'(\alpha) = 2 - 4\dfrac{d\beta}{d\alpha} = 2 - \dfrac{4}{k}\dfrac{\cos\alpha}{\cos\beta}$ and $D'(\alpha) = 0 \;\Leftrightarrow\; k\cos\beta = 2\cos\alpha$. So $k^2\cos^2\beta = 4\cos^2\alpha \;\Rightarrow$ $k^2 - k^2\sin^2\beta = 4 - 4\sin^2\alpha \;\Rightarrow\; k^2 - \sin^2\alpha = 4 - 4\sin^2\alpha \;\Rightarrow\; 3\sin^2\alpha = 4 - k^2 \;\Rightarrow\; \sin\alpha = \sqrt{\dfrac{4-k^2}{3}}$.

So for $k \approx 1.3318$ (red light) the minimum occurs when $\sin\alpha_1 = \sqrt{\dfrac{4 - (1.3318)^2}{3}}$ or $\alpha_1 \approx 1.038$ radians, so the rainbow angle is about $\pi - D(\alpha_1) \approx 42.3°$. For $k \approx 1.3435$ (violet light) the minimum occurs at $\alpha_2 \approx 1.026$ radians, and so the rainbow angle is about $\pi - D(\alpha_2) \approx 40.6°$.

3. At each reflection or refraction, the light is bent in a counterclockwise direction: the bend at A is $\alpha - \beta$, the bend at B is $\pi - 2\beta$, the bend at C is again $\pi - 2\beta$, and the bend at D is $\alpha - \beta$. So the total bend is $D(\alpha) = 2(\alpha - \beta) + 2(\pi - 2\beta) = 2\alpha - 6\beta + 2\pi$, as required. Now $\sin\alpha = k\sin\beta \;\Rightarrow\; \dfrac{d\beta}{d\alpha} = \dfrac{1}{k}\dfrac{\cos\alpha}{\cos\beta}$. So $D'(\alpha) = 2 - 6\dfrac{d\beta}{d\alpha} = 2 - \dfrac{6}{k}\dfrac{\cos\alpha}{\cos\beta}$ and $D'(\alpha) = 0 \;\Leftrightarrow\; k\cos\beta = 3\cos\alpha$. So $k^2\cos^2\beta = 9\cos^2\alpha \;\Rightarrow$ $k^2 - k^2\sin^2\beta = 9 - 9\sin^2\alpha \;\Rightarrow\; k^2 - \sin^2\alpha = 9 - 9\sin^2\alpha \;\Rightarrow\; \sin^2\alpha = \dfrac{9-k^2}{8} \;\Rightarrow\; \sin\alpha = \sqrt{\dfrac{9-k^2}{8}}$. If $k = \frac{4}{3}$, then the minimum occurs when $\sin\alpha_1 = \sqrt{\dfrac{9 - (4/3)^2}{8}}$ or $\alpha_1 \approx 1.254$ radians. Thus, the minimum *counterclockwise* rotation is $D(\alpha_1) \approx 231°$, which is equivalent to a *clockwise* rotation of $360° - 231° = 129°$ (see the figure). So the rainbow angle for the secondary rainbow is about $180° - 129° = 51°$, as required. In general, the rainbow angle for the secondary rainbow is $\pi - [2\pi - D(\alpha)] = D(\alpha) - \pi$.



4. In the primary rainbow, the rainbow angle gets smaller as k gets larger, as we found in Problem 2, so the colors appear from top to bottom in order of increasing k. But in the secondary rainbow, the rainbow angle gets larger as k gets larger. To see this, we find the minimum deviations for red light and for violet light in the secondary rainbow.

For $k \approx 1.3318$ (red light) the minimum occurs when $\sin\alpha_1 = \sqrt{\dfrac{9 - 1.3318^2}{8}}$ or $\alpha_1 \approx 1.254$ radians, and so the rainbow angle is $D(\alpha_1) - \pi \approx 50.6°$. For $k \approx 1.3435$ (violet light) the minimum occurs when $\sin\alpha_2 = \sqrt{\dfrac{9 - 1.3435^2}{8}}$ or $\alpha_2 \approx 1.248$ radians, and so the rainbow angle is $D(\alpha_2) - \pi \approx 53.6°$. Consequently, the rainbow angle is larger for colors with higher indices of refraction, and the colors appear from bottom to top in order of increasing k, the reverse of their order in the primary rainbow.

Note that our calculations above also explain why the secondary rainbow is more spread-out than the primary rainbow: in the primary rainbow, the difference between rainbow angles for red and violet light is about $1.7°$, whereas in the secondary rainbow it is about $3°$.

4.2 The Mean Value Theorem

1. $f(x) = x^2 - 4x + 1$, $[0, 4]$. Since f is a polynomial, it is continuous and differentiable on $\mathbb{R}$, so it is continuous on $[0, 4]$ and differentiable on $(0, 4)$. Also, $f(0) = 1 = f(4)$. $f'(c) = 0 \iff 2c - 4 = 0 \iff c = 2$, which is in the open interval $(0, 4)$, so $c = 2$ satisfies the conclusion of Rolle's Theorem.

2. $f(x) = x^3 - 3x^2 + 2x + 5$, $[0, 2]$. f is continuous on $[0, 2]$ and differentiable on $(0, 2)$. Also, $f(0) = 5 = f(2)$. $f'(c) = 0 \iff 3c^2 - 6c + 2 = 0 \iff c = \dfrac{6 \pm \sqrt{36 - 24}}{6} = 1 \pm \frac{1}{3}\sqrt{3}$, both in $(0, 2)$.

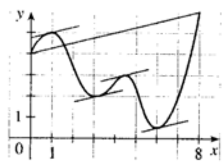
3. $f(x) = \sin 2\pi x$, $[-1, 1]$. f, being the composite of the sine function and the polynomial $2\pi x$, is continuous and differentiable on $\mathbb{R}$, so it is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. Also, $f(-1) = 0 = f(1)$. $f'(c) = 0 \iff 2\pi\cos 2\pi c = 0 \iff \cos 2\pi c = 0 \iff 2\pi c = \pm\frac{\pi}{2} + 2\pi n \iff c = \pm\frac{1}{4} + n$. If $n = 0$ or ± 1, then $c = \pm\frac{1}{4}, \pm\frac{3}{4}$ is in $(-1, 1)$.

4. $f(x) = x\sqrt{x + 6}$, $[-6, 0]$. f is continuous on its domain, $[-6, \infty)$, and differentiable on $(-6, \infty)$, so it is continuous on $[-6, 0]$ and differentiable on $(-6, 0)$. Also, $f(-6) = 0 = f(0)$. $f'(c) = 0 \iff \dfrac{3c + 12}{2\sqrt{c + 6}} = 0 \iff c = -4$, which is in $(-6, 0)$.

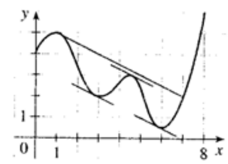
5. $f(x) = 1 - x^{2/3}$. $f(-1) = 1 - (-1)^{2/3} = 1 - 1 = 0 = f(1)$. $f'(x) = -\frac{2}{3}x^{-1/3}$, so $f'(c) = 0$ has no solution. This does not contradict Rolle's Theorem, since $f'(0)$ does not exist, and so f is not differentiable on $[-1, 1]$.

6. $f(x) = (x - 1)^{-2}$. $f(0) = (0 - 1)^{-2} = 1 = (2 - 1)^{-2} = f(2)$. $f'(x) = -2(x - 1)^{-3} \Rightarrow f'(x)$ is never 0. This does not contradict Rolle's Theorem since $f'(1)$ does not exist.

7. $\dfrac{f(8)-f(0)}{8-0}=\dfrac{6-4}{8}=\dfrac{1}{4}$. The values of c which satisfy $f'(c)=\frac{1}{4}$ seem to be about $c=0.8$, 3.2, 4.4, and 6.1.



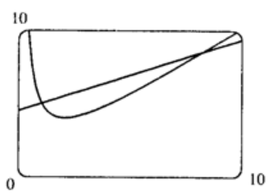
8. $\dfrac{f(7)-f(0)}{7-0}=\dfrac{2-4}{7}=-\dfrac{2}{7}$. The values of c which satisfy $f'(c)=-\frac{2}{7}$ seem to be about $c=1.2$, 2.8, 4.7, and 5.8.



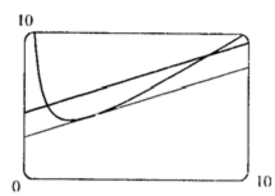
9. (a), (b) The equation of the secant line is

$$y-5=\frac{8.5-5}{8-1}(x-1)\quad\Leftrightarrow$$

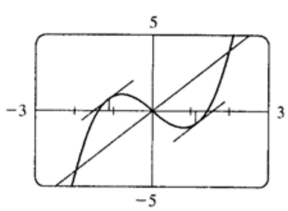
$$y=\tfrac{1}{2}x+\tfrac{9}{2}.$$



(c) $f(x)=x+4/x \quad\Rightarrow\quad f'(x)=1-4/x^2$. So $f'(c)=\frac{1}{2} \quad\Rightarrow\quad c^2=8 \quad\Rightarrow\quad c=2\sqrt{2}$, and $f(c)=2\sqrt{2}+\frac{4}{2\sqrt{2}}=3\sqrt{2}$. Thus, an equation of the tangent line is $y-3\sqrt{2}=\frac{1}{2}\left(x-2\sqrt{2}\right)$ $\Leftrightarrow\quad y=\frac{1}{2}x+2\sqrt{2}$.



10. (a)



It seems that the tangent lines are parallel to the secant at $x\approx\pm1.2$.

(b) The slope of the secant line is 2, and its equation is $y=2x$. $f(x)=x^3-2x \quad\Rightarrow\quad f'(x)=3x^2-2$, so we solve $f'(c)=2 \quad\Rightarrow\quad 3c^2=4 \quad\Rightarrow\quad c=\pm\frac{2\sqrt{3}}{3}\approx 1.155$. Our estimates were off by about 0.045 in each case.

11. $f(x)=3x^2+2x+5$, $[-1,1]$. f is continuous on $[-1,1]$ and differentiable on $(-1,1)$ since polynomials are continuous and differentiable on $\mathbb{R}$. $f'(c)=\dfrac{f(b)-f(a)}{b-a} \quad\Leftrightarrow\quad 6c+2=\dfrac{f(1)-f(-1)}{1-(-1)}=\dfrac{10-6}{2}=2 \quad\Leftrightarrow$ $6c=0 \quad\Leftrightarrow\quad c=0$, which is in $(-1,1)$.

12. $f(x)=x^3+x-1$, $[0,2]$. f is continuous on $[0,2]$ and differentiable on $(0,2)$. $f'(c)=\dfrac{f(2)-f(0)}{2-0} \quad\Leftrightarrow$ $3c^2+1=\dfrac{9-(-1)}{2} \quad\Leftrightarrow\quad 3c^2=5-1 \quad\Leftrightarrow\quad c^2=\frac{4}{3} \quad\Leftrightarrow\quad c=\pm\frac{2}{\sqrt{3}}$, but only $\frac{2}{\sqrt{3}}$ is in $(0,2)$.

13. $f(x) = \sqrt[3]{x}$, $[0, 1]$. f is continuous on $\mathbb{R}$ and differentiable on $(-\infty, 0) \cup (0, \infty)$, so f is continuous on $[0, 1]$ and differentiable on $(0, 1)$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \Leftrightarrow \dfrac{1}{3c^{2/3}} = \dfrac{f(1) - f(0)}{1 - 0} \Leftrightarrow \dfrac{1}{3c^{2/3}} = \dfrac{1 - 0}{1} \Leftrightarrow$ $3c^{2/3} = 1 \Leftrightarrow c^{2/3} = \frac{1}{3} \Leftrightarrow c^2 = \left(\frac{1}{3}\right)^3 = \frac{1}{27} \Leftrightarrow c = \pm\sqrt{\frac{1}{27}} = \pm\frac{\sqrt{3}}{9}$, but only $\frac{\sqrt{3}}{9}$ is in $(0, 1)$.

14. $f(x) = \dfrac{x}{x+2}$, $[1, 4]$. f is continuous on $[1, 4]$ and differentiable on $(1, 4)$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \Leftrightarrow$ $\dfrac{2}{(c+2)^2} = \dfrac{\frac{2}{3} - \frac{1}{3}}{4 - 1} \Leftrightarrow (c+2)^2 = 18 \Leftrightarrow c = -2 \pm 3\sqrt{2}$. $-2 + 3\sqrt{2} \approx 2.24$ is in $(1, 4)$.

15. $f(x) = |x - 1|$. $f(3) - f(0) = |3 - 1| - |0 - 1| = 1$. Since $f'(c) = -1$ if $c < 1$ and $f'(c) = 1$ if $c > 1$, $f'(c)(3 - 0) = \pm 3$ and so is never equal to 1. This does not contradict the Mean Value Theorem since $f'(1)$ does not exist.

16. $f(x) = \dfrac{x+1}{x-1}$. $f(2) - f(0) = 3 - (-1) = 4$. $f'(x) = \dfrac{1(x-1) - 1(x+1)}{(x+1)^2} = \dfrac{-2}{(x+1)^2}$. Since $f'(x) < 0$ for all x (except $x = -1$), $f'(c)(2 - 0)$ is always < 0 and hence cannot equal 4. This does not contradict the Mean Value Theorem since f is not continuous at $x = 1$.

17. $f(x) = x^5 + 10x + 3$. Since f is continuous and $f(-1) = -8$ and $f(0) = 3$, the equation $f(x) = 0$ has at least one root in $(-1, 0)$ by the Intermediate Value Theorem. Suppose that the equation has more than one root; say a and b are both roots with $a < b$. Then $f(a) = 0 = f(b)$ so by Rolle's Theorem $f'(x) = 5x^4 + 10 = 0$ has a root in (a, b). But this is impossible since clearly $f'(x) \geq 10 > 0$ for all real x.

18. $f(x) = 3x - 2 + \cos\left(\frac{\pi}{2}x\right)$. Since f is continuous and $f(0) = -1$ and $f(1) = 1$, the equation $f(x) = 0$ has at least one root in $(0, 1)$ by the Intermediate Value Theorem. Suppose it has more than one root; say $a < b$ are both roots. Then $f(a) = 0 = f(b)$, so by Rolle's Theorem, $f'(x) = 3 - \frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right) = 0$ has a root in (a, b). But this is impossible since $-\sin x \geq -1 \Rightarrow f'(x) \geq 3 - \frac{\pi}{2} > 0$ for all real x.

19. $f(x) = x^5 - 6x + c$. Suppose that $f(x) = 0$ has two roots a and b with $-1 \leq a < b \leq 1$. Then $f(a) = 0 = f(b)$, so by Rolle's Theorem there is a number d in (a, b) with $f'(d) = 0$. Now $0 = f'(d) = 5d^4 - 6 \Rightarrow d = \pm\sqrt[4]{\frac{6}{5}}$, which are both outside $[-1, 1]$ and hence outside (a, b). Thus, $f(x)$ can have at most one root in $[-1, 1]$.

20. $f(x) = x^4 + 4x + c$. Suppose that $f(x) = 0$ has three distinct real roots a, b, d where $a < b < d$. Then $f(a) = f(b) = f(d) = 0$. By Rolle's Theorem there are numbers c_1 and c_2 with $a < c_1 < b$ and $b < c_2 < d$ and $0 = f'(c_1) = f'(c_2)$, so $f'(x) = 0$ must have at least two real solutions. However $0 = f'(x) = 4x^3 + 4 = 4\left(x^3 + 1\right) = 4(x+1)\left(x^2 - x + 1\right)$ has as its only real solution $x = -1$. Thus, $f(x)$ can have at most two real roots.

21. (a) Suppose that a cubic polynomial $P(x)$ has roots $a_1 < a_2 < a_3 < a_4$, so $P(a_1) = P(a_2) = P(a_3) = P(a_4)$. By Rolle's Theorem there are numbers c_1, c_2, c_3 with $a_1 < c_1 < a_2$, $a_2 < c_2 < a_3$ and $a_3 < c_3 < a_4$ and $P'(c_1) = P'(c_2) = P'(c_3) = 0$. Thus, the second-degree polynomial $P'(x)$ has three distinct real roots, which is impossible.

(b) We prove by induction that a polynomial of degree n has at most n real roots. This is certainly true for $n = 1$. Suppose that the result is true for all polynomials of degree n and let $P(x)$ be a polynomial of degree $n + 1$. Suppose that $P(x)$ has more than $n + 1$ real roots, say $a_1 < a_2 < a_3 < \cdots < a_{n+1} < a_{n+2}$. Then $P(a_1) = P(a_2) = \cdots = P(a_{n+2}) = 0$. By Rolle's Theorem there are real numbers $c_1, \ldots, c_{n+1}$ with $a_1 < c_1 < a_2, \ldots, \Rightarrow a_{n+1} < c_{n+1} < a_{n+2}$ and $P'(c_1) = \cdots = P'(c_{n+1}) = 0$. Thus, the nth degree polynomial $P'(x)$ has at least $n + 1$ roots. This contradiction shows that $P(x)$ has at most $n + 1$ real roots.

22. (a) Suppose that $f(a) = f(b) = 0$ where $a < b$. By Rolle's Theorem applied to f on $[a, b]$ there is a number c such that $a < c < b$ and $f'(c) = 0$.

(b) Suppose that $f(a) = f(b) = f(c) = 0$ where $a < b < c$. By Rolle's Theorem applied to $f(x)$ on $[a, b]$ and $[b, c]$ there are numbers $a < d < b$ and $b < e < c$ with $f'(d) = 0$ and $f'(e) = 0$. By Rolle's Theorem applied to $f'(x)$ on $[d, e]$ there is a number g with $d < g < e$ such that $f''(g) = 0$.

(c) Suppose that f is n times differentiable on $\mathbb{R}$ and has $n + 1$ distinct real roots. Then $f^{(n)}$ has at least one real root.

23. By the Mean Value Theorem, $f(4) - f(1) = f'(c)(4 - 1)$ for some $c \in (1, 4)$. But for every $c \in (1, 4)$ we have $f'(c) \geq 2$. Putting $f'(c) \geq 2$ into the above equation and substituting $f(1) = 10$, we get $f(4) = f(1) + f'(c)(4 - 1) = 10 + 3f'(c) \geq 10 + 3 \cdot 2 = 16$. So the smallest possible value of $f(4)$ is 16.

24. By the Mean Value Theorem, $\dfrac{f(5) - f(2)}{5 - 2} = f'(c)$ for some $c \in (2, 5)$. Since $1 \leq f'(x) \leq 4$, we have $1 \leq \dfrac{f(5) - f(2)}{5 - 2} \leq 4$ or $1 \leq \dfrac{f(5) - f(2)}{3} \leq 4$ or $3 \leq f(5) - f(2) \leq 12$.

25. Suppose that such a function f exists. By the Mean Value Theorem there is a number $0 < c < 2$ with $f'(c) = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{5}{2}$. But this is impossible since $f'(x) \leq 2 < \frac{5}{2}$ for all x, so no such function can exist.

26. Let $h = f - g$. Then since f and g are continuous on $[a, b]$ and differentiable on (a, b), so is h, and thus h satisfies the assumptions of the Mean Value Theorem. Therefore, there is a number c with $a < c < b$ such that $h(b) = h(b) - h(a) = h'(c)(b - a)$. Since $h'(c) < 0$, $h'(c)(b - a) < 0$, so $f(b) - g(b) = h(b) < 0$ and hence $f(b) < g(b)$.

27. We use Exercise 26 with $f(x) = \sqrt{1 + x}$, $g(x) = 1 + \frac{1}{2}x$, and $a = 0$. Notice that $f(0) = 1 = g(0)$ and $f'(x) = \dfrac{1}{2\sqrt{1 + x}} < \dfrac{1}{2} = g'(x)$ for $x > 0$. So by Exercise 26, $f(b) < g(b) \Rightarrow \sqrt{1 + b} < 1 + \frac{1}{2}b$ for $b > 0$.
Another Method: Apply the Mean Value Theorem directly to either $f(x) = 1 + \frac{1}{2}x - \sqrt{1 + x}$ or $g(x) = \sqrt{1 + x}$ on $[0, b]$.

28. f satisfies the conditions for the Mean Value Theorem, so we use this theorem on the interval $[-b, b]$: $\dfrac{f(b) - f(-b)}{b - (-b)} = f'(c)$ for some $c \in (-b, b)$. But since f is odd, $f(-b) = -f(b)$. Substituting this into the above equation, we get $\dfrac{f(b) + f(b)}{2b} = f'(c) \Rightarrow \dfrac{f(b)}{b} = f'(c)$.

29. Let $f(x) = \sin x$ and let $b < a$. Then $f(x)$ is continuous on $[b, a]$ and differentiable on (b, a). By the Mean Value Theorem, there is a number $c \in (b, a)$ with $\sin a - \sin b = f(a) - f(b) = f'(c)(a - b) = (\cos c)(a - b)$. Thus, $|\sin a - \sin b| \leq |\cos c|\,|b - a| \leq |a - b|$. If $a < b$, then $|\sin a - \sin b| = |\sin b - \sin a| \leq |b - a| = |a - b|$. If $a = b$, both sides of the inequality are 0.

30. Suppose that $f'(x) = c$. Let $g(x) = cx$, so $g'(x) = c$. Then, by Corollary 7, $f(x) = g(x) + d$, where d is a constant, so $f(x) = cx + d$.

31. For $x > 0$, $f(x) = g(x)$, so $f'(x) = g'(x)$. For $x < 0$, $f'(x) = (1/x)' = -1/x^2$ and $g'(x) = (1 + 1/x)' = -1/x^2$, so again $f'(x) = g'(x)$. However, the domain of $g(x)$ is not an interval [it is $(-\infty, 0) \cup (0, \infty)$] so we cannot conclude that $f - g$ is constant (in fact it is not).

32. Let $v(t)$ be the velocity of the car t hours after 2:00 P.M. Then $\dfrac{v(1/6)-v(0)}{1/6-0} = \dfrac{50-30}{1/6} = 120$. By the Mean Value Theorem there is a number $0 < c < \frac{1}{6}$ with $v'(c) = 120$. Since $v'(t)$ is the acceleration at time t, the acceleration c hours after 2:00 P.M. is exactly 120 mi/h^2.

33. Let $g(t)$ and $h(t)$ be the position functions of the two runners and let $f(t) = g(t) - h(t)$. By hypothesis $f(0) = g(0) - h(0) = 0$ and $f(b) = g(b) - h(b) = 0$ where b is the finishing time. Then by Rolle's Theorem, there is a time $0 < c < b$ with $0 = f'(c) = g'(c) - h'(c)$. Hence, $g'(c) = h'(c)$, so at time c, both runners have the same velocity $g'(c) = h'(c)$.

34. Assume that f is differentiable (and hence continuous) on $\mathbb{R}$ and that $f'(x) \neq 1$ for all x. Suppose f has more than one fixed point. Then there are numbers a and b such that $a < b$, $f(a) = a$, and $f(b) = b$. Applying the Mean Value Theorem to the function f on $[a, b]$, we find that there is a number c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. But then $f'(c) = \dfrac{b-a}{b-a} = 1$, contradicting our assumption that $f'(x) \neq 1$ for every real number x. This shows that our supposition was wrong, that is, that f cannot have more than one fixed point.

4.3 How Derivatives Affect the Shape of a Graph

1. (a) f is increasing on $(0, 6)$ and $(8, 9)$.

(b) f is decreasing on $(6, 8)$.

(c) f is concave upward on $(2, 4)$ and $(7, 9)$.

(d) f is concave downward on $(0, 2)$ and $(4, 7)$.

(e) The points of inflection are $(2, 3)$, $(4, 4.5)$ and $(7, 4)$ (where the concavity changes).

2. (a) f is increasing on $(0, 2.8)$ and $(4, 5.5)$.

(b) f is decreasing on $(-1, 0)$, $(2.8, 4)$, $(5.5, 7)$, and $(7, 8)$.

(c) f is concave upward on $(-1, 2)$ and $(7, 8)$.

(d) f is concave downward on $(2, 4)$ and $(4, 7)$.

(e) The only point of inflection is $(2, 2)$. Note that 7 is not in the domain of this function.

3. (a) Use the Increasing/Decreasing (I/D) Test.

(b) Use the Concavity Test.

(c) At any value of x where the concavity changes, we have an inflection point at $(x, f(x))$.

4. (a) See the First Derivative Test.

(b) See the Second Derivative Test and the note that precedes Example 7.

5. (a) Since $f'(x) > 0$ on $(-\infty, 0)$ and $(3, \infty)$, f is increasing on the same intervals. $f'(x) < 0$ and f is decreasing on $(0, 3)$.

(b) Since $f'(x) = 0$ at $x = 0$ and f' changes from positive to negative there, f changes from increasing to decreasing and has a local maximum at $x = 0$. Since $f'(x) = 0$ at $x = 3$ and changes from negative to positive there, f changes from decreasing to increasing and has a local minimum at $x = 3$.

6. (a) $f'(x) > 0$ and f is increasing on $(-1, 3)$ and $(4, \infty)$. $f'(x) < 0$ and f is decreasing on $(-\infty, -1)$ and $(3, 4)$.

(b) Since $f'(x) = 0$ at $x = 3$ and f' changes from positive to negative there, f changes from increasing to decreasing and has a local maximum at $x = 3$. Since $f'(x) = 0$ at $x = -1$ and $x = 4$ and changes from negative to positive at both values, f changes from decreasing to increasing and has local minima at $x = -1$ and $x = 4$.

7. There is an inflection point at $x = 1$ because $f''(x)$ changes from negative to positive there, and one at $x = 7$ because $f''(x)$ changes from positive to negative there.

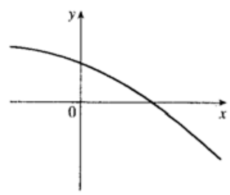
8. (a) f is increasing on the intervals where $f'(x) > 0$, namely, $(2, 4)$ and $(6, 9)$.

(b) f has a local maximum where it changes from increasing to decreasing, that is, where f' changes from positive to negative (at $x = 4$). Similarly, where f' changes from negative to positive, f has a local minimum (at $x = 2$ and at $x = 6$).

(c) When f' is increasing, its derivative f'' is positive and hence, f is concave upward. This happens on $(1, 3)$, $(5, 7)$, and $(8, 9)$. Similarly, f is concave downward when f' is decreasing — that is, on $(0, 1)$, $(3, 5)$, and $(7, 8)$.

(d) f has inflection points at $x = 1, 3, 5, 7$, and 8, since the direction of concavity changes at each of these values.

9. The function must be always decreasing and concave downward.



10. (a) The rate of increase of the population is initially very small, then gets larger until it reaches a maximum at about $t = 8$ hours, and decreases toward 0 as the population begins to level off.

(b) The rate of increase has its maximum value at $t = 8$ hours.

(c) The population function is concave upward on $(0, 8)$ and concave downward on $(8, 18)$.

(d) At $t = 8$, the population is about 350, so the inflection point is about $(8, 350)$.

11. (a) $f(x) = x^3 - 12x + 1 \quad \Rightarrow \quad f'(x) = 3x^2 - 12 = 3(x+2)(x-2)$. So $f'(x) > 0 \quad \Leftrightarrow \quad x > 2$ or $x < -2$ and $f'(x) < 0 \quad \Leftrightarrow \quad -2 < x < 2$. So f is increasing on $(-\infty, -2)$ and $(2, \infty)$ and decreasing on $(-2, 2)$.

(b) f changes from increasing to decreasing at $x = -2$ and from decreasing to increasing at $x = 2$. Thus, $f(-2) = 17$ is a local maximum and $f(2) = -15$ is a local minimum.

(c) $f''(x) = 6x$. $f''(x) > 0 \quad \Leftrightarrow \quad x > 0$ and $f''(x) < 0 \quad \Leftrightarrow \quad x < 0$. Thus, f is concave upward on $(0, \infty)$ and concave downward on $(-\infty, 0)$. There is an inflection point where the concavity changes, at $(0, f(0)) = (0, 1)$.

12. (a) $f(x) = 5 - 3x^2 + x^3 \quad \Rightarrow \quad f'(x) = -6x + 3x^2 = 3x(x-2)$. So $f'(x) > 0 \quad \Leftrightarrow \quad x < 0$ or $x > 2$ and $f'(x) < 0 \quad \Leftrightarrow \quad 0 < x < 2$. So f is increasing on $(-\infty, 0)$ and $(2, \infty)$ and f is decreasing on $(0, 2)$.

(b) f changes from increasing to decreasing at $x = 0$ and from decreasing to increasing at $x = 2$. Thus, $f(0) = 5$ is a local maximum and $f(2) = 1$ is a local minimum.

(c) $f''(x) = -6 + 6x = 6(x-1)$. $f''(x) > 0 \quad \Leftrightarrow \quad x > 1$ and $f''(x) < 0 \quad \Leftrightarrow \quad x < 1$. Thus, f is concave upward on $(1, \infty)$ and concave downward on $(-\infty, 1)$. There is an inflection point at $(1, 3)$.

13. (a) $f(x) = x^6 + 192x + 17 \Rightarrow f'(x) = 6x^5 + 192 = 6(x^5 + 32)$. So $f'(x) > 0 \Leftrightarrow x^5 > -32 \Leftrightarrow x > -2$ and $f'(x) < 0 \Leftrightarrow x < -2$. So f is increasing on $(-2, \infty)$ and decreasing on $(-\infty, -2)$.

(b) f changes from decreasing to increasing at its only critical number, $x = -2$. Thus, $f(-2) = -303$ is a local minimum.

(c) $f''(x) = 30x^4 \geq 0$ for all x, so the concavity of f doesn't change and there is no inflection point. f is concave upward on $(-\infty, \infty)$.

14. (a) $f(x) = x/(1+x)^2 \Rightarrow$

$$f'(x) = \frac{(1+x)^2(1) - (x)2(1+x)}{\left[(1+x)^2\right]^2} = \frac{(1+x)[(1+x) - 2x]}{(1+x)^4} = \frac{(1+x)(1-x)}{(1+x)^4} = \frac{1-x}{(1+x)^3}.$$ So

$f'(x) > 0 \Leftrightarrow -1 < x < 1$ and $f'(x) < 0 \Leftrightarrow x < -1$ or $x > 1$. So f is increasing on $(-1, 1)$ and f is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

(b) f changes from increasing to decreasing at $x = 1$. $x = -1$ is not in the domain of f. Thus, $f(1) = \frac{1}{4}$ is a local maximum.

(c) $$f''(x) = \frac{(1+x)^3(-1) - (1-x)3(1+x)^2}{\left[(1+x)^3\right]^2} = \frac{(1+x)^2[-1(1+x) - 3(1-x)]}{(1+x)^6} = \frac{2x-4}{(1+x)^4}.$$ $f''(x) > 0$ $\Leftrightarrow x > 2$ and $f''(x) < 0 \Leftrightarrow x < 2$ $(x \neq -1)$. Thus, f is concave upward on $(2, \infty)$ and f is concave downward on $(-\infty, -1)$ and $(-1, 2)$. There is an inflection point at $\left(2, \frac{2}{9}\right)$.

15. (a) $f(x) = x - 2\sin x$ on $(0, 3\pi) \Rightarrow f'(x) = 1 - 2\cos x$. $f'(x) > 0 \Leftrightarrow 1 - 2\cos x > 0 \Leftrightarrow \cos x < \frac{1}{2}$ $\Leftrightarrow \frac{\pi}{3} < x < \frac{5\pi}{3}$ or $\frac{7\pi}{3} < x < 3\pi$. $f'(x) < 0 \Leftrightarrow \cos x > \frac{1}{2} \Leftrightarrow 0 < x < \frac{\pi}{3}$ or $\frac{5\pi}{3} < x < \frac{7\pi}{3}$. So f is increasing on $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$ and $\left(\frac{7\pi}{3}, 3\pi\right)$, and f is decreasing on $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, \frac{7\pi}{3}\right)$.

(b) f changes from increasing to decreasing at $x = \frac{5\pi}{3}$, and from decreasing to increasing at $x = \frac{\pi}{3}$ and at $x = \frac{7\pi}{3}$. Thus, $f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3} \approx 6.97$ is a local maximum and $f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3} \approx -0.68$ and $f\left(\frac{7\pi}{3}\right) = \frac{7\pi}{3} - \sqrt{3} \approx 5.60$ are local minima.

(c) $f''(x) = 2\sin x > 0 \Leftrightarrow 0 < x < \pi$ and $2\pi < x < 3\pi$, $f''(x) < 0 \Leftrightarrow \pi < x < 2\pi$. Thus, f is concave upward on $(0, \pi)$ and $(2\pi, 3\pi)$, and f is concave downward on $(\pi, 2\pi)$. There are inflection points at (π, π) and $(2\pi, 2\pi)$.

16. (a) $f(x) = 2\sin x + \sin^2 x$ on $[0, 2\pi] \Rightarrow f'(x) = 2\cos x + 2\sin x\cos x = 2\cos x\,(1 + \sin x)$. $f'(x) > 0$ $\Leftrightarrow \cos x > 0$ (since $1 + \sin x \geq 0$ with equality when $x = \frac{3\pi}{2}$, a value where $\cos x = 0$) $\Leftrightarrow 0 \leq x < \frac{\pi}{2}$ or $\frac{3\pi}{2} < x \leq 2\pi$. So f is increasing on $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$, and f is decreasing on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

(b) Since f changes from increasing to decreasing at $x = \frac{\pi}{2}$, $f\left(\frac{\pi}{2}\right) = 3$ is a local maximum. Since f changes from decreasing to increasing at $x = \frac{3\pi}{2}$, $f\left(\frac{3\pi}{2}\right) = -1$ is a local minimum.

(c) $$\begin{aligned} f''(x) &= 2\cos x\,(\cos x) + (1 + \sin x)(-2\sin x) = 2\cos^2 x - 2\sin x - 2\sin^2 x \\ &= 2\left(1 - \sin^2 x\right) - 2\sin x - 2\sin^2 x = 2 - 2\sin x - 4\sin^2 x = 2(1 + \sin x)(1 - 2\sin x) \end{aligned}$$

$f''(x) > 0 \Leftrightarrow 1 - 2\sin x > 0 \Leftrightarrow \sin x < \frac{1}{2} \Leftrightarrow 0 \leq x < \frac{\pi}{6}$ or $\frac{5\pi}{6} < x \leq 2\pi$, so f is concave upward on $\left(0, \frac{\pi}{6}\right)$ and $\left(\frac{5\pi}{6}, 2\pi\right)$, and concave downward on $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$. There are inflection points at $\left(\frac{\pi}{6}, \frac{5}{4}\right)$ and $\left(\frac{5\pi}{6}, \frac{5}{4}\right)$.

17. $f(x) = x^5 - 5x + 3 \quad \Rightarrow \quad f'(x) = 5x^4 - 5 = 5(x^2+1)(x+1)(x-1)$.

First Derivative Test: $f'(x) < 0 \quad \Rightarrow \quad -1 < x < 1$ and $f'(x) > 0 \quad \Rightarrow \quad x > 1$ or $x < -1$. Since f' changes from positive to negative at $x = -1$, $f(-1) = 7$ is a local maximum; and since f' changes from negative to positive at $x = 1$, $f(1) = -1$ is a local minimum.

Second Derivative Test: $f''(x) = 20x^3$. $f'(x) = 0 \quad \Leftrightarrow \quad x = \pm 1$. $f''(-1) = -20 < 0 \quad \Rightarrow \quad f(-1) = 7$ is a local maximum. $f''(1) = 20 > 0 \quad \Rightarrow \quad f(1) = -1$ is a local minimum.

Preference: For this function, the two tests are equally easy.

18. $f(x) = \dfrac{x}{x^2+4} \quad \Rightarrow \quad f'(x) = \dfrac{(x^2+4)\cdot 1 - x(2x)}{(x^2+4)^2} = \dfrac{4-x^2}{(x^2+4)^2} = \dfrac{(2+x)(2-x)}{(x^2+4)^2}$.

First Derivative Test: $f'(x) > 0 \quad \Rightarrow \quad -2 < x < 2$ and $f'(x) < 0 \quad \Rightarrow \quad x > 2$ or $x < -2$. Since f' changes from positive to negative at $x = 2$, $f(2) = \frac{1}{4}$ is a local maximum; and since f' changes from negative to positive at $x = -2$, $f(-2) = -\frac{1}{4}$ is a local minimum.

Second Derivative Test:

$$f''(x) = \frac{(x^2+4)^2(-2x) - (4-x^2)\cdot 2(x^2+4)(2x)}{\left[(x^2+4)^2\right]^2} = \frac{-2x(x^2+4)\left[(x^2+4)+2(4-x^2)\right]}{(x^2+4)^4} = \frac{-2x(12-x^2)}{(x^2+4)^3}$$

$f'(x) = 0 \quad \Leftrightarrow \quad x = \pm 2$. $f''(-2) = \frac{1}{16} > 0 \quad \Rightarrow \quad f(-2) = -\frac{1}{4}$ is a local minimum. $f''(2) = -\frac{1}{16} < 0 \quad \Rightarrow$ $f(2) = \frac{1}{4}$ is a local maximum.

Preference: Since calculating the second derivative is fairly difficult, the First Derivative Test is easier to use for this function.

19. $f(x) = x + \sqrt{1-x} \quad \Rightarrow \quad f'(x) = 1 + \frac{1}{2}(1-x)^{-1/2}(-1) = 1 - \dfrac{1}{2\sqrt{1-x}}$. Note that f is defined for $1 - x \geq 0$, that is, for $x \leq 1$. $f'(x) = 0 \quad \Rightarrow \quad 2\sqrt{1-x} = 1 \quad \Rightarrow \quad \sqrt{1-x} = \frac{1}{2} \quad \Rightarrow \quad 1 - x = \frac{1}{4} \quad \Rightarrow \quad x = \frac{3}{4}$. f' does not exist at $x = 1$, but we can't have a local maximum or minimum at an endpoint.

First Derivative Test: $f'(x) > 0 \quad \Rightarrow \quad x < \frac{3}{4}$ and $f'(x) < 0 \quad \Rightarrow \quad \frac{3}{4} < x < 1$. Since f' changes from positive to negative at $x = \frac{3}{4}$, $f\left(\frac{3}{4}\right) = \frac{5}{4}$ is a local maximum.

Second Derivative Test: $f''(x) = -\frac{1}{2}\left(-\frac{1}{2}\right)(1-x)^{-3/2}(-1) = -\dfrac{1}{4\left(\sqrt{1-x}\right)^3}$. $f''\left(\frac{3}{4}\right) = -2 < 0 \quad \Rightarrow$ $f\left(\frac{3}{4}\right) = \frac{5}{4}$ is a local maximum.

Preference: The First Derivative Test may be slightly easier to apply in this case.

20. (a) $f(x) = x^4(x-1)^3 \quad \Rightarrow$

$f'(x) = x^4 \cdot 3(x-1)^2 + (x-1)^3 \cdot 4x^3 = x^3(x-1)^2[3x + 4(x-1)] = x^3(x-1)^2(7x-4)$

The critical numbers are 0, 1, and $\frac{4}{7}$.

(b) $f''(x) = 3x^2(x-1)^2(7x-4) + x^3 \cdot 2(x-1)(7x-4) + x^3(x-1)^2 \cdot 7$

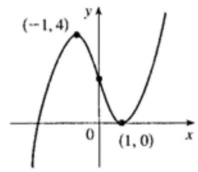
$= x^2(x-1)[3(x-1)(7x-4) + 2x(7x-4) + 7x(x-1)]$

Now $f''(0) = f''(1) = 0$, so the Second Derivative Test gives no information for $x = 0$ or $x = 1$.

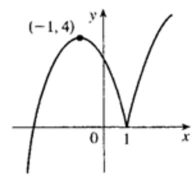
$f''\left(\frac{4}{7}\right) = \left(\frac{4}{7}\right)^2\left(\frac{4}{7}-1\right)\left[0 + 0 + 7\left(\frac{4}{7}\right)\left(\frac{4}{7}-1\right)\right] = \left(\frac{4}{7}\right)^2\left(-\frac{3}{7}\right)(4)\left(-\frac{3}{7}\right) > 0$, so there is a local minimum at $x = \frac{4}{7}$.

(c) f' is positive on $(-\infty, 0)$, negative on $\left(0, \frac{4}{7}\right)$, positive on $\left(\frac{4}{7}, 1\right)$, and positive on $(1, \infty)$. So f has a local maximum at $x = 0$, a local minimum at $x = \frac{4}{7}$, and no local maximum or minimum at $x = 1$.

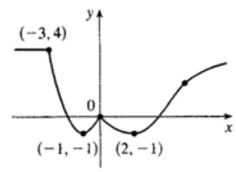
21. $f(-1) = 4$ and $f(1) = 0$ gives us two points to start with. $f'(-1) = f'(1) = 0 \Rightarrow$ horizontal tangents at $x = \pm 1$. $f'(x) < 0$ if $|x| < 1 \Rightarrow f$ is decreasing on $(-1, 1)$. $f'(x) > 0$ if $|x| > 1 \Rightarrow f$ is increasing on $(-\infty, -1)$ and $(1, \infty)$. $f''(x) < 0$ if $x < 0 \Rightarrow f$ is concave downward on $(-\infty, 0)$. $f''(x) > 0$ if $x > 0 \Rightarrow f$ is concave upward on $(0, \infty)$ and there is an inflection point at $x = 0$.



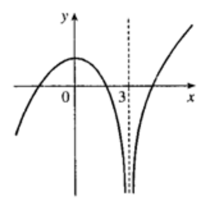
22. Since $f'(-1) = 0$ and $f'(1)$ does not exist, we have a horizontal tangent at $x = -1$ and a vertical tangent at $x = 1$. $f'(x) < 0$ if $|x| < 1 \Rightarrow f$ is decreasing on $(-1, 1)$, and $f'(x) > 0$ if $|x| > 1 \Rightarrow f$ is increasing on $(-\infty, -1)$ and $(1, \infty)$. $f''(x) < 0$ if $x \neq 1 \Rightarrow f$ is concave downward on $(-\infty, 1)$ and $(1, \infty)$.



23. Using the same principles as in Exercises 21 and 22, we sketch a possible graph.



24. $\lim_{x \to 3} f(x) = -\infty \Rightarrow$ there is a vertical asymptote at $x = 3$. $f'(0) = 0$ means that there is a horizontal tangent at $x = 0$. $f'(x) > 0$ if $x < 0$ or $x > 3$ and $f'(x) < 0$ if $0 < x < 3$ indicates that there is a local maximum at $x = 0$, since f is increasing on $(-\infty, 0)$ and decreasing on $(0, 3)$, and then increasing on $(3, \infty)$. $f''(x) < 0$ if $x \neq 3 \Rightarrow f$ is concave downward on $(-\infty, 3)$ and $(3, \infty)$.



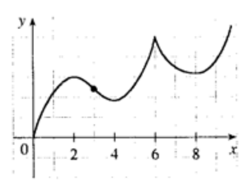
25. (a) f is increasing where f' is positive, that is, on $(0, 2)$, $(4, 6)$, and $(8, \infty)$; and decreasing where f' is negative, that is, on $(2, 4)$ and $(6, 8)$.

(b) f has local maxima where f' changes from positive to negative, at $x = 2$ and at $x = 6$, and local minima where f' changes from negative to positive, at $x = 4$ and at $x = 8$.

(c) f is concave upward where f' is increasing, that is, on $(3, 6)$ and $(6, \infty)$, and concave downward where f' is decreasing, that is, on $(0, 3)$.

(d) There is a point of inflection where f changes from being CD to being CU, that is, at $x = 3$.

(e)



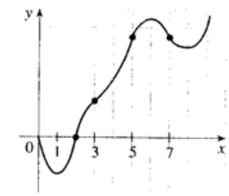
26. (a) f is increasing where f' is positive, on $(1, 6)$ and $(8, \infty)$, and decreasing where f' is negative, on $(0, 1)$ and $(6, 8)$.

(b) f has a local maximum where f' changes from positive to negative, at $x = 6$, and local minima where f' changes from negative to positive, at $x = 1$ and at $x = 8$.

(c) f is concave upward where f' is increasing, that is, on $(0, 2)$, $(3, 5)$, and $(7, \infty)$, and concave downward where f' is decreasing, that is, on $(2, 3)$ and $(5, 7)$.

(d) There are points of inflection where f changes its direction of concavity, at $x = 2$, $x = 3$, $x = 5$ and $x = 7$.

(e)

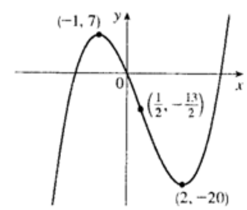


27. (a) $f(x) = 2x^3 - 3x^2 - 12x \quad \Rightarrow \quad f'(x) = 6x^2 - 6x - 12 = 6\left(x^2 - x - 2\right) = 6(x-2)(x+1)$. $f'(x) > 0$ $\Leftrightarrow \quad x < -1$ or $x > 2$ and $f'(x) < 0 \quad \Leftrightarrow \quad -1 < x < 2$. So f is increasing on $(-\infty, -1)$ and $(2, \infty)$, and f is decreasing on $(-1, 2)$.

(b) Since f changes from increasing to decreasing at $x = -1$, $f(-1) = 7$ is a local maximum value. Since f changes from decreasing to increasing at $x = 2$, $f(2) = -20$ is a local minimum value.

(c) $f''(x) = 6(2x - 1) \quad \Rightarrow \quad f''(x) > 0$ on $\left(\frac{1}{2}, \infty\right)$ and $f''(x) < 0$ on $\left(-\infty, \frac{1}{2}\right)$. So f is concave upward on $\left(\frac{1}{2}, \infty\right)$ and concave downward on $\left(-\infty, \frac{1}{2}\right)$. There is a change in concavity at $x = \frac{1}{2}$, and we have an inflection point at $\left(\frac{1}{2}, -\frac{13}{2}\right)$.

(d)

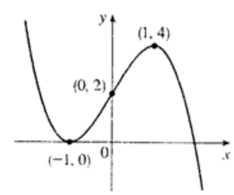


28. (a) $f(x) = 2 + 3x - x^3 \quad \Rightarrow \quad f'(x) = 3 - 3x^2 = -3\left(x^2 - 1\right) = -3(x+1)(x-1)$. $f'(x) > 0 \quad \Leftrightarrow$ $-1 < x < 1$ and $f'(x) < 0 \quad \Leftrightarrow \quad x < -1$ or $x > 1$. So f is increasing on $(-1, 1)$ and f is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

(b) $f(-1) = 0$ is a local minimum value and $f(1) = 4$ is a local maximum value.

(c) $f''(x) = -6x \quad \Rightarrow \quad f''(x) > 0$ on $(-\infty, 0)$ and $f''(x) < 0$ on $(0, \infty)$. So f is concave upward on $(-\infty, 0)$ and concave downward on $(0, \infty)$. There is an inflection point at $(0, 2)$.

(d)



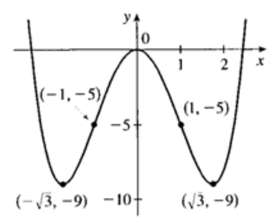
29. (a) $f(x) = x^4 - 6x^2 \Rightarrow f'(x) = 4x^3 - 12x = 4x(x^2 - 3) = 0$ when $x = 0, \pm\sqrt{3}$.

Interval	$4x$	$x^2 - 3$	$f'(x)$	f
$x < -\sqrt{3}$	$-$	$+$	$-$	decreasing on $(-\infty, -\sqrt{3})$
$-\sqrt{3} < x < 0$	$-$	$-$	$+$	increasing on $(-\sqrt{3}, 0)$
$0 < x < \sqrt{3}$	$+$	$-$	$-$	decreasing on $(0, \sqrt{3})$
$x > \sqrt{3}$	$+$	$+$	$+$	increasing on $(\sqrt{3}, \infty)$

(b) Local minima $f(\pm\sqrt{3}) = -9$, local maximum $f(0) = 0$

(c) $f''(x) = 12x^2 - 12 = 12(x^2 - 1) > 0 \Leftrightarrow x^2 > 1 \Leftrightarrow |x| > 1$ $\Leftrightarrow x > 1$ or $x < -1$, so f is CU on $(-\infty, -1)$, $(1, \infty)$ and CD on $(-1, 1)$. Inflection points at $(\pm 1, -5)$

(d)

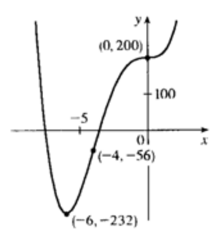


30. (a) $g(x) = 200 + 8x^3 + x^4 \Rightarrow g'(x) = 24x^2 + 4x^3 = 4x^2(6 + x) = 0$ when $x = -6$ and when $x = 0$. $g'(x) > 0 \Leftrightarrow x > -6$ $(x \neq 0)$ and $g'(x) < 0 \Leftrightarrow x < -6$, so g is decreasing on $(-\infty, -6)$ and g is increasing on $(-6, \infty)$, with a horizontal tangent at $x = 0$.

(b) $g(-6) = -232$ is a local minimum value. There is no local maximum value.

(c) $g''(x) = 48x + 12x^2 = 12x(4 + x) = 0$ when $x = -4$ and when $x = 0$. $g''(x) > 0 \Leftrightarrow x < -4$ or $x > 0$ and $g''(x) < 0 \Leftrightarrow$ $0 < x < 4$, so g is CU on $(-\infty, -4)$ and $(0, \infty)$, and g is CD on $(-4, 0)$. Inflection points at $(-4, -56)$ and $(0, 200)$

(d)



31. (a) $h(x) = 3x^5 - 5x^3 + 3 \Rightarrow h'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1) = 0$ when $x = 0, \pm 1$. $h'(x) > 0 \Leftrightarrow$ $x^2 > 1 \Leftrightarrow x > 1$ or $x < -1$, so h is increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 1)$.

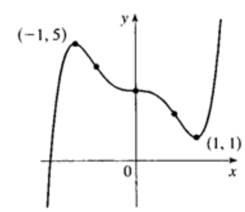
(b) Local maximum $h(-1) = 5$, local minimum $h(1) = 1$

(c) $h''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$

$$= 60x\left(x + \tfrac{1}{\sqrt{2}}\right)\left(x - \tfrac{1}{\sqrt{2}}\right) \Rightarrow$$

$h''(x) > 0$ when $x > \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}} < x < 0$, so h is CU on $\left(-\frac{1}{\sqrt{2}}, 0\right)$ and $\left(\frac{1}{\sqrt{2}}, \infty\right)$ and CD on $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ and $\left(0, \frac{1}{\sqrt{2}}\right)$. Inflection points at $\left(\pm\frac{1}{\sqrt{2}}, 3 \pm \frac{7}{8}\sqrt{2}\right)$ and $(0, 3)$

(d)



32. (a) $h(x) = (x^2 - 1)^3 \;\Rightarrow\; h'(x) = 6x(x^2 - 1)^2 \geq 0 \;\Leftrightarrow\; x > 0 \;(x \neq 1)$, so h is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

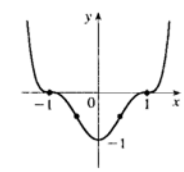
(b) $h(0) = -1$ is a local minimum.

(c) $h''(x) = 6(x^2 - 1)^2 + 24x^2(x^2 - 1) = 6(x^2 - 1)(5x^2 - 1)$. The roots ± 1 and $\pm\frac{1}{\sqrt{5}}$ divide $\mathbb{R}$ into five intervals.

Interval	$x^2 - 1$	$5x^2 - 1$	$h''(x)$	Concavity
$x < -1$	$+$	$+$	$+$	upward
$-1 < x < -\frac{1}{\sqrt{5}}$	$-$	$+$	$-$	downward
$-\frac{1}{\sqrt{5}} < x < \frac{1}{\sqrt{5}}$	$-$	$-$	$+$	upward
$\frac{1}{\sqrt{5}} < x < 1$	$-$	$+$	$-$	downward
$x > 1$	$+$	$+$	$+$	upward

From the table, we see that h is CU on $(-\infty, -1)$, $\left(-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ and $(1, \infty)$, and CD on $\left(-1, -\frac{1}{\sqrt{5}}\right)$ and $\left(\frac{1}{\sqrt{5}}, 1\right)$. Inflection points at $(\pm 1, 0)$ and $\left(\pm\frac{1}{\sqrt{5}}, -\frac{64}{125}\right)$

(d)



33. (a) $P(x) = x\sqrt{x^2 + 1} \;\Rightarrow$

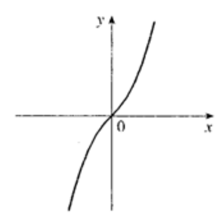
$$P'(x) = \sqrt{x^2 + 1} + \frac{x^2}{\sqrt{x^2 + 1}} = \frac{2x^2 + 1}{\sqrt{x^2 + 1}} > 0,$$ so P is increasing on $\mathbb{R}$.

(b) No maximum or minimum

(c) $$P''(x) = \frac{4x\sqrt{x^2 + 1} - (2x^2 + 1)\dfrac{x}{\sqrt{x^2 + 1}}}{x^2 + 1} = \frac{x(2x^2 + 3)}{(x^2 + 1)^{3/2}} > 0 \;\Leftrightarrow$$

$x > 0$ so P is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. IP at $(0, 0)$

(d)

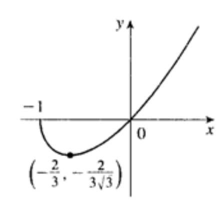


34. (a) $P(x) = x\sqrt{x + 1}$, domain $= [-1, \infty)$, $P'(x) = \sqrt{x + 1} + x\dfrac{1}{2\sqrt{x + 1}} = \dfrac{3x + 2}{2\sqrt{x + 1}} > 0$ when $x > -\frac{2}{3}$; $P'(x) < 0$ when $-1 < x < -\frac{2}{3}$, so P is increasing on $\left(-\frac{2}{3}, \infty\right)$ and decreasing on $\left(-1, -\frac{2}{3}\right)$.

(b) Local minimum $P\left(-\frac{2}{3}\right) = -\frac{2}{3\sqrt{3}}$

(c) $$P''(x) = \frac{3\left(2\sqrt{x + 1}\right) - (3x + 2)\left(1/\sqrt{x + 1}\right)}{4(x + 1)} = \frac{3x + 4}{4(x + 1)^{3/2}} > 0$$ when $x > -1$, so P is CU on $(-1, \infty)$. No inflection point

(d)

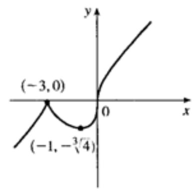


35. (a) $Q(x) = x^{1/3}(x+3)^{2/3} \;\Rightarrow\; Q'(x) = \frac{1}{3}x^{-2/3}(x+3)^{2/3} + x^{1/3}\left(\frac{2}{3}\right)(x+3)^{-1/3} = \dfrac{x+1}{x^{2/3}(x+3)^{1/3}}$. The critical numbers are -3, -1, and 0. Note that $x^{2/3} \geq 0$ for all x. So $Q'(x) > 0$ when $x < -3$ or $x > -1$ and $Q'(x) < 0$ when $-3 < x < -1 \;\Rightarrow\; Q$ is increasing on $(-\infty, -3)$ and $(-1, \infty)$ and decreasing on $(-3, -1)$.

(b) $Q(-3) = 0$ is a local maximum and $Q(-1) = -4^{1/3} \approx -1.6$ is a local minimum.

(c) $Q''(x) = -\dfrac{2}{x^{5/3}(x+3)^{4/3}} \;\Rightarrow\; Q''(x) > 0$ when $x < 0$, so Q is CU on $(-\infty, -3)$ and $(-3, 0)$ and CD on $(0, \infty)$. IP at $(0, 0)$

(d)

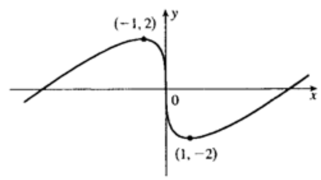


36. (a) $Q(x) = x - 3x^{1/3} \;\Rightarrow\; Q'(x) = 1 - \dfrac{1}{x^{2/3}} > 0 \;\Leftrightarrow\; x^{2/3} > 1 \;\Leftrightarrow\; x^2 > 1 \;\Leftrightarrow\; x < -1$ or $x > 1$, so Q is increasing on $(-\infty, -1)$, and $(1, \infty)$, and decreasing on $(-1, 1)$.

(b) $Q'(x) = 0 \;\Leftrightarrow\; x = \pm 1$; $Q(1) = -2$ is a local minimum, and $Q(-1) = 2$ is a local maximum.

(c) $Q''(x) = \frac{2}{3}x^{-5/3} > 0 \;\Leftrightarrow\; x > 0$, so Q is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. Inflection point at $(0, 0)$

(d)

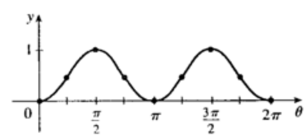


37. (a) $f(\theta) = \sin^2\theta \;\Rightarrow\; f'(\theta) = 2\sin\theta\cos\theta = \sin 2\theta > 0 \;\Leftrightarrow\; 2\theta \in (0, \pi) \cup (2\pi, 3\pi) \;\Leftrightarrow\; \theta \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right)$. So f is increasing on $\left(0, \frac{\pi}{2}\right)$ and $\left(\pi, \frac{3\pi}{2}\right)$, and decreasing on $\left(\frac{\pi}{2}, \pi\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$.

(b) Local minimum $f(\pi) = 0$, local maxima $f\left(\frac{\pi}{2}\right) = f\left(\frac{3\pi}{2}\right) = 1$

(c) $f''(\theta) = 2\cos 2\theta > 0 \;\Leftrightarrow\; 2\theta \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \cup \left(\frac{7\pi}{2}, 4\pi\right) \;\Leftrightarrow\; \theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right)$, so f is CU on these intervals and CD on $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$. IP at $\left(\frac{n\pi}{4}, \frac{1}{2}\right)$, $n = 1, 3, 5, 7$

(d)

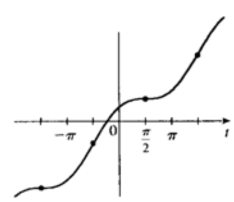


38. (a) $f(t) = t + \cos t \;\Rightarrow\; f'(t) = 1 - \sin t \geq 0$ for all t and $f'(t) = 0$ when $\sin t = 1 \;\Leftrightarrow\; t = -\frac{3\pi}{2}$ or $\frac{\pi}{2}$, so f is increasing on $(-\infty, \infty)$.

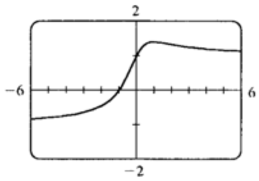
(b) No maximum or minimum

(c) $f''(t) = -\cos t > 0 \;\Leftrightarrow\; t \in \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, so f is CU on these intervals and CD on $\left(-2\pi, -\frac{3\pi}{2}\right)$, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and $\left(\frac{3\pi}{2}, 2\pi\right)$. Points of inflection at $t = \pm\frac{3\pi}{2}$, $\pm\frac{\pi}{2}$

(d)



39. (a)



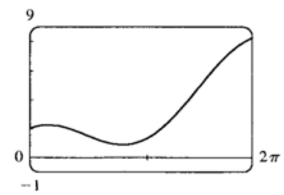
From the graph, we get us an estimate of $f(1) \approx 1.41$ as a local maximum, and no local minimum. $f(x) = \dfrac{x+1}{\sqrt{x^2+1}} \Rightarrow$ $f'(x) = \dfrac{1-x}{(x^2+1)^{3/2}}$. $f'(x) = 0 \Leftrightarrow x = 1$. $f(1) = \frac{2}{\sqrt{2}} = \sqrt{2}$ is the exact value.

(b) From the graph in part (a), f increases most rapidly somewhere between $x = -\frac{1}{2}$ and $x = -\frac{1}{4}$. To find the exact value, we need to find the maximum value of f', which we can do by finding the critical numbers of f'.

$f''(x) = \dfrac{2x^2 - 3x - 1}{(x^2+1)^{5/2}} = 0 \Leftrightarrow x = \dfrac{3 \pm \sqrt{17}}{4}$. $x = \dfrac{3+\sqrt{17}}{4}$ corresponds to the *minimum* value of f'.

The maximum value of f' is at $\left(\frac{3-\sqrt{17}}{4}, \sqrt{\frac{7}{6} - \frac{\sqrt{17}}{6}}\right) \approx (-0.28, 0.69)$.

40. (a)

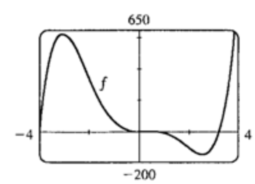


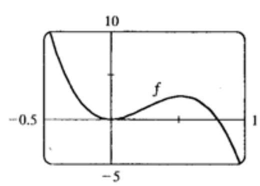
From the graph, we get estimates of $f(2.61) \approx 0.89$ as a local and absolute minimum, $f(0.53) \approx 2.26$ as a local maximum, and $f(2\pi) \approx 8.28$ as an absolute maximum. $f(x) = x + 2\cos x$ $(0 \le x \le 2\pi) \Rightarrow f'(x) = 1 - 2\sin x$. $f'(x) = 0 \Leftrightarrow$ $\sin x = \frac{1}{2} \Leftrightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$. $f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + \sqrt{3}$ is the exact value of the local maximum, $f\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} - \sqrt{3}$ is the exact value of the local and absolute minimum, and $f(2\pi) = 2\pi + 2$ is the exact value of the absolute maximum.

(b) From the graph in part (a), f increases most rapidly somewhere between $x = 4.5$ and $x = 5$. Now f increases most rapidly when $f'(x) = 1 - 2\sin x$ has its maximum value. $f''(x) = -2\cos x = 0 \Leftrightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$.

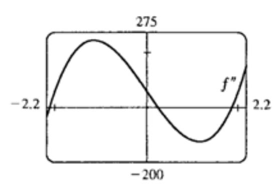
$f'(0) = f'(2\pi) = 1$, $f'\left(\frac{\pi}{2}\right) = -1$, and $f'\left(\frac{3\pi}{2}\right) = 3$. The maximum value of f' occurs at $\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$.

41. (a) From the graphs of $f(x) = 3x^5 - 40x^3 + 30x^2$, it seems that f is concave upward on $(-2, 0.25)$ and $(2, \infty)$, and concave downward on $(-\infty, -2)$ and $(0.25, 2)$, with inflection points at about $(-2, 350)$, $(0.25, 1)$, and $(2, -100)$.



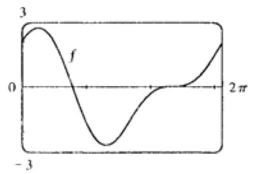


(b)



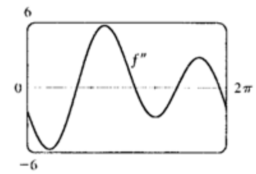
From the graph of $f''(x) = 60x^3 - 240x + 60$, it seems that f is CU on $(-2.1, 0.25)$ and $(1.9, \infty)$, and CD on $(-\infty, -2.1)$ and $(0.25, 2)$, with inflection points at about $(-2.1, 386)$, $(0.25, 1.3)$ and $(1.9, -87)$. (We have to check back on the graph of f to find the y-coordinates of the inflection points.)

42. (a)



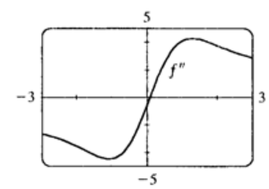
From the graphs of $f(x) = 2\cos x + \sin 2x$, it seems that f is CU on $(1.5, 3.5)$ and $(4.5, 6.0)$, and CD on $(0, 1.5)$, $(3.5, 4.5)$ and $(6.0, 2\pi)$, with inflection points at about $(1.5, 0.3)$, $(3.5, -1.3)$, $(4.5, 0.0)$ and $(6.0, 1.5)$.

(b)

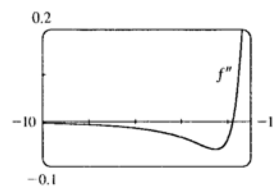


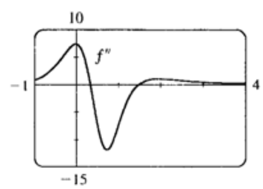
From the graph of $f''(x) = -2\cos x - 4\sin 2x$, it seems that f is CU on $(1.57, 3.39)$ and $(4.71, 6.03)$ and CD on $(0, 1.57)$, $(3.39, 4.71)$ and $(6.03, 2\pi)$, with inflection points at about $(1.57, 0.00)$, $(3.39, -1.45)$, $(4.71, 0.00)$ and $(6.03, 1.45)$.

43. In Maple, we define f and then use the command `plot(diff(diff(f,x),x),x=-3..3);`. In Mathematica, we define f and then use `Plot[Dt[Dt[f,x],x],{x,-3,3}]`. We see that $f'' > 0$ for $x > 0.1$ and $f'' < 0$ for $x < 0.1$. So f is concave up on $(0.1, \infty)$ and concave down on $(-\infty, 0.1)$.



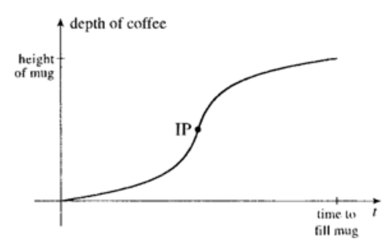
44. It appears that f'' is positive (and thus f is concave up) on $(-1.8, 0.3)$ and $(1.5, \infty)$ and negative (so f is concave down) on $(-\infty, -1.8)$ and $(0.3, 1.5)$.





45. Most students learn more in the third hour of studying than in the eighth hour, so $K(3) - K(2)$ is larger than $K(8) - K(7)$. In other words, as you begin studying for a test, the rate of knowledge gain is large and then starts to taper off, so $K'(t)$ decreases and the graph of K is concave downward.

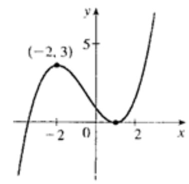
46. At first the depth increases slowly because the base of the mug is wide. But as the mug narrows, the coffee rises more quickly. Thus, the depth d increases at an increasing rate and its graph is concave upward. The rate of increase of d has a maximum when the mug is narrowed, that is, when the mug is half full. It is there that the inflection point occurs. Then the rate of increase of d starts to decrease as the mug widens and the graph becomes concave down.



47. $f(x) = \tan x - x \quad \Rightarrow \quad f'(x) = \sec^2 x - 1 > 0$ for $0 < x < \frac{\pi}{2}$ since $\sec^2 x > 1$ for $0 < x < \frac{\pi}{2}$. So f is increasing on $\left(0, \frac{\pi}{2}\right)$. Thus, $f(x) > f(0) = 0$ for $0 < x < \frac{\pi}{2} \quad \Rightarrow \quad \tan x - x > 0 \quad \Rightarrow \quad \tan x > x$ for $0 < x < \frac{\pi}{2}$.

48. Let $f(x) = 2\sqrt{x} - 3 + 1/x$. Then $f'(x) = 1/\sqrt{x} - 1/x^2 > 0$ for $x > 1$ since for $x > 1$, $x^2 > x > \sqrt{x}$. Hence, f is increasing, so for $x > 1$, $f(x) > f(1) = 0$ or $2\sqrt{x} - 3 + 1/x > 0$ for $x > 1$. Hence, $2\sqrt{x} > 3 - 1/x$ for $x > 1$.

49. $f(x) = ax^3 + bx^2 + cx + d \quad \Rightarrow \quad f(1) = a + b + c + d = 0$ and $f(-2) = -8a + 4b - 2c + d = 3$. Also $f'(1) = 3a + 2b + c = 0$ and $f'(-2) = 12a - 4b + c = 0$ by Fermat's Theorem. Solving these four equations, we get $a = \frac{2}{9}$, $b = \frac{1}{3}$, $c = -\frac{4}{3}$, $d = \frac{7}{9}$, so the function is $f(x) = \frac{1}{9}\left(2x^3 + 3x^2 - 12x + 7\right)$.



50. $f(x) = x^3 + ax^2 + bx + 2 \quad \Rightarrow \quad f'(x) = 3x^2 + 2ax + b$. If $x = -3$ is an extremum, then $f'(-3) = 27 - 6a + b = 0 \quad \Leftrightarrow \quad b = 6a - 27$. If $x = -1$ is an extremum, then $f'(-1) = 3 - 2a + b = 0$ $\Leftrightarrow \quad b = 2a - 3$. So $b = 2a - 3$ and $b = 6a - 27 \quad \Rightarrow \quad b = 9$, $a = 6$. Then $f'(x) = 3x^2 + 12x + 9 = 3(x+1)(x+3)$ and the First Derivative Test shows that f has a local maximum when $x = -3$ and a local minimum when $x = -1$.

51. We will make use of the converse of the Concavity Test; that is, if f is concave upward on I, then $f'' > 0$ on I. If f and g are CU on I, then $f'' > 0$ and $g'' > 0$ on I, so $(f+g)'' = f'' + g'' > 0$ on $I \quad \Rightarrow \quad f + g$ is CU on I.

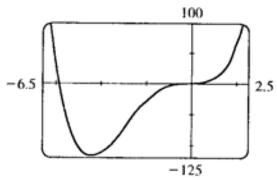
52. Since f is positive and CU on I, $f > 0$ and $f'' > 0$ on I. So $g(x) = [f(x)]^2 \quad \Rightarrow \quad g' = 2ff' \quad \Rightarrow$ $g'' = 2f'f' + 2ff'' = 2(f')^2 + 2ff'' > 0 \quad \Rightarrow \quad g$ is CU on I.

53. Since f and g are positive, increasing, and CU on I, we have $f > 0$, $f' > 0$, $f'' > 0$, $g > 0$, $g' > 0$, $g'' > 0$ on I. Then $(fg)' = f'g + fg' \quad \Rightarrow \quad (fg)'' = f''g + 2f'g' + fg'' > 0 \quad \Rightarrow \quad fg$ is CU on I.

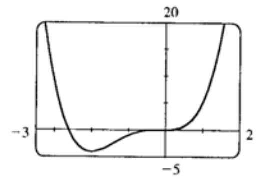
54. Since f and g are CU on $(-\infty, \infty)$, $f'' > 0$ and $g'' > 0$ on $(-\infty, \infty)$.
$h(x) = f(g(x)) \quad \Rightarrow \quad h'(x) = f'(g(x))\,g'(x) \quad \Rightarrow$
$h''(x) = f''(g(x))\,g'(x)\,g'(x) + f'(g(x))\,g''(x) = f''(g(x))\left[g'(x)\right]^2 + f'(g(x))\,g''(x) > 0$ if $f' > 0$.
So h is CU if f is increasing.

55. Let the cubic function be $f(x) = ax^3 + bx^2 + cx + d \quad \Rightarrow \quad f'(x) = 3ax^2 + 2bx + c \quad \Rightarrow$ $f''(x) = 6ax + 2b$. So f is CU when $6ax + 2b > 0 \quad \Leftrightarrow \quad x > -b/(3a)$, and CD when $x < -b/(3a)$, and so the only point of inflection occurs when $x = -b/(3a)$. If the graph has three x-intercepts x_1, x_2 and x_3, then the equation of $f(x)$ must factor as
$f(x) = a(x - x_1)(x - x_2)(x - x_3) = a\left[x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3\right]$.
So $b = -a(x_1 + x_2 + x_3)$. Hence, the x-coordinate of the point of inflection is
$-\dfrac{b}{3a} = -\dfrac{-a(x_1 + x_2 + x_3)}{3a} = \dfrac{x_1 + x_2 + x_3}{3}$.

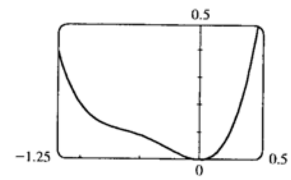
56. $P(x) = x^4 + cx^3 + x^2 \quad \Rightarrow \quad P'(x) = 4x^3 + 3cx^2 + 2x \quad \Rightarrow \quad P''(x) = 12x^2 + 6cx + 2$. The graph of $P''(x)$ is a parabola. If $P''(x)$ has two roots, then it changes sign twice and so has two inflection points. This happens when the discriminant of $P''(x)$ is positive, that is, $(6c)^2 - 4 \cdot 12 \cdot 2 > 0 \quad \Leftrightarrow \quad 36c^2 - 96 > 0 \quad \Leftrightarrow$ $|c| > \frac{2\sqrt{6}}{3}$. If $36c^2 - 96 = 0 \quad \Leftrightarrow \quad c = \pm\frac{2\sqrt{6}}{3}$, $P''(x)$ is 0 at one point, but there is still no inflection point since $P''(x)$ never changes sign, and if $36c^2 - 96 < 0 \quad \Leftrightarrow \quad |c| < \frac{2\sqrt{6}}{3}$, then $P''(x)$ never changes sign, and so there is no inflection point.



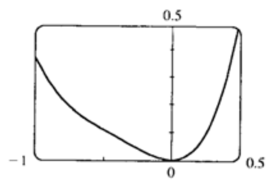
$c = 6$



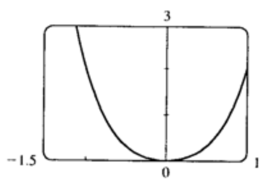
$c = 3$



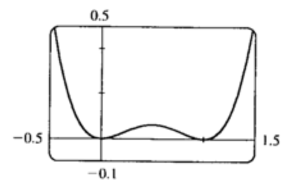
$c = 1.8$



$c = \frac{2\sqrt{6}}{3}$



$c = 0$



$c = -2$

For large positive c, the graph of f has two inflection points and a large dip to the left of the y-axis. As c decreases, the graph of f becomes flatter for $x < 0$, and eventually the dip rises above the x-axis, and then disappears entirely, along with the inflection points. As c continues to decrease, the dip and the inflection points reappear, to the right of the origin.

57. By hypothesis $g = f'$ is differentiable on an open interval containing c. Since $(c, f(c))$ is a point of inflection, the concavity changes at $x = c$, so $f'(x)$ changes signs at $x = c$. Hence, by the First Derivative Test, f' has a local extremum at $x = c$. Thus, by Fermat's Theorem $f''(c) = 0$.

58. $f(x) = x^4 \quad \Rightarrow \quad f'(x) = 4x^3 \quad \Rightarrow \quad f''(x) = 12x^2 \quad \Rightarrow \quad f''(0) = 0$. For $x < 0$, $f''(x) > 0$, so f is CU on $(-\infty, 0)$; for $x > 0$, $f''(x) > 0$, so f is also CU on $(0, \infty)$. Since f does not change concavity at 0, $(0, 0)$ is not an inflection point.

59. Using the fact that $|x| = \sqrt{x^2}$, we have that $g(x) = x\sqrt{x^2} \quad \Rightarrow \quad g'(x) = \sqrt{x^2} + \sqrt{x^2} = 2\sqrt{x^2} = 2|x| \quad \Rightarrow$ $g''(x) = 2x\left(x^2\right)^{-1/2} = \dfrac{2x}{|x|} < 0$ for $x < 0$ and $g''(x) > 0$ for $x > 0$, so $(0, 0)$ is an inflection point. But $g''(0)$ does not exist.

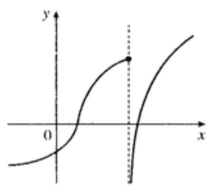
60. There must exist some interval containing c on which f''' is positive, since $f'''(c)$ is positive and f''' is continuous. On this interval, f'' is increasing (since f''' is positive), so $f'' = \left(f'\right)'$ changes from negative to positive at c. So by the First Derivative Test, f' has a local minimum at $x = c$ and thus cannot change sign there, so f has no maximum or minimum at c. But since f'' changes from negative to positive at c, f has a point of inflection at c (it changes from concave down to concave up).

4.4 Limits at Infinity; Horizontal Asymptotes

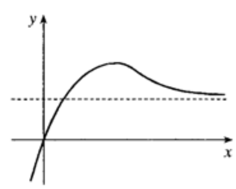
1. (a) As x becomes large, the values of $f(x)$ approach 5.

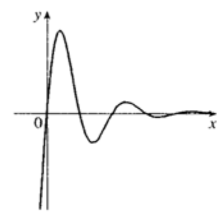
(b) As x becomes large negative, the values of $f(x)$ approach 3.

2. (a) The graph of a function can intersect a vertical asymptote in the sense that it can meet but not cross it.

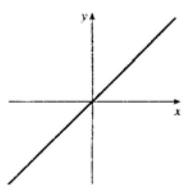


The graph of a function can intersect a horizontal asymptote. It can even intersect its horizontal asymptote an infinite number of times.

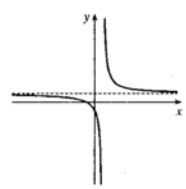




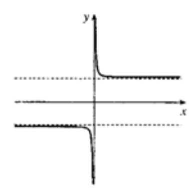
(b) The graph of a function can have 0, 1, or 2 horizontal asymptotes. Representative examples are shown.



No horizontal asymptote



One horizontal asymptote



Two horizontal asymptotes

3. (a) $\lim\limits_{x\to 2} f(x) = \infty$

(b) $\lim\limits_{x\to -1^-} f(x) = \infty$

(c) $\lim\limits_{x\to -1^+} f(x) = -\infty$

(d) $\lim\limits_{x\to \infty} f(x) = 1$

(e) $\lim\limits_{x\to -\infty} f(x) = 2$

(f) Vertical: $x = -1$, $x = 2$; Horizontal: $y = 1$, $y = 2$

4. (a) $\lim\limits_{x\to \infty} g(x) = 2$

(b) $\lim\limits_{x\to -\infty} g(x) = -2$

(c) $\lim\limits_{x\to 3} g(x) = \infty$

(d) $\lim\limits_{x\to 0} g(x) = -\infty$

(e) $\lim\limits_{x\to -2^+} g(x) = -\infty$

(f) Vertical: $x = -2$, $x = 0$, $x = 3$; Horizontal: $y = -2$, $y = 2$

5. If $f(x) = x^2/2^x$, then a calculator gives $f(0) = 0$, $f(1) = 0.5$, $f(2) = 1$, $f(3) = 1.125$, $f(4) = 1$, $f(5) = 0.78125$, $f(6) = 0.5625$, $f(7) = 0.3828125$, $f(8) = 0.25$, $f(9) = 0.158203125$, $f(10) = 0.09765625$, $f(20) \approx 0.00038147$, $f(50) \approx 2.2204 \times 10^{-12}$, $f(100) \approx 7.8886 \times 10^{-27}$. It appears that $\lim\limits_{x\to\infty} \left(x^2/2^x\right) = 0$.

6. (a) From a graph of $f(x) = (1 - 2/x)^x$ in a window of $[0, 10{,}000]$ by $[0, 0.2]$, we estimate that $\lim_{x\to\infty} f(x) = 0.14$ (to two decimal places.)

(b)

x	$f(x)$
10,000	0.135308
100,000	0.135333
1,000,000	0.135335

From the table, we estimate that $\lim_{x\to\infty} f(x) = 0.1353$ (to four decimal places.)

7. $$\lim_{x\to\infty} \frac{x+4}{x^2-2x+5} = \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{4}{x^2}}{1-\frac{2}{x}+\frac{5}{x^2}} = \frac{\lim_{x\to\infty}\left(\frac{1}{x}+\frac{4}{x^2}\right)}{\lim_{x\to\infty}\left(1-\frac{2}{x}+\frac{5}{x^2}\right)}$$
$$= \frac{\lim_{x\to\infty}\frac{1}{x}+4\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty}1-2\lim_{x\to\infty}\frac{1}{x}+5\lim_{x\to\infty}\frac{1}{x^2}} = \frac{0+4(0)}{1-2(0)+5(0)} = 0$$

8. $$\lim_{t\to\infty} \frac{7t^3+4t}{2t^3-t^2+3} = \lim_{t\to\infty} \frac{7+\frac{4}{t^2}}{2-\frac{1}{t}+\frac{3}{t^3}} = \frac{\lim_{t\to\infty}7+4\lim_{t\to\infty}\frac{1}{t^2}}{\lim_{t\to\infty}2-\lim_{t\to\infty}\frac{1}{t}+3\lim_{t\to\infty}\frac{1}{t^3}} = \frac{7+4(0)}{2-0+3(0)} = \frac{7}{2}$$

9. $$\lim_{x\to-\infty} \frac{(1-x)(2+x)}{(1+2x)(2-3x)} = \lim_{x\to-\infty} \frac{\left[\frac{1}{x}-1\right]\left[\frac{2}{x}+1\right]}{\left[\frac{1}{x}+2\right]\left[\frac{2}{x}-3\right]} = \frac{\left[\lim_{x\to-\infty}\frac{1}{x}-1\right]\left[\lim_{x\to-\infty}\frac{2}{x}+1\right]}{\left[\lim_{x\to-\infty}\frac{1}{x}+2\right]\left[\lim_{x\to-\infty}\frac{2}{x}-3\right]}$$
$$\overset{(5,4,1,2,7)}{=} \frac{(0-1)(0+1)}{(0+2)(0-3)} = \frac{1}{6}$$

10. $$\lim_{x\to\infty}\left[\frac{2x^2-1}{x+8x^2}\right]^{1/2} \overset{(11)}{=} \left[\lim_{x\to\infty}\frac{2-\frac{1}{x^2}}{\frac{1}{x}+8}\right]^{1/2} \overset{(5,1,2)}{=} \left[\frac{\lim_{x\to\infty}2-\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty}\frac{1}{x}+\lim_{x\to\infty}8}\right]^{1/2} = \left[\frac{2-0}{0+8}\right]^{1/2} = \tfrac{1}{2}$$ by (7) and Theorem 4.

11. $$\lim_{r\to\infty} \frac{r^4-r^2+1}{r^5+r^3-r} = \lim_{r\to\infty} \frac{\frac{1}{r}-\frac{1}{r^3}+\frac{1}{r^5}}{1+\frac{1}{r^2}-\frac{1}{r^4}} = \frac{\lim_{r\to\infty}\frac{1}{r}-\lim_{r\to\infty}\frac{1}{r^3}+\lim_{r\to\infty}\frac{1}{r^5}}{\lim_{r\to\infty}1+\lim_{r\to\infty}\frac{1}{r^2}-\lim_{r\to\infty}\frac{1}{r^4}} = \frac{0-0+0}{1+0-0} = 0$$

12. $$\lim_{t\to-\infty} \frac{6t^2+5t}{(1-t)(2t-3)} = \lim_{t\to-\infty} \frac{6t^2+5t}{-2t^2+5t-3} = \lim_{t\to-\infty} \frac{6+5/t}{-2+5/t-3/t^2}$$
$$= \frac{\lim_{t\to-\infty}6+5\lim_{t\to-\infty}(1/t)}{\lim_{t\to-\infty}(-2)+5\lim_{t\to-\infty}(1/t)-3\lim_{t\to-\infty}(1/t^2)} = \frac{6+5(0)}{-2+5(0)-3(0)} = -3$$

13. $$\lim_{x\to\infty} \frac{\sqrt{1+4x^2}}{4+x} = \lim_{x\to\infty} \frac{\sqrt{(1/x^2)+4}}{(4/x)+1} = \frac{\sqrt{0+4}}{0+1} = 2$$

14. $$\lim_{x\to-\infty} \frac{\sqrt{x^2+4x}}{4x+1} = \lim_{x\to-\infty} \frac{-\sqrt{1+4/x}}{4+1/x} = \frac{-\sqrt{1+0}}{4+0} = -\frac{1}{4}.$$

Note: In dividing numerator and denominator by x, we used the fact that for $x < 0$, $x = -\sqrt{x^2}$.

15. $\lim\limits_{x\to\infty} \dfrac{1-\sqrt{x}}{1+\sqrt{x}} = \lim\limits_{x\to\infty} \dfrac{(1/\sqrt{x})-1}{(1/\sqrt{x})+1} = \dfrac{0-1}{0+1} = -1$

16. $$\lim_{x\to\infty}\left(\sqrt{x^2+3x+1}-x\right) = \lim_{x\to\infty}\left(\sqrt{x^2+3x+1}-x\right)\frac{\sqrt{x^2+3x+1}+x}{\sqrt{x^2+3x+1}+x}$$
$$= \lim_{x\to\infty}\frac{x^2+3x+1-x^2}{\sqrt{x^2+3x+1}+x} = \lim_{x\to\infty}\frac{3x+1}{\sqrt{x^2+3x+1}+x}$$
$$= \lim_{x\to\infty}\frac{3+1/x}{\sqrt{1+(3/x)+(1/x^2)}+1} = \frac{3+0}{\sqrt{1+3\cdot 0+0}+1} = \frac{3}{2}$$

17. $$\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right) = \lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)\frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}$$
$$= \lim_{x\to\infty}\frac{(x^2+1)-(x^2-1)}{\sqrt{x^2+1}+\sqrt{x^2-1}} = \lim_{x\to\infty}\frac{2}{\sqrt{x^2+1}+\sqrt{x^2-1}}$$
$$= \lim_{x\to\infty}\frac{2/x}{\sqrt{1+(1/x^2)}+\sqrt{1-(1/x^2)}} = \frac{0}{\sqrt{1+0}+\sqrt{1-0}} = 0$$

18. $$\lim_{x\to-\infty}\left(x+\sqrt{x^2+2x}\right) = \lim_{x\to-\infty}\left(x+\sqrt{x^2+2x}\right)\left[\frac{x-\sqrt{x^2+2x}}{x-\sqrt{x^2+2x}}\right] = \lim_{x\to-\infty}\frac{x^2-(x^2+2x)}{x-\sqrt{x^2+2x}}$$
$$= \lim_{x\to-\infty}\frac{-2x}{x-\sqrt{x^2+2x}} = \lim_{x\to-\infty}\frac{-2}{1+\sqrt{1+2/x}} = \frac{-2}{1+\sqrt{1+2\cdot 0}} = -1$$

Note: In dividing numerator and denominator by x, we used the fact that for $x<0$, $x=-\sqrt{x^2}$.

19. $$\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right) = \lim_{x\to\infty}\frac{\left(\sqrt{9x^2+x}-3x\right)\left(\sqrt{9x^2+x}+3x\right)}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty}\frac{\left(\sqrt{9x^2+x}\right)^2-(3x)^2}{\sqrt{9x^2+x}+3x}$$
$$= \lim_{x\to\infty}\frac{(9x^2+x)-9x^2}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty}\frac{x/x}{\left(\sqrt{9x^2+x}+3x\right)/x} = \lim_{x\to\infty}\frac{1}{\sqrt{9+1/x}+3}$$
$$= \frac{1}{\sqrt{9}+3} = \frac{1}{3+3} = \frac{1}{6}$$

20. $\lim\limits_{x\to\infty}\cos x$ does not exist because, as x increases, $\cos x$ does not approach any one value, but oscillates between 1 and -1.

21. $\sqrt{x}$ is large when x is large, so $\lim\limits_{x\to\infty}\sqrt{x}=\infty$.

22. $\sqrt[3]{x}$ is large negative when x is large negative, so $\lim\limits_{x\to-\infty}\sqrt[3]{x}=-\infty$.

23. $\lim\limits_{x\to\infty}\left(x-\sqrt{x}\right) = \lim\limits_{x\to\infty}\sqrt{x}\left(\sqrt{x}-1\right) = \infty$ since $\sqrt{x}\to\infty$ and $\sqrt{x}-1\to\infty$ as $x\to\infty$.

24. $\lim\limits_{x\to\infty}\left(x+\sqrt{x}\right)=\infty$ since $x\to\infty$ and $\sqrt{x}\to\infty$.

25. $\lim\limits_{x\to-\infty}\left(x^3-5x^2\right)=-\infty$ since $x^3\to-\infty$ and $-5x^2\to-\infty$ as $x\to-\infty$.

Or: $\lim\limits_{x\to-\infty}\left(x^3-5x^2\right) = \lim\limits_{x\to-\infty}x^2(x-5) = -\infty$ since $x^2\to\infty$ and $x-5\to-\infty$.

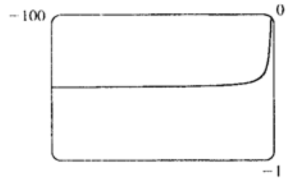
26. $\lim\limits_{x\to\infty}\left(x^2-x^4\right) = \lim\limits_{x\to\infty}x^2\left(1-x^2\right) = -\infty$ since $x^2\to\infty$ and $1-x^2\to-\infty$.

27. $\lim\limits_{x\to\infty} \dfrac{x^7-1}{x^6+1} = \lim\limits_{x\to\infty} \dfrac{1-1/x^7}{(1/x)+(1/x^7)} = \infty$ since $1-\dfrac{1}{x^7} \to 1$ while $\dfrac{1}{x}+\dfrac{1}{x^7} \to 0^+$ as $x \to \infty$.

Or: Divide numerator and denominator by x^6 instead of x^7.

28. If $t = \dfrac{1}{x}$, then $\lim\limits_{x\to\infty} x \sin\dfrac{1}{x} = \lim\limits_{t\to 0^+} \dfrac{1}{t}\sin t = \lim\limits_{t\to 0^+} \dfrac{\sin t}{t} = 1$.

29. (a)



From the graph of $f(x) = \sqrt{x^2+x+1}+x$, we estimate the value of $\lim\limits_{x\to-\infty} f(x)$ to be -0.5.

(b)

x	$f(x)$
$-10{,}000$	-0.4999625
$-100{,}000$	-0.4999962
$-1{,}000{,}000$	-0.4999996

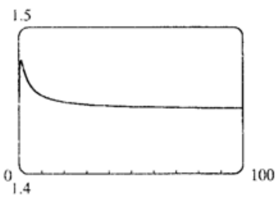
From the table, we estimate the limit to be -0.5.

(c) $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right) = \lim\limits_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right)\left[\dfrac{\sqrt{x^2+x+1}-x}{\sqrt{x^2+x+1}-x}\right] = \lim\limits_{x\to-\infty}\dfrac{(x^2+x+1)-x^2}{\sqrt{x^2+x+1}-x}$

$$= \lim_{x\to-\infty}\frac{(x+1)(1/x)}{\left(\sqrt{x^2+x+1}-x\right)(1/x)} = \lim_{x\to-\infty}\frac{1+(1/x)}{-\sqrt{1+(1/x)+(1/x^2)}-1} = \frac{1+0}{-\sqrt{1+0+0}-1} = -\frac{1}{2}$$

Note that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the radical by x, with $x < 0$, we get $\dfrac{1}{x}\sqrt{x^2+x+1} = -\dfrac{1}{\sqrt{x^2}}\sqrt{x^2+x+1} = -\sqrt{1+(1/x)+(1/x^2)}$.

30. (a)



From the graph of $f(x) = \sqrt{3x^2+8x+6} - \sqrt{3x^2+3x+1}$, we estimate (to one decimal place) the value of $\lim\limits_{x\to\infty} f(x)$ to be 1.4.

(b)

x	$f(x)$
10,000	1.44339
100,000	1.44338
1,000,000	1.44338

From the table, we estimate (to four decimal places) the limit to be 1.4434.

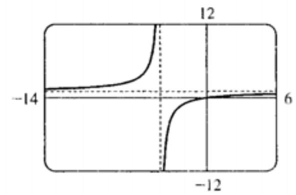
(c) $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty}\dfrac{\left(\sqrt{3x^2+8x+6}-\sqrt{3x^2+3x+1}\right)\left(\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}\right)}{\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}}$

$$= \lim_{x\to\infty}\frac{(3x^2+8x+6)-(3x^2+3x+1)}{\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}}$$

$$= \lim_{x\to\infty}\frac{(5x+5)(1/x)}{\left(\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}\right)(1/x)}$$

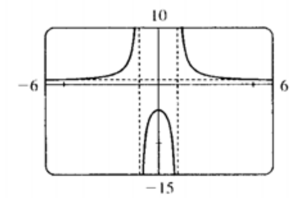
$$= \lim_{x\to\infty}\frac{5+5/x}{\sqrt{3+8/x+6/x^2}+\sqrt{3+3/x+1/x^2}} = \frac{5}{\sqrt{3}+\sqrt{3}} = \frac{5}{2\sqrt{3}} = \frac{5\sqrt{3}}{6} \approx 1.443376$$

31. $\lim\limits_{x\to\pm\infty} \dfrac{x}{x+4} = \lim\limits_{x\to\pm\infty} \dfrac{1}{1+4/x} = \dfrac{1}{1+0} = 1$, so $y = 1$ is a horizontal asymptote. $\lim\limits_{x\to-4^-} \dfrac{x}{x+4} = \infty$ and $\lim\limits_{x\to-4^+} \dfrac{x}{x+4} = -\infty$, so $x = -4$ is a vertical asymptote. The graph confirms these calculations.



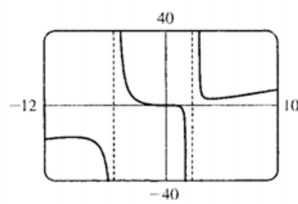
32. Since $x^2 - 1 \to 0$ and $y < 0$ for $-1 < x < 1$ and $y > 0$ for $x < -1$ and $x > 1$, we have $\lim\limits_{x\to1^-} \dfrac{x^2+4}{x^2-1} = -\infty$, $\lim\limits_{x\to1^+} \dfrac{x^2+4}{x^2-1} = \infty$, $\lim\limits_{x\to-1^-} \dfrac{x^2+4}{x^2-1} = \infty$, and $\lim\limits_{x\to-1^+} \dfrac{x^2+4}{x^2-1} = -\infty$, so $x = 1$ and $x = -1$ are vertical asymptotes. Also

$\lim\limits_{x\to\pm\infty} \dfrac{x^2+4}{x^2-1} = \lim\limits_{x\to\pm\infty} \dfrac{1+4/x^2}{1-1/x^2} = \dfrac{1+0}{1-0} = 1$, so $y = 1$ is a horizontal asymptote. The graph confirms these calculations.

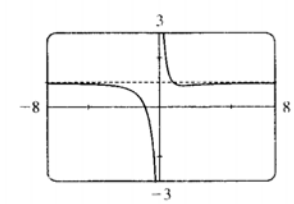


33. $\lim\limits_{x\to\pm\infty} \dfrac{x^3}{x^2+3x-10} = \lim\limits_{x\to\pm\infty} \dfrac{x}{1+(3/x)-(10/x^2)} = \pm\infty$, so there is no horizontal asymptote.

$\lim\limits_{x\to2^+} \dfrac{x^3}{x^2+3x-10} = \lim\limits_{x\to2^+} \dfrac{x^3}{(x+5)(x-2)} = \infty$, since $\dfrac{x^3}{(x+5)(x-2)} > 0$ for $x > 2$. Similarly, $\lim\limits_{x\to2^-} \dfrac{x^3}{x^2+3x-10} = -\infty$ and $\lim\limits_{x\to-5^-} \dfrac{x^3}{x^2+3x-10} = -\infty$, $\lim\limits_{x\to-5^+} \dfrac{x^3}{x^2+3x-10} = \infty$, so $x = 2$ and $x = -5$ are vertical asymptotes. The graph confirms these calculations.

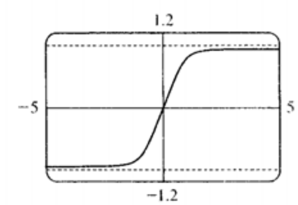


34. $\lim\limits_{x\to\pm\infty} \dfrac{x^3+1}{x^3+x} = \lim\limits_{x\to\pm\infty} \dfrac{1+1/x^3}{1+1/x^2} = 1$, so $y = 1$ is a horizontal asymptote. Since $y = \dfrac{x^3+1}{x^3+x} = \dfrac{x^3+1}{x\,(x^2+1)} > 0$ for $x > 0$ and $y < 0$ for $-1 < x < 0$, $\lim\limits_{x\to0^+} \dfrac{x^3+1}{x^3+x} = \infty$ and $\lim\limits_{x\to0^-} \dfrac{x^3+1}{x^3+x} = -\infty$, so $x = 0$ is a vertical asymptote.



35. $\lim\limits_{x\to\infty} \dfrac{x}{\sqrt[4]{x^4+1}} = \lim\limits_{x\to\infty} \dfrac{1}{\sqrt[4]{1+(1/x^4)}} = \dfrac{1}{\sqrt[4]{1+0}} = 1$ and

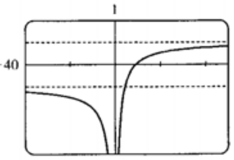
$\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt[4]{x^4+1}} = \lim\limits_{x\to-\infty} \dfrac{1}{-\sqrt[4]{1+(1/x^4)}} = \dfrac{1}{-\sqrt[4]{1+0}} = -1$, so $y = \pm1$ are horizontal asymptotes. There is no vertical asymptote.

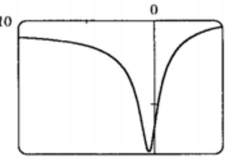


36. $\lim\limits_{x\to\infty} \dfrac{x-9}{\sqrt{4x^2+3x+2}} = \lim\limits_{x\to\infty} \dfrac{1-9/x}{\sqrt{4+(3/x)+(2/x^2)}} = \dfrac{1-0}{\sqrt{4+0+0}} = \dfrac{1}{2}$. Using the fact that $\sqrt{x^2} = |x| = -x$ for $x < 0$, we divide the numerator by $-x$ and the denominator by $\sqrt{x^2}$. Thus,

$$\lim_{x\to-\infty} \frac{x-9}{\sqrt{4x^2+3x+2}} = \lim_{x\to-\infty} \frac{-1+9/x}{\sqrt{4+(3/x)+(2/x^2)}} = \frac{-1+0}{\sqrt{4+0+0}} = -\frac{1}{2}.$$

The horizontal asymptotes are $y = \pm\frac{1}{2}$. The polynomial $4x^2+3x+2$ is positive for all x, so the denominator never approaches zero, and thus there is no vertical asymptote.





37. Let's look for a rational function.

(1) $\lim\limits_{x\to\pm\infty} f(x) = 0 \quad\Rightarrow\quad$ degree of numerator < degree of denominator

(2) $\lim\limits_{x\to 0} f(x) = -\infty \quad\Rightarrow\quad$ there is a factor of x^2 in the denominator (not just x, since that would produce a sign change at $x = 0$), and the function is negative near $x = 0$.

(3) $\lim\limits_{x\to 3^-} f(x) = \infty$ and $\lim\limits_{x\to 3^+} f(x) = -\infty \quad\Rightarrow\quad$ vertical asymptote at $x = 3$; there is a factor of $(x-3)$ in the denominator.

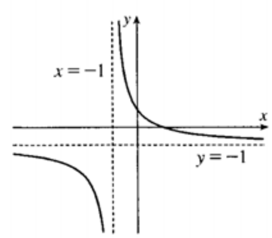
(4) $f(2) = 0 \quad\Rightarrow\quad 2$ is an x-intercept; there is at least one factor of $(x-2)$ in the numerator.

Combining all of this information, and putting in a negative sign to give us the desired left- and right-hand limits, gives us $f(x) = \dfrac{2-x}{x^2(x-3)}$ as one possibility.

38. The denominator of the rational function we are looking for must have factors $(x-1)$ and $(x-3)$. The degree of the numerator must equal the degree of the denominator, and the ratio of the leading coefficients must be 1. One possibility is $f(x) = \dfrac{x^2}{(x-1)(x-3)}$.

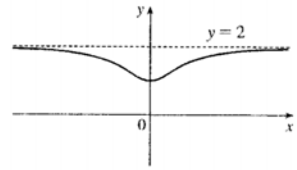
39. $y = \dfrac{1-x}{1+x}$ has domain $(-\infty,-1)\cup(-1,\infty)$.

$\lim\limits_{x\to\pm\infty} \dfrac{1-x}{1+x} = \lim\limits_{x\to\pm\infty} \dfrac{1/x-1}{1/x+1} = \dfrac{0-1}{0+1} = -1$, so $y = -1$ is a HA. The line $x = -1$ is a VA. $y' = \dfrac{(1+x)(-1)-(1-x)(1)}{(1+x)^2} = \dfrac{-2}{(1+x)^2} < 0$ for $x \neq 1$. Thus, $(-\infty,-1)$ and $(-1,\infty)$ are intervals of decrease.



$y'' = -2\cdot\dfrac{-2(1+x)}{\left[(1+x)^2\right]^2} = \dfrac{4}{(1+x)^3} < 0$ for $x < -1$ and $y'' > 0$ for $x > -1$, so the curve is CD on $(-\infty,-1)$ and CU on $(-1,\infty)$. Since $x = -1$ is not in the domain, there is no IP.

40. $y = \dfrac{1+2x^2}{1+x^2}$ has domain $\mathbb{R}$.



$\lim\limits_{x\to\pm\infty} \dfrac{1+2x^2}{1+x^2} = \lim\limits_{x\to\pm\infty} \dfrac{1/x^2+2}{1/x^2+1} = \dfrac{0+2}{0+1} = 2$, so $y = 2$ is a HA.

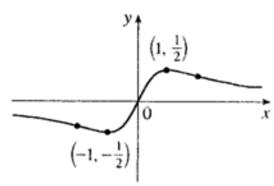
There is no VA. $y' = \dfrac{(1+x^2)(4x) - (1+2x^2)(2x)}{(1+x^2)^2} = \dfrac{2x}{(1+x^2)^2} > 0$

$\Leftrightarrow \quad x > 0$,

and $y' < 0 \quad \Leftrightarrow \quad x < 0$. Thus, y is increasing on $(0, \infty)$ and y is decreasing on $(-\infty, 0)$. There is a local (and absolute) minimum at $(0, 1)$. $y'' = \dfrac{(1+x^2)^2(2) - (2x)\cdot 2(1+x^2)(2x)}{\left[(1+x^2)^2\right]^2} = \dfrac{2-6x^2}{(1+x^2)^3} = 0 \quad \Leftrightarrow \quad x = \pm\frac{1}{\sqrt{3}}$.

$y'' > 0 \quad \Leftrightarrow \quad -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$, so the curve is CU on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and CD on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$. There are IP at $\left(\pm\frac{1}{\sqrt{3}}, \frac{5}{4}\right)$.

41. $\lim\limits_{x\to\pm\infty} \dfrac{x}{x^2+1} = \lim\limits_{x\to\pm\infty} \dfrac{1/x}{1+1/x^2} = \dfrac{0}{1+0} = 0$, so $y = 0$ is a horizontal asymptote.

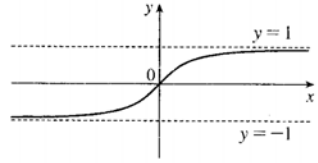


$y' = \dfrac{x^2+1-x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} = 0$ when $x = \pm 1$ and $y' > 0 \quad \Leftrightarrow$

$x^2 < 1 \quad \Leftrightarrow \quad -1 < x < 1$, so y is increasing on $(-1, 1)$ and decreasing on $(-\infty, -1)$ and $(1, \infty)$.

$y'' = \dfrac{(1+x^2)^2(-2x) - (1-x^2)\,2\,(x^2+1)\,2x}{(1+x^2)^4} = \dfrac{2x\,(x^2-3)}{(1+x^2)^3} > 0 \quad \Leftrightarrow \quad x > \sqrt{3}$ or $-\sqrt{3} < x < 0$, so y is CU on $\left(\sqrt{3}, \infty\right)$ and $\left(-\sqrt{3}, 0\right)$ and CD on $\left(-\infty, -\sqrt{3}\right)$ and $\left(0, \sqrt{3}\right)$.

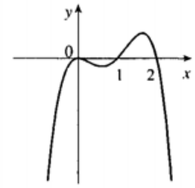
42. $y = \dfrac{x}{\sqrt{x^2+1}} = \dfrac{x/|x|}{\sqrt{1+1/x^2}}$ has domain $\mathbb{R}$. As $x \to \pm\infty$, $y \to \pm 1$, so $y = \pm 1$ are HA. There is no VA. $y = x\,(x^2+1)^{-1/2} \quad \Rightarrow$



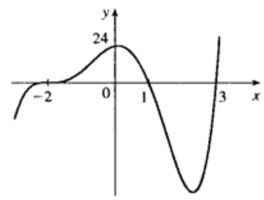
$$\begin{aligned} y' &= x\left(-\tfrac{1}{2}\right)\left(x^2+1\right)^{-3/2}(2x) + \left(x^2+1\right)^{-1/2}(1) \\ &= \left(x^2+1\right)^{-3/2}\left[-x^2 + \left(x^2+1\right)\right] \\ &= \left(x^2+1\right)^{-3/2} > 0 \text{ for all } x \end{aligned}$$

Thus, y is increasing for all x. $y'' = \left(-\frac{3}{2}\right)(x^2+1)^{-5/2}(2x) = \dfrac{-3x}{(x^2+1)^{5/2}} > 0$ for $x < 0$. So the curve is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. There is an inflection point at $(0, 0)$.

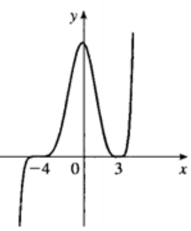
43. $y = f(x) = x^2(x-2)(1-x)$. The y-intercept is $f(0) = 0$, and the x-intercepts occur when $y = 0 \Rightarrow x = 0, 1, 2$. Notice that, since x^2 is always positive, the graph does not cross the x-axis at 0, but does cross the x-axis at 1 and 2. $\lim\limits_{x\to\infty} x^2(x-2)(1-x) = -\infty$, since the first two factors are large positive and the third large negative when x is large positive. $\lim\limits_{x\to-\infty} x^2(x-2)(1-x) = -\infty$ because the first and third factors are large positive and the second large negative as $x \to -\infty$.



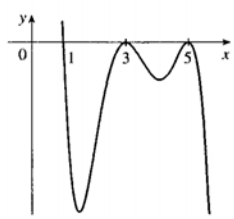
44. $y = (2+x)^3(1-x)(3-x)$. As $x \to \infty$, the first factor is large positive, and the second and third factors are large negative. Therefore, $\lim\limits_{x\to\infty} f(x) = \infty$. As $x \to -\infty$, the first factor is large negative, and the second and third factors are large positive. Therefore, $\lim\limits_{x\to-\infty} f(x) = -\infty$. Now the y-intercept is $f(0) = (2)^3(1)(3) = 24$ and the x-intercepts are the solutions to $f(x) = 0$ $\Rightarrow$ $x = -2$, 1 and 3, and the graph crosses the x-axis at all of these points.



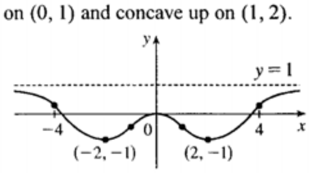
45. $y = f(x) = (x+4)^5(x-3)^4$. The y-intercept is $f(0) = 4^5(-3)^4 = 82{,}944$. The x-intercepts occur when $y = 0 \Rightarrow x = -4, 3$. Notice that the graph does not cross the x-axis at 3 because $(x-3)^4$ is always positive, but does cross the x-axis at -4. $\lim\limits_{x\to\infty}(x+4)^5(x-3)^4 = \infty$ since both factors are large positive when x is large positive. $\lim\limits_{x\to-\infty}(x+4)^5(x-3)^4 = -\infty$ since the first factor is large negative and the second factor is large positive when x is large negative.



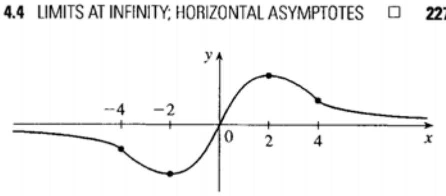
46. $y = (1-x)(x-3)^2(x-5)^2$. As $x \to \infty$, the first factor approaches $-\infty$ while the second and third factors approach ∞. Therefore, $\lim\limits_{x\to\infty}(x) = -\infty$. As $x \to -\infty$, the factors all approach ∞. Therefore, $\lim\limits_{x\to-\infty}(x) = \infty$. Now the y-intercept is $f(0) = (1)(-3)^2(-5)^2 = 225$ and the x-intercepts are the solutions to $f(x) = 0 \Rightarrow x = 1, 3$, and 5. Notice that $f(x)$ does not change sign at $x = 3$ or $x = 5$ because the factors $(x-3)^2$ and $(x-5)^2$ are always positive, so the graph does not cross the x-axis at $x = 3$ or $x = 5$, but does cross the x-axis at $x = 1$.



47. First we plot the points which are known to be on the graph: $(2, -1)$ and $(0, 0)$. We can also draw a short line segment of slope 0 at $x = 2$, since we are given that $f'(2) = 0$. Now we know that $f'(x) < 0$ (that is, the function is decreasing) on $(0, 2)$, and that $f''(x) < 0$ on $(0, 1)$ and $f''(x) > 0$ on $(1, 2)$. So we must join the points $(0, 0)$ and $(2, -1)$ in such a way that the curve is concave down on $(0, 1)$ and concave up on $(1, 2)$. The curve must be concave up and increasing on $(2, 4)$, and concave down and increasing on $(4, \infty)$. Now we just need to reflect the curve in the y-axis, since we are given that f is an even function. The diagram shows one possible function satisfying all of the given conditions.

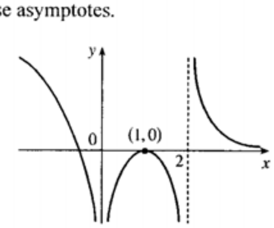


48. The diagram shows one possible function satisfying all of the given conditions.

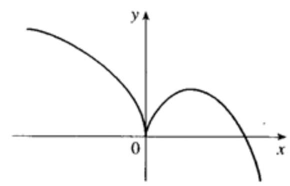


49. We are given that $f(1) = f'(1) = 0$. So we can draw a short horizontal line at the point $(1, 0)$ to represent this situation. We are given that $x = 0$ and $x = 2$ are vertical asymptotes, with $\lim_{x\to 0} f(x) = -\infty$, $\lim_{x\to 2^+} f(x) = \infty$ and $\lim_{x\to 2^-} f(x) = -\infty$, so we can draw the parts of the curve which approach these asymptotes.

On the interval $(-\infty, 0)$, the graph is concave down, and $f(x) \to \infty$ as $x \to -\infty$. Between the asymptotes the graph is concave down. On the interval $(2, \infty)$ the graph is concave up, and $f(x) \to 0$ as $x \to \infty$, so $y = 0$ is a horizontal asymptote. The diagram shows one possible function satisfying all of the given conditions.

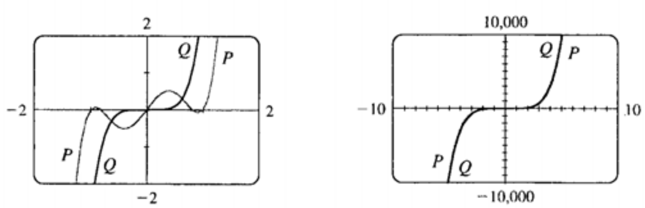


50. The diagram shows one possible function satisfying all of the given conditions.



51. Since $0 \le \sin^2 x \le 1$, we have $0 \le \dfrac{\sin^2 x}{x^2} \le \dfrac{1}{x^2}$ for all $x \ne 0$. Now $\lim_{x\to\infty} 0 = 0$ and $\lim_{x\to\infty} \dfrac{1}{x^2} = 0$, so by the Squeeze Theorem, $\lim_{x\to\infty} \dfrac{\sin^2 x}{x^2} = 0$.

52. (a) In both viewing rectangles, $\lim_{x\to\infty} P(x) = \lim_{x\to\infty} Q(x) = \infty$ and $\lim_{x\to-\infty} P(x) = \lim_{x\to-\infty} Q(x) = -\infty$. In the larger viewing rectangle, P and Q become less distinguishable.



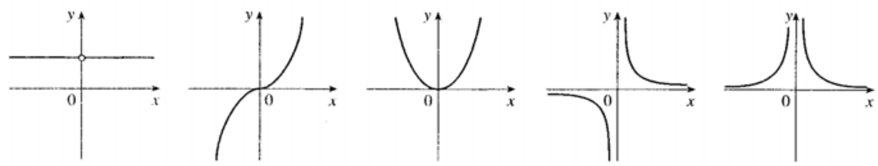
(b) $\lim_{x\to\infty} \dfrac{P(x)}{Q(x)} = \lim_{x\to\infty} \dfrac{3x^5 - 5x^3 + 2x}{3x^5} = \lim_{x\to\infty} \left(1 - \dfrac{5}{3} \cdot \dfrac{1}{x^2} + \dfrac{2}{3} \cdot \dfrac{1}{x^4}\right) = 1 - \frac{5}{3}(0) + \frac{2}{3}(0) = 1 \;\Rightarrow\; P$ and Q have the same end behavior.

53. Divide numerator and denominator by the highest power of x in $Q(x)$.

(a) If $\deg P < \deg Q$, then numerator $\to 0$ but denominator doesn't. So $\lim_{x\to\infty} [P(x)/Q(x)] = 0$.

(b) If $\deg P > \deg Q$, then numerator $\to \pm\infty$ but denominator doesn't, so $\lim_{x\to\infty} [P(x)/Q(x)] = \pm\infty$ (depending on the ratio of the leading coefficients of P and Q).

54.



(i) $n = 0$ (ii) $n > 0$ (n odd) (iii) $n > 0$ (n even) (iv) $n < 0$ (n odd) (v) $n < 0$ (n even)

From these sketches we see that

(a) $\displaystyle\lim_{x\to 0^+} x^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \\ \infty & \text{if } n < 0 \end{cases}$

(b) $\displaystyle\lim_{x\to 0^-} x^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \\ -\infty & \text{if } n < 0,\ n \text{ odd} \\ \infty & \text{if } n < 0,\ n \text{ even} \end{cases}$

(c) $\displaystyle\lim_{x\to\infty} x^n = \begin{cases} 1 & \text{if } n = 0 \\ \infty & \text{if } n > 0 \\ 0 & \text{if } n < 0 \end{cases}$

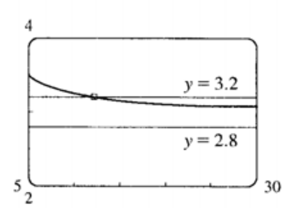
(d) $\displaystyle\lim_{x\to-\infty} x^n = \begin{cases} 1 & \text{if } n = 0 \\ -\infty & \text{if } n > 0,\ n \text{ odd} \\ \infty & \text{if } n > 0,\ n \text{ even} \\ 0 & \text{if } n < 0 \end{cases}$

55. $\displaystyle\lim_{x\to\infty} \frac{4x-1}{x} = \lim_{x\to\infty}\left(4 - \frac{1}{x}\right) = 4$, and $\displaystyle\lim_{x\to\infty} \frac{4x^2+3x}{x^2} = \lim_{x\to\infty}\left(4 + \frac{3}{x}\right) = 4$. Therefore, by the Squeeze Theorem, $\displaystyle\lim_{x\to\infty} f(x) = 4$.

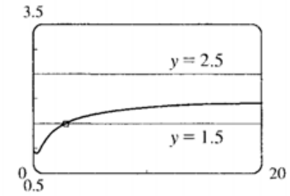
56. (a) After t minutes, $25t$ liters of brine with 30 g of salt per liter has been pumped into the tank, so it contains $(5000 + 25t)$ liters of water and $25t \cdot 30 = 750t$ grams of salt. Therefore, the salt concentration at time t will be $C(t) = \dfrac{750t}{5000 + 25t} = \dfrac{30t}{200 + t}\ \dfrac{\text{g}}{\text{L}}$.

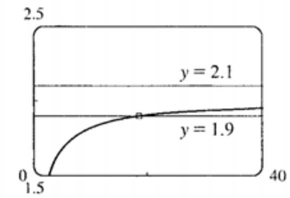
(b) $\displaystyle\lim_{t\to\infty} C(t) = \lim_{t\to\infty} \frac{30t}{200+t} = \lim_{t\to\infty} \frac{30t/t}{200/t + t/t} = \frac{30}{0+1} = 30$. So the salt concentration approaches that of the brine being pumped into the tank.

57. $\left|\dfrac{6x^2+5x-3}{2x^2-1} - 3\right| < 0.2 \quad\Leftrightarrow\quad 2.8 < \dfrac{6x^2+5x-3}{2x^2-1} < 3.2$. So we graph the three parts of this inequality on the same screen, and find that the curve $y = \dfrac{6x^2+5x-3}{2x^2-1}$ seems to lie between the lines $y = 2.8$ and $y = 3.2$ whenever $x > 12.8$. So we can choose $N = 13$ (or any larger number), so that the inequality holds whenever $x \geq N$.



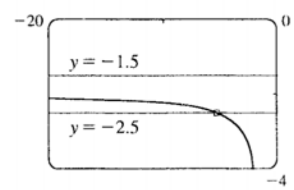
58. For $\varepsilon = 0.5$, we must find N such that whenever $x \geq N$, we have $\left|\dfrac{\sqrt{4x^2+1}}{x+1} - 2\right| < 0.5 \quad\Leftrightarrow\quad 1.5 < \dfrac{\sqrt{4x^2+1}}{x+1} < 2.5$. We graph the three parts of this inequality on the same screen, and find that it holds whenever $x \geq 3$. So we choose $N = 3$ (or any larger number). For $\varepsilon = 0.1$, we must have $1.9 < \dfrac{\sqrt{4x^2+1}}{x+1} < 2.1$, and the graphs show that this holds whenever $x \geq 19$. So we choose $N = 19$ (or any larger number).

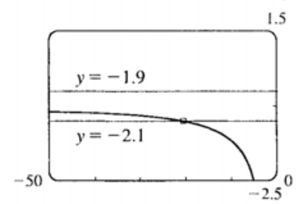




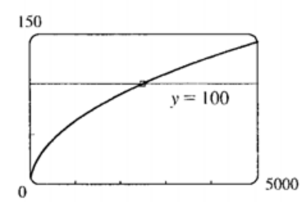
59. For $\varepsilon = 0.5$, we need to find N such that $\left|\dfrac{\sqrt{4x^2+1}}{x+1} - (-2)\right| < 0.5$ $\Leftrightarrow\quad -2.5 < \dfrac{\sqrt{4x^2+1}}{x+1} < -1.5$ whenever $x \leq N$. We graph the three parts of this inequality on the same screen, and see that the inequality holds for $x \leq -6$. So we choose $N = -6$ (or any smaller number).

For $\varepsilon = 0.1$, we need $-2.1 < \dfrac{\sqrt{4x^2+1}}{x+1} < -1.9$ whenever $x \leq N$. From the graph, it seems that this inequality holds for $x \leq -22$. So we choose any $N = -22$ (or any smaller number).





60. We need N such that $\dfrac{2x+1}{\sqrt{x+1}} > 100$ whenever $x \geq N$. From the graph, we see that this inequality holds for $x \geq 2500$. So we choose $N = 2500$ (or any larger number).



61. (a) $1/x^2 < 0.0001 \quad\Leftrightarrow\quad x^2 > 1/0.0001 = 10{,}000 \quad\Leftrightarrow\quad x > 100 \quad (x > 0)$

(b) If $\varepsilon > 0$ is given, then $1/x^2 < \varepsilon \quad\Leftrightarrow\quad x^2 > 1/\varepsilon \quad\Leftrightarrow\quad x > 1/\sqrt{\varepsilon}$. Let $N = 1/\sqrt{\varepsilon}$.

Then $x > N \quad\Rightarrow\quad x > \dfrac{1}{\sqrt{\varepsilon}} \quad\Rightarrow\quad \left|\dfrac{1}{x^2} - 0\right| = \dfrac{1}{x^2} < \varepsilon$, so $\lim\limits_{x\to\infty} \dfrac{1}{x^2} = 0$.

62. (a) $1/\sqrt{x} < 0.0001 \quad\Leftrightarrow\quad \sqrt{x} > 1/0.0001 = 10^4 \quad\Leftrightarrow\quad x > 10^8$.

(b) If $\varepsilon > 0$ is given, then $1/\sqrt{x} < \varepsilon \quad\Leftrightarrow\quad \sqrt{x} > 1/\varepsilon \quad\Leftrightarrow\quad x > 1/\varepsilon^2$. Let $N = 1/\varepsilon^2$.

Then $x > N \quad\Rightarrow\quad x > \dfrac{1}{\varepsilon^2} \quad\Rightarrow\quad \left|\dfrac{1}{\sqrt{x}} - 0\right| = \dfrac{1}{\sqrt{x}} < \varepsilon$, so $\lim\limits_{x\to\infty} \dfrac{1}{\sqrt{x}} = 0$.

63. For $x < 0$, $|1/x - 0| = -1/x$. If $\varepsilon > 0$ is given, then $-1/x < \varepsilon \Leftrightarrow x < -1/\varepsilon$.
Take $N = -1/\varepsilon$. Then $x < N \Rightarrow x < -1/\varepsilon \Rightarrow |(1/x) - 0| = -1/x < \varepsilon$, so $\lim\limits_{x \to -\infty} (1/x) = 0$.

64. Given $M > 0$, we need $N > 0$ such that $x > N \Rightarrow x^3 > M$. Now $x^3 > M \Leftrightarrow x > \sqrt[3]{M}$, so take $N = \sqrt[3]{M}$.
Then $x > N = \sqrt[3]{M} \Rightarrow x^3 > M$, so $\lim\limits_{x \to \infty} x^3 = \infty$.

65. Suppose that $\lim\limits_{x \to \infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding positive number N such that $|f(x) - L| < \varepsilon$ whenever $x > N$. If $t = 1/x$, then $x > N \Leftrightarrow 0 < t < 1/N$. Thus, for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $t < \delta$. This proves that $\lim\limits_{t \to 0^+} f(1/t) = L = \lim\limits_{x \to \infty} f(x)$.

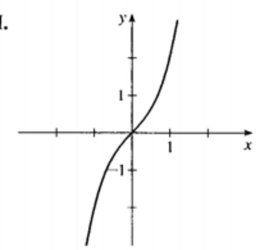
Now suppose that $\lim\limits_{x \to -\infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding negative number N such that $|f(x) - L| < \varepsilon$ whenever $x < N$. If $t = 1/x$, then $x < N \Leftrightarrow 1/N < 1/x < 0$. Thus, for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $-1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $-\delta < t < 0$. This proves that $\lim\limits_{t \to 0^-} f(1/t) = L = \lim\limits_{x \to -\infty} f(x)$.

66. Definition Let f be a function defined on some interval $(-\infty, a)$. Then $\lim\limits_{x \to -\infty} f(x) = -\infty$ means that for every negative number M there is a corresponding negative number N such that $f(x) < M$ whenever $x < N$.
Now we use the definition to prove that $\lim\limits_{x \to -\infty} \left(1 + x^3\right) = -\infty$. Given a negative number M, we need a negative number N such that $x < N \Rightarrow 1 + x^3 < M$. Now $1 + x^3 < M \Leftrightarrow x^3 < M - 1 \Leftrightarrow x < \sqrt[3]{M - 1}$. Thus, we take $N = \sqrt[3]{M - 1}$ and find that $x < N \Rightarrow 1 + x^3 < M$. This proves that $\lim\limits_{x \to -\infty} \left(1 + x^3\right) = -\infty$.

4.5 Summary of Curve Sketching

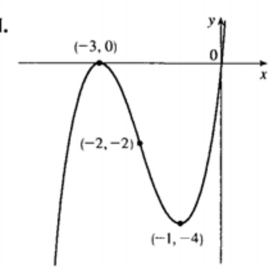
1. $y = f(x) = x^3 + x = x\left(x^2 + 1\right)$ **A.** f is a polynomial, so $D = \mathbb{R}$. **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$ **C.** $f(-x) = -f(x)$, so f is odd; the curve is symmetric about the origin. **D.** f is a polynomial, so there is no asymptote. **E.** $f'(x) = 3x^2 + 1 > 0$, so f is increasing on $(-\infty, \infty)$. **F.** There is no critical number and hence, no local maximum or minimum value. **G.** $f''(x) = 6x > 0$ on $(0, \infty)$ and $f''(x) < 0$ on $(-\infty, 0)$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. Since the concavity changes at $x = 0$, there is an inflection point at $(0, 0)$.

H.



2. $y = f(x) = x^3 + 6x^2 + 9x = x(x + 3)^2$ **A.** $D = \mathbb{R}$ **B.** x-intercepts are -3 and 0, y-intercept $= 0$ **C.** No symmetry **D.** No asymptote **E.** $f'(x) = 3x^2 + 12x + 9 = 3(x + 1)(x + 3) < 0 \Leftrightarrow -3 < x < -1$, so f is decreasing on $(-3, -1)$ and increasing on $(-\infty, -3)$ and $(-1, \infty)$. **F.** Local maximum $f(-3) = 0$, local minimum $f(-1) = -4$ **G.** $f''(x) = 6x + 12 = 6(x + 2) > 0 \Leftrightarrow x > -2$, so f is CU on $(-2, \infty)$ and CD on $(-\infty, -2)$. IP at $(-2, -2)$

H.



3. $y = f(x) = 2 - 15x + 9x^2 - x^3 = -(x-2)\left(x^2 - 7x + 1\right)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 2$; x-intercepts: $f(x) = 0 \Rightarrow x = 2$ or (by the quadratic formula) $x = \frac{7 \pm \sqrt{45}}{2}$ ($\approx 0.15, 6.85$) **C.** No symmetry **D.** No asymptote

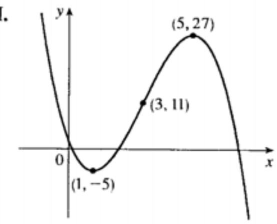
E. $f'(x) = -15 + 18x - 3x^2 = -3\left(x^2 - 6x + 5\right)$
$= -3(x-1)(x-5) > 0 \Leftrightarrow 1 < x < 5$

so f is increasing on $(1, 5)$ and decreasing on $(-\infty, 1)$ and $(5, \infty)$.

F. Local maximum $f(5) = 27$, local minimum $f(1) = -5$

G. $f''(x) = 18 - 6x = -6(x-3) > 0 \Leftrightarrow x < 3$, so f is CU on $(-\infty, 3)$ and CD on $(3, \infty)$. IP at $(3, 11)$

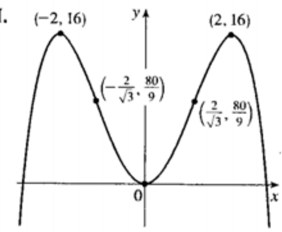
H.



4. $y = f(x) = 8x^2 - x^4 = x^2\left(8 - x^2\right)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow x = 0, \pm 2\sqrt{2}$ ($\approx \pm 2.83$) **C.** $f(-x) = f(x)$, so f is even and symmetric about the y-axis. **D.** No asymptote

E. $f'(x) = 16x - 4x^3 = 4x\left(4 - x^2\right) = 4x(2+x)(2-x) > 0 \Leftrightarrow x < -2$ or $0 < x < 2$, so f is increasing on $(-\infty, -2)$ and $(0, 2)$ and decreasing on $(-2, 0)$ and $(2, \infty)$. **F.** Local maxima $f(\pm 2) = 16$, local minimum $f(0) = 0$ **G.** $f''(x) = 16 - 12x^2 = 4\left(4 - 3x^2\right) = 0 \Leftrightarrow x = \pm\frac{2}{\sqrt{3}}$. $f''(x) > 0 \Leftrightarrow -\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$, so f is CU on $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ and CD on $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, \infty\right)$. IP at $\left(\pm\frac{2}{\sqrt{3}}, \frac{80}{9}\right)$

H.

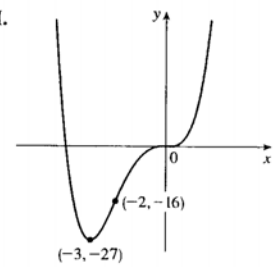


5. $y = f(x) = x^4 + 4x^3 = x^3(x+4)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Leftrightarrow x = -4, 0$ **C.** No symmetry **D.** No asymptote **E.** $f'(x) = 4x^3 + 12x^2 = 4x^2(x+3) > 0 \Leftrightarrow x > -3$, so f is increasing on $(-3, \infty)$ and decreasing on $(-\infty, -3)$.

F. Local minimum $f(-3) = -27$, no local maximum

G. $f''(x) = 12x^2 + 24x = 12x(x+2) < 0 \Leftrightarrow -2 < x < 0$, so f is CD on $(-2, 0)$ and CU on $(-\infty, -2)$ and $(0, \infty)$. IP at $(0, 0)$ and $(-2, -16)$

H.

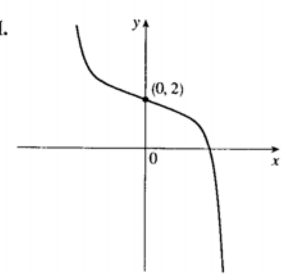


6. $y = f(x) = 2 - x - x^9$
$= -(x-1)\left(x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 2\right)$

A. $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 2$; x-intercept: $f(x) = 0 \Leftrightarrow x = 1$ (By part E below, f is decreasing on its domain, so it has only one x-intercept.) **C.** No symmetry **D.** No asymptote

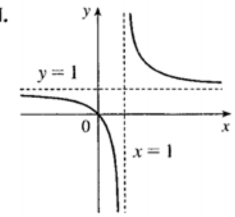
E. $f'(x) = -1 - 9x^8 = -1\left(9x^8 + 1\right) < 0$ for all x, so f is decreasing on $\mathbb{R}$. **F.** There is no extremum. **G.** $f''(x) = -72x^7 > 0 \Leftrightarrow x < 0$, so f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. IP at $(0, 2)$

H.



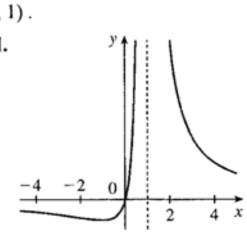
7. $y = f(x) = x/(x-1)$ **A.** $D = \{x \mid x \neq 1\} = (-\infty, 1) \cup (1, \infty)$ **B.** x-intercept= 0, y-intercept= $f(0) = 0$
C. No symmetry **D.** $\lim\limits_{x \to \pm\infty} \dfrac{x}{x-1} = 1$, so $y = 1$ is a HA. $\lim\limits_{x \to 1^-} \dfrac{x}{x-1} = -\infty$, $\lim\limits_{x \to 1^+} \dfrac{x}{x-1} = \infty$, so $x = 1$ is a VA.
E. $f'(x) = \dfrac{(x-1)-x}{(x-1)^2} = \dfrac{-1}{(x-1)^2} < 0$ for $x \neq 1$, so f is decreasing on $(-\infty, 1)$ and $(1, \infty)$. **F.** No extremum **G.** $f''(x) = \dfrac{2}{(x-1)^3} > 0$ $\Leftrightarrow$ $x > 1$, so f is CU on $(1, \infty)$ and CD on $(-\infty, 1)$. No IP

H.



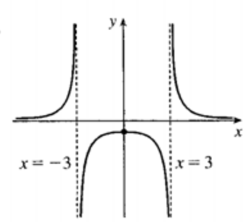
8. $y = x/(x-1)^2$ **A.** $D = \{x \mid x \neq 1\} = (-\infty, 1) \cup (1, \infty)$ **B.** x-intercept= 0, y-intercept= $f(0) = 0$
C. No symmetry **D.** $\lim\limits_{x \to \pm\infty} \dfrac{x}{(x-1)^2} = 0$, so $y = 0$ is a HA. $\lim\limits_{x \to 1} \dfrac{x}{(x-1)^2} = \infty$, so $x = 1$ is a VA.
E. $f'(x) = \dfrac{(x-1)^2(1) - x(2)(x-1)}{(x-1)^4} = \dfrac{-x-1}{(x-1)^3}$. This is negative on $(-\infty, -1)$ and $(1, \infty)$ and positive on $(-1, 1)$, so $f(x)$ is decreasing on $(-\infty, -1)$ and $(1, \infty)$ and increasing on $(-1, 1)$.
F. Local minimum $f(-1) = -\frac{1}{4}$, no local maximum.
G. $f''(x) = \dfrac{(x-1)^3(-1) + (x+1)(3)(x-1)^2}{(x-1)^6} = \dfrac{2(x+2)}{(x-1)^4}$. This is negative on $(-\infty, -2)$, and positive on $(-2, 1)$ and $(1, \infty)$. So f is CD on $(-\infty, -2)$ and CU on $(-2, 1)$ and $(1, \infty)$. f has an inflection point at $\left(-2, -\frac{2}{9}\right)$.

H.



9. $y = f(x) = 1/(x^2 - 9)$ **A.** $D = \{x \mid x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
B. y-intercept $= f(0) = -\frac{1}{9}$, no x-intercept **C.** $f(-x) = f(x)$ $\Rightarrow$ f is even; the curve is symmetric about the y-axis. **D.** $\lim\limits_{x \to \pm\infty} \dfrac{1}{x^2-9} = 0$, so $y = 0$ is a HA. $\lim\limits_{x \to 3^-} \dfrac{1}{x^2-9} = -\infty$, $\lim\limits_{x \to 3^+} \dfrac{1}{x^2-9} = \infty$, $\lim\limits_{x \to -3^-} \dfrac{1}{x^2-9} = \infty$, $\lim\limits_{x \to -3^+} \dfrac{1}{x^2-9} = -\infty$, so $x = 3$ and $x = -3$ are VA. **E.** $f'(x) = -\dfrac{2x}{(x^2-9)^2} > 0$ $\Leftrightarrow$ $x < 0$ $(x \neq -3)$ so f is increasing on $(-\infty, -3)$ and $(-3, 0)$ and decreasing on $(0, 3)$ and $(3, \infty)$.
F. Local maximum $f(0) = -\frac{1}{9}$.
G. $y'' = \dfrac{-2(x^2-9)^2 + (2x)2(x^2-9)(2x)}{(x^2-9)^4} = \dfrac{6(x^2+3)}{(x^2-9)^3} > 0$ $\Leftrightarrow$ $x^2 > 9$ $\Leftrightarrow$ $x > 3$ or $x < -3$, so f is CU on $(-\infty, -3)$ and $(3, \infty)$ and CD on $(-3, 3)$. No IP

H.



10. $y = f(x) = x/(x^2-9)$ **A.** $D = \{x \mid x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$. **C.** $f(-x) = -f(x)$, so f is odd; the curve is symmetric about the origin.

D. $\lim\limits_{x\to\pm\infty} \dfrac{x}{x^2-9} = 0$, so $y = 0$ is a HA. $\lim\limits_{x\to 3^+} \dfrac{x}{x^2-9} = \infty$, $\lim\limits_{x\to 3^-} \dfrac{x}{x^2-9} = -\infty$,

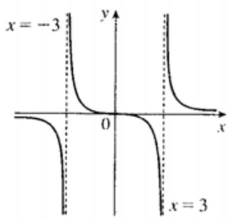
$\lim\limits_{x\to -3^+} \dfrac{x}{x^2-9} = \infty$, $\lim\limits_{x\to -3^-} \dfrac{x}{x^2-9} = -\infty$, so $x = 3$ and $x = -3$ are VA.

E. $f'(x) = \dfrac{(x^2-9) - x(2x)}{(x^2-9)^2} = -\dfrac{x^2+9}{(x^2-9)^2} < 0$ $(x \neq \pm 3)$ so f is decreasing on $(-\infty, -3)$, $(-3, 3)$, and $(3, \infty)$. **F.** No extremum

G. $$f''(x) = -\frac{2x(x^2-9)^2 - (x^2+9)\cdot 2(x^2-9)(2x)}{(x^2-9)^4}$$
$$= \frac{2x(x^2+27)}{(x^2-9)^3} > 0 \text{ when } -3 < x < 0 \text{ or } x > 3,$$

so f is CU on $(-3, 0)$ and $(3, \infty)$; CD on $(-\infty, -3)$ and $(0, 3)$. IP is $(0, 0)$.

H.



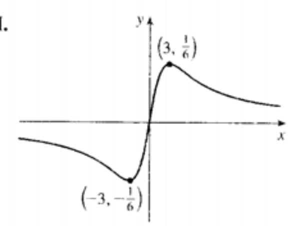
11. $y = f(x) = x/(x^2+9)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercept: $f(x) = 0 \Leftrightarrow x = 0$

C. $f(-x) = -f(x)$, so f is odd and the curve is symmetric about the origin. **D.** $\lim\limits_{x\to\pm\infty} [x/(x^2+9)] = 0$, so $y = 0$ is a HA; no VA **E.** $f'(x) = \dfrac{(x^2+9)(1) - x(2x)}{(x^2+9)^2} = \dfrac{9-x^2}{(x^2+9)^2} = \dfrac{(3+x)(3-x)}{(x^2+9)^2} > 0 \Leftrightarrow$ $-3 < x < 3$, so f is increasing on $(-3, 3)$ and decreasing on $(-\infty, -3)$ and $(3, \infty)$. **F.** Local minimum $f(-3) = -\frac{1}{6}$, local maximum $f(3) = \frac{1}{6}$

G. $$f''(x) = \frac{(x^2+9)^2(-2x) - (9-x^2)\cdot 2(x^2+9)(2x)}{\left[(x^2+9)^2\right]^2} = \frac{(2x)(x^2+9)\left[-(x^2+9) - 2(9-x^2)\right]}{(x^2+9)^4}$$
$$= \frac{2x(x^2-27)}{(x^2+9)^3} = 0 \Leftrightarrow x = 0, \pm\sqrt{27} = \pm 3\sqrt{3}$$

$f''(x) > 0 \Leftrightarrow -3\sqrt{3} < x < 0$ or $x > 3\sqrt{3}$, so f is CU on $\left(-3\sqrt{3}, 0\right)$ and $\left(3\sqrt{3}, \infty\right)$, and CD on $\left(-\infty, -3\sqrt{3}\right)$ and $\left(0, 3\sqrt{3}\right)$. There are three inflection points: $(0, 0)$ and $\left(\pm 3\sqrt{3}, \pm\frac{1}{12}\sqrt{3}\right)$.

H.



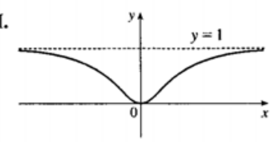
12. $y = f(x) = x^2/(x^2+9)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercept: $f(x) = 0 \Leftrightarrow x = 0$
C. $f(-x) = f(x)$, so f is even and symmetric about the y-axis. **D.** $\lim\limits_{x\to\pm\infty}\left[x^2/(x^2+9)\right] = 1$, so $y = 1$ is a HA; no VA **E.** $f'(x) = \dfrac{(x^2+9)(2x) - x^2(2x)}{(x^2+9)^2} = \dfrac{18x}{(x^2+9)^2} > 0 \Leftrightarrow x > 0$, so f is increasing on $(0,\infty)$ and decreasing on $(-\infty,0)$. **F.** Local minimum $f(0) = 0$; no local maximum

G. $f''(x) = \dfrac{(x^2+9)^2(18) - 18x\cdot 2(x^2+9)\cdot 2x}{\left[(x^2+9)^2\right]^2} = \dfrac{18(x^2+9)\left[(x^2+9)-4x^2\right]}{(x^2+9)^4} = \dfrac{18(9-3x^2)}{(x^2+9)^3}$

$= \dfrac{-54\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)}{(x^2+9)^3} > 0 \Leftrightarrow -\sqrt{3} < x < \sqrt{3}$

so f is CU on $\left(-\sqrt{3},\sqrt{3}\right)$ and CD on $\left(-\infty,-\sqrt{3}\right)$ and $\left(\sqrt{3},\infty\right)$. There are two inflection points: $\left(\pm\sqrt{3},\frac{1}{4}\right)$.

H.



13. $y = f(x) = \dfrac{1}{(x-1)(x+2)} = \dfrac{1}{x^2+x-2}$ **A.** $D = \{x \mid x \neq -2, 1\} = (-\infty,-2)\cup(-2,1)\cup(1,\infty)$

B. y-intercept: $f(0) = -\frac{1}{2}$; no x-intercept **C.** No symmetry

D. $\lim\limits_{x\to\pm\infty}\dfrac{1}{x^2+2x-2} = \lim\limits_{x\to\pm\infty}\dfrac{1/x^2}{1+1/x-2/x^2} = \dfrac{0}{1} = 0$, so $y = 0$ is a HA. $x = -2$ and $x = 1$ are VA. **E.** $f'(x) = \dfrac{(x^2+x-2)\cdot 0 - 1(2x+1)}{(x-1)^2(x+2)^2} = -\dfrac{2x+1}{(x-1)^2(x+2)^2} > 0 \Leftrightarrow x < -\frac{1}{2}\ (x \neq -2)$;

$f'(x) < 0 \Leftrightarrow x > -\frac{1}{2}\ (x \neq 1)$. So f is increasing on $(-\infty,-2)$ and $\left(-2,-\frac{1}{2}\right)$, and f is decreasing on $\left(-\frac{1}{2},1\right)$ and $(1,\infty)$. **F.** $f\left(-\frac{1}{2}\right) = -\frac{4}{9}$ is a local maximum.

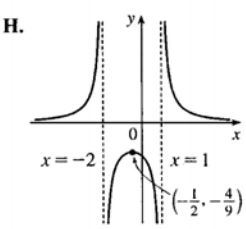
G. $f''(x) = \dfrac{(x^2+x-2)^2(-2) - [-(2x+1)](2)(x^2+x-2)(2x+1)}{\left[(x-1)^2(x+2)^2\right]^2}$

$= \dfrac{2(x^2+x-2)\left[-1(x^2+x-2)+(2x+1)^2\right]}{(x-1)^4(x+2)^4} = \dfrac{2(-x^2-x+2+4x^2+4x+1)}{(x-1)^3(x+2)^3}$

$= \dfrac{2(3x^2+3x+3)}{(x-1)^3(x+2)^3} = \dfrac{6(x^2+x+1)}{(x-1)^3(x+2)^3}$

The numerator is always positive, so the sign of f'' is determined by the denominator, which is negative only for $-2 < x < 1$. Thus, f is CD on $(-2,1)$ and CU on $(-\infty,-2)$ and $(1,\infty)$. No IP

H.

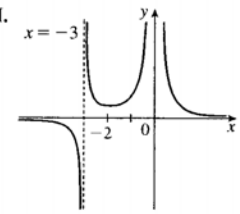


14. $y = f(x) = \dfrac{1}{x^2(x+3)}$ **A.** $D = \{x \mid x \neq 0, -3\} = (-\infty, -3) \cup (-3, 0) \cup (0, \infty)$ **B.** No intercept **C.** No symmetry **D.** $\lim\limits_{x\to\pm\infty} \dfrac{1}{x^2(x+3)} = 0$, so $y = 0$ is a HA. $\lim\limits_{x\to 0} \dfrac{1}{x^2(x+3)} = \infty$ and $\lim\limits_{x\to -3^+} \dfrac{1}{x^2(x+3)} = \infty$, $\lim\limits_{x\to -3^-} \dfrac{1}{x^2(x+3)} = -\infty$, so $x = 0$ and $x = -3$ are VA. **E.** $f'(x) = -\dfrac{3(x+2)}{x^3(x+3)^2} > 0 \iff -2 < x < 0$; $f'(x) < 0 \iff x < -2$ or $x > 0$. So f is increasing on $(-2, 0)$ and decreasing on $(-\infty, -3)$, $(-3, -2)$, and $(0, \infty)$. **F.** $f(-2) = \frac{1}{4}$ is a local minimum.

G.
$$f''(x) = -3\frac{x^3(x+3)^2 - (x+2)\left[3x^2(x+3)^2 + x^3 2(x+3)\right]}{x^6(x+3)^4} = \frac{6\left(2x^2+8x+9\right)}{x^4(x+3)^3}$$

Since $2x^2 + 8x + 9 > 0$ for all x, $f''(x) > 0 \iff x > -3$ $(x \neq 0)$, so f is CU on $(-3, 0)$ and $(0, \infty)$, and CD on $(-\infty, -3)$. No IP

H.



15. $y = f(x) = \dfrac{1+x^2}{1-x^2} = -1 + \dfrac{2}{1-x^2}$ **A.** $D = \{x \mid x \neq \pm 1\}$ **B.** No x-intercept, y-intercept $= f(0) = 1$ **C.** $f(-x) = f(x)$, so f is even and the curve is symmetric about the y-axis.

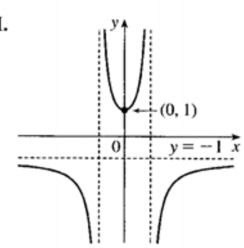
D. $\lim\limits_{x\to\pm\infty} \dfrac{1+x^2}{1-x^2} = \lim\limits_{x\to\pm\infty} \dfrac{(1/x^2)+1}{(1/x^2)-1} = -1$, so $y = -1$ is a HA. $\lim\limits_{x\to 1^-} \dfrac{1+x^2}{1-x^2} = \infty$, $\lim\limits_{x\to 1^+} \dfrac{1+x^2}{1-x^2} = -\infty$, $\lim\limits_{x\to -1^-} \dfrac{1+x^2}{1-x^2} = -\infty$, $\lim\limits_{x\to -1^+} \dfrac{1+x^2}{1-x^2} = \infty$. So $x = 1$ and $x = -1$ are VA.

E. $f'(x) = \dfrac{4x}{\left(1-x^2\right)^2} > 0 \iff x > 0$ $(x \neq 1)$, so f increases on $(0, 1)$ and $(1, \infty)$, and decreases on $(-\infty, -1)$ and $(-1, 0)$.

F. $f(0) = 1$ is a local minimum.

G. $y'' = \dfrac{4\left(1-x^2\right)^2 - 4x \cdot 2\left(1-x^2\right)(-2x)}{\left(1-x^2\right)^4} = \dfrac{4\left(1+3x^2\right)}{\left(1-x^2\right)^3} > 0 \iff$ $x^2 < 1 \iff -1 < x < 1$, so f is CU on $(-1, 1)$ and CD on $(-\infty, -1)$ and $(1, \infty)$. No IP

H.



16. $y = f(x) = \dfrac{x^3 - 1}{x^3 + 1}$ **A.** $D = \{x \mid x \neq -1\} = (-\infty, -1) \cup (-1, \infty)$ **B.** x-intercept $= 1$, y-intercept $= f(0) = -1$ **C.** No symmetry **D.** $\lim\limits_{x \to \pm\infty} \dfrac{x^3 - 1}{x^3 + 1} = \lim\limits_{x \to \pm\infty} \dfrac{1 - 1/x^3}{1 + 1/x^3} = 1$, so $y = 1$ is a HA. $\lim\limits_{x \to -1^-} \dfrac{x^3 - 1}{x^3 + 1} = \infty$ and $\lim\limits_{x \to -1^+} \dfrac{x^3 - 1}{x^3 + 1} = -\infty$, so $x = -1$ is a VA.

E. $f'(x) = \dfrac{(x^3 + 1)(3x^2) - (x^3 - 1)(3x^2)}{(x^3 + 1)^2} = \dfrac{6x^2}{(x^3 + 1)^2} > 0$ $(x \neq -1)$ so f is increasing on $(-\infty, -1)$ and $(-1, \infty)$. **F.** No extremum

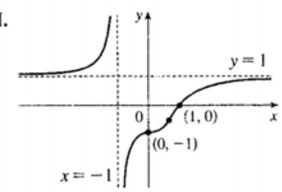
G. $y'' = \dfrac{12x(x^3 + 1)^2 - 6x^2 \cdot 2(x^3 + 1) \cdot 3x^2}{(x^3 + 1)^4}$

$= \dfrac{12x(1 - 2x^3)}{(x^3 + 1)^3} > 0 \Leftrightarrow x < -1 \text{ or } 0 < x < \frac{1}{\sqrt[3]{2}}$,

so f is CU on $(-\infty, -1)$ and $\left(0, \frac{1}{\sqrt[3]{2}}\right)$ and CD on $(-1, 0)$ and $\left(\frac{1}{\sqrt[3]{2}}, \infty\right)$.

IP $(0, -1)$, $\left(\frac{1}{\sqrt[3]{2}}, -\frac{1}{3}\right)$

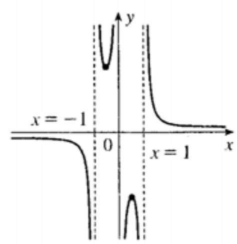
H.



17. $y = f(x) = \dfrac{1}{x^3 - x} = \dfrac{1}{x(x - 1)(x + 1)}$ **A.** $D = \{x \mid x \neq 0, \pm 1\}$ **B.** No intercept **C.** $f(-x) = -f(x)$, symmetric about $(0, 0)$ **D.** $\lim\limits_{x \to \pm\infty} \dfrac{1}{x^3 - x} = 0$, so $y = 0$ is a HA. $\lim\limits_{x \to 0^-} \dfrac{1}{x^3 - x} = \infty$, $\lim\limits_{x \to 0^+} \dfrac{1}{x^3 - x} = -\infty$, $\lim\limits_{x \to 1^-} \dfrac{1}{x^3 - x} = -\infty$, $\lim\limits_{x \to 1^+} \dfrac{1}{x^3 - x} = \infty$, $\lim\limits_{x \to -1^-} \dfrac{1}{x^3 - x} = -\infty$, $\lim\limits_{x \to -1^+} \dfrac{1}{x^3 - x} = \infty$. So $x = 0$, $x = 1$, and $x = -1$ are VA. **E.** $f'(x) = \dfrac{1 - 3x^2}{(x^3 - x)^2} \Rightarrow f'(x) > 0 \Leftrightarrow x^2 < \frac{1}{3} \Leftrightarrow -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$ $(x \neq 0)$, so f is increasing on $\left(-\frac{1}{\sqrt{3}}, 0\right)$, $\left(0, \frac{1}{\sqrt{3}}\right)$ and decreasing on $(-\infty, -1)$, $\left(-1, -\frac{1}{\sqrt{3}}\right)$, $\left(\frac{1}{\sqrt{3}}, 1\right)$, and $(1, \infty)$. **F.** Local minimum $f\left(-\frac{1}{\sqrt{3}}\right) = \frac{3\sqrt{3}}{2}$, local maximum $f\left(\frac{1}{\sqrt{3}}\right) = -\frac{3\sqrt{3}}{2}$

G. $f''(x) = \dfrac{2(6x^4 - 3x^2 + 1)}{(x^3 - x)^3}$. Since $6x^4 - 3x^2 + 1$ has negative discriminant as a quadratic in x^2, it is positive, so $f''(x) > 0 \Leftrightarrow x^3 - x > 0 \Leftrightarrow x > 1 \text{ or } -1 < x < 0$. f is CU on $(-1, 0)$ and $(1, \infty)$, and CD on $(-\infty, -1)$ and $(0, 1)$. No IP

H.

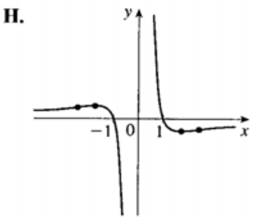


18. $y = f(x) = \dfrac{1-x^2}{x^3} = \dfrac{1}{x^3} - \dfrac{1}{x}$ **A.** $D = \{x \mid x \neq 0\}$ **B.** x-intercepts ± 1, no y-intercept

C. $f(-x) = -f(x)$, so the curve is symmetric about $(0, 0)$. **D.** $\lim\limits_{x\to\pm\infty} \dfrac{1-x^2}{x^3} = 0$, so $y = 0$ is a HA.

$\lim\limits_{x\to 0^+} \dfrac{1-x^2}{x^3} = \infty$, $\lim\limits_{x\to 0^-} \dfrac{1-x^2}{x^3} = -\infty$, so $x = 0$ is a VA. **E.** $f'(x) = -\dfrac{3}{x^4} + \dfrac{1}{x^2} = \dfrac{x^2-3}{x^4} > 0 \Leftrightarrow$ $|x| > \sqrt{3}$, so f is increasing on $\left(-\infty, -\sqrt{3}\right)$, $\left(\sqrt{3}, \infty\right)$ and decreasing on $\left(-\sqrt{3}, 0\right)$ and $\left(0, \sqrt{3}\right)$. **F.** $f\left(\sqrt{3}\right) = -\frac{2}{3\sqrt{3}}$ is a local minimum, $f\left(-\sqrt{3}\right) = \frac{2}{3\sqrt{3}}$ is a local maximum.

H.



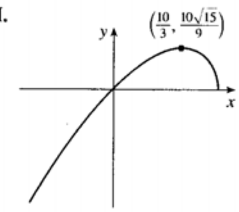
G. $f''(x) = \dfrac{12}{x^5} - \dfrac{2}{x^3} = \dfrac{2\left(6-x^2\right)}{x^5} > 0 \Leftrightarrow x < -\sqrt{6}$ or $0 < x < \sqrt{6}$, so f is CU on $\left(-\infty, -\sqrt{6}\right)$, $\left(0, \sqrt{6}\right)$ and CD on $\left(-\sqrt{6}, 0\right)$ and $\left(\sqrt{6}, \infty\right)$. IP $\left(\sqrt{6}, -\frac{5}{6\sqrt{6}}\right)$ and $\left(-\sqrt{6}, \frac{5}{6\sqrt{6}}\right)$.

19. $y = f(x) = x\sqrt{5-x}$ **A.** The domain is $\{x \mid 5 - x \geq 0\} = (-\infty, 5]$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Leftrightarrow x = 0, 5$ **C.** No symmetry **D.** No asymptote

E. $f'(x) = x \cdot \frac{1}{2}(5-x)^{-1/2}(-1) + (5-x)^{1/2} \cdot 1 = \frac{1}{2}(5-x)^{-1/2}[-x + 2(5-x)] = \dfrac{10-3x}{2\sqrt{5-x}} > 0 \Leftrightarrow$ $x < \frac{10}{3}$, so f is increasing on $\left(-\infty, \frac{10}{3}\right)$ and decreasing on $\left(\frac{10}{3}, 5\right)$.

F. Local maximum $f\left(\frac{10}{3}\right) = \frac{10}{9}\sqrt{15} \approx 4.3$; no local minimum

H.

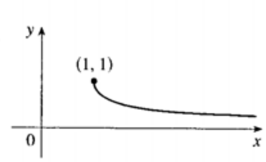


G.
$$f''(x) = \frac{2(5-x)^{1/2}(-3) - (10-3x) \cdot 2\left(\frac{1}{2}\right)(5-x)^{-1/2}(-1)}{\left(2\sqrt{5-x}\right)^2}$$
$$= \frac{(5-x)^{-1/2}[-6(5-x) + (10-3x)]}{4(5-x)} = \frac{3x-20}{4(5-x)^{3/2}}$$

$f''(x) < 0$ for $x < 5$, so f is CD on $(-\infty, 5)$. No IP

20. $y = f(x) = \sqrt{x} - \sqrt{x-1}$ **A.** $D = \{x \mid x \geq 0 \text{ and } x \geq 1\} = \{x \mid x \geq 1\} = [1, \infty)$ **B.** No intercepts **C.** No symmetry **D.** $\lim\limits_{x\to\infty}\left(\sqrt{x} - \sqrt{x-1}\right) = \lim\limits_{x\to\infty}\left(\sqrt{x} - \sqrt{x-1}\right)\dfrac{\sqrt{x}+\sqrt{x-1}}{\sqrt{x}+\sqrt{x-1}} = \lim\limits_{x\to\infty} \dfrac{1}{\sqrt{x}+\sqrt{x-1}} = 0$, so $y = 0$ is a HA. **E.** $f'(x) = \dfrac{1}{2\sqrt{x}} - \dfrac{1}{2\sqrt{x-1}} < 0$ for all $x > 1$, since $x - 1 < x \Rightarrow$ $\sqrt{x-1} < \sqrt{x}$, so f is decreasing on $(1, \infty)$. **F.** No local extremum

H.



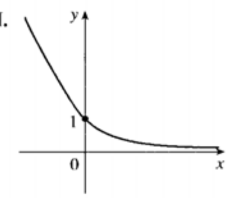
G. $f''(x) = -\dfrac{1}{4}\left[\dfrac{1}{x^{3/2}} - \dfrac{1}{(x-1)^{3/2}}\right] \Rightarrow f''(x) > 0$ for $x > 1$, so f is CU on $(1, \infty)$.

21. $y = f(x) = \sqrt{x^2+1} - x$ **A.** $D = \mathbb{R}$ **B.** No x-intercept, y-intercept $= 1$ **C.** No symmetry **D.** $\lim_{x\to-\infty}\left(\sqrt{x^2+1}-x\right) = \infty$ and

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right) = \lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} = \lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x} = 0,$$

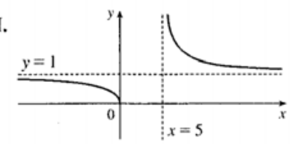
so $y = 0$ is a HA. **E.** $f'(x) = \dfrac{x}{\sqrt{x^2+1}} - 1 = \dfrac{x-\sqrt{x^2+1}}{\sqrt{x^2+1}} \Rightarrow$ $f'(x) < 0$, so f is decreasing on $\mathbb{R}$. **F.** No extremum

G. $f''(x) = \dfrac{1}{\left(x^2+1\right)^{3/2}} > 0$, so f is CU on $\mathbb{R}$. No IP

H. 

22. $y = f(x) = \sqrt{x/(x-5)}$ **A.** $D = \{x \mid x/(x-5) \geq 0\} = (-\infty, 0] \cup (5, \infty)$. **B.** Intercepts are 0. **C.** No symmetry **D.** $\lim_{x\to\pm\infty}\sqrt{\dfrac{x}{x-5}} = \lim_{x\to\pm\infty}\sqrt{\dfrac{1}{1-5/x}} = 1$, so $y = 1$ is a HA. $\lim_{x\to5^+}\sqrt{\dfrac{x}{x-5}} = \infty$, so $x = 5$ is a VA. **E.** $f'(x) = \dfrac{1}{2}\left(\dfrac{x}{x-5}\right)^{-1/2}\dfrac{(-5)}{(x-5)^2} = -\frac{5}{2}\left[x(x-5)^3\right]^{-1/2} < 0$, so f is decreasing on $(-\infty, 0)$ and $(5, \infty)$. **F.** No local extremum

G. $f''(x) = \frac{5}{4}\left[x(x-5)^3\right]^{-3/2}(x-5)^2(4x-5) > 0$ for $x > 5$, and $f''(x) < 0$ for $x < 0$, so f is CU on $(5, \infty)$ and CD on $(-\infty, 0)$. No IP

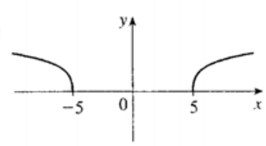
H. 

23. $y = f(x) = \sqrt[4]{x^2-25}$ **A.** $D = \left\{x \mid x^2 \geq 25\right\} = (-\infty, -5] \cup [5, \infty)$ **B.** x-intercepts are ±5, no y-intercept **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** $\lim_{x\to\pm\infty}\sqrt[4]{x^2-25} = \infty$, no asymptote

E. $f'(x) = \frac{1}{4}\left(x^2-25\right)^{-3/4}(2x) = \dfrac{x}{2\left(x^2-25\right)^{3/4}} > 0$ if $x > 5$, so f is increasing on $(5, \infty)$ and decreasing on $(-\infty, -5)$. **F.** No local extremum

$$\textbf{G. } y'' = \frac{2\left(x^2-25\right)^{3/4} - 3x^2\left(x^2-25\right)^{-1/4}}{4\left(x^2-25\right)^{3/2}} = -\frac{x^2+50}{4\left(x^2-25\right)^{7/4}} < 0$$

so f is CD on $(-\infty, -5)$ and $(5, \infty)$. No IP

H. 

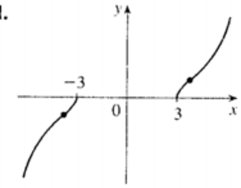
24. $y = f(x) = x\sqrt{x^2-9}$ **A.** $D = \left\{x \mid x^2 \geq 9\right\} = (-\infty, -3] \cup [3, \infty)$ **B.** x-intercepts are ±3, no y-intercept. **C.** $f(-x) = -f(x)$, so the curve is symmetric about the origin. **D.** $\lim_{x\to\infty}\sqrt{x^2-9} = \infty$, $\lim_{x\to-\infty}\sqrt{x^2-9} = -\infty$, no asymptote **E.** $f'(x) = \sqrt{x^2-9} + \dfrac{x^2}{\sqrt{x^2-9}} > 0$ for $x \in D$, so f is increasing on $(-\infty, -3)$ and $(3, \infty)$. **F.** No extremum

G. $f''(x) = \dfrac{x}{\sqrt{x^2-9}} + \dfrac{2x\sqrt{x^2-9} - x^2\left(x/\sqrt{x^2-9}\right)}{x^2-9}$

$= \dfrac{x\left(2x^2-27\right)}{\left(x^2-9\right)^{3/2}} > 0 \quad\Leftrightarrow\quad x > 3\sqrt{\frac{3}{2}}$ or $-3\sqrt{\frac{3}{2}} < x < 0$,

so f is CU on $\left(3\sqrt{\frac{3}{2}}, \infty\right)$ and $\left(-3\sqrt{\frac{3}{2}}, -3\right)$ and CD on $\left(-\infty, -3\sqrt{\frac{3}{2}}\right)$ and $\left(3, 3\sqrt{\frac{3}{2}}\right)$. IP $\left(\pm 3\sqrt{\frac{3}{2}}, \pm\frac{9\sqrt{3}}{2}\right)$

H.



25. $y = f(x) = \dfrac{\sqrt{1-x^2}}{x}$ **A.** $D = \{x \mid |x| \le 1, x \ne 0\} = [-1, 0) \cup (0, 1]$ **B.** x-intercepts ± 1, no y-intercept

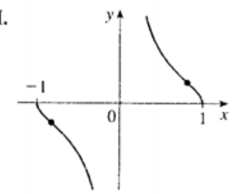
C. $f(-x) = -f(x)$, so the curve is symmetric about $(0, 0)$. **D.** $\lim\limits_{x\to 0^+} \dfrac{\sqrt{1-x^2}}{x} = \infty$, $\lim\limits_{x\to 0^-} \dfrac{\sqrt{1-x^2}}{x} = -\infty$,

so $x = 0$ is a VA. **E.** $f'(x) = \dfrac{\left(-x^2/\sqrt{1-x^2}\right) - \sqrt{1-x^2}}{x^2} = -\dfrac{1}{x^2\sqrt{1-x^2}} < 0$, so f is decreasing on $(-1, 0)$ and $(0, 1)$. **F.** No extremum

G. $f''(x) = \dfrac{2-3x^2}{x^3\left(1-x^2\right)^{3/2}} > 0 \quad\Leftrightarrow\quad -1 < x < -\sqrt{\frac{2}{3}}$ or

$0 < x < \sqrt{\frac{2}{3}}$, so f is CU on $\left(-1, -\sqrt{\frac{2}{3}}\right)$ and $\left(0, \sqrt{\frac{2}{3}}\right)$ and CD on $\left(-\sqrt{\frac{2}{3}}, 0\right)$ and $\left(\sqrt{\frac{2}{3}}, 1\right)$. IP $\left(\pm\sqrt{\frac{2}{3}}, \pm\frac{1}{\sqrt{2}}\right)$.

H.



26. $y = f(x) = \dfrac{x+1}{\sqrt{x^2+1}}$ **A.** $D = \mathbb{R}$ **B.** x-intercept -1, y-intercept 1 **C.** No symmetry

D. $\lim\limits_{x\to\infty} \dfrac{x+1}{\sqrt{x^2+1}} = 1$, and $\lim\limits_{x\to-\infty} \dfrac{x+1}{\sqrt{x^2+1}} = -1$, so horizontal asymptotes are $y = \pm 1$.

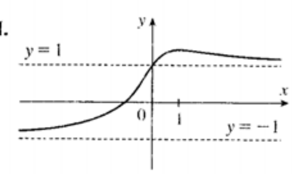
E. $f'(x) = \dfrac{\sqrt{x^2+1} - \dfrac{1}{2\sqrt{x^2+1}}(2x)(x+1)}{\left(x^2+1\right)} = \dfrac{1-x}{\left(x^2+1\right)^{3/2}} > 0 \quad\Leftrightarrow\quad x < 1$, so f is increasing on $(-\infty, 1)$, and decreasing on $(1, \infty)$. **F.** $f(1) = \sqrt{2}$ is a local maximum.

G. $f''(x) = \dfrac{-1\left(x^2+1\right)^{3/2} - \frac{3}{2}\left(x^2+1\right)^{1/2}(2x)(1-x)}{\left(x^2+1\right)^3} = \dfrac{2x^2-3x-1}{\left(x^2+1\right)^{5/2}}$.

$f''(x) = 0 \quad\Leftrightarrow\quad 2x^2 - 3x - 1 = 0 \quad\Leftrightarrow$

$x = \dfrac{3 \pm \sqrt{9 - 4(2)(-1)}}{2(2)} = \dfrac{3 \pm \sqrt{17}}{4}$. $f(x)$ is CU on $\left(-\infty, \frac{3-\sqrt{17}}{4}\right)$ and $\left(\frac{3+\sqrt{17}}{4}, \infty\right)$ and CD on $\left(\frac{3-\sqrt{17}}{4}, \frac{3+\sqrt{17}}{4}\right)$. IP at $x = \frac{3\pm\sqrt{17}}{4}$

H.



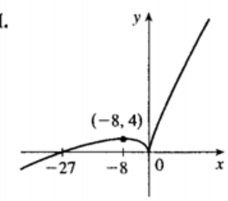
27. $y = f(x) = x + 3x^{2/3}$ **A.** $D = \mathbb{R}$ **B.** $y = x + 3x^{2/3} = x^{2/3}\left(x^{1/3} + 3\right) = 0$ if $x = 0$ or -27 (x-intercepts), y-intercept $= f(0) = 0$ **C.** No symmetry **D.** $\lim_{x\to\infty}\left(x + 3x^{2/3}\right) = \infty$,

$\lim_{x\to-\infty}\left(x + 3x^{2/3}\right) = \lim_{x\to-\infty} x^{2/3}\left(x^{1/3} + 3\right) = -\infty$, no asymptote

E. $f'(x) = 1 + 2x^{-1/3} = \left(x^{1/3} + 2\right)/x^{1/3} > 0 \Leftrightarrow x > 0$ or $x < -8$, so f increases on $(-\infty, -8)$, $(0, \infty)$ and decreases on $(-8, 0)$. **F.** Local maximum $f(-8) = 4$, local minimum $f(0) = 0$

G. $f''(x) = -\frac{2}{3}x^{-4/3} < 0$ $(x \neq 0)$ so f is CD on $(-\infty, 0)$ and $(0, \infty)$. No IP

H.



28. $y = f(x) = x^{5/3} - 5x^{2/3} = x^{2/3}(x - 5)$ **A.** $D = \mathbb{R}$ **B.** x-intercepts 0, 5, y-intercept 0 **C.** No symmetry

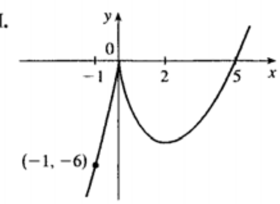
D. $\lim_{x\to\pm\infty} x^{2/3}(x - 5) = \pm\infty$, so there is no asymptote

E. $f'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5}{3}x^{-1/3}(x - 2) > 0 \Leftrightarrow x < 0$ or $x > 2$, so f is increasing on $(-\infty, 0)$, $(2, \infty)$ and decreasing on $(0, 2)$.

F. $f(0) = 0$ is a local maximum. $f(2) = -3\sqrt[3]{4}$ is a local minimum

G. $f''(x) = \frac{10}{9}x^{-1/3} + \frac{10}{9}x^{-4/3} = \frac{10}{9}x^{-4/3}(x + 1) > 0 \Leftrightarrow x > -1$, so f is CU on $(-1, 0)$ and $(0, \infty)$, CD on $(-\infty, -1)$. IP $(-1, -6)$

H.



29. $y = f(x) = x + \sqrt{|x|}$ **A.** $D = \mathbb{R}$ **B.** x-intercepts $= 0, -1$, y-intercept 0 **C.** No symmetry

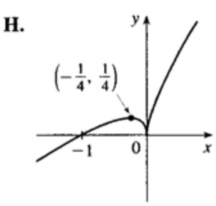
D. $\lim_{x\to\infty}\left(x + \sqrt{|x|}\right) = \infty$, $\lim_{x\to-\infty}\left(x + \sqrt{|x|}\right) = -\infty$. No asymptote **E.** For $x > 0$, $f(x) = x + \sqrt{x} \Rightarrow$

$f'(x) = 1 + \dfrac{1}{2\sqrt{x}} > 0$, so f increases on $(0, \infty)$.

For $x < 0$, $f(x) = x + \sqrt{-x} \to f'(x) = 1 - \dfrac{1}{2\sqrt{-x}} > 0 \Leftrightarrow$

$2\sqrt{-x} > 1 \Leftrightarrow -x > \frac{1}{4} \Leftrightarrow x < -\frac{1}{4}$, so f increases on $\left(-\infty, -\frac{1}{4}\right)$ and decreases on $\left(-\frac{1}{4}, 0\right)$. **F.** $f\left(-\frac{1}{4}\right) = \frac{1}{4}$ is a local maximum, $f(0) = 0$ is a local minimum. **G.** For $x > 0$, $f''(x) = -\frac{1}{4}x^{-3/2} \Rightarrow$ $f''(x) < 0$, so f is CD on $(0, \infty)$. For $x < 0$, $f''(x) = -\frac{1}{4}(-x)^{-3/2}$ $\Rightarrow f''(x) < 0$, so f is CD on $(-\infty, 0)$. No IP

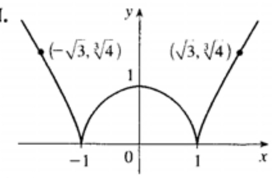
H.



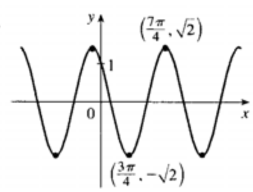
30. $y = f(x) = (x^2 - 1)^{2/3}$ **A.** $D = \mathbb{R}$ **B.** x-intercepts ± 1, y-intercept 1 **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** $\lim_{x \to \pm\infty} (x^2 - 1)^{2/3} = \infty$, no asymptote **E.** $f'(x) = \frac{4}{3}x(x^2 - 1)^{-1/3} \Rightarrow$ $f'(x) > 0 \Leftrightarrow x > 1$ or $-1 < x < 0$, $f'(x) < 0 \Leftrightarrow x < -1$ or $0 < x < 1$. So f is increasing on $(-1, 0)$, $(1, \infty)$ and decreasing on $(-\infty, -1)$, $(0, 1)$.
F. $f(-1) = f(1) = 0$ are local minima, $f(0) = 1$ is a local maximum.
G. $f''(x) = \frac{4}{3}(x^2 - 1)^{-1/3} + \frac{4}{3}x\left(-\frac{1}{3}\right)(x^2 - 1)^{-4/3}(2x)$

$$= \frac{4}{9}(x^2 - 3)(x^2 - 1)^{-4/3} > 0 \Leftrightarrow |x| > \sqrt{3}$$

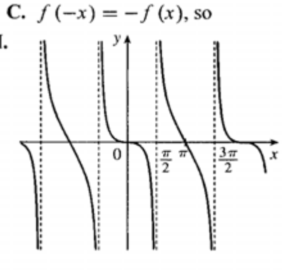
so f is CU on $\left(-\infty, -\sqrt{3}\right)$, $\left(\sqrt{3}, \infty\right)$ and CD on $\left(-\sqrt{3}, -1\right)$, $(-1, 1)$, $\left(1, \sqrt{3}\right)$. IP $\left(\pm\sqrt{3}, \sqrt[3]{4}\right)$

H.


31. $y = f(x) = \cos x - \sin x$ **A.** $D = \mathbb{R}$ **B.** $y = 0 \Leftrightarrow \cos x = \sin x \Leftrightarrow x = n\pi + \frac{\pi}{4}$, n an integer (x-intercepts), y-intercept $= f(0) = 1$. **C.** Periodic with period 2π **D.** No asymptote
E. $f'(x) = -\sin x - \cos x = 0 \Leftrightarrow \cos x = -\sin x \Leftrightarrow x = 2n\pi + \frac{3\pi}{4}$ or $2n\pi + \frac{7\pi}{4}$. $f'(x) > 0 \Leftrightarrow$ $\cos x < -\sin x \Leftrightarrow 2n\pi + \frac{3\pi}{4} < x < 2n\pi + \frac{7\pi}{4}$, so f is increasing on $\left(2n\pi + \frac{3\pi}{4}, 2n\pi + \frac{7\pi}{4}\right)$ and decreasing on $\left(2n\pi - \frac{\pi}{4}, 2n\pi + \frac{3\pi}{4}\right)$. **F.** Local maxima $f\left(2n\pi - \frac{\pi}{4}\right) = \sqrt{2}$, local minima $f\left(2n\pi + \frac{3\pi}{4}\right) = -\sqrt{2}$. **G.** $f''(x) = -\cos x + \sin x > 0 \Leftrightarrow$ $\sin x > \cos x \Leftrightarrow x \in \left(2n\pi + \frac{\pi}{4}, 2n\pi + \frac{5\pi}{4}\right)$, so f is CU on these intervals and CD on $\left(2n\pi - \frac{3\pi}{4}, 2n\pi + \frac{\pi}{4}\right)$. IP $\left(n\pi + \frac{\pi}{4}, 0\right)$

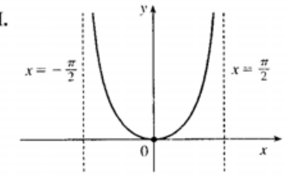
H.


32. $y = f(x) = \sin x - \tan x$ **A.** $D = \left\{x \mid x \neq (2n+1)\frac{\pi}{2}\right\}$ **B.** $y = 0 \Leftrightarrow \sin x = \tan x = \dfrac{\sin x}{\cos x} \Leftrightarrow$ $\sin x = 0$ or $\cos x = 1 \Leftrightarrow x = n\pi$ (x-intercepts), y-intercept $= f(0) = 0$ **C.** $f(-x) = -f(x)$, so the curve is symmetric about $(0, 0)$. Also periodic with period 2π
D. $\lim_{x \to \pi/2^-} (\sin x - \tan x) = -\infty$ and $\lim_{x \to \pi/2^+} (\sin x - \tan x) = \infty$, so $x = n\pi + \frac{\pi}{2}$ are VA. **E.** $f'(x) = \cos x - \sec^2 x \le 0$, so f decreases on each interval in its domain, that is, on $\left((2n-1)\frac{\pi}{2}, (2n+1)\frac{\pi}{2}\right)$.
F. No extremum **G.** $f''(x) = -\sin x - 2\sec^2 x \tan x =$ $-\sin x\left(1 + 2\sec^3 x\right)$. Note that $1 + 2\sec^3 x \neq 0$ since $\sec^3 x \neq -\frac{1}{2}$. $f''(x) > 0$ for $-\frac{\pi}{2} < x < 0$ and $\frac{3\pi}{2} < x < 2\pi$, so f is CU on $\left(\left(n - \frac{1}{2}\right)\pi, n\pi\right)$ and CD on $\left(n\pi, \left(n + \frac{1}{2}\right)\pi\right)$. IP $(n\pi, 0)$. Note also that $f'(0) = 0$ but $f'(\pi) = -2$.

H.


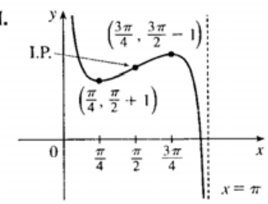
33. $y = f(x) = x \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ **A.** $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **B.** Intercepts are 0 **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** $\lim\limits_{x \to \pi/2^-} x \tan x = \infty$ and $\lim\limits_{x \to -\pi/2^+} x \tan x = \infty$, so $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are VA. **E.** $f'(x) = \tan x + x \sec^2 x > 0 \quad \Leftrightarrow \quad 0 < x < \frac{\pi}{2}$, so f increases on $\left(0, \frac{\pi}{2}\right)$ and decreases on $\left(-\frac{\pi}{2}, 0\right)$. **F.** Absolute minimum $f(0) = 0$. **G.** $y'' = 2\sec^2 x + 2x \tan x \sec^2 x > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, so f is CU on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. No IP

H.



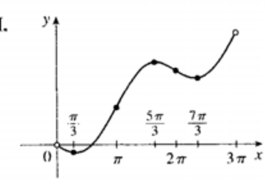
34. $y = f(x) = 2x + \cot x$, $0 < x < \pi$ **A.** $D = (0, \pi)$. **B.** No y-intercept **C.** No symmetry **D.** $\lim\limits_{x \to 0^+} (2x + \cot x) = \infty$, $\lim\limits_{x \to \pi^-} (2x + \cot x) = -\infty$, so $x = 0$ and $x = \pi$ are VA. **E.** $f'(x) = 2 - \csc^2 x > 0$ when $\csc^2 x < 2 \quad \Leftrightarrow \quad \sin x > \frac{1}{\sqrt{2}} \quad \Leftrightarrow \quad \frac{\pi}{4} < x < \frac{3\pi}{4}$, so f is increasing on $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ and decreasing on $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{3\pi}{4}, \pi\right)$. **F.** $f\left(\frac{\pi}{4}\right) = 1 + \frac{\pi}{2}$ is a local minimum, $f\left(\frac{3\pi}{4}\right) = \frac{3\pi}{2} - 1$ is a local maximum.
G. $f''(x) = -2\csc x\,(-\csc x \cot x) = 2\csc^2 x \cot x > 0 \Leftrightarrow \cot x > 0 \Leftrightarrow 0 < x < \frac{\pi}{2}$, so f is CU on $\left(0, \frac{\pi}{2}\right)$, CD on $\left(\frac{\pi}{2}, \pi\right)$. IP $\left(\frac{\pi}{2}, \pi\right)$

H.



35. $y = f(x) = x/2 - \sin x$, $0 < x < 3\pi$ **A.** $D = (0, 3\pi)$ **B.** No y-intercept. The x-intercept, approximately 1.9, can be found using Newton's Method. **C.** No symmetry **D.** No asymptote **E.** $f'(x) = \frac{1}{2} - \cos x > 0 \quad \Leftrightarrow \quad \cos x < \frac{1}{2} \quad \Leftrightarrow \quad \frac{\pi}{3} < x < \frac{5\pi}{3}$ or $\frac{7\pi}{3} < x < 3\pi$, so f is increasing on $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$ and $\left(\frac{7\pi}{3}, 3\pi\right)$ and decreasing on $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, \frac{7\pi}{3}\right)$. **F.** $f\left(\frac{\pi}{3}\right) = \frac{\pi}{6} - \frac{\sqrt{3}}{2}$ is a local minimum, $f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{6} + \frac{\sqrt{3}}{2}$ is a local maximum, $f\left(\frac{7\pi}{3}\right) = \frac{7\pi}{6} - \frac{\sqrt{3}}{2}$ is a local minimum. **G.** $f''(x) = \sin x > 0 \quad \Leftrightarrow \quad 0 < x < \pi$ or $2\pi < x < 3\pi$, so f is CU on $(0, \pi)$ and $(2\pi, 3\pi)$ and CD on $(\pi, 2\pi)$. IP $\left(\pi, \frac{\pi}{2}\right)$ and $(2\pi, \pi)$.

H.

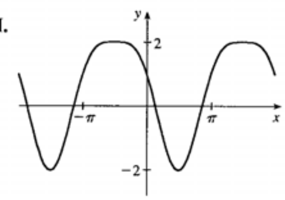


36. $y = f(x) = \cos^2 x - 2\sin x$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 1$ **C.** No symmetry, but f has period 2π.
D. No asymptote **E.** $y' = 2\cos x(-\sin x) - 2\cos x = -2\cos x(\sin x + 1)$. $y' = 0 \Leftrightarrow \cos x = 0$ or $\sin x = -1 \Leftrightarrow x = (2n+1)\frac{\pi}{2}$. $y' > 0$ when $\cos x < 0$ since $\sin x + 1 \geq 0$ for all x. So $y' > 0$ and f is increasing on $\left((4n+1)\frac{\pi}{2}, (4n+3)\frac{\pi}{2}\right)$; $y' < 0$ and f is decreasing on $\left((4n-1)\frac{\pi}{2}, (4n+1)\frac{\pi}{2}\right)$.
F. Local maxima are $f\left((4n+3)\frac{\pi}{2}\right) = 2$; local minima are $f\left((4n+1)\frac{\pi}{2}\right) = -2$.
G. $y' = -2\cos x(\sin x + 1) = -\sin 2x - 2\cos x \Rightarrow$

$$y'' = -2\cos 2x + 2\sin x = -2\left(1 - 2\sin^2 x\right) + 2\sin x$$
$$= 4\sin^2 x + 2\sin x - 2 = 2(2\sin x - 1)(\sin x + 1)$$

$y'' = 0 \Leftrightarrow \sin x = \frac{1}{2}$ or $-1 \Rightarrow x = \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi$, or $\frac{3\pi}{2} + 2n\pi$. $y'' > 0$ and f is CU on $\left(\frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi\right)$; $y'' \leq 0$ and f is CD on $\left(\frac{5\pi}{6} + 2n\pi, \frac{\pi}{6} + 2(n+1)\pi\right)$. There are inflection points at $\left(\frac{\pi}{6} + 2n\pi, -\frac{1}{4}\right)$ and $\left(\frac{5\pi}{6} + 2n\pi, -\frac{1}{4}\right)$.

H.

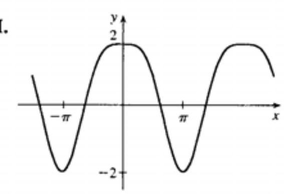


37. $y = f(x) = 2\cos x + \sin^2 x$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 2$ **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. Periodic with period 2π **D.** No asymptote
E. $f'(x) = -2\sin x + 2\sin x\cos x = 2\sin x(\cos x - 1) > 0 \Leftrightarrow \sin x < 0 \Leftrightarrow (2n-1)\pi < x < 2n\pi$, so f is increasing on $((2n-1)\pi, 2n\pi)$ and decreasing on $(2n\pi, (2n+1)\pi)$. **F.** $f(2n\pi) = 2$ is a local maximum. $f((2n+1)\pi) = -2$ is a local minimum.
G. $f''(x) = -2\cos x + 2\cos 2x = 2\left(2\cos^2 x - \cos x - 1\right)$

$$= 2(2\cos x + 1)(\cos x - 1) > 0$$

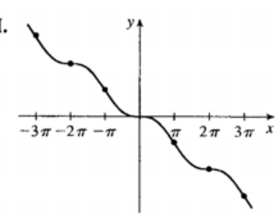
$\Leftrightarrow \cos x < -\frac{1}{2} \Leftrightarrow x \in \left(2n\pi + \frac{2\pi}{3}, 2n\pi + \frac{4\pi}{3}\right)$, so f is CU on these intervals and CD on $\left(2n\pi - \frac{2\pi}{3}, 2n\pi + \frac{2\pi}{3}\right)$. IP when $x = 2n\pi \pm \frac{2\pi}{3}$

H.



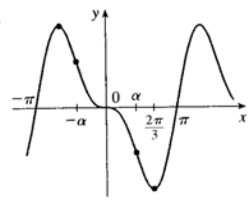
38. $f(x) = \sin x - x$ **A.** $D = \mathbb{R}$ **B.** x-intercept $= 0 = y$-intercept
C. $f(-x) = \sin(-x) - (-x) = -(\sin x - x) = -f(-x)$, so f is odd.
D. No asymptote **E.** $f'(x) = \cos x - 1 \leq 0$ for all x, so f is decreasing on $(-\infty, \infty)$. **F.** No local extremum **G.** $f''(x) = -\sin x \Rightarrow$ $f''(x) > 0 \Leftrightarrow \sin x < 0 \Leftrightarrow (2n-1)\pi < x < 2n\pi$, so f is CU on $((2n-1)\pi, 2n\pi)$ and CD on $(2n\pi, (2n+1)\pi)$, n an integer. Points of inflection occur when $x = n\pi$.

H.



39. $y = f(x) = \sin 2x - 2\sin x$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 0$. $y = 0 \Leftrightarrow$ $2\sin x = \sin 2x = 2\sin x \cos x \Leftrightarrow \sin x = 0$ or $\cos x = 1 \Leftrightarrow x = n\pi$ (x-intercepts) **C.** $f(-x) = -f(x)$, so the curve is symmetric about $(0, 0)$. *Note:* f is periodic with period 2π, so we determine D–G for $-\pi \le x \le \pi$. **D.** No asymptotes **E.** $f'(x) = 2\cos 2x - 2\cos x = 2\left(2\cos^2 x - 1 - \cos x\right) = 2\left(2\cos x + 1\right)(\cos x - 1) > 0 \Leftrightarrow \cos x < -\frac{1}{2}$ $\Leftrightarrow -\pi < x < -\frac{2\pi}{3}$ or $\frac{2\pi}{3} < x < \pi$, so f is increasing on $\left(-\pi, -\frac{2\pi}{3}\right)$ and $\left(\frac{2\pi}{3}, \pi\right)$ and decreasing on $\left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right)$. **F.** $f\left(-\frac{2\pi}{3}\right) = \frac{3\sqrt{3}}{2}$ is a local maximum, $f\left(\frac{2\pi}{3}\right) = -\frac{3\sqrt{3}}{2}$ is a local minimum. **G.** $f''(x) = -4\sin 2x + 2\sin x = 2\sin x\,(1 - 4\cos x) = 0$ when $x = 0$, $\pm\pi$ or $\cos x = \frac{1}{4}$. If $\alpha = \cos^{-1}\frac{1}{4}$, then f is CU on $(-\alpha, 0)$ and (α, π) and CD on $(-\pi, -\alpha)$ and $(0, \alpha)$. IP $(0, 0)$, $(\pi, 0)$, $\left(\alpha, -\frac{3\sqrt{15}}{8}\right)$, $\left(-\alpha, \frac{3\sqrt{15}}{8}\right)$.

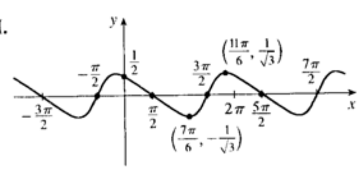
H.



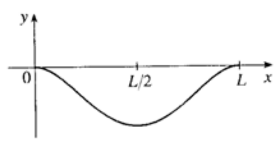
40. $y = f(x) = \cos x / (2 + \sin x)$ **A.** $D = \mathbb{R}$ *Note:* f is periodic with period 2π, so we determine B–G on $[0, 2\pi]$. **B.** x-intercepts $\frac{\pi}{2}, \frac{3\pi}{2}$, y-intercept$= f(0) = \frac{1}{2}$ **C.** No symmetry other than periodicity **D.** No asymptote **E.** $f'(x) = \dfrac{(2 + \sin x)(-\sin x) - \cos x\,(\cos x)}{(2 + \sin x)^2} = -\dfrac{2\sin x + 1}{(2 + \sin x)^2}$. $f'(x) > 0 \Leftrightarrow$ $2\sin x + 1 < 0 \Leftrightarrow \sin x < -\frac{1}{2} \Leftrightarrow \frac{7\pi}{6} < x < \frac{11\pi}{6}$, so f is increasing on $\left(\frac{7\pi}{6}, \frac{11\pi}{6}\right)$ and decreasing on $\left(0, \frac{7\pi}{6}\right)$, $\left(\frac{11\pi}{6}, 2\pi\right)$. **F.** $f\left(\frac{7\pi}{6}\right) = -\frac{1}{\sqrt{3}}$ is a local minimum, $f\left(\frac{11\pi}{6}\right) = \frac{1}{\sqrt{3}}$ is a local maximum.

G. $f''(x) = -\dfrac{(2 + \sin x)^2 (2\cos x) - (2\sin x + 1)\,2\,(2 + \sin x)\cos x}{(2 + \sin x)^4} = -\dfrac{2\cos x\,(1 - \sin x)}{(2 + \sin x)^3} > 0 \Leftrightarrow$ $\cos x < 0 \Leftrightarrow \frac{\pi}{2} < x < \frac{3\pi}{2}$, so f is CU on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and CD on $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$. IP $\left(\frac{\pi}{2}, 0\right)$, $\left(\frac{3\pi}{2}, 0\right)$.

H.



41. $y = -\dfrac{W}{24EI}x^4 + \dfrac{WL}{12EI}x^3 - \dfrac{WL^2}{24EI}x^2 = -\dfrac{W}{24EI}x^2\left(x^2 - 2Lx + L^2\right)$
$= \dfrac{-W}{24EI}x^2 (X - L)^2 = cx^2 (x - L)^2$
where $c = -\dfrac{W}{24EI}$ is a negative constant and $0 \le x \le L$. We sketch $f(x) = cx^2 (x - L)^2$ for $c = -1$. $f(0) = f(L) = 0$.



$f'(x) = cx^2[2(x-L)] + (x-L)^2(2cx) = 2cx(x-L)[x+(x-L)] = 2cx(x-L)(2x-L)$. So for $0 < x < L$, $f'(x) > 0 \Leftrightarrow x(x-L)(2x-L) < 0$ (since $c < 0$) $\Leftrightarrow L/2 < x < L$ and $f'(x) < 0 \Leftrightarrow 0 < x < L/2$. So f is increasing on $(L/2, L)$ and decreasing on $(0, L/2)$, and there is a local and absolute minimum at $(L/2, f(L/2)) = \left(L/2, cL^4/16\right)$.

$f'(x) = 2c[x(x-L)(2x-L)] \Rightarrow$

$f''(x) = 2c[1(x-L)(2x-L) + x(1)(2x-L) + x(x-L)(2)] = 2c\left(6x^2 - 6Lx + L^2\right) = 0 \Leftrightarrow$

$x = \dfrac{6L \pm \sqrt{12L^2}}{12} = \frac{1}{2}L \pm \frac{\sqrt{3}}{6}L$, and these are the x-coordinates of the two inflection points.

42. $F(x) = -\dfrac{k}{x^2} - \dfrac{k}{(x-2)^2}$, where $k > 0$ and $0 < x < 2$. $F'(x) = \dfrac{2k}{x^3} + \dfrac{2k}{(x-2)^3}$. $F'(x) = 0 \Leftrightarrow$

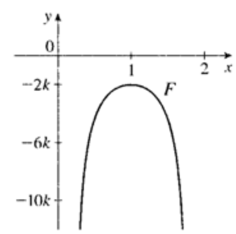
$x^3 = -(x-2)^3 \Leftrightarrow x = -(x-2) \Leftrightarrow 2x = 2 \Leftrightarrow x = 1$. For $0 < x < 1$, $\dfrac{2k}{x^3} > 2k$ and

$\dfrac{2k}{(x-2)^3} > -2k \Rightarrow F'(x) > 0$, and for $1 < x < 2$, $\dfrac{2k}{x^3} < 2k$ and

$\dfrac{2k}{(x-2)^3} < -2k \Rightarrow F'(x) < 0$.

So $F(x)$ is increasing on $(0, 1)$ and decreasing on $(1, 2)$, with an absolute and local maximum at $(1, F(1)) = (1, -2k)$.

The force is always negative, and is maximized (weakest) when the particle is equidistant from the other two particles.



43. $y = f(x) = x^3/\left(x^2-1\right)$ **A.** $D = \{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ **B.** x-intercept $= 0$, y-intercept $= 0$ **C.** $f(-x) = -f(x) \Rightarrow f$ is odd, so the curve is symmetric about the origin.

D. $\lim\limits_{x\to\infty} \dfrac{x^3}{x^2-1} = \infty$ but long division gives $\dfrac{x^3}{x^2-1} = x + \dfrac{x}{x^2-1}$ so $f(x) - x = \dfrac{x}{x^2-1} \to 0$ as $x \to \pm\infty$

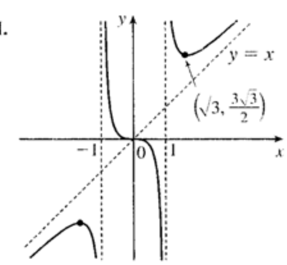
$\Rightarrow y = x$ is a slant asymptote. $\lim\limits_{x\to 1^-} \dfrac{x^3}{x^2-1} = -\infty$, $\lim\limits_{x\to 1^+} \dfrac{x^3}{x^2-1} = \infty$, $\lim\limits_{x\to -1^-} \dfrac{x^3}{x^2-1} = -\infty$,

$\lim\limits_{x\to -1^+} \dfrac{x^3}{x^2-1} = \infty$, so $x = 1$ and $x = -1$ are VA. **E.** $f'(x) = \dfrac{3x^2\left(x^2-1\right) - x^3(2x)}{\left(x^2-1\right)^2} = \dfrac{x^2\left(x^2-3\right)}{\left(x^2-1\right)^2} \Rightarrow$

$f'(x) > 0 \Leftrightarrow x^2 > 3 \Leftrightarrow x > \sqrt{3}$ or $x < -\sqrt{3}$, so f is increasing on $\left(-\infty, -\sqrt{3}\right)$ and $\left(\sqrt{3}, \infty\right)$ and decreasing on $\left(-\sqrt{3}, -1\right)$, $(-1, 1)$, and $\left(1, \sqrt{3}\right)$.

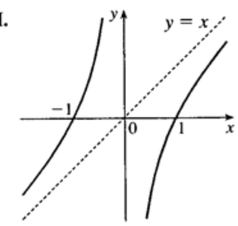
F. $f\left(-\sqrt{3}\right) = -\frac{3\sqrt{3}}{2}$ is a local maximum and $f\left(\sqrt{3}\right) = \frac{3\sqrt{3}}{2}$ is a local minimum. **G.** $y'' = \dfrac{2x\left(x^2+3\right)}{\left(x^2-1\right)^3} > 0 \Leftrightarrow x > 1$ or $-1 < x < 0$, so f is CU on $(-1, 0)$ and $(1, \infty)$ and CD on $(-\infty, -1)$ and $(0, 1)$. IP $(0, 0)$

H.



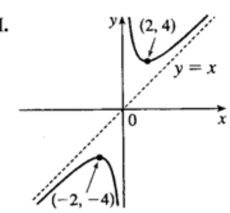
44. $y = f(x) = x - 1/x$ **A.** $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ **B.** x-intercepts ± 1, no y-intercept **C.** $f(-x) = -f(x)$, so the curve is symmetric about the origin. **D.** $\lim_{x \to \pm\infty} (x - 1/x) = \pm\infty$, so no HA. But $(x - 1/x) - x = -1/x \to 0$ as $x \to \pm\infty$, so $y = x$ is a slant asymptote. Also $\lim_{x \to 0^+} (x - 1/x) = -\infty$ and $\lim_{x \to 0^-} (x - 1/x) = \infty$, so $x = 0$ is a VA. **E.** $f'(x) = 1 + 1/x^2 > 0$, so f is increasing on $(-\infty, 0)$ and $(0, \infty)$. **F.** No extremum **G.** $f''(x) = -2/x^3 \Rightarrow f''(x) > 0 \Leftrightarrow x < 0$, so f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. No IP

H.



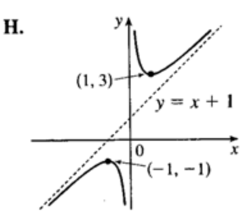
45. $y = f(x) = (x^2 + 4)/x = x + 4/x$ **A.** $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ **B.** No intercept **C.** $f(-x) = -f(x) \Rightarrow$ symmetry about the origin **D.** $\lim_{x \to \infty} (x + 4/x) = \infty$ but $f(x) - x = 4/x \to 0$ as $x \to \pm\infty$, so $y = x$ is a slant asymptote. $\lim_{x \to 0^+} (x + 4/x) = \infty$ and $\lim_{x \to 0^-} (x + 4/x) = -\infty$, so $x = 0$ is a VA. **E.** $f'(x) = 1 - 4/x^2 > 0 \Leftrightarrow x^2 > 4 \Leftrightarrow x > 2$ or $x < -2$, so f is increasing on $(-\infty, -2)$ and $(2, \infty)$ and decreasing on $(-2, 0)$ and $(0, 2)$. **F.** $f(-2) = -4$ is a local maximum and $f(2) = 4$ is a local minimum. **G.** $f''(x) = 8/x^3 > 0 \Leftrightarrow x > 0$ so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. No IP

H.



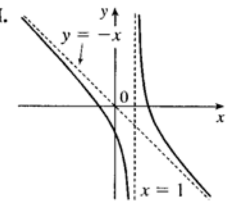
46. $y = f(x) = \dfrac{x^2 + x + 1}{x} = x + 1 + \dfrac{1}{x}$ **A.** $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ **B.** No intercept (x-intercepts would occur when $x^2 + x + 1 = 0$ but this equation has no real roots since $b^2 - 4ac = -3 < 0$.) **C.** No symmetry **D.** $\lim_{x \to \infty} (x + 1 + 1/x) = \pm\infty$, so no HA. But $(x + 1 + 1/x) - (x + 1) = 1/x \to 0$ as $x \to \pm\infty$, so $y = x + 1$ is a slant asymptote. Also $\lim_{x \to 0^+} (x + 1 + 1/x) = \infty$, $\lim_{x \to 0^-} (x + 1 + 1/x) = -\infty$, so $x = 0$ is a VA. **E.** f'$(x) = 1 - 1/x^2 > 0$ when $x^2 > 1 \Leftrightarrow x > 1$ or $x < -1$; $f'(x) < 0 \Leftrightarrow -1 < x < 1$. So f is increasing on $(-\infty, -1)$, $(1, \infty)$ and decreasing on $(-1, 0)$, $(0, 1)$. **F.** $f(1) = 3$ is a local minimum, $f(-1) = -1$ is a local maximum. **G.** $f''(x) = 2/x^3 > 0 \Leftrightarrow x > 0$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. No IP

H.



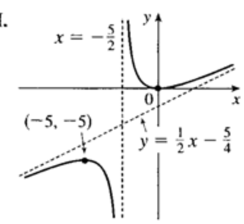
47. $y = \dfrac{1}{x-1} - x$ **A.** $D = \{x \mid x \neq 1\}$ **B.** $y = 0 \;\Leftrightarrow\; x = \dfrac{1}{x-1} \;\Leftrightarrow\; x^2 - x - 1 = 0 \;\Rightarrow\; x = \frac{1 \pm \sqrt{5}}{2}$ (x-intercepts), y-intercept $= f(0) = -1$ **C.** No symmetry **D.** $y - (-x) = \dfrac{1}{x-1} \to 0$ as $x \to \pm\infty$, so $y = -x$ is a slant asymptote. $\lim\limits_{x \to 1^+} \left(\dfrac{1}{x-1} - x \right) = \infty$ and $\lim\limits_{x \to 1^-} \left(\dfrac{1}{x-1} - x \right) = -\infty$, so $x = 1$ is a VA.
E. $f'(x) = -1 - 1/(x-1)^2 < 0$ for all $x \neq 1$, so f is decreasing on $(-\infty, 1)$ and $(1, \infty)$. **F.** No local extremum
G. $f''(x) = \dfrac{2}{(x-1)^3} > 0 \;\Leftrightarrow\; x > 1$, so f is CU on $(1, \infty)$ and CD on $(-\infty, 1)$. No IP

H.



48. $y = f(x) = \dfrac{x^2}{2x+5}$ **A.** $D = \left\{x \mid x \neq -\frac{5}{2}\right\} = \left(-\infty, -\frac{5}{2}\right) \cup \left(-\frac{5}{2}, \infty\right)$ **B.** Intercepts are 0. **C.** No symmetry **D.** $\lim\limits_{x \to \pm\infty} \dfrac{x^2}{2x+5} = \pm\infty$, so no HA. $\lim\limits_{x \to -5/2^+} \dfrac{x^2}{2x+5} = \infty$, $\lim\limits_{x \to -5/2^-} \dfrac{x^2}{2x+5} = -\infty$, so $x = -\frac{5}{2}$ is a VA. By long division, $\dfrac{x^2}{2x+5} = \dfrac{x}{2} - \dfrac{5}{4} + \dfrac{25/4}{2x+5}$, so $\dfrac{x^2}{2x+5} - \left(\dfrac{x}{2} - \dfrac{5}{4}\right) = \dfrac{25/4}{2x+5} \to 0$ as $x \to \pm\infty$, so $y = \frac{1}{2}x - \frac{5}{4}$ is a slant asymptote. **E.** $f'(x) = \dfrac{2x(x+5) - 2x^2}{(2x+5)^2} = \dfrac{2x(x+5)}{(2x+5)^2} \;\Rightarrow\; f'(x) > 0$ $\Leftrightarrow\; x < -5$ or $x > 0$; $f'(x) < 0 \;\Leftrightarrow\; -5 < x < 0$. So f is increasing on $(-\infty, -5)$ and $(0, \infty)$, decreasing on $\left(-5, -\frac{5}{2}\right)$ and $\left(-\frac{5}{2}, 0\right)$.
F. $f(0) = 0$ is a local minimum, $f(-5) = -5$ is a local maximum.
G. $f''(x) = \dfrac{(4x+10)(2x+5)^2 - (2x^2+10x)\cdot 2(2x+5)(2)}{(2x+5)^4} = \dfrac{50}{(2x+5)^3} > 0 \;\Leftrightarrow\; x > -\frac{5}{2}$, so f is CU on $\left(-\frac{5}{2}, \infty\right)$ and CD on $\left(-\infty, -\frac{5}{2}\right)$. No IP.

H.



49. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \;\Rightarrow\; y = \pm\dfrac{b}{a}\sqrt{x^2 - a^2}$. Now

$$\lim_{x \to \infty} \left[\frac{b}{a}\sqrt{x^2 - a^2} - \frac{b}{a}x \right] = \frac{b}{a} \cdot \lim_{x \to \infty} \left(\sqrt{x^2 - a^2} - x \right) \frac{\sqrt{x^2 - a^2} + x}{\sqrt{x^2 - a^2} + x} = \frac{b}{a} \cdot \lim_{x \to \infty} \frac{-a^2}{\sqrt{x^2 - a^2} + x} = 0,$$ which

shows that $y = \dfrac{b}{a}x$ is a slant asymptote. Similarly

$$\lim_{x \to \infty} \left[-\frac{b}{a}\sqrt{x^2 - a^2} - \left(-\frac{b}{a}x \right) \right] = -\frac{b}{a} \cdot \lim_{x \to \infty} \frac{-a^2}{\sqrt{x^2 - a^2} + x} = 0,$$ so $y = -\dfrac{b}{a}x$ is a slant asymptote.

50. $f(x) - x^2 = \dfrac{x^3+1}{x} - x^2 = \dfrac{x^3+1-x^3}{x} = \dfrac{1}{x}$, and $\lim\limits_{x\to\pm\infty} \dfrac{1}{x} = 0$. Therefore, $\lim\limits_{x\to\pm\infty} \left[f(x) - x^2\right] = 0$, and so the graph of f is asymptotic to that of $y = x^2$. For purposes of differentiation, we will use $f(x) = x^2 + 1/x$.

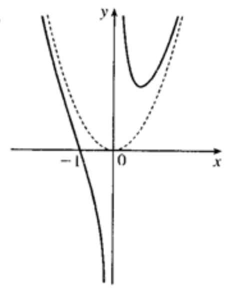
A. $D = \{x \mid x \neq 0\}$ **B.** No y-intercept; to find the x-intercept, we set $y = 0 \Leftrightarrow x = -1$. **C.** No symmetry

D. $\lim\limits_{x\to 0^+} \dfrac{x^3+1}{x} = \infty$ and $\lim\limits_{x\to 0^-} \dfrac{x^3+1}{x} = -\infty$, so $x = 0$ is a vertical asymptote. Also, the graph is asymptotic to the parabola $y = x^2$, as shown above. **E.** $f'(x) = 2x - 1/x^2 > 0 \Leftrightarrow x > \frac{1}{\sqrt[3]{2}}$, so f is increasing on $\left(\frac{1}{\sqrt[3]{2}}, \infty\right)$ and decreasing on $(-\infty, 0)$ and $\left(0, \frac{1}{\sqrt[3]{2}}\right)$.

F. Local minimum $f\left(\frac{1}{\sqrt[3]{2}}\right) = \frac{3\sqrt[3]{3}}{2}$, no local maximum

G. $f''(x) = 2 + 2/x^3 > 0 \Leftrightarrow x < -1$ or $x > 0$, so f is CU on $(-\infty, -1)$ and $(0, \infty)$, and CD on $(-1, 0)$. IP $(-1, 0)$.

H.

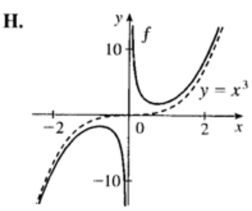


51. $\lim\limits_{x\to\pm\infty} \left[f(x) - x^3\right] = \lim\limits_{x\to\pm\infty} \dfrac{x^4+1}{x} - \dfrac{x^4}{x} = \lim\limits_{x\to\pm\infty} \dfrac{1}{x} = 0$, so the graph of f is asymptotic to that of $y = x^3$.

A. $D = \{x \mid x \neq 0\}$ **B.** No intercept **C.** f is symmetric about the origin. **D.** $\lim\limits_{x\to 0^-} \left(x^3 + \dfrac{1}{x}\right) = -\infty$ and $\lim\limits_{x\to 0^+} \left(x^3 + \dfrac{1}{x}\right) = \infty$, so $x = 0$ is a vertical asymptote, and as shown above, the graph of f is asymptotic to that of $y = x^3$. **E.** $f'(x) = 3x^2 - 1/x^2 > 0 \Leftrightarrow x^4 > \frac{1}{3} \Leftrightarrow |x| > \frac{1}{\sqrt[4]{3}}$, so f is increasing on $\left(-\infty, -\frac{1}{\sqrt[4]{3}}\right)$ and $\left(\frac{1}{\sqrt[4]{3}}, \infty\right)$ and decreasing on $\left(-\frac{1}{\sqrt[4]{3}}, 0\right)$ and $\left(0, \frac{1}{\sqrt[4]{3}}\right)$. **F.** Local maximum $f\left(-\frac{1}{\sqrt[4]{3}}\right) = -4 \cdot 3^{-5/4}$, local minimum $f\left(\frac{1}{\sqrt[4]{3}}\right) = 4 \cdot 3^{-5/4}$

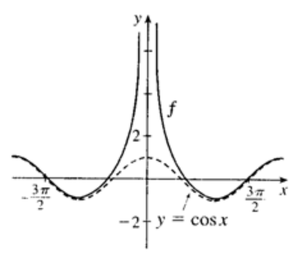
G. $f''(x) = 6x + 2/x^3 > 0 \Leftrightarrow x > 0$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. No IP

H.



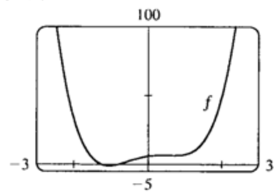
52. $\lim\limits_{x\to\pm\infty} [f(x) - \cos x] = \lim\limits_{x\to\pm\infty} 1/x^2 = 0$, so the graph of f is asymptotic to that of $\cos x$. The intercepts can only be found approximately.

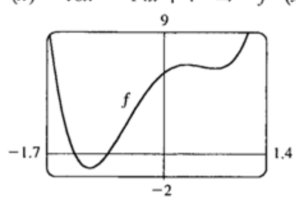
$f(x) = f(-x)$, so f is even. $\lim\limits_{x\to 0} \left(\cos x + \dfrac{1}{x^2}\right) = \infty$, so $x = 0$ is a vertical asymptote. We don't need to calculate the derivatives, since we know the asymptotic behavior of the curve.

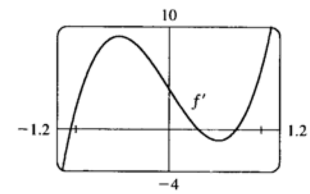


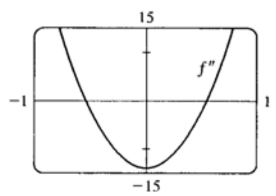
4.6 Graphing with Calculus *and* Calculators

1. $f(x) = 4x^4 - 7x^2 + 4x + 6 \Rightarrow f'(x) = 16x^3 - 14x + 4 \Rightarrow f''(x) = 48x^2 - 14$



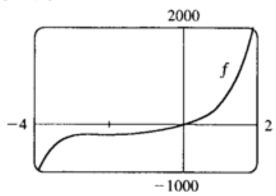


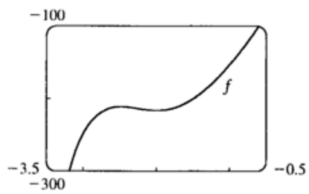


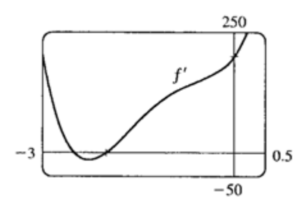


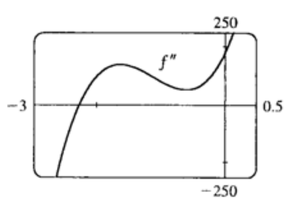
After finding suitable viewing rectangles (by ensuring that we have located all of the x-values where either $f' = 0$ or $f'' = 0$) we estimate from the graph of f' that f is increasing on $(-1.1, 0.3)$ and $(0.7, \infty)$ and decreasing on $(-\infty, -1.1)$ and $(0.3, 0.7)$, with a local maximum of $f(0.3) \approx 6.6$ and minima of $f(-1.1) \approx -1.0$ and $f(0.7) \approx 6.3$. We estimate from the graph of f'' that f is CU on $(-\infty, -0.5)$ and $(0.5, \infty)$ and CD on $(-0.5, 0.5)$, and that f has inflection points at about $(-0.5, 2.1)$ and $(0.5, 6.5)$.

2. $f(x) = 8x^5 + 45x^4 + 80x^3 + 90x^2 + 200x \Rightarrow f'(x) = 40x^4 + 180x^3 + 240x^2 + 180x + 200 \Rightarrow$
$f''(x) = 160x^3 + 540x^2 + 480x + 180$



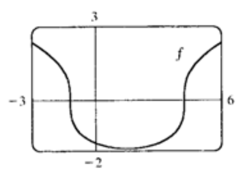






After finding suitable viewing rectangles, we estimate from the graph of f' that f is increasing on $(-\infty, -2.5)$ and $(-2.0, \infty)$ and decreasing on $(-2.5, -2.0)$. Maximum: $f(-2.5) \approx -211$. Minimum: $f(-2) \approx -216$. We estimate from the graph of f'' that f is CU on $(-2.3, \infty)$ and CD on $(-\infty, -2.3)$, and has an IP at $(-2.3, -213)$.

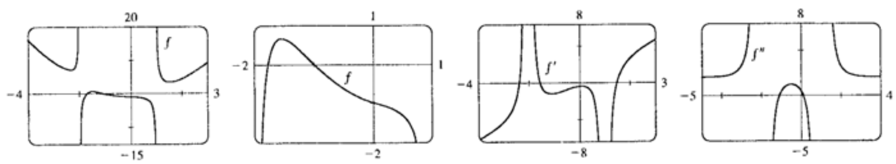
3. $f(x) = \sqrt[3]{x^2 - 3x - 5} \quad \Rightarrow \quad f'(x) = \dfrac{1}{3}\dfrac{2x-3}{(x^2-3x-5)^{2/3}} \quad \Rightarrow \quad f''(x) = -\dfrac{2}{9}\dfrac{x^2-3x+24}{(x^2-3x-5)^{5/3}}$



Note: With some CAS's, including Maple, it is necessary to define $f(x) = \dfrac{x^2-3x-5}{|x^2-3x-5|}\,|x^2-3x-5|^{1/3}$, since the CAS does not compute real cube roots of negative numbers. We estimate from the graph of f' that f is increasing on $(1.5, 4.2)$ and $(4.2, \infty)$, and decreasing on $(-\infty, -1.2)$ and $(-1.2, 1.5)$. f has no maximum. Minimum: $f(1.5) \approx -1.9$. From the graph of f'', we estimate that f is CU on $(-1.2, 4.2)$ and CD on $(-\infty, -1.2)$ and $(4.2, \infty)$. IP $(-1.2, 0)$ and $(4.2, 0)$.

4. $f(x) = \dfrac{x^4 + x^3 - 2x^2 + 2}{x^2 + x - 2}$, so $f'(x) = 2\dfrac{x^5 + 2x^4 - 3x^3 - 4x^2 + 2x - 1}{(x^2+x-2)^2}$ and

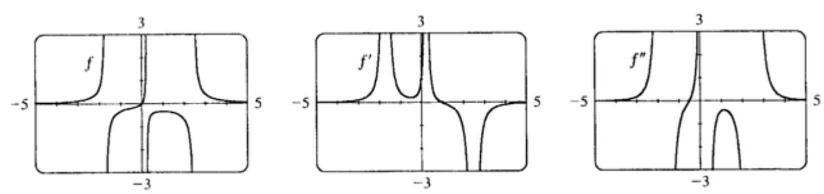
$f''(x) = 2\dfrac{x^6 + 3x^5 - 3x^4 - 11x^3 + 12x^2 + 18x - 2}{(x^2+x-2)^3}$.



We estimate from the graph of f' that f is increasing on $(-2.4, -2)$, $(-2, -1.5)$ and $(1.5, \infty)$ and decreasing on $(-\infty, -2.4)$, $(-1.5, 1)$ and $(1, 1.5)$. Local maximum: $f(-1.5) \approx 0.7$. Local minima: $f(-2.4) \approx 7.2$, $f(1.5) \approx 3.4$. From the graph of f'', we estimate that f is CU on $(-\infty, -2)$, $(-1.1, 0.1)$ and $(1, \infty)$ and CD on $(-2, -1.1)$ and $(0.1, 1)$. f has IP at $(-1.1, 0.2)$ and $(0.1, -1.1)$.

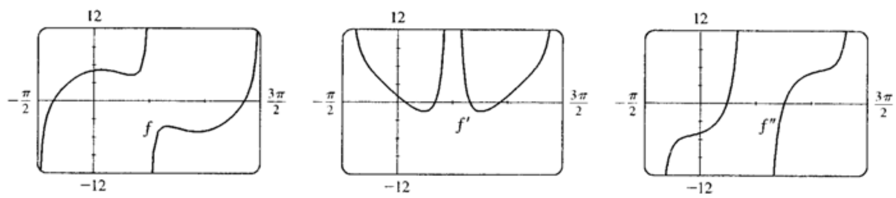
5. $f(x) = \dfrac{x}{x^3 - x^2 - 4x + 1} \quad \Rightarrow \quad f'(x) = \dfrac{-2x^3 + x^2 + 1}{(x^3 - x^2 - 4x + 1)^2} \quad \Rightarrow$

$f''(x) = \dfrac{2(3x^5 - 3x^4 + 5x^3 - 6x^2 + 3x + 4)}{(x^3 - x^2 - 4x + 1)^3}$



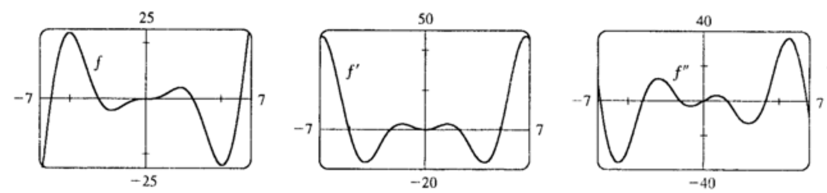
We estimate from the graph of f that $y = 0$ is a horizontal asymptote, and that there are vertical asymptotes at $x = -1.7$, $x = 0.24$, and $x = 2.46$. From the graph of f', we estimate that f is increasing on $(-\infty, -1.7)$, $(-1.7, 0.24)$, and $(0.24, 1)$, and that f is decreasing on $(1, 2.46)$ and $(2.46, \infty)$. There is a local maximum at $f(1) = -\frac{1}{3}$. From the graph of f'', we estimate that f is CU on $(-\infty, -1.7)$, $(-0.506, 0.24)$, and $(2.46, \infty)$, and that f is CD on $(-1.7, -0.506)$ and $(0.24, 2.46)$. There is an inflection point at $(-0.506, -0.192)$.

6. $f(x) = \tan x + 5\cos x \quad \Rightarrow \quad f'(x) = \sec^2 x - 5\sin x \quad \Rightarrow \quad f''(x) = 2\sec^2 x \tan x - 5\cos x$. Since f is periodic with period 2π, and defined for all x except odd multiples of $\frac{\pi}{2}$, we graph f and its derivatives on $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$.



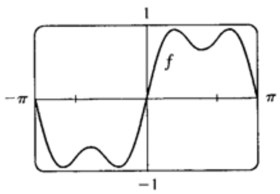
We estimate from the graph of f' that f is increasing on $\left(-\frac{\pi}{2}, 0.21\right)$, $\left(1.07, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, 2.07\right)$, and $\left(2.93, \frac{3\pi}{2}\right)$, and decreasing on $(0.21, 1.07)$ and $(2.07, 2.93)$. Local minima: $f(1.07) \approx 4.23$, $f(2.93) \approx -5.10$. Local maxima: $f(0.21) \approx 5.10$, $f(2.07) \approx -4.23$. From the graph of f'', we estimate that f is CU on $\left(0.76, \frac{\pi}{2}\right)$ and $\left(2.38, \frac{3\pi}{2}\right)$, and CD on $\left(-\frac{\pi}{2}, 0.76\right)$ and $\left(\frac{\pi}{2}, 2.38\right)$. f has IP at $(0.76, 4.57)$ and $(2.38, -4.57)$.

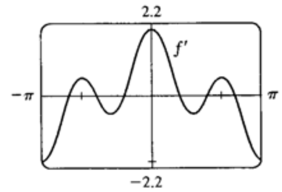
7. $f(x) = x^2 \sin x \quad \Rightarrow \quad f'(x) = 2x\sin x + x^2\cos x \quad \Rightarrow \quad f''(x) = 2\sin x + 4x\cos x - x^2\sin x$

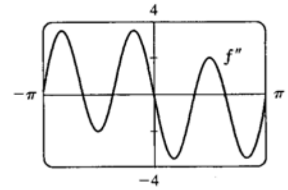


We estimate from the graph of f' that f is increasing on $(-7, -5.1)$, $(-2.3, 2.3)$, and $(5.1, 7)$ and decreasing on $(-5.1, -2.3)$, and $(2.3, 5.1)$. Local maxima: $f(-5.1) \approx 24.1$, $f(2.3) \approx 3.9$. Local minima: $f(-2.3) \approx -3.9$, $f(5.1) \approx -24.1$. From the graph of f'', we estimate that f is CU on $(-7, -6.8)$, $(-4.0, -1.5)$, $(0, 1.5)$, and $(4.0, 6.8)$, and CD on $(-6.8, -4.0)$, $(-1.5, 0)$, $(1.5, 4.0)$, and $(6.8, 7)$. f has IP at $(-6.8, -24.4)$, $(-4.0, 12.0)$, $(-1.5, -2.3)$, $(0, 0)$, $(1.5, 2.3)$, $(4.0, -12.0)$ and $(6.8, 24.4)$.

8. $f(x) = \sin x + \frac{1}{3}\sin 3x \quad \Rightarrow \quad f'(x) = \cos x + \cos 3x \quad \Rightarrow \quad f''(x) = -\sin x - 3\sin 3x$

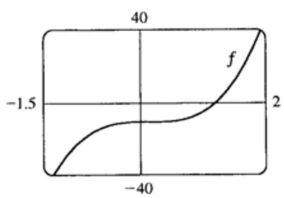


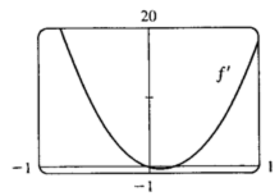


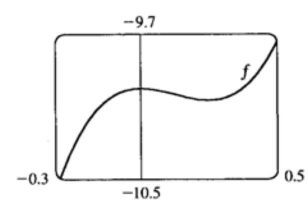


Note that f is periodic with period 2π, so we consider it on the interval $[-\pi, \pi]$. From the graph of f', we estimate that f is increasing on $(-2.4, -1.6)$, $(-0.8, 0.8)$, and $(1.6, 2.4)$ and decreasing on $(-\pi, -2.4)$, $(-1.6, -0.8)$, $(0.8, 1.6)$ and $(2.4, \pi)$. Maxima: $f(-1.6) \approx -0.7$, $f(0.8) \approx 0.9$, $f(2.4) \approx 0.9$. Minima: $f(-2.4) \approx -0.9$, $f(-0.8) \approx -0.9$, $f(1.6) \approx 0.7$. We estimate from the graph of f'' that f is CD on $(-2.0, -1.2)$, $(0, 1.2)$ and $(2.0, \pi)$ and CU on $(-\pi, -2.0)$, $(-1.2, 0)$ and $(1.2, 2)$. f has IP at $(-\pi, 0)$, $(-2.0, -0.8)$, $(-1.2, -0.8)$, $(0, 0)$, $(1.2, 0.8)$, $(2.0, 0.8)$, and $(\pi, 0)$.

9. $f(x) = 8x^3 - 3x^2 - 10 \quad \Rightarrow \quad f'(x) = 24x^2 - 6x \quad \Rightarrow \quad f''(x) = 48x - 6$

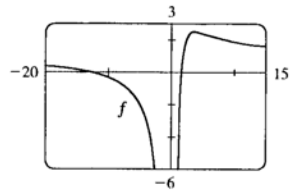


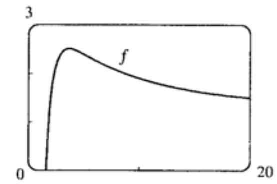




From the graphs, it appears that $f(x) = 8x^3 - 3x^2 - 10$ increases on $(-\infty, 0)$ and $(0.25, \infty)$ and decreases on $(0, 0.25)$; that f has a local maximum of $f(0) = -10.0$ and a local minimum of $f(0.25) \approx -10.1$; that f is CU on $(0.1, \infty)$ and CD on $(-\infty, 0.1)$; and that f has an IP at $(0.1, -10)$. $f(x) = 8x^3 - 3x^2 - 10 \quad \Rightarrow$ $f'(x) = 24x^2 - 6x = 6x(4x - 1)$, which is positive ($f$ is increasing) for $(-\infty, 0)$ and $\left(\frac{1}{4}, \infty\right)$, and negative ($f$ is decreasing) on $\left(0, \frac{1}{4}\right)$. By the FDT, f has a local maximum at $x = 0$: $f(0) = 8(0)^3 - 3(0)^2 - 10 = -10$; and f has a local minimum at $\frac{1}{4}$: $f\left(\frac{1}{4}\right) = \frac{1}{8} - \frac{3}{16} - 10 = -\frac{161}{16}$. $f'(x) = 24x^2 - 6x \quad \Rightarrow$ $f''(x) = 48x - 6 = 6(8x - 1)$, which positive ($f$ is CU) on $\left(\frac{1}{8}, \infty\right)$, and negative ($f$ is CD) on $\left(-\infty, \frac{1}{8}\right)$. f has an IP at $\left(\frac{1}{8}, f\left(\frac{1}{8}\right)\right) = \left(\frac{1}{8}, -\frac{321}{32}\right)$.

10.





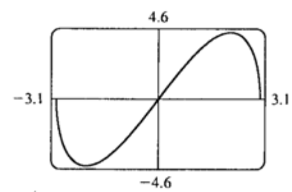
From the graphs, it appears that f increases on $(0, 3.6)$ and decreases on $(-\infty, 0)$ and $(3.6, \infty)$; that f has a local maximum of $f(3.6) \approx 2.5$ and no local minima; that f is CU on $(5.5, \infty)$ and CD on $(-\infty, 0)$ and $(0, 5.5)$; and that f has an IP at $(5.5, 2.3)$. $f(x) = \dfrac{x^2 + 11x - 20}{x^2} = 1 + \dfrac{11}{x} - \dfrac{20}{x^2} \Rightarrow$

$f'(x) = -11x^{-2} + 40x^{-3} = -x^{-3}(11x - 40)$, which is positive ($f$ is increasing) on $\left(0, \frac{40}{11}\right)$, and negative ($f$ is decreasing) on $(-\infty, 0)$ and on $\left(\frac{40}{11}, \infty\right)$. By the FDT, f has a local maximum at $x = \frac{40}{11}$:

$$f\left(\tfrac{40}{11}\right) = \frac{\left(\frac{40}{11}\right)^2 + 11\left(\frac{40}{11}\right) - 20}{\left(\frac{40}{11}\right)^2} = \frac{1600 + 11 \cdot 11 \cdot 40 - 20 \cdot 121}{1600} = \frac{201}{80}\text{; and } f \text{ has no local minimum.}$$

$f'(x) = -11x^{-2} + 40x^{-3} \Rightarrow f''(x) = 22x^{-3} - 120x^{-4} = 2x^{-4}(11x - 60)$, which is positive ($f$ is CU) on $\left(\frac{60}{11}, \infty\right)$, and negative ($f$ is CD) on $(-\infty, 0)$ and $\left(0, \frac{60}{11}\right)$. f has an IP at $\left(\frac{60}{11}, f\left(\frac{60}{11}\right)\right) = \left(\frac{60}{11}, \frac{211}{90}\right)$.

11.



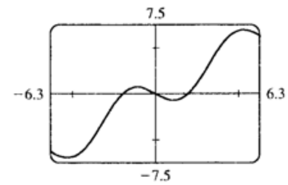
From the graph, it appears that f increases on $(-2.1, 2.1)$ and decreases on $(-3, -2.1)$ and $(2.1, 3)$; that f has a local maximum of $f(2.1) \approx 4.5$ and a local minimum of $f(-2.1) \approx -4.5$; that f is CU on $(-3.0, 0)$ and CD on $(0, 3.0)$, and that f has an IP at $(0, 0)$. $f(x) = x\sqrt{9 - x^2} \Rightarrow f'(x) = \dfrac{-x^2}{\sqrt{9 - x^2}} + \sqrt{9 - x^2} = \dfrac{9 - 2x^2}{\sqrt{9 - x^2}}$, which is positive ($f$ is increasing) on $\left(\frac{-3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$ and negative (f is decreasing) on $\left(-3, \frac{-3\sqrt{2}}{2}\right)$ and $\left(\frac{3\sqrt{2}}{2}, 3\right)$. By the FDT, f has a local maximum of $f\left(\frac{3\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2}\sqrt{9 - \left(\frac{3\sqrt{2}}{2}\right)^2} = \frac{9}{2}$; and f has a local minimum of $f\left(\frac{-3\sqrt{2}}{2}\right) = -\frac{9}{2}$ (since f is an odd function.) $f'(x) = \dfrac{-x^2}{\sqrt{9 - x^2}} + \sqrt{9 - x^2} \Rightarrow$

$$f''(x) = \frac{\sqrt{9 - x^2}(-2x) + x^2\left(\frac{1}{2}\right)(9 - x^2)^{-1/2}(-2x)}{9 - x^2} - x(9 - x^2)^{-1/2} = \frac{-2x - x^3(9 - x^2)^{-1} - x}{\sqrt{9 - x^2}}$$

$$= \frac{-3x}{\sqrt{9 - x^2}} - \frac{x^3}{(9 - x^2)^{3/2}} = \frac{x(2x^2 - 27)}{(9 - x^2)^{3/2}}$$

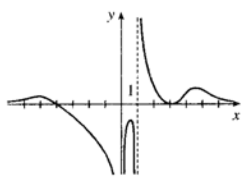
which is positive (f is CU) on $(-3, 0)$, and negative (f is CD) on $(0, 3)$. f has an IP at $(0, 0)$.

12.



From the graph, it appears that f increases on $(-5.2, -1.0)$ and $(1.0, 5.2)$ and decreases on $(-2\pi, -5.2)$, $(-1.0, 1.0)$, and $(5.2, 2\pi)$; that f has local maxima of $f(-1.0) \approx 0.7$ and $f(5.2) \approx 7.0$ and minima of $f(-5.2) \approx -7.0$ and $f(1.0) \approx -0.7$; that f is CU on $(-2\pi, -3.1)$ and $(0, 3.1)$ and CD on $(-3.1, 0)$ and $(3.1, 2\pi)$, and that f has IP at $(0, 0)$, $(-3.1, -3.1)$ and $(3.1, 3.1)$. $f(x) = x - 2\sin x \;\Rightarrow\; f'(x) = 1 - 2\cos x$, which is positive ($f$ is increasing) when $\cos x < \frac{1}{2}$, that is, on $\left(-\frac{5\pi}{3}, -\frac{\pi}{3}\right)$ and $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$, and negative ($f$ is decreasing) on $\left(-2\pi, -\frac{5\pi}{3}\right)$, $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, and $\left(\frac{5\pi}{3}, 2\pi\right)$. By the FDT, f has local maxima of $f\left(-\frac{\pi}{3}\right) = \frac{\pi}{3} + \sqrt{3}$ and $f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3}$, and local minima of $f\left(-\frac{5\pi}{3}\right) = -\frac{5\pi}{3} - \sqrt{3}$ and $f\left(\frac{\pi}{3}\right) = -\frac{\pi}{3} - \sqrt{3}$.
$f'(x) = 1 - 2\cos x \;\Rightarrow\; f''(x) = 2\sin x$, which is positive ($f$ is CU) on $(-2\pi, -\pi)$ and $(0, \pi)$ and negative (f is CD) on $(-\pi, 0)$ and $(\pi, 2\pi)$. f has IP at $(0, 0)$, $(-\pi, -\pi)$ and (π, π).

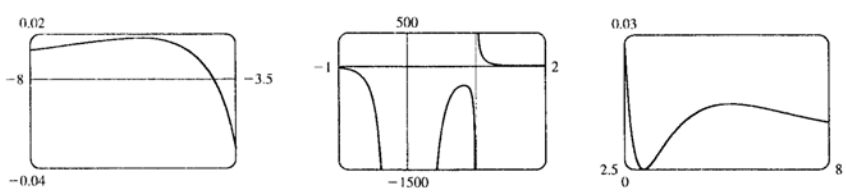
13.



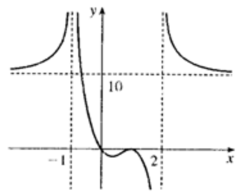
$f(x) = \dfrac{(x+4)(x-3)^2}{x^4(x-1)}$ has VA at $x = 0$ and at $x = 1$ since $\lim\limits_{x\to 0} f(x) = -\infty$, $\lim\limits_{x\to 1^-} f(x) = -\infty$ and $\lim\limits_{x\to 1^+} f(x) = \infty$.

$f(x) = \dfrac{(1 + 4/x)(1 - 3/x)^2}{x(x-1)} \to 0^+$ as $x \to \pm\infty$, so f is asymptotic to the x-axis. Since f is undefined at $x = 0$, it has no y-intercept.

$f(x) = 0 \;\Rightarrow\; (x+4)(x-3)^2 = 0 \;\Rightarrow\; x = -4$ or $x = 3$, so f has x-intercepts -4 and 3. Note, however, that the graph of f is only tangent to the x-axis and does not cross it at $x = 3$, since f is positive as $x \to 3^-$ and as $x \to 3^+$.

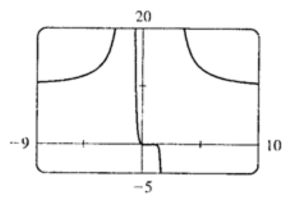


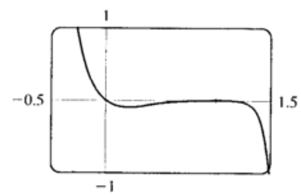
From these graphs, it appears that f has three maxima and one minimum. The maxima are approximately $f(-5.6) = 0.0182$, $f(0.82) = -281.5$ and $f(5.2) = 0.0145$ and we know (since the graph is tangent to the x-axis at $x = 3$) that the minimum is $f(3) = 0$.

14. 

$f(x) = \dfrac{10x\,(x-1)^4}{(x-2)^3\,(x+1)^2}$ has VA at $x = -1$ and at $x = 2$ since $\lim\limits_{x\to-1} f(x) = \infty$, $\lim\limits_{x\to2^-} f(x) = -\infty$ and $\lim\limits_{x\to2^+} f(x) = \infty$.

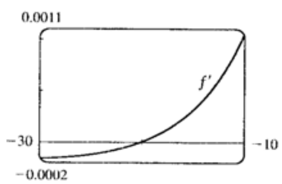
$f(x) = \dfrac{10\,(1-1/x)^4}{(1-2/x)^3\,(1+1/x)^2} \to 10$ as $x \to \pm\infty$, so f is asymptotic to the line $y = 10$. $f(0) = 0$, so f has a y-intercept at 0. $f(x) = 0 \Rightarrow 10x\,(x-1)^4 = 0 \Rightarrow x = 0$ or $x = 1$. So f has x-intercepts 0 and 1. Note, however, that f does not change sign at $x = 1$, so the graph is tangent to the x-axis and does not cross it.

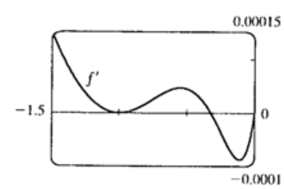


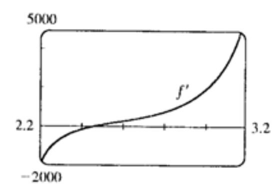


We know (since the graph is tangent to the x-axis at $x = 1$) that the maximum is $f(1) = 0$. From the graphs it appears that the minimum is about $f(0.2) = -0.1$.

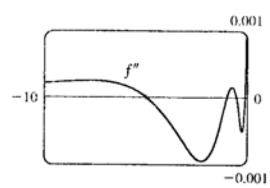
15. $f(x) = \dfrac{x^2\,(x+1)^3}{(x-2)^2\,(x-4)^4} \Rightarrow f'(x) = -\dfrac{x\,(x+1)^2\,(x^3+18x^2-44x-16)}{(x-2)^3\,(x-4)^5}$ (from CAS).

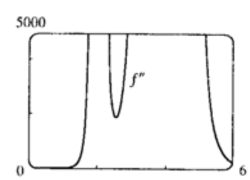






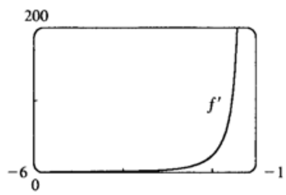
From the graphs of f', it seems that the critical points which indicate extrema occur at $x \approx -20$, -0.3, and 2.5, as estimated in Example 3. (There is another critical point at $x = -1$, but the sign of f' does not change there.) We differentiate again, obtaining $f''(x) = 2\dfrac{(x+1)\,(x^6+36x^5+6x^4-628x^3+684x^2+672x+64)}{(x-2)^4\,(x-4)^6}$.

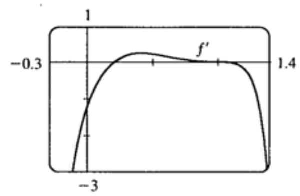


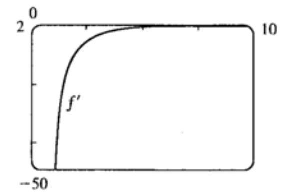


From the graphs of f'', it appears that f is CU on $(-\infty, -5.0)$, $(-1.0, -0.5)$, $(-0.1, 2.0)$, $(2.0, 4.0)$ and $(4.0, \infty)$ and CD on $(-5.0, -1.0)$ and $(-0.5, -0.1)$. We check back on the graphs of f to find the y-coordinates of the inflection points, and find that these points are approximately $(-5, -0.005)$, $(-1, 0)$, $(-0.5, 0.00001)$, and $(-0.1, 0.0000066)$.

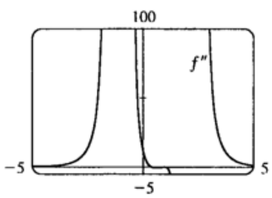
16. $f(x) = \dfrac{10x(x-1)^4}{(x-2)^3(x+1)^2} \quad \Rightarrow \quad f'(x) = -20\dfrac{(x-1)^3(5x-1)}{(x-2)^4(x+1)^3}$ (from CAS).

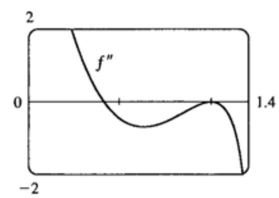






From the graphs of f', we estimate that f is increasing on $(-\infty, -1)$ and $(0.2, 1)$ and decreasing on $(-1, 0.2)$, $(1, 2)$ and $(2, \infty)$. Differentiating $f'(x)$, we get $f''(x) = 60\dfrac{(x-1)^2(5x^3 - 8x^2 + 17x - 6)}{(x-2)^5(x+1)^4}$.

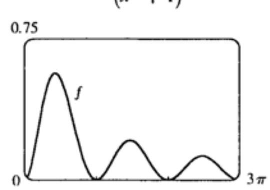


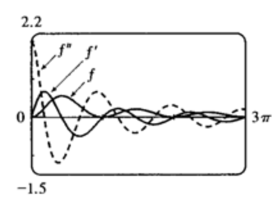


From the graphs of f'', it seems that f is CU on $(-\infty, -1.0)$, $(-1.0, 0.4)$ and $(2.0, \infty)$, and CD on $(0.4, 2)$.

17. $y = f(x) = \dfrac{\sin^2 x}{\sqrt{x^2+1}}$ with $0 \le x \le 3\pi$. From a CAS, $y' = \dfrac{\sin x\left[2\left(x^2+1\right)\cos x - x\sin x\right]}{\left(x^2+1\right)^{3/2}}$ and

$$y'' = \frac{\left(4x^4 + 6x^2 + 5\right)\cos^2 x - 4x\left(x^2+1\right)\sin x\cos x - 2x^4 - 2x^2 - 3}{\left(x^2+1\right)^{5/2}}.$$

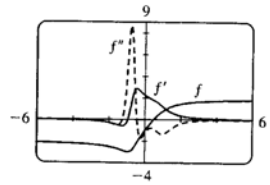




From the graph of f' and the formula for y', we determine that $y' = 0$ when $x = \pi, 2\pi, 3\pi$, or $x \approx 1.3, 4.6$, or 7.8. So f is increasing on $(0, 1.3)$, $(\pi, 4.6)$, and $(2\pi, 7.8)$. f is decreasing on $(1.3, \pi)$, $(4.6, 2\pi)$, and $(7.8, 3\pi)$. Local maxima: $f(1.3) \approx 0.6$, $f(4.6) \approx 0.21$, and $f(7.8) \approx 0.13$. Local minima: $f(\pi) = f(2\pi) = f(3\pi) = 0$. From the graph of f'', we see that $y'' = 0 \quad \Leftrightarrow \quad x \approx 0.6, 2.1, 3.8, 5.4, 7.0$, or 8.6. So f is CU on $(0, 0.6)$, $(2.1, 3.8)$, $(5.4, 7.0)$, and $(8.6, 3\pi)$. f is CD on $(0.6, 2.1)$, $(3.8, 5.4)$, and $(7.0, 8.6)$. There are IP at $(0.6, 0.25)$, $(2.1, 0.31)$, $(3.8, 0.10)$, $(5.4, 0.11)$, $(7.0, 0.061)$, and $(8.6, 0.065)$.

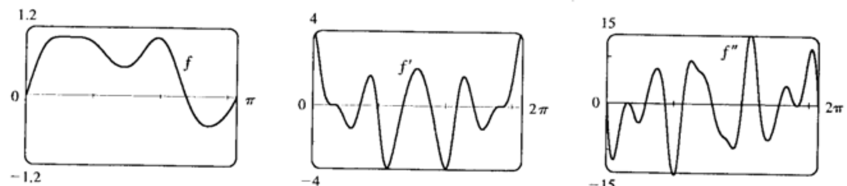
18. $f(x) = \dfrac{2x-1}{\sqrt[4]{x^4+x+1}} \quad \Rightarrow \quad f'(x) = \dfrac{4x^3+6x+9}{4(x^4+x+1)^{5/4}} \quad \Rightarrow$

$$f''(x) = -\frac{32x^6+96x^4+152x^3-48x^2+6x+21}{16(x^4+x+1)^{9/4}}$$

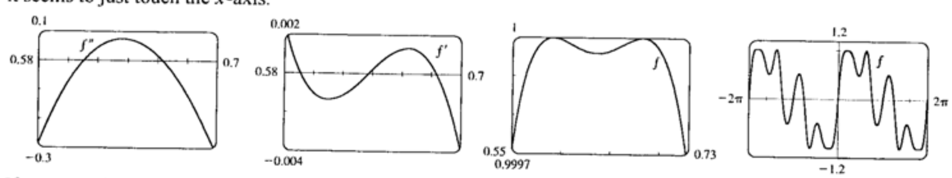


From the graph of f', f appears to be decreasing on $(-\infty, -0.94)$ and increasing on $(-0.94, \infty)$. There is a local minimum of $f(-0.94) \approx -3.01$. From the graph of f'', f appears to be CU on $(-1.25, -0.44)$ and CD on $(-\infty, -1.25)$ and $(-0.44, \infty)$. There are inflection points at $(-1.25, -2.87)$ and $(-0.44, -2.14)$.

19.

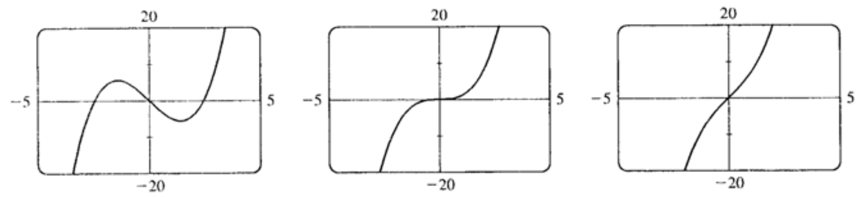


From the graph of $f(x) = \sin(x + \sin 3x)$ in the viewing rectangle $[0, \pi]$ by $[-1.2, 1.2]$, it looks like f has two maxima and two minima. If we calculate and graph $f'(x) = [\cos(x + \sin 3x)](1 + 3\cos 3x)$ on $[0, 2\pi]$, we see that the graph of f' appears to be almost tangent to the x-axis at about $x = 0.7$. The graph of $f'' = -[\sin(x + \sin 3x)](1 + 3\cos 3x)^2 + \cos(x + \sin 3x)(-9\sin 3x)$ is even more interesting near this x-value: it seems to just touch the x-axis.



If we zoom in on this place on the graph of f'', we see that f'' actually does cross the axis twice near $x = 0.65$, indicating a change in concavity for a very short interval. If we look at the graph of f' on the same interval, we see that it changes sign three times near $x = 0.65$, indicating that what we had thought was a broad extremum at about $x = 0.7$ actually consists of three extrema (two maxima and a minimum). These maxima are roughly $f(0.59) = 1$ and $f(0.68) = 1$, and the minimum is roughly $f(0.64) = 0.99996$. There are also a maximum of about $f(1.96) = 1$ and minima of about $f(1.46) = 0.49$ and $f(2.73) = -0.51$. The points of inflection on $(0, \pi)$ are about $(0.61, 0.99998)$, $(0.66, 0.99998)$, $(1.17, 0.72)$, $(1.75, 0.77)$, and $(2.28, 0.34)$. On $(\pi, 2\pi)$, they are about $(4.01, -0.34)$, $(4.54, -0.77)$, $(5.11, -0.72)$, $(5.62, -0.99998)$, and $(5.67, -0.99998)$. There are also IP at $(0, 0)$ and $(\pi, 0)$. Note that the function is odd and periodic with period 2π, and it is also rotationally symmetric about all points of the form $((2n+1)\pi, 0)$, n an integer.

20. $f(x) = x^3 + cx = x(x^2 + c) \quad\Rightarrow\quad f'(x) = 3x^2 + c \quad\Rightarrow\quad f''(x) = 6x$



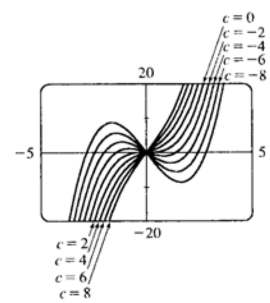
$c = -6$ $\qquad$ $c = 0$ $\qquad$ $c = 6$

x-intercepts: When $c \geq 0$, 0 is the only x-intercept. When $c < 0$, the x-intercepts are 0 and $\pm\sqrt{-c}$. y-intercept $= f(0) = 0$. f is odd, so the graph is symmetric with respect to the origin. $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so f is CD on $(-\infty, 0)$ and CU on $(0, \infty)$. The origin is the only inflection point.

If $c > 0$, then $f'(x) > 0$ for all x, so f is increasing and has no local maximum or minimum.

If $c = 0$, then $f'(x) \geq 0$ with equality at $x = 0$, so again f is increasing and has no local maximum or minimum.

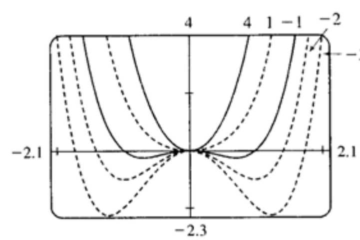
If $c < 0$, then $f'(x) = 3\left[x^2 - (-c/3)\right] = 3\left(x + \sqrt{-c/3}\right)\left(x - \sqrt{-c/3}\right)$, so $f'(x) > 0$ on $\left(-\infty, -\sqrt{-c/3}\right)$ and $\left(\sqrt{-c/3}, \infty\right)$; $f'(x) < 0$ on $\left(-\sqrt{-c/3}, \sqrt{-c/3}\right)$. It follows that $f\left(-\sqrt{-c/3}\right) = -\frac{2}{3}c\sqrt{-c/3}$ is a local maximum and $f\left(\sqrt{-c/3}\right) = \frac{2}{3}c\sqrt{-c/3}$ is a local minimum. As c decreases (toward more negative values), the local maximum and minimum move further apart.



There is no absolute maximum or minimum. The only transitional value of c corresponding to a change in character of the graph is $c = 0$.

21. $f(x) = x^4 + cx^2 = x^2(x^2 + c)$. Note that f is an even function. For $c \geq 0$, the only x-intercept is the point $(0, 0)$. We calculate $f'(x) = 4x^3 + 2cx = 4x\left(x^2 + \frac{1}{2}c\right) \quad\Rightarrow\quad f''(x) = 12x^2 + 2c$. If $c \geq 0$, $x = 0$ is the only critical point and there are no inflection points. As we can see from the examples, there is no change in the basic shape of the graph

for $c \geq 0$; it merely becomes steeper as c increases. For $c = 0$, the graph is the simple curve $y = x^4$. For $c < 0$, there are x-intercepts at 0 and at $\pm\sqrt{-c}$. Also, there is a maximum at $(0, 0)$, and there are minima at $\left(\pm\sqrt{-\frac{1}{2}c}, -\frac{1}{4}c^2\right)$. As $c \to -\infty$, the x-coordinates of these minima get larger in absolute value, and the minimum points move downward. There are inflection points at $\left(\pm\sqrt{-\frac{1}{6}c}, -\frac{5}{36}c^2\right)$, which also move away from the origin as $c \to -\infty$.



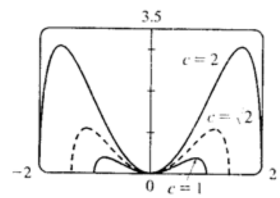
22. We need only consider the function $f(x) = x^2\sqrt{c^2 - x^2}$ for $c \ge 0$, because if c is replaced by $-c$, the function is unchanged. For $c = 0$, the graph consists of the single point $(0, 0)$.
The domain of f is $[-c, c]$, and the graph of f is symmetric about the y-axis.

$$f'(x) = 2x\sqrt{c^2 - x^2} + x^2\frac{-2x}{2\sqrt{c^2 - x^2}} = 2x\sqrt{c^2 - x^2} - \frac{x^3}{\sqrt{c^2 - x^2}} = \frac{2x\left(c^2 - x^2\right) - x^3}{\sqrt{c^2 - x^2}} = -\frac{3x\left(x^2 - \frac{2}{3}c^2\right)}{\sqrt{c^2 - x^2}}.$$

So we see that all members of the family of curves have horizontal tangents at $x = 0$, since $f'(0) = 0$ for all $c > 0$. Also, the tangents to all the curves become very steep as $x \to \pm c$, since $\lim\limits_{x \to -c^+} f'(x) = \infty$ and $\lim\limits_{x \to c^-} f'(x) = -\infty$. We set $f'(x) = 0$
$\Leftrightarrow$ $x = 0$ or $x^2 - \frac{2}{3}c^2 = 0$, so the absolute maxima are $f\left(\pm\sqrt{\frac{2}{3}}c\right) = \frac{2}{3\sqrt{3}}c^3$.

$$f''(x) = \frac{\left(-9x^2 + 2c^2\right)\sqrt{c^2 - x^2} - \left(-3x^3 + 2c^2x\right)\left(-x/\sqrt{c^2 - x^2}\right)}{c^2 - x^2} = \frac{6x^4 - 9c^2x^2 + 2c^4}{\left(c^2 - x^2\right)^{3/2}}.$$ Using the quadratic formula, we find that $f''(x) = 0 \;\Leftrightarrow\; x^2 = \dfrac{9c^2 \pm c^2\sqrt{33}}{12}$. Since $-c < x < c$, we take $x^2 = \frac{9 - \sqrt{33}}{12}c^2$, so the inflection points are

$$\left(\pm\sqrt{\frac{9 - \sqrt{33}}{12}}c, \frac{\left(9 - \sqrt{33}\right)\left(\sqrt{33} - 3\right)}{144}c^3\right).$$

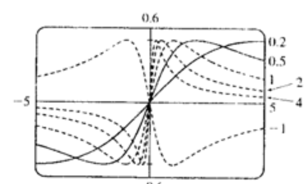


From these calculations we can see that the maxima and the points of inflection get both horizontally and vertically further from the origin as c increases. Since all of the functions have two maxima and two inflection points, we see that the basic shape of the curve does not change as c changes.

23. Note that $c = 0$ is a transitional value at which the graph consists of the x-axis. Also, we can see that if we substitute $-c$ for c, the function $f(x) = \dfrac{cx}{1 + c^2x^2}$ will be reflected in the x-axis, so we investigate only positive values of c (except $c = -1$, as a demonstration of this reflective property). Also, f is an odd function. $\lim\limits_{x \to \pm\infty} f(x) = 0$, so $y = 0$ is a horizontal asymptote for all c. We calculate

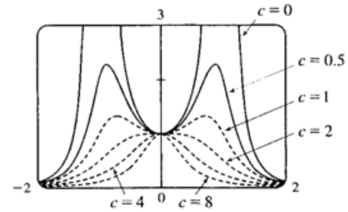
$$f'(x) = \frac{c\left(1 + c^2x^2\right) - cx\left(2c^2x\right)}{\left(1 + c^2x^2\right)^2} = -\frac{c\left(c^2x^2 - 1\right)}{\left(1 + c^2x^2\right)^2}.$$ $f'(x) = 0 \;\Leftrightarrow\; c^2x^2 - 1 = 0 \;\Leftrightarrow\; x = \pm 1/c$. So there is an absolute maximum of $f(1/c) = \frac{1}{2}$ and an absolute minimum of $f(-1/c) = -\frac{1}{2}$. These extrema have the same value regardless of c, but the maximum points move closer to the y-axis as c increases.

$$\begin{aligned} f''(x) &= \frac{\left(-2c^3x\right)\left(1 + c^2x^2\right)^2 - \left(-c^3x^2 + c\right)\left[2\left(1 + c^2x^2\right)\left(2c^2x\right)\right]}{\left(1 + c^2x^2\right)^4} \\ &= \frac{\left(-2c^3x\right)\left(1 + c^2x^2\right) + \left(c^3x^2 - c\right)\left(4c^2x\right)}{\left(1 + c^2x^2\right)^3} = \frac{2c^3x\left(c^2x^2 - 3\right)}{\left(1 + c^2x^2\right)^3} \end{aligned}$$



$f''(x) = 0 \;\Leftrightarrow\; x = 0$ or $\pm\sqrt{3}/c$, so there are inflection points at $(0, 0)$ and at $\left(\pm\sqrt{3}/c, \pm\sqrt{3}/4\right)$.
Again, the y-coordinate of the inflection points does not depend on c, but as c increases, both inflection points approach the y-axis.

24. Note that $f(x) = \dfrac{1}{(1-x^2)^2 + cx^2}$ is an even function, and also that $\lim\limits_{x\to\pm\infty} f(x) = 0$ for any value of c, so $y = 0$ is a horizontal asymptote. We calculate the derivatives:

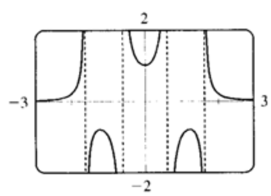


$$f'(x) = \frac{-4(1-x^2)x + 2cx}{\left[(1-x^2)^2 + cx^2\right]^2} = \frac{4x\left[x^2 + \left(\frac{1}{2}c - 1\right)\right]}{\left[(1-x^2)^2 + cx^2\right]^2}, \text{ and}$$

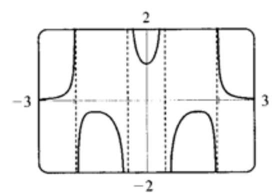
$$f''(x) = 2\frac{10x^6 + (9c-18)x^4 + (3c^2 - 12c + 6)x^2 + 2 - c}{\left[x^4 + (c-2)x^2 + 1\right]^3}.$$ We first consider the case $c > 0$. Then the denominator of f' is positive, that is, $(1-x^2)^2 + cx^2 > 0$ for all x, so f has domain $\mathbb{R}$ and also $f > 0$. If $\frac{1}{2}c - 1 \geq 0$, that is, $c \geq 2$, then the only critical point is $f(0) = 1$, a maximum. Graphing a few examples for $c \geq 2$ shows that there are two IP which approach the y-axis as $c \to \infty$.

$c = 2$ and $c = 0$ are transitional values of c at which the shape of the curve changes. For $0 < c < 2$, there are three critical points: $f(0) = 1$, a minimum, and $f\left(\pm\sqrt{1 - \frac{1}{2}c}\right) = \dfrac{1}{c(1 - c/4)}$, both maxima. As c decreases from 2 to 0, the maximum values get larger and larger, and the x-values at which they occur go from 0 to ± 1. Graphs show that there are four inflection points for $0 < c < 2$, and that they get farther away from the origin, both vertically and horizontally, as $c \to 0^+$. For $c = 0$, the function is simply asymptotic to the x-axis and to the lines $x = \pm 1$, approaching $+\infty$ from both sides of each. The y-intercept is 1, and $(0, 1)$ is a local minimum. There are no inflection points. Now if $c < 0$, we can write

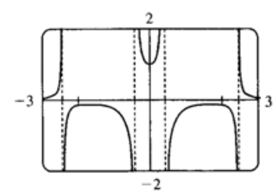
$$f(x) = \frac{1}{(1-x^2)^2 + cx^2} = \frac{1}{(1-x^2)^2 - \left(\sqrt{-c}\,x\right)^2} = \frac{1}{\left(x^2 - \sqrt{-c}\,x - 1\right)\left(x^2 + \sqrt{-c}\,x - 1\right)}.$$ So f has vertical asymptotes where $x^2 \pm \sqrt{-c}\,x - 1 = 0 \iff x = \left(-\sqrt{-c} \pm \sqrt{4-c}\right)/2$ or $x = \left(\sqrt{-c} \pm \sqrt{4-c}\right)/2$. As c decreases, the two exterior asymptotes move away from the origin, while the two interior ones move toward it. We graph a few examples to see the behavior of the graph near the asymptotes, and the nature of the critical points $x = 0$ and $x = \pm\sqrt{1 - \frac{1}{2}c}$:



$c = -1$



$c = -2$



$c = -4$

We see that there is one local minimum, $f(0) = 1$, and there are two local maxima, $f\left(\pm\sqrt{1 - \frac{1}{2}c}\right) = \dfrac{1}{c(1 - c/4)}$ as before. As c decreases, the x-values at which these maxima occur get larger, and the maximum values themselves approach 0, though they are always negative.

25. $f(x) = cx + \sin x \quad \Rightarrow \quad f'(x) = c + \cos x \quad \Rightarrow \quad f''(x) = -\sin x$

$f(-x) = -f(x)$, so f is an odd function and its graph is symmetric with respect to the origin.

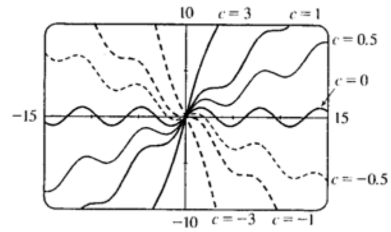
$f(x) = 0 \quad \Leftrightarrow \quad \sin x = -cx$, so 0 is always an x-intercept.

$f'(x) = 0 \quad \Leftrightarrow \quad \cos x = -c$, so there is no critical number when $|c| > 1$. If $|c| \leq 1$, then there are infinitely many critical numbers. If x_1 is the unique solution of $\cos x = -c$ in the interval $[0, \pi]$, then the critical numbers are $2n\pi \pm x_1$, where n ranges over the integers. (Special cases: When $c = 1$, $x_1 = 0$; when $c = 0$, $x = \frac{\pi}{2}$; and when $c = -1$, $x_1 = \pi$.)

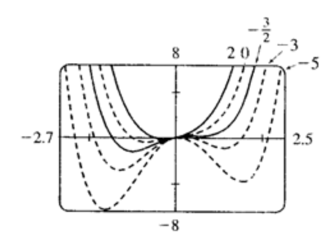
$f''(x) < 0 \quad \Leftrightarrow \quad \sin x > 0$, so f is CD on intervals of the form $(2n\pi, (2n+1)\pi)$. f is CU on intervals of the form $((2n-1)\pi, 2n\pi)$. The inflection points of f are the points $(2n\pi, 2n\pi c)$, where n is an integer.

If $c \geq 1$, then $f'(x) \geq 0$ for all x, so f is increasing and has no extremum. If $c \leq -1$, then $f'(x) \leq 0$ for all x, so f is decreasing and has no extremum. If $|c| < 1$, then $f'(x) > 0 \quad \Leftrightarrow \quad \cos x > -c \quad \Leftrightarrow \quad x$ is in an interval of the form $(2n\pi - x_1, 2n\pi + x_1)$ for some integer n. These are the intervals on which f is increasing. Similarly, we find that f is decreasing on the intervals of the form $(2n\pi + x_1, 2(n+1)\pi - x_1)$. Thus, f has local maxima at the points $2n\pi + x_1$, where f has the values $c(2n\pi + x_1) + \sin x_1 = c(2n\pi + x_1) + \sqrt{1-c^2}$, and f has local minima at the points $2n\pi - x_1$, where we have $f(2n\pi - x_1) = c(2n\pi - x_1) - \sin x_1 = c(2n\pi - x_1) - \sqrt{1-c^2}$.

The transitional values of c are -1 and 1. The inflection points move vertically, but not horizontally, when c changes. When $|c| \geq 1$, there is no extremum. For $|c| < 1$, the maxima are spaced 2π apart horizontally, as are the minima. The horizontal spacing between maxima and adjacent minima is regular (and equals π) when $c = 0$, but the horizontal space between a local maximum and the nearest local minimum shrinks as $|c|$ approaches 1.

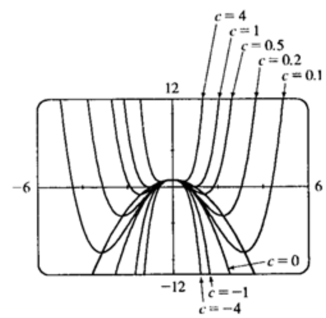


26. For $c = 0$, there is no inflection point; the curve is CU everywhere. If c increases, the curve simply becomes steeper, and there are still no inflection points. If c starts at 0 and decreases, a slight upward bulge appears near $x = 0$, so that there are two inflection points for any $c < 0$. This can be seen algebraically by calculating the second derivative: $f(x) = x^4 + cx^2 + x \quad \Rightarrow \quad f'(x) = 4x^3 + 2cx + 1 \quad \Rightarrow \quad f''(x) = 12x^2 + 2c$. Thus, $f''(x) > 0$ when $c > 0$. For $c < 0$, there are inflection points when $x = \pm\sqrt{-\frac{1}{6}c}$. For $c = 0$, the graph has one critical number, at the absolute minimum somewhere around $x = -0.6$. As c increases, the number of critical points does not change. If c instead decreases from 0, we see that the graph eventually sprouts another local minimum, to the right of the origin, somewhere between $x = 1$ and $x = 2$. Consequently, there is also a maximum near $x = 0$. After a bit of experimentation, we find that at $c = -1.5$, there appear to be two critical numbers: the absolute minimum at about $x = -1$, and a horizontal tangent with no extremum at about $x = 0.5$. For any c smaller than this there will be 3 critical points, as shown in the graphs with $c = -3$ and with $c = -5$. To prove this algebraically, we calculate $f'(x) = 4x^3 + 2cx + 1$. Now if we substitute our value of $c = -1.5$, the formula for $f'(x)$ becomes $4x^3 - 3x + 1 = (x+1)(2x-1)^2$. This has a double root at $x = \frac{1}{2}$, indicating that the function has two critical points: $x = -1$ and $x = \frac{1}{2}$, just as we had guessed from the graph.



27. (a) $f(x) = cx^4 - 2x^2 + 1$. For $c = 0$, $f(x) = -2x^2 + 1$, a parabola whose vertex, $(0, 1)$, is the absolute maximum. For $c > 0$, $f(x) = cx^4 - 2x^2 + 1$ opens upward with two minimum points. As $c \to 0$, the minimum points spread apart and move downward; they are below the x-axis for $0 < c < 1$ and above for $c > 1$. For $c < 0$, the graph opens downward, and has an absolute maximum at $x = 0$ and no local minimum.

(b) $f'(x) = 4cx^3 - 4x = 4cx\left(x^2 - 1/c\right)$ $(c \neq 0)$. If $c \leq 0$, 0 is the only critical number. $f''(x) = 12cx^2 - 4$, so $f''(0) = -4$ and there is a local maximum at $(0, f(0)) = (0, 1)$, which lies on $y = 1 - x^2$. If $c > 0$, the critical numbers are 0 and $\pm 1/\sqrt{c}$. As before, there is a local maximum at $(0, f(0)) = (0, 1)$, which lies on $y = 1 - x^2$. $f''\left(\pm 1/\sqrt{c}\right) = 12 - 4 = 8 > 0$, so there is a local minimum at $x = \pm 1/\sqrt{c}$. Here $f\left(\pm 1/\sqrt{c}\right) = c\left(1/c^2\right) - 2/c + 1 = -1/c + 1$. But $\left(\pm 1/\sqrt{c}, -1/c + 1\right)$ lies on $y = 1 - x^2$ since $1 - \left(\pm 1/\sqrt{c}\right)^2 = 1 - 1/c$.



28. (a) $f(x) = 2x^3 + cx^2 + 2x \Rightarrow f'(x) = 6x^2 + 2cx + 2 = 2\left(3x^2 + cx + 1\right)$. $f'(x) = 0 \Leftrightarrow$ $x = \dfrac{-c \pm \sqrt{c^2 - 12}}{6}$. So f has critical points $\Leftrightarrow c^2 - 12 \geq 0 \Leftrightarrow |c| \geq 2\sqrt{3}$. For $c = \pm 2\sqrt{3}$, $f'(x) \geq 0$ on $(-\infty, \infty)$, so f' does not change signs at $-c/6$, and there is no extremum. If $c^2 - 12 > 0$, then f' changes from positive to negative at $x = \dfrac{-c - \sqrt{c^2 - 12}}{6}$ and from negative to positive at $x = \dfrac{-c + \sqrt{c^2 - 12}}{6}$. So f has a local maximum at $x = \dfrac{-c - \sqrt{c^2 - 12}}{6}$ and a local minimum at $x = \dfrac{-c + \sqrt{c^2 - 12}}{6}$.

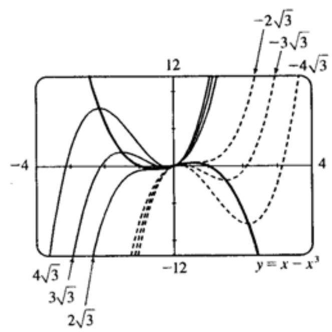
(b) Let x_0 be a critical number for $f(x)$. Then $f'(x_0) = 0 \Rightarrow$

$3x_0^2 + cx_0 + 1 = 0 \Leftrightarrow c = \dfrac{-1 - 3x_0^2}{x_0}$. Now

$$f(x_0) = 2x_0^3 + cx_0^2 + 2x_0 = 2x_0^3 + x_0^2\left(\frac{-1 - 3x_0^2}{x_0}\right) + 2x_0$$

$$= 2x_0^3 - x_0 - 3x_0^3 + 2x_0 = x_0 - x_0^3$$

So the point is $(x_0, y_0) = \left(x_0, x_0 - x_0^3\right)$; that is, the point lies on the curve $y = x - x^3$.



4.7 Optimization Problems

1. (a)

First Number	Second Number	Product
1	22	22
2	21	42
3	20	60
4	19	76
5	18	90
6	17	102
7	16	112
8	15	120
9	14	126
10	13	130
11	12	132

We needn't consider pairs where the first number is larger than the second, since we can just interchange the numbers in such cases. The answer appears to be 11 and 12, but we have considered only integers in the table.

(b) Call the two numbers x and y. Then $x + y = 23$, so $y = 23 - x$. Call the product P. Then $P = xy = x(23 - x) = 23x - x^2$, so we wish to maximize the function $P(x) = 23x - x^2$. Since $P'(x) = 23 - 2x$, we see that $P'(x) = 0 \Leftrightarrow x = \frac{23}{2} = 11.5$. Thus, the maximum value of P is $P(11.5) = (11.5)^2 = 132.25$ and it occurs when $x = y = 11.5$.

Or: Note that $P''(x) = -2 < 0$ for all x, so P is everywhere concave downward and the local maximum at $x = 11.5$ must be an absolute maximum.

2. The two numbers are $x + 100$ and x. Minimize $f(x) = (x + 100)x = x^2 + 100x$. $f'(x) = 2x + 100 = 0 \Rightarrow x = -50$. Since $f''(x) = 2 > 0$, there is an absolute minimum at $x = -50$. The two numbers are 50 and -50.

3. The two numbers are x and $\frac{100}{x}$, where $x > 0$. Minimize $f(x) = x + \frac{100}{x}$. $f'(x) = 1 - \frac{100}{x^2} = \frac{x^2 - 100}{x^2}$. The critical number is $x = 10$. Since $f'(x) < 0$ for $0 < x < 10$ and $f'(x) > 0$ for $x > 10$, there is an absolute minimum at $x = 10$. The numbers are 10 and 10.

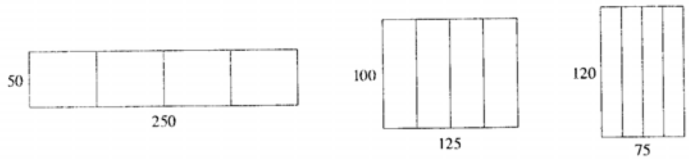
4. Let $x > 0$ and let $f(x) = x + 1/x$. We wish to minimize $f(x)$. Now $f'(x) = 1 - \frac{1}{x^2} = \frac{1}{x^2}\left(x^2 - 1\right) = \frac{1}{x^2}(x + 1)(x - 1)$, so the only critical number in $(0, \infty)$ is 1. $f'(x) < 0$ for $0 < x < 1$ and $f'(x) > 0$ for $x > 1$, so f has an absolute minimum at $x = 1$, and $f(1) = 2$.

Or: $f''(x) = 2/x^3 > 0$ for all $x > 0$, so f is concave upward everywhere and the critical point $(1, 2)$ must correspond to a local minimum for f.

5. If the rectangle has dimensions x and y, then its perimeter is $2x + 2y = 100$ m, so $y = 50 - x$. Thus, the area is $A = xy = x(50 - x)$. We wish to maximize the function $A(x) = x(50 - x) = 50x - x^2$, where $0 < x < 50$. Since $A'(x) = 50 - 2x = -2(x - 25)$, $A'(x) > 0$ for $0 < x < 25$ and $A'(x) < 0$ for $25 < x < 50$. Thus, A has an absolute maximum at $x = 25$, and $A(25) = 25^2 = 625\text{ m}^2$. The dimensions of the rectangle that maximize its area are $x = y = 25$ m. (The rectangle is a square.)

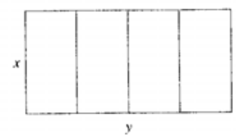
6. If the rectangle has dimensions x and y, then its area is $xy = 1000\text{ m}^2$, so $y = 1000/x$. The perimeter $P = 2x + 2y = 2x + 2000/x$. We wish to minimize the function $P(x) = 2x + 2000/x$ for $x > 0$. $P'(x) = 2 - 2000/x^2 = (2/x^2)(x^2 - 1000)$, so the only critical number in the domain of P is $x = \sqrt{1000}$. $P''(x) = 4000/x^3 > 0$, so P is concave upward throughout its domain and $P\left(\sqrt{1000}\right) = 4\sqrt{1000}$ is an absolute minimum value. The dimensions of the rectangle with minimal perimeter are $x = y = \sqrt{1000} = 10\sqrt{10}$ m. (The rectangle is a square.)

7. (a)



The areas of the three figures are 12,500, 12,500, and 9000 ft^2. There appears to be a maximum area of at least 12,500 ft^2.

(b) Let x denote the length of each of two sides and three dividers. Let y denote the length of the other two sides.



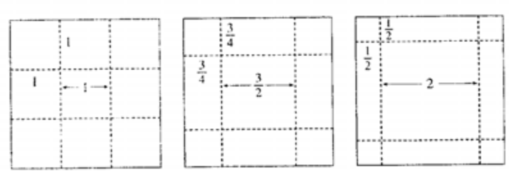
(c) Area $A = \text{length} \times \text{width} = y \cdot x$

(d) Length of fencing $= 750 \quad \Rightarrow \quad 5x + 2y = 750$

(e) $5x + 2y = 750 \quad \Rightarrow \quad y = 375 - \frac{5}{2}x \quad \Rightarrow \quad A(x) = \left(375 - \frac{5}{2}x\right)x = 375x - \frac{5}{2}x^2$

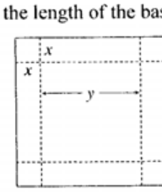
(f) $A'(x) = 375 - 5x = 0 \quad \Rightarrow \quad x = 75$. Since $A''(x) = -5 < 0$ there is an absolute maximum when $x = 75$. Then $y = \frac{375}{2} = 187.5$. The largest area is $75\left(\frac{375}{2}\right) = 14{,}062.5\text{ ft}^2$. These values of x and y are between the values in the first and second figures in part (a). Our original estimate was low.

8. (a)



The volumes of the resulting boxes are 1, 1.6875, and 2 ft^3. There appears to be a maximum volume of at least 2 ft^3.

(b) Let x denote the length of the side of the square being cut out. Let y denote the length of the base.

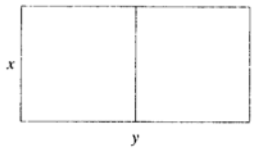


(c) Volume $V = \text{length} \times \text{width} \times \text{height} \quad \Rightarrow \quad V = y \cdot y \cdot x = xy^2$

(d) Length of cardboard $= 3 \quad \Rightarrow \quad x + y + x = 3 \quad \Rightarrow \quad y + 2x = 3$

(e) $y + 2x = 3 \quad \Rightarrow \quad y = 3 - 2x \quad \Rightarrow \quad V(x) = x(3 - 2x)^2$

(f) $V(x) = x(3 - 2x)^2 = x\left(4x^2 - 12x + 9\right) = 4x^3 - 12x^2 + 9x \quad \Rightarrow$
$V'(x) = 12x^2 - 24x + 9 = 3\left(4x^2 - 8x + 3\right) = 3(2x - 1)(2x - 3)$, so the critical numbers are $x = \frac{1}{2}$ and $x = \frac{3}{2}$. Now $0 \le x \le \frac{3}{2}$ and $V(0) = V\left(\frac{3}{2}\right) = 0$, so the maximum is $V\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)(2)^2 = 2\text{ ft}^3$, which is the value found from our third figure in part (a).

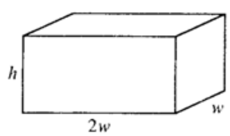
9. 

$xy = 1.5 \times 10^6$, so $y = 1.5 \times 10^6/x$. Minimize the amount of fencing, which is $3x + 2y = 3x + 2\left(1.5 \times 10^6/x\right) = 3x + 3 \times 10^6/x = F(x)$. $F'(x) = 3 - 3 \times 10^6/x^2 = 3\left(x^2 - 10^6\right)/x^2$. The critical number is $x = 10^3$ and $F'(x) < 0$ for $0 < x < 10^3$ and $F'(x) > 0$ if $x > 10^3$, so the absolute minimum occurs when $x = 10^3$ and $y = 1.5 \times 10^3$.

The field should be 1000 feet by 1500 feet with the middle fence parallel to the short side of the field.

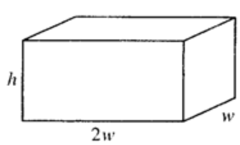
10. Let b be the area of the base of the box and h be its height, so $32{,}000 = hb^2$ or $h = 32{,}000/b^2$. The surface area of the open box is $b^2 + 4hb = b^2 + 4\left(32{,}000/b^2\right)b = b^2 + 4\,(32{,}000)\,/b$. So $V'(b) = 2b - 4\,(32{,}000)\,/b^2 = 2\left(b^3 - 64{,}000\right)/b^2 = 0 \quad\Leftrightarrow\quad b = \sqrt[3]{64{,}000} = 40$. This gives an absolute minimum since $V'(b) < 0$ if $b < 40$ and $V'(b) > 0$ if $b > 40$. The box should be $40 \times 40 \times 20$.

11. Let b be the base of the box and h the height. The surface area is $1200 = b^2 + 4hb \quad\Rightarrow\quad h = \left(1200 - b^2\right)/(4b)$. The volume is $V = b^2h = b^2\left(1200 - b^2\right)/4b = 300b - b^3/4 \quad\Rightarrow\quad V'(b) = 300 - \frac{3}{4}b^2$. $V'(b) = 0 \quad\Rightarrow$ $b = \sqrt{400} = 20$. Since $V'(b) > 0$ for $0 < b < 20$ and $V'(b) < 0$ for $b > 20$, there is an absolute maximum when $b = 20$ by the First Derivative Test for Absolute Maximum or Minimum Values. If $b = 20$, then $h = \left(1200 - 20^2\right)/(4 \cdot 20) = 10$, so the largest possible volume is $b^2h = (20)^2\,(10) = 4000\ \text{cm}^3$.

12. 

$10 = (2w)(w)\,h = 2w^2h$, so $h = 5/w^2$. The cost is $10\left(2w^2\right) + 6\left[2\,(2wh) + 2hw\right] = 20w^2 + 36wh$, so $C(w) = 20w^2 + 36w\left(5/w^2\right) = 20w^2 + 180/w$.

$C'(w) = 40w - 180/w^2 = 40\left(w^3 - \frac{9}{2}\right)\Big/\,w^2 \quad\Rightarrow\quad w = \sqrt[3]{\frac{9}{2}}$ is the critical number. There is an absolute minimum for $w = \sqrt[3]{\frac{9}{2}}$ since $C'(w) < 0$ for $0 < w < \sqrt[3]{\frac{9}{2}}$ and $C'(w) > 0$ for $w > \sqrt[3]{\frac{9}{2}}$. $C\left(\sqrt[3]{\frac{9}{2}}\right) = 20\left(\sqrt[3]{\frac{9}{2}}\right)^2 + \frac{180}{\sqrt[3]{9/2}} \approx \$163.54$.

13. 

$10 = (2w)(w)\,h = 2w^2h$, so $h = 5/w^2$. The cost is

$$\begin{aligned} C(w) &= 10\left(2w^2\right) + 6\left[2\,(2wh) + 2hw\right] + 6\left(2w^2\right) \\ &= 32w^2 + 36wh = 32w^2 + 180/w \end{aligned}$$

$C'(w) = 64w - 180/w^2 = 4\left(16w^3 - 45\right)\big/\,w^2 \quad\Rightarrow\quad w = \sqrt[3]{\frac{45}{16}}$ is the critical number. $C'(w) < 0$ for $0 < w < \sqrt[3]{\frac{45}{16}}$ and $C'(w) > 0$ for $w > \sqrt[3]{\frac{45}{16}}$. The minimum cost is $C\left(\sqrt[3]{\frac{45}{16}}\right) = 32\,(2.8125)^{2/3} + 180\Big/\sqrt[3]{2.8125} \approx \$191.28$.

14. (a) Let the rectangle have sides x and y and area A, so $A = xy$ or $y = A/x$. The problem is to minimize the perimeter $= 2x + 2y = 2x + 2A/x = P(x)$. Now $P'(x) = 2 - 2A/x^2 = 2\left(x^2 - A\right)/x^2$. So the critical number is $x = \sqrt{A}$. Since $P'(x) < 0$ for $0 < x < \sqrt{A}$ and $P'(x) > 0$ for $x > \sqrt{A}$, there is an absolute minimum at $x = \sqrt{A}$. The sides of the rectangle are $\sqrt{A}$ and $A\Big/\sqrt{A} = \sqrt{A}$, so the rectangle is a square.

(b) Let p be the perimeter and x and y the lengths of the sides, so $p = 2x + 2y \Rightarrow y = \frac{1}{2}p - x$. The area is $A(x) = x\left(\frac{1}{2}p - x\right) = \frac{1}{2}px - x^2$. Now $0 = A'(x) = \frac{1}{2}p - 2x \Rightarrow x = \frac{1}{4}p$. Since $A''(x) = -2 < 0$, there is an absolute maximum where $x = \frac{1}{4}p$ by the Second Derivative Test. The sides of the rectangle are $\frac{1}{4}p$ and $\frac{1}{2}p - \frac{1}{4}p = \frac{1}{4}p$, so the rectangle is a square.

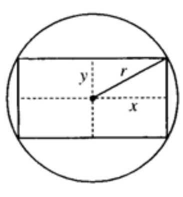
15. The distance from a point (x, y) on the line to the origin is $\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$. However, it is easier to work with the *square* of the distance, that is, $D(x) = \left(\sqrt{x^2 + y^2}\right)^2 = x^2 + y^2 = x^2 + (4x + 7)^2 = 17x^2 + 56x + 49$. Because the distance is positive, its minimum value will occur at the same point as the minimum value of D. $D'(x) = 34x + 56$, so $D'(x) = 0 \Leftrightarrow x = -\frac{28}{17}$. $D''(x) = 34 > 0$, so D is concave upward for all x. Thus, D has an absolute minimum at $x = -\frac{28}{17}$. The point closest to the origin is $\left(-\frac{28}{17}, \frac{7}{17}\right)$.

16. The square of the distance from a point (x, y) on the line $y = -6x + 9$ to the point $(-3, 1)$ is $D(x) = (x + 3)^2 + (y - 1)^2 = (x + 3)^2 + (-6x + 8)^2 = 37x^2 - 90x + 73$. $D'(x) = 74x - 90$, so $D'(x) = 0 \Leftrightarrow x = \frac{45}{37}$. $D''(x) = 74 > 0$, so D is concave upward for all x. Thus, D has an absolute minimum at $x = \frac{45}{37}$. The point on the line closest to $(-3, 1)$ is $\left(\frac{45}{37}, \frac{63}{37}\right)$.

17. By symmetry, the points are (x, y) and $(x, -y)$, where $y > 0$. The square of the distance is $D(x) = (x - 2)^2 + y^2 = (x - 2)^2 + \left(4 + x^2\right) = 2x^2 - 4x + 8$. So $D'(x) = 4x - 4 = 0 \Rightarrow x = 1$ and $y = \pm\sqrt{4 + 1} = \pm\sqrt{5}$. The points are $\left(1, \pm\sqrt{5}\right)$.

18. The square of the distance from a point (x, y) on the parabola $x = -y^2$ is $x^2 + (y + 3)^2 = y^4 + y^2 + 6y + 9 = D(y)$. Now $D'(y) = 4y^3 + 2y + 6 = 2(y + 1)\left(2y^2 - 2y + 3\right)$. Since $2y^2 - 2y + 3 = 0$ has no real roots, $y = -1$ is the only critical number. Then $x = -(-1)^2 = -1$, so the point is $(-1, -1)$.

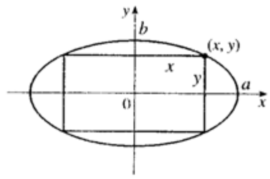
19.



Area of rectangle is $(2x)(2y) = 4xy$. Also $r^2 = x^2 + y^2$ so $y = \sqrt{r^2 - x^2}$, so the area is $A(x) = 4x\sqrt{r^2 - x^2}$. Now $A'(x) = 4\left(\sqrt{r^2 - x^2} - \dfrac{x^2}{\sqrt{r^2 - x^2}}\right) = 4\dfrac{r^2 - 2x^2}{\sqrt{r^2 - x^2}}$. The critical number is $x = \frac{1}{\sqrt{2}}r$. Clearly this gives a maximum.

$y = \sqrt{r^2 - \left(\frac{1}{\sqrt{2}}r\right)^2} = \sqrt{\frac{1}{2}r^2} = \frac{1}{\sqrt{2}}r = x$, which tells us that the rectangle is a square. The dimensions are $2x = \sqrt{2}r$ and $2y = \sqrt{2}r$.

20.

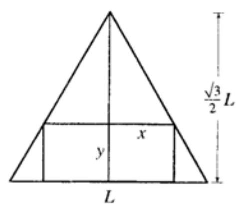


Area is $4xy$. Now the equation of the ellipse gives $y = \frac{b}{a}\sqrt{a^2 - x^2}$, so we maximize $A(x) = 4\frac{b}{a}x\sqrt{a^2 - x^2}$.

$$A'(x) = \frac{4b}{a}\sqrt{a^2 - x^2} + \frac{4bx}{a}\left[-\frac{2x}{2\sqrt{a^2 - x^2}}\right]$$
$$= \frac{4b}{a\sqrt{a^2 - x^2}}\left[a^2 - 2x^2\right]$$

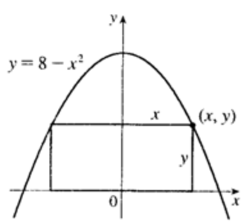
So the critical number is $x = \frac{1}{\sqrt{2}}a$, and this clearly gives a maximum. Then $y = \frac{1}{\sqrt{2}}b$, so the maximum area is $4\left(\frac{1}{\sqrt{2}}a\right)\left(\frac{1}{\sqrt{2}}b\right) = 2ab$.

21.



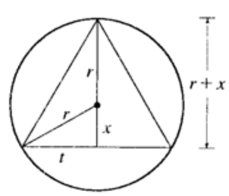
$\dfrac{\frac{\sqrt{3}}{2}L - y}{x} = \dfrac{\frac{\sqrt{3}}{2}L}{L/2} = \sqrt{3}$ (similar triangles) $\sqrt{3}x = \frac{\sqrt{3}}{2}L - y \;\Rightarrow$ $y = \frac{\sqrt{3}}{2}(L - 2x)$. The area of the inscribed rectangle is $A(x) = (2x)y = \sqrt{3}x(L - 2x)$ where $0 \le x \le L/2$. Now $0 = A'(x) = \sqrt{3}L - 4\sqrt{3}x \;\Rightarrow\; x = \sqrt{3}L\big/\left(4\sqrt{3}\right) = L/4$. Since $A(0) = A(L/2) = 0$, the maximum occurs when $x = L/4$, and $y = \frac{\sqrt{3}}{2}L - \frac{\sqrt{3}}{4}L = \frac{\sqrt{3}}{4}L$, so the dimensions are $L/2$ and $\frac{\sqrt{3}}{4}L$.

22.



The rectangle has area $A(x) = 2xy = 2x(8 - x^2) = 16x - 2x^3$, where $0 \le x \le 2\sqrt{2}$. Now $A'(x) = 16 - 6x^2 = 0 \;\Rightarrow\; x = 2\sqrt{\frac{2}{3}}$. Since $A(0) = A\left(2\sqrt{2}\right) = 0$, there is a maximum when $x = 2\sqrt{\frac{2}{3}}$. Then $y = \frac{16}{3}$, so the rectangle has dimensions $4\sqrt{\frac{2}{3}}$ and $\frac{16}{3}$.

23.

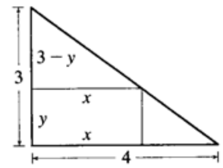


The area of the triangle is $A(x) = \frac{1}{2}(2t)(r + x) = t(r + x) = \sqrt{r^2 - x^2}\,(r + x)$. Then

$$0 = A'(x) = r\frac{-2x}{2\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2} + x\frac{-2x}{2\sqrt{r^2 - x^2}}$$
$$= -\frac{x^2 + rx}{\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2} \quad\Rightarrow$$

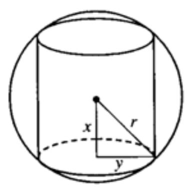
$\dfrac{x^2 + rx}{\sqrt{r^2 - x^2}} = \sqrt{r^2 - x^2} \;\Rightarrow\; x^2 + rx = r^2 - x^2 \;\Rightarrow\; 0 = 2x^2 + rx - r^2 = (2x - r)(x + r) \;\Rightarrow\; x = \frac{1}{2}r$ or $x = -r$. Now $A(r) = 0 = A(-r) \;\Rightarrow$ the maximum occurs where $x = \frac{1}{2}r$, so the triangle has height $r + \frac{1}{2}r = \frac{3}{2}r$ and base $2\sqrt{r^2 - \left(\frac{1}{2}r\right)^2} = 2\sqrt{\frac{3}{4}r^2} = \sqrt{3}r$.

24.



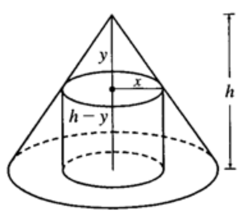
The rectangle has area xy. By similar triangles $\dfrac{3-y}{x}=\dfrac{3}{4} \Rightarrow$ $-4y+12=3x$ or $y=-\frac{3}{4}x+3$. So the area is $A(x)=x\left(-\frac{3}{4}x+3\right)=-\frac{3}{4}x^2+3x$ where $0\le x\le 4$. Now $0=A'(x)=-\frac{3}{2}x+3 \Rightarrow x=2$ and $y=\frac{3}{2}$. Since $A(0)=A(4)=0$, the maximum area is $A(2)=2\left(\frac{3}{2}\right)=3\text{ cm}^2$.

25.



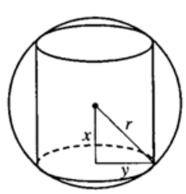
The cylinder has volume $V=\pi y^2(2x)$. Also $x^2+y^2=r^2 \Rightarrow$ $y^2=r^2-x^2$, so $V(x)=\pi\left(r^2-x^2\right)(2x)=2\pi\left(r^2x-x^3\right)$, where $0\le x\le r$. $V'(x)=2\pi\left(r^2-3x^2\right)=0 \Rightarrow x=r/\sqrt{3}$. Now $V(0)=V(r)=0$, so there is a maximum when $x=r\big/\sqrt{3}$ and $V\left(r\big/\sqrt{3}\right)=\pi\left(r^2-r^2/3\right)\left(2r\big/\sqrt{3}\right)=4\pi r^3\big/\left(3\sqrt{3}\right)$.

26.



By similar triangles, $y/x=h/r$, so $y=hx/r$. The volume of the cylinder is $\pi x^2(h-y)=\pi hx^2-(\pi h/r)x^3=V(x)$. Now $V'(x)=2\pi hx-(3\pi h/r)x^2=\pi hx(2-3x/r)$. So $0=V'(x) \Rightarrow$ $x=0$ or $x=\frac{2}{3}r$. The maximum clearly occurs when $x=\frac{2}{3}r$ and then the volume is $\pi\left(\frac{2}{3}r\right)^2h\left(1-\frac{2}{3}\right)=\frac{4}{27}\pi r^2h$.

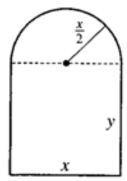
27.



The cylinder has surface area $2\pi y^2+2\pi y(2x)$. Now $x^2+y^2=r^2 \Rightarrow$ $y=\sqrt{r^2-x^2}$, so the surface area is $S(x)=2\pi\left(r^2-x^2\right)+4\pi x\sqrt{r^2-x^2}$, $0\le x\le r$.

$$\begin{aligned} S'(x)&=-4\pi x+4\pi\sqrt{r^2-x^2}-4\pi x^2/\sqrt{r^2-x^2}\\ &=\frac{4\pi\left(r^2-2x^2-x\sqrt{r^2-x^2}\right)}{\sqrt{r^2-x^2}}=0 \Rightarrow \end{aligned}$$

$x\sqrt{r^2-x^2}=r^2-2x^2$ $(\bigstar)$ $\Rightarrow$ $x^2\left(r^2-x^2\right)=r^4-4r^2x^2+4x^4 \Rightarrow 5x^4-5r^2x^2+r^4=0$. By the quadratic formula, $x^2=\frac{5\pm\sqrt{5}}{10}r^2$, but we reject the root with the $+$ sign since it doesn't satisfy $(\bigstar)$. So $x=\sqrt{\frac{5-\sqrt{5}}{10}}\,r$. Since $S(0)=S(r)=0$, the maximum occurs at the critical number and $x^2=\frac{5-\sqrt{5}}{10}r^2 \Rightarrow$ $y^2=\frac{5+\sqrt{5}}{10}r^2 \Rightarrow$ the surface area is $2\pi\left(\frac{5+\sqrt{5}}{10}\right)r^2+4\pi\sqrt{\frac{5-\sqrt{5}}{10}}\sqrt{\frac{5+\sqrt{5}}{10}}r^2=\pi r^2\left(1+\sqrt{5}\right)$.

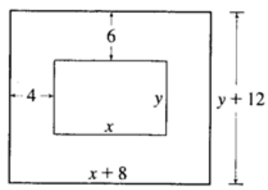
28. 

We are given $2y + x + \pi\left(\frac{x}{2}\right) = 30$, so $y = \frac{1}{2}\left(30 - x - \frac{\pi x}{2}\right)$. The area is $xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$, so

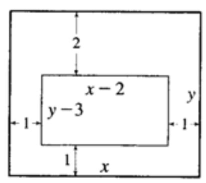
$$A(x) = x\left(15 - \frac{x}{2} - \frac{\pi x}{4}\right) + \tfrac{1}{8}\pi x^2 = 15x - \tfrac{1}{2}x^2 - \tfrac{\pi}{8}x^2.$$

$$A'(x) = 15 - \left(1 + \tfrac{\pi}{4}\right)x = 0 \quad\Rightarrow\quad x = \frac{15}{1 + \pi/4} = \frac{60}{4+\pi}.$$

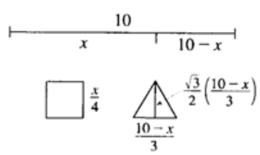
Clearly this gives a maximum, so the dimensions are $x = \dfrac{60}{4+\pi}$ ft and $y = 15 - \dfrac{30}{4+\pi} - \dfrac{15\pi}{4+\pi} = \dfrac{30}{4+\pi}$ ft, so the height of the rectangle is half the base.

29. 

$xy = 384 \;\Rightarrow\; y = 384/x$. Total area is
$A(x) = (8 + x)(12 + 384/x) = 12(40 + x + 256/x)$, so
$A'(x) = 12\left(1 - 256/x^2\right) = 0 \;\Rightarrow\; x = 16$. There is an absolute minimum when $x = 16$ since $A'(x) < 0$ for $0 < x < 16$ and $A'(x) > 0$ for $x > 16$. When $x = 16$, $y = 384/16 = 24$, so the dimensions are 24 cm and 36 cm.

30. 

$xy = 180$, so $y = 180/x$. The printed area is
$(x-2)(y-3) = (x-2)(180/x - 3) = 186 - 3x - 360/x = A(x)$.
$A'(x) = -3 + 360/x^2 = 0$ when $x^2 = 120 \;\Rightarrow\; x = 2\sqrt{30}$. This gives an absolute maximum since $A'(x) > 0$ for $0 < x < 2\sqrt{30}$ and $A'(x) < 0$ for $x > 2\sqrt{30}$. When $x = 2\sqrt{30}$, $y = 180/(2\sqrt{30})$, so the dimensions are $2\sqrt{30}$ in. and $90/\sqrt{30}$ in.

31. 

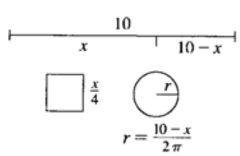
Let x be the length of the wire used for the square. The total area is

$$A(x) = \left(\frac{x}{4}\right)^2 + \frac{1}{2}\left(\frac{10-x}{3}\right)\frac{\sqrt{3}}{2}\left(\frac{10-x}{3}\right)$$
$$= \tfrac{1}{16}x^2 + \tfrac{\sqrt{3}}{36}(10-x)^2,\; 0 \le x \le 10$$

$A'(x) = \frac{1}{8}x - \frac{\sqrt{3}}{18}(10 - x) = 0 \;\Leftrightarrow\; \frac{9}{72}x + \frac{4\sqrt{3}}{72}x - \frac{40\sqrt{3}}{72} = 0 \;\Leftrightarrow\; x = \frac{40\sqrt{3}}{9+4\sqrt{3}}$. Now
$A(0) = \left(\frac{\sqrt{3}}{36}\right)100 \approx 4.81$, $A(10) = \frac{100}{16} = 6.25$ and $A\left(\frac{40\sqrt{3}}{9+4\sqrt{3}}\right) \approx 2.72$, so

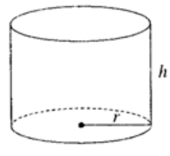
(a) The maximum area occurs when $x = 10$ m, and all the wire is used for the square.

(b) The minimum area occurs when $x = \frac{40\sqrt{3}}{9+4\sqrt{3}} \approx 4.35$ m.

32. 

Total area is $A(x) = \left(\frac{x}{4}\right)^2 + \pi\left(\frac{10-x}{2\pi}\right)^2 = \dfrac{x^2}{16} + \dfrac{(10-x)^2}{4\pi}$,
$0 \le x \le 10$. $A'(x) = \dfrac{x}{8} - \dfrac{10-x}{2\pi} = \left(\dfrac{1}{2\pi} + \dfrac{1}{8}\right)x - \dfrac{5}{\pi} = 0 \;\Rightarrow$
$x = 40/(4+\pi)$. $A(0) = 25/\pi \approx 7.96$, $A(10) = 6.25$, and $A(40/(4+\pi)) \approx 3.5$, so the maximum occurs when $x = 0$ m and the minimum occurs when $x = 40/(4+\pi)$ m.

33.



The volume is $V = \pi r^2 h$ and the surface area is

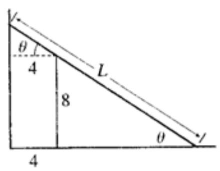
$$S(r) = \pi r^2 + 2\pi r h = \pi r^2 + 2\pi r\left(\frac{V}{\pi r^2}\right) = \pi r^2 + \frac{2V}{r}.$$

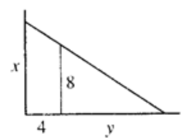
$$S'(r) = 2\pi r - \frac{2V}{r^2} = 0 \quad\Rightarrow\quad 2\pi r^3 = 2V \quad\Rightarrow\quad r = \sqrt[3]{\frac{V}{\pi}}\ \text{cm}.$$

This gives an absolute minimum since $S'(r) < 0$ for $0 < r < \sqrt[3]{\frac{V}{\pi}}$ and $S'(r) > 0$ for $r > \sqrt[3]{\frac{V}{\pi}}$. When $r = \sqrt[3]{\frac{V}{\pi}}$,

$$h = \frac{V}{\pi r^2} = \frac{V}{\pi (V/\pi)^{2/3}} = \sqrt[3]{\frac{V}{\pi}}\ \text{cm}.$$

34.



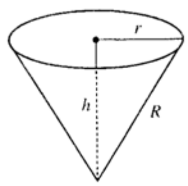


$L = 8\csc\theta + 4\sec\theta$, $0 < \theta < \frac{\pi}{2}$,

$\dfrac{dL}{d\theta} = -8\csc\theta\cot\theta + 4\sec\theta\tan\theta = 0$ when $\sec\theta\tan\theta = 2\csc\theta\cot\theta$ $\Leftrightarrow$ $\tan^3\theta = 2$ $\Leftrightarrow$ $\tan\theta = \sqrt[3]{2}$ $\Leftrightarrow$ $\theta = \tan^{-1}\sqrt[3]{2}$. $dL/d\theta < 0$ when $0 < \theta < \tan^{-1}\sqrt[3]{2}$, $dL/d\theta > 0$ when $\tan^{-1}\sqrt[3]{2} < \theta < \frac{\pi}{2}$, so L has an absolute minimum when $\theta = \tan^{-1}\sqrt[3]{2}$, so the shortest ladder has length $L = 8\dfrac{\sqrt{1 + 2^{2/3}}}{2^{1/3}} + 4\sqrt{1 + 2^{2/3}} \approx 16.65$ ft.

Another Method: Minimize $L^2 = x^2 + (4 + y)^2$, where $\dfrac{x}{4 + y} = \dfrac{8}{y}$.

35.



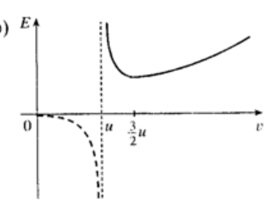
$h^2 + r^2 = R^2 \quad\Rightarrow\quad V = \frac{\pi}{3}r^2 h = \frac{\pi}{3}\left(R^2 - h^2\right)h = \frac{\pi}{3}\left(R^2 h - h^3\right)$.

$V'(h) = \frac{\pi}{3}\left(R^2 - 3h^2\right) = 0$ when $h = \frac{1}{\sqrt{3}}R$. This gives an absolute maximum, since $V'(h) > 0$ for $0 < h < \frac{1}{\sqrt{3}}R$ and $V'(h) < 0$ for $h > \frac{1}{\sqrt{3}}R$. Maximum volume is

$$V\left(\tfrac{1}{\sqrt{3}}R\right) = \tfrac{\pi}{3}\left(\tfrac{1}{\sqrt{3}}R^3 - \tfrac{1}{3\sqrt{3}}R^3\right) = \tfrac{2}{9\sqrt{3}}\pi R^3.$$

36. (a) $E(v) = \dfrac{aLv^3}{v - u} \quad\Rightarrow\quad E'(v) = aL\dfrac{(v - u)3v^2 - v^3}{(v - u)^2} = 0$ when $2v^3 = 3uv^2 \quad\Rightarrow\quad 2v = 3u \quad\Rightarrow\quad v = \frac{3}{2}u$. The First Derivative Test shows that this value of v gives the minimum value of E.

(b)

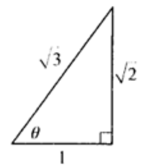


37. $S = 6sh - \frac{3}{2}s^2 \cot\theta + 3s^2 \frac{\sqrt{3}}{2} \csc\theta$

(a) $\dfrac{dS}{d\theta} = \frac{3}{2}s^2 \csc^2\theta - 3s^2 \frac{\sqrt{3}}{2} \csc\theta \cot\theta$ or $\frac{3}{2}s^2 \csc\theta \left(\csc\theta - \sqrt{3}\cot\theta\right)$.

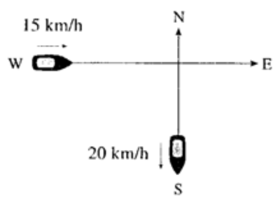
(b) $\dfrac{dS}{d\theta} = 0$ when $\csc\theta - \sqrt{3}\cot\theta = 0 \;\Rightarrow\; \dfrac{1}{\sin\theta} - \sqrt{3}\dfrac{\cos\theta}{\sin\theta} = 0 \;\Rightarrow\; \cos\theta = \frac{1}{\sqrt{3}}$. The First Derivative Test shows that the minimum surface area occurs when $\theta = \cos^{-1}\frac{1}{\sqrt{3}} \approx 55°$.

(c)



If $\cos\theta = \frac{1}{\sqrt{3}}$, then $\cot\theta = \frac{1}{\sqrt{2}}$ and $\csc\theta = \frac{\sqrt{3}}{\sqrt{2}}$, so the surface area is $S = 6sh - \frac{3}{2}s^2\frac{1}{\sqrt{2}} + 3s^2\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{\sqrt{2}} = 6sh - \frac{3}{2\sqrt{2}}s^2 + \frac{9}{2\sqrt{2}}s^2 = 6s\left(h + \frac{1}{2\sqrt{2}}s\right)$.

38.

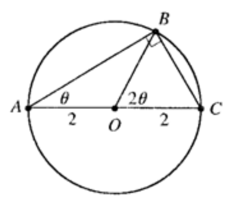


Let t be the time, in hours, after 2:00 P.M. The position of the boat heading south at time t is $(0, -20t)$. The position of the boat heading east at time t is $(-15 + 15t, 0)$. If $D(t)$ is the distance between the boats at time t, we minimize $f(t) = [D(t)]^2 = 20^2t^2 + 15^2(t-1)^2$.

$f'(t) = 800t + 450(t-1) = 1250t - 450 = 0$ when $t = \frac{450}{1250} = 0.36$ h. $0.36\text{ h} \times \frac{60\text{ min}}{\text{h}} = 21.6\text{ min} = 21$ min 36 s. Since $f''(t) > 0$, this gives a minimum, so the boats are closest together at 2:21:36 P.M.

39. Here $T(x) = \dfrac{\sqrt{x^2+25}}{6} + \dfrac{5-x}{8}$, $0 \le x \le 5 \;\Rightarrow\; T'(x) = \dfrac{x}{6\sqrt{x^2+25}} - \dfrac{1}{8} = 0 \;\Leftrightarrow\; 8x = 6\sqrt{x^2+25} \;\Leftrightarrow\; 16x^2 = 9\left(x^2+25\right) \;\Leftrightarrow\; x = \frac{15}{\sqrt{7}}$. But $\frac{15}{\sqrt{7}} > 5$, so T has no critical number. Since $T(0) \approx 1.46$ and $T(5) \approx 1.18$, he should row directly to B.

40.



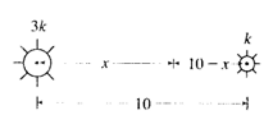
In isosceles triangle AOB, $\angle O = 180° - \theta - \theta$, so $\angle BOC = 2\theta$. The distance rowed is $4\cos\theta$ while the distance walked is the length of arc $BC = 2(2\theta) = 4\theta$. The time taken is given by

$T(\theta) = \dfrac{4\cos\theta}{2} + \dfrac{4\theta}{4} = 2\cos\theta + \theta$, $0 \le \theta \le \frac{\pi}{2}$.

$T'(\theta) = -2\sin\theta + 1 = 0 \;\Leftrightarrow\; \sin\theta = \frac{1}{2} \;\Rightarrow\; \theta = \frac{\pi}{6}$.

Check the value of T at $\theta = \frac{\pi}{6}$ and at the endpoints of the domain of T, that is, $\theta = 0$ and $\theta = \frac{\pi}{2}$. $T(0) = 2$, $T\left(\frac{\pi}{6}\right) = \sqrt{3} + \frac{\pi}{6} \approx 2.26$, and $T\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \approx 1.57$. Therefore, the minimum value of T is $\frac{\pi}{2}$ when $\theta = \frac{\pi}{2}$, that is, the woman should walk all the way. Note that $T''(\theta) = -2\cos\theta < 0$ for $0 \le \theta < \frac{\pi}{2}$, so $\theta = \frac{\pi}{6}$ gives a maximum time.

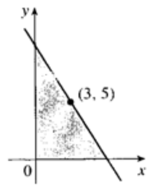
41.



The total illumination is $I(x) = \dfrac{3k}{x^2} + \dfrac{k}{(10-x)^2}$, $0 < x < 10$. Then

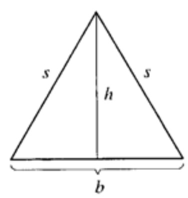
$I'(x) = \dfrac{-6k}{x^3} + \dfrac{2k}{(10-x)^3} = 0 \;\Rightarrow\; 6k(10-x)^3 = 2kx^3 \;\Rightarrow\; \sqrt[3]{3}(10-x) = x \;\Rightarrow\; x = \dfrac{10\sqrt[3]{3}}{1+\sqrt[3]{3}} \approx 5.9$ ft. This gives a minimum since $I''(x) > 0$ for $0 < x < 10$.

42.



The line with slope m (where $m < 0$) through $(3, 5)$ has equation $y - 5 = m(x - 3)$ or $y = mx + (5 - 3m)$. The y-intercept is $5 - 3m$ and the x-intercept is $-5/m + 3$. So the triangle has area $A(m) = \frac{1}{2}(5 - 3m)(-5/m + 3) = 15 - 25/(2m) - \frac{9}{2}m$. Now $A'(m) = \dfrac{25}{2m^2} - \dfrac{9}{2} = 0 \Leftrightarrow m = -\frac{5}{3}$ (since $m < 0$). $A''(m) = -\dfrac{25}{m^3} > 0$, so there is an absolute minimum when $m = -\frac{5}{3}$. Therefore, the equation of the line is $y - 5 = -\frac{5}{3}(x - 3)$ or $y = -\frac{5}{3}x + 10$.

43.

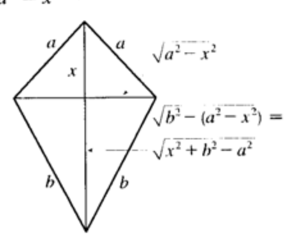


Here $s^2 = h^2 + b^2/4$, so $h^2 = s^2 - b^2/4$. The area is $A = \frac{1}{2}b\sqrt{s^2 - b^2/4}$. Let the perimeter be p, so $2s + b = p$ or $s = (p - b)/2 \Rightarrow$
$A(b) = \frac{1}{2}b\sqrt{(p - b)^2/4 - b^2/4} = b\sqrt{p^2 - 2pb}/4$. Now
$A'(b) = \dfrac{\sqrt{p^2 - 2pb}}{4} - \dfrac{bp/4}{\sqrt{p^2 - 2pb}} = \dfrac{-3pb + p^2}{4\sqrt{p^2 - 2pb}}$. Therefore, $A'(b) = 0$ $\Rightarrow -3pb + p^2 = 0 \Rightarrow b = p/3$. Since $A'(b) > 0$ for $b < p/3$ and $A'(b) < 0$ for $b > p/3$, there is an absolute maximum when $b = p/3$. But then $2s + p/3 = p$ so $s = p/3 \Rightarrow s = b \Rightarrow$ the triangle is equilateral.

44. The area is given by

$A(x) = \frac{1}{2}\left(2\sqrt{a^2 - x^2}\right)x + \frac{1}{2}\left(2\sqrt{a^2 - x^2}\right)\left(\sqrt{x^2 + b^2 - a^2}\right) = \sqrt{a^2 - x^2}\left(x + \sqrt{x^2 + b^2 - a^2}\right)$ for $0 \le x \le a$. Now $A'(x) = \dfrac{-x}{\sqrt{a^2 - x^2}}\left(x + \sqrt{x^2 + b^2 - a^2}\right) + \sqrt{a^2 - x^2}\left(1 + \dfrac{x}{\sqrt{x^2 + b^2 - a^2}}\right) = 0 \Leftrightarrow$

$$\frac{x}{\sqrt{a^2 - x^2}}\left(x + \sqrt{x^2 + b^2 - a^2}\right) = \sqrt{a^2 - x^2}\left(\frac{x + \sqrt{x^2 + b^2 - a^2}}{\sqrt{x^2 + b^2 - a^2}}\right).$$



Except for the trivial case where $x = 0$, $a = b$ and $A(x) = 0$, we have $x + \sqrt{x^2 + b^2 - a^2} > 0$. Hence, cancelling this factor gives

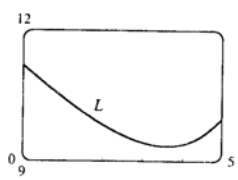
$\dfrac{x}{\sqrt{a^2 - x^2}} = \dfrac{\sqrt{a^2 - x^2}}{\sqrt{x^2 + b^2 - a^2}} \Rightarrow x\sqrt{x^2 + b^2 - a^2} = a^2 - x^2 \Rightarrow$
$x^2(x^2 + b^2 - a^2) = a^4 - 2a^2x^2 + x^4 \Rightarrow x^2(b^2 - a^2) = a^4 - 2a^2x^2$
$\Rightarrow x^2(b^2 + a^2) = a^4 \Rightarrow x = \dfrac{a^2}{\sqrt{a^2 + b^2}}.$

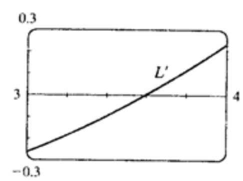
Now we must check the value of A at this point as well as at the endpoints of the domain to see which gives the maximum value. $A(0) = a\sqrt{b^2 - a^2}$, $A(a) = 0$ and

$$A\left(\frac{a^2}{\sqrt{a^2+b^2}}\right) = \sqrt{a^2 - \left(\frac{a^2}{\sqrt{a^2+b^2}}\right)^2}\left[\frac{a^2}{\sqrt{a^2+b^2}} + \sqrt{\left(\frac{a^2}{\sqrt{a^2+b^2}}\right)^2 + b^2 - a^2}\right]$$
$$= \frac{ab}{\sqrt{a^2+b^2}}\left[\frac{a^2}{\sqrt{a^2+b^2}} + \frac{b^2}{\sqrt{a^2+b^2}}\right] = \frac{ab\left(a^2+b^2\right)}{a^2+b^2} = ab$$

Since $b \geq \sqrt{b^2 - a^2}$, $A\left(a^2/\sqrt{a^2+b^2}\right) \geq A(0)$. So there is an absolute maximum when $x = \dfrac{a^2}{\sqrt{a^2+b^2}}$. In this case the horizontal piece should be $\dfrac{2ab}{\sqrt{a^2+b^2}}$ and the vertical piece should be $\dfrac{a^2+b^2}{\sqrt{a^2+b^2}} = \sqrt{a^2+b^2}$.

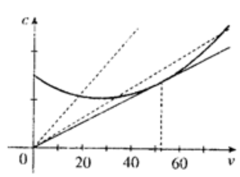
45. $L(x) = |AP| + |BP| + |CP| = x + \sqrt{(5-x)^2 + 2^2} + \sqrt{(5-x)^2 + 3^2}$
$= x + \sqrt{x^2 - 10x + 29} + \sqrt{x^2 - 10x + 34} \quad \Rightarrow$
$L'(x) = 1 + \dfrac{x-5}{\sqrt{x^2 - 10x + 29}} + \dfrac{x-5}{\sqrt{x^2 - 10x + 34}}$





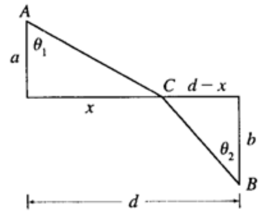
From the graphs of L and L', it seems that the minimum value of L is about $L(3.59) = 9.35$ m.

46. We note that since c is the consumption in gallons per hour, and v is the velocity in miles per hour, then $\dfrac{c}{v} = \dfrac{\text{gallons/hour}}{\text{miles/hour}} = \dfrac{\text{gallons}}{\text{mile}}$ gives us the consumption in gallons per mile, that is, the quantity G. To find the minimum, we calculate $\dfrac{dG}{dv} = \dfrac{d}{dv}\left(\dfrac{c}{v}\right) = \dfrac{v\dfrac{dc}{dv} - c\dfrac{dv}{dv}}{v^2} = \dfrac{v\dfrac{dc}{dv} - c}{v^2}$.



This is 0 when $v\dfrac{dc}{dv} - c = 0 \quad \Leftrightarrow \quad \dfrac{dc}{dv} = \dfrac{c}{v}$. This implies that the tangent line of $c(v)$ passes through the origin, and this occurs when $v \approx 53$ mi/h. Note that the slope of the secant line through the origin and a point $(v, c(v))$ on the graph is equal to $G(v)$, and it is intuitively clear that G is minimized in the case where the secant is in fact a tangent.

47.



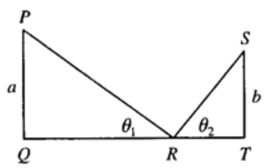
The total time is

$$T(x) = (\text{time from } A \text{ to } C) + (\text{time from } C \text{ to } B)$$
$$= \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(d-x)^2}}{v_2},\ 0 < x < d$$

$$T'(x) = \frac{x}{v_1\sqrt{a^2+x^2}} - \frac{d-x}{v_2\sqrt{b^2+(d-x)^2}} = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}$$

The minimum occurs when $T'(x) = 0 \Rightarrow \dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2}$.

48.



If $d = |QT|$, we minimize $f(\theta_1) = |PR| + |RS| = a\csc\theta_1 + b\csc\theta_2$.

Differentiating with respect to θ_1, and setting $\dfrac{df}{d\theta_1}$ equal to 0, we get

$$\frac{df}{d\theta_1} = 0 = -a\csc\theta_1\cot\theta_1 - b\csc\theta_2\cot\theta_2\frac{d\theta_2}{d\theta_1}.$$

So we need to find an expression for $\dfrac{d\theta_2}{d\theta_1}$. We can do this by observing that $|QT| = \text{constant} = a\cot\theta_1 + b\cot\theta_2$.

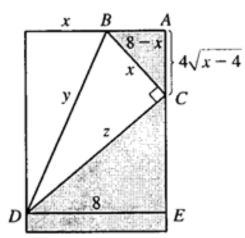
Differentiating this equation implicitly with respect to θ_1, we get $-a\csc^2\theta_1 - b\csc^2\theta_2\dfrac{d\theta_2}{d\theta_1} = 0$

$\Rightarrow \dfrac{d\theta_2}{d\theta_1} = -\dfrac{a\csc^2\theta_1}{b\csc^2\theta_2}$. We substitute this into the expression for $\dfrac{df}{d\theta_1}$ to get

$-a\csc\theta_1\cot\theta_1 - b\csc\theta_2\cot\theta_2\left(-\dfrac{a\csc^2\theta_1}{b\csc^2\theta_2}\right) = 0 \;\Leftrightarrow\; -a\csc\theta_1\cot\theta_1 + a\dfrac{\csc^2\theta_1\cot\theta_2}{\csc\theta_2} = 0 \;\Leftrightarrow$

$\cot\theta_1\csc\theta_2 = \csc\theta_1\cot\theta_2 \;\Leftrightarrow\; \dfrac{\cot\theta_1}{\csc\theta_1} = \dfrac{\cot\theta_2}{\csc\theta_2} \;\Leftrightarrow\; \cos\theta_1 = \cos\theta_2$. Since θ_1 and θ_2 are both acute, we have $\theta_1 = \theta_2$.

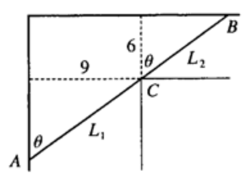
49.



$y^2 = x^2 + z^2$, but triangles CDE and BCA are similar, so $z/8 = x\big/\left(4\sqrt{x-4}\right) \Rightarrow z = 2x\big/\sqrt{x-4}$. Thus, we minimize $f(x) = y^2 = x^2 + 4x^2/(x-4) = x^3/(x-4)$, $4 < x \le 8$.

$f'(x) = \dfrac{3x^2(x-4) - x^3}{(x-4)^2} = \dfrac{2x^2(x-6)}{(x-4)^2} = 0$ when $x = 6$. $f'(x) < 0$ when $x < 6$, $f'(x) > 0$ when $x > 6$, so the minimum occurs when $x = 6$ in.

50.

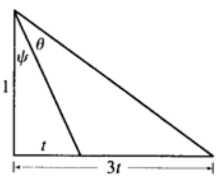


Paradoxically, we solve this maximum problem by solving a minimum problem. Let L be the length of the line ACB going from wall to wall touching the inner corner C. As $\theta \to 0$ or $\theta \to \frac{\pi}{2}$, we have $L \to \infty$ and there will be an angle that makes L a minimum. A pipe of this length will just fit around the corner.

From the diagram, $L = L_1 + L_2 = 9\csc\theta + 6\sec\theta \;\Rightarrow\; dL/d\theta = -9\csc\theta\cot\theta + 6\sec\theta\tan\theta = 0$ when $6\sec\theta\tan\theta = 9\csc\theta\cot\theta \;\Leftrightarrow\; \tan^3\theta = \frac{9}{6} = 1.5 \;\Leftrightarrow\; \tan\theta = \sqrt[3]{1.5}$. Then $\sec^2\theta = 1 + \left(\frac{3}{2}\right)^{2/3}$ and $\csc^2\theta = 1 + \left(\frac{3}{2}\right)^{-2/3}$, so the longest pipe has length $L = 9\left[1 + \left(\frac{3}{2}\right)^{-2/3}\right]^{1/2} + 6\left[1 + \left(\frac{3}{2}\right)^{2/3}\right]^{1/2} \approx 21.07$ ft.

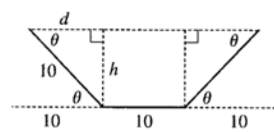
Or, use $\theta = \tan^{-1}\left(\sqrt[3]{1.5}\right) \approx 0.852 \;\Rightarrow\; L = 9\csc\theta + 6\sec\theta \approx 21.07$ ft.

51.



It suffices to maximize $\tan\theta$. Now $\dfrac{3t}{1} = \tan(\psi + \theta) = \dfrac{\tan\psi + \tan\theta}{1 - \tan\psi\tan\theta} = \dfrac{t + \tan\theta}{1 - t\tan\theta}$. So $3t(1 - t\tan\theta) = t + \tan\theta \;\Rightarrow\; 2t = \left(1 + 3t^2\right)\tan\theta \;\Rightarrow\; \tan\theta = \dfrac{2t}{1 + 3t^2}$. Let $f(t) = \tan\theta = \dfrac{2t}{1 + 3t^2} \;\Rightarrow$ $f'(t) = \dfrac{2\left(1 + 3t^2\right) - 2t(6t)}{\left(1 + 3t^2\right)^2} = \dfrac{2\left(1 - 3t^2\right)}{\left(1 + 3t^2\right)^2} = 0 \;\Leftrightarrow\; 1 - 3t^2 = 0 \;\Leftrightarrow\; t = \frac{1}{\sqrt{3}}$ since $t \geq 0$. Now $f'(t) > 0$ for $0 \leq t < \frac{1}{\sqrt{3}}$ and $f'(t) < 0$ for $t > \frac{1}{\sqrt{3}}$, so f has an absolute maximum when $t = \frac{1}{\sqrt{3}}$ and $\tan\theta = \dfrac{2\left(1/\sqrt{3}\right)}{1+3\left(1/\sqrt{3}\right)^2} = \frac{1}{\sqrt{3}} \;\Rightarrow\; \theta = \frac{\pi}{6}$. Substituting for t and θ in $3t = \tan(\psi + \theta)$ gives us $\sqrt{3} = \tan\left(\psi + \frac{\pi}{6}\right)$ $\Rightarrow\; \psi = \frac{\pi}{6}$.

52.

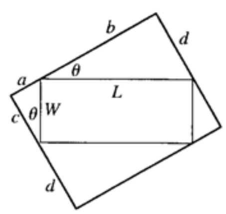


We maximize the cross-sectional area

$$A(\theta) = 10h + 2\left(\tfrac{1}{2}dh\right) = 10h + dh = 10(10\sin\theta) + (10\cos\theta)(10\sin\theta)$$
$$= 100(\sin\theta + \sin\theta\cos\theta),\; 0 \leq \theta \leq \tfrac{\pi}{2}$$
$$A'(\theta) = 100\left(\cos\theta + \cos^2\theta - \sin^2\theta\right) = 100\left(\cos\theta + 2\cos^2\theta - 1\right) = 100(2\cos\theta - 1)(\cos\theta + 1)$$
$$= 0 \text{ when } \cos\theta = \tfrac{1}{2} \;\Leftrightarrow\; \theta = \tfrac{\pi}{3}.\; (\cos\theta \neq -1 \text{ since } 0 \leq \theta \leq \tfrac{\pi}{2}.)$$

Now $A(0) = 0$, $A\left(\frac{\pi}{2}\right) = 100$ and $A\left(\frac{\pi}{3}\right) = 75\sqrt{3} \approx 129.9$, so the maximum occurs when $\theta = \frac{\pi}{3}$.

53.



$a = W\sin\theta$, $c = W\cos\theta$, $b = L\cos\theta$, $d = L\sin\theta$, so the area of the circumscribed rectangle is

$$A(\theta) = (a + b)(c + d) = (W\sin\theta + L\cos\theta)(W\cos\theta + L\sin\theta)$$
$$= LW\sin^2\theta + LW\cos^2\theta + \left(L^2 + W^2\right)\sin\theta\cos\theta$$
$$= LW + \tfrac{1}{2}\left(L^2 + W^2\right)\sin 2\theta,\; 0 \leq \theta \leq \tfrac{\pi}{2}$$

This expression shows, without calculus, that the maximum value of $A(\theta)$ occurs when $\sin 2\theta = 1 \;\Leftrightarrow\; 2\theta = \frac{\pi}{2}$ $\Rightarrow\; x = \frac{\pi}{4}$. So the maximum area is $A\left(\frac{\pi}{4}\right) = LW + \frac{1}{2}\left(L^2 + W^2\right) = \frac{1}{2}(L + W)^2$.

54. (a) Let D be the point such that $a = |AD|$. From the figure, $\sin\theta = \dfrac{b}{|BC|} \Rightarrow |BC| = b\csc\theta$ and

$\cos\theta = \dfrac{|BD|}{|BC|} = \dfrac{a - |AB|}{|BC|} \Rightarrow |BC| = (a - |AB|)\sec\theta$. Eliminating $|BC|$ gives

$(a - |AB|)\sec\theta = b\csc\theta \Rightarrow b\cot\theta = a - |AB| \Rightarrow |AB| = a - b\cot\theta$. The total resistance is

$R(\theta) = C\dfrac{|AB|}{r_1^4} + C\dfrac{|BC|}{r_2^4} = C\left(\dfrac{a - b\cot\theta}{r_1^4} + \dfrac{b\csc\theta}{r_2^4}\right)$.

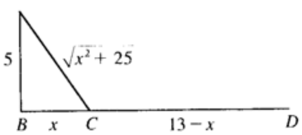
(b) $R'(\theta) = C\left(\dfrac{b\csc^2\theta}{r_1^4} - \dfrac{b\csc\theta\cot\theta}{r_2^4}\right) = bC\csc\theta\left(\dfrac{\csc\theta}{r_1^4} - \dfrac{\cot\theta}{r_2^4}\right)$.

$R'(\theta) = 0 \Leftrightarrow \dfrac{\csc\theta}{r_1^4} = \dfrac{\cot\theta}{r_2^4} \Leftrightarrow \dfrac{r_2^4}{r_1^4} = \dfrac{\cot\theta}{\csc\theta} = \cos\theta$.

$R'(\theta) > 0 \Leftrightarrow \dfrac{\csc\theta}{r_1^4} > \dfrac{\cot\theta}{r_2^4} \Rightarrow \cos\theta < \dfrac{r_2^4}{r_1^4}$ and $R'(\theta) < 0$ when $\cos\theta > \dfrac{r_2^4}{r_1^4}$, so there is an absolute minimum when $\cos\theta = r_2^4/r_1^4$.

(c) When $r_2 = \frac{2}{3}r_1$, we have $\cos\theta = \left(\frac{2}{3}\right)^4$, so $\theta = \cos^{-1}\left(\frac{2}{3}\right)^4 \approx 79°$.

55. (a)



If k = energy/km over land, then energy/km over water = $1.4k$. So the total energy is $E = 1.4k\sqrt{25 + x^2} + k(13 - x)$, $0 \le x \le 13$, and so $\dfrac{dE}{dx} = \dfrac{1.4kx}{(25 + x^2)^{1/2}} - k$. Set $\dfrac{dE}{dx} = 0$:

$1.4kx = k(25 + x^2)^{1/2} \Rightarrow 1.96x^2 = x^2 + 25 \Rightarrow 0.96x^2 = 25 \Rightarrow x = \frac{5}{\sqrt{0.96}} \approx 5.1$. Testing against the value of E at the endpoints: $E(0) = 1.4k(5) + 13k = 20k$, $E(5.1) \approx 17.9k$, $E(13) \approx 19.5k$. Thus, to minimize energy, the bird should fly to a point about 5.1 km from B.

(b) If W/L is large, the bird would fly to a point C that is closer to B than to D to minimize the energy used flying over water. If W/L is small, the bird would fly to a point C that is closer to D than to B to minimize the distance of the flight. $E = W\sqrt{25 + x^2} + L(13 - x) \Rightarrow \dfrac{dE}{dx} = \dfrac{Wx}{\sqrt{25 + x^2}} - L = 0$ when

$\dfrac{W}{L} = \dfrac{\sqrt{25 + x^2}}{x}$. By the same sort of argument as in part (a), this ratio will give the minimal expenditure of energy if the bird heads for the point x km from B.

(c) For flight direct to D, $x = 13$, so from part (b), $W/L = \frac{\sqrt{25 + 13^2}}{13} \approx 1.07$. There is no value of W/L for which the bird should fly directly to B. But note that $\lim\limits_{x \to 0^+} (W/L) = \infty$, so if the point at which E is a minimum is close to B, then W/L is large.

(d) Assuming that the birds instinctively choose the path that minimizes the energy expenditure, we can use the equation for $dE/dx = 0$ from part (a) with $1.4k = c$, $x = 4$, and $k = 1$: $c(4) = 1 \cdot (25 + 4^2)^{1/2} \Rightarrow$

$c = \sqrt{41}/4 \approx 1.6$.

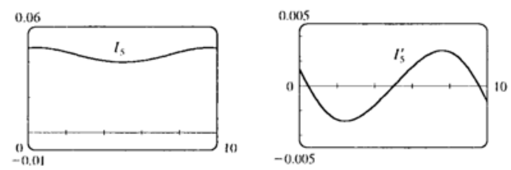
56. (a) $I(x) \propto \dfrac{\text{strength of source}}{(\text{distance from source})^2}$. Adding the intensities from the left and right lightbulbs,

$$I(x) = \frac{k}{x^2 + d^2} + \frac{k}{(10 - x)^2 + d^2} = \frac{k}{x^2 + d^2} + \frac{k}{x^2 - 20x + 100 + d^2}.$$

(b) The magnitude of the constant k won't affect the location of the point of maximum intensity, so for convenience we take $k = 1$. $I'(x) = -\dfrac{2x}{\left(x^2 + d^2\right)^2} - \dfrac{2(x - 10)}{\left(x^2 - 20x + 100 + d^2\right)^2}$.

Substituting $d = 5$ into the equations for $I(x)$ and $I'(x)$, we get

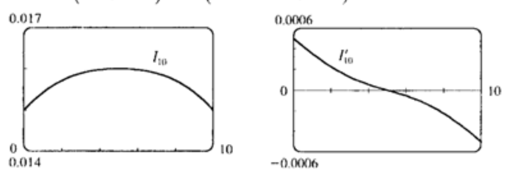
$$I_5(x) = \frac{1}{x^2 + 25} + \frac{1}{x^2 - 20x + 125} \quad \text{and} \quad I_5'(x) = -\frac{2x}{\left(x^2 + 25\right)^2} - \frac{2(x - 10)}{\left(x^2 - 20x + 125\right)^2}$$



From the graphs, it appears that $I_5(x)$ has a minimum at $x = 5$ m.

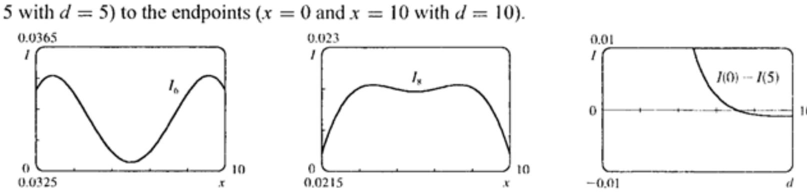
(c) Substituting $d = 10$ into the equations for $I(x)$ and $I'(x)$ gives $I_{10}(x) = \dfrac{1}{x^2 + 100} + \dfrac{1}{x^2 - 20x + 200}$ and

$$I_{10}'(x) = -\frac{2x}{\left(x^2 + 100\right)^2} - \frac{2(x - 10)}{\left(x^2 - 20x + 200\right)^2}.$$



From the graphs, it seems that for $d = 10$, the intensity is minimized at the endpoints, that is, $x = 0$ and $x = 10$. The midpoint is now the most brightly lit point!

(d) From the first figures in parts (b) and (c), we see that the minimal illumination changes from the midpoint ($x = 5$ with $d = 5$) to the endpoints ($x = 0$ and $x = 10$ with $d = 10$).



So we try $d = 6$ (see the first figure) and we see that the minimum value still occurs at $x = 5$. Next, we let $d = 8$ (see the second figure) and we see that the minimum value occurs at the endpoints. It appears that for some value of d between 6 and 8, we must have minima at both the midpoint and the endpoints, that is, $I(5)$ must equal $I(0)$. To find this value of d, we solve $I(0) = I(5)$ (with $k = 1$):

$$\frac{1}{d^2} + \frac{1}{100 + d^2} = \frac{1}{25 + d^2} + \frac{1}{25 + d^2} = \frac{2}{25 + d^2} \quad \Rightarrow$$

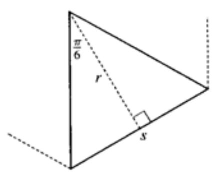
$$\left(25 + d^2\right)\left(100 + d^2\right) + d^2\left(25 + d^2\right) = 2d^2\left(100 + d^2\right) \quad \Rightarrow$$

$$2500 + 125d^2 + d^4 + 25d^2 + d^4 = 200d^2 + 2d^4 \quad \Rightarrow \quad 2500 = 50d^2 \quad \Rightarrow \quad d^2 = 50 \quad \Rightarrow$$

$d = 5\sqrt{2} \approx 7.071$ (for $0 \le d \le 10$). The third figure, a graph of $I(0) - I(5)$ with d independent, confirms that $I(0) - I(5) = 0$, that is, $I(0) = I(5)$, when $d = 5\sqrt{2}$. Thus the point of minimal illumination changes abruptly from the midpoint to the endpoints when $d = 5\sqrt{2}$.

Applied Project □ The Shape of a Can

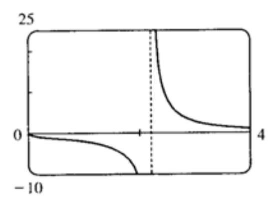
1. In this case, the amount of metal used in the making of each top or bottom is $(2r)^2 = 4r^2$. So the quantity we want to minimize is $A = 2\pi rh + 2\left(4r^2\right)$. But $V = \pi r^2 h \quad \Leftrightarrow \quad h = V/\pi r^2$. Substituting this expression for h and differentiating A with respect to r, we get $dA/dr = -2V/r^2 + 16r = 0 \quad \Rightarrow \quad 16r^3 = 2V = 2\pi r^2 h \quad \Leftrightarrow$ $\dfrac{h}{r} = \dfrac{8}{\pi}$. This gives a minimum because $\dfrac{d^2A}{dr^2} = 16 + \dfrac{4V}{r^3} > 0$.

2. 

We need to find the area of metal used up by each end, that is, the area of each hexagon. We subdivide the hexagon into six congruent triangles, each sharing one side (s in the diagram) with the hexagon. We calculate the length of $s = 2r\tan\frac{\pi}{6} = \frac{2}{\sqrt{3}}r$, so the area of each triangle is $\frac{1}{2}sr = \frac{1}{\sqrt{3}}r^2$, and the total area of the hexagon is $6 \cdot \frac{1}{\sqrt{3}}r^2 = 2\sqrt{3}r^2$. So the quantity we want to minimize is $A = 2\pi rh + 2 \cdot 2\sqrt{3}r^2$.

Substituting for h as in Problem 1 and differentiating, we get $\dfrac{dA}{dr} = -\dfrac{2V}{r^2} + 8\sqrt{3}r$. Setting this to 0, we get $8\sqrt{3}r^3 = 2V = 2\pi r^2 h \quad \Rightarrow \quad \dfrac{h}{r} = \dfrac{4\sqrt{3}}{\pi}$. Again this minimizes A because $\dfrac{d^2A}{dr^2} = 8\sqrt{3} + \dfrac{4V}{r^3} > 0$.

3. Let $C = 4\sqrt{3}r^2 + 2\pi rh + k(4\pi r + h) = 4\sqrt{3}r^2 + 2\pi r\left(\dfrac{V}{\pi r^2}\right) + k\left(4\pi r + \dfrac{V}{\pi r^2}\right)$. Then $\dfrac{dC}{dr} = 8\sqrt{3}r - \dfrac{2V}{r^2} + 4k\pi - \dfrac{2kV}{\pi r^3}$. Setting this equal to 0, dividing by 2 and substituting $\dfrac{V}{r^2} = \pi h$ and $\dfrac{V}{\pi r^3} = \dfrac{h}{r}$ in the second and fourth terms respectively, we get $0 = 4\sqrt{3}r - \pi h + 2k\pi - \dfrac{kh}{r} \quad \Leftrightarrow$ $k\left(2\pi - \dfrac{h}{r}\right) = \pi h - 4\sqrt{3}r \quad \Rightarrow \quad \dfrac{k}{r}\dfrac{2\pi - h/r}{\pi h/r - 4\sqrt{3}} = 1$. We now multiply by $\dfrac{\sqrt[3]{V}}{k}$, noting that $\dfrac{\sqrt[3]{V}}{k}\dfrac{k}{r} = \sqrt[3]{\dfrac{V}{r^3}} = \sqrt[3]{\dfrac{\pi h}{r}}$, and get $\dfrac{\sqrt[3]{V}}{k} = \sqrt[3]{\dfrac{\pi h}{r}} \cdot \dfrac{2\pi - h/r}{\pi h/r - 4\sqrt{3}}$.

4. 

We see from the graph that when the ratio $\sqrt[3]{V}/k$ is large, that is, either the volume of the can is large or the cost of joining (proportional to k) is small, the optimum value of h/r is about 2.21, but when $\sqrt[3]{V}/k$ is small, indicating small volume or expensive joining, the optimum value of h/r is larger. (The part of the graph for $\sqrt[3]{V}/k < 0$ has no physical meaning, but confirms the location of the asymptote.)

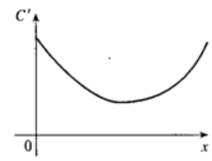
5. Our conclusion is usually true in practice. But there are exceptions, such as cans of tuna, which may have to do with the shape of a reasonable slice of tuna. And for a comfortable grip on a soda or beer can, the geometry of the human hand is a restriction on the radius. Other possible considerations are packaging, transportation and stocking constraints, aesthetic appeal and other marketing concerns. Also, there may be better models than ours which prescribe a differently-shaped can in special circumstances.

4.8 Applications to Economics

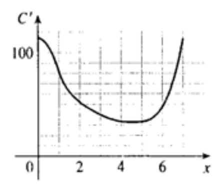
1. (a) $C(0)$ represents the fixed costs of production, such as rent, utilities, machinery etc., which are incurred even when nothing is produced.

(b) The inflection point is the point at which $C''(x)$ changes from negative to positive, that is, the marginal cost $C'(x)$ changes from decreasing to increasing. So the marginal cost is minimized.

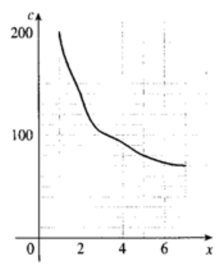
(c) The marginal cost function is $C'(x)$. We graph it as in Example 1 in Section 3.2.



2. (a) We graph C' as in Example 1 in Section 3.2.



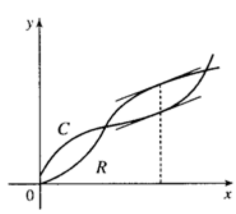
(b) By reading values of $C(x)$ from its graph, we can plot $c(x) = C(x)/x$.



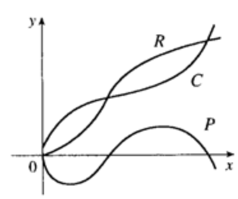
(c) Since the graph in part (b) is decreasing, we estimate that the minimum value of $c(x)$ occurs at $x = 7$. The average cost and the marginal cost are equal at that value.

3. $c(x) = 21.4 - 0.002x$ and $c(x) = C(x)/x \quad \Rightarrow \quad C(x) = 21.4x - 0.002x^2$. $C'(x) = 21.4 - 0.004x$ and $C'(1000) = 17.4$. This is an estimate of the cost of producing the 1001st unit.

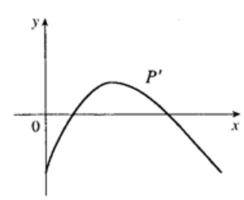
4. (a) Profit is maximized when the marginal revenue is equal to the marginal cost, that is, when R and C have equal slopes.



(b) $P(x) = R(x) - C(x)$ is sketched.



(c) The marginal profit function is defined as $P'(x)$.



5. (a) The cost function is $C(x) = 40{,}000 + 300x + x^2$, so the cost at a production level of 1000 is $C(1000) = \$1{,}340{,}000$. The average cost function is $c(x) = \dfrac{C(x)}{x} = \dfrac{40{,}000}{x} + 300 + x$ and $c(1000) = \$1340/\text{unit}$. The marginal cost function is $C'(x) = 300 + 2x$ and $C'(1000) = \$2300/\text{unit}$.

(b) We must have $C'(x) = c(x) \;\Leftrightarrow\; 300 + 2x = \dfrac{40{,}000}{x} + 300 + x \;\Leftrightarrow\; x = \dfrac{40{,}000}{x} \;\Rightarrow\; x^2 = 40{,}000$ $\Rightarrow\; x = \sqrt{40{,}000} = 200$. This gives a minimum value of the average cost function $c(x)$ since $c''(x) = \dfrac{80{,}000}{x^3} > 0$.

(c) The minimum average cost is $c(200) = \$700/\text{unit}$.

6. (a) $C(x) = 25{,}000 + 120x + 0.1x^2$, $C(1000) = \$245{,}000$. $c(x) = \dfrac{C(x)}{x} = \dfrac{25{,}000}{x} + 120 + 0.1x$, $c(1000) = \$245/\text{unit}$. $C'(x) = 120 + 0.2x$, $C'(1000) = \$320/\text{unit}$.

(b) We must have $C'(x) = c(x) \;\Leftrightarrow\; 120 + 0.2x = \dfrac{25{,}000}{x} + 120 + 0.1x \;\Leftrightarrow\; 0.1x = \dfrac{25{,}000}{x} \;\Rightarrow$ $0.1x^2 = 25{,}000 \;\Rightarrow\; x = \sqrt{250{,}000} = 500$. This gives a minimum since $c''(x) = \dfrac{50{,}000}{x^3} > 0$.

(c) The minimum average cost is $c(500) = \$220.00/\text{unit}$.

7. (a) $C(x) = 45 + \dfrac{x}{2} + \dfrac{x^2}{560}$, $C(1000) = \$2330.71$. $c(x) = \dfrac{45}{x} + \dfrac{1}{2} + \dfrac{x}{560}$, $c(1000) = \$2.33/\text{unit}$. $C'(x) = \dfrac{1}{2} + \dfrac{x}{280}$, $C'(1000) = \$4.07/\text{unit}$.

(b) We must have $C'(x) = c(x) \;\Rightarrow\; \dfrac{1}{2} + \dfrac{x}{280} = \dfrac{45}{x} + \dfrac{1}{2} + \dfrac{x}{560} \;\Rightarrow\; \dfrac{45}{x} = \dfrac{x}{560} \;\Rightarrow\; x^2 = (45)(560)$ $\Rightarrow\; x = \sqrt{25{,}200} \approx 159$. This is a minimum since $c''(x) = 90/x^3 > 0$.

(c) The minimum average cost is $c(159) = \$1.07/\text{unit}$.

8. (a) $C(x) = 2000 + 10x + 0.001x^3$, $C(1000) = \$1{,}012{,}000$. $c(x) = \dfrac{2000}{x} + 10 + 0.001x^2$, $c(1000) = \$1012/\text{unit}$. $C'(x) = 10 + 0.003x^2$, $C'(1000) = \$3010/\text{unit}$.

(b) We must have $C'(x) = c(x) \;\Leftrightarrow\; 10 + 0.003x^2 = \dfrac{2000}{x} + 10 + 0.001x^2 \;\Leftrightarrow\; \dfrac{2000}{x} = 0.002x^2 \;\Leftrightarrow$ $x^3 = 2000/0.002 = 1{,}000{,}000 \;\Leftrightarrow\; x = 100$. This is a minimum since $c''(x) = \dfrac{4000}{x^3} + 0.002 > 0$ for $x > 0$.

(c) The minimum average cost is $c(100) = \$40/\text{unit}$.

9. (a) $C(x) = 2\sqrt{x} + \dfrac{x^2}{8000}$, $C(1000) = \$188.25$. $c(x) = \dfrac{2}{\sqrt{x}} + \dfrac{x}{8000}$, $c(1000) = \$0.19/\text{unit}$. $C'(x) = \dfrac{1}{\sqrt{x}} + \dfrac{x}{4000}$, $C'(1000) = \$0.28/\text{unit}$.

(b) We must have $C'(x) = c(x) \;\Rightarrow\; \dfrac{1}{\sqrt{x}} + \dfrac{x}{4000} = \dfrac{2}{\sqrt{x}} + \dfrac{x}{8000} \;\Rightarrow\; \dfrac{x}{8000} = \dfrac{1}{\sqrt{x}} \;\Rightarrow\; x^{3/2} = 8000 \;\Rightarrow$ $x = (8000)^{2/3} = 400$. This is a minimum since $c''(x) = \frac{3}{2}x^{-5/2} > 0$.

(c) The minimum average cost is $c(400) = \$0.15/\text{unit}$.

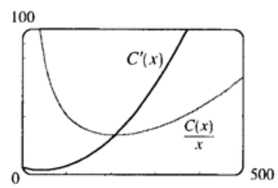
10. (a) $C(x) = 1000 + 96x + 2x^{3/2}$, $C(1000) = \$160{,}245.55$. $c(x) = \dfrac{1000}{x} + 96 + 2\sqrt{x}$, $c(1000) = \$160.25/\text{unit}$.
$C'(x) = 96 + 3\sqrt{x}$, $C'(1000) = \$190.87/\text{unit}$.

(b) We must have $C'(x) = c(x) \iff 96 + 3\sqrt{x} = 1000/x + 96 + 2\sqrt{x} \iff \sqrt{x} = 1000/x \iff$ $x^{3/2} = 1000 \iff x = (1000)^{2/3} = 100$. Since $c'(x) = \left(x^{3/2} - 1000\right)/x^2 < 0$ for $0 < x < 100$ and $c'(x) > 0$ for $x > 100$, there is an absolute minimum at $x = 100$.

(c) The minimum average cost is $c(100) = \$126/\text{unit}$.

11. (a) $C(x) = 3700 + 5x - 0.04x^2 + 0.0003x^3 \Rightarrow C'(x) = 5 - 0.08x + 0.0009x^2$ (marginal cost).
$c(x) = \dfrac{C(x)}{x} = \dfrac{3700}{x} + 5 - 0.04x + 0.0003x^2$ (average cost).

(b)



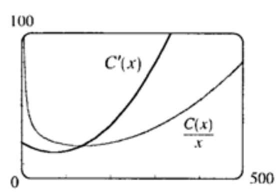
The graphs intersect at (208.51, 27.45), so the production level that minimizes average cost is about 209 units.

(c) $c'(x) = -\dfrac{3700}{x^2} - 0.04 + 0.0006x = 0 \Rightarrow$
$x_1 = 208.51$. $c(x_1) \approx \$27.45/\text{unit}$.

(d) $C''(x) = -0.08 + 0.0018x = 0 \Rightarrow$
$x_1 = \frac{800}{18} = 44.4\bar{4}$. $C'(x_1) = \$3.22/\text{unit}$.
$C'''(x) = 0.0018 > 0$ for all x, so this is the minimum marginal cost.

12. (a) $C(x) = 339 + 25x - 0.09x^2 + 0.0004x^3 \Rightarrow C'(x) = 25 - 0.18x + 0.0012x^2$ (marginal cost).
$c(x) = \dfrac{C(x)}{x} = \dfrac{339}{x} + 25 - 0.09x + 0.0004x^2$ (average cost).

(b)



The graphs intersect at (135.56, 22.65), so the production level that minimizes average cost is about 136 units.

(c) $c'(x) = -\dfrac{339}{x^2} - 0.09 + 0.0008x = 0 \Rightarrow$
$x_1 \approx 135.56$. $c(x_1) \approx \$22.65/\text{unit}$.

(d) $C''(x) = -0.18 + 0.0024x = 0 \Rightarrow$
$x = \frac{1800}{24} = 75$. $C'(75) = \$18.25/\text{unit}$.
$C'''(x) = 0.0024 > 0$ for all x, so this is the minimum marginal cost.

13. $C(x) = 680 + 4x + 0.01x^2$, $p(x) = 12 \Rightarrow R(x) = xp(x) = 12x$. If the profit is maximum, then $R'(x) = C'(x) \Rightarrow 12 = 4 + 0.02x \Rightarrow 0.02x = 8 \Rightarrow x = 400$. The profit is maximized if $P''(x) < 0$, but since $P''(x) = R''(x) - C''(x)$, we can just check the condition $R''(x) < C''(x)$. Now $R''(x) = 0 < 0.02 = C''(x)$, so $x = 400$ gives a maximum.

14. $C(x) = 680 + 4x + 0.01x^2$, $p(x) = 12 - x/500$. Then $R(x) = xp(x) = 12x - x^2/500$. If the profit is maximum, then $R'(x) = C'(x) \Leftrightarrow 12 - x/250 = 4 + 0.02x \Leftrightarrow 8 = 0.024x \Leftrightarrow x = 8/0.024 = \frac{1000}{3}$. The profit is maximized if $P''(x) < 0$, but since $P''(x) = R''(x) - C''(x)$, we can just check the condition $R''(x) < C''(x)$. Now $R''(x) = -\frac{1}{250} < 0.02 = C''(x)$, so $x = \frac{1000}{3}$ gives a maximum.

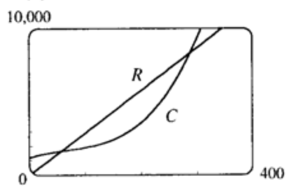
15. $C(x) = 1450 + 36x - x^2 + 0.001x^3$, $p(x) = 60 - 0.01x$. Then $R(x) = xp(x) = 60x - 0.01x^2$. If the profit is maximum, then $R'(x) = C'(x) \Leftrightarrow 60 - 0.02x = 36 - 2x + 0.003x^2 \Rightarrow 0.003x^2 - 1.98x - 24 = 0$. By the quadratic formula, $x = \dfrac{1.98 \pm \sqrt{(-1.98)^2 + 4(0.003)(24)}}{2(0.003)} = \dfrac{1.98 \pm \sqrt{4.2084}}{0.006}$. Since $x > 0$, $x \approx (1.98 + 2.05)/0.006 \approx 672$. Now $R''(x) = -0.02$ and $C''(x) = -2 + 0.006x \Rightarrow C''(672) = 2.032$ $\Rightarrow R''(672) < C''(672) \Rightarrow$ there is a maximum at $x = 672$.

16. $C(x) = 10{,}000 + 28x - 0.01x^2 + 0.002x^3$, $p(x) = 90 - 0.02x$. Then $R(x) = xp(x) = 90x - 0.02x^2$. If the profit is maximum, then $R'(x) = C'(x) \Leftrightarrow 90 - 0.04x = 28 - 0.02x + 0.006x^2 \Leftrightarrow$ $0.006x^2 + 0.02x - 62 = 0 \Leftrightarrow 3x^2 + 10x - 31{,}000 = 0 \Leftrightarrow (x - 100)(3x + 310) = 0 \Leftrightarrow x = 100$ (since $x > 0$). The profit is maximized if $P''(x) < 0$, but since $P''(x) = R''(x) - C''(x)$, we can just check the condition $R''(x) < C''(x)$. Now $R''(x) = -0.04 < -0.02 + 0.012x = C''(x)$ for $x > 0$, so there is a maximum at $x = 100$.

17. $C(x) = 0.001x^3 - 0.3x^2 + 6x + 900$. The marginal cost is $C'(x) = 0.003x^2 - 0.6x + 6$. $C'(x)$ is increasing when $C''(x) > 0 \Leftrightarrow 0.006x - 0.6 > 0 \Leftrightarrow x > 0.6/0.006 = 100$. So $C'(x)$ starts to increase when $x = 100$.

18. $C(x) = 0.0002x^3 - 0.25x^2 + 4x + 1500$. The marginal cost is $C'(x) = 0.0006x^2 - 0.50x + 4$. $C'(x)$ is increasing when $C''(x) > 0 \Leftrightarrow 0.0012x - 0.5 > 0 \Leftrightarrow x > 0.5/0.0012 \approx 417$. So $C'(x)$ starts to increase when $x = 417$.

19. (a) $C(x) = 1200 + 12x - 0.1x^2 + 0.0005x^3$. $R(x) = xp(x) = 29x - 0.00021x^2$.



Since the profit is maximized when $R'(x) = C'(x)$, we examine the curves R and C in the figure, looking for x-values at which the slopes of the tangent lines are equal. It appears that $x = 200$ is a good estimate.

(b) $R'(x) = C'(x) \Rightarrow 29 - 0.00042x = 12 - 0.2x + 0.0015x^2 \Rightarrow 0.0015x^2 - 0.19958x - 17 = 0 \Rightarrow$ $x \approx 192.06$ (for $x > 0$). As in Exercise 13, $R''(x) < C''(x) \Rightarrow -0.00042 < -0.2 + 0.003x \Leftrightarrow$ $0.003x > 0.19958 \Leftrightarrow x > 66.5$. Our value of 192 is in this range, so we have a maximum profit when we produce 192 yards of fabric.

20. (a) Cost = setup cost + manufacturing cost $\Rightarrow C(x) = 500 + m(x) = 500 + 20x - 5x^{3/4} + 0.01x^2$. We can solve $x(p) = 320 - 7.7p$ for p in terms of x to find the demand (or price) function. $x = 320 - 7.7p \Rightarrow$ $7.7p = 320 - x \Rightarrow p(x) = \dfrac{320 - x}{7.7}$. $R(x) = xp(x) = \dfrac{320x - x^2}{7.7}$.

(b) $C'(x) = R'(x) \Rightarrow 20 - \frac{15}{4}x^{-1/4} + 0.02x = \dfrac{320 - 2x}{7.7} \Rightarrow x \approx 81.53$ planes, and $p(x) = \$30.97$ million. The maximum profit associated with these values is about \$463.59 million.

21. (a) We are given that the demand function p is linear and $p(27{,}000) = 10$, $p(33{,}000) = 8$, so the slope is $\frac{10-8}{27{,}000-33{,}000} = -\frac{1}{3000}$ and an equation of the line is $y - 10 = \left(-\frac{1}{3000}\right)(x - 27{,}000) \Rightarrow$ $y = p(x) = -\frac{1}{3000}x + 19 = 19 - x/3000$.

(b) The revenue is $R(x) = xp(x) = 19x - x^2/3000 \Rightarrow R'(x) = 19 - x/1500 = 0$ when $x = 28{,}500$. Since $R''(x) = -1/1500 < 0$, the maximum revenue occurs when $x = 28{,}500 \Rightarrow$ the price is $p(28{,}500) = \$9.50$.

22. (a) Let $p(x)$ be the demand function. Then $p(x)$ is linear and $y = p(x)$ passes through $(20, 10)$ and $(18, 11)$, so the slope is $-\frac{1}{2}$ and an equation of the line is $y - 10 = -\frac{1}{2}(x - 20) \Leftrightarrow y = -\frac{1}{2}x + 20$. Thus, the demand is $p(x) = -\frac{1}{2}x + 20$ and the revenue is $R(x) = xp(x) = -\frac{1}{2}x^2 + 20x$.

(b) The cost is $C(x) = 6x$, so the profit is $P(x) = R(x) - C(x) = -\frac{1}{2}x^2 + 14x$. Then $0 = P'(x) = -x + 14$ $\Rightarrow x = 14$. Since $P''(x) = -1 < 0$, the selling price for maximum profit is $p(14) = 20 - \left(\frac{1}{2}\right)14 = \$13$.

23. (a) As in Example 3, we see that the demand function p is linear. We are given that $p(1000) = 450$ and deduce that $p(1100) = 440$, since a \$10 reduction in price increases sales by 100 per week. The slope for p is $\frac{440-450}{1100-1000} = -\frac{1}{10}$, so an equation is $p - 450 = -\frac{1}{10}(x - 1000)$ or $p(x) = -\frac{1}{10}x + 550$.

(b) $R(x) = xp(x) = 500x - x^2/10$. $R'(x) = 550 - x/5 = 0$ when $x = 5(550) = 2750$. $p(2750) = 275$, so the rebate should be $450 - 275 = \$175$.

(c) $C(x) = 68{,}000 + 150x \Rightarrow$
$P(x) = R(x) - C(x) = 550x - x^2/10 - 68{,}000 - 150x = 400x - x^2/10 - 68{,}000$, $P'(x) = 400 - x/5 = 0$ when $x = 2000$. $p(2000) = 350$. Therefore, the rebate to maximize profits should be $450 - 350 = \$100$.

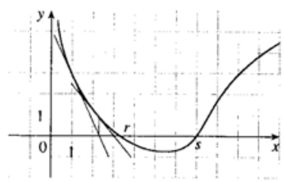
24. Let x denote the number of \$5 increases in rent. Then the price is $p(x) = 400 + 5x$, and the number of units occupied is $100 - x$. Now the revenue is

$$\begin{aligned} R(x) &= (\text{rental price per unit}) \times (\text{number of units rented}) \\ &= (400 + 5x)(100 - x) = -5x^2 + 100x + 40{,}000 \text{ for } 0 \le x \le 100 \quad \Rightarrow \end{aligned}$$

$R'(x) = -10x + 100 = 0 \Leftrightarrow x = 10$. This is a maximum since $R''(x) = -10 < 0$ for all x. Now we must check the value of $R(x) = (400 + 5x)(100 - x)$ at $x = 10$ and at the endpoints of the domain to see which gives the maximum value. $R(0) = 40{,}000$, $R(10) = (450)(90) = 40{,}500$, and $R(100) = (900)(0) = 0$. Thus, the maximum revenue of \$40,500/week occurs when 90 units are occupied at a rent of \$450/week.

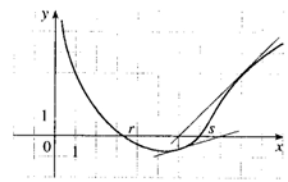
4.9 Newton's Method

1.



The tangent line at $x = 1$ intersects the x-axis at $x \approx 2.3$, so $x_2 \approx 2.3$. The tangent line at $x = 2.3$ intersects the x-axis at $x \approx 3$, so $x_3 \approx 3.0$.

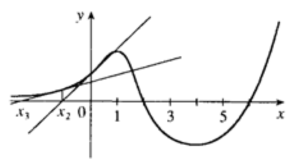
2.



The tangent line at $x = 9$ intersects the x-axis at $x \approx 6.0$, so $x_2 \approx 6.0$. The tangent line at $x = 6.0$ intersects the x-axis at $x \approx 8.0$, so $x_3 \approx 8.0$.

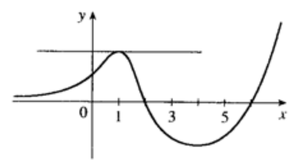
3. Since $x_1 = 3$ and $y = 5x - 4$ is tangent to $y = f(x)$ at $x = 3$, we simply need to find where the tangent line intersects the x-axis. $y = 0 \Rightarrow 5x_2 - 4 = 0 \Rightarrow x_2 = \frac{4}{5}$.

4. (a)



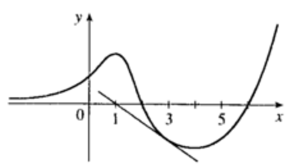
If $x_1 = 0$, then x_2 is negative, and x_3 is even more negative. The sequence of approximations does not converge, that is, Newton's method fails.

(b)



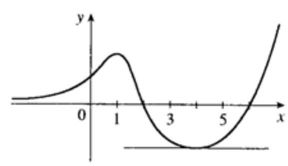
If $x_1 = 1$, the tangent line is horizontal and Newton's method fails.

(c)



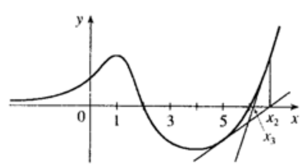
If $x_1 = 3$, then $x_2 = 1$ and we have the same situation as in part (b). Newton's method fails again.

(d)



If $x_1 = 4$, the tangent line is horizontal and Newton's method fails.

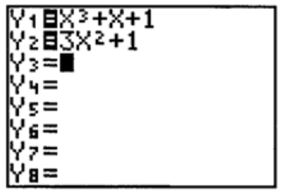
(e)

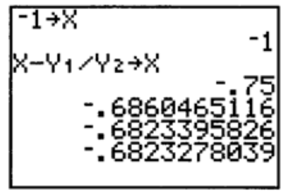


If $x_1 = 5$, then x_2 is greater than 6, x_3 gets closer to 6, and the sequence of approximations converges to 6. Newton's method succeeds!

5. $f(x) = x^3 + x + 1 \quad \Rightarrow \quad f'(x) = 3x^2 + 1$, so $x_{n+1} = x_n - \dfrac{x_n^3 + x_n + 1}{3x_n^2 + 1}$. $x_1 = -1 \quad \Rightarrow$

$x_2 = -1 - \dfrac{-1 - 1 + 1}{3 \cdot 1 + 1} = -0.75 \quad \Rightarrow \quad x_3 = -0.75 - \dfrac{(-0.75)^3 - 0.75 + 1}{3(-0.75)^2 + 1} \approx -0.6860$. Here is a quick and easy method for finding the iterations on a programmable calculator. (The screens shown are from the TI-82, but the method is similar on other calculators.) Assign $x^3 + x + 1$ to Y_1 and $3x^2 + 1$ to Y_2. Now store -1 in X and then enter $X - Y_1/Y_2 \rightarrow X$ to get -0.75. By successively pressing the ENTER key, you get the approximations $x_1, x_2, x_3, \ldots$.





6. $f(x) = x^3 - x^2 - 1 \quad \Rightarrow \quad f'(x) = 3x^2 - 2x$, so $x_{n+1} = x_n - \dfrac{x_n^3 - x_n^2 - 1}{3x_n^2 - 2x_n}$. $x_1 = 1 \quad \Rightarrow$

$x_2 = 1 - \dfrac{1 - 1 - 1}{3 - 2} = 2 \quad \Rightarrow \quad x_3 = 2 - \dfrac{2^3 - 2^2 - 1}{3 \cdot 2^2 - 2 \cdot 2} = 1.6250.$

7. $f(x) = x^4 - 20 \quad \Rightarrow \quad f'(x) = 4x^3$, so $x_{n+1} = x_n - \dfrac{x_n^4 - 20}{4x_n^3}$. $x_1 = 2 \quad \Rightarrow \quad x_2 = 2 - \dfrac{2^4 - 20}{4(2)^3} = 2.1250 \quad \Rightarrow$

$x_3 = 2.125 - \dfrac{(2.125)^4 - 20}{4(2.125)^3} = 2.1148.$

8. $f(x) = x^7 - 100 \quad \Rightarrow \quad f'(x) = 7x^6$, so $x_{n+1} = x_n - \dfrac{x_n^7 - 100}{7x_n^6}$. $x_1 = 2 \quad \Rightarrow \quad x_2 = 2 - \dfrac{128 - 100}{7 \cdot 64} = 1.9375$

$\Rightarrow \quad x_3 = 1.9375 - \dfrac{(1.9375)^7 - 100}{7(1.9375)^6} \approx 1.9308.$

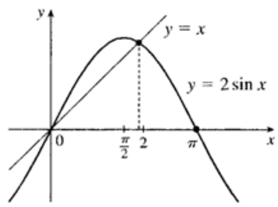
9. To approximate $x = \sqrt[3]{30}$ (so that $x^3 = 30$), we can take $f(x) = x^3 - 30$. So $f'(x) = 3x^2$, and thus, $x_{n+1} = x_n - \dfrac{x_n^3 - 30}{3x_n^2}$. Since $\sqrt[3]{27} = 3$ and 27 is close to 30, we'll use $x_1 = 3$. We need to find approximations until they agree to eight decimal places. $x_1 = 3 \quad \Rightarrow \quad x_2 \approx 3.11111111$, $x_3 \approx 3.10723734$, $x_4 \approx 3.10723251 \approx x_5$. So $\sqrt[3]{30} \approx 3.10723251$, to eight decimal places.

10. $f(x) = x^7 - 1000 \quad \Rightarrow \quad f'(x) = 7x^6$, so $x_{n+1} = x_n - \dfrac{x_n^7 - 1000}{7x_n^6}$. We need to find approximations until they agree to eight decimal places. $x_1 = 3 \quad \Rightarrow \quad x_2 \approx 2.76739173$, $x_3 \approx 2.69008741$, $x_4 \approx 2.68275645$, $x_5 \approx 2.68269580 \approx x_6$. Thus, $\sqrt[7]{1000} \approx 2.68269580$, to eight decimal places.

11. $f(x) = 2x^3 - 6x^2 + 3x + 1 \quad \Rightarrow \quad f'(x) = 6x^2 - 12x + 3 \quad \Rightarrow \quad x_{n+1} = x_n - \dfrac{2x_n^3 - 6x_n^2 + 3x_n + 1}{6x_n^2 - 12x_n + 3}$. We need to find approximations until they agree to six decimal places. $x_1 = 2.5 \quad \Rightarrow \quad x_2 \approx 2.285714$, $x_3 \approx 2.228824$, $x_4 \approx 2.224765$, $x_5 \approx 2.224745 \approx x_6$. So the root is 2.224745, to six decimal places.

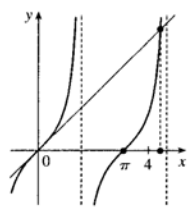
12. $f(x) = x^4 + x - 4 \;\Rightarrow\; f'(x) = 4x^3 + 1 \;\Rightarrow\; x_{n+1} = x_n - \dfrac{x_n^4 + x_n - 4}{4x_n^3 + 1}$. $x_1 = 1.5 \;\Rightarrow\; x_2 \approx 1.323276$, $x_3 \approx 1.285346$, $x_4 \approx 1.283784$, $x_5 \approx 1.283782 \approx x_6$. So the root is 1.283782, to six decimal places.

13.



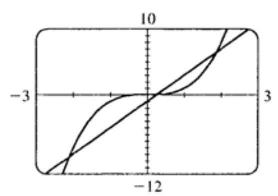
From the graph it appears that there is a root near 2, so we take $x_1 = 2$. Write the equation as $f(x) = 2\sin x - x = 0$. Then $f'(x) = 2\cos x - 1$, so $x_{n+1} = x_n - \dfrac{2\sin x_n - x_n}{2\cos x_n - 1} \;\Rightarrow\; x_1 = 2$, $x_2 \approx 1.900996$, $x_3 \approx 1.895512$, $x_4 \approx 1.895494 \approx x_5$. So the root is 1.895494, to six decimal places.

14.



From the graph, it appears there is a root near 4.5. So we take $x_1 = 4.5$. Write the equation as $f(x) = \tan x - x = 0$. Then $f'(x) = \sec^2 x - 1$, so $x_{n+1} = x_n - \dfrac{\tan x_n - x_n}{\sec^2 x_n - 1}$. $x_1 = 4.5$, $x_2 \approx 4.493614$, $x_3 \approx 4.493410$, $x_4 \approx 4.493409 \approx x_5$. To six decimal places, the root is 4.493409.

15.

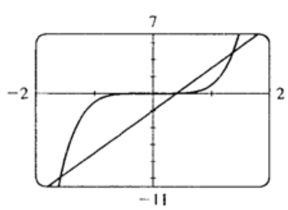


From the graph, we see that $y = x^3$ and $y = 4x - 1$ intersect three times. Good first approximations are $x = -2$, $x = 0$, and $x = 2$.
$f(x) = x^3 - 4x + 1 \;\Rightarrow\; f'(x) = 3x^2 - 4$, so
$x_{n+1} = x_n - \dfrac{x_n^3 - 4x_n + 1}{3x_n^2 - 4}$.

$x_1 = -2$	$x_1 = 0$	$x_1 = 2$
$x_2 = -2.125$	$x_2 = 0.25$	$x_2 = 1.875$
$x_3 \approx -2.114975$	$x_3 \approx 0.254098$	$x_3 \approx 1.860979$
$x_4 \approx -2.114908 \approx x_5$	$x_4 \approx 0.254102 \approx x_5$	$x_4 \approx 1.860806 \approx x_5$

To six decimal places, the roots are -2.114908, 0.254102, and 1.860806.

16.

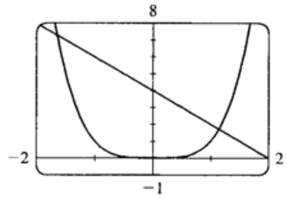


From the graph, we see that reasonable first approximations are $x = 0$ and $x = \pm 1.5$. $f(x) = x^5 - 5x + 2 \;\Rightarrow\; f'(x) = 5x^4 - 5$, so
$x_{n+1} = x_n - \dfrac{x_n^5 - 5x_n + 2}{5x_n^4 - 5}$.

$x_1 = -1.5$	$x_1 = 0.5$	$x_1 = 1.5$
$x_2 \approx -1.593846$	$x_2 = 0.4$	$x_2 \approx 1.396923$
$x_3 \approx -1.582241$	$x_3 \approx 0.402102 \approx x_4$	$x_3 \approx 1.373078$
$x_4 \approx -1.582036 \approx x_5$		$x_4 \approx 1.371885$
		$x_5 \approx 1.371882 \approx x_6$

To six decimal places, the roots are -1.582036, 0.402102, and 1.371882.

17.

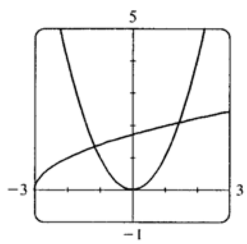


From the graph, we see that there appear to be points of intersection near $x = -1.7$ and $x = 1.2$. Now solving $x^4 = 4 - 2x$ is the same as solving $f(x) = x^4 + 2x - 4 = 0$. $f(x) = x^4 + 2x - 4 \Rightarrow f'(x) = 4x^3 + 2$, so $x_{n+1} = x_n - \dfrac{x_n^4 + 2x_n - 4}{4x_n^3 + 2}$.

$x_1 = -1.7$	$x_1 = 1.2$
$x_2 \approx -1.646063$	$x_2 \approx 1.146858$
$x_3 \approx -1.642945$	$x_3 \approx 1.143910$
$x_4 \approx -1.642935 \approx x_5$	$x_4 \approx 1.143901 \approx x_5$

To six decimal places, the roots are -1.642935 and 1.143901.

18.

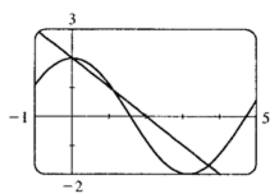


From the graph, we see that there appear to be points of intersection near $x = -1.2$ and $x = 1.5$. Solving $\sqrt{x+3} = x^2$ is the same as solving $f(x) = x^2 - \sqrt{x+3} = 0$. $f(x) = x^2 - \sqrt{x+3} \Rightarrow$ $f'(x) = 2x - \dfrac{1}{2\sqrt{x+3}}$, so $x_{n+1} = x_n - \dfrac{x_n^2 - \sqrt{x_n+3}}{2x_n - 1/(2\sqrt{x_n+3})}$.

$x_1 = -1.2$	$x_1 = 1.5$
$x_2 \approx -1.164526$	$x_2 \approx 1.453449$
$x_3 \approx -1.164035 \approx x_4$	$x_3 \approx 1.452627 \approx x_4$

To six decimal places, the roots are -1.164035 and 1.452627.

19.

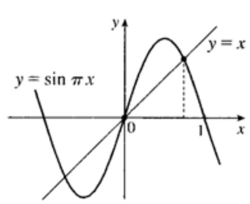


From the graph and by inspection, $x = 0$ is a root. Also, $y = 2\cos x$ and $y = 2 - x$ intersect at $x \approx 1$ and at $x \approx 3.5$. $f(x) = 2\cos x + x - 2 \Rightarrow$ $f'(x) = -2\sin x + 1$, so $x_{n+1} = x_n - \dfrac{2\cos x_n + x_n - 2}{-2\sin x_n + 1}$.

$x_1 = 1$	$x_1 = 3.5$
$x_2 \approx 1.118026$	$x_2 \approx 3.719159$
$x_3 \approx 1.109188$	$x_3 \approx 3.698331$
$x_4 \approx 1.109144 \approx x_5$	$x_4 \approx 3.698154 \approx x_5$

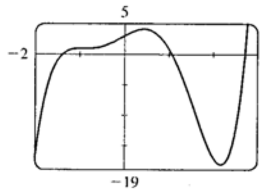
To six decimal places, the roots are 0, 1.109144, and 3.698154.

20.



Clearly $x = 0$ is a root. From the sketch, there appear to be roots near -0.75 and 0.75. Write the equation as $f(x) = \sin \pi x - x = 0$. Then $f'(x) = \pi \cos \pi x - 1$, so $x_{n+1} = x_n - \dfrac{\sin \pi x_n - x_n}{\pi \cos \pi x_n - 1}$. Taking $x_1 = 0.75$ we get $x_2 \approx 0.736685$, $x_3 \approx 0.736484 \approx x_4$. To six decimal places, the roots are 0, 0.736484 and -0.736484.

21.



$f(x) = x^5 - x^4 - 5x^3 - x^2 + 4x + 3 \quad \Rightarrow$

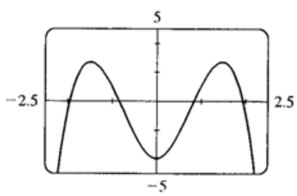
$f'(x) = 5x^4 - 4x^3 - 15x^2 - 2x + 4 \quad \Rightarrow$

$x_{n+1} = x_n - \dfrac{x_n^5 - x_n^4 - 5x_n^3 - x_n^2 + 4x_n + 3}{5x_n^4 - 4x_n^3 - 15x_n^2 - 2x_n + 4}$. From the graph of f, there appear to be roots near -1.4, 1.1, and 2.7.

$x_1 = -1.4$	$x_1 = 1.1$	$x_1 = 2.7$
$x_2 \approx -1.39210970$	$x_2 = 1.07780402$	$x_2 \approx 2.72046250$
$x_3 \approx -1.39194698$	$x_3 \approx 1.07739442$	$x_3 \approx 2.71987870$
$x_4 \approx -1.39194691 \approx x_5$	$x_4 \approx 1.07739428 \approx x_5$	$x_4 \approx 2.71987822 \approx x_5$

To eight decimal places, the roots are -1.39194691, 1.07739428, and 2.71987822.

22.



Solving $x^2(4 - x^2) = \dfrac{4}{x^2 + 1}$ is the same as solving

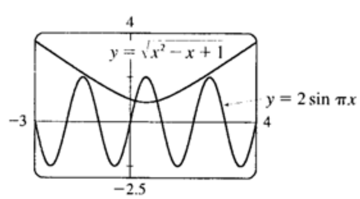
$f(x) = 4x^2 - x^4 - \dfrac{4}{x^2 + 1} = 0$. $f'(x) = 8x - 4x^3 + \dfrac{8x}{(x^2 + 1)^2} \quad \Rightarrow$

$x_{n+1} = x_n - \dfrac{4x_n^2 - x_n^4 - 4\big/(x_n^2 + 1)}{8x_n - 4x_n^3 + 8x_n\big/(x_n^2 + 1)^2}$. From the graph of $f(x)$, there appear to be roots near $x = \pm 1.9$ and $x = \pm 0.8$. Since f is even, we only need to find the positive roots.

$x_1 = 0.8$	$x_1 = 1.9$
$x_2 = 0.84287645$	$x_2 \approx 1.94689103$
$x_3 \approx 0.84310820$	$x_3 \approx 1.94383891$
$x_4 \approx 0.84310821 \approx x_5$	$x_4 \approx 1.94382538 \approx x_5$

To eight decimal places, the roots are ± 0.84310821 and ± 1.94382538.

23.



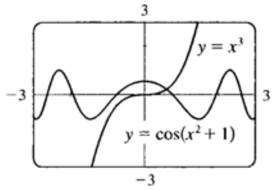
From the graph, we see that there are roots of this equation near 0.2 and 0.8. $f(x) = \sqrt{x^2 - x + 1} - 2\sin \pi x \quad \Rightarrow$

$f'(x) = \dfrac{2x - 1}{2\sqrt{x^2 - x + 1}} - 2\pi \cos \pi x$, so

$x_{n+1} = x_n - \dfrac{\sqrt{x_n^2 - x_n + 1} - 2\sin \pi x_n}{\dfrac{2x_n - 1}{2\sqrt{x_n^2 - x_n + 1}} - 2\pi \cos \pi x_n}$.

Taking $x_1 = 0.2$, we get $x_2 \approx 0.15212015$, $x_3 \approx 0.15438067$, $x_4 \approx 0.15438500 \approx x_5$. Taking $x_1 = 0.8$, we get $x_2 \approx 0.84787985$, $x_3 \approx 0.84561933$, $x_4 \approx 0.84561500 \approx x_5$. So, to eight decimal places, the roots of the equation are 0.15438500 and 0.84561500.

24.



From the graph, we see that the only root of this equation is near 0.6. $f(x) = \cos(x^2+1) - x^3 \Rightarrow f'(x) = -2x\sin(x^2+1) - 3x^2$, so $x_{n+1} = x_n + \dfrac{\cos(x_n^2+1) - x_n^3}{2x_n \sin(x_n^2+1) + 3x_n^2}$. Taking $x_1 = 0.6$, we get $x_2 \approx 0.59699955$, $x_3 \approx 0.59698777 \approx x_3$. To eight decimal places, the root of the equation is 0.59698777.

25. (a) $f(x) = x^2 - a \Rightarrow f'(x) = 2x$, so Newton's method gives

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = x_n - \tfrac{1}{2}x_n + \frac{a}{2x_n} = \tfrac{1}{2}x_n + \frac{a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

(b) Using (a) with $a = 1000$ and $x_1 = \sqrt{900} = 30$, we get $x_2 \approx 31.666667$, $x_3 \approx 31.622807$, and $x_4 \approx 31.622777 \approx x_5$. So $\sqrt{1000} \approx 31.622777$.

26. (a) $f(x) = \dfrac{1}{x} - a \Rightarrow f'(x) = -\dfrac{1}{x^2}$, so $x_{n+1} = x_n - \dfrac{1/x_n - a}{-1/x_n^2} = x_n + x_n - ax_n^2 = 2x_n - ax_n^2$.

(b) Using (a) with $a = 1.6894$ and $x_1 = \frac{1}{2} = 0.5$, we get $x_2 = 0.5754$, $x_3 \approx 0.588485$, and $x_4 \approx 0.588789 \approx x_5$. So $1/1.6984 \approx 0.588789$.

27. $f(x) = x^3 - 3x + 6 \Rightarrow f'(x) = 3x^2 - 3$. If $x_1 = 1$, then $f'(x_1) = 0$ and the tangent line used for approximating x_2 is horizontal. Attempting to find x_2 results in trying to divide by zero.

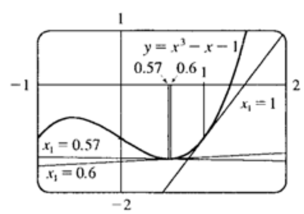
28. $x^3 - x = 1 \Leftrightarrow x^3 - x - 1 = 0$. $f(x) = x^3 - x - 1 \Rightarrow f'(x) = 3x^2 - 1$, so $x_{n+1} = x_n - \dfrac{x_n^3 - x_n - 1}{3x_n^2 - 1}$.

(a) $x_1 = 1$, $x_2 = 1.5$, $x_3 \approx 1.347826$, $x_4 \approx 1.325200$, $x_5 \approx 1.324718 \approx x_6$

(b) $x_1 = 0.6$, $x_2 = 17.9$, $x_3 \approx 11.946802$, $x_4 \approx 7.985520$, $x_5 \approx 5.356909$, $x_6 \approx 3.624996$, $x_7 \approx 2.505589$, $x_8 \approx 1.820129$, $x_9 \approx 1.461044$, $x_{10} \approx 1.339323$, $x_{11} \approx 1.324913$, $x_{12} \approx 1.324718 \approx x_{13}$

(c) $x_1 = 0.57$, $x_2 \approx -54.165455$, $x_3 \approx -36.114293$, $x_4 \approx -24.082094$, $x_5 \approx -16.063387$, $x_6 \approx -10.721483$, $x_7 \approx -7.165534$, $x_8 \approx -4.801704$, $x_9 \approx -3.233425$, $x_{10} \approx -2.193674$, $x_{11} \approx -1.496867$, $x_{12} \approx -0.997546$, $x_{13} \approx -0.496305$, $x_{14} \approx -2.894162$, $x_{15} \approx -1.967962$, $x_{16} \approx -1.341355$, $x_{17} \approx -0.870187$, $x_{18} \approx -0.249949$, $x_{19} \approx -1.192219$, $x_{20} \approx -0.731952$, $x_{21} \approx 0.355213$, $x_{22} \approx -1.753322$, $x_{23} \approx -1.189420$, $x_{24} \approx -0.729123$, $x_{25} \approx 0.377844$, $x_{26} \approx -1.937872$, $x_{27} \approx -1.320350$, $x_{28} \approx -0.851919$, $x_{29} \approx -0.200959$, $x_{30} \approx -1.119386$, $x_{31} \approx -0.654291$, $x_{32} \approx 1.547009$, $x_{33} \approx 1.360050$, $x_{34} \approx 1.325828$, $x_{35} \approx 1.324719$, $x_{36} \approx 1.324718 \approx x_{37}$.

(d)

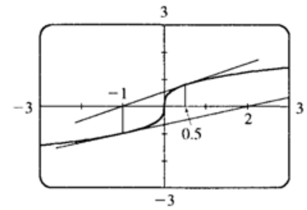


From the figure, we see that the tangent line corresponding to $x_1 = 1$ results in a sequence of approximations that converges quite quickly ($x_5 \approx x_6$). The tangent line corresponding to $x_1 = 0.6$ is close to being horizontal, so x_2 is quite far from the root. But the sequence still converges — just a little more slowly ($x_{12} \approx x_{13}$). Lastly, the tangent line corresponding to $x_1 = 0.57$ is very nearly horizontal, x_2 is farther away from the root, and the sequence takes more iterations to converge ($x_{36} \approx x_{37}$).

29. For $f(x) = x^{1/3}$, $f'(x) = \frac{1}{3}x^{-2/3}$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} = x_n - 3x_n = -2x_n.$$ Therefore,

each successive approximation becomes twice as large as the previous one in absolute value, so the sequence of approximations fails to converge to the root, which is 0. In the figure, we have $x_1 = 0.5$, $x_2 = -2(0.5) = -1$, and $x_3 = -2(-1) = 2$.

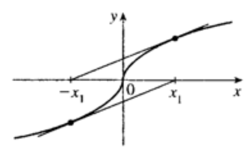


30. According to Newton's Method, for $x_n > 0$,

$$x_{n+1} = x_n - \frac{\sqrt{x_n}}{1/(2\sqrt{x_n})} = x_n - 2x_n = -x_n$$ and for $x_n < 0$,

$$x_{n+1} = x_n - \frac{-\sqrt{-x_n}}{1/\left(2\sqrt{-x_n}\right)} = x_n - [-2(-x_n)] = -x_n.$$ So we can see that

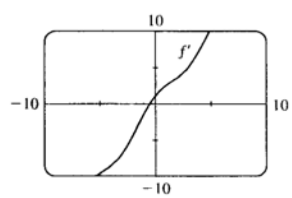
after choosing any value x_1 the subsequent values will alternate between $-x_1$ and x_1 and never approach the root.



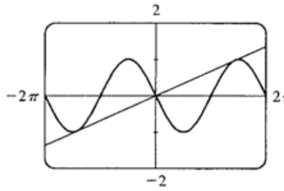
31. (a) $f(x) = 3x^4 - 28x^3 + 6x^2 + 24x \Rightarrow f'(x) = 12x^3 - 84x^2 + 12x + 24 \Rightarrow$ $f''(x) = 36x^2 - 168x + 12$. Now to solve $f'(x) = 0$, try $x_1 = \frac{1}{2} \Rightarrow x_2 = x_1 - \dfrac{f'(x_1)}{f''(x_1)} = \dfrac{2}{3} \Rightarrow$ $x_3 \approx 0.6455 \Rightarrow x_4 \approx 0.6452 \Rightarrow x_5 \approx 0.6452$. Now try $x_1 = 6 \Rightarrow x_2 = 7.12 \Rightarrow x_3 \approx 6.8353$ $\Rightarrow x_4 \approx 6.8102 \Rightarrow x_5 \approx 6.8100$. Finally try $x_1 = -0.5 \Rightarrow x_2 \approx -0.4571 \Rightarrow x_3 \approx -0.4552 \Rightarrow$ $x_4 \approx -0.4552$. Therefore, $x = -0.455$, 6.810 and 0.645 are all critical numbers correct to three decimal places.

(b) $f(-1) = 13$, $f(7) = -1939$, $f(6.810) \approx -1949.07$, $f(-0.455) \approx -6.912$, $f(0.645) \approx 10.982$. Therefore, $f(6.810) \approx -1949.07$ is the absolute minimum correct to two decimal places.

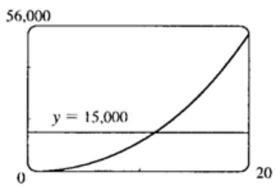
32. $f(x) = x^2 + \sin x \Rightarrow f'(x) = 2x + \cos x$. $f'(x)$ exists for all x, so to find the minimum of f, we can examine the zeros of f'. From the graph of f', we see that a good choice for x_1 is $x_1 = -0.5$. Use $g(x) = 2x + \cos x$ and $g'(x) = 2 - \sin x$ to obtain $x_2 \approx -0.4506$, $x_3 \approx -0.4502 \approx x_4$. Since $f''(x) = 2 - \sin x > 0$ for all x, $f(-0.4502) \approx -0.2325$ is the absolute minimum.



33. $y = x^3 + \cos x \Rightarrow y' = 3x^2 - \sin x \Rightarrow y'' = 6x - \cos x \Rightarrow y''' = 6 + \sin x$. Now to solve $y'' = 0$, try $x_1 = 0$, and then $x_2 = x_1 - \dfrac{y''(x_1)}{y'''(x_1)} \approx 0.1677 \Rightarrow x_3 \approx 0.1643 \Rightarrow x_4 \approx 0.1644 \approx x_5$. For $x < 0.1644$, $y'' < 0$, and for $x > 0.1644$, $y'' > 0$. Therefore, the point of inflection, correct to three decimal places, is $(0.164, f(0.164)) = (0.164, 0.991)$.

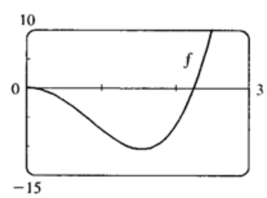
34. 

$f(x) = -\sin x \Rightarrow f'(x) = -\cos x$. At $x = a$, the slope of the tangent line is $f'(a) = -\cos a$. The line through the origin and $(a, f(a))$ is $y = \dfrac{-\sin a - 0}{a - 0}x$. If this line is to be tangent to f at $x = a$, then its slope must equal $f'(a)$. Thus, $\dfrac{-\sin a}{a} = -\cos a \Rightarrow \tan a = a$. To solve this equation using Newton's method, let $g(x) = \tan x - x$, $g'(x) = \sec^2 x - 1$, and $x_{n+1} = x_n - \dfrac{\tan x_n - x_n}{\sec^2 x_n - 1}$ with $x_1 = 4.5$ (estimated from the figure). $x_2 \approx 4.493614$, $x_3 \approx 4.493410$, $x_4 \approx 4.493409 \approx x_5$. Thus, the slope of the line that has the largest slope is $f'(x_5) \approx 0.217234$.

35. 

The volume of the silo, in terms of its radius, is $V(r) = \pi r^2(30) + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = 30\pi r^2 + \frac{2}{3}\pi r^3$. From a graph of V, we see that $V(r) = 15{,}000$ at $r \approx 11$ ft. Now we use Newton's method to solve the equation $V(r) - 15{,}000 = 0$. $\dfrac{dV}{dr} = 60\pi r + 2\pi r^2$, so $r_{n+1} = r_n - \dfrac{30\pi r_n^2 + \frac{2}{3}\pi r_n^3 - 15{,}000}{60\pi r_n + 2\pi r_n^2}$. Taking $r_1 = 11$, we get $r_2 = 11.2853$, $r_3 = 11.2807 \approx r_4$. So in order for the silo to hold 15,000 ft^3 of grain, its radius must be about 11.2807 ft.

36. Let the radius of the circle be r. Using $s = r\theta$, we have $5 = r\theta$ and so $r = 5/\theta$. From the Law of Cosines we get $4^2 = r^2 + r^2 - 2 \cdot r \cdot r \cdot \cos\theta \Leftrightarrow 16 = 2r^2(1 - \cos\theta) = 2(5/\theta)^2(1 - \cos\theta)$.
Multiplying by θ^2 gives $16\theta^2 = 50(1 - \cos\theta)$, so we take $f(\theta) = 16\theta^2 + 50\cos\theta - 50$ and $f'(\theta) = 32\theta - 50\sin\theta$. The formula for Newton's method is $\theta_{n+1} = \theta_n - \dfrac{16\theta_n^2 + 50\cos\theta_n - 50}{32\theta_n - 50\sin\theta_n}$. From the graph of f, we can use $\theta_1 = 2.2$, giving us $\theta_2 \approx 2.2662$, $\theta_3 \approx 2.2622 \approx \theta_4$. So correct to four decimal places, the angle is 2.2622 radians $\approx 130°$.



37. In this case, $A = 18{,}000$, $R = 375$, and $n = 5(12) = 60$. So the formula becomes $18{,}000 = \dfrac{375}{x}\left[1 - (1 + x)^{-60}\right]$
$\Leftrightarrow 48x = 1 - (1 + x)^{-60} \Leftrightarrow 48x(1 + x)^{60} - (1 + x)^{60} + 1 = 0$.
Let the LHS be called $f(x)$, so that

$$\begin{aligned} f'(x) &= 48x(60)(1 + x)^{59} + 48(1 + x)^{60} - 60(1 + x)^{59} \\ &= 12(1 + x)^{59}[4x(60) + 4(1 + x) - 5] = 12(1 + x)^{59}(244x - 1) \end{aligned}$$

$x_{n+1} = x_n - \dfrac{48x_n(1 + x_n)^{60} - (1 + x_n)^{60} + 1}{12(1 + x_n)^{59}(244x_n - 1)}$. An interest rate of 1%/month seems like a reasonable estimate for $x = i$. So let $x_1 = 1\% = 0.01$, and we get $x_2 = 0.0082202$, $x_3 \approx 0.0076802$, $x_4 \approx 0.0076291$, $x_5 \approx 0.0076286 \approx x_6$. Thus, the dealer is charging a monthly interest rate of 0.76286% (or 9.55%/year, compounded monthly).

38. (a) $p(x) = x^5 - (2+r)x^4 + (1+2r)x^3 - (1-r)x^2 + 2(1-r)x + r - 1 \Rightarrow$
$p'(x) = 5x^4 - 4(2+r)x^3 + 3(1+2r)x^2 - 2(1-r)x + 2(1-r)$. So we use
$x_{n+1} = x_n - \dfrac{x_n^5 - (2+r)x_n^4 + (1+2r)x_n^3 - (1-r)x_n^2 + 2(1-r)x_n + r - 1}{5x_n^4 - 4(2+r)x_n^3 + 3(1+2r)x_n^2 - 2(1-r)x_n + 2(1-r)}$. We substitute in the value
$r \approx 3.04042 \times 10^{-6}$ in order to evaluate the approximations numerically. The libration point L_1 is slightly less than 1 AU from the sun, so we take $x_1 = 0.95$ as our first approximation, and get $x_2 \approx 0.96682$, $x_3 \approx 0.97770$, $x_4 \approx 0.98451$, $x_5 \approx 0.98830$, $x_6 \approx 0.98976$, $x_7 \approx 0.98998$, $x_8 \approx 0.98999 \approx x_9$. So, to five decimal places, L_1 is located 0.98999 AU from the sun (or 0.01001 AU from Earth).

(b) In this case we use Newton's method with the function
$p(x) - 2rx^2 = x^5 - (2+r)x^4 + (1+2r)x^3 - (1+r)x^2 + 2(1-r)x + r - 1 \Rightarrow$
$\left[p(x) - 2rx^2\right]' = 5x^4 - 4(2+r)x^3 + 3(1+2r)x^2 - 2(1+r)x + 2(1-r)$. So
$x_{n+1} = x_n - \dfrac{x_n^5 - (2+r)x_n^4 + (1+2r)x_n^3 - (1+r)x_n^2 + 2(1-r)x_n + r - 1}{5x_n^4 - 4(2+r)x_n^3 + 3(1+2r)x_n^2 - 2(1+r)x_n + 2(1-r)}$. Again, we substitute
$r \approx 3.04042 \times 10^{-6}$. L_2 is slightly more than 1 AU from the sun and, judging from the result of part (a), probably less than 0.02 AU from Earth. So we take $x_1 = 1.02$ and get $x_2 \approx 1.01422$, $x_3 \approx 1.01118$, $x_4 \approx 1.01018$, $x_5 \approx 1.01008 \approx x_6$. So, to five decimal places, L_2 is located 1.01008 AU from the sun (or 0.01008 AU from Earth).

4.10 Antiderivatives

1. b is the antiderivative of f. For small x, f is negative, so the graph of its antiderivative must be decreasing. But both a and c are increasing for small x, so only b can be f's antiderivative. Also, f is positive where b is increasing, which supports our conclusion.

2. We know right away that c cannot be f's antiderivative, since the slope of c is not zero at the x-value where $f = 0$. Now f is positive when a is increasing and negative when a is decreasing, so a is the antiderivative of f.

3. $f(x) = 6x^2 - 8x + 3 \Rightarrow F(x) = 6\dfrac{x^{2+1}}{2+1} - 8\dfrac{x^{1+1}}{1+1} + 3x + C = 2x^3 - 4x^2 + 3x + C$

Check: $F'(x) = 2 \cdot 3x^2 - 4 \cdot 2x + 3 + 0 = 6x^2 - 8x + 3 = f(x)$

4. $f(x) = 4 + x^2 - 5x^3 \Rightarrow F(x) = 4x + \frac{1}{3}x^3 - \frac{5}{4}x^4 + C$

5. $f(x) = 1 - x^3 + 5x^5 - 3x^7 \Rightarrow F(x) = x - \dfrac{x^{3+1}}{3+1} + 5\dfrac{x^{5+1}}{5+1} - 3\dfrac{x^{7+1}}{7+1} + C = x - \frac{1}{4}x^4 + \frac{5}{6}x^6 - \frac{3}{8}x^8 + C$

6. $f(x) = x^{20} + 4x^{10} + 8 \Rightarrow F(x) = \frac{1}{21}x^{21} + \frac{4}{11}x^{11} + 8x + C$

7. $f(x) = 5x^{1/4} - 7x^{3/4} \Rightarrow F(x) = 5\dfrac{x^{1/4+1}}{\frac{1}{4}+1} - 7\dfrac{x^{3/4+1}}{\frac{3}{4}+1} + C = 5\dfrac{x^{5/4}}{5/4} - 7\dfrac{x^{7/4}}{7/4} + C = 4x^{5/4} - 4x^{7/4} + C$

8. $f(x) = 2x + 3x^{1.7} \Rightarrow F(x) = x^2 + \frac{3}{2.7}x^{2.7} + C = x^2 + \frac{10}{9}x^{2.7} + C$

9. $f(x) = \sqrt{x} + \sqrt[3]{x} = x^{1/2} + x^{1/3} \Rightarrow F(x) = \frac{1}{3/2}x^{3/2} + \frac{1}{4/3}x^{4/3} + C = \frac{2}{3}x^{3/2} + \frac{3}{4}x^{4/3} + C$

10. $f(x) = \sqrt[3]{x^2} - \sqrt{x^3} = x^{2/3} - x^{3/2} \Rightarrow F(x) = \frac{1}{5/3}x^{5/3} - \frac{1}{5/2}x^{5/2} + C = \frac{3}{5}x^{5/3} - \frac{2}{5}x^{5/2} + C$

11. $f(x) = \dfrac{10}{x^9} = 10x^{-9}$ has domain $(-\infty, 0) \cup (0, \infty)$, so $F(x) = \begin{cases} \dfrac{10x^{-8}}{-8} + C_1 = -\dfrac{5}{4x^8} + C_1 & \text{if } x < 0 \\ -\dfrac{5}{4x^8} + C_2 & \text{if } x > 0 \end{cases}$

See Example 1(c) for a similar exercise.

12. $f(x) = 3x^{-2} - 5x^{-4}$ has domain $(-\infty, 0) \cup (0, \infty)$, so

$$F(x) = \begin{cases} \dfrac{3x^{-1}}{-1} - \dfrac{5x^{-3}}{-3} + C_1 = -\dfrac{3}{x} + \dfrac{5}{3x^3} + C_1 & \text{if } x < 0 \\ -\dfrac{3}{x} + \dfrac{5}{3x^3} + C_2 & \text{if } x > 0 \end{cases}$$

13. $g(t) = \dfrac{t^3 + 2t^2}{\sqrt{t}} = t^{5/2} + 2t^{3/2} \quad \Rightarrow \quad G(t) = \dfrac{t^{7/2}}{7/2} + \dfrac{2t^{5/2}}{5/2} + C = \frac{2}{7}t^{7/2} + \frac{4}{5}t^{5/2} + C$

Note that g has domain $(0, \infty)$.

14. $f(x) = x^{2/3} + 2x^{-1/3}$ has domain $(-\infty, 0) \cup (0, \infty)$, so

$$F(x) = \begin{cases} \dfrac{x^{5/3}}{5/3} + \dfrac{2x^{2/3}}{2/3} + C_1 = \frac{3}{5}x^{5/3} + 3x^{2/3} + C_1 & \text{if } x > 0 \\ \frac{3}{5}x^{5/3} + 3x^{2/3} + C_2 & \text{if } x < 0 \end{cases}$$

15. $h(x) = x^3 + 5\sin x \quad \Rightarrow \quad H(x) = \frac{1}{4}x^4 + 5(-\cos x) + C = \frac{1}{4}x^4 - 5\cos x + C$

16. $f(t) = 3\cos t - 4\sin t \quad \Rightarrow \quad F(t) = 3(\sin t) - 4(-\cos t) + C = 3\sin t + 4\cos t + C$

17. $f(t) = 4\sqrt{t} - \sec t \tan t \quad \Rightarrow \quad F(t) = \frac{4}{3/2}t^{3/2} - \sec t + C = \frac{8}{3}t^{3/2} - \sec t + C_n$ on the interval $\left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right)$.

18. $f(\theta) = 6\theta^2 - 7\sec^2\theta \quad \Rightarrow \quad F(\theta) = 2\theta^3 - 7\tan\theta + C_n$ on the interval $\left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right)$.

19. $f''(x) = 6x + 12x^2 \quad \Rightarrow \quad f'(x) = 3x^2 + 4x^3 + C \quad \Rightarrow \quad f(x) = x^3 + x^4 + Cx + D$

20. $f''(x) = 2 + x^3 + x^6 \quad \Rightarrow \quad f'(x) = 2x + \frac{1}{4}x^4 + \frac{1}{7}x^7 + C \quad \Rightarrow \quad f(x) = x^2 + \frac{1}{20}x^5 + \frac{1}{56}x^8 + Cx + D$

21. $f''(x) = 1 + x^{4/5} \quad \Rightarrow \quad f'(x) = x + \frac{5}{9}x^{9/5} + C \quad \Rightarrow$

$f(x) = \frac{1}{2}x^2 + \frac{5}{9} \cdot \frac{5}{14}x^{14/5} + Cx + D = \frac{1}{2}x^2 + \frac{25}{126}x^{14/5} + Cx + D$

22. $f''(x) = \cos x \quad \Rightarrow \quad f'(x) = \sin x + C \quad \Rightarrow \quad f(x) = -\cos x + Cx + D$

23. $f'''(t) = 60t^2 \quad \Rightarrow \quad f''(t) = 20t^3 + C \quad \Rightarrow \quad f'(t) = 5t^4 + Ct + D \quad \Rightarrow \quad f(t) = t^5 + \frac{1}{2}Ct^2 + Dt + E$

24. $f'''(t) = t - \sqrt{t} \quad \Rightarrow \quad f''(t) = \frac{1}{2}t^2 - \frac{2}{3}t^{3/2} + C \quad \Rightarrow \quad f'(t) = \frac{1}{6}t^3 - \frac{4}{15}t^{5/2} + Ct + D \quad \Rightarrow$

$f(t) = \frac{1}{24}t^4 - \frac{8}{105}t^{7/2} + \frac{1}{2}Ct^2 + Dt + E$

25. $f'(x) = 1 - 6x \quad \Rightarrow \quad f(x) = x - 3x^2 + C$. $f(0) = C$ and $f(0) = 8 \quad \Rightarrow \quad C = 8$, so $f(x) = x - 3x^2 + 8$.

26. $f'(x) = 8x^3 + 12x + 3 \quad \Rightarrow \quad f(x) = 2x^4 + 6x^2 + 3x + C$. $f(1) = 11 + C$ and $f(1) = 6 \quad \Rightarrow \quad 11 + C = 6$ $\Rightarrow \quad C = -5$, so $f(x) = 2x^4 + 6x^2 + 3x - 5$.

27. $f'(x) = 3\sqrt{x} - 1/\sqrt{x} = 3x^{1/2} - x^{-1/2} \quad \Rightarrow \quad f(x) = 3\left(\frac{1}{3/2}\right)x^{3/2} - \frac{1}{1/2}x^{1/2} + C \quad \Rightarrow$

$2 = f(1) = 2 - 2 + C = C \quad \Rightarrow \quad f(x) = 2x^{3/2} - 2x^{1/2} + 2$

28. $f'(x) = 1 + x^{-2}, x > 0 \quad \Rightarrow \quad f(x) = x - 1/x + C$. Now $f(1) = 1 - 1 + C = 1 \quad \Rightarrow \quad C = 1$, so $f(x) = 1 + x - 1/x$.

29. $f'(x) = 3\cos x + 5\sin x \quad \Rightarrow \quad f(x) = 3\sin x - 5\cos x + C \quad \Rightarrow \quad 4 = f(0) = -5 + C \quad \Rightarrow \quad C = 9 \quad \Rightarrow$ $f(x) = 3\sin x - 5\cos x + 9$

30. $f'(x) = 3x^{-2} \Rightarrow f(x) = \begin{cases} -3/x + C_1 & \text{if } x > 0 \\ -3/x + C_2 & \text{if } x < 0 \end{cases}$ $f(1) = -3 + C_1 = 0 \Rightarrow C_1 = 3$,

$f(-1) = 3 + C_2 = 0 \Rightarrow C_2 = -3$. So $f(x) = \begin{cases} -3/x + 3 & \text{if } x > 0 \\ -3/x - 3 & \text{if } x < 0 \end{cases}$

31. $f''(x) = x \Rightarrow f'(x) = \frac{1}{2}x^2 + C \Rightarrow 2 = f'(0) = C \Rightarrow f'(x) = \frac{1}{2}x^2 + 2 \Rightarrow$
$f(x) = \frac{1}{6}x^3 + 2x + D \Rightarrow -3 = f(0) = D \Rightarrow f(x) = \frac{1}{6}x^3 + 2x - 3$

32. $f''(x) = 20x^3 - 10 \Rightarrow f'(x) = 5x^4 - 10x + C \Rightarrow -5 = f'(1) = 5 - 10 + C \Rightarrow C = 0 \Rightarrow$
$f'(x) = 5x^4 - 10x \Rightarrow f(x) = x^5 - 5x^2 + D \Rightarrow 1 = f(1) = 1 - 5 + D \Rightarrow D = 5 \Rightarrow$
$f(x) = x^5 - 5x^2 + 5$

33. $f''(x) = x^2 + 3\cos x \Rightarrow f'(x) = \frac{1}{3}x^3 + 3\sin x + C \Rightarrow 3 = f'(0) = C \Rightarrow f'(x) = \frac{1}{3}x^3 + 3\sin x + 3$
$\Rightarrow f(x) = \frac{1}{12}x^4 - 3\cos x + 3x + D \Rightarrow 2 = f(0) = -3 + D \Rightarrow D = 5 \Rightarrow$
$f(x) = \frac{1}{12}x^4 - 3\cos x + 3x + 5$

34. $f''(x) = x + x^{1/2} \Rightarrow f'(x) = \frac{1}{2}x^2 + \frac{2}{3}x^{3/2} + C \Rightarrow 2 = f'(1) = \frac{1}{2} + \frac{2}{3} + C \Rightarrow C = \frac{5}{6} \Rightarrow$
$f'(x) = \frac{1}{2}x^2 + \frac{2}{3}x^{3/2} + \frac{5}{6} \Rightarrow f(x) = \frac{1}{6}x^3 + \frac{4}{15}x^{5/2} + \frac{5}{6}x + D \Rightarrow 1 = f(1) = \frac{1}{6} + \frac{4}{15} + \frac{5}{6} + D \Rightarrow$
$D = -\frac{4}{15} \Rightarrow f(x) = \frac{1}{6}x^3 + \frac{4}{15}x^{5/2} + \frac{5}{6}x - \frac{4}{15}$

35. $f''(x) = 6x + 6 \Rightarrow f'(x) = 3x^2 + 6x + C \Rightarrow f(x) = x^3 + 3x^2 + Cx + D$. $4 = f(0) = D$ and
$3 = f(1) = 1 + 3 + C + D = 4 + C + 4 \Rightarrow C = -5$, so $f(x) = x^3 + 3x^2 - 5x + 4$.

36. $f''(x) = 12x^2 - 6x + 2 \Rightarrow f'(x) = 4x^3 - 3x^2 + 2x + C \Rightarrow f(x) = x^4 - x^3 + x^2 + Cx + D$.
$1 = f(0) = D$ and $11 = f(2) = 16 - 8 + 4 + 2C + D = 13 + 2C \Rightarrow C = -1$, so
$f(x) = x^4 - x^3 + x^2 - x + 1$.

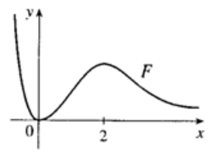
37. $f''(x) = x^{-3} \Rightarrow f'(x) = -\frac{1}{2}x^{-2} + C \Rightarrow f(x) = \frac{1}{2}x^{-1} + Cx + D \Rightarrow 0 = f(1) = \frac{1}{2} + C + D$ and
$0 = f(2) = \frac{1}{4} + 2C + D$. Solving these equations, we get $C = \frac{1}{4}$, $D = -\frac{3}{4}$, so $f(x) = 1/(2x) + \frac{1}{4}x - \frac{3}{4}$.

38. $f'''(x) = \sin x \Rightarrow f''(x) = -\cos x + C \Rightarrow 1 = f''(0) = -1 + C = 1 \Rightarrow C = 2$, so
$f''(x) = -\cos x + 2 \Rightarrow f'(x) = -\sin x + 2x + D \Rightarrow 1 = f'(0) = D \Rightarrow f'(x) = -\sin x + 2x + 1$
$\Rightarrow f(x) = \cos x + x^2 + x + E \Rightarrow 1 = f(0) = 1 + E \Rightarrow E = 0$, so $f(x) = \cos x + x^2 + x$.

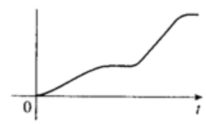
39. Given $f'(x) = 2x + 1$, we have $f(x) = x^2 + x + C$. Since f passes through $(1, 6)$, $6 = f(1) = 1^2 + 1 + C \Rightarrow$
$C = 4$. Therefore, $f(x) = x^2 + x + 4$ and $f(2) = 2^2 + 2 + 4 = 10$.

40. $f'(x) = x^3 \Rightarrow f(x) = \frac{1}{4}x^4 + C$. $x + y = 0 \Rightarrow y = -x \Rightarrow m = -1$. Now $m = f'(x) \Rightarrow$
$x^3 = -1 \Rightarrow x = -1 \Rightarrow y = 1$ (from the equation of the tangent line), so $(-1, 1)$ is a point on the graph of f. From f, $1 = (-1)^4/4 + C \Rightarrow C = \frac{3}{4}$. Therefore, the function is $f(x) = \frac{1}{4}x^4 + \frac{3}{4}$.

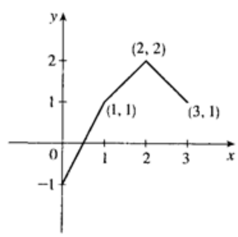
41. The graph of F will have a minimum at 0 and a maximum at 2, since $f = F'$ goes from negative to positive at $x = 0$, and from positive to negative at $x = 2$.



42. The position function is the antiderivative of the velocity function, so its graph will have be horizontal where the velocity function is equal to 0.



43.



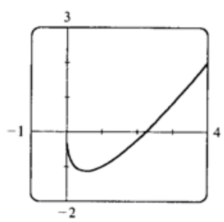
$$f'(x) = \begin{cases} 2 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 < x < 2 \\ -1 & \text{if } 2 < x \le 3 \end{cases} \quad \Rightarrow \quad f(x) = \begin{cases} 2x + C & \text{if } 0 \le x < 1 \\ x + D & \text{if } 1 < x < 2 \\ -x + E & \text{if } 2 < x \le 3 \end{cases}$$

$f(0) = -1 \;\Rightarrow\; 2(0) + C = -1 \;\Rightarrow\; C = -1$. Starting at the point $(0, -1)$ and moving to the right on a line with slope 2 gets us to the point $(1, 1)$. The slope for $1 < x < 2$ is 1, so we get to the point $(2, 2)$. The line connecting $(1, 1)$ to $(2, 2)$ is $y = x$, so $D = 0$. The slope for $2 < x \le 3$ is -1, so we get to $(3, 1)$. $f(3) = 1 \;\Rightarrow\; -3 + E = 1 \;\Rightarrow\; E = 4$.

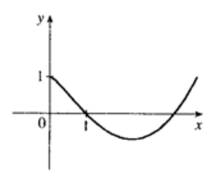
Thus, $f(x) = \begin{cases} 2x - 1 & \text{if } 0 \le x < 1 \\ x & \text{if } 1 < x < 2 \\ -x + 4 & \text{if } 2 < x \le 3 \end{cases}$

Note that f is continuous, but $f'(x)$ does not exist at $x = 1$ or at $x = 2$.

44. (a)



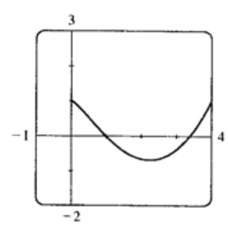
(b) Since $F(0) = 1$, we can start our graph at $(0, 1)$. f has a minimum at about $x = 0.5$, so its derivative is zero there. f is decreasing on $(0, 0.5)$, so its derivative is negative and hence, F is CD on $(0, 0.5)$ and has an IP at $x \approx 0.5$. On $(0.5, 2.2)$, f is negative and increasing (f' is positive), so F is decreasing and CU. On $(2.2, \infty)$, f is positive and increasing, so F is increasing and CU.



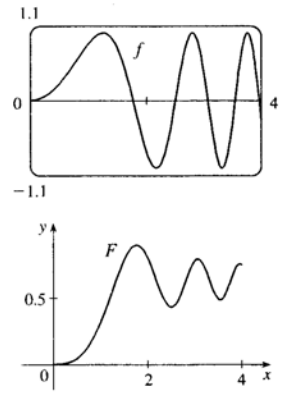
(c) $f(x) = 2x - 3\sqrt{x} \;\Rightarrow$

$F(x) = x^2 - 3 \cdot \frac{2}{3}x^{3/2} + C$. $F(0) = C$ and $F(0) = 1 \;\Rightarrow\; C = 1$, so $F(x) = x^2 - 2x^{3/2} + 1$.

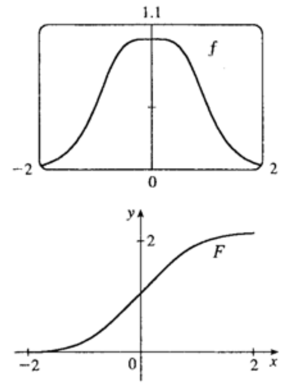
(d)



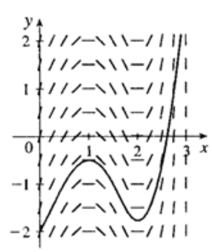
45.



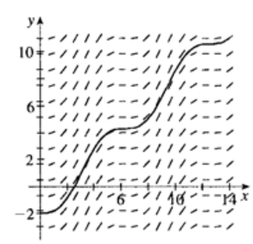
46.



47.



48.

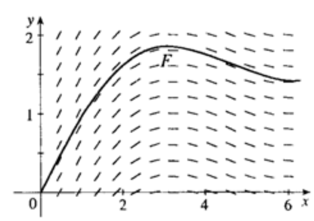


49.

x	$f(x)$
0	1
0.5	0.959
1.0	0.841
1.5	0.665
2.0	0.455
2.5	0.239
3.0	0.047

x	$f(x)$
3.5	−0.100
4.0	−0.189
4.5	−0.217
5.0	−0.192
5.5	−0.128
6.0	−0.047

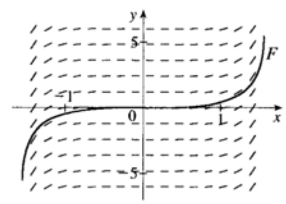
We compute slopes (values of f) as in the table and draw a direction field as in Example 6. Then we use the direction field to graph F starting at $(0, 0)$.



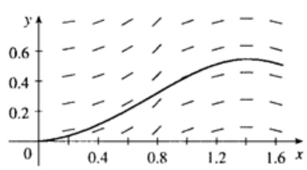
50.

x	$f(x)$
0	0
± 0.2	0.041
± 0.4	0.169
± 0.6	0.410
± 0.8	0.824
± 1.0	1.557
± 1.2	3.087
± 1.4	8.117
± 1.5	21.152

We compute slopes (values of f) as in the table and draw a direction field as in Example 6. Then we use the direction field to graph F starting at $(0, 0)$ and extending in both directions. Note that if f is an even function, then the antiderivative F that passes through the origin is an odd function.

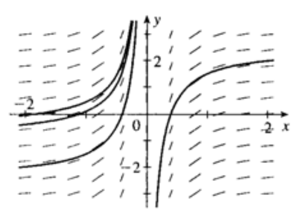


51.



Remember that the values of f are the slopes of F at any x. For example, at $x = 1.4$, the slope of F is $f(1.4) = 0$.

52. (a)



(b) The general antiderivative of $f(x) = x^{-2}$ is

$$F(x) = \begin{cases} -1/x + C_1 & \text{if } x < 0 \\ -1/x + C_2 & \text{if } x > 0 \end{cases}$$ since $f(x)$ is not defined at $x = 0$. The graph of the general antiderivatives of $f(x)$ looks like the graph in part (a), as expected.

53. $v(t) = s'(t) = \sin t - \cos t \;\Rightarrow\; s(t) = -\cos t - \sin t + C$. $s(0) = -1 + C$ and $s(0) = 0 \;\Rightarrow\; C = 1$, so $s(t) = -\cos t - \sin t + 1$.

54. $v(t) = s'(t) = 1.5\sqrt{t} \;\Rightarrow\; s(t) = t^{3/2} + C$. $s(4) = 8 + C$ and $s(4) = 10 \;\Rightarrow\; C = 2$, so $s(t) = t^{3/2} + 2$.

55. $a(t) = v'(t) = t - 2 \;\Rightarrow\; v(t) = \frac{1}{2}t^2 - 2t + C$. $v(0) = C$ and $v(0) = 3 \;\Rightarrow\; C = 3$, so $v(t) = \frac{1}{2}t^2 - 2t + 3$ and $s(t) = \frac{1}{6}t^3 - t^2 + 3t + D$. $s(0) = D$ and $s(0) = 1 \;\Rightarrow\; D = 1$, and $s(t) = \frac{1}{6}t^3 - t^2 + 3t + 1$.

56. $a(t) = v'(t) = \cos t + \sin t \;\Rightarrow\; v(t) = \sin t - \cos t + C \;\Rightarrow\; 5 = v(0) = -1 + C \;\Rightarrow\; C = 6$, so $v(t) = \sin t - \cos t + 6 \;\Rightarrow\; s(t) = -\cos t - \sin t + 6t + D \;\Rightarrow\; 0 = s(0) = -1 + D \;\Rightarrow\; D = 1$, so $s(t) = -\cos t - \sin t + 6t + 1$.

57. $a(t) = v'(t) = 10\sin t + 3\cos t \;\Rightarrow\; v(t) = -10\cos t + 3\sin t + C \;\Rightarrow$ $s(t) = -10\sin t - 3\cos t + Ct + D$. $s(0) = -3 + D = 0$ and $s(2\pi) = -3 + 2\pi C + D = 12 \;\Rightarrow\; D = 3$ and $C = \frac{6}{\pi}$. Thus, $s(t) = -10\sin t - 3\cos t + \frac{6}{\pi}t + 3$.

58. $a(t) = v'(t) = 10 + 3t - 3t^2 \;\Rightarrow\; v(t) = 10t + \frac{3}{2}t^2 - t^3 + C \;\Rightarrow\; s(t) = 5t^2 + \frac{1}{2}t^3 - \frac{1}{4}t^4 + Ct + D \;\Rightarrow$ $0 = s(0) = D$ and $10 = s(2) = 20 + 4 - 4 + 2C \;\Rightarrow\; C = -5$, so $s(t) = -5t + 5t^2 + \frac{1}{2}t^3 - \frac{1}{4}t^4$.

59. (a) We first observe that since the stone is dropped 450 m above the ground, $v(0) = 0$ and $s(0) = 450$.
$v'(t) = a(t) = -9.8 \Rightarrow v(t) = -9.8t + C$, but $C = v(0) = 0$, so $v(t) = -9.8t \Rightarrow$
$s(t) = -4.9t^2 + D \Rightarrow D = s(0) = 450 \Rightarrow s(t) = 450 - 4.9t^2$.

(b) It reaches the ground when $0 = s(t) = 450 - 4.9t^2 \Rightarrow t^2 = 450/4.9 \Rightarrow t_1 = \sqrt{450/4.9} \approx 9.58$ s.

(c) $v(t_1) = -9.8\sqrt{450/4.9} \approx -93.9$ m/s

(d) This is just reworking parts (a) and (b) with $v(0) = -5$. $v(t) = -9.8t + C \Rightarrow -5 = 0 + C \Rightarrow$
$v(t) = -9.8t - 5$. $s(t) = -4.9t^2 - 5t + D \Rightarrow 450 = s(0) = D \Rightarrow s(t) = -4.9t^2 - 5t + 450$.
$s(t) = 0 \Rightarrow t = \left(5 \pm \sqrt{8845}\right) / (-9.8) \Rightarrow t_1 \approx 9.09$ s.

60. $v'(t) = a(t) = a \Rightarrow v(t) = at + C$ and $v_0 = v(0) = C \Rightarrow v(t) = at + v_0 \Rightarrow s(t) = \frac{1}{2}at^2 + v_0t + D$
$\Rightarrow s_0 = s(0) = D \Rightarrow s(t) = \frac{1}{2}at^2 + v_0t + s_0$

61. By Exercise 60, $s(t) = -4.9t^2 + v_0t + s_0$ and $v(t) = s'(t) = -9.8t + v_0$. So
$[v(t)]^2 = (-9.8t + v_0)^2 = (9.8)^2 t^2 - 19.6v_0t + v_0^2 = v_0^2 - 19.6\left(v_0t - 4.9t^2\right)$. But $-4.9t^2 + v_0t$ is just $s(t)$ without the s_0 term, that is, $s(t) - s_0$. Thus, $[v(t)]^2 = v_0^2 - 19.6[s(t) - s_0]$.

62. For the first ball, $s_1(t) = -16t^2 + 48t + 432$ from Example 8. For the second ball, $a(t) = -32 \Rightarrow$
$v(t) = -32t + C$, but $v(1) = -32(1) + C = 24 \Rightarrow C = 56$, so $v(t) = -32t + 56 \Rightarrow$
$s(t) = -16t^2 + 56t + D$, but $s(1) = -16(1)^2 + 56(1) + D = 432 \Rightarrow D = 392$, and
$s_2(t) = -16t^2 + 56t + 392$. The balls pass each other when $s_1(t) = s_2(t) \Rightarrow$
$-16t^2 + 48t + 432 = -16t^2 + 56t + 392 \Leftrightarrow 8t = 40 \Leftrightarrow t = 5$ s.
Another Solution: From Exercise 60, we have $s_1(t) = -16t^2 + 48t + 432$ and $s_2(t) = -16t^2 + 24t + 432$. We now want to solve $s_1(t) = s_2(t-1) \Rightarrow -16t^2 + 48t + 432 = -16(t-1)^2 + 24(t-1) + 432 \Rightarrow$
$48t = 32t - 16 + 24t - 24 \Rightarrow 40 = 8t \Rightarrow t = 5$ s.

63. Marginal cost $= 1.92 - 0.002x = C'(x) \Rightarrow C(x) = 1.92x - 0.001x^2 + K$. But
$C(1) = 1.92 - 0.001 + K = 562 \Rightarrow K = 560.081$. Therefore, $C(x) = 1.92x - 0.001x^2 + 560.081 \Rightarrow$
$C(100) = 742.081$, so the cost of producing 100 items is \$742.08.

64. Let the mass, measured from one end, be $m(x)$. Then $m(0) = 0$ and $\rho = \dfrac{dm}{dx} = x^{-1/2} \Rightarrow m(x) = 2x^{1/2} + C$ and $m(0) = C = 0$, so $m(x) = 2\sqrt{x}$. Thus, the mass of the rod is $m(100) = 2\sqrt{100} = 20$ g.

65. Taking the upward direction to be positive we have that for $0 \le t \le 10$ (using the subscript 1 to refer to $0 \le t \le 10$), $a_1(t) = -(9 - 0.9t) = v_1'(t) \Rightarrow v_1(t) = -9t + 0.45t^2 + v_0$, but $v_1(0) = v_0 = -10 \Rightarrow$
$v_1(t) = -9t + 0.45t^2 - 10 = s_1'(t) \Rightarrow s_1(t) = -\frac{9}{2}t^2 + 0.15t^3 - 10t + s_0$. But $s_1(0) = 500 = s_0 \Rightarrow$
$s_1(t) = -\frac{9}{2}t^2 + 0.15t^3 - 10t + 500$. $s_1(10) = 100$, so it takes more than 10 seconds for the raindrop to fall. Now for $t > 10$, $a(t) = 0 = v'(t) \Rightarrow v(t) = \text{constant} = v_1(10) = -9(10) + 0.45(10)^2 - 10 = -55 \Rightarrow$
$v(t) = -55$. At 55 ft/s, it will take $100/55 \approx 1.8$ s to fall the last 100 ft. Hence, the total time is 11.8 s.

66. $v'(t) = a(t) = -40$. The initial velocity is 50 mi/h $= \frac{50 \cdot 5280}{3600} = \frac{220}{3}$ ft/s, so $v(t) = -40t + \frac{220}{3}$. The car stops when $v(t) = 0 \Leftrightarrow t = \frac{220}{3 \cdot 40} = \frac{11}{6}$. Since $s(t) = -20t^2 + \frac{220}{3}t$, the distance covered is
$s\left(\frac{11}{6}\right) = -20\left(\frac{11}{6}\right)^2 + \frac{220}{3} \cdot \frac{11}{6} = \frac{605}{9} \approx 67.2$ ft.

67. $a(t) = k$, the initial velocity is 30 mi/h $= 30 \cdot \frac{5280}{3600} = 44$ ft/s, and the final velocity is
50 mi/h$= 50 \cdot \frac{5280}{3600} = \frac{220}{3}$ ft/s. So $v(t) = kt + C$ and $v(0) = 44 \Rightarrow C = 44$. Thus, $v(t) = kt + 44 \Rightarrow$
$\frac{220}{3} = v(5) = 5k + 44 \Rightarrow k = \frac{88}{15} \approx 5.87$ ft/s^2.

68. $a(t) = -40 \Rightarrow v(t) = -40t + v_0$ where v_0 is the car's speed (in ft/s) when the brakes were applied. The car stops when $-40t + v_0 = 0 \Leftrightarrow t = \frac{1}{40}v_0$. Now $s(t) = \frac{1}{2}(-40)t^2 + v_0 t = -20t^2 + v_0 t$. The car travels 160 ft in the time that it takes to stop, so $s\left(\frac{1}{40}v_0\right) = 160 \Rightarrow 160 = -20\left(\frac{1}{40}v_0\right)^2 + v_0\left(\frac{1}{40}v_0\right) = \frac{1}{80}v_0^2 \Rightarrow$ $v_0^2 = 12{,}800 \Rightarrow v_0 = 80\sqrt{2} \approx 113$ ft/s (about 77 mi/h).

69. Using Exercise 60 with $a = -32$, $v_0 = 0$, and $s_0 = h$ (the height of the cliff), we know that the height at time t is $s(t) = -16t^2 + h$. $v(t) = s'(t) = -32t \Rightarrow -32t = -120 \Rightarrow t = 3.75$, so $0 = s(3.75) = -16(3.75)^2 + h \Rightarrow h = 16(3.75)^2 = 225$ ft.

70. (a) For $0 \le t \le 3$ we have $a(t) = 60t \Rightarrow v(t) = 30t^2 + C \Rightarrow v(0) = 0 = C \Rightarrow v(t) = 30t^2$, so $s(t) = 10t^3 + C \Rightarrow s(0) = 0 = C \Rightarrow s(t) = 10t^3$. Note that $v(3) = 270$ and $s(3) = 270$.

For $3 < t \le 17$: $a(t) = -g = -32$ ft/s $\Rightarrow v(t) = -32(t-3) + C \Rightarrow v(3) = 270 = C \Rightarrow$ $v(t) = -32(t-3) + 270 \Rightarrow s(t) = -16(t-3)^2 + 270(t-3) + C \Rightarrow s(3) = 270 = C \Rightarrow$ $s(t) = -16(t-3)^2 + 270(t-3) + 270$. Note that $v(17) = -178$ and $s(17) = 914$.

For $17 < t \le 22$: The velocity increases linearly from -178 ft/s to -18 ft/s during this period, so $\frac{\Delta v}{\Delta t} = \frac{-18-(-178)}{22-17} = \frac{160}{5} = 32$. Thus, $v(t) = 32(t-17) - 178 \Rightarrow$ $s(t) = 16(t-17)^2 - 178(t-17) + 914$ and $s(22) = 424$ ft.

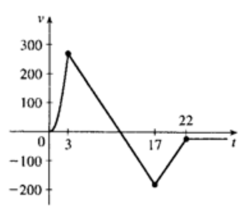
For $t > 22$: $v(t) = -18 \Rightarrow s(t) = -18(t-22) + C$. But $s(22) = 424 = C \Rightarrow$ $s(t) = -18(t-22) + 424$.

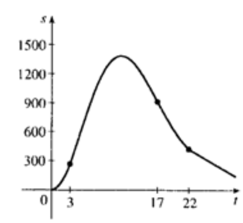
Therefore, until the rocket lands, we have

$$v(t) = \begin{cases} 30t^2 & \text{if } 0 \le t \le 3 \\ -32(t-3) + 270 & \text{if } 3 < t \le 17 \\ 32(t-17) - 178 & \text{if } 17 < t \le 22 \\ -18 & \text{if } t > 22 \end{cases}$$

and

$$s(t) = \begin{cases} 10t^3 & \text{if } 0 \le t \le 3 \\ -16(t-3)^2 + 270(t-3) + 270 & \text{if } 3 < t \le 17 \\ 16(t-17)^2 - 178(t-17) + 914 & \text{if } 17 < t \le 22 \\ -18(t-22) + 424 & \text{if } t > 22 \end{cases}$$





(b) To find the maximum height, set $v(t)$ on $3 < t \le 17$ equal to 0. $-32(t-3) + 270 = 0 \Rightarrow t_1 = 11.4375$ s and the maximum height is $s(t_1) = -16(t_1-3)^2 + 270(t_1-3) + 270 = 1409.0625$ ft.

(c) To find the time to land, set $s(t) = -18(t-22) + 424 = 0$. Then $t - 22 = \frac{424}{18} = 23.\overline{5}$, so $t \approx 45.6$ s.

71. (a) First note that 90 mi/h $= 90 \times \frac{5280}{3600}$ ft/s $= 132$ ft/s. Then $a(t) = 4\text{ ft/s}^2 \Rightarrow v(t) = 4t + C$, but $v(0) = 0$ $\Rightarrow C = 0$. Now $4t = 132$ when $t = \frac{132}{4} = 33$ s, so it takes 33 s to reach 132 ft/s. Therefore, taking $s(0) = 0$, we have $s(t) = 2t^2$, $0 \le t \le 33$. So $s(33) = 2178$ ft. 15 minutes $= 15(60) = 900$ s, so for $33 \le t \le 933$ we have $v(t) = 132$ ft/s $\Rightarrow s(933) = 132(900) + 2178 = 120{,}978$ ft $= 22.9125$ mi.

(b) As in part (a), the train accelerates for 33 s and travels 2178 ft while doing so. Similarly, it decelerates for 33 s and travels 2178 ft at the end of its trip. During the remaining $900 - 66 = 834$ s it travels at 132 ft/s, so the distance traveled is $132 \cdot 834 = 110{,}088$ ft. Thus, the total distance is $2178 + 110{,}088 + 2178 = 114{,}444$ ft $= 21.675$ mi.

(c) 45 mi $= 45(5280) = 237{,}600$ ft. Subtract $2(2178)$ to take care of the speeding up and slowing down, and we have 233,244 ft at 132 ft/s for a trip of $233{,}244/132 = 1767$ s at 90 mi/h. The total time is $1767 + 2(33) = 1833$ s or 30.55 min.

(d) $37.5(60) = 2250$ s. $2250 - 2(33) = 2184$ s at maximum speed. $2184(132) + 2(2178) = 292{,}644$ total feet or $292{,}644/5280 = 55.425$ mi.

4 Review

CONCEPT CHECK

1. A function f has an **absolute maximum** at $x = c$ if $f(c)$ is the largest function value on the entire domain of f, whereas f has a **local maximum** at c if $f(c)$ is the largest function value when x is near c. See Figure 4 in Section 4.1.

2. (a) See Theorem 4.1.3.

(b) See the Closed Interval Method before Example 8 in Section 4.1.

3. (a) See Theorem 4.1.4.

(b) See Definition 4.1.6.

4. (a) See Rolle's Theorem at the beginning of Section 4.2.

(b) See the Mean Value Theorem in Section 4.2. Geometrical interpretation — there is some point P on the graph of a function f [on the interval (a, b)] where the tangent line is parallel to the secant line that connects $(a, f(a))$ and $(b, f(b))$.

5. (a) See the I/D Test before Example 1 in Section 4.3.

(b) See the Concavity Test just before Example 4 in Section 4.3.

6. (a) See the First Derivative Test after Example 1 in Section 4.3.

(b) See the Second Derivative Test before Example 6 in Section 4.3.

(c) See the note before Example 7 in Section 4.3.

7. (a) See Definitions 4.4.1 and 4.4.5.

(b) See Definitions 4.4.2 and 4.4.6.

(c) See Definition 4.4.7.

(d) See Definition 4.4.3.

8. Without calculus you could get misleading graphs that fail to show the most interesting features of a function. See the discussions on pages 264 and 271.

9. (a) See Figure 3 in Section 4.9.

(b) $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

(c) $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

(d) Newton's method is likely to fail or to work very slowly when $f'(x_1)$ is close to 0.

10. (a) See the definition at the beginning of Section 4.10.

(b) If F_1 and F_2 are both antiderivatives of f on an interval I, then they differ by a constant.

TRUE-FALSE QUIZ

1. False. For example, take $f(x) = x^3$, then $f'(x) = 3x^2$ and $f'(0) = 0$, but $f(0) = 0$ is not a maximum or minimum; $(0, 0)$ is an inflection point.

2. False. For example, $f(x) = |x|$ has an absolute minimum at 0, but $f'(0)$ does not exist.

3. False. For example, $f(x) = x$ is continuous on $(0, 1)$ but attains neither a maximum nor a minimum value on $(0, 1)$. Don't confuse this with f being continuous on the *closed* interval $[a, b]$, which would make the statement true.

4. True. By the Mean Value Theorem, $f'(c) = \dfrac{f(1) - f(-1)}{1 - (-1)} = \dfrac{0}{2} = 0$. Note that $|c| < 1 \quad \Leftrightarrow \quad c \in (-1, 1)$.

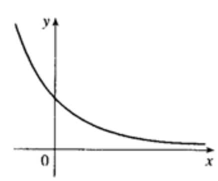
5. True by the ID Test.

6. False. For example, the curve $y = f(x) = 1$ has no inflection points but $f''(c) = 0$ for all c.

7. False. $f'(x) = g'(x) \quad \Rightarrow \quad f(x) = g(x) + C$. For example, $f(x) = x + 2$, $g(x) = x + 1 \quad \Rightarrow$ $f'(x) = g'(x) = 1$, but $f(x) \neq g(x)$.

8. False. Assume there is a function f such that $f(1) = -2$ and $f(3) = 0$. Then by the Mean Value Theorem there exists a number $c \in (1, 3)$ such that $f'(c) = \dfrac{f(3) - f(1)}{3 - 1} = \dfrac{0 - (-2)}{2} = 1$. But $f'(x) > 1$ for all x, a contradiction.

9. True. The graph of one such function is sketched.



10. False. At any point $(a, f(a))$, we know that $f'(a) < 0$. So since the tangent line at $(a, f(a))$ is not horizontal, it must cross the x-axis — at $x = b$, say. But since $f''(x) > 0$ for all x, the graph of f must lie above all of its tangents; in particular, $f(b) > 0$. But this is a contradiction, since we are given that $f(x) < 0$ for all x.

11. True. Let $x_1 < x_2$ where $x_1, x_2 \in I$. Then $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$ (since f and g are increasing on I), so $(f + g)(x_1) = f(x_1) + g(x_1) < f(x_2) + g(x_2) = (f + g)(x_2)$.

12. False. $f(x) = x$ and $g(x) = 2x$ are both increasing on $(0, 1)$, but $f(x) - g(x) = -x$ is not increasing on $(0, 1)$.

13. False. Take $f(x) = x$ and $g(x) = x - 1$. Then both f and g are increasing on $(0, 1)$. But $f(x)g(x) = x(x-1)$ is not increasing on $(0, 1)$.

14. True. Let $x_1 < x_2$ where $x_1, x_2 \in I$. Then $0 < f(x_1) < f(x_2)$ and $0 < g(x_1) < g(x_2)$ (since f and g are both positive and increasing). Hence, $f(x_1)g(x_1) < f(x_2)g(x_1) < f(x_2)g(x_2)$. So fg is increasing on I.

15. True. Let $x_1, x_2 \in I$ and $x_1 < x_2$. Then $f(x_1) < f(x_2)$ (f is increasing) $\Rightarrow$ $\dfrac{1}{f(x_1)} > \dfrac{1}{f(x_2)}$ (f is positive) $\Rightarrow$ $g(x_1) > g(x_2)$ $\Rightarrow$ $g(x) = 1/f(x)$ is decreasing on I.

16. False. The most general antiderivative is $F(x) = -1/x + C_1$ for $x < 0$ and $F(x) = -1/x + C_2$ for $x > 0$ (see Example 1 in Section 4.10).

17. True. If $f'(x)$ exists and is nonzero for all x, then $f'(x)$ is either positive everywhere or negative everywhere. Hence, f is either strictly increasing everywhere or strictly decreasing everywhere, so $f(0)$ cannot equal $f(1)$.

EXERCISES

1. $f(x) = 10 + 27x - x^3$, $0 \le x \le 4$. $f'(x) = 27 - 3x^2 = -3(x^2 - 9) = -3(x+3)(x-3) = 0$ only when $x = 3$ (since -3 is not in the domain). $f'(x) > 0$ for $x < 3$ and $f'(x) < 0$ for $x > 3$, so $f(3) = 64$ is a local and absolute maximum value. Checking the endpoints, we find $f(0) = 10$ and $f(4) = 54$. Thus, $f(0) = 10$ is the absolute minimum value.

2. $f(x) = x - \sqrt{x}$, $0 \le x \le 4$. $f'(x) = 1 - 1/(2\sqrt{x}) = 0$ $\Leftrightarrow$ $2\sqrt{x} = 1$ $\Rightarrow$ $x = \frac{1}{4}$. $f'(x)$ does not exist $\Leftrightarrow$ $x = 0$. $f'(x) > 0$ for $0 < x < \frac{1}{4}$ and $f'(x) < 0$ for $\frac{1}{4} < x < 4$, so $f\left(\frac{1}{4}\right) = -\frac{1}{4}$ is a local and absolute minimum value. $f(0) = 0$ and $f(4) = 2$, so $f(4) = 2$ is the absolute maximum value.

3. $f(x) = \dfrac{x}{x^2 + x + 1}$, $-2 \le x \le 0$. $f'(x) = \dfrac{(x^2+x+1)(1) - x(2x+1)}{(x^2+x+1)^2} = \dfrac{1 - x^2}{(x^2+x+1)^2} = 0$ $\Leftrightarrow$ $x = -1$ (since 1 is not in the domain). $f'(x) < 0$ for $-2 < x < -1$ and $f'(x) > 0$ for $-1 < x < 0$, so $f(-1) = -1$ is a local and absolute minimum value. $f(-2) = -\frac{2}{3}$ and $f(0) = 0$, $f(0) = 0$ is an absolute maximum value.

4. $f(x) = \sqrt{x^2 + 4x + 8}$, $-3 \le x \le 0$. $f'(x) = (x+2)/\sqrt{x^2+4x+8} = 0$ when $x = -2$, and $f'(x) < 0$ for $x < -2$, $f'(x) > 0$ for $x > -2$. So $f(-2) = 2$ is a local and absolute minimum. Also $f(-3) = \sqrt{5}$, $f(0) = 2\sqrt{2}$, so $f(0) = 2\sqrt{2}$ is an absolute maximum.

5. $f(x) = x - \sqrt{2}\sin x$, $0 \le x \le \pi$. $f'(x) = 1 - \sqrt{2}\cos x = 0$ $\Rightarrow$ $\cos x = \frac{1}{\sqrt{2}}$ $\Rightarrow$ $x = \frac{\pi}{4}$. $f''\left(\frac{\pi}{4}\right) = \sqrt{2}\sin\frac{\pi}{4} = 1 > 0$, so $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - 1$ is a local minimum. Also $f(0) = 0$ and $f(\pi) = \pi$, so the absolute minimum is $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - 1$, the absolute maximum is $f(\pi) = \pi$.

6. $f(x) = 2x + 2\cos x - 4\sin x - \cos 2x$, $0 \le x \le \pi$.

$$f'(x) = 2 - 2\sin x - 4\cos x + 2\sin 2x = 2(1 - \sin x - 2\cos x + 2\sin x\cos x)$$
$$= 2(\sin x - 1)(2\cos x - 1) = 0 \text{ when } \sin x = 1 \text{ or } \cos x = \tfrac{1}{2}, \text{ so } x = \tfrac{\pi}{2} \text{ or } \tfrac{\pi}{3}.$$

$f'(x) < 0$ for $0 < x < \frac{\pi}{3}$, $f'(x) \ge 0$ for $\frac{\pi}{3} < x < \pi$, so $f\left(\frac{\pi}{3}\right) = \frac{2\pi}{3} + \frac{3}{2} - 2\sqrt{3}$ is a local minimum. Also $f\left(\frac{\pi}{3}\right) \approx 0.13$, $f(0) = 1$, $f(\pi) = 2\pi - 3 \approx 3.28$, so $f\left(\frac{\pi}{3}\right) = \frac{2\pi}{3} + \frac{3}{2} - 2\sqrt{3}$ is an absolute minimum and $f(\pi) = 2\pi - 3$ is an absolute maximum.

7. $\lim\limits_{x\to\infty} \dfrac{3x^4+x-5}{6x^4-2x^2+1} = \lim\limits_{x\to\infty} \dfrac{3+\dfrac{1}{x^3}-\dfrac{5}{x^4}}{6-\dfrac{2}{x^2}+\dfrac{1}{x^4}} = \dfrac{3+0+0}{6-0+0} = \dfrac{1}{2}$

8. If $t = \dfrac{1}{x}$, then $\lim\limits_{x\to\infty} x\tan\dfrac{1}{x} = \lim\limits_{t\to 0^+} \dfrac{\tan t}{t} = \lim\limits_{t\to 0^+} \dfrac{\sin t}{t}\dfrac{1}{\cos t} = 1\cdot 1 = 1.$

9. $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2-9}}{2x-6} = \lim\limits_{x\to\infty} \dfrac{\sqrt{1-9/x^2}}{2-6/x} = \dfrac{\sqrt{1-0}}{2-0} = \dfrac{1}{2}$

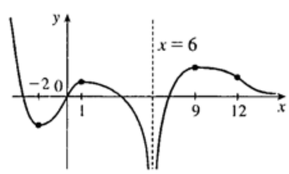
10. $0 \le \cos^2 x \le 1 \quad\Rightarrow\quad 0 \le \dfrac{\cos^2 x}{x^2} \le \dfrac{1}{x^2}$ and $\lim\limits_{x\to-\infty} 0 = 0$, $\lim\limits_{x\to-\infty} \dfrac{1}{x^2} = 0$, so by the Squeeze Theorem,

$\lim\limits_{x\to-\infty} \dfrac{\cos^2 x}{x^2} = 0.$

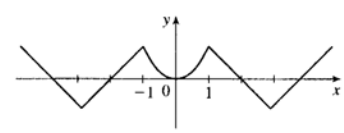
11. $\lim\limits_{x\to\infty} \left(\sqrt[3]{x} - \frac{1}{3}x\right) = \lim\limits_{x\to\infty} \sqrt[3]{x}\left(1-\frac{1}{3}x^{2/3}\right) = -\infty$, since $\sqrt[3]{x}\to\infty$ and $1-\frac{1}{3}x^{2/3}\to-\infty$.

12.
$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right) &= \lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} \\ &= \lim_{x\to\infty}\frac{\left(x^2+x+1\right)-\left(x^2-x\right)}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} = \lim_{x\to\infty}\frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} \\ &= \lim_{x\to\infty}\frac{2+1/x}{\sqrt{1+(1/x)+\left(1/x^2\right)}+\sqrt{1-1/x}} = \frac{2+0}{\sqrt{1+0+0}+\sqrt{1-0}} = 1\end{aligned}$$

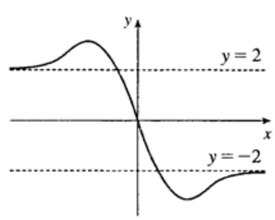
13.



14. For $0 < x < 1$, $f'(x) = 2x$, so $f(x) = x^2 + C$. Since $f(0) = 0$, $f(x) = x^2$ on $[0, 1]$. For $1 < x < 3$, $f'(x) = -1$, so $f(x) = -x + D$. $1 = f(1) = -1 + D \;\Rightarrow\; D = 2$, so $f(x) = 2 - x$. For $x > 3$, $f'(x) = 1$, so $f(x) = x + E$. $-1 = f(3) = 3 + E \;\Rightarrow\; E = -4$, so $f(x) = x - 4$. Since f is even, its graph is symmetric about the y-axis.



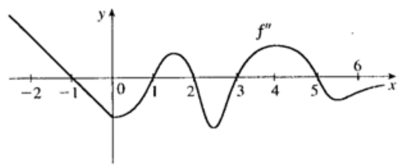
15.



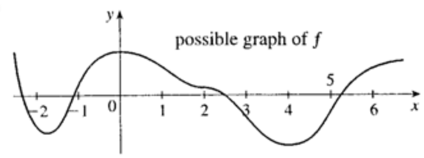
16. (a) Using the Test for Monotonic Functions we know that f is increasing on $(-2, 0)$ and $(4, \infty)$ because $f' > 0$ on $(-2, 0)$ and $(4, \infty)$, and that f is decreasing on $(-\infty, -2)$ and $(0, 4)$ because $f' < 0$ on $(-\infty, -2)$ and $(0, 4)$.

(b) Using the First Derivative Test, we know that f has a local maximum at $x = 0$ because f' changes from positive to negative at $x = 0$, and that f has a local minimum at $x = 4$ because f' changes from negative to positive at $x = 4$.

(c)

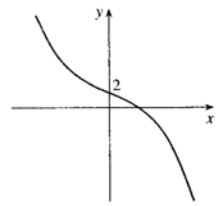


(d)



17. $y = f(x) = 2 - 2x - x^3$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 2$. The x-intercept (approximately 0.770917) can be found using Newton's Method. **C.** No symmetry **D.** No asymptote
E. $f'(x) = -2 - 3x^2 = -1\left(3x^2 + 2\right) < 0$, so f is decreasing on $\mathbb{R}$.
F. No local extremum **G.** $f''(x) = -6x < 0$ on $(0, \infty)$ and $f''(x) > 0$ on $(-\infty, 0)$, so f is CD on $(0, \infty)$ and CU on $(-\infty, 0)$. There is an IP at $(0, 2)$.

H.

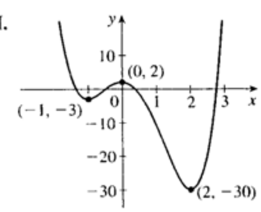


18. $y = f(x) = 3x^4 - 4x^3 - 12x^2 + 2$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 2$ **C.** No symmetry
D. $\lim\limits_{x \to \pm\infty} \left(3x^4 - 4x^3 - 12x^2 + 2\right) = \infty$, no asymptote
E. $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x - 2)(x + 1) = 0$ when $x = -1, 0, 2$

Interval	$12x$	$x - 2$	$x + 1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	increasing on $(-1, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$

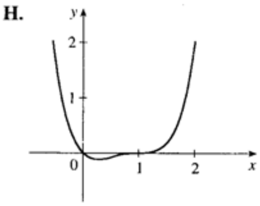
F. $f(-1) = -3$ is a local minimum, $f(0) = 2$ is a local maximum, $f(2) = -30$ is a local minimum. **G.** $f''(x) = 12\left(3x^2 - 2x - 2\right) = 0$ $\Rightarrow$ $x = \frac{1 \pm \sqrt{7}}{3}$. $f''(x) > 0$ $\Leftrightarrow$ $x > \frac{1+\sqrt{7}}{3}$ or $x < \frac{1-\sqrt{7}}{3}$, so f is CU on $\left(-\infty, \frac{1-\sqrt{7}}{3}\right)$ and $\left(\frac{1+\sqrt{7}}{3}, \infty\right)$ and CD on $\left(\frac{1-\sqrt{7}}{3}, \frac{1+\sqrt{7}}{3}\right)$. IP at $x = \frac{1 \pm \sqrt{7}}{3}$

H.



19. $y = f(x) = x^4 - 3x^3 + 3x^2 - x = x(x-1)^3$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0$ $\Leftrightarrow$ $x = 0$ or $x = 1$ **C.** No symmetry **D.** f is a polynomial function and hence, it has no asymptote. **E.** $f'(x) = 4x^3 - 9x^2 + 6x - 1$. Since the sum of the coefficients is 0, 1 is a root of f', so $f'(x) = (x-1)(4x^2 - 5x + 1) = (x-1)^2(4x-1)$. $f'(x) < 0$ $\Rightarrow$ $x < \frac{1}{4}$, so f is decreasing on $\left(-\infty, \frac{1}{4}\right)$ and f is increasing on $\left(\frac{1}{4}, \infty\right)$. **F.** $f'(x)$ does not change sign at $x = 1$, so there is not a local extremum there. $f\left(\frac{1}{4}\right) = -\frac{27}{256}$ is a local minimum. **G.** $f''(x) = 12x^2 - 18x + 6 = 6(2x-1)(x-1)$. $f''(x) = 0$ $\Leftrightarrow$ $x = \frac{1}{2}$ or 1. $f''(x) < 0$ $\Leftrightarrow$ $\frac{1}{2} < x < 1$ $\Rightarrow$ f is CD on $\left(\frac{1}{2}, 1\right)$ and CU on $\left(-\infty, \frac{1}{2}\right)$ and $(1, \infty)$. There are inflection points at $\left(\frac{1}{2}, -\frac{1}{16}\right)$ and $(1, 0)$.

H.



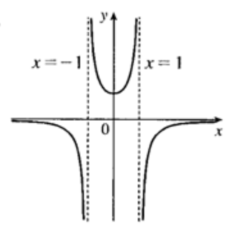
20. $y = f(x) = \dfrac{1}{1-x^2} = \dfrac{1}{(1+x)(1-x)}$ **A.** $D = \{x \mid x \neq \pm 1\}$ **B.** y-intercept: $f(0) = 1$; no x-intercept **C.** $f(-x) = f(x)$, so f is even and the graph of f is symmetric about the y-axis. **D.** Vertical asymptotes: $x = \pm 1$. Horizontal asymptote: $y = 0$ **E.** $y' = \dfrac{2x}{(1-x^2)^2} = 0$ $\Leftrightarrow$ $x = 0$, so f is decreasing on $(-\infty, -1)$ and $(-1, 0)$, and increasing on $(0, 1)$ and $(1, \infty)$. **F.** Local minimum $f(0) = 1$; no local maximum

$$\textbf{G. } f''(x) = \frac{(1-x^2)^2 \cdot 2 - 2x \cdot 2(1-x^2)(-2x)}{(1-x^2)^4}$$

$$= \frac{2(1-x^2) + 8x^2}{(1-x^2)^3} = \frac{6x^2 + 2}{(1-x^2)^3} < 0 \quad \Leftrightarrow \quad x^2 > 1,$$

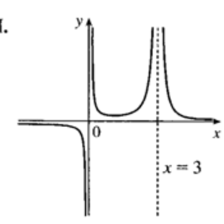
so f is CD on $(-\infty, -1)$ and $(1, \infty)$, and CU on $(-1, 1)$. No IP

H.



21. $y = f(x) = \dfrac{1}{x(x-3)^2}$ **A.** $D = \{x \mid x \neq 0, 3\} = (-\infty, 0) \cup (0, 3) \cup (3, \infty)$ **B.** No intercepts. **C.** No symmetry. **D.** $\displaystyle\lim_{x \to \pm\infty} \frac{1}{x(x-3)^2} = 0$, so $y = 0$ is a HA. $\displaystyle\lim_{x \to 0^+} \frac{1}{x(x-3)^2} = \infty$, $\displaystyle\lim_{x \to 0^-} \frac{1}{x(x-3)^2} = -\infty$, $\displaystyle\lim_{x \to 3} \frac{1}{x(x-3)^2} = \infty$, so $x = 0$ and $x = 3$ are VA. **E.** $f'(x) = -\dfrac{(x-3)^2 + 2x(x-3)}{x^2(x-3)^4} = \dfrac{3(1-x)}{x^2(x-3)^3}$ $\Rightarrow$ $f'(x) > 0$ $\Leftrightarrow$ $1 < x < 3$, so f is increasing on $(1, 3)$ and decreasing on $(-\infty, 0)$, $(0, 1)$, and $(3, \infty)$. **F.** $f(1) = \frac{1}{4}$ is a local minimum. **G.** $f''(x) = \dfrac{6(2x^2 - 4x + 3)}{x^3(x-3)^4}$. Note that $2x^2 - 4x + 3 > 0$ for all x since it has negative discriminant. So $f''(x) > 0$ $\Leftrightarrow$ $x > 0$ $\Rightarrow$ f is CU on $(0, 3)$ and $(3, \infty)$ and CD on $(-\infty, 0)$. No IP

H.

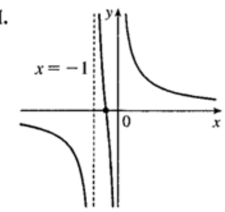


22. $y = f(x) = \frac{1}{x} + \frac{1}{x+1} = \frac{2x+1}{x(x+1)}$ **A.** $D = \{x \mid x \neq 0, -1\}$ **B.** No y-intercept, x-intercept$= -\frac{1}{2}$ **C.** No symmetry **D.** $\lim_{x\to\pm\infty} f(x) = 0$, so $y = 0$ is a HA. $\lim_{x\to 0^+} \frac{2x+1}{x(x+1)} = \infty$, $\lim_{x\to 0^-} \frac{2x+1}{x(x+1)} = -\infty$, $\lim_{x\to -1^+} \frac{2x+1}{x(x+1)} = \infty$, $\lim_{x\to -1^-} \frac{2x+1}{x(x+1)} = -\infty$, so $x = 0$, $x = -1$ are VA.

E. $f'(x) = -\frac{1}{x^2} - \frac{1}{(x+1)^2} < 0$, so f is decreasing on $(-\infty, -1)$, $(-1, 0)$ and $(0, \infty)$. **F.** No extremum

G. $f''(x) = \frac{2}{x^3} + \frac{2}{(x+1)^3} = \frac{2(2x+1)(x^2+x+1)}{x^3(x+1)^3}$. $f''(x) > 0$ $\Leftrightarrow$ $x > 0$ or $-1 < x < -\frac{1}{2}$, so f is CU on $(0, \infty)$ and $\left(-1, -\frac{1}{2}\right)$ and CD on $(-\infty, -1)$ and $\left(-\frac{1}{2}, 0\right)$. IP $\left(-\frac{1}{2}, 0\right)$

H.

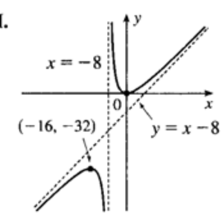


23. $y = f(x) = \frac{x^2}{x+8} = x - 8 + \frac{64}{x+8}$ **A.** $D = \{x \mid x \neq -8\}$ **B.** Intercepts are 0 **C.** No symmetry

D. $\lim_{x\to\infty} \frac{x^2}{x+8} = \infty$, but $f(x) - (x-8) = \frac{64}{x+8}$ $\Rightarrow$ 0 as $x \to \infty$, so $y = x - 8$ is a slant asymptote. $\lim_{x\to -8^+} \frac{x^2}{x+8} = \infty$ and $\lim_{x\to -8^-} \frac{x^2}{x+8} = -\infty$, so $x = -8$ is a VA.

E. $f'(x) = 1 - \frac{64}{(x+8)^2} = \frac{x(x+16)}{(x+8)^2} > 0$ $\Leftrightarrow$ $x > 0$ or $x < -16$, so f is increasing on $(-\infty, -16)$ and $(0, \infty)$ and decreasing on $(-16, -8)$ and $(-8, 0)$. **F.** $f(-16) = -32$ is a local maximum, $f(0) = 0$ is a local minimum. **G.** $f''(x) = 128/(x+8)^3 > 0$ $\Leftrightarrow$ $x > -8$, so f is CU on $(-8, \infty)$ and CD on $(-\infty, -8)$. No IP

H.

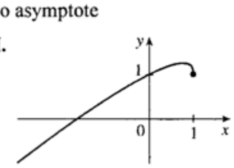


24. $y = f(x) = x + \sqrt{1-x}$ **A.** $D = \{x \mid x \leq 1\} = (-\infty, 1]$ **B.** y-intercept $= 1$; x-intercepts occur when $x + \sqrt{1-x} = 0$ $\Rightarrow$ $\sqrt{1-x} = -x$ $\Rightarrow$ $1 - x = x^2$ $\Rightarrow$ $x^2 + x - 1 = 0$ $\Rightarrow$ $x = \frac{-1\pm\sqrt{5}}{2}$, but the larger root is extraneous, so the only x-intercept is $\frac{-1-\sqrt{5}}{2}$. **C.** No symmetry **D.** No asymptote

E. $f'(x) = 1 - 1/\left(2\sqrt{1-x}\right) = 0$ $\Leftrightarrow$ $2\sqrt{1-x} = 1$ $\Leftrightarrow$ $1 - x = \frac{1}{4}$ $\Leftrightarrow$ $x = \frac{3}{4}$ and $f'(x) > 0$ $\Leftrightarrow$ $x < \frac{3}{4}$, so f is increasing on $\left(-\infty, \frac{3}{4}\right)$, decreasing on $\left(\frac{3}{4}, 1\right)$. **F.** $f\left(\frac{3}{4}\right) = \frac{5}{4}$ is a local maximum.

G. $f''(x) = -\frac{1}{4(1-x)^{3/2}} < 0$ $\Leftrightarrow$ $x < 1$, so f is CD on $(-\infty, 1)$. No IP

H.

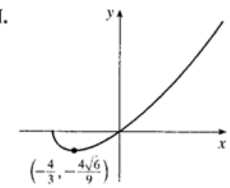


25. $y = f(x) = x\sqrt{2+x}$ **A.** $D = [-2, \infty)$ **B.** y-intercept: $f(0) = 0$; x-intercepts: -2 and 0 **C.** No symmetry **D.** No asymptote **E.** $f'(x) = \dfrac{x}{2\sqrt{2+x}} + \sqrt{2+x} = \dfrac{1}{2\sqrt{2+x}}[x + 2(2+x)] = \dfrac{3x+4}{2\sqrt{2+x}} = 0$ when $x = -\frac{4}{3}$, so f is decreasing on $\left(-2, -\frac{4}{3}\right)$ and increasing on $\left(-\frac{4}{3}, \infty\right)$. **F.** Local minimum $f\left(-\frac{4}{3}\right) = -\frac{4}{3}\sqrt{\frac{2}{3}} = -\frac{4\sqrt{6}}{9} \approx -1.09$, no local maximum

G. $$f''(x) = \frac{2\sqrt{2+x}\cdot 3 - (3x+4)\dfrac{1}{\sqrt{2+x}}}{4(2+x)} = \frac{6(2+x) - (3x+4)}{4(2+x)^{3/2}} = \frac{3x+8}{4(2+x)^{3/2}}$$

$f''(x) > 0$ for $x > -2$, so f is CU everywhere. No IP

H.

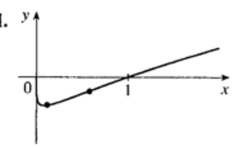


26. $y = f(x) = \sqrt{x} - \sqrt[3]{x}$ **A.** $D = [0, \infty)$ **B.** y-intercept 0, x-intercepts 0, 1 **C.** No symmetry **D.** $\lim\limits_{x\to\infty}\left(x^{1/2} - x^{1/3}\right) = \lim\limits_{x\to\infty}\left[x^{1/3}\left(x^{1/6} - 1\right)\right] = \infty$, no asymptote

E. $f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{3}x^{-2/3} = \dfrac{3x^{1/6} - 2}{6x^{2/3}} > 0 \iff 3x^{1/6} > 2 \iff x > \left(\frac{2}{3}\right)^6$, so f is increasing on $\left(\left(\frac{2}{3}\right)^6, \infty\right)$ and decreasing on $\left(0, \left(\frac{2}{3}\right)^6\right)$. **F.** $f\left(\left(\frac{2}{3}\right)^6\right) = -\frac{4}{27}$ is a local minimum.

G. $f''(x) = -\frac{1}{4}x^{-3/2} + \frac{2}{9}x^{-5/3} = \dfrac{8 - 9x^{1/6}}{36x^{5/3}} > 0 \iff x^{1/6} < \frac{8}{9} \iff x < \left(\frac{8}{9}\right)^6$, so f is CU on $\left(0, \left(\frac{8}{9}\right)^6\right)$ and CD on $\left(\left(\frac{8}{9}\right)^6, \infty\right)$. IP $\left(\frac{8}{9}, -\frac{64}{729}\right)$

H.

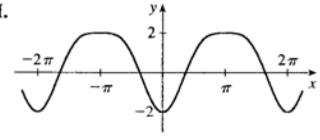


27. $y = f(x) = \sin^2 x - 2\cos x$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = -2$ **C.** $f(-x) = f(x)$, so f is symmetric with respect to the y-axis. f has period 2π. **D.** No asymptote
E. $y' = 2\sin x\cos x + 2\sin x = 2\sin x(\cos x + 1)$. $y' = 0 \iff \sin x = 0$ or $\cos x = -1 \iff x = n\pi$ or $x = (2n+1)\pi$. $y' > 0$ when $\sin x > 0$, since $\cos x + 1 \geq 0$ for all x. Therefore, $y' > 0$ (and so f is increasing) on $(2n\pi, (2n+1)\pi)$; $y' < 0$ (and so f is decreasing) on $((2n-1)\pi, 2n\pi)$. **F.** Local maxima are $f((2n+1)\pi) = 2$; local minima are $f(2n\pi) = -2$. **G.** $y' = \sin 2x + 2\sin x \Rightarrow$

$$y'' = 2\cos 2x + 2\cos x = 2\left(2\cos^2 x - 1\right) + 2\cos x = 4\cos^2 x + 2\cos x - 2$$
$$= 2\left(2\cos^2 x + \cos x - 1\right) = 2(2\cos x - 1)(\cos x + 1)$$

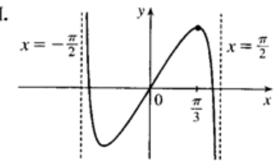
$y'' = 0 \iff \cos x = \frac{1}{2}$ or $-1 \iff x = 2n\pi \pm \frac{\pi}{3}$ or $x = (2n+1)\pi$. $y'' > 0$ (and so f is CU) on $\left(2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3}\right)$; $y'' \leq 0$ (and so f is CD) on $\left(2n\pi + \frac{\pi}{3}, 2n\pi + \frac{5\pi}{3}\right)$. There are inflection points at $\left(2n\pi \pm \frac{\pi}{3}, -\frac{1}{4}\right)$.

H.



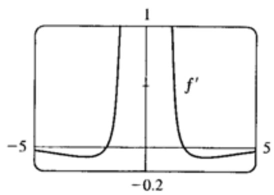
28. $y = f(x) = 4x - \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ **A.** $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. **B.** y-intercept $= f(0) = 0$
C. $f(-x) = -f(x)$, so the curve is symmetric about $(0, 0)$. **D.** $\lim\limits_{x \to \pi/2^-} (4x - \tan x) = -\infty$,
$\lim\limits_{x \to -\pi/2^+} (4x - \tan x) = \infty$, so $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are VA. **E.** $f'(x) = 4 - \sec^2 x > 0 \iff \sec x < 2 \iff$
$\cos x > \frac{1}{2} \iff -\frac{\pi}{3} < x < \frac{\pi}{3}$, so f is increasing on $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ and decreasing on $\left(-\frac{\pi}{2}, -\frac{\pi}{3}\right)$ and $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$. **F.** $f\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} - \sqrt{3}$ is a local maximum, $f\left(-\frac{\pi}{3}\right) = \sqrt{3} - \frac{4\pi}{3}$ is a local minimum. **G.** $f''(x) = -2\sec^2 x \tan x > 0 \iff \tan x < 0 \iff$
$-\frac{\pi}{2} < x < 0$, so f is CU on $\left(-\frac{\pi}{2}, 0\right)$ and CD on $\left(0, \frac{\pi}{2}\right)$. IP $(0, 0)$

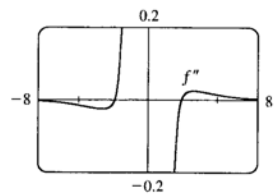
H.

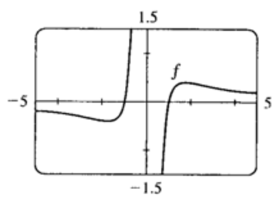


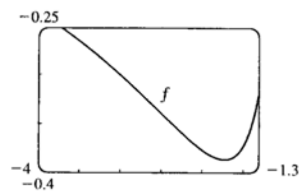
29. $f(x) = \dfrac{x^2 - 1}{x^3} \Rightarrow f'(x) = \dfrac{x^3(2x) - (x^2 - 1)3x^2}{x^6} = \dfrac{3 - x^2}{x^4} \Rightarrow$

$f''(x) = \dfrac{x^4(-2x) - (3 - x^2)4x^3}{x^8} = \dfrac{2x^2 - 12}{x^5}$









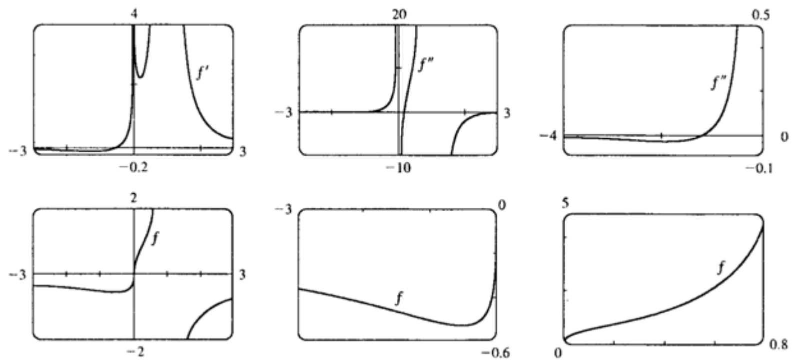
Estimates: From the graphs of f' and f'', it appears that f is increasing on $(-1.73, 0)$ and $(0, 1.73)$ and decreasing on $(-\infty, -1.73)$ and $(1.73, \infty)$; f has a local maximum of about $f(1.73) = 0.38$ and a local minimum of about $f(-1.7) = -0.38$; f is CU on $(-2.45, 0)$ and $(2.45, \infty)$, and CD on $(-\infty, -2.45)$ and $(0, 2.45)$; and f has inflection points at about $(-2.45, -0.34)$ and $(2.45, 0.34)$.

Exact: Now $f'(x) = \dfrac{3 - x^2}{x^4}$ is positive for $0 < x^2 < 3$, that is, f is increasing on $\left(-\sqrt{3}, 0\right)$ and $\left(0, \sqrt{3}\right)$; and $f'(x)$ is negative (and so f is decreasing) on $\left(-\infty, -\sqrt{3}\right)$ and $\left(\sqrt{3}, \infty\right)$. $f'(x) = 0$ when $x = \pm\sqrt{3}$. f' goes from positive to negative at $x = \sqrt{3}$, so f has a local maximum of $f(\sqrt{3}) = \dfrac{\left(\sqrt{3}\right)^2 - 1}{\left(\sqrt{3}\right)^3} = \frac{2\sqrt{3}}{9}$; and since f is odd, we know that maxima on the interval $(0, \infty)$ correspond to minima on $(-\infty, 0)$, so f has a local minimum of $f(-\sqrt{3}) = -\frac{2\sqrt{3}}{9}$. Also, $f''(x) = \dfrac{2x^2 - 12}{x^5}$ is positive (so f is CU) on $\left(-\sqrt{6}, 0\right)$ and $\left(\sqrt{6}, \infty\right)$, and negative (so f is CD) on $\left(-\infty, -\sqrt{6}\right)$ and $\left(0, \sqrt{6}\right)$. There are IP at $\left(\sqrt{6}, \frac{5\sqrt{6}}{36}\right)$ and $\left(-\sqrt{6}, -\frac{5\sqrt{6}}{36}\right)$.

30. $f(x) = \dfrac{\sqrt[3]{x}}{1-x} = x^{1/3}(1-x)^{-1} \quad \Rightarrow$

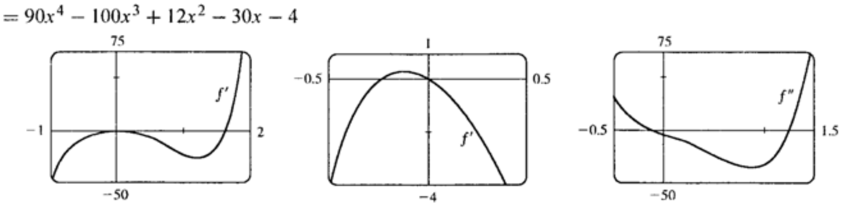
$$f'(x) = x^{1/3}(-1)(1-x)^{-2}(-1) + (1-x)^{-1}\left(\tfrac{1}{3}\right)x^{-2/3} = \frac{x^{-2/3}}{3}\frac{1+2x}{(x-1)^2} \quad \Rightarrow$$

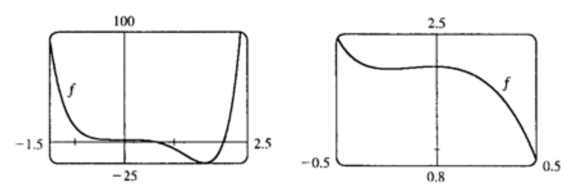
$$f''(x) = \frac{x^{-2/3}}{3}\frac{(x-1)^2(2) - (1+2x)(2)(x-1)}{(x-1)^4} + \frac{1+2x}{(x-1)^2}\left(\frac{-2x^{-5/3}}{9}\right) = -\frac{2x^{-5/3}}{9}\frac{5x^2+5x-1}{(x-1)^3}$$



From the graphs, it appears that f is increasing on $(-0.50, 1)$ and $(1, \infty)$, with a vertical asymptote at $x = 1$, and decreasing on $(-\infty, -0.50)$; f has no local maximum, but a local minimum of about $f(-0.50) = -0.53$; f is CU on $(-1.17, 0)$ and $(0.17, 1)$ and CD on $(-\infty, -1.17)$, $(0, 0.17)$ and $(1, \infty)$; and f has inflection points at about $(-1.17, -0.49)$, $(0, 0)$ and $(0.17, 0.67)$. Note also that $\lim\limits_{x\to\pm\infty} f(x) = 0$, so $y = 0$ is a horizontal asymptote.

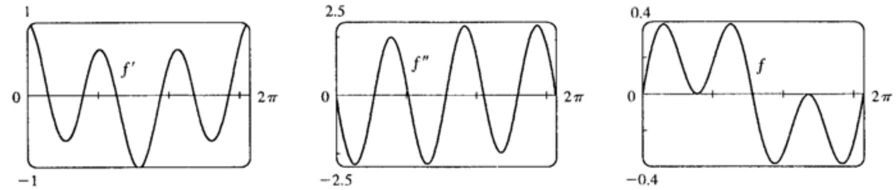
31. $f(x) = 3x^6 - 5x^5 + x^4 - 5x^3 - 2x^2 + 2$, $f'(x) = 18x^5 - 25x^4 + 4x^3 - 15x^2 - 4x$,
$f''(x) = 90x^4 - 100x^3 + 12x^2 - 30x - 4$





From the graphs of f' and f'', it appears that f is increasing on $(-0.23, 0)$ and $(1.62, \infty)$ and decreasing on $(-\infty, -0.23)$ and $(0, 1.62)$; f has a local maximum of about $f(0) = 2$ and local minima of about $f(-0.23) = 1.96$ and $f(1.62) = -19.2$; f is CU on $(-\infty, -0.12)$ and $(1.24, \infty)$ and CD on $(-0.12, 1.24)$; and f has inflection points at about $(-0.12, 1.98)$ and $(1.2, -12.1)$.

32. $f(x) = \sin x \cos^2 x \quad \Rightarrow \quad f'(x) = \cos^3 x - 2\sin^2 x \cos x \quad \Rightarrow \quad f''(x) = -7\sin x\, \cos^2 x + 2\sin^3 x$



From the graphs of f' and f'', it appears that f is increasing on $(0, 0.62)$, $(1.57, 2.53)$, $(3.76, 4.71)$ and $(5.67, 2\pi)$ and decreasing on $(0.62, 1.57)$, $(2.53, 3.76)$ and $(4.71, 5.67)$; f has local maxima of about $f(0.62) = f(2.53) = 0.38$ and $f(4.71) = 0$ and local minima of about $f(1.57) = 0$ and $f(3.76) = f(5.67) = -0.38$; f is CU on $(1.08, 2.06)$, $(3.14, 4.22)$ and $(5.20, 2\pi)$ and CD on $(0, 1.08)$, $(2.06, 3.14)$ and $(4.22, 5.20)$; and f has inflection points at about $(0, 0)$, $(1.08, 0.20)$, $(2.06, 0.20)$, $(3.14, 0)$, $(4.22, -0.20)$, $(5.20, -0.20)$ and $(2\pi, 0)$.

33. $f(x) = x^{101} + x^{51} + x - 1 = 0$. Since f is continuous and $f(0) = -1$ and $f(1) = 2$, the equation has at least one root in $(0, 1)$, by the Intermediate Value Theorem. Suppose the equation has two roots, a and b, with $a < b$. Then $f(a) = 0 = f(b)$, so by the Mean Value Theorem, $f'(x) = \dfrac{f(b) - f(a)}{b - a} = \dfrac{0}{b - a} = 0$, so $f'(x)$ has a root in (a, b). But this is impossible since $f'(x) = 101x^{100} + 51x^{50} + 1 \geq 1$ for all x.

34. By the Mean Value Theorem, $f'(c) = \dfrac{f(4) - f(0)}{4 - 0} \quad \Leftrightarrow \quad 4f'(c) = f(4) - 1$ for some c with $0 < c < 4$. Since $2 \leq f'(c) \leq 5$, we have $4(2) \leq 4f'(c) \leq 4(5) \quad \Leftrightarrow \quad 4(2) \leq f(4) - 1 \leq 4(5) \quad \Leftrightarrow \quad 8 \leq f(4) - 1 \leq 20 \quad \Leftrightarrow$ $9 \leq f(4) \leq 21$.

35. Since f is continuous on $[32, 33]$ and differentiable on $(32, 33)$, then by the Mean Value Theorem there exists a number c in $(32, 33)$ such that $f'(c) = \frac{1}{5}c^{-4/5} = \dfrac{\sqrt[5]{33} - \sqrt[5]{32}}{33 - 32} = \sqrt[5]{33} - 2$, but $\frac{1}{5}c^{-4/5} > 0 \quad \Rightarrow \quad \sqrt[5]{33} - 2 > 0$ $\Rightarrow \quad \sqrt[5]{33} > 2$. Also f' is decreasing, so that $f'(c) < f'(32) = \frac{1}{5}(32)^{-4/5} = 0.0125 \quad \Rightarrow$ $0.0125 > f'(c) = \sqrt[5]{33} - 2 \quad \Rightarrow \quad \sqrt[5]{33} < 2.0125$. Therefore, $2 < \sqrt[5]{33} < 2.0125$.

36. For $(1, 6)$ to be on the curve $y = x^3 + ax^2 + bx + 1$, we have that $6 = a + b + 2 \quad \Rightarrow \quad b = 4 - a$. Now $y' = 3x^2 + 2ax + b$ and $y'' = 6x + 2a$. Also, for $(1, 6)$ to be an inflection point it must be true that $y''(1) = 6(1) + 2a = 0 \quad \Rightarrow \quad a = -3 \quad \Rightarrow \quad b = 4 - (-3) = 7$.

37. (a) $g(x) = f\left(x^2\right) \quad \Rightarrow \quad g'(x) = 2xf'\left(x^2\right)$ by the Chain Rule. Since $f'(x) > 0$ for all $x \neq 0$, we must have $f'\left(x^2\right) > 0$ for $x \neq 0$, so $g'(x) = 0 \quad \Leftrightarrow \quad x = 0$. Now $g'(x)$ changes sign (from negative to positive) at $x = 0$, since one of its factors, $f'\left(x^2\right)$, is positive for all x, and its other factor, $2x$, changes from negative to positive at this point, so by the First Derivative Test, f has a local and absolute minimum at $x = 0$.

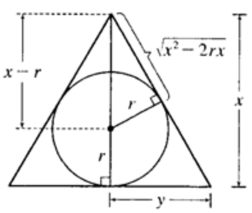
(b) $g'(x) = 2xf'\left(x^2\right) \quad \Rightarrow \quad g''(x) = 2\left[xf''\left(x^2\right)(2x) + f'\left(x^2\right)\right] = 4x^2 f''\left(x^2\right) + 2f'\left(x^2\right)$ by the Product Rule and the Chain Rule. But $x^2 > 0$ for all $x \neq 0$, $f''\left(x^2\right) > 0$ (since f is CU for $x > 0$), and $f'\left(x^2\right) > 0$ for all $x \neq 0$, so since all of its factors are positive, $g''(x) > 0$ for $x \neq 0$. Whether $g''(0)$ is positive or 0 doesn't matter (since the sign of g'' does not change there); g is concave upward on $\mathbb{R}$.

38. Call the two integers x and y. Then $x + 4y = 1000$, so $x = 1000 - 4y$. Their product is $P = xy = (1000 - 4y)\,y$, so our problem is to maximize the function $P(y) = 1000y - 4y^2$, where $0 < y < 250$ and y is an integer. $P'(y) = 1000 - 8y$, so $P'(y) = 0 \Leftrightarrow y = 125$. $P''(y) = -8 < 0$, so $P(125) = 62{,}500$ is an absolute maximum. Since the optimal y turned out to be an integer, we have found the desired pair of numbers, namely $x = 1000 - 4(125) = 500$ and $y = 125$.

39. If $B = 0$, the line is vertical and the distance from $x = -\dfrac{C}{A}$ to (x_1, y_1) is $\left|x_1 + \dfrac{C}{A}\right| = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$, so assume $B \neq 0$. The square of the distance from (x_1, y_1) to the line is $f(x) = (x - x_1)^2 + (y - y_1)^2$ where $Ax + By + C = 0$, so we minimize $f(x) = (x - x_1)^2 + \left(-\dfrac{A}{B}x - \dfrac{C}{B} - y_1\right)^2 \Rightarrow$ $f'(x) = 2(x - x_1) + 2\left(-\dfrac{A}{B}x - \dfrac{C}{B} - y_1\right)\left(-\dfrac{A}{B}\right)$. $f'(x) = 0 \Rightarrow x = \dfrac{B^2x_1 - ABy_1 - AC}{A^2 + B^2}$ and this gives a minimum since $f''(x) = 2\left(1 + \dfrac{A^2}{B^2}\right) > 0$. Substituting this value of x into $f(x)$ and simplifying gives $f(x) = \dfrac{(Ax_1 + By_1 + C)^2}{A^2 + B^2}$, so the minimum distance is $\sqrt{f(x)} = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.

40. If $d(x)$ is the distance from the point $(x, 8/x)$ on the hyperbola to $(3, 0)$, then $[d(x)]^2 = (x - 3)^2 + 64/x^2 = f(x)$. $f'(x) = 2(x - 3) - 128/x^3 = 0 \Rightarrow x^4 - 3x^3 - 64 = 0 \Rightarrow$ $(x - 4)(x^3 + x^2 + 4x + 16) = 0 \Rightarrow x = 4$ since the solution must have $x > 0$. Then $y = \frac{8}{4} = 2$, so the point is $(4, 2)$.

41.



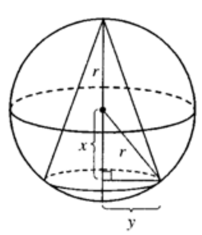
By similar triangles, $\dfrac{y}{x} = \dfrac{r}{\sqrt{x^2 - 2rx}}$, so the area of the triangle is $A(x) = \frac{1}{2}(2y)x = xy = \dfrac{rx^2}{\sqrt{x^2 - 2rx}} \Rightarrow$

$$A'(x) = \frac{2rx\sqrt{x^2 - 2rx} - rx^2(x - r)\big/\sqrt{x^2 - 2rx}}{x^2 - 2rx} = \frac{rx^2(x - 3r)}{(x^2 - 2rx)^{3/2}}$$

$= 0$ when $x = 3r$.

$A'(x) < 0$ when $2r < x < 3r$, $A'(x) > 0$ when $x > 3r$. So $x = 3r$ gives a minimum and $A(3r) = r(9r^2)\big/\left(\sqrt{3}r\right) = 3\sqrt{3}r^2$.

42.



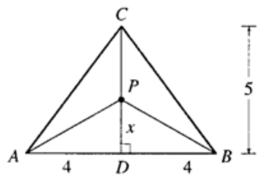
The volume of the cone is

$$V = \tfrac{1}{3}\pi y^2(r + x) = \tfrac{1}{3}\pi\left(r^2 - x^2\right)(r + x),\ -r \le x \le r.$$

$$\begin{aligned} V'(x) &= \tfrac{\pi}{3}\left[\left(r^2 - x^2\right)(1) + (r + x)(-2x)\right] \\ &= \tfrac{\pi}{3}\left[(r + x)(r - x - 2x)\right] = \tfrac{\pi}{3}(r + x)(r - 3x) \\ &= 0 \text{ when } x = -r \text{ or } x = r/3. \end{aligned}$$

Now $V(r) = 0 = V(-r)$, so the maximum occurs at $x = r/3$ and the volume is $V\left(\dfrac{r}{3}\right) = \dfrac{\pi}{3}\left(r^2 - \dfrac{r^2}{9}\right)\left(\dfrac{4r}{3}\right) = \dfrac{32\pi r^3}{81}$.

43.



We minimize

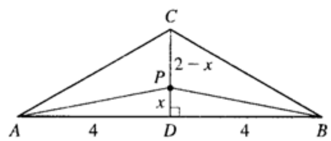
$L(x) = |PA| + |PB| + |PC| = 2\sqrt{x^2+16} + (5-x)$,

$0 \le x \le 5$. $L'(x) = 2x \Big/ \sqrt{x^2+16} - 1 = 0 \quad \Leftrightarrow$

$2x = \sqrt{x^2+16} \quad \Leftrightarrow \quad 4x^2 = x^2 + 16 \quad \Leftrightarrow \quad x = \frac{4}{\sqrt{3}}$.

$L(0) = 13$, $L\left(\frac{4}{\sqrt{3}}\right) \approx 11.9$, $L(5) \approx 12.8$, so the minimum occurs when $x = \frac{4}{\sqrt{3}} \approx 2.3$.

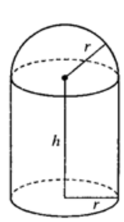
44.



If $|CD| = 2$, $L(x)$ changes from $(5-x)$ to $(2-x)$ with $0 \le x \le 2$. But we still get $L'(x) = 0 \quad \Leftrightarrow \quad x = \frac{4}{\sqrt{3}}$, which isn't in the interval $[0, 2]$. Now $L(0) = 10$ and $L(2) = 2\sqrt{20} = 4\sqrt{5} \approx 8.9$. The minimum occurs when $P = C$.

45. $v = K\sqrt{\dfrac{L}{C} + \dfrac{C}{L}} \quad \Rightarrow \quad \dfrac{dv}{dL} = \dfrac{K}{2\sqrt{(L/C) + (C/L)}}\left(\dfrac{1}{C} - \dfrac{C}{L^2}\right) = 0 \quad \Leftrightarrow \quad \dfrac{1}{C} = \dfrac{C}{L^2} \quad \Leftrightarrow \quad L^2 = C^2 \quad \Leftrightarrow$ $L = C$. This gives the minimum velocity since $v' < 0$ for $0 < L < C$ and $v' > 0$ for $L > C$.

46.



We minimize the surface area $S = \pi r^2 + 2\pi rh + 2\pi r^2 = 3\pi r^2 + 2\pi rh$. Solving $V = \pi r^2 h + \frac{2}{3}\pi r^3$ for h, we get $h = \dfrac{V}{\pi r^2} - \frac{2}{3}r$, so

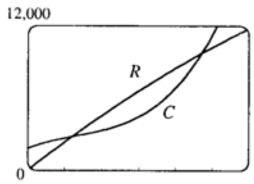
$$S(r) = 3\pi r^2 + 2\pi r\left[\frac{V}{\pi r^2} - \frac{2}{3}r\right] = \frac{5}{3}\pi r^2 + \frac{2V}{r}.$$

$$S'(r) = -\frac{2V}{r^2} + \frac{10}{3}\pi r = \frac{\frac{10}{3}\pi r^3 - 2V}{r^2} = 0 \quad \Leftrightarrow \quad \frac{10}{3}\pi r^3 = 2V$$

$\Leftrightarrow \quad r^3 = \dfrac{3V}{5\pi} \quad \Leftrightarrow \quad r = \sqrt[3]{\dfrac{3V}{5\pi}}$. This gives an absolute minimum since $S'(r) < 0$ for $0 < r < \sqrt[3]{\dfrac{3V}{5\pi}}$ and $S'(r) > 0$ for $r > \sqrt[3]{\dfrac{3V}{5\pi}}$. Thus, $r = \sqrt[3]{\dfrac{3V}{5\pi}} = h$.

47. Let x = selling price of ticket. Then $12 - x$ is the amount the ticket price has been lowered, so the number of tickets sold is $11{,}000 + 1000(12 - x) = 23{,}000 - 1000x$. The revenue is $R(x) = x(23{,}000 - 1000x) = 23{,}000x - 1000x^2$, so $R'(x) = 23{,}000 - 2000x = 0$ when $x = 11.5$. Since $R''(x) = -2000 < 0$, the maximum revenue occurs when the ticket prices are \$11.50.

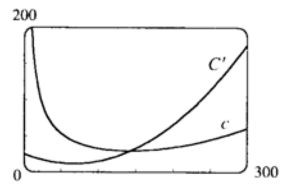
48. (a) $C(x) = 1800 + 25x - 0.2x^2 + 0.001x^3$ and $R(x) = xp(x) = 48.2x - 0.03x^2$. The profit is maximized when $C'(x) = R'(x)$.



From the figure, we estimate that the tangents are parallel when $x \approx 160$.

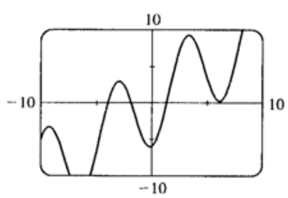
(b) $C'(x) = 25 - 0.4x + 0.003x^2$ and $R'(x) = 48.2 - 0.06x$. $C'(x) = R'(x) \Rightarrow 0.003x^2 - 0.34x - 23.2 = 0 \Rightarrow x_1 \approx 161.3$ $(x > 0)$. $R''(x) = -0.06$ and $C''(x) = -0.4 + 0.006x$, so $R''(x_1) = -0.06 < C''(x_1) \approx 0.57 \Rightarrow$ profit is maximized by producing 161 units.

(c) $c(x) = \dfrac{C(x)}{x} = \dfrac{1800}{x} + 25 - 0.2x + 0.001x^2$ is the average cost. Since the average cost is minimized when the marginal cost equals the average cost, we graph $c(x)$ and $C'(x)$ and estimate the point of intersection. From the figure, $C'(x) = c(x) \Leftrightarrow x \approx 144$.



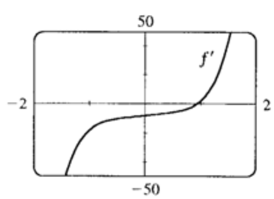
49. $f(x) = x^4 + x - 1 \Rightarrow f'(x) = 4x^3 + 1 \Rightarrow x_{n+1} = x_n - \dfrac{x_n^4 + x_n - 1}{4x_n^3 + 1}$. If $x_1 = 0.5$ then $x_2 \approx 0.791667$, $x_3 \approx 0.729862$, $x_4 \approx 0.724528$, $x_5 \approx 0.724492$ and $x_6 \approx 0.724492$, so, to six decimal places, the root is 0.724492.

50. $f(x) = x - 6\cos x \Rightarrow f'(x) = 1 + 6\sin x \Rightarrow x_{n+1} = x_n - \dfrac{x_n - 6\cos x_n}{1 + 6\sin x_n}$.



From the graph of f, it appears that there are roots near 1, −2 and −4. If $x_1 = 1$, then $x_2 \approx 1.370620$, $x_3 \approx 1.344812$, $x_4 \approx 1.344751 \approx x_5$. If $x_1 = -2$, then $x_2 \approx -1.888486$, $x_3 \approx -1.891518$, $x_4 \approx -1.891520 \approx x_5$. If $x_1 = -4$, then $x_2 \approx -3.985898$, $x_3 \approx -3.985826 \approx x_4$. So, to six decimal places, the roots are 1.344751, −1.891520 and −3.985826.

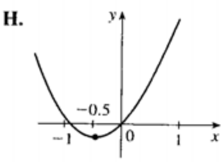
51. $f(x) = x^6 + 2x^2 - 8x + 3 \Rightarrow f'(x) = 6x^5 + 4x - 8$. We want to find the minimum of f, so we examine the graph of f' looking for values at which f' changes from negative to positive.



From the graph, we see that this occurs at $x \approx 1$. So we will use Newton's method with $g(x) = f'(x) = 6x^5 + 4x - 8$, $g'(x) = 30x^4 + 4$, and $x_1 = 1$. $x_{n+1} = x_n - \dfrac{6x_n^5 + 4x_n - 8}{30x_n^4 + 4}$ gives us $x_2 \approx 0.941176$, $x_3 \approx 0.934068$, $x_4 \approx 0.933975 \approx x_5$. Thus, $f(x_5) \approx -2.063421$ is the absolute minimum value of f.

52. $y = f(x) = x^2 + \sin x$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 0$, x-intercepts occur when $x^2 + \sin x = 0$. $x = 0$ is a solution, and using Newton's Method try $x_1 = -1$ to get $x_2 \approx -0.8914$, $x_3 \approx -0.8770$, $x_4 \approx -0.8767$, $x_5 \approx -0.8767$. So the x-intercepts are $x \approx -0.88$, and $x = 0$. **C.** No symmetry **D.** No asymptote
E. $f'(x) = 2x + \cos x$ and to find intervals of increase or decrease solve $2x + \cos x = 0$. Use Newton's Method with $x_1 = -0.5$ to get $x_2 \approx -0.4506$, $x_3 \approx -0.4502$, $x_4 \approx -0.4502$. So the root is $\alpha \approx -0.45$. For $x > \alpha$, $f'(x) > 0$, and for $x < \alpha$, $f'(x) < 0$. So f is increasing on (α, ∞), and decreasing on $(-\infty, \alpha)$. **F.** $f(\alpha) \approx -0.23$ is a local minimum.
G. $f''(x) = 2 - \sin x > 0$ for all x, so f is CU on $(-\infty, \infty)$. No IP

H.



53. $f'(x) = \sqrt{x^5} - 4/\sqrt[5]{x} = x^{5/2} - 4x^{-1/5} \Rightarrow f(x) = \frac{2}{7}x^{7/2} - 4\left(\frac{5}{4}x^{4/5}\right) + C = \frac{2}{7}x^{7/2} - 5x^{4/5} + C$

54. $f'(x) = 8x - 3\sec^2 x \Rightarrow f(x) = 8\left(\frac{1}{2}x^2\right) - 3\tan x + C = 4x^2 - 3\tan x + C$

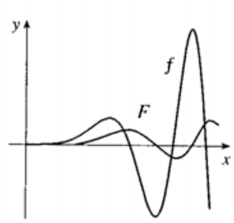
55. $f'(x) = (1 + x)/\sqrt{x} = x^{-1/2} + x^{1/2} \Rightarrow f(x) = 2x^{1/2} + \frac{2}{3}x^{3/2} + C \Rightarrow 0 = f(1) = 2 + \frac{2}{3} + C \Rightarrow C = -\frac{8}{3} \Rightarrow f(x) = 2x^{1/2} + \frac{2}{3}x^{3/2} - \frac{8}{3}$

56. $f'(x) = 1 + 2\sin x - \cos x \Rightarrow f(x) = x - 2\cos x - \sin x + C \Rightarrow 3 = f(0) = -2 + C \Rightarrow C = 5$, so $f(x) = x - 2\cos x - \sin x + 5$.

57. $f''(x) = x^3 + x \Rightarrow f'(x) = \frac{1}{4}x^4 + \frac{1}{2}x^2 + C \Rightarrow 1 = f'(0) = C \Rightarrow f'(x) = \frac{1}{4}x^4 + \frac{1}{2}x^2 + 1 \Rightarrow f(x) = \frac{1}{20}x^5 + \frac{1}{6}x^3 + x + D \Rightarrow -1 = f(0) = D \Rightarrow f(x) = \frac{1}{20}x^5 + \frac{1}{6}x^3 + x - 1$

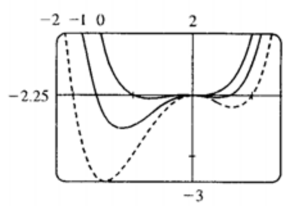
58. $f''(x) = x^4 - 4x^2 + 3x - 2 \Rightarrow f'(x) = \frac{1}{5}x^5 - \frac{4}{3}x^3 + \frac{3}{2}x^2 - 2x + C \Rightarrow$
$f(x) = \frac{1}{30}x^6 - \frac{1}{3}x^4 + \frac{1}{2}x^3 - x^2 + Cx + D$. $0 = f(0) = D \Rightarrow f(x) = \frac{1}{30}x^6 - \frac{1}{3}x^4 + \frac{1}{2}x^3 - x^2 + Cx$.
$1 = f(1) = \frac{1}{30} - \frac{1}{3} + \frac{1}{2} - 1 + C \Rightarrow C = \frac{9}{5}$, so $f(x) = \frac{1}{30}x^6 - \frac{1}{3}x^4 + \frac{1}{2}x^3 - x^2 + \frac{9}{5}x$.

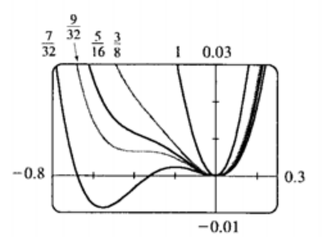
59.



60. $f(x) = x^4 + x^3 + cx^2 \quad \Rightarrow \quad f'(x) = 4x^3 + 3x^2 + 2cx$. This is 0 when $x\left(4x^2 + 3x + 2c\right) = 0 \quad \Leftrightarrow \quad x = 0$ or $4x^2 + 3x + 2c = 0$. Using the quadratic formula, we find that the roots of this last equation are $x = \dfrac{-3 \pm \sqrt{9 - 32c}}{8}$. Now if $9 - 32c < 0 \quad \Leftrightarrow \quad c > \frac{9}{32}$, then $x = 0$ is the only critical point, a minimum. If $c = \frac{9}{32}$, then there are two critical points (a minimum at $x = 0$, and a horizontal tangent with no maximum or minimum at $x = -\frac{3}{8}$) and if $c < \frac{9}{32}$, then there are three critical points except when $c = 0$, in which case the root with the $+$ sign coincides with the critical point at $x = 0$. For $0 < c < \frac{9}{32}$, there is a minimum at $x = -\dfrac{3}{8} - \dfrac{\sqrt{9 - 32c}}{8}$, a maximum at $x = -\dfrac{3}{8} + \dfrac{\sqrt{9 - 32c}}{8}$, and a minimum at $x = 0$. For $c = 0$, there is a minimum at $x = -\frac{3}{4}$ and a horizontal tangent with no extremum at $x = 0$, and for $c < 0$, there is a maximum at $x = 0$, and there are minima at $x = -\dfrac{3}{8} \pm \dfrac{\sqrt{9 - 32c}}{8}$. Now we calculate $f''(x) = 12x^2 + 6x + 2c$. The roots of this equation are $x = \dfrac{-6 \pm \sqrt{36 - 4 \cdot 12 \cdot 2c}}{24}$. So if $36 - 96c \le 0 \quad \Leftrightarrow c \ge \frac{3}{8}$, then there is no inflection point. If $c < \frac{3}{8}$, then there are two inflection points at $x = -\dfrac{1}{4} \pm \dfrac{\sqrt{9 - 24c}}{12}$.

Value of c	No. of CP	No. of IP
$c < 0$	3	2
$c = 0$	2	2
$0 < c < \frac{9}{32}$	3	2
$c = \frac{9}{32}$	2	2
$\frac{9}{32} < c < \frac{3}{8}$	1	2
$c \ge \frac{3}{8}$	1	0

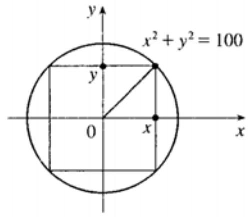




61. Choosing the positive direction to be upward, we have $a(t) = -9.8 \quad \Rightarrow \quad v(t) = -9.8t + v_0$, but $v(0) = 0 = v_0$ $\Rightarrow \quad v(t) = -9.8t = s'(t) \quad \Rightarrow \quad s(t) = -4.9t^2 + s_0$, but $s(0) = s_0 = 500 \quad \Rightarrow \quad s(t) = -4.9t^2 + 500$. When $s = 0$, $-4.9t^2 + 500 = 0 \quad \Rightarrow \quad t_1 = \sqrt{\frac{500}{4.9}} \approx 10.1 \quad \Rightarrow \quad v(t_1) = -9.8\sqrt{\frac{500}{4.9}} \approx -98.995$ m/s. Therefore, the canister will not burst.

62. Let $s_A(t)$ and $s_B(t)$ be the position functions for cars A and B and let $f(t) = s_A(t) - s_B(t)$. Since A passed B twice, there must be three values of t such that $f(t) = 0$. Then by three applications of Rolle's Theorem (see Exercise 4.2.22), there is a number c such that $f''(c) = 0$. So $s_A''(c) = s_B''(c)$, that is, A and B had equal accelerations at $t = c$.

63. (a)

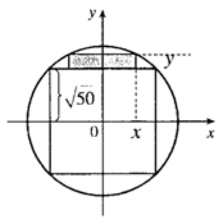


The cross-sectional area is $A = 2x \cdot 2y = 4xy = 4x\sqrt{100 - x^2}$, $0 \le x \le 10$, so

$$\frac{dA}{dx} = 4x\left(\tfrac{1}{2}\right)\left(100 - x^2\right)^{-1/2}(-2x) + \left(100 - x^2\right)^{1/2} \cdot 4$$
$$= \frac{-4x^2}{\left(100 - x^2\right)^{1/2}} + 4\left(100 - x^2\right)^{1/2}$$
$$= 0 \text{ when } -x^2 + \left(100 - x^2\right) = 0 \quad \Rightarrow$$

$x^2 = 50 \quad \Rightarrow \quad x = \sqrt{50} \approx 7.07 \quad \Rightarrow \quad y = \sqrt{100 - \left(\sqrt{50}\right)^2} = \sqrt{50}$. Since $A(0) = A(10) = 0$, the rectangle of maximum area is a square.

(b)



The cross-sectional area of each rectangular plank (shaded in the figure) is

$A = 2x\left(y - \sqrt{50}\right) = 2x\left[\sqrt{100 - x^2} - \sqrt{50}\right]$, $0 \le x \le \sqrt{50}$, so

$$\frac{dA}{dx} = 2\left(\sqrt{100 - x^2} - \sqrt{50}\right) + 2x\left(\tfrac{1}{2}\right)\left(100 - x^2\right)^{-1/2}(-2x)$$
$$= 2\left(100 - x^2\right)^{1/2} - 2\sqrt{50} - \frac{2x^2}{\left(100 - x^2\right)^{1/2}}$$

Set $\dfrac{dA}{dx} = 0$: $\left(100 - x^2\right) - \sqrt{50}\left(100 - x^2\right)^{1/2} - x^2 = 0 \quad \Rightarrow \quad 100 - 2x^2 = \sqrt{50}\left(100 - x^2\right)^{1/2} \quad \Rightarrow$

$10{,}000 - 400x^2 + 4x^4 = 50\left(100 - x^2\right) \quad \Rightarrow \quad 2500 - 175x^2 + 2x^4 = 0 \quad \Rightarrow$

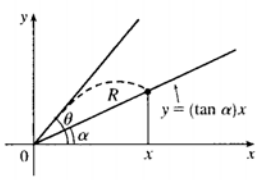
$x^2 = \dfrac{175 \pm \sqrt{10{,}625}}{4} \approx 69.52$ or $17.98 \quad \Rightarrow \quad x \approx 8.34$ or 4.24.

But $8.34 > \sqrt{50}$, so $x_1 \approx 4.24 \quad \Rightarrow \quad y - \sqrt{50} = \sqrt{100 - x_1^2} - \sqrt{50} \approx 1.99$. Each plank should have dimensions about $8\frac{1}{2}$ inches by 2 inches.

(c) From the figure in part (a), the width is $2x$ and the depth is $2y$, so the strength is

$S = k(2x)(2y)^2 = 8kxy^2 = 8kx\left(100 - x^2\right) = 800kx - 8kx^3$, $0 \le x \le 10$. $dS/dx = 800k - 24kx^2 = 0$ when $24kx^2 = 800k \quad \Rightarrow \quad x^2 = \frac{100}{3} \quad \Rightarrow \quad x = \frac{10}{\sqrt{3}} \quad \Rightarrow \quad y = \sqrt{\frac{200}{3}} = \frac{10\sqrt{2}}{\sqrt{3}} = \sqrt{2}x$ and $S(0) = S(10) = 0$, so the maximum strength occurs when $x = \frac{10}{\sqrt{3}}$. The dimensions should be $\frac{20}{\sqrt{3}} \approx 11.55$ inches by $\frac{20\sqrt{2}}{\sqrt{3}} \approx 16.33$ inches.

64. (a)



$y = (\tan\theta)x - \dfrac{g}{2v^2\cos^2\theta}x^2$. The parabola intersects the line when

$$(\tan\alpha)x = (\tan\theta)x - \frac{g}{2v^2\cos^2\theta}x^2 \quad \Rightarrow$$
$$x = \frac{(\tan\theta - \tan\alpha)\,2v^2\cos^2\theta}{g} \quad \Rightarrow$$

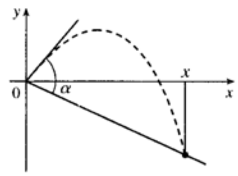
$$R(\theta) = \frac{x}{\cos\alpha} = \left(\frac{\sin\theta}{\cos\theta} - \frac{\sin\alpha}{\cos\alpha}\right)\frac{2v^2\cos^2\theta}{g\cos\alpha} = \left(\frac{\sin\theta}{\cos\theta} - \frac{\sin\alpha}{\cos\alpha}\right)(\cos\theta\cos\alpha)\frac{2v^2\cos\theta}{g\cos^2\alpha}$$
$$= (\sin\theta\cos\alpha - \sin\alpha\cos\theta)\frac{2v^2\cos\theta}{g\cos^2\alpha} = \sin(\theta - \alpha)\frac{2v^2\cos\theta}{g\cos^2\alpha}$$

(b) $R'(\theta) = \dfrac{2v^2}{g\cos^2\alpha}\left[\cos\theta\cdot\cos(\theta-\alpha)+\sin(\theta-\alpha)(-\sin\theta)\right] = \dfrac{2v^2}{g\cos^2\alpha}\cos\left[\theta+(\theta-\alpha)\right]$

$= \dfrac{2v^2}{g\cos^2\alpha}\cos(2\theta-\alpha) = 0$ when $\cos(2\theta-\alpha)=0 \;\Rightarrow\; 2\theta-\alpha=\frac{\pi}{2} \;\Rightarrow$

$\theta = \dfrac{\pi/2+\alpha}{2} = \dfrac{\pi}{4}+\dfrac{\alpha}{2}$. The First Derivative Test shows that this gives a maximum value for $R(\theta)$. [This could be done without calculus by applying the formula for $\sin x\cos y$ to $R(\theta)$.]

(c)



Replacing α by $-\alpha$ in part (a), we get $R(\theta) = \dfrac{2v^2\cos\theta\sin(\theta+\alpha)}{g\cos^2\alpha}$. Proceeding as in part (b), or simply by replacing α by $-\alpha$ in the result of part (b), we see that $R(\theta)$ is maximized when $\theta = \dfrac{\pi}{4}-\dfrac{\alpha}{2}$.

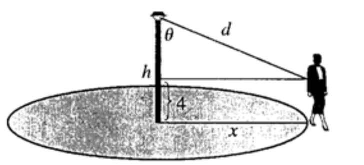
65. (a) $I = \dfrac{k\cos\theta}{d^2} = \dfrac{k(h/d)}{d^2} = k\dfrac{h}{d^3} = k\dfrac{h}{\left(\sqrt{40^2+h^2}\right)^3} = k\dfrac{h}{(1600+h^2)^{3/2}} \;\Rightarrow$

$$\frac{dI}{dh} = k\frac{(1600+h^2)^{3/2} - h\frac{3}{2}(1600+h^2)^{1/2}\cdot 2h}{(1600+h^2)^3} = \frac{k(1600+h^2)^{1/2}(1600+h^2-3h^2)}{(1600+h^2)^3}$$

$$= \frac{k(1600-2h^2)}{(1600+h^2)^{5/2}}$$

Set $dI/dh = 0$: $1600-2h^2=0 \;\Rightarrow\; h^2=800 \;\Rightarrow\; h=\sqrt{800}=20\sqrt{2}$. By the First Derivative Test, I has a local maximum at $h=20\sqrt{2}\approx 28$ ft.

(b)



$\dfrac{dx}{dt} = 4$ ft/s

$$I = \frac{k\cos\theta}{d^2} = \frac{k[(h-4)/d]}{d^2} = \frac{k(h-4)}{d^3} = \frac{k(h-4)}{\left[(h-4)^2+x^2\right]^{3/2}} = k(h-4)\left[(h-4)^2+x^2\right]^{-3/2}$$

$$\frac{dI}{dt} = \frac{dI}{dx}\cdot\frac{dx}{dt} = k(h-4)\left(-\tfrac{3}{2}\right)\left[(h-4)^2+x^2\right]^{-5/2}\cdot 2x\cdot\frac{dx}{dt}$$

$$= k(h-4)(-3x)\left[(h-4)^2+x^2\right]^{-5/2}\cdot 4 = \frac{-12xk(h-4)}{\left[(h-4)^2+x^2\right]^{5/2}}$$

$$\left.\frac{dI}{dt}\right|_{x=40} = -\frac{480k(h-4)}{\left[(h-4)^2+1600\right]^{5/2}}$$

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Problems Plus

1. Let $f(x) = \sin x - \cos x$ on $[0, 2\pi]$ since f has period 2π. $f'(x) = \cos x + \sin x = 0 \Leftrightarrow \cos x = -\sin x \Leftrightarrow \tan x = -1 \Leftrightarrow x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$. Evaluating f at its critical numbers and endpoints, we get $f(0) = -1$, $f\left(\frac{3\pi}{4}\right) = \sqrt{2}$, $f\left(\frac{7\pi}{4}\right) = -\sqrt{2}$, and $f(2\pi) = -1$. So f has absolute maximum value $\sqrt{2}$ and absolute minimum value $-\sqrt{2}$. Thus, $-\sqrt{2} \le \sin x - \cos x \le \sqrt{2} \Rightarrow |\sin x - \cos x| \le \sqrt{2}$.

2. $x^2y^2(4-x^2)(4-y^2) = x^2(4-x^2)y^2(4-y^2) = f(x)f(y)$, where $f(t) = t^2(4-t^2)$. We will show that $0 \le f(t) \le 4$ for $|t| \le 2$, which gives $0 \le f(x)f(y) \le 16$ for $|x| \le 2$ and $|y| \le 2$.
$f(t) = 4t^2 - t^4 \Rightarrow f'(t) = 8t - 4t^3 = 4t(2-t^2) = 0 \Rightarrow t = 0$ or $\pm\sqrt{2}$. $f(0) = 0$, $f\left(\pm\sqrt{2}\right) = 2(4-2) = 4$, and $f(2) = 0$. So 0 is the absolute minimum value of $f(t)$ on $[-2, 2]$ and 4 is the absolute maximum value of $f(t)$ on $[-2, 2]$. We conclude that $0 \le f(t) \le 4$ for $|t| \le 2$ and hence, $0 \le f(x)f(y) \le 4^2$ or $0 \le x^2(4-x^2)y^2(4-y^2) \le 16$.

3. First we show that $x(1-x) \le \frac{1}{4}$ for all x. Let $f(x) = x(1-x) = x - x^2$. Then $f'(x) = 1 - 2x$. This is 0 when $x = \frac{1}{2}$ and $f'(x) > 0$ for $x < \frac{1}{2}$, $f'(x) < 0$ for $x > \frac{1}{2}$, so the absolute maximum of f is $f\left(\frac{1}{2}\right) = \frac{1}{4}$. Thus, $x(1-x) \le \frac{1}{4}$ for all x.
Now suppose that the given assertion is false, that is, $a(1-b) > \frac{1}{4}$ and $b(1-a) > \frac{1}{4}$. Multiply these inequalities: $a(1-b)b(1-a) > \frac{1}{16} \Rightarrow [a(1-a)][b(1-b)] > \frac{1}{16}$. But we know that $a(1-a) \le \frac{1}{4}$ and $b(1-b) \le \frac{1}{4} \Rightarrow [a(1-a)][b(1-b)] \le \frac{1}{16}$. Thus, we have a contradiction, so the given assertion is proved.

4. Let $P(a, 1-a^2)$ be the point of contact. The equation of the tangent line at P is $y - (1-a^2) = (-2a)(x-a)$ $\Rightarrow y - 1 + a^2 = -2ax + 2a^2 \Rightarrow y = -2ax + a^2 + 1$. To find the x-intercept, put $y = 0$: $2ax = a^2 + 1$ $\Rightarrow x = \dfrac{a^2+1}{2a}$. To find the y-intercept, put $x = 0$: $y = a^2 + 1$. Therefore, the area of the triangle is $\dfrac{1}{2}\left(\dfrac{a^2+1}{2a}\right)(a^2+1) = \dfrac{(a^2+1)^2}{4a}$. Therefore, we minimize the function $A(a) = \dfrac{(a^2+1)^2}{4a}$, $0 < a \le 1$.
$A'(a) = \dfrac{(4a)2(a^2+1)(2a) - (a^2+1)^2(4)}{16a^2} = \dfrac{(a^2+1)[4a^2 - (a^2+1)]}{4a^2} = \dfrac{(a^2+1)(3a^2-1)}{4a^2}$. $A'(a) = 0$ when $3a^2 - 1 = 0 \Rightarrow a = \frac{1}{\sqrt{3}}$. $A'(a) < 0$ for $a < \frac{1}{\sqrt{3}}$, $A'(a) > 0$ for $a > \frac{1}{\sqrt{3}}$. So by the First Derivative Test, there is an absolute minimum when $a = \frac{1}{\sqrt{3}}$. The required point is $\left(\frac{1}{\sqrt{3}}, \frac{2}{3}\right)$.

5. Differentiating $x^2 + xy + y^2 = 12$ implicitly with respect to x gives $2x + y + x\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{2x+y}{x+2y}$. At a highest or lowest point, $\dfrac{dy}{dx} = 0 \Leftrightarrow y = -2x$. Substituting this into the original equation gives $x^2 + x(-2x) + (-2x)^2 = 12$, so $3x^2 = 12$ and $x = \pm 2$. If $x = 2$, then $y = -2x = -4$, and if $x = -2$ then $y = 4$. Thus, the highest and lowest points are $(-2, 4)$ and $(2, -4)$.

6. (a) $V'(t)$ is the rate of change of the volume of the water with respect to time. $H'(t)$ is the rate of change of the height of the water with respect to time. Since the volume and the height are increasing, $V'(t)$ and $H'(t)$ are positive.

(b) $V'(t)$ is constant, so $V''(t)$ is zero (the slope of a constant function is 0).

(c) At first, the height H of the water increases quickly because the tank is narrow. But as the sphere widens, the rate of increase of the height slows down, reaching a minimum at $t = t_2$. Thus, the height is increasing at a decreasing rate on $(0, t_2)$, so its graph is concave downward and $H''(t_1) < 0$. As the sphere narrows for $t > t_2$, the rate of increase of the height begins to increase, and the graph of H is concave upward. Thereore, $H''(t_2) = 0$ and $H''(t_3) > 0$.

7. $f(x) = \dfrac{1}{1+|x|} + \dfrac{1}{1+|x-2|}$

$$= \begin{cases} \dfrac{1}{1-x} + \dfrac{1}{1-(x-2)} & \text{if } x < 0 \\ \dfrac{1}{1+x} + \dfrac{1}{1-(x-2)} & \text{if } 0 \le x < 2 \\ \dfrac{1}{1+x} + \dfrac{1}{1+(x-2)} & \text{if } x \ge 2 \end{cases} \Rightarrow f'(x) = \begin{cases} \dfrac{1}{(1-x)^2} + \dfrac{1}{(3-x)^2} & \text{if } x < 0 \\ \dfrac{-1}{(1+x)^2} + \dfrac{1}{(3-x)^2} & \text{if } 0 < x < 2 \\ \dfrac{-1}{(1+x)^2} - \dfrac{1}{(x-1)^2} & \text{if } x > 2 \end{cases}$$

We see that $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 2$. For $0 < x < 2$, we have

$f'(x) = \dfrac{1}{(3-x)^2} - \dfrac{1}{(x+1)^2} = \dfrac{(x^2+2x+1)-(x^2-6x+9)}{(3-x)^2(x+1)^2} = \dfrac{8(x-1)}{(3-x)^2(x+1)^2}$, so $f'(x) < 0$ for $0 < x < 1$, $f'(1) = 0$ and $f'(x) > 0$ for $1 < x < 2$. We have shown that $f'(x) > 0$ for $x < 0$; $f'(x) < 0$ for $0 < x < 1$; $f'(x) > 0$ for $1 < x < 2$; and $f'(x) < 0$ for $x > 2$. Therefore, by the First Derivative Test, the local maxima of f are at $x = 0$ and $x = 2$, where f takes the value $\frac{4}{3}$. Therefore, $\frac{4}{3}$ is the absolute maximum value of f.

8. If $f''(x) > 0$ for all x, then f' is increasing on $(-\infty, \infty)$, so $f'(0)$ must be greater than $f'(-1)$. But $f'(0) = 0 < \frac{1}{2} = f'(-1)$, so such a function cannot exist.

9. $A = (x_1, x_1^2)$ and $B = (x_2, x_2^2)$, where x_1 and x_2 are the solutions of the quadratic equation $x^2 = mx + b$. Let $P = (x, x^2)$ and set $A_1 = (x_1, 0)$, $B_1 = (x_2, 0)$, and $P_1 = (x, 0)$. Let $f(x)$ denote the area of triangle PAB. Then $f(x)$ can be expressed in terms of the areas of three trapezoids as follows:

$$\begin{aligned} f(x) &= \text{area}(A_1ABB_1) - \text{area}(A_1APP_1) - \text{area}(B_1BPP_1) \\ &= \tfrac{1}{2}\left(x_1^2 + x_2^2\right)(x_2 - x_1) - \tfrac{1}{2}\left(x_1^2 + x^2\right)(x - x_1) - \tfrac{1}{2}\left(x^2 + x_2^2\right)(x_2 - x) \end{aligned}$$

After expanding and canceling terms, we get

$f(x) = \frac{1}{2}\left(x_2x_1^2 - x_1x_2^2 - xx_1^2 + x_1x^2 - x_2x^2 + xx_2^2\right) = \frac{1}{2}\left[x_1^2(x_2 - x) + x_2^2(x - x_1) + x^2(x_1 - x_2)\right]$

$f'(x) = \frac{1}{2}\left[-x_1^2 + x_2^2 + 2x(x_1 - x_2)\right]$. $f''(x) = \frac{1}{2}[2(x_1 - x_2)] = x_1 - x_2 < 0$ since $x_2 > x_1$.

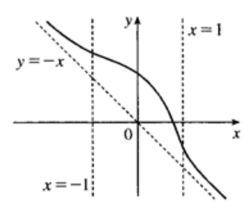
$f'(x) = 0 \;\Rightarrow\; 2x(x_1 - x_2) = x_1^2 - x_2^2 \;\Rightarrow\; x_P = \frac{1}{2}(x_1 + x_2)$.

$$\begin{aligned}
f(x_P) &= \tfrac{1}{2}\left(x_1^2\left[\tfrac{1}{2}(x_2 - x_1)\right] + x_2^2\left[\tfrac{1}{2}(x_2 - x_1)\right] + \tfrac{1}{4}(x_1 + x_2)^2(x_1 - x_2)\right) \\
&= \tfrac{1}{2}\left[\tfrac{1}{2}(x_2 - x_1)\left(x_1^2 + x_2^2\right) - \tfrac{1}{4}(x_2 - x_1)(x_1 + x_2)^2\right] \\
&= \tfrac{1}{8}(x_2 - x_1)\left[2\left(x_1^2 + x_2^2\right) - \left(x_1^2 + 2x_1x_2 + x_2^2\right)\right] \\
&= \tfrac{1}{8}(x_2 - x_1)\left(x_1^2 - 2x_1x_2 + x_2^2\right) = \tfrac{1}{8}(x_2 - x_1)(x_1 - x_2)^2 = \tfrac{1}{8}(x_2 - x_1)(x_2 - x_1)^2 \\
&= \tfrac{1}{8}(x_2 - x_1)^3
\end{aligned}$$

To put this in terms of m and b, we solve the system $y = x_1^2$ and $y = mx_1 + b$, giving us $x_1^2 - mx_1 - b = 0 \;\Rightarrow$ $x_1 = \frac{1}{2}\left(m - \sqrt{m^2 + 4b}\right)$. Similarly, $x_2 = \frac{1}{2}\left(m + \sqrt{m^2 + 4b}\right)$. The area is then $\frac{1}{8}(x_2 - x_1)^3 = \frac{1}{8}\left(\sqrt{m^2 + 4b}\right)^3$, and is attained at the point $P\left(x_P, x_P^2\right) = P\left(\frac{1}{2}m, \frac{1}{4}m^2\right)$.

Note: Another way to get an expression for $f(x)$ is to use the formula for an area of a triangle in terms of the coordinates of the vertices: $f(x) = \frac{1}{2}\left[\left(x_2x_1^2 - x_1x_2^2\right) + \left(x_1x^2 - xx_1^2\right) + \left(xx_2^2 - x_2x^2\right)\right]$.

10. If $f'(x) < 0$ for all x, $f''(x) > 0$ for $|x| > 1$, $f''(x) < 0$ for $|x| < 1$, and $\lim_{x\to\pm\infty}[f(x) + x] = 0$, then f is decreasing everywhere, concave up on $(-\infty, -1)$ and $(1, \infty)$, concave down on $(-1, 1)$, and approaches the line $y = -x$ as $x \to \pm\infty$. An example of such a graph is sketched.



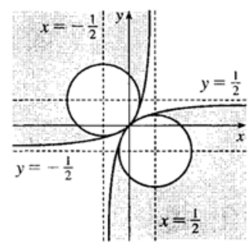
11. $f(x) = \left(a^2 + a - 6\right)\cos 2x + (a - 2)x + \cos 1 \;\Rightarrow\; f'(x) = -\left(a^2 + a - 6\right)\sin 2x\,(2) + (a - 2)$.

The derivative exists for all x, so the only possible critical points will occur where $f'(x) = 0 \;\Leftrightarrow$ $2(a - 2)(a + 3)\sin 2x = a - 2 \;\Leftrightarrow\;$ either $a = 2$ or $2(a + 3)\sin 2x = 1$, with the latter implying that $\sin 2x = \dfrac{1}{2(a + 3)}$. Since the range of $\sin 2x$ is $[-1, 1]$, this equation has no solution whenever either $\dfrac{1}{2(a + 3)} > 1$ or $\dfrac{1}{2(a + 3)} > 1$. Solving these inequalities, we get $-\frac{7}{2} < a < -\frac{5}{2}$.

12. To sketch the region $\{(x, y) \mid 2xy \le |x - y| \le x^2 + y^2\}$, we consider two cases.

Case 1: $x \ge y$ This is the case in which (x, y) lies on or below the line $y = x$. The double inequality becomes $2xy \le x - y \le x^2 + y^2$. The right-hand inequality holds if and only if $x^2 - x + y^2 + y \ge 0 \iff \left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 \ge \frac{1}{2} \iff (x, y)$ lies on or outside the circle with radius $\frac{1}{\sqrt{2}}$ centered at $\left(\frac{1}{2}, -\frac{1}{2}\right)$. The left-hand inequality holds if and only if $2xy - x + y \le 0 \iff xy - \frac{1}{2}x + \frac{1}{2}y \le 0 \iff \left(x + \frac{1}{2}\right)\left(y - \frac{1}{2}\right) \le -\frac{1}{4} \iff (x, y)$ lies on or below the hyperbola $\left(x + \frac{1}{2}\right)\left(y - \frac{1}{2}\right) = -\frac{1}{4}$, which passes through the origin and approaches the lines $y = \frac{1}{2}$ and $x = -\frac{1}{2}$ asymptotically.

Case 2: $y \ge x$ This is the case in which (x, y) lies on or above the line $y = x$. The double inequality becomes $2xy \le y - x \le x^2 + y^2$. The right-hand inequality holds if and only if $x^2 + x + y^2 - y \ge 0 \iff \left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 \ge \frac{1}{2} \iff (x, y)$ lies on or outside the circle of radius $\frac{1}{\sqrt{2}}$ centered at $\left(-\frac{1}{2}, \frac{1}{2}\right)$. The left-hand inequality holds if and only if $2xy + x - y \le 0 \iff xy + \frac{1}{2}x - \frac{1}{2}y \le 0 \iff \left(x - \frac{1}{2}\right)\left(y + \frac{1}{2}\right) \le -\frac{1}{4} \iff (x, y)$ lies on or above the left-hand branch of the hyperbola $\left(x - \frac{1}{2}\right)\left(y + \frac{1}{2}\right) = -\frac{1}{4}$, which passes through the origin and approaches the lines $y = -\frac{1}{2}$ and $x = \frac{1}{2}$ asymptotically. Therefore, the region of interest consists of the points on or above the left branch of the hyperbola $\left(x - \frac{1}{2}\right)\left(y + \frac{1}{2}\right) = -\frac{1}{4}$ that are on or outside the circle $\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$, together with the points on or below the right branch of the hyperbola $\left(x + \frac{1}{2}\right)\left(y - \frac{1}{2}\right) = -\frac{1}{4}$ that are on or outside the circle $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1}{2}$. Note that the inequalities are unchanged when x and y are interchanged, so the region is symmetric about the line $y = x$. So we need only have analyzed case 1 and then reflected that region about the line $y = x$, instead of considering case 2.



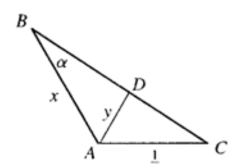
13. (a) Let $y = |AD|$, $x = |AB|$, and $1/x = |AC|$, so that $|AB| \cdot |AC| = 1$. We compute the area $\mathcal{A}$ of $\triangle ABC$ in two ways. First,

$\mathcal{A} = \frac{1}{2}|AB||AC|\sin\frac{2\pi}{3} = \frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$. Second,

$\mathcal{A} = (\text{area of } \triangle ABD) + (\text{area of } \triangle ACD)$

$= \frac{1}{2}|AB||AD|\sin\frac{\pi}{3} + \frac{1}{2}|AD||AC|\sin\frac{\pi}{3}$

$= \frac{1}{2}xy\frac{\sqrt{3}}{2} + \frac{1}{2}y(1/x)\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}y(x + 1/x)$



Equating the two expressions for the area, we get $\frac{\sqrt{3}}{4}y\left(x + \frac{1}{x}\right) = \frac{\sqrt{3}}{4} \iff y = \dfrac{1}{x + 1/x} = \dfrac{x}{x^2 + 1}$, $x > 0$.

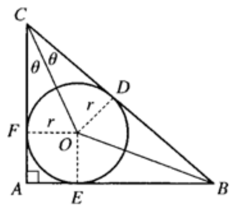
Another Method: Use the Law of Sines on the triangles ABD and ABC. In $\triangle ABD$, we have $\angle A + \angle B + \angle D = 180° \iff 60° + \alpha + \angle D = 180° \iff \angle D = 120° - \alpha$. Thus,

$\dfrac{x}{y} = \dfrac{\sin(120° - \alpha)}{\sin\alpha} = \dfrac{\sin 120° \cos\alpha - \cos 120° \sin\alpha}{\sin\alpha} = \dfrac{\frac{\sqrt{3}}{2}\cos\alpha + \frac{1}{2}\sin\alpha}{\sin\alpha} \Rightarrow \dfrac{x}{y} = \frac{\sqrt{3}}{2}\cot\alpha + \frac{1}{2}$, and by a similar argument with $\triangle ABC$, $\frac{\sqrt{3}}{2}\cot\alpha = x^2 + \frac{1}{2}$. Eliminating $\cot\alpha$ gives $\dfrac{x}{y} = \left(x^2 + \frac{1}{2}\right) + \frac{1}{2} \Rightarrow$

$y = \dfrac{x}{x^2 + 1}$, $x > 0$.

(b) We differentiate our expression for y with respect to x to find the maximum:
$\dfrac{dy}{dx}=\dfrac{(x^2+1)-x\,(2x)}{(x^2+1)^2}=\dfrac{1-x^2}{(x^2+1)^2}=0$ when $x=1$. This indicates a maximum by the First Derivative Test, since $y'(x)>0$ for $0<x<1$ and $y'(x)<0$ for $x>1$, so the maximum value of y is $y(1)=\frac{1}{2}$.

14. (a)



From geometry, two tangents to a circle from a given point have the same length, so $|CF|=|CD|$, $|AE|=|AF|$, and $|BD|=|BE|$. Thus,

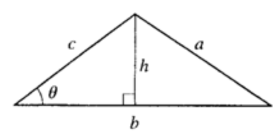
$$\begin{aligned}|CD|&=\tfrac{1}{2}\left(|CD|+|CF|\right)=\tfrac{1}{2}\left[(|BC|-|BD|)+(|AC|-|AF|)\right]\\&=\tfrac{1}{2}\left[|AC|+|BC|-(|AF|+|BD|)\right]\\&=\tfrac{1}{2}\left[|AC|+|BC|-(|AE|+|BE|)\right]=\tfrac{1}{2}\left(|AC|+|BC|-|AB|\right)\end{aligned}$$

(b) Using the result from part (a) and the fact that $a=|BC|$, we have $\tan\theta=\dfrac{r}{|CD|}$ $\Rightarrow$
$\dfrac{r}{\tan\theta}=|CD|=\frac{1}{2}\left(|AC|+|BC|-|AB|\right)=\frac{1}{2}\left(a\cos 2\theta+a-a\sin 2\theta\right)$ $\Leftrightarrow$

$$\begin{aligned}r&=\tfrac{1}{2}a\tan\theta\left(2\cos^2\theta-2\sin\theta\cos\theta\right)=\tfrac{1}{2}a\sin\theta\,(2\cos\theta-2\sin\theta)\\&=\tfrac{1}{2}a\left(2\sin\theta\cos\theta-2\sin^2\theta\right)=\tfrac{1}{2}a\,(\sin 2\theta+\cos 2\theta-1)\end{aligned}$$

(c) We differentiate r with respect to θ and set $dr/d\theta=0$ to find the maximum values:
$dr/d\theta=\frac{1}{2}a\,(2\cos 2\theta-2\sin 2\theta)=a\,(\cos 2\theta-\sin 2\theta)$. Since $0<\theta<\frac{\pi}{4}$, $dr/d\theta=0$ $\Leftrightarrow$ $\cos 2\theta=\sin 2\theta$ $\Leftrightarrow$ $1=\tan 2\theta$ $\Leftrightarrow$ $2\theta=\frac{\pi}{4}$ $\Leftrightarrow$ $\theta=\frac{\pi}{8}$. This gives a maximum by the First Derivative Test, since $dr/d\theta>0$ for $0<\theta<\frac{\pi}{8}$, and $dr/d\theta>0$ for $\frac{\pi}{8}<\theta<\frac{\pi}{4}$. The maximum value is
$r\left(\frac{\pi}{8}\right)=\frac{1}{2}a\left(\sin\frac{\pi}{4}+\cos\frac{\pi}{4}-1\right)=\frac{1}{2}\left(\sqrt{2}-1\right)a\approx 0.207a$.

15. (a) $A=\frac{1}{2}bh$ with $\sin\theta=h/c$, so $A=\frac{1}{2}bc\sin\theta$. But A is a constant, so differentiating this equation with respect to t, we get $\dfrac{dA}{dt}=0=\dfrac{1}{2}\left[bc\cos\theta\dfrac{d\theta}{dt}+b\dfrac{dc}{dt}\sin\theta+\dfrac{db}{dt}c\sin\theta\right]$

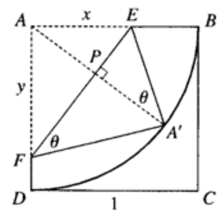


$\Rightarrow$ $bc\cos\theta\dfrac{d\theta}{dt}=-\sin\theta\left[b\dfrac{dc}{dt}+c\dfrac{db}{dt}\right]$ $\Rightarrow$ $\dfrac{d\theta}{dt}=-\tan\theta\left[\dfrac{1}{c}\dfrac{dc}{dt}+\dfrac{1}{b}\dfrac{db}{dt}\right]$.

(b) We use the Law of Cosines to get the length of side a in terms of those of b and c, and then we differentiate implicitly with respect to t: $a^2=b^2+c^2-2bc\cos\theta$ $\Rightarrow$
$2a\dfrac{da}{dt}=2b\dfrac{db}{dt}+2c\dfrac{dc}{dt}-2\left[bc\,(-\sin\theta)\dfrac{d\theta}{dt}+b\dfrac{dc}{dt}\cos\theta+\dfrac{db}{dt}c\cos\theta\right]$ $\Rightarrow$
$\dfrac{da}{dt}=\dfrac{1}{a}\left(b\dfrac{db}{dt}+c\dfrac{dc}{dt}+bc\sin\theta\dfrac{d\theta}{dt}-b\dfrac{dc}{dt}\cos\theta-c\dfrac{db}{dt}\cos\theta\right)$. Now we substitute our value of a from the Law of Cosines and the value of $d\theta/dt$ from part (a), and simplify (primes signify differentiation by t):

$$\begin{aligned}\frac{da}{dt}&=\frac{bb'+cc'+bc\sin\theta\left[-\tan\theta\left(c'/c+b'/b\right)\right]-\left(bc'+cb'\right)(\cos\theta)}{\sqrt{b^2+c^2-2bc\cos\theta}}\\&=\frac{bb'+cc'-\left[\sin^2\theta\left(bc'+cb'\right)+\cos^2\theta\left(bc'+cb'\right)\right]/\cos\theta}{\sqrt{b^2+c^2-2bc\cos\theta}}=\frac{bb'+cc'-\left(bc'+cb'\right)\sec\theta}{\sqrt{b^2+c^2-2bc\cos\theta}}\end{aligned}$$

16. Let $x = |AE|$, $y = |AF|$ as shown. The area of the triangle AEF is therefore $\mathcal{A} = \frac{1}{2}xy$. We need a relationship between x and y, so we can take the derivative $d\mathcal{A}/dx$ and find the maxima and minima of $\mathcal{A}$. Now let A' be the point on which A ends up after the fold has been performed, and let P be the intersection of AA' and EF. Note that AA' is perpendicular to EF since we are reflecting A through the line EF to get to A', and that $|AP| = |PA'|$ for the same reason. But $|AA'| = 1$, since AA' is a radius of the circle.



So since $|AP| + |PA'| = |AA'|$, we have $AP = \frac{1}{2}$. So another way to express the area of the triangle is $\mathcal{A} = \frac{1}{2}|EF||AP| = \frac{1}{2}\sqrt{x^2+y^2}\left(\frac{1}{2}\right) = \frac{1}{4}\sqrt{x^2+y^2}$. Equating the two expressions for $\mathcal{A}$, we get $\frac{1}{2}xy = \frac{1}{4}\sqrt{x^2+y^2} \;\Rightarrow\; x^2y^2 = \frac{1}{4}(x^2+y^2) \;\Rightarrow\; y^2(4x^2-1) = x^2 \;\Rightarrow\; y = x/\sqrt{4x^2-1}$. (Note that we could also have derived this result from the similarity of $\triangle A'PE$ and $\triangle A'FE$, that is, $\dfrac{\frac{1}{2}}{\frac{1}{2}\sqrt{4x^2-1}} = \dfrac{y}{x}$.) Now we can substitute and calculate $\dfrac{d\mathcal{A}}{dx}$: $\mathcal{A} = \dfrac{1}{2}\dfrac{x^2}{\sqrt{4x^2-1}} \;\Rightarrow$

$\dfrac{d\mathcal{A}}{dx} = \dfrac{1}{2}\left[\dfrac{\sqrt{4x^2-1}\,(2x) - x^2\left(\frac{1}{2}\right)(4x^2-1)^{-1/2}(8x)}{4x^2-1}\right]$. This is 0 when $2x\sqrt{4x^2-1} - 4x^3(4x^2-1)^{-1/2} = 0$ $\Leftrightarrow\; (4x^2-1) - 2x^2 = 0 \;\Leftrightarrow\; 2x^2 = 1 \;\Leftrightarrow\; x = \frac{1}{\sqrt{2}}$. So this is one possible value for an extremum. We must also test the endpoints of the interval over which x ranges. The largest value that x can attain is 1, and the smallest value of x occurs when $y = 1 \;\Leftrightarrow\; 1 = x\big/\sqrt{4x^2-1} \;\Leftrightarrow\; x^2 = 4x^2 - 1 \;\Leftrightarrow\; 3x^2 = 1 \;\Leftrightarrow$ $x = \frac{1}{\sqrt{3}}$. This will give the same value of $\mathcal{A}$ as will $x = 1$, since the geometric situation is the same (reflected through the line $y = x$). We calculate $\mathcal{A}\left(\frac{1}{\sqrt{2}}\right) = \dfrac{1}{2}\dfrac{(1/\sqrt{2})^2}{\sqrt{4(1/\sqrt{2})^2 - 1}} = \dfrac{1}{4}$, and

$\mathcal{A}(1) = \dfrac{1}{2}\left[\dfrac{1^2}{\sqrt{4(1)^2-1}}\right] = \dfrac{1}{2\sqrt{3}}$. So the maximum area is $\mathcal{A}(1) = \mathcal{A}\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{2\sqrt{3}}$ and the minimum area is $\mathcal{A}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4}$.

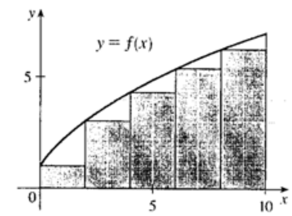
Another Method: Use the angle θ (see diagram above) as a variable:

$\mathcal{A} = \frac{1}{2}xy = \frac{1}{2}\left(\frac{1}{2}\sec\theta\right)\left(\frac{1}{2}\csc\theta\right) = \dfrac{1}{8\sin\theta\cos\theta} = \dfrac{1}{4\sin 2\theta}$. $\mathcal{A}$ is minimized when $\sin 2\theta$ is maximal, that is, when $\sin 2\theta = 1 \;\Rightarrow\; 2\theta = \frac{\pi}{2} \;\Rightarrow\; \theta = \frac{\pi}{4}$. Also note that $A'E = x = \frac{1}{2}\sec\theta \le 1 \;\Rightarrow\; \sec\theta \le 2 \;\Rightarrow$ $\cos\theta \ge \frac{1}{2} \;\Rightarrow\; \theta \le \frac{\pi}{3}$, and similarly, $A'F = y = \frac{1}{2}\csc\theta \le 1 \;\Rightarrow\; \csc\theta \le 2 \;\Rightarrow\; \sin\theta \le \frac{1}{2} \;\Rightarrow\; \theta \ge \frac{\pi}{6}$. As above, we find that $\mathcal{A}$ is maximized at these endpoints: $\mathcal{A}\left(\frac{\pi}{6}\right) = \dfrac{1}{4\sin\frac{\pi}{3}} = \dfrac{1}{2\sqrt{3}} = \dfrac{1}{4\sin\frac{2\pi}{3}} = \mathcal{A}\left(\frac{\pi}{3}\right)$; and minimized at $\theta = \frac{\pi}{4}$: $\mathcal{A}\left(\frac{\pi}{4}\right) = \dfrac{1}{4\sin\frac{\pi}{2}} = \dfrac{1}{4}$.

5 Integrals

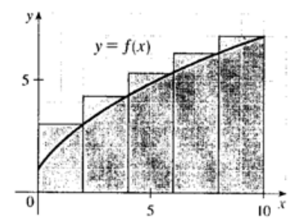
5.1 Areas and Distances

1. (a) Since f is *increasing*, we can obtain a *lower* estimate by using *left* endpoints.



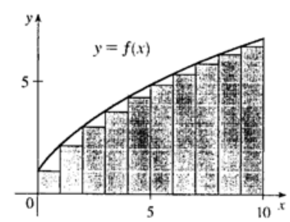
$$L_5 = \sum_{i=1}^{5} f(x_{i-1})\,\Delta x \quad [\Delta x = \tfrac{10-0}{5} = 2]$$
$$= f(x_0)\cdot 2 + f(x_1)\cdot 2 + f(x_2)\cdot 2 + f(x_3)\cdot 2 + f(x_4)\cdot 2$$
$$= 2\,[f(0) + f(2) + f(4) + f(6) + f(8)]$$
$$\approx 2\,(1 + 3 + 4.3 + 5.4 + 6.3) = 2\,(20) = 40$$

Since f is *increasing*, we can obtain an *upper* estimate by using *right* endpoints.

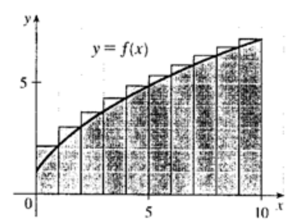


$$R_5 = \sum_{i=1}^{5} f(x_i)\,\Delta x$$
$$= 2\,[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]$$
$$= 2\,[f(2) + f(4) + f(6) + f(8) + f(10)]$$
$$\approx 2\,(3 + 4.3 + 5.4 + 6.3 + 7) = 2\,(26) = 52$$

(b) $$L_{10} = \sum_{i=1}^{10} f(x_{i-1})\,\Delta x \quad [\Delta x = \tfrac{10-0}{10} = 1]$$
$$= 1\,[f(x_0) + f(x_1) + \cdots + f(x_9)]$$
$$= f(0) + f(1) + \cdots + f(9)$$
$$\approx 1 + 2.1 + 3 + 3.7 + 4.3 + 4.9 + 5.4 + 5.8 + 6.3 + 6.7$$
$$= 43.2$$



$$R_{10} = \sum_{i=1}^{10} f(x_i)\,\Delta x = f(1) + f(2) + \cdots + f(10)$$
$$= L_{10} + 1\cdot f(10) - 1\cdot f(0) \quad \begin{bmatrix}\text{add rightmost rectangle,}\\ \text{subtract leftmost}\end{bmatrix}$$
$$= 43.2 + 7 - 1 = 49.2$$



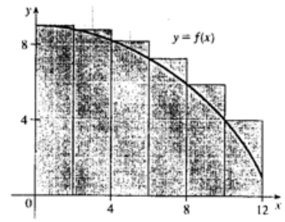
2. (a) (i) $L_6 = \sum_{i=1}^{6} f(x_{i-1})\,\Delta x \quad [\Delta x = \frac{12-0}{6} = 2]$

$= 2\,[f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]$

$= 2\,[f(0) + f(2) + f(4) + f(6) + f(8) + f(10)]$

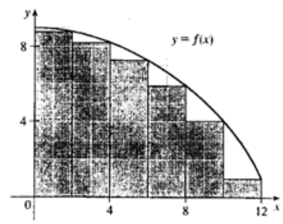
$\approx 2\,(9 + 8.8 + 8.2 + 7.3 + 5.9 + 4.1)$

$= 2\,(43.3) = 86.6$



(ii) $R_6 = L_6 + 2 \cdot f(12) - 2 \cdot f(0)$

$\approx 86.6 + 2\,(1) - 2\,(9) = 70.6$

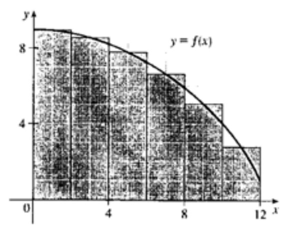


(iii) $M_6 = \sum_{i=1}^{6} f(x_i^*)\,\Delta x$

$= 2\,[f(1) + f(3) + f(5) + f(7) + f(9) + f(11)]$

$\approx 2\,(8.9 + 8.5 + 7.8 + 6.6 + 5.1 + 2.8)$

$= 2\,(39.7) = 79.4$



(b) Since f is *decreasing*, we obtain an *overestimate* by using *left* endpoints, that is, L_6.

(c) R_6 gives us an underestimate.

(d) M_6 gives the best estimate, since the area of each rectangle appears to be closer to the true area than the overestimates and underestimates in L_6 and R_6.

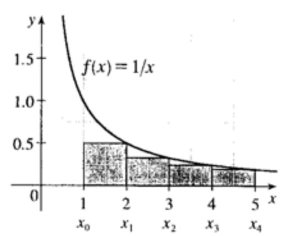
3. (a) $R_4 = \sum_{i=1}^{4} f(x_i)\,\Delta x \quad [\Delta x = \frac{5-1}{4} = 1]$

$= f(x_1) \cdot 1 + f(x_2) \cdot 1 + f(x_3) \cdot 1 + f(x_4) \cdot 1$

$= f(2) + f(3) + f(4) + f(5)$

$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60} = 1.28\overline{3}$

Since f is decreasing on $[1, 5]$, R_4 is an underestimate.



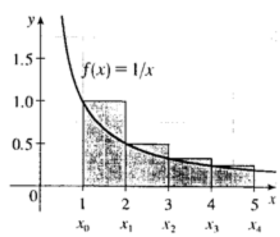
(b) $L_4 = \sum_{i=1}^{4} f(x_{i-1})\,\Delta x$

$= f(1) + f(2) + f(3) + f(4)$

$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} = 2.08\overline{3}$

L_4 is an overestimate. Alternatively, we could just add the leftmost rectangle and subtract the rightmost; that is,

$L_4 = R_4 + f(1) \cdot 1 - f(5) \cdot 1.$



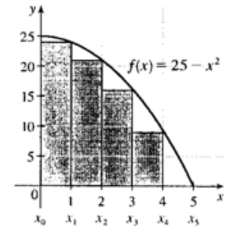
4. (a) $R_5 = \sum_{i=1}^{5} f(x_i)\,\Delta x \quad [\Delta x = \frac{5-0}{5} = 1]$

$= f(x_1)\cdot 1 + f(x_2)\cdot 1 + f(x_3)\cdot 1 + f(x_4)\cdot 1 + f(x_5)\cdot 1$

$= f(1) + f(2) + f(3) + f(4) + f(5)$

$= 24 + 21 + 16 + 9 + 0 = 70$

Since f is decreasing on $[0, 5]$, R_5 is an underestimate.

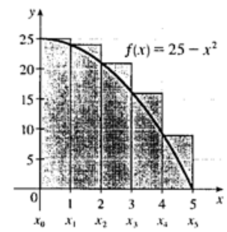


(b) $L_5 = \sum_{i=1}^{5} f(x_{i-1})\,\Delta x$

$= f(0) + f(1) + f(2) + f(3) + f(4)$

$= 25 + 24 + 21 + 16 + 9 = 95$

L_5 is an overestimate.



5. (a) $f(x) = x^3 + 2$ and $\Delta x = \dfrac{2-(-1)}{3} = 1 \quad \Rightarrow$

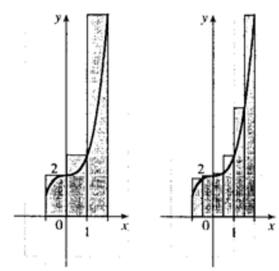
$R_3 = 1\cdot f(0) + 1\cdot f(1) + 1\cdot f(2) = 1\cdot 2 + 1\cdot 3 + 1\cdot 10 = 15.$

$\Delta x = \dfrac{2-(-1)}{6} = 0.5 \quad \Rightarrow$

$R_6 = 0.5\,[f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5) + f(2)]$

$= 0.5\,(1.875 + 2 + 2.125 + 3 + 5.375 + 10)$

$= 0.5\,(24.375) = 12.1875$

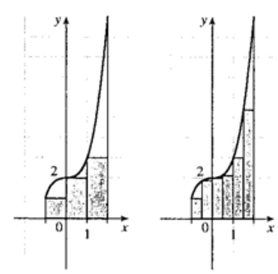


(b) $L_3 = 1\cdot f(-1) + 1\cdot f(0) + 1\cdot f(1) = 1\cdot 1 + 1\cdot 2 + 1\cdot 3 = 6.$

$L_6 = 0.5\,[f(-1) + f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5)]$

$= 0.5\,(1 + 1.875 + 2 + 2.125 + 3 + 5.375)$

$= 0.5\,(15.375) = 7.6875$



(c) $M_3 = 1\cdot f(-0.5) + 1\cdot f(0.5) + 1\cdot f(1.5)$

$= 1\cdot 1.875 + 1\cdot 2.125 + 1\cdot 5.375 = 9.375.$

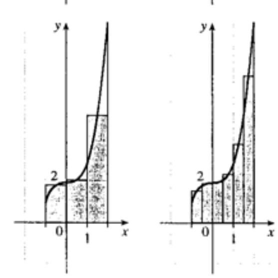
$M_6 = 0.5\,[f(-0.75) + f(-0.25) + f(0.25)$

$\quad + f(0.75) + f(1.25) + f(1.75)]$

$= 0.5\,(1.578125 + 1.984375 + 2.015625$

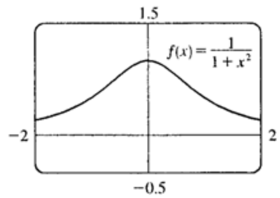
$\quad + 2.421875 + 3.953125 + 7.359375)$

$= 0.5\,(19.3125) = 9.65625$



(d) M_6 appears to be the best estimate.

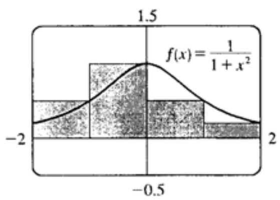
6. (a)

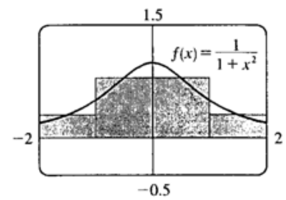


(b) $f(x) = 1/\left(1+x^2\right)$ and $\Delta x = \frac{2-(-2)}{4} = 1 \quad \Rightarrow$

(i) $R_4 = \sum_{i=1}^{4} f(x_i)\,\Delta x$
$= f(-1)\cdot 1 + f(0)\cdot 1$
$\quad + f(1)\cdot 1 + f(2)\cdot 1$
$= \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{5} = \frac{11}{5} = 2.2$

(ii) $M_4 = \sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x \quad \left[\overline{x}_i = \frac{1}{2}(x_{i-1}+x_i)\right]$
$= f(-1.5)\cdot 1 + f(-0.5)\cdot 1$
$\quad + f(0.5)\cdot 1 + f(1.5)\cdot 1$
$= \frac{4}{13} + \frac{4}{5} + \frac{4}{5} + \frac{4}{13} = \frac{144}{65} \approx 2.2154$





(c) $n = 8$, so $\Delta x = \frac{2-(-2)}{8} = \frac{1}{2}$.

$$R_8 = \tfrac{1}{2}\left[f(-1.5) + f(-1) + f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5) + f(2)\right]$$
$$= \tfrac{1}{2}\left[\tfrac{4}{13} + \tfrac{1}{2} + \tfrac{4}{5} + 1 + \tfrac{4}{5} + \tfrac{1}{2} + \tfrac{4}{13} + \tfrac{1}{5}\right] = \tfrac{287}{130} \approx 2.2077$$

$$M_8 = \tfrac{1}{2}\left[f(-1.75) + f(-1.25) + f(-0.75) + f(-0.25) + f(0.25) + f(0.75) + f(1.25) + f(1.75)\right]$$
$$= \tfrac{1}{2}\left[2\left(\tfrac{16}{65} + \tfrac{16}{41} + \tfrac{16}{25} + \tfrac{16}{17}\right)\right] \approx 2.2176$$

7. Here is one possible algorithm (ordered sequence of operations) for calculating the sums:

1 Let SUM = 0, X_MIN = 0, X_MAX = π, N = 10 (or 30 or 50, depending on which sum we are calculating), DELTA_X = (X_MAX − X_MIN) /N, and RIGHT_ENDPOINT = X_MIN + DELTA_X.

2 Repeat steps 2a, 2b in sequence until RIGHT_ENDPOINT > X_MAX.

2a Add sin (RIGHT_ENDPOINT) to SUM.

2b Add DELTA_X to RIGHT_ENDPOINT.

At the end of this procedure, (DELTA_X) · (SUM) is equal to the answer we are looking for. We find that $R_{10} = \frac{\pi}{10}\sum_{i=1}^{10}\sin\left(\frac{i\pi}{10}\right) \approx 1.9835$, $R_{30} = \frac{\pi}{30}\sum_{i=1}^{30}\sin\left(\frac{i\pi}{30}\right) \approx 1.9982$, and $R_{50} = \frac{\pi}{50}\sum_{i=1}^{50}\sin\left(\frac{i\pi}{50}\right) \approx 1.9993$. It appears that the exact area is 2.

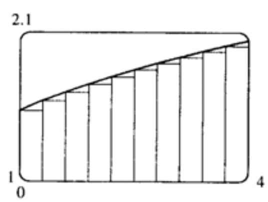
8. We can use the algorithm from Exercise 7 with X_MIN $= 1$, X_MAX $= 2$, and $1/(\text{RIGHT\_ENDPOINT})^2$ instead of sin (RIGHT_ENDPOINT) in step 2a. We find that $R_{10} = \frac{1}{10}\sum_{i=1}^{10}\frac{1}{(1+i/10)^2} \approx 0.4640$, $R_{30} = \frac{1}{30}\sum_{i=1}^{30}\frac{1}{(1+i/30)^2} \approx 0.4877$, and $R_{50} = \frac{1}{50}\sum_{i=1}^{50}\frac{1}{(1+i/50)^2} \approx 0.4926$. It appears that the exact area is $\frac{1}{2}$.

9. In Maple, we have to perform a number of steps before getting a numerical answer. After loading the student package [command: `with(student);`] we use the command `left_sum:=leftsum(x^(1/2),x=1..4,10` [or `30`, or `50`]`);` which gives us the expression in summation notation. To get a numerical approximation to the sum, we use `evalf(left_sum);`. Mathematica does not have a special command for these sums, so we must type them in manually. For example, the first left sum is given by `(3/10)*Sum[Sqrt[1+3(i-1)/10],{i,1,10}]`, and we use the `N` command on the resulting output to get a numerical approximation.

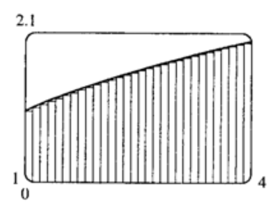
In Derive, we use the `LEFT_RIEMANN` command to get the left sums, but must define the right sums ourselves. (We can define a new function using `LEFT_RIEMANN` with k ranging from 1 to n instead of from 0 to $n-1$.)

(a) With $f(x) = \sqrt{x}$, $1 \le x \le 4$, the left sums are of the form $L_n = \frac{3}{n}\sum_{i=1}^{n}\sqrt{1+\frac{3(i-1)}{10}}$. Specifically, $L_{10} \approx 4.5148$, $L_{30} \approx 4.6165$, and $L_{50} \approx 4.6366$. The right sums are of the form $R_n = \frac{3}{n}\sum_{i=1}^{n}\sqrt{1+\frac{3i}{n}}$. Specifically, $R_{10} \approx 4.8148$, $R_{30} \approx 4.7165$, and $R_{50} \approx 4.6966$.

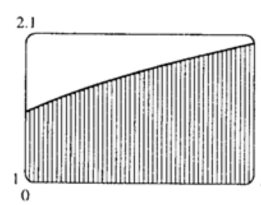
(b) In Maple, we use the `leftbox` and `rightbox` commands (with the same arguments as `leftsum` and `rightsum` above) to generate the graphs.



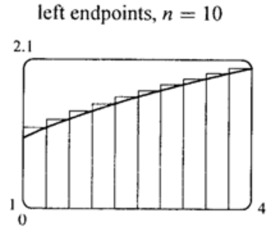
left endpoints, $n = 10$



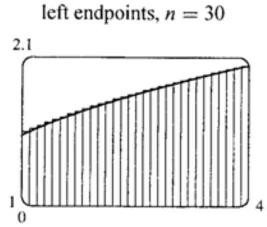
left endpoints, $n = 30$



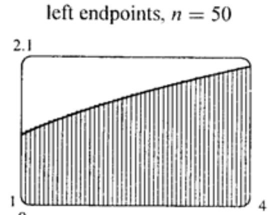
left endpoints, $n = 50$



right endpoints, $n = 10$



right endpoints, $n = 30$



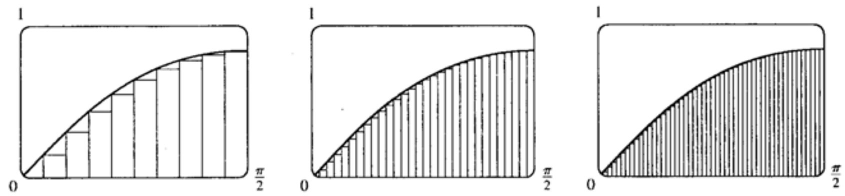
right endpoints, $n = 50$

(c) We know that since $\sqrt{x}$ is an increasing function on $(1, 4)$, all of the left sums are smaller than the actual area, and all of the right sums are larger than the actual area. Since the left sum with $n = 50$ is about $4.637 > 4.6$ and the right sum with $n = 50$ is about $4.697 < 4.7$, we conclude that $4.6 < L_{50} <$ actual area $< R_{50} < 4.7$, so the actual area is between 4.6 and 4.7.

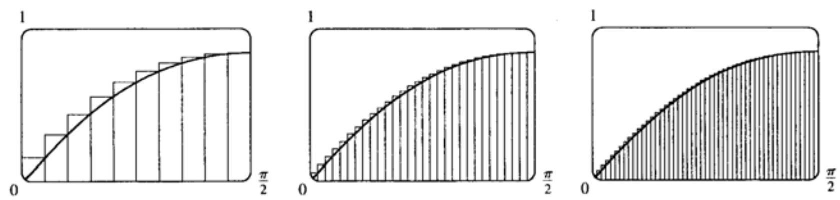
10. See the solution to Exercise 9 for the CAS commands for evaluating the sums.

(a) With $f(x) = \sin(\sin x)$, $0 \le x \le \frac{\pi}{2}$, the left sums are of the form $L_n = \dfrac{\pi}{2n} \sum\limits_{i=1}^{n} \sin\left(\sin \dfrac{\pi(i-1)}{2n}\right)$. In particular, $L_{10} \approx 0.8251$, $L_{30} \approx 0.8710$, and $L_{50} \approx 0.8799$. The right sums are of the form $R_n = \dfrac{\pi}{2n} \sum\limits_{i=1}^{n} \sin\left(\sin \dfrac{\pi i}{2n}\right)$. In particular, $R_{10} \approx 0.9573$, $R_{30} \approx 0.9150$, and $R_{50} \approx 0.9064$.

(b) In Maple, we use the `leftbox` and `rightbox` commands (with the same arguments as `leftsum` and `rightsum` above) to generate the graphs.



left endpoints, $n = 10$ left endpoints, $n = 30$ left endpoints, $n = 50$



right endpoints, $n = 10$ right endpoints, $n = 30$ right endpoints, $n = 50$

(c) We know that since $\sin(\sin x)$ is an increasing function on $\left(0, \frac{\pi}{2}\right)$ [this is true because its derivative, $-\cos(\sin x)(-\cos x)$, is positive on that interval], all of the left sums are smaller than the actual area, and all of the right sums are larger than the actual area. Since the left sum with $n = 50$ is about $0.8799 > 0.87$ and the right sum with $n = 50$ is about $0.9064 < 0.91$, we conclude that $0.87 < L_{50} <$ actual area $< R_{50} < 0.91$, so the actual area is between 0.87 and 0.91.

11. Since v is an increasing function, L_6 will give us a lower estimate and R_6 will give us an upper estimate.

$L_6 = (0 \text{ ft/s})(0.5 \text{ s}) + (6.2)(0.5) + (10.8)(0.5) + (14.9)(0.5) + (18.1)(0.5) + (19.4)(0.5)$

$\quad = 0.5(69.4) = 34.7 \text{ ft}$

$R_6 = 0.5(6.2 + 10.8 + 14.9 + 18.1 + 19.4 + 20.2) = 0.5(89.6) = 44.8 \text{ ft}$

12. We can find an upper estimate by using the final velocity for each time interval. Thus, the distance d traveled after 62 seconds can be approximated by

$d = \sum_{i=1}^{6} v(t_i)\,\Delta t_i = (185 \text{ ft/s})(10 \text{ s}) + 319 \cdot 5 + 447 \cdot 5 + 742 \cdot 12 + 1325 \cdot 27 + 1445 \cdot 3 = 54{,}694 \text{ ft}$

13. For a decreasing function, using left endpoints gives us an overestimate and using right endpoints results in an underestimate. We will use M_6 to get an estimate. $\Delta t = 1$, so

$$\begin{aligned} M_6 &= 1\,[v(0.5) + v(1.5) + v(2.5) + v(3.5) + v(4.5) + v(5.5)] \\ &\approx 55 + 40 + 28 + 18 + 10 + 4 = 155 \text{ ft} \end{aligned}$$

For a very rough check on the above calculation, we can draw a line from $(0, 70)$ to $(6, 0)$ and calculate the area of the triangle: $\frac{1}{2}(70)(6) = 210$. This is clearly an overestimate, so our midpoint estimate of 155 is reasonable.

14. For an increasing function, using left endpoints gives us an underestimate and using right endpoints results in an overestimate. We will use M_6 to get an estimate. $\Delta t = \frac{30-0}{6} = 5\text{ s} = \frac{5}{3600}\text{ h} = \frac{1}{720}\text{ h}$.

$$M_6 = \tfrac{1}{720}\left[v(2.5) + v(7.5) + v(12.5) + v(17.5) + v(22.5) + v(27.5)\right]$$
$$= \tfrac{1}{720}(31.25 + 66 + 88 + 103.5 + 113.75 + 119.25) = \tfrac{1}{720}(521.75) \approx 0.725$$

For a very rough check on the above calculation, we can draw a line from $(0, 0)$ to $(30, 120)$ and calculate the area of the triangle: $\frac{1}{2}(30)(120) = 1800$. Divide by 3600 to get 0.5, which is clearly an underestimate, making our midpoint estimate of 0.725 seem reasonable. Of course, answers will vary due to different readings of the graph.

15. $f(x) = \sqrt[3]{x},\ 0 \le x \le 8 \quad\Rightarrow\quad \Delta x = \dfrac{8-0}{n} = \dfrac{8}{n},\ x_i = 0 + i\,\Delta x = \dfrac{8i}{n} \quad\Rightarrow$

$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x = \displaystyle\sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \sqrt[3]{\frac{8i}{n}} \cdot \frac{8}{n}$. Thus,

$A = \displaystyle\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{8}{n} \sum_{i=1}^{n} \sqrt[3]{\frac{8i}{n}}$.

16. $f(x) = 5 + \sqrt[3]{x},\ 1 \le x \le 8 \quad\Rightarrow\quad \Delta x = \dfrac{8-1}{n} = \dfrac{7}{n},\ x_i = 1 + i\,\Delta x = 1 + \dfrac{7i}{n} \quad\Rightarrow$

$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x = \displaystyle\sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \left(5 + \sqrt[3]{1 + \frac{7i}{n}}\right) \cdot \frac{7}{n}$. Thus,

$A = \displaystyle\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{7}{n} \sum_{i=1}^{n} \left(5 + \sqrt[3]{1 + \frac{7i}{n}}\right)$.

17. $f(x) = x + \sin x,\ \pi \le x \le 2\pi \quad\Rightarrow\quad \Delta x = \dfrac{2\pi - \pi}{n} = \dfrac{\pi}{n},\ x_i = \pi + i\,\Delta x = \pi + \dfrac{\pi i}{n} \quad\Rightarrow$

$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x = \displaystyle\sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \left[\pi + \frac{\pi i}{n} + \sin\left(\pi + \frac{\pi i}{n}\right)\right] \cdot \frac{\pi}{n}$. Thus,

$A = \displaystyle\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{\pi}{n} \sum_{i=1}^{n} \left[\pi + \frac{\pi i}{n} + \sin\left(\pi + \frac{\pi i}{n}\right)\right]$.

18. The two most obvious ways to interpret $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1 + \frac{3i}{n}}$ are: as the area of the region under the graph of $\sqrt{x}$ on the interval $[1, 4]$ [see Exercise 9(a)], or as the area of the region lying under the graph of $\sqrt{x+1}$ on the interval $[0, 3]$, since for $y = \sqrt{x+1}$ on $[0, 3]$ with partition points $x_i = \dfrac{i}{3n}$, $\Delta x = \dfrac{1}{3n}$ and $x_i^* = x_i$, the expression for the area is $A = \displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \frac{3i}{n}}\left(\frac{3}{n}\right)$.

19. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan\frac{\pi i}{4n}$ can be interpreted as the area of the region lying under the graph of $y = \tan x$ on the interval $\left[0, \frac{\pi}{4}\right]$, since for $y = \tan x$ on $\left[0, \frac{\pi}{4}\right]$ with partition points $x_i = \left(\dfrac{\pi}{4}\right)\dfrac{i}{n}$, $\Delta x = \dfrac{\pi}{4n}$ and $x_i^* = x_i$, the expression for the area is $A = \displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \tan\left(\frac{\pi i}{4n}\right)\frac{\pi}{4n}$. Note that this answer is not unique, since the expression for the area is the same for the function $y = \tan(k\pi + x)$ on the interval $\left[k\pi, k\pi + \frac{\pi}{4}\right]$ where k is any integer.

20. (a) $\Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{i}{n}$. $A = \lim\limits_{n\to\infty} R_n = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\,\Delta x = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \left(\dfrac{i}{n}\right)^3 \cdot \dfrac{1}{n}$.

(b) $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \dfrac{i^3}{n^3} \cdot \dfrac{1}{n} = \lim\limits_{n\to\infty} \dfrac{1}{n^4} \sum\limits_{i=1}^{n} i^3 = \lim\limits_{n\to\infty} \dfrac{1}{n^4} \left[\dfrac{n(n+1)}{2}\right]^2 = \lim\limits_{n\to\infty} \dfrac{(n+1)^2}{4n^2} = \dfrac{1}{4} \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^2 = \dfrac{1}{4}$

21. (a) $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{2i}{n}$.

$A = \lim\limits_{n\to\infty} R_n = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\,\Delta x = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \left(\dfrac{2i}{n}\right)^5 \cdot \dfrac{2}{n} = \lim\limits_{n\to\infty} \dfrac{64}{n^6} \sum\limits_{i=1}^{n} i^5$.

(b) $\sum\limits_{i=1}^{n} i^5 = \dfrac{n^2(n+1)^2\left(2n^2+2n-1\right)}{12}$

(c) $$\lim_{n\to\infty} \frac{64}{n^6} \cdot \frac{n^2(n+1)^2\left(2n^2+2n-1\right)}{12} = \frac{64}{12} \lim_{n\to\infty} \frac{\left(n^2+2n+1\right)\left(2n^2+2n-1\right)}{n^2 \cdot n^2}$$
$$= \frac{16}{3} \lim_{n\to\infty} \left(1 + \frac{2}{n} + \frac{1}{n^2}\right)\left(2 + \frac{2}{n} - \frac{1}{n^2}\right) = \tfrac{16}{3} \cdot 1 \cdot 2 = \tfrac{32}{3}$$

22. (a) $y = f(x) = x^4 + 5x^2 + x,\ 2 \le x \le 7 \quad\Rightarrow\quad \Delta x = \dfrac{7-2}{n} = \dfrac{5}{n},\ x_i = 2 + i\,\Delta x = 2 + \dfrac{5i}{n} \quad\Rightarrow$

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{5}{n} \sum_{i=1}^{n} \left[\left(2 + \tfrac{5i}{n}\right)^4 + 5\left(2 + \tfrac{5i}{n}\right)^2 + \left(2 + \tfrac{5i}{n}\right)\right]$$

(b) $R_n = \dfrac{5}{n} \cdot \dfrac{4723n^4 + 7845n^3 + 3475n^2 - 125}{6n^3}$

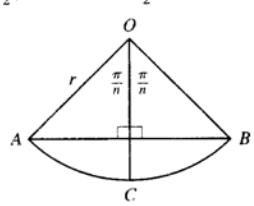
(c) $A = \lim\limits_{n\to\infty} R_n = \dfrac{23{,}615}{6} = 3935.8\overline{3}$.

23. $\Delta x = \dfrac{b-0}{n} = \dfrac{b}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{bi}{n}$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \cos\left(\frac{bi}{n}\right) \cdot \frac{b}{n} = \lim_{n\to\infty} \left[\frac{b \sin\left(b\left(\frac{1}{2n} + 1\right)\right)}{2n \sin\left(\frac{b}{2n}\right)} - \frac{b}{2n}\right] = \sin b$$

If $b = \frac{\pi}{2}$, then $A = \sin \frac{\pi}{2} = 1$.

24. (a)



The diagram shows one of the n congruent triangles, $\triangle AOB$, with central angle $2\pi/n$. O is the center of the circle and AB is one of the sides of the polygon. Radius OC is drawn so as to bisect $\angle AOB$. It follows that OC intersects AB at right angles and bisects AB. Thus, $\triangle AOB$ is divided into two right triangles with legs of length $r\sin(\pi/n)$ and $r\cos(\pi/n)$.

$\triangle AOB$ has area $2 \cdot \frac{1}{2}[r\sin(\pi/n)][r\cos(\pi/n)] = r^2\sin(\pi/n)\cos(\pi/n) = \frac{1}{2}r^2\sin(2\pi/n)$, so $A_n = n \cdot \text{area}(\triangle AOB) = \frac{1}{2}nr^2\sin(2\pi/n)$.

(b) $\lim\limits_{n\to\infty} A_n = \lim\limits_{n\to\infty} \frac{1}{2}nr^2\sin(2\pi/n) = \lim\limits_{n\to\infty} \dfrac{\sin(2\pi/n)}{2\pi/n}\pi r^2$. Let $\theta = \dfrac{2\pi}{n}$. Then as $n\to\infty$, $\theta\to 0$, so $\lim\limits_{n\to\infty} \dfrac{\sin(2\pi/n)}{2\pi/n}\pi r^2 = \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta}\pi r^2 = (1)\,\pi r^2 = \pi r^2$.

5.2 The Definite Integral

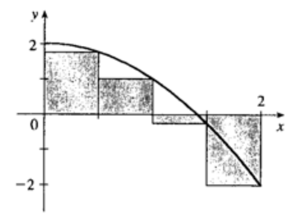
1. $R_4 = \sum_{i=1}^{4} f(x_i)\,\Delta x \quad [x_i^* = x_i \text{ is a right endpoint and } \Delta x = 0.5]$

$= 0.5\,[f(0.5) + f(1) + f(1.5) + f(2)] \quad [f(x) = 2 - x^2]$

$= 0.5\,[1.75 + 1 + (-0.25) + (-2)]$

$= 0.5\,(0.5) = 0.25$

The Riemann sum represents the area of the two rectangles above the x-axis minus the area of the two rectangles below the x-axis.



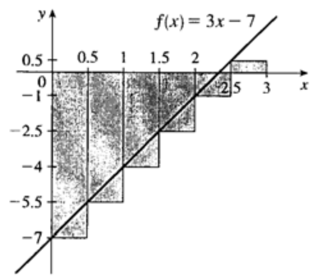
2. $L_6 = \sum_{i=1}^{6} f(x_{i-1})\,\Delta x \quad [x_i^* = x_{i-1} \text{ is a left endpoint and } \Delta x = 0.5]$

$= 0.5\,[f(0) + f(0.5) + f(1)$
$\qquad + f(1.5) + f(2) + f(2.5)] \quad [f(x) = 3x - 7]$

$= 0.5\,[-7 + (-5.5) + (-4) + (-2.5) + (-1) + 0.5]$

$= 0.5\,(-19.5) = -9.75$

The Riemann sum represents the area of the rectangle above the x-axis minus the area of the five rectangles below the x-axis.

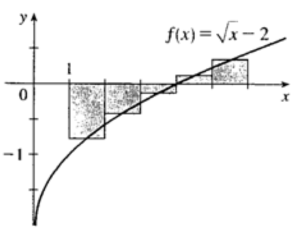


3. $M_5 = \sum_{i=1}^{5} f(\overline{x}_i)\,\Delta x \quad [x_i^* = \overline{x}_i = \frac{1}{2}(x_{i-1} + x_i) \text{ is a midpoint and } \Delta x = 1]$

$= 1\,[f(1.5) + f(2.5) + f(3.5)$
$\qquad + f(4.5) + f(5.5)] \quad [f(x) = \sqrt{x} - 2]$

≈ -0.856759

The Riemann sum represents the area of the two rectangles above the x-axis minus the area of the three rectangles below the x-axis.

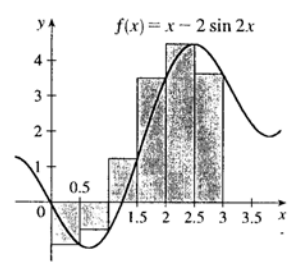


4. (a) $R_6 = \sum_{i=1}^{6} f(x_i)\,\Delta x \quad [x_i^* = x_i \text{ is a right endpoint and } \Delta x = 0.5]$

$= 0.5\,[f(0.5) + f(1) + f(1.5) + f(2)$
$\qquad + f(2.5) + f(3)] \quad [f(x) = x - 2\sin 2x]$

≈ 5.353254

The Riemann sum represents the area of the four rectangles above the x-axis minus the area of the two rectangles below the x-axis.



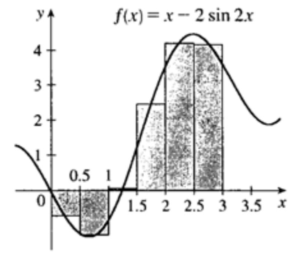
(b) $M_6 = \sum_{i=1}^{6} f(\overline{x}_i)\,\Delta x \quad [x_i^* = \overline{x}_i \text{ is a midpoint and } \Delta x = 0.5]$

$= 0.5\,[f(0.25) + f(0.75) + f(1.25) + f(1.75)$

$+ f(2.25) + f(2.75)] \quad [f(x) = x - 2\sin 2x]$

≈ 4.458461

The Riemann sum represents the area of the four rectangles above the x-axis minus the area of the two rectangles below the x-axis.



5. (a) Using the right endpoints to approximate $\int_0^8 f(x)\,dx$, we have

$$\sum_{i=1}^{4} f(x_i)\,\Delta x = 2\,[f(2) + f(4) + f(6) + f(8)] \approx 2\,(1 + 2 - 2 + 1) = 4.$$

(b) Using the left endpoints to approximate $\int_0^8 f(x)\,dx$, we have

$$\sum_{i=1}^{4} f(x_{i-1})\,\Delta x = 2\,[f(0) + f(2) + f(4) + f(6)] \approx 2\,(2 + 1 + 2 - 2) = 6.$$

(c) Using the midpoint of each subinterval to approximate $\int_0^8 f(x)\,dx$, we have

$$\sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x = 2\,[f(1) + f(3) + f(5) + f(7)] \approx 2\,(3 + 2 + 1 - 1) = 10.$$

6. (a) Using the right endpoints to approximate $\int_{-3}^{3} g(x)\,dx$, we have

$$\begin{aligned}\sum_{i=1}^{6} g(x_i)\,\Delta x &= 1\,[g(-2) + g(-1) + g(0) + g(1) + g(2) + g(3)]\\ &\approx 1 - 0.5 - 1.5 - 1.5 - 0.5 + 2.5 = -0.5\end{aligned}$$

(b) Using the left endpoints to approximate $\int_{-3}^{3} g(x)\,dx$, we have

$$\begin{aligned}\sum_{i=1}^{6} g(x_{i-1})\,\Delta x &= 1\,[g(-3) + g(-2) + g(-1) + g(0) + g(1) + g(2)]\\ &\approx 2 + 1 - 0.5 - 1.5 - 1.5 - 0.5 = -1\end{aligned}$$

(c) Using the midpoint of each subinterval to approximate $\int_{-3}^{3} g(x)\,dx$, we have

$$\begin{aligned}\sum_{i=1}^{6} g(\overline{x}_i)\,\Delta x &= 1\,[g(-2.5) + g(-1.5) + g(-0.5) + g(0.5) + g(1.5) + g(2.5)]\\ &\approx 1.5 + 0 - 1 - 1.75 - 1 + 0.5 = -1.75\end{aligned}$$

7. Since f is increasing, $L_5 \le \int_0^{25} f(x)\,dx \le R_5$.

$$\begin{aligned}\text{Lower estimate} = L_5 &= \sum_{i=1}^{5} f(x_{i-1})\,\Delta x = 5\,[f(0) + f(5) + f(10) + f(15) + f(20)]\\ &= 5\,(-42 - 37 - 25 - 6 + 15) = 5\,(-95) = -475\end{aligned}$$

$$\begin{aligned}\text{Upper estimate} = R_5 &= \sum_{i=1}^{5} f(x_i)\,\Delta x = 5\,[f(5) + f(10) + f(15) + f(20) + f(25)]\\ &= 5\,(-37 - 25 - 6 + 15 + 36) = 5\,(-17) = -85\end{aligned}$$

8. (a) Using the right endpoints to approximate $\int_0^6 f(x)\,dx$, we have

$$\sum_{i=1}^{3} f(x_i)\,\Delta x = 2\,[f(2) + f(4) + f(6)] = 2\,(8.3 + 2.3 - 10.5) = 0.2$$

(b) Using the left endpoints to approximate $\int_0^6 f(x)\,dx$, we have

$$\sum_{i=1}^{3} f(x_{i-1})\,\Delta x = 2\,[f(0) + f(2) + f(4)] = 2\,(9.3 + 8.3 + 2.3) = 39.8$$

(c) Using the midpoint of each interval to approximate $\int_0^6 f(x)\,dx$, we have

$$\sum_{i=1}^{3} f(\overline{x}_i)\,\Delta x = 2\,[f(1) + f(3) + f(5)] = 2\,(9 + 6.5 - 7.6) = 15.8.$$

The estimate using the right endpoints must be less than $\int_0^6 f(x)\,dx$, since if we take x_i^* to be the right endpoint x_i of each interval, then $f(x_i) \le f(x)$ for all x on $[x_{i-1}, x_i]$, which implies that $f(x_i)\,\Delta x \le \int_{x_{i-1}}^{x_i} f(x)\,dx$, and so the sum $\sum_{i=1}^{3} [f(x_i)\,\Delta x] \le \sum_{i=1}^{3} \left[\int_{x_{i-1}}^{x_i} f(x)\,dx\right] = \int_0^6 f(x)\,dx$. Similarly, if we take x_i^* to be the left endpoint x_{i-1} of each interval, then $f(x_{i-1}) \ge f(x)$ for all x on $[x_{i-1}, x_i]$, and so $\sum_{i=1}^{3} [f(x_{i-1})\,\Delta x] \ge \int_0^6 f(x)\,dx$. We cannot say anything about the midpoint estimate.

9. $\Delta x = (10 - 0)/5 = 2$, so the endpoints are 0, 2, 4, 6, 8, and 10, and the midpoints are 1, 3, 5, 7, and 9. The Midpoint Rule gives

$$\int_0^{10} \sin\sqrt{x}\,dx \approx \sum_{i=1}^{5} f(\overline{x}_i)\,\Delta x = 2\left(\sin\sqrt{1} + \sin\sqrt{3} + \sin\sqrt{5} + \sin\sqrt{7} + \sin\sqrt{9}\right) \approx 6.4643.$$

10. $\Delta x = (\pi - 0)/6 = \frac{\pi}{6}$, so the endpoints are $0, \frac{\pi}{6}, \frac{2\pi}{6}, \frac{3\pi}{6}, \frac{4\pi}{6}, \frac{5\pi}{6}$, and $\frac{6\pi}{6}$, and the midpoints are $\frac{\pi}{12}, \frac{3\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{9\pi}{12}$, and $\frac{11\pi}{12}$. The Midpoint Rule gives

$$\int_0^{\pi} \sec(x/3)\,dx \approx \sum_{i=1}^{6} f(\overline{x}_i)\,\Delta x = \tfrac{\pi}{6}\left(\sec\tfrac{\pi}{36} + \sec\tfrac{3\pi}{36} + \sec\tfrac{5\pi}{36} + \sec\tfrac{7\pi}{36} + \sec\tfrac{9\pi}{36} + \sec\tfrac{11\pi}{36}\right) \approx 3.9379.$$

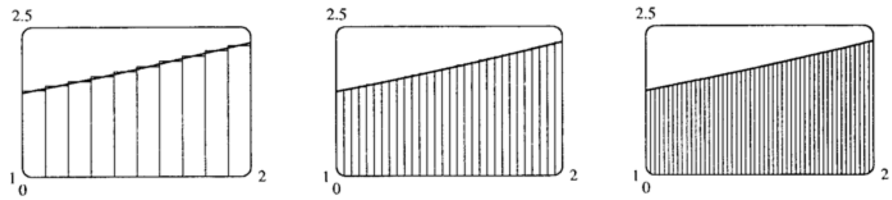
11. $\Delta x = (2 - 1)/10 = 0.1$, so the endpoints are $1.0, 1.1, \ldots, 2.0$ and the midpoints are $1.05, 1.15, \ldots, 1.95$. The Midpoint Rule gives

$$\int_1^2 \sqrt{1+x^2}\,dx \approx \sum_{i=1}^{10} f(\overline{x}_i)\,\Delta x = 0.1\left[\sqrt{1+(1.05)^2} + \sqrt{1+(1.15)^2} + \cdots + \sqrt{1+(1.95)^2}\right] \approx 1.8100.$$

12. $\Delta x = (4 - 2)/4 = 0.5$, so the endpoints are 2, 2.5, 3, 3.5, and 4, and the midpoints are 2.25, 2.75, 3.25, and 3.75. The Midpoint Rule gives

$$\begin{aligned}\int_2^4 \frac{x}{x^2+1}\,dx &\approx \sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x \qquad [f(x) = x/(x^2+1)]\\ &= 0.5\,[f(2.25) + f(2.75) + f(3.25) + f(3.75)] \approx 0.6112\end{aligned}$$

13. In Maple, we use the command `with(student);` to load the sum and box commands, then `m:=middlesum(sqrt(1+x^2),x=1..2,10);` which gives us the sum in summation notation, then `M:=evalf(m);` which gives $M_{10} \approx 1.81001414$, confirming the result of Exercise 11. The command `middlebox(sqrt(1+x^2),x=1..2,10);` generates the graph. Repeating for $n = 20$ and $n = 30$ gives $M_{20} \approx 1.81007263$ and $M_{30} \approx 1.81008347$.



14. See the solution to Exercise 5.1.7 for a possible algorithm to calculate the sums. With $\Delta x = 0.01$ and subinterval endpoints $1, 1.01, 1.02, \ldots, 1.99, 2$, we calculate that the left Riemann sum is $L_{100} = \sum\limits_{i=1}^{100} \sqrt{1 + (x_{i-1})^2}\,\Delta x \approx 1.80598$, and the right Riemann sum is $R_{100} = \sum\limits_{i=1}^{100} \sqrt{1 + (x_i)^2}\,\Delta x \approx 1.81420$, so since $\sqrt{1+x^2}$ is an increasing function, we must have $L_{100} \le \int_1^2 \sqrt{1+x^2}\,dx \le R_{100}$, so $1.805 < L_{100} \le \int_1^2 \sqrt{1+x^2}\,dx \le R_{100} < 1.815$.
Therefore, the approximate value 1.8100 in Exercise 11 must be accurate to two decimal places.

15. On $[0, \pi]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} x_i \sin x_i \,\Delta x = \int_0^\pi (x \sin x)\,dx$.

16. On $[1, 5]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \dfrac{x_i}{1+x_i}\,\Delta x = \displaystyle\int_1^5 \frac{x}{1+x}\,dx$.

17. On $[0, 1]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \left[2\left(x_i^*\right)^2 - 5x_i^*\right]\Delta x = \int_0^1 (2x^2 - 5x)\,dx$.

18. On $[1, 4]$, $\lim\limits_{n\to\infty} \sqrt{x_i^*}\,\Delta x = \int_1^4 \sqrt{x}\,dx$.

19. Note that $\Delta x = \dfrac{5-(-1)}{n} = \dfrac{6}{n}$ and $x_i = -1 + i\,\Delta x = -1 + \dfrac{6i}{n}$.

$$\begin{aligned}
\int_{-1}^{5} (1+3x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left[1 + 3\left(-1 + \frac{6i}{n}\right)\right]\frac{6}{n} \\
&= \lim_{n\to\infty} \frac{6}{n} \sum_{i=1}^{n} \left[-2 + \frac{18i}{n}\right] = \lim_{n\to\infty} \frac{6}{n}\left[\sum_{i=1}^{n}(-2) + \sum_{i=1}^{n} \frac{18i}{n}\right] \\
&= \lim_{n\to\infty} \frac{6}{n}\left[-2n + \frac{18}{n}\sum_{i=1}^{n} i\right] = \lim_{n\to\infty} \frac{6}{n}\left[-2n + \frac{18}{n}\cdot\frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty}\left[-12 + \frac{108}{n^2}\cdot\frac{n(n+1)}{2}\right] = \lim_{n\to\infty}\left[-12 + 54\frac{n+1}{n}\right] \\
&= \lim_{n\to\infty}\left[-12 + 54\left(1 + \frac{1}{n}\right)\right] = -12 + 54\cdot 1 = 42
\end{aligned}$$

20. $\int_1^5 \left(2+3x-x^2\right)dx = \lim\limits_{n\to\infty}\frac{4}{n}\sum\limits_{i=1}^{n}\left[2+3\left(1+\frac{4i}{n}\right)-\left(1+\frac{4i}{n}\right)^2\right] = \lim\limits_{n\to\infty}\frac{4}{n}\sum\limits_{i=1}^{n}\left[-\frac{16i^2}{n^2}+\frac{4i}{n}+4\right]$

$$= \lim_{n\to\infty}\left[-\frac{64}{n^3}\sum_{i=1}^{n}i^2+\frac{16}{n^2}\sum_{i=1}^{n}i+\frac{16}{n}\sum_{i=1}^{n}1\right]$$

$$= \lim_{n\to\infty}\left[-\frac{64}{n^3}\frac{n(n+1)(2n+1)}{6}+\frac{16}{n^2}\frac{n(n+1)}{2}+\frac{16}{n}n\right]$$

$$= \lim_{n\to\infty}\left[-\frac{32}{3}\cdot 1\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+8\cdot 1\left(1+\frac{1}{n}\right)+16\right] = -\frac{64}{3}+8+16=\frac{8}{3}$$

21. $\int_0^2 \left(2-x^2\right)dx = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i)\,\Delta x \quad [\Delta x = 2/n \text{ and } x_i = 2i/n] \quad = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\left(2-\frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)$

$$= \lim_{n\to\infty}\frac{2}{n}\left(2n-\frac{8}{n^3}\sum_{i=1}^{n}i^2\right) = \lim_{n\to\infty}\left[4-\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\left(4-\frac{4}{3}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}\right) = \lim_{n\to\infty}\left[4-\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = 4-\tfrac{4}{3}\cdot 2=\tfrac{4}{3}$$

22. $\int_0^5 \left(1+2x^3\right)dx = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i)\,\Delta x \quad [\Delta x = 5/n \text{ and } x_i = 5i/n]$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+2\cdot\frac{125i^3}{n^3}\right)\left(\frac{5}{n}\right) = \lim_{n\to\infty}\frac{5}{n}\left(1\cdot n+\frac{250}{n^3}\sum_{i=1}^{n}i^3\right)$$

$$= \lim_{n\to\infty}\left[5+\frac{1250}{n^4}\cdot\frac{n^2(n+1)^2}{4}\right] = \lim_{n\to\infty}\left[5+312.5\cdot\frac{(n+1)^2}{n^2}\right]$$

$$= \lim_{n\to\infty}\left[5+312.5\left(1+\frac{1}{n}\right)^2\right] = 5+312.5 = 317.5$$

23. Note that $x_i = 1+i\,\Delta x = 1+i\,(1/n) = 1+i/n$.

$$\int_1^2 x^3\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^3\left(\frac{1}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{n+i}{n}\right)^3$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}\left(n^3+3n^2i+3ni^2+i^3\right) = \lim_{n\to\infty}\frac{1}{n^4}\left[n\cdot n^3+3n^2\sum_{i=1}^{n}i+3n\sum_{i=1}^{n}i^2+\sum_{i=1}^{n}i^3\right]$$

$$= \lim_{n\to\infty}\left[1+\frac{3}{n^2}\cdot\frac{n(n+1)}{2}+\frac{3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{1}{n^4}\cdot\frac{n^2(n+1)^2}{4}\right]$$

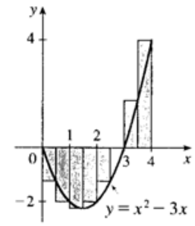
$$= \lim_{n\to\infty}\left[1+\frac{3}{2}\cdot\frac{n+1}{n}+\frac{1}{2}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}+\frac{1}{4}\cdot\frac{(n+1)^2}{n^2}\right]$$

$$= \lim_{n\to\infty}\left[1+\frac{3}{2}\left(1+\frac{1}{n}\right)+\frac{1}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+\frac{1}{4}\left(1+\frac{1}{n}\right)^2\right] = 1+\frac{3}{2}+\frac{1}{2}\cdot 2+\frac{1}{4} = 3.75$$

24. (a) $\Delta x = (4-0)/8 = 0.5$ and $x_i^* = x_i = 0.5i$.

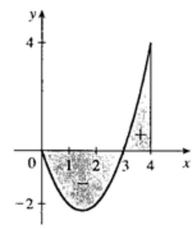
$$\int_0^4 \left(x^2-3x\right)dx \approx \sum_{i=1}^{8} f\left(x_i^*\right)\Delta x$$
$$= 0.5\left\{\left[0.5^2 - 3\,(0.5)\right] + \left[1.0^2 - 3\,(1.0)\right] + \cdots + \left[3.5^2 - 3\,(3.5)\right] + \left[4.0^2 - 3\,(4.0)\right]\right\}$$
$$= \tfrac{1}{2}\left(-\tfrac{5}{4} - 2 - \tfrac{9}{4} - 2 - \tfrac{5}{4} + 0 + \tfrac{7}{4} + 4\right) = -1.5$$

(b)



(c) $$\int_0^4 \left(x^2-3x\right)dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{4i}{n}\right)^2 - 3\left(\frac{4i}{n}\right)\right]\left(\frac{4}{n}\right)$$
$$= \lim_{n\to\infty}\frac{4}{n}\left[\frac{16}{n^2}\sum_{i=1}^{n} i^2 - \frac{12}{n}\sum_{i=1}^{n} i\right]$$
$$= \lim_{n\to\infty}\left[\frac{64}{n^3}\cdot\frac{n\,(n+1)\,(2n+1)}{6} - \frac{48}{n^2}\cdot\frac{n\,(n+1)}{2}\right]$$
$$= \lim_{n\to\infty}\left[\frac{32}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - 24\left(1+\frac{1}{n}\right)\right]$$
$$= \tfrac{32}{3}\cdot 2 - 24 = -\tfrac{8}{3}$$

(d) $\int_0^4 \left(x^2-3x\right)dx = A_1 - A_2$, where A_1 is the area marked $+$ and A_2 is the area marked $-$.



25. $$\int_a^b x\,dx = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}\left[a + \frac{b-a}{n}i\right] = \lim_{n\to\infty}\left[\frac{a\,(b-a)}{n}\sum_{i=1}^{n}1 + \frac{(b-a)^2}{n^2}\sum_{i=1}^{n} i\right]$$
$$= \lim_{n\to\infty}\left[\frac{a\,(b-a)}{n}n + \frac{(b-a)^2}{n^2}\cdot\frac{n\,(n+1)}{2}\right] = a\,(b-a) + \lim_{n\to\infty}\frac{(b-a)^2}{2}\left(1+\frac{1}{n}\right)$$
$$= a\,(b-a) + \tfrac{1}{2}\,(b-a)^2 = (b-a)\left(a + \tfrac{1}{2}b - \tfrac{1}{2}a\right) = (b-a)\,\tfrac{1}{2}\,(b+a) = \tfrac{1}{2}\left(b^2-a^2\right)$$

26. $$\int_a^b x^2\,dx = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}\left[a + \frac{b-a}{n}i\right]^2 = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}\left[a^2 + 2a\frac{b-a}{n}i + \frac{(b-a)^2}{n^2}i^2\right]$$
$$= \lim_{n\to\infty}\left[\frac{(b-a)^3}{n^3}\sum_{i=1}^{n} i^2 + \frac{2a\,(b-a)^2}{n^2}\sum_{i=1}^{n} i + \frac{a^2\,(b-a)}{n}\sum_{i=1}^{n} 1\right]$$
$$= \lim_{n\to\infty}\left[\frac{(b-a)^3}{n^3}\,\frac{n\,(n+1)\,(2n+1)}{6} + \frac{2a\,(b-a)^2}{n^2}\,\frac{n\,(n+1)}{2} + \frac{a^2\,(b-a)}{n}n\right]$$
$$= \lim_{n\to\infty}\left[\frac{(b-a)^3}{6}\cdot 1\cdot\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + a\,(b-a)^2\cdot 1\cdot\left(1+\frac{1}{n}\right) + a^2\,(b-a)\right]$$
$$= \frac{(b-a)^3}{3} + a\,(b-a)^2 + a^2\,(b-a) = \frac{b^3 - 3ab^2 + 3a^2b - a^3}{3} + ab^2 - 2a^2b + a^3 + a^2b - a^3$$
$$= \frac{b^3}{3} - \frac{a^3}{3} - ab + a^2b + ab^2 - a^2b = \frac{b^3 - a^3}{3}$$

27. $\Delta x = (\pi - 0)/n = \pi/n$ and $x_i^* = x_i = \pi i/n$.

$$\int_0^{\pi} \sin 5x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} (\sin 5x_i)\left(\frac{\pi}{n}\right) = \pi \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sin\left(\frac{5\pi i}{n}\right) = \pi \lim_{n\to\infty} \frac{1}{n} \cot\left(\frac{5\pi}{2n}\right) = \pi\left(\tfrac{2}{5\pi}\right) = \tfrac{2}{5}$$

28. $\Delta x = (10-2)/n = 8/n$ and $x_i^* = x_i = 2 + 8i/n$.

$$\begin{aligned}\int_2^{10} x^6\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2+\frac{8i}{n}\right)^6 \left(\frac{8}{n}\right) = 8 \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(2+\frac{8i}{n}\right)^6 \\ &= 8 \lim_{n\to\infty} \frac{1}{n} \cdot \frac{64\left(58{,}593n^6 + 164{,}052n^5 + 131{,}208n^4 - 27{,}776n^2 + 2048\right)}{21n^5} \\ &= 8\left(\tfrac{1{,}249{,}984}{7}\right) = \tfrac{9{,}999{,}872}{7} \approx 1{,}428{,}553.1\end{aligned}$$

29. (a) Think of $\int_0^2 f(x)\,dx$ as the area of a trapezoid with bases 1 and 3 and height 2.
$\int_0^2 f(x)\,dx = \frac{1}{2}(1+3)\,2 = 4$.

(b) $$\begin{aligned}\int_0^5 f(x)\,dx &= \underset{\text{trapezoid}}{\int_0^2 f(x)\,dx} + \underset{\text{rectangle}}{\int_2^3 f(x)\,dx} + \underset{\text{triangle}}{\int_3^5 f(x)\,dx} \\ &= \tfrac{1}{2}(1+3)\,2 + 3\cdot 1 + \tfrac{1}{2}\cdot 2\cdot 3 = 10\end{aligned}$$

(c) $\int_5^7 f(x)\,dx$ is the negative of the area of the triangle with base 2 and height 3. $\int_5^7 f(x)\,dx = -\frac{1}{2}\cdot 2\cdot 3 = -3$.

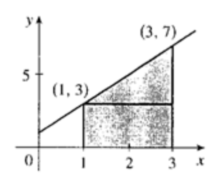
(d) As in parts (a) and (c), $\int_7^9 f(x)\,dx = -\frac{1}{2}(3+2)\,2 = -5$. Now
$\int_0^9 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx + \int_7^9 f(x)\,dx = 10 - 3 - 5 = 2$.

30. (a) $\int_0^2 g(x)\,dx = \frac{1}{2}\cdot 4\cdot 2 = 4$ (area of a triangle)

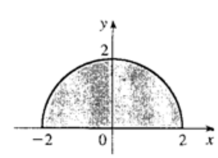
(b) $\int_2^6 g(x)\,dx = -\frac{1}{2}\pi(2)^2 = -2\pi$ (negative of the area of a semicircle)

(c) $\int_6^7 g(x)\,dx = \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$ (area of a triangle)
$\int_0^7 g(x)\,dx = \int_0^2 g(x)\,dx + \int_2^6 g(x)\,dx + \int_6^7 g(x)\,dx = 4 - 2\pi + \frac{1}{2} = 4.5 - 2\pi$

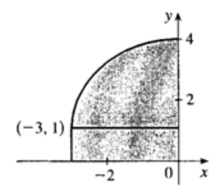
31. $\int_1^3 (1+2x)\,dx$ can be interpreted as the area under the graph of $f(x) = 1 + 2x$ between $x = 1$ and $x = 3$. This is equal to the area of the rectangle plus the area of the triangle, so $\int_1^3 (1+2x)\,dx = A = 2\cdot 3 + \frac{1}{2}\cdot 2\cdot 4 = 10$.
Or: Use the formula for the area of a trapezoid: $a = \frac{1}{2}(2)(3+7) = 10$.



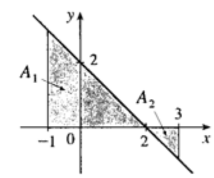
32. $\int_{-2}^{2} \sqrt{4-x^2}\,dx$ can be interpreted as the area under the graph of $f(x) = \sqrt{4-x^2}$ between $x = -2$ and $x = 2$. This is equal to half the area of the circle with radius 2, so $\int_{-2}^{2} \sqrt{4-x^2}\,dx = \frac{1}{2}\pi\cdot 2^2 = 2\pi$.



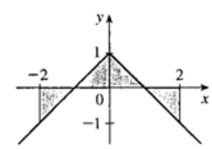
33. $\int_{-3}^{0}\left(1+\sqrt{9-x^2}\right)dx$ can be interpreted as the area under the graph of $f(x)=1+\sqrt{9-x^2}$ between $x=-3$ and $x=0$. This is equal to one-quarter the area of the circle with radius 3, plus the area of the rectangle, so $\int_{-3}^{0}\left(1+\sqrt{9-x^2}\right)dx=\frac{1}{4}\pi\cdot 3^2+1\cdot 3=3+\frac{9}{4}\pi$.



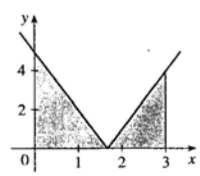
34. $\int_{-1}^{3}(2-x)\,dx$ can be interpreted as A_1-A_2, where A_1 and A_2 are the areas of the triangles shown. Thus, $\int_{-1}^{3}(2-x)\,dx=\frac{1}{2}\cdot 3\cdot 3-\frac{1}{2}\cdot 1\cdot 1=4$.



35. $\int_{-2}^{2}(1-|x|)\,dx$ can be interpreted as the area of the middle triangle minus the areas of the outside ones, so $\int_{-2}^{2}(1-|x|)\,dx=\frac{1}{2}\cdot 2\cdot 1-2\cdot\frac{1}{2}\cdot 1\cdot 1=0$.



36. $\int_{0}^{3}|3x-5|\,dx$ can be interpreted as the area under the graph of the function $f(x)=|3x-5|$ between $x=0$ and $x=3$. This is equal to the sum of the areas of the two triangles, so $\int_{0}^{3}|3x-5|\,dx=\frac{1}{2}\cdot\frac{5}{3}\cdot 5+\frac{1}{2}\left(3-\frac{5}{3}\right)4=\frac{41}{6}$.



37. $\int_{9}^{4}\sqrt{t}\,dt=-\int_{4}^{9}\sqrt{t}\,dt$ (because we reversed the limits of integration) $=-\frac{38}{3}$

38. $\int_{1}^{1}x^2\cos x\,dx=0$ since the limits of integration are equal.

39. $\int_{0}^{1}(5-6x^2)\,dx=\int_{0}^{1}5\,dx-6\int_{0}^{1}x^2\,dx=5(1-0)-6\left(\frac{1}{3}\right)=5-2=3$

40. $\int_{2}^{5}(1+3x^4)\,dx=\int_{2}^{5}1\,dx+\int_{2}^{5}3x^4\,dx=1(5-2)+3\int_{2}^{5}x^4\,dx=1(3)+3(618.6)=1858.8$

41.
$$\begin{aligned}\int_{1}^{4}(2x^2-3x+1)\,dx&=2\int_{1}^{4}x^2\,dx-3\int_{1}^{4}x\,dx+\int_{1}^{4}1\,dx\\&=2\cdot\tfrac{1}{3}(4^3-1^3)-3\cdot\tfrac{1}{2}(4^2-1^2)+1(4-1)=\tfrac{45}{2}=22.5\end{aligned}$$

42.
$$\begin{aligned}\int_{0}^{\pi/2}(2\cos x-5x)\,dx&=\int_{0}^{\pi/2}2\cos x\,dx-\int_{0}^{\pi/2}5x\,dx=2\int_{0}^{\pi/2}\cos x\,dx-5\int_{0}^{\pi/2}x\,dx\\&=2(1)-5\frac{(\pi/2)^2-0^2}{2}=2-\frac{5\pi^2}{8}\end{aligned}$$

43. $\int_{1}^{3}f(x)\,dx+\int_{3}^{6}f(x)\,dx+\int_{6}^{12}f(x)\,dx=\int_{1}^{6}f(x)\,dx+\int_{6}^{12}f(x)\,dx=\int_{1}^{12}f(x)\,dx$

44. $\int_{2}^{10}f(x)\,dx-\int_{2}^{7}f(x)\,dx=\int_{2}^{7}f(x)\,dx+\int_{7}^{10}f(x)\,dx-\int_{2}^{7}f(x)\,dx=\int_{7}^{10}f(x)\,dx$

45. $\int_{2}^{5}f(x)\,dx+\int_{5}^{8}f(x)\,dx=\int_{2}^{8}f(x)\,dx \;\Rightarrow\; \int_{2}^{5}f(x)\,dx+2.5=1.7 \;\Rightarrow\; \int_{2}^{5}f(x)\,dx=-0.8$

46. $\int_{0}^{1}f(t)\,dt+\int_{1}^{3}f(t)\,dt+\int_{3}^{4}f(t)\,dt=\int_{0}^{4}f(t)\,dt \;\Rightarrow\; 2+\int_{1}^{3}f(t)\,dt+1=-6 \;\Rightarrow$
$\int_{1}^{3}f(t)\,dt=-6-2-1=-9$

47. $0\le\sin x<1$ on $\left[0,\frac{\pi}{4}\right]$, so $\sin^3 x\le\sin^2 x$ on $\left[0,\frac{\pi}{4}\right]$. Hence, $\int_{0}^{\pi/4}\sin^3 x\,dx\le\int_{0}^{\pi/4}\sin^2 x\,dx$ (Property 7).

48. $5 - x \geq 3 \geq x + 1$ on $[1, 2]$, so $\sqrt{5-x} \geq \sqrt{x+1}$ and $\int_1^2 \sqrt{5-x}\,dx \geq \int_1^2 \sqrt{x+1}\,dx$.

49. If $-1 \leq x \leq 1$, then $0 \leq x^2 \leq 1$ and $1 \leq 1 + x^2 \leq 2$, so $1 \leq \sqrt{1+x^2} \leq \sqrt{2}$ and
$1\,[1-(-1)] \leq \int_{-1}^1 \sqrt{1+x^2}\,dx \leq \sqrt{2}\,[1-(-1)]$ [Property 8]; that is, $2 \leq \int_{-1}^1 \sqrt{1+x^2}\,dx \leq 2\sqrt{2}$.

50. $\frac{1}{2} \leq \sin x \leq 1$ for $\frac{\pi}{6} \leq x \leq \frac{\pi}{2}$, so $\frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) \leq \int_{\pi/6}^{\pi/2} \sin x\,dx \leq 1\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$ (Property 8); that is,
$\frac{\pi}{6} \leq \int_{\pi/6}^{\pi/2} \sin x\,dx \leq \frac{\pi}{3}$.

51. If $1 \leq x \leq 2$, then $\frac{1}{2} \leq \frac{1}{x} \leq 1$, so $\frac{1}{2}(2-1) \leq \int_1^2 \frac{1}{x}\,dx \leq 1\,(2-1)$ or $\frac{1}{2} \leq \int_1^2 \frac{1}{x}\,dx \leq 1$.

52. If $0 \leq x \leq 2$, then $0 \leq x^3 \leq 8$, so $1 \leq x^3 + 1 \leq 9$ and $1 \leq \sqrt{x^3+1} \leq 3$. Thus,
$1\,(2-0) \leq \int_0^2 \sqrt{x^3+1}\,dx \leq 3\,(2-0)$; that is, $2 \leq \int_0^2 \sqrt{x^3+1}\,dx \leq 6$.

53. If $f(x) = x^2 + 2x$, $-3 \leq x \leq 0$, then $f'(x) = 2x + 2 = 0$ when $x = -1$, and $f(-1) = -1$. At the endpoints, $f(-3) = 3$, $f(0) = 0$. Thus, the absolute minimum is $m = -1$ and the absolute maximum is $M = 3$. Thus,
$-1\,[0-(-3)] \leq \int_{-3}^0 (x^2+2x)\,dx \leq 3\,[0-(-3)]$ or $-3 \leq \int_{-3}^0 (x^2+2x)\,dx \leq 9$.

54. If $\frac{\pi}{4} \leq x \leq \frac{\pi}{3}$, then $\frac{1}{2} \leq \cos x \leq \frac{\sqrt{2}}{2}$, so $\frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \leq \int_{\pi/4}^{\pi/3} \cos x\,dx \leq \frac{\sqrt{2}}{2}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$ or
$\frac{\pi}{24} \leq \int_{\pi/4}^{\pi/3} \cos x\,dx \leq \frac{\sqrt{2}\pi}{24}$.

55. For $-1 \leq x \leq 1$, $0 \leq x^4 \leq 1$ and $1 \leq \sqrt{1+x^4} \leq \sqrt{2}$, so $1\,[1-(-1)] \leq \int_{-1}^1 \sqrt{1+x^4}\,dx \leq \sqrt{2}\,[1-(-1)]$ or
$2 \leq \int_{-1}^1 \sqrt{1+x^4}\,dx \leq 2\sqrt{2}$.

56. If $\frac{1}{4}\pi \leq x \leq \frac{3}{4}\pi$, then $\frac{\sqrt{2}}{2} \leq \sin x \leq 1$ and $\frac{1}{2} \leq \sin^2 x \leq 1$, so $\frac{1}{2}\left(\frac{3}{4}\pi - \frac{1}{4}\pi\right) \leq \int_{\pi/4}^{3\pi/4} \sin^2 x\,dx \leq 1\left(\frac{3}{4}\pi - \frac{1}{4}\pi\right)$;
that is, $\frac{1}{4}\pi \leq \int_{\pi/4}^{3\pi/4} \sin^2 x\,dx \leq \frac{1}{2}\pi$.

57. $\sqrt{x^4+1} \geq \sqrt{x^4} = x^2$, so $\int_1^3 \sqrt{x^4+1}\,dx \geq \int_1^3 x^2\,dx = \frac{1}{3}\left(3^3 - 1^3\right) = \frac{26}{3}$.

58. $0 \leq \sin x \leq 1$ for $0 \leq x \leq \frac{\pi}{2}$, so $x \sin x \leq x \quad\Rightarrow\quad \int_0^{\pi/2} x \sin x\,dx \leq \int_0^{\pi/2} x\,dx = \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - 0^2\right] = \frac{\pi^2}{8}$.

59. Using a regular partition and right endpoints as in the proof of Property 2, we calculate

$$\int_a^b cf(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n cf(x_i)\,\Delta x_i = \lim_{n\to\infty} c \sum_{i=1}^n f(x_i)\,\Delta x_i = c \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\,\Delta x_i = c\int_a^b f(x)\,dx.$$

60. As in the proof of Property 2, we write $\int_a^b f(x)\,dx = \lim\limits_{n\to\infty} \sum\limits_{i=1}^n f(x_i)\,\Delta x$. Now $f(x_i) \geq 0$ and $\Delta x \geq 0$, so
$f(x_i)\,\Delta x \geq 0$ and therefore $\sum\limits_{i=1}^n f(x_i)\,\Delta x \geq 0$. But the limit of nonnegative quantities is nonnegative by
Theorem 2.3.2, so $\int_a^b f(x)\,dx \geq 0$.

61. Since $-|f(x)| \leq f(x) \leq |f(x)|$, it follows from Property 7 that

$$-\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx \quad\Rightarrow\quad \left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

Note that the definite integral is a real number, and so the following property applies: $-a \leq b \leq a \quad\Rightarrow\quad |b| \leq a$ for all real numbers b and nonnegative numbers a.

62. $\left|\int_0^{2\pi} f(x) \sin 2x\,dx\right| \leq \int_0^{2\pi} |f(x)\sin 2x|\,dx$ (by Exercise 61) $= \int_0^{2\pi} |f(x)|\,|\sin 2x|\,dx \leq \int_0^{2\pi} |f(x)|\,dx$ by Property 7, since $|\sin 2x| \leq 1 \quad\Rightarrow\quad |f(x)|\,|\sin 2x| \leq |f(x)|$.

63. $\lim\limits_{n\to\infty} \sum\limits_{i=1}^n \frac{i^4}{n^5} = \lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{i=1}^n \left(\frac{i}{n}\right)^4 = \int_0^1 x^4\,dx$

64. $\displaystyle\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+(i/n)^2}=\int_0^1\frac{dx}{1+x^2}$

65. Choose $x_i = 1+\dfrac{i}{n}$ and $x_i^* = \sqrt{x_{i-1}x_i} = \sqrt{\left(1+\dfrac{i-1}{n}\right)\left(1+\dfrac{i}{n}\right)}$. Then

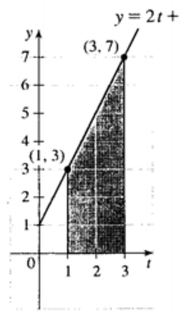
$$\int_1^2 x^{-2}dx = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\left(1+\frac{i-1}{n}\right)\left(1+\frac{i}{n}\right)} = \lim_{n\to\infty} n\sum_{i=1}^{n}\frac{1}{(n+i-1)(n+i)}$$

$$= \lim_{n\to\infty} n\sum_{i=1}^{n}\left(\frac{1}{n+i-1}-\frac{1}{n+1}\right) \quad \text{(by the hint)}$$

$$= \lim_{n\to\infty} n\left(\sum_{i=0}^{n-1}\frac{1}{n+i}-\sum_{i=1}^{n}\frac{1}{n+i}\right) = \lim_{n\to\infty} n\left(\frac{1}{n}-\frac{1}{2n}\right) = \lim_{n\to\infty}\left(1-\tfrac{1}{2}\right)=\tfrac{1}{2}$$

Discovery Project □ Area Functions

1. (a)



Area of trapezoid $= \frac{1}{2}(b_1+b_2)h$

$= \frac{1}{2}(3+7)2$

$= 10$ square units

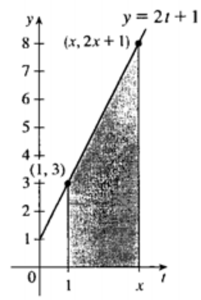
Or:

Area of rectangle + area of triangle

$= b_r h_r + \frac{1}{2}b_t h_t = (2)(3) + \frac{1}{2}(2)(4)$

$= 10$ square units

(b)



As in part (a),

$A(x) = \frac{1}{2}[3+(2x+1)](x-1)$

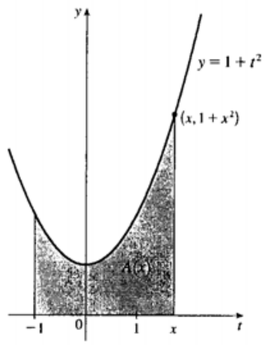
$= \frac{1}{2}(2x+4)(x-1)$

$= (x+2)(x-1)$

$= x^2+x-2$ square units.

(c) $A'(x) = 2x+1$. This is the y-coordinate of the point $(x, 2x+1)$ on the given line.

2. (a)



(b) $A(x) = \int_{-1}^{x} (1+t^2)\,dt = \int_{-1}^{x} 1\,dt + \int_{-1}^{x} t^2\,dt$ (Property 2)

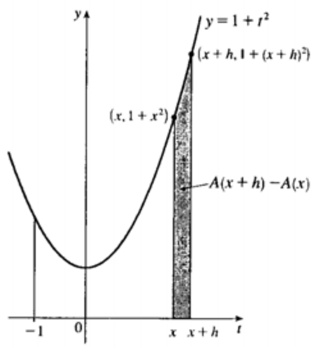
$= 1[x-(-1)] + \dfrac{x^3-(-1)^3}{3}$ $\begin{pmatrix}\text{Property 1 and}\\ \text{Exercise 5.2.26}\end{pmatrix}$

$= x + 1 + \frac{1}{3}x^3 + \frac{1}{3}$

$= \frac{1}{3}x^3 + x + \frac{4}{3}$

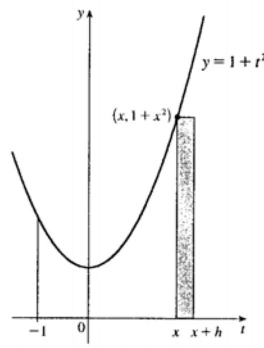
(c) $A'(x) = x^2 + 1$. This is the y-coordinate of the point $(x, 1+x^2)$ on the given curve.

(d)



$A(x+h) - A(x)$ is the area under the curve $y = 1 + t^2$ from $t = x$ to $t = x + h$.

(e)



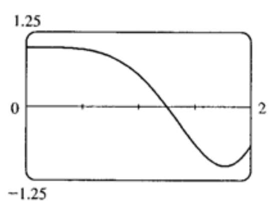
An approximating rectangle is shown in the figure. It has height $1 + x^2$, width h, and area $h(1+x^2)$, so

$A(x+h) - A(x) \approx h(1+x^2) \quad \Rightarrow$

$\dfrac{A(x+h) - A(x)}{h} \approx 1 + x^2.$

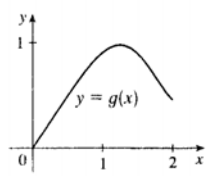
(f) Part (e) says that the average rate of change of A is approximately $1 + x^2$. As h approaches 0, the quotient approaches the instantaneous rate of change — namely, $A'(x)$. So the result of part (c), $A'(x) = x^2 + 1$, is geometrically plausible.

3. (a) $f(x) = \cos(x^2)$

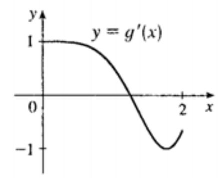


(b) $g(x)$ starts to decrease at that value of x where $\cos(t^2)$ changes from positive to negative, that is, at about $x = 1.25$.

(c) $g(x) = \int_0^x \cos(t^2)\,dt$. Using an integration command, we find that $g(0) = 0$, $g(0.2) \approx 0.200$, $g(0.4) \approx 0.399$, $g(0.6) \approx 0.592$, $g(0.8) \approx 0.768$, $g(1.0) \approx 0.905$, $g(1.2) \approx 0.974$, $g(1.4) \approx 0.950$, $g(1.6) \approx 0.826$, $g(1.8) \approx 0.635$, and $g(2.0) \approx 0.461$.



(d) We sketch the graph of g' using the method of Example 1 in Section 3.2. The graphs of $g'(x)$ and $f(x)$ look alike, so we guess that $g'(x) = f(x)$.



4. In Problems 1 and 2, we showed that if $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$, for the functions $f(t) = 2t + 1$ and $f(t) = 1 + t^2$. In Problem 3 we guessed that the same is true for $f(t) = \cos(t^2)$, based on visual evidence. So we conjecture that $g'(x) = f(x)$ for any continuous function f. This turns out to be true and is proved in Section 5.3 (the Fundamental Theorem of Calculus).

5.3 The Fundamental Theorem of Calculus

1. (a) $g(0) = \int_0^0 f(t)\,dt = 0$, $g(1) = \int_0^1 f(t)\,dt = 1 \cdot 2 = 2$,

$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = g(1) + \int_1^2 f(t)\,dt = 2 + 1 \cdot 2 + \frac{1}{2} \cdot 1 \cdot 2 = 5$,

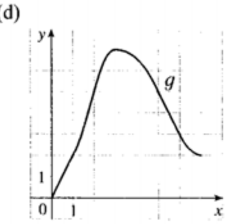
$g(3) = \int_0^3 f(t)\,dt = g(2) + \int_2^3 f(t)\,dt = 5 + \frac{1}{2} \cdot 1 \cdot 4 = 7$,

$g(6) = g(3) + \int_3^6 f(t)\,dt = 7 + \left[-\left(\frac{1}{2} \cdot 2 \cdot 2 + 1 \cdot 2\right)\right] = 7 - 4 = 3$

(b) g is increasing on $(0, 3)$ because as x increases from 0 to 3, we keep adding more area.

(c) g has a maximum value when we start subtracting area, that is, at $x = 3$.

(d)



2. (a) $g(-3) = \int_{-3}^{-3} f(t)\,dt = 0$, $g(3) = \int_{-3}^{3} f(t)\,dt = \int_{-3}^{0} f(t)\,dt + \int_0^3 f(t)\,dt = 0$ by symmetry, since the area above the x-axis is the same as the area below the axis.

(b) From the graph, it appears that to the nearest $\frac{1}{2}$,

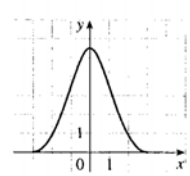
$g(-2) = \int_{-3}^{-2} f(t)\,dt \approx 1$, $g(-1) = \int_{-3}^{-1} f(t)\,dt \approx 3\frac{1}{2}$, and

$g(0) = \int_{-3}^{0} f(t)\,dt \approx 5\frac{1}{2}$.

(c) g is increasing on $(-3, 0)$ because as x increases from -3 to 0, we keep adding more area.

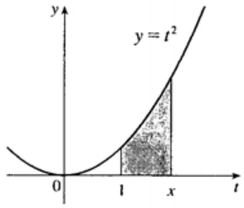
(d) g has a maximum value when we start subtracting area, that is, at $x = 0$.

(e)



(f) The graph of $g'(x)$ is the same as that of $f(x)$, as indicated by FTC1.

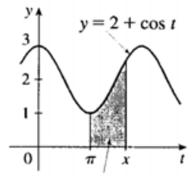
3.



(a) By FTC1 with $f(t) = t^2$ and $a = 1$, $g(x) = \int_1^x t^2\,dt \;\Rightarrow$
$g'(x) = f(x) = x^2$.

(b) Using FTC2, $g(x) = \int_1^x t^2\,dt = \left[\frac{1}{3}t^3\right]_1^x = \frac{1}{3}x^3 - \frac{1}{3} \;\Rightarrow$
$g'(x) = x^2$.

4.



$g(x)$ for $x > \pi$

(a) $g(x) = \int_\pi^x (2 + \cos t)\,dt \;\Rightarrow\; g'(x) = 2 + \cos x$

(b) $g(x) = \int_\pi^x (2 + \cos t)\,dt = [2t + \sin t]_\pi^x$
$= (2x + \sin x) - (2\pi + 0) = 2x + \sin x - 2\pi$
so $g'(x) = 2 + \cos x$.

5. $f(t) = \sqrt{1 + 2t}$ and $g(x) = \int_0^x \sqrt{1 + 2t}\,dt$, so by FTC1, $g'(x) = f(x) = \sqrt{1 + 2x}$.

6. $f(t) = \left(2 + t^4\right)^5$ and $g(x) = \int_1^x \left(2 + t^4\right)^5 dt$, so $g'(x) = f(x) = \left(2 + x^4\right)^5$.

7. $f(t) = t^2 \sin t$ and $g(y) = \int_2^y t^2 \sin t\,dt$, so by FTC1, $g'(y) = f(y) = y^2 \sin y$.

8. $f(x) = \dfrac{1}{x + x^2}$ and $g(u) = \displaystyle\int_3^u \frac{1}{x + x^2}\,dx$, so $g'(u) = f(u) = \dfrac{1}{u + u^2}$.

9. $F(x) = \int_x^2 \cos\left(t^2\right) dt = -\int_2^x \cos\left(t^2\right) dt \;\Rightarrow\; F'(x) = -\cos\left(x^2\right)$

10. $F(x) = \int_x^{10} \tan\theta\,d\theta = -\int_{10}^x \tan\theta\,d\theta \;\Rightarrow\; F'(x) = -\tan x$

11. Let $u = \dfrac{1}{x}$. Then $\dfrac{du}{dx} = -\dfrac{1}{x^2}$. Also, $\dfrac{dh}{dx} = \dfrac{dh}{du}\dfrac{du}{dx}$, so
$$h'(x) = \frac{d}{dx}\int_2^{1/x} \sin^4 t\,dt = \frac{d}{du}\int_2^u \sin^4 t\,dt \cdot \frac{du}{dx} = \sin^4 u\,\frac{du}{dx} = \frac{-\sin^4(1/x)}{x^2}.$$

12. Let $u = x^2$. Then $\dfrac{du}{dx} = 2x$. Also, $\dfrac{dh}{dx} = \dfrac{dh}{du}\dfrac{du}{dx}$, so
$$h'(x) = \frac{d}{dx}\int_0^{x^2} \sqrt{1 + r^3}\,dr = \frac{d}{du}\int_0^u \sqrt{1 + r^3}\,dr \cdot \frac{du}{dx} = \sqrt{1 + u^3}\,(2x) = 2x\sqrt{1 + \left(x^2\right)^3} = 2x\sqrt{1 + x^6}.$$

13. Let $u = \sqrt{x}$. Then $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so
$$y' = \frac{d}{dx}\int_3^{\sqrt{x}} \frac{\cos t}{t}\,dt = \frac{d}{du}\int_3^u \frac{\cos t}{t}\,dt \cdot \frac{du}{dx} = \frac{\cos u}{u} \cdot \frac{1}{2\sqrt{x}} = \frac{\cos\sqrt{x}}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{\cos\sqrt{x}}{2x}.$$

14. Let $u = \cos x$. Then $\dfrac{du}{dx} = -\sin x$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so
$$y' = \frac{d}{dx}\int_1^{\cos x} (t + \sin t)\,dt = \frac{d}{du}\int_1^u (t + \sin t)\,dt \cdot \frac{du}{dx}$$
$$= (u + \sin u) \cdot (-\sin x) = -\sin x\,[\cos x + \sin(\cos x)]$$

15. Let $w = 1 - 3x$. Then $\dfrac{dw}{dx} = -3$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{dw}\dfrac{dw}{dx}$, so

$$\begin{aligned} y' &= \frac{d}{dx}\int_{1-3x}^{1} \frac{u^3}{1+u^2}\,du = \frac{d}{dw}\int_{w}^{1} \frac{u^3}{1+u^2}du \cdot \frac{dw}{dx} \\ &= -\frac{d}{dw}\int_{1}^{w} \frac{u^3}{1+u^2}\,du \cdot \frac{dw}{dx} = -\frac{w^3}{1+w^2}(-3) = \frac{3\,(1-3x)^3}{1+(1-3x)^2} \end{aligned}$$

16. Let $u = \dfrac{1}{x^2}$. Then $\dfrac{du}{dx} = -\dfrac{2}{x^3}$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$y' = \frac{d}{dx}\int_{1/x^2}^{0} \sin^3 t\,dt = \frac{d}{du}\int_{u}^{0} \sin^3 t\,dt \cdot \frac{du}{dx} = -\frac{d}{du}\int_{0}^{u} \sin^3 t\,dt \cdot \frac{du}{dx} = -\sin^3 u\left(-\frac{2}{x^3}\right) = \frac{2\sin^3\left(1/x^2\right)}{x^3}.$$

17. $\displaystyle\int_{-1}^{3} x^5\,dx = \left[\frac{x^6}{6}\right]_{-1}^{3} = \frac{3^6}{6} - \frac{(-1)^6}{6} = \frac{729-1}{6} = \frac{364}{3}$

18. $\int_1^2 x^{-2}\,dx = \left[-x^{-1}\right]_1^2 = [-1/x]_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$

19. $\int_2^8 (4x+3)\,dx = \left[\frac{4}{2}x^2 + 3x\right]_2^8 = \left[(2\cdot 8^2 + 3\cdot 8) - (2\cdot 2^2 + 3\cdot 2)\right] = 152 - 14 = 138$

20. $\int_0^4 \left(1 + 3y - y^2\right)dy = \left[y + \frac{3}{2}y^2 - \frac{1}{3}y^3\right]_0^4 = \left[\left(4 + \frac{3}{2}\cdot 16 - \frac{1}{3}\cdot 64\right) - (0)\right] = \frac{20}{3}$

21. $\displaystyle\int_0^4 \sqrt{x}\,dx = \int_0^4 x^{1/2}\,dx = \left[\frac{x^{3/2}}{3/2}\right]_0^4 = \left[\frac{2x^{3/2}}{3}\right]_0^4 = \frac{2\,(4)^{3/2}}{3} - 0 = \frac{16}{3}$

22. $\displaystyle\int_0^1 x^{3/7}\,dx = \left[\frac{x^{10/7}}{10/7}\right]_0^1 = \left[\tfrac{7}{10}x^{10/7}\right]_0^1 = \frac{7}{10} - 0 = \frac{7}{10}$

23. $\displaystyle\int_1^2 \frac{3}{t^4}\,dt = 3\int_1^2 t^{-4}\,dt = 3\left[\frac{t^{-3}}{-3}\right]_1^2 = \frac{3}{-3}\left[\frac{1}{t^3}\right]_1^2 = -1\left(\tfrac{1}{8} - 1\right) = \tfrac{7}{8}$

24. $\displaystyle\int_{-1}^{1} \frac{3}{t^4}\,dt$ does not exist since $f(t) = \dfrac{3}{t^4}$ has an infinite discontinuity at 0.

25. $\int_3^3 \sqrt{x^5+2}\,dx = 0$ since the lower and upper limits are equal.

26. $\int_\pi^{2\pi} \cos\theta\,d\theta = [\sin\theta]_\pi^{2\pi} = \sin 2\pi - \sin\pi = 0 - 0 = 0$

27. $\displaystyle\int_{-4}^{2} \frac{2}{x^6}\,dx$ does not exist since $f(x) = \dfrac{2}{x^6}$ has an infinite discontinuity at 0.

28. $\displaystyle\int_1^4 \frac{1}{\sqrt{x}}\,dx = \int_1^4 x^{-1/2}\,dx = \left[2x^{1/2}\right]_1^4 = 2\sqrt{4} - 2\sqrt{1} = 4 - 2 = 2$

29. $\int_{\pi/4}^{\pi/3} \sin t\,dt = [-\cos t]_{\pi/4}^{\pi/3} = -\cos\frac{\pi}{3} + \cos\frac{\pi}{4} = -\frac{1}{2} + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{2}$

30. $\int_0^1 \left(3 + x\sqrt{x}\right)dx = \int_0^1 \left(3 + x^{3/2}\right)dx = \left[3x + \frac{2}{5}x^{5/2}\right]_0^1 = \left[\left(3 + \frac{2}{5}\right) - 0\right] = \frac{17}{5}$

31. $\int_{\pi/2}^{\pi} \sec x\,\tan x\,dx$ does not exist since $\sec x\,\tan x$ has an infinite discontinuity at $\frac{\pi}{2}$.

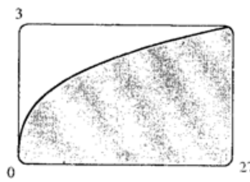
32. $\int_{\pi/4}^{\pi} \sec^2\theta\,d\theta$ does not exist since $\sec^2\theta$ has an infinite discontinuity at $\frac{\pi}{2}$.

33. $\int_0^2 f(x)\,dx = \int_0^1 x^4 dx + \int_1^2 x^5 dx = \left[\frac{1}{5}x^5\right]_0^1 + \left[\frac{1}{6}x^6\right]_1^2 = \left(\frac{1}{5} - 0\right) + \left(\frac{64}{6} - \frac{1}{6}\right) = 10.7$

34. $\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{0} x\,dx + \int_0^{\pi} \sin x\,dx = \left[\frac{1}{2}x^2\right]_{-\pi}^{0} - [\cos x]_0^{\pi} = \left(0 - \frac{\pi^2}{2}\right) - (\cos\pi - \cos 0)$

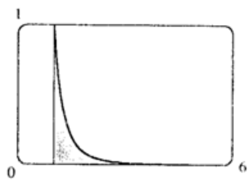
$= -\frac{\pi^2}{2} - (-1-1) = 2 - \frac{\pi^2}{2}$

35. From the graph, it appears that the area is about 60. The actual area is $\int_0^{27} x^{1/3} dx = \left[\frac{3}{4}x^{4/3}\right]_0^{27} = \frac{3}{4}\cdot 81 - 0 = \frac{243}{4} = 60.75$. This is $\frac{3}{4}$ of the area of the viewing rectangle.

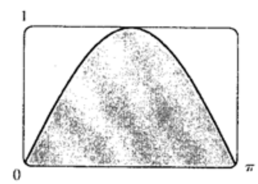


36. From the graph, it appears that the area is about $\frac{1}{3}$. The actual area is

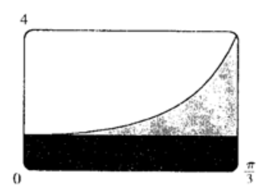
$$\int_1^6 x^{-4} dx = \left[\frac{x^{-3}}{-3}\right]_1^6 = \left[\frac{-1}{3x^3}\right]_1^6 = -\frac{1}{3\cdot 216} + \frac{1}{3} = \frac{215}{648} \approx 0.3318.$$



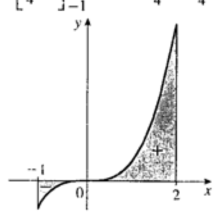
37. It appears that the area under the graph is about $\frac{2}{3}$ of the area of the viewing rectangle, or about $\frac{2}{3}\pi \approx 2.1$. The actual area is $\int_0^{\pi} \sin x\,dx = [-\cos x]_0^{\pi} = -\cos\pi + \cos 0 = -(-1) + 1 = 2$.



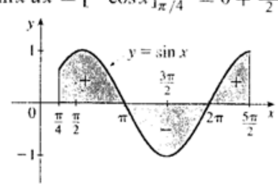
38. Splitting up the region as shown, we estimate that the area under the graph is $\frac{\pi}{3} + \frac{1}{4}\left(3\cdot\frac{\pi}{3}\right) \approx 1.8$. The actual area is $\int_0^{\pi/3} \sec^2 x\,dx = [\tan x]_0^{\pi/3} = \sqrt{3} - 0 = \sqrt{3} \approx 1.73$.



39. $\int_{-1}^{2} x^3\,dx = \left[\frac{1}{4}x^4\right]_{-1}^{2} = 4 - \frac{1}{4} = \frac{15}{4} = 3.75$



40. $\int_{\pi/4}^{5\pi/2} \sin x\,dx = [-\cos x]_{\pi/4}^{5\pi/2} = 0 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$



41. $g(x) = \int_{2x}^{3x} \frac{u^2-1}{u^2+1}\,du = \int_{2x}^{0} \frac{u^2-1}{u^2+1}\,du + \int_{0}^{3x} \frac{u^2-1}{u^2+1}\,du = -\int_{0}^{2x} \frac{u^2-1}{u^2+1}\,du + \int_{0}^{3x} \frac{u^2-1}{u^2+1}\,du \;\Rightarrow$

$g'(x) = -\frac{(2x)^2-1}{(2x)^2+1} \cdot \frac{d}{dx}(2x) + \frac{(3x)^2-1}{(3x)^2+1} \cdot \frac{d}{dx}(3x) = -2 \cdot \frac{4x^2-1}{4x^2+1} + 3 \cdot \frac{9x^2-1}{9x^2+1}$

42. $g(x) = \int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt = \int_{\tan x}^{1} \frac{dt}{\sqrt{2+t^4}} + \int_{1}^{x^2} \frac{dt}{\sqrt{2+t^4}} = -\int_{1}^{\tan x} \frac{dt}{\sqrt{2+t^4}} + \int_{1}^{x^2} \frac{dt}{\sqrt{2+t^4}} \;\Rightarrow$

$g'(x) = \frac{-1}{\sqrt{2+\tan^4 x}} \frac{d}{dx}(\tan x) + \frac{1}{\sqrt{2+x^8}} \frac{d}{dx}\left(x^2\right) = -\frac{\sec^2 x}{\sqrt{2+\tan^4 x}} + \frac{2x}{\sqrt{2+x^8}}$

43. $y = \int_{\sqrt{x}}^{x^3} \sqrt{t}\sin t\,dt = \int_{\sqrt{x}}^{1} \sqrt{t}\sin t\,dt + \int_{1}^{x^3} \sqrt{t}\sin t\,dt = -\int_{1}^{\sqrt{x}} \sqrt{t}\sin t\,dt + \int_{1}^{x^3} \sqrt{t}\sin t\,dt \;\Rightarrow$

$y' = -\sqrt[4]{x}\left(\sin\sqrt{x}\right) \cdot \frac{d}{dx}\left(\sqrt{x}\right) + x^{3/2}\sin\left(x^3\right) \cdot \frac{d}{dx}\left(x^3\right) = -\frac{\sqrt[4]{x}\sin\sqrt{x}}{2\sqrt{x}} + x^{3/2}\sin\left(x^3\right)\left(3x^2\right)$

$= 3x^{7/2}\sin\left(x^3\right) - \frac{\sin\sqrt{x}}{2\sqrt[4]{x}}$

44. $y = \int_{\cos x}^{5x} \cos\left(u^2\right)du = \int_{0}^{5x} \cos\left(u^2\right)du - \int_{0}^{\cos x} \cos\left(u^2\right)du \;\Rightarrow$

$y' = \cos\left(25x^2\right) \cdot \frac{d}{dx}(5x) - \cos\left(\cos^2 x\right) \cdot \frac{d}{dx}(\cos x) = \cos\left(25x^2\right) \cdot 5 - \cos\left(\cos^2 x\right) \cdot (-\sin x)$

$= 5\cos\left(25x^2\right) + \sin x\cos\left(\cos^2 x\right)$

45. $F(x) = \int_{1}^{x} f(t)\,dt \;\Rightarrow\; F'(x) = f(x) = \int_{1}^{x^2} \frac{\sqrt{1+u^4}}{u}\,du \;\Rightarrow$

$F''(x) = f'(x) = \frac{\sqrt{1+\left(x^2\right)^4}}{x^2} \cdot \frac{d}{dx}\left(x^2\right) = \frac{2\sqrt{1+x^8}}{x}$. So $F''(2) = \sqrt{1+2^8} = \sqrt{257}$.

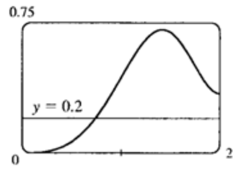
46. For the curve to be concave upward, we must have $y'' > 0$. $y = \int_{0}^{x} \frac{1}{1+t+t^2}\,dt \;\Rightarrow\; y' = \frac{1}{1+x+x^2} \;\Rightarrow$

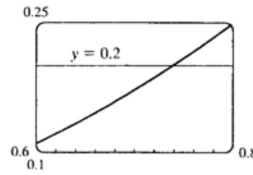
$y'' = \frac{-(1+2x)}{\left(1+x+x^2\right)^2}$. For this expression to be positive, we must have $(1+2x) < 0$, since $\left(1+x+x^2\right)^2 > 0$ for all x. $(1+2x) < 0 \;\Leftrightarrow\; x < -\frac{1}{2}$. Thus, the curve is concave upward on $\left(-\infty, -\frac{1}{2}\right)$.

47. (a) The Fresnel Function $S(x) = \int_0^x \sin(\frac{\pi}{2}t^2)dt$ has local maximum values where $0 = S'(x) = \sin(\frac{\pi}{2}x^2)$ and S' changes from positive to negative. For $x > 0$, this happens when $\frac{\pi}{2}x^2 = (2n-1)\pi$ [odd multiples of π] $\Leftrightarrow$ $x = \sqrt{2(2n-1)}$, n any positive integer. For $x < 0$, S' changes from positive to negative where $\frac{\pi}{2}x^2 = 2n\pi$ [even multiples of π] $\Leftrightarrow$ $x = -2\sqrt{n}$, since if $x < 0$, then as x increases, x^2 decreases. S' does not change sign at $x = 0$.

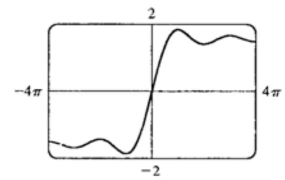
(b) S is concave upward on those intervals where $S''(x) > 0$. Differentiating our expression for $S'(x)$, we get $S''(x) = \cos(\frac{\pi}{2}x^2)\left(2\frac{\pi}{2}x\right) = \pi x\cos(\frac{\pi}{2}x^2)$. For $x > 0$, $S''(x) > 0$ where $\cos(\frac{\pi}{2}x^2) > 0 \;\Leftrightarrow\; 0 < \frac{\pi}{2}x^2 < \frac{\pi}{2}$ or $\left(2n-\frac{1}{2}\right)\pi < \frac{\pi}{2}x^2 < \left(2n+\frac{1}{2}\right)\pi$, n any integer $\Leftrightarrow$ $0 < x < 1$ or $\sqrt{4n-1} < x < \sqrt{4n+1}$, n any positive integer. For $x < 0$, as x increases, x^2 decreases, so the intervals of upward concavity for $x < 0$ are $(-\sqrt{4n-1}, -\sqrt{4n-3})$, n any positive integer. To summarize: S is concave upward on the intervals $(0,1)$, $(-\sqrt{3},-1)$, $(\sqrt{3},\sqrt{5})$, $(-\sqrt{7},-\sqrt{5})$, $(\sqrt{7},3)$, ...

(c) In Maple, we use `plot({int(sin(Pi*t^2/2),t=0..x),0.2},x=0..2);`. Note that Maple recognizes the Fresnel function, calling it `FresnelS(x)`. In Mathematica, we use `Plot[{Integrate[Sin[Pi*t^2/2],{t,0,x}],0.2},{x,0,2}]`. In Derive, we load the utility file `FRESNEL` and plot `FRESNEL_SIN(x)`. From the graphs, we see that $\int_0^x \sin(\frac{\pi}{2}t^2)dt = 0.2$ at $x \approx 0.74$.





48. (a) In Maple, we should start by setting `si:=int(sin(t)/t,t=0..x);`. In Mathematica, the command is `si=Integrate[Sin[t]/t,{t,0,x}]`. Note that both systems recognize this function; Maple calls it `Si(x)` and Mathematica calls it `SinIntegral[x]`. In Maple, the command to generate the graph is `plot(si,x=-4*Pi..4*Pi);`. In Mathematica, it is `Plot[si,{x,-4*Pi,4*Pi}]`. In Derive, we load the utility file `EXP_INT` and plot `SI(x)`.



(b) Si (x) has local maximum values where Si$'(x)$ changes from positive to negative, passing through 0. From the Fundamental Theorem we know that Si$'(x) = \frac{d}{dx}\int_0^x \frac{\sin t}{t}\,dt = \frac{\sin x}{x}$, so we must have $\sin x = 0$ for a maximum, and for $x > 0$ we must have $x = (2n-1)\pi$, n any positive integer, for Si$'$ to be changing from positive to negative at x. For $x < 0$, we must have $x = 2n\pi$, n any positive integer, for a maximum, since the denominator of Si$'(x)$ is negative for $x < 0$. Thus, the local maxima occur at $x = \pi, -2\pi, 3\pi, -4\pi, 5\pi, -6\pi, \ldots$.

(c) To find the first inflection point, we solve Si$''(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2} = 0$. We can see from the graph that the first inflection point lies somewhere between $x = 3$ and $x = 5$. Using a root finder gives the value $x \approx 4.4934$. To find the y-coordinate of the inflection point, we evaluate Si $(4.4934) \approx 1.6556$. So the coordinates of the first inflection point to the right of the origin are about $(4.4934, 1.6556)$. Alternatively, we could graph $S''(x)$ and estimate the first positive x-value at which it changes sign.

(d) It seems from the graph that the function has horizontal asymptotes at $y \approx 1.5$, with $\lim_{x\to\pm\infty}$ Si $(x) \approx \pm 1.5$ respectively. Using the limit command, we get $\lim_{x\to\infty}$ Si $(x) = \frac{\pi}{2}$. Since Si (x) is an odd function, $\lim_{x\to-\infty}$ Si $(x) = -\frac{\pi}{2}$. So Si (x) has the horizontal asymptotes $y = \pm\frac{\pi}{2}$.

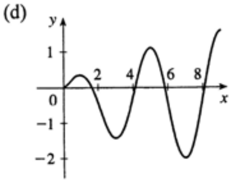
(e) We use the `fsolve` command in Maple (or `FindRoot` in Mathematica) to find that the solution is $x \approx 1.1$. Or, as in Exercise 47(c), we graph $y = $ Si (x) and $y = 1$ on the same screen to see where they intersect.

49. (a) By FTC1, $g'(x) = f(x)$. So $g'(x) = f(x) = 0$ at $x = 1, 3, 5, 7$, and 9. g has local maxima at $x = 1$ and 5 (since $f = g'$ changes from positive to negative there) and local minima at $x = 3$ and 7. There is no local maximum or minimum at $x = 9$, since f is not defined for $x > 9$.

(b) We can see from the graph that $\left|\int_0^1 f\,dt\right| < \left|\int_1^3 f\,dt\right| < \left|\int_3^5 f\,dt\right| < \left|\int_5^7 f\,dt\right| < \left|\int_7^9 f\,dt\right|$. So $g(1) = \left|\int_0^1 f\,dt\right|$, $g(5) = \int_0^5 f\,dt = g(1) - \left|\int_1^3 f\,dt\right| + \left|\int_3^5 f\,dt\right|$, and $g(9) = \int_0^9 f\,dt = g(5) - \left|\int_5^7 f\,dt\right| + \left|\int_7^9 f\,dt\right|$. Thus, $g(1) < g(5) < g(9)$, and so the absolute maximum of $g(x)$ occurs at $x = 9$.

(c) g is concave downward on those intervals where $g'' < 0$. But $g'(x) = f(x)$, so $g''(x) = f'(x)$, which is negative on (approximately) $\left(\frac{1}{2}, 2\right)$, $(4, 6)$ and $(8, 9)$. So g is concave downward on these intervals.

(d)

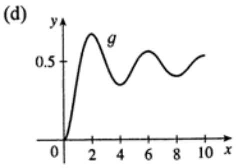


50. (a) By FTC1, $g'(x) = f(x)$. So $g'(x) = f(x) = 0$ at $x = 2, 4, 6, 8$, and 10. g has local maxima at $x = 2$ and 6 (since $f = g'$ changes from positive to negative there) and local minima at $x = 4$ and 8. There is no local maximum or minimum at $x = 10$, since f is not defined for $x > 10$.

(b) We can see from the graph that $\left|\int_0^2 f\,dt\right| > \left|\int_2^4 f\,dt\right| > \left|\int_4^6 f\,dt\right| > \left|\int_6^8 f\,dt\right| > \left|\int_8^{10} f\,dt\right|$. So $g(2) = \left|\int_0^2 f\,dt\right|$, $g(6) = \int_0^6 f\,dt = g(2) - \left|\int_2^4 f\,dt\right| + \left|\int_4^6 f\,dt\right|$, and $g(10) = \int_0^{10} f\,dt = g(6) - \left|\int_6^8 f\,dt\right| + \left|\int_8^{10} f\,dt\right|$. Thus, $g(2) > g(6) > g(10)$, and so the absolute maximum of $g(x)$ occurs at $x = 2$.

(c) g is concave downward on those intervals where $g'' < 0$. But $g'(x) = f(x)$, so $g''(x) = f'(x)$, which is negative on $(1, 3)$, $(5, 7)$ and $(9, 10)$. So g is concave downward on these intervals.

(d)



51. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4} = \lim_{n\to\infty} \frac{1-0}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 = \int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$

52. $\displaystyle \lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}\right) = \lim_{n\to\infty} \frac{1-0}{n} \sum_{i=1}^{n} \sqrt{\frac{i}{n}} = \int_0^1 \sqrt{x}\,dx = \left[\frac{2x^{3/2}}{3}\right]_0^1 = \frac{2}{3} - 0 = \frac{2}{3}$

53. Suppose $h < 0$. Since f is continuous on $[x+h, x]$, the Extreme Value Theorem says that there are numbers u and v in $[x+h, x]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x+h, x]$. By Property 8 of integrals, $m(-h) \le \int_{x+h}^{x} f(t)\,dt \le M(-h)$; that is, $f(u)(-h) \le -\int_x^{x+h} f(t)\,dt \le f(v)(-h)$. Since $-h > 0$, we can divide this inequality by $-h$: $f(u) \le \dfrac{1}{h}\displaystyle\int_x^{x+h} f(t)\,dt \le f(v)$. By Equation 2, $\dfrac{g(x+h) - g(x)}{h} = \dfrac{1}{h}\displaystyle\int_x^{x+h} f(t)\,dt$ for $h \ne 0$, and hence $f(u) \le \dfrac{g(x+h) - g(x)}{h} \le f(v)$, which is Equation 3 in the case where $h < 0$.

54. $\dfrac{d}{dx}\displaystyle\int_{g(x)}^{h(x)} f(t)\,dt = \dfrac{d}{dx}\left[\int_{g(x)}^{a} f(t)\,dt + \int_{a}^{h(x)} f(t)\,dt\right]$ (where a is in the domain of f)

$$= \frac{d}{dx}\left[-\int_a^{g(x)} f(t)\,dt\right] + \frac{d}{dx}\left[\int_a^{h(x)} f(t)\,dt\right] = -f(g(x))\,g'(x) + f(h(x))\,h'(x)$$

$$= f(h(x))\,h'(x) - f(g(x))\,g'(x)$$

55. (a) Let $f(x) = \sqrt{x} \Rightarrow f'(x) = 1/(2\sqrt{x}) > 0$ for $x > 0 \Rightarrow f$ is increasing on $(0, \infty)$. If $x \geq 0$, then $x^3 \geq 0$, so $1 + x^3 \geq 1$ and since f is increasing, this means that $f(1 + x^3) \geq f(1) \Rightarrow \sqrt{1 + x^3} \geq 1$ for $x \geq 0$. Next let $g(t) = t^2 - t \Rightarrow g'(t) = 2t - 1 \Rightarrow g'(t) > 0$ when $t \geq 1$. Thus, g is increasing on $(1, \infty)$. And since $g(1) = 0$, $g(t) \geq 0$ when $t \geq 1$. Now let $t = \sqrt{1 + x^3}$, where $x \geq 0$. $\sqrt{1 + x^3} \geq 1$ (from above) $\Rightarrow t \geq 1 \Rightarrow g(t) \geq 0 \Rightarrow (1 + x^3) - \sqrt{1 + x^3} \geq 0$ for $x \geq 0$. Therefore, $1 \leq \sqrt{1 + x^3} \leq 1 + x^3$ for $x \geq 0$.

(b) From part (a) and Property 7: $\int_0^1 1\,dx \leq \int_0^1 \sqrt{1 + x^3}\,dx \leq \int_0^1 (1 + x^3)\,dx \Leftrightarrow$ $[x]_0^1 \leq \int_0^1 \sqrt{1 + x^3}\,dx \leq \left[x + \frac{1}{4}x^4\right]_0^1 \Leftrightarrow 1 \leq \int_0^1 \sqrt{1 + x^3}\,dx \leq 1 + \frac{1}{4} = 1.25.$

56. (a) If $x < 0$, then $g(x) = \int_0^x f(t)\,dt = \int_0^x 0\,dt = 0$.

If $0 \leq x \leq 1$, then $g(x) = \int_0^x f(t)\,dt = \int_0^x t\,dt = \left[\frac{1}{2}t^2\right]_0^x = \frac{1}{2}x^2$.

If $1 < x \leq 2$, then

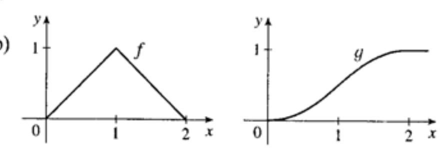
$$g(x) = \int_0^x f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^x f(t)\,dt = g(1) + \int_1^x (2 - t)\,dt = \tfrac{1}{2}(1)^2 + \left[2t - \tfrac{1}{2}t^2\right]_1^x$$

$$= \tfrac{1}{2} + \left(2x - \tfrac{1}{2}x^2\right) - \left(2 - \tfrac{1}{2}\right) = 2x - \tfrac{1}{2}x^2 - 1.$$

If $x > 2$, then $g(x) = \int_0^x f(t)\,dt = g(2) + \int_2^x 0\,dt = 1 + 0 = 1$. So

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2}x^2 & \text{if } 0 \leq x \leq 1 \\ 2x - \frac{1}{2}x^2 - 1 & \text{if } 1 < x \leq 2 \\ 1 & \text{if } x > 2 \end{cases}$$

(b)



(c) f is not differentiable at its corners at $x = 0$, 1, and 2. f is differentiable on $(-\infty, 0)$, $(0, 1)$, $(1, 2)$ and $(2, \infty)$. g is differentiable on $(-\infty, \infty)$.

57. Using FTC1, we differentiate both sides of $6 + \displaystyle\int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x}$ to get $\dfrac{f(x)}{x^2} = 2\dfrac{1}{2\sqrt{x}} \Rightarrow f(x) = x^{3/2}$.

To find a, we substitute $x = a$ in the original equation to obtain $6 + \displaystyle\int_a^a \frac{f(t)}{t^2}\,dt = 2\sqrt{a} \Rightarrow 6 + 0 = 2\sqrt{a} \Rightarrow$ $a = 9$.

58. By the Fundamental Theorem of Calculus, $\int_1^2 (h')'(u)\,du = h'(2) - h'(1) = 5 - 2 = 3$. The other information is unnecessary.

59. (a) Let $F(t)=\int_0^t f(s)\,ds$. Then, by FTC1, $F'(t)=f(t)=$ rate of depreciation, so $F(t)$ represents the loss in value over the interval $[0,t]$.

(b) $C(t)=\dfrac{A+F(t)}{t}$ represents the average expenditure per unit of t during the interval $[0,t]$, assuming that there has been only one overhaul during that time period. The company wants to minimize average expenditure.

(c) $C(t)=\dfrac{1}{t}\left[A+\displaystyle\int_0^t f(s)\,ds\right]$. Using FTC1, we have $C'(t)=-\dfrac{1}{t^2}\left[A+\displaystyle\int_0^t f(s)\,ds\right]+\dfrac{1}{t}f(t)$. $C'(t)=0$ $\Rightarrow$ $t\,f(t)=A+\displaystyle\int_0^t f(s)\,ds$ $\Rightarrow$ $f(t)=\dfrac{1}{t}\left[A+\displaystyle\int_0^t f(s)\,ds\right]=C(t)$.

60. (a) $C(t)=(1/t)\int_0^t[f(s)+g(s)]\,ds$. Using FTC1, we have

$C'(t)=\dfrac{1}{t}[f(t)+g(t)]-\dfrac{1}{t^2}\int_0^t[f(s)+g(s)]\,ds$. Set $C'(t)=0$:

$\dfrac{1}{t}[f(t)+g(t)]-\dfrac{1}{t^2}\int_0^t[f(s)+g(s)]\,ds=0$ $\Rightarrow$ $[f(t)+g(t)]-\dfrac{1}{t}\int_0^t[f(s)+g(s)]\,ds=0$ $\Rightarrow$ $[f(t)+g(t)]-C(t)=0$ or $C(t)=f(t)+g(t)$.

(b) For $0\le t\le 30$, we have $D(t)=\displaystyle\int_0^t\left(\frac{V}{15}-\frac{V}{450}s\right)ds=\left[\frac{V}{15}s-\frac{V}{900}s^2\right]_0^t=\frac{V}{15}t-\frac{V}{900}t^2$. So $D(t)=V$ $\Rightarrow$ $60t-t^2=900$ $\Rightarrow$ $t^2-60t+900=0$ $\Rightarrow$ $(t-30)^2=0$ $\Rightarrow$ $T=30$.

(c)
$$C(t)=\frac{1}{t}\int_0^t\left(\frac{V}{15}-\frac{V}{450}s+\frac{V}{12{,}900}s^2\right)ds=\frac{1}{t}\left[\frac{V}{15}s-\frac{V}{900}s^2+\frac{V}{38{,}700}s^3\right]_0^t$$
$$=\frac{1}{t}\left(\frac{V}{15}t-\frac{V}{900}t^2+\frac{V}{38{,}700}t^3\right)=\frac{V}{15}-\frac{V}{900}t+\frac{V}{38{,}700}t^2\quad\Rightarrow$$

$C'(t)=-\dfrac{V}{900}+\dfrac{V}{19{,}350}t=0$ when $\dfrac{1}{19{,}350}t=\dfrac{1}{900}$ $\Rightarrow$ $t=21.5$.

$C(21.5)=\dfrac{V}{15}-\dfrac{V}{900}(21.5)+\dfrac{V}{38{,}700}(21.5)^2\approx 0.05472V$, $C(0)=\dfrac{V}{15}\approx 0.06667V$, and

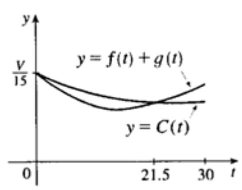
$C(30)=\dfrac{V}{15}-\dfrac{V}{900}(30)+\dfrac{V}{38{,}700}(30)^2\approx 0.05659V$, so the absolute minimum is $C(21.5)\approx 0.05472V$.

(d) As in part (c), we have $C(t)=\dfrac{V}{15}-\dfrac{V}{900}t+\dfrac{V}{38{,}700}t^2$, so

$C(t)=f(t)+g(t)$ $\Leftrightarrow$ $\dfrac{V}{15}-\dfrac{V}{900}t+\dfrac{V}{38{,}700}t^2=\dfrac{V}{15}-\dfrac{V}{450}t+\dfrac{V}{12{,}900}t^2$

$\Leftrightarrow$ $t^2\left(\dfrac{1}{12{,}900}-\dfrac{1}{38{,}700}\right)=t\left(\dfrac{1}{450}-\dfrac{1}{900}\right)$ $\Leftrightarrow$

$t=\dfrac{1/900}{2/38{,}700}=\dfrac{43}{2}=21.5$. This is the value of t that we obtained as the critical number of C in part (c), so we have verified the result of (a) in this case.



61. $\int_4^8(1/x)\,dx=[\ln x]_4^8=\ln 8-\ln 4=\ln\frac{8}{4}=\ln 2$

62. $\int_{\ln 3}^{\ln 6}8e^x\,dx=[8e^x]_{\ln 3}^{\ln 6}=8\left(e^{\ln 6}-e^{\ln 3}\right)=8(6-3)=24$

63. $\displaystyle\int_8^9 2^t\,dt=\left[\frac{1}{\ln 2}2^t\right]_8^9=\frac{1}{\ln 2}\left(2^9-2^8\right)=\frac{2^8}{\ln 2}$

64. $\int_{-e^2}^{-e} (3/x)\,dx = [3\ln|x|]_{-e^2}^{-e} = 3\ln e - 3\ln\left(e^2\right) = 3\cdot 1 - 3\cdot 2 = -3$

65. $\displaystyle\int_1^{\sqrt{3}} \frac{6}{1+x^2}\,dx = 6\left[\tan^{-1}x\right]_1^{\sqrt{3}} = 6\tan^{-1}\sqrt{3} - 6\tan^{-1}1 = 6\cdot\frac{\pi}{3} - 6\cdot\frac{\pi}{4} = \frac{\pi}{2}$

66. $\displaystyle\int_0^{0.5} \frac{dx}{\sqrt{1-x^2}} = \left[\sin^{-1}x\right]_0^{0.5} = \sin^{-1}\frac{1}{2} - \sin^{-1}0 = \frac{\pi}{6}$

5.4 Indefinite Integrals and the Total Change Theorem

1. $\displaystyle\frac{d}{dx}\left[\sqrt{x^2+1}+C\right] = \tfrac{1}{2}\left(x^2+1\right)^{-1/2}\cdot 2x = \frac{x}{\sqrt{x^2+1}}$

2. $\displaystyle\frac{d}{dx}[x\sin x + \cos x + C] = x\cos x + (\sin x)\cdot 1 - \sin x = x\cos x$

3. $\displaystyle\frac{d}{dx}\left[\frac{x}{a^2\sqrt{a^2-x^2}}+C\right] = \frac{1}{a^2}\frac{\sqrt{a^2-x^2} - x\,(-x)\Big/\sqrt{a^2-x^2}}{a^2-x^2} = \frac{1}{a^2}\frac{\left(a^2-x^2\right)+x^2}{\left(a^2-x^2\right)^{3/2}} = \frac{1}{\sqrt{\left(a^2-x^2\right)^3}}$

4. $\displaystyle\frac{d}{dx}\left[-\frac{\sqrt{x^2+a^2}}{a^2x}+C\right] = -\frac{1}{a^2}\frac{d}{dx}\left[\frac{\sqrt{x^2+a^2}}{x}\right] = -\frac{x\left(x/\sqrt{x^2+a^2}\right) - \sqrt{x^2+a^2}\cdot 1}{a^2x^2}$

$$= -\frac{x^2-\left(x^2+a^2\right)}{a^2x^2\sqrt{x^2+a^2}} = \frac{1}{x^2\sqrt{x^2+a^2}}$$

5. $\displaystyle\int x^{-3/4}\,dx = \frac{x^{-3/4+1}}{-3/4+1}+C = \frac{x^{1/4}}{1/4}+C = 4x^{1/4}+C$

6. $\displaystyle\int \sqrt[3]{x}\,dx = \int x^{1/3}\,dx = \frac{x^{4/3}}{4/3}+C = \tfrac{3}{4}x^{4/3}+C$

7. $\displaystyle\int\left(x^3+6x+1\right)dx = \frac{x^4}{4}+6\frac{x^2}{2}+x+C = \tfrac{1}{4}x^4+3x^2+x+C$

8. $\displaystyle\int x\left(1+2x^4\right)dx = \int\left(x+2x^5\right)dx = \frac{x^2}{2}+2\frac{x^2}{6}+C = \tfrac{1}{2}x^2+\tfrac{1}{3}x^6+C$

9. $\displaystyle\int(1-t)\left(2+t^2\right)dt = \int\left(2-2t+t^2-t^3\right)dt = 2t-2\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+C = 2t-t^2+\tfrac{1}{3}t^3-\tfrac{1}{4}t^4+C$

10. $\displaystyle\int\left(u^2+1+\frac{1}{u^2}\right)du = \int\left(u^2+1+u^{-2}\right)du = \frac{u^3}{3}+u+\frac{u^{-1}}{-1}+C = \tfrac{1}{3}u^3+u-\frac{1}{u}+C$

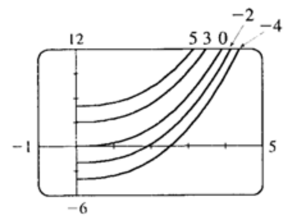
11. $\displaystyle\int\left(2-\sqrt{x}\right)^2dx = \int\left(4-4\sqrt{x}+x\right)dx = 4x-4\frac{x^{3/2}}{3/2}+\frac{x^2}{2}+C = 4x-\tfrac{8}{3}x^{3/2}+\tfrac{1}{2}x^2+C$

12. $\int(\sin\theta+3\cos\theta)\,d\theta = -\cos\theta+3\sin\theta+C$

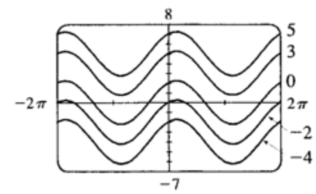
13. $\displaystyle\int\frac{\sin x}{1-\sin^2x}\,dx = \int\frac{\sin x}{\cos^2x}\,dx = \int\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}\,dx = \int\sec x\tan x\,dx = \sec x+C$

14. $\displaystyle\int\frac{\sin 2x}{\sin x}\,dx = \int\frac{2\sin x\cos x}{\sin x}\,dx = \int 2\cos x\,dx = 2\sin x+C$

15. $\int x\sqrt{x}\,dx = \int x^{3/2}\,dx = \frac{2}{5}x^{5/2} + C$



16. $\int(\cos x - 2\sin x)\,dx = \sin x + 2\cos x + C$



17. $\int_0^1 (1 - 2x - 3x^2)\,dx = \left[x - 2\cdot\frac{1}{2}x^2 - 3\cdot\frac{1}{3}x^3\right]_0^1 = \left[x - x^2 - x^3\right]_0^1 = (1 - 1 - 1) - 0 = -1$

18. $\int_1^2 (5x^2 - 4x + 3)\,dx = \left[5\cdot\frac{1}{3}x^3 - 4\cdot\frac{1}{2}x^2 + 3x\right]_1^2 = 5\cdot\frac{8}{3} - 4\cdot 2 + 6 - \left(\frac{5}{3} - 2 + 3\right) = \frac{26}{3}$

19. $\int_{-3}^0 (5y^4 - 6y^2 + 14)\,dy = \left[5\left(\frac{1}{5}y^5\right) - 6\left(\frac{1}{3}y^3\right) + 14y\right]_{-3}^0 = \left[y^5 - 2y^3 + 14y\right]_{-3}^0 = 0 - (-243 + 54 - 42) = 231$

20. $\int_0^1 (y^9 - 2y^5 + 3y)\,dy = \left[\frac{1}{10}y^{10} - 2\left(\frac{1}{6}y^6\right) + 3\left(\frac{1}{2}y^2\right)\right]_0^1 = \left(\frac{1}{10} - \frac{1}{3} + \frac{3}{2}\right) - 0 = \frac{19}{15}$

21. $\displaystyle\int_1^3 \left(\frac{1}{t^2} - \frac{1}{t^4}\right)dt = \int_1^3 \left(t^{-2} - t^{-4}\right)dt = \left[\frac{t^{-1}}{-1} - \frac{t^{-3}}{-3}\right]_1^3 = \left[\frac{1}{3t^3} - \frac{1}{t}\right]_1^3 = \left(\frac{1}{81} - \frac{1}{3}\right) - \left(\frac{1}{3} - 1\right) = \frac{28}{81}$

22. $\displaystyle\int_1^2 \frac{t^6 - t^2}{t^4}\,dt = \int_1^2 \left(t^2 - t^{-2}\right)dt = \left[\frac{t^3}{3} - \frac{t^{-1}}{-1}\right]_1^2 = \left[\frac{t^3}{3} + \frac{1}{t}\right]_1^2 = \left(\frac{8}{3} + \frac{1}{2}\right) - \left(\frac{1}{3} + 1\right) = \frac{11}{6}$

23. $\displaystyle\int_1^2 \frac{x^2 + 1}{\sqrt{x}}\,dx = \int_1^2 \left(x^{3/2} + x^{-1/2}\right)dx = \left[\frac{x^{5/2}}{5/2} + \frac{x^{1/2}}{1/2}\right]_1^2 = \left[\frac{2}{5}x^{5/2} + 2x^{1/2}\right]_1^2$

$= \left(\frac{2}{5}4\sqrt{2} + 2\sqrt{2}\right) - \left(\frac{2}{5} + 2\right) = \frac{18\sqrt{2} - 12}{5} = \frac{6}{5}\left(3\sqrt{2} - 2\right)$

24. $\int_0^2 (x^3 - 1)^2\,dx = \int_0^2 (x^6 - 2x^3 + 1)\,dx = \left[\frac{1}{7}x^7 - 2\left(\frac{1}{4}x^4\right) + x\right]_0^2 = \left(\frac{128}{7} - 2\cdot 4 + 2\right) - 0 = \frac{86}{7}$

25. $\int_0^1 u\left(\sqrt{u} + \sqrt[3]{u}\right)du = \int_0^1 \left(u^{3/2} + u^{4/3}\right)du = \left[\frac{u^{5/2}}{5/2} + \frac{u^{7/3}}{7/3}\right]_0^1 = \left[\frac{2}{5}u^{5/2} + \frac{3}{7}u^{7/3}\right]_0^1 = \frac{2}{5} + \frac{3}{7} = \frac{29}{35}$

26. $\displaystyle\int_1^2 \left(x + \frac{1}{x}\right)^2 dx = \int_1^2 \left(x^2 + 2 + x^{-2}\right)dx = \left[\frac{x^3}{3} + 2x + \frac{x^{-1}}{-1}\right]_1^2 = \left[\frac{x^3}{3} + 2x - \frac{1}{x}\right]_1^2$

$= \left(\frac{8}{3} + 4 - \frac{1}{2}\right) - \left(\frac{1}{3} + 2 - 1\right) = \frac{29}{6}$

27. $\int_1^4 \sqrt{5/x}\,dx = \sqrt{5}\int_1^4 x^{-1/2}\,dx = \sqrt{5}\left[2\sqrt{x}\right]_1^4 = \sqrt{5}\,(2\cdot 2 - 2\cdot 1) = 2\sqrt{5}$

28. $\int_{-1}^2 |x - x^2|\,dx = \int_{-1}^0 (x^2 - x)\,dx + \int_0^1 (x - x^2)\,dx + \int_1^2 (x^2 - x)\,dx$

$= \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^0 + \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2$

$= 0 - \left(-\frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) - 0 + \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - \frac{1}{2}\right) = \frac{11}{6}$

29. $\int_{-2}^{3} |x^2 - 1|\, dx = \int_{-2}^{-1} (x^2 - 1)\, dx + \int_{-1}^{1} (1 - x^2)\, dx + \int_{1}^{3} (x^2 - 1)\, dx$

$$= \left[\frac{x^3}{3} - x\right]_{-2}^{-1} + \left[x - \frac{x^3}{3}\right]_{-1}^{1} + \left[\frac{x^3}{3} - x\right]_{1}^{3}$$

$$= \left(-\tfrac{1}{3} + 1\right) - \left(-\tfrac{8}{3} + 2\right) + \left(1 - \tfrac{1}{3}\right) - \left(-1 + \tfrac{1}{3}\right) + (9 - 3) - \left(\tfrac{1}{3} - 1\right) = \tfrac{28}{3}$$

30. $\int_{1}^{-1} (x - 1)(3x + 2)\, dx = -\int_{-1}^{1} (3x^2 - x - 2)\, dx = -\left[3\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^{1} = \left[-x^3 + \frac{x^2}{2} + 2x\right]_{-1}^{1}$

$$= \left(-1 + \tfrac{1}{2} + 2\right) - \left(1 + \tfrac{1}{2} - 2\right) = 2$$

31. $\displaystyle\int_{1}^{4} \left(\sqrt{t} - \frac{2}{\sqrt{t}}\right) dt = \int_{1}^{4} \left(t^{1/2} - 2t^{-1/2}\right) dt = \left[\frac{t^{3/2}}{3/2} - 2\frac{t^{1/2}}{1/2}\right]_{1}^{4} = \left[\tfrac{2}{3}t^{3/2} - 4t^{1/2}\right]_{1}^{4}$

$$= \left(\tfrac{2}{3} \cdot 8 - 4 \cdot 2\right) - \left(\tfrac{2}{3} - 4\right) = \tfrac{2}{3}$$

32. $\displaystyle\int_{1}^{8} \left(\sqrt[3]{r} + \frac{1}{\sqrt[3]{r}}\right) dr = \int_{1}^{8} \left(r^{1/3} + r^{-1/3}\right) dr = \left[\frac{r^{4/3}}{4/3} + \frac{r^{2/3}}{2/3}\right]_{1}^{8} = \left[\tfrac{3}{4}r^{4/3} + \tfrac{3}{2}r^{2/3}\right]_{1}^{8}$

$$= \left(\tfrac{3}{4} \cdot 16 + \tfrac{3}{2} \cdot 4\right) - \left(\tfrac{3}{4} + \tfrac{3}{2}\right) = \tfrac{63}{4}$$

33. $\int_{-1}^{0} (x + 1)^3\, dx = \int_{-1}^{0} \left(x^3 + 3x^2 + 3x + 1\right) dx = \left[\frac{x^4}{4} + 3\frac{x^3}{3} + 3\frac{x^2}{2} + x\right]_{-1}^{0} = 0 - \left[\tfrac{1}{4} - 1 + \tfrac{3}{2} - 1\right]$

$$= 2 - \tfrac{7}{4} = \tfrac{1}{4}$$

34. $\displaystyle\int_{-5}^{-2} \frac{x^4 - 1}{x^2 + 1}\, dx = \int_{-5}^{-2} \left(x^2 - 1\right) dx = \left[\frac{x^3}{3} - x\right]_{-5}^{-2} = \left(-\frac{8}{3} + 2\right) - \left(\frac{-125}{3} + 5\right) = 36$

35. $\int_{\pi/6}^{\pi/3} \csc^2\theta\, d\theta = [-\cot\theta]_{\pi/6}^{\pi/3} = -\cot\frac{\pi}{3} + \cot\frac{\pi}{6} = -\frac{1}{3}\sqrt{3} + \sqrt{3} = \frac{2}{3}\sqrt{3}$

36. $\int_{0}^{\pi/2} (\cos\theta + 2\sin\theta)\, d\theta = [\sin\theta - 2\cos\theta]_{0}^{\pi/2} = (1 - 2 \cdot 0) - (0 - 2 \cdot 1) = 3$

37. $\displaystyle\int_{0}^{\pi/4} \frac{1 + \cos^2\theta}{\cos^2\theta}\, d\theta = \int_{0}^{\pi/4} \left(\frac{1}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta}\right) d\theta = \int_{0}^{\pi/4} \left(\sec^2\theta + 1\right) d\theta = [\tan\theta + \theta]_{0}^{\pi/4}$

$$= \left(1 + \tfrac{\pi}{4}\right) - (0 + 0) = 1 + \tfrac{\pi}{4}$$

38. $\int_{\pi/3}^{\pi/2} \csc x \cot x\, dx = [-\csc x]_{\pi/3}^{\pi/2} = -\csc\frac{\pi}{2} + \csc\frac{\pi}{3} = -1 + \frac{2}{3}\sqrt{3}$

39. $\int_{0}^{1} \left(\sqrt[4]{x^5} + \sqrt[5]{x^4}\right) dx = \int_{0}^{1} \left(x^{5/4} + x^{4/5}\right) dx = \left[\frac{x^{9/4}}{9/4} + \frac{x^{9/5}}{9/5}\right]_{0}^{1} = \left[\tfrac{4}{9}x^{9/4} + \tfrac{5}{9}x^{9/5}\right]_{0}^{1} = \tfrac{4}{9} + \tfrac{5}{9} - 0 = 1$

40. $\displaystyle\int_{1}^{8} \frac{x - 1}{\sqrt[3]{x^2}}\, dx = \int_{1}^{8} \left(x^{1/3} - x^{-2/3}\right) dx = \left[\frac{x^{4/3}}{4/3} - \frac{x^{1/3}}{1/3}\right]_{1}^{8} = \left[\frac{3}{4}x^{4/3} - 3x^{1/3}\right]_{1}^{8}$

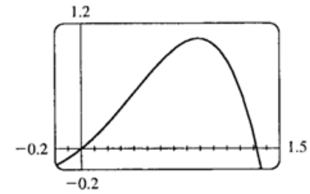
$$= \left(\tfrac{3}{4} \cdot 16 - 3 \cdot 2\right) - \left(\tfrac{3}{4} - 3\right) = \tfrac{33}{4}$$

41. $\int_{-1}^{2} (x - 2|x|)\, dx = \int_{-1}^{0} 3x\, dx + \int_{0}^{2} (-x)\, dx = 3\left[\tfrac{1}{2}x^2\right]_{-1}^{0} - \left[\tfrac{1}{2}x^2\right]_{0}^{2} = \left(3 \cdot 0 - 3 \cdot \tfrac{1}{2}\right) - (2 - 0) = -\tfrac{7}{2} = -3.5$

42. $\int_{0}^{2} \left(x^2 - |x - 1|\right) dx = \int_{0}^{1} \left(x^2 + x - 1\right) dx + \int_{1}^{2} \left(x^2 - x + 1\right) dx = \left[\frac{x^3}{3} + \frac{x^2}{2} - x\right]_{0}^{1} + \left[\frac{x^3}{3} - \frac{x^2}{2} + x\right]_{1}^{2}$

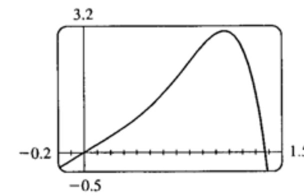
$$= \left(\tfrac{1}{3} + \tfrac{1}{2} - 1\right) - 0 + \left(\tfrac{8}{3} - 2 + 2\right) - \left(\tfrac{1}{3} - \tfrac{1}{2} + 1\right) = \tfrac{5}{3}$$

43. The graph shows that $y = x + x^2 - x^4$ has x-intercepts at $x = 0$ and at $x \approx 1.32$. So the area of the region below the curve and above the x-axis is about



$$\int_0^{1.32} (x + x^2 - x^4)\,dx = \left[\tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\right]_0^{1.32}$$
$$= \left[\tfrac{1}{2}(1.32)^2 + \tfrac{1}{3}(1.32)^3 - \tfrac{1}{5}(1.32)^5\right] - 0$$
$$\approx 0.84$$

44. The graph shows that $y = 2x + 3x^4 - 2x^6$ has x-intercepts at $x = 0$ and at $x \approx 1.37$. So the area of the region below the curve and above the x-axis is about



$$\int_0^{1.37} (2x + 3x^4 - 2x^6)\,dx = \left[x^2 + \tfrac{3}{5}x^5 - \tfrac{2}{7}x^7\right]_0^{1.37}$$
$$= \left[(1.37)^2 + \tfrac{3}{5}(1.37)^5 - \tfrac{2}{7}(1.37)^7\right] - 0$$
$$\approx 2.18$$

45. $A = \int_0^2 (2y - y^2)\,dy = \left[y^2 - \frac{1}{3}y^3\right]_0^2 = \left(4 - \frac{8}{3}\right) - 0 = \frac{4}{3}$

46. $y = \sqrt[4]{x} \quad\Rightarrow\quad x = y^4$, so $A = \int_0^1 y^4\,dy = \left[\frac{1}{5}y^5\right]_0^1 = \frac{1}{5}$.

47. If $w'(t)$ is the rate of change of weight in pounds per year, then $w(t)$ represents the weight in pounds of the child at age t. We know from the Total Change Theorem that $\int_5^{10} w'(t)\,dt = w(10) - w(5)$, so the integral represents the increase in the child's weight between the ages of 5 and 10.

48. $\int_a^b I(t)\,dt = \int_a^b Q'(t)\,dt = Q(b) - Q(a)$ by the Total Change Theorem, so it represents the change in the charge Q from time $t = a$ to $t = b$.

49. Since $r(t)$ is the rate at which oil leaks, we can write $r(t) = -V'(t)$, where $V(t)$ is the volume of oil at time t. [Note that the minus sign is needed because V is decreasing, so $V'(t)$ is negative, but $r(t)$ is positive.] Thus, by the Total Change Theorem, $\int_0^{120} r(t)\,dt = -\int_0^{120} V'(t)\,dt = -[V(120) - V(0)] = V(0) - V(120)$, which is the number of gallons of oil that leaked from the tank in the first two hours.

50. By the Total Change Theorem, $\int_0^{15} n'(t)\,dt = n(15) - n(0) = n(15) - 100$ represents the increase in the bee population in 15 weeks. So $100 + \int_0^{15} n'(t)\,dt = n(15)$ represents the total bee population after 15 weeks.

51. By the Total Change Theorem, $\int_{1000}^{5000} R'(x)\,dx = R(5000) - R(1000)$, so it represents the increase in revenue when production is increased from 1000 units to 5000 units.

52. The slope of the trail is the rate of change of the elevation E, so $f(x) = E'(x)$. By the Total Change Theorem, $\int_3^5 f(x)\,dx = \int_3^5 E'(x)\,dx = E(5) - E(3)$ is the change in the elevation E between $x = 3$ miles and $x = 5$ miles from the start of the trail.

53. (a) displacement $= \int_0^3 (3t - 5)\,dt = \left[\frac{3}{2}t^2 - 5t\right]_0^3 = \frac{27}{2} - 15 = -\frac{3}{2}$ m

(b) distance traveled $= \int_0^3 |3t - 5|dt = \int_0^{5/3} (5 - 3t)\,dt + \int_{5/3}^3 (3t - 5)\,dt$

$$= \left[5t - \tfrac{3}{2}t^2\right]_0^{5/3} + \left[\tfrac{3}{2}t^2 - 5t\right]_{5/3}^3 = \tfrac{25}{3} - \tfrac{3}{2}\cdot\tfrac{25}{9} + \tfrac{27}{2} - 15 - \left(\tfrac{3}{2}\cdot\tfrac{25}{9} - \tfrac{25}{3}\right) = \tfrac{41}{6}\text{ m}$$

54. (a) displacement $= \int_1^6 (t^2 - 2t - 8)\,dt = \left[\frac{1}{3}t^3 - t^2 - 8t\right]_1^6 = (72 - 36 - 48) - \left(\frac{1}{3} - 1 - 8\right) = -\frac{10}{3}$ m

(b) distance traveled $= \int_1^6 |t^2 - 2t - 8|\,dt = \int_1^6 |(t-4)(t+2)|\,dt$

$$= \int_1^4 (-t^2 + 2t + 8)\,dt + \int_4^6 (t^2 - 2t - 8)\,dt = \left[-\tfrac{1}{3}t^3 + t^2 + 8t\right]_1^4 + \left[\tfrac{1}{3}t^3 - t^2 - 8t\right]_4^6$$

$$= \left(-\tfrac{64}{3} + 16 + 32\right) - \left(-\tfrac{1}{3} + 1 + 8\right) + (72 - 36 - 48) - \left(\tfrac{64}{3} - 16 - 32\right) = \tfrac{98}{3}\text{ m}$$

55. (a) $v'(t) = a(t) = t + 4 \Rightarrow v(t) = \frac{1}{2}t^2 + 4t + C \Rightarrow v(0) = C = 5 \Rightarrow v(t) = \frac{1}{2}t^2 + 4t + 5$ m/s

(b) distance traveled $= \int_0^{10} |v(t)|\,dt = \int_0^{10} \left|\frac{1}{2}t^2 + 4t + 5\right| dt = \int_0^{10} \left(\frac{1}{2}t^2 + 4t + 5\right) dt$

$$= \left[\tfrac{1}{6}t^3 + 2t^2 + 5t\right]_0^{10} = \tfrac{500}{3} + 200 + 50 = 416\tfrac{2}{3}\text{ m}$$

56. (a) $v'(t) = 2t + 3 \Rightarrow v(t) = t^2 + 3t + C \Rightarrow v(0) = C = -4 \Rightarrow v(t) = t^2 + 3t - 4$

(b) distance traveled $= \int_0^3 |t^2 + 3t - 4|\,dt = \int_0^3 |(t+4)(t-1)|\,dt$

$$= \int_0^1 (-t^2 - 3t + 4)\,dt + \int_1^3 (t^2 + 3t - 4)\,dt$$

$$= \left[-\tfrac{1}{3}t^3 - \tfrac{3}{2}t^2 + 4t\right]_0^1 + \left[\tfrac{1}{3}t^3 + \tfrac{3}{2}t^2 - 4t\right]_1^3$$

$$= \left(-\tfrac{1}{3} - \tfrac{3}{2} + 4\right) + \left(9 + \tfrac{27}{2} - 12\right) - \left(\tfrac{1}{3} + \tfrac{3}{2} - 4\right) = \tfrac{89}{6}\text{ m}$$

57. Since $m'(x) = \rho(x)$, $m = \int_0^4 \rho(x)\,dx = \int_0^4 (9 + 2\sqrt{x})\,dx = \left[9x + \frac{4}{3}x^{3/2}\right]_0^4 = 36 + \frac{32}{3} - 0 = \frac{140}{3} = 46\frac{2}{3}$ kg.

58. $n(10) - n(4) = \int_4^{10} (200 + 50t)\,dt = \left[200t + 25t^2\right]_4^{10} = 2000 + 2500 - (800 + 400) = 3300$

59. Let s be the position of the car. We know from Equation 2 that $s(100) - s(0) = \int_0^{100} v(t)\,dt$. We use the Midpoint Rule for $0 \le t \le 100$ with $n = 5$. Note that the length of each of the five time intervals is 20 seconds $= \frac{1}{180}$ hour. So the distance traveled is

$$\int_0^{100} v(t)\,dt \approx \tfrac{1}{180}[v(10) + v(30) + v(50) + v(70) + v(90)] = \tfrac{1}{180}(38 + 58 + 51 + 53 + 47)$$

$$= \tfrac{247}{180} \approx 1.4\text{ miles}$$

60. The total percentage increase in the CPI is $\int_{1981}^{1997} r(t)\,dt$. Using the Midpoint Rule with $n = 8$ and $\Delta t = 2$ gives us

$$\int_{1981}^{1997} r(t)\,dt \approx 2[r(1982) + r(1984) + \cdots + r(1996)]$$

$$= 2(6.2 + 4.3 + 1.9 + 4.1 + 5.4 + 3.0 + 2.6 + 2.9) = 60.8$$

61. $\int_{2000}^{4000} C'(x)\,dx = \int_{2000}^{4000} (3 - 0.01x + 0.000006x^2)\,dx = \left[3x - 0.005x^2 + 0.000002x^3\right]_{2000}^{4000}$

$= 60{,}000 - 2{,}000 = \$58{,}000$

62. Let w be the amount of water in the tank. We are given that the rate of water leaving the tank is $r(t) = -dw/dt$. So by the Total Change Theorem, the total loss of water from the tank after four hours is $w(0) - w(4) = -[w(4) - w(0)] = -\int_0^4 w'(t)\,dt = \int_0^4 r(t)\,dt$. We use the Midpoint Rule with $n = 4$ and $\Delta t = 1$: $\int_0^4 r(t)\,dt \approx \sum_{i=1}^{4} r(\bar{t}_i)(1) = r(0.5) + r(1.5) + r(2.5) + r(3.5) \approx 5.9 + 5.4 + 4.7 + 3.6 = 19.6$ L.

63. (a) We can find the area between the Lorenz curve and the line $y = x$ by subtracting the area under $y = L(x)$ from the area under $y = x$. Thus,

$$\text{coefficient of inequality} = \frac{\text{area between Lorenz curve and straight line}}{\text{area under straight line}} = \frac{\int_0^1 [x - L(x)]\,dx}{\int_0^1 x\,dx}$$

$$= \frac{\int_0^1 [x - L(x)]\,dx}{\left[x^2/2\right]_0^1} = \frac{\int_0^1 [x - L(x)]\,dx}{1/2} = 2\int_0^1 [x - L(x)]\,dx$$

(b) $L(x) = \frac{5}{12}x^2 + \frac{7}{12}x \Rightarrow L\left(\frac{1}{2}\right) = \frac{5}{48} + \frac{7}{24} = \frac{19}{48} = 0.3958\overline{3}$, so the bottom 50% of the households receive about 40% of the income.

$$\text{coefficient of inequality} = 2\int_0^1 [x - L(x)]\,dx = 2\int_0^1 \left(x - \tfrac{5}{12}x^2 - \tfrac{7}{12}x\right) dx = 2\int_0^1 \tfrac{5}{12}\left(x - x^2\right) dx$$

$$= \tfrac{5}{6}\left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{5}{36}$$

64. (a) From Exercise 4.1.68(a), $v(t) = 0.00146t^3 - 0.11553t^2 + 24.98169t - 21.26872$.

(b) $h(125) - h(0) = \int_0^{125} v(t)\,dt = \left[0.000365t^4 - 0.03851t^3 + 12.490845t^2 - 21.26872t\right]_0^{125} \approx 206{,}407$ ft

65. $$\int_1^e \frac{x^2 + x + 1}{x}\,dx = \int_1^e \left(x + 1 + \frac{1}{x}\right) dx = \left[\frac{1}{2}x^2 + x + \ln x\right]_1^e$$
$$= \left(\tfrac{1}{2}e^2 + e + \ln e\right) - \left(\tfrac{1}{2} + 1 + \ln 1\right) = \tfrac{1}{2}e^2 + e - \tfrac{1}{2}$$

66. $$\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx = \int_4^9 \left(x + 2 + \frac{1}{x}\right) dx = \left[\frac{1}{2}x^2 + 2x + \ln x\right]_4^9$$
$$= \tfrac{81}{2} + 18 + \ln 9 - (8 + 8 + \ln 4) = \tfrac{85}{2} + \ln\tfrac{9}{4}$$

67. $$\int \left(x^2 + 1 + \frac{1}{1 + x^2}\right) dx = \frac{x^3}{3} + x + \tan^{-1} x + C$$

68. $B = 3A \Rightarrow \int_0^b e^x\,dx = 3\int_0^a e^x\,dx \Rightarrow \left[e^x\right]_0^b = 3\left[e^x\right]_0^a \Rightarrow e^b - 1 = 3\left(e^a - 1\right) \Rightarrow e^b = 3e^a - 2 \Rightarrow b = \ln\left(3e^a - 2\right)$

5.5 The Substitution Rule

1. Let $u = 3x$. Then $du = 3\,dx$, so $\int \cos 3x\,dx = \int \cos u\left(\frac{1}{3}\,du\right) = \frac{1}{3}\sin u + C = \frac{1}{3}\sin 3x + C$.

2. Let $u = 4 + x^2$. Then $du = 2x\,dx$, so $\int x\left(4 + x^2\right)^{10} dx = \int u^{10}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\cdot\frac{1}{11}u^{11} + C = \frac{1}{22}\left(4 + x^2\right)^{11} + C$.

3. Let $u = x^3 + 1$. Then $du = 3x^2\,dx$, so

$$\int x^2\sqrt{x^3 + 1}\,dx = \int \sqrt{u}\left(\tfrac{1}{3}\,du\right) = \frac{1}{3}\frac{u^{3/2}}{3/2} + C = \tfrac{1}{3}\cdot\tfrac{2}{3}u^{3/2} + C = \tfrac{2}{9}\left(x^3 + 1\right)^{3/2} + C$$

4. Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}\,dx$, so $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \sin u\,(2\,du) = 2(-\cos u) + C = -2\cos\sqrt{x} + C$.

5. Let $u = 1 + 2x$. Then $du = 2\,dx$, so

$$\int \frac{4}{(1 + 2x)^3}\,dx = 4\int u^{-3}\left(\tfrac{1}{2}\,du\right) = 2\frac{u^{-2}}{-2} + C = -\frac{1}{u^2} + C = -\frac{1}{(1 + 2x)^2} + C$$

6. Let $u = \sin\theta$. Then $du = \cos\theta\, d\theta$, so $\int \sin^3\theta\cos\theta\, d\theta = \int u^3\, du = \frac{1}{4}u^4 + C = \frac{1}{4}\sin^4\theta + C$.

7. Let $u = x^2 + 3$. Then $du = 2x\,dx$, so $\int 2x\left(x^2+3\right)^4 dx = \int u^4\,du = \frac{1}{5}u^5 + C = \frac{1}{5}\left(x^2+3\right)^5 + C$.

8. Let $u = 1 - x^4$. Then $du = -4x^3\,dx$, so

$$\int x^3\left(1-x^4\right)^5 dx = \int u^5\left(-\tfrac{1}{4}\,du\right) = -\tfrac{1}{4}\left(\tfrac{1}{6}u^6\right) + C = -\tfrac{1}{24}\left(1-x^4\right)^6 + C$$

9. Let $u = x - 1$. Then $du = dx$, so $\int \sqrt{x-1}\,dx = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(x-1)^{3/2} + C$.

10. Let $u = 2 - x$. Then $du = -dx$, so $\int (2-x)^6\,dx = \int u^6\,(-du) = -\frac{1}{7}u^7 + C = -\frac{1}{7}(2-x)^7 + C$.

11. Let $u = 1 + x + 2x^2$. Then $du = (1+4x)\,dx$, so

$$\int \frac{1+4x}{\sqrt{1+x+2x^2}}\,dx = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2} + C = 2\sqrt{1+x+2x^2} + C$$

12. Let $u = x^2 + 1$. Then $du = 2x\,dx$, so

$$\int x\left(x^2+1\right)^{3/2} dx = \int u^{3/2}\left(\tfrac{1}{2}\,du\right) = \frac{1}{2}\frac{u^{5/2}}{5/2} + C = \tfrac{1}{5}u^{5/2} + C = \tfrac{1}{5}\left(x^2+1\right)^{5/2} + C$$

13. Let $u = t + 1$. Then $du = dt$, so $\displaystyle\int \frac{2}{(t+1)^6}\,dt = 2\int u^{-6}\,du = -\frac{2}{5}u^{-5} + C = -\frac{2}{5(t+1)^5} + C$.

14. Let $u = 1 - 3t$. Then $du = -3\,dt$, so

$$\int \frac{1}{(1-3t)^4}\,dt = \int u^{-4}\left(-\frac{1}{3}\,du\right) = -\frac{1}{3}\left(\frac{u^{-3}}{-3}\right) + C = \frac{1}{9u^3} + C = \frac{1}{9(1-3t)^3} + C$$

15. Let $u = 1 - 2y$. Then $du = -2\,dy$, so

$$\int (1-2y)^{1.3}\,dy = \int u^{1.3}\left(-\tfrac{1}{2}\,du\right) = -\frac{1}{2}\left(\frac{u^{2.3}}{2.3}\right) + C = -\frac{(1-2y)^{2.3}}{4.6} + C$$

16. Let $u = 3 - 5y$. Then $du = -5\,dy$, so

$$\int \sqrt[5]{3-5y}\,dy = \int u^{1/5}\left(-\tfrac{1}{5}\,du\right) = -\tfrac{1}{5}\cdot\tfrac{5}{6}u^{6/5} + C = -\tfrac{1}{6}(3-5y)^{6/5} + C$$

17. Let $u = 2\theta$. Then $du = 2\,d\theta$, so $\int \cos 2\theta\,d\theta = \int \cos u\left(\frac{1}{2}\,du\right) = \frac{1}{2}\sin u + C = \frac{1}{2}\sin 2\theta + C$.

18. Let $u = 3\theta$. Then $du = 3\,d\theta$, so $\int \sec^2 3\theta\,d\theta = \int \sec^2 u\left(\frac{1}{3}\,du\right) = \frac{1}{3}\tan u + C = \frac{1}{3}\tan 3\theta + C$.

19. Let $u = t^2$. Then $du = 2t\,dt$, so $\int t\sin\left(t^2\right)dt = \int \sin u\left(\frac{1}{2}\,du\right) = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos\left(t^2\right) + C$.

20. Let $u = 1 + \sqrt{x}$. Then $du = \dfrac{dx}{2\sqrt{x}}$, so $\displaystyle\int \frac{(1+\sqrt{x})^9}{\sqrt{x}}\,dx = \int u^9\cdot 2\,du = 2\frac{u^{10}}{10} + C = \frac{(1+\sqrt{x})^{10}}{5} + C$.

21. Let $u = 1 + \sec x$. Then $du = \sec x \tan x \, dx$, so

$$\int \sec x \tan x \sqrt{1 + \sec x}\, dx = \int u^{1/2}\, du = \tfrac{2}{3} u^{3/2} + C = \tfrac{2}{3}(1 + \sec x)^{3/2} + C$$

22. Let $u = 1 - t^3$. Then $du = -3t^2\, dt$, so

$$\int t^2 \cos\left(1 - t^3\right) dt = \int \cos u \left(-\tfrac{1}{3}\, du\right) = -\tfrac{1}{3} \sin u + C = -\tfrac{1}{3} \sin\left(1 - t^3\right) + C$$

23. Let $u = \cos x$. Then $du = -\sin x\, dx$, so $\int \cos^4 x \sin x\, dx = \int u^4 (-du) = -\frac{1}{5} u^5 + C = -\frac{1}{5} \cos^5 x + C$.

24. Let $u = ax^2 + 2bx + c$. Then $du = 2(ax + b)\, dx$, so

$$\int \frac{(ax + b)\, dx}{\sqrt{ax^2 + 2bx + c}} = \int \frac{\frac{1}{2}\, du}{\sqrt{u}} = \tfrac{1}{2} \int u^{-1/2}\, du = u^{1/2} + C = \sqrt{ax^2 + 2bx + c} + C$$

25. Let $u = \cot x$. Then $du = -\csc^2 x\, dx$, so $\int \sqrt{\cot x} \csc^2 x\, dx = \int \sqrt{u}\,(-du) = -\dfrac{u^{3/2}}{3/2} + C = -\frac{2}{3} (\cot x)^{3/2} + C$.

26. Let $u = \dfrac{\pi}{x}$. Then $du = -\dfrac{\pi}{x^2}\, dx$, so $\displaystyle\int \frac{\cos(\pi/x)}{x^2}\, dx = \int \cos u \left(-\frac{1}{\pi}\, du\right) = -\frac{1}{\pi} \sin u + C = -\frac{1}{\pi} \sin \frac{\pi}{x} + C$.

27. Let $u = \sec x$. Then $du = \sec x \tan x\, dx$, so

$$\int \sec^3 x \tan x\, dx = \int \sec^2 x\, (\sec x \tan x)\, dx = \int u^2\, du = \tfrac{1}{3} u^3 + C = \tfrac{1}{3} \sec^3 x + C$$

28. Let $u = x^3 + 1$. Then $x^3 = u - 1$ and $du = 3x^2\, dx$, so

$$\begin{aligned} \int \sqrt[3]{x^3 + 1}\, x^5\, dx &= \int u^{1/3} (u - 1)\, \tfrac{1}{3}\, du = \tfrac{1}{3} \int \left(u^{4/3} - u^{1/3}\right) du = \tfrac{1}{3} \left(\tfrac{3}{7} u^{7/3} - \tfrac{3}{4} u^{4/3}\right) + C \\ &= \tfrac{1}{7} \left(x^3 + 1\right)^{7/3} - \tfrac{1}{4} \left(x^3 + 1\right)^{4/3} + C \end{aligned}$$

29. Let $u = b + cx^{a+1}$. Then $du = (a + 1)\, cx^a\, dx$, so

$$\int x^a \sqrt{b + cx^{a+1}}\, dx = \int u^{1/2} \frac{1}{(a + 1)\, c}\, du = \frac{1}{(a + 1)\, c} \left(\tfrac{2}{3} u^{3/2}\right) + C = \frac{2}{3c\,(a + 1)} \left(b + cx^{a+1}\right)^{3/2} + C$$

30. Let $u = \sin x$. Then $du = \cos x\, dx$, so $\int \cos x \cos(\sin x)\, dx = \int \cos u\, du = \sin u + C = \sin(\sin x) + C$.

31. Let $u = x + 2$. Then $du = dx$, so

$$\begin{aligned} \int \frac{x}{\sqrt[4]{x + 2}}\, dx &= \int \frac{u - 2}{\sqrt[4]{u}}\, du = \int \left(u^{3/4} - 2u^{-1/4}\right) du = \tfrac{4}{7} u^{7/4} - 2 \cdot \tfrac{4}{3} u^{3/4} + C \\ &= \tfrac{4}{7} (x + 2)^{7/4} - \tfrac{8}{3} (x + 2)^{3/4} + C \end{aligned}$$

32. Let $u = 1 - x$. Then $x = 1 - u$ and $dx = -du$, so

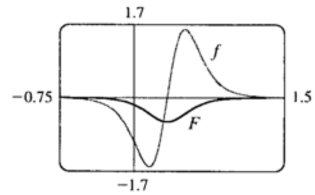
$$\begin{aligned} \int \frac{x^2}{\sqrt{1 - x}}\, dx &= \int \frac{(1 - u)^2}{\sqrt{u}} (-du) = -\int \frac{1 - 2u + u^2}{\sqrt{u}}\, du = -\int \left(u^{-1/2} - 2u^{1/2} + u^{3/2}\right) du \\ &= -\left(2u^{1/2} - 2 \cdot \tfrac{2}{3} u^{3/2} + \tfrac{2}{5} u^{5/2}\right) + C = -2\sqrt{1 - x} + \tfrac{4}{3} (1 - x)^{3/2} - \tfrac{2}{5} (1 - x)^{5/2} + C \end{aligned}$$

In Exercises 33–36, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

33. $f(x) = \dfrac{3x-1}{(3x^2-2x+1)^4}$. $u = 3x^2 - 2x + 1 \Rightarrow du = 2(3x-1)\,dx$, so

$$\int \frac{3x-1}{(3x^2-2x+1)^4}\,dx = \int \frac{1}{u^4}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int u^{-4}\,du$$

$$= -\tfrac{1}{6}u^{-3} + C = -\frac{1}{6(3x^2-2x+1)^3} + C$$

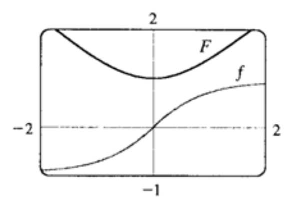


Notice that at $x = \frac{1}{3}$, f changes from negative to positive, and F has a local minimum.

34. $f(x) = \dfrac{x}{\sqrt{x^2+1}}$. $u = x^2 + 1 \Rightarrow du = 2x\,dx$, so

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sqrt{u}}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int u^{-1/2}\,du$$

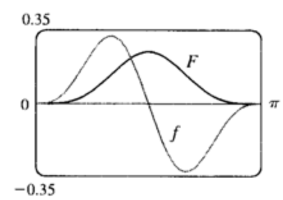
$$= u^{1/2} + C = \sqrt{x^2+1} + C$$



Note that at $x = 0$, f changes from negative to positive and F has a local minimum.

35. $f(x) = \sin^3 x \cos x$. $u = \sin x \Rightarrow du = \cos x\,dx$, so

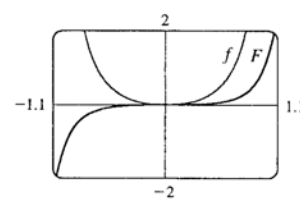
$$\int \sin^3 x \cos x\,dx = \int u^3\,du = \tfrac{1}{4}u^4 + C = \tfrac{1}{4}\sin^4 x + C$$



Note that at $x = \frac{\pi}{2}$, f changes from positive to negative and F has a local maximum. Also, both f and F are periodic with period π, so at $x = 0$ and at $x = \pi$, f changes from negative to positive and F has local minima.

36. $f(\theta) = \tan^2\theta \sec^2\theta$. $u = \tan\theta \Rightarrow du = \sec^2\theta\,d\theta$, so

$$\int \tan^2\theta \sec^2\theta\,d\theta = \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\tan^3\theta + C$$



Note that f is positive and F is increasing.

37. Let $u = x - 1$, so $du = dx$. When $x = 0$, $u = -1$; when $x = 2$, $u = 1$. Therefore, $\int_0^2 (x-1)^{25}\,dx = \int_{-1}^1 u^{25}\,du = 0$ by Theorem 6(b), since $f(u) = u^{25}$ is an odd function.

38. Let $u = 4 + 3x$, so $du = 3\,dx$. When $x = 0$, $u = 4$; when $x = 7$, $u = 25$. Therefore,

$$\int_0^7 \sqrt{4+3x}\,dx = \int_4^{25} \sqrt{u}\left(\tfrac{1}{3}\,du\right) = \frac{1}{3}\left[\frac{u^{3/2}}{3/2}\right]_4^{25} = \tfrac{2}{9}\left(25^{3/2} - 4^{3/2}\right) = \tfrac{2}{9}(125 - 8) = \tfrac{234}{9} = 26$$

39. Let $u = 1 + 2x^3$, so $du = 6x^2\,dx$. When $x = 0$, $u = 1$; when $x = 1$, $u = 3$. Therefore,

$$\int_0^1 x^2\left(1+2x^3\right)^5\,dx = \int_1^3 u^5\left(\tfrac{1}{6}\,du\right) = \tfrac{1}{6}\left[\tfrac{1}{6}u^6\right]_1^3 = \tfrac{1}{36}\left(3^6 - 1^6\right) = \tfrac{1}{36}(729 - 1) = \tfrac{728}{36} = \tfrac{182}{9}$$

40. Let $u = x^2$, so $du = 2x\,dx$. When $x = 0$, $u = 0$; when $x = \sqrt{\pi}$, $u = \pi$. Therefore,

$$\int_0^{\sqrt{\pi}} x\cos\left(x^2\right)dx = \int_0^{\pi}\cos u\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\left[\sin u\right]_0^{\pi} = \tfrac{1}{2}\left(\sin\pi - \sin 0\right) = \tfrac{1}{2}\left(0-0\right) = 0$$

41. Let $u = \pi t$, so $du = \pi\,dt$. When $t = 0$, $u = 0$; when $t = 1$, $u = \pi$. Therefore,

$$\int_0^1 \cos\pi t\,dt = \int_0^{\pi}\cos u\left(\tfrac{1}{\pi}\,du\right) = \tfrac{1}{\pi}\left[\sin u\right]_0^{\pi} = \tfrac{1}{\pi}\left(0-0\right) = 0$$

42. Let $u = 4t$, so $du = 4\,dt$. When $t = 0$, $u = 0$; when $t = \frac{\pi}{4}$, $u = \pi$. Therefore,

$$\int_0^{\pi/4}\sin 4t\,dt = \int_0^{\pi}\sin u\left(\tfrac{1}{4}\,du\right) = -\tfrac{1}{4}\left[\cos u\right]_0^{\pi} = -\tfrac{1}{4}\left(-1-1\right) = \tfrac{1}{2}$$

43. Let $u = 1 + \dfrac{1}{x}$, so $du = -\dfrac{dx}{x^2}$. When $x = 1$, $u = 2$; when $x = 4$, $u = \frac{5}{4}$. Therefore,

$$\int_1^4 \frac{1}{x^2}\sqrt{1+\frac{1}{x}}\,dx = \int_2^{5/4} u^{1/2}\left(-du\right) = \int_{5/4}^2 u^{1/2}\,du$$

$$= \left[\tfrac{2}{3}u^{3/2}\right]_{5/4}^2 = \tfrac{2}{3}\left(2\sqrt{2} - \tfrac{5\sqrt{5}}{8}\right) = \tfrac{4\sqrt{2}}{3} - \tfrac{5\sqrt{5}}{12}$$

44. $\displaystyle\int_0^2 \frac{dx}{(2x-3)^2}$ does not exist since $\dfrac{1}{(2x-3)^2}$ has an infinite discontinuity at $x = \frac{3}{2}$.

45. Let $u = \cos\theta$, so $du = -\sin\theta\,d\theta$. When $\theta = 0$, $u = 1$; when $\theta = \frac{\pi}{3}$, $u = \frac{1}{2}$. Therefore,

$$\int_0^{\pi/3}\frac{\sin\theta}{\cos^2\theta}\,d\theta = \int_1^{1/2}\frac{-du}{u^2} = \int_{1/2}^1 u^{-2}\,du = \left[-\frac{1}{u}\right]_{1/2}^1 = -1+2 = 1$$

46. $\displaystyle\int_{-\pi/2}^{\pi/2}\frac{x^2\sin x}{1+x^6}\,dx = 0$ by Theorem 6(b), since $f(x) = \dfrac{x^2\sin x}{1+x^6}$ is an odd function.

47. Let $u = 1 + 2x$, so $du = 2\,dx$. When $x = 0$, $u = 1$; when $x = 13$, $u = 27$. Therefore,

$$\int_0^{13}\frac{dx}{\sqrt[3]{(1+2x)^2}} = \int_1^{27}u^{-2/3}\left(\tfrac{1}{2}\,du\right) = \left[\tfrac{1}{2}\cdot 3u^{1/3}\right]_1^{27} = \tfrac{3}{2}\left(3-1\right) = 3$$

48. $\int_{-\pi/3}^{\pi/3}\sin^5\theta\,d\theta = 0$ since $f(\theta) = \sin^5\theta$ is an odd function.

49. Let $u = x - 1$, so $du = dx$. When $x = 1$, $u = 0$; when $x = 2$, $u = 1$. Therefore,

$$\int_1^2 x\sqrt{x-1}\,dx = \int_0^1 (u+1)\sqrt{u}\,du = \int_0^1\left(u^{3/2}+u^{1/2}\right)du = \left[\tfrac{2}{5}u^{5/2}+\tfrac{2}{3}u^{3/2}\right]_0^1 = \tfrac{2}{5}+\tfrac{2}{3} = \tfrac{16}{15}$$

50. Let $u = 1 + 2x$, so $x = \frac{1}{2}(u-1)$ and $du = 2\,dx$. When $x = 0$, $u = 1$; when $x = 4$, $u = 9$. Therefore,

$$\int_0^4\frac{x\,dx}{\sqrt{1+2x}} = \int_1^9\frac{\frac{1}{2}(u-1)}{\sqrt{u}}\frac{du}{2} = \tfrac{1}{4}\int_1^9\left(u^{1/2}-u^{-1/2}\right)du = \tfrac{1}{4}\left[\tfrac{2}{3}u^{3/2}-2u^{1/2}\right]_1^9$$

$$= \tfrac{1}{4}\cdot\tfrac{2}{3}\left[u^{3/2}-3u^{1/2}\right]_1^9 = \tfrac{1}{6}\left[(27-9)-(1-3)\right] = \tfrac{20}{6} = \tfrac{10}{3}$$

51. $\displaystyle\int_0^4\frac{dx}{(x-2)^3}$ does not exist since $\dfrac{1}{(x-2)^3}$ has an infinite discontinuity at $x = 2$.

52. Let $u = a^2 - x^2$, so $du = -2x\,dx$. When $x = 0$, $u = a^2$; when $x = a$, $u = 0$. Therefore,

$$\int_0^a x\sqrt{a^2-x^2}\,dx = \int_{a^2}^0 u^{1/2}\left(-\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int_0^{a^2}u^{1/2}\,du = \tfrac{1}{2}\cdot\left[\tfrac{2}{3}u^{3/2}\right]_0^{a^2} = \tfrac{1}{3}a^3$$

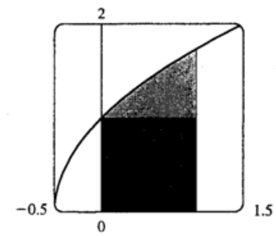
53. Let $u = x^2 + a^2$, so $du = 2x\,dx$. When $x = 0$, $u = a^2$; when $x = a$, $u = 2a^2$. Therefore,

$$\int_0^a x\sqrt{x^2+a^2}\,dx = \int_{a^2}^{2a^2} u^{1/2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\left[\tfrac{2}{3}u^{3/2}\right]_{a^2}^{2a^2} = \left[\tfrac{1}{3}u^{3/2}\right]_{a^2}^{2a^2} = \tfrac{1}{3}\left(2\sqrt{2}-1\right)a^3$$

54. $\int_{-a}^{a} x\sqrt{x^2+a^2}\,dx = 0$ by Theorem 6(b), since $f(x) = x\sqrt{x^2+a^2}$ is an odd function.

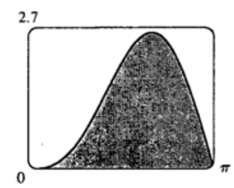
55. From the graph, it appears that the area under the curve is about $1 + \left(\text{a little more than } \tfrac{1}{2}\cdot 1\cdot 0.7\right)$, or about 1.4. The exact area is given by $A = \int_0^1 \sqrt{2x+1}\,dx$. Let $u = 2x+1$, so $du = 2\,dx$, the limits change to $2\cdot 0 + 1 = 1$ and $2\cdot 1 + 1 = 3$, and

$A = \int_1^3 \sqrt{u}\left(\tfrac{1}{2}\,du\right) = \left[\tfrac{1}{3}u^{3/2}\right]_1^3 = \tfrac{1}{3}\left(3\sqrt{3}-1\right) = \sqrt{3} - \tfrac{1}{3} \approx 1.399.$



56. From the graph, it appears that the area under the curve is almost $\tfrac{1}{2}\cdot\pi\cdot 2.6$, or about 4. The exact area is given by

$$\begin{aligned} A &= \int_0^\pi (2\sin x - \sin 2x)\,dx \\ &= -2\left[\cos x\right]_0^\pi - \int_0^\pi \sin 2x\,dx \\ &= -2(-1-1) - 0 \quad \text{(by symmetry of the graph of } y = \sin 2x\text{)} \\ &= 4 \end{aligned}$$



57. We split the integral: $\int_{-2}^{2}(x+3)\sqrt{4-x^2}\,dx = \int_{-2}^{2} x\sqrt{4-x^2}\,dx + \int_{-2}^{2} 3\sqrt{4-x^2}\,dx$. The first integral is 0 by Theorem 6(b), since $f(x) = x\sqrt{4-x^2}$ is an odd function and we are integrating from $x = -2$ to $x = 2$. The second integral we interpret as three times the area of a semicircle with radius 2, so the original integral is equal to $0 + 3\cdot\tfrac{1}{2}\left(\pi\cdot 2^2\right) = 6\pi$.

58. Let $u = x^2$. Then $du = 2x\,dx$ and the limits are unchanged ($0^2 = 0$ and $1^2 = 1$), so $I = \int_0^1 x\sqrt{1-x^4}\,dx = \tfrac{1}{2}\int_0^1 \sqrt{1-u^2}\,du$. But this integral can be interpreted as the area of a quarter-circle with radius 1. So $I = \tfrac{1}{2}\cdot\tfrac{1}{4}\left(\pi\cdot 1^2\right) = \tfrac{1}{8}\pi$.

59. The volume of inhaled air in the lungs at time t is

$$V(t) = \int_0^t f(u)\,du = \int_0^t \frac{1}{2}\sin\left(\frac{2}{5}\pi u\right)du = \int_0^{2\pi t/5} \frac{1}{2}\sin v\left(\tfrac{5}{2\pi}\,dv\right) \quad \left[\text{substitute } v = \tfrac{2\pi}{5}u,\ dv = \tfrac{2\pi}{5}\,du\right]$$

$$= \tfrac{5}{4\pi}\left[-\cos v\right]_0^{2\pi t/5} = \tfrac{5}{4\pi}\left[-\cos\left(\tfrac{2}{5}\pi t\right) + 1\right] = \tfrac{5}{4\pi}\left[1 - \cos\left(\tfrac{2}{5}\pi t\right)\right] \text{ liters}$$

60. Number of calculators $= x(4) - x(2) = \int_2^4 5000\left[1 - 100(t+10)^{-2}\right]dt$

$$= 5000\left[t + 100(t+10)^{-1}\right]_2^4 = 5000\left[\left(4 + \tfrac{100}{14}\right) - \left(2 + \tfrac{100}{12}\right)\right] \approx 4048$$

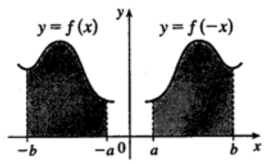
61. Let $u = 2x$. Then $du = 2\,dx$, so $\int_0^2 f(2x)\,dx = \int_0^4 f(u)\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int_0^4 f(u)\,du = \tfrac{1}{2}(10) = 5$.

62. Let $u = x^2$. Then $du = 2x\,dx$, so $\int_0^3 x f\left(x^2\right)dx = \int_0^9 f(u)\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int_0^9 f(u)\,du = \tfrac{1}{2}(4) = 2$.

63. (a) Let $u = -x$. Then $du = -dx$. When $x = a$, $u = -a$; when $x = b$, $u = -b$. So

$\int_a^b f(-x)\,dx = \int_{-a}^{-b} f(u)\,(-du) = \int_{-b}^{-a} f(u)\,du = \int_{-b}^{-a} f(x)\,dx$

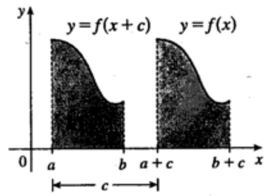
From the diagram, we see that the equality follows from the fact that we are reflecting the graph of f, and the limits of integration, about the y-axis.



(b) Let $u = x + c$. Then $du = dx$. When $x = a$, $u = a + c$; when $x = b$, $u = b + c$. So

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(u)\,du = \int_{a+c}^{b+c} f(x)\,dx$$

From the diagram, we see that the equality follows from the fact that we are translating the graph of f, and the limits of integration, by a distance c.



64. The area under the graph of $y = \sin\sqrt{x}$ from 0 to 4 is $A_1 = \int_0^4 \sin\sqrt{x}\,dx$. The area under the graph of $y = 2x\sin x$ from 0 to 2 is $A_2 = \int_0^2 2x\sin x\,dx$ $[u = x^2, du = 2x\,dx, \sqrt{u} = x \text{ for } 0 \le x \le 2]$ $= \int_0^4 \sin\sqrt{u}\,du$. Since the integration variable is immaterial, $A_1 = A_2$.

65. Let $u = 1 - x$. Then $du = -dx$, so
$\int_0^1 x^a (1-x)^b\,dx = -\int_1^0 (1-u)^a u^b\,du = \int_0^1 u^b (1-u)^a\,du = \int_0^1 x^b (1-x)^a\,dx$.

66. Let $u = \pi - x$. Then $du = -dx$. When $x = \pi$, $u = 0$ and when $x = 0$, $u = \pi$. So

$$\begin{aligned}\int_0^\pi x f(\sin x)\,dx &= -\int_\pi^0 (\pi - u) f(\sin(\pi - u))\,du = \int_0^\pi (\pi - u) f(\sin u)\,du \\ &= \pi\int_0^\pi f(\sin u)\,du - \int_0^\pi u f(\sin u)\,du = \pi\int_0^\pi f(\sin x)\,dx - \int_0^\pi x f(\sin x)\,dx\end{aligned}$$

$\Rightarrow \quad 2\int_0^\pi x f(\sin x)\,dx = \pi\int_0^\pi f(\sin x)\,dx \quad \Rightarrow \quad \int_0^\pi x f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$.

67. Let $u = 5 - 3x$. Then $du = -3\,dx$, so $\displaystyle\int \frac{dx}{5-3x} = -\frac{1}{3}\int \frac{1}{u}\,du = -\frac{1}{3}\ln|u| + C = -\frac{1}{3}\ln|5-3x| + C$.

68. Let $u = x^2 + 1$. Then $du = 2x\,dx$, so $\displaystyle\int \frac{x}{x^2+1}\,dx = \int \frac{\frac{1}{2}du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left(x^2+1\right) + C$ or $\ln\sqrt{x^2+1} + C$.

69. Let $u = \ln x$. Then $du = \dfrac{dx}{x}$, so $\displaystyle\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C$.

70. Let $u = \tan^{-1} x$. Then $du = \dfrac{dx}{1+x^2}$, so $\displaystyle\int \frac{\tan^{-1} x}{1+x^2}\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{\left(\tan^{-1} x\right)^2}{2} + C$.

71. Let $u = 1 + e^x$. Then $du = e^x\,dx$, so $\int e^x\sqrt{1+e^x}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C$.
Or: Let $u = \sqrt{1+e^x}$. Then $u^2 = 1 + e^x$ and $2u\,du = e^x\,dx$, so
$\int e^x\sqrt{1+e^x}\,dx = \int u \cdot 2u\,du = \frac{2}{3}u^3 + C = \frac{2}{3}(1+e^x)^{3/2} + C$.

72. Let $u = e^x$. Then $du = e^x\,dx$, so $\int e^x \sin(e^x)\,dx = \int \sin u\,du = -\cos u + C = -\cos(e^x) + C$.

73. Let $u = \ln x$. Then $du = \dfrac{dx}{x}$, so $\displaystyle\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$.

74. Let $u = e^x + 1$. Then $du = e^x\,dx$, so $\int \frac{e^x}{e^x+1}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln\left(e^x+1\right) + C$.

75. Let $u = x^2 + 2x$. Then $du = 2(x+1)\,dx$, so $\int \frac{x+1}{x^2+2x}\,dx = \int \frac{\frac{1}{2}\,du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left|x^2+2x\right| + C$.

76. Let $u = \cos x$. Then $du = -\sin x\,dx$, so $\int \frac{\sin x}{1+\cos^2 x}\,dx = \int \frac{-du}{1+u^2} = -\tan^{-1} u + C = -\tan^{-1}(\cos x) + C$.

77. $\int \frac{1+x}{1+x^2}\,dx = \int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx = \tan^{-1} x + \frac{1}{2}\int \frac{2x\,dx}{1+x^2} = \tan^{-1} x + \frac{1}{2}\ln\left(1+x^2\right) + C$

(In the last step, we evaluate $\int du/u$ where $u = 1 + x^2$.)

78. Let $u = x^2$. Then $du = 2x\,dx$, so $\int \frac{x}{1+x^4}\,dx = \int \frac{\frac{1}{2}\,du}{1+u^2} = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1}\left(x^2\right) + C$.

79. Let $u = 2x + 3$, so $du = 2\,dx$. When $x = 0$, $u = 3$; when $x = 3$, $u = 9$. Therefore,

$$\int_0^3 \frac{dx}{2x+3} = \int_3^9 \frac{\frac{1}{2}\,du}{u} = \left[\tfrac{1}{2}\ln u\right]_3^9 = \tfrac{1}{2}(\ln 9 - \ln 3) = \tfrac{1}{2}\ln\tfrac{9}{3} = \tfrac{1}{2}\ln 3 \quad (\text{or } \ln\sqrt{3})$$

80. Let $u = -x^2$, so $du = -2x\,dx$. When $x = 0$, $u = 0$; when $x = 1$, $u = -1$. Therefore,

$$\int_0^1 xe^{-x^2}\,dx = \int_0^{-1} e^u\left(-\tfrac{1}{2}\,du\right) = -\tfrac{1}{2}\left[e^u\right]_0^{-1} = -\tfrac{1}{2}\left(e^{-1} - e^0\right) = \tfrac{1}{2}(1 - 1/e)$$

81. Let $u = \ln x$, so $du = \frac{dx}{x}$. When $x = e$, $u = 1$; when $x = e^4$; $u = 4$. Therefore,

$$\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \int_1^4 u^{-1/2}\,du = 2\left[u^{1/2}\right]_1^4 = 2(2-1) = 2$$

82. Let $u = \sin^{-1} x$, so $du = \frac{dx}{\sqrt{1-x^2}}$. When $x = 0$, $u = 0$; when $x = \frac{1}{2}$, $u = \frac{\pi}{6}$. Therefore,

$$\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/6} u\,du = \left[\frac{u^2}{2}\right]_0^{\pi/6} = \frac{\pi^2}{72}$$

83. $\frac{x\sin x}{1+\cos^2 x} = x\cdot\frac{\sin x}{2-\sin^2 x} = xf(\sin x)$, where $f(t) = \frac{t}{2-t^2}$. By Exercise 66,

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^\pi xf(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx$$

Let $u = \cos x$. Then $du = -\sin x\,dx$. When $x = \pi$, $u = -1$ and when $x = 0$, $u = 1$. So

$$\frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx = -\frac{\pi}{2}\int_1^{-1} \frac{du}{1+u^2} = \frac{\pi}{2}\int_{-1}^1 \frac{du}{1+u^2} = \frac{\pi}{2}\left[\tan^{-1} u\right]_{-1}^1$$

$$= \frac{\pi}{2}\left[\tan^{-1} 1 - \tan^{-1}(-1)\right] = \frac{\pi}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{4}$$

5 Review

CONCEPT CHECK

1. (a) $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ is an expression for a Riemann sum of a function f. x_i^* is a point in the ith subinterval $[x_{i-1}, x_i]$ and Δx is the length of the subintervals.

(b) See Figure 1 in Section 5.2.

(c) In Section 5.2, see Figure 3 and the paragraph to the right of it.

2. (a) See Definition 5.2.2.

(b) See Figure 2 in Section 5.2.

(c) In Section 5.2, see Figure 4 and the paragraph to the right of it.

3. See page 343.

4. (a) See the Total Change Theorem on page 350.

(b) $\int_{t_1}^{t_2} r(t)\,dt$ represents the change in the amount of water in the reservoir between time t_1 and time t_2.

5. (a) $\int_{60}^{120} v(t)\,dt$ represents the change in position of the particle from $t = 60$ to $t = 120$ seconds.

(b) $\int_{60}^{120} |v(t)|\,dt$ represents the total distance traveled by the particle from $t = 60$ to 120 seconds.

(c) $\int_{60}^{120} a(t)\,dt$ represents the change in the velocity of the particle from $t = 60$ to $t = 120$ seconds.

6. (a) $\int f(x)\,dx$ is the family of functions $\{F \mid F' = f\}$. Any two such functions differ by a constant.

(b) The connection is given by the Evaluation Theorem: $\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b$ if f is continuous..

7. The precise version of this statement is given by the Fundamental Theorem of Calculus. See the statement of this theorem and the paragraph that follows it on page 343.

8. See the Substitution Rule (5.5.4). This says that it is permissible to operate with the dx after an integral sign as if it were a differential.

TRUE-FALSE QUIZ

1. True by Property 2 of the Integral in Section 5.2.

2. False. Try $a = 0$, $b = 2$, $f(x) = g(x) = 1$ as a counterexample.

3. True by Property 3 of the Integral in Section 5.2.

4. False. You can't take a variable outside the integral sign. For example, using $f(x) = 1$ on $[0, 1]$,

$$\int_0^1 x\,f(x)\,dx = \int_0^1 x\,dx = \left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{1}{2} \text{ (a constant) while } x\int_0^1 1\,dx = x\,[x]_0^1 = x \cdot 1 = x \text{ (a variable).}$$

5. False. For example, let $f(x) = x^2$. Then $\int_0^1 \sqrt{x^2}\,dx = \int_0^1 x\,dx = \frac{1}{2}$, but $\sqrt{\int_0^1 x^2\,dx} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$.

6. True by the Total Change Theorem.

7. True by Comparison Property 7 of the Integral in Section 5.2.

8. False. For example, let $a = 0$, $b = 1$, $f(x) = 3$, $g(x) = x$. $f(x) > g(x)$ for each x in $(0, 1)$, but $f'(x) = 0 < 1 = g'(x)$ for $x \in (0, 1)$.

9. True. The integrand is an odd function that is continuous on $[-1, 1]$, so the result follows from Equation 5.5.6(b).

10. True. $\int_{-5}^{5} (ax^2 + bx + c)\,dx = \int_{-5}^{5} (ax^2 + c)\,dx + \int_{-5}^{5} bx\,dx$
$= 2\int_0^5 (ax^2 + c)\,dx$ [by 5.5.6(a)] $+ 0$ [by 5.5.6(b)]

11. False. The function $f(x) = 1/x^4$ is not bounded on the interval $[-2, 1]$. It has an infinite discontinuity at $x = 0$, so it is not integrable on the interval. (If the integral were to exist, a positive value would be expected, by Comparison Property 6 of Integrals.)

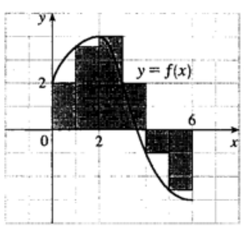
12. False. See the remarks and Figure 4 before Example 1 in Section 5.2, and notice that $y = x - x^3 < 0$ for $1 < x \leq 2$.

13. False. For example, the function $y = |x|$ is continuous on $\mathbb{R}$, but has no derivative at $x = 0$.

14. True by FTC1.

EXERCISES

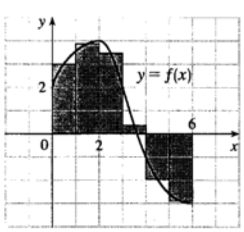
1. (a)



$L_6 = \sum_{i=1}^{6} f(x_{i-1})\,\Delta x \quad [\Delta x = \frac{6-0}{6} = 1]$
$= f(x_0)\cdot 1 + f(x_1)\cdot 1 + f(x_2)\cdot 1 + f(x_3)\cdot 1 + f(x_4)\cdot 1 + f(x_5)\cdot 1$
$\approx 2 + 3.5 + 4 + 2 + (-1) + (-2.5) = 8$

The Riemann sum represents the sum of the areas of the first four rectangles and the negatives of the areas of the last two rectangles.

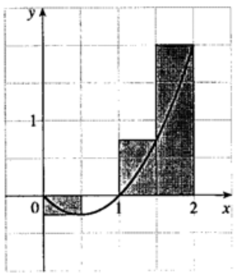
(b)



$M_6 = \sum_{i=1}^{6} f(\bar{x}_i)\,\Delta x \quad [\Delta x = \frac{6-0}{6} = 1]$
$= f(\bar{x}_1)\cdot 1 + f(\bar{x}_2)\cdot 1 + f(\bar{x}_3)\cdot 1 + f(\bar{x}_4)\cdot 1 + f(\bar{x}_5)\cdot 1 + f(\bar{x}_6)\cdot 1$
$= f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5)$
$= 3 + 3.9 + 3.3 + 0.2 + (-2) + (-2.8) = 5.6$

The Riemann sum represents the sum of the areas of the first four rectangles and the negatives of the areas of the last two rectangles.

2. (a)



$f(x) = x^2 - x$ and $\Delta x = \frac{2-0}{4} = 0.5 \Rightarrow$

$$R_4 = 0.5f(0.5) + 0.5f(1) + 0.5f(1.5) + 0.5f(2)$$
$$= 0.5(-0.25 + 0 + 0.75 + 2) = 1.25$$

The Riemann sum represents the sum of the areas of the third and fourth rectangles and the negative of the area of the first rectangle. (The second rectangle vanishes.)

(b) $\int_0^2 (x^2 - x)\,dx = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\,\Delta x \quad [\Delta x = 2/n \text{ and } x_i = 2i/n]$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i^2}{n^2} - \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty} \frac{2}{n}\left[\frac{4}{n^2}\sum_{i=1}^{n} i^2 - \frac{2}{n}\sum_{i=1}^{n} i\right]$$

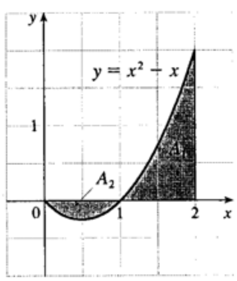
$$= \lim_{n\to\infty} \left[\frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{4}{n^2} \cdot \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \left[\frac{4}{3} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n} - 2 \cdot \frac{n+1}{n}\right]$$

$$= \lim_{n\to\infty} \left[\frac{4}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) - 2\left(1 + \frac{1}{n}\right)\right] = \frac{4}{3} \cdot 1 \cdot 2 - 2 \cdot 1 = \frac{2}{3}$$

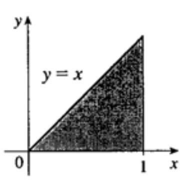
(c) $\int_0^2 (x^2 - x)\,dx = \left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_0^2 = \left(\frac{8}{3} - 2\right) = \frac{2}{3}$

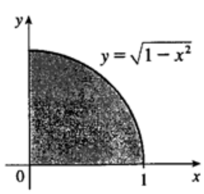
(d)



$\int_0^2 (x^2 - x)\,dx = A_1 - A_2$ where A_1 and A_2 are the areas shown in the diagram.

3. $\int_0^1 \left(x + \sqrt{1 - x^2}\right) dx = \int_0^1 x\,dx + \int_0^1 \sqrt{1 - x^2}\,dx = I_1 + I_2.$





I_1 can be interpreted as the area of the triangle shown in the figure and I_2 can be interpreted as the area of the quarter-circle. Area $= \frac{1}{2}(1)(1) + \frac{1}{4}(\pi)(1)^2 = \frac{1}{2} + \frac{\pi}{4}$.

4. On $[0, \pi]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \sin x_i\,\Delta x = \int_0^\pi \sin x\,dx = [-\cos x]_0^\pi = -(-1) - (-1) = 2.$

5. $\int_0^6 f(x)\,dx = \int_0^4 f(x)\,dx + \int_4^6 f(x)\,dx \quad\Rightarrow\quad 10 = 7 + \int_4^6 f(x)\,dx \quad\Rightarrow\quad \int_4^6 f(x)\,dx = 10 - 7 = 3$

6. (a) $\int_1^5 \left(x + 2x^5\right) dx = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\,\Delta x \quad \left[\Delta x = \dfrac{5-1}{n} = \dfrac{4}{n}, x_i = 1 + \dfrac{4i}{n}\right]$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(1 + \frac{4i}{n}\right) + 2\left(1 + \frac{4i}{n}\right)^5\right] \cdot \frac{4}{n}$$

$$= \lim_{n\to\infty} \frac{1305n^4 + 3126n^3 + 2080n^2 - 256}{n^3} \cdot \frac{4}{n} = 5220$$

(b) $\int_1^5 \left(x + 2x^5\right) dx = \left[\frac{1}{2}x^2 + \frac{2}{6}x^6\right]_1^5 = \left(\frac{25}{2} + \frac{15{,}625}{3}\right) - \left(\frac{1}{2} + \frac{1}{3}\right) = 12 + 5208 = 5220$

7. First note that either a or b must be the graph of $\int_0^x f(t)\,dt$, since $\int_0^0 f(t)\,dt = 0$, and $c(0) \neq 0$. Now notice that $b > 0$ when c is increasing, and that $c > 0$ when a is increasing. It follows that c is the graph of $f(x)$, b is the graph of $f'(x)$, and a is the graph of $\int_0^x f(t)\,dt$.

8. (a) By FTC2, we have $\displaystyle\int_0^{\pi/2} \frac{d}{dx}\left(\sin\frac{x}{2}\cos\frac{x}{3}\right) dx = \left[\sin\frac{x}{2}\cos\frac{x}{3}\right]_0^{\pi/2} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - 0 \cdot 1 = \frac{\sqrt{6}}{4}$.

(b) $\displaystyle\frac{d}{dx}\int_0^{\pi/2} \sin\frac{x}{2}\cos\frac{x}{3}\,dx = 0$, since the definite integral is a constant.

(c) $\displaystyle\frac{d}{dx}\int_x^{\pi/2} \sin\frac{t}{2}\cos\frac{t}{3}\,dt = \frac{d}{dx}\left(-\int_{\pi/2}^{x} \sin\frac{t}{2}\cos\frac{t}{3}\,dt\right) = -\frac{d}{dx}\int_{\pi/2}^{x} \sin\frac{t}{2}\cos\frac{t}{2}\,dt = -\sin\frac{x}{2}\cos\frac{x}{3}$, by FTC1.

9. $\int_1^2 \left(8x^3 + 3x^2\right) dx = \left[\frac{8}{4}x^4 + \frac{3}{3}x^3\right]_1^2 = \left(2 \cdot 2^4 + 2^3\right) - (2 + 1) = 37$

10. $\int_0^b \left(x^3 + 4x - 1\right) dx = \left[\frac{1}{4}x^4 + 2x^2 - x\right]_0^b = \frac{1}{4}b^4 + 2b^2 - b$

11. $\int_0^1 \left(1 - x^9\right) dx = \left[x - \frac{1}{10}x^{10}\right]_0^1 = 1 - \frac{1}{10} = \frac{9}{10}$

12. Let $u = 1 - x$. Then $du = -dx$, so $\int_0^1 (1 - x)^9\,dx = \int_1^0 u^9\,(-du) = \int_0^1 u^9\,du = \frac{1}{10}\left[u^{10}\right]_0^1 = \frac{1}{10}$

13. $\int_1^8 \sqrt[3]{x}\,(x - 1)\,dx = \int_1^8 \left(x^{4/3} - x^{1/3}\right) dx = \left[\frac{3}{7}x^{7/3} - \frac{3}{4}x^{4/3}\right]_1^8 = \left(\frac{3}{7} \cdot 128 - \frac{3}{4} \cdot 16\right) - \left(\frac{3}{7} - \frac{3}{4}\right) = \frac{1209}{28}$

14. $\displaystyle\int_1^4 \frac{x^2 - x + 1}{\sqrt{x}}\,dx = \int_1^4 \left(x^{3/2} - x^{1/2} + x^{-1/2}\right) dx = \left[\tfrac{2}{5}x^{5/2} - \tfrac{2}{3}x^{3/2} + 2x^{1/2}\right]_1^4$

$$= \left(\tfrac{2}{5} \cdot 32 - \tfrac{2}{3} \cdot 8 + 4\right) - \left(\tfrac{2}{5} - \tfrac{2}{3} + 2\right) = \tfrac{146}{15}$$

15. Let $u = 1 + 2x^3$. Then $du = 6x^2\,dx$, so

$\int_0^2 x^2\left(1 + 2x^3\right)^3 dx = \int_1^{17} u^3\left(\frac{1}{6}\,du\right) = \left[\frac{1}{24}u^4\right]_1^{17} = \frac{1}{24}\left(17^4 - 1\right) = 3480.$

16. Let $u = 16 - 3x$. Then $x = \frac{1}{3}(16 - u)$, $dx = -\frac{1}{3}du$, so

$$\int_0^4 x\sqrt{16 - 3x}\,dx = \int_{16}^4 u^{1/2}\left(\frac{16 - u}{3}\right)\left(-\tfrac{1}{3}\,du\right) = \frac{1}{9}\int_4^{16}\left(16u^{1/2} - u^{3/2}\right) du$$

$$= \tfrac{1}{9}\left[16 \cdot \tfrac{2}{3}u^{3/2} - \tfrac{2}{5}u^{5/2}\right]_4^{16} = \tfrac{1}{9}\left[\tfrac{32}{3} \cdot 64 - \tfrac{2}{5} \cdot 1024 - \tfrac{32}{3} \cdot 8 + \tfrac{2}{5} \cdot 32\right] = \tfrac{3008}{135}$$

17. Let $u = 2x + 3$. Then $du = 2\,dx$, so $\displaystyle\int_3^{11} \frac{dx}{\sqrt{2x + 3}} = \int_9^{25} u^{-1/2}\left(\tfrac{1}{2}\,du\right) = \left[u^{1/2}\right]_9^{25} = 5 - 3 = 2.$

18. $\displaystyle\int_0^2 \frac{x\,dx}{(x^2-1)^2}$ does not exist since the integrand has an infinite discontinuity at $x = 1$.

19. $\displaystyle\int_{-2}^{-1} \frac{dx}{(2x+3)^4}$ does not exist since the integrand has an infinite discontinuity at $x = -\frac{3}{2}$.

20. $\displaystyle\int_{-1}^{1} \frac{x+x^3+x^5}{1+x^2+x^4}\,dx = \int_{-1}^{1} x\,dx = 0$ by Theorem 5.5.6(b), since the integrand is odd.

21. Let $u = 2 + x^5$. Then $du = 5x^4\,dx$, so

$$\int \frac{x^4\,dx}{(2+x^5)^6} = \int u^{-6}\left(\frac{1}{5}\,du\right) = \frac{1}{5}\left(\frac{u^{-5}}{-5}\right) + C = -\frac{1}{25u^5} + C = -\frac{1}{25\,(2+x^5)^5} + C.$$

22. Let $u = 2x - x^2$. Then $du = 2\,(1-x)\,dx$, so

$\int (1-x)\sqrt{2x-x^2}\,dx = \int u^{1/2}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{3}\left(2x-x^2\right)^{3/2} + C.$

23. Let $u = \pi x$. Then $du = \pi\,dx$, so $\displaystyle\int \sin \pi x\,dx = \int \frac{\sin u\,du}{\pi} = \frac{-\cos u}{\pi} + C = -\frac{\cos \pi x}{\pi} + C.$

24. Let $u = 3t$. Then $du = 3\,dt$, so $\int \csc^2 3t\,dt = \int \csc^2 u\left(\frac{1}{3}\,du\right) = -\frac{1}{3}\cot u + C = -\frac{1}{3}\cot 3t + C.$

25. Let $u = \dfrac{1}{t}$. Then $du = -\dfrac{1}{t^2}\,dt$, so $\displaystyle\int \frac{\cos(1/t)}{t^2}\,dt = \int \cos u\,(-du) = -\sin u + C = -\sin\left(\frac{1}{t}\right) + C.$

26. Let $u = \cos x$. Then $du = -\sin x\,dx$, so

$\int \sin x \cos(\cos x)\,dx = -\int \cos u\,du = -\sin u + C = -\sin(\cos x) + C.$

27. Let $u = 2\theta$. Then $du = 2\,d\theta$, so

$$\begin{aligned}\textstyle\int_0^{\pi/8} \sec 2\theta \tan 2\theta\,d\theta = \int_0^{\pi/4} \sec u \tan u\left(\frac{1}{2}\,du\right) &= \tfrac{1}{2}\left[\sec u\right]_0^{\pi/4} = \tfrac{1}{2}\left(\sec\tfrac{\pi}{4} - \sec 0\right)\\ &= \tfrac{1}{2}\left(\sqrt{2}-1\right) = \tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}\end{aligned}$$

28. Let $u = 1 + \tan t$. Then $du = \sec^2 t\,dt$, so $\int_0^{\pi/4} (1+\tan t)^3 \sec^2 t\,dt = \int_1^2 u^3\,du = \frac{1}{4}\left[u^4\right]_1^2 = \frac{1}{4}\left(2^4 - 1^4\right) = \frac{15}{4}.$

29. $\int_0^{2\pi} |\sin x|\,dx = \int_0^{\pi} \sin x\,dx - \int_{\pi}^{2\pi} \sin x\,dx = 2\int_0^{\pi} \sin x\,dx = -2\left[\cos x\right]_0^{\pi} = -2\left[(-1) - 1\right] = 4$

30. $\int_0^8 |x^2 - 6x + 8|\,dx = \int_0^8 |(x-2)(x-4)|\,dx$

$= \int_0^2 (x^2-6x+8)\,dx - \int_2^4 (x^2-6x+8)\,dx + \int_4^8 (x^2-6x+8)\,dx$

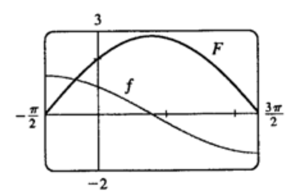
$= \left[\frac{1}{3}x^3 - 3x^2 + 8x\right]_0^2 - \left[\frac{1}{3}x^3 - 3x^2 + 8x\right]_2^4 + \left[\frac{1}{3}x^3 - 3x^2 + 8x\right]_4^8$

$= \left(\frac{8}{3} - 12 + 16\right) - 0 - \left(\frac{64}{3} - 48 + 32\right) + \left(\frac{8}{3} - 12 + 16\right) + \left(\frac{512}{3} - 192 + 64\right) - \left(\frac{64}{3} - 48 + 32\right) = \frac{136}{3}$

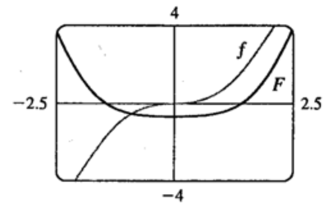
In Exercises 31 and 32, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

31. $u = 1 + \sin x \;\Rightarrow\; du = \cos x\,dx$, so

$\displaystyle\int \frac{\cos x\,dx}{\sqrt{1+\sin x}} = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{1+\sin x} + C.$



32. $u = x^2 + 1 \Rightarrow x^2 = u - 1$ and $x\,dx = \frac{1}{2}\,du$, so



$$\int \frac{x^3}{\sqrt{x^2+1}}\,dx = \int \frac{(u-1)}{\sqrt{u}}\left(\tfrac{1}{2}\,du\right) = \frac{1}{2}\int \left(u^{1/2} - u^{-1/2}\right)du$$

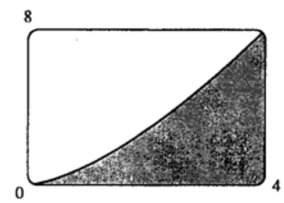
$$= \tfrac{1}{2}\left(\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right) + C$$

$$= \tfrac{1}{3}\left(x^2+1\right)^{3/2} - \left(x^2+1\right)^{1/2} + C$$

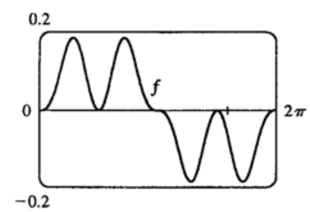
$$= \tfrac{1}{3}\sqrt{x^2+1}\left(x^2-2\right) + C$$

33. From the graph, it appears that the area under the curve $y = x\sqrt{x}$ between $x = 0$ and $x = 4$ is somewhat less than half the area of an 8×4 rectangle, so perhaps about 13 or 14. To find the exact value, we evaluate

$\int_0^4 x\sqrt{x}\,dx = \int_0^4 x^{3/2}\,dx = \left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{2}{5}(4)^{5/2} = \frac{64}{5} = 12.8.$



34. From the graph, it seems as though $\int_0^{2\pi} \cos^2 x \sin^3 x\,dx = 0$. To evaluate the integral, we write the integral as $I = \int_0^{2\pi} \cos^2 x\left(1 - \cos^2 x\right)\sin x\,dx$ and let $u = \cos x \Rightarrow du = -\sin x\,dx$. Thus, $I = \int_1^1 u^2\left(1-u^2\right)(-du) = 0$.



35. By FTC1, $F(x) = \int_1^x \sqrt{1+t^4}\,dt \Rightarrow F'(x) = \sqrt{1+x^4}$.

36. $F(x) = \int_\pi^x \tan\left(s^2\right)ds \Rightarrow F'(x) = \tan\left(x^2\right)$

37. $g(x) = \displaystyle\int_0^{x^3} \frac{t\,dt}{\sqrt{1+t^3}}$. Let $y = g(u)$ and $u = x^3$.

Then $g'(x) = \dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{u}{\sqrt{1+u^3}}3x^2 = \dfrac{x^3}{\sqrt{1+x^9}}3x^2 = \dfrac{3x^5}{\sqrt{1+x^9}}$.

38. $g(x) = \int_1^{\cos x} \sqrt[3]{1-t^2}\,dt$. Let $y = g(x)$ and $u = \cos x$. Then

$$g'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \sqrt[3]{1-u^2}\,(-\sin x) = \sqrt[3]{1-\cos^2 x}\,(-\sin x)$$

$$= -\sin x\sqrt[3]{\sin^2 x} = -(\sin x)^{5/3}$$

39. $y = \displaystyle\int_{\sqrt{x}}^{x} \frac{\cos\theta}{\theta}\,d\theta = \int_1^x \frac{\cos\theta}{\theta}\,d\theta + \int_{\sqrt{x}}^1 \frac{\cos\theta}{\theta}\,d\theta = \int_1^x \frac{\cos\theta}{\theta}\,d\theta - \int_1^{\sqrt{x}} \frac{\cos\theta}{\theta}\,d\theta \Rightarrow$

$y' = \dfrac{\cos x}{x} - \dfrac{\cos\sqrt{x}}{\sqrt{x}}\dfrac{1}{2\sqrt{x}} = \dfrac{2\cos x - \cos\sqrt{x}}{2x}$

40. $y = \int_{2x}^{3x+1} \sin(t^4)\,dt = \int_0^{3x+1} \sin(t^4)\,dt - \int_0^{2x} \sin(t^4)\,dt \quad \Rightarrow \quad y' = 3\sin\left[(3x+1)^4\right] - 2\sin\left[(2x)^4\right]$

41. If $1 \le x \le 3$, then $2 \le \sqrt{x^2+3} \le 2\sqrt{3}$, so $2(3-1) \le \int_1^3 \sqrt{x^2+3}\,dx \le 2\sqrt{3}(3-1)$; that is, $4 \le \int_1^3 \sqrt{x^2+3}\,dx \le 4\sqrt{3}$.

42. If $3 \le x \le 5$, then $4 \le x+1 \le 6$ and $\dfrac{1}{6} \le \dfrac{1}{x+1} \le \dfrac{1}{4}$, so $\frac{1}{6}(5-3) \le \displaystyle\int_3^5 \frac{1}{x+1}\,dx \le \frac{1}{4}(5-3)$; that is, $\frac{1}{3} \le \displaystyle\int_3^5 \frac{1}{x+1}\,dx \le \frac{1}{2}$.

43. $0 \le x \le 1 \quad \Rightarrow \quad 0 \le \cos x \le 1 \quad \Rightarrow \quad x^2 \cos x \le x^2 \quad \Rightarrow \quad \int_0^1 x^2 \cos x\,dx \le \int_0^1 x^2\,dx = \frac{1}{3}\left[x^3\right]_0^1 = \frac{1}{3}$ [Property 7].

44. On the interval $\left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right]$, x is increasing and $\sin x$ is decreasing, so $\dfrac{\sin x}{x}$ is decreasing. Therefore, the largest value of $\dfrac{\sin x}{x}$ on $\left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right]$ is $\dfrac{\sin(\pi/4)}{\pi/4} = \dfrac{\sqrt{2}/2}{\pi/4} = \dfrac{2\sqrt{2}}{\pi}$. By Property 8 with $M = \dfrac{2\sqrt{2}}{\pi}$ we get $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\sin x}{x}\,dx \le \frac{2\sqrt{2}}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

45. Let $f(x) = \sqrt{1+x^3}$ on $[0, 1]$. The Midpoint Rule with $n = 5$ gives

$$\begin{aligned}\int_0^1 \sqrt{1+x^3}\,dx &\approx \tfrac{1}{5}\left[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)\right] \\ &= \tfrac{1}{5}\left[\sqrt{1+(0.1)^3} + \sqrt{1+(0.3)^3} + \cdots + \sqrt{1+(0.9)^3}\right] \approx 1.110\end{aligned}$$

46. (a) displacement $= \int_0^5 (t^2 - t)\,dt = \left[\frac{1}{3}t^3 - \frac{1}{2}t^2\right]_0^5 = \frac{125}{3} - \frac{25}{2} = \frac{175}{6} = 29.1\overline{6}$

(b)
$$\begin{aligned}\text{distance traveled} &= \int_0^5 \left|t^2 - t\right|dt = \int_0^5 |t(t-1)|\,dt = \int_0^1 (t - t^2)\,dt + \int_1^5 (t^2 - t)\,dt \\ &= \left[\tfrac{1}{2}t^2 - \tfrac{1}{3}t^3\right]_0^1 + \left[\tfrac{1}{3}t^3 - \tfrac{1}{2}t^2\right]_1^5 \\ &= \tfrac{1}{2} - \tfrac{1}{3} - 0 + \left(\tfrac{125}{3} - \tfrac{25}{2}\right) - \left(\tfrac{1}{3} - \tfrac{1}{2}\right) = 29.5\end{aligned}$$

47.
$$\begin{aligned}\text{Total percentage increase} = \int_{1991}^{1997} r(t)\,dt \approx M_3 &= \tfrac{1997-1991}{3}\left[r(1992) + r(1994) + r(1996)\right] \\ &= 2(7.4 + 4.8 + 3.0) = 30.4\end{aligned}$$

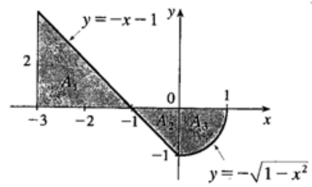
48.
$$\begin{aligned}\text{Distance covered} = \int_0^{5.0} v(t)\,dt \approx M_5 &= \tfrac{5.0-0}{5}\left[v(0.5) + v(1.5) + v(2.5) + v(3.5) + v(4.5)\right] \\ &= 1(4.67 + 8.86 + 10.22 + 10.67 + 10.81) = 45.23 \text{ m}\end{aligned}$$

49. We use the Midpoint Rule with $n = 6$ and $\Delta t = \frac{24-0}{6} = 4$.

$$\begin{aligned}\text{Increase in bee population} = \int_0^{24} r(t)\,dt \approx M_6 &= 4\left[f(2) + f(6) + f(10) + f(14) + f(18) + f(22)\right] \\ &\approx 4\left[60.15 + 976.81 + 7023.11 + 8550.88 + 1374.96 + 152.74\right] \\ &= 4(18{,}138.65) = 72{,}554.6\end{aligned}$$

For the answer in the back of the text, we used the estimates 50, 1000, 7000, 8550, 1350, and 150 to get 72,400.

50. $A_1 = \frac{1}{2}bh = \frac{1}{2}(2)(2) = 2$, $A_2 = \frac{1}{2}bh = \frac{1}{2}(1)(1) = \frac{1}{2}$, and since $y = -\sqrt{1-x^2}$ for $0 \le x \le 1$ represents a quarter-circle with radius 1, $A_3 = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(1)^2 = \frac{\pi}{4}$. So $\int_{-3}^{1} f(x)\,dx = A_1 - A_2 - A_3 = 2 - \frac{1}{2} - \frac{\pi}{4} = \frac{1}{4}(6-\pi)$.

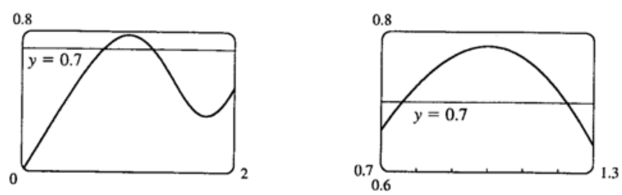


51. By the Fundamental Theorem of Calculus, we know that $F(x) = \int_a^x t^2 \sin(t^2)\,dt$ is an antiderivative of $f(x) = x^2 \sin(x^2)$. This integral cannot be expressed in any simpler form. Since $\int_a^a f\,dt = 0$ for any a, we can take $a = 1$, and then $F(1) = 0$, as required. So $F(x) = \int_1^x t^2 \sin(t^2)\,dt$ is the desired function.

52. (a) C is increasing on those intervals where C' is positive. By the Fundamental Theorem of Calculus, $C'(x) = \dfrac{d}{dx}\left[\int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt\right] = \cos\left(\frac{\pi}{2}x^2\right)$. This is positive when $\frac{\pi}{2}x^2$ is in the interval $\left(\left(2n-\frac{1}{2}\right)\pi, \left(2n+\frac{1}{2}\right)\pi\right)$, n any integer. This implies that $\left(2n-\frac{1}{2}\right)\pi < \frac{\pi}{2}x^2 < \left(2n+\frac{1}{2}\right)\pi \iff$ $0 \le |x| \le 1$ or $\sqrt{4n-1} < |x| < \sqrt{4n+1}$, n any positive integer. So C is increasing on the intervals $[-1, 1]$, $\left[\sqrt{3}, \sqrt{5}\right]$, $\left[-\sqrt{5}, -\sqrt{3}\right]$, $\left[\sqrt{7}, 3\right]$, $\left[-3, -\sqrt{7}\right]$, ...

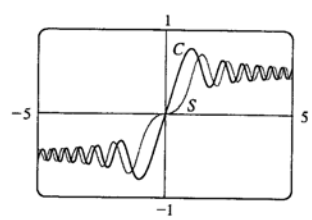
(b) C is concave upward on those intervals where $C'' > 0$. We differentiate C' to find C'': $C'(x) = \cos\left(\frac{\pi}{2}x^2\right) \Rightarrow$ $C''(x) = -\sin\left(\frac{\pi}{2}x^2\right)\left(\frac{\pi}{2}\cdot 2x\right) = -\pi x \sin\left(\frac{\pi}{2}x^2\right)$. For $x > 0$, this is positive where $(2n-1)\pi < \frac{\pi}{2}x^2 < 2n\pi$, n any positive integer $\iff$ $\sqrt{2(2n-1)} < x < 2\sqrt{n}$, n any positive integer. Since there is a factor of $-x$ in C'', the intervals of upward concavity for $x < 0$ are $\left(-\sqrt{2(2n+1)}, -2\sqrt{n}\right)$, n any nonnegative integer. That is, C is concave upward on $\left(-\sqrt{2}, 0\right)$, $\left(\sqrt{2}, 2\right)$, $\left(-\sqrt{6}, -2\right)$, $\left(\sqrt{6}, 2\sqrt{2}\right)$, ...

(c)



From the graphs, we can determine that $\int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt = 0.7$ at $x \approx 0.76$ and $x \approx 1.22$.

(d)



The graphs of $S(x)$ and $C(x)$ have similar shapes, except that S's flattens out near the origin, while C's does not. Note that for $x > 0$, C is increasing where S is concave up, and C is decreasing where S is concave down. Similarly, S is increasing where C is concave down, and S is decreasing where C is concave up. For $x < 0$, these relationships are reversed; that is, C is increasing where S is concave down, and S is increasing where C is concave up. See Example 5.3.2 and Exercise 5.3.47 for a discussion of $S(x)$.

53. $\int_0^x f(t)\,dt = x\sin x + \int_0^x \dfrac{f(t)}{1+t^2}\,dt \;\Rightarrow\; f(x) = x\cos x + \sin x + \dfrac{f(x)}{1+x^2}$ (by differentiation) $\Rightarrow$

$f(x)\left(1 - \dfrac{1}{1+x^2}\right) = x\cos x + \sin x \;\Rightarrow\; f(x)\left(\dfrac{x^2}{1+x^2}\right) = x\cos x + \sin x \;\Rightarrow$

$f(x) = \dfrac{1+x^2}{x^2}\,(x\cos x + \sin x)$

54. From the given equation, $\int_a^x f(t)\,dt = \sin x - \frac{1}{2}$. Differentiating both sides using FTC1 gives $f(x) = \cos x$. We put $x = a$ into the first equation to get $0 = \sin a - \frac{1}{2}$, so $a = \frac{\pi}{6}$ satisfies the given equation.

55. Let $u = f(x)$ and $du = f'(x)\,dx$. So $2\int_a^b f(x) f'(x)\,dx = 2\int_{f(a)}^{f(b)} u\,du = \left[u^2\right]_{f(a)}^{f(b)} = [f(b)]^2 - [f(a)]^2$.

56. Let $F(x) = \int_2^x \sqrt{1+t^3}\,dt$. Then $F'(2) = \lim\limits_{h\to 0} \dfrac{F(2+h) - F(2)}{h} = \lim\limits_{h\to 0} \dfrac{1}{h}\int_2^{2+h} \sqrt{1+t^3}\,dt$, and

$F'(x) = \sqrt{1+x^3}$, so $\lim\limits_{h\to 0} \dfrac{1}{h}\int_2^{2+h} \sqrt{1+t^3}\,dt = F'(2) = \sqrt{1+2^3} = \sqrt{9} = 3$.

57. Let $u = 1 - x$. Then $du = -dx$, so $\int_0^1 f(1-x)\,dx = \int_1^0 f(u)(-du) = \int_0^1 f(u)\,du = \int_0^1 f(x)\,dx$.

58. $\lim\limits_{n\to\infty} \dfrac{1}{n}\left[\left(\dfrac{1}{n}\right)^9 + \left(\dfrac{2}{n}\right)^9 + \left(\dfrac{3}{n}\right)^9 + \cdots + \left(\dfrac{n}{n}\right)^9\right] = \lim\limits_{n\to\infty} \dfrac{1-0}{n}\sum_{i=1}^{n}\left(\dfrac{i}{n}\right)^9 = \int_0^1 x^9\,dx = \left[\dfrac{x^{10}}{10}\right]_0^1 = \dfrac{1}{10}$

The limit is based on Riemann sums using right endpoints and subintervals of equal length.

Problems Plus

1. Differentiating both sides of the equation $x \sin \pi x = \int_0^{x^2} f(t)\,dt$ (using FTC1 and the Chain Rule for the right side) gives $\sin \pi x + \pi x \cos \pi x = 2xf(x^2)$. Letting $x = 2$ so that $f(x^2) = f(4)$, we obtain $\sin 2\pi + 2\pi \cos 2\pi = 4f(4)$, so $f(4) = \frac{1}{4}(0 + 2\pi \cdot 1) = \frac{\pi}{2}$.

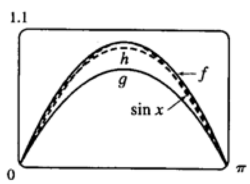
2. (a) Let $f(x) = ax^2 + bx + c$. $f(0) = f(\pi) = 0$, so we know that $f(0) = 0 \;\Rightarrow\; c = 0$, and $f(\pi) = 0 \;\Leftrightarrow\; a\pi^2 + b\pi = 0 \;\Leftrightarrow\; b = -a\pi$. So $f(x) = ax^2 - a\pi x$. Now we want the maximum value of $f(x)$ on $[0, \pi]$ to be the same as that of $\sin x$, that is, 1. So we find the value of x at which f has a maximum by differentiating and setting $f'(x) = 0 \;\Leftrightarrow\; 2ax - a\pi = 0 \;\Leftrightarrow\; x = \frac{\pi}{2}$. Now $f\left(\frac{\pi}{2}\right) = a\left(\frac{\pi}{2}\right)^2 - a\pi\left(\frac{\pi}{2}\right) = -\frac{1}{4}\pi^2 a$. We set this equal to 1, in order to find a: $-\frac{1}{4}\pi^2 a = 1 \;\Leftrightarrow\; a = -\frac{4}{\pi^2} \;\Rightarrow\; b = -a\pi = \frac{4}{\pi}$. Thus, the desired function is $f(x) = -\frac{4}{\pi^2}x^2 + \frac{4}{\pi}x$.

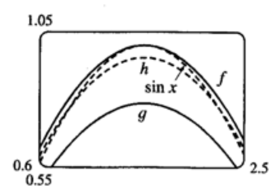
Alternate Solution (without calculus): Use $f(x) = ax(x - \pi)$.

(b) Once again, $g(0) = g(\pi) = 0 \;\Rightarrow\; g(x) = ax^2 - a\pi x$. We want $g'(0) = \left[\frac{d}{dx}(\sin x)\right]_0 = \cos 0 = 1$. We calculated $g'(x)$ in part (a), so we set $g'(0) = 1 \;\Leftrightarrow\; 2a(0) - a\pi = 1 \;\Leftrightarrow\; a = -\frac{1}{\pi}$. We also want $g'(\pi) = \left[\frac{d}{dx}(\sin x)\right]_\pi = \cos \pi = -1$, so we check that $g'(\pi) = -1$ with $a = -\frac{1}{\pi}$:
$g'(\pi) = 2\left(-\frac{1}{\pi}\right)\pi - \left(-\frac{1}{\pi}\right)\pi = -1$. Thus, the desired function is $g(x) = -\frac{1}{\pi}x^2 + x$.

(c) Again, $h(x) = ax^2 - a\pi x$. Now we want the area under the curves of $h(x)$ and $\sin x$ to be the same; that is, $\int_0^\pi h(x)\,dx = \int_0^\pi \sin x\,dx = [-\cos x]_0^\pi = (1) - (-1) = 2$. We integrate h between 0 and π and set the result equal to 2: $\int_0^\pi (ax^2 - a\pi x)\,dx = \left[\frac{1}{3}ax^3 - \frac{1}{2}a\pi x^2\right]_0^\pi = \frac{1}{3}a\pi^3 - \frac{1}{2}a\pi^3 = -\frac{1}{6}a\pi^3 = 2 \;\Leftrightarrow\; a = -12/\pi^3$.
Thus, the desired function is $h(x) = -\dfrac{12}{\pi^3}x^2 + \dfrac{12}{\pi^2}x$.

(d)





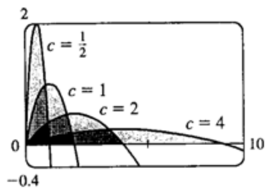
3. For $1 \le x \le 2$, we have $x^4 \le 2^4 = 16$, so $1 + x^4 \le 17$ and $\dfrac{1}{1+x^4} \ge \dfrac{1}{17}$. Thus,
$\displaystyle\int_1^2 \frac{1}{1+x^4}\,dx \ge \int_1^2 \frac{1}{17}\,dx = \frac{1}{17}$. Also $1 + x^4 > x^4$ for $1 \le x \le 2$, so $\dfrac{1}{1+x^4} < \dfrac{1}{x^4}$ and
$\displaystyle\int_1^2 \frac{1}{1+x^4}\,dx < \int_1^2 x^{-4}\,dx = \left[\frac{x^{-3}}{-3}\right]_1^2 = -\frac{1}{24} + \frac{1}{3} = \frac{7}{24}$. Thus, we have the estimate
$\displaystyle\frac{1}{17} \le \int_1^2 \frac{1}{1+x^4}\,dx \le \frac{7}{24}$.

4. By FTC2, $\int_0^1 f'(x)\,dx = f(1) - f(0) = 1 - 0 = 1$.

5. Such a function cannot exist. $f'(x) > 3$ for all x means that f is differentiable (and hence continuous) for all x. So by FTC2, $\int_1^4 f'(x)\,dx = f(4) - f(1) = 7 - (-1) = 8$. However, if $f'(x) > 3$ for all x, then $\int_1^4 f'(x)\,dx \geq 3 \cdot (4-1) = 9$ by Comparison Property 8 in Section 5.2.

Another Solution: By the Mean Value Theorem, there exists a number $c \in (1,4)$ such that $f'(c) = \dfrac{f(4) - f(1)}{4-1} = \dfrac{7-(-1)}{3} = \dfrac{8}{3} \Rightarrow 8 = 3f'(c)$. But $f'(x) > 3 \Rightarrow 3f'(c) > 9$, so such a function cannot exist.

6. (a)

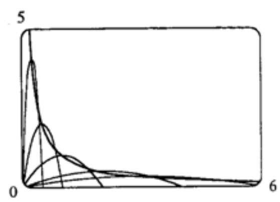


From the graph of $f(x) = \dfrac{2cx - x^2}{c^3}$, it appears that the areas are equal; that is, the area enclosed is independent of c.

(b) We first find the x-intercepts of the curve, to determine the limits of integration: $y = 0 \Leftrightarrow 2cx - x^2 = 0 \Leftrightarrow x = 0$ or $x = 2c$. Now we integrate the function between these limits to find the enclosed area:

$$A = \int_0^{2c} \frac{2cx - x^2}{c^3}\,dx = \frac{1}{c^3}\left[cx^2 - \tfrac{1}{3}x^3\right]_0^{2c} = \frac{1}{c^3}\left[c(2c)^2 - \tfrac{1}{3}(2c)^3\right] = \frac{1}{c^3}\left[4c^3 - \tfrac{8}{3}c^3\right] = \tfrac{4}{3}, \text{ a constant.}$$

(c)



The vertices of the family of parabolas seem to determine a branch of a hyperbola.

(d) For a particular c, the vertex is the point where the maximum occurs. We have seen that the x-intercepts are 0 and $2c$, so by symmetry, the maximum occurs at $x = c$, and its value is $\dfrac{2c(c) - c^2}{c^3} = \dfrac{1}{c}$. So we are interested in the curve consisting of all points of the form $\left(c, \dfrac{1}{c}\right)$, $c > 0$. This is the part of the hyperbola $y = 1/x$ lying in the first quadrant.

7. $f(x) = \displaystyle\int_0^{g(x)} \frac{1}{\sqrt{1+t^3}}\,dt$, where $g(x) = \int_0^{\cos x} \left[1 + \sin\left(t^2\right)\right] dt$. Using FTC1 and the Chain Rule (twice) we have $f'(x) = \dfrac{1}{\sqrt{1 + [g(x)]^3}}\, g'(x) = \dfrac{1}{\sqrt{1 + [g(x)]^3}}\left[1 + \sin\left(\cos^2 x\right)\right](-\sin x)$. Now

$g\left(\frac{\pi}{2}\right) = \int_0^0 \left[1 + \sin\left(t^2\right)\right] dt = 0$, so $f'\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{1+0}}(1 + \sin 0)(-1) = 1 \cdot 1 \cdot (-1) = -1$.

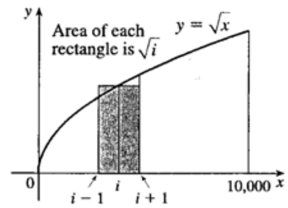
8. If $f(x) = \int_0^x x^2 \sin\left(t^2\right) dt = x^2 \int_0^x \sin\left(t^2\right) dt$, then $f'(x) = 2x \int_0^x \sin\left(t^2\right) dt + x^2 \sin\left(x^2\right)$, by the Product Rule and FTC1.

9. $f(x) = 2 + x - x^2 = (-x+2)(x+1) = 0 \Leftrightarrow x = 2$ or $x = -1$. $f(x) \geq 0$ for $x \in [-1, 2]$ and $f(x) < 0$ everywhere else. The integral $\int_a^b \left(2 + x - x^2\right) dx$ has a maximum on the interval where the integrand is positive, which is $[-1, 2]$. So $a = -1$, $b = 2$. (Any larger interval gives a smaller integral since $f(x) < 0$ outside $[-1, 2]$. Any smaller interval also gives a smaller integral since $f(x) \geq 0$ in $[-1, 2]$.)

10. This sum can be interpreted as a Riemann sum, with the right endpoints of the subintervals as sample points and with $a = 0$, $b = 10{,}000$, and $f(x) = \sqrt{x}$. So we approximate

$$\sum_{i=1}^{10{,}000} \sqrt{i} \approx \lim_{n\to\infty} \frac{10{,}000}{n} \sum_{i=1}^{n} \sqrt{\frac{10{,}000i}{n}} = \int_0^{10{,}000} \sqrt{x}\,dx = \left[\tfrac{2}{3}x^{3/2}\right]_0^{10{,}000} = \tfrac{2}{3}(1{,}000{,}000) \approx 666{,}667.$$

Alternate Method: We can use graphical methods as follows:



From the figure we see that $\int_{i-1}^{i} \sqrt{x}\,dx < \sqrt{i} < \int_{i}^{i+1} \sqrt{x}\,dx$, so

$\int_0^{10{,}000} \sqrt{x}\,dx < \sum_{i=1}^{10{,}000} \sqrt{i} < \int_1^{10{,}001} \sqrt{x}\,dx$. Since $\int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$,

we get $\int_0^{10{,}000} \sqrt{x}\,dx = 666{,}666.\overline{6}$ and

$\int_1^{10{,}001} \sqrt{x}\,dx = \frac{2}{3}\left[(10{,}001)^{3/2} - 1\right] \approx 666{,}766.$

Hence, $666{,}666.\overline{6} < \sum_{i=1}^{10{,}000} \sqrt{i} < 666{,}766$. We can estimate the sum by averaging these bounds:

$\sum_{i=1}^{10{,}000} \approx \frac{666{,}666.\overline{6} + 666{,}766}{2} \approx 666{,}716$. The actual value is about 666,716.46.

11. (a) We can split the integral $\int_0^n [\![x]\!]\,dx$ into the sum $\sum_{i=1}^{n}\left[\int_{i-1}^{i} [\![x]\!]\,dx\right]$. But on each of the intervals $[i-1, i)$ of integration, $[\![x]\!]$ is a constant function, namely $i - 1$. So the ith integral in the sum is equal to $(i-1)[i-(i-1)] = (i-1)$. So the original integral is equal to $\sum_{i=1}^{n}(i-1) = \sum_{i=1}^{n-1} i = \dfrac{(n-1)n}{2}$.

(b) We can write $\int_a^b [\![x]\!]\,dx = \int_0^b [\![x]\!]\,dx - \int_0^a [\![x]\!]\,dx$.

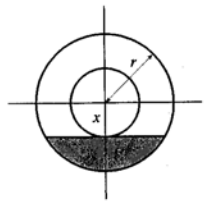
Now $\int_0^b [\![x]\!]\,dx = \int_0^{[\![b]\!]} [\![x]\!]\,dx + \int_{[\![b]\!]}^b [\![x]\!]\,dx$. The first of these integrals is equal to $\frac{1}{2}([\![b]\!]-1)[\![b]\!]$, by part (a), and since $[\![x]\!] = [\![b]\!]$ on $[[\![b]\!], b]$, the second integral is just $[\![b]\!](b - [\![b]\!])$. So $\int_0^b [\![x]\!]\,dx = \frac{1}{2}([\![b]\!]-1)[\![b]\!] + [\![b]\!](b-[\![b]\!]) = \frac{1}{2}[\![b]\!](2b - [\![b]\!] - 1)$ and similarly $\int_0^a [\![x]\!]\,dx = \frac{1}{2}[\![a]\!](2a - [\![a]\!] - 1)$. Therefore $\int_a^b [\![x]\!]\,dx = \frac{1}{2}[\![b]\!](2b - [\![b]\!] - 1) - \frac{1}{2}[\![a]\!](2a - [\![a]\!] - 1)$.

12. By FTC1, $\dfrac{d}{dx}\int_0^x \left(\int_1^{\sin t} \sqrt{1+u^4}\,du\right)dt = \int_1^{\sin x} \sqrt{1+u^4}\,du$. Again using FTC1,

$$\frac{d^2}{dx^2}\int_0^x \left(\int_1^{\sin t} \sqrt{1+u^4}\,du\right)dt = \frac{d}{dx}\int_1^{\sin x} \sqrt{1+u^4}\,du = \sqrt{1+\sin^4 x}\cos x.$$

13. Differentiating the equation $\int_0^x f(t)\,dt = [f(x)]^2$ using FTC1 gives $f(x) = 2f(x)f'(x) \Rightarrow f(x)\left[2f'(x) - 1\right] = 0$, so $f(x) = 0$ or $f'(x) = \frac{1}{2}$. $f'(x) = \frac{1}{2} \Rightarrow f(x) = \frac{1}{2}x + C$. To find C we substitute into the original equation to get $\int_0^x \left(\frac{1}{2}t + C\right)dt = \left(\frac{1}{2}x + C\right)^2 \Leftrightarrow \frac{1}{4}x^2 + Cx = \frac{1}{4}x^2 + Cx + C^2$. It follows that $C = 0$, so $f(x) = \frac{1}{2}x$. Therefore, $f(x) = 0$ or $f(x) = \frac{1}{2}x$.

14.



Let x be the distance between the center of the disk and the surface of the liquid. The wetted circular region has area $\pi r^2 - \pi x^2$ while the unexposed wetted region (shaded in the diagram) has area $2\int_x^r \sqrt{r^2 - t^2}\,dt$, so the exposed wetted region has area $A(x) = \pi r^2 - \pi x^2 - 2\int_x^r \sqrt{r^2 - t^2}\,dt$, $0 \le x \le r$. By FTC1, we have $A'(x) = -2\pi x + 2\sqrt{r^2 - x^2}$, so $A'(x) = 0$ when $\pi x = \sqrt{r^2 - x^2} \;\Rightarrow\; \pi^2 x^2 = r^2 - x^2$ and hence $(1 + \pi^2)x^2 = r^2$, so $x = \dfrac{r}{\sqrt{1 + \pi^2}}$.

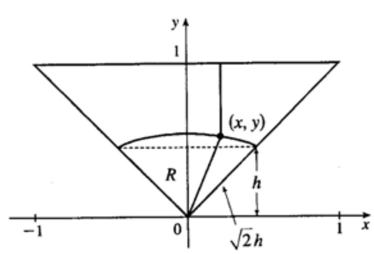
Now $A(0) = \frac{1}{2}\pi r^2$ and $A(r) = 0$, while $A\left(\dfrac{r}{\sqrt{1+\pi^2}}\right) = \frac{1}{2}\pi r^2 + r^2 \sin^{-1}\dfrac{1}{\sqrt{1+\pi^2}}$ (after some simplification) $> \frac{1}{2}\pi r^2$, so there is an absolute maximum when $x = \dfrac{r}{\sqrt{1+\pi^2}}$.

15. Note that $\frac{d}{dx}\left(\int_0^x \left[\int_0^u f(t)\,dt\right] du\right) = \int_0^x f(t)\,dt$ by FTC1, while

$$\begin{aligned}\tfrac{d}{dx}\left[\int_0^x f(u)(x-u)\,du\right] &= \tfrac{d}{dx}\left[x\int_0^x f(u)\,du\right] - \tfrac{d}{dx}\left[\int_0^x f(u)\,u\,du\right] = \int_0^x f(u)\,du + xf(x) - f(x)x \\ &= \int_0^x f(u)\,du\end{aligned}$$

Hence, $\int_0^x f(u)(x-u)\,du = \int_0^x \left[\int_0^u f(t)\,dt\right] du + C$. Setting $x = 0$ gives $C = 0$.

16.



We restrict our attention to the triangle shown. A point in this triangle is closer to the side shown than to any other side, so if we find the area of the region R consisting of all points in the triangle that are closer to the center than to that side, we can multiply this area by 4 to find the total area. We find the equation of the set of points which are equidistant from the center and the side: the distance of the point (x, y) from the side is $1 - y$, and its distance from the center is $\sqrt{x^2 + y^2}$. So the distances are equal if $\sqrt{x^2 + y^2} = 1 - y \;\Leftrightarrow\; x^2 + y^2 = 1 - 2y + y^2 \;\Leftrightarrow\; y = \frac{1}{2}(1 - x^2)$. Note that the area we are interested in is equal to the area of a triangle plus a crescent-shaped area. To find these areas, we have to find the y-coordinate h of the horizontal line separating them. From the diagram, $1 - h = \sqrt{2}h \;\Leftrightarrow\; h = \frac{1}{1+\sqrt{2}} = \sqrt{2} - 1$. We calculate the areas in terms of h, and substitute afterward.

The area of the triangle is $\frac{1}{2}(2h)(h) = h^2$, and the area of the crescent-shaped section is $\int_{-h}^{h}\left[\frac{1}{2}(1 - x^2) - h\right]dx = 2\left[\left(\frac{1}{2} - h\right)x - \frac{1}{6}x^3\right]_0^h = h - 2h^2 - \frac{1}{3}h^3$. So the area of the whole region is

$$\begin{aligned}4\left[\left(h - 2h^2 - \tfrac{1}{3}h^3\right) + h^2\right] &= 4h\left(1 - h - \tfrac{1}{3}h^2\right) = 4\left(\sqrt{2} - 1\right)\left[1 - \left(\sqrt{2} - 1\right) - \tfrac{1}{3}\left(\sqrt{2} - 1\right)^2\right] \\ &= 4\left(\sqrt{2} - 1\right)\left(1 - \tfrac{1}{3}\sqrt{2}\right) = \tfrac{4}{3}\left(4\sqrt{2} - 5\right).\end{aligned}$$

17. $\displaystyle\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}}+\frac{1}{\sqrt{n}\sqrt{n+2}}+\cdots+\frac{1}{\sqrt{n}\sqrt{n+n}}\right)=\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{n}{n+1}}+\sqrt{\frac{n}{n+2}}+\cdots+\sqrt{\frac{n}{n+n}}\right)$

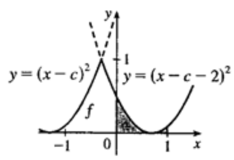
$\displaystyle=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{\sqrt{1+1/n}}+\frac{1}{\sqrt{1+2/n}}+\cdots+\frac{1}{\sqrt{1+1}}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{i}{n}\right)\ \left(\text{where } f(x)=\frac{1}{\sqrt{1+x}}\right)$

$\displaystyle=\int_0^1\frac{1}{\sqrt{1+x}}\,dx=\left[2\sqrt{1+x}\right]_0^1=2\left(\sqrt{2}-1\right)$

18. Note that the graphs of $(x-c)^2$ and $[(x-c)-2]^2$ intersect when $|x-c|=|x-c-2|\ \Leftrightarrow\ c-x=x-c-2$ $\Leftrightarrow\ x=c+1$. The integration will proceed differently depending on the value of c.

Case 1: $-2\le c<-1$ In this case, $f_c(x)=(x-c-2)^2$ for $x\in[0,1]$, so

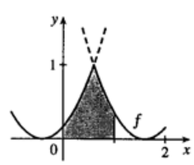
$$g(c)=\int_0^1(x-c-2)^2\,dx=\tfrac{1}{3}\left[(x-c-2)^3\right]_0^1=\tfrac{1}{3}\left[(-c-1)^3-(-c-2)^3\right]$$
$$=\tfrac{1}{3}\left(3c^2+9c+7\right)=c^2+3c+\tfrac{7}{3}=\left(c+\tfrac{3}{2}\right)^2+\tfrac{1}{12}$$



This is a parabola; its maximum for $-2\le c<-1$ is $g(-2)=\frac{1}{3}$, and its minimum is $g\left(-\frac{3}{2}\right)=\frac{1}{12}$.

Case 2: $-1\le c<0$ In this case, $f_c(x)=\begin{cases}(x-c)^2 & \text{if } 0\le x\le c+1\\ (x-c-2)^2 & \text{if } c+1<x\le 1\end{cases}$ Therefore,

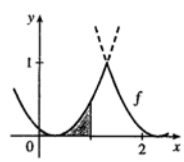
$$g(c)=\int_0^1 f_c(x)\,dx=\int_0^{c+1}(x-c)^2\,dx+\int_{c+1}^1(x-c-2)^2\,dx$$
$$=\tfrac{1}{3}\left[(x-c)^3\right]_0^{c+1}+\tfrac{1}{3}\left[(x-c-2)^3\right]_{c+1}^1=\tfrac{1}{3}\left[1+c^3+(-c-1)^3-(-1)\right]$$
$$=-c^2-c+\tfrac{1}{3}=-\left(c+\tfrac{1}{2}\right)^2+\tfrac{7}{12}$$



Again, this is a parabola, whose maximum for $-1\le c<0$ is $g\left(-\frac{1}{2}\right)=\frac{7}{12}$, and whose minimum on this c-interval is $g(-1)=\frac{1}{3}$.

Case 3: $0\le c\le 2$ In this case, $f_c(x)=(x-c)^2$ for $x\in[0,1]$, so

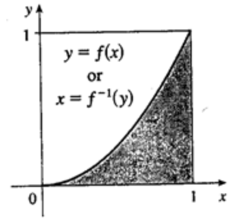
$$g(c)=\int_0^1(x-c)^2\,dx=\tfrac{1}{3}\left[(x-c)^3\right]_0^1=\tfrac{1}{3}\left[(1-c)^3-(-c)^3\right]=c^2-c+\tfrac{1}{3}=\left(c-\tfrac{1}{2}\right)^2+\tfrac{1}{12}$$



This parabola has a maximum of $g(2)=\frac{7}{3}$ and a minimum of $g\left(\frac{1}{2}\right)=\frac{1}{12}$.

We conclude that $g(c)$ has an absolute maximum of $g(2)=\frac{7}{3}$, and absolute minima of $g\left(-\frac{3}{2}\right)=g\left(\frac{1}{2}\right)=\frac{1}{12}$.

19. The shaded region has area $\int_0^1 f(x)\,dx = \frac{1}{3}$. The integral $\int_0^1 f^{-1}(y)\,dy$ gives the area of the unshaded region, which we know to be $1 - \frac{1}{3} = \frac{2}{3}$. So $\int_0^1 f^{-1}(y)\,dy = \frac{2}{3}$.



20. (a) To find $B_1(x)$, we use the fact that $B_1'(x) = B_0(x) \Rightarrow B_1(x) = \int B_0(x)\,dx = \int 1\,dx = x + C$. Now we impose the condition that $\int_0^1 B_1(x)\,dx = 0 \Rightarrow 0 = \int_0^1 (x + C)\,dx = \left[\frac{1}{2}x^2\right]_0^1 + [Cx]_0^1 = \frac{1}{2} + C \Rightarrow C = -\frac{1}{2}$. So $B_1(x) = x - \frac{1}{2}$. Similarly $B_2(x) = \int B_1(x)\,dx = \int\left(x - \frac{1}{2}\right)dx = \frac{1}{2}x^2 - \frac{1}{2}x + D$. But $\int_0^1 B_2(x)\,dx = 0 \Rightarrow 0 = \int_0^1 \left(\frac{1}{2}x^2 - \frac{1}{2}x + D\right)dx = \frac{1}{6} - \frac{1}{4} + D \Rightarrow D = \frac{1}{12}$, so $B_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}$. $B_3(x) = \int B_2(x)\,dx = \int\left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}\right)dx = \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{12}x + E$. But $\int_0^1 B_3(x)\,dx = 0 \Rightarrow 0 = \int_0^1 \left(\frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{12}x + E\right)dx = \frac{1}{24} - \frac{1}{12} + \frac{1}{24} + E \Rightarrow E = 0$. So $B_3(x) = \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{12}x$. $B_4(x) = \int B_3(x)\,dx = \int\left(\frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{12}x\right)dx = \frac{1}{24}x^4 - \frac{1}{12}x^3 + \frac{1}{24}x^2 + F$. But $\int_0^1 B_4(x)\,dx = 0 \Rightarrow 0 = \int_0^1 \left(\frac{1}{24}x^4 - \frac{1}{12}x^3 + \frac{1}{24}x^2 + F\right)dx = \frac{1}{120} - \frac{1}{48} + \frac{1}{72} + F \Rightarrow F = -\frac{1}{720}$. So $B_4(x) = \frac{1}{24}x^4 - \frac{1}{12}x^3 + \frac{1}{24}x^2 - \frac{1}{720}$.

(b) By FTC2, $B_n(1) - B_n(0) = \int_0^1 B_n'(x)\,dx = \int_0^1 B_{n-1}(x)\,dx = 0$ for $n - 1 \geq 1$, by definition. Thus, $B_n(0) = B_n(1)$ for $n \geq 2$.

(c) We know that $B_n(x) = \dfrac{1}{n!}\displaystyle\sum_{k=0}^{n}\binom{n}{k} b_k x^{n-k}$. If we set $x = 1$ in this expression, and use the fact that $B_n(1) = B_n(0) = \dfrac{b_n}{n!}$ for $n \geq 2$, we get $b_n = \displaystyle\sum_{k=0}^{n}\binom{n}{k} b_k$. Now if we expand the right-hand side, we get $b_n = \binom{n}{0}b_0 + \binom{n}{1}b_1 + \cdots + \binom{n}{n-2}b_{n-2} + \binom{n}{n-1}b_{n-1} + \binom{n}{n}b_n$. We cancel the b_n terms, move the b_{n-1} term to the LHS and divide by $-\binom{n}{n-1} = -n$: $b_{n-1} = -\frac{1}{n}\left[\binom{n}{0}b_0 + \binom{n}{1}b_1 + \cdots + \binom{n}{n-2}b_{n-2}\right]$ for $n \geq 2$, as required.

(d) We use mathematical induction. For $n = 0$: $B_0(1 - x) = 1$ and $(-1)^0 B_0(x) = 1$, so the equation holds for $n = 0$ since $b_0 = 1$. Now if $B_k(1 - x) = (-1)^k B_k(x)$, then since $\frac{d}{dx}B_{k+1}(1 - x) = B_{k+1}'(1 - x)\frac{d}{dx}(1 - x) = -B_k(1 - x)$, we have $\frac{d}{dx}B_{k+1}(1 - x) = (-1)(-1)^k B_k(x) = (-1)^{k+1} B_k(x)$. Integrating, we get $B_{k+1}(1 - x) = (-1)^{k+1} B_{k+1}(x) + C$. But the constant of integration must be 0, since if we substitute $x = 0$ in the equation, we get $B_{k+1}(1) = (-1)^{k+1} B_{k+1}(0) + C$, and if we substitute $x = 1$ we get $B_{k+1}(0) = (-1)^{k+1} B_{k+1}(1) + C$, and these two equations together imply that $B_{k+1}(0) = (-1)^{k+1}\left[(-1)^{k+1} B_{k+1}(0) + C\right] + C = B_{k+1}(0) + 2C \Leftrightarrow C = 0$. So the equation holds for all n, by induction. Now if the power of -1 is odd, then we have $B_{2n+1}(1 - x) = -B_{2n+1}(x)$. In particular, $B_{2n+1}(1) = -B_{2n+1}(0)$. But from part (b), we know that $B_k(1) = B_k(0)$ for $k > 1$. The only possibility is that $B_{2n+1}(0) = B_{2n+1}(1) = 0$ for all $n > 0$, and this implies that $b_{2n+1} = (2n + 1)!\, B_{2n+1}(0) = 0$ for $n > 0$.

(e) From part (a), we know that $b_0 = 0!\,B_0(0) = 1$, and similarly $b_1 = -\frac{1}{2}$, $b_2 = \frac{1}{6}$, $b_3 = 0$ and $b_4 = -\frac{1}{30}$.

We use the formula to find

$$b_6 = b_{7-1} = -\frac{1}{7}\left[\binom{7}{0}b_0 + \binom{7}{1}b_1 + \binom{7}{2}b_2 + \binom{7}{3}b_3 + \binom{7}{4}b_4 + \binom{7}{5}b_5\right]$$

The b_3 and b_5 terms are 0, so this is equal to

$$-\frac{1}{7}\left[1 + 7\left(-\frac{1}{2}\right) + \frac{7\cdot 6}{2\cdot 1}\left(\frac{1}{6}\right) + \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1}\left(-\frac{1}{30}\right)\right] = -\frac{1}{7}\left(1 - \frac{7}{2} + \frac{7}{2} - \frac{7}{6}\right) = \frac{1}{42}$$

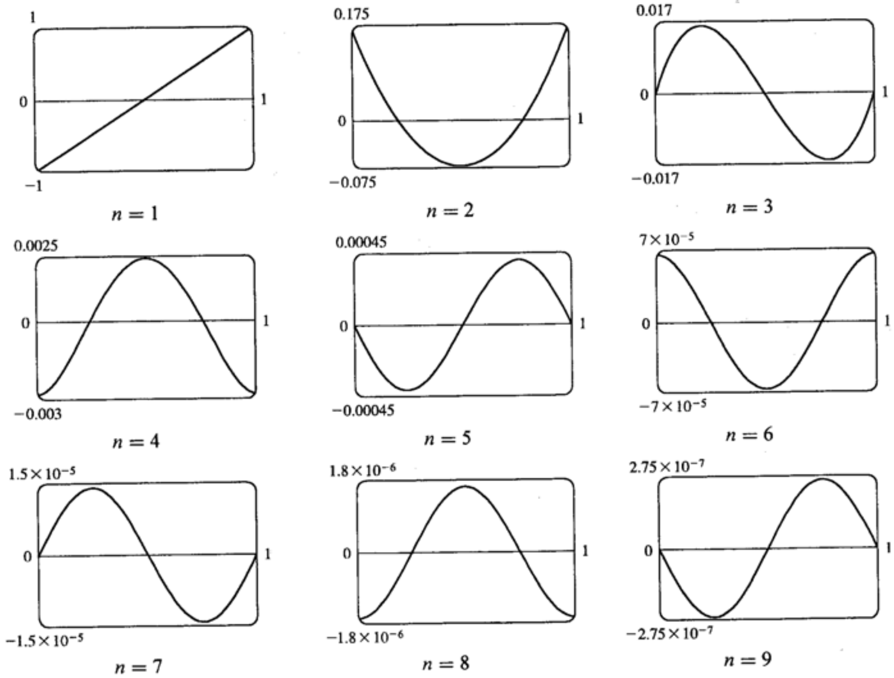
Similarly,

$$\begin{aligned} b_8 &= -\frac{1}{9}\left[\binom{9}{0}b_0 + \binom{9}{1}b_1 + \binom{9}{2}b_2 + \binom{9}{4}b_4 + \binom{9}{6}b_6\right] \\ &= -\frac{1}{9}\left[1 + 9\left(-\frac{1}{2}\right) + \frac{9\cdot 8}{2\cdot 1}\left(\frac{1}{6}\right) + \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2\cdot 1}\left(-\frac{1}{30}\right) + \frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1}\left(\frac{1}{42}\right)\right] \\ &= -\frac{1}{9}\left(1 - \frac{9}{2} + 6 - \frac{21}{5} + 2\right) = -\frac{1}{30} \end{aligned}$$

Now we can calculate

$$\begin{aligned} B_5(x) &= \frac{1}{5!}\sum_{k=0}^{5}\binom{5}{k}b_k x^{5-k} \\ &= \frac{1}{120}\left[x^5 + 5\left(-\frac{1}{2}\right)x^4 + \frac{5\cdot 4}{2\cdot 1}\left(\frac{1}{6}\right)x^3 + 5\left(-\frac{1}{30}\right)x\right] \\ &= \tfrac{1}{120}\left(x^5 - \tfrac{5}{2}x^4 + \tfrac{5}{3}x^3 - \tfrac{1}{6}x\right) \\ B_6(x) &= \frac{1}{720}\left[x^6 + 6\left(-\frac{1}{2}\right)x^5 + \frac{6\cdot 5}{2\cdot 1}\left(\frac{1}{6}\right)x^4 + \frac{6\cdot 5}{2\cdot 1}\left(-\frac{1}{30}\right)x^2 + \frac{1}{42}\right] \\ &= \tfrac{1}{720}\left(x^6 - 3x^5 + \tfrac{5}{2}x^4 - \tfrac{1}{2}x^2 + \tfrac{1}{42}\right) \\ B_7(x) &= \frac{1}{5040}\left[x^7 + 7\left(-\frac{1}{2}\right)x^6 + \frac{7\cdot 6}{2\cdot 1}\left(\frac{1}{6}\right)x^5 + \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1}\left(-\frac{1}{30}\right)x^3 + 7\left(\frac{1}{42}\right)x\right] \\ &= \tfrac{1}{5040}\left(x^7 - \tfrac{7}{2}x^6 + \tfrac{7}{2}x^5 - \tfrac{7}{6}x^3 + \tfrac{1}{6}x\right) \\ B_8(x) &= \frac{1}{40{,}320}\left[x^8 + 8\left(-\frac{1}{2}\right)x^7 + \frac{8\cdot 7}{2\cdot 1}\left(\frac{1}{6}\right)x^6 + \frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2\cdot 1}\left(-\frac{1}{30}\right)x^4 + \frac{8\cdot 7}{2\cdot 1}\left(\frac{1}{42}\right)x^2 + \left(-\frac{1}{30}\right)\right] \\ &= \tfrac{1}{40{,}320}\left(x^8 - 4x^7 + \tfrac{14}{3}x^6 - \tfrac{7}{3}x^4 + \tfrac{2}{3}x^2 - \tfrac{1}{30}\right) \\ B_9(x) &= \frac{1}{362{,}880}\left[x^9 + 9\left(-\frac{1}{2}\right)x^8 + \frac{9\cdot 8}{2\cdot 1}\left(\frac{1}{6}\right)x^7 + \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2\cdot 1}\left(-\frac{1}{30}\right)x^5\right. \\ &\qquad\left. + \frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1}\left(\frac{1}{42}\right)x^3 + 9\left(-\frac{1}{30}\right)x\right] \\ &= \tfrac{1}{362{,}880}\left(x^9 - \tfrac{9}{2}x^8 + 6x^7 - \tfrac{21}{5}x^5 + 2x^3 - \tfrac{3}{10}x\right) \end{aligned}$$

(f)



There are four basic shapes for the graphs of B_n (excluding B_1), and as n increases, they repeat in a cycle of four. For $n = 4m$, the shape resembles that of the graph of $-\cos 2\pi x$; For $n = 4m + 1$, that of $-\sin 2\pi x$; for $n = 4m + 2$, that of $\cos 2\pi x$; and for $n = 4m + 3$, that of $\sin 2\pi x$.

(g) For $k = 0$: $B_1(x+1) - B_1(x) = x + 1 - \frac{1}{2} - \left(x - \frac{1}{2}\right) = 1$, and $\dfrac{x^0}{0!} = 1$, so the equation holds for $k = 0$.

We now assume that $B_n(x+1) - B_n(x) = \dfrac{x^{n-1}}{(n-1)!}$. We integrate this equation with respect to x:

$\displaystyle\int [B_n(x+1) - B_n(x)]\,dx = \int \frac{x^{n-1}}{(n-1)!}\,dx$. But we can evaluate the LHS using the definition $B_{n+1}(x) = \int B_n(x)\,dx$, and the RHS is a simple integral. The equation becomes

$B_{n+1}(x+1) - B_{n+1}(x) = \dfrac{1}{(n-1)!}\left(\dfrac{1}{n}x^n\right) = \dfrac{1}{n!}x^n$, since by part (b) $B_{n+1}(1) - B_{n+1}(0) = 0$, and so the constant of integration must vanish. So the equation holds for all k, by induction.

(h) The result from part (g) implies that $p^k = k!\left[B_{k+1}(p+1) - B_{k+1}(p)\right]$. If we sum both sides of this equation from $p = 0$ to $p = n$ (note that k is fixed in this process), we get

$\displaystyle\sum_{p=0}^{n} p^k = k! \sum_{p=0}^{n}\left[B_{k+1}(p+1) - B_{k+1}(p)\right]$. But the RHS is just a telescoping sum, so the equation becomes

$1^k + 2^k + 3^k + \cdots + n^k = k!\left[B_{k+1}(n+1) - B_{k+1}(0)\right]$. But from the definition of Bernoulli polynomials (and using the Fundamental Theorem of Calculus), the RHS is equal to $k!\int_0^{n+1} B_k(x)\,dx$.

(i) If we let $k = 3$ and then substitute from part (a), the formula in part (h) becomes

$$1^3 + 2^3 + \cdots + n^3 = 3!\,[B_4(n+1) - B_4(0)]$$
$$= 6\left[\tfrac{1}{24}(n+1)^4 - \tfrac{1}{12}(n+1)^3 + \tfrac{1}{24}(n+1)^2 - \tfrac{1}{720} - \left(\tfrac{1}{24} - \tfrac{1}{12} + \tfrac{1}{24} - \tfrac{1}{720}\right)\right]$$
$$= \frac{(n+1)^2\,[1 + (n+1)^2 - 2(n+1)]}{4} = \frac{(n+1)^2\,[1 - (n+1)]^2}{4} = \left[\frac{n(n+1)}{2}\right]^2$$

(j) $1^k + 2^k + 3^k + \cdots + n^k = k! \displaystyle\int_0^{n+1} B_k(x)\,dx$ [by part (h)]

$$= k! \int_0^{n+1} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} b_j x^{k-j}\,dx = \int_0^{n+1} \sum_{j=0}^{k} \binom{k}{j} b_j x^{k-j}\,dx$$

Now view $\displaystyle\sum_{j=0}^{k} \binom{k}{j} b_j x^{k-j}$ as $(x+b)^k$, as explained in the problem. Then

$$1^k + 2^k + 3^k + \cdots + n^k \text{ ``}{=}\text{''} \int_0^{n+1} (x+b)^k\,dx = \left[\frac{(x+b)^{k+1}}{k+1}\right]_0^{n+1} = \frac{(n+1+b)^{k+1} - b^{k+1}}{k+1}$$

(k) We expand the RHS of the formula in (j), turning the b^i into b_i, and remembering that $b_{2i+1} = 0$ for $i > 0$:

$$1^5 + 2^5 + \cdots + n^5 = \tfrac{1}{6}\left[(n+1+b)^6 - b^6\right]$$
$$= \tfrac{1}{6}\left[(n+1)^6 + 6(n+1)^5 b_1 + \tfrac{6\cdot 5}{2\cdot 1}(n+1)^4 b_2 + \tfrac{6\cdot 5}{2\cdot 1}(n+1)^2 b_4\right]$$
$$= \tfrac{1}{6}\left[(n+1)^6 - 3(n+1)^5 + \tfrac{5}{2}(n+1)^4 - \tfrac{1}{2}(n+1)^2\right]$$
$$= \tfrac{1}{12}(n+1)^2\left[2(n+1)^4 - 6(n+1)^3 + 5(n+1)^2 - 1\right]$$
$$= \tfrac{1}{12}(n+1)^2\,[(n+1) - 1]^2\left[2(n+1)^2 - 2(n+1) - 1\right]$$
$$= \tfrac{1}{12}n^2(n+1)^2\left(2n^2 + 2n - 1\right)$$

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6 Applications of Integration

6.1 Areas between Curves

1. $A = \int_{-1}^{1}\left[(x^2+3)-x\right]dx = 2\int_0^1 (x^2+3)\,dx$ [by Theorem 5.5.6(a)] $= 2\left[\frac{1}{3}x^3+3x\right]_0^1 = 2\left(\frac{1}{3}+3\right) = \frac{20}{3}$

2. $A = \int_0^6\left[2x-(x^2-4x)\right]dx = \int_0^6 (6x-x^2)\,dx = \left[3x^2-\frac{1}{3}x^3\right]_0^6 = 108-72 = 36$

3. $A = \int_{-1}^{1}\left[(1-y^4)-(y^3-y)\right]dy = 2\int_0^1 (1-y^4)\,dx$ [by Theorem 5.5.6(a)]

$= 2\left[-\frac{1}{5}y^5+y\right]_0^1 = 2\left(-\frac{1}{5}+1\right) = \frac{8}{5}$

4. $A = \int_{-1}^{2}\left[y^2-(y-5)\right]dy = \left[\frac{1}{3}y^3-\frac{1}{2}y^2+5y\right]_{-1}^{2} = \left(\frac{8}{3}-2+10\right)-\left(-\frac{1}{3}-\frac{1}{2}-5\right) = 16.5$

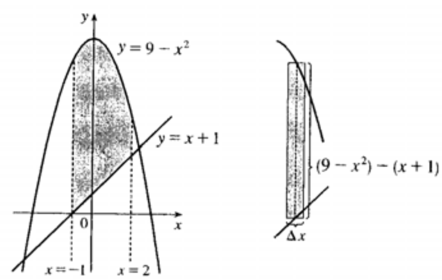
5. $A = \int_{-1}^{2}\left[(9-x^2)-(x+1)\right]dx$

$= \int_{-1}^{2}(8-x-x^2)\,dx$

$= \left[8x-\dfrac{x^2}{2}-\dfrac{x^3}{3}\right]_{-1}^{2}$

$= \left(16-2-\frac{8}{3}\right)-\left(-8-\frac{1}{2}+\frac{1}{3}\right)$

$= 22-3+\frac{1}{2} = \frac{39}{2}$

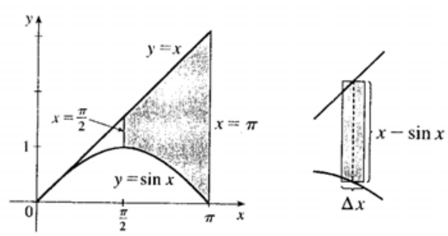


6. $A = \int_{\pi/2}^{\pi}(x-\sin x)\,dx$

$= \left[\dfrac{x^2}{2}+\cos x\right]_{\pi/2}^{\pi}$

$= \left(\dfrac{\pi^2}{2}-1\right)-\left(\dfrac{\pi^2}{8}+0\right)$

$= \dfrac{3\pi^2}{8}-1$

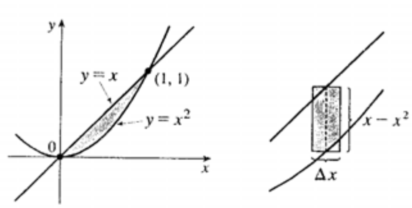


7. $A = \int_0^1 (x-x^2)\,dx$

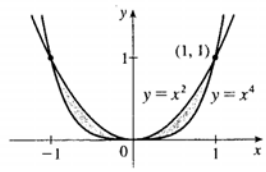
$= \left[\frac{1}{2}x^2-\frac{1}{3}x^3\right]_0^1$

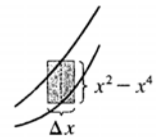
$= \frac{1}{2}-\frac{1}{3}$

$= \frac{1}{6}$



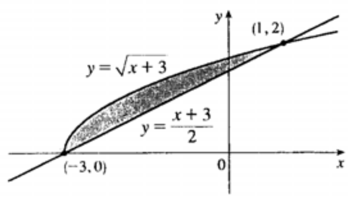
8. $A = \int_{-1}^{1} (x^2 - x^4)\,dx$
$= 2\int_0^1 (x^2 - x^4)\,dx$
$= 2\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1$
$= 2\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{4}{15}$

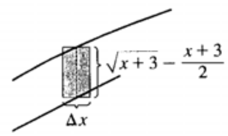




9. First find the points of intersection: $\sqrt{x+3} = \dfrac{x+3}{2} \;\Rightarrow\; \left(\sqrt{x+3}\right)^2 = \left(\dfrac{x+3}{2}\right)^2 \;\Rightarrow\; x + 3 = \frac{1}{4}(x+3)^2$
$\Rightarrow\; 4(x+3) - (x+3)^2 = 0 \;\Rightarrow\; (x+3)[4 - (x+3)] = 0 \;\Rightarrow\; (x+3)(1-x) = 0 \;\Rightarrow\; x = -3$ or 1.
So

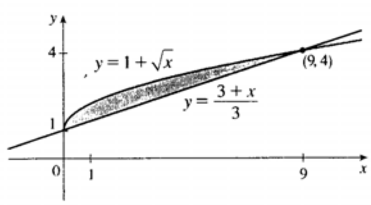
$$A = \int_{-3}^{1}\left(\sqrt{x+3} - \frac{x+3}{2}\right)dx = \left[\frac{2}{3}(x+3)^{3/2} - \frac{(x+3)^2}{4}\right]_{-3}^{1} = \left(\frac{16}{3} - 4\right) - (0 - 0) = \frac{4}{3}$$

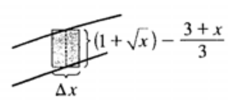




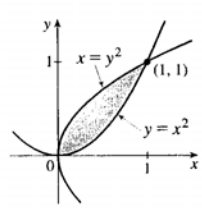
10. $1 + \sqrt{x} = \dfrac{3+x}{3} = 1 + \dfrac{x}{3} \;\Rightarrow\; \sqrt{x} = \dfrac{x}{3} \;\Rightarrow\; x = \dfrac{x^2}{9} \;\Rightarrow\; 9x - x^2 = 0 \;\Rightarrow\; x(9-x) = 0 \;\Rightarrow\; x = 0$
or 9, so

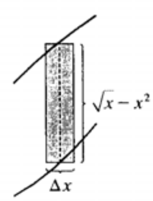
$$A = \int_0^9 \left[\left(1 + \sqrt{x}\right) - \left(\frac{3+x}{x}\right)\right]dx = \int_0^9 \left[\left(1 + \sqrt{x}\right) - \left(1 + \frac{x}{3}\right)\right]dx$$
$$= \int_0^9 \left(\sqrt{x} - \frac{1}{3}x\right)dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{6}x^2\right]_0^9 = 18 - \frac{27}{2} = \frac{9}{2}$$





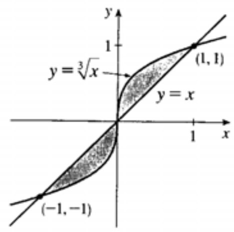
11. $A = \int_0^1 \left(\sqrt{x} - x^2\right)dx$
$= \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1$
$= \frac{2}{3} - \frac{1}{3}$
$= \frac{1}{3}$

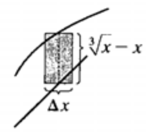




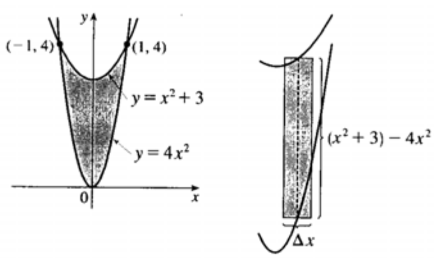
12. $x = \sqrt[3]{x} \Rightarrow x^3 = x \Rightarrow x^3 - x = 0 \Rightarrow x\left(x^2 - 1\right) = 0 \Rightarrow x(x+1)(x-1) = 0 \Rightarrow x = -1, 0,$ or 1, so

$$A = \int_{-1}^{1} \left|\sqrt[3]{x} - x\right| dx = \int_{-1}^{0} \left(x - \sqrt[3]{x}\right) dx + \int_0^1 \left(\sqrt[3]{x} - x\right) dx = 2\int_0^1 \left(x^{1/3} - x\right) dx \text{ [by symmetry]}$$
$$= 2\left[\tfrac{3}{4}x^{4/3} - \tfrac{1}{2}x^2\right]_0^1 = 2\left(\tfrac{3}{4} - \tfrac{1}{2}\right) = \tfrac{1}{2}$$



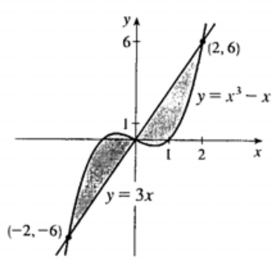


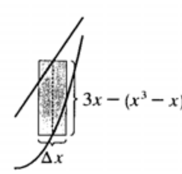
13. $A = \int_{-1}^{1} \left[\left(x^2 + 3\right) - 4x^2\right] dx$
$= 2\int_0^1 \left(3 - 3x^2\right) dx$
$= 2\left[3x - x^3\right]_0^1 = 2(3 - 1) = 4$



14. $x^3 - x = 3x \Rightarrow x^3 - 4x = 0 \Rightarrow x\left(x^2 - 4\right) = 0 \Rightarrow x(x+2)(x-2) = 0 \Rightarrow x = 0, -2,$ or 2. By symmetry,

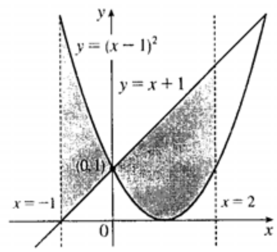
$$A = \int_{-2}^{2} \left|3x - \left(x^3 - x\right)\right| dx = 2\int_0^2 \left[3x - \left(x^3 - x\right)\right] dx = 2\int_0^2 \left(4x - x^3\right) dx = 2\left[2x^2 - \tfrac{1}{4}x^4\right]_0^2$$
$$= 2(8 - 4) = 8$$

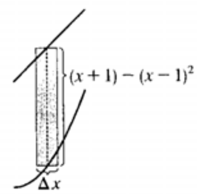




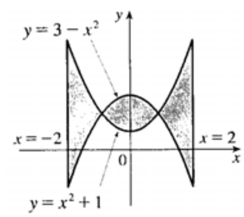
15. $x+1=(x-1)^2 \;\Rightarrow\; x+1=x^2-2x+1 \;\Rightarrow\; 0=x^2-3x \;\Rightarrow\; 0=x(x-3) \;\Rightarrow\; x=0$ or 3.

$$A=\int_{-1}^{2}\left|(x+1)-(x-1)^2\right|dx=\int_{-1}^{0}\left[(x-1)^2-(x+1)\right]dx+\int_0^2\left[(x+1)-(x-1)^2\right]dx$$
$$=\int_{-1}^{0}\left(x^2-3x\right)dx+\int_0^2\left(3x-x^2\right)dx=\left[\tfrac{1}{3}x^3-\tfrac{3}{2}x^2\right]_{-1}^{0}+\left[\tfrac{3}{2}x^2-\tfrac{1}{3}x^3\right]_0^2$$
$$=0-\left(-\tfrac{1}{3}-\tfrac{3}{2}\right)+\left(6-\tfrac{8}{3}\right)-0=\tfrac{31}{6}$$

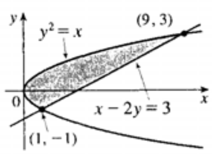


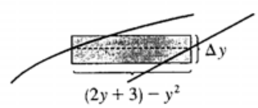


16. $$A=\int_{-2}^{-1}\left[\left(x^2+1\right)-\left(3-x^2\right)\right]dx+\int_{-1}^{1}\left[\left(3-x^2\right)-\left(x^2+1\right)\right]dx$$
$$+\int_1^2\left[\left(x^2+1\right)-\left(3-x^2\right)\right]dx$$
$$=\int_{-2}^{-1}\left(2x^2-2\right)dx+\int_{-1}^{1}\left(2-2x^2\right)dx+\int_1^2\left(2x^2-2\right)dx$$
$$=2\int_0^1\left(2-2x^2\right)dx+2\int_1^2\left(2x^2-2\right)dx \quad \text{[by symmetry]}$$
$$=2\left[2x-\tfrac{2}{3}x^3\right]_0^1+2\left[\tfrac{2}{3}x^3-2x\right]_1^2=2\left(2-\tfrac{2}{3}\right)+2\left(\tfrac{16}{3}-4\right)-2\left(\tfrac{2}{3}-2\right)=8$$

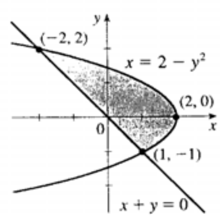


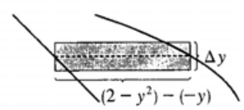
17. $$A=\int_{-1}^{3}\left[(2y+3)-y^2\right]dy$$
$$=\left[y^2+3y-\tfrac{1}{3}y^3\right]_{-1}^{3}$$
$$=(9+9-9)-\left(1-3+\tfrac{1}{3}\right)$$
$$=\tfrac{32}{3}$$



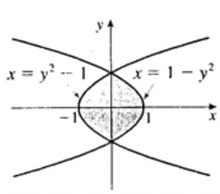


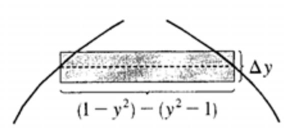
18. $$A=\int_{-1}^{2}\left[2-y^2-(-y)\right]dy$$
$$=\int_{-1}^{2}\left(-y^2+y+2\right)dy$$
$$=\left[-\tfrac{1}{3}y^3+\tfrac{1}{2}y^2+2y\right]_{-1}^{2}$$
$$=\left(-\tfrac{8}{3}+2+4\right)-\left(\tfrac{1}{3}+\tfrac{1}{2}-2\right)=\tfrac{9}{2}$$





19. $$A=\int_{-1}^{1}\left[\left(1-y^2\right)-\left(y^2-1\right)\right]dy$$
$$=\int_{-1}^{1}2\left(1-y^2\right)dy$$
$$=4\int_0^1\left(1-y^2\right)dy$$
$$=4\left[y-\tfrac{1}{3}y^3\right]_0^1=4\left(1-\tfrac{1}{3}\right)=\tfrac{8}{3}$$





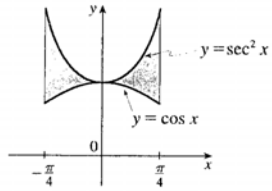
20. $A = \int_{-\pi/4}^{\pi/4} \left(\sec^2 x - \cos x\right) dx$

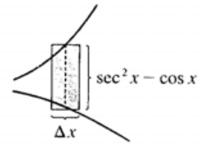
$= 2\int_0^{\pi/4} \left(\sec^2 x - \cos x\right) dx$

$= 2\left[\tan x - \sin x\right]_0^{\pi/4}$

$= 2\left(1 - \frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2}$

$= 2\left(1 - \frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2} \approx 0.59$



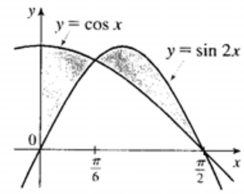


21. Notice that $\cos x = \sin 2x = 2\sin x\cos x \iff 2\sin x = 1$ or $\cos x = 0 \iff$ $x = \frac{\pi}{6}$ or $\frac{\pi}{2}$.

$$A = \int_0^{\pi/6} (\cos x - \sin 2x)\,dx + \int_{\pi/6}^{\pi/2} (\sin 2x - \cos x)\,dx$$

$$= \left[\sin x + \tfrac{1}{2}\cos 2x\right]_0^{\pi/6} + \left[-\tfrac{1}{2}\cos 2x - \sin x\right]_{\pi/6}^{\pi/2}$$

$$= \tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2} - \left(0 + \tfrac{1}{2}\cdot 1\right) + \left(\tfrac{1}{2} - 1\right) - \left(-\tfrac{1}{2}\cdot\tfrac{1}{2} - \tfrac{1}{2}\right) = \tfrac{1}{2}$$



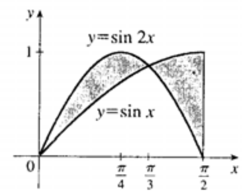
22. $\sin x = \sin 2x = 2\sin x\cos x$ when $\sin x = 0$ and when $\cos x = \frac{1}{2}$; that is, when $x = 0$ or $\frac{\pi}{3}$.

$$A = \int_0^{\pi/3} (\sin 2x - \sin x)\,dx + \int_{\pi/3}^{\pi/2} (\sin x - \sin 2x)\,dx$$

$$= \left[-\tfrac{1}{2}\cos 2x + \cos x\right]_0^{\pi/3} + \left[\tfrac{1}{2}\cos 2x - \cos x\right]_{\pi/3}^{\pi/2}$$

$$= \left[-\tfrac{1}{2}\left(-\tfrac{1}{2}\right) + \tfrac{1}{2}\right] - \left(-\tfrac{1}{2} + 1\right)$$

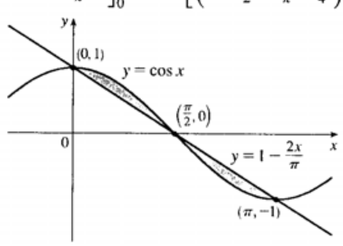
$$+ \left(-\tfrac{1}{2} - 0\right) - \left[\tfrac{1}{2}\left(-\tfrac{1}{2}\right) - \tfrac{1}{2}\right] = \tfrac{1}{2}$$

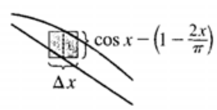


23. From the graph, we see that the curves intersect at $x = 0$, $x = \frac{\pi}{2}$, and $x = \pi$. By symmetry,

$$A = \int_0^{\pi} \left|\cos x - \left(1 - \frac{2x}{\pi}\right)\right| dx = 2\int_0^{\pi/2} \left[\cos x - \left(1 - \frac{2x}{\pi}\right)\right] dx = 2\int_0^{\pi/2} \left(\cos x - 1 + \frac{2x}{\pi}\right) dx$$

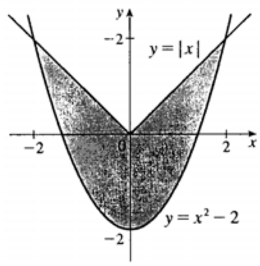
$$= 2\left[\sin x - x + \tfrac{1}{\pi}x^2\right]_0^{\pi/2} = 2\left[\left(1 - \tfrac{\pi}{2} + \tfrac{1}{\pi}\cdot\tfrac{\pi^2}{4}\right) - 0\right] = 2\left(1 - \tfrac{\pi}{2} + \tfrac{\pi}{4}\right) = 2 - \tfrac{\pi}{2}$$

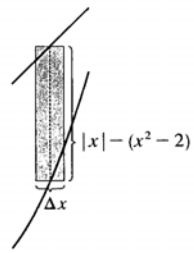




24. For $x > 0$, $x = x^2 - 2 \;\Rightarrow\; 0 = x^2 - x - 2 \;\Rightarrow\; 0 = (x-2)(x+1) \;\Rightarrow\; x = 2$. By symmetry,

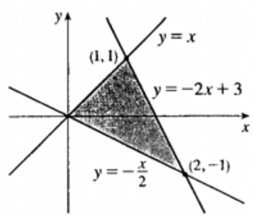
$$\int_{-2}^{2}\left[|x| - \left(x^2 - 2\right)\right]dx = 2\int_0^2\left[x - \left(x^2-2\right)\right]dx = 2\int_0^2\left(x - x^2 + 2\right)dx = 2\left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + 2x\right]_0^2$$
$$= 2\left(2 - \tfrac{8}{3} + 4\right) = \tfrac{20}{3}$$

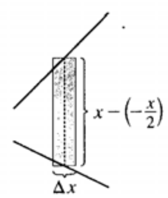




25. Graph the three functions $y = x$, $y = -\frac{1}{2}x$, and $y = -2x + 3$; then determine the points of intersection: $(0,0)$, $(1,1)$, and $(2,-1)$.

$$A = \int_0^1\left[x - \left(-\tfrac{1}{2}x\right)\right]dx + \int_1^2\left[(3-2x) - \left(-\tfrac{1}{2}x\right)\right]dx = \int_0^1 \tfrac{3}{2}x\,dx + \int_1^2\left(3 - \tfrac{3}{2}x\right)dx$$
$$= \left[\tfrac{3}{4}x^2\right]_0^1 + \left[3x - \tfrac{3}{4}x^2\right]_1^2 = \left(\tfrac{3}{4} - 0\right) + (6-3) - \left(3 - \tfrac{3}{4}\right) = \tfrac{3}{2}$$



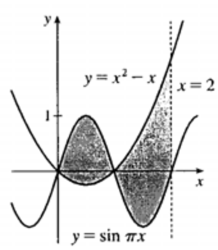


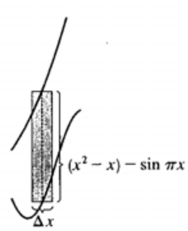
26. $A = \int_0^1\left[\sin\pi x - \left(x^2 - x\right)\right]dx + \int_1^2\left[\left(x^2-x\right) - \sin\pi x\right]dx$

$= \left[-\frac{1}{\pi}\cos\pi x - \frac{1}{3}x^3 + \frac{1}{2}x^2\right]_0^1 + \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{\pi}\cos\pi x\right]_1^2$

$= \left(\frac{1}{\pi} - \frac{1}{3} + \frac{1}{2}\right) - \left(-\frac{1}{\pi}\right) + \left(\frac{8}{3} - 2 + \frac{1}{\pi}\right) - \left(\frac{1}{3} - \frac{1}{2} - \frac{1}{\pi}\right)$

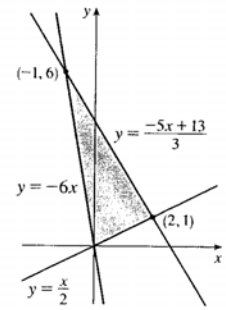
$= \frac{4}{\pi} + 1$



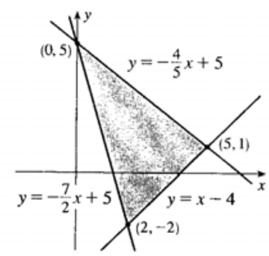


27. An equation of the line through $(0, 0)$ and $(2, 1)$ is $y = \frac{1}{2}x$; through $(0, 0)$ and $(-1, 6)$ is $y = -6x$; through $(2, 1)$ and $(-1, 6)$ is $y = -\frac{5}{3}x + \frac{13}{3}$.

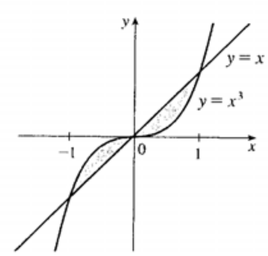
$$\begin{aligned} A &= \int_{-1}^{0}\left[\left(-\tfrac{5}{3}x + \tfrac{13}{3}\right) - (-6x)\right]dx + \int_{0}^{2}\left[\left(-\tfrac{5}{3}x + \tfrac{13}{3}\right) - \tfrac{1}{2}x\right]dx \\ &= \int_{-1}^{0}\left(\tfrac{13}{3}x + \tfrac{13}{3}\right)dx + \int_{0}^{2}\left(-\tfrac{13}{6}x + \tfrac{13}{3}\right)dx \\ &= \tfrac{13}{3}\int_{-1}^{0}(x+1)\,dx + \tfrac{13}{3}\int_{0}^{2}\left(-\tfrac{1}{2}x + 1\right)dx \\ &= \tfrac{13}{3}\left[\tfrac{1}{2}x^2 + x\right]_{-1}^{0} + \tfrac{13}{3}\left[-\tfrac{1}{4}x^2 + x\right]_{0}^{2} \\ &= \tfrac{13}{3}\left[0 - \left(\tfrac{1}{2} - 1\right)\right] + \tfrac{13}{3}\left[(-1+2) - 0\right] = \tfrac{13}{3}\cdot\tfrac{1}{2} + \tfrac{13}{3}\cdot 1 = \tfrac{13}{2} \end{aligned}$$



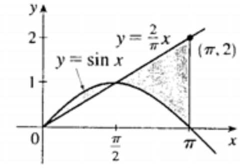
28. $$\begin{aligned} A &= \int_{0}^{2}\left[\left(-\tfrac{4}{5}x + 5\right) - \left(-\tfrac{7}{2}x + 5\right)\right]dx + \int_{2}^{5}\left[\left(-\tfrac{4}{5}x + 5\right) - (x - 4)\right]dx \\ &= \int_{0}^{2}\tfrac{27}{10}x\,dx + \int_{2}^{5}\left(-\tfrac{9}{5}x + 9\right)dx = \left[\tfrac{27}{20}x^2\right]_{0}^{2} + \left[-\tfrac{9}{10}x^2 + 9x\right]_{2}^{5} \\ &= \left(\tfrac{27}{5} - 0\right) + \left(-\tfrac{45}{2} + 45\right) - \left(-\tfrac{18}{5} + 18\right) = \tfrac{27}{2} \end{aligned}$$



29. $$\begin{aligned} A &= \int_{-1}^{1}\left|x^3 - x\right|dx = 2\int_{0}^{1}\left(x - x^3\right)dx \quad \text{[by symmetry]} \\ &= 2\left[\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right]_{0}^{1} = 2\left(\tfrac{1}{2} - \tfrac{1}{4}\right) = \tfrac{1}{2} \end{aligned}$$



30. $$\begin{aligned} \int_{0}^{\pi}\left|\sin x - \frac{2}{\pi}x\right|dx &= \int_{0}^{\pi/2}\left(\sin x - \frac{2}{\pi}x\right)dx + \int_{\pi/2}^{\pi}\left(\frac{2}{\pi}x - \sin x\right)dx \\ &= \left[-\cos x - \frac{x^2}{\pi}\right]_{0}^{\pi/2} + \left[\frac{x^2}{\pi} + \cos x\right]_{\pi/2}^{\pi} \\ &= -\tfrac{\pi}{4} + 1 + (\pi - 1) - \tfrac{\pi}{4} = \tfrac{\pi}{2} \end{aligned}$$



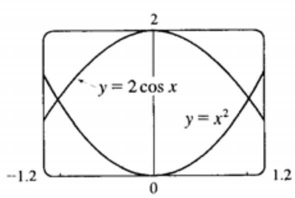
31. Let $f(x) = \sqrt{1+x^3} - (1-x)$, $\Delta x = \frac{2-0}{4} = \frac{1}{2}$.

$$\begin{aligned} A &= \int_0^2 \left[\sqrt{1+x^3} - (1-x)\right] dx \approx \tfrac{1}{2}\left[f\left(\tfrac{1}{4}\right) + f\left(\tfrac{3}{4}\right) + f\left(\tfrac{5}{4}\right) + f\left(\tfrac{7}{4}\right)\right] \\ &= \tfrac{1}{2}\left[\left(\tfrac{\sqrt{65}}{8} - \tfrac{3}{4}\right) + \left(\tfrac{\sqrt{91}}{8} - \tfrac{1}{4}\right) + \left(\tfrac{3\sqrt{21}}{8} + \tfrac{1}{4}\right) + \left(\tfrac{\sqrt{407}}{8} + \tfrac{3}{4}\right)\right] \\ &= \tfrac{1}{16}\left(\sqrt{65} + \sqrt{91} + 3\sqrt{21} + \sqrt{407}\right) \approx 3.22 \end{aligned}$$

32. Let $f(x) = x - x\tan x$, and $\Delta x = \frac{\pi/4 - 0}{4} = \frac{\pi}{16}$. Then

$$\begin{aligned} A &= \textstyle\int_0^{\pi/4} (x - x\tan x)\, dx \approx \frac{\pi}{16}\left[f\left(\frac{\pi}{32}\right) + f\left(\frac{3\pi}{32}\right) + f\left(\frac{5\pi}{32}\right) + f\left(\frac{7\pi}{32}\right)\right] \\ &\approx \tfrac{\pi}{16}\left[\tfrac{\pi}{32}\left(1 - \tan\tfrac{\pi}{32}\right) + \tfrac{3\pi}{32}\left(1 - \tan\tfrac{3\pi}{32}\right) + \tfrac{5\pi}{32}\left(1 - \tan\tfrac{5\pi}{32}\right) + \tfrac{7\pi}{32}\left(1 - \tan\tfrac{7\pi}{32}\right)\right] \\ &= \tfrac{\pi^2}{512}\left[16 - \tan\tfrac{\pi}{32} - 3\tan\tfrac{3\pi}{32} - 5\tan\tfrac{5\pi}{32} - 7\tan\tfrac{7\pi}{32}\right] \approx 0.1267 \end{aligned}$$

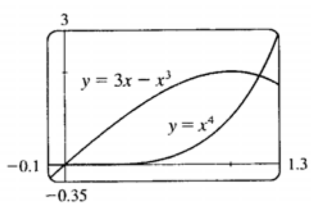
33.



From the graph, we see that the curves intersect at $x \approx \pm 1.02$, with $2\cos x > x^2$ on $(-1.02, 1.02)$. So the area of the region bounded by the curves is

$$\begin{aligned} A &\approx \textstyle\int_{-1.02}^{1.02} \left(2\cos x - x^2\right) dx = 2\int_0^{1.02} \left(2\cos x - x^2\right) dx \\ &= 2\left[2\sin x - \tfrac{1}{3}x^3\right]_0^{1.02} \approx 2.70 \end{aligned}$$

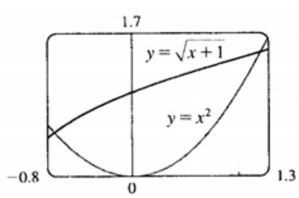
34.



From the graph, we see that the curves intersect at $x = 0$ and at $x \approx 1.17$, with $3x - x^3 > x^4$ on $(0, 1.17)$. So the area between the curves is

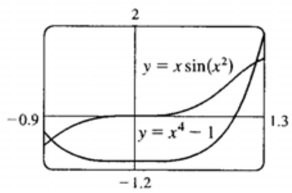
$$\begin{aligned} A &\approx \textstyle\int_0^{1.17} \left[\left(3x - x^3\right) - x^4\right] dx = \left[\frac{3}{2}x^2 - \frac{1}{4}x^4 - \frac{1}{5}x^5\right]_0^{1.17} \\ &\approx 1.15 \end{aligned}$$

35.



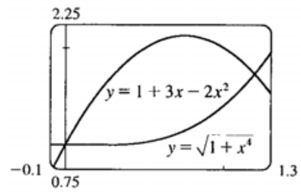
From the graph, we see that the curves intersect at $x \approx -0.72$ and at $x \approx 1.22$, with $\sqrt{x+1} > x^2$ on $(-0.72, 1.22)$. So the area between the curves is

$$\begin{aligned} A &\approx \textstyle\int_{-0.72}^{1.22} \left(\sqrt{x+1} - x^2\right) dx = \left[\frac{2}{3}(x+1)^{3/2} - \frac{1}{3}x^3\right]_{-0.72}^{1.22} \\ &\approx 1.38 \end{aligned}$$

36. 

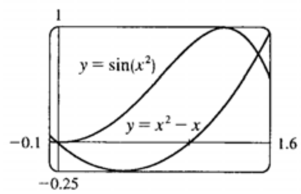
From the graph, we see that the curves intersect at $x \approx -0.83$ and $x \approx 1.22$, with $x\sin\left(x^2\right) > x^4 - 1$ on $(-0.83, 1.22)$. So the area of the region bounded by the curves is

$$A \approx \int_{-0.83}^{1.22} \left[x\sin\left(x^2\right) - \left(x^4 - 1\right)\right] dx$$
$$= \left[-\tfrac{1}{2}\cos\left(x^2\right) - \tfrac{1}{5}x^5 + x\right]_{-0.83}^{1.22} \approx 1.78$$

37. 

From the graph, we see that the curves intersect at $x = 0$ and at $x \approx 1.19$, with $1 + 3x - 2x^2 > \sqrt{1 + x^4}$ on $(0, 1.19)$. So, using the Midpoint Rule with $f(x) = 1 + 3x - 2x^2 - \sqrt{1 + x^4}$ on $[0, 1.19]$ with $n = 4$, we calculate the approximate area between the curves:

$$A \approx \int_0^{1.19} \left(1 + 3x - 2x^2 - \sqrt{1 + x^4}\right) dx$$
$$\approx \tfrac{1.19}{4}\left[f\left(\tfrac{1.19}{8}\right) + f\left(\tfrac{3\cdot 1.19}{8}\right) + f\left(\tfrac{5\cdot 1.19}{8}\right) + f\left(\tfrac{7\cdot 1.19}{8}\right)\right] \approx 0.83$$

38. 

From the graph, we see that the curves intersect at $x = 0$ and at $x \approx 1.5$, with $\sin\left(x^2\right) > x^2 - x$ on $(0, 1.51)$. So, using the Midpoint Rule with $f(x) = \sin\left(x^2\right) - x^2 + x$ on $(0, 1.51)$ with $n = 4$, we calculate that the area between the curves is

$$A \approx \int_0^{1.51} \left[\sin\left(x^2\right) - \left(x^2 - x\right)\right] dx$$
$$\approx \tfrac{1.51}{4}\left[f\left(\tfrac{1.51}{8}\right) + f\left(\tfrac{3\cdot 1.51}{8}\right) + f\left(\tfrac{5\cdot 1.51}{8}\right) + f\left(\tfrac{7\cdot 1.51}{8}\right)\right] \approx 0.81$$

39. 1 second $= \frac{1}{3600}$ hour, so 10 s $= \frac{1}{360}$ h. With the given data, we can take $n = 5$ to use the Midpoint Rule. $\Delta t = \frac{1/360 - 0}{5} = \frac{1}{1800}$, so

$$\begin{aligned}
\text{distance}_{\text{Kelly}} - \text{distance}_{\text{Chris}} &= \int_0^{1/360} v_K\,dt - \int_0^{1/360} v_C\,dt = \int_0^{1/360} (v_K - v_C)\,dt \\
&\approx M_5 = \tfrac{1}{1800}\left[(v_K - v_C)(1) + (v_K - v_C)(3) + (v_K - v_C)(5)\right. \\
&\qquad\qquad \left. + (v_K - v_C)(7) + (v_K - v_C)(9)\right] \\
&= \tfrac{1}{1800}\left[(22 - 20) + (52 - 46) + (71 - 62) + (86 - 75) + (98 - 86)\right] \\
&= \tfrac{1}{1800}(2 + 6 + 9 + 11 + 12) = \tfrac{1}{1800}(40) = \tfrac{1}{45} \text{ mile, or } 117\tfrac{1}{3} \text{ feet}
\end{aligned}$$

40. If x = distance from left end of pool and $w = w(x)$ = width at x, then the Midpoint Rule with $n = 4$ and $\Delta x = \dfrac{b - a}{n} = \dfrac{8 \cdot 2 - 0}{4} = 4$ gives Area $= \int_0^{16} w\,dx \approx 4\,(6.2 + 6.8 + 5.0 + 4.8) = 4\,(22.8) = 91.2\ \text{m}^2$

41. We know that the area under curve A between $t = 0$ and $t = x$ is $\int_0^x v_A(t)\,dt = s_A(x)$, where $v_A(t)$ is the velocity of car A and s_A is its displacement. Similarly, the area under curve B between $t = 0$ and $t = x$ is $\int_0^x v_B(t)\,dt = s_B(x)$.

(a) After one minute, the area under curve A is greater than the area under curve B. So A is ahead after one minute.

(b) Its numerical value is $s_A(1) - s_B(1)$, which is the distance by which A is ahead of B after 1 minute.

(c) After two minutes, car B is traveling faster than car A and has gained some ground, but the area under curve A from $t = 0$ to $t = 2$ is still greater than the corresponding area for curve B, so car A is still ahead.

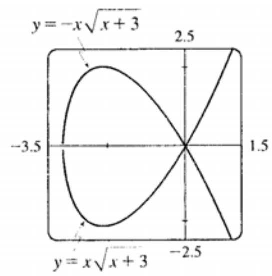
(d) From the graph, it appears that the area between curves A and B for $0 \le t \le 1$ (when car A is going faster), which corresponds to the distance by which car A is ahead, seems to be about 3 squares. Therefore, the cars will be side by side at the time x where the area between the curves for $1 \le t \le x$ (when car B is going faster) is the same as the area for $0 \le t \le 1$. From the graph, it appears that this time is $x \approx 2.2$. So the cars are side by side when $t \approx 2.2$ minutes.

42. The area under $R'(x)$ from $x = 50$ to $x = 100$ represents the change in revenue, and the area under $C'(x)$ from $x = 50$ to $x = 100$ represents the change in cost. The shaded region represents the difference between these two values, that is, the increase in profit as the production level increases from 50 units to 100 units. We use the Midpoint Rule with $n = 5$ and $\Delta x = 10$:

$$\begin{aligned} M_5 &= \Delta x \left[R'(55) - C'(55) + R'(65) - C'(65) + R'(75) - C'(75) \right. \\ &\qquad\qquad\qquad\qquad \left. + R'(85) - C'(85) + R'(95) - C'(95)\right] \\ &\approx 10\,(2.40 - 0.85 + 2.20 - 0.90 + 2.00 - 1.00 + 1.80 - 1.10 + 1.70 - 1.20) \\ &= 10\,(5.05) = 50.5 \text{ thousand dollars} \end{aligned}$$

Using M_1 would give us $50\,(2 - 1) = 50$ thousand dollars.

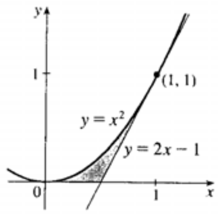
43.



To graph this function, we must first express it as a combination of explicit functions of y; namely, $y = \pm x\sqrt{x+3}$. We can see from the graph that the loop extends from $x = -3$ to $x = 0$, and that by symmetry, the area we seek is just twice the area under the top half of the curve on this interval, the equation of the top half being $y = -x\sqrt{x+3}$. So the area is $A = 2\int_{-3}^{0}\left(-x\sqrt{x+3}\right)dx$. We substitute $u = x + 3$, so $du = dx$ and the limits change to 0 and 3, and we get

$$\begin{aligned} A &= -2\int_0^3 \left[(u-3)\sqrt{u}\right]du = -2\int_0^3 \left(u^{3/2} - 3u^{1/2}\right)du \\ &= -2\left[\tfrac{2}{5}u^{5/2} - 2u^{3/2}\right]_0^3 = -2\left[\tfrac{2}{5}\left(3^2\sqrt{3}\right) - 2\left(3\sqrt{3}\right)\right] = \tfrac{24}{5}\sqrt{3} \end{aligned}$$

44.



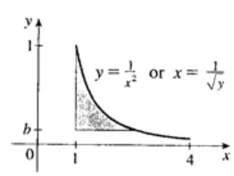
We start by finding the equation of the tangent line to $y = x^2$ at the point $(1, 1)$: $y' = 2x$, so the slope of the tangent is $2(1) = 2$, and its equation is therefore $y - 1 = 2(x - 1) \quad \Leftrightarrow \quad y = 2x - 1$. We would need two integrals to integrate with respect to x, but only one to integrate with respect to y.

$$A = \int_0^1 \left[\tfrac{1}{2}(y+1) - \sqrt{y}\right] dy = \left[\tfrac{1}{4}y^2 + \tfrac{1}{2}y - \tfrac{2}{3}y^{3/2}\right]_0^1$$
$$= \tfrac{1}{4} + \tfrac{1}{2} - \tfrac{2}{3} = \tfrac{1}{12}$$

45. By the symmetry of the problem, we consider only the first quadrant where $y = x^2 \quad \Rightarrow \quad x = \sqrt{y}$. We are looking for a number b such that $\int_0^4 x\,dy = 2\int_0^b x\,dy \quad \Rightarrow \quad \int_0^4 \sqrt{y}\,dy = 2\int_0^b \sqrt{y}\,dy \quad \Rightarrow \quad \frac{2}{3}\left[y^{3/2}\right]_0^4 = \frac{4}{3}\left[y^{3/2}\right]_0^b$ $\Rightarrow \quad \frac{2}{3}(8 - 0) = \frac{4}{3}\left(b^{3/2} - 0\right) \quad \Rightarrow \quad b^{3/2} = 4 \quad \Rightarrow \quad b = 4^{2/3} \approx 2.52$.

46. (a) We want to choose a so that $\displaystyle\int_1^a \frac{1}{x^2}\,dx = \int_a^4 \frac{1}{x^2}\,dx \quad \Rightarrow \quad \left[\frac{-1}{x}\right]_1^a = \left[\frac{-1}{x}\right]_a^4 \quad \Rightarrow \quad 1 - \frac{1}{a} = \frac{1}{a} - \frac{1}{4} \quad \Rightarrow$ $\displaystyle\frac{2}{a} = \frac{5}{4} \quad \Rightarrow \quad a = \frac{8}{5}$.

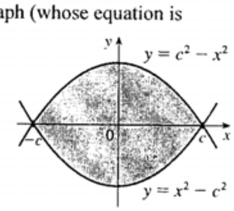
(b) The area under the curve $y = 1/x^2$ from $x = 1$ to $x = 4$ is $\frac{3}{4}$ [take $a = 4$ in the first integral in part (a)], so that b must be greater than $\frac{1}{16}$, since the area under the line $y = \frac{1}{16}$ from $x = 1$ to $x = 4$ is only $\frac{3}{16}$, which is less than half of $\frac{3}{4}$. We want to choose b so that the upper area in the diagram is half of the total area under the curve $y = \dfrac{1}{x^2}$ from $x = 1$ to $x = 4$. This implies that $\int_b^1 \left(1/\sqrt{y} - 1\right) dy = \frac{1}{2}\cdot\frac{3}{4} \quad \Rightarrow \quad \left[2\sqrt{y} - y\right]_b^1 = \frac{3}{8}$ $\Rightarrow \quad 1 - 2\sqrt{b} + b = \frac{3}{8} \quad \Rightarrow \quad b - 2\sqrt{b} + \frac{5}{8} = 0$. Letting $c = \sqrt{b}$, we get $c^2 - 2c + \frac{5}{8} = 0 \quad \Rightarrow \quad 8c^2 - 16c + 5 = 0$. Thus, $c = \frac{16 \pm \sqrt{256 - 160}}{16} = 1 \pm \frac{\sqrt{6}}{4}$. But $c = \sqrt{b} < 1 \quad \Rightarrow \quad c = 1 - \frac{\sqrt{6}}{4} \quad \Rightarrow$ $b = c^2 = 1 + \frac{3}{8} - \frac{\sqrt{6}}{2} = \frac{1}{8}\left(11 - 4\sqrt{6}\right) \approx 0.1503$.

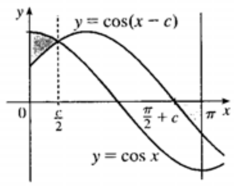


47. We first assume that $c > 0$, since c can be replaced by $-c$ in both equations without changing the graphs, and if $c = 0$ the curves do not enclose a region. We see from the graph that the enclosed area lies between $x = -c$ and $x = c$, and by symmetry, it is equal to twice the area under the top half of the graph (whose equation is $y = c^2 - x^2$). The enclosed area is

$$2\int_{-c}^{c}\left(c^2 - x^2\right)dx = 4\int_0^c\left(c^2 - x^2\right)dx = 4\left[c^2x - \tfrac{1}{3}x^3\right]_0^c$$
$$= 4\left(c^3 - \tfrac{1}{3}c^3\right) = \tfrac{8}{3}c^3$$

which is equal to 576 when $c = \sqrt[3]{216} = 6$. Note that $c = -6$ is another solution, since the graphs are the same.



48. 

We see something close to the desired situation in the diagram. The point of intersection of $y = \cos x$ and $y = \cos(x - c)$ occurs where $x = c/2$ [since $\cos(c/2) = \cos(c/2 - c)$ by the evenness of the cosine function] and the point where $\cos(x - c)$ crosses the x-axis is $x = \frac{\pi}{2} + c$, since $\cos\left(\left(\frac{\pi}{2} + c\right) - c\right) = 0$. So we require that $\int_0^{c/2} [\cos x - \cos(x - c)]\, dx = -\int_{\pi/2+c}^{\pi} \cos(x - c)\, dx$ (the negative sign on the RHS is needed since the second area is beneath the x-axis)

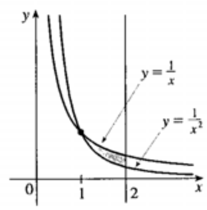
$\Leftrightarrow \quad [\sin x - \sin(x - c)]_0^{c/2} = -[\sin(x - c)]_{\pi/2+c}^{\pi} \quad \Rightarrow$

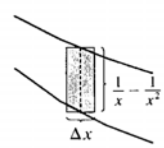
$[\sin(c/2) - \sin(-c/2)] - [-\sin(-c)] = \sin\left(\left(\frac{\pi}{2} + c\right) - c\right) - \sin(\pi - c) \quad \Leftrightarrow \quad 2\sin(c/2) - \sin c = 1 - \sin c$. [Here we have used the oddness of the sine function, and the fact that $\sin(\pi - c) = \sin c$]. So $2\sin(c/2) = 1 \quad \Leftrightarrow$ $c/2 = \frac{\pi}{6} \quad \Leftrightarrow \quad c = \frac{\pi}{3}$.

49. $A = \displaystyle\int_1^2 \left(\frac{1}{x} - \frac{1}{x^2}\right) dx = \left[\ln x + \frac{1}{x}\right]_1^2$

$= \left(\ln 2 + \frac{1}{2}\right) - (\ln 1 + 1)$

$= \ln 2 - \frac{1}{2} \approx 0.19$

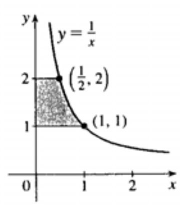


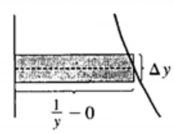


50. $A = \int_1^2 (1/y)\, dy = [\ln y]_1^2$

$= \ln 2 - \ln 1$

$= \ln 2 \approx 0.69$



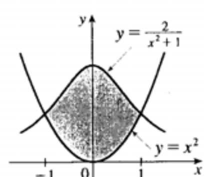


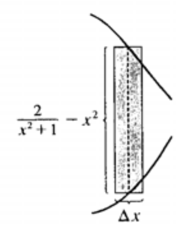
51. $A = \displaystyle\int_{-1}^1 \left(\frac{2}{x^2 + 1} - x^2\right) dx$

$= 2\displaystyle\int_0^1 \left(\frac{2}{x^2 + 1} - x^2\right) dx$

$= 2\left[2\tan^{-1} x - \frac{1}{3}x^3\right]_0^1 = 2\left(2 \cdot \frac{\pi}{4} - \frac{1}{3}\right)$

$= \pi - \frac{2}{3} \approx 2.47$



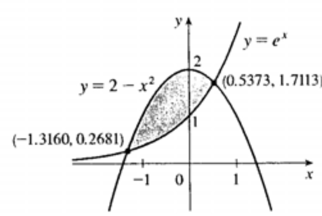


52. The graphs intersect at $a \approx -1.315974$ and $b \approx 0.537274$, so

$$A = \int_a^b \left[(2 - x^2) - e^x\right] dx$$

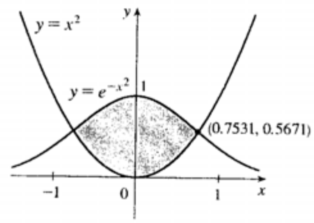
$$= \left[2x - \tfrac{1}{3}x^3 - e^x\right]_a^b$$

$$\approx 1.452014$$

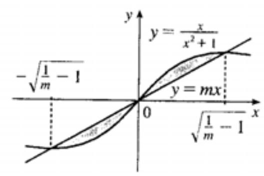


53. A typical graphing calculator solution is as follows. Assign `X^2` to `Y1` ($y_1 = x^2$) and `Exp(-X^2)` to `Y2` ($y_2 = e^{-x^2}$). Graph the functions and find (and store) the x-coordinates of the points of intersection. In this case, we have some symmetry, so we need to find only one point of intersection. Store $x \approx 0.75308916$ in memory location `B`. Now use the appropriate integration command to approximate the area:

$A = 2\int_0^B (y_2 - y_1)\,dx =$ `2*Int(Y2-Y1,X,0,B)` ≈ 0.979263.



54. The curve and the line will determine a region when they intersect at two or more points. So we solve the equation $x/\left(x^2+1\right) = mx \Rightarrow$ $x = 0$ or $mx^2 + m - 1 = 0 \Rightarrow x = 0$ or $x^2 = \dfrac{1-m}{m} \Rightarrow$ $x = 0$ or $x = \pm\sqrt{\dfrac{1}{m} - 1}$. Note that if $m = 1$, this has only the solution $x = 0$, and no region is determined. But if $1/m - 1 > 0 \Leftrightarrow 1/m > 1 \Leftrightarrow 0 < m < 1$, then there are two solutions. [Another way of seeing this is to observe that the slope of the tangent to $y = x/\left(x^2+1\right)$ at the origin is $y' = 1$ and therefore we must have $0 < m < 1$.] Note that we cannot just integrate between the positive and negative roots, since the curve and the line cross at the origin. Since mx and $x\,/\left(x^2+1\right)$ are both odd functions, the total area is twice the area between the curves on the interval $\left[0, \sqrt{1/m - 1}\right]$. So the total area enclosed is

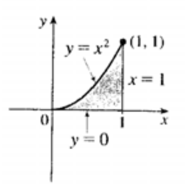


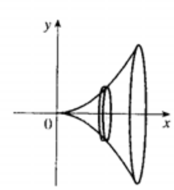
$$\begin{aligned} 2\int_0^{\sqrt{1/m-1}} \left[\frac{x}{x^2+1} - mx\right] dx &= 2\left[\tfrac{1}{2}\ln\left(x^2+1\right) - \tfrac{1}{2}mx^2\right]_0^{\sqrt{1/m-1}} \\ &= \left[\ln\left(1/m - 1 + 1\right) - m\left(1/m - 1\right)\right] - (\ln 1 - 0) \\ &= \ln(1/m) + m - 1 = m - \ln m - 1 \end{aligned}$$

6.2 Volumes

1. A cross-section is circular with radius x^2, so its area is $A(x) = \pi\left(x^2\right)^2$.

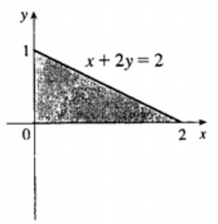
$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi\left(x^2\right)^2 dx = \pi\int_0^1 x^4 dx = \pi\left[\tfrac{1}{5}x^5\right]_0^1 = \tfrac{\pi}{5}$$

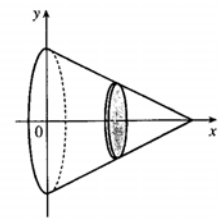




2. $x + 2y = 2 \quad \Leftrightarrow \quad y = 1 - \frac{1}{2}x$, so a cross-section is circular with radius $1 - \frac{1}{2}x$, and its area is $A(x) = \pi\left(1 - \frac{1}{2}x\right)^2$.

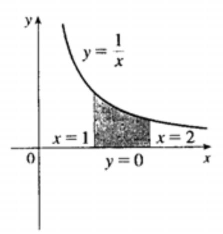
$$V = \int_0^2 \pi y^2\,dx = \pi\int_0^2 \left(1 - \tfrac{1}{2}x\right)^2 dx = \pi\int_0^2 \left(1 - x + \tfrac{1}{4}x^2\right) dx = \pi\left[x - \tfrac{1}{2}x^2 + \tfrac{1}{12}x^3\right]_0^2$$
$$= \pi\left(2 - 2 + \tfrac{2}{3}\right) = \tfrac{2}{3}\pi$$

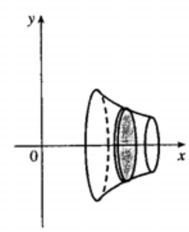




3. A cross-section is circular with radius $1/x$, so its area is $A(x) = \pi\,(1/x)^2$.

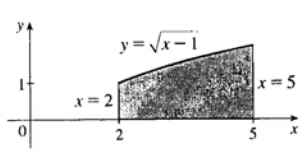
$$V = \int_1^2 A(x)\,dx = \int_1^2 \pi\left(\frac{1}{x}\right)^2 dx = \pi\int_1^2 \frac{1}{x^2}\,dx = \pi\left[-\frac{1}{x}\right]_1^2 = \pi\left[-\tfrac{1}{2} - (-1)\right] = \tfrac{\pi}{2}$$

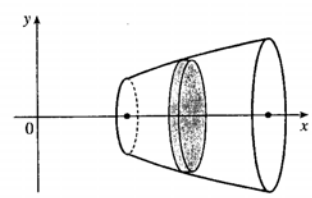




4. A cross-section is circular with radius $\sqrt{x - 1}$, so its area is $A(x) = \pi\left(\sqrt{x-1}\right)^2 = \pi\,(x - 1)$.

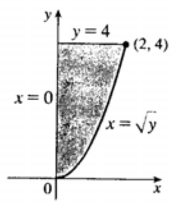
$$V = \int_2^5 A(x)\,dx = \int_2^5 \pi\,(x - 1)\,dx = \pi\left[\tfrac{1}{2}x^2 - x\right]_2^5 = \pi\left(\tfrac{25}{2} - 5 - \tfrac{4}{2} + 2\right) = \tfrac{15}{2}\pi$$

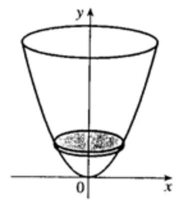




5. A cross-section is circular with radius $\sqrt{y}$, so its area is $A(y) = \pi\left(\sqrt{y}\right)^2$.

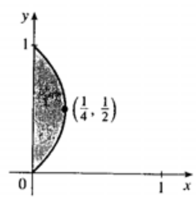
$$V = \int_0^4 A(y)\,dy = \int_0^4 \pi\left(\sqrt{y}\right)^2\,dy = \pi\int_0^4 y\,dy = \pi\left[\tfrac{1}{2}y^2\right]_0^4 = 8\pi$$

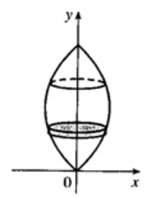




6. A cross-section is circular with radius $y - y^2$, so its area is $A(y) = \pi\left(y - y^2\right)^2$.

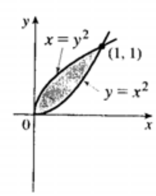
$$V = \int_0^1 A(y)\,dy = \int_0^1 \pi\left(y - y^2\right)^2\,dy = \pi\int_0^1 \left(y^4 - 2y^3 + y^2\right)dy = \pi\left[\tfrac{1}{5}y^5 - \tfrac{1}{2}y^4 + \tfrac{1}{3}y^3\right]_0^1$$
$$= \pi\left(\tfrac{1}{5} - \tfrac{1}{2} + \tfrac{1}{3}\right) = \tfrac{\pi}{30}$$

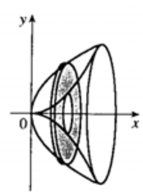




7. A cross-section is an annulus with inner radius x^2 and outer radius $\sqrt{x}$, so its area is $A(x) = \pi\left(\sqrt{x}\right)^2 - \pi\left(x^2\right)^2 = \pi\left(x - x^4\right)$.

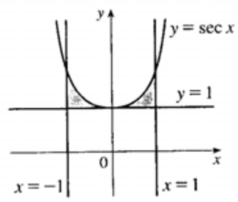
$$V = \int_0^1 A(x)\,dx = \pi\int_0^1 \left(x - x^4\right)dx = \pi\left[\tfrac{1}{2}x^2 - \tfrac{1}{5}x^5\right]_0^1 = \pi\left(\tfrac{1}{2} - \tfrac{1}{5}\right) = \tfrac{3\pi}{10}$$

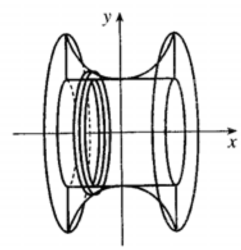




8. A cross-section is an annulus with inner radius 1 and outer radius $\sec x$, so its area is $A(x) = \pi(\sec x)^2 - \pi(1)^2 = \pi\left(\sec^2 x - 1\right)$.

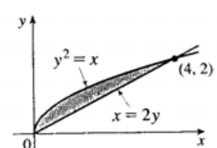
$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \pi\left(\sec^2 x - 1\right)dx = 2\pi\int_0^1 \left(\sec^2 x - 1\right)dx = 2\pi\left[\tan x - x\right]_0^1 = 2\pi(\tan 1 - 1)$$
$$\approx 3.5023$$

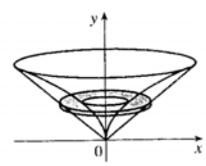




9. A cross-section is an annulus with inner radius y^2 and outer radius $2y$, so its area is $A(y) = \pi(2y)^2 - \pi\left(y^2\right)^2 = \pi\left(4y^2 - y^4\right)$.

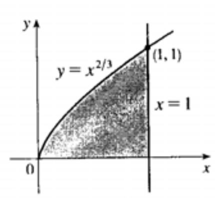
$$V = \int_0^2 A(y)\,dy = \pi\int_0^2 \left(4y^2 - y^4\right)dy = \pi\left[\tfrac{4}{3}y^3 - \tfrac{1}{5}y^5\right]_0^2 = \pi\left(\tfrac{32}{3} - \tfrac{32}{5}\right) = \tfrac{64\pi}{15}$$

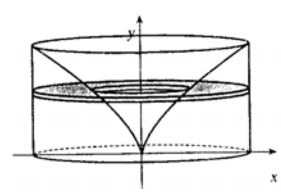




10. $y = x^{2/3} \iff x = y^{3/2}$, so a cross-section is an annulus with inner radius $y^{3/2}$ and outer radius 1, and its area is $A(y) = \pi(1)^2 - \pi\left(y^{3/2}\right)^2 = \pi\left(1 - y^3\right)$.

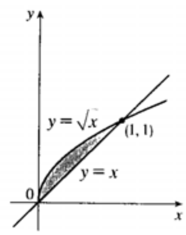
$$V = \int_0^1 A(y)\,dy = \pi\int_0^1 \left(1 - y^3\right)dy = \pi\left[y - \tfrac{1}{4}y^4\right]_0^1 = \tfrac{3}{4}\pi$$

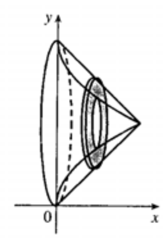




11. A cross-section is an annulus with inner radius $1 - \sqrt{x}$ and outer radius $1 - x$, so its area is
$A(x) = \pi(1-x)^2 - \pi\left(1-\sqrt{x}\right)^2 = \pi\left[\left(1 - 2x + x^2\right) - \left(1 - 2\sqrt{x} + x\right)\right] = \pi\left(-3x + x^2 + 2\sqrt{x}\right)$.

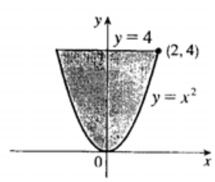
$$V = \int_0^1 A(x)\,dx = \pi\int_0^1\left(-3x + x^2 + 2\sqrt{x}\right)dx = \pi\left[-\tfrac{3}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{4}{3}x^{3/2}\right]_0^1 = \pi\left(-\tfrac{3}{2} + \tfrac{5}{3}\right) = \tfrac{\pi}{6}$$

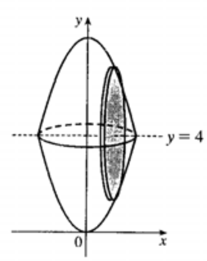




12. A cross-section is circular with radius $4 - x^2$, so its area is $A(x) = \pi\left(4 - x^2\right)^2 = \pi\left(16 - 8x^2 + x^4\right)$.

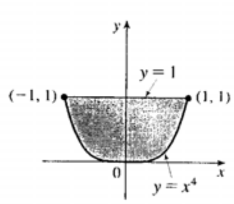
$$V = \int_{-2}^{2} A(x)\,dx = 2\int_0^2 A(x)\,dx = 2\pi\int_0^2\left(16 - 8x^2 + x^4\right)dx = 2\pi\left[16x - \tfrac{8}{3}x^3 + \tfrac{1}{5}x^5\right]_0^2$$
$$= 2\pi\left(32 - \tfrac{64}{3} + \tfrac{32}{5}\right) = 64\pi\left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right) = 64\pi \cdot \tfrac{8}{15} = \tfrac{512\pi}{15}$$

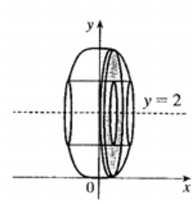




13. A cross-section is an annulus with inner radius $2 - 1$ and outer radius $2 - x^4$, so its area is
$A(x) = \pi\left(2 - x^4\right)^2 - \pi(2-1)^2 = \pi\left(3 - 4x^4 + x^8\right)$.

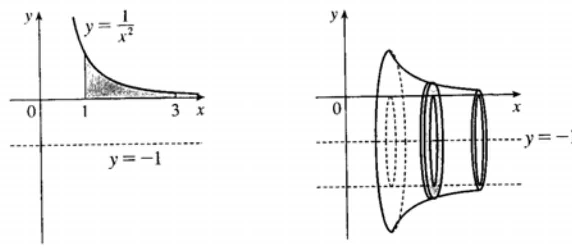
$$V = \int_{-1}^{1} A(x)\,dx = 2\int_0^1 A(x)\,dx = 2\pi\int_0^1\left(3 - 4x^4 + x^8\right)dx = 2\pi\left[3x - \tfrac{4}{5}x^5 + \tfrac{1}{9}x^9\right]_0^1$$
$$= 2\pi\left(3 - \tfrac{4}{5} + \tfrac{1}{9}\right) = \tfrac{208}{45}\pi$$





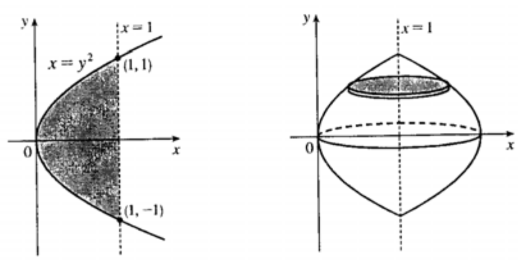
14. $V = \int_1^3 \pi \left\{ \left[\frac{1}{x^2} - (-1) \right]^2 - [0 - (-1)]^2 \right\} dx = \pi \int_1^3 \left[\left(\frac{1}{x^2} + 1 \right)^2 - 1^2 \right] dx = \pi \int_1^3 \left(\frac{1}{x^4} + \frac{2}{x^2} \right) dx$

$= \pi \left[-\frac{1}{3x^3} - \frac{2}{x} \right]_1^3 = \pi \left[\left(-\frac{1}{81} - \frac{2}{3} \right) - \left(-\frac{1}{3} - 2 \right) \right] = \frac{134\pi}{81}$



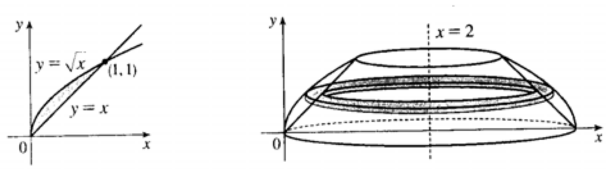
15. $V = \int_{-1}^1 \pi \left(1 - y^2\right)^2 dy = 2 \int_0^1 \pi \left(1 - y^2\right)^2 dy = 2\pi \int_0^1 \left(1 - 2y^2 + y^4\right) dy$

$= 2\pi \left[y - \frac{2}{3}y^3 + \frac{1}{5}y^5 \right]_0^1 = 2\pi \cdot \frac{8}{15} = \frac{16}{15}\pi$



16. $V = \int_0^1 \pi \left[\left(2 - y^2\right)^2 - (2 - y)^2 \right] dy = \pi \int_0^1 \left(4 - 4y^2 + y^4 - 4 + 4y - y^2\right) dy$

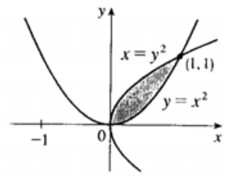
$= \pi \int_0^1 \left(y^4 - 5y^2 + 4y\right) dy = \pi \left[\frac{1}{5}y^5 - \frac{5}{3}y^3 + 2y^2 \right]_0^1 = \pi \left(\frac{1}{5} - \frac{5}{3} + 2 \right) = \frac{8}{15}\pi$

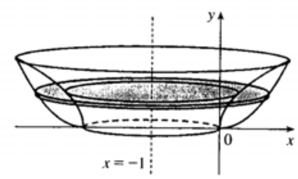


17. $V = \int_0^1 \pi \left\{ \left[\sqrt{y} - (-1)\right]^2 - \left[y^2 - (-1)\right]^2 \right\} dy = \int_0^1 \pi \left[\left(\sqrt{y} + 1\right)^2 - \left(y^2 + 1\right)^2 \right] dy$

$= \pi \int_0^1 \left(y + 2\sqrt{y} + 1 - y^4 - 2y^2 - 1\right) dy = \pi \int_0^1 \left(y + 2\sqrt{y} - y^4 - 2y^2\right) dy$

$= \pi \left[\frac{1}{2}y^2 + \frac{4}{3}y^{3/2} - \frac{1}{5}y^5 - \frac{2}{3}y^3 \right]_0^1 = \pi \left(\frac{1}{2} + \frac{4}{3} - \frac{1}{5} - \frac{2}{3} \right) = \frac{29}{30}\pi$



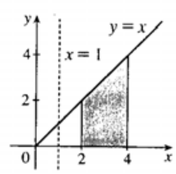


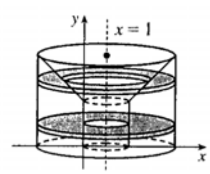
18. For $0 \leq y < 2$, a cross-section is an annulus with inner radius $2 - 1$ and outer radius $4 - 1$, the area of which is $A_1(y) = \pi(4-1)^2 - \pi(2-1)^2$. For $2 \leq y \leq 4$, a cross-section is an annulus with inner radius $y - 1$ and outer radius $4 - 1$, the area of which is $A_2(y) = \pi(4-1)^2 - \pi(y-1)^2$.

$$V = \int_0^2 A(y)\,dy = \pi \int_0^2 \left[(4-1)^2 - (2-1)^2\right] dy + \pi \int_2^4 \left[(4-1)^2 - (y-1)^2\right] dy$$

$$= \pi\,[8y]_0^2 + \pi \int_2^4 \left(8 + 2y - y^2\right) dy = 16\pi + \pi \left[8y + y^2 - \frac{1}{3}y^3\right]_2^4$$

$$= 16\pi + \pi \left[\left(32 + 16 - \frac{64}{3}\right) - \left(16 + 4 - \frac{8}{3}\right) \right] = \frac{76}{3}\pi$$





19. $V = \pi \int_0^8 \left(\frac{1}{4}x\right)^2 dx = \frac{\pi}{16}\left[\frac{1}{3}x^3\right]_0^8 = \frac{32}{3}\pi$

20. $V = \pi \int_0^2 \left[8^2 - (4y)^2\right] dy = \pi \left[64y - \frac{16}{3}y^3\right]_0^2 = \pi \left(128 - \frac{128}{3}\right) = \frac{256}{3}\pi$

21. $V = \pi \int_0^2 (8 - 4y)^2\,dy = \pi \left[64y - 32y^2 + \frac{16}{3}y^3\right]_0^2 = \pi \left(128 - 128 + \frac{128}{3}\right) = \frac{128}{3}\pi$

22. $V = \pi \int_0^8 \left[2^2 - \left(2 - \frac{1}{4}x\right)^2\right] dx = \pi \int_0^8 \left(x - \frac{1}{16}x^2\right) dx = \pi \left[\frac{1}{2}x^2 - \frac{1}{48}x^3\right]_0^8 = \pi \left(32 - \frac{32}{3}\right) = \frac{64}{3}\pi$

23. $V = \pi \int_0^8 \left[\left(\sqrt[3]{x}\right)^2 - \left(\frac{1}{4}x\right)^2\right] = \pi \int_0^8 \left(x^{2/3} - \frac{1}{16}x^2\right) dx = \pi \left[\frac{3}{5}x^{5/3} - \frac{1}{48}x^3\right]_0^8 = \pi \left(\frac{96}{5} - \frac{32}{3}\right) = \frac{128}{15}\pi$

24. $V = \pi \int_0^2 \left[(4y)^2 - \left(y^3\right)^2\right] dy = \pi \int_0^2 \left(16y^2 - y^6\right) dy = \pi \left[\frac{16}{3}y^3 - \frac{1}{7}y^7\right]_0^2 = \pi \left(\frac{128}{3} - \frac{128}{7}\right) = \frac{512}{21}\pi$

25. $V = \pi \int_0^8 \left[\left(2 - \frac{1}{4}x\right)^2 - \left(2 - \sqrt[3]{x}\right)^2\right] dx = \pi \int_0^8 \left(-x + \frac{1}{16}x^2 + 4x^{1/3} - x^{2/3}\right) dx$

$= \pi \left[-\frac{1}{2}x^2 + \frac{1}{48}x^3 + 3x^{4/3} - \frac{3}{5}x^{5/3}\right]_0^8 = \pi \left(-32 + \frac{32}{3} + 48 - \frac{96}{5}\right) = \frac{112}{15}\pi$

26. $V = \pi \int_0^2 \left[\left(8 - y^3\right)^2 - (8 - 4y)^2\right] dy = \pi \int_0^2 \left(-16y^3 + y^6 + 64y - 16y^2\right) dy$

$= \pi \left[-4y^4 + \frac{1}{7}y^7 + 32y^2 - \frac{16}{3}y^3\right]_0^2 = \pi \left(-64 + \frac{128}{7} + 128 - \frac{128}{3}\right) = \frac{832}{21}\pi$

27. $V = \pi \int_0^8 \left(2^2 - x^{2/3}\right) dx = \pi \left[4x - \frac{3}{5}x^{5/3}\right]_0^8 = \pi \left(32 - \frac{96}{5}\right) = \frac{64}{5}\pi$

28. $V = \pi \int_0^2 \left(y^3\right)^2 dy = \pi \left[\frac{1}{7}y^7\right]_0^2 = \frac{128}{7}\pi$

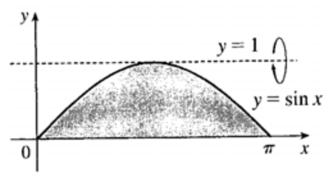
29. $V = \pi \int_0^8 \left(2 - \sqrt[3]{x}\right)^2 dx = \pi \int_0^8 \left(4 - 4x^{1/3} + x^{2/3}\right) dx = \pi \left[4x - 3x^{4/3} + \frac{3}{5}x^{5/3}\right]_0^8 = \pi \left(32 - 48 + \frac{96}{5}\right) = \frac{16}{5}\pi$

30. $V = \pi \int_0^2 \left[8^2 - \left(8 - y^3\right)^2\right] dy = \pi \int_0^2 \left(16y^3 - y^6\right) dy = \pi \left[4y^4 - \frac{1}{7}y^7\right]_0^2 = \pi \left(64 - \frac{128}{7}\right) = \frac{320}{7}\pi$

31. $V = \pi \int_0^{\pi/4} \left(1^2 - \tan^2 x\right) dx$

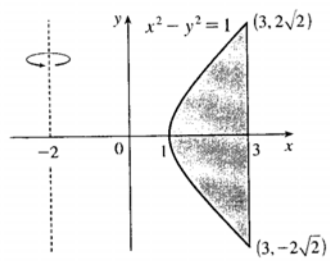
32. $V = \pi \int_0^2 \left[5^2 - \left(y^2 + 1\right)^2\right] dy = \pi \int_0^2 \left(24 - y^4 - 2y^2\right) dy$

33. $V = \pi \int_0^\pi \left[(1 - 0)^2 - (1 - \sin x)^2\right] dx = \pi \int_0^\pi \left[1^2 - (1 - \sin x)^2\right] dx$



34. $V = \pi \int_0^\pi \left[(\sin x + 2)^2 - 2^2\right] dx$

35. $V = \pi \int_{-\sqrt{8}}^{\sqrt{8}} \left\{[3 - (-2)]^2 - \left[\sqrt{y^2 + 1} - (-2)\right]^2\right\} dy = \pi \int_{-2\sqrt{2}}^{2\sqrt{2}} \left[5^2 - \left(\sqrt{1 + y^2} + 2\right)^2\right] dy$



36. The points of intersection of the two curves are $(3, 0)$ and $\left(-\frac{9}{4}, \frac{7}{2}\right)$. Therefore

$$V = \pi \int_0^{7/2} \left[\left[4 - (y - 1)^2 + 5\right]^2 - \left(3 - \frac{3}{2}y + 5\right)^2\right] dy = \pi \int_0^{7/2} \left[\left[9 - (y - 1)^2\right]^2 - \left(8 - \frac{3}{2}y\right)^2\right] dy$$

37. We see from the graph in Exercise 6.1.35 that the x-coordinates of the points of intersection are $x \approx -0.72$ and $x \approx 1.22$, with $\sqrt{x + 1} > x^2$ on $(-0.72, 1.22)$, so the volume of revolution is about

$$\pi \int_{-0.72}^{1.22} \left[\left(\sqrt{x + 1}\right)^2 - \left(x^2\right)^2\right] dx = \pi \int_{-0.72}^{1.22} \left(x + 1 - x^4\right) dx = \pi \left[\frac{1}{2}x^2 + x - \frac{1}{5}x^5\right]_{-0.72}^{1.22} \approx 5.80.$$

38. We see from the graph in Exercise 6.1.34 that the x-coordinates of the points of intersection are $x = 0$ and $x \approx 1.17$, with $3x - x^3 > x^4$ on $(0, 1.17)$, so the volume of revolution is about

$$\pi \int_0^{1.17} \left[\left(3x - x^3\right)^2 - \left(x^4\right)^2 \right] dx = \pi \int_0^{1.17} \left(9x^2 - 6x^4 + x^6 - x^8\right) dx$$
$$= \pi \left[3x^3 - \tfrac{6}{5}x^5 + \tfrac{1}{7}x^7 - \tfrac{1}{9}x^9\right]_0^{1.17} \approx 6.74$$

39. $\pi \int_0^{\pi/2} \cos^2 x\, dx$ describes the volume of the solid obtained by rotating the region $\mathcal{R} = \left\{(x, y) \mid 0 \le x \le \frac{\pi}{2}, 0 \le y \le \cos x\right\}$ of the xy-plane about the x-axis.

40. $\pi \int_2^5 y\, dy = \pi \int_2^5 \left(\sqrt{y}\right)^2 dy$ describes the volume of the solid obtained by rotating the region $\mathcal{R} = \left\{(x, y) \mid 2 \le y \le 5, 0 \le x \le \sqrt{y}\right\}$ of the xy-plane about the y-axis.

41. $\pi \int_0^1 \left(y^4 - y^8\right) dy = \pi \int_0^1 \left[\left(y^2\right)^2 - \left(y^4\right)^2\right] dy$ describes the volume of the solid obtained by rotating the region $\mathcal{R} = \left\{(x, y) \mid 0 \le y \le 1, y^4 \le x \le y^2\right\}$ of the xy-plane about the y-axis.

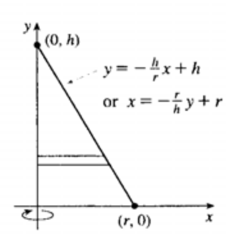
42. $\pi \int_0^{\pi/2} \left[(1 + \cos x)^2 - 1^2\right] dx$ describes the volume of the solid obtained by rotating the region $\mathcal{R} = \left\{(x, y) \mid 0 \le x \le \frac{\pi}{2}, 1 \le y \le 1 + \cos x\right\}$ of the xy-plane about the x-axis.
Or: The solid could be obtained by rotating the region $\mathcal{R}' = \left\{(x, y) \mid 0 \le x \le \frac{\pi}{2}, 0 \le y \le \cos x\right\}$ about the line $y = -1$.

43. $V = \int_0^{15} A(x)\, dx \approx M_5 = 3\left[A(1.5) + A(4.5) + A(7.5) + A(10.5) + A(13.5)\right]$
$= 3(18 + 79 + 106 + 128 + 39) = 3 \cdot 370 = 1110 \text{ cm}^3$

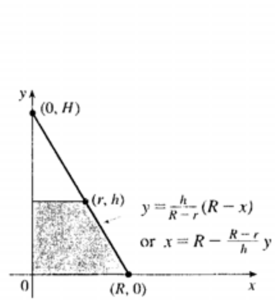
44. We use the Midpoint Rule with $\Delta x = 2$ and $n = 5$.

$$V = \int_0^{10} A(x)\, dx \approx M_5 = \Delta x\left[A(1) + A(3) + A(5) + A(7) + A(9)\right]$$
$$= 2(0.65 + 0.61 + 0.59 + 0.55 + 0.50) = 5.80 \text{ m}^3$$

45. $V = \pi \int_0^h \left(-\frac{r}{h}y + r\right)^2 dy = \pi \int_0^h \left[\frac{r^2}{h^2}y^2 - \frac{2r^2}{h}y + r^2\right] dy$
$= \pi \left[\frac{r^2}{3h^2}y^3 - \frac{r^2}{h}y^2 + r^2 y\right]_0^h = \pi\left(\frac{1}{3}r^2 h - r^2 h + r^2 h\right) = \frac{1}{3}\pi r^2 h$



46. $V = \pi \int_0^h \left(R - \frac{R-r}{h}y\right)^2 dy$



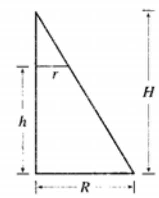
$$= \pi \int_0^h \left[R^2 - \frac{2R(R-r)}{h}y + \left(\frac{R-r}{h}\right)^2 y^2\right] dy$$

$$= \pi \left[R^2 y - \frac{R(R-r)}{h}y^2 + \frac{1}{3}\left(\frac{R-r}{h}\right)^2 y^3\right]_0^h$$

$$= \pi \left[R^2 h - R(R-r)h + \tfrac{1}{3}(R-r)^2 h\right]$$

$$= \tfrac{1}{3}\pi h\left[3Rr + \left(R^2 - 2Rr + r^2\right)\right] = \tfrac{1}{3}\pi h\left(R^2 + Rr + r^2\right)$$

Another Solution: $\frac{H}{R} = \frac{H-h}{r}$ by similar triangles. Therefore, $Hr = HR - hR$ $\Rightarrow$ $hR = H(R-r)$ $\Rightarrow$ $H = \frac{hR}{R-r}$. Now



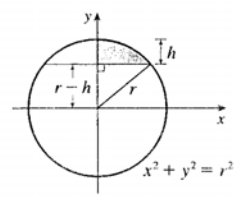
$$V = \tfrac{1}{3}\pi R^2 H - \tfrac{1}{3}\pi r^2 (H-h) \quad \text{(by Exercise 45)}$$

$$= \tfrac{1}{3}\pi R^2 \frac{hR}{R-r} - \tfrac{1}{3}\pi r^2 \frac{rh}{R-r} = \frac{\pi h}{3}\,\frac{R^3 - r^3}{R-r} = \tfrac{1}{3}\pi h\left(R^2 + Rr + r^2\right)$$

$$= \tfrac{1}{3}\left[\pi R^2 + \pi r^2 + \sqrt{\left(\pi R^2\right)\left(\pi r^2\right)}\right] h = \tfrac{1}{3}\left(A_1 + A_2 + \sqrt{A_1 A_2}\right) h$$

where A_1 and A_2 are the areas of the bases of the frustum. (See Exercise 48 for a related result.)

47. $x^2 + y^2 = r^2 \;\Rightarrow\; x^2 = r^2 - y^2 \;\Rightarrow$



$$V = \pi \int_{r-h}^{r} \left(r^2 - y^2\right) dy = \pi \left[r^2 y - \frac{y^3}{3}\right]_{r-h}^{r}$$

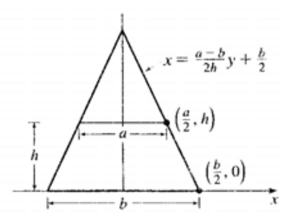
$$= \pi \left(\left[r^3 - \frac{r^3}{3}\right] - \left[r^2(r-h) - \frac{(r-h)^3}{3}\right]\right)$$

$$= \pi \left(\tfrac{2}{3}r^3 - \tfrac{1}{3}(r-h)\left[3r^2 - (r-h)^2\right]\right)$$

$$= \tfrac{1}{3}\pi \left(2r^3 - (r-h)\left[3r^2 - \left(r^2 - 2rh + h^2\right)\right]\right)$$

$$= \tfrac{1}{3}\pi h^2 (3r - h), \text{ or, equivalently, } \pi h^2 \left(r - \frac{h}{3}\right)$$

48. $V = \int_0^h A(y)\,dy = \int_0^h \left[2\left(\frac{a-b}{2h}y + \frac{b}{2}\right)\right]\left[2\left(\frac{a-b}{2h}y + \frac{b}{2}\right)\right] dy$



$$= \int_0^h \left[\frac{a-b}{h}y + b\right]^2 dy = \int_0^h \left[\frac{(a-b)^2}{h^2}y^2 + \frac{2b(a-b)}{h}y + b^2\right] dy$$

$$= \left[\frac{(a-b)^2}{3h^2}y^3 + \frac{b(a-b)}{h}y^2 + b^2 y\right]_0^h$$

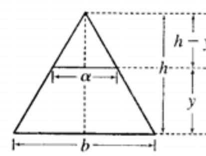
$$= \tfrac{1}{3}(a-b)^2 h + b(a-b)h + b^2 h = \tfrac{1}{3}\left(a^2 - 2ab + b^2 + 3ab\right) h$$

$$= \tfrac{1}{3}\left(a^2 + ab + b^2\right) h$$

[Note that this can be written as $\frac{1}{3}\left(A_1 + A_2 + \sqrt{A_1 A_2}\right) h$, as in Exercise 46.]

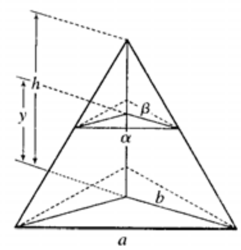
49. For a cross-section at height y, we see from similar triangles that $\dfrac{\alpha/2}{b/2} = \dfrac{h-y}{h}$, so $\alpha = b\left(1 - \frac{y}{h}\right)$. Similarly, for cross-sections having $2b$ as their base and β replacing α, $\beta = 2b\left(1 - \frac{y}{h}\right)$. So

$$V = \int_0^h A(y)\,dy = \int_0^h \left[b\left(1 - \tfrac{y}{h}\right)\right]\left[2b\left(1 - \tfrac{y}{h}\right)\right] dy = \int_0^h 2b^2 \left(1 - \tfrac{y}{h}\right)^2 dy$$
$$= 2b^2 \int_0^h \left(1 - \frac{2y}{h} + \frac{y^2}{h^2}\right) dy = 2b^2 \left[y - \frac{y^2}{h} + \frac{y^3}{3h^2}\right]_0^h = 2b^2 \left[h - h + \tfrac{1}{3}h\right]$$
$$= \tfrac{2}{3}b^2 h \quad \left(= \tfrac{1}{3}Bh \text{ where } B \text{ is the area of the base, as with any pyramid.}\right)$$



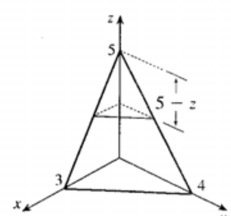
50. Consider the triangle consisting of two vertices of the base and the center of the base. This triangle is similar to the corresponding triangle at a height y, so $a/b = \alpha/\beta \;\Rightarrow\; \alpha = a\beta/b$. Also by similar triangles, $b/h = \beta/(h-y)$ $\Rightarrow\; \beta = b(h-y)/h$. These two equations imply that $\alpha = a(1 - y/h)$, and since the cross-section is an equilateral triangle, it has area

$$A(y) = \frac{\alpha}{2} \cdot \frac{\sqrt{3}\alpha}{2} = \frac{a^2(1-y/h)^2}{4}\sqrt{3}, \text{ so}$$
$$V = \int_0^h A(y)\,dy = \frac{a^2\sqrt{3}}{4}\int_0^h \left(1 - \frac{y}{h}\right)^2 dy$$
$$= \frac{a^2\sqrt{3}}{4}\left[-\frac{h}{3}\left(1 - \frac{y}{h}\right)^3\right]_0^h = -\tfrac{\sqrt{3}}{12}a^2 h(-1) = \tfrac{\sqrt{3}}{12}a^2 h$$



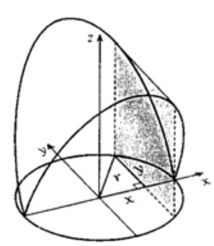
51. A cross-section at height z is a triangle similar to the base, so its area is

$$A(z) = \tfrac{1}{2} \cdot 3\left(\frac{5-z}{5}\right) \cdot 4\left(\frac{5-z}{5}\right) = 6\left(1 - \frac{z}{5}\right)^2, \text{ so}$$
$$V = \textstyle\int_0^5 A(z)\,dz = 6\int_0^5 (1 - z/5)^2\,dz = 6\left[(-5)\tfrac{1}{3}\left(1 - \tfrac{1}{5}z\right)^3\right]_0^5$$
$$= -10(-1) = 10 \text{ cm}^3$$

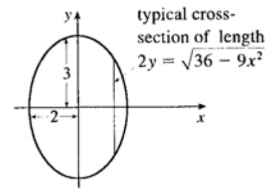


52. A cross-section is shaded in the diagram.

$$A(x) = (2y)^2 = \left(2\sqrt{r^2 - x^2}\right)^2, \text{ so}$$
$$V = \textstyle\int_{-r}^r A(x)\,dx = 2\int_0^r 4\left(r^2 - x^2\right) dx = 8\left[r^2 x - \tfrac{1}{3}x^3\right]_0^r$$
$$= \tfrac{16}{3}r^3$$

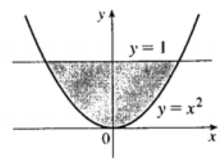


53. If l is a leg of the isosceles right triangle and $2y$ is the hypotenuse, then $l^2 + l^2 = (2y)^2 \Rightarrow l = \sqrt{2}y$.



$$V = \int_{-2}^{2} A(x)\,dx = 2\int_0^2 A(x)\,dx = 2\int_0^2 \tfrac{1}{2}\left(\sqrt{2}y\right)^2 dx = 2\int_0^2 y^2\,dx$$
$$= \tfrac{1}{2}\int_0^2 (36 - 9x^2)\,dx = \tfrac{9}{2}\int_0^2 (4 - x^2)\,dx = \tfrac{9}{2}\left[4x - \tfrac{1}{3}x^3\right]_0^2$$
$$= \tfrac{9}{2}\left(8 - \tfrac{8}{3}\right) = 24$$

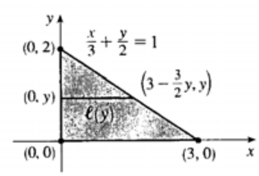
54. The cross-section of the base corresponding to the coordinate y has length $2x = 2\sqrt{y}$. The corresponding equilateral triangle with side s has area $A(y) = s^2\left(\frac{\sqrt{3}}{4}\right) = (2x)^2\left(\frac{\sqrt{3}}{4}\right) = \left(2\sqrt{y}\right)^2\left(\frac{\sqrt{3}}{4}\right) = y\sqrt{3}$. Therefore,



$$V = \int_0^1 A(y)\,dy = \int_0^1 y\sqrt{3}dy = \sqrt{3}\left[\tfrac{1}{2}y^2\right]_0^1 = \tfrac{\sqrt{3}}{2}$$

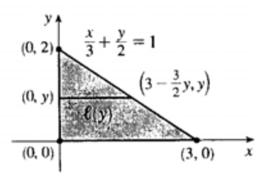
55. The square has area $A(y) = \left(2\sqrt{y}\right)^2 = 4y$, so $V = \int_0^1 A(y)\,dy = \int_0^1 4y\,dy = \left[2y^2\right]_0^1 = 2$.

56. A typical cross-section perpendicular to the y-axis in the base has length $\ell(y) = 3 - \frac{3}{2}y$. This length is the diameter of a cross-sectional semicircle in S, so



$$V = \int_0^2 A(y)\,dy = \int_0^2 \frac{\pi}{2}\left[\frac{\ell(y)}{2}\right]^2 dy = \frac{\pi}{8}\int_0^2 \left(3 - \tfrac{3}{2}y\right)^2 dy$$
$$= \tfrac{9\pi}{8}\int_0^2 \left(1 - \tfrac{1}{2}y\right)^2 dy = \tfrac{9\pi}{8}\int_0^2 \left(1 - y + \tfrac{1}{4}y^2\right) dy$$
$$= \tfrac{9\pi}{8}\left[y - \tfrac{1}{2}y^2 + \tfrac{1}{12}y^3\right]_0^2 = \tfrac{9\pi}{8}\left(2 - 2 + \tfrac{2}{3}\right) = \tfrac{3\pi}{4}$$

57. A typical cross-section perpendicular to the y-axis in the base has length $\ell(y) = 3 - \frac{3}{2}y$. This length is the leg of an isosceles right triangle, so



$$A(y) = \tfrac{1}{2}\left[\ell(y)\right]^2 \quad \left(\tfrac{1}{2}bh \text{ with base} = \text{height}\right)$$
$$= \tfrac{1}{2}\left[3\left(1 - \tfrac{1}{2}y\right)\right]^2 = \tfrac{9}{2}\left(1 - \tfrac{1}{2}y\right)^2 = \tfrac{9}{2}\left(1 - y + \tfrac{1}{4}y^2\right)$$

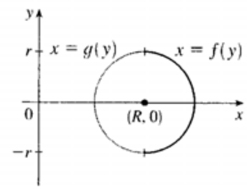
Thus,

$$V = \int_0^2 A(y)\,dy = \tfrac{9}{2}\int_0^2 \left(1 - y + \tfrac{1}{4}y^2\right) dy = \tfrac{9}{2}\left[y - \tfrac{1}{2}y^2 + \tfrac{1}{12}y^3\right]_0^2 = \tfrac{9}{2}\left[\left(2 - 2 + \tfrac{2}{3}\right) - 0\right] = \tfrac{9}{2}\cdot\tfrac{2}{3} = 3$$

58. (a) $V = \int_{-r}^{r} A(x)\,dx = 2\int_0^r A(x)\,dx = 2\int_0^r \frac{1}{2}h\left(2\sqrt{r^2 - x^2}\right) dx = 2h\int_0^r \sqrt{r^2 - x^2}dx$

(b) Observe that the integral represents one quarter of the area of a circle of radius r, so $V = 2h \cdot \frac{1}{4}\pi r^2 = \frac{1}{2}\pi hr^2$.

59. (a) The torus is obtained by rotating the circle $(x-R)^2+y^2=r^2$ about the y-axis. Solving for y, we see that the right half of the circle is given by $x=R+\sqrt{r^2-y^2}=f(y)$ and the left half by $x=R-\sqrt{r^2-y^2}=g(y)$. So



$$V=\pi\int_{-r}^{r}\left([f(y)]^2-[g(y)]^2\right)dy=2\pi\int_0^r 4R\sqrt{r^2-y^2}\,dy$$
$$=8\pi R\int_0^r\sqrt{r^2-y^2}\,dy$$

(b) Observe that the integral represents a quarter of the area of a circle with radius r, so $8\pi R\int_0^r\sqrt{r^2-y^2}\,dy=8\pi R\cdot\frac{1}{4}\left(\pi r^2\right)=2\pi^2r^2R$.

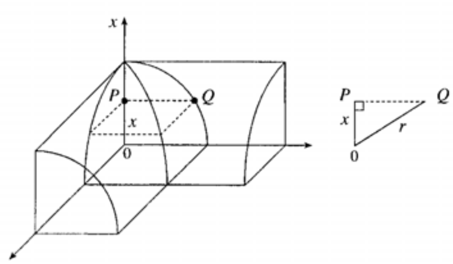
60. The cross-sections perpendicular to the y-axis in Figure 17 are rectangles. The rectangle corresponding to the coordinate y has a base of length $2\sqrt{16-y^2}$ in the xy-plane and a height of $\frac{1}{\sqrt{3}}y$, since $\angle BAC=30^\circ$ and $|BC|=\frac{1}{\sqrt{3}}|AB|$. Thus, $A(y)=\frac{2}{\sqrt{3}}y\sqrt{16-y^2}$ and

$$V=\int_0^4 A(y)\,dy=\int_0^4 A(y)\,dy=\frac{2}{\sqrt{3}}\int_0^4\sqrt{16-y^2}\,y\,dy$$
$$=\frac{2}{\sqrt{3}}\int_{16}^{0}u^{1/2}\left(-\tfrac{1}{2}\,du\right)\quad[\text{Put } u=16-y^2,\text{ so } du=-2y\,dy]$$
$$=\frac{1}{\sqrt{3}}\int_0^{16}u^{1/2}\,du=\frac{1}{\sqrt{3}}\frac{2}{3}\left[u^{3/2}\right]_0^{16}=\frac{2}{3\sqrt{3}}(64)=\frac{128}{3\sqrt{3}}$$

61. (a) $\text{Volume}(S_1)=\int_0^h A(z)\,dz=\text{Volume}(S_2)$ since the cross-sectional area $A(z)$ at height z is the same for both solids.

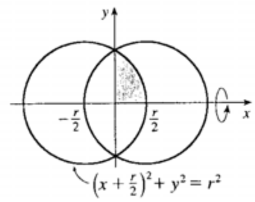
(b) By Cavalieri's Principle, the volume of the cylinder in the figure is the same as that of a right circular cylinder with radius r and height h, that is, πr^2h.

62. Each cross-section of the solid S in a plane perpendicular to the x-axis is a square (since the edges of the cut lie on the cylinders, which are perpendicular). One-quarter of this square and one-eighth of S are shown. The area of this quarter-square is $|PQ|^2=r^2-x^2$. Therefore, $A(x)=4\left(r^2-x^2\right)$ and the volume of S is



$$V=\int_{-r}^{r}A(x)\,dx=4\int_{-r}^{r}\left(r^2-x^2\right)dx$$
$$=8\int_0^r\left(r^2-x^2\right)dx=8\left[r^2x-\tfrac{1}{3}x^3\right]_0^r=\tfrac{16}{3}r^3$$

63. The volume is obtained by rotating the area common to two circles of radius r, as shown. The volume of the right half is



$$V_{\text{right}} = \pi \int_0^{r/2} y^2\,dx = \pi \int_0^{r/2} \left[r^2 - \left(\tfrac{1}{2}r + x\right)^2\right] dx$$

$$= \pi \left[r^2 x - \tfrac{1}{3}\left(\tfrac{1}{2}r + x\right)^3\right]_0^{r/2} = \pi \left[\left(\tfrac{1}{2}r^3 - \tfrac{1}{3}r^3\right) - \left(0 - \tfrac{1}{24}r^3\right)\right] = \tfrac{5}{24}\pi r^3$$

So by symmetry, the total volume is twice this, or $\frac{5}{12}\pi r^3$.

Another Solution: We observe that the volume is the twice the volume of a cap of a sphere, so we can use the formula from Exercise 47 with $h = \frac{1}{2}r$: $V = 2 \cdot \frac{1}{3}\pi r h^2 (3r - h) = \frac{2}{3}\pi \left(\frac{1}{2}r\right)^2 \left(3r - \frac{1}{2}r\right) = \frac{5}{12}\pi r^3$.

64. We consider two cases: one in which the ball is not completely submerged and the other in which it is.

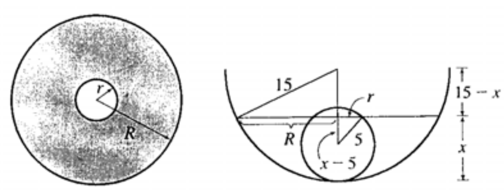
Case 1: $0 \le h \le 10$ The ball will not be completely submerged, and so a cross-section of the water parallel to the surface will be the shaded area shown in the first diagram. We can find the area of the cross-section at height x above the bottom of the bowl by using the Pythagorean Theorem: $R^2 = 15^2 - (15 - x)^2$ and $r^2 = 5^2 - (x - 5)^2$, so $A(x) = \pi\left(R^2 - r^2\right) = 20\pi x$. The volume of water when it has depth h is then

$V(h) = \int_0^h A(x)\,dx = \int_0^h 20\pi x\,dx = \left[10\pi x^2\right]_0^h = 10\pi h^2,\ 0 \le h \le 10$.

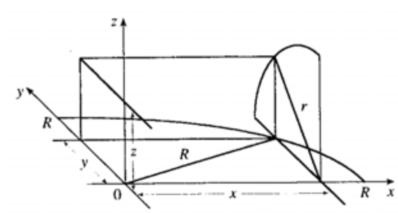
Case 2: $10 < h \le 15$ In this case we can find the volume by simply subtracting the volume displaced by the ball from the total volume inside the bowl underneath the surface of the water. The total volume underneath the surface is just the volume of a cap of the bowl, so we use the formula from Exercise 47:

$V_{\text{cap}}(h) = \frac{1}{3}\pi h^2 (45 - h)$. The volume of the small sphere is $V_{\text{ball}} = \frac{4}{3}\pi (5)^3 = \frac{500}{3}\pi$, so the total volume is

$V_{\text{cap}} - V_{\text{ball}} = \frac{1}{3}\pi \left(45h^2 - h^3 - 500\right)\ \text{cm}^3$.



65. Take the x-axis to be the axis of the cylindrical hole of radius r. A quarter of the cross-section through y, perpendicular to the y-axis, is the rectangle shown. Using Pythagoras twice, we see that the dimensions of this rectangle are $x = \sqrt{R^2 - y^2}$ and $z = \sqrt{r^2 - y^2}$, so $\frac{1}{4}A(y) = xz = \sqrt{r^2 - y^2}\sqrt{R^2 - y^2}$, and



$$V = \int_{-r}^{r} A(y)\,dy = \int_{-r}^{r} 4\sqrt{r^2 - y^2}\sqrt{R^2 - y^2}\,dy$$

$$= 8\int_0^r \sqrt{r^2 - y^2}\sqrt{R^2 - y^2}\,dy$$

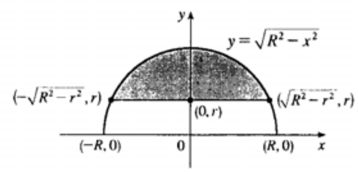
66. The line $y = r$ intersects the semicircle $y = \sqrt{R^2 - x^2}$ when $r = \sqrt{R^2 - x^2} \Rightarrow r^2 = R^2 - x^2 \Rightarrow x^2 = R^2 - r^2 \Rightarrow x = \pm\sqrt{R^2 - r^2}$. Rotating the shaded region about the x-axis gives us

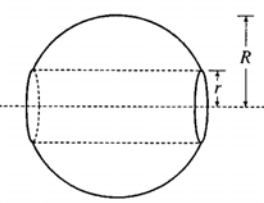
$$V = \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \pi \left[\left(\sqrt{R^2 - x^2}\right)^2 - r^2\right] dx = 2\pi \int_0^{\sqrt{R^2-r^2}} \left(R^2 - x^2 - r^2\right) dx \;\; \text{[by symmetry]}$$

$$= 2\pi \int_0^{\sqrt{R^2-r^2}} \left[\left(R^2 - r^2\right) - x^2\right] dx = 2\pi \left[\left(R^2 - r^2\right)x - \tfrac{1}{3}x^3\right]_0^{\sqrt{R^2-r^2}}$$

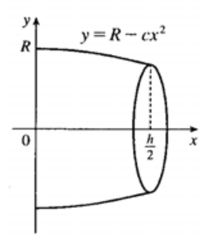
$$= 2\pi \left[\left(R^2 - r^2\right)^{3/2} - \tfrac{1}{3}\left(R^2 - r^2\right)^{3/2}\right] = 2\pi \cdot \tfrac{2}{3}\left(R^2 - r^2\right)^{3/2} = \tfrac{4\pi}{3}\left(R^2 - r^2\right)^{3/2}$$

Our answer makes sense in limiting cases. As $r \to 0$, $V \to \frac{4}{3}\pi R^3$, which is the volume of the full sphere. As $r \to R$, $V \to 0$, which makes sense because the hole's radius is approaching that of the sphere.





67. (a) The radius of the barrel is the same at each end by symmetry, since the function $y = R - cx^2$ is even. Since the barrel is obtained by rotating the function y about the x-axis, this radius is equal to the value of y at $x = \frac{1}{2}h$, which is $R - c\left(\frac{1}{2}h\right)^2 = R - d = r$.



(b) The barrel is symmetric about the y-axis, so its volume is twice the volume of that part of the barrel for $x > 0$. Also, the barrel is a volume of rotation, so

$$V = 2\int_0^{h/2} \pi\left(y^2\right) dx = 2\pi \int_0^{h/2} \left(R - cx^2\right)^2 dx = 2\pi \left[R^2 x - \tfrac{2}{3}Rcx^3 + \tfrac{1}{5}c^2x^5\right]_0^{h/2}$$

$$= 2\pi\left(\tfrac{1}{2}R^2h - \tfrac{1}{12}Rch^3 + \tfrac{1}{160}c^2h^5\right)$$

Trying to make this look more like the expression we want, we rewrite it as $V = \frac{1}{3}\pi h\left[2R^2 + \left(R^2 - \frac{1}{2}Rch^2 + \frac{3}{80}c^2h^4\right)\right]$. But

$R^2 - \frac{1}{2}Rch^2 + \frac{3}{80}c^2h^4 = \left(R - \frac{1}{4}ch^2\right)^2 - \frac{1}{40}c^2h^4 = (R - d)^2 - \frac{2}{5}\left(\frac{1}{4}ch^2\right)^2 = r^2 - \frac{2}{5}d^2$.

Substituting this back into V, we see that $V = \frac{1}{3}\pi h\left(2R^2 + r^2 - \frac{2}{5}d^2\right)$, as required.

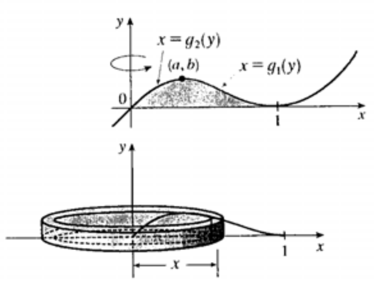
68. It suffices to consider the case where $\mathcal{R}$ is bounded by the curves $y = f(x)$ and $y = g(x)$ for $a \le x \le b$, where $g(x) \le f(x)$ for all x in $[a, b]$, since other regions can be decomposed into subregions of this type. We are concerned with the volume obtained when $\mathcal{R}$ is rotated about the line $y = -k$, which is equal to

$$\begin{aligned} V_2 &= \pi \int_a^b \left([f(x) + k]^2 - [g(x) + k]^2\right) dx = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2\right) dx + 2\pi k \int_a^b [f(x) - g(x)]\, dx \\ &= V_1 + 2\pi k A \end{aligned}$$

69. We are given that the rate of change of the volume of water is $\dfrac{dV}{dt} = -kA(x)$, where k is some positive constant and $A(x)$ is the area of the surface when the water has depth x. Now we are concerned with the rate of change of the depth of the water with respect to time, that is, $\dfrac{dx}{dt}$. But by the Chain Rule, $\dfrac{dV}{dt} = \dfrac{dV}{dx}\dfrac{dx}{dt}$, so the first equation can be written $\dfrac{dV}{dx}\dfrac{dx}{dt} = -kA(x)$ (★). Also, we know that the total volume of water up to a depth x is $V(x) = \int_0^x A(s)\, ds$, where $A(s)$ is the area of a cross-section of the water at a depth s. Differentiating this equation with respect to x, we get $\dfrac{dV}{dx} = A(x)$. Substituting this into equation ★, we get $A(x)\dfrac{dx}{dt} = -kA(x)$ $\Rightarrow$ $\dfrac{dx}{dt} = -k$, a constant.

6.3 Volumes by Cylindrical Shells

1.

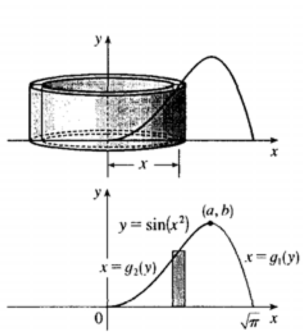


If we were to use the "washer" method, we would first have to locate the local maximum point (a, b) of $y = x(x-1)^2$ using the methods of Chapter 4. Then we would have to solve the equation $y = x(x-1)^2$ for x in terms of y to obtain the functions $x = g_1(y)$ and $x = g_2(y)$ shown in the first figure. This step would be difficult because it involves the cubic formula. Finally we would find the volume using $V = \pi \int_0^b \left\{[g_1(y)]^2 - [g_2(y)]^2\right\} dy$.

Using shells, we find that a typical approximating shell has radius x, so its circumference is $2\pi x$. Its height is y, that is, $x(x-1)^2$. So the total volume is

$$V = \int_0^1 2\pi x \left[x(x-1)^2\right] dx = 2\pi \int_0^1 \left(x^4 - 2x^3 + x^2\right) dx = 2\pi \left[\frac{x^5}{5} - 2\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1 = \frac{\pi}{15}$$

2.

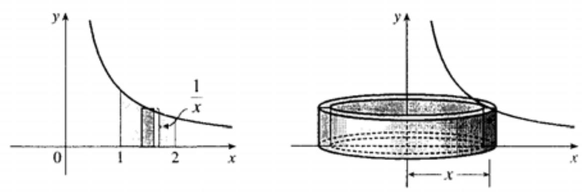


A typical cylindrical shell has circumference $2\pi x$ and height $\sin\left(x^2\right)$. $V = \int_0^{\sqrt{\pi}} 2\pi x \sin\left(x^2\right) dx$. Let $u = x^2$. Then $du = 2x\,dx$, so

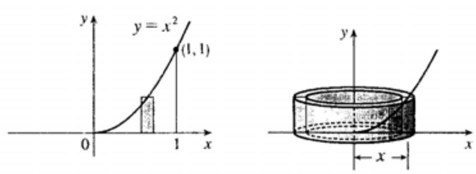
$$V = \pi \int_0^{\pi} \sin u\,du = \pi\left[-\cos u\right]_0^{\pi}$$
$$= \pi\left[1 - (-1)\right] = 2\pi$$

For slicing, we would first have to locate the local maximum point (a, b) of $y = \sin\left(x^2\right)$ using the methods of Chapter 4. Then we would have to solve the equation $y = \sin\left(x^2\right)$ for x in terms of y to obtain the functions $x = g_1(y)$ and $x = g_2(y)$ shown in the second figure. This step would be difficult because it involves the cubic formula. Finally we would find the volume using $V = \pi \int_0^b \left\{[g_1(y)]^2 - [g_2(y)]^2\right\} dy$.

3. $V = \displaystyle\int_1^2 2\pi x \cdot \frac{1}{x}\,dx = 2\pi \int_1^2 1\,dx$
$= 2\pi\left[x\right]_1^2 = 2\pi\,(2-1) = 2\pi$

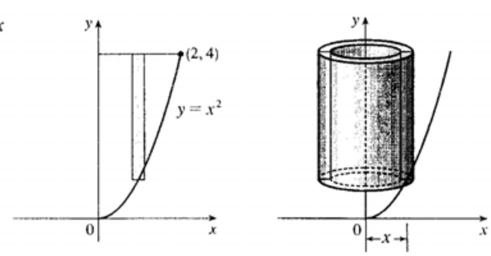


4. $V = \int_0^1 2\pi x \cdot x^2\,dx = 2\pi \int_0^1 x^3\,dx$
$= 2\pi\left[\frac{1}{4}x^4\right]_0^1 = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$



5. $V = \int_0^2 2\pi x\left(4 - x^2\right) dx = 2\pi \int_0^2 \left(4x - x^3\right) dx$
$= 2\pi\left[2x^2 - \frac{1}{4}x^4\right]_0^2 = 2\pi\,(8-4)$
$= 8\pi$

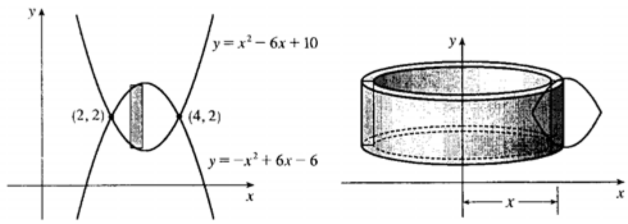
Note: If we integrated from -2 to 2, we would be generating the volume twice.



6. $V = \int_2^4 2\pi x \left[\left(-x^2 + 6x - 6\right) - \left(x^2 - 6x + 10\right)\right] dx = 2\pi \int_2^4 x\left(-2x^2 + 12x - 16\right) dx$

$= 4\pi \int_2^4 \left(-x^3 + 6x^2 - 8x\right) dx = 4\pi \left[-\frac{1}{4}x^4 + 2x^3 - 4x^2\right]_2^4$

$= 4\pi\left[(-64 + 128 - 64) - (-4 + 16 - 16)\right] = 16\pi$

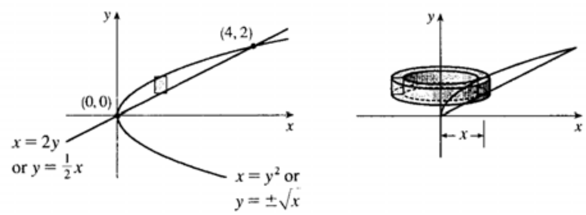


7. $V = \int_0^4 2\pi x\left(\sqrt{x} - \frac{1}{2}x\right) dx$

$= 2\pi \int_0^4 x^{3/2}\, dx - \pi \int_0^4 x^2\, dx$

$= 2\pi \left[\frac{2}{5}x^{5/2}\right]_0^4 - \pi\left[\frac{1}{3}x^3\right]_0^4$

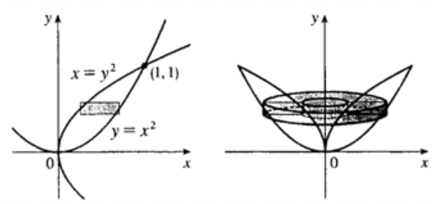
$= \frac{4}{5}\pi\,(32) - \frac{64}{3}\pi = \frac{64}{15}\pi$



8. By slicing:

$V = \int_0^1 \pi\left[\left(\sqrt{y}\right)^2 - \left(y^2\right)^2\right] dy = \pi \int_0^1 \left(y - y^4\right) dy$

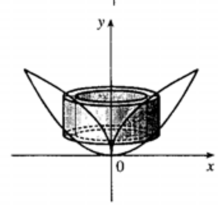
$= \pi\left[\frac{1}{2}y^2 - \frac{1}{5}y^5\right]_0^1 = \pi\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3\pi}{10}$



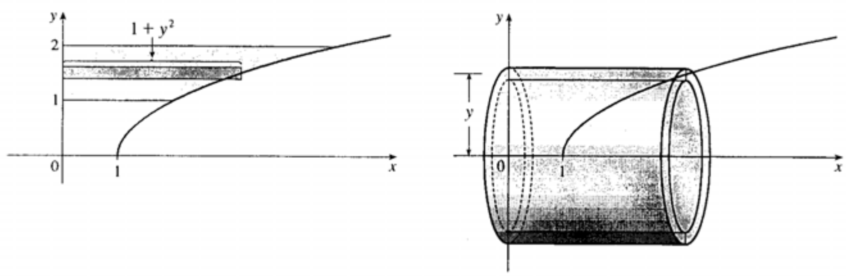
By cylindrical shells:

$V = \int_0^1 2\pi x\left(\sqrt{x} - x^2\right) dx = 2\pi \int_0^1 \left(x^{3/2} - x^3\right) dx$

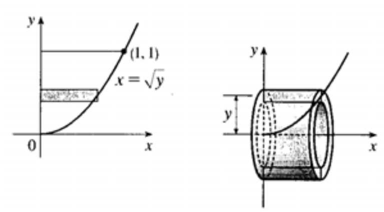
$= 2\pi\left[\frac{2}{5}x^{5/2} - \frac{1}{4}x^4\right]_0^1 = 2\pi\left(\frac{2}{5} - \frac{1}{4}\right) = 2\pi\left(\frac{3}{20}\right) = \frac{3\pi}{10}$



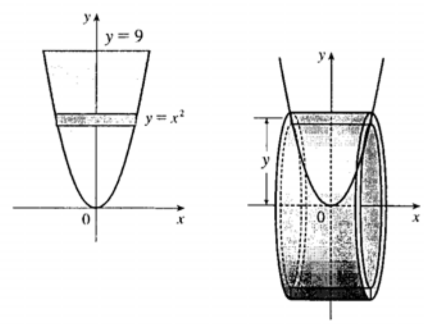
9. $V = \int_1^2 2\pi y\left(1+y^2\right)dy = 2\pi \int_1^2 \left(y+y^3\right)dy = 2\pi\left[\frac{1}{2}y^2+\frac{1}{4}y^4\right]_1^2$
$= 2\pi\left[(2+4)-\left(\frac{1}{2}+\frac{1}{4}\right)\right] = 2\pi\left(\frac{21}{4}\right) = \frac{21\pi}{2}$



10. $V = \int_0^1 2\pi y\sqrt{y}\,dy = 2\pi \int_0^1 y^{3/2}\,dy$
$= 2\pi\left[\frac{2}{5}y^{5/2}\right]_0^1 = \frac{4\pi}{5}$

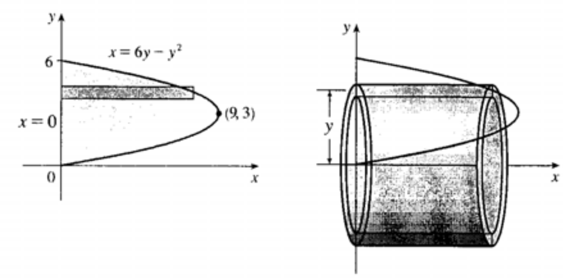


11. $V = \int_0^9 2\pi y \cdot 2\sqrt{y}\,dy$
$= 4\pi \int_0^9 y^{3/2}\,dy$
$= 4\pi\left[\frac{2}{5}y^{5/2}\right]_0^9$
$= \frac{8}{5}\pi\,(243-0)$
$= \frac{1944}{5}\pi$

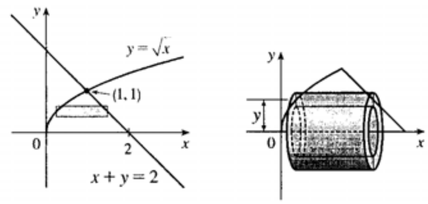


12. The two curves intersect at (0, 0) and (0, 6), so

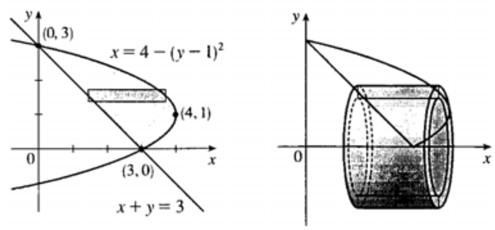
$V = \int_0^6 2\pi y\left(-y^2+6y\right)dy$
$= 2\pi\left[-\frac{1}{4}y^4+2y^3\right]_0^6$
$= 2\pi\,(-324+432) = 216\pi$



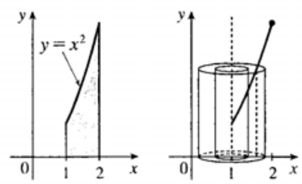
13. $V = \int_0^1 2\pi y\left[(2-y) - y^2\right]dy$
$= 2\pi\left[y^2 - \frac{1}{3}y^3 - \frac{1}{4}y^4\right]_0^1$
$= 2\pi\left(1 - \frac{1}{3} - \frac{1}{4}\right) = \frac{5}{6}\pi$



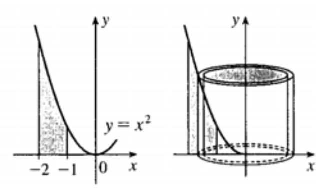
14. $V = \int_0^3 2\pi y\left[4 - (y-1)^2 - (3-y)\right]dy$
$= 2\pi\int_0^3 y\left(-y^2 + 3y\right)dy$
$= 2\pi\int_0^3\left(-y^3 + 3y^2\right)dy = 2\pi\left[-\frac{1}{4}y^4 + y^3\right]_0^3$
$= 2\pi\left(-\frac{81}{4} + 27\right) = 2\pi\left(\frac{27}{4}\right) = \frac{27\pi}{2}$



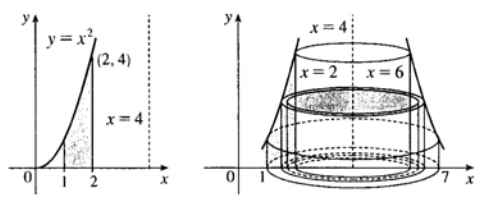
15. $V = \int_1^2 2\pi\,(x-1)\,x^2\,dx = 2\pi\left[\frac{1}{4}x^4 - \frac{1}{3}x^3\right]_1^2$
$= 2\pi\left[\left(4 - \frac{8}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right)\right] = \frac{17}{6}\pi$



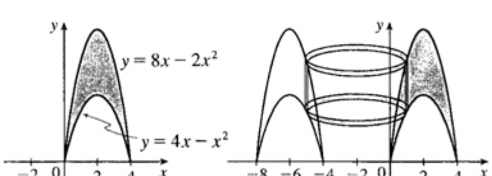
16. $V = \int_{-2}^{-1} 2\pi\,(-x)\cdot x^2\,dx = 2\pi\left[-\frac{1}{4}x^4\right]_{-2}^{-1}$
$= 2\pi\left[\left(-\frac{1}{4}\right) - (-4)\right] = \frac{15}{2}\pi$



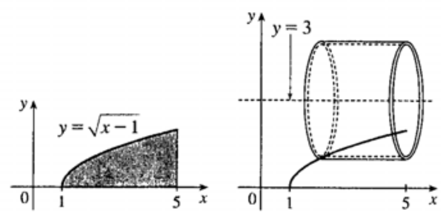
17. $V = \int_1^2 2\pi\,(4-x)\,x^2\,dx = 2\pi\left[\frac{4}{3}x^3 - \frac{1}{4}x^4\right]_1^2$
$= 2\pi\left[\left(\frac{32}{3} - 4\right) - \left(\frac{4}{3} - \frac{1}{4}\right)\right] = \frac{67}{6}\pi$



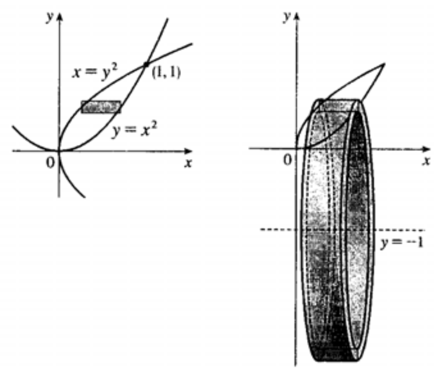
18. $V = \int_0^4 2\pi\,(2+x)\left[\left(8x - 2x^2\right) - \left(4x - x^2\right)\right]dx$
$= \int_0^4 2\pi\,(2+x)\left(4x - x^2\right)dx$
$= 2\pi\int_0^4\left(8x + 2x^2 - x^3\right)dx$
$= 2\pi\left[4x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^4$
$= 2\pi\left(64 + \frac{128}{3} - 64\right) = \frac{256}{3}\pi$



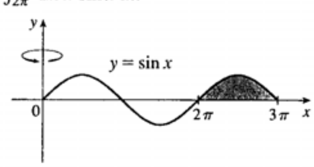
19. $V = \int_0^2 2\pi\,(3-y)\,(5-x)\,dy$
$= \int_0^2 2\pi\,(3-y)\left(5-y^2-1\right)dy$
$= \int_0^2 2\pi\left(12-4y-3y^2+y^3\right)dy$
$= 2\pi\left[12y-2y^2-y^3+\frac{1}{4}y^4\right]_0^2$
$= 2\pi\,(24-8-8+4) = 24\pi$



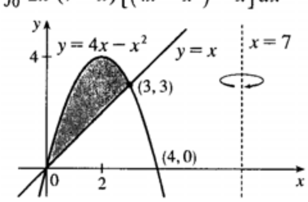
20. $V = \int_0^1 2\pi\,(y+1)\left(\sqrt{y}-y^2\right)dy$
$= 2\pi\int_0^1\left(y^{3/2}+y^{1/2}-y^3-y^2\right)dy$
$= 2\pi\left[\frac{2}{5}y^{5/2}+\frac{2}{3}y^{3/2}-\frac{1}{4}y^4-\frac{1}{3}y^3\right]_0^1$
$= 2\pi\left(\frac{2}{5}+\frac{2}{3}-\frac{1}{4}-\frac{1}{3}\right) = 2\pi\left(\frac{29}{60}\right) = \frac{29\pi}{30}$



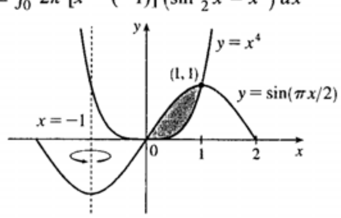
21. $V = \int_{2\pi}^{3\pi} 2\pi x \sin x\,dx$



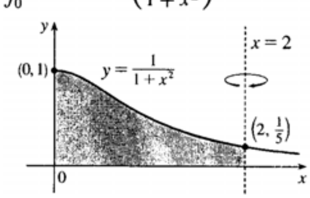
22. $V = \int_0^3 2\pi\,(7-x)\left[\left(4x-x^2\right)-x\right]dx$



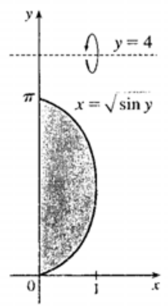
23. $V = \int_0^1 2\pi\,[x-(-1)]\left(\sin\frac{\pi}{2}x-x^4\right)dx$



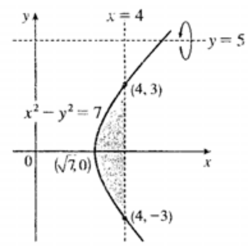
24. $V = \displaystyle\int_0^2 2\pi\,(2-x)\left(\frac{1}{1+x^2}\right)dx$



25. $V = \int_0^{\pi} 2\pi\,(4-y)\,\sqrt{\sin y}\,dy$



26. $V = \int_{-3}^{3} 2\pi\,(5-y)\left(4-\sqrt{y^2+7}\right)dy$



27. $\Delta x = \dfrac{\pi/4 - 0}{4} = \dfrac{\pi}{16}$.

$V = \int_0^{\pi/4} 2\pi x \tan x\,dx \approx 2\pi \cdot \frac{\pi}{16}\left(\frac{\pi}{32}\tan\frac{\pi}{32} + \frac{3\pi}{32}\tan\frac{3\pi}{32} + \frac{5\pi}{32}\tan\frac{5\pi}{32} + \frac{7\pi}{32}\tan\frac{7\pi}{32}\right) \approx 1.142$

28. $\Delta x = \dfrac{12-2}{5} = 2$, $n = 5$ and $x_i^* = 2 + (2i+1)$, where $i = 0, 1, 2, 3, 4$. The values of $f(x)$ are taken directly from the diagram.

$$\begin{aligned} V &= \int_2^{12} 2\pi x f(x)\,dx \approx 2\pi\,[3f(3) + 5f(5) + 7f(7) + 9f(9) + 11f(11)] \cdot 2 \\ &\approx 2\pi\,[3(2) + 5(4) + 7(4) + 9(2) + 11(1)]\,2 = 332\pi \end{aligned}$$

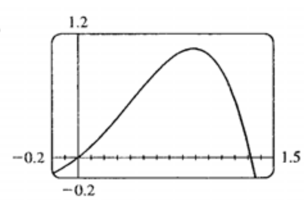
29. The solid is obtained by rotating the region bounded by the curve $y = \cos x$ and the line $y = 0$, from $x = 0$ to $x = \frac{\pi}{2}$, about the y-axis.

30. The solid is obtained by rotating the region bounded by the curve $x = \sqrt{y}$ and the lines $y = 9$ and $x = 0$ about the x-axis.

31. The solid is obtained by rotating the region in the first quadrant bounded by the curves $y = x^2$ and $y = x^6$ about the y-axis.

32. The solid is obtained by rotating the region under the curve $y = \sin^4 x$, above $y = 0$, from $x = 0$ to $x = \pi$, about the line $x = 4$.

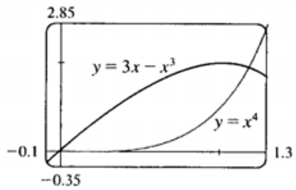
33.



From the graph, it appears that the curves intersect at $x = 0$ and at $x \approx 1.32$, with $x + x^2 - x^4 > 0$ on $(0, 1.32)$. So the volume of the solid obtained by rotating the region about the y-axis is

$V \approx 2\pi \int_0^{1.32} x\left(x + x^2 - x^4\right)dx = 2\pi\left[\frac{1}{3}x^3 + \frac{1}{4}x^4 - \frac{1}{6}x^6\right]_0^{1.32} \approx 4.05.$

34.



From the graph, it appears that the curves intersect at $x = 0$ and at $x \approx 1.17$, with $3x - x^3 > x^4$ on $(0, 1.17)$. So the volume of the solid obtained by rotation about the y-axis is

$$V \approx 2\pi \int_0^{1.17} x\left[(3x - x^3) - x^4\right] dx = 2\pi\left[x^3 - \tfrac{1}{5}x^5 - \tfrac{1}{6}x^6\right]_0^{1.17} \approx 4.62.$$

35. Use disks:

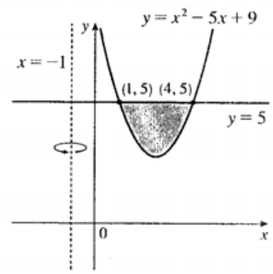
$$\begin{aligned} V &= \int_{-2}^{1} \pi\left(x^2 + x - 2\right)^2 dx = \pi\int_{-2}^{1}\left(x^4 + 2x^3 - 3x^2 - 4x + 4\right) dx \\ &= \pi\left[\tfrac{1}{5}x^5 + \tfrac{1}{2}x^4 - x^3 - 2x^2 + 4x\right]_{-2}^{1} = \pi\left[\left(\tfrac{1}{5} + \tfrac{1}{2} - 1 - 2 + 4\right) - \left(-\tfrac{32}{5} + 8 + 8 - 8 - 8\right)\right] \\ &= \pi\left(\tfrac{33}{5} + \tfrac{3}{2}\right) = \tfrac{81}{10}\pi \end{aligned}$$

36. Use shells:

$$\begin{aligned} V &= \int_1^2 2\pi x\left(-x^2 + 3x - 2\right) dx = 2\pi\int_1^2\left(-x^3 + 3x^2 - 2x\right) dx \\ &= 2\pi\left[-\tfrac{1}{4}x^4 + x^3 - x^2\right]_1^2 = 2\pi\left[(-4 + 8 - 4) - \left(-\tfrac{1}{4} + 1 - 1\right)\right] = \tfrac{\pi}{2} \end{aligned}$$

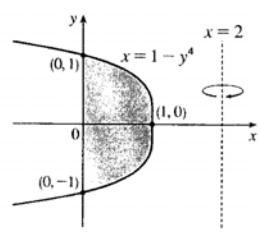
37. Use shells:

$$\begin{aligned} V &= \int_1^4 2\pi\,[x - (-1)]\left[5 - \left(x^2 - 5x + 9\right)\right] dx \\ &= 2\pi\int_1^4 (x + 1)\left(-x^2 + 5x - 4\right) dx \\ &= 2\pi\int_1^4\left(-x^3 + 4x^2 + x - 4\right) dx = 2\pi\left[-\tfrac{1}{4}x^4 + \tfrac{4}{3}x^3 + \tfrac{1}{2}x^2 - 4x\right]_1^4 \\ &= 2\pi\left[\left(-64 + \tfrac{256}{3} + 8 - 16\right) - \left(-\tfrac{1}{4} + \tfrac{4}{3} + \tfrac{1}{2} - 4\right)\right] \\ &= 2\pi\left(\tfrac{63}{4}\right) = \tfrac{63\pi}{2} \end{aligned}$$



38. Use washers:

$$\begin{aligned} V &= \int_{-1}^{1} \pi\left\{[2 - 0]^2 - \left[2 - \left(1 - y^4\right)\right]^2\right\} dy \\ &= 2\pi\int_0^1\left[4 - \left(1 + y^4\right)^2\right] dy \quad \text{[by symmetry]} \\ &= 2\pi\int_0^1\left[4 - \left(1 + 2y^4 + y^8\right)\right] dy = 2\pi\int_0^1\left(3 - 2y^4 - y^8\right) dy \\ &= 2\pi\left[3y - \tfrac{2}{5}y^5 - \tfrac{1}{9}y^9\right]_0^1 = 2\pi\left(3 - \tfrac{2}{5} - \tfrac{1}{9}\right) = 2\pi\left(\tfrac{112}{45}\right) = \tfrac{224\pi}{45} \end{aligned}$$



39. Use disks: $V = \pi\displaystyle\int_0^2\left[\sqrt{1 - (y - 1)^2}\right]^2 dy = \pi\int_0^2\left(2y - y^2\right) dy = \pi\left[y^2 - \tfrac{1}{3}y^3\right]_0^2 = \pi\left(4 - \frac{8}{3}\right) = \frac{4}{3}\pi$

40. Using shells, we have

$$V = 2\pi \int_0^2 y \cdot 2\sqrt{1-(y-1)^2}\,dy = 4\pi \int_{-1}^{1} (u+1)\sqrt{1-u^2}\,du \quad [\text{Put } u = y-1]$$
$$= 4\pi \int_{-1}^{1} \sqrt{1-u^2}\,du - 4\pi \int_{-1}^{1} u\sqrt{1-u^2}\,du$$

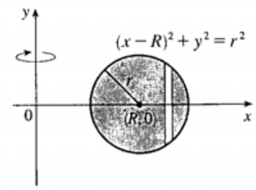
The first definite integral is the area of a semicircle of radius 1, that is, $\frac{\pi}{2}$. The second equals zero because its integrand is an odd function. Thus, $V = 4\pi\frac{\pi}{2} - 4\pi \cdot 0 = 2\pi^2$.

41. $V = 2\int_0^r 2\pi x\sqrt{r^2-x^2}\,dx = -2\pi \int_0^r \left(r^2-x^2\right)^{1/2}(-2x)\,dx = \left[-2\pi \cdot \frac{2}{3}\left(r^2-x^2\right)^{3/2}\right]_0^r = -\frac{4}{3}\pi\left(0-r^3\right) = \frac{4}{3}\pi r^3$

42. $V = \int_{R-r}^{R+r} 2\pi x \cdot 2\sqrt{r^2-(x-R)^2}\,dx$

$= \int_{-r}^{r} 4\pi\,(u+R)\sqrt{r^2-u^2}\,du \quad [\text{Put } u = x-R]$

$= 4\pi R \int_{-r}^{r} \sqrt{r^2-u^2}\,du + 4\pi \int_{-r}^{r} u\sqrt{r^2-u^2}\,du$

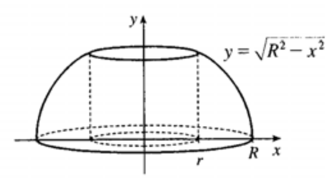


The first integral is the area of a semicircle of radius r, that is, $\frac{1}{2}\pi r^2$, and the second is zero since the integrand is an odd function. Thus,

$V = 4\pi R\left(\frac{1}{2}\pi r^2\right) + 4\pi \cdot 0 = 2\pi Rr^2$.

43. $V = 2\pi \displaystyle\int_0^r x\left(-\frac{h}{r}x+h\right)dx = 2\pi h \int_0^r \left(-\frac{x^2}{r}+x\right)dx = 2\pi h\left[-\frac{x^3}{3r}+\frac{x^2}{2}\right]_0^r = 2\pi h\frac{r^2}{6} = \frac{\pi r^2 h}{3}$

44. By symmetry, the volume of a napkin ring obtained by drilling a hole of radius r through a sphere with radius R is twice the volume obtained by rotating the area above the x-axis and below the curve $y = \sqrt{R^2-x^2}$ (the equation of the top half of the cross-section of the sphere), between $x = r$ and $x = R$, about the y-axis. This volume is equal to



$$2\int_{\text{inner radius}}^{\text{outer radius}} 2\pi rh\,dx = 2 \cdot 2\pi \int_r^R x\sqrt{R^2-x^2}\,dx = 4\pi\left[-\frac{1}{3}\left(R^2-x^2\right)^{3/2}\right]_r^R = \frac{4}{3}\pi\left(R^2-r^2\right)^{3/2}$$

But by the Pythagorean Theorem, $R^2 - r^2 = \left(\frac{1}{2}h\right)^2$, so the volume of the napkin ring is $\frac{4}{3}\pi\left(\frac{1}{2}h\right)^3 = \frac{1}{6}\pi h^3$, which is independent of both R and r; that is, the amount of wood in a napkin ring of height h is the same regardless of the size of the sphere used. Note that most of this calculation has been done already, but with more difficulty, in Exercise 6.2.66.

Another Solution: The height of the missing cap is the radius of the sphere minus half the height of the cut-out cylinder, that is, $R - \frac{1}{2}h$. Using Exercise 6.2.47,

$$V_{\text{napkin ring}} = V_{\text{sphere}} - V_{\text{cylinder}} - 2V_{\text{cap}} = \frac{4}{3}\pi R^3 - \pi r^2 h - 2 \cdot \frac{\pi}{3}\left(R-\frac{1}{2}h\right)^2\left[3R - \left(R-\frac{1}{2}h\right)\right] = \frac{1}{6}\pi h^3$$

6.4 Work

1. By Equation 2, $W = Fd = (900)(8) = 7200$ J.

2. $F = mg = (60)(9.8) = 588$ N; $W = Fd = 588 \cdot 2 = 1176$ J

3. By Equation 4,

$$W = \int_a^b f(x)\,dx = \int_0^9 \frac{10}{(1+x)^2}\,dx = 10\int_1^{10} \frac{1}{u^2}\,du \quad (u = 1+x,\ du = dx)$$
$$= 10\left[-\frac{1}{u}\right]_1^{10} = 10\left(-\tfrac{1}{10}+1\right) = 9 \text{ ft-lb}$$

4. $W = \int_1^2 \cos\left(\frac{1}{3}\pi x\right) dx = \frac{3}{\pi}\left[\sin\left(\frac{1}{3}\pi x\right)\right]_1^2 = \frac{3}{\pi}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = 0$ N-m $= 0$ J.

Interpretation: From $x = 1$ to $x = \frac{3}{2}$, the force does work equal to $\int_1^{3/2} \cos\left(\frac{1}{3}\pi x\right) dx = \frac{3}{\pi}\left(1 - \frac{\sqrt{3}}{2}\right)$ J in accelerating the particle and increasing its kinetic energy. From $x = \frac{3}{2}$ to $x = 2$, the force opposes the motion of the particle, decreasing its kinetic energy. This is negative work, equal in magnitude but opposite in sign to the work done from $x = 1$ to $x = \frac{3}{2}$.

5. $10 = f(x) = kx = \frac{1}{3}k$ (4 inches $= \frac{1}{3}$ foot), so $k = 30$ lb/ft and $f(x) = 30x$. Now 6 inches $= \frac{1}{2}$ foot, so $W = \int_0^{1/2} 30x\,dx = \left[15x^2\right]_0^{1/2} = \frac{15}{4}$ ft-lb.

6. $25 = f(x) = kx = k(0.1)$ (10 cm $= 0.1$ m), so $k = 250$ N/m and $f(x) = 250x$. Now 5 cm $= 0.05$ m, so $W = \int_0^{0.05} 250x\,dx = \left[125x^2\right]_0^{0.05} = 125(0.0025) = 0.3125 \approx 0.31$ J.

7. If $\int_0^{0.12} kx\,dx = 2$ J, then $2 = \left[\frac{1}{2}kx^2\right]_0^{0.12} = \frac{1}{2}k(0.0144) = 0.0072k$ and $k = \frac{2}{0.0072} = \frac{2500}{9} \approx 277.78$. Thus, the work needed to stretch the spring from 35 cm to 40 cm is
$\int_{0.05}^{0.10} \frac{2500}{9}x\,dx = \left[\frac{1250}{9}x^2\right]_{1/20}^{1/10} = \frac{1250}{9}\left(\frac{1}{100} - \frac{1}{400}\right) = \frac{25}{24} \approx 1.04$ J.

8. If $12 = \int_0^1 kx\,dx = \left[\frac{1}{2}kx^2\right]_0^1 = \frac{1}{2}k$, then $k = 24$ and the work required is
$\int_0^{3/4} 24x\,dx = \left[12x^2\right]_0^{3/4} = 12 \cdot \frac{9}{16} = \frac{27}{4} = 6.75$ ft-lb.

9. $f(x) = kx$, so $30 = \frac{2500}{9}x$ and $x = \frac{270}{2500}$ m $= 10.8$ cm

10. Let L be the natural length of the spring in meters.
Then $6 = \int_{0.10-L}^{0.12-L} kx\,dx = \left[\frac{1}{2}kx^2\right]_{0.10-L}^{0.12-L} = \frac{1}{2}k\left[(0.12-L)^2 - (0.10-L)^2\right]$ and
$10 = \int_{0.12-L}^{0.14-L} kx\,dx = \left[\frac{1}{2}kx^2\right]_{0.12-L}^{0.14-L} = \frac{1}{2}k\left[(0.14-L)^2 - (0.12-L)^2\right]$. In other words,
$12 = k(0.0044 - 0.04L)$ and $20 = k(0.0052 - 0.04L)$. Subtracting the first equation from the second gives $8 = 0.0008k$, so $k = 10{,}000$. Now the second equation becomes $20 = 52 - 400L$, so $L = \frac{32}{400}$ m $= 8$ cm.

In Exercises 11–16, n is the number of subintervals of length Δx, and x_i^* is a sample point in the ith subinterval $[x_{i-1}, x_i]$.

11. First notice that the exact height of the building does not matter (as long as it is more than 50 ft). The portion of the rope from x ft to $(x + \Delta x)$ ft below the top of the building weighs $\frac{1}{2}\Delta x$ lb and must be lifted x_i^* ft, so its contribution to the total work is $\frac{1}{2}x_i^*\Delta x$ ft-lb. The total work is

$$W = \lim_{n\to\infty}\sum_{i=1}^{n}\tfrac{1}{2}x_i^*\Delta x = \int_0^{50}\tfrac{1}{2}x\,dx = \left[\tfrac{1}{4}x^2\right]_0^{50} = \tfrac{2500}{4} = 625 \text{ ft-lb}$$

12. Each part of the top 10 ft of cable is lifted a distance x_i^* equal to its distance from the top. The cable weighs $\frac{60}{40} = 1.5$ lb/ft, so the work done on the ith subinterval is $\frac{3}{2}x_i^*\Delta x$. The remaining 30 ft of cable is lifted 10 ft. Thus,

$$W = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\tfrac{3}{2}x_i^*\Delta x + \tfrac{3}{2}\cdot 10\Delta x\right) = \int_0^{10}\tfrac{3}{2}x\,dx + \int_{10}^{40}\tfrac{3}{2}\cdot 10\,dx = \left[\tfrac{3}{4}x^2\right]_0^{10} + [15x]_{10}^{40}$$

$$= \tfrac{3}{4}(100) + 15(30) = 75 + 450 = 525 \text{ ft-lb}$$

13. The work needed to lift the cable is $\lim_{n\to\infty}\sum_{i=1}^{n} 2x_i^*\Delta x = \int_0^{500} 2x\,dx = \left[x^2\right]_0^{500} = 250{,}000$ ft-lb. The work needed to lift the coal is 800 lb · 500 ft = 400,000 ft-lb. Thus, the total work required is $250{,}000 + 400{,}000 = 650{,}000$ ft-lb.

14. The work needed to lift the bucket itself is 4 lb · 80 ft = 320 ft-lb. At time t (in seconds) the bucket is $x_i^* = 2t$ ft above its original 80 ft depth, but it now holds only $(40 - 0.2t)$ lb of water. In terms of distance, the bucket holds $\left[40 - 0.2\left(\frac{1}{2}x_i^*\right)\right]$ lb of water when it is x_i^* ft above its original 80 ft depth. Moving this amount of water a distance Δx requires $\left(40 - \frac{1}{10}x_i^*\right)\Delta x$ ft-lb of work. Thus, the work needed to lift the water is

$$W = \lim_{n\to\infty}\sum_{i=1}^{n}\left(40 - \tfrac{1}{10}x_i^*\right)\Delta x = \int_0^{80}\left(40 - \tfrac{1}{10}x\right)dx = \left[40x - \tfrac{1}{20}x^2\right]_0^{80} = (3200 - 320) \text{ ft-lb}$$

Adding the work of lifting the bucket gives a total of 3200 ft-lb of work.

15. A "slice" of water Δx m thick and lying at a depth of x_i^* m (where $0 \le x_i^* \le \frac{1}{2}$) has volume $2\Delta x$ m^3, a mass of $2000\Delta x$ kg, weighs about $(9.8)(2000\Delta x) = 19{,}600\Delta x$ N, and thus requires about $19{,}600x_i^*\Delta x$ J of work for its removal. So

$$W = \lim_{n\to\infty}\sum_{i=1}^{n} 19{,}600x_i^*\Delta x = \int_0^{1/2} 19{,}600x\,dx = \left[9800x^2\right]_0^{1/2} = 2450 \text{ J}$$

16. A horizontal cylindrical slice of water Δx ft thick has a volume of $\pi r^2 h = \pi\cdot 12^2\cdot\Delta x$ ft^3 and weighs about $(62.5 \text{ lb/ft}^3)(144\pi\,\Delta x \text{ ft}^3) = 9000\pi\,\Delta x$ lb. If the slice lies x_i^* ft below the edge of the pool (where $1 \le x_i^* \le 5$), then the work needed to pump it out is about $9000\pi x_i^*\Delta x$. Thus,

$$W = \lim_{n\to\infty}\sum_{i=1}^{n} 9000\pi x_i^*\Delta x = \int_1^5 9000\pi x\,dx = \left[4500\pi x^2\right]_1^5 = 4500\pi(25 - 1) = 108{,}000\pi \text{ ft-lb}$$

17. A rectangular "slice" of water Δx m thick and lying x ft above the bottom has width x ft and volume $8x\,\Delta x$ m^3. It weighs about $(9.8\times 10^3)(8x\,\Delta x)$ N, and must be lifted $(5 - x)$ m by the pump, so the work needed is about $(9.8\times 10^3)(5 - x)(8x\,\Delta x)$J. The total work required is

$$W \approx \int_0^3 (9.8\times 10^3)(5 - x)\,8x\,dx = (9.8\times 10^3)\int_0^3 (40x - 8x^2)\,dx = (9.8\times 10^3)\left[20x^2 - \tfrac{8}{3}x^3\right]_0^3$$

$$= \left(9.8\times 10^3\right)(180 - 72) = \left(9.8\times 10^3\right)(108) = 1058.4\times 10^3 \approx 1.06\times 10^6 \text{ J}$$

18. For convenience, measure depth x from the middle of the tank, so that $-1.5 \le x \le 1.5$ m. Lifting a slice of water of thickness Δx at depth x requires a work contribution of $\Delta W \approx (9.8 \times 10^3)\left(2\sqrt{(1.5)^2 - x^2}\right)(6\Delta x)(2.5 + x)$, so

$$W \approx \int_{-1.5}^{1.5} (9.8 \times 10^3)\, 12\sqrt{2.25 - x^2}\,(2.5 + x)\,dx$$
$$= \left(9.8 \times 10^3\right)\left[60\int_0^{3/2}\sqrt{\tfrac{9}{4} - x^2}\,dx + 12\int_{-3/2}^{3/2} x\sqrt{\tfrac{9}{4} - x^2}\,dx\right]$$

The second integral is 0 because its integrand is an odd function, and the first integral represents the area of a quarter-circle of radius $\frac{3}{2}$. Therefore,

$$W \approx \left(9.8 \times 10^3\right) 60\int_0^{3/2}\sqrt{\tfrac{9}{4} - x^2}\,dx = (9.8 \times 10^3)(60)\left(\tfrac{9}{8}\right)\left(\tfrac{\pi}{2}\right) = 330{,}750\pi \approx 1.04 \times 10^6 \text{ J}$$

19. Measure depth x downward from the flat top of the tank, so that $0 \le x \le 2$ ft. Then $\Delta W = (62.5)\left(2\sqrt{4 - x^2}\right)(8\Delta x)(x + 1)$ ft-lb, so

$$W \approx (62.5)(16)\int_0^2 (x + 1)\sqrt{4 - x^2}\,dx = 1000\left(\int_0^2 x\sqrt{4 - x^2}\,dx + \int_0^2 \sqrt{4 - x^2}\,dx\right)$$
$$= 1000\left[\int_0^4 u^{1/2}\left(\tfrac{1}{2}\,du\right) + \tfrac{1}{4}\pi\left(2^2\right)\right] \quad (\text{Put } u = 4 - x^2, \text{ so } du = -2x\,dx)$$
$$= 1000\left(\left[\tfrac{1}{2} \cdot \tfrac{2}{3}u^{3/2}\right]_0^4 + \pi\right) = 1000\left(\tfrac{8}{3} + \pi\right) \approx 5.8 \times 10^3 \text{ ft-lb}$$

Note: The second integral represents the area of a quarter-circle of radius 2.

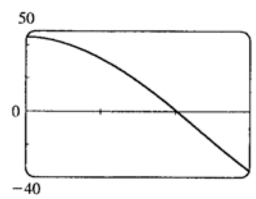
20. Let x be depth in feet, so that $0 \le x \le 5$. Then $\Delta W = (62.5)\,\pi\left(\sqrt{5^2 - x^2}\right)^2 \Delta x \cdot x$ ft-lb and

$$W \approx 62.5\pi\int_0^5 x\left(25 - x^2\right)dx = 62.5\pi\left[\tfrac{25}{2}x^2 - \tfrac{1}{4}x^4\right]_0^5 = 62.5\pi\left(\tfrac{625}{2} - \tfrac{625}{4}\right) = 62.5\pi\left(\tfrac{625}{4}\right)$$
$$\approx 3.07 \times 10^4 \text{ ft-lb}$$

21. If only 4.7×10^5 J of work is done, then only the water above a certain level (call it h) will be pumped out. So we use the same formula as in Exercise 17, except that the work is fixed, and we are trying to find the lower limit of integration: $4.7 \times 10^5 \approx \int_h^3 (9.8 \times 10^3)(5 - x)\,8x\,dx = (9.8 \times 10^3)\left[20x^2 - \frac{8}{3}x^3\right]_h^3 \quad \Leftrightarrow$

$\frac{4.7}{9.8} \times 10^2 \approx 48 = \left(20 \cdot 3^2 - \frac{8}{3} \cdot 3^3\right) - \left(20h^2 - \frac{8}{3}h^3\right) \quad \Leftrightarrow$

$2h^3 - 15h^2 + 45 = 0$. To find the solution of this equation, we plot $2h^3 - 15h^2 + 45$ between $h = 0$ and $h = 3$. We see that the equation is satisfied for $h \approx 2.0$. So the depth of water remaining in the tank is about 2.0 m.



22. $W \approx (9.8 \times 920)\int_0^{3/2} 12\sqrt{\frac{9}{4} - x^2}\left(\frac{5}{2} + x\right)dx = 9016\left[30\int_0^{3/2}\sqrt{\frac{9}{4} - x^2}\,dx + 12\int_0^{3/2} x\sqrt{\frac{9}{4} - x^2}\,dx\right]$.

Here $\int_0^{3/2}\sqrt{\frac{9}{4} - x^2}\,dx = \frac{1}{4}\pi\left(\frac{3}{2}\right)^2 = \frac{9\pi}{16}$ and

$\int_0^{3/2} x\sqrt{\frac{9}{4} - x^2}\,dx = \int_0^{9/4} \frac{1}{2}u^{1/2}\,du$ (where $u = \frac{9}{4} - x^2$, so $du = -2x\,dx$) $= \left[\frac{1}{3}u^{3/2}\right]_0^{9/4} = \frac{1}{3}\left(\frac{27}{8}\right) = \frac{9}{8}$, so

$W \approx 9016\left[30 \cdot \frac{9}{16}\pi + 12 \cdot \frac{9}{8}\right] = 9016\left(\frac{135}{8}\pi + \frac{27}{2}\right) \approx 6.00 \times 10^5$ J.

23. $V = \pi r^2 x$, so V is a function of x and P can also be regarded as a function of x. If $V_1 = \pi r^2 x_1$ and $V_2 = \pi r^2 x_2$, then

$W = \int_{x_1}^{x_2} F(x)\,dx = \int_{x_1}^{x_2} \pi r^2 P(V(x))\,dx = \int_{x_1}^{x_2} P(V(x))\,dV(x)$ [Put $V(x) = \pi r^2 x$, so $dV(x) = \pi r^2\,dx$]

$= \int_{V_1}^{V_2} P(V)\,dV$ by the Substitution Rule.

24. $160\text{ lb/in}^2 = 160 \cdot 144\text{ lb/ft}^2$, $100\text{ in}^3 = \frac{100}{1728}\text{ ft}^3$, and $800\text{ in}^3 = \frac{800}{1728}\text{ ft}^3$.

$k = PV^{1.4} = (160 \cdot 144)\left(\frac{100}{1728}\right)^{1.4} = 23{,}040\left(\frac{25}{432}\right)^{1.4} \approx 426.5$. Therefore, $P \approx 426.5V^{-1.4}$ and

$$W = \int_{100/1728}^{800/1728} 426.5V^{-1.4}dV = 426.5\left[\frac{1}{-0.4}V^{-0.4}\right]_{25/432}^{25/54} = (426.5)(2.5)\left[\left(\frac{432}{25}\right)^{0.4} - \left(\frac{54}{25}\right)^{0.4}\right]$$

$\approx 1.88 \times 10^3$ ft-lb

25. $W = \displaystyle\int_a^b F(r)\,dr = \int_a^b G\frac{m_1 m_2}{r^2}\,dr = Gm_1 m_2\left[\frac{-1}{r}\right]_a^b = Gm_1 m_2\left(\frac{1}{a} - \frac{1}{b}\right)$

26. By Exercise 25, $W = GMm\left(\dfrac{1}{R} - \dfrac{1}{R + 1{,}000{,}000}\right)$ where M = mass of earth in kg, R = radius of earth in m, and m = mass of satellite in kg. (Note that 1000 km = 1,000,000 m.) Thus,

$W = (6.67 \times 10^{-11})(5.98 \times 10^{24})(1000) \times \left(\dfrac{1}{6.37 \times 10^6} - \dfrac{1}{7.37 \times 10^6}\right) \approx 8.50 \times 10^9$ J.

6.5 Average Value of a Function

1. $f_{\text{ave}} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx = \dfrac{1}{1-(-1)}\int_{-1}^{1} x^2\,dx = \frac{1}{2} \cdot 2\int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$

2. $f_{\text{ave}} = \frac{1}{2-0}\int_0^2 (x - x^2)\,dx = \frac{1}{2}\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^2 = \frac{1}{2}\left(2 - \frac{8}{3}\right) = -\frac{1}{3}$

3. $g_{\text{ave}} = \frac{1}{\pi/2 - 0}\int_0^{\pi/2} \cos x\,dx = \frac{2}{\pi}\left[\sin x\right]_0^{\pi/2} = \frac{2}{\pi}(1 - 0) = \frac{2}{\pi}$

4. $g_{\text{ave}} = \frac{1}{4-1}\int_1^4 \sqrt{x}\,dx = \frac{1}{3}\left[\frac{2}{3}x^{3/2}\right]_1^4 = \frac{2}{9}\left[x^{3/2}\right]_1^4 = \frac{2}{9}(8 - 1) = \frac{14}{9}$

5. $f_{\text{ave}} = \frac{1}{5-0}\int_0^5 t\sqrt{1+t^2}\,dt = \frac{1}{5}\int_1^{26} \sqrt{u}\left(\frac{1}{2}\,du\right)$ [$u = 1 + t^2$, $du = 2t\,dt$]

$= \frac{1}{10}\int_1^{26} u^{1/2}\,du = \frac{1}{10} \cdot \frac{2}{3}\left[u^{3/2}\right]_1^{26} = \frac{1}{15}\left(26^{3/2} - 1\right)$

6. $f_{\text{ave}} = \frac{1}{\pi/4 - 0}\int_0^{\pi/4} \sec\theta\tan\theta\,d\theta = \frac{4}{\pi}\left[\sec\theta\right]_0^{\pi/4} = \frac{4}{\pi}\left(\sqrt{2} - 1\right)$

7. $h_{\text{ave}} = \frac{1}{\pi - 0}\int_0^{\pi} \cos^4 x\sin x\,dx = \frac{1}{\pi}\int_1^{-1} u^4(-du)$ [$u = \cos x$, $du = -\sin x\,dx$] $= \frac{1}{\pi}\int_{-1}^{1} u^4\,du$

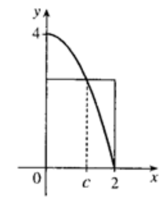
$= \frac{1}{\pi} \cdot 2\int_0^1 u^4 du = \frac{2}{\pi}\left[\frac{1}{5}u^5\right]_0^1 = \frac{2}{5\pi}$

8. $h_{\text{ave}} = \dfrac{1}{6-1}\displaystyle\int_1^6 \frac{3}{(1+r)^2}\,dr = \frac{1}{5}\int_2^7 3u^{-2}\,du$ [$u = 1 + r$, $du = dr$] $= -\frac{3}{5}\left[u^{-1}\right]_2^7 = -\frac{3}{5}\left(\frac{1}{7} - \frac{1}{2}\right)$

$= \frac{3}{5}\left(\frac{1}{2} - \frac{1}{7}\right) = \frac{3}{5} \cdot \frac{5}{14} = \frac{3}{14}$

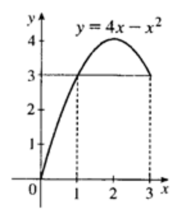
9. (a) $f_{\text{ave}} = \frac{1}{2-0}\int_0^2 (4-x^2)\,dx$
$= \frac{1}{2}\left[4x - \frac{1}{3}x^3\right]_0^2$
$= \frac{1}{2}\left(8 - \frac{8}{3}\right) = \frac{8}{3}$

(b) $f_{\text{ave}} = f(c) \iff \frac{8}{3} = 4 - c^2 \iff c^2 = \frac{4}{3} \iff c = \frac{2}{\sqrt{3}} \approx 1.15$

(c)


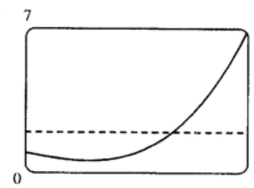
10. (a) $f_{\text{ave}} = \frac{1}{3-0}\int_0^3 (4x-x^2)\,dx$
$= \frac{1}{3}\left[2x^2 - \frac{1}{3}x^3\right]_0^3$
$= \frac{1}{3}(18-9) = 3$

(b) $f_{\text{ave}} = f(c) \iff 3 = 4c - c^2 \iff c^2 - 4c + 3 = 0 \iff c = 1$ or 3

(c)


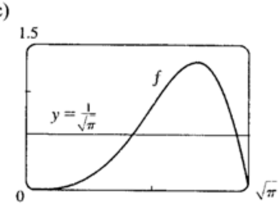
11. (a) $f_{\text{ave}} = \frac{1}{2-0}\int_0^2 (x^3 - x + 1)\,dx$
$= \frac{1}{2}\left[\frac{1}{4}x^4 - \frac{1}{2}x^2 + x\right]_0^2$
$= \frac{1}{2}(4 - 2 + 2) = 2$

(b) From the graph, $f(x) = 2$ at $x \approx 1.32$.

(c)


12. (a) $f_{\text{ave}} = \frac{1}{\sqrt{\pi}-0}\int_0^{\sqrt{\pi}} \left[x\sin(x^2)\right]dx$
$= \frac{1}{\sqrt{\pi}}\left[-\frac{1}{2}\cos(x^2)\right]_0^{\sqrt{\pi}}$
$= -\frac{1}{2\sqrt{\pi}}(\cos\pi - \cos 0) = \frac{1}{\sqrt{\pi}} \approx 0.56$

(b) From the graph, $f(x) = \frac{1}{\sqrt{\pi}}$ at $x \approx 0.85$ and at $x \approx 1.67$.

(c)


13. Since f is continuous on $[1, 3]$, by the Mean Value Theorem for Integrals there exists a number c in $[1, 3]$ such that $\int_1^3 f(x)\,dx = f(c)(3-1) \Rightarrow 8 = 2f(c)$; that is, there is a number c such that $f(c) = \frac{8}{2} = 4$.

14. The requirement is that $\dfrac{1}{b-0}\displaystyle\int_0^b f(x)\,dx = 3$. The LHS of this equation is equal to $\dfrac{1}{b}\displaystyle\int_0^b \left(2 + 6x - 3x^2\right)dx = \dfrac{1}{b}\left[2x + 3x^2 - x^3\right]_0^b = 2 + 3b - b^2$, so we solve the equation $2 + 3b - b^2 = 3 \iff b^2 - 3b + 1 = 0 \iff b = \dfrac{3 \pm \sqrt{(-3)^2 - 4\cdot1\cdot1}}{2\cdot1} = \dfrac{3\pm\sqrt{5}}{2}$. Both roots are valid since they are both positive.

15. $T_{\text{ave}} = \frac{1}{12}\int_0^{12}\left[50 + 14\sin\frac{1}{12}\pi t\right]dt = \frac{1}{12}\left[50t - 14\cdot\frac{12}{\pi}\cos\frac{1}{12}\pi t\right]_0^{12}$

$= \frac{1}{12}\left[50\cdot 12 + 14\cdot\frac{12}{\pi} + 14\cdot\frac{12}{\pi}\right] = \left(50 + \frac{28}{\pi}\right)^\circ\text{F} \approx 59^\circ\text{F}$

16. $T_{\text{ave}} = \frac{1}{5}\int_0^5 4x\,dx = \frac{1}{5}\left[2x^2\right]_0^5 = 10^\circ\text{C}$

17. $\rho_{\text{ave}} = \dfrac{1}{8}\displaystyle\int_0^8 \frac{12}{\sqrt{x+1}}\,dx = \frac{3}{2}\int_0^8 (x+1)^{-1/2}\,dx = \left[3\sqrt{x+1}\right]_0^8 = 9 - 3 = 6\text{ kg/m}$

18. $s(t) = \frac{1}{2}gt^2 \Rightarrow v(t) = s'(t) = gt \Rightarrow v_T = v(T) = gT$, and $s(T) = \frac{1}{2}gT^2$. Also, $s'(t) = gt = g\sqrt{2s/g} = \sqrt{2gs} = v(s)$. The average of the velocities with respect to time during the interval $[0, T]$ is $v_{\text{ave}} = \dfrac{1}{T}\displaystyle\int_0^T v(t)\,dt = \frac{1}{T}[s(T) - s(0)]$ (by FTC) $= \dfrac{1}{2}\cdot\dfrac{1}{T}gT^2 = \frac{1}{2}gT = \frac{1}{2}v_T$. But with respect to s,

$$v_{\text{ave}} = \frac{1}{s(T)}\int_0^{s(T)} v(s)\,ds = \frac{1}{s(T)}\int_0^{s(T)}\sqrt{2gs}\,ds = \frac{2}{gT^2}\sqrt{2g}\int_0^{s(T)} s^{1/2}\,ds$$
$$= \frac{2\sqrt{2}}{\sqrt{g}T^2}\left(\tfrac{2}{3}\right)\left[s^{3/2}\right]_0^{gT^2/2} = \frac{4\sqrt{2}}{3\sqrt{g}T^2}\left(\tfrac{1}{2}gT^2\right)^{3/2} = \tfrac{2}{3}gT = \tfrac{2}{3}v_T$$

19. $V_{\text{ave}} = \frac{1}{5}\int_0^5 V(t)\,dt = \frac{1}{5}\int_0^5 \frac{5}{4\pi}\left[1 - \cos\left(\frac{2}{5}\pi t\right)\right]dt = \frac{1}{4\pi}\int_0^5\left[1 - \cos\left(\frac{2}{5}\pi t\right)\right]dt$

$= \frac{1}{4\pi}\left[t - \frac{5}{2\pi}\sin\left(\frac{2}{5}\pi t\right)\right]_0^5 = \frac{1}{4\pi}[(5-0) - 0] = \frac{5}{4\pi} \approx 0.4\text{ L}$

20. $v_{\text{ave}} = \dfrac{1}{R-0}\displaystyle\int_0^R v(r)\,dr = \frac{1}{R}\int_0^R \frac{P}{4\eta l}\left(R^2 - r^2\right)dr = \frac{P}{4\eta lR}\left[R^2 r - \tfrac{1}{3}r^3\right]_0^R = \frac{P}{4\eta lR}\left(\tfrac{2}{3}\right)R^3 = \frac{PR^2}{6\eta l}$.

Since $v(r)$ is decreasing on $(0, R]$, $v_{\max} = v(0) = \dfrac{PR^2}{4\eta l}$. Thus, $v_{\text{ave}} = \frac{2}{3}v_{\max}$.

21. Let $F(x) = \int_a^x f(t)\,dt$ for x in $[a, b]$. Then F is continuous on $[a, b]$ and differentiable on (a, b), so by the Mean Value Theorem there is a number c in (a, b) such that $F(b) - F(a) = F'(c)(b-a)$. But $F'(x) = f(x)$ by the Fundamental Theorem of Calculus. Therefore, $\int_a^b f(t)\,dt - 0 = f(c)(b-a)$.

22. $f_{\text{ave}}[a, b] = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^c f(x)\,dx + \frac{1}{b-a}\int_c^b f(x)\,dx$

$= \dfrac{c-a}{b-a}\left[\dfrac{1}{c-a}\displaystyle\int_a^c f(x)\,dx\right] + \dfrac{b-c}{b-a}\left[\dfrac{1}{b-c}\displaystyle\int_c^b f(x)\,dx\right] = \dfrac{c-a}{b-a}f_{\text{ave}}[a, c] + \dfrac{b-c}{b-a}f_{\text{ave}}[c, b]$

6 Review

CONCEPT CHECK

1. (a) See Section 6.1, Figure 2 and Equations 6.1.1 and 6.1.2.

(b) Instead of using "top minus bottom" and integrating from left to right, we use "right minus left" and integrate from bottom to top. See Figures 11 and 12 in Section 6.1.

2. The numerical value of the area represents the number of meters by which Sue is ahead of Kathy after 1 minute.

3. (a) See the discussion in Section 6.2, near Figures 2 and 3, ending in the Definition of Volume.

(b) See the discussion between Examples 5 and 6 in Section 6.2. If the cross-section is a disk, find the radius in terms of x or y and use $A = \pi\,(\text{radius})^2$. If the cross-section is a washer, find the inner radius r_{in} and outer radius r_{out} and use $A = \pi\left(r_{\text{out}}^2\right) - \pi\left(r_{\text{in}}^2\right)$.

4. (a) $V = 2\pi rh\,\Delta r =$ (circumference) (height) (thickness)

(b) For a typical shell, find the circumference and height in terms of x or y and calculate $V = \int_a^b$ (circumference) (height) (dx or dy), where a and b are the limits on x or y.

(c) Sometimes slicing produces washers or disks whose radii are difficult (or impossible) to find explicitly. On other occasions, the cylindrical shell method leads to an easier integral than slicing does.

5. $\int_0^6 f(x)\,dx$ represents the amount of work done. Its units are newton-meters, or joules.

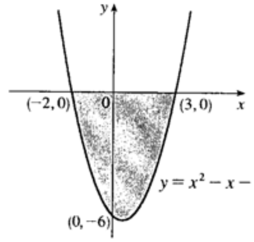
6. (a) See the boxed equation preceding Example 1 in Section 6.5.

(b) The Mean Value Theorem for Integrals says that there is a number c at which the value of f is exactly equal to the average value of the function, that is, $f(c) = f_{\text{ave}}$. For a geometric interpretation of the Mean Value Theorem for Integrals , see Figure 2 in Section 6.5 and the discussion which accompanies it.

EXERCISES

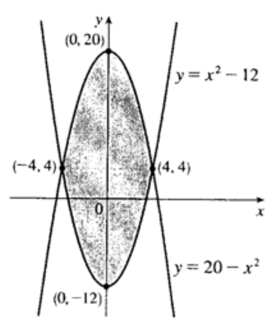
1. $0 = x^2 - x - 6 = (x-3)(x+2) \iff x = 3$ or -2. So

$$A = \int_{-2}^{3}\left[0 - \left(x^2 - x - 6\right)\right]dx = \int_{-2}^{3}\left(-x^2 + x + 6\right)dx$$
$$= \left[-\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + 6x\right]_{-2}^{3}$$
$$= \left(-9 + \tfrac{9}{2} + 18\right) - \left(\tfrac{8}{3} + 2 - 12\right)$$
$$= \tfrac{125}{6}$$



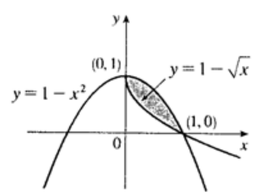
2. $20 - x^2 = x^2 - 12 \iff 32 = 2x^2 \iff x^2 = 16 \iff x = \pm 4$. So

$$A = \int_{-4}^{4}\left[\left(20 - x^2\right) - \left(x^2 - 12\right)\right]dx = \int_{-4}^{4}\left(32 - 2x^2\right)dx$$
$$= 2\int_0^4\left(32 - 2x^2\right)dx \quad \text{[even function]}$$
$$= 2\left[32x - \tfrac{2}{3}x^3\right]_0^4$$
$$= 2\left(128 - \tfrac{128}{3}\right) = \tfrac{512}{3}$$



3. $1 - x^2 = 1 - \sqrt{x} \iff -x^2 = -\sqrt{x} \iff x^2 = \sqrt{x} \Rightarrow x^4 = x$
$\Rightarrow x^4 - x = 0 \Rightarrow x\left(x^3 - 1\right) = 0 \Rightarrow x(x-1)\left(x^2 + x + 1\right) = 0$
$\Rightarrow x = 0$ or 1. So

$$A = \int_0^1\left[\left(1 - x^2\right) - \left(1 - \sqrt{x}\right)\right]dx = \int_0^1\left(\sqrt{x} - x^2\right)dx$$
$$= \left[\tfrac{2}{3}x^{3/2} - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{2}{3} - \tfrac{1}{3} = \tfrac{1}{3}$$

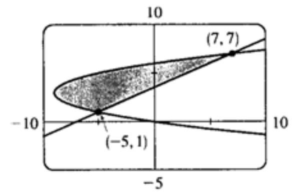


4. $A = \int_1^7 \left[(2y-7) - (y^2-6y)\right] dy$

$= \int_1^7 \left(-y^2 + 8y - 7\right) dy$

$= \left[-\frac{1}{3}y^3 + 4y^2 - 7y\right]_1^7$

$= -\frac{343}{3} + 196 - 49 - \left(-\frac{1}{3} + 4 - 7\right) = 36$



5. $A = \int_0^\pi |\sin x - (-\cos x)|\, dx = \int_0^{3\pi/4} (\sin x + \cos x)\, dx - \int_{3\pi/4}^{\pi} (\sin x + \cos x)\, dx$

$= [\sin x - \cos x]_0^{3\pi/4} - [-\cos x + \sin x]_{3\pi/4}^{\pi}$

$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0-1) - (1+0) + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \sqrt{2} + 1 - 1 + \sqrt{2} = 2\sqrt{2}$

6. The curves intersect at $(1, 1)$, so the area is

$$A = \int_0^2 \left|x^3 - (x^2 - 4x + 4)\right| dx = \int_0^1 \left(-x^3 + x^2 - 4x + 4\right) dx + \int_1^2 \left(x^3 - x^2 + 4x - 4\right) dx$$

$$= \left[-\frac{1}{4}x^4 + \frac{1}{3}x^3 - 2x^2 + 4x\right]_0^1 + \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 + 2x^2 - 4x\right]_1^2$$

$$= -\frac{1}{4} + \frac{1}{3} - 2 + 4 + 4 - \frac{8}{3} + 8 - 8 - \frac{1}{4} + \frac{1}{3} - 2 + 4 = 5.5$$

7. $V = \int_1^3 \pi \left(\sqrt{x-1}\right)^2 dx = \pi \int_1^3 (x-1)\, dx = \pi \left[\frac{1}{2}x^2 - x\right]_1^3 = \pi \left[\left(\frac{9}{2} - 3\right) - \left(\frac{1}{2} - 1\right)\right] = 2\pi$

8. $V = \int_0^1 \pi \left[\left(x^2\right)^2 - \left(x^3\right)^2\right] dx = \pi \int_0^1 \left(x^4 - x^6\right) dx = \pi \left[\frac{1}{5}x^5 - \frac{1}{7}x^7\right]_0^1 = \pi \left(\frac{1}{5} - \frac{1}{7}\right) = \frac{2\pi}{35}$

9. $V = \int_1^3 2\pi y \left(-y^2 + 4y - 3\right) dy = 2\pi \int_1^3 \left(-y^3 + 4y^2 - 3y\right) dy = 2\pi \left[-\frac{1}{4}y^4 + \frac{4}{3}y^3 - \frac{3}{2}y^2\right]_1^3$

$= 2\pi \left[\left(-\frac{81}{4} + 36 - \frac{27}{2}\right) - \left(-\frac{1}{4} + \frac{4}{3} - \frac{3}{2}\right)\right] = \frac{16\pi}{3}$

10. $V = \int_0^8 \pi \left(y^{1/3}\right)^2 dy = \pi \int_0^8 y^{2/3}\, dy = \pi \left[\frac{3}{5}y^{5/3}\right]_0^8 = \frac{96\pi}{5}$

11. $V = \int_a^{a+h} 2\pi x \cdot 2\sqrt{x^2 - a^2}\, dx = 2\pi \int_0^{2ah+h^2} u^{1/2}\, du$ (Put $u = x^2 - a^2$, so $du = 2x\, dx$)

$= 2\pi \left[\frac{2}{3}u^{3/2}\right]_0^{2ah+h^2} = \frac{4}{3}\pi \left(2ah + h^2\right)^{3/2}$

12. $V = \int_{3\pi/2}^{5\pi/2} 2\pi x \cos x\, dx$ (by the method of cylindrical shells)

13. $V = \int_0^1 \pi \left[\left(1 - x^3\right)^2 - \left(1 - x^2\right)^2\right] dx$

14. $V = \int_0^2 2\pi \left(8 - x^3\right)(2 - x)\, dx$

15. (a) $V = \int_0^1 \pi \left[(x)^2 - \left(x^2\right)^2\right] dx = \int_0^1 \pi \left(x^2 - x^4\right) dx = \pi \left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \pi \left[\frac{1}{3} - \frac{1}{5}\right] = \frac{2\pi}{15}$

(b) $V = \int_0^1 \pi \left[\left(\sqrt{y}\right)^2 - y^2\right] dy = \int_0^1 \pi \left(y - y^2\right) dy = \pi \left[\frac{1}{2}y^2 - \frac{1}{3}y^3\right]_0^1 = \pi \left[\frac{1}{2} - \frac{1}{3}\right] = \frac{\pi}{6}$

(c) $V = \int_0^1 \pi \left[\left(2 - x^2\right)^2 - (2 - x)^2\right] dx = \int_0^1 \pi \left(x^4 - 5x^2 + 4x\right) dx = \pi \left[\frac{1}{5}x^5 - \frac{5}{3}x^3 + 2x^2\right]_0^1$

$= \pi \left[\frac{1}{5} - \frac{5}{3} + 2\right] = \frac{8\pi}{15}$

16. (a) $A = \int_0^1 (2x - x^2 - x^3)\,dx = \left[x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = 1 - \frac{1}{3} - \frac{1}{4} = \frac{5}{12}$

(b) A cross-section is an annulus with inner radius x^3 and outer radius $2x - x^2$, so its area is $\pi\,(2x - x^2)^2 - \pi\,(x^3)^2$.

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi\left[(2x - x^2)^2 - (x^3)^2\right]dx = \int_0^1 \pi\,(4x^2 - 4x^3 + x^4 - x^6)\,dx$$

$$= \pi\left[\tfrac{4}{3}x^3 - x^4 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7\right]_0^1 = \pi\left(\tfrac{4}{3} - 1 + \tfrac{1}{5} - \tfrac{1}{7}\right) = \tfrac{41\pi}{105}$$

(c) $V = \int_0^1 2\pi x\,(2x - x^2 - x^3)\,dx = \int_0^1 2\pi\,(2x^2 - x^3 - x^4)\,dx = 2\pi\left[\frac{2}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5\right]_0^1$

$= 2\pi\left(\frac{2}{3} - \frac{1}{4} - \frac{1}{5}\right) = \frac{13\pi}{30}$ (by the method of cylindrical shells)

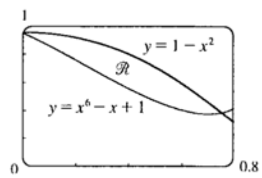
17. (a) Using the Midpoint Rule on $[0, 1]$ with $f(x) = \tan(x^2)$ and $n = 4$, we estimate

$$A = \int_0^1 \tan(x^2)\,dx \approx \tfrac{1}{4}\left[\tan\left(\left(\tfrac{1}{8}\right)^2\right) + \tan\left(\left(\tfrac{3}{8}\right)^2\right) + \tan\left(\left(\tfrac{5}{8}\right)^2\right) + \tan\left(\left(\tfrac{7}{8}\right)^2\right)\right] \approx \tfrac{1}{4}(1.53) \approx 0.38$$

(b) Using the Midpoint Rule on $[0, 1]$ with $f(x) = \pi\tan^2(x^2)$ (for disks) and $n = 4$, we estimate

$$V = \int_0^1 f(x)\,dx \approx \tfrac{1}{4}\pi\left[\tan^2\left(\left(\tfrac{1}{8}\right)^2\right) + \tan^2\left(\left(\tfrac{3}{8}\right)^2\right) + \tan^2\left(\left(\tfrac{5}{8}\right)^2\right) + \tan^2\left(\left(\tfrac{7}{8}\right)^2\right)\right] \approx \tfrac{\pi}{4}(1.114) \approx 0.87$$

18. (a)



From the graph, it appears that the curves intersect at $x = 0$ and at $x \approx 0.75$, with $1 - x^2 > x^6 - x + 1$ on $(0, 0.75)$.

(b) We estimate

$$A \approx \int_0^{0.75}\left[(1 - x^2) - (x^6 - x + 1)\right]dx = \left[-\tfrac{1}{3}x^3 - \tfrac{1}{7}x^7 + \tfrac{1}{2}x^2\right]_0^{0.75} \approx 0.12$$

(c) Using disks, we estimate

$$V \approx \pi\int_0^{0.75}\left[(1 - x^2)^2 - (x^6 - x + 1)^2\right]dx = \pi\int_0^{0.75}(-x^{12} + 2x^7 - 2x^6 + x^4 - 3x^2 + 2x)\,dx$$

$$= \pi\left[-\tfrac{1}{13}x^{13} + \tfrac{1}{4}x^8 - \tfrac{2}{7}x^7 + \tfrac{1}{5}x^5 - x^3 + x^2\right]_0^{0.75} \approx 0.54$$

(d) Using shells, we estimate

$$V \approx 2\pi\int_0^{0.75} x\left[(1 - x^2) - (x^6 - x + 1)\right]dx = 2\pi\int_0^{0.75}(-x^3 - x^7 + x^2)\,dx$$

$$= \left[-\tfrac{1}{4}x^4 - \tfrac{1}{8}x^8 + \tfrac{1}{3}x^3\right]_0^{0.75} \approx 0.31$$

19. The solid is obtained by rotating the region $\mathcal{R} = \left\{(x, y) \mid 0 \le x \le \frac{\pi}{2}, 0 \le y \le \cos x\right\}$ about the y-axis.

20. The solid is obtained by rotating the region $\mathcal{R} = \left\{(x, y) \mid 0 \le x \le \frac{\pi}{2}, 0 \le y \le \sqrt{2}\cos x\right\}$ about the x-axis.

21. The solid is obtained by rotating the region under the curve $y = \sin x$, above $y = 0$, from $x = 0$ to $x = \pi$, about the x-axis.

22. The solid is obtained by rotating the region under the curve $y = \sin x$, above $y = 0$, from $x = 0$ to $x = \pi$, about the y-axis.

23. Take the base to be the disk $x^2 + y^2 \le 9$. Then $V = \int_{-3}^{3} A(x)\,dx$, where $A(x_0)$ is the area of the isosceles right triangle whose hypotenuse lies along the line $x = x_0$ in the xy-plane. $A(x) = \frac{1}{2}\left(\sqrt{2}\sqrt{9-x^2}\right)^2 = 9 - x^2$, so $V = 2\int_0^3 A(x)\,dx = 2\int_0^3 (9 - x^2)\,dx = 2\left[9x - \frac{1}{3}x^3\right]_0^3 = 2(27 - 9) = 36.$

24. $V = \int_{-1}^{1} A(x)\,dx = 2\int_0^1 A(x)\,dx = 2\int_0^1 \left[(2 - x^2) - x^2\right]^2 dx = 2\int_0^1 \left[2(1 - x^2)\right]^2 dx$
$= 8\int_0^1 (1 - 2x^2 + x^4)\,dx = 8\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_0^1 = 8\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{64}{15}$

25. Equilateral triangles with sides measuring $\frac{1}{4}x$ meters have height $\frac{1}{4}x \sin 60° = \frac{\sqrt{3}}{8}x$. Therefore, $A(x) = \frac{1}{2} \cdot \frac{1}{4}x \cdot \frac{\sqrt{3}}{8}x = \frac{\sqrt{3}}{64}x^2$. $V = \int_0^{20} A(x)\,dx = \frac{\sqrt{3}}{64}\int_0^{20} x^2\,dx = \frac{\sqrt{3}}{64}\left[\frac{1}{3}x^3\right]_0^{20} = \frac{8000\sqrt{3}}{64 \cdot 3} = \frac{125\sqrt{3}}{3}\ \text{m}^3$

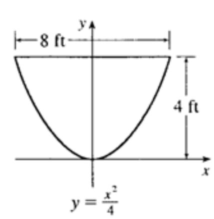
26. (a) By the symmetry of the problem, we consider only the solid to the right of the origin. The semicircular cross-sections perpendicular to the x-axis have radius $1 - x$, so $A(x) = \frac{1}{2}\pi(1 - x)^2$. Now we can calculate $V = 2\int_0^1 A(x)\,dx = 2\int_0^1 \frac{1}{2}\pi(1 - x)^2\,dx = \int_0^1 \pi(1 - x)^2\,dx = -\frac{\pi}{3}\left[(1 - x)^3\right]_0^1 = \frac{\pi}{3}$

(b) Cut the solid with a plane perpendicular to the x-axis and passing through the y-axis. Fold the half of the solid in the region $x \le 0$ under the xy-plane so that the point $(-1, 0)$ comes around and touches the point $(1, 0)$. The resulting solid is a right circular cone of radius 1 with vertex at $(1, 0, 0)$ and with its base in the yz-plane, centered at the origin. The volume of this cone is $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \cdot 1^2 \cdot 1 = \frac{\pi}{3}$.

27. $f(x) = kx \;\Rightarrow\; 30\ \text{N} = k(15 - 12)\ \text{cm} \;\Rightarrow\; k = 10\ \text{N/cm} = 1000\ \text{N/m}$. $20\ \text{cm} - 12\ \text{cm} = 0.08\ \text{m} \;\Rightarrow$
$W = \int_0^{0.08} kx\,dx = 1000\int_0^{0.08} x\,dx = 500\left[x^2\right]_0^{0.08} = 500(0.08)^2 = 3.2\ \text{N-m} = 3.2\ \text{J}.$

28. The work needed to raise the elevator alone is 1600 lb × 30 ft = 48,000 ft-lb. The work needed to raise the bottom 170 ft of cable is 170 ft × 10 lb/ft × 30 ft = 51,000 ft-lb. The work needed to raise the top 30 ft of cable is $\int_0^{30} 10x\,dx = \left[5x^2\right]_0^{30} = 5 \cdot 900 = 4500$ ft-lb. Adding these, we see that the total work needed is $48{,}000 + 51{,}000 + 4{,}500 = 103{,}500$ ft-lb.

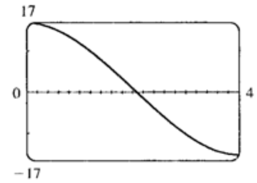
29. (a) The parabola has equation $y = ax^2$ with vertex at the origin and passing through $(4, 4)$. $4 = a \cdot 4^2 \;\Rightarrow\; a = \frac{1}{4} \;\Rightarrow\; y = \frac{1}{4}x^2 \;\Rightarrow\; x^2 = 4y$
$\Rightarrow\; x = 2\sqrt{y}$. Each circular disk has radius $2\sqrt{y}$ and is moved $4 - y$ ft.



$$W = \int_0^4 \pi\left(2\sqrt{y}\right)^2 62.5(4 - y)\,dy = 250\pi\int_0^4 y(4 - y)\,dy$$
$$= 250\pi\left[2y^2 - \tfrac{1}{3}y^3\right]_0^4 = 250\pi\left(32 - \tfrac{64}{3}\right) = \tfrac{8000\pi}{3} \approx 8377.6\ \text{ft-lb}$$

(b) In part (a) we knew the final water level (0) but not the amount of work done. Here we use the same equation, except with the work fixed, and the lower limit of integration (that is, the final water level — call it h) unknown: $W = 4000 \;\Leftrightarrow\; 250\pi\left[2y^2 - \frac{1}{3}y^3\right]_h^4 = 4000 \;\Leftrightarrow$

$\frac{16}{\pi} = \left[\left(32 - \frac{64}{3}\right) - \left(2h^2 - \frac{1}{3}h^3\right)\right] \;\Leftrightarrow\; h^3 - 6h^2 + 32 - \frac{48}{\pi} = 0.$



We plot the graph of the function $f(h) = h^3 - 6h^2 + 32 - \frac{48}{\pi}$ on the interval $[0, 4]$ to see where it is 0. From the graph, $f(h) = 0$ for $h \approx 2.06$. So the depth of water remaining is about 2.06 ft.

30. $f_{\text{ave}} = \frac{1}{2-0}\int_0^2 x^2\sqrt{1+x^3}\,dx = \frac{1}{2}\cdot\frac{1}{3}\int_1^9 \sqrt{u}\,du \quad [u = 1 + x^3,\ du = 3x^2\,dx]$

$= \frac{1}{6}\left[\frac{2}{3}u^{3/2}\right]_1^9 = \frac{1}{9}\left(9^{3/2} - 1^{3/2}\right) = \frac{1}{9}(27 - 1) = \frac{26}{9}$

31. $\lim\limits_{h\to 0} f_{\text{ave}} = \lim\limits_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim\limits_{h\to 0} \frac{F(x+h) - F(x)}{h}$, where $F(x) = \int_a^x f(t)\,dt$. But we recognize this limit as being $F'(x)$ by the definition of a derivative. Therefore, $\lim\limits_{h\to 0} f_{\text{ave}} = F'(x) = f(x)$ by FTC1.

32. (a) $\mathcal{R}_1$ is the region below the graph of $y = x^2$ and above the x-axis between $x = 0$ and $x = b$, and $\mathcal{R}_2$ is the region to the left of the graph of $x = \sqrt{y}$ and to the right of the y-axis between $y = 0$ and $y = b^2$. So the area of $\mathcal{R}_1$ is $A_1 = \int_0^b x^2\,dx = \left[\frac{1}{3}x^3\right]_0^b = \frac{1}{3}b^3$, and the area of $\mathcal{R}_2$ is $A_2 = \int_0^{b^2}\sqrt{y}\,dy = \left[\frac{2}{3}y^{3/2}\right]_0^{b^2} = \frac{2}{3}b^3$. So there is no solution to $A_1 = A_2$ for $b \neq 0$.

(b) Using disks, we calculate the volume of rotation of $\mathcal{R}_1$ about the x-axis to be $V_{1,x} = \pi\int_0^b \left(x^2\right)^2 dx = \frac{1}{5}\pi b^5$. Using cylindrical shells, we calculate the volume of rotation of $\mathcal{R}_1$ about the y-axis to be

$V_{1,y} = 2\pi\int_0^b x\left(x^2\right)dx = 2\pi\left[\frac{1}{4}x^4\right]_0^b = \frac{1}{2}\pi b^4$. So $V_{1,x} = V_{1,y} \;\Leftrightarrow\; \frac{1}{5}\pi b^5 = \frac{1}{2}\pi b^4 \;\Leftrightarrow\; 2b = 5 \;\Leftrightarrow$

$b = \frac{5}{2}$. So the volumes of rotation about the x- and y-axes are the same for $b = \frac{5}{2}$.

(c) We use cylindrical shells to calculate the volume of rotation of $\mathcal{R}_2$ about the x-axis:

$\mathcal{R}_{2,x} = 2\pi\int_0^{b^2} y\left(\sqrt{y}\right)dy = 2\pi\left[\frac{2}{5}y^{5/2}\right]_0^{b^2} = \frac{4}{5}\pi b^5$. We already know the volume of rotation of $\mathcal{R}_1$ about the x-axis from part (b), and $\mathcal{R}_{1,x} = \mathcal{R}_{2,x} \;\Leftrightarrow\; \frac{1}{5}\pi b^5 = \frac{4}{5}\pi b^5$, which has no solution for $b \neq 0$.

(d) We use disks to calculate the volume of rotation of $\mathcal{R}_2$ about the y-axis:

$\mathcal{R}_{2,y} = \pi\int_0^{b^2}\left(\sqrt{y}\right)^2 dy = \pi\left[\frac{1}{2}y^2\right]_0^{b^2} = \frac{1}{2}\pi b^4$. We know the volume of rotation of $\mathcal{R}_1$ about the y-axis from part (b), and $\mathcal{R}_{1,y} = \mathcal{R}_{2,y} \;\Leftrightarrow\; \frac{1}{2}\pi b^4 = \frac{1}{2}\pi b^4$. But this equation is true for all b, so the volumes of rotation about the y-axis are equal for all values of b.

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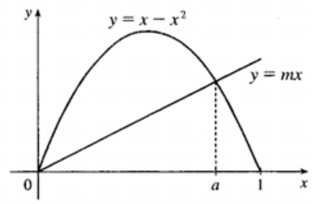
Problems Plus

1. The area under the graph of f from 0 to t is equal to $\int_0^t f(x)\,dx$, so the requirement is that $\int_0^t f(x)\,dx = t^3$ for all t. We differentiate this equation with respect to t (with the help of FTC1) to get $f(t) = 3t^2$. This function is positive and continuous, as required.

2. The total area of the region bounded by the parabola and the x-axis is

$\int_0^1 (x - x^2)\,dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{1}{6}$.



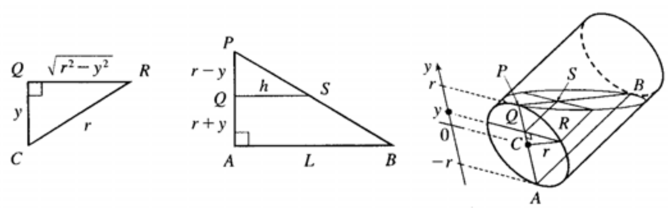
Let the slope of the line we are looking for be m. Then the area above this line but below the parabola is $\int_0^a \left[(x - x^2) - mx\right] dx$, where a is the x-coordinate of the point of intersection of the line and the parabola. We find the point of intersection by solving the equation $x - x^2 = mx \quad \Leftrightarrow \quad 1 - x = m \quad \Leftrightarrow \quad x = 1 - m$. So the value of a is $1 - m$, and

$$\begin{aligned}\int_0^{1-m} \left[(x - x^2) - mx\right] dx &= \int_0^{1-m} \left[(1 - m)x - x^2\right] dx = \left[\tfrac{1}{2}(1 - m)x^2 - \tfrac{1}{3}x^3\right]_0^{1-m} \\ &= \tfrac{1}{2}(1 - m)(1 - m)^2 - \tfrac{1}{3}(1 - m)^3 = \tfrac{1}{6}(1 - m)^3\end{aligned}$$

We want this to be half of $\frac{1}{6}$, so $\frac{1}{6}(1 - m)^3 = \frac{1}{12} \quad \Rightarrow \quad m = 1 - \frac{1}{\sqrt[3]{2}}$. So the slope of the required line is $1 - \frac{1}{\sqrt[3]{2}} \approx 0.206$.

3. The volume generated from $x = 0$ to $x = b$ is $\int_0^b \pi\,[f(x)]^2\,dx$. Hence, we are given that $b^2 = \int_0^b \pi\,[f(x)]^2\,dx$ for all $b > 0$. Differentiating both sides of this equation using the Fundamental Theorem of Calculus gives $2b = \pi\,[f(b)]^2 \quad \Rightarrow \quad f(b) = \sqrt{2b/\pi}$, since f is positive. Therefore, $f(x) = \sqrt{2x/\pi}$.

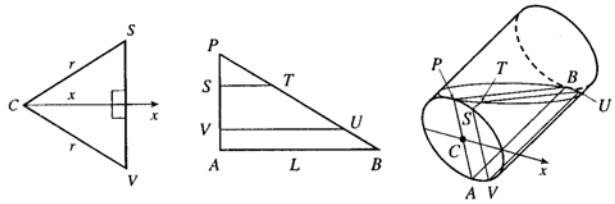
4. (a) Take slices perpendicular to the line through the center C of the bottom of the glass and the point P where the top surface of the water meets the bottom of the glass.



A typical rectangular cross-section y units above the axis of the graph has width $2\,|QR| = 2\sqrt{r^2 - y^2}$ and length $h = |QS| = \dfrac{L}{2r}(r - y)$. $\left[\text{Triangles } PQS \text{ and } PAB \text{ are similar, so } \dfrac{h}{L} = \dfrac{|PQ|}{|PA|} = \dfrac{r - y}{2r}.\right]$ Thus,

$$\begin{aligned}V &= \int_{-r}^{r} 2\sqrt{r^2 - y^2} \cdot \frac{L}{2r}(r - y)\,dy = L\int_{-r}^{r} \left(1 - \frac{y}{r}\right)\sqrt{r^2 - y^2}\,dy \\ &= L\int_{-r}^{r} \sqrt{r^2 - y^2}\,dy + \frac{L}{2r}\int_{-r}^{r} (-2y)\sqrt{r^2 - y^2}\,dy \\ &= L \cdot \frac{\pi r^2}{2} + \frac{L}{2r} \cdot 0 \quad \left[\begin{array}{c}\text{the first integral is the area of a semicircle of radius } r, \\ \text{and the second has an odd integrand}\end{array}\right] \quad = \frac{\pi r^2 L}{2}\end{aligned}$$

(b) Slice parallel to the plane through the axis of the glass and the point of contact P. (This is the plane determined by P, B, and C in the figure.) $STUV$ is a typical trapezoidal slice. With respect to an x-axis with origin at C as shown, if S and V have coordinate x, then $|SV| = 2\sqrt{r^2 - x^2}$. Projecting the trapezoid $STUV$ onto the plane of the triangle PAB, we see that $|AP| = 2r$, $|SV| = 2\sqrt{r^2 - x^2}$, and
$|SP| = |VA| = \frac{1}{2}(|AP| - |SV|) = r - \sqrt{r^2 - x^2}$.

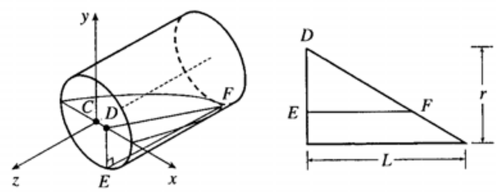


By similar triangles, $\dfrac{|ST|}{|SP|} = \dfrac{|AB|}{|AP|}$, so $|ST| = \left(r - \sqrt{r^2 - x^2}\right) \cdot \dfrac{L}{2r}$. In the same way, we find that
$\dfrac{|VU|}{|VP|} = \dfrac{|AB|}{|AP|}$, so $|VU| = |VP| \cdot \dfrac{L}{2r} = (|AP| - |VA|) \cdot \dfrac{L}{2r} = \left(r + \sqrt{r^2 - x^2}\right) \cdot \dfrac{L}{2r}$.
The area $A(x)$ of the trapezoid $STUV$ is $\frac{1}{2}|SV| \cdot (|ST| + |VU|)$; that is,
$A(x) = \frac{1}{2} \cdot 2\sqrt{r^2 - x^2} \cdot \left[\left(r - \sqrt{r^2 - x^2}\right) \cdot \dfrac{L}{2r} + \left(r + \sqrt{r^2 - x^2}\right) \cdot \dfrac{L}{2r}\right] = L\sqrt{r^2 - x^2}$. Thus,
$V = \int_{-r}^{r} A(x)\,dx = L\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx = L \cdot \dfrac{\pi r^2}{2} = \dfrac{\pi r^2 L}{2}$.

(c) See the computation of V in part (a) or part (b).

(d) The volume of the water is exactly half the volume of the cylindrical glass, so $V = \frac{1}{2}\pi r^2 L$.

(e)



Choose x-, y-, and z-axes as shown in the figure. Then slices perpendicular to the x-axis are triangular, slices perpendicular to the y-axis are rectangular, and slices perpendicular to the z-axis are segments of circles. Using triangular slices, we find that the area $A(x)$ of a typical slice DEF, where D has coordinate x, is given by
$A(x) = \frac{1}{2}|DE| \cdot |EF| = \frac{1}{2}|DE| \cdot \left(\dfrac{L}{r}|DE|\right) = \dfrac{L}{2r}|DE|^2 = \dfrac{L}{2r}(r^2 - x^2)$. Thus,

$$V = \int_{-r}^{r} A(x)\,dx = \frac{L}{2r}\int_{-r}^{r}\left(r^2 - x^2\right)dx = \frac{L}{r}\int_{0}^{r}\left(r^2 - x^2\right)dx = \frac{L}{r}\left[r^2 x - \frac{x^3}{3}\right]_0^r$$
$$= \frac{L}{r}\left(r^3 - \frac{r^3}{3}\right) = \frac{L}{r} \cdot \frac{2}{3}r^3 = \frac{2}{3}r^2 L$$

5. (a) $V = \pi h^2 (r - h/3) = \frac{1}{3}\pi h^2 (3r - h)$. See the solution to Exercise 6.2.47.

(b) The smaller segment has height $h = 1 - x$ and so by part (a) its volume is $V = \frac{1}{3}\pi (1 - x)^2 [3(1) - (1 - x)] = \frac{1}{3}\pi (x - 1)^2 (x + 2)$. This volume must be $\frac{1}{3}$ of the total volume of the sphere, which is $\frac{4}{3}\pi (1)^3$. So $\frac{1}{3}\pi (x - 1)^2 (x + 2) = \frac{1}{3}\left(\frac{4}{3}\pi\right) \Rightarrow \left(x^2 - 2x + 1\right)(x + 2) = \frac{4}{3} \Rightarrow$ $x^3 - 3x + 2 = \frac{4}{3} \Rightarrow 3x^3 - 9x + 2 = 0$. Using Newton's method with $f(x) = 3x^3 - 9x + 2$, $f'(x) = 9x^2 - 9$, we get $x_{n+1} = x_n - \dfrac{3x_n^3 - 9x_n + 2}{9x_n^2 - 9}$. Taking $x_1 = 0$, we get $x_2 \approx 0.2222$, and $x_3 \approx 0.2261 \approx x_4$, so, correct to four decimal places, $x \approx 0.2261$.

(c) With $r = 0.5$ and $s = 0.75$, the equation $x^3 - 3rx^2 + 4r^3 s = 0$ becomes $x^3 - 3(0.5)x^2 + 4(0.5)^3(0.75) = 0$ $\Rightarrow x^3 - \frac{3}{2}x^2 + 4\left(\frac{1}{8}\right)\frac{3}{4} = 0 \Rightarrow 8x^3 - 12x^2 + 3 = 0$. We use Newton's method with $f(x) = 8x^3 - 12x^2 + 3$, $f'(x) = 24x^2 - 24x$, so $x_{n+1} = x_n - \dfrac{8x_n^3 - 12x_n^2 + 3}{24x_n^2 - 24x_n}$. Take $x_1 = 0.5$. Then $x_2 \approx 0.6667$, and $x_3 \approx 0.6736 \approx x_4$. So to four decimal places the depth is 0.6736 m.

(d) (i) From part (a) with $r = 5$ in., the volume of water in the bowl is $V = \frac{1}{3}\pi h^2 (3r - h) = \frac{1}{3}\pi h^2 (15 - h) = 5\pi h^2 - \frac{1}{3}\pi h^3$. We are given that $\dfrac{dV}{dt} = 0.2\ \text{m}^3/\text{s}$ and we want to find $\dfrac{dh}{dt}$ when $h = 3$. Now $\dfrac{dV}{dt} = 10\pi h \dfrac{dh}{dt} - \pi h^2 \dfrac{dh}{dt}$, so $\dfrac{dh}{dt} = \dfrac{0.2}{\pi\left(10h - h^2\right)}$. When $h = 3$, we have $\dfrac{dh}{dt} = \dfrac{0.2}{\pi\left(10 \cdot 3 - 3^2\right)} = \dfrac{1}{105\pi} \approx 0.003$ in/s.

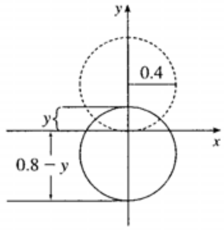
(ii) From part (a), the volume of water required to fill the bowl from the instant that the water is 4 in. deep is $V = \frac{1}{2} \cdot \frac{4}{3}\pi (5)^3 - \frac{1}{3}\pi (4)^2 (15 - 4) = \frac{2}{3} \cdot 125\pi - \frac{16}{3} \cdot 11\pi = \frac{74}{3}\pi$. To find the time required to fill the bowl we divide this volume by the rate: Time $= \frac{74\pi/3}{0.2} = \frac{370\pi}{3} \approx 387\ \text{s} \approx 6.5$ min

6. (a) The volume above the surface is $\int_0^{L-h} A(y)\,dy = \int_{-h}^{L-h} A(y)\,dy - \int_{-h}^{0} A(y)\,dy$. So the proportion of volume above the surface is $\dfrac{\int_0^{L-h} A(y)\,dy}{\int_{-h}^{L-h} A(y)\,dy} = \dfrac{\int_{-h}^{L-h} A(y)\,dy - \int_{-h}^{0} A(y)\,dy}{\int_{-h}^{L-h} A(y)\,dy}$. Now by Archimedes' Principle, we have $F = W \Rightarrow \rho_f g \int_{-h}^{0} A(y)\,dy = \rho_0 g \int_{-h}^{L-h} A(y)\,dy$, so $\int_{-h}^{0} A(y)\,dy = (\rho_0/\rho_f) \int_{-h}^{L-h} A(y)\,dy$. Therefore, $\dfrac{\int_0^{L-h} A(y)\,dy}{\int_{-h}^{L-h} A(y)\,dy} = \dfrac{\int_{-h}^{L-h} A(y)\,dy - (\rho_0/\rho_f)\int_{-h}^{L-h} A(y)\,dy}{\int_{-h}^{L-h} A(y)\,dy} = \dfrac{\rho_f - \rho_0}{\rho_f}$, so the percentage of volume above the surface is $100\left(\dfrac{\rho_f - \rho_0}{\rho_f}\right)\%$.

(b) For an iceberg, the percentage of volume above the surface is $100\left(\frac{1030 - 917}{1030}\right)\% \approx 11\%$.

(c) No, the water does not overflow. Let V_i be the volume of the ice cube, and let V_w be the volume of the water which results from the melting. Then by the formula derived in part (a), the volume of ice above the surface of the water is $\left[(\rho_f - \rho_0)/\rho_f\right] V_i$, so the volume below the surface is $V_i - \left[(\rho_f - \rho_0)/\rho_f\right] V_i = (\rho_0/\rho_f) V_i$. Now the mass of the ice cube is the same as the mass of the water which is created when it melts, namely $m = \rho_0 V_i = \rho_f V_w \Rightarrow V_w = (\rho_0/\rho_f) V_i$. So when the ice cube melts, the volume of the resulting water is the same as the underwater volume of the ice cube, and so the water does not overflow.

(d)

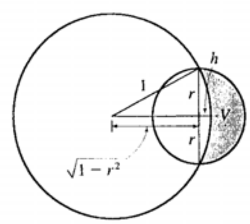


The figure shows the instant when the height of the exposed part of the ball is y. Using the formula in part (a) with $r = 0.4$ and $h = 0.8 - y$, we see that the volume of the submerged part of the sphere is $\frac{1}{3}\pi\,(0.8 - y)^2\,[1.2 - (0.8 - y)]$, so its weight is $1000g \cdot \frac{1}{3}\pi s^2\,(1.2 - s)$, where $s = 0.8 - y$. Then the work done to submerge the sphere is

$$\begin{aligned} W &= \int_0^{0.8} g\,\tfrac{1000}{3}\pi s^2\,(1.2 - s)\,ds = g\,\tfrac{1000}{3}\pi \int_0^{0.8} \left(1.2s^2 - s^3\right) ds \\ &= g\,\tfrac{1000}{3}\pi \left[0.4s^3 - \tfrac{1}{4}s^4\right]_0^{0.8} = g\,\tfrac{1000}{3}\pi\,(0.2048 - 0.1024) \\ &= 9.8\,\tfrac{1000}{3}\pi\,(0.1024) \approx 1.05 \times 10^3 \text{ joules} \end{aligned}$$

7. We are given that the rate of change of the volume of water is $\dfrac{dV}{dt} = -kA\,(x)$, where k is some positive constant and $A\,(x)$ is the area of the surface when the water has depth x. Now we are concerned with the rate of change of the depth of the water with respect to time, that is, $\dfrac{dx}{dt}$. But by the Chain Rule, $\dfrac{dV}{dt} = \dfrac{dV}{dx}\dfrac{dx}{dt}$, so the first equation can be written $\dfrac{dV}{dx}\dfrac{dx}{dt} = -kA\,(x)$ (★). Also, we know that the total volume of water up to a depth x is $V\,(x) = \int_0^x A\,(s)\,ds$, where $A\,(s)$ is the area of a cross-section of the water at a depth s. Differentiating this equation with respect to x, we get $dV/dx = A\,(x)$. Substituting this into equation ★, we get $A\,(x)\,(dx/dt) = -kA\,(x) \;\Rightarrow\; dx/dt = -k$, a constant.

8. A typical sphere of radius r is shown in the figure. We wish to maximize the shaded volume V, which can be thought of as the volume of a hemisphere of radius r minus the volume of the spherical cap with height $h = 1 - \sqrt{1 - r^2}$ and radius 1.



$$\begin{aligned} V &= \tfrac{1}{2} \cdot \tfrac{4}{3}\pi r^3 - \tfrac{1}{3}\pi\left(1 - \sqrt{1 - r^2}\right)^2\left[3\,(1) - \left(1 - \sqrt{1 - r^2}\right)\right] \quad \text{[by Problem 5(a)]} \\ &= \tfrac{1}{3}\pi\left[2r^3 - \left(2 - 2\sqrt{1 - r^2} - r^2\right)\left(2 + \sqrt{1 - r^2}\right)\right] \\ &= \tfrac{1}{3}\pi\left[2r^3 - 2 + \left(r^2 + 2\right)\sqrt{1 - r^2}\right] \end{aligned}$$

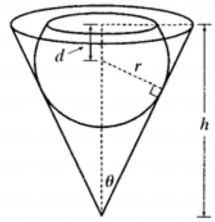
$$\begin{aligned} V' &= \tfrac{1}{3}\pi\left[6r^2 + \frac{(r^2 + 2)\,(-r)}{\sqrt{1 - r^2}} + \sqrt{1 - r^2}\,(2r)\right] = \tfrac{1}{3}\pi\left[\frac{6r^2\sqrt{1 - r^2} - r\left(r^2 + 2\right) + 2r\left(1 - r^2\right)}{\sqrt{1 - r^2}}\right] \\ &= \tfrac{1}{3}\pi\left(\frac{6r^2\sqrt{1 - r^2} - 3r^3}{\sqrt{1 - r^2}}\right) = \frac{\pi r^2\left(2\sqrt{1 - r^2} - r\right)}{\sqrt{1 - r^2}} \end{aligned}$$

$V'\,(r) = 0 \;\Leftrightarrow\; 2\sqrt{1 - r^2} = r \;\Leftrightarrow\; 4 - 4r^2 = r^2 \;\Leftrightarrow\; r^2 = \frac{4}{5} \;\Leftrightarrow\; r = \frac{2}{\sqrt{5}} \approx 0.89$. Since $V'\,(r) > 0$ for $0 < r < \frac{2}{\sqrt{5}}$ and $V'\,(r) < 0$ for $\frac{2}{\sqrt{5}} < r < 1$, we know that V attains a maximum at $r = \frac{2}{\sqrt{5}}$.

9. We must find expressions for the areas A and B, and then set them equal and see what this says about the curve C. If $P = (a, 2a^2)$, then area A is just $\int_0^a (2x^2 - x^2)\,dx = \int_0^a x^2\,dx = \frac{1}{3}a^3$. To find area B, we use y as the variable of integration. So we find the equation of the middle curve as a function of y: $y = 2x^2 \Leftrightarrow x = \sqrt{y/2}$, since we are concerned with the first quadrant only. We can express area B as

$\int_0^{2a^2} \left[\sqrt{y/2} - C(y)\right] dy = \left[\frac{4}{3}(y/2)^{3/2}\right]_0^{2a^2} - \int_0^{2a^2} C(y)\,dy = \frac{4}{3}a^3 - \int_0^{2a^2} C(y)\,dy$, where $C(y)$ is the function with graph C. Setting $A = B$, we get $\frac{1}{3}a^3 = \frac{4}{3}a^3 - \int_0^{2a^2} C(y)\,dy \Leftrightarrow \int_0^{2a^2} C(y)\,dy = a^3$. Now we differentiate this equation with respect to a using the Chain Rule and the Fundamental Theorem: $C(2a^2)(4a) = 3a^2 \Rightarrow C(y) = \frac{3}{4}\sqrt{y/2}$, where $y = 2a^2$. Now we can solve for y: $x = \frac{3}{4}\sqrt{y/2} \Rightarrow x^2 = \frac{9}{16}(y/2) \Rightarrow y = \frac{32}{9}x^2$.

10.



We want to find the volume of that part of the sphere which is below the surface of the water. As we can see from the diagram, this region is a cap of a sphere with radius r and height $r + d$. If we can find an expression for d in terms of h, r and θ, then we can determine the volume of the region [see Problem 5(a)], and then differentiate with respect to r to find the maximum. We see that $\sin\theta = \dfrac{r}{h-d} \Leftrightarrow h - d = \dfrac{r}{\sin\theta} \Leftrightarrow d = h - r\csc\theta$. Now we can use the formula from Problem 5(a) to find the volume of water displaced:

$$\begin{aligned} V &= \tfrac{1}{3}\pi h^2(3r - h) = \tfrac{1}{3}\pi (r+d)^2[3r - (r+d)] = \tfrac{1}{3}\pi(r + h - r\csc\theta)^2(2r - h + r\csc\theta) \\ &= \tfrac{\pi}{3}[r(1-\csc\theta) + h]^2[r(2+\csc\theta) - h] \end{aligned}$$

Now we differentiate with respect to r:

$$\begin{aligned} dV/dr &= \tfrac{\pi}{3}\Big([r(1-\csc\theta)+h]^2(2+\csc\theta) + 2[r(1-\csc\theta)+h](1-\csc\theta)[r(2+\csc\theta)+h]\Big) \\ &= \tfrac{\pi}{3}[r(1-\csc\theta)+h]\left([r(1-\csc\theta)+h](2+\csc\theta) + 2(1-\csc\theta)[r(2+\csc\theta)-h]\right) \\ &= \tfrac{\pi}{3}[r(1-\csc\theta)+h]\left(3(2+\csc\theta)(1-\csc\theta)r + [(2+\csc\theta) - 2(1-\csc\theta)]h\right) \\ &= \tfrac{\pi}{3}[r(1-\csc\theta)+h][3(2+\csc\theta)(1-\csc\theta)r + 3h\csc\theta] \end{aligned}$$

This is 0 when $r = \dfrac{h}{\csc\theta - 1}$ and when $r = \dfrac{h\csc\theta}{(\csc\theta+2)(\csc\theta-1)}$. Now since $V\left(\dfrac{h}{\csc\theta - 1}\right) = 0$ (the first factor vanishes; this corresponds to $d = -r$), the maximum volume of water is displaced when $r = \dfrac{h\csc\theta}{(\csc\theta - 1)(\csc\theta+2)}$. (Our intuition tells that a maximum value does exist, and it must occur at a critical number.) Multiplying numerator and denominator by $\sin^2\theta$, we get an alternative form of the answer: $r = \dfrac{h\sin\theta}{\sin\theta + \cos 2\theta}$.

11. (a) Stacking disks along the y-axis gives us $V = \int_0^h \pi\,[f(y)]^2\,dy$.

(b) Using the Chain Rule, $\dfrac{dV}{dt} = \dfrac{dV}{dh} \cdot \dfrac{dh}{dt} = \pi\,[f(h)]^2\,\dfrac{dh}{dt}$.

(c) $kA\sqrt{h} = \pi\,[f(h)]^2\,\dfrac{dh}{dt}$. Set $\dfrac{dh}{dt} = C$: $\pi\,[f(h)]^2\,C = kA\sqrt{h} \;\Rightarrow\; [f(h)]^2 = \dfrac{kA}{\pi C}\sqrt{h} \;\Rightarrow$ $f(h) = \sqrt{\dfrac{kA}{\pi C}}\,h^{1/4}$, that is, $f(y) = \sqrt{\dfrac{kA}{\pi C}}\,y^{1/4}$. The advantage of having $\dfrac{dh}{dt} = C$ is that the markings on the container are evenly spaced.

12. (a) We first use the cylindrical shell method to express the volume V in terms of h, r, and ω:

$$\begin{aligned} V &= \int_0^r 2\pi xy\,dx = \int_0^r 2\pi x\left[h + \frac{\omega^2 x^2}{2g}\right]dx = 2\pi\int_0^r\left(hx + \frac{\omega^2 x^3}{2g}\right)dx \\ &= 2\pi\left[\frac{hx^2}{2} + \frac{\omega^2 x^4}{8g}\right]_0^r = 2\pi\left[\frac{hr^2}{2} + \frac{\omega^2 r^4}{8g}\right] = \pi hr^2 + \frac{\omega^2 r^4}{4g} \quad \Rightarrow \end{aligned}$$

$$h = \frac{V - (\pi\omega^2 r^4)/(4g)}{\pi r^2} = \frac{4gV - \pi\omega^2 r^4}{4\pi g r^2}.$$

(b) The surface touches the bottom when $h = 0 \;\Rightarrow\; 4gV - \pi\omega^2 r^4 = 0 \;\Rightarrow\; \omega^2 = \dfrac{4gV}{\pi r^4} \;\Rightarrow\; \omega = \dfrac{2\sqrt{gV}}{\sqrt{\pi}\,r^2}$.

To spill over the top, $y(r) > L \;\Leftrightarrow$

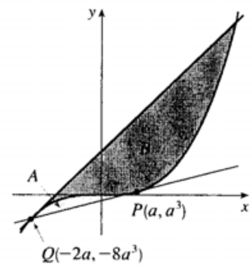
$$\begin{aligned} L &> h + \frac{\omega^2 r^2}{2g} = \frac{4gV - \pi\omega^2 r^4}{4\pi g r^2} + \frac{\omega^2 r^2}{2g} = \frac{4gV}{4\pi g r^2} - \frac{\pi\omega^2 r^2}{4\pi g r^2} + \frac{\omega^2 r^2}{2g} \\ &= \frac{V}{\pi r^2} - \frac{\omega^2 r^2}{4g} + \frac{\omega^2 r^2}{2g} = \frac{V}{\pi r^2} + \frac{\omega^2 r^2}{4g} \quad \Leftrightarrow \end{aligned}$$

$\dfrac{\omega^2 r^2}{4g} > L - \dfrac{V}{\pi r^2} = \dfrac{\pi r^2 L - V}{\pi r^2} \;\Leftrightarrow\; \omega^2 > \dfrac{4g\left(\pi r^2 L - V\right)}{\pi r^4}$. So for spillage, the angular speed should be $\omega > \dfrac{2\sqrt{g\left(\pi r^2 L - V\right)}}{r^2\sqrt{\pi}}$.

(c) (i) Here we have $r = 2$, $L = 7$, $h = 7 - 5 = 2$. When $x = 1$, $y = 7 - 4 = 3$. Therefore, $3 = 2 + \dfrac{\omega^2 \cdot 1^2}{2 \cdot 32}$ $\Rightarrow\; 1 = \dfrac{\omega^2}{2 \cdot 32} \;\Rightarrow\; \omega^2 = 64 \;\Rightarrow\; \omega = 8$ rad/s. $V = \pi\,(2)\,(2)^2 + \dfrac{\pi \cdot 8^2 \cdot 2^4}{4g} = 8\pi + 8\pi = 16\pi$ ft^2.

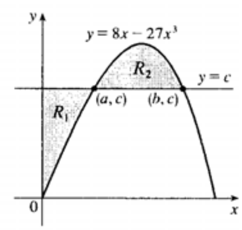
(ii) At the wall, $x = 2$, so $y = 2 + \dfrac{8^2 \cdot 2^2}{2 \cdot 32} = 6$ and the surface is $7 - 6 = 1$ ft below the top of the tank.

13. We assume that P lies in the region of positive x. Since $y = x^3$ is an odd function, this assumption will not affect the result of the calculation. Let $P = (a, a^3)$. The slope of the tangent to the curve $y = x^3$ at P is $3a^2$, and so the equation of the tangent is $y - a^3 = 3a^2(x - a) \Leftrightarrow y = 3a^2x - 2a^3$. We solve this simultaneously with $y = x^3$ to find the other point of intersection: $x^3 = 3a^2x - 2a^3 \Leftrightarrow (x - a)^2(x + 2a) = 0$. So $Q = (-2a, -8a^3)$ is the other point of intersection. The equation of the tangent at Q is $y - (-8a^3) = 12a^2[x - (-2a)] \Leftrightarrow y = 12a^2x + 16a^3$.



By symmetry, this tangent will intersect the curve again at $x = -2(-2a) = 4a$. The curve lies above the first tangent, and below the second, so we are looking for a relationship between $A = \int_{-2a}^{a}\left[x^3 - (3a^2x - 2a^3)\right]dx$ and $B = \int_{-2a}^{4a}\left[(12a^2x + 16a^3) - x^3\right]dx$. We calculate $A = \left[\frac{1}{4}x^4 - \frac{3}{2}a^2x^2 + 2a^3x\right]_{-2a}^{a} = \frac{3}{4}a^4 - (-6a^4) = \frac{27}{4}a^4$, and $B = \left[6a^2x^2 + 16a^3x - \frac{1}{4}x^4\right]_{-2a}^{4a} = 96a^4 - (-12a^4) = 108a^4$. We see that $B = 16A = 2^4A$. This is because our calculation of area B was essentially the same as that of area A, with a replaced by $-2a$, so if we replace a with $-2a$ in our expression for A, we get $\frac{27}{4}(-2a)^4 = 108a^4 = B$.

14. Let a and b be the x-coordinates of the points where the line intersects the curve. From the figure,



$$R_1 = R_2 \quad \Rightarrow$$

$$\int_0^a\left[c - (8x - 27x^3)\right]dx = \int_a^b\left[(8x - 27x^3) - c\right]dx$$

$$\left[cx - 4x^2 + \tfrac{27}{4}x^4\right]_0^a = \left[4x^2 - \tfrac{27}{4}x^4 - cx\right]_a^b$$

$$ac - 4a^2 + \tfrac{27}{4}a^4 = \left(4b^2 - \tfrac{27}{4}b^4 - bc\right) - \left(4a^2 - \tfrac{27}{4}a^4 - ac\right)$$

$$0 = 4b^2 - \tfrac{27}{4}b^4 - bc = 4b^2 - \tfrac{27}{4}b^4 - b\left(8b - 27b^3\right)$$

$$= 4b^2 - \tfrac{27}{4}b^4 - 8b^2 + 27b^4 = \tfrac{81}{4}b^4 - 4b^2$$

$$= b^2\left(\tfrac{81}{4}b^2 - 4\right)$$

So for $b > 0$, $b^2 = \frac{16}{81} \Rightarrow b = \frac{4}{9}$. Thus, $c = 8b - 27b^3 = 8\left(\frac{4}{9}\right) - 27\left(\frac{64}{729}\right) = \frac{32}{9} - \frac{64}{27} = \frac{32}{27}$.

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7 Inverse Functions: Exponential, Logarithmic and Inverse Trigonometric Functions

7.1 Inverse Functions

1. (a) See Definition 1.

(b) It must pass the Horizontal Line Test.

2. (a) $f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$ for any y in B. The domain of f^{-1} is B and the range of f^{-1} is A.

(b) See the steps in (5).

(c) Reflect the graph of f about the line $y = x$.

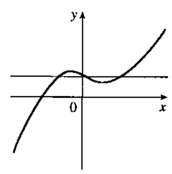
3. f is not one-to-one because $2 \neq 6$, but $f(2) = f(6)$.

4. f is one-to-one since for any two different domain values, there are different range values.

5. A football will attain every height h up to its maximum height twice: once on the way up, and again on the way down. Thus, even if t_1 does not equal t_2, $f(t_1)$ may equal $f(t_2)$, so f is not 1-1.

6. f is not 1-1 because eventually we all stop growing and therefore, there are two times at which we have the same height.

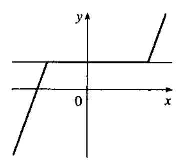
7. The diagram shows that there is a horizontal line which intersects the graph more than once, so the function is not one-to-one.



8. The function is one-to-one because no horizontal line intersects the graph more than once.

9. The function is one-to-one because no horizontal line intersects the graph more than once.

10. The diagram shows that there is a horizontal line which intersects the graph more than once, so the function is not one-to-one.



11. $f(x) = 4 - 3x \quad \Rightarrow \quad f'(x) = -3 < 0 \quad \Rightarrow \quad f$ is decreasing and hence one-to-one.

12. $f(x) = 3x^2 + 5x - 4 \quad \Rightarrow \quad f'(x) = 6x + 5 \quad \Rightarrow \quad f'(x) < 0$ and f is decreasing for $x < -\frac{5}{6}$ while $f'(x) > 0$ and f is increasing for $x > -\frac{5}{6}$. Thus, any horizontal line $y = k$ [where $k > f\left(-\frac{5}{6}\right)$] will intersect the graph of f more than once, so f fails the horizontal line test and is not 1-1.

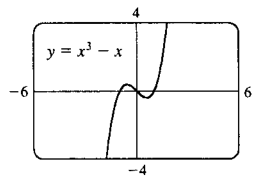
13. $x_1 \neq x_2 \quad \Rightarrow \quad \sqrt{x_1} \neq \sqrt{x_2} \quad \Rightarrow \quad g(x_1) \neq g(x_2)$, so g is 1-1.

14. $g(x) = |x| \quad \Rightarrow \quad g(-1) = 1 = g(1)$, so g is not one-to-one.

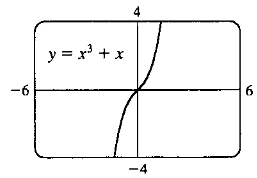
15. $h(x) = x^4 + 5 \quad \Rightarrow \quad h(1) = 6 = h(-1)$, so h is not 1-1.

16. $x_1 \neq x_2 \quad \Rightarrow \quad x_1^4 \neq x_2^4$ (since $x \geq 0$) $\quad \Rightarrow \quad x_1^4 + 5 \neq x_2^4 + 5 \quad \Rightarrow \quad h(x_1) \neq h(x_2)$, so h is 1-1.

17. f does not pass the Horizontal Line Test, so f is not 1-1.



18. f passes the Horizontal Line Test, so f is 1-1.



19. Since $f(2) = 9$ and f is 1-1, we know that $f^{-1}(9) = 2$. Remember, if the point $(2, 9)$ is on the graph of f, then the point $(9, 2)$ is on the graph of f^{-1}.

20. $f(x) = x + \cos x \Rightarrow f'(x) = 1 - \sin x \geq 0$, with equality only if $x = \frac{\pi}{2} + 2n\pi$. So f is increasing on $\mathbb{R}$, and hence, 1-1. By inspection, $f(0) = 0 + \cos 0 = 1$, so $f^{-1}(1) = 0$.

21. $h(x) = x + \sqrt{x} \Rightarrow h'(x) = 1 + 1/\left(2\sqrt{x}\right) > 0$ on $(0, \infty)$. So h is increasing and hence, 1-1. By inspection, $h(4) = 4 + \sqrt{4} = 6$, so $h^{-1}(6) = 4$.

22. (a) f is 1-1 because it passes the Horizontal Line Test.

(b) Domain of $f = [-3, 3] =$ Range of f^{-1}. Range of $f = [-2, 2] =$ Domain of f^{-1}.

(c) Since $f(-2) = 1$, $f^{-1}(1) = -2$.

23. We solve $C = \frac{5}{9}(F - 32)$ for F: $\frac{9}{5}C = F - 32 \Rightarrow F = \frac{9}{5}C + 32$. This gives us the Fahrenheit temperature F as a function of the Celsius temperature C. $F \geq -459.67 \Rightarrow \frac{9}{5}C + 32 \geq -459.67 \Rightarrow \frac{9}{5}C \geq -491.67 \Rightarrow C \geq -273.15$, the domain of the inverse function.

24. $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}} \Rightarrow 1 - \dfrac{v^2}{c^2} = \dfrac{m_0^2}{m^2} \Rightarrow \dfrac{v^2}{c^2} = 1 - \dfrac{m_0^2}{m^2} \Rightarrow v^2 = c^2\left(1 - \dfrac{m_0^2}{m^2}\right) \Rightarrow v = c\sqrt{1 - \dfrac{m_0^2}{m^2}}$.

This formula gives us the velocity v of the particle in terms of its mass m, that is, $v = f^{-1}(m)$.

25. $y = f(x) = 4x + 7 \Rightarrow 4x = y - 7 \Rightarrow x = (y - 7)/4$. Interchange x and y: $y = (x - 7)/4$. So $f^{-1}(x) = (x - 7)/4$.

26. $y = f(x) = \dfrac{x - 2}{x + 2} \Rightarrow xy + 2y = x - 2 \Rightarrow x(1 - y) = 2(y + 1) \Rightarrow x = \dfrac{2(1 + y)}{1 - y}$.

Interchange x and y: $y = \dfrac{2(1 + x)}{1 - x}$. So $f^{-1}(x) = \dfrac{2(1 + x)}{1 - x}$.

27. $y = f(x) = \dfrac{1 + 3x}{5 - 2x} \Rightarrow 5y - 2xy = 1 + 3x \Rightarrow 5y - 1 = 3x + 2xy \Rightarrow x(3 + 2y) = 5y - 1 \Rightarrow x = \dfrac{5y - 1}{2y + 3}$. Interchange x and y: $y = \dfrac{5x - 1}{2x + 3}$. So $f^{-1}(x) = \dfrac{5x - 1}{2x + 3}$.

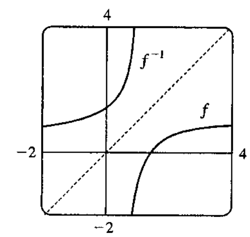
28. $y = f(x) = 5 - 4x^3 \Rightarrow 4x^3 = 5 - y \Rightarrow x^3 = (5 - y)/4 \Rightarrow x = \left(\dfrac{5 - y}{4}\right)^{1/3}$.

Interchange x and y: $y = \left(\dfrac{5 - x}{4}\right)^{1/3}$. So $f^{-1}(x) = \left(\dfrac{5 - x}{4}\right)^{1/3}$.

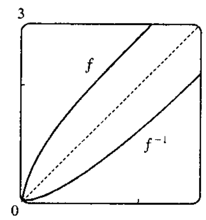
29. $y = f(x) = \sqrt{2 + 5x} \Rightarrow y^2 = 2 + 5x$ and $y \geq 0 \Rightarrow 5x = y^2 - 2 \Rightarrow x = \dfrac{y^2 - 2}{5}$, $y \geq 0$. Interchange x and y: $y = \dfrac{x^2 - 2}{5}$, $x \geq 0$. So $f^{-1}(x) = \dfrac{x^2 - 2}{5}$, $x \geq 0$.

30. $y = f(x) = x^2 + x \Rightarrow x^2 + x - y = 0 \Rightarrow x = \frac{1}{2}\left(-1 \pm \sqrt{1+4y}\right)$ by the quadratic formula. But $x \geq -\frac{1}{2}$ $\Rightarrow x = \frac{1}{2}\left(-1 + \sqrt{1+4y}\right)$.
Interchange x and y: $y = \frac{1}{2}\left(-1 + \sqrt{1+4x}\right)$. So $f^{-1}(x) = \frac{1}{2}\left(-1 + \sqrt{1+4x}\right)$.

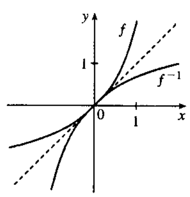
31. $y = f(x) = 1 - \dfrac{2}{x^2} \Rightarrow 1 - y = \dfrac{2}{x^2} \Rightarrow x^2 = \dfrac{2}{1-y} \Rightarrow$
$x = \sqrt{\dfrac{2}{1-y}}$, since $x > 0$. Interchange x and y: $y = \sqrt{\dfrac{2}{1-x}}$. So
$f^{-1}(x) = \sqrt{\dfrac{2}{1-x}}$.



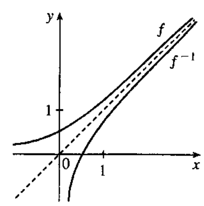
32. $y = f(x) = \sqrt{x^2 + 2x}, x > 0 \Rightarrow y > 0$ and $y^2 = x^2 + 2x \Rightarrow$
$x^2 + 2x - y^2 = 0$. Now we use the quadratic formula:
$x = \dfrac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-y^2)}}{2 \cdot 1} = -1 \pm \sqrt{1+y^2}$. But $x > 0$, so the negative root is inadmissible. Interchange x and y: $y = -1 + \sqrt{1+x^2}$. So
$f^{-1}(x) = -1 + \sqrt{1+x^2}, x > 0$.

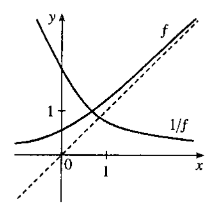


33. The function f is one-to-one, so its inverse exists and the graph of its inverse can be obtained by reflecting the graph of f through the line $y = x$.



34. For the graph of $1/f$, the y-coordinates are simply the reciprocals of f. For example, if $f(0) = \frac{1}{2}$, then $1/f(0) = 2$. If we draw the horizontal line $y = 1$, we see that the only place where the graphs intersect is on that line.





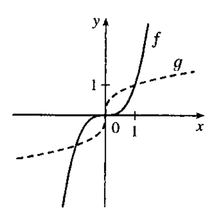
35. (a) $x_1 \neq x_2 \Rightarrow x_1^3 \neq x_2^3 \Rightarrow f(x_1) \neq f(x_2)$, so f is one-to-one.

(b) $f'(x) = 3x^2$ and $f(2) = 8 \Rightarrow g(8) = 2$, so $g'(8) = 1/f'(g(8)) = 1/f'(2) = \frac{1}{12}$.

(c) $y = x^3 \Rightarrow x = y^{1/3}$. Interchanging x and y gives $y = x^{1/3}$, so $f^{-1}(x) = x^{1/3}$. Domain(g) = range(f) = $\mathbb{R}$. Range(g) = domain(f) = $\mathbb{R}$.

(d) $g(x) = x^{1/3} \Rightarrow g'(x) = \frac{1}{3}x^{-2/3} \Rightarrow g'(8) = \frac{1}{3}\left(\frac{1}{4}\right) = \frac{1}{12}$ as in part (b).

(e)



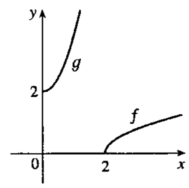
36. (a) $x_1 \neq x_2 \Rightarrow x_1 - 2 \neq x_2 - 2 \Rightarrow \sqrt{x_1 - 2} \neq \sqrt{x_2 - 2} \Rightarrow f(x_1) \neq f(x_2)$, so f is 1-1.

(b) $f(6) = 2$, so $g(2) = 6$. Also $f'(x) = \dfrac{1}{2\sqrt{x-2}}$, so $g'(2) = \dfrac{1}{f'(g(2))} = \dfrac{1}{f'(6)} = \dfrac{1}{1/4} = 4$.

(c) $y = \sqrt{x-2} \Rightarrow y^2 = x - 2 \Rightarrow x = y^2 + 2$. Interchange x and y: $y = x^2 + 2$. So $g(x) = x^2 + 2$.

(d) Domain $= [0, \infty)$, range $= [2, \infty)$. $g'(x) = 2x \Rightarrow g'(2) = 4$.

(e)



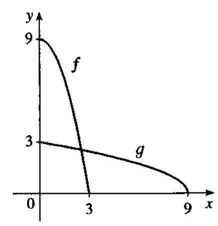
37. (a) Since $x \geq 0$, $x_1 \neq x_2 \Rightarrow x_1^2 \neq x_2^2 \Rightarrow 9 - x_1^2 \neq 9 - x_2^2 \Rightarrow f(x_1) \neq f(x_2)$, so f is 1-1.

(b) $f'(x) = -2x$ and $f(1) = 8 \Rightarrow g(8) = 1$, so $g'(8) = \dfrac{1}{f'(g(8))} = \dfrac{1}{f'(1)} = \dfrac{1}{(-2)} = -\dfrac{1}{2}$.

(c) $y = 9 - x^2 \Rightarrow x^2 = 9 - y \Rightarrow x = \sqrt{9-y}$. Interchange x and y: $y = \sqrt{9-x}$, so $f^{-1}(x) = \sqrt{9-x}$.
Domain(g) = range(f) = $[0, 9]$. Range(g) = domain(f) = $[0, 3]$.

(d) $g'(x) = -1/\left(2\sqrt{9-x}\right) \Rightarrow g'(8) = -\frac{1}{2}$ as in part (b).

(e)



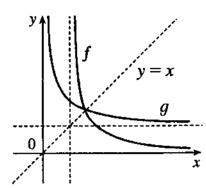
38. (a) $x_1 \neq x_2 \Rightarrow x_1 - 1 \neq x_2 - 1 \Rightarrow \dfrac{1}{x_1 - 1} \neq \dfrac{1}{x_2 - 1} \Rightarrow$
$f(x_1) \neq f(x_2)$, so f is 1-1.

(b) $g(2) = \frac{3}{2}$ since $f\left(\frac{3}{2}\right) = 2$. Also $f'(x) = -1/(x-1)^2$, so
$g'(2) = 1/f'\left(\frac{3}{2}\right) = \frac{1}{-4} = -\frac{1}{4}$.

(c) $y = 1/(x-1) \Rightarrow x - 1 = 1/y \Rightarrow x = 1 + 1/y$. Interchange x and y: $y = 1 + 1/x$. So $g(x) = 1 + 1/x$, $x > 0$ (since $y > 1$).
Domain $= (0, \infty)$, range $= (1, \infty)$.

(d) $g'(x) = -1/x^2$, so $g'(2) = -\frac{1}{4}$.

(e)



39. $f(0) = 1 \Rightarrow f^{-1}(1) = 0$, and $f(x) = x^3 + x + 1 \Rightarrow f'(x) = 3x^2 + 1$ and $f'(0) = 1$. Thus,
$(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))} = \dfrac{1}{f'(0)} = \dfrac{1}{1} = 1$.

40. $f(1) = 2 \Rightarrow f^{-1}(2) = 1$, and $f(x) = x^5 - x^3 + 2x \Rightarrow f'(x) = 5x^4 - 3x^2 + 2$ and $f'(1) = 4$. Thus,
$(f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))} = \dfrac{1}{f'(1)} = \dfrac{1}{4}$.

41. $f(0) = 3 \Rightarrow f^{-1}(3) = 0$, and $f(x) = 3 + x^2 + \tan(\pi x/2) \Rightarrow f'(x) = 2x + \frac{\pi}{2}\sec^2(\pi x/2)$ and
$f'(0) = 1 \cdot \frac{\pi}{2} = \frac{\pi}{2}$. Thus, $(f^{-1})'(3) = 1/f'(f^{-1}(3)) = 1/f'(0) = 2/\pi$.

42. $f(1) = 2 \;\Rightarrow\; f^{-1}(2) = 1$, and $f(x) = \sqrt{x^3 + x^2 + x + 1} \;\Rightarrow\; f'(x) = \dfrac{3x^2 + 2x + 1}{2\sqrt{x^3 + x^2 + x + 1}}$ and

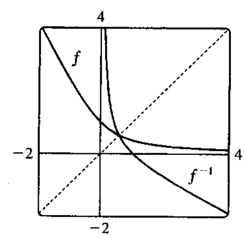
$f'(1) = \dfrac{3 + 2 + 1}{2\sqrt{1 + 1 + 1 + 1}} = \dfrac{3}{2}$. Thus, $(f^{-1})'(2) = 1/f'(f^{-1}(2)) = 1/f'(1) = \frac{2}{3}$.

43. $f(4) = 5 \;\Rightarrow\; g(5) = 4$. Thus, $g'(5) = \dfrac{1}{f'(g(5))} = \dfrac{1}{f'(4)} = \dfrac{1}{2/3} = \dfrac{3}{2}$.

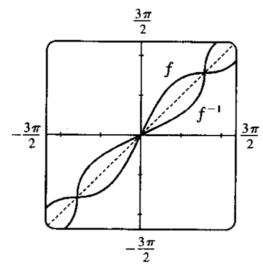
44. $f(3) = 2 \;\Rightarrow\; g(2) = 3$. Thus, $g'(2) = \dfrac{1}{f'(g(2))} = \dfrac{1}{f'(3)} = 9$. Hence, $G(x) = \dfrac{1}{g(x)} \;\Rightarrow$

$G'(x) = -\dfrac{g'(x)}{[g(x)]^2} \;\Rightarrow\; G'(2) = -\dfrac{g'(2)}{[g(2)]^2} = -\dfrac{9}{(3)^2} = -1$.

45. Since $f'(x) = \dfrac{2x}{2\sqrt{x^2+1}} - 1 = \dfrac{x - \sqrt{x^2+1}}{\sqrt{x^2+1}}$ is negative for all x, we know that f is a decreasing function on $\mathbb{R}$, and hence is 1-1. We could also use the Horizontal Line Test to show that f is 1-1. Parametric equations for the graph of f are $x = t$, $y = \sqrt{t^2+1} - t$; for the graph of f^{-1} they are $x = \sqrt{t^2+1} - t$, $y = t$.

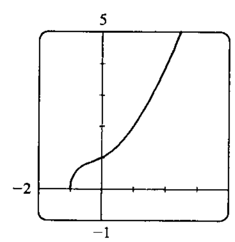


46. Since $f'(x) = 1 + \cos x \geq 0$ for all x, f is increasing and is therefore one-to-one. We can also use the Horizontal Line Test to show that f is 1-1. Parametric equations for the graph of f are $x = t$, $y = t + \sin t$; for the graph of f^{-1} they are $x = t + \sin t$, $y = t$.



47. We see that the graph of $y = f(x) = \sqrt{x^3 + x^2 + x + 1}$ is increasing, so f is 1-1. Enter $x = \sqrt{y^3 + y^2 + y + 1}$ and use your CAS to solve the equation for y. Using Derive, we get two (irrelevant) solutions involving imaginary expressions, as well as one which can be simplified to the following:

$$y = f^{-1}(x) = -\frac{\sqrt[3]{4}}{6}\left(\sqrt[3]{D - 27x^2 + 20} - \sqrt[3]{D + 27x^2 - 20} + \sqrt[3]{2}\right)$$



where $D = 3\sqrt{3}\sqrt{27x^4 - 40x^2 + 16}$. Maple and Mathematica each give two complex expressions and one real expression, and the real expression is equivalent to that given by Derive. For example, Maple's expression simplifies to $\dfrac{1}{6}\dfrac{M^{2/3} - 8 - 2M^{1/3}}{2M^{1/3}}$, where $M = 108x^2 + 12\sqrt{48 - 120x^2 + 81x^4} - 80$.

48. Since $\sin(2n\pi) = 0$, $h(x) = \sin x$ is not one-to-one. $h'(x) = \cos x > 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so h is increasing and hence 1-1 on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Let $y = f^{-1}(x) = \sin^{-1} x$ so that $\sin y = x$. Differentiating $\sin y = x$ implicitly with respect to x gives us $\cos y \dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\cos y}$. Now $\cos^2 y + \sin^2 y = 1 \Rightarrow \cos y = \pm\sqrt{1 - \sin^2 y}$, but since $\cos y > 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - \sin^2 y}} = \dfrac{1}{\sqrt{1 - x^2}}$.

49. (a) If the point (x, y) is on the graph of $y = f(x)$, then the point $(x - c, y)$ is that point shifted c units to the left. Since f is 1-1, the point (y, x) is on the graph of $y = f^{-1}(x)$ and the point corresponding to $(x - c, y)$ on the graph of f is $(y, x - c)$ on the graph of f^{-1}. Thus, the curve's reflection is shifted *down* the same number of units as the curve itself is shifted to the left. So an expression for the inverse function is $g^{-1}(x) = f^{-1}(x) - c$.

(b) If we compress (or stretch) a curve horizontally, the curve's reflection in the line $y = x$ is compressed (or stretched) *vertically* by the same factor. Using this geometric principle, we see that the inverse of $h(x) = f(cx)$ can be expressed as $h^{-1}(x) = (1/c)\, f^{-1}(x)$.

50. (a) We know that $g'(x) = \dfrac{1}{f'(g(x))}$. Thus,

$$g''(x) = -\frac{g'(x)\, f''(g(x))}{[f'(g(x))]^2} = -\frac{f''(g(x))}{f'(g(x))\,[f'(g(x))]^2} = -\frac{f''(g(x))}{[f'(g(x))]^3}.$$

(b) f is increasing $\Rightarrow f'(g(x)) > 0 \Rightarrow [f'(g(x))]^3 > 0$. f is concave upward $\Rightarrow f''(g(x)) > 0$. So $g''(x) = -\dfrac{f''(g(x))}{[f'(g(x))]^3} < 0$, which implies that g (f's inverse) is concave downward.

7.2 Exponential Functions and Their Derivatives

1. (a) $f(x) = a^x$, $a > 0$

(b) $\mathbb{R}$

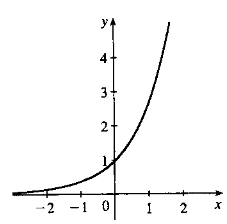
(c) $(0, \infty)$

(d) See Figures 6(c), 6(b), and 6(a), respectively.

2. (a) See Definition 8 and the paragraph which follows it.

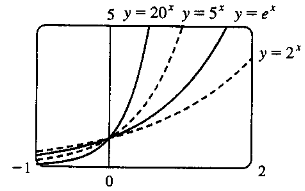
(b) $e \approx 2.71828$

(c)

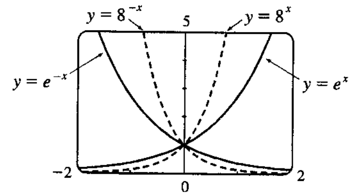


The function value at $x = 0$ is 1 and the slope at $x = 0$ is 1.

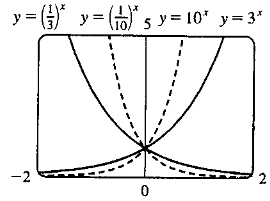
3. All of these graphs approach 0 as $x \to -\infty$, all of them pass through the point $(0, 1)$, and all of them are increasing and approach ∞ as $x \to \infty$. The larger the base, the faster the function increases for $x > 0$, and the faster it approaches 0 as $x \to -\infty$.



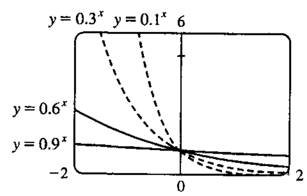
4. The graph of e^{-x} is the reflection of the graph of e^x about the y-axis, and the graph of 8^{-x} is the reflection of that of 8^x about the y-axis. The graph of 8^x increases more quickly than that of e^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.



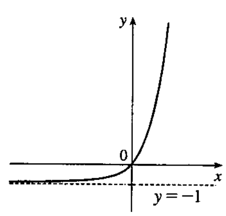
5. The functions with bases greater than 1 (3^x and 10^x) are increasing, while those with bases less than 1 $\left[\left(\frac{1}{3}\right)^x \text{ and } \left(\frac{1}{10}\right)^x\right]$ are decreasing. The graph of $\left(\frac{1}{3}\right)^x$ is the reflection of that of 3^x about the y-axis, and the graph of $\left(\frac{1}{10}\right)^x$ is the reflection of that of 10^x about the y-axis. The graph of 10^x increases more quickly than that of 3^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.



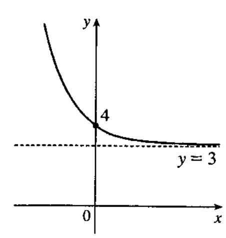
6. Each of the graphs approaches ∞ as $x \to -\infty$, and each approaches 0 as $x \to \infty$. The smaller the base, the faster the function grows as $x \to -\infty$, and the faster it approaches 0 as $x \to \infty$.



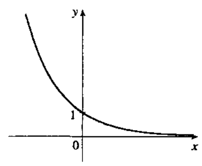
7. We start with the graph of $y = 10^x$ (Figure 3) and shift it 1 unit downward.



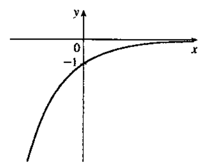
8. We start with the graph of $y = 2^x$ (Figure 3), reflect it about the y-axis ($y = 2^{-x}$), and then shift 3 units upward.



9. We start with the graph of $y = 2^x$ (Figure 3), reflect it about the y-axis ($y = 2^{-x}$), and then reflect about the x-axis.

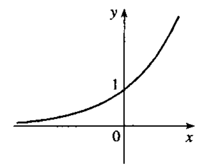


$y = 2^{-x}$

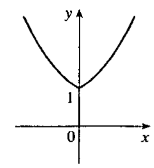


$y = -2^{-x}$

10. We reflect the part of $y = 2^x$ for $x > 0$ through the y-axis to obtain the part of $y = 2^{|x|}$ for $x < 0$.

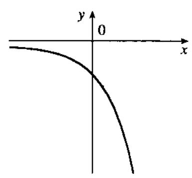


$y = 2^x$

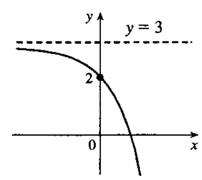


$y = 2^{|x|}$

11. We start with the graph of $y = e^x$ (Figure 13), reflect it about the x-axis, and then shift 3 units upward. Note the horizontal asymptote at $y = 3$.

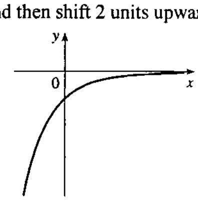


$y = -e^x$

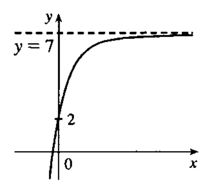


$y = 3 - e^x$

12. We start with the graph of $y = e^x$ (Figure 13), reflect it about the y-axis, and then about the x-axis (or just rotate 180° to handle both reflections) to obtain the graph of $y = -e^{-x}$. Now shift this graph 1 unit upward, vertically stretch by a factor of 5, and then shift 2 units upward.



$y = -e^{-x}$



$y = 2 + 5\left(1 - e^{-x}\right)$

13. (a) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units downward, we subtract 2 from the original function to get $y = e^x - 2$.

(b) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units to the right, we replace x with $x - 2$ in the original function to get $y = e^{(x-2)}$.

(c) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis, we multiply the original function by -1 to get $y = -e^x$.

(d) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the y-axis, we replace x with $-x$ in the original function to get $y = e^{-x}$.

(e) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis and then about the y-axis, we first multiply the original function by -1 (to get $y = -e^x$) and then replace x with $-x$ in this equation to get $y = -e^{-x}$.

14. (a) This reflection consists of first reflecting the graph through the x-axis (giving the graph with equation $y = -e^x$) and then shifting this graph $2 \cdot 4 = 8$ units upward. So the equation is $y = -e^x + 8$.

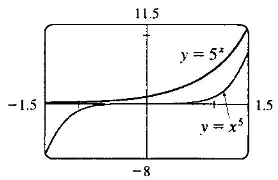
(b) This reflection consists of first reflecting the graph through the y-axis (giving the graph with equation $y = e^{-x}$) and then shifting this graph $2 \cdot 2 = 4$ units to the right. So the equation is $y = e^{-(x-4)}$.

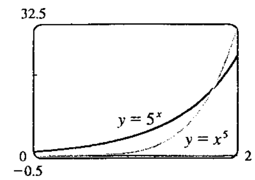
15. Use $y = Ca^x$ with the points $(1, 6)$ and $(3, 24)$. $6 = Ca^1$ and $24 = Ca^3 \Rightarrow 24 = \left(\frac{6}{a}\right) a^3 \Rightarrow 4 = a^2 \Rightarrow$ $a = 2$ (since $a > 0$) and $C = 3$. The function is $f(x) = 3 \cdot 2^x$.

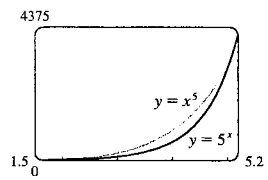
16. Given the y-intercept $(0, 2)$, we have $y = Ca^x = 2a^x$. Using the point $\left(2, \frac{2}{9}\right)$ gives us $\frac{2}{9} = 2a^2 \Rightarrow \frac{1}{9} = a^2$ $\Rightarrow a = \frac{1}{3}$ (since $a > 0$). The function is $f(x) = 2\left(\frac{1}{3}\right)^x$ or $f(x) = 2(3)^{-x}$.

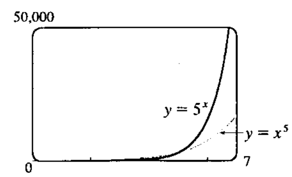
17. 2 ft $= 24$ in, $f(24) = 24^2$ in $= 576$ in $= 48$ ft. $g(24) = 2^{24}$ in $= 2^{24}/(12 \cdot 5280)$ mi ≈ 265 mi

18. We see from the graphs that for x less than about 1.8, $g(x) = 5^x > f(x) = x^5$, and then near the point $(1.8, 17.1)$ the curves intersect. Then $f(x) > g(x)$ from $x \approx 1.8$ until $x = 5$. At $(5, 3125)$ there is another point of intersection, and for $x > 5$ we see that $g(x) > f(x)$. In fact, g increases much more rapidly than f beyond that point.

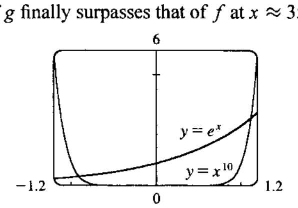


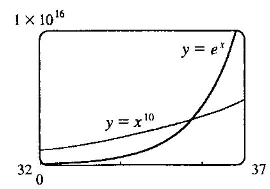




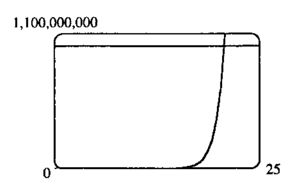


19. The graph of g finally surpasses that of f at $x \approx 35.8$.





20. We graph $y = e^x$ and $y = 1{,}000{,}000{,}000$ and determine where $e^x = 1 \times 10^9$. This seems to be true at $x \approx 20.723$, so $e^x > 1 \times 10^9$ for $x > 20.723$.



21. $\lim_{x\to\infty} (1.001)^x = \infty$ by (3), since $1.001 > 1$.

22. Let $t = -x^2$. As $x \to \infty$, $t \to -\infty$. So $\lim_{x\to\infty} e^{-x^2} = \lim_{t\to-\infty} e^t = 0$ by (11).

23. Divide numerator and denominator by e^{3x}: $\lim_{x\to\infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \lim_{x\to\infty} \dfrac{1 - e^{-6x}}{1 + e^{-6x}} = \dfrac{1-0}{1+0} = 1$

24. Divide numerator and denominator by e^{-3x}: $\lim_{x\to-\infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \lim_{x\to-\infty} \dfrac{e^{6x} - 1}{e^{6x} + 1} = \dfrac{0-1}{0+1} = -1$

25. Let $t = 3/(2-x)$. As $x \to 2^+$, $t \to -\infty$. So $\lim_{x\to 2^+} e^{3/(2-x)} = \lim_{t\to-\infty} e^t = 0$ by (11).

26. Let $t = 3/(2-x)$. As $x \to 2^-$, $t \to \infty$. So $\lim_{x\to 2^-} e^{3/(2-x)} = \lim_{t\to\infty} e^t = \infty$ by (11).

27. $\lim_{x\to\pi/2^-} \dfrac{2}{1 + e^{\tan x}} = 0$ since $\tan x \to \infty \;\Rightarrow\; e^{\tan x} \to \infty$.

28. As $x \to 0^-$, $\cot x = \dfrac{\cos x}{\sin x} \to -\infty$, so $e^{\cot x} \to 0$ and $\lim_{x\to 0} \dfrac{2}{1 + e^{\cot x}} = \dfrac{2}{1+0} = 2$.

29. By the Product Rule, $f(x) = x^2 e^x \;\Rightarrow\; f'(x) = x^2 \dfrac{d}{dx}(e^x) + e^x \dfrac{d}{dx}(x^2) = x^2 e^x + e^x(2x) = xe^x(x+2)$.

30. By the Quotient Rule, $y = \dfrac{e^x}{1+x} \;\Rightarrow\; y' = \dfrac{(1+x)e^x - e^x(1)}{(1+x)^2} = \dfrac{e^x + xe^x - e^x}{(x+1)^2} = \dfrac{xe^x}{(x+1)^2}$.

31. $y = e^{-mx} \;\Rightarrow\; y' = e^{-mx} \frac{d}{dx}(-mx) = e^{-mx}(-m) = -me^{-mx}$

32. $g(x) = e^{-5x} \cos 3x \;\Rightarrow\; g'(x) = -5e^{-5x} \cos 3x - 3e^{-5x} \sin 3x$

33. $f(x) = e^{\sqrt{x}} \;\Rightarrow\; f'(x) = e^{\sqrt{x}} / \left(2\sqrt{x}\right)$

34. By the Product Rule, $g(x) = \sqrt{x} e^x = x^{1/2} e^x \;\Rightarrow\; g'(x) = x^{1/2}(e^x) + e^x \left(\frac{1}{2} x^{-1/2}\right) = \frac{1}{2} x^{-1/2} e^x (2x+1)$.

35. $h(t) = \sqrt{1 - e^t} \;\Rightarrow\; h'(t) = -e^t / \left(2\sqrt{1 - e^t}\right)$

36. $h(\theta) = e^{\sin 5\theta} \;\Rightarrow\; h'(\theta) = 5\cos(5\theta)\, e^{\sin 5\theta}$

37. $y = e^{x \cos x} \;\Rightarrow\; y' = e^{x \cos x}(\cos x - x \sin x)$

38. $y = \cos(e^{\pi x}) \;\Rightarrow\; y' = -\sin(e^{\pi x}) \cdot e^{\pi x} \cdot \pi = -\pi e^{\pi x} \sin(e^{\pi x})$

39. $y = e^{e^x} \;\Rightarrow\; y' = e^{e^x} \cdot \frac{d}{dx}(e^x) = e^{e^x} \cdot e^x$ or $e^{e^x + x}$

40. $y = \sqrt{1 + xe^{-2x}} \;\Rightarrow\; y' = \frac{1}{2}\left(1 + xe^{-2x}\right)^{-1/2}\left[x\left(-2e^{-2x}\right) + e^{-2x}\right] = \dfrac{e^{-2x}(-2x+1)}{2\sqrt{1 + xe^{-2x}}}$

41. $y = \dfrac{e^{3x}}{1 + e^x} \;\Rightarrow\; y' = \dfrac{3e^{3x}(1 + e^x) - e^{3x}(e^x)}{(1+e^x)^2} = \dfrac{3e^{3x} + 3e^{4x} - e^{4x}}{(1+e^x)^2} = \dfrac{3e^{3x} + 2e^{4x}}{(1+e^x)^2}$

42. $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}} \;\Rightarrow$

$$y' = \frac{\left(e^x - e^{-x}\right)\left(e^x - e^{-x}\right) - \left(e^x + e^{-x}\right)\left(e^x + e^{-x}\right)}{\left(e^x - e^{-x}\right)^2} = \frac{\left(e^{2x} - 2 + e^{-2x}\right) - \left(e^{2x} + 2 + e^{-2x}\right)}{\left(e^x - e^{-x}\right)^2} = -\frac{4}{\left(e^x - e^{-x}\right)^2}$$

43. $y = f(x) = e^{-x} \sin x \;\Rightarrow\; f'(x) = -e^{-x} \sin x + e^{-x} \cos x \;\Rightarrow\; f'(\pi) = e^{-\pi}(\cos \pi - \sin \pi) = -e^{-\pi}$, so an equation of the tangent line at $(\pi, 0)$ is $y - 0 = -e^{-\pi}(x - \pi)$, or $y = -e^{-\pi} x + \pi e^{-\pi}$, or $x + e^{\pi} y = \pi$.

44. $y = \dfrac{e^x}{x} \;\Rightarrow\; y' = \dfrac{xe^x - e^x}{x^2} = \dfrac{e^x(x-1)}{x^2}$. At $x = 1$, $y' = 0$, so an equation of the tangent line at $(1, e)$ is $y - e = 0(x-1)$, or $y = e$.

45. $\cos(x-y) = xe^x \;\Rightarrow\; -\sin(x-y)(1-y') = e^x + xe^x \;\Rightarrow\; y' = 1 + \dfrac{e^x(1+x)}{\sin(x-y)}$

46. Using implicit differentiation, $2e^{xy} = x + y \;\Rightarrow\; (y + xy')2e^{xy} = 1 + y' \;\Rightarrow\; y'(2xe^{xy} - 1) = 1 - 2ye^{xy}$ $\Rightarrow\; y' = (1 - 2ye^{xy})/(2xe^{xy} - 1)$. So at $(0, 2)$, $m = y' = 3$, and an equation of the tangent line is $y - 2 = 3(x-0) \;\Rightarrow\; y = 3x + 2$.

47. $y = e^{2x} + e^{-3x} \;\Rightarrow\; y' = 2e^{2x} - 3e^{-3x} \;\Rightarrow\; y'' = 4e^{2x} + 9e^{-3x}$, so $y'' + y' - 6y = (4e^{2x} + 9e^{-3x}) + (2e^{2x} - 3e^{-3x}) - 6(e^{2x} + e^{-3x}) = 0$.

48. $y = Ae^{-x} + Bxe^{-x} \;\Rightarrow\; y' = -Ae^{-x} + Be^{-x} - Bxe^{-x} = (B-A)e^{-x} - Bxe^{-x} \;\Rightarrow$
$y'' = (A-B)e^{-x} - Be^{-x} + Bxe^{-x} = (A-2B)e^{-x} + Bxe^{-x}$, so
$y'' + 2y' + y = (A-2B)e^{-x} + Bxe^{-x} + 2\left[(B-A)e^{-x} - Bxe^{-x}\right] + Ae^{-x} + Bxe^{-x} = 0$.

49. $y = e^{rx} \;\Rightarrow\; y' = re^{rx} \;\Rightarrow\; y'' = r^2e^{rx}$, so
$y'' + 5y' - 6y = r^2e^{rx} + 5re^{rx} - 6e^{rx} = e^{rx}(r^2 + 5r - 6) = e^{rx}(r+6)(r-1) = 0 \;\Rightarrow\; (r+6)(r-1) = 0$ $\Rightarrow\; r = 1$ or -6.

50. $y = e^{\lambda x} \;\Rightarrow\; y' = \lambda e^{\lambda x} \;\Rightarrow\; y'' = \lambda^2 e^{\lambda x}$. Thus, $y + y' = y'' \;\Leftrightarrow\; e^{\lambda x} + \lambda e^{\lambda x} = \lambda^2 e^{\lambda x} \;\Leftrightarrow$ $e^{\lambda x}(\lambda^2 - \lambda - 1) = 0 \;\Leftrightarrow\; \lambda = \frac{1 \pm \sqrt{5}}{2}$, since $e^{\lambda x} \neq 0$.

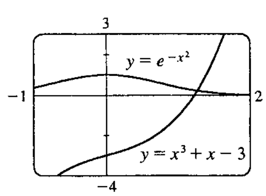
51. $f(x) = e^{2x} \;\Rightarrow\; f'(x) = 2e^{2x} \;\Rightarrow\; f''(x) = 2 \cdot 2e^{2x} = 2^2e^{2x} \;\Rightarrow\; f'''(x) = 2^2 \cdot 2e^{2x} = 2^3e^{2x} \;\Rightarrow\; \cdots$ $\Rightarrow\; f^{(n)}(x) = 2^n e^{2x}$

52. $f(x) = xe^{-x}$, $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$, $f''(x) = -e^{-x} + (1-x)(-e^{-x}) = (x-2)e^{-x}$. Similarly, $f'''(x) = (3-x)e^{-x}$, $f^{(4)}(x) = (x-4)e^{-x}$, $\ldots$, $f^{(1000)}(x) = (x - 1000)e^{-x}$.

53. (a) $f(x) = e^x + x$ is continuous on $\mathbb{R}$ and $f(-1) = e^{-1} - 1 < 0 < 1 = f(0)$, so by the Intermediate Value Theorem, $e^x + x = 0$ has a root in $(-1, 0)$.

(b) $f(x) = e^x + x \;\Rightarrow\; f'(x) = e^x + 1$, so $x_{n+1} = x_n - \dfrac{e^{x_n} + x_n}{e^{x_n} + 1}$. Using $x_1 = -0.5$, we get $x_2 \approx -0.566311$, $x_3 \approx -0.567143 \approx x_4$, so the root is -0.567143 to six decimal places.

54.



From the graph, it appears that the curves intersect at about $x \approx 1.2$ or 1.3. We use Newton's Method with $f(x) = x^3 + x - 3 - e^{-x^2}$, so $f'(x) = 3x^2 + 1 - 2xe^{-x^2}$, and the formula is $x_{n+1} = x_n - f(x_n)/f'(x_n)$. We take $x_1 = 1.2$, and the formula gives $x_2 \approx 1.252462$, $x_3 \approx 1.251045$, and $x_4 \approx x_5 \approx 1.251044$. So the root of the equation, correct to six decimal places, is $x = 1.251044$.

55. (a) $m(t) = 24 \cdot 2^{-t/25} \quad \Rightarrow \quad m(40) = 24 \cdot 2^{-40/25} \approx 7.92$ mg

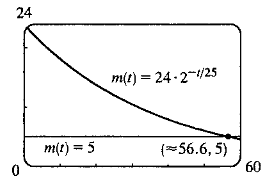
(b) $m'(t) = 24\dfrac{d}{dt}\left(2^{-t/25}\right)$

$\approx 24(0.69)\, 2^{-t/25}\dfrac{d}{dt}\left(-\dfrac{t}{25}\right)$

$= 24(0.69)\left(-\frac{1}{25}\right)2^{-t/25}$

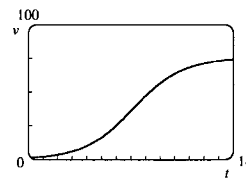
so $m'(40) \approx -\frac{24}{25}(0.69)\,2^{-40/25} \approx -0.22$ mg/yr.

(c)



From the graph, we can determine that $m(t) = 5 \quad \Rightarrow \quad t \approx 56.6$ y.

56. From the graph, we estimate that the most rapid increase in the number of VCRs occurs at about $t = 7$. To maximize the first derivative, we need to determine the values for which the second derivative is 0. $V(t) = \dfrac{75}{1 + 74e^{-0.6t}} \quad \Rightarrow$



$$V'(t) = -\frac{75\left[74e^{-0.6t}(-0.6)\right]}{\left(1 + 74e^{-0.6t}\right)^2} = \frac{3330e^{-0.6t}}{\left(1 + 74e^{-0.6t}\right)^2} \quad \Rightarrow$$

$$V''(t) = \frac{\left(1 + 74e^{-0.6t}\right)^2\left[3330e^{-0.6t}(-0.6)\right] - \left(3330e^{-0.6t}\right)2\left(1 + 74e^{-0.6t}\right)\left[74e^{-0.6t}(-0.6)\right]}{\left[\left(1 + 74e^{-0.6t}\right)^2\right]^2}$$

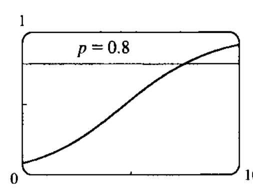
$$= \frac{\left(1 + 74e^{-0.6t}\right)\left[3330e^{-0.6t}(-0.6)\right]\left[\left(1 + 74e^{-0.6t}\right) - 2\left(74e^{-0.6t}\right)\right]}{\left(1 + 74e^{-0.6t}\right)^4} = \frac{-1998e^{-0.6t}\left(1 - 74e^{-0.6t}\right)}{\left(1 + 74e^{-0.6t}\right)^3}$$

$V''(t) = 0 \quad \Leftrightarrow \quad 1 = 74e^{-0.6t} \quad \Leftrightarrow \quad e^{0.6t} = 74 \quad \Leftrightarrow \quad 0.6t = \ln 74 \quad \Leftrightarrow \quad t = \frac{5}{3}\ln 74 \approx 7.173$ years, which corresponds to early September 1987.

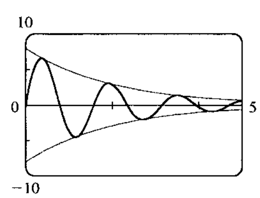
57. (a) $\displaystyle\lim_{t\to\infty} p(t) = \lim_{t\to\infty}\frac{1}{1 + ae^{-kt}} = \frac{1}{1 + a \cdot 0} = 1$, since $k > 0 \quad \Rightarrow$ $-kt \to -\infty$ as $t \to \infty$.

(b) $\dfrac{dp}{dt} = -\left(1 + ae^{-kt}\right)^{-2}\left(-kae^{-kt}\right) = \dfrac{kae^{-kt}}{\left(1 + ae^{-kt}\right)^2}$

(c) From the graph, it seems that $p(t) = 0.8$ (indicating that 80% of the population has heard the rumor) when $t \approx 7.4$ hours



58. (a)



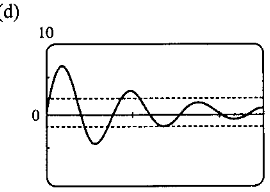
The displacement function is squeezed between the other two functions. This is because $-1 \leq \sin 4t \leq 1 \quad \Rightarrow$ $-8e^{-t/2} \leq 8e^{-t/2}\sin 4t \leq 8e^{-t/2}$.

(b) The maximum value of the displacement is about 6.6 cm, occurring at $t \approx 0.36$ s. It occurs just before the graph of the displacement function touches the graph of $8e^{-t/2}$ (when $t = \frac{\pi}{8} \approx 0.39$).

(c) The velocity of the object is the derivative of its displacement function, that is,

$\frac{d}{dt}\left(8e^{-t/2}\sin 4t\right) = 8\left[e^{-t/2}\cos 4t\,(4) + \sin 4t\left(-\frac{1}{2}\right)e^{-t/2}\right]$. If the displacement is zero, then we must have $\sin 4t = 0$ (since the exponential term in the displacement function is always positive). The first time that $\sin 4t = 0$ after $t = 0$ occurs at $t = \frac{\pi}{4}$. Substituting this into our expression for the velocity, and noting that the second term vanishes, we get $v\left(\frac{\pi}{4}\right) = 8e^{-\pi/8}\cos\left(4\cdot\frac{\pi}{4}\right)\cdot 4 = -32e^{-\pi/8} \approx -21.6$ cm/s.

(d)

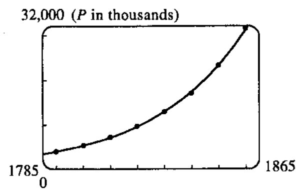


The graph indicates that the displacement is less than 2 cm from equilibrium whenever t is larger than about 2.8.

59. (a) Using a calculator or CAS, we obtain the model $Q = ab^t$ with $a = 100.0124369$ and $b = 0.000045145933$. We can change this model to one with base e and exponent $\ln b$: $Q = ae^{t\ln b} = 100.0124369e^{-10.00553063t}$.

(b) $Q'(t) = ab^t \ln b$. $Q'(0.04) \approx -670.63\ \mu$A. The result of Example 2 in Section 2.1 was $-670\ \mu$A.

60. (a) $P = ab^t$ or $P = ae^{t\ln b}$ with $a = 4.502714 \times 10^{-20}$ and $b = 1.029953851$, where P is measured in thousands of people. The fit appears to be very good.



(b) **For 1800:** $m_1 = \frac{5308 - 3929}{1800 - 1790} = 137.9$, $m_2 = \frac{7240 - 5308}{1810 - 1800} = 193.2$.

So $P'(1800) \approx (m_1 + m_2)/2 = 165.55$ thousand people/year.

For 1850: $m_1 = \frac{23{,}192 - 17{,}063}{1850 - 1840} = 612.9$, $m_2 = \frac{31{,}443 - 23{,}192}{1860 - 1850} = 825.1$.

So $P'(1850) \approx (m_1 + m_2)/2 = 719$ thousand people/year.

(c) $P'(t) = ab^t \ln b$. $P'(1800) \approx 156.85$ and $P'(1850) \approx 686.07$.

(d) $P(1870) \approx 41{,}946.56$. The difference of 3.4 million people is most likely due to the Civil War (1861–1865).

61. $f(x) = x - e^x \Rightarrow f'(x) = 1 - e^x = 0 \Leftrightarrow e^x = 1 \Leftrightarrow x = 0$. Now $f'(x) > 0$ for all $x < 0$ and $f'(x) < 0$ for all $x > 0$, so the absolute maximum value is $f(0) = 0 - 1 = -1$.

62. $g(x) = \frac{e^x}{x} \Rightarrow g'(x) = \frac{xe^x - e^x}{x^2} = 0 \Leftrightarrow e^x(x-1) = 0 \Rightarrow x = 1$. Now $g'(x) > 0 \Leftrightarrow \frac{xe^x - e^x}{x^2} > 0 \Leftrightarrow x - 1 > 0 \Leftrightarrow x > 1$ and $g'(x) < 0 \Leftrightarrow \frac{xe^x - e^x}{x^2} < 0 \Leftrightarrow x - 1 < 0 \Leftrightarrow x < 1$. Thus there is an absolute minimum value of $g(1) = e$ at $x = 1$.

63. (a) $f(x) = xe^x \Rightarrow f'(x) = e^x + xe^x = e^x(1+x) > 0 \Leftrightarrow 1 + x > 0 \Leftrightarrow x > -1$, so f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$.

(b) $f''(x) = e^x(1+x) + e^x = e^x(2+x) > 0 \Leftrightarrow 2 + x > 0 \Leftrightarrow x > -2$, so f is CU on $(-2, \infty)$ and CD on $(-\infty, -2)$.

(c) f has an inflection point at $\left(-2, -2e^{-2}\right)$.

64. (a) $f(x) = x^2e^x \Rightarrow f'(x) = 2xe^x + x^2e^x = (x^2+2x)e^x$. $f'(x) > 0 \Leftrightarrow x(x+2) > 0 \Leftrightarrow x < -2$ or $x > 0$, $f'(x) < 0 \Leftrightarrow -2 < x < 0$, so f is increasing on $(-\infty, -2)$ and $(0, \infty)$ and decreasing on $(-2, 0)$.

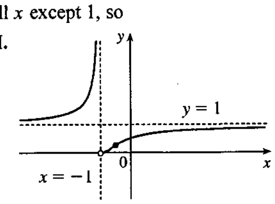
(b) $f''(x) = (2x+2)e^x + (x^2+2x)e^x = (x^2+4x+2)e^x = 0 \Leftrightarrow x^2+4x+2 = 0 \Leftrightarrow x = -2 \pm \sqrt{2}$. $f''(x) > 0$ when $x > -2+\sqrt{2}$ or $x < -2-\sqrt{2}$, so f is CU on $\left(-\infty, -2-\sqrt{2}\right)$ and $\left(-2+\sqrt{2}, \infty\right)$ and CD on $\left(-2-\sqrt{2}, -2+\sqrt{2}\right)$.

(c) f has inflection points at $\left(-2+\sqrt{2}, \left(6-4\sqrt{2}\right)e^{-2+\sqrt{2}}\right)$ and $\left(-2-\sqrt{2}, \left(6+4\sqrt{2}\right)e^{-2-\sqrt{2}}\right)$.

65. $y = f(x) = e^{-1/(x+1)}$ **A.** $D = \{x \mid x \neq -1\} = (-\infty, -1) \cup (-1, \infty)$ **B.** No x-intercept; y-intercept $= f(0) = e^{-1}$ **C.** No symmetry **D.** $\lim\limits_{x\to\pm\infty} e^{-1/(x+1)} = 1$ since $-1/(x+1) \to 0$, so $y = 1$ is a HA. $\lim\limits_{x\to-1^+} e^{-1/(x+1)} = 0$ since $-1/(x+1) \to -\infty$, $\lim\limits_{x\to-1^-} e^{-1/(x+1)} = \infty$ since $-1/(x+1) \to \infty$, so $x = -1$ is a VA. **E.** $f'(x) = e^{-1/(x+1)}/(x+1)^2 \Rightarrow f'(x) > 0$ for all x except 1, so f is increasing on $(-\infty, -1)$ and $(-1, \infty)$. **F.** No extrema

G. $f''(x) = \dfrac{e^{-1/(x+1)}}{(x+1)^4} + \dfrac{e^{-1/(x+1)}(-2)}{(x+1)^3} = -\dfrac{e^{-1/(x+1)}(2x+1)}{(x+1)^4}$ $\Rightarrow f''(x) > 0 \Leftrightarrow 2x+1 < 0 \Leftrightarrow x < -\frac{1}{2}$, so f is CU on $(-\infty, -1)$ and $\left(-1, -\frac{1}{2}\right)$, and CD on $\left(-\frac{1}{2}, \infty\right)$. f has an IP at $\left(-\frac{1}{2}, e^{-2}\right)$.

H.



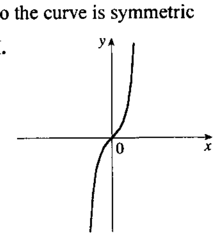
66. $y = f(x) = xe^{x^2}$ **A.** $D = \mathbb{R}$ **B.** Both intercepts are 0. **C.** $f(-x) = -f(x)$, so the curve is symmetric about the origin. **D.** $\lim\limits_{x\to\infty} xe^{x^2} = \infty$, $\lim\limits_{x\to-\infty} xe^{x^2} = -\infty$, no asymptote

E. $f'(x) = e^{x^2} + xe^{x^2}(2x) = e^{x^2}\left(1+2x^2\right) > 0$, so f is increasing on $\mathbb{R}$.

F. No extremum

G. $f''(x) = e^{x^2}(2x)\left(1+2x^2\right) + e^{x^2}(4x) = e^{x^2}(2x)\left(3+2x^2\right) > 0 \Leftrightarrow$ $x > 0$, so f is CU on $(0, \infty)$and CD on $(-\infty, 0)$. f has an inflection point at $(0, 0)$.

H.

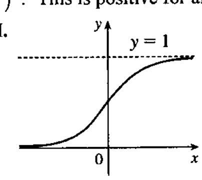


67. $y = 1/\left(1+e^{-x}\right)$ **A.** $D = \mathbb{R}$ **B.** No x-intercepts; y-intercept $= f(0) = \frac{1}{2}$. **C.** No symmetry **D.** $\lim\limits_{x\to\infty} 1/\left(1+e^{-x}\right) = \frac{1}{1+0} = 1$ and $\lim\limits_{x\to-\infty} 1/\left(1+e^{-x}\right) = 0$ (since $\lim\limits_{x\to-\infty} e^{-x} = \infty$), so f has horizontal asymptotes $y = 0$ and $y = 1$. **E.** $f'(x) = -\left(1+e^{-x}\right)^{-2}\left(-e^{-x}\right) = e^{-x}/\left(1+e^{-x}\right)^2$. This is positive for all x, so f is increasing on $\mathbb{R}$. **F.** No extrema **G.** $f''(x) =$

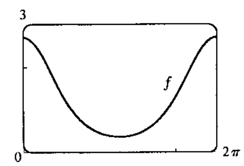
$$\frac{\left(1+e^{-x}\right)^2\left(-e^{-x}\right) - e^{-x}(2)\left(1+e^{-x}\right)\left(-e^{-x}\right)}{\left(1+e^{-x}\right)^4} = \frac{e^{-x}\left(e^{-x}-1\right)}{\left(1+e^{-x}\right)^3}.$$

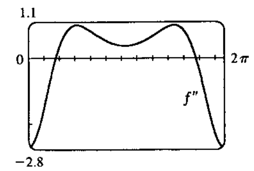
The second factor in the numerator is negative for $x > 0$ and positive for $x < 0$, and the other factors are always positive, so f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. f has an inflection point at $\left(0, \frac{1}{2}\right)$.

H.



68. The function $f(x) = e^{\cos x}$ is periodic with period 2π, so we consider it only on the interval $[0, 2\pi]$. We see that it has local maxima of about $f(0) \approx 2.72$ and $f(2\pi) \approx 2.72$, and a local minimum of about $f(3.14) \approx 0.37$. To find the exact values, we calculate $f'(x) = -\sin x e^{\cos x}$. This is 0 when $-\sin x = 0 \Leftrightarrow x = 0, \pi$ or 2π (since we are only considering $x \in [0, 2\pi]$). Also $f'(x) > 0 \Leftrightarrow \sin x < 0 \Leftrightarrow 0 < x < \pi$. So $f(0) = f(2\pi) = e$ (both maxima) and $f(\pi) = e^{\cos \pi} = 1/e$ (minimum). To find the inflection points, we calculate and graph $f''(x) = \frac{d}{dx}\left(-\sin x\, e^{\cos x}\right) = -\cos x\, e^{\cos x} - \sin x\left(e^{\cos x}\right)(-\sin x) = e^{\cos x}\left(\sin^2 x - \cos x\right)$. From the graph of $f''(x)$, we see that f has inflection points at $x \approx 0.90$ and at $x \approx 5.38$. These x-coordinates correspond to inflection points $(0.90, 1.86)$ and $(5.38, 1.86)$.

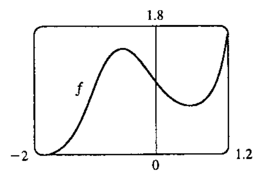


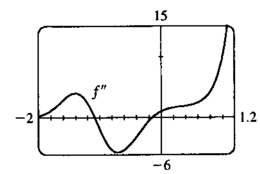


69. $f(x) = e^{x^3-x} \to 0$ as $x \to -\infty$, and $f(x) \to \infty$ as $x \to \infty$. From the graph, it appears that f has a local minimum of about $f(0.58) = 0.68$, and a local maximum of about $f(-0.58) = 1.47$. To find the exact values, we calculate $f'(x) = \left(3x^2 - 1\right)e^{x^3-x}$, which is 0 when $3x^2 - 1 = 0 \Leftrightarrow x = \pm\frac{1}{\sqrt{3}}$. The negative root corresponds to the local maximum $f\left(-\frac{1}{\sqrt{3}}\right) = e^{(-1/\sqrt{3})^3 - (-1/\sqrt{3})} = e^{2\sqrt{3}/9}$, and the positive root corresponds to the local minimum $f\left(\frac{1}{\sqrt{3}}\right) = e^{(1/\sqrt{3})^3 - (1/\sqrt{3})} = e^{-2\sqrt{3}/9}$. To estimate the inflection points, we calculate and graph

$f''(x) = \frac{d}{dx}\left[\left(3x^2 - 1\right)e^{x^3-x}\right] = \left(3x^2 - 1\right)e^{x^3-x}\left(3x^2 - 1\right) + e^{x^3-x}(6x) = e^{x^3-x}\left(9x^4 - 6x^2 + 6x + 1\right)$.

From the graph, it appears that $f''(x)$ changes sign (and thus f has inflection points) at $x \approx -0.15$ and $x \approx -1.09$. From the graph of f, we see that these x-values correspond to inflection points at about $(-0.15, 1.15)$ and $(-1.09, 0.82)$.





70. (a) As $|x| \to \infty$, $t = -x^2/\left(2\sigma^2\right) \to -\infty$, and $e^t \to 0$. The HA is $y = 0$. Since t takes on its maximum value at $x = 0$, so does e^t. Showing this result using derivatives, we have $f(x) = e^{-x^2/(2\sigma^2)} \Rightarrow$

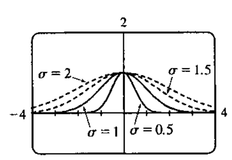
$f'(x) = e^{-x^2/(2\sigma^2)}\left(-x/\sigma^2\right)$. $f'(x) = 0 \Leftrightarrow x = 0$. Because f' changes from positive to negative at $x = 0$, $f(0) = 1$ is a local maximum. For inflection points, we find

$$f''(x) = -\frac{1}{\sigma^2}\left[e^{-x^2/(2\sigma^2)} \cdot 1 + xe^{-x^2/(2\sigma^2)}\left(-x/\sigma^2\right)\right] = \frac{-1}{\sigma^2}e^{-x^2/(2\sigma^2)}\left(1 - x^2/\sigma^2\right).$$

$f''(x) = 0 \Leftrightarrow x^2 = \sigma^2 \Leftrightarrow x = \pm\sigma$. $f''(x) < 0 \Leftrightarrow x^2 < \sigma^2 \Leftrightarrow -\sigma < x < \sigma$. So f is CD on $(-\sigma, \sigma)$ and CU on $(-\infty, -\sigma)$ and (σ, ∞). IP at $\left(\pm\sigma, e^{-1/2}\right)$.

(b) Since we have IP at $x = \pm\sigma$, the inflection points move away from the y-axis as σ increases.

(c)



From the graph, we see that as σ increases, the graph tends to spread out and there is more area between the curve and the x-axis.

71. Let $u = -3x$. Then $du = -3\,dx$, so
$\int_0^5 e^{-3x}\,dx = -\frac{1}{3}\int_0^{-15} e^u\,du = -\frac{1}{3}\left[e^u\right]_0^{-15} = -\frac{1}{3}\left(e^{-15} - e^0\right) = \frac{1}{3}\left(1 - e^{-15}\right)$.

72. Let $u = -x^2$, so $du = -2x\,dx$. When $x = 0$, $u = 0$; when $x = 1$, $u = -1$. Therefore,

$$\int_0^1 xe^{-x^2}\,dx = \int_0^{-1} e^u\left(-\tfrac{1}{2}\,du\right) = -\tfrac{1}{2}\left[e^u\right]_0^{-1} = -\tfrac{1}{2}\left(e^{-1} - e^0\right) = \tfrac{1}{2}\left(1 - 1/e\right)$$

73. Let $u = 1 + e^x$. Then $du = e^x\,dx$, so $\int e^x\sqrt{1+e^x}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}\left(1+e^x\right)^{3/2} + C$.

74. Let $u = \tan x$. Then $du = \sec^2 x\,dx$, so $\int \sec^2 x\, e^{\tan x}\,dx = \int e^u\,du = e^u + C = e^{\tan x} + C$.

75. $\int \dfrac{e^x + 1}{e^x}\,dx = \int \left(1 + e^{-x}\right)dx = x - e^{-x} + C$

76. Let $u = \dfrac{1}{x}$. Then $du = -\dfrac{1}{x^2}\,dx$, so $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx = -\int e^u\,du = -e^u + C = -e^{1/x} + C$.

77. Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}\,dx$, so $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C$.

78. Let $u = e^x$. Then $du = e^x\,dx$, so $\int e^x \sin\left(e^x\right)dx = \int \sin u\,du = -\cos u + C = -\cos\left(e^x\right) + C$.

79. Area $= \int_0^1 \left(e^{3x} - e^x\right)dx = \left[\frac{1}{3}e^{3x} - e^x\right]_0^1 = \left(\frac{1}{3}e^3 - e\right) - \left(\frac{1}{3} - 1\right) = \frac{1}{3}e^3 - e + \frac{2}{3} \approx 4.644$

80. $f''(x) = 3e^x + 5\sin x \;\Rightarrow\; f'(x) = 3e^x - 5\cos x + C \;\Rightarrow\; 2 = f'(0) = 3 - 5 + C \;\Rightarrow\; C = 4$, so
$f'(x) = 3e^x - 5\cos x + 4 \;\Rightarrow\; f(x) = 3e^x - 5\sin x + 4x + D \;\Rightarrow\; 1 = f(0) = 3 + D \;\Rightarrow\; D = -2$, so
$f(x) = 3e^x - 5\sin x + 4x - 2$.

81. $V = \int_0^1 \pi\left(e^x\right)^2 dx = \int_0^1 \pi e^{2x}\,dx = \frac{1}{2}\left[\pi e^{2x}\right]_0^1 = \frac{\pi}{2}\left(e^2 - 1\right)$

82. $V = \int_0^1 2\pi x e^{-x^2}\,dx$. Let $u = x^2$. Thus $du = 2x\,dx$, so $V = \pi\int_0^1 e^{-u}\,du = \pi\left[-e^{-u}\right]_0^1 = \pi\left(1 - 1/e\right)$.

83. We use Theorem 7.1.7. Note that $f(0) = 3 + 0 + e^0 = 4$, so $f^{-1}(4) = 0$. Also $f'(x) = 1 + e^x$. Therefore,
$\left(f^{-1}\right)'(4) = \dfrac{1}{f'\left(f^{-1}(4)\right)} = \dfrac{1}{f'(0)} = \dfrac{1}{1 + e^0} = \dfrac{1}{2}$.

84. We recognize this limit as the definition of the derivative of the function $f(x) = e^{\sin x}$ at $x = \pi$, since it is of the form $\displaystyle\lim_{x\to\pi}\frac{f(x) - f(\pi)}{x - \pi}$. Therefore, the limit is equal to $f'(\pi) = (\cos\pi)\,e^{\sin\pi} = -1\cdot e^0 = -1$.

85. (a) Let $f(x) = e^x - 1 - x$. Now $f(0) = e^0 - 1 = 0$, and for $x \geq 0$, we have $f'(x) = e^x - 1 \geq 0$. Now, since $f(0) = 0$ and f is increasing on $[0, \infty)$, $f(x) \geq 0$ for $x \geq 0 \;\Rightarrow\; e^x - 1 - x \geq 0 \;\Rightarrow\; e^x \geq 1 + x$.

(b) For $0 \leq x \leq 1$, $x^2 \leq x$, so $e^{x^2} \leq e^x$ (since e^x is increasing.) Hence [from (a)] $1 + x^2 \leq e^{x^2} \leq e^x$. So
$\frac{4}{3} = \int_0^1 \left(1 + x^2\right)dx \leq \int_0^1 e^{x^2}\,dx \leq \int_0^1 e^x\,dx = e - 1 < e \;\Rightarrow\; \frac{4}{3} \leq \int_0^1 e^{x^2}\,dx \leq e$.

86. (a) Let $f(x) = e^x - 1 - x - \frac{1}{2}x^2$. Thus, $f'(x) = e^x - 1 - x$, which is positive for $x \geq 0$ by Exercise 85(a). Thus $f(x)$ is increasing on $(0, \infty)$, so on that interval, $0 = f(0) \leq f(x) = e^x - 1 - x - \frac{1}{2}x^2 \;\Rightarrow$
$e^x \geq 1 + x + \frac{1}{2}x^2$.

(b) Using the same argument as in Exercise 85(b), from part (a) we have $1 + x^2 + \frac{1}{2}x^4 \leq e^{x^2} \leq e^x$ (for $0 \leq x \leq 1$)
$\Rightarrow\; \int_0^1 \left(1 + x^2 + \frac{1}{2}x^4\right)dx \leq \int_0^1 e^{x^2}\,dx \leq \int_0^1 e^x\,dx \;\Rightarrow\; \frac{43}{30} \leq \int_0^1 e^{x^2}\,dx \leq e - 1$.

87. (a) By Exercise 85(a), the result holds for $n = 1$. Suppose that $e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!}$ for $x \geq 0$. Let $f(x) = e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^k}{k!} - \frac{x^{k+1}}{(k+1)!}$. Then $f'(x) = e^x - 1 - x - \cdots - \frac{x^k}{k!} \geq 0$ by assumption. Hence $f(x)$ is increasing on $(0, \infty)$. So $0 \leq x$ implies that $0 = f(0) \leq f(x) = e^x - 1 - x - \cdots - \frac{x^k}{k!} - \frac{x^{k+1}}{(k+1)!}$, and hence $e^x \geq 1 + x + \cdots + \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!}$ for $x \geq 0$. Therefore, for $x \geq 0$, $e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ for every positive integer n, by mathematical induction.

(b) Taking $n = 4$ and $x = 1$ in (a), we have $e = e^1 \geq 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 2.708\overline{3} > 2.7$.

(c) $e^x \geq 1 + x + \cdots + \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!} \quad \Rightarrow \quad \frac{e^x}{x^k} \geq \frac{1}{x^k} + \frac{1}{x^{k-1}} + \cdots + \frac{1}{k!} + \frac{x}{(k+1)!} \geq \frac{x}{(k+1)!}$. But $\lim\limits_{x\to\infty} \frac{x}{(k+1)!} = \infty$, so $\lim\limits_{x\to\infty} \frac{e^x}{x^k} = \infty$.

7.3 Logarithmic Functions

1. (a) It is defined as the inverse of the exponential function with base a, that is, $\log_a x = y \quad \Leftrightarrow \quad a^y = x$.

(b) $(0, \infty)$ (c) $\mathbb{R}$ (d) See Figure 1.

2. (a) The logarithm with base e, denoted $\ln x$ (b) The logarithm with base 10, denoted $\log x$

(c) See Figure 3.

3. (a) $\log_{10} 1000 = 3$ because $10^3 = 1000$. *Or:* $\log_{10} 1000 = \log_{10} 10^3 = 3$ by (2).

(b) $\log_{16} 4 = \frac{1}{2}$ because $16^{1/2} = 4$. *Or:* $\log_{16} 4 = \log_{16} 16^{1/2} = \frac{1}{2}$ by (2).

4. (a) $\log_8 2 = \frac{1}{3}$ since $8^{1/3} = 2$. (b) $\ln e^{\sqrt{2}} = \sqrt{2}$

5. (a) $\log_5 \frac{1}{25} = \log_5 5^{-2} = -2$ by (2). (b) $e^{\ln 15} = 15$ by (6).

6. (a) $\log_{10} 0.1 = \log_{10} 10^{-1} = -1$ (b) $\log_3 108 - \log_3 4 = \log_3 \frac{108}{4} = \log_3 27 = 3$

7. (a) $\log_{10} 1.25 + \log_{10} 80 = \log_{10}(1.25 \cdot 80) = \log_{10} 100 = \log_{10} 10^2 = 2$

(b) $\log_5 10 + \log_5 20 - 3\log_5 2 = \log_5(10 \cdot 20) - \log_5 2^3 = \log_5 \frac{200}{8} = \log_5 25 = \log_5 5^2 = 2$

8. (a) $2^{(\log_2 3 + \log_2 5)} = 2^{\log_2 15} = 15$ [*Or:* $2^{(\log_2 3 + \log_2 5)} = 2^{\log_2 3} \cdot 2^{\log_2 5} = 3 \cdot 5 = 15$]

(b) $e^{3\ln 2} = e^{\ln(2^3)} = e^{\ln 8} = 8$ [*Or:* $e^{3\ln 2} = \left(e^{\ln 2}\right)^3 = 2^3 = 8$]

9. $\log_2\left(\frac{x^3 y}{z^2}\right) = \log_2(x^3 y) - \log_2 z^2 = \log_2 x^3 + \log_2 y - \log_2 z^2 = 3\log_2 x + \log_2 y - 2\log_2 z$ (assuming that the variables are positive)

10. $\ln\sqrt{a(b^2 + c^2)} = \ln\left(a(b^2 + c^2)\right)^{1/2} = \frac{1}{2}\ln\left(a(b^2 + c^2)\right) = \frac{1}{2}\left[\ln a + \ln(b^2 + c^2)\right]$

$= \frac{1}{2}\ln a + \frac{1}{2}\ln(b^2 + c^2)$

11. $\ln(uv)^{10} = 10\ln(uv) = 10(\ln u + \ln v) = 10\ln u + 10\ln v$

12. $\ln \dfrac{3x^2}{(x+1)^5} = \ln 3x^2 - \ln (x+1)^5 = \ln 3 + \ln x^2 - 5\ln(x+1) = \ln 3 + 2\ln x - 5\ln(x+1)$

13. $\log_{10} a - \log_{10} b + \log_{10} c = \log_{10} \dfrac{a}{b} + \log_{10} c = \log_{10}\left(\dfrac{a}{b}\cdot c\right) = \log_{10} \dfrac{ac}{b}$

14. $\ln(x+y) + \ln(x-y) - 2\ln z = \ln((x+y)(x-y)) - \ln z^2 = \ln\left(x^2 - y^2\right) - \ln z^2 = \ln \dfrac{x^2-y^2}{z^2}$

15. $2\ln 4 - \ln 2 = \ln 4^2 - \ln 2 = \ln 16 - \ln 2 = \ln \frac{16}{2} = \ln 8$

16. $\ln 3 + \frac{1}{3}\ln 8 = \ln 3 + \ln 8^{1/3} = \ln 3 + \ln 2 = \ln(3\cdot 2) = \ln 6$

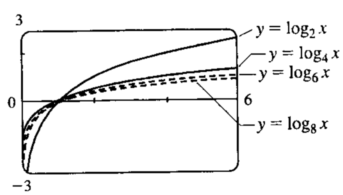
17. $\frac{1}{2}\ln x - 5\ln\left(x^2+1\right) = \ln x^{1/2} - \ln\left(x^2+1\right)^5 = \ln \dfrac{\sqrt{x}}{\left(x^2+1\right)^5}$

18. $\ln x + a\ln y - b\ln z = \ln x + \ln y^a - \ln z^b = \ln\left(xy^a/z^b\right)$

19. (a) $\log_2 5 = \dfrac{\ln 5}{\ln 2} \approx 2.321928$ (b) $\log_5 26.05 = \dfrac{\ln 26.05}{\ln 5} \approx 2.025563$

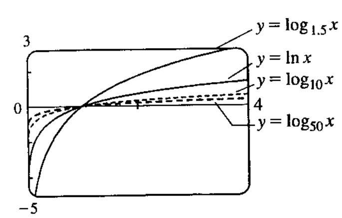
(c) $\log_3 e = \dfrac{1}{\ln 3} \approx 0.910239$

20.

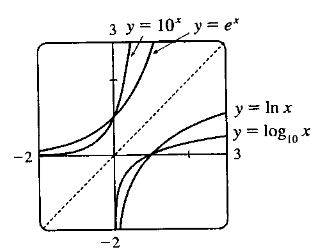


To graph the functions, we use $\log_2 x = \dfrac{\ln x}{\ln 2}$, $\log_4 x = \dfrac{\ln x}{\ln 4}$, etc. These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The smaller the base, the larger the rate of increase of the function (for $x > 1$) and the closer the approach to the y-axis (as $x \to 0^+$).

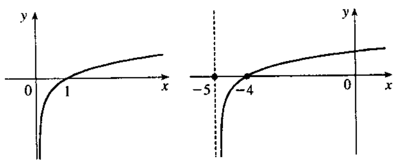
21. To graph these functions, we use $\log_{1.5} x = \dfrac{\ln x}{\ln 1.5}$ and $\log_{50} x = \dfrac{\ln x}{\ln 50}$. These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The functions with larger bases increase extremely slowly, and the ones with smaller bases do so somewhat more quickly. The functions with large bases approach the y-axis more closely as $x \to 0^+$.



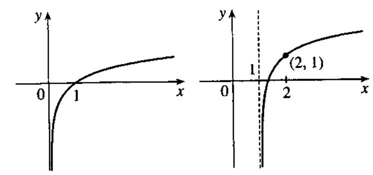
22. We see that the graph of $\ln x$ is the reflection of the graph of e^x about the line $y = x$, and that the graph of $\log_{10} x$ is the reflection of the graph of 10^x about the same line. The graph of 10^x increases more quickly than that of e^x. Also note that $\log_{10} x \to \infty$ as $x \to \infty$ more slowly than $\ln x$.



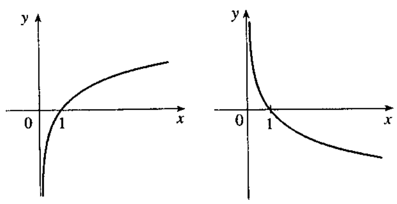
23. $y = \log_{10} x$ $\quad$ $y = \log_{10}(x+5)$



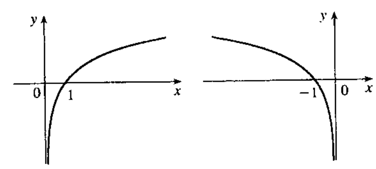
24. $y = \log_5 x$ $\quad$ $y = 1 + \log_5(x-1)$



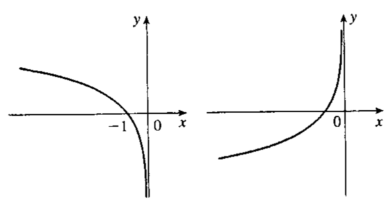
25. $y = \ln x$ $\quad$ $y = -\ln x$



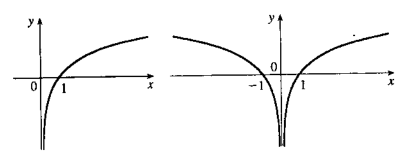
26. $y = \ln x$ $\quad$ $y = \ln(-x)$



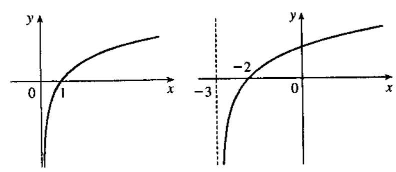
27. $y = \ln(-x)$ $\quad$ $y = -\ln(-x)$



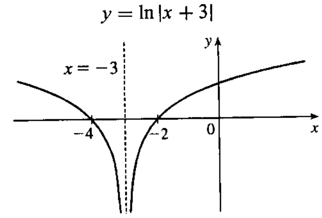
28. $y = \ln x$ $\quad$ $y = \ln|x|$



29. $y = \ln x$ $\quad$ $y = \ln(x+3)$



30. $y = \ln|x+3|$



31. (a) $e^x = 16 \iff \ln e^x = \ln 16 \iff x = \ln 16 = \ln 2^4 = 4\ln 2$

(b) $\ln x = -1 \iff e^{\ln x} = e^{-1} \iff x = 1/e$

32. (a) $\ln(2x-1) = 3 \iff e^{\ln(2x-1)} = e^3 \iff 2x - 1 = e^3 \iff x = \frac{1}{2}(e^3+1)$

(b) $e^{3x-4} = 2 \iff \ln(e^{3x-4}) = \ln 2 \iff 3x - 4 = \ln 2 \iff x = \frac{1}{3}(\ln 2 + 4)$

33. (a) $5^{x-3} = 10 \iff \log_{10} 5^{x-3} = \log_{10} 10 \iff (x-3)\log_{10} 5 = 1 \iff x - 3 = 1/\log_{10} 5 \iff$
$x = 3 + 1/\log_{10} 5$

(b) $\log_{10}(x+1) = 4 \iff x + 1 = 10^4 \iff x = 10{,}000 - 1 = 9999$

34. (a) $e^{3x+1} = k \Leftrightarrow 3x + 1 = \ln k \Leftrightarrow x = \frac{1}{3}(\ln k - 1)$

(b) $\log_2(mx) = c \Leftrightarrow mx = 2^c \Leftrightarrow x = 2^c/m$

35. $\ln(\ln x) = 1 \Leftrightarrow e^{\ln(\ln x)} = e^1 \Leftrightarrow \ln x = e^1 = e \Leftrightarrow e^{\ln x} = e^e \Leftrightarrow x = e^e$

36. $e^{e^x} = 10 \Leftrightarrow \ln\left(e^{e^x}\right) = \ln 10 \Leftrightarrow e^x \ln e = e^x = \ln 10 \Leftrightarrow \ln e^x = \ln(\ln 10) \Leftrightarrow x = \ln(\ln 10)$

37. $\ln(x+6) + \ln(x-3) = \ln 5 + \ln 2 \Leftrightarrow \ln((x+6)(x-3)) = \ln 10 \Leftrightarrow (x+6)(x-3) = 10 \Leftrightarrow x^2 + 3x - 18 = 10 \Leftrightarrow x^2 + 3x - 28 = 0 \Leftrightarrow (x+7)(x-4) = 0 \Leftrightarrow x = -7$ or 4. However, $x = -7$ is not a solution since $\ln(-7+6)$ is not defined. So $x = 4$ is the only solution.

38. $\ln x + \ln(x-1) = \ln(x(x-1)) = 1 \Leftrightarrow x(x-1) = e^1 \Leftrightarrow x^2 - x - e = 0$. The quadratic formula gives $x = \frac{1}{2}\left(1 \pm \sqrt{1+4e}\right)$, but we reject the negative root since the natural logarithm is not defined for $x < 0$. So $x = \frac{1}{2}\left(1 + \sqrt{1+4e}\right)$.

39. $e^{ax} = Ce^{bx} \Leftrightarrow \ln e^{ax} = \ln\left(Ce^{bx}\right) \Leftrightarrow ax = \ln C + bx \Leftrightarrow (a-b)x = \ln C \Leftrightarrow x = \dfrac{\ln C}{a-b}$

40. $7e^x - e^{2x} = 12 \Leftrightarrow (e^x)^2 - 7e^x + 12 = 0 \Leftrightarrow (e^x - 3)(e^x - 4) = 0$, so we have either $e^x = 3 \Leftrightarrow x = \ln 3$, or $e^x = 4 \Leftrightarrow x = \ln 4$.

41. $e^{2+5x} = 100 \Rightarrow \ln\left(e^{2+5x}\right) = \ln 100 \Rightarrow 2 + 5x = \ln 100 \Rightarrow 5x = \ln 100 - 2 \Rightarrow x = \frac{1}{5}(\ln 100 - 2) \approx 0.5210$

42. $\ln\left(1+\sqrt{x}\right) = 2 \Rightarrow 1 + \sqrt{x} = e^2 \Rightarrow \sqrt{x} = e^2 - 1 \Rightarrow x = \left(e^2-1\right)^2 \approx 40.8200$

43. $\ln(e^x - 2) = 3 \Rightarrow e^x - 2 = e^3 \Rightarrow e^x = e^3 + 2 \Rightarrow x = \ln\left(e^3+2\right) \approx 3.0949$

44. $3^{1/(x-4)} = 7 \Rightarrow \ln 3^{1/(x-4)} = \ln 7 \Rightarrow \dfrac{1}{x-4}\ln 3 = \ln 7 \Rightarrow x - 4 = \dfrac{\ln 3}{\ln 7} \Rightarrow x = 4 + \dfrac{\ln 3}{\ln 7} \approx 4.5646$

45. 3 ft = 36 in, so we need x such that $\log_2 x = 36 \Leftrightarrow x = 2^{36} = 68{,}719{,}476{,}736$. In miles, this is $68{,}719{,}476{,}736 \text{ in} \cdot \dfrac{1 \text{ ft}}{12 \text{ in}} \cdot \dfrac{1 \text{ mi}}{5280 \text{ ft}} \approx 1{,}084{,}587.7$ mi.

46. (a) $v(t) = ce^{-kt} \Rightarrow a(t) = v'(t) = -kce^{-kt} = -kv(t)$

(b) $v(0) = ce^0 = c$, so c is the initial velocity.

(c) $v(t) = ce^{-kt} = c/2 \Rightarrow e^{-kt} = \frac{1}{2} \Rightarrow -kt = \ln\frac{1}{2} = -\ln 2 \Rightarrow t = (\ln 2)/k$

47. If I is the intensity of the 1989 San Francisco earthquake, then $\log_{10}(I/S) = 7.1 \Rightarrow \log_{10}(16I/S) = \log_{10} 16 + \log_{10}(I/S) = \log_{10} 16 + 7.1 \approx 8.3$.

48. Let I_1 and I_2 be the intensities of the music and the mower. Then $10\log_{10}\left(\dfrac{I_1}{I_0}\right) = 120$ and $10\log_{10}\left(\dfrac{I_2}{I_0}\right) = 106$, so $\log_{10}\left(\dfrac{I_1}{I_2}\right) = \log_{10}\left(\dfrac{I_1/I_0}{I_2/I_0}\right) = \log_{10}\left(\dfrac{I_1}{I_0}\right) - \log_{10}\left(\dfrac{I_2}{I_0}\right) = 12 - 10.6 = 1.4 \Rightarrow \dfrac{I_1}{I_2} = 10^{1.4} \approx 25$.

49. (a) $n = 100 \cdot 2^{t/3} \Rightarrow \dfrac{n}{100} = 2^{t/3} \Rightarrow \log_2\left(\dfrac{n}{100}\right) = \dfrac{t}{3} \Rightarrow t = 3\log_2\left(\dfrac{n}{100}\right)$. This function tells us how long it will take to obtain n bacteria (given the number n).

(b) $n = 50{,}000 \Rightarrow t = 3\log_2 \dfrac{50{,}000}{100} = 3\log_2 500 = 3\left(\dfrac{\ln 500}{\ln 2}\right) \approx 26.9$ hours

50. (a) $Q = Q_0\left(1 - e^{-t/a}\right) \Rightarrow \dfrac{Q}{Q_0} = 1 - e^{-t/a} \Rightarrow e^{-t/a} = 1 - \dfrac{Q}{Q_0} \Rightarrow -\dfrac{t}{a} = \ln\left(1 - \dfrac{Q}{Q_0}\right) \Rightarrow t = -a\ln(1 - Q/Q_0)$. This gives us the time t necessary to obtain a given charge Q.

(b) $Q = 0.9Q_0$ and $a = 2 \Rightarrow t = -2\ln(1 - 0.9(Q_0/Q_0)) = -2\ln 0.1 \approx 4.6$ seconds.

51. Let $t = 2 - x$. As $x \to 2^-$, $t \to 0^+$. $\lim\limits_{x\to 2^-} \ln(2-x) = \lim\limits_{t\to 0^+} \ln t = -\infty$ by (8).

52. Let $t = x^2 - 5x + 6$. As $x \to 3^+$, $t = (x-2)(x-3) \to 0^+$. $\lim\limits_{x\to 3^+} \log_{10}(x^2 - 5x + 6) = \lim\limits_{t\to 0^+} \log_{10} t = -\infty$ by (4).

53. Let $t = 1 + x^2$. As $x \to \infty$, $t \to \infty$. $\lim\limits_{x\to\infty} \ln(1+x^2) = \lim\limits_{t\to\infty} \ln t = \infty$ by (8).

54. $\lim\limits_{x\to 0^+} \ln(\sin x) = -\infty$ since $\sin x \to 0^+$ as $x \to 0^+$.

55. $\lim\limits_{x\to(\pi/2)^-} \log_{10}(\cos x) = -\infty$ since $\cos x \to 0^+$ as $x \to (\pi/2)^-$.

56. $\lim\limits_{x\to\infty} \dfrac{\ln x}{1+\ln x} = \lim\limits_{x\to\infty} \dfrac{1}{(1/\ln x)+1} = \dfrac{1}{0+1} = 1$

57. $\lim\limits_{x\to\infty} \ln\left(1+e^{-x^2}\right) = \ln\left(1 + \lim\limits_{x\to\infty} e^{-x^2}\right) = \ln(1+0) = 0$

58. $\lim\limits_{x\to\infty} [\ln(2+x) - \ln(1+x)] = \lim\limits_{x\to\infty} \ln\left(\dfrac{2+x}{1+x}\right) = \lim\limits_{x\to\infty} \ln\left(\dfrac{2/x+1}{1/x+1}\right) = \ln\dfrac{1}{1} = \ln 1 = 0$

59. The domain of $f(x) = \log_2(5x-3)$ is $\{x \mid 5x - 3 > 0\} = \left\{x \mid x > \frac{3}{5}\right\} = \left(\frac{3}{5}, \infty\right)$. Since $5x - 3$ takes on all positive values for x in $\left(\frac{3}{5}, \infty\right)$, the range of f is $\mathbb{R}$.

60. $g(x) = \ln(4 - x^2)$. Domain $(g) = \{x \mid 4 - x^2 > 0\} = \{x \mid |x| < 2\} = (-2, 2)$. Since $4 - x^2 \leq 4$, we have $\ln(4 - x^2) \leq \ln 4$. Also $\lim\limits_{x\to 2^-} g(x) = -\infty$, so range $(g) = (-\infty, \ln 4]$.

61. $F(t) = \sqrt{t}\ln(t^2 - 1)$. Domain $(F) = \{t \mid t \geq 0 \text{ and } t^2 - 1 > 0\} = \{t \mid t > 1\} = (1, \infty)$. Range $(F) = \mathbb{R}$.

62. The domain of $G(t) = \ln(e^t - 2)$ is $\{t \mid e^t - 2 > 0\} = \{t \mid e^t > 2\} = \{t \mid t > \ln 2\} = (\ln 2, \infty)$. Since $e^t - 2$ takes on all positive values for t in $(\ln 2, \infty)$, the range of G is $\mathbb{R}$.

63. $y = \ln(x+3) \;\Rightarrow\; e^y = e^{\ln(x+3)} = x + 3 \;\Rightarrow\; x = e^y - 3$.
Interchange x and y: the inverse function is $y = e^x - 3$.

64. $y = 2^{10^x} \;\Rightarrow\; \log_2 y = 10^x \;\Rightarrow\; \log_{10}(\log_2 y) = x$.
Interchange x and y: $y = \log_{10}(\log_2 x)$ is the inverse function.

65. $y = e^{\sqrt{x}} \;\Rightarrow\; \ln y = \ln e^{\sqrt{x}} = \sqrt{x} \;\Rightarrow\; x = (\ln y)^2$. Also note that $\sqrt{x} \geq 0 \;\Rightarrow\; y = e^{\sqrt{x}} \geq 1$. Interchange x and y: the inverse function is $y = (\ln x)^2$, $x \geq 1$.

66. $y = (\ln x)^2$, $x \geq 1$, $\ln x = \sqrt{y} \;\Rightarrow\; x = e^{\sqrt{y}}$. Interchange x and y: $y = e^{\sqrt{x}}$ is the inverse function.

67. $y = \dfrac{10^x}{10^x + 1} \;\Rightarrow\; 10^x y + y = 10^x \;\Rightarrow\; 10^x(1-y) = y \;\Rightarrow\; 10^x = \dfrac{y}{1-y} \;\Rightarrow\; x = \log_{10}\left(\dfrac{y}{1-y}\right)$.
Interchange x and y: $y = \log_{10}\left(\dfrac{x}{1-x}\right)$ is the inverse function.

68. $y = \dfrac{1+e^x}{1-e^x} \;\Rightarrow\; y - ye^x = 1 + e^x \;\Rightarrow\; e^x(y+1) = y - 1 \;\Rightarrow\; e^x = \dfrac{y-1}{y+1} \;\Rightarrow\; x = \ln\left(\dfrac{y-1}{y+1}\right)$.
Interchange x and y: $y = \ln\left(\dfrac{x-1}{x+1}\right)$ is the inverse function.

69. $y = e^x - 2e^{-x}$, so $y' = e^x + 2e^{-x}$, $y'' = e^x - 2e^{-x}$. $y'' > 0 \iff e^x - 2e^{-x} > 0 \iff e^x > 2e^{-x} \iff e^{2x} > 2 \iff 2x > \ln 2 \iff x > \frac{1}{2}\ln 2$. Therefore, y is concave upward on $\left(\frac{1}{2}\ln 2, \infty\right)$.

70. $f(x) = e^x + e^{-2x}$, $f'(x) = e^x - 2e^{-2x} > 0 \iff e^x > 2e^{-2x} \iff e^{3x} > 2 \iff 3x > \ln 2 \iff x > \frac{1}{3}\ln 2$. Thus, f is increasing on $\left(\frac{1}{3}\ln 2, \infty\right)$.

71. (a) We have to show that $-f(x) = f(-x)$.

$$\begin{aligned} -f(x) &= -\ln\left(x + \sqrt{x^2+1}\right) = \ln\left(\left(x + \sqrt{x^2+1}\right)^{-1}\right) = \ln\frac{1}{x + \sqrt{x^2+1}} \\ &= \ln\left(\frac{1}{x + \sqrt{x^2+1}} \cdot \frac{x - \sqrt{x^2+1}}{x - \sqrt{x^2+1}}\right) = \ln\frac{x - \sqrt{x^2+1}}{x^2 - x^2 - 1} \\ &= \ln\left(\sqrt{x^2+1} - x\right) = f(-x) \end{aligned}$$

Thus, f is an odd function.

(b) Let $y = \ln\left(x + \sqrt{x^2+1}\right)$. Then $e^y = x + \sqrt{x^2+1} \iff (e^y - x)^2 = x^2 + 1 \iff e^{2y} - 2xe^y + x^2 = x^2 + 1 \iff 2xe^y = e^{2y} - 1 \iff x = \dfrac{e^{2y} - 1}{2e^y} = \frac{1}{2}\left(e^y - e^{-y}\right)$. Thus, the inverse function is $f^{-1}(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$.

72. Let $\left(a, e^{-a}\right)$ be the point where the tangent meets the curve. The tangent has slope $-e^{-a}$ and is perpendicular to the line $2x - y = 8$, which has slope 2. So $-e^{-a} = -\frac{1}{2} \Rightarrow e^{-a} = \frac{1}{2} \Rightarrow e^a = 2 \Rightarrow a = \ln(e^a) = \ln 2$. Thus, the point on the curve is $\left(\ln 2, \frac{1}{2}\right)$ and the equation of the tangent is $y - \frac{1}{2} = -\frac{1}{2}(x - \ln 2)$ or $x + 2y = 1 + \ln 2$.

73. Let $x = \log_{10} 99$, $y = \log_9 82$. Then $10^x = 99 < 10^2 \Rightarrow x < 2$, and $9^y = 82 > 9^2 \Rightarrow y > 2$. Therefore, $y = \log_9 82$ is larger.

74. (a) $\lim\limits_{x\to\infty} x^{\ln x} = \lim\limits_{x\to\infty} \left(e^{\ln x}\right)^{\ln x} = \lim\limits_{x\to\infty} e^{(\ln x)^2} = \infty$ since $(\ln x)^2 \to \infty$ as $x \to \infty$.

(b) $\lim\limits_{x\to 0^+} x^{-\ln x} = \lim\limits_{x\to 0^+} \left(e^{\ln x}\right)^{-\ln x} = \lim\limits_{x\to 0^+} e^{-(\ln x)^2} = 0$ since $-(\ln x)^2 \to -\infty$ as $x \to 0^+$.

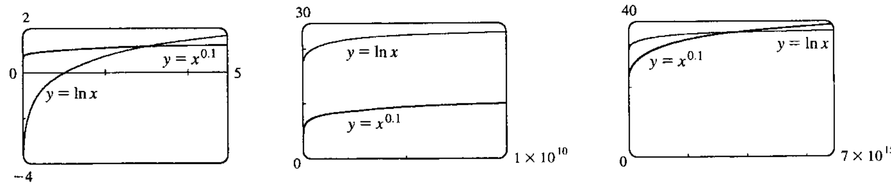
(c) $\lim\limits_{x\to 0^+} x^{1/x} = \lim\limits_{x\to 0^+} \left(e^{\ln x}\right)^{1/x} = \lim\limits_{x\to 0^+} e^{(\ln x/x)} = 0$ since $\dfrac{\ln x}{x} \to -\infty$ as $x \to 0^+$.

(d) $\lim\limits_{x\to\infty} (\ln 2x)^{-\ln x} = \lim\limits_{x\to\infty} \left[e^{\ln(\ln 2x)}\right]^{-\ln x} = \lim\limits_{x\to\infty} e^{-\ln x \ln(\ln 2x)} = 0$ since $-\ln x \ln(\ln 2x) \to -\infty$ as $x \to \infty$.

75. (a) Let $\varepsilon > 0$ be given. We need N such that $|a^x - 0| < \varepsilon$ when $x < N$. But $a^x < \varepsilon \iff x < \log_a \varepsilon$. Let $N = \log_a \varepsilon$. Then $x < N \Rightarrow x < \log_a \varepsilon \Rightarrow |a^x - 0| = a^x < \varepsilon$, so $\lim\limits_{x\to -\infty} a^x = 0$.

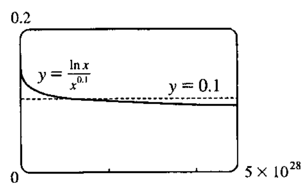
(b) Let $M > 0$ be given. We need N such that $a^x > M$ when $x > N$. But $a^x > M \iff x > \log_a M$. Let $N = \log_a M$. Then $x > N \Rightarrow x > \log_a M \Rightarrow a^x > M$, so $\lim\limits_{x\to\infty} a^x = \infty$.

76. (a)



From the graphs, we see that $f(x) = x^{0.1} > g(x) = \ln x$ for approximately $0 < x < 3.06$, and then $g(x) > f(x)$ for $3.06 < x < 3.43 \times 10^{15}$ (approximately). At that point, the graph of f finally surpasses the graph of g for good.

(b)



(c) From the graph at left, it seems that $\dfrac{\ln x}{x^{0.1}} < 0.1$ whenever $x > 1.3 \times 10^{28}$ (approximately). So we can take $N = 1.3 \times 10^{28}$, or any larger number.

77. $\ln(x^2 - 2x - 2) \le 0 \Rightarrow 0 < x^2 - 2x - 2 \le 1$. Now $x^2 - 2x - 2 \le 1$ gives $x^2 - 2x - 3 \le 0$ and hence $(x-3)(x+1) \le 0$. So $-1 \le x \le 3$. Now $0 < x^2 - 2x - 2 \Rightarrow x < 1 - \sqrt{3}$ or $x > 1 + \sqrt{3}$. Therefore, $\ln(x^2 - 2x - 2) \le 0 \Leftrightarrow -1 \le x < 1 - \sqrt{3}$ or $1 + \sqrt{3} < x \le 3$.

78. (a) The primes less than 25 are 2, 3, 5, 7, 11, 13, 17, 19, and 23. There are 9 of them, so $\pi(25) = 9$. We use the sieve of Eratosthenes, and arrive at the figure at right. There are 25 numbers left over, so $\pi(100) = 25$.

	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	57	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	~~79~~	~~80~~
~~81~~	~~82~~	83	~~84~~	~~35~~	~~86~~	~~87~~	~~88~~	89	~~90~~
91	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

(b) Let $f(n) = \dfrac{\pi(n)}{n/\ln n}$. We compute $f(100) = \dfrac{25}{100/\ln 100} \approx 1.15$, $f(1000) \approx 1.16$, $f(10^4) \approx 1.13$, $f(10^5) \approx 1.10$, $f(10^6) \approx 1.08$, and $f(10^7) \approx 1.07$.

(c) By the Prime Number Theorem, the number of primes less than a billion, that is, $\pi(10^9)$, should be close to $10^9/\ln 10^9 \approx 48{,}254{,}942$. In fact, $\pi(10^9) = 50{,}847{,}543$, so our estimate is off by about 5.1%. Do not attempt this calculation at home.

7.4 Derivatives of Logarithmic Functions

1. The differentiation formula for logarithmic functions, $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a}$, is simplest when $a = e$ because $\ln e = 1$.

2. $f(x) = \ln(2 - x) \Rightarrow f'(x) = \dfrac{1}{2 - x}\dfrac{d}{dx}(2 - x) = \dfrac{-1}{2 - x} = \dfrac{1}{x - 2}$

3. $f(\theta) = \ln(\cos\theta) \Rightarrow f'(\theta) = \dfrac{1}{\cos\theta}\dfrac{d}{d\theta}(\cos\theta) = \dfrac{-\sin\theta}{\cos\theta} = -\tan\theta$

4. $f(x) = \cos(\ln x) \Rightarrow f'(x) = -\sin(\ln x) \cdot \dfrac{1}{x} = \dfrac{-\sin(\ln x)}{x}$

5. $f(x) = \log_3(x^2 - 4) \Rightarrow f'(x) = \dfrac{1}{(x^2 - 4)\ln 3}(2x) = \dfrac{2x}{(x^2 - 4)\ln 3}$

6. $f(x) = \log_{10}\left(\dfrac{x}{x-1}\right) = \log_{10} x - \log_{10}(x - 1) \Rightarrow f'(x) = \dfrac{1}{x \ln 10} - \dfrac{1}{(x-1)\ln 10}$ or $-\dfrac{1}{x(x-1)\ln 10}$

7. $F(x) = \ln\sqrt{x} = \ln x^{1/2} = \frac{1}{2}\ln x \Rightarrow F'(x) = \dfrac{1}{2}\left(\dfrac{1}{x}\right) = \dfrac{1}{2x}$

8. $G(x) = \sqrt[3]{\ln x} = (\ln x)^{1/3} \Rightarrow G'(x) = \frac{1}{3}(\ln x)^{-2/3} \cdot \dfrac{1}{x} = \dfrac{1}{3x(\ln x)^{2/3}}$

9. $f(x) = \sqrt{x}\ln x \Rightarrow f'(x) = \dfrac{1}{2\sqrt{x}}\ln x + \sqrt{x}\left(\dfrac{1}{x}\right) = \dfrac{\ln x}{2\sqrt{x}} + \dfrac{1}{\sqrt{x}} = \dfrac{\ln x + 2}{2\sqrt{x}}$

10. $f(t) = \dfrac{1 + \ln t}{1 - \ln t} \Rightarrow$

$$f'(t) = \frac{(1 - \ln t)(1/t) - (1 + \ln t)(-1/t)}{(1 - \ln t)^2} = \frac{(1/t)[(1 - \ln t) + (1 + \ln t)]}{(1 - \ln t)^2} = \frac{2}{t(1 - \ln t)^2}$$

11. $g(x) = \ln\dfrac{a - x}{a + x} = \ln(a - x) - \ln(a + x) \Rightarrow$

$$g'(x) = \frac{1}{a - x}(-1) - \frac{1}{a + x} = \frac{-(a + x) - (a - x)}{(a - x)(a + x)} = \frac{-2a}{a^2 - x^2}$$

12. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right) \Rightarrow h'(x) = \dfrac{1}{x + \sqrt{x^2 - 1}}\left(1 + \dfrac{x}{\sqrt{x^2 - 1}}\right) = \dfrac{1}{\sqrt{x^2 - 1}}$

13. $F(x) = e^x \ln x \Rightarrow F'(x) = e^x \ln x + e^x\left(\dfrac{1}{x}\right) = e^x\left(\ln x + \dfrac{1}{x}\right)$

14. $h(y) = \ln(y^3 \sin y) = 3\ln y + \ln(\sin y) \Rightarrow h'(y) = \dfrac{3}{y} + \dfrac{1}{\sin y}(\cos y) = \dfrac{3}{y} + \cot y$

15. $y = \dfrac{\ln x}{1 + x} \Rightarrow y' = \dfrac{(1 + x)(1/x) - (\ln x)(1)}{(1 + x)^2} = \dfrac{\dfrac{1 + x}{x} - \dfrac{x \ln x}{x}}{(1 + x)^2} = \dfrac{1 + x - x\ln x}{x(1 + x)^2}$

16. $y = (\ln \tan x)^2 \Rightarrow y' = 2(\ln \tan x) \cdot \dfrac{1}{\tan x} \cdot \sec^2 x = \dfrac{2(\ln \tan x)\sec^2 x}{\tan x}$

17. $h(t) = t^3 - 3^t \Rightarrow h'(t) = 3t^2 - 3^t \ln 3$

18. $y = 10^{\tan\theta} \Rightarrow y' = 10^{\tan\theta}(\ln 10)(\sec^2\theta)$

19. $y = \ln|x^3 - x^2| \Rightarrow y' = \dfrac{1}{x^3 - x^2}(3x^2 - 2x) = \dfrac{x(3x - 2)}{x^2(x - 1)} = \dfrac{3x - 2}{x(x - 1)}$

20. $G(u) = \ln\sqrt{\dfrac{3u + 2}{3u - 2}} = \frac{1}{2}[\ln(3u + 2) - \ln(3u - 2)] \Rightarrow G'(u) = \dfrac{1}{2}\left(\dfrac{3}{3u + 2} - \dfrac{3}{3u - 2}\right) = \dfrac{-6}{9u^2 - 4}$

21. $y = \ln(e^{-x}(1 + x)) = \ln(e^{-x}) + \ln(1 + x) = -x + \ln(1 + x) \Rightarrow y' = -1 + \dfrac{1}{1 + x} = -\dfrac{x}{1 + x}$

22. $y = \ln(x + \ln x) \Rightarrow y' = \dfrac{1}{x + \ln x}\left(1 + \dfrac{1}{x}\right) = \dfrac{x + 1}{x(x + \ln x)}$

23. Using Formula 7 and the Chain Rule, $y = 5^{-1/x} \Rightarrow y' = 5^{-1/x}(\ln 5)\left[-1 \cdot (-x^{-2})\right] = 5^{-1/x}(\ln 5)/x^2$

24. $y = 2^{3^{x^2}} \Rightarrow y' = 2^{3^{x^2}} (\ln 2) \dfrac{d}{dx}\left(3^{x^2}\right) = 2^{3^{x^2}} (\ln 2)\, 3^{x^2} (\ln 3)(2x)$

25. $y = x \ln x \Rightarrow y' = \ln x + x(1/x) = \ln x + 1 \Rightarrow y'' = 1/x$

26. $y = \ln\left(1 + x^2\right) \Rightarrow y' = \dfrac{1}{1+x^2} \cdot 2x = \dfrac{2x}{x^2+1} \Rightarrow$

$$y'' = \frac{(x^2+1)(2) - (2x)(2x)}{(x^2+1)^2} = \frac{2x^2 + 2 - 4x^2}{(x^2+1)^2} = \frac{2 - 2x^2}{(x^2+1)^2}$$

27. $y = \log_{10} x \Rightarrow y' = \dfrac{1}{x \ln 10} = \dfrac{1}{\ln 10}\left(\dfrac{1}{x}\right) \Rightarrow y'' = \dfrac{1}{\ln 10}\left(-\dfrac{1}{x^2}\right) = -\dfrac{1}{x^2 \ln 10}$

28. $y = \ln(\sec x + \tan x) \Rightarrow y' = \dfrac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x \Rightarrow y'' = \sec x \tan x$

29. $f(x) = \ln(2x+1) \Rightarrow f'(x) = \dfrac{1}{2x+1} \cdot 2 = \dfrac{2}{2x+1}$. $\text{Dom}(f) = \{x \mid 2x + 1 > 0\} = \left(-\tfrac{1}{2}, \infty\right)$.

30. $f(x) = \dfrac{1}{1 + \ln x} \Rightarrow f'(x) = -\dfrac{1/x}{(1+\ln x)^2}$ (Reciprocal Rule) $= -\dfrac{1}{x(1+\ln x)^2}$.
$\text{Dom}(f) = \{x \mid x > 0 \text{ and } \ln x \neq -1\} = \{x \mid x > 0 \text{ and } x \neq 1/e\} = (0, 1/e) \cup (1/e, \infty)$.

31. $f(x) = x^2 \ln\left(1 - x^2\right) \Rightarrow f'(x) = 2x \ln\left(1 - x^2\right) + \dfrac{x^2(-2x)}{1 - x^2} = 2x \ln\left(1 - x^2\right) - \dfrac{2x^3}{1 - x^2}$.
$\text{Dom}(f) = \left\{x \mid 1 - x^2 > 0\right\} = \{x \mid |x| < 1\} = (-1, 1)$.

32. $f(x) = \ln \ln \ln x \Rightarrow f'(x) = \dfrac{1}{\ln \ln x} \cdot \dfrac{1}{\ln x} \cdot \dfrac{1}{x}$.
$\text{Dom}(f) = \{x \mid \ln \ln x > 0\} = \{x \mid \ln x > 1\} = \{x \mid x > e\} = (e, \infty)$.

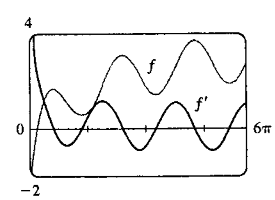
33. $f(x) = \dfrac{x}{\ln x} \Rightarrow f'(x) = \dfrac{\ln x - x(1/x)}{(\ln x)^2} = \dfrac{\ln x - 1}{(\ln x)^2} \Rightarrow f'(e) = \dfrac{1 - 1}{1^2} = 0$

34. $f(x) = x^2 \ln x \Rightarrow f'(x) = 2x \ln x + x^2\left(\dfrac{1}{x}\right) = 2x \ln x + x \Rightarrow f'(1) = 2 \ln 1 + 1 = 1$

35. $y = f(x) = \ln \ln x \Rightarrow f'(x) = \dfrac{1}{\ln x}\left(\dfrac{1}{x}\right) \Rightarrow f'(e) = \dfrac{1}{e}$, so an equation of the tangent line at $(e, 0)$ is
$y - 0 = \dfrac{1}{e}(x - e)$, or $y = \dfrac{1}{e}x - 1$, or $x - ey = e$.

36. $y = f(x) = \ln\left(x^2 + 1\right) \Rightarrow f'(x) = \dfrac{1}{x^2+1} \cdot 2x = \dfrac{2x}{x^2+1} \Rightarrow f'(1) = 1$, so an equation of the tangent line at $(1, \ln 2)$ is $y - \ln 2 = 1(x - 1)$, or $y = x + \ln 2 - 1$.

37. $f(x) = \sin x + \ln x \Rightarrow f'(x) = \cos x + 1/x$. This is reasonable, because the graph shows that f increases when $f'(x)$ is positive, and $f'(x) = 0$ when f has a horizontal tangent.

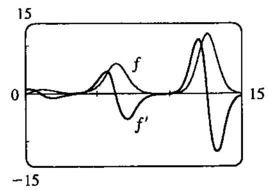


38. $f(x) = x^{\cos x} = e^{\ln x \cos x} \Rightarrow$

$f'(x) = e^{\ln x \cos x}\left[\ln x\,(-\sin x) + \cos x\left(\frac{1}{x}\right)\right]$

$= x^{\cos x}\left[\frac{\cos x}{x} - \sin x \ln x\right]$

This is reasonable, because the graph shows that f increases when $f'(x)$ is positive.



39. $y = (2x+1)^5 (x^4-3)^6 \Rightarrow \ln y = \ln\left((2x+1)^5 (x^4-3)^6\right) \Rightarrow \ln y = 5\ln(2x+1) + 6\ln(x^4-3) \Rightarrow$

$\frac{1}{y}y' = 5 \cdot \frac{1}{2x+1} \cdot 2 + 6 \cdot \frac{1}{x^4-3} \cdot 4x^3 \Rightarrow$

$y' = y\left(\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right) = y \cdot \frac{10(x^4-3) + 24x^3(2x+1)}{(2x+1)(x^4-3)} = (2x+1)^5(x^4-3)^6 \cdot \frac{58x^4+24x^3-30}{(2x+1)(x^4-3)}$

$= 2(2x+1)^4(x^4-3)^5(29x^4+12x^3-15)$

40. $y = \sqrt{x}e^{x^2}(x^2+1)^{10} \Rightarrow \ln y = \frac{1}{2}\ln x + x^2 + 10\ln(x^2+1) \Rightarrow \frac{1}{y}y' = \frac{1}{2}\cdot\frac{1}{x} + 2x + 10 \cdot \frac{1}{x^2+1}\cdot 2x$

$\Rightarrow y' = \sqrt{x}e^{x^2}(x^2+1)^{10}\left(\frac{1}{2x} + 2x + \frac{20x}{x^2+1}\right)$

41. $y = \frac{\sin^2 x \tan^4 x}{(x^2+1)^2} \Rightarrow \ln y = \ln(\sin^2 x \tan^4 x) - \ln(x^2+1)^2 = 2\ln\sin x + 4\ln\tan x - 2\ln(x^2+1) \Rightarrow$

$\frac{1}{y}y' = 2\cdot\frac{1}{\sin x}\cdot\cos x + 4\cdot\frac{1}{\tan x}\cdot\sec^2 x - 2\cdot\frac{1}{x^2+1}\cdot 2x \Rightarrow$

$y' = \frac{\sin^2 x \tan^4 x}{(x^2+1)^2}\left(2\cot x + \frac{4\sec^2 x}{\tan x} - \frac{4x}{x^2+1}\right)$

42. $y = \sqrt[4]{\frac{x^2+1}{x^2-1}} \Rightarrow \ln y = \frac{1}{4}\ln(x^2+1) - \frac{1}{4}\ln(x^2-1) \Rightarrow \frac{1}{y}y' = \frac{1}{4}\cdot\frac{1}{x^2+1}\cdot 2x - \frac{1}{4}\cdot\frac{1}{x^2-1}\cdot 2x \Rightarrow$

$y' = \sqrt[4]{\frac{x^2+1}{x^2-1}}\cdot\frac{1}{2}\left(\frac{x}{x^2+1} - \frac{x}{x^2-1}\right) = \frac{1}{2}\sqrt[4]{\frac{x^2+1}{x^2-1}}\left(\frac{-2x}{x^4-1}\right) = \frac{x}{1-x^4}\sqrt[4]{\frac{x^2+1}{x^2-1}}$

43. $y = x^x \Rightarrow \ln y = x\ln x \Rightarrow y'/y = \ln x + x(1/x) \Rightarrow y' = x^x(\ln x + 1)$

44. $y = x^{1/x} \Rightarrow \ln y = \frac{1}{x}\ln x \Rightarrow \frac{y'}{y} = -\frac{1}{x^2}\ln x + \frac{1}{x}\left(\frac{1}{x}\right) \Rightarrow y' = x^{1/x}\frac{1-\ln x}{x^2}$

45. $y = x^{\sin x} \Rightarrow \ln y = \sin x \ln x \Rightarrow \frac{y'}{y} = \cos x \ln x + \frac{\sin x}{x} \Rightarrow y' = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)$

46. $y = (\sin x)^x \Rightarrow \ln y = x\ln(\sin x) \Rightarrow y'/y = \ln(\sin x) + x(\cos x)/(\sin x) \Rightarrow$
$y' = (\sin x)^x[\ln(\sin x) + x\cot x]$

47. $y = (\ln x)^x \Rightarrow \ln y = x\ln\ln x \Rightarrow \frac{y'}{y} = \ln\ln x + x\cdot\frac{1}{\ln x}\cdot\frac{1}{x} \Rightarrow y' = (\ln x)^x\left(\ln\ln x + \frac{1}{\ln x}\right)$

48. $y = x^{\ln x} \Rightarrow \ln y = \ln x \ln x = (\ln x)^2 \Rightarrow \frac{y'}{y} = 2\ln x\left(\frac{1}{x}\right) \Rightarrow y' = x^{\ln x}\left(\frac{2\ln x}{x}\right)$

49. $y = x^{e^x} \Rightarrow \ln y = e^x \ln x \Rightarrow \frac{y'}{y} = e^x\ln x + \frac{e^x}{x} \Rightarrow y' = x^{e^x}e^x\left(\ln x + \frac{1}{x}\right)$

50. $y = (\ln x)^{\cos x} \quad \Rightarrow \quad \ln y = \cos x \ln(\ln x) \quad \Rightarrow \quad \dfrac{y'}{y} = \cos x \cdot \dfrac{1}{\ln x} \cdot \dfrac{1}{x} + (\ln \ln x)(-\sin x) \quad \Rightarrow$

$y' = (\ln x)^{\cos x}\left(\dfrac{\cos x}{x \ln x} - \sin x \ln \ln x\right)$

51. $y = \ln\left(x^2 + y^2\right) \quad \Rightarrow \quad y' = \dfrac{2x + 2yy'}{x^2 + y^2} \quad \Rightarrow \quad x^2 y' + y^2 y' = 2x + 2yy' \quad \Rightarrow \quad y' = \dfrac{2x}{x^2 + y^2 - 2y}$

52. $x^y = y^x \quad \Rightarrow \quad y \ln x = x \ln y \quad \Rightarrow \quad y \cdot \dfrac{1}{x} + y' \ln x = x \cdot \dfrac{1}{y} \cdot y' + \ln y \quad \Rightarrow \quad y' \ln x - \dfrac{x}{y} y' = \ln y - \dfrac{y}{x} \quad \Rightarrow$

$y' = \dfrac{\ln y - y/x}{\ln x - x/y}$

53. $f(x) = \ln(x-1) \quad \Rightarrow \quad f'(x) = 1/(x-1) = (x-1)^{-1} \quad \Rightarrow \quad f''(x) = -(x-1)^{-2}$

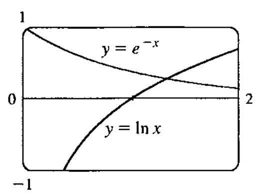
$\Rightarrow \quad f'''(x) = 2(x-1)^{-3} \quad \Rightarrow \quad f^{(4)}(x) = -2 \cdot 3(x-1)^{-4} \quad \Rightarrow \quad \cdots \quad \Rightarrow$

$f^{(n)}(x) = (-1)^{n-1} \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (n-1)(x-1)^{-n} = (-1)^{n-1} \dfrac{(n-1)!}{(x-1)^n}$

54. $y = x^8 \ln x$, so

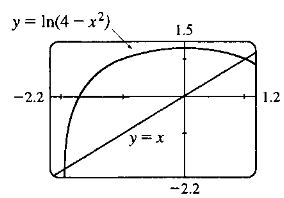
$$D^9 y = D^8\left(8x^7 \ln x + x^7\right) = D^8\left(8x^7 \ln x\right) = D^7\left(8 \cdot 7x^6 \ln x + 8x^6\right) = D^7\left(8 \cdot 7x^6 \ln x\right)$$
$$= D^6\left(8 \cdot 7 \cdot 6x^5 \ln x\right) = \cdots = D\left(8!\, x^0 \ln x\right) = 8!/x$$

55.



From the graph, it appears that the only root of the equation occurs at about $x = 1.3$. So we use Newton's Method with this as our initial approximation, and with $f(x) = \ln x - e^{-x} \quad \Rightarrow \quad f'(x) = 1/x + e^{-x}$. The formula is $x_{n+1} = x_n - f(x_n)/f'(x_n)$, and we calculate $x_1 = 1.3$, $x_2 \approx 1.309760$, $x_3 \approx x_4 \approx 1.309800$. So, correct to six decimal places, the root of the equation $\ln x = e^{-x}$ is $x = 1.309800$.

56.



We use Newton's Method with $f(x) = \ln\left(4 - x^2\right) - x$ and $f'(x) = \dfrac{1}{4 - x^2}(-2x) - 1 = 1 - \dfrac{2x}{4 - x^2}$. The formula is $x_{n+1} = x_n - f(x_n)/f'(x_n)$. From the graphs it seems that the roots occur at approximately $x = -1.9$ and $x = 1.1$. However, if we use $x_1 = -1.9$ as an initial approximation to the first root, we get $x_2 \approx -2.009611$, and $f(x) = \ln(x-2)^2 - x$ is undefined at this point, making it impossible to calculate x_3. We must use a more accurate first estimate, such as $x_1 = -1.95$. With this approximation, we get $x_1 = -1.95$, $x_2 \approx -1.1967495$, $x_3 \approx -1.964760$, $x_4 \approx x_5 \approx -1.964636$. Calculating the second root gives $x_1 = 1.1$, $x_2 \approx 1.058649$, $x_3 \approx 1.058007$, $x_4 \approx x_5 \approx 1.058006$. So, correct to six decimal places, the two roots of the equation $\ln\left(4 - x^2\right) = x$ are $x = -1.964636$ and $x = 1.058006$.

57. $f(x) = \dfrac{\ln x}{\sqrt{x}} \quad \Rightarrow \quad f'(x) = \dfrac{\sqrt{x}\,(1/x) - (\ln x)\left[1/\left(2\sqrt{x}\right)\right]}{x} = \dfrac{2 - \ln x}{2x^{3/2}} \quad \Rightarrow$

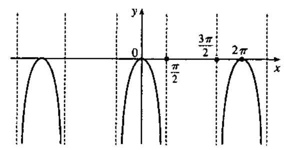
$f''(x) = \dfrac{2x^{3/2}(-1/x) - (2 - \ln x)\left(3x^{1/2}\right)}{4x^3} = \dfrac{3\ln x - 8}{4x^{5/2}} > 0 \quad \Leftrightarrow \quad \ln x > \frac{8}{3} \quad \Leftrightarrow \quad x > e^{8/3}$, so f is CU on $\left(e^{8/3}, \infty\right)$ and CD on $\left(0, e^{8/3}\right)$. The inflection point is $\left(e^{8/3}, \frac{8}{3}e^{-4/3}\right)$.

58. $f(x) = x \ln x$, $f'(x) = \ln x + 1 = 0$ when $\ln x = -1 \Leftrightarrow x = e^{-1}$. $f'(x) > 0 \Leftrightarrow \ln x + 1 > 0 \Leftrightarrow \ln x > -1 \Leftrightarrow x > 1/e$. $f'(x) < 0 \Leftrightarrow \ln x + 1 < 0 \Leftrightarrow x < 1/e$. Therefore, there is an absolute minimum value of $f(1/e) = (1/e)\ln(1/e) = -1/e$.

59. $y = f(x) = \ln(\cos x)$

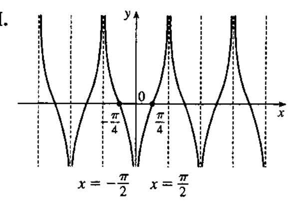
A. $D = \{x \mid \cos x > 0\} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \cup \cdots = \left\{x \mid 2n\pi - \frac{\pi}{2} < x < 2n\pi + \frac{\pi}{2}, n = 0, \pm 1, \pm 2, \ldots\right\}$
B. x-intercepts occur when $\ln(\cos x) = 0 \Leftrightarrow \cos x = 1 \Leftrightarrow x = 2n\pi$, y-intercept $= f(0) = 0$.
C. $f(-x) = f(x)$, so the curve is symmetric about the y-axis. $f(x + 2\pi) = f(x)$, f has period 2π, so in parts D–G we consider only $-\frac{\pi}{2} < x < \frac{\pi}{2}$. **D.** $\lim\limits_{x \to \pi/2^-} \ln(\cos x) = -\infty$ and $\lim\limits_{x \to -\pi/2^+} \ln(\cos x) = -\infty$, so $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are VA. No HA.
E. $f'(x) = (1/\cos x)(-\sin x) = -\tan x > 0 \Leftrightarrow -\frac{\pi}{2} < x < 0$, so f is increasing on $\left(-\frac{\pi}{2}, 0\right)$ and decreasing on $\left(0, \frac{\pi}{2}\right)$.
F. $f(0) = 0$ is a local maximum. **G.** $f''(x) = -\sec^2 x < 0 \Rightarrow f$ is CD on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. No IP.

H.



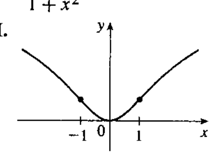
60. $y = \ln(\tan^2 x)$ **A.** $D = \{x \mid x \neq n\pi/2\}$ **B.** x-intercepts $n\pi + \frac{\pi}{4}$, no y-intercept. **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. Also $f(x + \pi) = f(x)$, so f is periodic with period π, and we consider parts D–G only for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. **D.** $\lim\limits_{x \to 0} \ln(\tan^2 x) = -\infty$ and $\lim\limits_{x \to \pi/2^-} \ln(\tan^2 x) = \infty$, $\lim\limits_{x \to -\pi/2^+} \ln(\tan^2 x) = \infty$, so $x = 0$, $x = \pm\frac{\pi}{2}$ are VA. **E.** $f'(x) = \dfrac{2\tan x \sec^2 x}{\tan^2 x} = 2\dfrac{\sec^2 x}{\tan x} > 0 \Leftrightarrow \tan x > 0 \Leftrightarrow 0 < x < \frac{\pi}{2}$,
so f is increasing on $\left(0, \frac{\pi}{2}\right)$ and decreasing on $\left(-\frac{\pi}{2}, 0\right)$. **F.** No maximum or minimum **G.** $f'(x) = \dfrac{2}{\sin x \cos x} = \dfrac{4}{\sin 2x} \Rightarrow$
$f''(x) = \dfrac{-8\cos 2x}{\sin^2 2x} < 0 \Leftrightarrow \cos 2x > 0 \Leftrightarrow -\frac{\pi}{4} < x < \frac{\pi}{4}$, so f is CD on $\left(-\frac{\pi}{4}, 0\right)$ and $\left(0, \frac{\pi}{4}\right)$ and CU on $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. IP are $\left(\pm\frac{\pi}{4}, 0\right)$.

H.

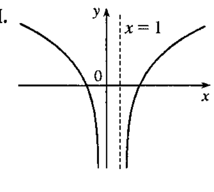


61. $y = f(x) = \ln(1 + x^2)$ **A.** $D = \mathbb{R}$ **B.** Both intercepts are 0. **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** $\lim\limits_{x \to \pm\infty} \ln(1 + x^2) = \infty$, no asymptotes. **E.** $f'(x) = \dfrac{2x}{1 + x^2} > 0 \Leftrightarrow x > 0$, so f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.
F. $f(0) = 0$ is a local and absolute minimum.
G. $f''(x) = \dfrac{2(1 + x^2) - 2x(2x)}{(1 + x^2)^2} = \dfrac{2(1 - x^2)}{(1 + x^2)^2} > 0 \Leftrightarrow |x| < 1$, so f is CU on $(-1, 1)$, CD on $(-\infty, -1)$ and $(1, \infty)$. IP $(1, \ln 2)$ and $(-1, \ln 2)$.

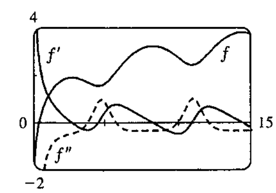
H.



62. $y = f(x) = \ln(x^2 - x)$ **A.** $\{x \mid x^2 - x > 0\} = \{x \mid x < 0 \text{ or } x > 1\} = (-\infty, 0) \cup (1, \infty)$. **B.** x-intercepts occur when $x^2 - x = 1 \Leftrightarrow x^2 - x - 1 = 0 \Leftrightarrow x = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$. No y-intercept **C.** No symmetry **D.** $\lim\limits_{x\to\infty} \ln(x^2 - x) = \infty$, no HA. $\lim\limits_{x\to 0^-} \ln(x^2 - x) = -\infty$, $\lim\limits_{x\to 1^+} \ln(x^2 - x) = -\infty$, so $x = 0$ and $x = 1$ are VA. **E.** $f'(x) = \dfrac{2x-1}{x^2-x} > 0$ when $x > 1$ and $f'(x) < 0$ when $x < 0$, so f is increasing on $(1, \infty)$ and decreasing on $(-\infty, 0)$. **F.** No extrema **G.** $f''(x) = \dfrac{2(x^2-x) - (2x-1)^2}{(x^2-x)^2} = \dfrac{-2x^2+2x-1}{(x^2-x)^2}$ $\Rightarrow$ $f''(x) < 0$ for all x since $-2x^2 + 2x - 1$ has a negative discriminant. So f is CD on $(-\infty, 0)$ and $(1, \infty)$. No IP. **H.**

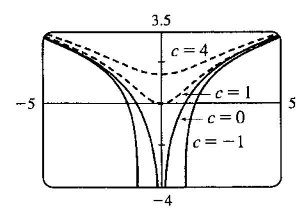


63. We use the CAS to calculate $f'(x) = \dfrac{2 + \sin x + x\cos x}{2x + x\sin x}$ and $f''(x) = \dfrac{2x^2\sin x + 4\sin x - \cos^2 x + x^2 + 5}{x^2(\cos^2 x - 4\sin x - 5)}$. From the graphs, it seems that $f' > 0$ (and so f is increasing) on approximately the intervals $(0, 2.7)$, $(4.5, 8.2)$ and $(10.9, 14.3)$. It seems that f'' changes sign (indicating inflection points) at $x \approx 3.8, 5.7, 10.0$ and 12.0.



Looking back at the graph of f, this implies that the inflection points have approximate coordinates $(3.8, 1.7)$, $(5.7, 2.1)$, $(10.0, 2.7)$, and $(12.0, 2.9)$.

64. We see that if $c \le 0$, $f(x) = \ln(x^2 + c)$ is only defined for $x^2 > -c \Rightarrow |x| > \sqrt{-c}$, and $\lim\limits_{x\to\sqrt{-c}^+} f(x) = \lim\limits_{x\to -\sqrt{-c}^-} f(x) = -\infty$, since $\ln y \to -\infty$ as $y \to 0$. Thus, for $c < 0$, there are vertical asymptotes at $x = \pm\sqrt{c}$, and as c decreases (that is, $|c|$ increases), the asymptotes get further apart. For $c = 0$, $\lim\limits_{x\to 0} f(x) = -\infty$, so there is a vertical asymptote at $x = 0$. If $c > 0$, there is no asymptote. To find the maxima, minima, and inflection points, we differentiate: $f(x) = \ln(x^2 + c) \Rightarrow f'(x) = \dfrac{1}{x^2+c}(2x)$, so by the First Derivative Test there is a local and absolute minimum at $x = 0$. Differentiating again, we get $f''(x) = \dfrac{1}{x^2+c}(2) + 2x\left[-(x^2+c)^{-2}(2x)\right] = \dfrac{2(c-x^2)}{(x^2+c)^2}$. Now if $c \le 0$, this is always negative, so f is concave down on both of the intervals on which it is defined. If $c > 0$, then f'' changes sign when $c = x^2 \Leftrightarrow x = \pm\sqrt{c}$. So for $c > 0$ there are inflection points at $\pm\sqrt{c}$, and as c increases, the inflection points get further apart.



65. $\displaystyle\int_1^9 \frac{1}{2x}\,dx = \frac{1}{2}\int_1^9 \frac{1}{x}\,dx = \tfrac{1}{2}\left[\ln|x|\right]_1^9 = \tfrac{1}{2}(\ln 9 - \ln 1) = \tfrac{1}{2}\ln 9 - 0 = \ln 9^{1/2} = \ln 3$

66. $\displaystyle\int_{-e^2}^{-e} \frac{3}{x}\,dx = \left[3\ln|x|\right]_{-e^2}^{-e} = 3\ln e - 3\ln\left(e^2\right) = 3 - 6 = -3$

67. Let $u = 2x + 3$. Then $du = 2\,dx$, so

$$\int_0^3 \frac{dx}{2x+3} = \int_3^9 \frac{\frac{1}{2}\,du}{u} = \tfrac{1}{2}\ln u\Big]_3^9 = \tfrac{1}{2}(\ln 9 - \ln 3) = \tfrac{1}{2}\left(\ln 3^2 - \ln 3\right) = \tfrac{1}{2}(2\ln 3 - \ln 3) = \tfrac{1}{2}\ln 3 \text{ (or } \ln\sqrt{3}\text{)}.$$

68. $\displaystyle\int_4^9 \left[\sqrt{x} + \frac{1}{\sqrt{x}}\right]^2 dx = \int_4^9 \left(x + 2 + \frac{1}{x}\right) dx = \left[\tfrac{1}{2}x^2 + 2x + \ln x\right]_4^9 = \frac{81}{2} + 18 + \ln 9 - (8 + 8 + \ln 4) = \frac{85}{2} + \ln\frac{9}{4}$

69. $\displaystyle\int_1^e \frac{x^2+x+1}{x}\,dx = \int_1^e \left(x + 1 + \frac{1}{x}\right) dx = \left[\tfrac{1}{2}x^2 + x + \ln x\right]_1^e = \left(\tfrac{1}{2}e^2 + e + 1\right) - \left(\tfrac{1}{2} + 1 + 0\right) = \tfrac{1}{2}e^2 + e - \tfrac{1}{2}$

70. Let $u = \ln x$. Then $du = \dfrac{1}{x}\,dx$, so $\displaystyle\int_e^6 \frac{dx}{x\ln x} = \int_1^{\ln 6} \frac{1}{u}\,du = [\ln|u|]_1^{\ln 6} = \ln\ln 6 - \ln 1 = \ln\ln 6$

71. Let $u = 5 - 3x$. Then $du = -3\,dx$, so $\displaystyle\int \frac{dx}{5-3x} = -\frac{1}{3}\int\frac{1}{u}\,du = -\tfrac{1}{3}\ln|u| + C = -\tfrac{1}{3}\ln|5-3x| + C.$

72. Let $u = x^3 + 3x + 1$. Then $du = 3\left(x^2+1\right)dx$, so

$$\int \frac{x^2+1}{x^3+3x+1}\,dx = \frac{1}{3}\int\frac{du}{u} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln\left|x^3+3x+1\right| + C.$$

73. Let $u = 1 + x^4$. Then $du = 4x^3\,dx$, so

$$\int \frac{x^3}{1+x^4}\,dx = \frac{1}{4}\int\frac{1}{u}\,du = \tfrac{1}{4}\ln|u| + C = \tfrac{1}{4}\ln\left|1+x^4\right| + C = \tfrac{1}{4}\ln\left(1+x^4\right) + C \text{ (since } 1 + x^4 > 0\text{)}.$$

74. Let $u = 2 + \sin x$. Then $du = \cos x\,dx$, so

$$\int \frac{\cos x}{2+\sin x}\,dx = \int\frac{1}{u}\,du = \ln|u| + C = \ln|2+\sin x| + C = \ln(2+\sin x) + C \text{ (since } 2 + \sin x > 0\text{)}.$$

75. Let $u = \ln x$. Then $du = \dfrac{dx}{x}$ $\Rightarrow$ $\displaystyle\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}(\ln x)^3 + C.$

76. Let $u = e^x + 1$. Then $du = e^x\,dx$, so $\displaystyle\int \frac{e^x}{e^x+1}\,dx = \int\frac{du}{u} = \ln|u| + C = \ln\left(e^x+1\right) + C.$

77. $\displaystyle\int_1^2 10^t\,dt = \left[\frac{10^t}{\ln 10}\right]_1^2 = \frac{10^2}{\ln 10} - \frac{10^1}{\ln 10} = \frac{100-10}{\ln 10} = \frac{90}{\ln 10}$

78. Let $u = x^2$. Then $du = 2x\,dx$, so $\int x2^{x^2}\,dx = \tfrac{1}{2}\int 2^u\,du = \dfrac{1}{2}\dfrac{2^u}{\ln 2} + C = \dfrac{1}{2\ln 2}2^{x^2} + C.$

79. (a) $\dfrac{d}{dx}(\ln|\sin x| + C) = \dfrac{1}{\sin x}\cos x = \cot x$

(b) Let $u = \sin x$. Then $du = \cos x\,dx$, so $\int\cot x\,dx = \displaystyle\int\frac{\cos x}{\sin x}\,dx = \int\frac{du}{u} = \ln|u| + C = \ln|\sin x| + C.$

80. Let $u = x - 2$. Then the area is

$$A = -\int_{-4}^{-1}\frac{2}{x-2}\,dx = -2\int_{-6}^{-3}\frac{du}{u} = [-2\ln|u|]_{-6}^{-3} = -2\ln 3 + 2\ln 6 = 2\ln 2 \approx 1.386.$$

81. The cross-sectional area is $\pi\left(1/\sqrt{x+1}\right)^2 = \pi/(x+1)$. Therefore, the volume is

$$\int_0^1 \frac{\pi}{x+1}\,dx = \pi\,[\ln(x+1)]_0^1 = \pi\ln 2 - \ln 1 = \pi\ln 2.$$

82. Using cylindrical shells, we get $V = \int_0^3 \frac{2\pi x}{x^2+1}\,dx = \pi\left[\ln\left(1+x^2\right)\right]_0^3 = \pi \ln 10$.

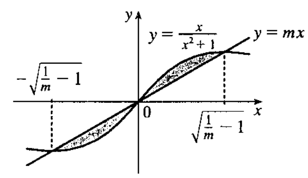
83. The domain of $f(x) = 1/x$ is $(-\infty, 0) \cup (0, \infty)$, so its general antiderivative is $F(x) = \begin{cases} \ln x + C_1 & \text{if } x > 0 \\ \ln|x| + C_2 & \text{if } x < 0 \end{cases}$

84. $f''(x) = x^{-2}, x > 0 \Rightarrow f'(x) = -1/x + C \Rightarrow f(x) = -\ln x + Cx + D$. $0 = f(1) = C + D$ and $0 = f(2) = -\ln 2 + 2C + D = -\ln 2 + 2C - C = -\ln 2 + C \Rightarrow C = \ln 2$ and $D = -\ln 2$. So $f(x) = -\ln x + (\ln 2)x - \ln 2$.

85. $f(x) = 2x + \ln x \Rightarrow f'(x) = 2 + 1/x$. If $g = f^{-1}$, then $f(1) = 2 \Rightarrow g(2) = 1$, so $g'(2) = 1/f'(g(2)) = 1/f'(1) = \frac{1}{3}$.

86. $f(x) = e^x + \ln x \Rightarrow f'(x) = e^x + 1/x$. $h = f^{-1}$ and $f(1) = e \Rightarrow h(e) = 1$, so $h'(e) = 1/f'(1) = 1/(e+1)$.

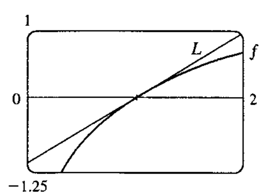
87. The curve and the line will determine a region when they intersect at two or more points. So we solve the equation $x/\left(x^2+1\right) = mx \Rightarrow x = 0$ or $mx^2 + m - 1 = 0 \Rightarrow x = 0$ or $x = \frac{\pm\sqrt{-4(m)(m-1)}}{2m} = \pm\sqrt{\frac{1}{m} - 1}$. Note that if $m = 1$, this has only the solution $x = 0$, and no region is determined. But if $1/m - 1 > 0 \Leftrightarrow 1/m > 1 \Leftrightarrow 0 < m < 1$, then there are two solutions. [Another way of seeing this is to observe that the slope of the tangent to $y = x/\left(x^2+1\right)$ at the origin is $y' = 1$ and therefore we must have $0 < m < 1$.] Note that we cannot just integrate between the positive and negative roots, since the curve and the line cross at the origin. Since mx and $x/\left(x^2+1\right)$ are both odd functions, the total area is twice the area between the curves on the interval $\left[0, \sqrt{1/m - 1}\right]$. So the total area enclosed is



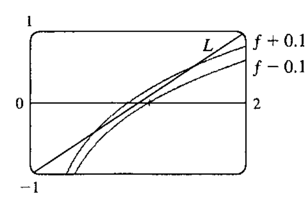
$$\begin{aligned} 2\int_0^{\sqrt{1/m-1}} \left[\frac{x}{x^2+1} - mx\right] dx &= 2\left[\tfrac{1}{2}\ln\left(x^2+1\right) - \tfrac{1}{2}mx^2\right]_0^{\sqrt{1/m-1}} \\ &= \left[\ln\left(\frac{1}{m} - 1 + 1\right) - m\left(\frac{1}{m} - 1\right)\right] - (\ln 1 - 0) \\ &= \ln\left(\frac{1}{m}\right) + m - 1 = m - \ln m - 1 \end{aligned}$$

88. (a) Let $f(x) = \ln x \Rightarrow f'(x) = 1/x \Rightarrow f''(x) = -1/x^2$. The linear approximation to $\ln x$ near 1 is $\ln x \approx f(1) + f'(1)(x-1) = \ln 1 + \frac{1}{1}(x-1) = x - 1$.

(b)



(c)



From the graph, it appears that the linear approximation is accurate to within 0.1 for x between about 0.62 and 1.51.

89. If $f(x) = \ln(1+x)$, then $f'(x) = \dfrac{1}{1+x}$, so $f'(0) = 1$.

Thus, $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = \lim\limits_{x\to 0} \dfrac{f(x)}{x} = \lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = f'(0) = 1$.

90. Let $m = n/x$. Then $n = xm$, and as $n \to \infty$, $m \to \infty$.

Therefore, $\lim\limits_{n\to\infty} \left(1 + \dfrac{x}{n}\right)^n = \lim\limits_{m\to\infty} \left(1 + \dfrac{1}{m}\right)^{mx} = \left[\lim\limits_{m\to\infty} \left(1 + \dfrac{1}{m}\right)^m\right]^x = e^x$ by Equation 9.

7.2* The Natural Logarithmic Function

1. $\ln \dfrac{x^3 y}{z^2} = \ln x^3 y - \ln z^2 = \ln x^3 + \ln y - \ln z^2 = 3\ln x + \ln y - 2\ln z$

2. $\ln \sqrt{a\left(b^2 + c^2\right)} = \ln \left(a\left(b^2+c^2\right)\right)^{1/2} = \frac{1}{2} \ln \left(a\left(b^2+c^2\right)\right) = \frac{1}{2}\left[\ln a + \ln\left(b^2+c^2\right)\right]$

$= \frac{1}{2}\ln a + \frac{1}{2}\ln\left(b^2+c^2\right)$

3. $\ln(uv)^{10} = 10\ln(uv) = 10(\ln u + \ln v) = 10\ln u + 10\ln v$

4. $\ln \dfrac{3x^2}{(x+1)^5} = \ln 3x^2 - \ln(x+1)^5 = \ln 3 + \ln x^2 - 5\ln(x+1) = \ln 3 + 2\ln x - 5\ln(x+1)$

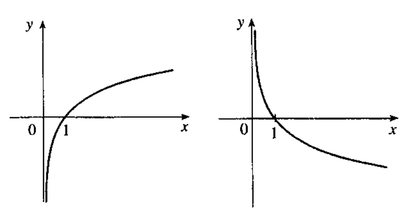
5. $2\ln 4 - \ln 2 = \ln 4^2 - \ln 2 = \ln 16 - \ln 2 = \ln \frac{16}{2} = \ln 8$

6. $\ln 3 + \frac{1}{3}\ln 8 = \ln 3 + \ln 8^{1/3} = \ln 3 + \ln 2 = \ln(3\cdot 2) = \ln 6$

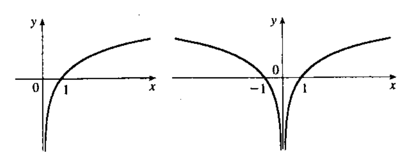
7. $\frac{1}{2}\ln x - 5\ln\left(x^2+1\right) = \ln x^{1/2} - \ln\left(x^2+1\right)^5 = \ln \dfrac{\sqrt{x}}{\left(x^2+1\right)^5}$

8. $\ln x + a\ln y - b\ln z = \ln x + \ln y^a - \ln z^b = \ln\left(xy^a/z^b\right)$

9. $y = \ln x$ $\quad$ $y = -\ln x$

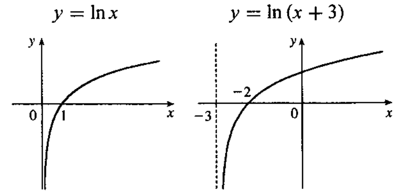


10. $y = \ln x$ $\quad$ $y = \ln|x|$

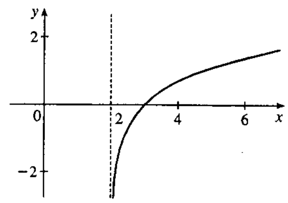


11.

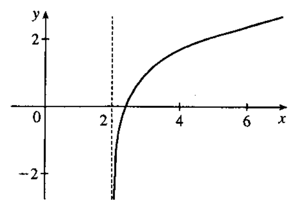
$y = \ln x$ $\quad$ $y = \ln(x+3)$



12. $y = \ln(x-2)$



$y = 1 + \ln(x-2)$



13. $f(x) = \sqrt{x}\ln x \quad \Rightarrow \quad f'(x) = \dfrac{1}{2\sqrt{x}}\ln x + \sqrt{x}\left(\dfrac{1}{x}\right) = \dfrac{\ln x + 2}{2\sqrt{x}}$

14. $f(x) = \ln(2-x) \quad \Rightarrow \quad f'(x) = \dfrac{1}{2-x}\dfrac{d}{dx}(2-x) = \dfrac{-1}{2-x} = \dfrac{1}{x-2}$

15. $f(\theta) = \ln(\cos\theta) \quad \Rightarrow \quad f'(\theta) = \dfrac{1}{\cos\theta}\dfrac{d}{d\theta}(\cos\theta) = \dfrac{-\sin\theta}{\cos\theta} = -\tan\theta$

16. $f(x) = \cos(\ln x) \quad \Rightarrow \quad f'(x) = -\sin(\ln x)\cdot\dfrac{1}{x} = \dfrac{-\sin(\ln x)}{x}$

17. $g(x) = \ln\dfrac{a-x}{a+x} = \ln(a-x) - \ln(a+x) \quad \Rightarrow$

$$g'(x) = \frac{1}{a-x}(-1) - \frac{1}{a+x} = \frac{-(a+x)-(a-x)}{(a-x)(a+x)} = \frac{-2a}{a^2-x^2}$$

18. $h(x) = \ln\left(x + \sqrt{x^2-1}\right) \quad \Rightarrow \quad h'(x) = \dfrac{1}{x+\sqrt{x^2-1}}\left(1 + \dfrac{x}{\sqrt{x^2-1}}\right) = \dfrac{1}{\sqrt{x^2-1}}$

19. $F(x) = \ln\sqrt{x} = \ln x^{1/2} = \frac{1}{2}\ln x \quad \Rightarrow \quad F'(x) = \dfrac{1}{2}\left(\dfrac{1}{x}\right) = \dfrac{1}{2x}$

20. $G(x) = \sqrt[3]{\ln x} = (\ln x)^{1/3} \quad \Rightarrow \quad G'(x) = \frac{1}{3}(\ln x)^{-2/3}\cdot\dfrac{1}{x} = \dfrac{1}{3x(\ln x)^{2/3}}$

21. $h(y) = \ln(y^3\sin y) = 3\ln y + \ln(\sin y) \quad \Rightarrow \quad h'(y) = \dfrac{3}{y} + \dfrac{1}{\sin y}(\cos y) = \dfrac{3}{y} + \cot y$

22. $f(t) = \dfrac{1+\ln t}{1-\ln t} \quad \Rightarrow$

$$f'(t) = \frac{(1-\ln t)(1/t) - (1+\ln t)(-1/t)}{(1-\ln t)^2} = \frac{(1/t)[(1-\ln t)+(1+\ln t)]}{(1-\ln t)^2} = \frac{2}{t(1-\ln t)^2}$$

23. $y = \dfrac{\ln x}{1+x} \quad \Rightarrow \quad y' = \dfrac{(1+x)(1/x) - (\ln x)(1)}{(1+x)^2} = \dfrac{\dfrac{1+x}{x} - \dfrac{x\ln x}{x}}{(1+x)^2} = \dfrac{1+x-x\ln x}{x(1+x)^2}$

24. $y = (\ln\tan x)^2 \quad \Rightarrow \quad y' = 2(\ln\tan x)\cdot\dfrac{1}{\tan x}\cdot\sec^2 x = \dfrac{2(\ln\tan x)\sec^2 x}{\tan x}$

25. $y = \ln|x^3 - x^2| \quad \Rightarrow \quad y' = \dfrac{1}{x^3-x^2}(3x^2-2x) = \dfrac{x(3x-2)}{x^2(x-1)} = \dfrac{3x-2}{x(x-1)}$

26. $G(u) = \ln\sqrt{\dfrac{3u+2}{3u-2}} = \frac{1}{2}[\ln(3u+2) - \ln(3u-2)] \quad \Rightarrow \quad G'(u) = \dfrac{1}{2}\left(\dfrac{3}{3u+2} - \dfrac{3}{3u-2}\right) = \dfrac{-6}{9u^2-4}$

27. $y = \ln\left(\dfrac{x+1}{x-1}\right)^{3/5} = \frac{3}{5}[\ln(x+1) - \ln(x-1)] \quad \Rightarrow \quad y' = \dfrac{3}{5}\left(\dfrac{1}{x+1} - \dfrac{1}{x-1}\right) = \dfrac{-6}{5(x^2-1)}$

28. $y = \ln(x + \ln x) \Rightarrow y' = \dfrac{1}{x + \ln x}\left(1 + \dfrac{1}{x}\right) = \dfrac{x+1}{x(x + \ln x)}$

29. $y = \tan[\ln(ax + b)] \Rightarrow y' = \sec^2(\ln(ax+b))\dfrac{a}{ax+b}$

30. $y = \ln|\tan 2x| \Rightarrow y' = \dfrac{2\sec^2 2x}{\tan 2x}$

31. $y = x\ln x \Rightarrow y' = \ln x + x(1/x) = \ln x + 1 \Rightarrow y'' = 1/x$

32. $y = \ln(\sec x + \tan x) \Rightarrow y' = \dfrac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x \Rightarrow y'' = \sec x \tan x$

33. $f(x) = \ln(2x + 1) \Rightarrow f'(x) = \dfrac{1}{2x+1} \cdot 2 = \dfrac{2}{2x+1}$. $\operatorname{Dom}(f) = \{x \mid 2x + 1 > 0\} = \left(-\frac{1}{2}, \infty\right)$.

34. $f(x) = \dfrac{1}{1 + \ln x} \Rightarrow f'(x) = -\dfrac{1/x}{(1+\ln x)^2}$ (Reciprocal Rule) $= -\dfrac{1}{x(1+\ln x)^2}$.
$\operatorname{Dom}(f) = \{x \mid x > 0 \text{ and } \ln x \neq -1\} = \{x \mid x > 0 \text{ and } x \neq 1/e\} = (0, 1/e) \cup (1/e, \infty)$.

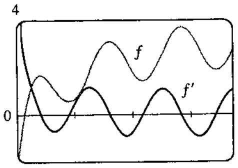
35. $f(x) = x^2 \ln(1 - x^2) \Rightarrow f'(x) = 2x\ln(1 - x^2) + \dfrac{x^2(-2x)}{1 - x^2} = 2x\ln(1 - x^2) - \dfrac{2x^3}{1 - x^2}$.
$\operatorname{Dom}(f) = \{x \mid 1 - x^2 > 0\} = \{x \mid |x| < 1\} = (-1, 1)$.

36. $f(x) = \ln\ln\ln x \Rightarrow f'(x) = \dfrac{1}{\ln\ln x} \cdot \dfrac{1}{\ln x} \cdot \dfrac{1}{x}$.
$\operatorname{Dom}(f) = \{x \mid \ln\ln x > 0\} = \{x \mid \ln x > 1\} = \{x \mid x > e\} = (e, \infty)$.

37. $f(x) = \dfrac{x}{\ln x} \Rightarrow f'(x) = \dfrac{\ln x - x(1/x)}{(\ln x)^2} = \dfrac{\ln x - 1}{(\ln x)^2} \Rightarrow f'(e) = \dfrac{1 - 1}{1^2} = 0$

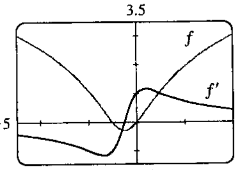
38. $f(x) = x^2 \ln x \Rightarrow f'(x) = 2x\ln x + x^2\left(\dfrac{1}{x}\right) = 2x\ln x + x \Rightarrow f'(1) = 2\ln 1 + 1 = 1$

39. $f(x) = \sin x + \ln x \Rightarrow f'(x) = \cos x + 1/x$. This is reasonable, because the graph shows that f increases when $f'(x)$ is positive, and $f'(x) = 0$ when f has a horizontal tangent.



40. $f(x) = \ln(x^2 + x + 1) \Rightarrow f'(x) = \dfrac{1}{x^2 + x + 1}(2x + 1)$.

Notice from the graph that f is increasing when $f'(x)$ is positive.



41. $y = f(x) = \ln\ln x \Rightarrow f'(x) = \dfrac{1}{\ln x}\left(\dfrac{1}{x}\right) \Rightarrow f'(e) = \dfrac{1}{e}$, so an equation of the tangent line at $(e, 0)$ is $y - 0 = \dfrac{1}{e}(x - e)$, or $y = \dfrac{1}{e}x - 1$, or $x - ey = e$.

42. $y = f(x) = \ln(x^2+1) \Rightarrow f'(x) = \dfrac{1}{x^2+1} \cdot 2x = \dfrac{2x}{x^2+1} \Rightarrow f'(1) = 1$, so an equation of the tangent line at $(1, \ln 2)$ is $y - \ln 2 = 1(x-1)$, or $y = x + \ln 2 - 1$.

43. $y = \ln(x^2+y^2) \Rightarrow y' = \dfrac{2x+2yy'}{x^2+y^2} \Rightarrow x^2y' + y^2y' = 2x + 2yy' \Rightarrow y' = \dfrac{2x}{x^2+y^2-2y}$

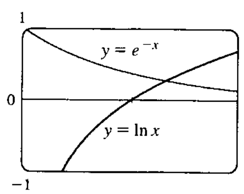
44. $\ln xy = \ln x + \ln y = y \sin x \Rightarrow 1/x + y'/y = y\cos x + y' \sin x \Rightarrow y'(1/y - \sin x) = y\cos x - 1/x \Rightarrow$
$y' = \dfrac{y\cos x - 1/x}{1/y - \sin x} = \left(\dfrac{y}{x}\right)\dfrac{xy\cos x - 1}{1 - y\sin x}$

45. $f(x) = \ln(x-1) \Rightarrow f'(x) = 1/(x-1) = (x-1)^{-1} \Rightarrow f''(x) = -(x-1)^{-2}$
$\Rightarrow f'''(x) = 2(x-1)^{-3} \Rightarrow f^{(4)}(x) = -2 \cdot 3(x-1)^{-4} \Rightarrow \cdots \Rightarrow$
$f^{(n)}(x) = (-1)^{n-1} \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (n-1)(x-1)^{-n} = (-1)^{n-1}\dfrac{(n-1)!}{(x-1)^n}$

46. $y = x^8 \ln x$, so

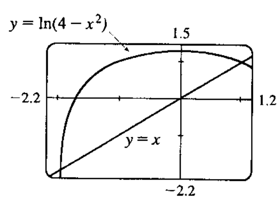
$$D^9 y = D^8\left(8x^7 \ln x + x^7\right) = D^8\left(8x^7 \ln x\right) = D^7\left(8 \cdot 7x^6 \ln x + 8x^6\right) = D^7\left(8 \cdot 7x^6 \ln x\right)$$
$$= D^6\left(8 \cdot 7 \cdot 6x^5 \ln x\right) = \cdots = D\left(8!\,x^0 \ln x\right) = 8!/x$$

47.



From the graph, it appears that the only root of the equation occurs at about $x = 1.3$. So we use Newton's Method with this as our initial approximation, and with $f(x) = \ln x - e^{-x} \Rightarrow f'(x) = 1/x + e^{-x}$. The formula is $x_{n+1} = x_n - f(x_n)/f'(x_n)$, and we calculate $x_1 = 1.3$, $x_2 \approx 1.309760$, $x_3 \approx x_4 \approx 1.309800$. So, correct to six decimal places, the root of the equation $\ln x = e^{-x}$ is $x = 1.309800$.

48.



We use Newton's Method with $f(x) = \ln(4-x^2) - x$ and $f'(x) = \dfrac{1}{4-x^2}(-2x) - 1 = 1 - \dfrac{2x}{4-x^2}$. The formula is $x_{n+1} = x_n - f(x_n)/f'(x_n)$. From the graphs it seems that the roots occur at approximately $x = -1.9$ and $x = 1.1$. However, if we use $x_1 = -1.9$ as an initial approximation to the first root, we get $x_2 \approx -2.009611$, and $f(x) = \ln(x-2)^2 - x$ is undefined at this point, making it impossible to calculate x_3. We must use a more accurate first estimate, such as $x_1 = -1.95$. With this approximation, we get $x_1 = -1.95$, $x_2 \approx -1.1967495$, $x_3 \approx -1.964760$, $x_4 \approx x_5 \approx -1.964636$. Calculating the second root gives $x_1 = 1.1$, $x_2 \approx 1.058649$, $x_3 \approx 1.058007$, $x_4 \approx x_5 \approx 1.058006$. So, correct to six decimal places, the two roots of the equation $\ln(4-x^2) = x$ are $x = -1.964636$ and $x = 1.058006$.

49. $y = f(x) = \ln(\cos x)$

A. $D = \{x \mid \cos x > 0\} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \cup \cdots = \left\{x \mid 2n\pi - \frac{\pi}{2} < x < 2n\pi + \frac{\pi}{2}, n = 0, \pm 1, \pm 2, \ldots\right\}$

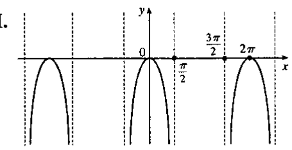
B. x-intercepts occur when $\ln(\cos x) = 0 \iff \cos x = 1 \iff x = 2n\pi$, y-intercept $= f(0) = 0$.

C. $f(-x) = f(x)$, so the curve is symmetric about the y-axis. $f(x + 2\pi) = f(x)$, f has period 2π, so in parts D–G we consider only $-\frac{\pi}{2} < x < \frac{\pi}{2}$. **D.** $\lim\limits_{x\to\pi/2^-} \ln(\cos x) = -\infty$ and $\lim\limits_{x\to-\pi/2^+} \ln(\cos x) = -\infty$, so $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are VA. No HA.

E. $f'(x) = (1/\cos x)(-\sin x) = -\tan x > 0 \iff -\frac{\pi}{2} < x < 0$, so f is increasing on $\left(-\frac{\pi}{2}, 0\right)$ and decreasing on $\left(0, \frac{\pi}{2}\right)$.

F. $f(0) = 0$ is a local maximum. **G.** $f''(x) = -\sec^2 x < 0 \Rightarrow$ f is CD on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. No IP.

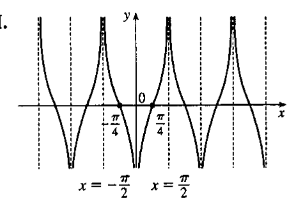
H.



50. $y = \ln(\tan^2 x)$ **A.** $D = \{x \mid x \neq n\pi/2\}$ **B.** x-intercepts $n\pi + \frac{\pi}{4}$, no y-intercept. **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. Also $f(x + \pi) = f(x)$, so f is periodic with period π, and we consider parts D–G only for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. **D.** $\lim\limits_{x\to 0} \ln(\tan^2 x) = -\infty$ and $\lim\limits_{x\to\pi/2^-} \ln(\tan^2 x) = \infty$, $\lim\limits_{x\to-\pi/2^+} \ln(\tan^2 x) = \infty$, so $x = 0$, $x = \pm\frac{\pi}{2}$ are VA. **E.** $f'(x) = \dfrac{2\tan x \sec^2 x}{\tan^2 x} = 2\dfrac{\sec^2 x}{\tan x} > 0 \iff$ $\tan x > 0 \iff 0 < x < \frac{\pi}{2}$, so f is increasing on $\left(0, \frac{\pi}{2}\right)$ and decreasing on $\left(-\frac{\pi}{2}, 0\right)$. **F.** No maximum or minimum **G.** $f'(x) = \dfrac{2}{\sin x \cos x} = \dfrac{4}{\sin 2x} \Rightarrow$

$f''(x) = \dfrac{-8\cos 2x}{\sin^2 2x} < 0 \iff \cos 2x > 0 \iff -\frac{\pi}{4} < x < \frac{\pi}{4}$, so f is CD on $\left(-\frac{\pi}{4}, 0\right)$ and $\left(0, \frac{\pi}{4}\right)$ and CU on $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. IP are $\left(\pm\frac{\pi}{4}, 0\right)$.

H.

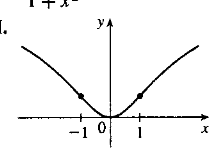


51. $y = f(x) = \ln(1 + x^2)$ **A.** $D = \mathbb{R}$ **B.** Both intercepts are 0. **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** $\lim\limits_{x\to\pm\infty} \ln(1 + x^2) = \infty$, no asymptotes. **E.** $f'(x) = \dfrac{2x}{1 + x^2} > 0$ $\iff x > 0$, so f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

F. $f(0) = 0$ is a local and absolute minimum.

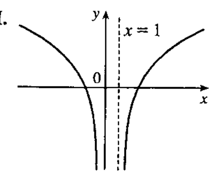
G. $f''(x) = \dfrac{2(1 + x^2) - 2x(2x)}{(1 + x^2)^2} = \dfrac{2(1 - x^2)}{(1 + x^2)^2} > 0 \iff$ $|x| < 1$, so f is CU on $(-1, 1)$, CD on $(-\infty, -1)$ and $(1, \infty)$. IP $(1, \ln 2)$ and $(-1, \ln 2)$.

H.

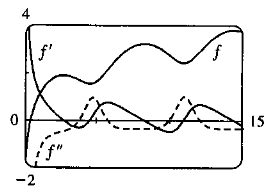


52. $y = f(x) = \ln(x^2 - x)$ **A.** $\{x \mid x^2 - x > 0\} = \{x \mid x < 0 \text{ or } x > 1\} = (-\infty, 0) \cup (1, \infty)$. **B.** x-intercepts occur when $x^2 - x = 1 \Leftrightarrow x^2 - x - 1 = 0 \Leftrightarrow x = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$. No y-intercept **C.** No symmetry **D.** $\lim\limits_{x\to\infty} \ln(x^2 - x) = \infty$, no HA. $\lim\limits_{x\to 0^-} \ln(x^2 - x) = -\infty$, $\lim\limits_{x\to 1^+} \ln(x^2 - x) = -\infty$, so $x = 0$ and $x = 1$ are VA. **E.** $f'(x) = \dfrac{2x - 1}{x^2 - x} > 0$ when $x > 1$ and $f'(x) < 0$ when $x < 0$, so f is increasing on $(1, \infty)$ and decreasing on $(-\infty, 0)$.
F. No extrema **G.** $f''(x) = \dfrac{2(x^2 - x) - (2x - 1)^2}{(x^2 - x)^2} = \dfrac{-2x^2 + 2x - 1}{(x^2 - x)^2}$
$\Rightarrow f''(x) < 0$ for all x since $-2x^2 + 2x - 1$ has a negative discriminant. So f is CD on $(-\infty, 0)$ and $(1, \infty)$. No IP.

H.

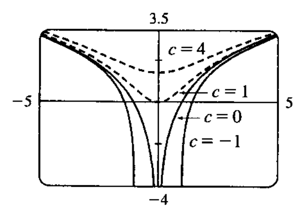


53. We use the CAS to calculate $f'(x) = \dfrac{2 + \sin x + x\cos x}{2x + x\sin x}$ and
$f''(x) = \dfrac{2x^2 \sin x + 4\sin x - \cos^2 x + x^2 + 5}{x^2(\cos^2 x - 4\sin x - 5)}$. From the graphs, it seems that $f' > 0$ (and so f is increasing) on approximately the intervals $(0, 2.7)$, $(4.5, 8.2)$ and $(10.9, 14.3)$. It seems that f'' changes sign (indicating inflection points) at $x \approx 3.8, 5.7, 10.0$ and 12.0.



Looking back at the graph of f, this implies that the inflection points have approximate coordinates $(3.8, 1.7)$, $(5.7, 2.1)$, $(10.0, 2.7)$, and $(12.0, 2.9)$.

54. We see that if $c \le 0$, $f(x) = \ln(x^2 + c)$ is only defined for $x^2 > -c \Rightarrow |x| > \sqrt{-c}$, and $\lim\limits_{x\to\sqrt{-c}^+} f(x) = \lim\limits_{x\to-\sqrt{-c}^-} f(x) = -\infty$, since $\ln y \to -\infty$ as $y \to 0$. Thus, for $c < 0$, there are vertical asymptotes at $x = \pm\sqrt{c}$, and as c decreases (that is, $|c|$ increases), the asymptotes get further apart. For $c = 0$, $\lim\limits_{x\to 0} f(x) = -\infty$, so there is a vertical asymptote at $x = 0$. If $c > 0$, there is no asymptote. To find the maxima, minima, and inflection points, we differentiate: $f(x) = \ln(x^2 + c) \Rightarrow f'(x) = \dfrac{1}{x^2 + c}(2x)$, so by the First Derivative Test there is a local and absolute minimum at $x = 0$. Differentiating again, we get
$f''(x) = \dfrac{1}{x^2 + c}(2) + 2x\left[-(x^2 + c)^{-2}(2x)\right] = \dfrac{2(c - x^2)}{(x^2 + c)^2}$. Now if $c \le 0$, this is always negative, so f is concave down on both of the intervals on which it is defined. If $c > 0$, then f'' changes sign when $c = x^2 \Leftrightarrow x = \pm\sqrt{c}$. So for $c > 0$ there are inflection points at $\pm\sqrt{c}$, and as c increases, the inflection points get further apart.



55. $\displaystyle\int_1^9 \frac{1}{2x}\,dx = \frac{1}{2}\int_1^9 \frac{1}{x}\,dx = \frac{1}{2}\left[\ln|x|\right]_1^9 = \frac{1}{2}(\ln 9 - \ln 1) = \frac{1}{2}\ln 9 - 0 = \ln 9^{1/2} = \ln 3$

56. $\displaystyle\int_{-e^2}^{-e} \frac{3}{x}\,dx = \left[3\ln|x|\right]_{-e^2}^{-e} = 3\ln e - 3\ln\left(e^2\right) = 3 - 6 = -3$

57. Let $u = 2x + 3$. Then $du = 2\,dx$, so

$$\int_0^3 \frac{dx}{2x+3} = \int_3^9 \frac{\frac{1}{2}\,du}{u} = \tfrac{1}{2}\ln u\Big]_3^9 = \tfrac{1}{2}(\ln 9 - \ln 3) = \tfrac{1}{2}\left(\ln 3^2 - \ln 3\right) = \tfrac{1}{2}(2\ln 3 - \ln 3) = \tfrac{1}{2}\ln 3 \text{ (or } \ln\sqrt{3}).$$

58. $\displaystyle\int_4^9 \left[\sqrt{x} + \frac{1}{\sqrt{x}}\right]^2 dx = \int_4^9 \left(x + 2 + \frac{1}{x}\right) dx = \left[\tfrac{1}{2}x^2 + 2x + \ln x\right]_4^9 = \frac{81}{2} + 18 + \ln 9 - (8 + 8 + \ln 4) = \frac{85}{2} + \ln\frac{9}{4}$

59. $\displaystyle\int_1^e \frac{x^2+x+1}{x}\,dx = \int_1^e \left(x + 1 + \frac{1}{x}\right) dx = \left[\tfrac{1}{2}x^2 + x + \ln x\right]_1^e = \left(\tfrac{1}{2}e^2 + e + 1\right) - \left(\tfrac{1}{2} + 1 + 0\right) = \tfrac{1}{2}e^2 + e - \tfrac{1}{2}$

60. Let $u = \ln x$. Then $du = \dfrac{1}{x}\,dx$, so $\displaystyle\int_e^6 \frac{dx}{x\ln x} = \int_1^{\ln 6} \frac{1}{u}\,du = [\ln|u|]_1^{\ln 6} = \ln\ln 6 - \ln 1 = \ln\ln 6$

61. Let $u = 5 - 3x$. Then $du = -3\,dx$, so $\displaystyle\int \frac{dx}{5-3x} = -\frac{1}{3}\int \frac{1}{u}\,du = -\tfrac{1}{3}\ln|u| + C = -\tfrac{1}{3}\ln|5-3x| + C.$

62. Let $u = x^3 + 3x + 1$. Then $du = 3\left(x^2+1\right)dx$, so

$$\int \frac{x^2+1}{x^3+3x+1}\,dx = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln\left|x^3+3x+1\right| + C.$$

63. Let $u = 1 + x^4$. Then $du = 4x^3\,dx$, so

$$\int \frac{x^3}{1+x^4}\,dx = \frac{1}{4}\int \frac{1}{u}\,du = \tfrac{1}{4}\ln|u| + C = \tfrac{1}{4}\ln\left|1+x^4\right| + C = \tfrac{1}{4}\ln\left(1+x^4\right) + C \text{ (since } 1 + x^4 > 0).$$

64. Let $u = 2 + \sin x$. Then $du = \cos x\,dx$, so

$$\int \frac{\cos x}{2+\sin x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|2+\sin x| + C = \ln(2+\sin x) + C \text{ (since } 2 + \sin x > 0).$$

65. Let $u = \ln x$. Then $du = \dfrac{dx}{x} \Rightarrow \displaystyle\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}(\ln x)^3 + C.$

66. Let $u = 1 + \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}\,dx$, so $\displaystyle\int \frac{\left(1+\sqrt{x}\right)^4}{\sqrt{x}}\,dx = 2\int u^4\,du = \tfrac{2}{5}u^5 + C = \tfrac{2}{5}\left(1+\sqrt{x}\right)^5 + C.$

67. (a) $\dfrac{d}{dx}(\ln|\sin x| + C) = \dfrac{1}{\sin x}\cos x = \cot x$

(b) Let $u = \sin x$. Then $du = \cos x\,dx$, so $\displaystyle\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C.$

68. Let $u = x - 2$. Then the area is

$$A = -\int_{-4}^{-1} \frac{2}{x-2}\,dx = -2\int_{-6}^{-3} \frac{du}{u} = [-2\ln|u|]_{-6}^{-3} = -2\ln 3 + 2\ln 6 = 2\ln 2 \approx 1.386.$$

69. The cross-sectional area is $\pi\left(1/\sqrt{x+1}\right)^2 = \pi/(x+1)$. Therefore, the volume is

$$\int_0^1 \frac{\pi}{x+1}\,dx = \pi\,[\ln(x+1)]_0^1 = \pi\ln 2 - \ln 1 = \pi\ln 2.$$

70. Using cylindrical shells, we get $\displaystyle V = \int_0^3 \frac{2\pi x}{x^2+1}\,dx = \pi\left[\ln\left(1+x^2\right)\right]_0^3 = \pi\ln 10.$

71. $y = (2x+1)^5 (x^4-3)^6 \Rightarrow \ln y = \ln\left((2x+1)^5 (x^4-3)^6\right) \Rightarrow \ln y = 5\ln(2x+1) + 6\ln(x^4-3) \Rightarrow$

$\frac{1}{y}y' = 5 \cdot \frac{1}{2x+1} \cdot 2 + 6 \cdot \frac{1}{x^4-3} \cdot 4x^3 \Rightarrow$

$y' = y\left(\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right) = y \cdot \frac{10(x^4-3) + 24x^3(2x+1)}{(2x+1)(x^4-3)} = (2x+1)^5(x^4-3)^6 \cdot \frac{58x^4+24x^3-30}{(2x+1)(x^4-3)}$

$= 2(2x+1)^4(x^4-3)^5(29x^4+12x^3-15)$

72. $y = \frac{(x^3+1)^4 \sin^2 x}{x^{1/3}} \Rightarrow \ln|y| = 4\ln|x^3+1| + 2\ln|\sin x| - \frac{1}{3}\ln|x|$. So $\frac{y'}{y} = 4\frac{3x^2}{x^3+1} + 2\frac{\cos x}{\sin x} - \frac{1}{3x} \Rightarrow$

$y' = \frac{(x^3+1)^4 \sin^2 x}{x^{1/3}}\left(\frac{12x^2}{x^3+1} + 2\cot x - \frac{1}{3x}\right)$.

73. $y = \frac{\sin^2 x \tan^4 x}{(x^2+1)^2} \Rightarrow \ln y = \ln(\sin^2 x \tan^4 x) - \ln(x^2+1)^2 = 2\ln\sin x + 4\ln\tan x - 2\ln(x^2+1) \Rightarrow$

$\frac{1}{y}y' = 2 \cdot \frac{1}{\sin x} \cdot \cos x + 4 \cdot \frac{1}{\tan x} \cdot \sec^2 x - 2 \cdot \frac{1}{x^2+1} \cdot 2x \Rightarrow$

$y' = \frac{\sin^2 x \tan^4 x}{(x^2+1)^2}\left(2\cot x + \frac{4\sec^2 x}{\tan x} - \frac{4x}{x^2+1}\right)$

74. $y = \sqrt[4]{\frac{x^2+1}{x^2-1}} \Rightarrow \ln y = \frac{1}{4}\ln(x^2+1) - \frac{1}{4}\ln(x^2-1) \Rightarrow \frac{1}{y}y' = \frac{1}{4} \cdot \frac{1}{x^2+1} \cdot 2x - \frac{1}{4} \cdot \frac{1}{x^2-1} \cdot 2x \Rightarrow$

$y' = \sqrt[4]{\frac{x^2+1}{x^2-1}} \cdot \frac{1}{2}\left(\frac{x}{x^2+1} - \frac{x}{x^2-1}\right) = \frac{1}{2}\sqrt[4]{\frac{x^2+1}{x^2-1}}\left(\frac{-2x}{x^4-1}\right) = \frac{x}{1-x^4}\sqrt[4]{\frac{x^2+1}{x^2-1}}$

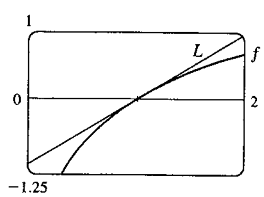
75. The domain of $f(x) = 1/x$ is $(-\infty, 0) \cup (0, \infty)$, so its general antiderivative is $F(x) = \begin{cases} \ln x + C_1 & \text{if } x > 0 \\ \ln|x| + C_2 & \text{if } x < 0 \end{cases}$

76. $f''(x) = x^{-2}, x > 0 \Rightarrow f'(x) = -1/x + C \Rightarrow f(x) = -\ln x + Cx + D$. $0 = f(1) = C + D$ and $0 = f(2) = -\ln 2 + 2C + D = -\ln 2 + 2C - C = -\ln 2 + C \Rightarrow C = \ln 2$ and $D = -\ln 2$. So $f(x) = -\ln x + (\ln 2)x - \ln 2$.

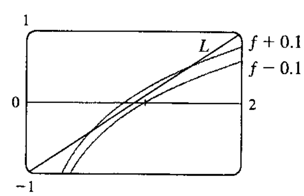
77. $f(x) = 2x + \ln x \Rightarrow f'(x) = 2 + 1/x$. If $g = f^{-1}$, then $f(1) = 2 \Rightarrow g(2) = 1$, so $g'(2) = 1/f'(g(2)) = 1/f'(1) = \frac{1}{3}$.

78. (a) Let $f(x) = \ln x \Rightarrow f'(x) = 1/x \Rightarrow f''(x) = -1/x^2$. The linear approximation to $\ln x$ near 1 is $\ln x \approx f(1) + f'(1)(x-1) = \ln 1 + \frac{1}{1}(x-1) = x - 1$.

(b)

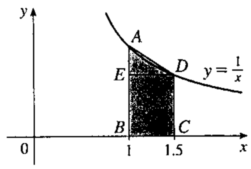


(c)



From the graph, it appears that the linear approximation is accurate to within 0.1 for x between about 0.62 and 1.51.

79. (a)



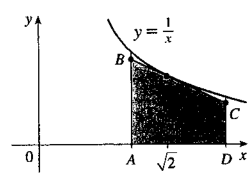
We interpret ln 1.5 as the area under the curve $y = 1/x$ from $x = 1$ to $x = 1.5$. The area of the rectangle $BCDE$ is $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$. The area of the trapezoid $ABCD$ is $\frac{1}{2} \cdot \frac{1}{2}\left(1 + \frac{2}{3}\right) = \frac{5}{12}$. Thus, by comparing areas, we observe that $\frac{1}{3} < \ln 1.5 < \frac{5}{12}$.

(b) With $f(t) = 1/t$, $n = 10$, and $\Delta x = 0.05$, we have

$$\begin{aligned}\ln 1.5 &= \textstyle\int_1^{1.5} (1/t)\,dt \approx (0.05)\,[f(1.025) + f(1.075) + \cdots + f(1.475)] \\ &= (0.05)\left[\tfrac{1}{1.025} + \tfrac{1}{1.075} + \cdots + \tfrac{1}{1.475}\right] \approx 0.4054\end{aligned}$$

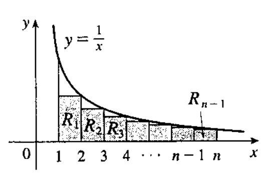
80. (a) $y = \dfrac{1}{t}$, $y' = -\dfrac{1}{t^2}$. The slope of AD is $\dfrac{1/2 - 1}{2 - 1} = -\dfrac{1}{2}$. Let c be the t-coordinate of the point on $y = \dfrac{1}{t}$ with slope $-\dfrac{1}{2}$. Then $-\dfrac{1}{c^2} = -\dfrac{1}{2} \;\Rightarrow\; c^2 = 2 \;\Rightarrow\; c = \sqrt{2}$ since $c > 0$. Therefore the tangent line is given by $y - \frac{1}{\sqrt{2}} = -\frac{1}{2}\left(t - \sqrt{2}\right) \;\Rightarrow\; y = -\frac{1}{2}t + \sqrt{2}$.

(b)

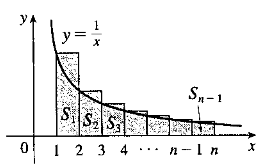


Since the graph of $y = 1/t$ is concave upward, the graph lies above the tangent line, that is, above the line segment BC. Now $|AB| = -\frac{1}{2} + \sqrt{2}$ and $|CD| = -1 + \sqrt{2}$. So the area of the trapezoid $ABCD$ is $\frac{1}{2}\left[\left(-\frac{1}{2} + \sqrt{2}\right) + \left(-1 + \sqrt{2}\right) 1\right] = -\frac{3}{4} + \sqrt{2} \approx 0.6642$. So $\ln 2 >$ area of trapezoid $ABCD > 0.66$.

81.



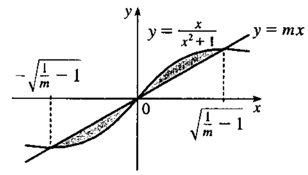
The area of R_i is $\dfrac{1}{i+1}$ and so $\dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} < \displaystyle\int_1^n \frac{1}{t}\,dt = \ln n$.



The area of S_i is $\dfrac{1}{i}$ and so $1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n-1} > \displaystyle\int_1^n \frac{1}{t}\,dt = \ln n$.

82. If $f(x) = \ln(x^r)$, then $f'(x) = (1/x^r)\left(rx^{r-1}\right) = r/x$. But if $g(x) = r\ln x$, then $g'(x) = r/x$. So f and g must differ by a constant: $\ln(x^r) = r\ln x + C$. Put $x = 1$: $\ln(1^r) = r\ln 1 + C \;\Rightarrow\; C = 0$, so $\ln(x^r) = r\ln x$.

83. The curve and the line will determine a region when they intersect at two or more points. So we solve the equation $x/\left(x^2+1\right) = mx \Rightarrow$ $x = 0$ or $mx^2 + m - 1 = 0 \Rightarrow x = 0$ or $x = \dfrac{\pm\sqrt{-4(m)(m-1)}}{2m} = \pm\sqrt{\dfrac{1}{m} - 1}$. Note that if $m = 1$, this has only the solution $x = 0$, and no region is determined. But if $1/m - 1 > 0 \Leftrightarrow 1/m > 1 \Leftrightarrow 0 < m < 1$, then there are two solutions. [Another way of seeing this is to observe that the slope of the tangent to $y = x/\left(x^2+1\right)$ at the origin is $y' = 1$ and therefore we must have $0 < m < 1$.] Note that we cannot just integrate between the positive and negative roots, since the curve and the line cross at the origin. Since mx and $x/\left(x^2+1\right)$ are both odd functions, the total area is twice the area between the curves on the interval $\left[0, \sqrt{1/m - 1}\right]$. So the total area enclosed is



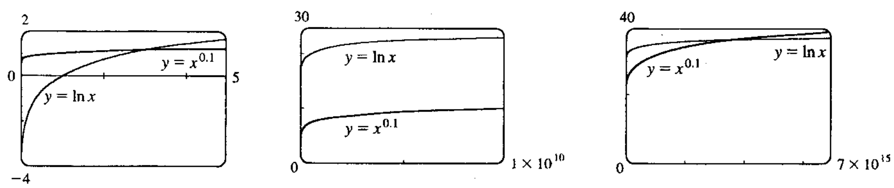
$$\begin{aligned}
2\int_0^{\sqrt{1/m-1}} \left[\frac{x}{x^2+1} - mx\right] dx &= 2\left[\tfrac{1}{2}\ln\left(x^2+1\right) - \tfrac{1}{2}mx^2\right]_0^{\sqrt{1/m-1}} \\
&= \left[\ln\left(\frac{1}{m} - 1 + 1\right) - m\left(\frac{1}{m} - 1\right)\right] - (\ln 1 - 0) \\
&= \ln\left(\frac{1}{m}\right) + m - 1 = m - \ln m - 1
\end{aligned}$$

84. $\lim\limits_{x\to\infty}[\ln(2+x) - \ln(1+x)] = \lim\limits_{x\to\infty}\ln\left(\dfrac{2+x}{1+x}\right) = \lim\limits_{x\to\infty}\ln\left(\dfrac{2/x+1}{1/x+1}\right) = \ln\dfrac{1}{1} = \ln 1 = 0$

85. If $f(x) = \ln(1+x)$, then $f'(x) = \dfrac{1}{1+x}$, so $f'(0) = 1$.

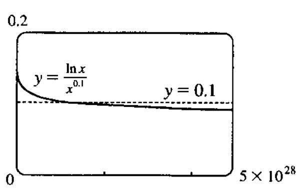
Thus, $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x} = \lim\limits_{x\to 0}\dfrac{f(x)}{x} = \lim\limits_{x\to 0}\dfrac{f(x) - f(0)}{x - 0} = f'(0) = 1$.

86. (a)



From the graphs, we see that $f(x) = x^{0.1} > g(x) = \ln x$ for approximately $0 < x < 3.06$, and then $g(x) > f(x)$ for $3.06 < x < 3.43 \times 10^{15}$ (approximately). At that point, the graph of f finally surpasses the graph of g for good.

(b)



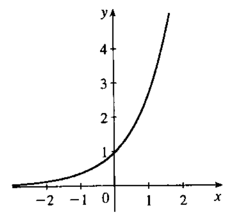
(c) From the graph at left, it seems that $\dfrac{\ln x}{x^{0.1}} < 0.1$ whenever $x > 1.3 \times 10^{28}$ (approximately). So we can take $N = 1.3 \times 10^{28}$, or any larger number.

7.3* The Natural Exponential Function

1. (a) e is the number such that $\ln e = 1$.

(b) $e \approx 2.71828$

(c)



The function value at $x = 0$ is 1 and the slope at $x = 0$ is 1.

2. (a) $e^{\ln 6} = 6$ (b) $\ln \sqrt{e} = \ln\left(e^{1/2}\right) = \frac{1}{2}$

3. (a) $\ln e^{\sqrt{2}} = \sqrt{2}$ (b) $e^{3\ln 2} = \left(e^{\ln 2}\right)^3 = 2^3 = 8$

4. (a) $\ln e^{\sin x} = \sin x$ (b) $e^{x+\ln x} = e^x e^{\ln x} = xe^x$

5. (a) $e^x = 16 \;\Leftrightarrow\; \ln e^x = \ln 16 \;\Leftrightarrow\; x = \ln 16 = 4\ln 2$

(b) $\ln x = -1 \;\Leftrightarrow\; e^{\ln x} = e^{-1} \;\Leftrightarrow\; x = 1/e$

6. (a) $\ln(2x-1) = 3 \;\Leftrightarrow\; e^{\ln(2x-1)} = e^3 \;\Leftrightarrow\; 2x - 1 = e^3 \;\Leftrightarrow\; x = \frac{1}{2}\left(e^3 + 1\right)$

(b) $e^{3x+1} = k \;\Leftrightarrow\; 3x + 1 = \ln k \;\Leftrightarrow\; x = \frac{1}{3}(\ln k - 1)$

7. $\ln(\ln x) = 1 \;\Leftrightarrow\; e^{\ln(\ln x)} = e^1 \;\Leftrightarrow\; \ln x = e^1 = e \;\Leftrightarrow\; e^{\ln x} = e^e \;\Leftrightarrow\; x = e^e$

8. $e^{e^x} = 10 \;\Leftrightarrow\; \ln\left(e^{e^x}\right) = \ln 10 \;\Leftrightarrow\; e^x \ln e = e^x = \ln 10 \;\Leftrightarrow\; \ln e^x = \ln(\ln 10) \;\Leftrightarrow\; x = \ln(\ln 10)$

9. $\ln(x+6) + \ln(x-3) = \ln 5 + \ln 2 \;\Leftrightarrow\; \ln((x+6)(x-3)) = \ln 10 \;\Leftrightarrow\; (x+6)(x-3) = 10 \;\Leftrightarrow\; x^2 + 3x - 18 = 10 \;\Leftrightarrow\; x^2 + 3x - 28 = 0 \;\Leftrightarrow\; (x+7)(x-4) = 0 \;\Leftrightarrow\; x = -7$ or 4. However, $x = -7$ is not a solution since $\ln(-7+6)$ is not defined. So $x = 4$ is the only solution.

10. $\ln x + \ln(x-1) = \ln(x(x-1)) = 1 \;\Leftrightarrow\; x(x-1) = e^1 \;\Leftrightarrow\; x^2 - x - e = 0$. The quadratic formula gives $x = \frac{1}{2}\left(1 \pm \sqrt{1+4e}\right)$, but we reject the negative root since the natural logarithm is not defined for $x < 0$. So $x = \frac{1}{2}\left(1 + \sqrt{1+4e}\right)$.

11. $e^{ax} = Ce^{bx} \;\Leftrightarrow\; \ln e^{ax} = \ln\left(Ce^{bx}\right) \;\Leftrightarrow\; ax = \ln C + bx \;\Leftrightarrow\; (a-b)x = \ln C \;\Leftrightarrow\; x = \dfrac{\ln C}{a-b}$

12. $7e^x - e^{2x} = 12 \;\Leftrightarrow\; (e^x)^2 - 7e^x + 12 = 0 \;\Leftrightarrow\; (e^x - 3)(e^x - 4) = 0$, so we have either $e^x = 3 \;\Leftrightarrow\; x = \ln 3$, or $e^x = 4 \;\Leftrightarrow\; x = \ln 4$.

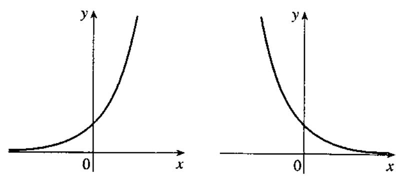
13. $e^{2+5x} = 100 \;\Rightarrow\; \ln\left(e^{2+5x}\right) = \ln 100 \;\Rightarrow\; 2 + 5x = \ln 100 \;\Rightarrow\; 5x = \ln 100 - 2 \;\Rightarrow\; x = \frac{1}{5}(\ln 100 - 2) \approx 0.5210$

14. $\ln\left(1 + \sqrt{x}\right) = 2 \;\Rightarrow\; 1 + \sqrt{x} = e^2 \;\Rightarrow\; \sqrt{x} = e^2 - 1 \;\Rightarrow\; x = \left(e^2 - 1\right)^2 \approx 40.8200$

15. $\ln(e^x - 2) = 3 \;\Rightarrow\; e^x - 2 = e^3 \;\Rightarrow\; e^x = e^3 + 2 \;\Rightarrow\; x = \ln\left(e^3 + 2\right) \approx 3.0949$

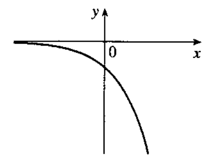
16. $e^{1/(x-4)} = 7 \;\Rightarrow\; \ln e^{1/(x-4)} = \ln 7 \;\Rightarrow\; \dfrac{1}{x-4} = \ln 7 \;\Rightarrow\; \dfrac{1}{\ln 7} = x - 4 \;\Rightarrow\; x = 4 + \dfrac{1}{\ln 7} \approx 4.5139$

17.



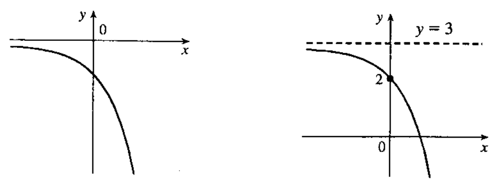
$y = e^x$ $\qquad$ $y = e^{-x}$

18.



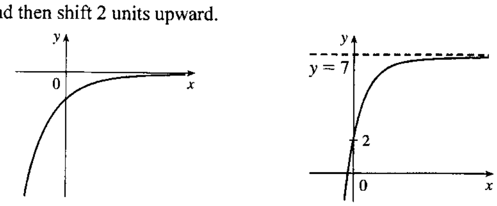
$y = -e^x$

19. We start with the graph of $y = e^x$ (Figure 13), reflect it about the x-axis, and then shift 3 units upward. Note the horizontal asymptote at $y = 3$.



$y = -e^x$ $\qquad$ $y = 3 - e^x$

20. We start with the graph of $y = e^x$ (Figure 13), reflect it about the y-axis, and then about the x-axis (or just rotate 180° to handle both reflections) to obtain the graph of $y = -e^{-x}$. Now shift this graph 1 unit upward, vertically stretch by a factor of 5, and then shift 2 units upward.



$y = -e^{-x}$ $\qquad$ $y = 2 + 5\left(1 - e^{-x}\right)$

21. (a) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units downward, we subtract 2 from the original function to get $y = e^x - 2$.

(b) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units to the right, we replace x with $x - 2$ in the original function to get $y = e^{(x-2)}$.

(c) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis, we multiply the original function by -1 to get $y = -e^x$.

(d) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the y-axis, we replace x with $-x$ in the original function to get $y = e^{-x}$.

(e) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis and then about the y-axis, we first multiply the original function by -1 (to get $y = -e^x$) and then replace x with $-x$ in this equation to get $y = -e^{-x}$.

22. (a) This reflection consists of first reflecting the graph through the x-axis (giving the graph with equation $y = -e^x$) and then shifting this graph $2 \cdot 4 = 8$ units upward. So the equation is $y = -e^x + 8$.

(b) This reflection consists of first reflecting the graph through the y-axis (giving the graph with equation $y = e^{-x}$) and then shifting this graph $2 \cdot 2 = 4$ units to the right. So the equation is $y = e^{-(x-4)}$.

23. Divide numerator and denominator by e^{3x}: $\lim_{x\to\infty}\frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}=\lim_{x\to\infty}\frac{1-e^{-6x}}{1+e^{-6x}}=\frac{1-0}{1+0}=1$

24. Divide numerator and denominator by e^{-3x}: $\lim_{x\to-\infty}\frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}=\lim_{x\to-\infty}\frac{e^{6x}-1}{e^{6x}+1}=\frac{0-1}{0+1}=-1$

25. Let $t=3/(2-x)$. As $x\to 2^+$, $t\to-\infty$. So $\lim_{x\to 2^+}e^{3/(2-x)}=\lim_{t\to-\infty}e^t=0$ by (11).

26. Let $t=3/(2-x)$. As $x\to 2^-$, $t\to\infty$. So $\lim_{x\to 2^-}e^{3/(2-x)}=\lim_{t\to\infty}e^t=\infty$ by (11).

27. $\lim_{x\to\pi/2^-}\frac{2}{1+e^{\tan x}}=0$ since $\tan x\to\infty \;\Rightarrow\; e^{\tan x}\to\infty$.

28. As $x\to 0^-$, $\cot x=\frac{\cos x}{\sin x}\to-\infty$, so $e^{\cot x}\to 0$ and $\lim_{x\to 0}\frac{2}{1+e^{\cot x}}=\frac{2}{1+0}=2$.

29. By the Product Rule, $f(x)=x^2e^x \;\Rightarrow\; f'(x)=x^2\frac{d}{dx}(e^x)+e^x\frac{d}{dx}(x^2)=x^2e^x+e^x(2x)=xe^x(x+2)$.

30. By the Quotient Rule, $y=\frac{e^x}{1+x} \;\Rightarrow\; y'=\frac{(1+x)e^x-e^x(1)}{(1+x)^2}=\frac{e^x+xe^x-e^x}{(x+1)^2}=\frac{xe^x}{(x+1)^2}$.

31. $y=e^{-mx} \;\Rightarrow\; y'=e^{-mx}\frac{d}{dx}(-mx)=e^{-mx}(-m)=-me^{-mx}$

32. $g(x)=e^{-5x}\cos 3x \;\Rightarrow\; g'(x)=-5e^{-5x}\cos 3x-3e^{-5x}\sin 3x$

33. $f(x)=e^{\sqrt{x}} \;\Rightarrow\; f'(x)=e^{\sqrt{x}}/\left(2\sqrt{x}\right)$

34. $y=e^x\ln x \;\Rightarrow\; y'=e^x\left(\frac{1}{x}\right)+(\ln x)(e^x)=e^x\left(\ln x+\frac{1}{x}\right)$

35. $h(t)=\sqrt{1-e^t} \;\Rightarrow\; h'(t)=-e^t/\left(2\sqrt{1-e^t}\right)$

36. $h(\theta)=e^{\sin 5\theta} \;\Rightarrow\; h'(\theta)=5\cos(5\theta)\,e^{\sin 5\theta}$

37. $y=e^{x\cos x} \;\Rightarrow\; y'=e^{x\cos x}(\cos x-x\sin x)$

38. $y=\cos(e^{\pi x}) \;\Rightarrow\; y'=-\sin(e^{\pi x})\cdot e^{\pi x}\cdot\pi=-\pi e^{\pi x}\sin(e^{\pi x})$

39. $y=e^{e^x} \;\Rightarrow\; y'=e^{e^x}\cdot\frac{d}{dx}(e^x)=e^{e^x}\cdot e^x$ or e^{e^x+x}

40. $y=\sqrt{1+xe^{-2x}} \;\Rightarrow\; y'=\frac{1}{2}\left(1+xe^{-2x}\right)^{-1/2}\left[x\left(-2e^{-2x}\right)+e^{-2x}\right]=\frac{e^{-2x}(-2x+1)}{2\sqrt{1+xe^{-2x}}}$

41. $y=\frac{e^{3x}}{1+e^x} \;\Rightarrow\; y'=\frac{3e^{3x}(1+e^x)-e^{3x}(e^x)}{(1+e^x)^2}=\frac{3e^{3x}+3e^{4x}-e^{4x}}{(1+e^x)^2}=\frac{3e^{3x}+2e^{4x}}{(1+e^x)^2}$

42. $y=\frac{e^x+e^{-x}}{e^x-e^{-x}} \;\Rightarrow$

$$y'=\frac{\left(e^x-e^{-x}\right)\left(e^x-e^{-x}\right)-\left(e^x+e^{-x}\right)\left(e^x+e^{-x}\right)}{\left(e^x-e^{-x}\right)^2}=\frac{\left(e^{2x}-2+e^{-2x}\right)-\left(e^{2x}+2+e^{-2x}\right)}{\left(e^x-e^{-x}\right)^2}=-\frac{4}{\left(e^x-e^{-x}\right)^2}$$

43. $y=f(x)=e^{-x}\sin x \;\Rightarrow\; f'(x)=-e^{-x}\sin x+e^{-x}\cos x \;\Rightarrow\; f'(\pi)=e^{-\pi}(\cos\pi-\sin\pi)=-e^{-\pi}$, so an equation of the tangent line at $(\pi,0)$ is $y-0=-e^{-\pi}(x-\pi)$, or $y=-e^{-\pi}x+\pi e^{-\pi}$, or $x+e^{\pi}y=\pi$.

44. $y=\frac{e^x}{x} \;\Rightarrow\; y'=\frac{xe^x-e^x}{x^2}=\frac{e^x(x-1)}{x^2}$. At $x=1$, $y'=0$, so an equation of the tangent line at $(1,e)$ is $y-e=0(x-1)$, or $y=e$.

45. $\cos(x-y)=xe^x \;\Rightarrow\; -\sin(x-y)\left(1-y'\right)=e^x+xe^x \;\Rightarrow\; y'=1+\frac{e^x(1+x)}{\sin(x-y)}$

46. $y = Ae^{-x} + Bxe^{-x} \Rightarrow y' = -Ae^{-x} + Be^{-x} - Bxe^{-x} = (B - A)e^{-x} - Bxe^{-x} \Rightarrow$
$y'' = (A - B)e^{-x} - Be^{-x} + Bxe^{-x} = (A - 2B)e^{-x} + Bxe^{-x}$, so
$y'' + 2y' + y = (A - 2B)e^{-x} + Bxe^{-x} + 2\left[(B - A)e^{-x} - Bxe^{-x}\right] + Ae^{-x} + Bxe^{-x} = 0.$

47. $y = e^{rx} \Rightarrow y' = re^{rx} \Rightarrow y'' = r^2e^{rx}$, so
$y'' + 5y' - 6y = r^2e^{rx} + 5re^{rx} - 6e^{rx} = e^{rx}\left(r^2 + 5r - 6\right) = e^{rx}(r + 6)(r - 1) = 0 \Rightarrow (r + 6)(r - 1) = 0$
$\Rightarrow r = 1$ or -6.

48. $y = e^{\lambda x} \Rightarrow y' = \lambda e^{\lambda x} \Rightarrow y'' = \lambda^2 e^{\lambda x}$. Thus, $y + y' = y'' \Leftrightarrow e^{\lambda x} + \lambda e^{\lambda x} = \lambda^2 e^{\lambda x} \Leftrightarrow$
$e^{\lambda x}\left(\lambda^2 - \lambda - 1\right) = 0 \Leftrightarrow \lambda = \frac{1 \pm \sqrt{5}}{2}$, since $e^{\lambda x} \neq 0$.

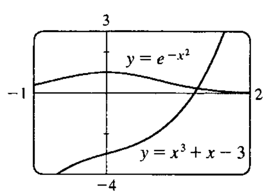
49. $f(x) = e^{2x} \Rightarrow f'(x) = 2e^{2x} \Rightarrow f''(x) = 2 \cdot 2e^{2x} = 2^2e^{2x} \Rightarrow f'''(x) = 2^2 \cdot 2e^{2x} = 2^3e^{2x} \Rightarrow \cdots$
$\Rightarrow f^{(n)}(x) = 2^n e^{2x}$

50. $f(x) = xe^{-x}$, $f'(x) = e^{-x} - xe^{-x} = (1 - x)e^{-x}$, $f''(x) = -e^{-x} + (1 - x)\left(-e^{-x}\right) = (x - 2)e^{-x}$. Similarly,
$f'''(x) = (3 - x)e^{-x}$, $f^{(4)}(x) = (x - 4)e^{-x}, \ldots, f^{(1000)}(x) = (x - 1000)e^{-x}$.

51. (a) $f(x) = e^x + x$ is continuous on $\mathbb{R}$ and $f(-1) = e^{-1} - 1 < 0 < 1 = f(0)$, so by the Intermediate Value Theorem, $e^x + x = 0$ has a root in $(-1, 0)$.

(b) $f(x) = e^x + x \Rightarrow f'(x) = e^x + 1$, so $x_{n+1} = x_n - \dfrac{e^{x_n} + x_n}{e^{x_n} + 1}$. Using $x_1 = -0.5$, we get $x_2 \approx -0.566311$, $x_3 \approx -0.567143 \approx x_4$, so the root is -0.567143 to six decimal places.

52.



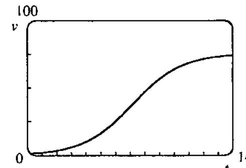
From the graph, it appears that the curves intersect at about $x \approx 1.2$ or 1.3. We use Newton's Method with $f(x) = x^3 + x - 3 - e^{-x^2}$, so $f'(x) = 3x^2 + 1 - 2xe^{-x^2}$, and the formula is $x_{n+1} = x_n - f(x_n)/f'(x_n)$. We take $x_1 = 1.2$, and the formula gives $x_2 \approx 1.252462$, $x_3 \approx 1.251045$, and $x_4 \approx x_5 \approx 1.251044$. So the root of the equation, correct to six decimal places, is $x = 1.251044$.

53. (a) $m(t) = 24 \cdot e^{-(\ln 2)t/25} = 24 \cdot 2^{-t/25} \Rightarrow m(40) = 24 \cdot 2^{-40/25} \approx 7.92$ mg.

(b) $m'(t) = 24\frac{d}{dt}\left[e^{-(\ln 2)t/25}\right] = 24 \cdot e^{-(\ln 2)t/25}\left(-\frac{\ln 2}{25}\right)$, so $m'(40) = 24e^{-(\ln 2)(40)/25}\left(-\frac{\ln 2}{25}\right) \approx -0.22$ mg/yr

(c) $m(t) = 5 \Rightarrow 24e^{-(\ln 2)t/25} = 5 \Rightarrow e^{-(\ln 2)t/25} = \frac{5}{24} \Rightarrow -(\ln 2)t/25 = \ln\frac{5}{24} \Rightarrow$
$t = -25\dfrac{\ln\frac{5}{24}}{\ln 2} \approx 56.6$ yr

54. From the graph, we estimate that the most rapid increase in the number of VCRs occurs at about $t = 7$. To maximize the first derivative, we need to determine the values for which the second derivative is 0. $V(t) = \dfrac{75}{1 + 74e^{-0.6t}} \Rightarrow$



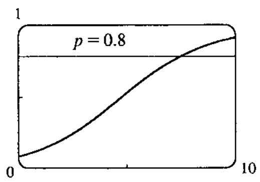
$$V'(t) = -\frac{75\left[74e^{-0.6t}(-0.6)\right]}{\left(1 + 74e^{-0.6t}\right)^2} = \frac{3330e^{-0.6t}}{\left(1 + 74e^{-0.6t}\right)^2} \Rightarrow$$

$$V''(t) = \frac{\left(1 + 74e^{-0.6t}\right)^2\left[3330e^{-0.6t}(-0.6)\right] - \left(3330e^{-0.6t}\right)2\left(1 + 74e^{-0.6t}\right)\left[74e^{-0.6t}(-0.6)\right]}{\left[\left(1 + 74e^{-0.6t}\right)^2\right]^2}$$

$$= \frac{\left(1 + 74e^{-0.6t}\right)\left[3330e^{-0.6t}(-0.6)\right]\left[\left(1 + 74e^{-0.6t}\right) - 2\left(74e^{-0.6t}\right)\right]}{\left(1 + 74e^{-0.6t}\right)^4} = \frac{-1998e^{-0.6t}\left(1 - 74e^{-0.6t}\right)}{\left(1 + 74e^{-0.6t}\right)^3}$$

$V''(t) = 0 \Leftrightarrow 1 = 74e^{-0.6t} \Leftrightarrow e^{0.6t} = 74 \Leftrightarrow 0.6t = \ln 74 \Leftrightarrow t = \frac{5}{3}\ln 74 \approx 7.173$ years, which corresponds to early September 1987.

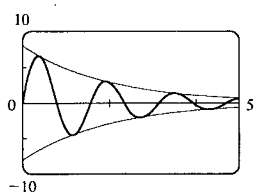
55. (a) $\lim\limits_{t\to\infty} p(t) = \lim\limits_{t\to\infty} \dfrac{1}{1 + ae^{-kt}} = \dfrac{1}{1 + a \cdot 0} = 1$, since $k > 0 \Rightarrow -kt \to -\infty$ as $t \to \infty$.



(b) $\dfrac{dp}{dt} = -\left(1 + ae^{-kt}\right)^{-2}\left(-kae^{-kt}\right) = \dfrac{kae^{-kt}}{\left(1 + ae^{-kt}\right)^2}$

(c) From the graph, it seems that $p(t) = 0.8$ (indicating that 80% of the population has heard the rumor) when $t \approx 7.4$ hours

56. (a)



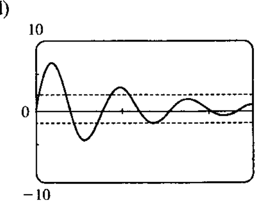
The displacement function is squeezed between the other two functions. This is because $-1 \le \sin 4t \le 1 \Rightarrow -8e^{-t/2} \le 8e^{-t/2}\sin 4t \le 8e^{-t/2}$.

(b) The maximum value of the displacement is about 6.6 cm, occurring at $t \approx 0.36$ s. It occurs just before the graph of the displacement function touches the graph of $8e^{-t/2}$ (when $t = \frac{\pi}{8} \approx 0.39$).

(c) The velocity of the object is the derivative of its displacement function, that is,

$\dfrac{d}{dt}\left(8e^{-t/2}\sin 4t\right) = 8\left[e^{-t/2}\cos 4t\,(4) + \sin 4t\left(-\frac{1}{2}\right)e^{-t/2}\right]$. If the displacement is zero, then we must have $\sin 4t = 0$ (since the exponential term in the displacement function is always positive). The first time that $\sin 4t = 0$ after $t = 0$ occurs at $t = \frac{\pi}{4}$. Substituting this into our expression for the velocity, and noting that the second term vanishes, we get $v\left(\frac{\pi}{4}\right) = 8e^{-\pi/8}\cos\left(4 \cdot \frac{\pi}{4}\right) \cdot 4 = -32e^{-\pi/8} \approx -21.6$ cm/s.

(d)



The graph indicates that the displacement is less than 2 cm from equilibrium whenever t is larger than about 2.8.

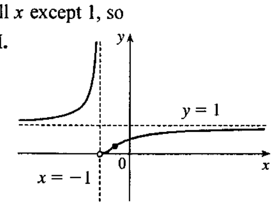
57. $f(x) = x - e^x \Rightarrow f'(x) = 1 - e^x = 0 \Leftrightarrow e^x = 1 \Leftrightarrow x = 0$. Now $f'(x) > 0$ for all $x < 0$ and $f'(x) < 0$ for all $x > 0$, so the absolute maximum value is $f(0) = 0 - 1 = -1$.

58. $g(x) = \dfrac{e^x}{x} \Rightarrow g'(x) = \dfrac{xe^x - e^x}{x^2} = 0 \Leftrightarrow e^x(x-1) = 0 \Rightarrow x = 1$. Now $g'(x) > 0 \Leftrightarrow$ $\dfrac{xe^x - e^x}{x^2} > 0 \Leftrightarrow x - 1 > 0 \Leftrightarrow x > 1$ and $g'(x) < 0 \Leftrightarrow \dfrac{xe^x - e^x}{x^2} < 0 \Leftrightarrow x - 1 < 0 \Leftrightarrow x < 1$. Thus there is an absolute minimum value of $g(1) = e$ at $x = 1$.

59. $y = e^x - 2e^{-x}$, so $y' = e^x + 2e^{-x}$, $y'' = e^x - 2e^{-x}$. $y'' > 0 \Leftrightarrow e^x - 2e^{-x} > 0 \Leftrightarrow e^x > 2e^{-x} \Leftrightarrow$ $e^{2x} > 2 \Leftrightarrow 2x > \ln 2 \Leftrightarrow x > \frac{1}{2}\ln 2$. Therefore, y is concave upward on $\left(\frac{1}{2}\ln 2, \infty\right)$.

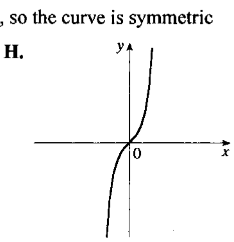
60. $f(x) = e^x + e^{-2x}$, $f'(x) = e^x - 2e^{-2x} > 0 \Leftrightarrow e^x > 2e^{-2x} \Leftrightarrow e^{3x} > 2 \Leftrightarrow 3x > \ln 2 \Leftrightarrow$ $x > \frac{1}{3}\ln 2$. Thus, f is increasing on $\left(\frac{1}{3}\ln 2, \infty\right)$.

61. $y = f(x) = e^{-1/(x+1)}$ **A.** $D = \{x \mid x \neq -1\} = (-\infty, -1) \cup (-1, \infty)$ **B.** No x-intercept; y-intercept $= f(0) = e^{-1}$ **C.** No symmetry **D.** $\lim\limits_{x\to\pm\infty} e^{-1/(x+1)} = 1$ since $-1/(x+1) \to 0$, so $y = 1$ is a HA. $\lim\limits_{x\to -1^+} e^{-1/(x+1)} = 0$ since $-1/(x+1) \to -\infty$, $\lim\limits_{x\to -1^-} e^{-1/(x+1)} = \infty$ since $-1/(x+1) \to \infty$, so $x = -1$ is a VA. **E.** $f'(x) = e^{-1/(x+1)}/(x+1)^2 \Rightarrow f'(x) > 0$ for all x except 1, so f is increasing on $(-\infty, -1)$ and $(-1, \infty)$. **F.** No extrema **H.**



G. $f''(x) = \dfrac{e^{-1/(x+1)}}{(x+1)^4} + \dfrac{e^{-1/(x+1)}(-2)}{(x+1)^3} = -\dfrac{e^{-1/(x+1)}(2x+1)}{(x+1)^4}$ $\Rightarrow f''(x) > 0 \Leftrightarrow 2x + 1 < 0 \Leftrightarrow x < -\frac{1}{2}$, so f is CU on $(-\infty, -1)$ and $\left(-1, -\frac{1}{2}\right)$, and CD on $\left(-\frac{1}{2}, \infty\right)$. f has an IP at $\left(-\frac{1}{2}, e^{-2}\right)$.

62. $y = f(x) = xe^{x^2}$ **A.** $D = \mathbb{R}$ **B.** Both intercepts are 0. **C.** $f(-x) = -f(x)$, so the curve is symmetric about the origin. **D.** $\lim\limits_{x\to\infty} xe^{x^2} = \infty$, $\lim\limits_{x\to-\infty} xe^{x^2} = -\infty$, no asymptote **H.**



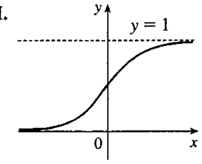
E. $f'(x) = e^{x^2} + xe^{x^2}(2x) = e^{x^2}\left(1 + 2x^2\right) > 0$, so f is increasing on $\mathbb{R}$.
F. No extremum
G. $f''(x) = e^{x^2}(2x)\left(1 + 2x^2\right) + e^{x^2}(4x) = e^{x^2}(2x)\left(3 + 2x^2\right) > 0 \Leftrightarrow$ $x > 0$, so f is CU on $(0, \infty)$and CD on $(-\infty, 0)$. f has an inflection point at $(0, 0)$.

63. $y = 1/\left(1+e^{-x}\right)$ **A.** $D = \mathbb{R}$ **B.** No x-intercepts; y-intercept $= f(0) = \frac{1}{2}$. **C.** No symmetry **D.** $\lim\limits_{x\to\infty} 1/\left(1+e^{-x}\right) = \frac{1}{1+0} = 1$ and $\lim\limits_{x\to-\infty} 1/\left(1+e^{-x}\right) = 0$ (since $\lim\limits_{x\to-\infty} e^{-x} = \infty$), so f has horizontal asymptotes $y = 0$ and $y = 1$. **E.** $f'(x) = -\left(1+e^{-x}\right)^{-2}\left(-e^{-x}\right) = e^{-x}/\left(1+e^{-x}\right)^2$. This is positive for all x, so f is increasing on $\mathbb{R}$. **F.** No extrema **G.** $f''(x) =$

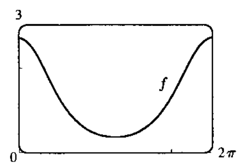
$$\frac{\left(1+e^{-x}\right)^2\left(-e^{-x}\right) - e^{-x}(2)\left(1+e^{-x}\right)\left(-e^{-x}\right)}{\left(1+e^{-x}\right)^4} = \frac{e^{-x}\left(e^{-x}-1\right)}{\left(1+e^{-x}\right)^3}.$$

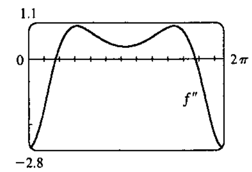
The second factor in the numerator is negative for $x > 0$ and positive for $x < 0$, and the other factors are always positive, so f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. f has an inflection point at $\left(0, \frac{1}{2}\right)$.

H.

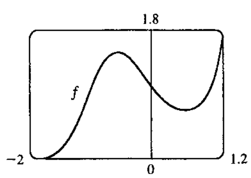


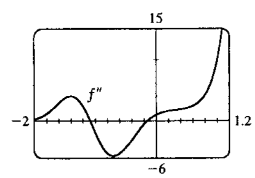
64. The function $f(x) = e^{\cos x}$ is periodic with period 2π, so we consider it only on the interval $[0, 2\pi]$. We see that it has local maxima of about $f(0) \approx 2.72$ and $f(2\pi) \approx 2.72$, and a local minimum of about $f(3.14) \approx 0.37$. To find the exact values, we calculate $f'(x) = -\sin x e^{\cos x}$. This is 0 when $-\sin x = 0 \Leftrightarrow x = 0, \pi$ or 2π (since we are only considering $x \in [0, 2\pi]$). Also $f'(x) > 0 \Leftrightarrow \sin x < 0 \Leftrightarrow 0 < x < \pi$. So $f(0) = f(2\pi) = e$ (both maxima) and $f(\pi) = e^{\cos\pi} = 1/e$ (minimum). To find the inflection points, we calculate and graph $f''(x) = \frac{d}{dx}\left(-\sin x\, e^{\cos x}\right) = -\cos x\, e^{\cos x} - \sin x\left(e^{\cos x}\right)(-\sin x) = e^{\cos x}\left(\sin^2 x - \cos x\right)$. From the graph of $f''(x)$, we see that f has inflection points at $x \approx 0.90$ and at $x \approx 5.38$. These x-coordinates correspond to inflection points $(0.90, 1.86)$ and $(5.38, 1.86)$.





65. $f(x) = e^{x^3-x} \to 0$ as $x \to -\infty$, and $f(x) \to \infty$ as $x \to \infty$. From the graph, it appears that f has a local minimum of about $f(0.58) = 0.68$, and a local maximum of about $f(-0.58) = 1.47$. To find the exact values, we calculate $f'(x) = \left(3x^2-1\right)e^{x^3-x}$, which is 0 when $3x^2 - 1 = 0 \Leftrightarrow x = \pm\frac{1}{\sqrt{3}}$. The negative root corresponds to the local maximum $f\left(-\frac{1}{\sqrt{3}}\right) = e^{(-1/\sqrt{3})^3-(-1/\sqrt{3})} = e^{2\sqrt{3}/9}$, and the positive root corresponds to the local minimum $f\left(\frac{1}{\sqrt{3}}\right) = e^{(1/\sqrt{3})^3-(1/\sqrt{3})} = e^{-2\sqrt{3}/9}$. To estimate the inflection points, we calculate and graph $f''(x) = \frac{d}{dx}\left[\left(3x^2-1\right)e^{x^3-x}\right] = \left(3x^2-1\right)e^{x^3-x}\left(3x^2-1\right) + e^{x^3-x}(6x) = e^{x^3-x}\left(9x^4 - 6x^2 + 6x + 1\right)$. From the graph, it appears that $f''(x)$ changes sign (and thus f has inflection points) at $x \approx -0.15$ and $x \approx -1.09$. From the graph of f, we see that these x-values correspond to inflection points at about $(-0.15, 1.15)$ and $(-1.09, 0.82)$.

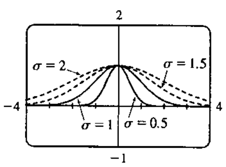




66. (a) As $|x| \to \infty$, $t = -x^2/(2\sigma^2) \to -\infty$, and $e^t \to 0$. The HA is $y = 0$. Since t takes on its maximum value at $x = 0$, so does e^t. Showing this result using derivatives, we have $f(x) = e^{-x^2/(2\sigma^2)} \Rightarrow$
$f'(x) = e^{-x^2/(2\sigma^2)}\left(-x/\sigma^2\right)$. $f'(x) = 0 \Leftrightarrow x = 0$. Because f' changes from positive to negative at $x = 0$, $f(0) = 1$ is a local maximum. For inflection points, we find
$f''(x) = -\frac{1}{\sigma^2}\left[e^{-x^2/(2\sigma^2)} \cdot 1 + xe^{-x^2/(2\sigma^2)}\left(-x/\sigma^2\right)\right] = \frac{-1}{\sigma^2}e^{-x^2/(2\sigma^2)}\left(1 - x^2/\sigma^2\right)$.
$f''(x) = 0 \Leftrightarrow x^2 = \sigma^2 \Leftrightarrow x = \pm\sigma$. $f''(x) < 0 \Leftrightarrow x^2 < \sigma^2 \Leftrightarrow -\sigma < x < \sigma$. So f is CD on $(-\sigma, \sigma)$ and CU on $(-\infty, -\sigma)$ and (σ, ∞). IP at $\left(\pm\sigma, e^{-1/2}\right)$.

(b) Since we have IP at $x = \pm\sigma$, the inflection points move away from the y-axis as σ increases.

(c)



From the graph, we see that as σ increases, the graph tends to spread out and there is more area between the curve and the x-axis.

67. Let $u = -3x$. Then $du = -3\,dx$, so
$\int_0^5 e^{-3x}\,dx = -\frac{1}{3}\int_0^{-15} e^u\,du = -\frac{1}{3}\left[e^u\right]_0^{-15} = -\frac{1}{3}\left(e^{-15} - e^0\right) = \frac{1}{3}\left(1 - e^{-15}\right)$.

68. Let $u = -x^2$, so $du = -2x\,dx$. When $x = 0$, $u = 0$; when $x = 1$, $u = -1$. Therefore,

$$\int_0^1 xe^{-x^2}\,dx = \int_0^{-1} e^u\left(-\tfrac{1}{2}\,du\right) = -\tfrac{1}{2}\left[e^u\right]_0^{-1} = -\tfrac{1}{2}\left(e^{-1} - e^0\right) = \tfrac{1}{2}\left(1 - 1/e\right)$$

69. Let $u = 1 + e^x$. Then $du = e^x\,dx$, so $\int e^x\sqrt{1+e^x}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}\left(1 + e^x\right)^{3/2} + C$.

70. Let $u = \tan x$. Then $du = \sec^2 x\,dx$, so $\int \sec^2 x\, e^{\tan x}\,dx = \int e^u\,du = e^u + C = e^{\tan x} + C$.

71. $\int \frac{e^x + 1}{e^x}\,dx = \int\left(1 + e^{-x}\right)dx = x - e^{-x} + C$

72. Let $u = \frac{1}{x}$. Then $du = -\frac{1}{x^2}\,dx$, so $\int \frac{e^{1/x}}{x^2}\,dx = -\int e^u\,du = -e^u + C = -e^{1/x} + C$.

73. Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}\,dx$, so $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C$.

74. Let $u = e^x$. Then $du = e^x\,dx$, so $\int e^x \sin\left(e^x\right)dx = \int \sin u\,du = -\cos u + C = -\cos\left(e^x\right) + C$.

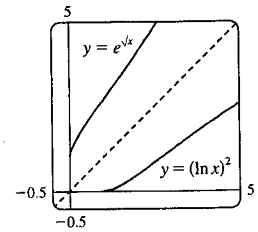
75. Area $= \int_0^1 \left(e^{3x} - e^x\right)dx = \left[\frac{1}{3}e^{3x} - e^x\right]_0^1 = \left(\frac{1}{3}e^3 - e\right) - \left(\frac{1}{3} - 1\right) = \frac{1}{3}e^3 - e + \frac{2}{3} \approx 4.644$

76. $f''(x) = 3e^x + 5\sin x \Rightarrow f'(x) = 3e^x - 5\cos x + C \Rightarrow 2 = f'(0) = 3 - 5 + C \Rightarrow C = 4$, so
$f'(x) = 3e^x - 5\cos x + 4 \Rightarrow f(x) = 3e^x - 5\sin x + 4x + D \Rightarrow 1 = f(0) = 3 + D \Rightarrow D = -2$, so
$f(x) = 3e^x - 5\sin x + 4x - 2$.

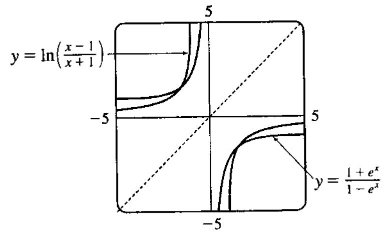
77. $V = \int_0^1 \pi\left(e^x\right)^2 dx = \int_0^1 \pi e^{2x}\,dx = \frac{1}{2}\left[\pi e^{2x}\right]_0^1 = \frac{\pi}{2}\left(e^2 - 1\right)$

78. $V = \int_0^1 2\pi x e^{-x^2}\,dx$. Let $u = x^2$. Thus $du = 2x\,dx$, so $V = \pi\int_0^1 e^{-u}\,du = \pi\left[-e^{-u}\right]_0^1 = \pi\left(1 - 1/e\right)$.

79. $y = e^{\sqrt{x}} \Rightarrow \ln y = \ln e^{\sqrt{x}} = \sqrt{x} \Rightarrow x = (\ln y)^2$.
Interchange x and y: the inverse function is $y = (\ln x)^2$.
The domain of the inverse function is the range of the original function $y = e^{\sqrt{x}}$, that is, $[1, \infty)$.



80. $y = \dfrac{1 + e^x}{1 - e^x} \Rightarrow y - ye^x = 1 + e^x \Rightarrow$
$e^x(y + 1) = y - 1 \Rightarrow e^x = \dfrac{y - 1}{y + 1} \Rightarrow$
$x = \ln\left(\dfrac{y - 1}{y + 1}\right)$. Interchange x and y: $y = \ln\left(\dfrac{x - 1}{x + 1}\right)$ is the inverse function.



81. We use Theorem 7.1.7. Note that $f(0) = 3 + 0 + e^0 = 4$, so $f^{-1}(4) = 0$. Also $f'(x) = 1 + e^x$. Therefore,
$(f^{-1})'(4) = \dfrac{1}{f'(f^{-1}(4))} = \dfrac{1}{f'(0)} = \dfrac{1}{1 + e^0} = \dfrac{1}{2}$.

82. We recognize this limit as the definition of the derivative of the function $f(x) = e^{\sin x}$ at $x = \pi$, since it is of the form $\lim\limits_{x \to \pi} \dfrac{f(x) - f(\pi)}{x - \pi}$. Therefore, the limit is equal to $f'(\pi) = (\cos\pi)\, e^{\sin\pi} = -1 \cdot e^0 = -1$.

83. Using the second law of logarithms and Equation 5, we have $\ln(e^x/e^y) = \ln e^x - \ln e^y = x - y = \ln(e^{x-y})$. Since ln is a one-to-one function, it follows that $e^x/e^y = e^{x-y}$.

84. Using the third law of logarithms and Equation 5, we have $\ln e^{rx} = rx = r \ln e^x = \ln(e^x)^r$. Since ln is a one-to-one function, it follows that $e^{rx} = (e^x)^r$.

85. (a) Let $f(x) = e^x - 1 - x$. Now $f(0) = e^0 - 1 = 0$, and for $x \geq 0$, we have $f'(x) = e^x - 1 \geq 0$. Now, since $f(0) = 0$ and f is increasing on $[0, \infty)$, $f(x) \geq 0$ for $x \geq 0 \Rightarrow e^x - 1 - x \geq 0 \Rightarrow e^x \geq 1 + x$.

(b) For $0 \leq x \leq 1$, $x^2 \leq x$, so $e^{x^2} \leq e^x$ (since e^x is increasing.) Hence [from (a)] $1 + x^2 \leq e^{x^2} \leq e^x$. So
$\frac{4}{3} = \int_0^1 (1 + x^2)\,dx \leq \int_0^1 e^{x^2}\,dx \leq \int_0^1 e^x\,dx = e - 1 < e \Rightarrow \frac{4}{3} \leq \int_0^1 e^{x^2}\,dx \leq e$.

86. (a) Let $f(x) = e^x - 1 - x - \frac{1}{2}x^2$. Thus, $f'(x) = e^x - 1 - x$, which is positive for $x \geq 0$ by Exercise 85(a). Thus $f(x)$ is increasing on $(0, \infty)$, so on that interval, $0 = f(0) \leq f(x) = e^x - 1 - x - \frac{1}{2}x^2 \Rightarrow$
$e^x \geq 1 + x + \frac{1}{2}x^2$.

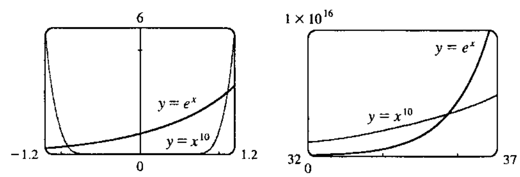
(b) Using the same argument as in Exercise 85(b), from part (a) we have $1 + x^2 + \frac{1}{2}x^4 \leq e^{x^2} \leq e^x$ (for $0 \leq x \leq 1$)
$\Rightarrow \int_0^1 \left(1 + x^2 + \frac{1}{2}x^4\right) dx \leq \int_0^1 e^{x^2}\,dx \leq \int_0^1 e^x\,dx \Rightarrow \frac{43}{30} \leq \int_0^1 e^{x^2}\,dx \leq e - 1$.

87. (a) By Exercise 85(a), the result holds for $n = 1$. Suppose that $e^x \geq 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^k}{k!}$ for $x \geq 0$. Let $f(x) = e^x - 1 - x - \dfrac{x^2}{2!} - \cdots - \dfrac{x^k}{k!} - \dfrac{x^{k+1}}{(k+1)!}$. Then $f'(x) = e^x - 1 - x - \cdots - \dfrac{x^k}{k!} \geq 0$ by assumption. Hence $f(x)$ is increasing on $(0, \infty)$. So $0 \leq x$ implies that $0 = f(0) \leq f(x) = e^x - 1 - x - \cdots - \dfrac{x^k}{k!} - \dfrac{x^{k+1}}{(k+1)!}$, and hence $e^x \geq 1 + x + \cdots + \dfrac{x^k}{k!} + \dfrac{x^{k+1}}{(k+1)!}$ for $x \geq 0$. Therefore, for $x \geq 0$, $e^x \geq 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!}$ for every positive integer n, by mathematical induction.

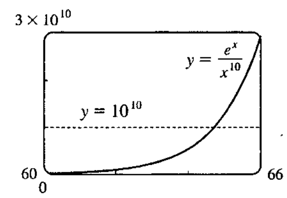
(b) Taking $n = 4$ and $x = 1$ in (a), we have $e = e^1 \geq 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 2.708\overline{3} > 2.7$.

(c) $e^x \geq 1 + x + \cdots + \dfrac{x^k}{k!} + \dfrac{x^{k+1}}{(k+1)!} \Rightarrow \dfrac{e^x}{x^k} \geq \dfrac{1}{x^k} + \dfrac{1}{x^{k-1}} + \cdots + \dfrac{1}{k!} + \dfrac{x}{(k+1)!} \geq \dfrac{x}{(k+1)!}$. But $\displaystyle\lim_{x\to\infty} \frac{x}{(k+1)!} = \infty$, so $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^k} = \infty$.

88. (a) The graph of g finally surpasses that of f at $x \approx 35.8$.



(b)



(c) From the graph in part (b), it seems that $e^x/x^{10} > 10^{10}$ whenever $x > 65$, approximately. So we can take $N \geq 65$.

7.4* General Logarithmic and Exponential Functions

1. (a) $a^x = e^{x \ln a}$

(b) The domain of $f(x) = a^x$ is $\mathbb{R}$.

(c) The range of $f(x) = a^x$ $(a \neq 1)$ is $(0, \infty)$.

(d) (i) See Figure 1. (ii) See Figure 3. (iii) See Figure 2.

2. (a) $\log_a x$ is the number y such that $a^y = x$.

(b) The domain of $f(x) = \log_a x$ is $(0, \infty)$.

(c) The range of $f(x) = \log_a x$ is $\mathbb{R}$.

(d) See Figure 9.

3. $10^\pi = e^{\pi \ln 10}$

4. $x^{\sqrt{2}} = e^{\sqrt{2} \ln x}$

5. $2^{\cos x} = e^{(\cos x) \ln 2}$

6. $(\sin x)^{\ln x} = e^{(\ln x) \ln(\sin x)}$

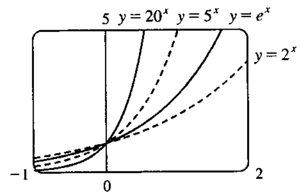
7. $\log_{10} 1000 = 3$ because $10^3 = 1000$.

8. $\log_5 \frac{1}{25} = -2$ because $5^{-2} = \frac{1}{25}$.

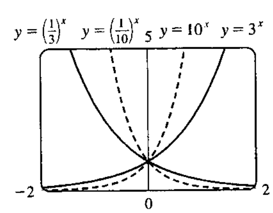
9. $2^{(\log_2 3+\log_2 5)} = 2^{\log_2 15} = 15$ [*Or:* $2^{(\log_2 3+\log_2 5)} = 2^{\log_2 3} \cdot 2^{\log_2 5} = 3 \cdot 5 = 15$]

10. $\log_3 3^{\sqrt{5}} = \sqrt{5}$

11. All of these graphs approach 0 as $x \to -\infty$, all of them pass through the point $(0, 1)$, and all of them are increasing and approach ∞ as $x \to \infty$. The larger the base, the faster the function increases for $x > 0$, and the faster it approaches 0 as $x \to -\infty$.



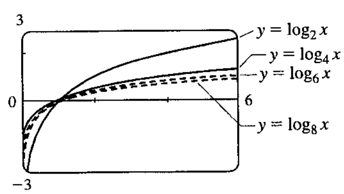
12. The functions with bases greater than 1 (3^x and 10^x) are increasing, while those with bases less than 1 $\left[\left(\frac{1}{3}\right)^x \text{ and } \left(\frac{1}{10}\right)^x\right]$ are decreasing. The graph of $\left(\frac{1}{3}\right)^x$ is the reflection of that of 3^x about the y-axis, and the graph of $\left(\frac{1}{10}\right)^x$ is the reflection of that of 10^x about the y-axis. The graph of 10^x increases more quickly than that of 3^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.



13. (a) $\log_2 5 = \dfrac{\ln 5}{\ln 2} \approx 2.321928$ (b) $\log_5 26.05 = \dfrac{\ln 26.05}{\ln 5} \approx 2.025563$

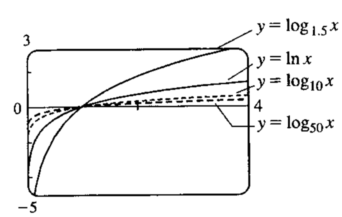
(c) $\log_3 e = \dfrac{1}{\ln 3} \approx 0.910239$

14.

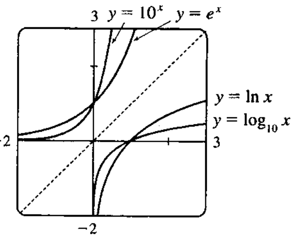


To graph the functions, we use $\log_2 x = \dfrac{\ln x}{\ln 2}$, $\log_4 x = \dfrac{\ln x}{\ln 4}$, etc. These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The smaller the base, the larger the rate of increase of the function (for $x > 1$) and the closer the approach to the y-axis (as $x \to 0^+$).

15. To graph these functions, we use $\log_{1.5} x = \dfrac{\ln x}{\ln 1.5}$ and $\log_{50} x = \dfrac{\ln x}{\ln 50}$. These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The functions with larger bases increase extremely slowly, and the ones with smaller bases do so somewhat more quickly. The functions with large bases approach the y-axis more closely as $x \to 0^+$.



16. We see that the graph of $\ln x$ is the reflection of the graph of e^x about the line $y = x$, and that the graph of $\log_{10} x$ is the reflection of the graph of 10^x about the same line. The graph of 10^x increases more quickly than that of e^x. Also note that $\log_{10} x \to \infty$ as $x \to \infty$ more slowly than $\ln x$.



17. Use $y = Ca^x$ with the points $(1, 6)$ and $(3, 24)$. $6 = Ca^1$ and $24 = Ca^3 \;\Rightarrow\; 24 = \left(\dfrac{6}{a}\right)a^3 \;\Rightarrow\; 4 = a^2 \;\Rightarrow$ $a = 2$ (since $a > 0$) and $C = 3$. The function is $f(x) = 3 \cdot 2^x$.

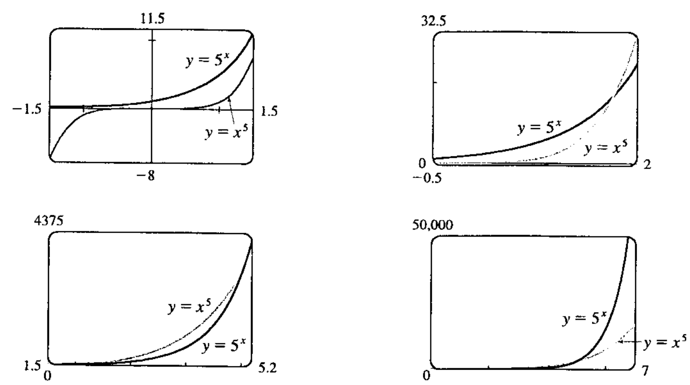
18. Given the y-intercept $(0, 2)$, we have $y = Ca^x = 2a^x$. Using the point $\left(2, \frac{2}{9}\right)$ gives us $\frac{2}{9} = 2a^2 \;\Rightarrow\; \frac{1}{9} = a^2$ $\Rightarrow\; a = \frac{1}{3}$ (since $a > 0$). The function is $f(x) = 2\left(\frac{1}{3}\right)^x$ or $f(x) = 2\,(3)^{-x}$.

19. (a) $2 \text{ ft} = 24 \text{ in}$, $f(24) = 24^2 \text{ in} = 576 \text{ in} = 48 \text{ ft}$. $g(24) = 2^{24} \text{ in} = 2^{24}/(12 \cdot 5280) \text{ mi} \approx 265 \text{ mi}$

(b) $3 \text{ ft} = 36 \text{ in}$, so we need x such that $\log_2 x = 36 \;\Leftrightarrow\; x = 2^{36} = 68{,}719{,}476{,}736$. In miles, this is

$$68{,}719{,}476{,}736 \text{ in} \cdot \frac{1 \text{ mi}}{5280 \text{ ft}} \cdot \frac{1 \text{ ft}}{12 \text{ in}} \approx 1{,}084{,}587.7 \text{ mi}.$$

20. We see from the graphs that for x less than about 1.8, $g(x) = 5^x > f(x) = x^5$, and then near the point $(1.8, 17.1)$ the curves intersect. Then $f(x) > g(x)$ from $x \approx 1.8$ until $x = 5$. At $(5, 3125)$ there is another point of intersection, and for $x > 5$ we see that $g(x) > f(x)$. In fact, g increases much more rapidly than f beyond that point.



21. $\displaystyle\lim_{x \to (\pi/2)^-} \log_{10}(\cos x) = -\infty$ since $\cos x \to 0^+$ as $x \to (\pi/2)^-$.

22. Since $(1.1)^x$ is continuous, we know that $\ln\left(\displaystyle\lim_{x \to -\infty} (1.1)^x\right) = \lim_{x \to -\infty} \ln\left((1.1)^x\right) = \lim_{x \to -\infty} x \ln 1.1 = -\infty$.

Therefore $\displaystyle\lim_{x \to -\infty} (1.1)^x = \lim_{y \to -\infty} e^{\ln y} = 0$.

23. $h(t) = t^3 - 3^t \;\Rightarrow\; h'(t) = 3t^2 - 3^t \ln 3$

24. $y = 10^{\tan\theta} \;\Rightarrow\; y' = 10^{\tan\theta}(\ln 10)\left(\sec^2\theta\right)$

25. Using Formula 4 and the Chain Rule, $y = 5^{-1/x} \Rightarrow y' = 5^{-1/x}(\ln 5)\left[-1 \cdot \left(-x^{-2}\right)\right] = 5^{-1/x}(\ln 5)/x^2$

26. $y = 2^{3^{x^2}} \Rightarrow y' = 2^{3^{x^2}}(\ln 2)\frac{d}{dx}\left(3^{x^2}\right) = 2^{3^{x^2}}(\ln 2)\,3^{x^2}(\ln 3)(2x)$

27. $f(x) = \log_3\left(x^2 - 4\right) \Rightarrow f'(x) = \frac{1}{\left(x^2-4\right)\ln 3}(2x) = \frac{2x}{\left(x^2-4\right)\ln 3}$

28. $f(x) = \log_{10}\left(\frac{x}{x-1}\right) = \log_{10}x - \log_{10}(x-1) \Rightarrow f'(x) = \frac{1}{x\ln 10} - \frac{1}{(x-1)\ln 10}$ or $-\frac{1}{x(x-1)\ln 10}$

29. $y = x^x \Rightarrow \ln y = x\ln x \Rightarrow y'/y = \ln x + x(1/x) \Rightarrow y' = x^x(\ln x + 1)$

30. $y = x^{1/x} \Rightarrow \ln y = \frac{1}{x}\ln x \Rightarrow \frac{y'}{y} = -\frac{1}{x^2}\ln x + \frac{1}{x}\left(\frac{1}{x}\right) \Rightarrow y' = x^{1/x}\frac{1-\ln x}{x^2}$

31. $y = x^{\sin x} \Rightarrow \ln y = \sin x\ln x \Rightarrow \frac{y'}{y} = \cos x\ln x + \frac{\sin x}{x} \Rightarrow y' = x^{\sin x}\left(\cos x\ln x + \frac{\sin x}{x}\right)$

32. $y = (\sin x)^x \Rightarrow \ln y = x\ln(\sin x) \Rightarrow y'/y = \ln(\sin x) + x(\cos x)/(\sin x) \Rightarrow$
$y' = (\sin x)^x[\ln(\sin x) + x\cot x]$

33. $y = (\ln x)^x \Rightarrow \ln y = x\ln\ln x \Rightarrow \frac{y'}{y} = \ln\ln x + x\cdot\frac{1}{\ln x}\cdot\frac{1}{x} \Rightarrow y' = (\ln x)^x\left(\ln\ln x + \frac{1}{\ln x}\right)$

34. $y = x^{\ln x} \Rightarrow \ln y = \ln x\ln x = (\ln x)^2 \Rightarrow \frac{y'}{y} = 2\ln x\left(\frac{1}{x}\right) \Rightarrow y' = x^{\ln x}\left(\frac{2\ln x}{x}\right)$

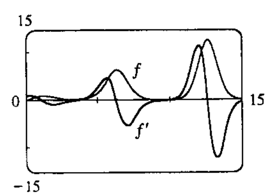
35. $y = x^{e^x} \Rightarrow \ln y = e^x\ln x \Rightarrow \frac{y'}{y} = e^x\ln x + \frac{e^x}{x} \Rightarrow y' = x^{e^x}e^x\left(\ln x + \frac{1}{x}\right)$

36. $y = (\ln x)^{\cos x} \Rightarrow \ln y = \cos x\ln(\ln x) \Rightarrow \frac{y'}{y} = \cos x\cdot\frac{1}{\ln x}\cdot\frac{1}{x} + (\ln\ln x)(-\sin x) \Rightarrow$
$y' = (\ln x)^{\cos x}\left(\frac{\cos x}{x\ln x} - \sin x\ln\ln x\right)$

37. $y = 10^x \Rightarrow y' = 10^x\ln 10$, so at $(1, 10)$, the slope of the tangent line is $10^1\ln 10 = 10\ln 10$, and its equation is $y - 10 = 10\ln 10\,(x-1)$, or $y = (10\ln 10)x + 10(1 - \ln 10)$.

38. $f(x) = x^{\cos x} = e^{\ln x\cos x} \Rightarrow$

$$f'(x) = e^{\ln x\cos x}\left[\ln x(-\sin x) + \cos x\left(\frac{1}{x}\right)\right] = x^{\cos x}\left[\frac{\cos x}{x} - \sin x\ln x\right]$$



This is reasonable, because the graph shows that f increases when $f'(x)$ is positive.

39. $\int_1^2 10^t\,dt = \left[\frac{10^t}{\ln 10}\right]_1^2 = \frac{10^2}{\ln 10} - \frac{10^1}{\ln 10} = \frac{100-10}{\ln 10} = \frac{90}{\ln 10}$

40. $\int\left(x^5 + 5^x\right)dx = \frac{1}{6}x^6 + \frac{1}{\ln 5}5^x + C$

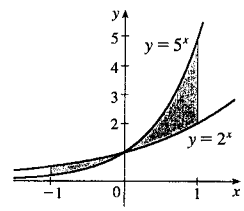
41. $\int\frac{\log_{10}x}{x}\,dx = \int\frac{(\ln x)/(\ln 10)}{x}\,dx = \frac{1}{\ln 10}\int\frac{\ln x}{x}\,dx$. Now put $u = \ln x$, so $du = \frac{1}{x}\,dx$, and the expression becomes $\frac{1}{\ln 10}\int u\,du = \frac{1}{\ln 10}\left(\frac{1}{2}u^2 + C_1\right) = \frac{1}{2\ln 10}(\ln x)^2 + C$.

Or: The substitution $u = \log_{10}x$ gives $du = \frac{dx}{x\ln 10}$ and we get $\int\frac{\log_{10}x}{x}\,dx = \frac{1}{2}\ln 10\left(\log_{10}x\right)^2 + C$.

42. Let $u = x^2$. Then $du = 2x\,dx$, so $\int x2^{x^2}\,dx = \frac{1}{2}\int 2^u\,du = \frac{1}{2}\frac{2^u}{\ln 2} + C = \frac{1}{2\ln 2}2^{x^2} + C$.

43.

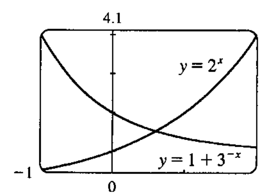
$$A = \int_{-1}^{0}(2^x - 5^x)\,dx + \int_0^1 (5^x - 2^x)\,dx$$
$$= \left[\frac{2^x}{\ln 2} - \frac{5^x}{\ln 5}\right]_{-1}^{0} + \left[\frac{5^x}{\ln 5} - \frac{2^x}{\ln 2}\right]_0^1$$
$$= \left(\frac{1}{\ln 2} - \frac{1}{\ln 5}\right) - \left(\frac{1/2}{\ln 2} - \frac{1/5}{\ln 5}\right) + \left(\frac{5}{\ln 5} - \frac{2}{\ln 2}\right) - \left(\frac{1}{\ln 5} - \frac{1}{\ln 2}\right)$$
$$= \frac{16}{5\ln 5} - \frac{1}{2\ln 2}$$



44. We use the formula for disks (Equation 5.2.2). The volume is $V = \int_0^1 \pi\left[10^{-x}\right]^2 dx = \pi\int_0^1 10^{-2x}\,dx$. To evaluate the integral, we let $u = -2x \Rightarrow du = -2\,dx$, $x = 0 \Rightarrow u = 0$, and $x = 1 \Rightarrow u = -2$, so we have

$$V = -\frac{\pi}{2}\int_0^{-2} 10^u\,du = -\frac{\pi}{2}\left[\frac{1}{\ln 10}10^u\right]_0^{-2} = -\frac{\pi}{2\ln 10}\left(10^{-2} - 1\right) = \frac{99\pi}{200\ln 10}$$

45. We see that the graphs of $y = 2^x$ and $y = 1 + 3^{-x}$ intersect at $x \approx 0.6$. We let $f(x) = 2^x - 1 - 3^{-x}$ and calculate $f'(x) = 2^x \ln 2 + 3^{-x}\ln 3$, and using the formula $x_{n+1} = x_n - f(x_n)/f'(x_n)$ (Newton's Method), we get $x_1 = 0.6$, $x_2 \approx x_3 \approx 0.600967$. So, correct to six decimal places, the root occurs at $x = 0.600967$.



46. $x^y = y^x \Rightarrow y\ln x = x\ln y \Rightarrow y\cdot\frac{1}{x} + y'\ln x = x\cdot\frac{1}{y}\cdot y' + \ln y \Rightarrow y'\ln x - \frac{x}{y}y' = \ln y - \frac{y}{x} \Rightarrow$
$y' = \dfrac{\ln y - y/x}{\ln x - x/y}$

47. $y = \dfrac{10^x}{10^x + 1} \Leftrightarrow (10^x + 1)y = 10^x \Leftrightarrow y = 10^x(1 - y) \Leftrightarrow 10^x = \dfrac{y}{1 - y} \Leftrightarrow$
$\log_{10} 10^x = \log_{10}\left(\dfrac{y}{1-y}\right) \Leftrightarrow x = \log_{10} y - \log_{10}(1 - y)$. Interchange x and y:
$y = \log_{10} x - \log_{10}(1 - x)$ is the inverse function.

48. $\lim\limits_{x\to 0^+} x^{-\ln x} = \lim\limits_{x\to 0^+}\left(e^{\ln x}\right)^{-\ln x} = \lim\limits_{x\to 0^+} e^{-(\ln x)^2} = 0$ since $-(\ln x)^2 \to -\infty$ as $x \to 0^+$.

49. If I is the intensity of the 1989 San Francisco earthquake, then $\log_{10}(I/S) = 7.1 \Rightarrow$
$\log_{10}(16I/S) = \log_{10} 16 + \log_{10}(I/S) = \log_{10} 16 + 7.1 \approx 8.3$.

50. Let I_1 and I_2 be the intensities of the music and the mower. Then $10\log_{10}\left(\frac{I_1}{I_0}\right) = 120$ and $10\log_{10}\left(\frac{I_2}{I_0}\right) = 106$, so $\log_{10}\left(\frac{I_1}{I_2}\right) = \log_{10}\left(\frac{I_1/I_0}{I_2/I_0}\right) = \log_{10}\left(\frac{I_1}{I_0}\right) - \log_{10}\left(\frac{I_2}{I_0}\right) = 12 - 10.6 = 1.4 \Rightarrow \frac{I_1}{I_2} = 10^{1.4} \approx 25$.

51. We find I with the loudness formula from Exercise 50, substituting $I_0 = 10^{-12}$ and $L = 50$: $50 = 10\log_{10}\dfrac{I}{10^{-12}}$ $\Leftrightarrow$ $5 = \log_{10}\dfrac{I}{10^{-12}}$ $\Leftrightarrow$ $10^5 = \dfrac{I}{10^{-12}}$ $\Leftrightarrow$ $I = 10^{-7}$ watt/m^2. Now we differentiate L with respect to I: $L = 10\log_{10}\dfrac{I}{I_0}$ $\Rightarrow$ $\dfrac{dL}{dI} = 10\dfrac{1}{(I/I_0)\ln 10}\left(\dfrac{1}{I_0}\right) = \dfrac{10}{\ln 10}\left(\dfrac{1}{I}\right)$. Substituting $I = 10^{-7}$, we get $L'(50) = \dfrac{10}{\ln 10}\left(\dfrac{1}{10^{-7}}\right) = \dfrac{10^8}{\ln 10} \approx 4.34 \times 10^7\ \dfrac{\text{dB}}{\text{watt/m}^2}$.

52. (a) $I(x) = I_0 a^x \Rightarrow I'(x) = I_0(\ln a)\,a^x = (I_0 a^x)\ln a = I(x)\ln a$

(b) We substitute $I_0 = 8$, $a = 0.38$ and $x = 20$ into the first expression for $I'(x)$ above: $I'(20) = 8(\ln 0.38)(0.38)^{20} \approx -3.05 \times 10^{-8}$.

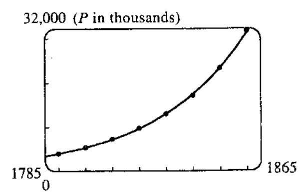
(c) The average value of the function $I(x)$ between $x = 0$ and $x = 20$ is

$$\frac{\int_0^{20} I(x)dx}{20-0} = \frac{1}{20}\int_0^{20} 8(0.38)^x\,dx = \frac{2}{5}\left[\frac{(0.38)^x}{\ln 0.38}\right]_0^{20} = \frac{2(0.38^{20}-1)}{5\ln 0.38} \approx 0.41.$$

53. (a) Using a calculator or CAS, we obtain the model $Q = ab^t$ with $a = 100.0124369$ and $b = 0.000045145933$. We can change this model to one with base e and exponent $\ln b$: $Q = ae^{t\ln b} = 100.0124369e^{-10.00553063t}$.

(b) $Q'(t) = ab^t\ln b$. $Q'(0.04) \approx -670.63\ \mu$A. The result of Example 2 in Section 2.1 was $-670\ \mu$A.

54. (a) $P = ab^t$ or $P = ae^{t\ln b}$ with $a = 4.502714 \times 10^{-20}$ and $b = 1.029953851$, where P is measured in thousands of people. The fit appears to be very good.



(b) **For 1800:** $m_1 = \dfrac{5308-3929}{1800-1790} = 137.9$, $m_2 = \dfrac{7240-5308}{1810-1800} = 193.2$.

So $P'(1800) \approx (m_1+m_2)/2 = 165.55$ thousand people/year.

For 1850: $m_1 = \dfrac{23{,}192-17{,}063}{1850-1840} = 612.9$, $m_2 = \dfrac{31{,}443-23{,}192}{1860-1850} = 825.1$.

So $P'(1850) \approx (m_1+m_2)/2 = 719$ thousand people/year.

(c) $P'(t) = ab^t\ln b$. $P'(1800) \approx 156.85$ and $P'(1850) \approx 686.07$.

(d) $P(1870) \approx 41{,}946.56$. The difference of 3.4 million people is most likely due to the Civil War (1861–1865).

55. Using Definition 1 and the second law of exponents for e^x, we have

$$a^{x-y} = e^{(x-y)\ln a} = e^{x\ln a - y\ln a} = \frac{e^{x\ln a}}{e^{y\ln a}} = \frac{a^x}{a^y}.$$

56. Using Definition 1, the first law of logarithms, and the first law of exponents for e^x, we have

$$(ab)^x = e^{x\ln(ab)} = e^{x(\ln a+\ln b)} = e^{x\ln a + x\ln b} = e^{x\ln a}e^{x\ln b} = a^x b^x.$$

57. Let $\log_a x = r$ and $\log_a y = s$. Then $a^r = x$ and $a^s = y$.

(a) $xy = a^r a^s = a^{r+s} \Rightarrow \log_a(xy) = r+s = \log_a x + \log_a y$

(b) $\dfrac{x}{y} = \dfrac{a^r}{a^s} = a^{r-s} \Rightarrow \log_a\dfrac{x}{y} = r-s = \log_a x - \log_a y$

(c) $x^y = (a^r)^y = a^{ry} \Rightarrow \log_a(x^y) = ry = y\log_a x$

58. Let $m = n/x$. Then $n = xm$, and as $n \to \infty$, $m \to \infty$.

Therefore, $\lim\limits_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim\limits_{m\to\infty}\left(1+\frac{1}{m}\right)^{mx} = \left[\lim\limits_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right]^x = e^x$ by Equation 9.

7.5 Inverse Trigonometric Functions

1. (a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ since $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(b) $\cos^{-1}(-1) = \pi$ since $\cos\pi = -1$ and π is in $[0, \pi]$.

2. (a) $\arctan(-1) = -\frac{\pi}{4}$ since $\tan\left(-\frac{\pi}{4}\right) = -1$ and $-\frac{\pi}{4}$ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(b) $\csc^{-1} 2 = \frac{\pi}{6}$ since $\csc\frac{\pi}{6} = 2$ and $\frac{\pi}{6}$ is in $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$.

3. (a) $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$ since $\tan\frac{\pi}{3} = \sqrt{3}$ and $\frac{\pi}{3}$ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(b) $\arcsin\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$ since $\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ and $-\frac{\pi}{4}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

4. (a) $\sec^{-1}\sqrt{2} = \frac{\pi}{4}$ since $\sec\frac{\pi}{4} = \sqrt{2}$ and $\frac{\pi}{4}$ is in $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$.

(b) $\arcsin 1 = \frac{\pi}{2}$ since $\sin\frac{\pi}{2} = 1$ and $\frac{\pi}{2}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

5. (a) $\sin(\sin^{-1} 0.7) = 0.7$ since 0.7 is in $[-1, 1]$.

(b) $\tan^{-1}\left(\tan\frac{4\pi}{3}\right) = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$ since $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

6. (a) $\sin^{-1}(\sin 1) = 1$ since $-\frac{\pi}{2} \le 1 \le \frac{\pi}{2}$.

(b) $\tan(\cos^{-1} 0.5) = \tan\frac{\pi}{3} = \sqrt{3}$

7. Let $\theta = \cos^{-1}\frac{4}{5}$, so $\cos\theta = \frac{4}{5}$. Then $\sin\left(\cos^{-1}\frac{4}{5}\right) = \sin\theta = \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25}} = \frac{3}{5}$.

8. Let $\theta = \arctan 2$, so $\tan\theta = 2 \Rightarrow \sec^2\theta = 1 + \tan^2\theta = 1 + 4 = 5 \Rightarrow \sec\theta = \sqrt{5} \Rightarrow$ $\sec(\arctan 2) = \sec\theta = \sqrt{5}$.

9. Let $\theta = \sin^{-1}\frac{5}{13}$. Then $\sin\theta = \frac{5}{13}$, so $\cos\left(2\sin^{-1}\frac{5}{13}\right) = \cos 2\theta = 1 - 2\sin^2\theta = 1 - 2\left(\frac{5}{13}\right)^2 = \frac{119}{169}$.

10. Let $x = \sin^{-1}\frac{1}{3}$ and $y = \sin^{-1}\frac{2}{3}$. Then $\sin x = \frac{1}{3}$,

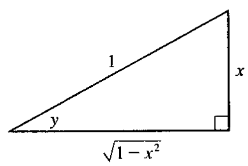
$\cos x = \sqrt{1-\left(\frac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3}$, $\sin y = \frac{2}{3}$, $\cos y = \sqrt{1-\left(\frac{2}{3}\right)^2} = \frac{\sqrt{5}}{3}$, so

$\sin\left(\sin^{-1}\frac{1}{3} + \sin^{-1}\frac{2}{3}\right) = \sin(x+y) = \sin x\cos y + \cos x\sin y = \frac{1}{3}\left(\frac{\sqrt{5}}{3}\right) + \frac{2\sqrt{2}}{3}\left(\frac{2}{3}\right) = \frac{1}{9}\left(\sqrt{5} + 4\sqrt{2}\right)$

11. Let $y = \sin^{-1} x$. Then $-\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow \cos y \ge 0$, so $\cos(\sin^{-1} x) = \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$

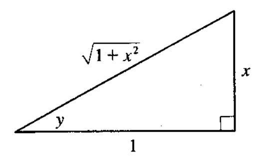
12. Let $y = \sin^{-1} x$. Then $\sin y = x$, so from the triangle we see that

$$\tan\left(\sin^{-1} x\right) = \tan y = \frac{x}{\sqrt{1 - x^2}}.$$



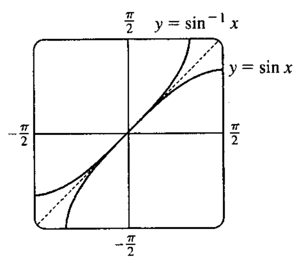
13. Let $y = \tan^{-1} x$. Then $\tan y = x$, so from the triangle we see that

$$\sin\left(\tan^{-1} x\right) = \sin y = \frac{x}{\sqrt{1 + x^2}}.$$



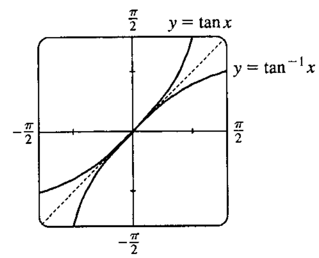
14. Let $y = \cos^{-1} x$. Then $\cos y = x \Rightarrow \sin y = \sqrt{1 - x^2}$ since $0 \le y \le \pi$. So $\sin\left(2\cos^{-1} x\right) = \sin 2y = 2\sin y\cos y = 2x\sqrt{1 - x^2}$.

15.



The graph of $\sin^{-1} x$ is the reflection of the graph of $\sin x$ about the line $y = x$.

16.



The graph of $\tan^{-1} x$ is the reflection of the graph of $\tan x$ about the line $y = x$.

17. Let $y = \cos^{-1} x$. Then $\cos y = x$ and $0 \le y \le \pi \Rightarrow -\sin y \dfrac{dy}{dx} = 1 \Rightarrow$

$$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1 - \cos^2 y}} = -\frac{1}{\sqrt{1 - x^2}}.$$ (Note that $\sin y \ge 0$ for $0 \le y \le \pi$.)

18. (a) Let $a = \sin^{-1} x$ and $b = \cos^{-1} x$. Then $\cos a = \sqrt{1 - \sin^2 a} = \sqrt{1 - x^2}$ since $\cos a \ge 0$ for $-\frac{\pi}{2} \le a \le \frac{\pi}{2}$. Similarly, $\sin b = \sqrt{1 - x^2}$. So

$$\begin{aligned}\sin\left(\sin^{-1} x + \cos^{-1} x\right) &= \sin(a + b) = \sin a\cos b + \cos a\sin b = x \cdot x + \sqrt{1 - x^2}\sqrt{1 - x^2} \\ &= x^2 + \left(1 - x^2\right) = 1\end{aligned}$$

But $\frac{\pi}{2} \le \sin^{-1} x + \cos^{-1} x \le \frac{\pi}{2}$, and so $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.

(b) We differentiate the result of part (a) with respect to x, and get $\dfrac{1}{\sqrt{1 - x^2}} + \dfrac{d}{dx}\cos^{-1} x = 0 \Rightarrow$

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}.$$

19. Let $y = \cot^{-1} x$. Then $\cot y = x \Rightarrow -\csc^2 y \dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dx} = -\dfrac{1}{\csc^2 y} = -\dfrac{1}{1 + \cot^2 y} = -\dfrac{1}{1 + x^2}$.

20. Let $y = \sec^{-1} x$. Then $\sec y = x$ and $y \in \left(0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right)$. Differentiate with respect to x:

$\sec y \tan y \left(\frac{dy}{dx}\right) = 1 \;\Rightarrow\; \frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y\sqrt{\sec^2 y - 1}} = \frac{1}{x\sqrt{x^2 - 1}}$. Note that $\tan^2 y = \sec^2 y - 1$

$\Rightarrow\; \tan y = \sqrt{\sec^2 y - 1}$ since $\tan y > 0$ when $0 < y < \frac{\pi}{2}$ or $\pi < y < \frac{3\pi}{2}$.

21. Let $y = \csc^{-1} x$. Then $\csc y = x \;\Rightarrow\; -\csc y \cot y \frac{dy}{dx} = 1 \;\Rightarrow$

$\frac{dy}{dx} = -\frac{1}{\csc y \cot y} = -\frac{1}{\csc y\sqrt{\csc^2 y - 1}} = -\frac{1}{x\sqrt{x^2 - 1}}$. Note that $\cot y \geq 0$ on the domain of $\csc^{-1} x$.

22. $y = \left(\sin^{-1} x\right)^2 \;\Rightarrow\; y' = 2\left(\sin^{-1} x\right)\frac{d}{dx}\left(\sin^{-1} x\right) \;\Rightarrow\; y' = \frac{2\sin^{-1} x}{\sqrt{1 - x^2}}$

23. $y = \sin^{-1}\left(x^2\right) \;\Rightarrow\; y' = \frac{1}{\sqrt{1 - \left(x^2\right)^2}}\frac{d}{dx}\left(x^2\right) = \frac{2x}{\sqrt{1 - x^4}}$

24. $h(x) = \sqrt{1 - x^2}\arcsin x \;\Rightarrow\; h'(x) = \sqrt{1 - x^2}\cdot\frac{1}{\sqrt{1 - x^2}} + \arcsin x\left[\frac{1}{2}\left(1 - x^2\right)^{-1/2}(-2x)\right] = 1 - \frac{x\arcsin x}{\sqrt{1 - x^2}}$

25. $y = \tan^{-1}\left(e^x\right) \;\Rightarrow\; y' = \frac{1}{1 + \left(e^x\right)^2}\frac{d}{dx}\left(e^x\right) = \frac{e^x}{1 + e^{2x}}$

26. $f(x) = (\arctan x)\ln x \;\Rightarrow\; f'(x) = \arctan x\cdot\frac{1}{x} + \ln x\cdot\frac{1}{1 + x^2} = \frac{\arctan x}{x} + \frac{\ln x}{1 + x^2}$

27. $H(x) = \left(1 + x^2\right)\arctan x \;\Rightarrow\; H'(x) = (2x)\arctan x + \left(1 + x^2\right)\frac{1}{1 + x^2} = 1 + 2x\arctan x$

28. $h(t) = e^{\sec^{-1} t} \;\Rightarrow\; h'(t) = e^{\sec^{-1} t}\frac{d}{dt}\left(\sec^{-1} t\right) = \frac{e^{\sec^{-1} t}}{t\sqrt{t^2 - 1}}$

29. $g(t) = \sin^{-1}\left(\frac{4}{t}\right) \;\Rightarrow\; g'(t) = \frac{1}{\sqrt{1 - (4/t)^2}}\left(-\frac{4}{t^2}\right) = -\frac{4}{\sqrt{t^4 - 16t^2}}$

30. $y = x\cos^{-1} x - \sqrt{1 - x^2} \;\Rightarrow\; y' = \cos^{-1} x - \frac{x}{\sqrt{1 - x^2}} + \frac{x}{\sqrt{1 - x^2}} = \cos^{-1} x$

31. $y = \arctan(\cos\theta) \;\Rightarrow\; y' = \frac{1}{1 + (\cos\theta)^2}(-\sin\theta) = -\frac{\sin\theta}{1 + \cos^2\theta}$

32. $y = \tan^{-1}\left(x - \sqrt{x^2 + 1}\right) \;\Rightarrow$

$$y' = \frac{1}{1 + \left(x - \sqrt{x^2 + 1}\right)^2}\left(1 - \frac{x}{\sqrt{x^2 + 1}}\right) = \frac{\sqrt{x^2 + 1} - x}{2\left(1 + x^2 - x\sqrt{x^2 + 1}\right)\sqrt{x^2 + 1}}$$

$$= \frac{\sqrt{x^2 + 1} - x}{2\left(\sqrt{x^2 + 1} + x^2\sqrt{x^2 + 1} - x^3 - x\right)} = \frac{1}{2\left(1 + x^2\right)}$$

33. $y = x^2\cot^{-1}(3x) \;\Rightarrow\; y' = 2x\cot^{-1}(3x) + x^2\left[-\frac{1}{1 + (3x)^2}\right](3) = 2x\cot^{-1}(3x) - \frac{3x^2}{1 + 9x^2}$

34. $y = \tan^{-1}\frac{x}{a} + \frac{1}{2}\ln(x - a) - \frac{1}{2}\ln(x + a) \;\Rightarrow$

$$y' = \frac{a}{x^2 + a^2} + \frac{1/2}{x - a} - \frac{1/2}{x + a} = \frac{a}{x^2 + a^2} + \frac{a}{x^2 - a^2} = \frac{2ax^2}{x^4 - a^4}$$

35. $y = \arccos\left(\dfrac{b + a\cos x}{a + b\cos x}\right) \Rightarrow$

$$y' = -\frac{1}{\sqrt{1 - \left(\dfrac{b + a\cos x}{a + b\cos x}\right)^2}} \frac{(a + b\cos x)(-a\sin x) - (b + a\cos x)(-b\sin x)}{(a + b\cos x)^2}$$

$$= \frac{1}{\sqrt{a^2 + b^2\cos^2 x - b^2 - a^2\cos^2 x}} \frac{(a^2 - b^2)\sin x}{|a + b\cos x|}$$

$$= \frac{1}{\sqrt{a^2 - b^2}\sqrt{1 - \cos^2 x}} \frac{(a^2 - b^2)\sin x}{|a + b\cos x|} = \frac{\sqrt{a^2 - b^2}}{|a + b\cos x|} \frac{\sin x}{|\sin x|}$$

But $0 \le x \le \pi$, so $|\sin x| = \sin x$. Also $a > b > 0 \Rightarrow b\cos x \ge -b > -a$, so $a + b\cos x > 0$.

Thus $y' = \dfrac{\sqrt{a^2 - b^2}}{a + b\cos x}$.

36. $f(x) = \arcsin(e^x) \Rightarrow f'(x) = \dfrac{1}{\sqrt{1 - (e^x)^2}} \cdot e^x = \dfrac{e^x}{\sqrt{1 - e^{2x}}}$.

Domain $(f) = \{x \mid -1 \le e^x \le 1\} = \{x \mid 0 < e^x \le 1\} = (-\infty, 0]$.

Domain $(f') = \{x \mid 1 - e^{2x} > 0\} = \{x \mid e^{2x} < 1\} = \{x \mid 2x < 0\} = (-\infty, 0)$.

37. $g(x) = \cos^{-1}(3 - 2x) \Rightarrow g'(x) = -\dfrac{1}{\sqrt{1 - (3 - 2x)^2}}(-2) = \dfrac{2}{\sqrt{1 - (3 - 2x)^2}}$.

Domain $(g) = \{x \mid -1 \le 3 - 2x \le 1\} = \{x \mid -4 \le -2x \le -2\} = \{x \mid 2 \ge x \ge 1\} = [1, 2]$.

Domain $(g') = \{x \mid 1 - (3 - 2x)^2 > 0\} = \{x \mid (3 - 2x)^2 < 1\} = \{x \mid |3 - 2x| < 1\} = \{x \mid -1 < 3 - 2x < 1\}$
$= \{x \mid -4 < -2x < -2\} = \{x \mid 2 > x > 1\} = (1, 2)$

38. $f(x) = x\tan^{-1} x \Rightarrow f'(x) = \tan^{-1} x + \dfrac{x}{1 + x^2} \Rightarrow f'(1) = \frac{\pi}{4} + \frac{1}{2}$

39. $g(x) = x\sin^{-1}\left(\frac{x}{4}\right) + \sqrt{16 - x^2} \Rightarrow g'(x) = \sin^{-1}\left(\frac{x}{4}\right) + \dfrac{x}{4\sqrt{1 - (x/4)^2}} - \dfrac{x}{\sqrt{16 - x^2}} = \sin^{-1}\left(\frac{x}{4}\right) \Rightarrow$

$g'(2) = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

40. $y = 3\arccos\frac{x}{2} \Rightarrow y' = 3\left[-\dfrac{1}{\sqrt{1 - (x/2)^2}}\right]\left(\frac{1}{2}\right)$, so at $(1, \pi)$, $y' = -\dfrac{3}{2\sqrt{1 - \frac{1}{4}}} = -\sqrt{3}$. An equation of the tangent line is $y - \pi = -\sqrt{3}(x - 1)$, or $y = -\sqrt{3}x + \pi + \sqrt{3}$.

41. $f(x) = e^x - x^2\arctan x \Rightarrow$

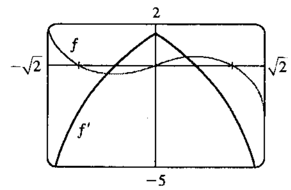
$$f'(x) = e^x - \left[x^2\left(\frac{1}{1 + x^2}\right) + 2x\arctan x\right]$$

$$= e^x - \frac{x^2}{1 + x^2} - 2x\arctan x$$

This is reasonable because the graphs show that f is increasing when $f'(x)$ is positive.

42. $f(x) = x \arcsin(1 - x^2) \Rightarrow$

$$f'(x) = \arcsin(1 - x^2) + x\left[\frac{-2x}{\sqrt{1 - (1 - x^2)^2}}\right] = \arcsin(1 - x^2) - \frac{2x^2}{\sqrt{2x^2 - x^4}}$$



This is reasonable because the graphs show that f is increasing when $f'(x)$ is positive, and that f has an inflection point when f' changes from increasing to decreasing.

43. $\lim\limits_{x \to -1^+} \sin^{-1} x = \sin^{-1}(-1) = -\frac{\pi}{2}$

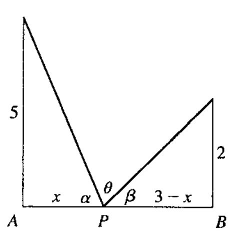
44. Let $t = \dfrac{1 + x^2}{1 + 2x^2}$. As $x \to \infty$, $t = \dfrac{1 + x^2}{1 + 2x^2} = \dfrac{1/x^2 + 1}{1/x^2 + 2} \to \dfrac{1}{2}$.

$\lim\limits_{x \to \infty} \arccos\left(\dfrac{1 + x^2}{1 + 2x^2}\right) = \lim\limits_{t \to 1/2} \arccos t = \arccos \frac{1}{2} = \frac{\pi}{3}$.

45. Let $t = e^x$. As $x \to \infty$, $t \to \infty$. $\lim\limits_{x \to \infty} \arctan(e^x) = \lim\limits_{t \to \infty} \arctan t = \frac{\pi}{2}$ by (8).

46. Let $t = \ln x$. As $x \to 0^+$, $t \to -\infty$. $\lim\limits_{x \to 0^+} \tan^{-1}(\ln x) = \lim\limits_{t \to -\infty} \tan^{-1} t = -\frac{\pi}{2}$ by (8).

47.



$\tan\alpha = \dfrac{5}{x}$, $\tan\beta = \dfrac{2}{3 - x}$ $\Rightarrow$ $\theta = \pi - \tan^{-1}\left(\dfrac{5}{x}\right) - \tan^{-1}\left(\dfrac{2}{3 - x}\right)$

$$\Rightarrow \quad \frac{d\theta}{dx} = -\frac{1}{1 + \left(\frac{5}{x}\right)^2}\left(-\frac{5}{x^2}\right) - \frac{1}{1 + \left(\frac{2}{3 - x}\right)^2}\left[\frac{2}{(3 - x)^2}\right]. \quad \frac{d\theta}{dx} = 0$$

$$\Rightarrow \quad \frac{5}{x^2 + 25} = \frac{2}{x^2 - 6x + 13} \quad \Rightarrow \quad 2x^2 + 50 = 5x^2 - 30x + 65 \quad \Rightarrow$$

$x^2 - 10x + 5 = 0 \Rightarrow x = 5 \pm 2\sqrt{5}$. We reject the root with the $+$ sign, since it is larger than 3. $d\theta/dx > 0$ for $x < 5 - 2\sqrt{5}$ and $d\theta/dx < 0$ for $x > 5 - 2\sqrt{5}$, so θ is maximized when $|AP| = x = 5 - 2\sqrt{5} \approx 0.53$.

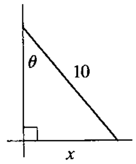
48. Let x be the distance from the observer to the wall. Then, from the given figure,

$\theta = \tan^{-1}\left(\dfrac{h + d}{x}\right) - \tan^{-1}\left(\dfrac{d}{x}\right), x > 0 \Rightarrow$

$$\frac{d\theta}{dx} = \frac{1}{1 + [(h + d)/x]^2}\left[-\frac{h + d}{x^2}\right] - \frac{1}{1 + (d/x)^2}\left[-\frac{d}{x^2}\right] = -\frac{h + d}{x^2 + (h + d)^2} + \frac{d}{x^2 + d^2}$$

$$= \frac{d[x^2 + (h + d)^2] - (h + d)(x^2 + d^2)}{[x^2 + (h + d)^2](x^2 + d^2)} = \frac{h^2 d + hd^2 - hx^2}{[x^2 + (h + d)^2](x^2 + d^2)} = 0 \quad \Leftrightarrow$$

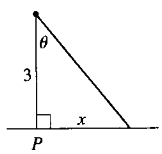
$hx^2 = h^2 d + hd^2 \Leftrightarrow x^2 = hd + d^2 \Leftrightarrow x = \sqrt{d(h + d)}$. Since $d\theta/dx > 0$ for all $x < \sqrt{d(h + d)}$ and $d\theta/dx < 0$ for all $x > \sqrt{d(h + d)}$, the absolute maximum occurs when $x = \sqrt{d(h + d)}$.

49. 

$\dfrac{dx}{dt} = 2$ ft/s, $\sin\theta = \dfrac{x}{10} \;\Rightarrow\; \theta = \sin^{-1}\left(\dfrac{x}{10}\right)$, $\dfrac{d\theta}{dx} = \dfrac{1/10}{\sqrt{1-(x/10)^2}}$,

$$\frac{d\theta}{dt} = \frac{d\theta}{dx}\frac{dx}{dt} = \frac{1/10}{\sqrt{1-(x/10)^2}}\,(2)\ \text{rad/s},$$

$$\left.\frac{d\theta}{dt}\right]_{x=6} = \frac{2/10}{\sqrt{1-(6/10)^2}}\ \text{rad/s} = \tfrac{1}{4}\ \text{rad/s}$$

50. 

$\dfrac{d\theta}{dt} = 4$ rev/min $= 8\pi\cdot 60$ rad/h. From the diagram, we see that $\tan\theta = \dfrac{x}{3}$ $\Rightarrow\; \theta = \tan^{-1}\left(\dfrac{x}{3}\right)$. Thus, $8\pi\cdot 60 = \dfrac{d\theta}{dt} = \dfrac{d\theta}{dx}\dfrac{dx}{dt} = \dfrac{1/3}{1+(x/3)^2}\dfrac{dx}{dt}$. So

$\dfrac{dx}{dt} = 8\pi\cdot 60\cdot 3\left[1+\left(\dfrac{x}{3}\right)^2\right]$ km/h, and at $x = 1$,

$\dfrac{dx}{dt} = 8\pi\cdot 60\cdot 3\left[1+\frac{1}{9}\right]$ km/h $= 1600\pi$ km/h.

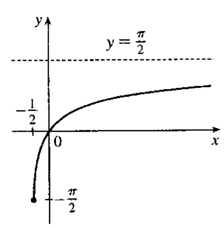
51. $y = f(x) = \sin^{-1}(x/(x+1))$ **A.** $D = \{x \mid -1 \le x/(x+1) \le 1\}$. For $x > -1$ we have $-x-1 \le x \le x+1$ $\Leftrightarrow\; 2x \ge -1 \;\Leftrightarrow\; x \ge -\frac{1}{2}$, so $D = \left[-\frac{1}{2}, \infty\right)$. **B.** Intercepts are 0 **C.** No symmetry

D. $\displaystyle\lim_{x\to\infty} \sin^{-1}\left(\frac{x}{x+1}\right) = \lim_{x\to\infty} \sin^{-1}\left(\frac{1}{1+1/x}\right) = \sin^{-1} 1 = \frac{\pi}{2}$, so $y = \frac{\pi}{2}$ is a HA.

E. $f'(x) = \dfrac{1}{\sqrt{1-[x/(x+1)]^2}}\,\dfrac{(x+1)-x}{(x+1)^2} = \dfrac{1}{(x+1)\sqrt{2x+1}} > 0$, so f is increasing on $\left(-\frac{1}{2}, \infty\right)$. **F.** No local maximum or minimum, $f\left(-\frac{1}{2}\right) = \sin^{-1}(-1) = -\frac{\pi}{2}$ is an absolute minimum **G.** $f''(x) =$

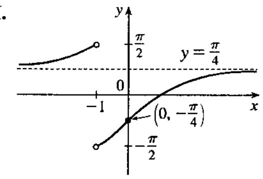
$$\frac{\sqrt{2x+1} + (x+1)/\sqrt{2x+1}}{(x+1)^2(2x+1)} = -\frac{3x+2}{(x+1)^2(2x+1)^{3/2}} < 0 \text{ on } D, \text{ so } f$$

is CD on $\left(-\frac{1}{2}, \infty\right)$.

H. 

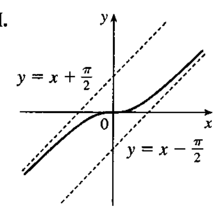
52. $y = f(x) = \tan^{-1}\left(\frac{x-1}{x+1}\right)$ **A.** $D = \{x \mid x \neq -1\}$ **B.** x-intercept$= 1$,
y-intercept $= f(0) = \tan^{-1}(-1) = -\frac{\pi}{4}$ **C.** No symmetry
D. $\lim\limits_{x\to\pm\infty} \tan^{-1}\left(\frac{x-1}{x+1}\right) = \lim\limits_{x\to\pm\infty} \tan^{-1}\left(\frac{1-1/x}{1+1/x}\right) = \tan^{-1} 1 = \frac{\pi}{4}$, so $y = \frac{\pi}{4}$ is a HA. Also
$\lim\limits_{x\to -1^+} \tan^{-1}\left(\frac{x-1}{x+1}\right) = -\frac{\pi}{2}$ and $\lim\limits_{x\to -1^-} \tan^{-1}\left(\frac{x-1}{x+1}\right) = \frac{\pi}{2}$.
E. $f'(x) = \frac{1}{1+[(x-1)/(x+1)]^2} \frac{(x+1)-(x-1)}{(x+1)^2} =$
$\frac{2}{(x+1)^2+(x-1)^2} = \frac{1}{x^2+1} > 0$, so f is increasing on $(-\infty,-1)$ and
$(-1,\infty)$. **F.** No maximum or minimum
G. $f''(x) = -2x/\left(x^2+1\right)^2 > 0 \iff x < 0$, so f is CU on $(-\infty,-1)$
and $(-1,0)$,and CD on $(0,\infty)$. IP is $\left(0,-\frac{\pi}{4}\right)$.

H.



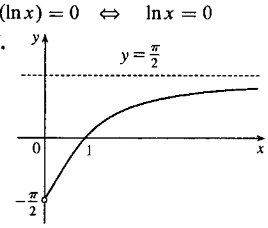
53. $y = f(x) = x - \tan^{-1} x$ **A.** $D = \mathbb{R}$ **B.** Intercepts are 0 **C.** $f(-x) = -f(x)$, so the curve is symmetric about the origin. **D.** $\lim\limits_{x\to\infty}\left(x - \tan^{-1} x\right) = \infty$ and $\lim\limits_{x\to-\infty}\left(x - \tan^{-1} x\right) = -\infty$, no HA.
But $f(x) - \left(x - \frac{\pi}{2}\right) = -\tan^{-1} x + \frac{\pi}{2} \to 0$ as $x \to \infty$, and
$f(x) - \left(x + \frac{\pi}{2}\right) = -\tan^{-1} x - \frac{\pi}{2} \to 0$ as $x \to -\infty$, so $y = x \pm \frac{\pi}{2}$ are
slant asymptotes. **E.** $f'(x) = 1 - \frac{1}{x^2+1} = \frac{x^2}{x^2+1} > 0$, so f is
increasing on $\mathbb{R}$. **F.** No extrema
G. $f''(x) = \frac{\left(1+x^2\right)(2x) - x^2(2x)}{\left(1+x^2\right)^2} = \frac{2x}{\left(1+x^2\right)^2} > 0 \iff x > 0$, so
f is CU on $(0,\infty)$, CD on $(-\infty,0)$. IP $(0,0)$

H.

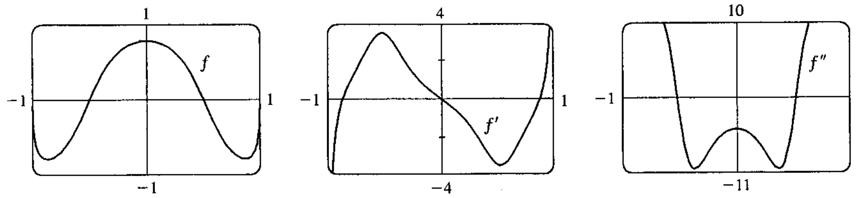


54. $y = \tan^{-1}(\ln x)$ **A.** $D = (0,\infty)$ **B.** No y-intercept, x-intercept when $\tan^{-1}(\ln x) = 0 \iff \ln x = 0$
$\iff x = 1$. **C.** No symmetry **D.** $\lim\limits_{x\to\infty} \tan^{-1}(\ln x) = \frac{\pi}{2}$, so $y = \frac{\pi}{2}$ is a
HA. Also $\lim\limits_{x\to 0^+} \tan^{-1}(\ln x) = -\frac{\pi}{2}$. **E.** $f'(x) = \frac{1}{x\left[1+(\ln x)^2\right]} > 0$,
so f is increasing on $(0,\infty)$. **F.** No maximum or minimum
G. $f''(x) = \frac{-\left[1+(\ln x)^2 + x(2\ln x/x)\right]}{x^2\left[1+(\ln x)^2\right]^2} = -\frac{(1+\ln x)^2}{x^2[1+(\ln x)^2]^2} < 0$, so
f is CD on $(0,\infty)$.

H.

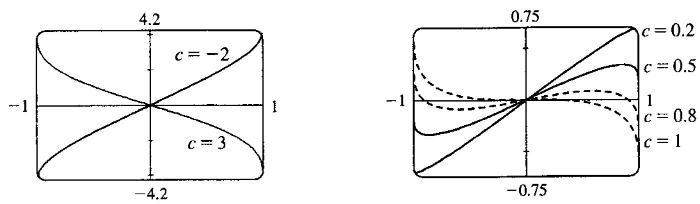


55. $f(x) = \arctan[\cos(3\arcsin x)]$. We use a CAS to compute f' and f'', and to graph f, f', and f'':



From the graph of f', it appears that the only maximum occurs at $x = 0$ and there are minima at $x = \pm 0.87$. From the graph of f'', it appears that there are inflection points at $x = \pm 0.52$.

56. First note that the function $f(x) = x - c\sin^{-1} x$ is only defined on the interval $[-1, 1]$, since $\sin^{-1}$ is only defined on that interval. We differentiate to get $f'(x) = 1 - c/\sqrt{1 - x^2}$. Now if $c \le 0$, then $f'(x) \ge 1$, so there is no extremum and f is increasing on its domain. If $c > 1$, then $f'(x) < 0$, so there is no local extremum and f is decreasing on its domain, and if $c = 1$, then there is still no extremum, since $f'(x)$ does not change sign at $x = 0$. So we can only have local extrema if $0 < c < 1$. In this case, f is increasing where $f'(x) > 0 \iff \sqrt{1 - x^2} > c \iff |x| < \sqrt{1 - c^2}$, and decreasing where $\sqrt{1 - c^2} < |x| \le 1$. f has a maximum at $x = \sqrt{1 - c^2}$ and a minimum at $x = -\sqrt{1 - c^2}$.



57. $f(x) = 2x + 5/\sqrt{1 - x^2} \;\Rightarrow\; F(x) = x^2 + 5\sin^{-1} x + C$

58. $f'(x) = 4 - 3(1 + x^2)^{-1} \;\Rightarrow\; f(x) = 4x - 3\tan^{-1} x + C \;\Rightarrow\; f\left(\frac{\pi}{4}\right) = \pi - 3 + C = 0 \;\Rightarrow\; C = 3 - \pi$, so $f(x) = 4x - 3\tan^{-1} x + 3 - \pi$.

59. $\displaystyle\int_1^{\sqrt{3}} \frac{6}{1 + x^2}\,dx = \left[6\tan^{-1} x\right]_1^{\sqrt{3}} = 6\left(\tan^{-1}\sqrt{3} - \tan^{-1} 1\right) = 6\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{2}$

60. $\displaystyle\int_0^{0.5} \frac{dx}{\sqrt{1 - x^2}} = \left[\sin^{-1} x\right]_0^{0.5} = \sin^{-1}\tfrac{1}{2} - \sin^{-1} 0 = \tfrac{\pi}{6}$

61. Let $u = t^2$. Then $du = 2t\,dt$, so $\displaystyle\int \frac{t}{\sqrt{1 - t^4}}\,dt = \frac{1}{2}\int \frac{1}{\sqrt{1 - u^2}}\,du = \tfrac{1}{2}\sin^{-1} u + C = \tfrac{1}{2}\sin^{-1}\left(t^2\right) + C$.

62. Let $u = \tan^{-1} x$. Then $du = dx/\left(1 + x^2\right)$, so $\displaystyle\int \frac{\tan^{-1} x}{1 + x^2}\,dx = \int u\,du = \tfrac{1}{2}u^2 + C = \tfrac{1}{2}\left(\tan^{-1} x\right)^2 + C$.

63. Let $u = \sin^{-1} x$. Then $du = \dfrac{1}{\sqrt{1 - x^2}}\,dx$, so $\displaystyle\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1 - x^2}}\,dx = \int_0^{\pi/6} u\,du = \left.\frac{u^2}{2}\right]_0^{\pi/6} = \frac{1}{2}\left(\frac{\pi}{6}\right)^2 = \frac{\pi^2}{72}$.

64. Let $u = -\cos x$. Then $du = \sin x\,dx$, so

$$\int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x}\,dx = \int_{-1}^{0} \frac{1}{1 + u^2}\,du = \left[\tan^{-1} u\right]_{-1}^{0} = \tan^{-1} 0 - \tan^{-1}(-1) = 0 - \left(-\tfrac{\pi}{4}\right) = \tfrac{\pi}{4}.$$

65. $\int \frac{x+9}{x^2+9}\,dx = \int \frac{x}{x^2+9}\,dx + 9\int \frac{1}{x^2+9}\,dx = \frac{1}{2}\ln\left(x^2+9\right) + 3\tan^{-1}\frac{x}{3} + C$

(Let $u = x^2 + 9$ in the first integral; use Equation 14 in the second.)

66. Let $u = \frac{1}{2}x$. Then $du = \frac{1}{2}\,dx \Rightarrow$

$$\int \frac{dx}{x\sqrt{x^2-4}} = \int \frac{dx}{2x\sqrt{(x/2)^2-1}} = \int \frac{2\,du}{4u\sqrt{u^2-1}} = \frac{1}{2}\int \frac{du}{u\sqrt{u^2-1}} = \frac{1}{2}\sec^{-1}u + C = \frac{1}{2}\sec^{-1}\left(\frac{1}{2}x\right) + C.$$

67. Let $u = 4x$. Then $du = 4\,dx$, so

$$\int_0^{\sqrt{3}/4} \frac{dx}{1+16x^2} = \frac{1}{4}\int_0^{\sqrt{3}} \frac{1}{1+u^2}\,du = \frac{1}{4}\left[\tan^{-1}u\right]_0^{\sqrt{3}} = \frac{1}{4}\left(\tan^{-1}\sqrt{3} - \tan^{-1}0\right) = \frac{1}{4}\left(\frac{\pi}{3} - 0\right) = \frac{\pi}{12}.$$

68. Let $u = e^{2x}$. Then $du = 2e^{2x}\,dx \Rightarrow \int \frac{e^{2x}\,dx}{\sqrt{1-e^{4x}}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}u + C = \frac{1}{2}\sin^{-1}\left(e^{2x}\right) + C$

69. Let $u = e^x$. Then $du = e^x\,dx$, so $\int \frac{e^x\,dx}{e^{2x}+1} = \int \frac{du}{u^2+1} = \tan^{-1}u + C = \tan^{-1}\left(e^x\right) + C.$

70. Let $u = \ln x$. Then $du = (1/x)\,dx \Rightarrow$

$$\int \frac{dx}{x\left[4+(\ln x)^2\right]} = \int \frac{du}{4+u^2} = \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\ln x\right) + C.$$

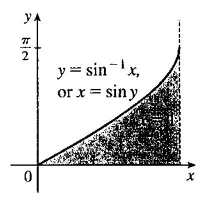
71. Let $u = x/a$. Then $du = dx/a$, so

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{dx}{a\sqrt{1-(x/a)^2}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C = \sin^{-1}\frac{x}{a} + C.$$

72. We use the disk method: $A = \int_0^2 \pi\left[\frac{1}{\sqrt{x^2+4}}\right]^2 dx = \pi\int_0^2 \frac{1}{x^2+4}\,dx$. By Formula 14, this is equal to

$\pi\left[\frac{1}{2}\tan^{-1}(x/2)\right]_0^2 = \frac{\pi}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi^2}{8}.$

73.



The integral represents the area below the curve $y = \sin^{-1}x$ on the interval $x \in [0, 1]$. The bounding curves are $y = \sin^{-1}x \Leftrightarrow x = \sin y$, $y = 0$ and $x = 1$. We see that y ranges between $\sin^{-1}0 = 0$ and $\sin^{-1}1 = \frac{\pi}{2}$. So we have to integrate the function $x = 1 - \sin y$ between $y = 0$ and $y = \frac{\pi}{2}$:

$\int_0^1 \sin^{-1}x\,dx = \int_0^{\pi/2}(1 - \sin y)\,dy = \left(\frac{\pi}{2} + \cos\frac{\pi}{2}\right) - (0 + \cos 0) = \frac{\pi}{2} - 1.$

74. Let $a = \arctan x$ and $b = \arctan y$. Then by the addition formula for the tangent (see endpapers),

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - (\tan a)(\tan b)} = \frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} \Rightarrow \tan(a+b) = \frac{x+y}{1-xy} \Rightarrow$$

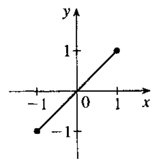
$\arctan x + \arctan y = a + b = \arctan\left(\frac{x+y}{1-xy}\right)$, since $-\frac{\pi}{2} < \arctan x + \arctan y < \frac{\pi}{2}$.

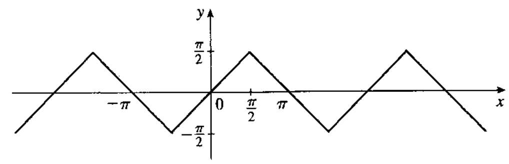
75. (a) $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right) = \arctan 1 = \frac{\pi}{4}$

(b) $2\arctan\frac{1}{3} + \arctan\frac{1}{7} = \left(\arctan\frac{1}{3} + \arctan\frac{1}{3}\right) + \arctan\frac{1}{7} = \arctan\left(\frac{\frac{1}{3}+\frac{1}{3}}{1-\frac{1}{3}\cdot\frac{1}{3}}\right) + \arctan\frac{1}{7}$

$= \arctan\frac{3}{4} + \arctan\frac{1}{7} = \arctan\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}\right) = \arctan 1 = \frac{\pi}{4}$

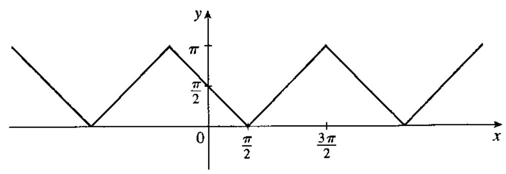
76. (a) $f(x) = \sin(\sin^{-1} x)$ (b) $g(x) = \sin^{-1}(\sin x)$





(c) $g'(x) = \dfrac{d}{dx}\sin^{-1}(\sin x) = \dfrac{1}{\sqrt{1-\sin^2 x}}\cos x = \dfrac{\cos x}{\sqrt{\cos^2 x}} = \dfrac{\cos x}{|\cos x|}$

(d) $h(x) = \cos^{-1}(\sin x)$, so $h'(x) = -\dfrac{\cos x}{\sqrt{1-\sin^2 x}} = \dfrac{\cos x}{|\cos x|}$



77. Let $f(x) = 2\sin^{-1} x - \cos^{-1}\left(1-2x^2\right)$. Then

$$f'(x) = \frac{2}{\sqrt{1-x^2}} - \frac{4x}{\sqrt{1-\left(1-2x^2\right)^2}} = \frac{2}{\sqrt{1-x^2}} - \frac{4x}{2x\sqrt{1-x^2}} = 0 \text{ (since } x \geq 0)$$

Thus $f'(x) = 0$ for all $x \in [0, 1)$. Thus $f(x) = C$. To find C let $x = 0$. Thus $2\sin^{-1}(0) - \cos^{-1}(1) = 0 = C$. Therefore we see that $f(x) = 2\sin^{-1} x - \cos^{-1}\left(1-2x^2\right) = 0 \Rightarrow 2\sin^{-1} x = \cos^{-1}\left(1-2x^2\right)$.

78. Let $f(x) = \sin^{-1}\left(\dfrac{x-1}{x+1}\right) - 2\tan^{-1}\sqrt{x} + \frac{\pi}{2}$. Note that the domain of f is $[0, \infty)$. Thus

$$f'(x) = \frac{1}{\sqrt{1-\left(\dfrac{x-1}{x+1}\right)^2}} \frac{(x+1)-(x-1)}{(x+1)^2} - \frac{2}{1+x}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}\,(x+1)} - \frac{1}{\sqrt{x}\,(x+1)} = 0$$

Then $f(x) = C$. To find C, we let $x = 0 \Rightarrow \sin^{-1}(-1) - 2\tan^{-1}(0) + \frac{\pi}{2} = C \Rightarrow -\frac{\pi}{2} - 0 + \frac{\pi}{2} = 0 = C$. Thus, $f(x) = 0 \Rightarrow \sin^{-1}\left(\dfrac{x-1}{x+1}\right) = 2\tan^{-1}\sqrt{x} - \frac{\pi}{2}$.

79. $y = \sec^{-1} x \Rightarrow \sec y = x \Rightarrow \sec y \tan y \dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\sec y \tan y}$. Now $\tan^2 y = \sec^2 y - 1 = x^2 - 1$, so $\tan y = \pm\sqrt{x^2-1}$. For $y \in \left[0, \frac{\pi}{2}\right)$, $x \geq 1$, so $\sec y = x = |x|$ and $\tan y \geq 0$ $\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x\sqrt{x^2-1}} = \dfrac{1}{|x|\sqrt{x^2-1}}$. For $y \in \left(\frac{\pi}{2}, \pi\right]$, $x \leq -1$, so $|x| = -x$ and $\tan y = -\sqrt{x^2-1} \Rightarrow$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{x\left(-\sqrt{x^2-1}\right)} = \frac{1}{(-x)\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}$$

80. (a) Since $|\arctan(1/x)| < \frac{\pi}{2}$, we have $0 \le |x \arctan(1/x)| \le \frac{\pi}{2}|x| \to 0$ as $x \to 0$. So, by the Squeeze Theorem, $\lim_{x\to 0} f(x) = 0 = f(0)$, so f is continuous at 0.

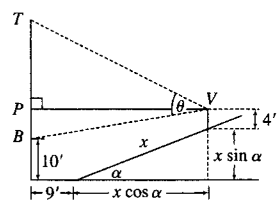
(b) Here $\dfrac{f(x)-f(0)}{x-0} = \dfrac{x\arctan(1/x) - 0}{x} = \arctan\left(\dfrac{1}{x}\right)$. So (see Exercise 40 in Section 3.2 for a discussion of left- and right-hand derivatives)

$f'_{-}(0) = \lim\limits_{x\to 0^-} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^-} \arctan\left(\dfrac{1}{x}\right) = \lim\limits_{y\to -\infty} \arctan y = -\dfrac{\pi}{2}$, while

$f'_{+}(0) = \lim\limits_{x\to 0^+} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^+} \arctan\left(\dfrac{1}{x}\right) = \lim\limits_{y\to \infty} \arctan y = \dfrac{\pi}{2}$. So $f'(0)$ does not exist.

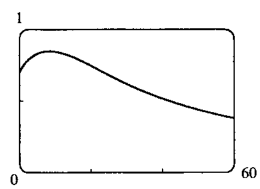
Applied Project □ Where to Sit at the Movies

1.



$|VP| = 9 + x\cos\alpha$, $|PT| = 35 - (4 + x\sin\alpha) = 31 - x\sin\alpha$, and $|PB| = (4 + x\sin\alpha) - 10 = x\sin\alpha - 6$. So using the Pythagorean Theorem, we have $|VT| = \sqrt{|VP|^2 + |PT|^2} = \sqrt{(9 + x\cos\alpha)^2 + (31 - x\sin\alpha)^2} = a$, and $|VB| = \sqrt{|VP|^2 + |PB|^2} = \sqrt{(9 + x\cos\alpha)^2 + (x\sin\alpha - 6)^2} = b$. Using the Law of Cosines on $\triangle VBT$, we get $25^2 = a^2 + b^2 - 2ab\cos\theta \iff \cos\theta = \dfrac{a^2 + b^2 - 625}{2ab} \iff \theta = \arccos\left(\dfrac{a^2+b^2-625}{2ab}\right)$, as required.

2. From the graph of θ, it appears that the value of x which maximizes θ is $x \approx 8.25$ ft. Assuming that the first row is at $x = 0$, the row closest to this value of x is the fourth row, at $x = 9$ ft, and from the graph, the viewing angle in this row seems to be about 0.85 radians, or about 49°.



3. With a CAS, we type in the definition of θ, substitute in the proper values of a and b in terms of x and $\alpha = 20° = \frac{\pi}{9}$ radians, and then use the differentiation command to find the derivative. We use a numerical root finder and find that the root of the equation $d\theta/dx = 0$ is $x \approx 8.253062$, as approximated above.

4. From the graph in Problem 2, it seems that the average value of the function on the interval [0, 60] is about 0.6. We can use a CAS to approximate $\frac{1}{60}\int_0^{60} \theta(x)\,dx \approx 0.625 \approx 36°$. (The calculation is much faster if we reduce the number of digits of accuracy required.) The minimum value is $\theta(60) \approx 0.38$ and, from Problem 2, the maximum value is about 0.85.

7.6 Hyperbolic Functions

1. (a) $\sinh 0 = \frac{1}{2}\left(e^0 - e^0\right) = 0$ (b) $\cosh 0 = \frac{1}{2}\left(e^0 + e^0\right) = \frac{1}{2}(1+1) = 1$

2. (a) $\tanh 0 = \dfrac{\left(e^0 - e^{-0}\right)/2}{\left(e^0 + e^{-0}\right)/2} = 0$ (b) $\tanh 1 = \dfrac{e^1 - e^{-1}}{e^1 + e^{-1}} = \dfrac{e^2 - 1}{e^2 + 1} \approx 0.76159$

3. (a) $\sinh(\ln 2) = \dfrac{e^{\ln 2} - e^{-\ln 2}}{2} = \dfrac{2 - \frac{1}{2}}{2} = \dfrac{3}{4}$ (b) $\sinh 2 = \frac{1}{2}\left(e^2 - e^{-2}\right) \approx 3.62686$

4. (a) $\cosh 3 = \frac{1}{2}\left(e^3 + e^{-3}\right) \approx 10.06766$ (b) $\cosh(\ln 3) = \dfrac{e^{\ln 3} + e^{-\ln 3}}{2} = \dfrac{3 + \frac{1}{3}}{2} = \dfrac{5}{3}$

5. (a) $\operatorname{sech} 0 = \dfrac{1}{\cosh 0} = \dfrac{1}{1} = 1$ (b) $\cosh^{-1} 1 = 0$ because $\cosh 0 = 1$.

6. (a) $\sinh 1 = \frac{1}{2}\left(e^1 - e^{-1}\right) \approx 1.17520$

   (b) Using Equation 3, we have $\sinh^{-1} 1 = \ln\left(1 + \sqrt{1^2 + 1}\right) = \ln\left(1 + \sqrt{2}\right) \approx 0.88137$.

7. $\sinh(-x) = \frac{1}{2}\left[e^{-x} - e^{-(-x)}\right] = \frac{1}{2}\left(e^{-x} - e^{x}\right) = -\frac{1}{2}\left(e^{x} - e^{-x}\right) = -\sinh x$

8. $\cosh(-x) = \frac{1}{2}\left[e^{-x} + e^{-(-x)}\right] = \frac{1}{2}\left(e^{-x} + e^{x}\right) = \frac{1}{2}\left(e^{x} + e^{-x}\right) = \cosh x$

9. $\cosh x + \sinh x = \frac{1}{2}\left(e^x + e^{-x}\right) + \frac{1}{2}\left(e^x - e^{-x}\right) = \frac{1}{2}\left(2e^x\right) = e^x$

10. $\cosh x - \sinh x = \frac{1}{2}\left(e^x + e^{-x}\right) - \frac{1}{2}\left(e^x - e^{-x}\right) = \frac{1}{2}\left(2e^{-x}\right) = e^{-x}$

11.
$$\begin{aligned}\sinh x \cosh y + \cosh x \sinh y &= \left[\tfrac{1}{2}\left(e^x - e^{-x}\right)\right]\left[\tfrac{1}{2}\left(e^y + e^{-y}\right)\right] + \left[\tfrac{1}{2}\left(e^x + e^{-x}\right)\right]\left[\tfrac{1}{2}\left(e^y - e^{-y}\right)\right] \\ &= \tfrac{1}{4}\left[\left(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}\right) + \left(e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y}\right)\right] \\ &= \tfrac{1}{4}\left(2e^{x+y} - 2e^{-x-y}\right) = \tfrac{1}{2}\left[e^{x+y} - e^{-(x+y)}\right] = \sinh(x+y)\end{aligned}$$

12.
$$\begin{aligned}\cosh x \cosh y + \sinh x \sinh y &= \left[\tfrac{1}{2}\left(e^x + e^{-x}\right)\right]\left[\tfrac{1}{2}\left(e^y + e^{-y}\right)\right] + \left[\tfrac{1}{2}\left(e^x - e^{-x}\right)\right]\left[\tfrac{1}{2}\left(e^y - e^{-y}\right)\right] \\ &= \tfrac{1}{4}\left[\left(e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y}\right) + \left(e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y}\right)\right] \\ &= \tfrac{1}{4}\left(2e^{x+y} + 2e^{-x-y}\right) = \tfrac{1}{2}\left[e^{x+y} + e^{-(x+y)}\right] = \cosh(x+y)\end{aligned}$$

13. Divide both sides of the identity $\cosh^2 x - \sinh^2 x = 1$ by $\sinh^2 x$:

$$\frac{\cosh^2 x}{\sinh^2 x} - 1 = \frac{1}{\sinh^2 x} \quad \Leftrightarrow \quad \coth^2 x - 1 = \operatorname{csch}^2 x.$$

14.
$$\begin{aligned}\tanh(x+y) &= \frac{\sinh(x+y)}{\cosh(x+y)} = \frac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y + \sinh x \sinh y} = \frac{\dfrac{\sinh x \cosh y}{\cosh x \cosh y} + \dfrac{\cosh x \sinh y}{\cosh x \cosh y}}{\dfrac{\cosh x \cosh y}{\cosh x \cosh y} + \dfrac{\sinh x \sinh y}{\cosh x \cosh y}} \\ &= \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}\end{aligned}$$

15. By Exercise 11, $\sinh 2x = \sinh(x+x) = \sinh x \cosh x + \cosh x \sinh x = 2 \sinh x \cosh x$.

16. Putting $y = x$ in the result from Exercise 12, we have

    $\cosh 2x = \cosh(x+x) = \cosh x \cosh x + \sinh x \sinh x = \cosh^2 x + \sinh^2 x$.

17. $\tanh(\ln x) = \dfrac{\sinh(\ln x)}{\cosh(\ln x)} = \dfrac{\left(e^{\ln x} - e^{-\ln x}\right)/2}{\left(e^{\ln x} + e^{-\ln x}\right)/2} = \dfrac{x - 1/x}{x + 1/x} = \dfrac{x^2 - 1}{x^2 + 1}$

18. $\dfrac{1+(\sinh x)/\cosh x}{1-(\sinh x)/\cosh x} = \dfrac{\cosh x + \sinh x}{\cosh x - \sinh x} = \dfrac{\frac{1}{2}\left(e^x+e^{-x}\right)+\frac{1}{2}\left(e^x-e^{-x}\right)}{\frac{1}{2}\left(e^x+e^{-x}\right)-\frac{1}{2}\left(e^x-e^{-x}\right)}$

$= \dfrac{e^x+e^{-x}+e^x-e^{-x}}{e^x+e^{-x}-e^x+e^{-x}} = \dfrac{2e^x}{2e^{-x}} = e^{2x}$

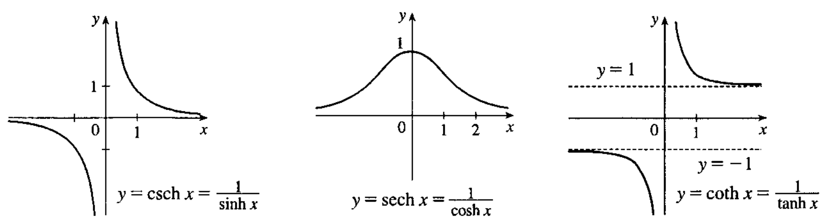
Or: Using the results of Exercises 9 and 10, $\dfrac{\cosh x + \sinh x}{\cosh x - \sinh x} = \dfrac{e^x}{e^{-x}} = e^{2x}$

19. By Exercise 9, $(\cosh x + \sinh x)^n = (e^x)^n = e^{nx} = \cosh nx + \sinh nx$.

20. $\sinh x = \frac{3}{4} \Rightarrow \operatorname{csch} x = 1/\sinh x = \frac{4}{3}$. $\cosh^2 x = \sinh^2 x + 1 = \frac{9}{16} + 1 = \frac{25}{16} \Rightarrow \cosh x = \frac{5}{4}$ (since $\cosh x > 0$). $\operatorname{sech} x = 1/\cosh x = \frac{4}{5}$, $\tanh x = \sinh x/\cosh x = \frac{3/4}{5/4} = \frac{3}{5}$, and $\coth x = 1/\tanh x = \frac{5}{3}$.

21. $\tanh x = \frac{4}{5} > 0$, so $x > 0$. $\coth x = 1/\tanh x = \frac{5}{4}$, $\operatorname{sech}^2 x = 1 - \tanh^2 x = 1 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} \Rightarrow \operatorname{sech} x = \frac{3}{5}$ (since $\operatorname{sech} x > 0$), $\cosh x = 1/\operatorname{sech} x = \frac{5}{3}$, $\sinh x = \tanh x \cosh x = \frac{4}{5}\cdot\frac{5}{3} = \frac{4}{3}$, and $\operatorname{csch} x = 1/\sinh x = \frac{3}{4}$.

22.



23. (a) $\lim\limits_{x\to\infty} \tanh x = \lim\limits_{x\to\infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \lim\limits_{x\to\infty} \dfrac{1 - e^{-2x}}{1 + e^{-2x}} = \dfrac{1-0}{1+0} = 1$

(b) $\lim\limits_{x\to-\infty} \tanh x = \lim\limits_{x\to-\infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \lim\limits_{x\to-\infty} \dfrac{e^{2x} - 1}{e^{2x} + 1} = \dfrac{0-1}{0+1} = -1$

(c) $\lim\limits_{x\to\infty} \sinh x = \lim\limits_{x\to\infty} \dfrac{e^x - e^{-x}}{2} = \infty$

(d) $\lim\limits_{x\to-\infty} \sinh x = \lim\limits_{x\to-\infty} \dfrac{e^x - e^{-x}}{2} = -\infty$

(e) $\lim\limits_{x\to\infty} \operatorname{sech} x = \lim\limits_{x\to\infty} \dfrac{2}{e^x + e^{-x}} = 0$

(f) $\lim\limits_{x\to\infty} \coth x = \lim\limits_{x\to\infty} \dfrac{e^x + e^{-x}}{e^x - e^{-x}} = \lim\limits_{x\to\infty} \dfrac{1 + e^{-2x}}{1 - e^{-2x}} = \dfrac{1+0}{1-0} = 1$ [*Or:* Use part (a)]

(g) $\lim\limits_{x\to0^+} \coth x = \lim\limits_{x\to0^+} \dfrac{\cosh x}{\sinh x} = \infty$, since $\sinh x \to 0$ and $\coth x > 0$.

(h) $\lim\limits_{x\to0^-} \coth x = \lim\limits_{x\to0^-} \dfrac{\cosh x}{\sinh x} = -\infty$, since $\sinh x \to 0$ and $\coth x < 0$.

(i) $\lim\limits_{x\to-\infty} \operatorname{csch} x = \lim\limits_{x\to-\infty} \dfrac{2}{e^x - e^{-x}} = 0$

24. (a) $\dfrac{d}{dx}\cosh x = \dfrac{d}{dx}\left[\frac{1}{2}\left(e^x + e^{-x}\right)\right] = \frac{1}{2}\left(e^x - e^{-x}\right) = \sinh x$

(b) $\dfrac{d}{dx}\tanh x = \dfrac{d}{dx}\left[\dfrac{\sinh x}{\cosh x}\right] = \dfrac{\cosh x\cosh x - \sinh x\sinh x}{\cosh^2 x} = \dfrac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \dfrac{1}{\cosh^2 x} = \operatorname{sech}^2 x$

(c) $\dfrac{d}{dx}\operatorname{csch} x = \dfrac{d}{dx}\left[\dfrac{1}{\sinh x}\right] = -\dfrac{\cosh x}{\sinh^2 x} = -\dfrac{1}{\sinh x}\cdot\dfrac{\cosh x}{\sinh x} = -\operatorname{csch} x\coth x$

(d) $\dfrac{d}{dx}\operatorname{sech} x = \dfrac{d}{dx}\left[\dfrac{1}{\cosh x}\right] = -\dfrac{\sinh x}{\cosh^2 x} = -\dfrac{1}{\cosh x}\cdot\dfrac{\sinh x}{\cosh x} = -\operatorname{sech} x\tanh x$

(e) $\dfrac{d}{dx}\coth x = \dfrac{d}{dx}\left[\dfrac{\cosh x}{\sinh x}\right] = \dfrac{\sinh x\sinh x - \cosh x\cosh x}{\sinh^2 x} = \dfrac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = -\dfrac{1}{\sinh^2 x}$
$= -\operatorname{csch}^2 x$

25. Let $y = \sinh^{-1} x$. Then $\sinh y = x$ and, by Example 1(a), $\cosh y = \sqrt{1+\sinh^2 y} = \sqrt{1+x^2}$. So by Exercise 9, $e^y = \sinh y + \cosh y = x + \sqrt{1+x^2} \;\Rightarrow\; y = \ln\left(x + \sqrt{1+x^2}\right)$.

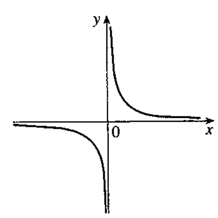
26. Let $y = \cosh^{-1} x$. Then $\cosh y = x$ and $y \geq 0$, so $\sinh y = \sqrt{\cosh^2 y - 1} = \sqrt{x^2-1}$. So, by Exercise 9, $e^y = \cosh y + \sinh y = x + \sqrt{x^2-1} \;\Rightarrow\; y = \ln\left(x + \sqrt{x^2-1}\right)$.
Another Method: Write $x = \cosh y = \frac{1}{2}\left(e^y + e^{-y}\right)$ and solve a quadratic, as in Example 3.

27. (a) Let $y = \tanh^{-1} x$. Then $x = \tanh y = \dfrac{e^y - e^{-y}}{e^y + e^{-y}} = \dfrac{e^{2y}-1}{e^{2y}+1} \;\Rightarrow\; xe^{2y} + x = e^{2y} - 1 \;\Rightarrow\; e^{2y} = \dfrac{1+x}{1-x}$
$\Rightarrow\; 2y = \ln\left(\dfrac{1+x}{1-x}\right) \;\Rightarrow\; y = \frac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$.

(b) Let $y = \tanh^{-1} x$. Then $x = \tanh y$, so from Exercise 18 we have $e^{2y} = \dfrac{1+\tanh y}{1-\tanh y} = \dfrac{1+x}{1-x} \;\Rightarrow$
$2y = \ln\left(\dfrac{1+x}{1-x}\right) \;\Rightarrow\; y = \frac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$.

28. (a) (i) $y = \operatorname{csch}^{-1} x \;\Leftrightarrow\; \operatorname{csch} y = x \quad (x \neq 0)$

(ii) We sketch the graph of csch^{-1} by reflecting the graph of csch (see Exercise 22) about the line $y = x$.

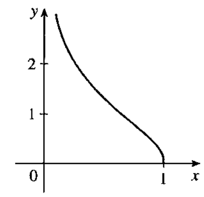


(iii) Let $y = \operatorname{csch}^{-1} x$. Then $x = \operatorname{csch} y = \dfrac{2}{e^y - e^{-y}} \;\Rightarrow\; xe^y - xe^{-y} = 2$
$\Rightarrow\; x\left(e^y\right)^2 - 2e^y - x = 0 \;\Rightarrow\; e^y = \dfrac{1 \pm \sqrt{x^2+1}}{x}$.

But $e^y > 0$, so for $x > 0$, $e^y = \dfrac{1+\sqrt{x^2+1}}{x}$ and for $x < 0$, $e^y = \dfrac{1-\sqrt{x^2+1}}{x}$.

Thus, $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{x^2+1}}{|x|}\right)$.

(b) (i) $y = \operatorname{sech}^{-1} x \quad \Leftrightarrow \quad \operatorname{sech} y = x$ and $y > 0$.



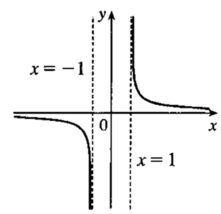
(ii) We sketch the graph of sech^{-1} by reflecting the graph of sech (see Exercise 22) about the line $y = x$.

(iii) Let $y = \operatorname{sech}^{-1} x$, so $x = \operatorname{sech} y = \dfrac{2}{e^y + e^{-y}} \quad \Rightarrow \quad xe^y + xe^{-y} = 2 \quad \Rightarrow$

$x\,(e^y)^2 - 2e^y + x = 0 \quad \Leftrightarrow \quad e^y = \dfrac{1 \pm \sqrt{1 - x^2}}{x}$.

But $y > 0 \quad \Rightarrow \quad e^y > 1$. This rules out the minus sign because $\dfrac{1 - \sqrt{1 - x^2}}{x} > 1 \quad \Leftrightarrow \quad 1 - \sqrt{1 - x^2} > x$ $\quad \Leftrightarrow \quad 1 - x > \sqrt{1 - x^2} \quad \Leftrightarrow \quad 1 - 2x + x^2 > 1 - x^2 \quad \Leftrightarrow \quad x^2 > x \quad \Leftrightarrow \quad x > 1$, but $x = \operatorname{sech} y \le 1$. Thus, $e^y = \dfrac{1 + \sqrt{1 - x^2}}{x} \quad \Rightarrow \quad \operatorname{sech}^{-1} x = \ln\left(\dfrac{1 + \sqrt{1 - x^2}}{x}\right)$.

(c) (i) $y = \coth^{-1} x \quad \Leftrightarrow \quad \coth y = x$



(ii) We sketch the graph of $\coth^{-1}$ by reflecting the graph of coth (see Exercise 22) about the line $y = x$.

(iii) Let $y = \coth^{-1} x$. Then $x = \coth y = \dfrac{e^y + e^{-y}}{e^y - e^{-y}} \quad \Rightarrow$

$xe^y - xe^{-y} = e^y + e^{-y} \quad \Rightarrow \quad (x - 1)\,e^y = (x + 1)\,e^{-y} \quad \Rightarrow$

$e^{2y} = \dfrac{x + 1}{x - 1} \quad \Rightarrow \quad 2y = \ln \dfrac{x + 1}{x - 1} \quad \Rightarrow \quad \coth^{-1} x = \frac{1}{2} \ln \dfrac{x + 1}{x - 1}$

29. (a) Let $y = \cosh^{-1} x$. Then $\cosh y = x$ and $y \ge 0 \quad \Rightarrow \quad \sinh y \dfrac{dy}{dx} = 1 \quad \Rightarrow$

$\dfrac{dy}{dx} = \dfrac{1}{\sinh y} = \dfrac{1}{\sqrt{\cosh^2 y - 1}} = \dfrac{1}{\sqrt{x^2 - 1}}$ (since $\sinh y \ge 0$ for $y \ge 0$). *Or:* Use Formula 4.

(b) Let $y = \tanh^{-1} x$. Then $\tanh y = x \quad \Rightarrow \quad \operatorname{sech}^2 y \dfrac{dy}{dx} = 1 \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{\operatorname{sech}^2 y} = \dfrac{1}{1 - \tanh^2 y} = \dfrac{1}{1 - x^2}$.

Or: Use Formula 5.

(c) Let $y = \operatorname{csch}^{-1} x$. Then $\operatorname{csch} y = x \quad \Rightarrow \quad -\operatorname{csch} y \coth y \dfrac{dy}{dx} = 1 \quad \Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch} y \coth y}$.

By Exercise 13, $\coth y = \pm\sqrt{\operatorname{csch}^2 y + 1} = \pm\sqrt{x^2 + 1}$. If $x > 0$, then $\coth y > 0$, so $\coth y = \sqrt{x^2 + 1}$. If $x < 0$, then $\coth y < 0$, so $\coth y = -\sqrt{x^2 + 1}$. In either case we have

$\dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch} y \coth y} = -\dfrac{1}{|x|\sqrt{x^2 + 1}}$.

(d) Let $y = \operatorname{sech}^{-1} x$. Then $\operatorname{sech} y = x \quad \Rightarrow \quad -\operatorname{sech} y \tanh y \dfrac{dy}{dx} = 1 \quad \Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{1}{\operatorname{sech} y \tanh y} = -\dfrac{1}{\operatorname{sech} y\sqrt{1 - \operatorname{sech}^2 y}} = -\dfrac{1}{x\sqrt{1 - x^2}}$. (Note that $y > 0$ and so $\tanh y > 0$.)

(e) Let $y = \coth^{-1} x$. Then $\coth y = x \quad \Rightarrow \quad -\operatorname{csch}^2 y \dfrac{dy}{dx} = 1 \quad \Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch}^2 y} = \dfrac{1}{1 - \coth^2 y} = \dfrac{1}{1 - x^2}$ by Exercise 13.

30. $f(x) = \tanh 4x \quad \Rightarrow \quad f'(x) = 4\,\mathrm{sech}^2\,4x$

31. $f(x) = x\cosh x \quad \Rightarrow \quad f'(x) = x(\cosh x)' + (\cosh x)(x)' = x\sinh x + \cosh x$

32. $g(x) = \sinh^2 x \quad \Rightarrow \quad g'(x) = 2\sinh x \cosh x$

33. $h(x) = \sinh\left(x^2\right) \quad \Rightarrow \quad h'(x) = \cosh\left(x^2\right)\cdot 2x = 2x\cosh\left(x^2\right)$

34. $F(x) = \sinh x \tanh x \quad \Rightarrow \quad F'(x) = \sinh x\,\mathrm{sech}^2\,x + \tanh x \cosh x$

35. $G(x) = \dfrac{1-\cosh x}{1+\cosh x} \quad \Rightarrow$

$$G'(x) = \frac{(1+\cosh x)(-\sinh x) - (1-\cosh x)(\sinh x)}{(1+\cosh x)^2} = \frac{-\sinh x - \sinh x\cosh x - \sinh x + \sinh x\cosh x}{(1+\cosh x)^2}$$
$$= \frac{-2\sinh x}{(1+\cosh x)^2}$$

36. $f(t) = e^t\,\mathrm{sech}\,t \quad \Rightarrow \quad f'(t) = e^t(-\mathrm{sech}\,t\tanh t) + (\mathrm{sech}\,t)\,e^t = e^t\,\mathrm{sech}\,t\,(1-\tanh t)$

37. $h(t) = \coth\sqrt{1+t^2} \quad \Rightarrow \quad h'(t) = -\mathrm{csch}^2\sqrt{1+t^2}\cdot\frac{1}{2}\left(1+t^2\right)^{-1/2}(2t) = -\dfrac{t\,\mathrm{csch}^2\sqrt{1+t^2}}{\sqrt{1+t^2}}$

38. $f(t) = \ln(\sinh t) \quad \Rightarrow \quad f'(t) = \dfrac{1}{\sinh t}\cosh t = \coth t$

39. $H(t) = \tanh\left(e^t\right) \quad \Rightarrow \quad H'(t) = \mathrm{sech}^2\left(e^t\right)\cdot e^t = e^t\,\mathrm{sech}^2\left(e^t\right)$

40. $y = \sinh(\cosh x) \quad \Rightarrow \quad y' = \cosh(\cosh x)\cdot\sinh x$

41. $y = e^{\cosh 3x} \quad \Rightarrow \quad y' = e^{\cosh 3x}\cdot\sinh 3x\cdot 3 = 3e^{\cosh 3x}\sinh 3x$

42. $y = x^2\sinh^{-1}(2x) \quad \Rightarrow \quad y' = x^2\cdot\dfrac{1}{\sqrt{1+(2x)^2}}\cdot 2 + \sinh^{-1}(2x)\cdot 2x = 2x\left[\dfrac{x}{\sqrt{1+4x^2}} + \sinh^{-1}(2x)\right]$

43. $y = \tanh^{-1}\sqrt{x} \quad \Rightarrow \quad y' = \dfrac{1}{1-\left(\sqrt{x}\right)^2}\cdot\frac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}\,(1-x)}$

44. $y = x\tanh^{-1}x + \ln\sqrt{1-x^2} = x\tanh^{-1}x + \frac{1}{2}\ln\left(1-x^2\right) \quad \Rightarrow$

$$y' = \tanh^{-1}x + \frac{x}{1-x^2} + \frac{1}{2}\left(\frac{1}{1-x^2}\right)(-2x) = \tanh^{-1}x$$

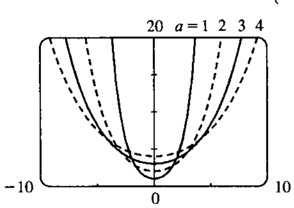
45. $y = x\sinh^{-1}(x/3) - \sqrt{9+x^2} \quad \Rightarrow$

$$y' = \sinh^{-1}\left(\frac{x}{3}\right) + x\frac{1/3}{\sqrt{1+(x/3)^2}} - \frac{2x}{2\sqrt{9+x^2}} = \sinh^{-1}\left(\frac{x}{3}\right) + \frac{x}{\sqrt{9+x^2}} - \frac{x}{\sqrt{9+x^2}} = \sinh^{-1}\left(\frac{x}{3}\right)$$

46. $y = \mathrm{sech}^{-1}\sqrt{1-x^2} \quad \Rightarrow \quad y' = -\dfrac{1}{\sqrt{1-x^2}\sqrt{1-\left(1-x^2\right)}}\dfrac{-2x}{2\sqrt{1-x^2}} = \dfrac{x}{\left(1-x^2\right)|x|}$

47. $y = \coth^{-1}\sqrt{x^2+1} \quad \Rightarrow \quad y' = \dfrac{1}{1-\left(x^2+1\right)}\dfrac{2x}{2\sqrt{x^2+1}} = -\dfrac{1}{x\sqrt{x^2+1}}$

48.



For $y = a\cosh(x/a)$ with $a > 0$, we have the y-intercept equal to a. As a increases, the graph flattens.

49. (a) $y = 20\cosh(x/20) - 15 \;\Rightarrow\; y' = 20\sinh(x/20)\cdot\frac{1}{20} = \sinh(x/20)$. Since the right pole is positioned at $x = 7$, we have $y' = \sinh\frac{7}{20} \approx 0.3572$.

(b) If θ is the angle between the tangent line and the x-axis, then $\tan\theta =$ slope of the line $= \sinh\frac{7}{20}$, so $\theta = \tan^{-1}\left(\sinh\frac{7}{20}\right) \approx 0.343$ rad $\approx 19.66°$. Thus, the angle between the line and the pole is about $90° - 19.66° = 70.34°$.

50. We differentiate the function twice, then substitute into the differential equation: $y = \dfrac{T}{\rho g}\cosh\dfrac{\rho g x}{T} \;\Rightarrow$

$$\frac{dy}{dx} = \frac{T}{\rho g}\sinh\left(\frac{\rho g x}{T}\right)\frac{\rho g}{T} = \sinh\frac{\rho g x}{T} \;\Rightarrow\; \frac{d^2y}{dx^2} = \cosh\left(\frac{\rho g x}{T}\right)\frac{\rho g}{T} = \frac{\rho g}{T}\cosh\frac{\rho g x}{T}.$$

We evaluate the two sides separately: LHS $= \dfrac{d^2y}{dx^2} = \dfrac{\rho g}{T}\cosh\dfrac{\rho g x}{T}$,

RHS $= \dfrac{\rho g}{T}\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \dfrac{\rho g}{T}\sqrt{1 + \sinh^2\dfrac{\rho g x}{T}} = \dfrac{\rho g}{T}\cosh\dfrac{\rho g x}{T}$, by the identity proved in Example 1(a).

51. (a) $y = A\sinh mx + B\cosh mx \;\Rightarrow\; y' = mA\cosh mx + mB\sinh mx \;\Rightarrow$
$y'' = m^2A\sinh mx + m^2B\cosh mx = m^2y$

(b) From part (a), a solution of $y'' = 9y$ is $y(x) = A\sinh 3x + B\cosh 3x$. So
$-4 = y(0) = A\sinh 0 + B\cosh 0 = B$, so $B = -4$. Now $y'(x) = 3A\cosh 3x - 12\sinh 3x \;\Rightarrow$
$6 = y'(0) = 3A \;\Rightarrow\; A = 2$, so $y = 2\sinh 3x - 4\cosh 3x$.

52. $\displaystyle\lim_{x\to\infty}\frac{\sinh x}{e^x} = \lim_{x\to\infty}\frac{e^x - e^{-x}}{2e^x} = \lim_{x\to\infty}\frac{1 - e^{-2x}}{2} = \frac{1-0}{2} = \frac{1}{2}$

53. The tangent to $y = \cosh x$ has slope 1 when $y' = \sinh x = 1 \;\Rightarrow\; x = \sinh^{-1} 1 = \ln\left(1+\sqrt{2}\right)$, by Equation 3.
Since $\sinh x = 1$ and $y = \cosh x = \sqrt{1+\sinh^2 x}$, we have $\cosh x = \sqrt{2}$. The point is $\left(\ln\left(1+\sqrt{2}\right), \sqrt{2}\right)$.

54. Let $u = 1 + 4x$. Then $du = 4\,dx$, so $\int \sinh(1+4x)\,dx = \frac{1}{4}\int \sinh u\,du = \frac{1}{4}\cosh u + C = \frac{1}{4}\cosh(1+4x) + C$.

55. Let $u = \cosh x$. Then $du = \sinh x\,dx$, so $\int \sinh x\cosh^2 x\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\cosh^3 x + C$.

56. Let $u = \cosh x$. Then $du = \sinh x\,dx$, and $\int \tanh x\,dx = \displaystyle\int \frac{\sinh x}{\cosh x}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln(\cosh x) + C$.

57. Let $u = \sinh x$, so $du = \cosh x\,dx$, and $\int \coth x\,dx = \displaystyle\int \frac{\cosh x}{\sinh x}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sinh x| + C$.

58. Let $u = 2 + \tanh x$. Then $du = \operatorname{sech}^2 x\,dx$, so

$\displaystyle\int \frac{\operatorname{sech}^2 x}{2+\tanh x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|2+\tanh x| + C = \ln(2+\tanh x) + C$ (since $2 + \tanh x > 1$).

59. Let $u = \frac{1}{2}x$ so that $x = 2u$. Then $dx = 2\,du$, so

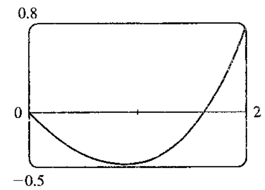
$$\int_0^2 \frac{1}{\sqrt{4+x^2}}\,dx = \int_0^1 \frac{1}{\sqrt{4+4u^2}}\,(2\,du) = \int_0^1 \frac{1}{\sqrt{1+u^2}}\,du = \Big[\sinh^{-1} u\Big]_0^1 = \sinh^{-1} 1 - \sinh^{-1} 0$$
$$= \sinh^{-1} 1$$

We could use Equation 3 to write this as $\ln\left(1+\sqrt{2}\right)$.

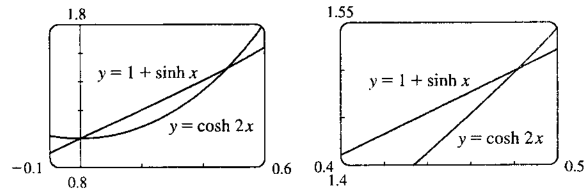
60. $\displaystyle\int_2^3 \frac{1}{\sqrt{x^2-1}}\,dx = \Big[\cosh^{-1}x\Big]_2^3 = \cosh^{-1}3 - \cosh^{-1}2$. Using Equation 4, we could write this as $\ln\left(3+2\sqrt{2}\right) - \ln\left(2+\sqrt{3}\right) = \ln\left[\left(3+2\sqrt{2}\right)/\left(2+\sqrt{3}\right)\right]$.

61. $\displaystyle\int_0^{1/2} \frac{1}{1-x^2}\,dx = \Big[\tanh^{-1}x\Big]_0^{1/2} = \tanh^{-1}\frac{1}{2} = \frac{1}{2}\ln\left(\frac{1+1/2}{1-1/2}\right)$ (from Equation 5) $= \frac{1}{2}\ln 3$.

62. We want $\int_0^1 \sinh cx\,dx = 1$. To calculate the integral, we put $u = cx$, so $du = c\,dx$, the upper limit becomes c, and the equation becomes $\displaystyle\frac{1}{c}\int_0^c \sinh u\,du = 1 \Leftrightarrow \frac{1}{c}[\cosh c - 1] = 1 \Leftrightarrow \cosh c - 1 = c$. We plot the function $f(c) = \cosh c - c - 1$, and see that its positive root lies at approximately $c = 1.62$. So the equation $\int_0^1 \sinh cx\,dx = 1$ holds for $c \approx 1.62$.



63. (a) From the graphs, we estimate that the two curves $y = \cosh 2x$ and $y = 1 + \sinh x$ intersect at $x = 0$ and at $x \approx 0.481$.



(b) We have found the two roots of the equation $\cosh 2x = 1 + \sinh x$ to be $x = 0$ and $x \approx 0.481$. Note from the first graph that $1 + \sinh x > \cosh 2x$ on the interval $(0, 0.481)$, so the area between the two curves is

$$A \approx \int_0^{0.481} (1 + \sinh x - \cosh 2x)\,dx = \Big[x + \cosh x - \tfrac{1}{2}\sinh 2x\Big]_0^{0.481}$$
$$= [0.481 + \cosh 0.481 - \tfrac{1}{2}\sinh(2\cdot 0.481)] - [0 + \cosh 0 - \tfrac{1}{2}\sinh(2\cdot 0)] \approx 0.0402$$

64. The area of the triangle with vertices O, P, and $(\cosh t, 0)$ is $\frac{1}{2}\sinh t\cosh t$, and the area under the curve $x^2 - y^2 = 1$, from $x = 1$ to $x = \cosh t$, is $\int_1^{\cosh t}\sqrt{x^2-1}\,dx$. Therefore, the area of the shaded region is $A(t) = \frac{1}{2}\sinh t\cosh t - \int_1^{\cosh t}\sqrt{x^2-1}\,dx$. So, by FTC1,

$$A'(t) = \tfrac{1}{2}\left(\cosh^2 t + \sinh^2 t\right) - \sqrt{\cosh^2 t - 1}\,\sinh t = \tfrac{1}{2}\left(\cosh^2 t + \sinh^2 t\right) - \sqrt{\sinh^2 t}\,\sinh t$$
$$= \tfrac{1}{2}\left(\cosh^2 t + \sinh^2 t\right) - \sinh^2 t = \tfrac{1}{2}\left(\cosh^2 t - \sinh^2 t\right) = \tfrac{1}{2}(1) = \tfrac{1}{2}$$

Thus $A(t) = \frac{1}{2}t + C$, since $A'(t) = \frac{1}{2}$. To calculate C, we let $t = 0$. Thus, $A(0) = \frac{1}{2}\sinh 0\cosh 0 - \int_1^{\cosh 0}\sqrt{x^2-1}\,dx = \frac{1}{2}(0) + C \Rightarrow C = 0$. Thus $A(t) = \frac{1}{2}t$.

65. $$\cosh x = \frac{e^x + e^{-x}}{2} = \tfrac{1}{2}\left[e^{\ln(\sec\theta+\tan\theta)} + e^{-\ln(\sec\theta+\tan\theta)}\right] = \tfrac{1}{2}\left[\sec\theta + \tan\theta + \left[e^{\ln(\sec\theta+\tan\theta)}\right]^{-1}\right]$$
$$= \tfrac{1}{2}\left[\sec\theta + \tan\theta + (\sec\theta+\tan\theta)^{-1}\right] = \tfrac{1}{2}\left[\sec\theta + \tan\theta + 1/(\sec\theta+\tan\theta)\right]$$
$$= \frac{(\sec\theta+\tan\theta)^2 + 1}{2(\sec\theta+\tan\theta)} = \frac{\sec^2\theta + 2\sec\theta\tan\theta + \tan^2\theta + 1}{2(\sec\theta+\tan\theta)} = \frac{\sec^2\theta + 2\sec\theta\tan\theta + \sec^2\theta}{2(\sec\theta+\tan\theta)}$$
$$= \frac{2\sec^2\theta + 2\sec\theta\tan\theta}{2(\sec\theta+\tan\theta)} = \frac{2\sec\theta(\sec\theta+\tan\theta)}{2(\sec\theta+\tan\theta)} = \sec\theta$$

7.7 Indeterminate Forms and l'Hospital's Rule

1. (a) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ is an indeterminate form of type $\dfrac{0}{0}$.

(b) $\lim\limits_{x\to a} \dfrac{f(x)}{p(x)} = 0$ because the numerator approaches 0 while the denominator becomes large.

(c) $\lim\limits_{x\to a} \dfrac{h(x)}{p(x)} = 0$ because the numerator approaches a finite number while the denominator becomes large.

(d) If $\lim\limits_{x\to a} p(x) = \infty$ and $f(x) \to 0$ through positive values, then $\lim\limits_{x\to a} \dfrac{p(x)}{f(x)} = \infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x^2$.] If $f(x) \to 0$ through negative values, then $\lim\limits_{x\to a} \dfrac{p(x)}{f(x)} = -\infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = -x^2$.] If $f(x) \to 0$ through both positive and negative values, then the limit might not exist. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x$.] It is not possible to evaluate this limit.

(e) $\lim\limits_{x\to a} \dfrac{p(x)}{q(x)}$ is an indeterminate form of type $\dfrac{\infty}{\infty}$.

2. (a) $\lim\limits_{x\to a} [f(x)\, p(x)]$ is an indeterminate form of type $0 \cdot \infty$.

(b) When x is near a, $p(x)$ is large and $h(x)$ is near 1, so $h(x)\, p(x)$ is large. Thus, $\lim\limits_{x\to a} [h(x)\, p(x)] = \infty$.

(c) When x is near a, $p(x)$ and $q(x)$ are both large, so $p(x)\, q(x)$ is large. Thus, $\lim\limits_{x\to a} [p(x)\, q(x)] = \infty$.

3. (a) When x is near a, $f(x)$ is near 0 and $p(x)$ is large, so $f(x) - p(x)$ is large negative. Thus, $\lim\limits_{x\to a} [f(x) - p(x)] = -\infty$.

(b) $\lim\limits_{x\to a} [p(x) - q(x)]$ is an indeterminate form of type $\infty - \infty$.

(c) When x is near a, $p(x)$ and $q(x)$ are both large, so $p(x)\, q(x)$ is large. Thus, $\lim\limits_{x\to a} [p(x) + q(x)] = \infty$.

4. (a) $\lim\limits_{x\to a} [f(x)]^{g(x)}$ is an indeterminate form of type 0^0.

(b) If $y = f(x)^{p(x)}$, then $\ln y = p(x) \ln f(x)$. When x is near a, $p(x) \to \infty$ and $\ln f(x) \to -\infty$, so $\ln y \to -\infty$. Therefore, $\lim\limits_{x\to a} [f(x)]^{p(x)} = \lim\limits_{x\to a} y = \lim\limits_{x\to a} e^{\ln y} = 0$, provided f^p is defined.

(c) $\lim\limits_{x\to a} [h(x)]^{p(x)}$ is an indeterminate form of type 1^∞.

(d) $\lim\limits_{x\to a} [p(x)]^{f(x)}$ is an indeterminate form of type ∞^0.

(e) If $y = [p(x)]^{q(x)}$, then $\ln y = q(x) \ln p(x)$. When x is near a, $q(x) \to \infty$ and $\ln p(x) \to \infty$, so $\ln y \to \infty$. Therefore, $\lim\limits_{x\to a} [p(x)]^{q(x)} = \lim\limits_{x\to a} y = \lim\limits_{x\to a} e^{\ln y} = \infty$.

(f) $\lim\limits_{x\to a} \sqrt[q(x)]{p(x)} = \lim\limits_{x\to a} [p(x)]^{1/q(x)}$ is an indeterminate form of type ∞^0.

5. $\lim\limits_{x\to -1} \dfrac{x^2 - 1}{x + 1} = \lim\limits_{x\to -1} \dfrac{(x+1)(x-1)}{x+1} = \lim\limits_{x\to -1} (x - 1) = -2$

6. $\lim\limits_{x\to -2} \dfrac{x+2}{x^2 + 3x + 2} = \lim\limits_{x\to -2} \dfrac{x+2}{(x+1)(x+2)} = \lim\limits_{x\to -2} \dfrac{1}{x+1} = -1$

7. $\lim\limits_{x\to 1}\dfrac{x^9-1}{x^5-1}=\lim\limits_{x\to 1}\dfrac{(x-1)\left(x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1\right)}{(x-1)\left(x^4+x^3+x^2+x+1\right)}$

$=\lim\limits_{x\to 1}\dfrac{x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1}{x^4+x^3+x^2+x+1}=\dfrac{9}{5}$

8. $\lim\limits_{x\to 1}\dfrac{x^a-1}{x^b-1}\overset{\text{H}}{=}\lim\limits_{x\to 1}\dfrac{ax^{a-1}}{bx^{b-1}}=\dfrac{a}{b}$

9. $\lim\limits_{x\to 0}\dfrac{e^x-1}{\sin x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{e^x}{\cos x}=\dfrac{1}{1}=1$

10. $\lim\limits_{x\to 0}\dfrac{x+\tan x}{\sin x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{1+\sec^2 x}{\cos x}=\dfrac{1+1^2}{1}=2$

11. $\lim\limits_{x\to 0}\dfrac{\sin x}{x^3}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{\cos x}{3x^2}=\infty$

12. $\lim\limits_{x\to\pi}\dfrac{\tan x}{x}=\dfrac{\tan\pi}{\pi}=\dfrac{0}{\pi}=0$

13. $\lim\limits_{x\to 0}\dfrac{\tan px}{\tan qx}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{p\sec^2 px}{q\sec^2 qx}=\dfrac{p\,(1)^2}{q\,(1)^2}=\dfrac{p}{q}$

14. $\lim\limits_{x\to 3\pi/2}\dfrac{\cos x}{x-3\pi/2}\overset{\text{H}}{=}\lim\limits_{x\to 3\pi/2}\dfrac{-\sin x}{1}=-\sin\frac{3\pi}{2}=1$

15. $\lim\limits_{x\to\infty}\dfrac{\ln x}{x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{1/x}{1}=0$

16. $\lim\limits_{x\to\infty}\dfrac{e^x}{x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{1}=\lim\limits_{x\to\infty}e^x=\infty$

17. $\lim\limits_{x\to 0^+}[(\ln x)/x]=-\infty$ since $\ln x\to-\infty$ as $x\to 0^+$ and dividing by small values of x just increases the magnitude of the quotient $(\ln x)/x$. L'Hospital's Rule does not apply.

18. $\lim\limits_{x\to\infty}\dfrac{\ln\ln x}{x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1}=\lim\limits_{x\to\infty}\dfrac{1}{x\ln x}=0$

19. $\lim\limits_{t\to 0}\dfrac{5^t-3^t}{t}\overset{\text{H}}{=}\lim\limits_{t\to 0}\dfrac{5^t\ln 5-3^t\ln 3}{1}=\ln 5-\ln 3=\ln\frac{5}{3}$

20. $\lim\limits_{t\to 16}\dfrac{\sqrt[4]{t}-2}{t-16}=\lim\limits_{t\to 16}\dfrac{\sqrt[4]{t}-2}{\left(\sqrt{t}+4\right)\left(\sqrt{t}-4\right)}=\lim\limits_{t\to 16}\dfrac{\sqrt[4]{t}-2}{\left(\sqrt{t}+4\right)\left(\sqrt[4]{t}+2\right)\left(\sqrt[4]{t}-2\right)}$

$=\lim\limits_{t\to 16}\dfrac{1}{\left(\sqrt{t}+4\right)\left(\sqrt[4]{t}+2\right)}=\dfrac{1}{(4+4)\,(2+2)}=\dfrac{1}{32}$

21. $\lim\limits_{x\to 0}\dfrac{e^x-1-x}{x^2}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{e^x-1}{2x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{e^x}{2}=\dfrac{1}{2}$

22. $\lim\limits_{x\to 0}\dfrac{e^x-1-x-x^2/2}{x^3}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{e^x-1-x}{3x^2}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{e^x-1}{6x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{e^x}{6}=\dfrac{1}{6}$

23. $\lim\limits_{x\to\infty}\dfrac{e^x}{x^3}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{3x^2}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{6x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{6}=\infty$

24. $\lim\limits_{x\to\infty}\dfrac{(\ln x)^3}{x^2}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{3\,(\ln x)^2\,(1/x)}{2x}=\lim\limits_{x\to\infty}\dfrac{3\,(\ln x)^2}{2x^2}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{6\,(\ln x)\,(1/x)}{4x}$

$=\lim\limits_{x\to\infty}\dfrac{3\ln x}{2x^2}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{3/x}{4x}=\lim\limits_{x\to\infty}\dfrac{3}{4x^2}=0$

25. $\lim\limits_{x\to 0}\dfrac{\sin^{-1}x}{x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{1/\sqrt{1-x^2}}{1}=\lim\limits_{x\to 0}\dfrac{1}{\sqrt{1-x^2}}=\dfrac{1}{1}=1$

26. $\lim\limits_{x\to 0}\dfrac{\sin x}{\sinh x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{\cos x}{\cosh x}=\dfrac{1}{1}=1$

27. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{\sin x}{2x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{\cos x}{2}=\dfrac{1}{2}$

28. $\lim\limits_{x\to 0}\dfrac{\sin x-x}{x^3}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{\cos x-1}{3x^2}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{-\sin x}{6x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{-\cos x}{6}=-\dfrac{1}{6}$

29. $\lim\limits_{x\to 0}\dfrac{\sin x}{e^x}=\dfrac{0}{1}=0$. L'Hospital's Rule does not apply.

30. $\lim\limits_{x\to 0}\dfrac{\cos mx-\cos nx}{x^2}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{-m\sin mx+n\sin nx}{2x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{-m^2\cos mx+n^2\cos nx}{2}=\frac{1}{2}\left(n^2-m^2\right)$

31. $\lim\limits_{x\to 0}\dfrac{\tan\alpha x}{x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{\alpha\sec^2\alpha x}{1}=\alpha$

32. $\lim\limits_{x\to 0}\dfrac{x}{\tan^{-1}(4x)}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{1}{\dfrac{1}{1+(4x)^2}\cdot 4}=\lim\limits_{x\to 0}\dfrac{1+16x^2}{4}=\dfrac{1}{4}$

33. $\lim\limits_{x\to\infty}\dfrac{x}{\ln\left(1+2e^x\right)}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{1}{\dfrac{1}{1+2e^x}\cdot 2e^x}=\lim\limits_{x\to\infty}\dfrac{1+2e^x}{2e^x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{2e^x}{2e^x}=1$

34. $\lim\limits_{x\to 0}\dfrac{x+\tan 2x}{x-\tan 2x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{1+2\sec^2 2x}{1-2\sec^2 2x}=\dfrac{1+2\,(1)^2}{1-2\,(1)^2}=-3$

35. $\lim\limits_{x\to 0}\dfrac{\tan 2x}{\tanh 3x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{2\sec^2 2x}{3\operatorname{sech}^2 3x}=\dfrac{2}{3}$

36. $\lim\limits_{x\to 0}\dfrac{1-e^{-2x}}{\sec x}=\dfrac{1-1}{1}=0$. L'Hospital's Rule does not apply.

37. $\lim\limits_{x\to 0}\dfrac{2x-\sin^{-1}x}{2x+\cos^{-1}x}=\dfrac{2\,(0)-0}{2\,(0)+\pi/2}=0$. L'Hospital's Rule does not apply.

38. $\lim\limits_{x\to 0}\dfrac{2x-\sin^{-1}x}{2x+\tan^{-1}x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{2-1/\sqrt{1-x^2}}{2+1/\left(1+x^2\right)}=\dfrac{2-1}{2+1}=\dfrac{1}{3}$

39. $\lim\limits_{x\to 0^+}\sqrt{x}\ln x=\lim\limits_{x\to 0^+}\dfrac{\ln x}{x^{-1/2}}\overset{\text{H}}{=}\lim\limits_{x\to 0^+}\dfrac{1/x}{-\frac{1}{2}x^{-3/2}}=\lim\limits_{x\to 0^+}\left(-2\sqrt{x}\right)=0$

40. $\lim\limits_{x\to -\infty}x^2e^x=\lim\limits_{x\to -\infty}\dfrac{x^2}{e^{-x}}\overset{\text{H}}{=}\lim\limits_{x\to -\infty}\dfrac{2x}{-e^{-x}}\overset{\text{H}}{=}\lim\limits_{x\to -\infty}\dfrac{2}{e^{-x}}=\lim\limits_{x\to -\infty}2e^x=0$

41. $\lim\limits_{x\to\infty}e^{-x}\ln x=\lim\limits_{x\to\infty}\dfrac{\ln x}{e^x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{1/x}{e^x}=\lim\limits_{x\to\infty}\dfrac{1}{xe^x}=0$

42. $\lim\limits_{x\to(\pi/2)^-}\sec 7x\cos 3x=\lim\limits_{x\to(\pi/2)^-}\dfrac{\cos 3x}{\cos 7x}=\lim\limits_{x\to(\pi/2)^-}\dfrac{-3\sin 3x}{-7\sin 7x}=\dfrac{3\,(-1)}{7\,(-1)}=\dfrac{3}{7}$

43. $\lim\limits_{x\to\infty}x^3e^{-x^2}=\lim\limits_{x\to\infty}\dfrac{x^3}{e^{x^2}}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{3x^2}{2xe^{x^2}}=\lim\limits_{x\to\infty}\dfrac{3x}{2e^{x^2}}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{3}{4xe^{x^2}}=0$

44. $\lim\limits_{x\to 0^+}\sqrt{x}\sec x=0\cdot 1=0$

45. $\lim_{x\to\pi}(x-\pi)\cot x = \lim_{x\to\pi}\frac{x-\pi}{\tan x} \overset{\text{H}}{=} \lim_{x\to\pi}\frac{1}{\sec^2 x} = \frac{1}{(-1)^2} = 1$

46. $\lim_{x\to 1^+}(x-1)\tan(\pi x/2) = \lim_{x\to 1^+}\frac{x-1}{\cot(\pi x/2)} \overset{\text{H}}{=} \lim_{x\to 1^+}\frac{1}{-\csc^2(\pi x/2)\frac{\pi}{2}} = -\frac{2}{\pi}$

47. $\lim_{x\to 0}\left(\frac{1}{x^4}-\frac{1}{x^2}\right) = \lim_{x\to 0}\frac{1-x^2}{x^4} = \infty$

48. $\lim_{x\to 0}(\csc x-\cot x) = \lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right) = \lim_{x\to 0}\frac{1-\cos x}{\sin x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{\sin x}{\cos x} = 0$

49. $\lim_{x\to 0}\left(\frac{1}{x}-\csc x\right) = \lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right) = \lim_{x\to 0}\frac{\sin x - x}{x\sin x}$

$$\overset{\text{H}}{=} \lim_{x\to 0}\frac{\cos x - 1}{\sin x + x\cos x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{-\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$$

50. $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) = \lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x} \overset{\text{H}}{=} \lim_{x\to 1}\frac{1-1/x}{\ln x + (x-1)(1/x)}$

$$= \lim_{x\to 1}\frac{x-1}{x\ln x + x - 1} \overset{\text{H}}{=} \lim_{x\to 1}\frac{1}{\ln x + 1 + 1} = \frac{1}{0+2} = \frac{1}{2}$$

51. $\lim_{x\to\infty}\left(x-\sqrt{x^2-1}\right) = \lim_{x\to\infty}\left(x-\sqrt{x^2-1}\right)\frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = \lim_{x\to\infty}\frac{x^2-\left(x^2-1\right)}{x+\sqrt{x^2-1}} = \lim_{x\to\infty}\frac{1}{x+\sqrt{x^2-1}} = 0$

52. $\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right) = \lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$

$$= \lim_{x\to\infty}\frac{\left(x^2+x+1\right)-\left(x^2-x\right)}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} = \lim_{x\to\infty}\frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$$

$$= \lim_{x\to\infty}\frac{2+1/x}{\sqrt{1+1/x+1/x^2}+\sqrt{1-1/x}} = \frac{2}{1+1} = 1$$

53. $\lim_{x\to\infty}\left(\frac{1}{x}-\frac{1}{e^x-1}\right) = \lim_{x\to\infty}\frac{1}{x} - \lim_{x\to\infty}\frac{1}{e^x-1}$ (since both limits exist) $= 0 - 0 = 0$

54. $\lim_{x\to\infty}\left(xe^{1/x}-x\right) = \lim_{x\to\infty}x\left(e^{1/x}-1\right) = \lim_{x\to\infty}\frac{e^{1/x}-1}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{e^{1/x}\left(-1/x^2\right)}{-1/x^2} = \lim_{x\to\infty}e^{1/x} = e^0 = 1$

55. $y = x^{\sin x} \Rightarrow \ln y = \sin x\ln x$, so

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\sin x\ln x = \lim_{x\to 0^+}\frac{\ln x}{\csc x} \overset{\text{H}}{=} \lim_{x\to 0^+}\frac{1/x}{-\csc x\cot x} = -\left(\lim_{x\to 0^+}\frac{\sin x}{x}\right)\left(\lim_{x\to 0^+}\tan x\right)$$

$$= -1\cdot 0 = 0 \Rightarrow$$

$\lim_{x\to 0^+}x^{\sin x} = \lim_{x\to 0^+}e^{\ln y} = e^0 = 1.$

56. $y = (\sin x)^{\tan x} \Rightarrow \ln y = \tan x\ln(\sin x)$, so

$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\tan x\ln(\sin x) = \lim_{x\to 0^+}\frac{\ln(\sin x)}{\cot x} \overset{\text{H}}{=} \lim_{x\to 0^+}\frac{(\cos x)/\sin x}{-\csc^2 x} = \lim_{x\to 0^+}(-\sin x\cos x) = 0 \Rightarrow$

$\lim_{x\to 0^+}(\sin x)^{\tan x} = \lim_{x\to 0^+}e^{\ln y} = e^0 = 1.$

57. $y = (1-2x)^{1/x} \Rightarrow \ln y = \frac{1}{x}\ln(1-2x)$, so $\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(1-2x)}{x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{-2/(1-2x)}{1} = -2 \Rightarrow$

$\lim_{x\to 0}(1-2x)^{1/x} = \lim_{x\to 0}e^{\ln y} = e^{-2}.$

58. $y = (1 + a/x)^{bx} \quad \Rightarrow \quad \ln y = bx \ln\left(1 + \dfrac{a}{x}\right)$, so

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{b \ln(1 + a/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{b\left(\dfrac{1}{1 + a/x}\right)\left(-\dfrac{a}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty} \frac{ab}{1 + a/x} = ab \quad \Rightarrow$$

$$\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^{bx} = \lim_{x\to\infty} e^{\ln y} = e^{ab}.$$

59. $y = \left(1 + \dfrac{3}{x} + \dfrac{5}{x^2}\right)^x \quad \Rightarrow \quad \ln y = x \ln\left(1 + \dfrac{3}{x} + \dfrac{5}{x^2}\right) \quad \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1 + \dfrac{3}{x} + \dfrac{5}{x^2}\right)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\left(-\dfrac{3}{x^2} - \dfrac{10}{x^3}\right)\Big/\left(1 + \dfrac{3}{x} + \dfrac{5}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty} \frac{3 + 10/x}{1 + 3/x + 5/x^2} = 3, \text{ so}$$

$$\lim_{x\to\infty} \left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^3.$$

60. $y = x^{(\ln 2)/(1 + \ln x)} \quad \Rightarrow \quad \ln y = \dfrac{\ln 2}{1 + \ln x} \ln x \quad \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{(\ln 2)(\ln x)}{1 + \ln x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{(\ln 2)(1/x)}{1/x} = \lim_{x\to\infty} \ln 2 = \ln 2, \text{ so}$$

$$\lim_{x\to\infty} x^{(\ln 2)/(1 + \ln x)} = \lim_{x\to\infty} e^{\ln y} = e^{\ln 2} = 2.$$

61. $y = x^{1/x} \quad \Rightarrow \quad \ln y = (1/x) \ln x \quad \Rightarrow \quad \lim\limits_{x\to\infty} \ln y = \lim\limits_{x\to\infty} \dfrac{\ln x}{x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{1} = 0 \quad \Rightarrow$

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^0 = 1$$

62. $y = (e^x + x)^{1/x} \quad \Rightarrow \quad \ln y = \dfrac{1}{x} \ln(e^x + x)$, so

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(e^x + x)}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x + 1}{e^x + x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{e^x + 1} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{e^x} = 1 \quad \Rightarrow$$

$$\lim_{x\to\infty} (e^x + x)^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^1 = e.$$

63. $y = \left(\dfrac{x}{x + 1}\right)^x \quad \Rightarrow \quad \ln y = x \ln\left(\dfrac{x}{x + 1}\right) \quad \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x \ln\left(\frac{x}{x + 1}\right) = \lim_{x\to\infty} \frac{\ln x - \ln(x + 1)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x - 1/(x + 1)}{-1/x^2}$$

$$= \lim_{x\to\infty} \left(-x + \frac{x^2}{x + 1}\right) = \lim_{x\to\infty} \frac{-x}{x + 1} = -1$$

so $\lim\limits_{x\to\infty} \left(\dfrac{x}{x + 1}\right)^x = \lim\limits_{x\to\infty} e^{\ln y} = e^{-1}$

Or: $\lim\limits_{x\to\infty} \left(\dfrac{x}{x + 1}\right)^x = \lim\limits_{x\to\infty} \left[\left(\dfrac{x + 1}{x}\right)^{-1}\right]^x = \left[\lim\limits_{x\to\infty} \left(1 + \dfrac{1}{x}\right)^x\right]^{-1} = e^{-1}$

64. Let $y = (\cos 3x)^{5/x}$. Then $\ln y = \dfrac{5}{x} \ln(\cos 3x) \quad \Rightarrow \quad \lim\limits_{x\to 0} \ln y = 5 \lim\limits_{x\to 0} \dfrac{\ln(\cos 3x)}{x} \overset{\text{H}}{=} 5 \lim\limits_{x\to 0} \dfrac{-3 \tan 3x}{1} = 0$, so $\lim\limits_{x\to 0} (\cos 3x)^{5/x} = e^0 = 1$.

65. $y = (-\ln x)^x \quad \Rightarrow \quad \ln y = x\ln(-\ln x)$, so

$$\lim_{x\to0^+}\ln y = \lim_{x\to0^+} x\ln(-\ln x) = \lim_{x\to0^+}\frac{\ln(-\ln x)}{1/x} \overset{H}{=} \lim_{x\to0^+}\frac{(1/-\ln x)(-1/x)}{-1/x^2} = \lim_{x\to0^+}\frac{-x}{\ln x} = 0 \quad\Rightarrow$$

$$\lim_{x\to0^+}(-\ln x)^x = e^0 = 1.$$

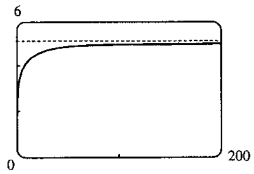
66. Let $y = \left(\dfrac{2x-3}{2x+5}\right)^{2x+1}$. Then $\ln y = (2x+1)\ln\left(\dfrac{2x-3}{2x+5}\right) \quad\Rightarrow$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln(2x-3)-\ln(2x+5)}{1/(2x+1)} \overset{H}{=} \lim_{x\to\infty}\frac{2/(2x-3)-2/(2x+5)}{-2/(2x+1)^2} = \lim_{x\to\infty}\frac{-8(2x+1)^2}{(2x-3)(2x+5)}$$

$$= \lim_{x\to\infty}\frac{-8(2+1/x)^2}{(2-3/x)(2+5/x)} = -8$$

$$\Rightarrow \quad \lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1} = e^{-8}.$$

67.

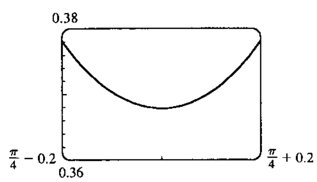


From the graph, it appears that $\lim\limits_{x\to\infty} x[\ln(x+5)-\ln x] = 5$. Now

$$\lim_{x\to\infty} x[\ln(x+5)-\ln x] = \lim_{x\to\infty}\frac{\ln(x+5)-\ln x}{1/x}$$

$$\overset{H}{=} \lim_{x\to\infty}\frac{1/(x+5)-1/x}{-1/x^2} = \lim_{x\to\infty}\frac{5x^2}{x(x+5)} = 5$$

68.

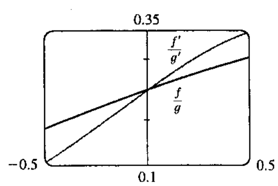


From the graph, it appears that $\lim\limits_{x\to\pi/4}(\tan x)^{\tan 2x} \approx 0.368$. (Note that $\frac{\pi}{4}\approx 0.785$.) Let $y = (\tan x)^{\tan 2x}$, so $\ln y = \tan 2x\ln(\tan x)$. Then by l'Hospital's Rule,

$$\lim_{x\to\pi/4}\ln y = \lim_{x\to\pi/4}\frac{\ln(\tan x)}{\cot 2x} = \lim_{x\to\pi/4}\frac{\sec^2 x/\tan x}{-2\csc^2 2x} = \frac{2/1}{-2(1)} = -1, \text{ so}$$

$$\lim_{x\to\pi/4}(\tan x)^{\tan 2x} = \lim_{x\to\pi/4} e^{\ln y} = e^{-1} = 1/e \approx 0.3679.$$

69.

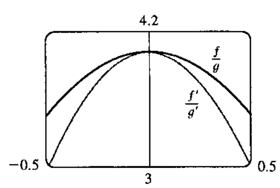


From the graph, it appears that

$$\lim_{x\to0}\frac{f(x)}{g(x)} = \lim_{x\to0}\frac{f'(x)}{g'(x)} = 0.25. \text{ We calculate}$$

$$\lim_{x\to0}\frac{f(x)}{g(x)} = \lim_{x\to0}\frac{e^x-1}{x^3+4x} \overset{H}{=} \lim_{x\to0}\frac{e^x}{3x^2+4} = \frac{1}{4}.$$

70.



From the graph, it appears that $\lim\limits_{x\to0}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to0}\dfrac{f'(x)}{g'(x)} = 4$.

We calculate

$$\lim_{x\to0}\frac{f(x)}{g(x)} = \lim_{x\to0}\frac{2x\sin x}{\sec x-1} \overset{H}{=} \lim_{x\to0}\frac{2(x\cos x+\sin x)}{\sec x\tan x}$$

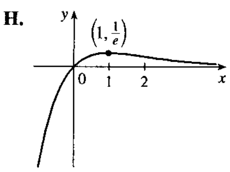
$$\overset{H}{=} \lim_{x\to0}\frac{2(-x\sin x+\cos x+\cos x)}{\sec x(\sec^2 x)+\tan x(\sec x\tan x)} = \frac{4}{1} = 4$$

71. $y = f(x) = xe^{-x}$ **A.** $D = \mathbb{R}$ **B.** Intercepts are 0 **C.** No symmetry

D. $\lim\limits_{x\to\infty} xe^{-x} = \lim\limits_{x\to\infty} \dfrac{x}{e^x} \overset{H}{=} \lim\limits_{x\to\infty} \dfrac{1}{e^x} = 0$, so $y = 0$ is a HA. $\lim\limits_{x\to-\infty} xe^{-x} = -\infty$

E. $f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x) > 0 \iff x < 1$, so f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. **F.** Absolute maximum $f(1) = 1/e$ **G.** $f''(x) = e^{-x}(x-2) > 0 \iff x > 2$, so f is CU on $(2, \infty)$ and CD on $(-\infty, 2)$. IP is $(2, 2/e^2)$.

H.



72. $f(x) = (\ln x)/x$ **A.** $D = (0, \infty)$ **B.** x-intercept= 1 **C.** No symmetry **D.** $\lim\limits_{x\to\infty} \dfrac{\ln x}{x} \overset{H}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{1} = 0$, so $y = 0$ is a horizontal asymptote. Also $\lim\limits_{x\to0^+} \dfrac{\ln x}{x} = -\infty$ since $\ln x \to -\infty$ and $x \to 0^+$, so $x = 0$ is a vertical asymptote. **E.** $f'(x) = \dfrac{1 - \ln x}{x^2} = 0$ when $\ln x = 1 \iff x = e$. $f'(x) > 0 \iff 1 - \ln x > 0 \iff \ln x < 1 \iff 0 < x < e$. $f'(x) < 0 \iff x > e$. So f is increasing on $(0, e)$ and decreasing on (e, ∞).

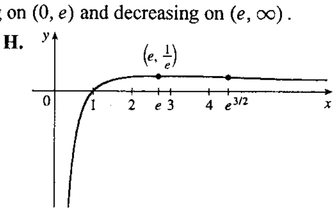
F. Thus $f(e) = 1/e$ is a local (and absolute) maximum.

G. $f''(x) = \dfrac{(-1/x)x^2 - (1 - \ln x)(2x)}{x^4} = \dfrac{2\ln x - 3}{x^3}$, so

$f''(x) > 0 \iff 2\ln x - 3 > 0 \iff \ln x > \frac{3}{2} \iff x > e^{3/2}$.

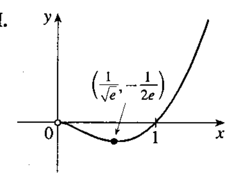
$f''(x) < 0 \iff 0 < x < e^{3/2}$. So f is CU on $(e^{3/2}, \infty)$ and CD on $(0, e^{3/2})$. Inflection point: $\left(e^{3/2}, \frac{3}{2}e^{-3/2}\right)$

H.



73. $y = f(x) = x^2 \ln x$ **A.** $D = (0, \infty)$ **B.** x-intercept when $\ln x = 0 \iff x = 1$, no y-intercept **C.** No symmetry **D.** $\lim\limits_{x\to\infty} x^2 \ln x = \infty$, $\lim\limits_{x\to0^+} x^2 \ln x = \lim\limits_{x\to0^+} \dfrac{\ln x}{1/x^2} \overset{H}{=} \lim\limits_{x\to0^+} \dfrac{1/x}{-2/x^3} = \lim\limits_{x\to0^+}\left(-\dfrac{x^2}{2}\right) = 0$, no asymptote **E.** $f'(x) = 2x\ln x + x = x(2\ln x + 1) > 0 \iff \ln x > -\frac{1}{2} \iff x > e^{-1/2}$, so f is increasing on $(1/\sqrt{e}, \infty)$, decreasing on $(0, 1/\sqrt{e})$. **F.** $f(1/\sqrt{e}) = -1/(2e)$ is an absolute minimum. **G.** $f''(x) = 2\ln x + 3 > 0 \iff \ln x > -\frac{3}{2} \iff x > e^{-3/2}$, so f is CU on $(e^{-3/2}, \infty)$ and CD on $(0, e^{-3/2})$. IP is $(e^{-3/2}, -3/(2e^3))$

H.

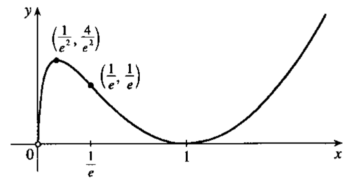


74. $y = f(x) = x(\ln x)^2$ **A.** $D = (0, \infty)$ **B.** x-intercept$= 1$, no y-intercept **C.** No symmetry **D.** $\lim\limits_{x\to\infty} x(\ln x)^2 = \infty$,

$\lim\limits_{x\to0^+} x(\ln x)^2 = \lim\limits_{x\to0^+} \dfrac{(\ln x)^2}{1/x} \overset{\text{H}}{=} \lim\limits_{x\to0^+} \dfrac{2(\ln x)(1/x)}{-1/x^2} = \lim\limits_{x\to0^+} \dfrac{2\ln x}{-1/x} \overset{\text{H}}{=} \lim\limits_{x\to0^+} \dfrac{2/x}{1/x^2} = \lim\limits_{x\to0^+} 2x = 0$, no asymptote **E.** $f'(x) = (\ln x)^2 + 2\ln x = (\ln x)(\ln x + 2) = 0$ when $\ln x = 0 \iff x = 1$ and when $\ln x = -2 \iff x = e^{-2}$. $f'(x) > 0$ when $0 < x < e^{-2}$ and when $x > 1$, so f is increasing on $(0, e^{-2})$ and $(1, \infty)$ and decreasing on $(e^{-2}, 1)$.

F. $f(e^{-2}) = 4e^{-2}$ is a local maximum, $f(1) = 0$ is a local minimum. **G.** $f''(x) = 2(\ln x)(1/x) + 2/x = (2/x)(\ln x + 1) = 0$ when $\ln x = -1 \iff x = e^{-1}$. $f''(x) > 0 \iff x > 1/e$, so f is CU on $(1/e, \infty)$, CD on $(0, 1/e)$. IP $(1/e, 1/e)$

H.

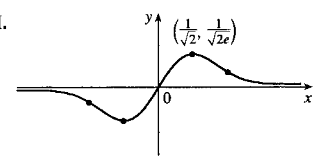


75. $y = f(x) = xe^{-x^2}$ **A.** $D = \mathbb{R}$ **B.** Intercepts are 0 **C.** $f(-x) = -f(x)$, so the curve is symmetric about the origin. **D.** $\lim\limits_{x\to\pm\infty} xe^{-x^2} = \lim\limits_{x\to\pm\infty} \dfrac{x}{e^{x^2}} \overset{\text{H}}{=} \lim\limits_{x\to\pm\infty} \dfrac{1}{2xe^{x^2}} = 0$, so $y = 0$ is a HA.

E. $f'(x) = e^{-x^2} - 2x^2e^{-x^2} = e^{-x^2}(1 - 2x^2) > 0 \iff x^2 < \frac{1}{2} \iff |x| < \frac{1}{\sqrt{2}}$, so f is increasing on $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and decreasing on $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \infty\right)$. **F.** $f\left(\frac{1}{\sqrt{2}}\right) = 1/\sqrt{2e}$ is a local maximum, $f\left(-\frac{1}{\sqrt{2}}\right) = -1/\sqrt{2e}$ is a local minimum.

G. $f''(x) = -2xe^{-x^2}(1 - 2x^2) - 4xe^{-x^2} = 2xe^{-x^2}(2x^2 - 3) > 0 \iff x > \sqrt{\frac{3}{2}}$ or $-\sqrt{\frac{3}{2}} < x < 0$, so f is CU on $\left(\sqrt{\frac{3}{2}}, \infty\right)$ and $\left(-\sqrt{\frac{3}{2}}, 0\right)$ and CD on $\left(-\infty, -\sqrt{\frac{3}{2}}\right)$ and $\left(0, \sqrt{\frac{3}{2}}\right)$. IP are $(0, 0)$ and $\left(\pm\sqrt{\frac{3}{2}}, \pm\sqrt{\frac{3}{2}}e^{-3/2}\right)$.

H.

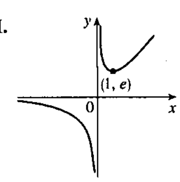


76. $y = f(x) = e^x/x$ **A.** $D = \{x \mid x \neq 0\}$ **B.** No intercepts **C.** No symmetry **D.** $\lim\limits_{x\to\infty} \dfrac{e^x}{x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{e^x}{1} = \infty$, $\lim\limits_{x\to-\infty} \dfrac{e^x}{x} = 0$, so $y = 0$ is a HA. $\lim\limits_{x\to0^+} \dfrac{e^x}{x} = \infty$, $\lim\limits_{x\to0^-} \dfrac{e^x}{x} = -\infty$, so $x = 0$ is a VA.

E. $f'(x) = \dfrac{xe^x - e^x}{x^2} > 0 \iff (x - 1)e^x > 0 \iff x > 1$, so f is increasing on $(1, \infty)$, and decreasing on $(-\infty, 0)$ and $(0, 1)$. **F.** $f(1) = e$ is a local minimum.

G. $f''(x) = \dfrac{x^2(xe^x) - 2x(xe^x - e^x)}{x^4} = \dfrac{e^x(x^2 - 2x + 2)}{x^3} > 0 \iff x > 0$ since $x^2 - 2x + 2 > 0$ for all x. So f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. No IP.

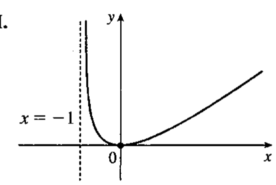
H.



77. $y = f(x) = x - \ln(1+x)$ **A.** $D = \{x \mid x > -1\} = (-1, \infty)$ **B.** Intercepts are 0 **C.** No symmetry

D. $\lim_{x \to -1^+} [x - \ln(1+x)] = \infty$, so $x = -1$ is a VA. $\lim_{x \to \infty} [x - \ln(1+x)] = \lim_{x \to \infty} x \left[1 - \frac{\ln(1+x)}{x}\right] = \infty$,

since $\lim_{x \to \infty} \frac{\ln(1+x)}{x} \overset{\text{H}}{=} \lim_{x \to \infty} \frac{1/(1+x)}{1} = 0$. **H.**



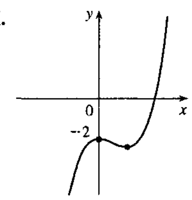
E. $f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0 \Leftrightarrow x > 0$ since $x + 1 > 0$. So f is increasing on $(0, \infty)$ and decreasing on $(-1, 0)$. **F.** $f(0) = 0$ is an absolute minimum. **G.** $f''(x) = 1/(1+x)^2 > 0$, so f is CU on $(-1, \infty)$.

78. $y = f(x) = e^x - 3e^{-x} - 4x$ **A.** $D = \mathbb{R}$ **B.** y-intercept$= -2$ **C.** No symmetry

D. $\lim_{x \to \infty} \left(e^x - 3e^{-x} - 4x\right) = \lim_{x \to \infty} x \left(\frac{e^x}{x} - 3\frac{e^{-x}}{x} - 4\right) = \infty$,

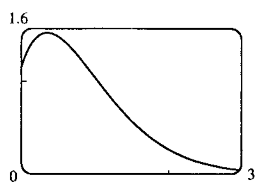
since $\lim_{x \to \infty} \frac{e^x}{x} \overset{\text{H}}{=} \lim_{x \to \infty} \frac{e^x}{1} = \infty$. Similarly, $\lim_{x \to -\infty} \left(e^x - 3e^{-x} - 4x\right) = -\infty$.

E. $f'(x) = e^x + 3e^{-x} - 4 = e^{-x}\left(e^{2x} - 4e^x + 3\right) = e^{-x}(e^x - 3)(e^x - 1) > 0$ $\Leftrightarrow$ $e^x > 3$ or $e^x < 1$ $\Leftrightarrow$ $x > \ln 3$ or $x < 0$. So f is increasing on $(-\infty, 0)$ and $(\ln 3, \infty)$ and decreasing on $(0, \ln 3)$. **F.** $f(0) = -2$ is a local maximum and $f(\ln 3) = 2 - 4\ln 3$ is a local minimum. **H.**



G. $f''(x) = e^x - 3e^{-x} = e^{-x}\left(e^{2x} - 3\right) > 0 \Leftrightarrow e^{2x} > 3 \Leftrightarrow$ $x > \frac{1}{2}\ln 3$, so f is CU on $\left(\frac{1}{2}\ln 3, \infty\right)$ and CD on $\left(-\infty, \frac{1}{2}\ln 3\right)$. IP at $x = \frac{1}{2}\ln 3$.

79. (a)

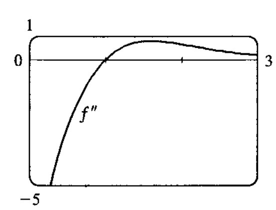


(b) $y = f(x) = x^{-x}$. We note that $\ln f(x) = \ln x^{-x} = -x \ln x = -\frac{\ln x}{1/x}$, so $\lim_{x \to 0^+} \ln f(x) \overset{\text{H}}{=} \lim_{x \to 0^+} -\frac{1/x}{-x^{-2}} = \lim_{x \to 0^+} x = 0$. Thus $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{\ln f(x)} = e^0 = 1$.

(c) From the graph, it appears that there is a local and absolute maximum of about $f(0.37) \approx 1.44$. To find the exact value, we differentiate: $f(x) = x^{-x} = e^{-x \ln x}$ $\Rightarrow$ $f'(x) = e^{-x \ln x}\left[-x\left(\frac{1}{x}\right) + \ln x\,(-1)\right] = -x^{-x}(1 + \ln x)$. This is 0 only when $1 + \ln x = 0$ $\Leftrightarrow$ $x = e^{-1}$. Also $f'(x)$ changes from positive to negative at e^{-1}. So the maximum value is $f(1/e) = (1/e)^{-1/e} = e^{1/e}$.

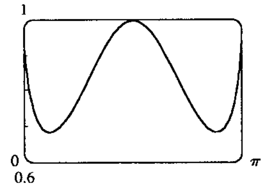
(d) We differentiate again to get

$$f''(x) = -x^{-x}(1/x) + (1 + \ln x)^2\left(x^{-x}\right) = x^{-x}\left[(1 + \ln x)^2 - 1/x\right]$$



From the graph of $f''(x)$, it seems that $f''(x)$ changes from negative to positive at $x = 1$, so we estimate that f has an IP at $x = 1$.

80. (a)



Note that the function is only defined for $\sin x > 0$, and is periodic with period 2π, so we consider it only on $[0, \pi]$.

(b) $\ln f(x) = \ln(\sin x)^{\sin x} = \sin x \ln(\sin x) = \dfrac{\ln(\sin x)}{\csc x}$, so

$\lim\limits_{x\to 0^+} \ln f(x) \overset{\text{H}}{=} \lim\limits_{x\to 0^+} \dfrac{\cot x}{-\csc x \cot x} = \lim\limits_{x\to 0^+} -\sin x = 0$. Thus

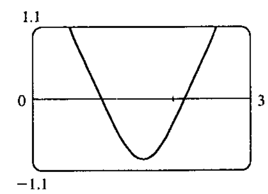
$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} e^{\ln f(x)} = e^0 = 1$.

(c) From the graph, it seems that there are local minima at about $f(0.38) = f(2.76) \approx 0.69$, and a local maximum of about $f(1.57) = 1$. To find the exact values, we differentiate: $f(x) = (\sin x)^{\sin x} = e^{\sin x \ln(\sin x)}$

$\Rightarrow \quad f'(x) = e^{\sin x \ln(\sin x)}\left[\sin x\left(\dfrac{1}{\sin x}\right)\cos x + \ln(\sin x)\cos x\right] = \cos x\,[1 + \ln(\sin x)]\,(\sin x)^{\sin x}$. This is 0 when $\cos x = 0 \quad \Leftrightarrow \quad x = \frac{\pi}{2}$ and when $1 + \ln(\sin x) = 0 \quad \Leftrightarrow \quad \ln(\sin x) = -1 \quad \Leftrightarrow \quad \sin x = 1/e$. This occurs at $x = \sin^{-1} 1/e$ and at $x = \pi - \sin^{-1} 1/e$ [since $\sin x = \sin(\pi - x)$]. So the local maximum is $f\left(\frac{\pi}{2}\right) = \left(\sin\frac{\pi}{2}\right)^{\sin(\pi/2)} = 1^1 = 1$, and the local minima are $f\left(\sin^{-1}(1/e)\right) = f\left(\pi - \sin^{-1}(1/e)\right) = (1/e)^{1/e} = e^{-1/e}$.

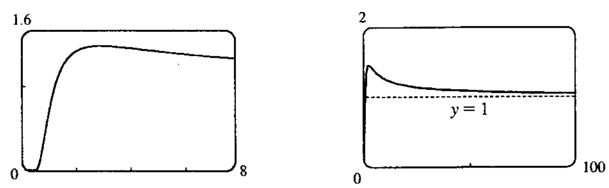
(d) We differentiate again to get

$$\begin{aligned} f''(x) &= [\cos x\,(1 + \ln \sin x)]^2\,(\sin x)^{\sin x} \\ &\quad + \cos x\,(\csc x \cos x)\,(\sin x)^{\sin x} \\ &\quad - \sin x[1 + \ln(\sin x)]\,(\sin x)^{\sin x} \end{aligned}$$



From the graph of $f''(x)$, it seems that $f''(x)$ changes sign at $x \approx 0.94$ and at $x \approx 2.20$, and so f has inflection points at approximately those x-values.

81. (a)



(b) $\ln f(x) = \ln x^{1/x} = \dfrac{1}{x}\ln x$, so $\lim\limits_{x\to\infty} \ln f(x) \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{1} = 0$. Therefore $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} e^{\ln f(x)} = 1$.

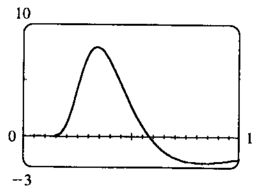
Also $\lim\limits_{x\to 0^+} \ln f(x) = \lim\limits_{x\to 0^+}\left(\dfrac{1}{x}\ln x\right) = -\infty$. So $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} e^{\ln f(x)} = 0$.

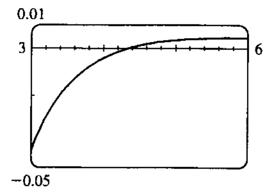
(c) From the graph, it appears that f has a local maximum at about $f(2.7) \approx 1.44$. To find the exact value, we differentiate: $f(x) = x^{1/x} = e^{(1/x)\ln x} \quad \Rightarrow$

$f'(x) = e^{(1/x)\ln x}\left[\dfrac{1}{x}\left(\dfrac{1}{x}\right) + \ln x\left(-x^{-2}\right)\right] = (1 - \ln x)\,x^{1/x}x^{-2}$. This is 0 only when $1 - \ln x = 0$

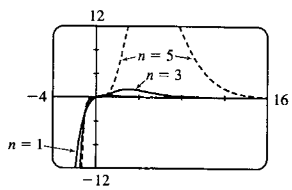
$\Rightarrow \quad x = e$, and $f'(x)$ changes from positive to negative there. So the local maximum value is $f(e) = e^{1/e}$.

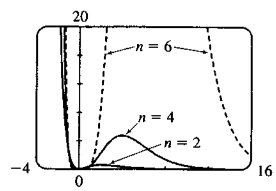
(d) We differentiate again to get $f''(x) = (1 - \ln x)\, x^{1/x}\left(-2x^{-3}\right) + (1 - \ln x)^2 x^{1/x} x^{-4} + (-1/x)\, x^{1/x} x^{-2}$.
From the graphs it appears that $f''(x)$ changes sign at $x \approx 0.58$ and at $x \approx 4.4$, so f has inflection points there.





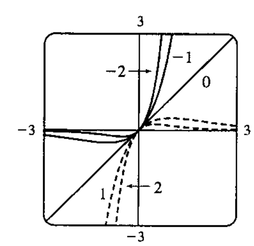
82.





The first figure shows representative examples of $f(x) = x^n e^{-x}$ with n odd. n is even in the second figure. All curves pass through the origin and approach $y = 0$ as $x \to \infty$. $f'(x) = \dfrac{x^n (n - x)}{xe^x} = 0 \iff x = n$ or $x = 0$ (the latter for $n > 1$). At $x = 0$, we have a local minimum for n even. At $x = n$, we have a local maximum for all n. As n increases, $(n, f(n))$ gets farther away from the origin. $f''(x) = \dfrac{x^n\left(x^2 - 2nx + n^2 - n\right)}{x^2 e^x} = 0 \iff$ $x = n \pm \sqrt{n}$ or $x = 0$ (the latter for $n > 2$). As n increases, the IP move farther away from the origin — they are symmetric about the line $x = n$.

83. If $c < 0$, then $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{x}{e^{cx}} \overset{\text{H}}{=} \lim\limits_{x\to-\infty} \dfrac{1}{ce^{cx}} = 0$, and $\lim\limits_{x\to\infty} f(x) = \infty$. If $c > 0$, then $\lim\limits_{x\to-\infty} f(x) = -\infty$, and $\lim\limits_{x\to\infty} f(x) \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1}{ce^{cx}} = 0$. If $c = 0$, then $f(x) = x$, so $\lim\limits_{x\to\pm\infty} f(x) = \pm\infty$ respectively. So we see that $c = 0$ is a transitional value. We now exclude the case $c = 0$, since we know how the function behaves in that case. To find the maxima and minima of f, we differentiate: $f(x) = xe^{-cx} \Rightarrow$ $f'(x) = x\left(-ce^{-cx}\right) + e^{-cx} = (1 - cx)\, e^{-cx}$. This is 0 when $1 - cx = 0 \iff x = 1/c$. If $c < 0$ then this represents a minimum of $f(1/c) = 1/(ce)$, since $f'(x)$ changes from negative to positive at $x = 1/c$; and if $c > 0$, it represents a maximum. As $|c|$ increases, the maximum or minimum gets closer to the origin. To find the inflection points, we differentiate again: $f'(x) = e^{-cx}(1 - cx) \Rightarrow$ $f''(x) = e^{-cx}(-c) + (1 - cx)\left(-ce^{-cx}\right) = (cx - 2)\, ce^{-cx}$. This changes sign when $cx - 2 = 0 \iff x = 2/c$. So as $|c|$ increases, the points of inflection get closer to the origin.



84. We see that both numerator and denominator approach 0, so we can use l'Hospital's Rule:

$$\lim_{x\to a}\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{aax}}{a-\sqrt[4]{ax^3}}\overset{\text{H}}{=}\lim_{x\to a}\frac{\frac{1}{2}\left(2a^3x-x^4\right)^{-1/2}\left(2a^3-4x^3\right)-a\left(\frac{1}{3}\right)(aax)^{-2/3}a^2}{-\frac{1}{4}\left(ax^3\right)^{-3/4}\left(3ax^2\right)}$$

$$=\frac{\frac{1}{2}\left(2a^3a-a^4\right)^{-1/2}\left(2a^3-4a^3\right)-\frac{1}{3}a^3\left(a^2a\right)^{-2/3}}{-\frac{1}{4}\left(aa^3\right)^{-3/4}\left(3aa^2\right)}$$

$$=\frac{\left(a^4\right)^{-1/2}\left(-a^3\right)-\frac{1}{3}a^3\left(a^3\right)^{-2/3}}{-\frac{3}{4}a^3\left(a^4\right)^{-3/4}}=\frac{-a-\frac{1}{3}a}{-\frac{3}{4}}=\frac{4}{3}\left(\frac{4}{3}a\right)=\frac{16}{9}a$$

85. First we will find $\lim\limits_{n\to\infty}\left(1+\dfrac{i}{n}\right)^{nt}$, which is of the form 1^{∞}. $y=\left(1+\dfrac{i}{n}\right)^{nt}$ $\Rightarrow$ $\ln y=nt\ln\left(1+\dfrac{i}{n}\right)$, so

$$\lim_{n\to\infty}\ln y=\lim_{n\to\infty}nt\ln\left(1+\frac{i}{n}\right)=t\lim_{n\to\infty}\frac{\ln(1+i/n)}{1/n}\overset{\text{H}}{=}t\lim_{n\to\infty}\frac{\left(-i/n^2\right)}{(1+i/n)\left(-1/n^2\right)}=t\lim_{n\to\infty}\frac{i}{1+i/n}=ti\quad\Rightarrow$$

$\lim\limits_{n\to\infty}y=e^{it}$. Thus, as $n\to\infty$, $A=A_0\left(1+\dfrac{i}{n}\right)^{nt}\to A_0e^{it}$.

86. (a) $\lim\limits_{t\to\infty}v=\lim\limits_{t\to\infty}\dfrac{mg}{c}\left(1-e^{-ct/m}\right)=\dfrac{mg}{c}\lim\limits_{t\to\infty}\left(1-e^{-ct/m}\right)$

$=\dfrac{mg}{c}(1-0)$ [because $-ct/m\to-\infty$ as $t\to\infty$] $=\dfrac{mg}{c}$

which is the speed the object approaches as time goes on, the so-called limiting velocity.

(b) $\lim\limits_{m\to\infty}v=\lim\limits_{m\to\infty}\dfrac{mg}{c}\left(1-e^{-ct/m}\right)=\dfrac{g}{c}\lim\limits_{m\to\infty}\dfrac{1-e^{-ct/m}}{1/m}\overset{\text{H}}{=}\dfrac{g}{c}\lim\limits_{m\to\infty}\dfrac{-e^{-ct/m}\left(ct/m^2\right)}{-1/m^2}$

$=\dfrac{g}{c}\lim\limits_{m\to\infty}cte^{-ct/m}=\dfrac{g}{c}(ct)=gt$ [because $-ct/m\to 0$ as $m\to\infty$]

The speed of a very heavy falling object is approximately proportional to the elapsed time — it doesn't depend on the mass.

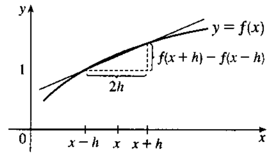
87. Both numerator and denominator approach 0 as $x\to 0$, so we use l'Hospital's Rule (and FTC1):

$$\lim_{x\to 0}\frac{S(x)}{x^3}=\lim_{x\to 0}\frac{\int_0^x\sin\left(\pi t^2/2\right)dt}{x^3}\overset{\text{H}}{=}\lim_{x\to 0}\frac{\sin\left(\pi x^2/2\right)}{3x^2}\overset{\text{H}}{=}\lim_{x\to 0}\frac{\pi x\cos\left(\pi x^2/2\right)}{6x}=\frac{\pi}{6}\cdot\cos 0=\frac{\pi}{6}$$

88. Both numerator and denominator approach 0 as $a\to 0$, so we use l'Hospital's Rule. (Note that we are differentiating with respect to a, since that is the quantity which is changing.) We also use the Fundamental Theorem of Calculus, Part 1: $\lim\limits_{a\to 0}T(x,t)=\lim\limits_{a\to 0}\dfrac{C\int_0^a e^{-(x-u)^2/(4kt)}\,du}{a\sqrt{4\pi kt}}\overset{\text{H}}{=}\lim\limits_{a\to 0}\dfrac{Ce^{-(x-a)^2/(4kt)}}{\sqrt{4\pi kt}}=\dfrac{Ce^{-x^2/(4kt)}}{\sqrt{4\pi kt}}$.

89. Since $\lim\limits_{h\to 0}[f(x+h)-f(x-h)]=f(x)-f(x)=0$ (f is differentiable and hence continuous) and $\lim\limits_{h\to 0}2h=0$, we use l'Hospital's Rule:

$$\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}\overset{\text{H}}{=}\lim_{h\to 0}\frac{f'(x+h)(1)-f'(x-h)(-1)}{2}=\frac{f'(x)+f'(x)}{2}=\frac{2f'(x)}{2}=f'(x)$$



$\dfrac{f(x+h)-f(x-h)}{2h}$ is the slope of the secant line between $(x-h, f(x-h))$ and $(x+h, f(x+h))$. As $h \to 0$, this line gets closer to the tangent line and its slope approaches $f'(x)$.

90. Since $\lim\limits_{h\to 0}[f(x+h)-2f(x)+f(x-h)] = f(x)-2f(x)+f(x) = 0$ (f is differentiable and hence continuous) and $\lim\limits_{h\to 0} h^2 = 0$, we can apply l'Hospital's Rule:

$$\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2} \overset{\text{H}}{=} \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h} = f''(x)$$

At the last step, we have applied the result of Exercise 89 to $f'(x)$.

91. $\lim\limits_{x\to\infty}\dfrac{e^x}{x^n} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{e^x}{nx^{n-1}} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{e^x}{n(n-1)x^{n-2}} \overset{\text{H}}{=} \cdots \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{e^x}{n!} = \infty$

92. $\lim\limits_{x\to\infty}\dfrac{\ln x}{x^p} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{1/x}{px^{p-1}} = \lim\limits_{x\to\infty}\dfrac{1}{px^p} = 0$ since $p > 0$.

93. $\lim\limits_{x\to 0^+} x^\alpha \ln x = \lim\limits_{x\to 0^+}\dfrac{\ln x}{x^{-\alpha}} \overset{\text{H}}{=} \lim\limits_{x\to 0^+}\dfrac{1/x}{-\alpha x^{-\alpha-1}} = \lim\limits_{x\to 0^+}\dfrac{x^\alpha}{-\alpha} = 0$ since $\alpha > 0$.

94. Using l'Hospital's Rule and FTC1, we have

$$\lim_{x\to 0}\frac{\int_0^x \sin(t^2)\,dt}{x^3} = \lim_{x\to 0}\frac{\frac{d}{dx}\int_0^x \sin(t^2)\,dt}{3x^2} = \lim_{x\to 0}\frac{\sin(x^2)}{3x^2} = \frac{1}{3}\lim_{x\to 0}\frac{\sin(x^2)}{x^2} = \frac{1}{3}$$

95. Let the radius of the circle be r. We see that $A(\theta)$ is the area of the whole figure (a sector of the circle with radius 1), minus the area of $\triangle OPR$. But the area of the sector of the circle is $\frac{1}{2}r^2\theta$ (see endpapers), and the area of the triangle is $\frac{1}{2}r\,|PQ| = \frac{1}{2}r\,(r\sin\theta) = \frac{1}{2}r^2\sin\theta$. So we have $A(\theta) = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta = \frac{1}{2}r^2(\theta - \sin\theta)$. Now by elementary trigonometry, $B(\theta) = \frac{1}{2}|QR|\,|PQ| = \frac{1}{2}(r - |OQ|)\,|PQ| = \frac{1}{2}r\,(1-\cos\theta)\,(r\sin\theta)$. So the limit we want is

$$\begin{aligned}
\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} &= \lim_{\theta\to 0^+}\frac{\frac{1}{2}r^2(\theta-\sin\theta)}{\frac{1}{2}r^2(1-\cos\theta)\sin\theta} \overset{\text{H}}{=} \lim_{\theta\to 0^+}\frac{1-\cos\theta}{(1-\cos\theta)\cos\theta + \sin\theta\,(\sin\theta)} \\
&= \lim_{\theta\to 0^+}\frac{1-\cos\theta}{\cos\theta - \cos^2\theta + \sin^2\theta} \overset{\text{H}}{=} \lim_{\theta\to 0^+}\frac{\sin\theta}{-\sin\theta + 4\sin\theta\,(\cos\theta)} \\
&= \frac{1}{-1+4\cos 0} = \frac{1}{3}
\end{aligned}$$

96. The area $A(t) = \int_0^t \sin(x^2)\,dx$, and the area $B(t) = \frac{1}{2}t\sin(t^2)$. Since $\lim\limits_{t\to 0^+} A(t) = 0 = \lim\limits_{t\to 0^+} B(t)$, we can use l'Hospital's Rule:

$$\begin{aligned}
\lim_{t\to 0^+}\frac{A(t)}{B(t)} &\overset{\text{H}}{=} \lim_{t\to 0^+}\frac{\sin(t^2)}{\frac{1}{2}\sin(t^2) + \frac{1}{2}t\left[2t\cos(t^2)\right]} \quad \text{(by FTC1)} \overset{\text{H}}{=} \lim_{t\to 0^+}\frac{2t\cos(t^2)}{t\cos(t^2) - 2t^3\sin(t^2) + 2t\cos(t^2)} \\
&= \lim_{t\to 0^+}\frac{2\cos(t^2)}{3\cos(t^2) - 2t^2\sin(t^2)} = \frac{2}{3-0} = \frac{2}{3}
\end{aligned}$$

97. (a) We show that $\lim\limits_{x\to 0} \dfrac{f(x)}{x^n} = 0$ for every integer $n \geq 0$. Let $y = \dfrac{1}{x^2}$. Then

$$\lim_{x\to 0} \frac{f(x)}{x^{2n}} = \lim_{x\to 0} \frac{e^{-1/x^2}}{(x^2)^n} = \lim_{y\to\infty} \frac{y^n}{e^y} \overset{\text{H}}{=} \lim_{y\to\infty} \frac{ny^{n-1}}{e^y} \overset{\text{H}}{=} \cdots \overset{\text{H}}{=} \lim_{y\to\infty} \frac{n!}{e^y} = 0 \;\Rightarrow$$

$$\lim_{x\to 0} \frac{f(x)}{x^n} = \lim_{x\to 0} x^n \frac{f(x)}{x^{2n}} = \lim_{x\to 0} x^n \lim_{x\to 0} \frac{f(x)}{x^{2n}} = 0. \text{ Thus, } f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x} = 0.$$

(b) Using the Chain Rule and the Quotient Rule we see that $f^{(n)}(x)$ exists for $x \neq 0$. In fact, we prove by induction that for each $n \geq 0$, there is a polynomial p_n and a non-negative integer k_n with $f^{(n)}(x) = p_n(x) f(x)/x^{k_n}$ for $x \neq 0$. This is true for $n = 0$; suppose it is true for the nth derivative. Then

$$\begin{aligned} f^{(n+1)}(x) &= \left[x^{k_n}\left[p_n'(x) f(x) + p_n(x) f'(x)\right] - k_n x^{k_n-1} p_n(x) f(x)\right] x^{-2k_n} \\ &= \left[x^{k_n} p_n'(x) + p_n(x)\left(2/x^3\right) - k_n x^{k_n-1} p_n(x)\right] f(x)\, x^{-2k_n} \\ &= \left[x^{k_n+3} p_n'(x) + 2p_n(x) - k_n x^{k_n+2} p_n(x)\right] f(x)\, x^{-(2k_n+3)} \end{aligned}$$

which has the desired form.

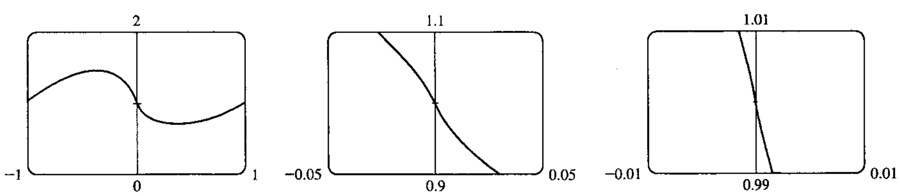
Now we show by induction that $f^{(n)}(0) = 0$ for all n. By part (a), $f'(0) = 0$. Suppose that $f^{(n)}(0) = 0$. Then

$$\begin{aligned} f^{(n+1)}(0) &= \lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \lim_{x\to 0} \frac{f^{(n)}(x)}{x} = \lim_{x\to 0} \frac{p_n(x) f(x)/x^{k_n}}{x} = \lim_{x\to 0} \frac{p_n(x) f(x)}{x^{k_n+1}} \\ &= \lim_{x\to 0} p_n(x) \lim_{x\to 0} \frac{f(x)}{x^{k_n+1}} = p_n(0)\cdot 0 = 0 \end{aligned}$$

98. (a) For f to be continuous, we need $\lim\limits_{x\to 0} f(x) = f(0) = 1$. We note that for $x \neq 0$, $\ln f(x) = \ln|x|^x = x \ln|x|$.

So $\lim\limits_{x\to 0} \ln f(x) = \lim\limits_{x\to 0} \dfrac{\ln|x|}{1/x} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{1/x}{-1/x^2} = 0$. Therefore, $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} e^{\ln f(x)} = e^0 = 1$. So f is continuous at 0.

(b) From the graphs, it seems that $f(x)$ is differentiable at 0.



(c) To find f', we use logarithmic differentiation: $\ln f(x) = x\ln|x| \;\Rightarrow\; \dfrac{f'(x)}{f(x)} = x\left(\dfrac{1}{x}\right) + \ln|x| \;\Rightarrow$

$f'(x) = f(x)(1 + \ln|x|) = |x|^x(1 + \ln|x|)$, $x \neq 0$. Now $f'(x) \to -\infty$ as $x \to 0$ [since $|x|^x \to 1$ and $(1 + \ln|x|) \to -\infty$], so the curve has a vertical tangent at $(0, 1)$ and is therefore not differentiable there. The fact cannot be seen in the graphs in part (b) because $\ln|x| \to -\infty$ very slowly as $x \to 0$.

7 Review

CONCEPT CHECK

1. (a) See Definition 1 in Section 7.1. It must pass the Horizontal Line Test.

(b) See Definition 2 in Section 7.1. The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

(c) $g'(a) = (f^{-1})'(a) = \dfrac{1}{f'(g(a))}$

2. (a) The function $f(x) = e^x$ has domain $\mathbb{R}$ and range $(0, \infty)$.

(b) The function $f(x) = \ln x$ has domain $(0, \infty)$ and range $\mathbb{R}$.

(c) The graphs are reflections of one another about the line $y = x$. See Figure 7.3.3 or Figure 7.3*.1.

(d) $\log_a x = \dfrac{\ln x}{\ln a}$

3. (a) See Definition 7.5.1. Domain $= [-1, 1]$, Range $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(b) See Definition 7.5.7. Domain $= \mathbb{R}$, Range $= \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

4. $\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$, $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

5. (a) $y = e^x \Rightarrow y' = e^x$

(b) $y = a^x \Rightarrow y' = a^x \ln a$

(c) $y = \ln x \Rightarrow y' = 1/x$

(d) $y = \log_a x \Rightarrow y' = 1/(x \ln a)$

(e) $y = \sin^{-1} x \Rightarrow y' = 1/\sqrt{1 - x^2}$

(f) $y = \cos^{-1} x \Rightarrow y' = -1/\sqrt{1 - x^2}$

(g) $y = \tan^{-1} x \Rightarrow y' = 1/\left(1 + x^2\right)$

(h) $y = \sinh x \Rightarrow y' = \cosh x$

(i) $y = \cosh x \Rightarrow y' = \sinh x$

(j) $y = \tanh x \Rightarrow y' = \operatorname{sech}^2 x$

(k) $y = \sinh^{-1} x \Rightarrow y' = 1/\sqrt{1 + x^2}$

(l) $y = \cosh^{-1} x \Rightarrow y' = 1/\sqrt{x^2 - 1}$

(m) $y = \tanh^{-1} x \Rightarrow y' = 1/\left(1 - x^2\right)$

6. (a) e is the number such that $\lim\limits_{h \to 0} \dfrac{e^h - 1}{h} = 1$.

(b) $e = \lim\limits_{x \to 0} (1 + x)^{1/x}$

(c) The differentiation formula for $y = a^x$ [$y' = a^x \ln a$] is simplest when $a = e$ because $\ln e = 1$.

(d) The differentiation formula for $y = \log_a x$ [$y' = 1/(x \ln a)$] is simplest when $a = e$ because $\ln e = 1$.

7. (a) See page 486.

(b) Write fg as $\dfrac{f}{1/g}$ or $\dfrac{g}{1/f}$.

(c) Convert the difference into a quotient using a common denominator, rationalizing, factoring, or some other method.

(d) Convert the power to a product by taking the natural logarithm of both sides of $y = f^g$ or by writing f^g as $e^{g \ln f}$.

TRUE-FALSE QUIZ

1. False. For example, $\cos \frac{\pi}{2} = \cos\left(-\frac{\pi}{2}\right)$, so $\cos x$ is not 1-1.

2. False, since the range of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so $\tan^{-1}(-1) = -\frac{\pi}{4}$.

3. True, since $\ln x$ is an increasing function on $(0, \infty)$.

4. True, by Equation 7.4*.1.

5. True. We can divide by e^x since $e^x \neq 0$ for every x.

6. False. For example, $\ln(1+1) = \ln 2$, but $\ln 1 + \ln 1 = 0$. In fact $\ln a + \ln b = \ln(ab)$.

7. False. Let $x = e$. Then $(\ln x)^6 = (\ln e)^6 = 1^6 = 1$, but $6 \ln x = 6 \ln e = 6 \cdot 1 = 6 \neq 1 = (\ln x)^6$.

8. False. $\dfrac{d}{dx} 10^x = 10^x \ln 10$

9. False. $\ln 10$ is a constant, so its derivative is 0.

10. True. $y = e^{3x} \Rightarrow \ln y = 3x \Rightarrow x = \frac{1}{3}\ln y \Rightarrow$ the inverse function is $y = \frac{1}{3}\ln x$.

11. False. The "-1" is not an exponent; it is an indication of an inverse function.

12. False. For example, $\tan^{-1} 20$ is defined; $\sin^{-1} 20$ and $\cos^{-1} 20$ are not.

13. True. See Figure 2 in Section 7.6.

14. True. $\ln \frac{1}{10} = -\ln 10 = -\int_1^{10} (1/x)\,dx$, by Equation 7.4.4 or by Definition 7.2*.1.

15. True. $\int_2^{16} (1/x)\,dx = \ln x]_2^{16} = \ln 16 - \ln 2 = \ln \frac{16}{2} = \ln 8 = \ln 2^3 = 3 \ln 2$

16. False. L'Hospital's Rule does not apply since $\lim\limits_{x \to \pi^-} \dfrac{\tan x}{1 - \cos x} = \dfrac{0}{2} = 0$.

EXERCISES

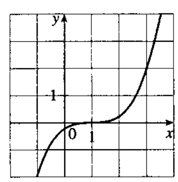
1. No. f is not 1-1 because the graph of f fails the Horizontal Line Test.

2. (a) g is one-to-one because it passes the Horizontal Line Test.

(b) When $y = 2$, $x \approx 0.2$. So $g^{-1}(2) \approx 0.2$.

(c) The range of g is $[-1, 3.5]$, which is the same as the domain of g^{-1}.

(d) We reflect the graph of g through the line $y = x$ to obtain the graph of g^{-1}.

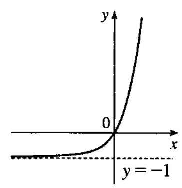


3. (a) $f^{-1}(3) = 7$ since $f(7) = 3$.

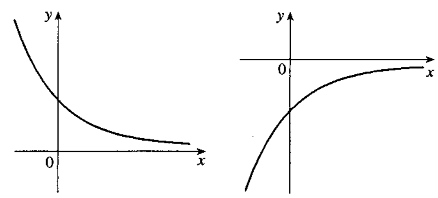
(b) $(f^{-1})'(3) = \dfrac{1}{f'(f^{-1}(3))} = \dfrac{1}{f'(7)} = \dfrac{1}{8}$

4. $y = \dfrac{x+1}{2x+1}$. Interchanging x and y gives us $x = \dfrac{y+1}{2y+1} \Rightarrow 2xy + x = y + 1 \Rightarrow 2xy - y = 1 - x \Rightarrow$ $y(2x - 1) = 1 - x \Rightarrow y = \dfrac{1-x}{2x-1} = f^{-1}(x)$.

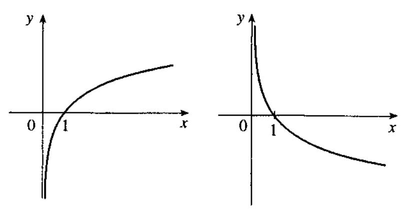
5. $y = 5^x - 1$



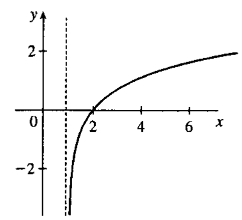
6. $y = e^{-x}$ $y = -e^{-x}$



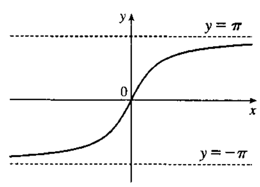
7. $y = \ln x$ $y = -\ln x$



8. $y = \ln(x - 1)$



9. $y = 2\arctan x$



10. We have seen that if $a > 1$, then $a^x > x^a$ for sufficiently large x. (See Exercise 7.2.18.) In general, we could show that $\lim\limits_{x\to\infty} (a^x/x^a) = \infty$ by using l'Hospital's Rule repeatedly. Also, $\log_a x$ increases much more slowly than either x^a or a^x. [Compare the graph of $\log_a x$ with those of x^a and a^x, or use l'Hospital's Rule to show that $\lim\limits_{x\to\infty} \left[(\log_a x)/x^a\right] = 0$.] So for large x, $\log_a x < x^a < a^x$.

11. (a) $e^{2\ln 3} = \left(e^{\ln 3}\right)^2 = 3^2 = 9$

(b) $\log_{10} 25 + \log_{10} 4 = \log_{10}(25 \cdot 4) = \log_{10} 100 = \log_{10} 10^2 = 2$

12. (a) $\ln e^{\pi} = \pi$

(b) $\tan\left(\arcsin \frac{1}{2}\right) = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$

13. $\ln x = \frac{1}{3} \Rightarrow x = e^{1/3}$

14. $e^x = \frac{1}{3} \Rightarrow x = \ln \frac{1}{3} = \ln 1 - \ln 3 = -\ln 3$

15. $e^{e^x} = 17 \Rightarrow \ln e^{e^x} = \ln 17 \Rightarrow e^x = \ln 17 \Rightarrow \ln e^x = \ln(\ln 17) \Rightarrow x = \ln\ln 17$

16. $\ln\left(1 + e^{-x}\right) = 3 \Rightarrow 1 + e^{-x} = e^3 \Rightarrow e^{-x} = e^3 - 1 \Rightarrow \ln e^{-x} = \ln\left(e^3 - 1\right) \Rightarrow -x = \ln\left(e^3 - 1\right)$
$\Rightarrow x = -\ln\left(e^3 - 1\right)$

17. $\log_{10}(e^x) = 1 \Rightarrow e^x = 10 \Rightarrow x = \ln(e^x) = \ln 10$
Or: $1 = \log_{10}(e^x) = x \log_{10} e \Rightarrow x = 1/\log_{10} e = \ln 10$

18. $1 = \ln(x+1) - \ln(x) = \ln\left(\frac{x+1}{x}\right) \Rightarrow \frac{x+1}{x} = e \Rightarrow ex = x + 1 \Rightarrow x = \frac{1}{e-1}$

19. $\tan^{-1} x = 1 \quad\Rightarrow\quad \tan\tan^{-1} x = \tan 1 \quad\Rightarrow\quad x = \tan 1 \ (\approx 1.5574)$

20. $\sin x = 0.3 \quad\Rightarrow\quad x = \sin^{-1} 0.3 = \alpha$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. The reference angle for α is $\pi - \alpha$, so all solutions are $x = \alpha + 2n\pi$ and $x = \pi - \alpha + 2n\pi$ [or $(2n+1)\pi - \alpha$]

21. $f(t) = t^2 \ln t \quad\Rightarrow\quad f'(t) = t^2 \cdot \frac{1}{t} + (\ln t)(2t) = t + 2t\ln t$ or $t(1 + 2\ln t)$

22. $g(t) = \dfrac{e^t}{1+e^t} \quad\Rightarrow\quad g'(t) = \dfrac{(1+e^t)\,e^t - e^t\,(e^t)}{(1+e^t)^2} = \dfrac{e^t}{(1+e^t)^2}$

23. $h(\theta) = e^{\tan 2\theta} \quad\Rightarrow\quad h'(\theta) = e^{\tan 2\theta} \cdot \sec^2 2\theta \cdot 2 = 2\sec^2 2\theta\, e^{\tan 2\theta}$

24. $h(u) = 10^{\sqrt{u}} \quad\Rightarrow\quad h'(u) = 10^{\sqrt{u}} \cdot \ln 10 \cdot \dfrac{1}{2\sqrt{u}} = \dfrac{(\ln 10)\,10^{\sqrt{u}}}{2\sqrt{u}}$

25. $y = \dfrac{\sqrt{x+1}\,(2-x)^5}{(x+3)^7} \quad\Rightarrow\quad \ln|y| = \frac{1}{2}\ln(x+1) + 5\ln|2-x| - 7\ln(x+3) \quad\Rightarrow$

$\dfrac{y'}{y} = \dfrac{1}{2(x+1)} + \dfrac{-5}{2-x} - \dfrac{7}{x+3} \quad\Rightarrow\quad y' = \dfrac{\sqrt{x+1}\,(2-x)^5}{(x+3)^7}\left[\dfrac{1}{2(x+1)} - \dfrac{5}{2-x} - \dfrac{7}{x+3}\right]$

26. $y = \ln(\csc 5x) \quad\Rightarrow\quad y' = \dfrac{-5\csc 5x \cot 5x}{\csc 5x} = -5\cot 5x$

27. $y = e^{cx}(c\sin x - \cos x) \quad\Rightarrow\quad y' = ce^{cx}(c\sin x - \cos x) + e^{cx}(c\cos x + \sin x) = \left(c^2+1\right)e^{cx}\sin x$

28. $y = \sin^{-1}(e^x) \quad\Rightarrow\quad y' = e^x/\sqrt{1-e^{2x}}$

29. $y = \ln\left(\sec^2 x\right) = 2\ln|\sec x| \quad\Rightarrow\quad y' = (2/\sec x)(\sec x\tan x) = 2\tan x$

30. $y = \ln\left(x^2 e^x\right) = 2\ln|x| + x \quad\Rightarrow\quad y' = 2/x + 1$

31. $y = xe^{-1/x} \quad\Rightarrow\quad y' = e^{-1/x} + xe^{-1/x}\left(1/x^2\right) = e^{-1/x}(1 + 1/x)$

32. $y = \ln|\csc 3x + \cot 3x| \quad\Rightarrow\quad y' = \dfrac{-3\csc 3x\cot 3x - 3\csc^2 3x}{\csc 3x + \cot 3x} = -3\csc 3x$

33. $y = x^{\cos x} = e^{\cos x \ln x} \quad\Rightarrow\quad y' = e^{\cos x\ln x}\left[\cos x \cdot \dfrac{1}{x} + (\ln x)(-\sin x)\right] = x^{\cos x}\left(\dfrac{\cos x}{x} - \sin x\ln x\right)$

34. $y = x^r e^{sx} \quad\Rightarrow\quad y' = rx^{r-1}e^{sx} + sx^r e^{sx}$

35. $y = 2^{-t^2} \quad\Rightarrow\quad y' = 2^{-t^2}(\ln 2)(-2t) = (-2\ln 2)\,t\,2^{-t^2}$

36. $y = e^{\cos x} + \cos(e^x) \quad\Rightarrow\quad y' = -\sin x e^{\cos x} - e^x\sin(e^x)$

37. $H(v) = v\tan^{-1} v \quad\Rightarrow\quad H'(v) = v \cdot \dfrac{1}{1+v^2} + \tan^{-1} v \cdot 1 = \dfrac{v}{1+v^2} + \tan^{-1} v$

38. $F(z) = \log_{10}\left(1+z^2\right) \quad\Rightarrow\quad F'(z) = \dfrac{1}{(\ln 10)\left(1+z^2\right)} \cdot 2z = \dfrac{2z}{(\ln 10)\left(1+z^2\right)}$

39. $y = \ln\dfrac{1}{x} + \dfrac{1}{\ln x} = -\ln x + (\ln x)^{-1} \quad\Rightarrow\quad y' = -\dfrac{1}{x} - \dfrac{1}{x(\ln x)^2}$

40. $xe^y = y - 1 \quad\Rightarrow\quad e^y + xe^y y' = y' \quad\Rightarrow\quad y' = e^y/(1 - xe^y)$

41. $y = \ln(\cosh 3x) \quad\Rightarrow\quad y' = (1/\cosh 3x)(\sinh 3x)(3) = 3\tanh 3x$

42. $y = \ln\left|x^2 - 4\right| - \ln|2x+5| \quad\Rightarrow\quad y' = \dfrac{2x}{x^2-4} - \dfrac{2}{2x+5}$

43. $y = \cosh^{-1}(\sinh x) \quad\Rightarrow\quad y' = (\cosh x)/\sqrt{\sinh^2 x - 1}$

44. $y = x\tanh^{-1}\sqrt{x} \;\Rightarrow\; y' = \tanh^{-1}\sqrt{x} + x\dfrac{1}{1-(\sqrt{x})^2}\dfrac{1}{2\sqrt{x}} = \tanh^{-1}\sqrt{x} + \dfrac{\sqrt{x}}{2(1-x)}$

45. $f(x) = e^{g(x)} \;\Rightarrow\; f'(x) = e^{g(x)}g'(x)$

46. $f(x) = g(e^x) \;\Rightarrow\; f'(x) = g'(e^x)\,e^x$

47. $f(x) = \ln|g(x)| \;\Rightarrow\; f'(x) = \dfrac{1}{g(x)}g'(x) = \dfrac{g'(x)}{g(x)}$

48. $f(x) = g(\ln x) \;\Rightarrow\; f'(x) = g'(\ln x)\cdot\dfrac{1}{x} = \dfrac{g'(\ln x)}{x}$

49. $f(x) = 2^x \;\Rightarrow\; f'(x) = 2^x\ln 2 \;\Rightarrow\; f''(x) = 2^x(\ln 2)^2 \;\Rightarrow\; \cdots \;\Rightarrow\; f^{(n)}(x) = 2^x(\ln 2)^n$

50. $f(x) = \ln(2x) = \ln 2 + \ln x \;\Rightarrow\; f'(x) = x^{-1},\ f''(x) = -x^{-2},\ f'''(x) = 2x^{-3},\ f^{(4)}(x) = -2\cdot 3x^{-4}, \ldots,$
$f^{(n)}(x) = (-1)^{n-1}(n-1)!\,x^{-n}$

51. We first show it is true for $n = 1$: $f'(x) = e^x + xe^x = (x+1)e^x$. We now assume it is true for $n = k$: $f^{(k)}(x) = (x+k)e^x$. With this assumption, we must show it is true for $n = k+1$:
$f^{(k+1)}(x) = \dfrac{d}{dx}\left[f^{(k)}(x)\right] = \dfrac{d}{dx}\left[(x+k)e^x\right] = e^x + (x+k)e^x = [x+(k+1)]e^x$.
Therefore, $f^{(n)}(x) = (x+n)e^x$ by mathematical induction.

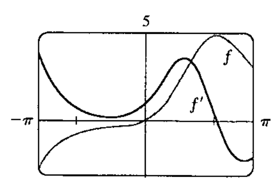
52. Using implicit differentiation, $y = x + \arctan y \;\Rightarrow\; y' = 1 + \dfrac{1}{1+y^2}y' \;\Rightarrow\; y'\left(1 - \dfrac{1}{1+y^2}\right) = 1 \;\Rightarrow$
$y'\left(\dfrac{y^2}{1+y^2}\right) = 1 \;\Rightarrow\; y' = \dfrac{1+y^2}{y^2} = \dfrac{1}{y^2} + 1.$

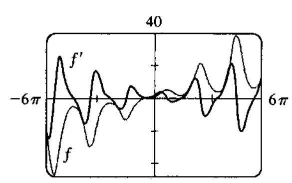
53. $y = f(x) = \ln\left(e^x + e^{2x}\right) \;\Rightarrow\; f'(x) = \dfrac{e^x + 2e^{2x}}{e^x + e^{2x}} \;\Rightarrow\; f'(0) = \frac{3}{2}$, so the tangent line at $(0, \ln 2)$ is
$y - \ln 2 = \frac{3}{2}x$ or $3x - 2y + \ln 4 = 0$.

54. $y = f(x) = x\ln x \;\Rightarrow\; f'(x) = \ln x + 1$, so the slope of the tangent at (e, e) is $f'(e) = 2$ and an equation is $y - e = 2(x - e)$ or $y = 2x - e$.

55. $y = [\ln(x+4)]^2 \;\Rightarrow\; y' = 2\dfrac{\ln(x+4)}{x+4} = 0 \;\Leftrightarrow\; \ln(x+4) = 0 \;\Leftrightarrow\; x + 4 = 1 \;\Leftrightarrow\; x = -3$, so the tangent is horizontal at $(-3, 0)$.

56. $f(x) = xe^{\sin x} \;\Rightarrow\; f'(x) = x\left[e^{\sin x}(\cos x)\right] + e^{\sin x}(1) = e^{\sin x}(x\cos x + 1)$. As a check on our work, we notice from the graphs that $f'(x) > 0$ when f is increasing. Also, we see in the larger viewing rectangle a certain similarity in the graphs of f and f': the sizes of the oscillations of f and f' are linked.





57. (a) The line $x - 4y = 1$ has slope $\frac{1}{4}$. The tangent to $y = e^x$ has slope $\frac{1}{4}$ when $y' = e^x = \frac{1}{4} \Rightarrow$ $x = \ln\frac{1}{4} = -\ln 4$, so an equation is $y - \frac{1}{4} = \frac{1}{4}(x + \ln 4)$ or $y = \frac{1}{4}x + \frac{1}{4}(\ln 4 + 1)$.

(b) The slope of the tangent at the point (a, e^a) is $\left.\frac{d}{dx}e^x\right]_{x=a} = e^a$. An equation of the tangent line is thus $y - e^a = e^a(x - a)$. We substitute $x = 0$, $y = 0$ into this equation, since we want the line to pass through the origin: $0 - e^a = e^a(0 - a) \Leftrightarrow -e^a = e^a(-a) \Leftrightarrow a = 1$. So an equation of the tangent is $y - e = e(x - 1)$, or $y = ex$.

58. (a) $\lim\limits_{t\to\infty} C(t) = \lim\limits_{t\to\infty}\left[K\left(e^{-at} - e^{-bt}\right)\right] = K\lim\limits_{t\to\infty}\left(e^{-at} - e^{-bt}\right) = K(0 - 0) = 0$ because $-at \to -\infty$ and $-bt \to -\infty$ as $t \to \infty$.

(b) $C'(t) = K\left(-ae^{-at} + be^{-bt}\right)$

(c) $C'(t) = 0 \Rightarrow be^{-bt} = ae^{-at} \Rightarrow \frac{b}{a} = e^{(-a+b)t} \Rightarrow \ln\frac{b}{a} = (b - a)t \Rightarrow t = \frac{\ln(b/a)}{b - a}$

59. If $y = -3x$, then as $x \to \infty$, $y \to -\infty$. $\lim\limits_{x\to\infty} e^{-3x} = \lim\limits_{y\to-\infty} e^y = 0$ by (7.2.11).

60. $\lim\limits_{x\to 10^-} \ln\left(100 - x^2\right) = -\infty$ since as $x \to 10^-$, $\left(100 - x^2\right) \to 0^+$.

61. Let $t = 2/(x - 3)$. As $x \to 3^-$, $t \to -\infty$. $\lim\limits_{x\to 3^-} e^{2/(x-3)} = \lim\limits_{t\to-\infty} e^t = 0$

62. If $y = x^3 - x = x\left(x^2 - 1\right)$, then as $x \to \infty$, $y \to \infty$. $\lim\limits_{x\to\infty}\arctan\left(x^3 - x\right) = \lim\limits_{y\to\infty}\arctan y = \frac{\pi}{2}$ by (7.5.8).

63. Let $t = \sinh x$. As $x \to 0^+$, $t \to 0^+$. $\lim\limits_{x\to 0^+}\ln(\sinh x) = \lim\limits_{t\to 0^+}\ln t = -\infty$

64. $-1 \le \sin x \le 1 \Rightarrow -e^{-x} \le e^{-x}\sin x \le e^{-x}$. Now $\lim\limits_{x\to\infty}\left(\pm e^{-x}\right) = 0$, so by the Squeeze Theorem, $\lim\limits_{x\to\infty} e^{-x}\sin x = 0$.

65. $\lim\limits_{x\to\infty}\frac{(1 + 2^x)/2^x}{(1 - 2^x)/2^x} = \lim\limits_{x\to\infty}\frac{1/2^x + 1}{1/2^x - 1} = \frac{0 + 1}{0 - 1} = -1$

66. Let $t = x/4$, so $x = 4t$. As $x \to \infty$, $t \to \infty$. $\lim\limits_{x\to\infty}\left(1 + \frac{4}{x}\right)^x = \lim\limits_{t\to\infty}\left(1 + \frac{1}{t}\right)^{4t} = \left[\lim\limits_{t\to\infty}\left(1 + \frac{1}{t}\right)^t\right]^4 = e^4$

67. $\lim\limits_{x\to\pi}\frac{\sin x}{x^2 - \pi^2} \overset{\text{H}}{=} \lim\limits_{x\to\pi}\frac{\cos x}{2x} = -\frac{1}{2\pi}$

68. $\lim\limits_{x\to 0}\frac{e^{ax} - e^{bx}}{x} \overset{\text{H}}{=} \lim\limits_{x\to 0}\frac{ae^{ax} - be^{bx}}{1} = a - b$

69. $\lim\limits_{x\to\infty}\frac{\ln(\ln x)}{\ln x} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\frac{1/(x\ln x)}{1/x} = \lim\limits_{x\to\infty}\frac{1}{\ln x} = 0$

70. $\lim\limits_{x\to 0}\frac{1 + \sin x - \cos x}{1 - \sin x - \cos x} \overset{\text{H}}{=} \lim\limits_{x\to 0}\frac{\cos x + \sin x}{-\cos x + \sin x} = \frac{1 + 0}{-1 + 0} = -1$

71. $\lim\limits_{x\to 0}\frac{\ln(1 - x) + x + \frac{1}{2}x^2}{x^3} \overset{\text{H}}{=} \lim\limits_{x\to 0}\frac{-\frac{1}{1 - x} + 1 + x}{3x^2} \overset{\text{H}}{=} \lim\limits_{x\to 0}\frac{-\frac{1}{(1 - x)^2} + 1}{6x} \overset{\text{H}}{=} \lim\limits_{x\to 0}\frac{-\frac{2}{(1 - x)^3}}{6} = -\frac{2}{6} = -\frac{1}{3}$

72. $\lim\limits_{x\to\pi/2}\left(\frac{\pi}{2} - x\right)\tan x = \lim\limits_{x\to\pi/2}\frac{\pi/2 - x}{\cot x} \overset{\text{H}}{=} \lim\limits_{x\to\pi/2}\frac{-1}{-\csc^2 x} = \lim\limits_{x\to\pi/2}\sin^2 x = 1$

73. $\lim\limits_{x\to0^+}\sin x\,(\ln x)^2 = \lim\limits_{x\to0^+}\dfrac{(\ln x)^2}{\csc x} \overset{\text{H}}{=} \lim\limits_{x\to0^+}\dfrac{2\ln x/x}{-\csc x\cot x} = -2\lim\limits_{x\to0}\dfrac{\sin x}{x}\lim\limits_{x\to0}\dfrac{\ln x}{\cot x} = -2\lim\limits_{x\to0}\dfrac{\ln x}{\cot x}$

$\overset{\text{H}}{=} -2\lim\limits_{x\to0}\dfrac{1/x}{-\csc^2 x} = 2\lim\limits_{x\to0}\dfrac{\sin^2 x}{x} = 2\lim\limits_{x\to0}\dfrac{\sin x}{x}\lim\limits_{x\to0}\sin x = 2\cdot1\cdot0 = 0$

74. $\lim\limits_{x\to0}\left(\csc^2 x - x^{-2}\right) = \lim\limits_{x\to0}\left[\dfrac{1}{\sin^2 x} - \dfrac{1}{x^2}\right] = \lim\limits_{x\to0}\dfrac{x^2-\sin^2 x}{x^2\sin^2 x} \overset{\text{H}}{=} \lim\limits_{x\to0}\dfrac{2x-\sin 2x}{2x\sin^2 x + x^2\sin 2x}$

$\overset{\text{H}}{=} \lim\limits_{x\to0}\dfrac{2-2\cos 2x}{2\sin^2 x + 4x\sin 2x + 2x^2\cos 2x} \overset{\text{H}}{=} \lim\limits_{x\to0}\dfrac{4\sin 2x}{6\sin 2x + 12x\cos 2x - 4x^2\sin 2x}$

$\overset{\text{H}}{=} \lim\limits_{x\to0}\dfrac{8\cos 2x}{24\cos 2x - 32x\sin 2x - 8x^2\cos 2x} = \dfrac{8}{24} = \dfrac{1}{3}$

75. Let $y = x^{\tan x}$ so $\ln y = \tan x\ln x$. Then

$$\lim_{x\to0^+}\ln y = \lim_{x\to0^+}(\tan x\ln x) = \lim_{x\to0^+}\frac{\ln x}{\cot x} \overset{\text{H}}{=} \lim_{x\to0^+}\frac{1/x}{-\csc^2 x} = \lim_{x\to0^+}\frac{\sin x}{x}(-\sin x)$$

$$= \lim_{x\to0^+}\frac{\sin x}{x}\cdot\lim_{x\to0^+}(-\sin x) = 1\cdot0 = 0$$

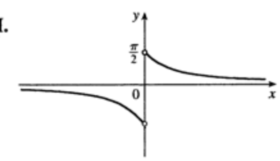
so $\lim\limits_{x\to0^+} y = e^0 = 1$.

76. Let $y = x^{1/(1-x)}$. Then $\ln y = \dfrac{\ln x}{1-x}$, so $\lim\limits_{x\to1}\ln y = \lim\limits_{x\to1}\dfrac{\ln x}{1-x} \overset{\text{H}}{=} \lim\limits_{x\to1}\dfrac{1/x}{-1} = -1 \;\Rightarrow\; \lim\limits_{x\to1}x^{1/(1-x)} = e^{-1}$.

77. $y = f(x) = \tan^{-1}(1/x)$ **A.** $D = \{x \mid x \neq 0\}$ **B.** No intercept **C.** $f(-x) = -f(x)$, so the curve is symmetric about the origin. **D.** $\lim\limits_{x\to\pm\infty}\tan^{-1}(1/x) = \tan^{-1}0 = 0$, so $y=0$ is a HA. $\lim\limits_{x\to0^+}\tan^{-1}(1/x) = \frac{\pi}{2}$ and $\lim\limits_{x\to0^-}\tan^{-1}(1/x) = -\frac{\pi}{2}$ since $\dfrac{1}{x}\to\pm\infty$ as $x\to0^\pm$.

E. $f'(x) = \dfrac{1}{1+(1/x)^2}\left(-1/x^2\right) = \dfrac{-1}{x^2+1} \;\Rightarrow\; f'(x) < 0$, so f is decreasing on $(-\infty,0)$ and $(0,\infty)$. **F.** No extremum **G.** $f''(x) = \dfrac{2x}{\left(x^2+1\right)^2} > 0 \;\Leftrightarrow\; x > 0$, so f is CU on $(0,\infty)$ and CD on $(-\infty,0)$.

H.



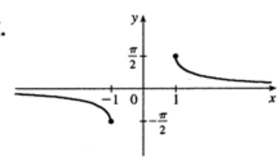
78. $y = f(x) = \sin^{-1}(1/x)$ **A.** $D = \{x \mid -1 \le 1/x \le 1\} = (-\infty,-1]\cup[1,\infty)$. **B.** No intercept **C.** $f(-x) = -f(x)$, symmetric about the origin **D.** $\lim\limits_{x\to\pm\infty}\sin^{-1}(1/x) = \sin^{-1}(0) = 0$, so $y=0$ is a HA.

E. $f'(x) = \dfrac{1}{\sqrt{1-(1/x)^2}}\left(-\dfrac{1}{x^2}\right) = \dfrac{-1}{\sqrt{x^4-x^2}} < 0$, so f is decreasing on $(-\infty,-1)$ and $(1,\infty)$.

F. No local extremum, but $f(1) = \frac{\pi}{2}$ is the absolute maximum and $f(-1) = -\frac{\pi}{2}$ is the absolute minimum.

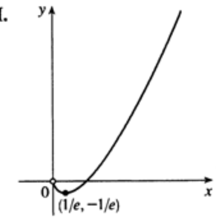
G. $f''(x) = \dfrac{4x^3-2x}{2\left(x^4-x^2\right)^{3/2}} = \dfrac{x\left(2x^2-1\right)}{\left(x^4-x^2\right)^{3/2}} > 0$ for $x > 1$ and $f''(x) < 0$ for $x < -1$, so f is CU on $(1,\infty)$ and CD on $(-\infty,-1)$. No IP

H.



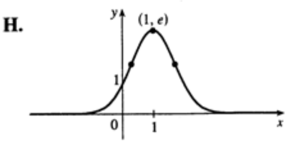
79. $y = f(x) = x \ln x$ **A.** $D = (0, \infty)$ **B.** No y-intercept; x-intercept 1. **C.** No symmetry **D.** No asymptote [Note that the graph approaches the point $(0, 0)$ as $x \to 0^+$.]
E. $f'(x) = x(1/x) + (\ln x)(1) = 1 + \ln x$, so $f'(x) \to -\infty$ as $x \to 0^+$ and $f'(x) \to \infty$ as $x \to \infty$. $f'(x) = 0 \iff \ln x = -1 \iff x = e^{-1} = 1/e$. $f'(x) > 0$ for $x > 1/e$, so f is decreasing on $(0, 1/e)$ and increasing on $(1/e, \infty)$. **F.** Local minimum: $f(1/e) = -1/e$. No local maximum. **G.** $f''(x) = 1/x$, so $f''(x) > 0$ for $x > 0$. The graph is CU on $(0, \infty)$ and there is no IP.

H.



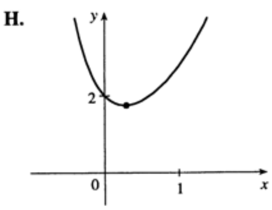
80. $y = f(x) = e^{2x-x^2}$ **A.** $D = \mathbb{R}$ **B.** y-intercept 1; no x-intercept **C.** No symmetry **D.** $\lim\limits_{x\to\pm\infty} e^{2x-x^2} = 0$, so $y = 0$ is a HA. **E.** $y = f(x) = e^{2x-x^2} \Rightarrow f'(x) = 2(1-x)e^{2x-x^2} > 0 \iff x < 1$, so f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. **F.** $f(1) = e$ is a local and absolute maximum.
G. $f''(x) = 2(2x^2 - 4x + 1)e^{2x-x^2} = 0 \iff x = 1 \pm \frac{\sqrt{2}}{2}$.
$f''(x) > 0 \iff x < 1 - \frac{\sqrt{2}}{2}$ or $x > 1 + \frac{\sqrt{2}}{2}$, so f is CU on $\left(-\infty, 1 - \frac{\sqrt{2}}{2}\right)$ and $\left(1 + \frac{\sqrt{2}}{2}, \infty\right)$, and CD on $\left(1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}\right)$.
IP $\left(1 \pm \frac{\sqrt{2}}{2}, \sqrt{e}\right)$

H.



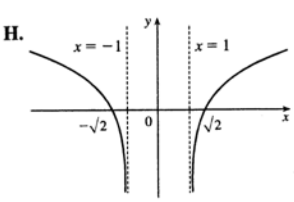
81. $y = f(x) = e^x + e^{-3x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept 2; no x-intercept **C.** No symmetry
D. $\lim\limits_{x\to\pm\infty} (e^x + e^{-3x}) = \infty$, no asymptote **E.** $y = f(x) = e^x + e^{-3x} \Rightarrow$
$f'(x) = e^x - 3e^{-3x} = e^{-3x}(e^{4x} - 3) > 0 \iff e^{4x} > 3 \iff$
$4x > \ln 3 \iff x > \frac{1}{4}\ln 3$, so f is increasing on $\left(\frac{1}{4}\ln 3, \infty\right)$ and decreasing on $\left(-\infty, \frac{1}{4}\ln 3\right)$. **F.** Absolute minimum $f\left(\frac{1}{4}\ln 3\right) = 3^{1/4} + 3^{-3/4} \approx 1.75$. **G.** $f''(x) = e^x + 9e^{-3x} > 0$, so f is CU on $(-\infty, \infty)$. No IP.

H.

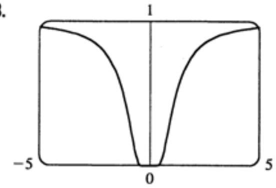


82. $y = f(x) = \ln(x^2 - 1)$ **A.** $D = (-\infty, -1) \cup (1, \infty)$ **B.** No y-intercept; x-intercepts $\pm\sqrt{2}$ **C.** Symmetric about the y-axis **D.** $\lim\limits_{x\to\pm\infty} \ln(x^2 - 1) = \infty$, $\lim\limits_{x\to 1^+} \ln(x^2 - 1) = -\infty$, $\lim\limits_{x\to -1^-} \ln(x^2 - 1) = -\infty$, so $x = 1$ and $x = -1$ are VA. **E.** $y = f(x) = \ln(x^2 - 1) \Rightarrow f'(x) = \dfrac{2x}{x^2 - 1} > 0$ for $x > 1$ and $f'(x) < 0$ for $x < -1$, so f is increasing on $(1, \infty)$ and decreasing on $(-\infty, -1)$. Note that the domain of f is $|x| > 1$. **F.** No extremum
G. $f''(x) = -2\dfrac{x^2 + 1}{(x^2 - 1)^2} < 0$, so f is CD on $(-\infty, -1)$ and $(1, \infty)$.
No IP

H.



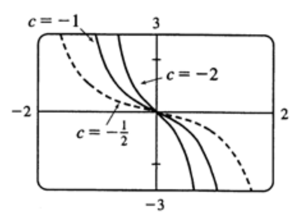
83.

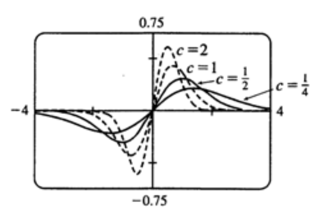


From the graph, we estimate the points of inflection to be about $(\pm 0.8, 0.2)$. $f(x) = e^{-1/x^2} \Rightarrow f'(x) = 2x^{-3}e^{-1/x^2} \Rightarrow f''(x) = 2\left[x^{-3}\left(2x^{-3}\right)e^{-1/x^2} + e^{-1/x^2}\left(-3x^{-4}\right)\right] = 2x^{-6}e^{-1/x^2}\left(2-3x^2\right)$. This is 0 when $2 - 3x^2 = 0 \Leftrightarrow x = \pm\sqrt{\frac{2}{3}}$, so the inflection points are $\left(\pm\sqrt{\frac{2}{3}}, e^{-3/2}\right)$.

84. We exclude the case $c = 0$, since in that case $f(x) = 0$ for all x. To find the maxima and minima, we differentiate: $f(x) = cxe^{-cx^2} \Rightarrow f'(x) = c\left[xe^{-cx^2}(-2cx) + e^{-cx^2}(1)\right] = ce^{-cx^2}\left(-2cx^2+1\right)$. This is 0 where $-2cx^2 + 1 = 0 \Leftrightarrow x = \pm 1 \big/ \sqrt{2c}$. So if $c > 0$, there are two maxima or minima, whose x-coordinates approach 0 as c increases. The negative root gives a minimum and the positive root gives a maximum, by the First Derivative Test. By substituting back into the equation, we see that $f\left(\pm 1 \big/ \sqrt{2c}\right) = c\left(\pm 1 \big/ \sqrt{2c}\right)e^{-c\left(\pm 1/\sqrt{2c}\right)^2} = \pm\sqrt{c/2e}$. So as c increases, the extreme points become more pronounced. Note that if $c > 0$, then $\lim\limits_{x\to\pm\infty} f(x) = 0$. If $c < 0$, then there are no extreme values, and $\lim\limits_{x\to\pm\infty} f(x) = \mp\infty$.

To find the points of inflection, we differentiate again: $f'(x) = ce^{-cx^2}\left(-2cx^2+1\right) \Rightarrow$ $f''(x) = c\left[e^{-cx^2}(-4cx) + \left(-2cx^2+1\right)\left(-2cxe^{-cx^2}\right)\right] = -2c^2xe^{-cx^2}\left(3-2cx^2\right)$. This is 0 at $x = 0$ and where $3 - 2cx^2 = 0 \Leftrightarrow x = \pm\sqrt{3/(2c)} \Rightarrow$ IP at $\left(\pm\sqrt{3/(2c)}, \pm\sqrt{3c/2}e^{-3/2}\right)$. If $c > 0$ there are three inflection points, and as c increases, the x-coordinates of the nonzero inflection points approach 0. If $c < 0$, there is only one inflection point, the origin.





85. $s(t) = Ae^{-ct}\cos(\omega t + \delta) \Rightarrow$

$$
\begin{aligned}
v(t) = s'(t) &= -cAe^{-ct}\cos(\omega t+\delta) + Ae^{-ct}\left[-\omega\sin(\omega t+\delta)\right] \\
&= -Ae^{-ct}\left[c\cos(\omega t+\delta) + \omega\sin(\omega t+\delta)\right] \Rightarrow \\
a(t) = v'(t) &= cAe^{-ct}\left[c\cos(\omega t+\delta) + \omega\sin(\omega t+\delta)\right] + \left(-Ae^{-ct}\right)\left[-\omega c\sin(\omega t+\delta) + \omega^2\cos(\omega t+\delta)\right] \\
&= Ae^{-ct}\left[\left(c^2-\omega^2\right)\cos(\omega t+\delta) + 2c\omega\sin(\omega t+\delta)\right]
\end{aligned}
$$

86. (a) Let $f(x) = \ln x + x - 3$. Then $f'(x) = 1/x + 1 > 0$ (for $x > 0$) and $f(2) \approx -0.307$ and $f(e) \approx 0.718$. f is differentiable on $(2, e)$, continuous on $[2, e]$ and $f(2) < 0$, $f(e) > 0$. Therefore, by the Intermediate Value Theorem there exists a number c in $(2, e)$ such that $f(c) = 0$. Thus, there is one root. But $f'(x) > 0$ for $x \in (2, e)$, so f is increasing on $(2, e)$, which means that there is exactly one root.

(b) We use Newton's Method with $f(x) = \ln x + x - 3$, $f'(x) = 1/x + 1$, and $x_1 = 2$.
$x_2 = x_1 - \dfrac{\ln x_1 + x_1 - 3}{1/x_1 + 1} = 2 - \dfrac{\ln 2 + 2 - 3}{1/2 + 1} \approx 2.20457$. Similarly, $x_3 \approx 2.20794$, $x_4 = 2.20794$. Thus, the root of the equation, correct to four decimal places, is 2.2079.

87. Let $P(t) = \dfrac{64}{1 + 31e^{-0.7944t}} = \dfrac{A}{1 + Be^{ct}} = A\left(1 + Be^{ct}\right)^{-1}$, where $A = 64$, $B = 31$, and $c = -0.7944$.

$$P'(t) = -A\left(1 + Be^{ct}\right)^{-2}\left(Bce^{ct}\right) = -ABce^{ct}\left(1 + Be^{ct}\right)^{-2}$$

$$P''(t) = -ABce^{ct}\left[-2\left(1 + Be^{ct}\right)^{-3}\left(Bce^{ct}\right)\right] + \left(1 + Be^{ct}\right)^{-2}\left(-ABc^2e^{ct}\right)$$

$$= -ABc^2e^{ct}\left(1 + Be^{ct}\right)^{-3}\left[-2Be^{ct} + \left(1 + Be^{ct}\right)\right] = -\frac{ABc^2e^{ct}\left(1 - Be^{ct}\right)}{\left(1 + Be^{ct}\right)^3}$$

The population is increasing most rapidly when its graph changes from CU to CD; that is, when $P''(t) = 0$ in this case. $P''(t) = 0 \quad\Rightarrow\quad Be^{ct} = 1 \quad\Rightarrow\quad e^{ct} = \dfrac{1}{B} \quad\Rightarrow$
$ct = \ln\dfrac{1}{B} \quad\Rightarrow\quad t = \dfrac{\ln(1/B)}{c} = \dfrac{\ln(1/31)}{-0.7944} \approx 4.32$ days. Note that
$P\left(\dfrac{1}{c}\ln\dfrac{1}{B}\right) = \dfrac{A}{1 + Be^{c(1/c)\ln(1/B)}} = \dfrac{A}{1 + Be^{\ln(1/B)}} = \dfrac{A}{1 + B(1/B)} = \dfrac{A}{1+1} = \dfrac{A}{2}$, one-half the limit of P as $t \to \infty$.

88. $\int_0^3 e^{-2t}\,dt = -\frac{1}{2}\left[e^{-2t}\right]_0^3 = -\frac{1}{2}\left(e^{-6} - e^0\right) = \frac{1}{2}\left(1 - e^{-6}\right)$

89. $\displaystyle\int_0^1 \frac{1}{x^2+1}\,dx = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}1 - \tan^{-1}0 = \tfrac{\pi}{4} - 0 = \tfrac{\pi}{4}$

90. $\displaystyle\int_2^5 \frac{dr}{1+2r} = \tfrac{1}{2}\left[\ln|1+2r|\right]_2^5 = \tfrac{1}{2}(\ln 11 - \ln 5) = \tfrac{1}{2}\ln\tfrac{11}{5}$

91. $\int_{\ln 2}^{\ln 8} e^s\,ds = \left[e^s\right]_{\ln 2}^{\ln 8} = e^{\ln 8} - e^{\ln 2} = 8 - 2 = 6$

92. Let $u = \sin x$. Then $du = \cos x\,dx$, so

$$\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx = \int_0^1 \frac{1}{1+u^2}\,du = \left[\tan^{-1}u\right]_0^1 = \tan^{-1}1 - \tan^{-1}0 = \tfrac{\pi}{4} - 0 = \tfrac{\pi}{4}$$

93. $\displaystyle\int_2^4 \frac{1+x-x^2}{x^2}\,dx = \int_2^4\left(x^{-2} + \frac{1}{x} - 1\right)dx = \left[-\frac{1}{x} + \ln x - x\right]_2^4$

$$= \left(-\tfrac{1}{4} + \ln 4 - 4\right) - \left(-\tfrac{1}{2} + \ln 2 - 2\right) = \ln 2 - \tfrac{7}{4}$$

94. Let $u = \ln x$. Then $du = \dfrac{dx}{x} \quad\Rightarrow\quad \displaystyle\int\frac{\cos(\ln x)}{x}\,dx = \int\cos u\,du = \sin u + C = \sin(\ln x) + C$.

95. Let $u = \sqrt{x}$. Then $du = \dfrac{dx}{2\sqrt{x}} \quad\Rightarrow\quad \displaystyle\int\frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C$.

96. Let $u = x^2$. Then $du = 2x\,dx \quad\Rightarrow\quad \displaystyle\int\frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int\frac{du}{\sqrt{1-u^2}} = \tfrac{1}{2}\sin^{-1}u + C = \tfrac{1}{2}\sin^{-1}\left(x^2\right) + C$.

97. Let $u = \ln(\cos x)$. Then $du = \dfrac{-\sin x}{\cos x}\,dx = -\tan x\,dx \Rightarrow$

$\int \tan x \ln(\cos x)\,dx = -\int u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}[\ln(\cos x)]^2 + C.$

98. Let $u = \ln(e^x + 1)$. Then $du = \left[e^x/(e^x+1)\right]dx \Rightarrow$

$\displaystyle\int \frac{e^x}{(e^x+1)\ln(e^x+1)}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln\ln(e^x+1) + C.$

99. Let $u = 1 + x^4$. Then $du = 4x^3\,dx \Rightarrow \displaystyle\int \frac{x^3}{1+x^4}\,dx = \frac{1}{4}\int \frac{1}{u}\,du = \tfrac{1}{4}\ln|u| + C = \frac{1}{4}\ln\left(1+x^4\right) + C.$

100. $\int \sinh au\,du = \dfrac{1}{a}\cosh au + C$

101. Let $u = 1 + \sec\theta$, so $du = \sec\theta\tan\theta\,d\theta \Rightarrow \displaystyle\int \frac{\sec\theta\tan\theta}{1+\sec\theta}\,d\theta = \int \frac{1}{u}\,du = \ln|u| + C = \ln|1+\sec\theta| + C.$

102. $1 + e^{2x} > e^{2x} \Rightarrow \sqrt{1+e^{2x}} > \sqrt{e^{2x}} = e^x \Rightarrow \int_0^1 \sqrt{1+e^{2x}}\,dx \geq \int_0^1 e^x\,dx = e^x\big]_0^1 = e - 1$

103. $\cos x \leq 1 \Rightarrow e^x \cos x \leq e^x \Rightarrow \int_0^1 e^x \cos x\,dx \leq \int_0^1 e^x\,dx = e^x\big]_0^1 = e - 1$

104. For $0 \leq x \leq 1$, $0 \leq \sin^{-1} x \leq \frac{\pi}{2}$, so $\int_0^1 x \sin^{-1} x\,dx \leq \int_0^1 x\left(\frac{\pi}{2}\right)dx = \frac{\pi}{4}x^2\big]_0^1 = \frac{\pi}{4}$.

105. $f'(x) = \dfrac{d}{dx}\displaystyle\int_1^{\sqrt{x}} \frac{e^s}{s}\,ds = \frac{e^{\sqrt{x}}}{\sqrt{x}}\frac{d}{dx}\sqrt{x} = \frac{e^{\sqrt{x}}}{\sqrt{x}}\frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2x}$

106. $f'(x) = \dfrac{d}{dx}\int_{\ln x}^{2x} e^{-t^2}\,dt = -\dfrac{d}{dx}\int_0^{\ln x} e^{-t^2}\,dt + \dfrac{d}{dx}\int_0^{2x} e^{-t^2}\,dt = -e^{-(\ln x)^2}\left(\dfrac{1}{x}\right) + e^{-(2x)^2}(2) = -\dfrac{e^{-(\ln x)^2}}{x} + 2e^{-4x^2}$

107. $f_{\text{ave}} = \dfrac{1}{4-1}\int_1^4 \dfrac{1}{x}\,dx = \frac{1}{3}\ln x\Big]_1^4 = \frac{1}{3}\ln 4$

108. $A = \int_{-2}^0 (e^{-x} - e^x)\,dx + \int_0^1 (e^x + e^{-x})\,dx = \left[-e^{-x} - e^x\right]_{-2}^0 + \left[e^x + e^{-x}\right]_0^1$

$= (-1-1) - \left(-e^2 - e^{-2}\right) + \left(e + e^{-1}\right) - (1+1) = e^2 + e + e^{-1} + e^{-2} - 4$

109. $V = \displaystyle\int_0^1 \frac{2\pi x}{1+x^4}\,dx$ by cylindrical shells. Let $u = x^2 \Rightarrow du = 2x\,dx$. Then

$V = \displaystyle\int_0^1 \frac{\pi}{1+u^2}\,du = \pi\left[\tan^{-1} u\right]_0^1 = \pi\left(\tan^{-1} 1 - \tan^{-1} 0\right) = \pi\left(\frac{\pi}{4}\right) = \frac{\pi^2}{4}.$

110. $f(x) = x + x^2 + e^x \Rightarrow f'(x) = 1 + 2x + e^x$ and $f(0) = 1 \Rightarrow g(1) = 0$, so

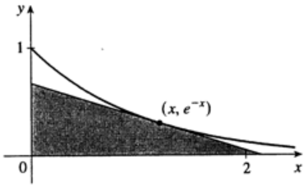
$g'(1) = \dfrac{1}{f'(g(1))} = \dfrac{1}{f'(0)} = \dfrac{1}{2}.$

111. $f(x) = \ln x + \tan^{-1} x \Rightarrow f(1) = \ln 1 + \tan^{-1} 1 = \frac{\pi}{4} \Rightarrow g\left(\frac{\pi}{4}\right) = 1.$

$f'(x) = \dfrac{1}{x} + \dfrac{1}{1+x^2}$, so $g'\left(\frac{\pi}{4}\right) = \dfrac{1}{f'(1)} = \dfrac{1}{3/2} = \dfrac{2}{3}.$

112.

The area of such a rectangle is just the product of its sides, that is, $A(x) = x \cdot e^{-x}$. We want to find the maximum of this function, so we differentiate: $A'(x) = x\left(-e^{-x}\right) + e^{-x}(1) = e^{-x}(1-x)$. This is 0 only at $x = 1$, and changes from positive to negative there, so by the First Derivative Test this gives a local maximum. So the largest area is $A(1) = 1/e$.

113. 

We find the equation of a tangent to the curve $y = e^{-x}$, so that we can find the x- and y-intercepts of this tangent, and then we can find the area of the triangle. The slope of the tangent at the point (a, e^{-a}) is given by $\left.\frac{d}{dx}e^{-x}\right]_{x=a} = -e^{-a}$, and so the equation of the tangent is $y - e^{-a} = -e^{-a}(x-a) \Leftrightarrow y = e^{-a}(a - x + 1)$. The y-intercept of this line is $y = e^{-a}(a - 0 + 1) = e^{-a}(a+1)$. To find the x-intercept we set $y = 0 \Rightarrow e^{-a}(a - x + 1) = 0 \Rightarrow x = a + 1$. So the area of the triangle is $A(a) = \frac{1}{2}\left[e^{-a}(a+1)\right](a+1) = \frac{1}{2}e^{-a}(a+1)^2$. We differentiate this with respect to a: $A'(a) = \frac{1}{2}\left[e^{-a}(2)(a+1) + (a+1)^2 e^{-a}(-1)\right] = \frac{1}{2}e^{-a}(1-a^2)$. This is 0 at $a = \pm 1$, and the root $a = 1$ gives a maximum, by the First Derivative Test. So the maximum area of the triangle is $A(1) = \frac{1}{2}e^{-1}(1+1)^2 = 2e^{-1} = 2/e$.

114. Using Formula 5.2.3 with $a = 0$ and $b = 1$, we have $\displaystyle\int_0^1 e^x\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} e^{i/n}$. This series is a geometric series with $a = r = e^{1/n}$, so $\displaystyle\sum_{i=1}^{n} e^{i/n} = e^{1/n}\frac{e^{n/n}-1}{e^{1/n}-1} = e^{1/n}\frac{e-1}{e^{1/n}-1} \Rightarrow$

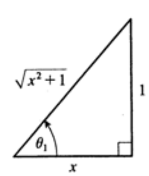
$\displaystyle\int_0^1 e^x\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} e^{i/n} = \lim_{n\to\infty}(e-1)\,e^{1/n}\frac{1/n}{e^{1/n}-1}$. As $n \to \infty$, $1/n \to 0^+$, so $e^{1/n} \to e^0 = 1$. Let $t = 1/n$. Then $e^{1/n} - 1 = e^t - 1 \to 0^+$, so l'Hospital's Rule gives $\displaystyle\lim_{t\to 0}\frac{t}{e^t - 1} = \lim_{t\to 0}\frac{1}{e^t} = 1$ and we have $\int_0^1 e^x\,dx = \left[\lim_{t\to 0^+}(e-1)\,e^t\right]\left[\lim_{t\to 0^+}\frac{t}{e^t-1}\right] = e - 1$.

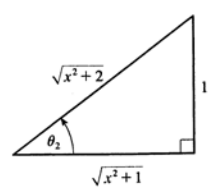
115. $\displaystyle\lim_{x\to -1} F(x) = \lim_{x\to -1}\frac{b^{x+1} - a^{x+1}}{x+1} \overset{\text{H}}{=} \lim_{x\to -1}\frac{b^{x+1}\ln b - a^{x+1}\ln a}{1} = \ln b - \ln a = F(-1)$, so F is continuous at -1.

116. Let $\theta_1 = \operatorname{arccot} x$, so $\cot\theta_1 = x = x/1$. So $\sin(\operatorname{arccot} x) = \sin\theta_1 = \dfrac{1}{\sqrt{x^2+1}}$.

Let $\theta_2 = \arctan\left[\dfrac{1}{\sqrt{x^2+1}}\right]$, so $\tan\theta_2 = \dfrac{1}{\sqrt{x^2+1}}$.

Hence, $\cos(\arctan(\sin(\operatorname{arccot} x))) = \cos\theta_2 = \dfrac{\sqrt{x^2+1}}{\sqrt{x^2+2}} = \sqrt{\dfrac{x^2+1}{x^2+2}}$.





117. Differentiating both sides of the given equation, using the Fundamental Theorem for each side, gives $f(x) = e^{2x} + 2xe^{2x} + e^{-x}f(x)$. So $f(x)\left(1 - e^{-x}\right) = e^{2x} + 2xe^{2x}$. Hence $f(x) = \dfrac{e^{2x}(1+2x)}{1-e^{-x}}$.

118. (a) Let $f(x) = x - \ln x - 1$, so $f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$. Since $x > 0$, $f'(x) < 0$ for $0 < x < 1$ and $f'(x) > 0$ for $x > 1$. So there is an absolute minimum at $x = 1$ with $f(1) = 0$.
So for $x > 0$, $x \neq 1$, $x - \ln x - 1 = f(x) > f(1) = 0$, and hence $\ln x < x - 1$.

(b) Here let $f(x) = \ln x - \frac{x-1}{x} = \ln x - 1 + \frac{1}{x}$. So $f'(x) = \frac{1}{x} - \frac{1}{x^2} = \frac{x-1}{x^2}$. As in (a), we see that there is an absolute minimum value at $x = 1$ and that $f(1) = 0$. So for $x > 0$, $x \neq 1$,
$\ln x - \frac{x-1}{x} = f(x) > f(1) = 0$ and hence $\frac{x-1}{x} < \ln x$.

(c) Let $b > a > 0$, so $b/a > 1$. Letting $x = b/a$ in the inequalities in (a) and (b) gives
$\frac{b-a}{b} = \frac{b/a - 1}{b/a} < \ln\frac{b}{a} < \frac{b}{a} - 1 = \frac{b-a}{a}$. Noting that $\ln\frac{b}{a} = \ln b - \ln a$, the result follows after dividing through by $b - a$.

(d) Let $f(x) = \ln x$. From the given diagram, we see that
(slope of tangent at $x = b$) $<$ (slope of secant line) $<$ (slope of tangent at $x = a$). Since $f'(x) = \frac{1}{x}$, we therefore have $\frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$. To make this geometric argument more rigorous, we could use the Mean Value Theorem: For any a and b with $0 < a < b$, there exists some $c \in (a, b)$ for which
$f'(c) = \frac{1}{c} = \frac{\ln b - \ln a}{b - a}$. But $\frac{1}{x}$ is a decreasing function on $(0, \infty)$, so $\frac{1}{b} < \frac{1}{c} = \frac{\ln b - \ln a}{b - a} < \ln\frac{1}{a}$.

(e) Since $\frac{1}{b} < \frac{1}{x} < \frac{1}{a}$ for $a < x < b$, Property 8 says that $\frac{1}{b}(b - a) < \int_a^b \frac{1}{x}\,dx < \frac{1}{a}(b - a) \Rightarrow$
$\frac{1}{b}(b - a) < \ln b - \ln a < \frac{1}{a}(b - a) \Rightarrow \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$. (Note from the proof of Property 8 that we are justified in making all of the inequalities strict.)

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Problems Plus

1. Let $y = f(x) = e^{-x^2}$. The area of the rectangle under the curve from $-x$ to x is $A(x) = 2xe^{-x^2}$ where $x \geq 0$. We maximize $A(x)$: $A'(x) = 2e^{-x^2} - 4x^2e^{-x^2} = 2e^{-x^2}(1 - 2x^2) = 0 \Rightarrow x = \frac{1}{\sqrt{2}}$. This gives a maximum since $A'(x) > 0$ for $0 \leq x < \frac{1}{\sqrt{2}}$ and $A'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$. We next determine the points of inflection of $f(x)$. Notice that $f'(x) = -2xe^{-x^2} = -A(x)$. So $f''(x) = -A'(x)$ and hence, $f''(x) < 0$ for $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ and $f''(x) > 0$ for $x < -\frac{1}{\sqrt{2}}$ and $x > \frac{1}{\sqrt{2}}$. So $f(x)$ changes concavity at $x = \pm\frac{1}{\sqrt{2}}$, and the two vertices of the rectangle of largest area are at the inflection points.

2. We use proof by contradiction. Suppose that $\log_2 5$ is a rational number. Then $\log_2 5 = m/n$ where m and n are positive integers $\Rightarrow$ $2^{m/n} = 5$ $\Rightarrow$ $2^m = 5^n$. But this is impossible since 2^m is even and 5^n is odd. So $\log_2 5$ is irrational.

3. Consider the statement that $\frac{d^n}{dx^n}(e^{ax}\sin bx) = r^n e^{ax}\sin(bx + n\theta)$. For $n = 1$,

$\frac{d}{dx}(e^{ax}\sin bx) = ae^{ax}\sin bx + be^{ax}\cos bx$, and

$$\begin{aligned} re^{ax}\sin(bx+\theta) &= re^{ax}[\sin bx\cos\theta + \cos bx\sin\theta] = re^{ax}\left(\frac{a}{r}\sin bx + \frac{b}{r}\cos bx\right) \\ &= ae^{ax}\sin bx + be^{ax}\cos bx \end{aligned}$$

since $\tan\theta = b/a \Rightarrow \sin\theta = b/r$ and $\cos\theta = a/r$.

So the statement is true for $n = 1$. Assume it is true for $n = k$. Then

$$\begin{aligned} \frac{d^{k+1}}{dx^{k+1}}(e^{ax}\sin bx) &= \frac{d}{dx}\left[r^k e^{ax}\sin(bx+k\theta)\right] = r^k ae^{ax}\sin(bx+k\theta) + r^k e^{ax} b\cos(bx+k\theta) \\ &= r^k e^{ax}[a\sin(bx+k\theta) + b\cos(bx+k\theta)] \end{aligned}$$

But

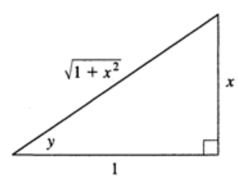
$$\begin{aligned} \sin[bx+(k+1)\theta] &= \sin[(bx+k\theta)+\theta] = \sin(bx+k\theta)\cos\theta + \sin\theta\cos(bx+k\theta) \\ &= \frac{a}{r}\sin(bx+k\theta) + \frac{b}{r}\cos(bx+k\theta) \end{aligned}$$

Hence, $a\sin(bx+k\theta) + b\cos(bx+k\theta) = r\sin[bx+(k+1)\theta]$. So

$$\begin{aligned} \frac{d^{k+1}}{dx^{k+1}}(e^{ax}\sin bx) &= r^k e^{ax}[a\sin(bx+k\theta) + b\sin(bx+k\theta)] = r^k e^{ax}[r\sin(bx+(k+1)\theta)] \\ &= r^{k+1}e^{ax}[\sin(bx+(k+1)\theta)] \end{aligned}$$

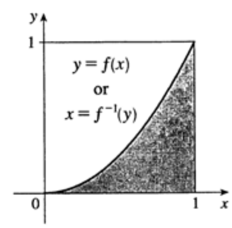
Therefore, the statement is true for all n by mathematical induction.

4. Let $y = \tan^{-1} x$. Then $\tan y = x$, so from the triangle we see that $\sin(\tan^{-1} x) = \sin y = \frac{x}{\sqrt{1+x^2}}$. Using this fact we have that $\sin(\tan^{-1}(\sinh x)) = \frac{\sinh x}{\sqrt{1+\sinh^2 x}} = \frac{\sinh x}{\cosh x} = \tanh x$. Hence, $\sin^{-1}(\tanh x) = \sin^{-1}(\sin(\tan^{-1}(\sinh x))) = \tan^{-1}(\sinh x)$.



5. We first show that $\frac{x}{1+x^2} < \tan^{-1} x$ for $x > 0$. Let $f(x) = \tan^{-1} x - \frac{x}{1+x^2}$. Then $f'(x) = \frac{1}{1+x^2} - \frac{1(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{(1+x^2)-(1-x^2)}{(1+x^2)^2} = \frac{2x^2}{(1+x^2)^2} > 0$ for $x > 0$. So $f(x)$ is increasing on $(0, \infty)$. Hence, $0 < x \Rightarrow 0 = f(0) < f(x) = \tan^{-1} x - \frac{x}{1+x^2}$. So $\frac{x}{1+x^2} < \tan^{-1} x$ for $0 < x$. We next show that $\tan^{-1} x < x$ for $x > 0$. Let $h(x) = x - \tan^{-1} x$. Then $h'(x) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2} > 0$. Hence, $h(x)$ is increasing on $(0, \infty)$. So for $0 < x$, $0 = h(0) < h(x) = x - \tan^{-1} x$. Hence, $\tan^{-1} x < x$ for $x > 0$, and we conclude that $\frac{x}{1+x^2} < \tan^{-1} x < x$ for $x > 0$.

6. The shaded region has area $\int_0^1 f(x)\,dx = \frac{1}{3}$. The integral $\int_0^1 f^{-1}(y)\,dy$ gives the area of the unshaded region, which we know to be $1 - \frac{1}{3} = \frac{2}{3}$. So $\int_0^1 f^{-1}(y)\,dy = \frac{2}{3}$.



7. By the Fundamental Theorem of Calculus, $f'(x) = \sqrt{1+x^3} > 0$ for $x > -1$. So f is increasing on $(-1, \infty)$ and hence is one-to-one. Note that $f(1) = 0$, so $f^{-1}(1) = 0 \Rightarrow (f^{-1})'(0) = 1/f'(1) = \frac{1}{\sqrt{2}}$.

8. $y = \frac{x}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \arctan \frac{\sin x}{a + \sqrt{a^2-1} + \cos x}$. Let $k = a + \sqrt{a^2-1}$. Then

$$y' = \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{1}{1 + \sin^2 x/(k+\cos x)^2} \cdot \frac{\cos x\,(k+\cos x) + \sin^2 x}{(k+\cos x)^2}$$
$$= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{k\cos x + \cos^2 x + \sin^2 x}{(k+\cos x)^2 + \sin^2 x} = \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{k\cos x + 1}{k^2 + 2k\cos x + 1}$$
$$= \frac{k^2 + 2k\cos x + 1 - 2k\cos x - 2}{\sqrt{a^2-1}\,(k^2 + 2k\cos x + 1)} = \frac{k^2-1}{\sqrt{a^2-1}\,(k^2 + 2k\cos x + 1)}$$

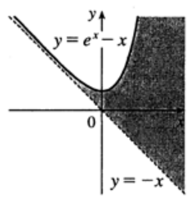
But $k^2 = 2a^2 + 2a\sqrt{a^2-1} - 1 = 2a\left(a + \sqrt{a^2-1}\right) - 1 = 2ak - 1$, so $k^2 + 1 = 2ak$, and $k^2 - 1 = 2(ak-1)$. So $y' = \frac{2(ak-1)}{\sqrt{a^2-1}\,(2ak + 2k\cos x)} = \frac{ak-1}{\sqrt{a^2-1}\,k\,(a+\cos x)}$. But $ak - 1 = a^2 + a\sqrt{a^2-1} - 1 = k\sqrt{a^2-1}$, so $y' = 1/(a + \cos x)$.

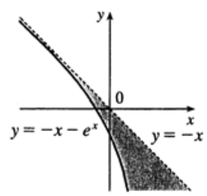
9. If $L = \lim\limits_{x\to\infty} \left(\dfrac{x+a}{x-a}\right)^x$, then L has the indeterminate form 1^∞, so

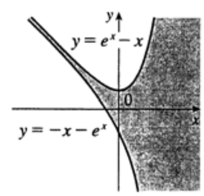
$$\begin{aligned}
\ln L &= \lim_{x\to\infty} \ln \left(\frac{x+a}{x-a}\right)^x = \lim_{x\to\infty} x \ln \left(\frac{x+a}{x-a}\right) = \lim_{x\to\infty} \frac{\ln(x+a) - \ln(x-a)}{1/x} \\
&\overset{\text{H}}{=} \lim_{x\to\infty} \frac{\dfrac{1}{x+a} - \dfrac{1}{x-a}}{-1/x^2} = -\lim_{x\to\infty} \frac{(x-a)\,x^2 - (x+a)\,x^2}{(x+a)\,(x-a)} = -\lim_{x\to\infty} \frac{-2ax^2}{x^2-a^2} \\
&= \lim_{x\to\infty} \frac{2a}{1 - a^2/x^2} = 2a.
\end{aligned}$$

Hence, $\ln L = 2a$, so $L = e^{2a}$. From the original equation, we want $L = e^1 \;\Rightarrow\; 2a = 1 \;\Rightarrow\; a = \frac{1}{2}$.

10. *Case (i) (first graph):* For $x + y \geq 0$, that is, $y \geq -x$, $x + y = |x + y| \leq e^x \;\Rightarrow\; y \leq e^x - x$. Note that $y = e^x - x$ is always above the line $y = -x$ and that $y = -x$ is a slant asymptote.
Case (ii) (second graph): For $x + y < 0$, $-x - y = |x + y| \leq e^x \;\Rightarrow\; y \geq -x - e^x$. Note that $-x - e^x$ is always below the line $y = -x$ and $y = -x$ is a slant asymptote.
Putting the two pieces together gives the third graph.







11. Both sides of the inequality are positive, so $\cosh(\sinh x) < \sinh(\cosh x) \;\Leftrightarrow\; \cosh^2(\sinh x) < \sinh^2(\cosh x)$
$\Leftrightarrow\; \sinh^2(\sinh x) + 1 < \sinh^2(\cosh x) \;\Leftrightarrow\; 1 < [\sinh(\cosh x) - \sinh(\sinh x)]\,[\sinh(\cosh x) + \sinh(\sinh x)]$
$\Leftrightarrow\; 1 < \left[\sinh\left(\dfrac{e^x + e^{-x}}{2}\right) - \sinh\left(\dfrac{e^x - e^{-x}}{2}\right)\right]\left[\sinh\left(\dfrac{e^x + e^{-x}}{2}\right) + \sinh\left(\dfrac{e^x - e^{-x}}{2}\right)\right] \;\Leftrightarrow$
$1 < \left[2\cosh(e^x/2)\sinh\left(e^{-x}/2\right)\right]\left[2\sinh(e^x/2)\cosh\left(e^{-x}/2\right)\right]$ (use the addition formulas and cancel)
$\Leftrightarrow\; 1 < [2\sinh(e^x/2)\cosh(e^x/2)]\left[2\sinh\left(e^{-x}/2\right)\cosh\left(e^{-x}/2\right)\right] \;\Leftrightarrow\; 1 < \sinh e^x \sinh e^{-x}$, by the half-angle formula. Now both e^x and e^{-x} are positive, and $\sinh y > y$ for $y > 0$, since $\sinh 0 = 0$ and $(\sinh y - y)' = \cosh y - 1 > 0$ for $x > 0$, so $1 = e^x e^{-x} < \sinh e^x \sinh e^{-x}$. So, following this chain of reasoning backward, we arrive at the desired result.
Another Method: Using Formula 3.7.3, we have

$$\begin{aligned}
\sinh^{-1}(\cosh(\sinh x)) &= \ln\left(\cosh(\sinh x) + \sqrt{1 + \cosh^2(\sinh x)}\right) = \ln(\cosh(\sinh x) + \sinh(\cosh x)) \\
&= \ln\left(e^{\sinh x}\right) = \sinh x
\end{aligned}$$

But $\sinh x < \cosh x$, so $\sinh^{-1}(\cosh(\sinh x)) < \cosh x$. Since sinh is an increasing function, we can apply it to both sides of the inequality and get $\cosh(\sinh x) < \sinh(\cosh x)$.

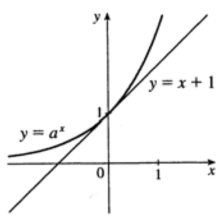
12. First, we recognize some symmetry in the inequality: $\dfrac{e^{x+y}}{xy} \geq e^2 \quad \Leftrightarrow \quad \dfrac{e^x}{x} \cdot \dfrac{e^y}{y} \geq e \cdot e$. This suggests that we need to show that $\dfrac{e^x}{x} \geq e$ for $x > 0$. If we can do this, then the inequality $\dfrac{e^y}{y} \geq e$ is true, and the given inequality follows.

$f(x) = \dfrac{e^x}{x} \quad \Rightarrow \quad f'(x) = \dfrac{xe^x - e^x}{x^2} = \dfrac{e^x(x-1)}{x^2} = 0 \quad \Rightarrow \quad x = 1$. By the First Derivative Test, we have a minimum of $f(1) = e$, so $e^x/x \geq e$ for all x.

13. Suppose that the curve $y = a^x$ intersects the line $y = x$. Then $a^{x_0} = x_0$ for some $x_0 > 0$, and hence $a = x_0^{1/x_0}$. We find the maximum value of $g(x) = x^{1/x}, > 0$, because if a is larger than the maximum value of this function, then the curve $y = a^x$ does not intersect the line $y = x$.

$g'(x) = e^{(1/x)\ln x}\left(-\dfrac{1}{x^2}\ln x + \dfrac{1}{x}\cdot\dfrac{1}{x}\right) = x^{1/x}\left(\dfrac{1}{x^2}\right)(1 - \ln x)$. This is 0 only where $x = e$, and for $0 < x < e$, $f'(x) > 0$, while for $x > e$, $f'(x) < 0$, so g has an absolute maximum of $g(e) = e^{1/e}$. So if $y = a^x$ intersects $y = x$, we must have $0 < a \leq e^{1/e}$. Conversely, suppose that $0 < a \leq e^{1/e}$. Then $a^e \leq e$, so the graph of $y = a^x$ lies below or touches the graph of $y = x$ at $x = e$. Also $a^0 = 1 > 0$, so the graph of $y = a^x$ lies above that of $y = x$ at $x = 0$. Therefore, by the Intermediate Value Theorem, the graphs of $y = a^x$ and $y = x$ must intersect somewhere between $x = 0$ and $x = e$.

14.



We see that at $x = 0$, $f(x) = a^x = 1 + x = 1$, so if $y = a^x$ is to lie above $y = 1 + x$, the two curves must just touch at $(0, 1)$, that is, we must have $f'(0) = 1$.

[To see this analytically, note that $a^x \geq 1 + x \quad \Rightarrow \quad a^x - 1 \geq x \quad \Rightarrow \quad \dfrac{a^x - 1}{x} \geq 1$ for $x > 0$, so $f'(0) = \lim\limits_{x\to 0^+} \dfrac{a^x - 1}{x} \geq 1$. Similarly, for $x < 0$, $a^x - 1 \geq x \quad \Rightarrow \quad \dfrac{a^x - 1}{x} \leq 1$, so $f'(0) = \lim\limits_{x\to 0^-} \dfrac{a^x - 1}{x} \leq 1$. Since $1 \leq f'(0) \leq 1$, we must have $f'(0) = 1$.]

But $f'(x) = a^x \ln a \quad \Rightarrow \quad f'(0) = \ln a$, so we have $\ln a = 1 \quad \Leftrightarrow \quad a = e$.

Another Method: The inequality certainly holds for $x \leq -1$, so consider $x > -1, x \neq 0$. Then $a^x \geq 1 + x \quad \Rightarrow \quad a \geq (1+x)^{1/x}$ for $x > 0 \quad \Rightarrow \quad a \geq \lim\limits_{x\to 0^+}(1+x)^{1/x} = e$, by Equation 3.7.5. Also, $a^x \geq 1 + x \quad \Rightarrow \quad a \leq (1+x)^{1/x}$ for $x < 0 \quad \Rightarrow \quad a \leq \lim\limits_{x\to 0^-}(1+x)^{1/x} = e$. So since $e \leq a \leq e$, we must have $a = e$.

8 Techniques of Integration

8.1 Integration by Parts

1. Let $u = \ln x$, $dv = x\,dx \Rightarrow du = dx/x$, $v = \frac{1}{2}x^2$. Then by Equation 2,

$$\begin{aligned}\int x \ln x\,dx &= \tfrac{1}{2}x^2 \ln x - \int \tfrac{1}{2}x^2\,(dx/x) = \tfrac{1}{2}x^2 \ln x - \tfrac{1}{2}\int x\,dx = \tfrac{1}{2}x^2 \ln x - \tfrac{1}{2}\cdot\tfrac{1}{2}x^2 + C \\ &= \tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2 + C\end{aligned}$$

2. Let $u = \theta$, $dv = \sec^2\theta\,d\theta \Rightarrow du = d\theta$, $v = \tan\theta$. Then
$\int \theta \sec^2\theta\,d\theta = \theta\tan\theta - \int \tan\theta\,d\theta = \theta\tan\theta - \ln|\sec\theta| + C$.

3. Let $u = x$, $dv = e^{2x}\,dx \Rightarrow du = dx$, $v = \frac{1}{2}e^{2x}$. Then by Equation 2,
$\int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$.

4. Let $u = x$, $dv = \cos x\,dx \Rightarrow du = dx$, $v = \sin x$. Then by Equation 2,
$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$.

5. Let $u = x$, $dv = \sin 4x\,dx \Rightarrow du = dx$, $v = -\frac{1}{4}\cos 4x$. Then
$\int x\sin 4x\,dx = -\frac{1}{4}x\cos 4x - \int \left(-\frac{1}{4}\cos 4x\right)dx = -\frac{1}{4}x\cos 4x + \frac{1}{16}\sin 4x + C$.

6. Let $u = \sin^{-1}x$, $dv = dx \Rightarrow du = \dfrac{dx}{\sqrt{1-x^2}}$, $v = x$. Then $\int \sin^{-1}x\,dx = x\sin^{-1}x - \displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$.
Setting $t = 1 - x^2$, we get $dt = -2x\,dx$, so $-\displaystyle\int \frac{x\,dx}{\sqrt{1-x^2}} = \int t^{-1/2}\cdot\tfrac{1}{2}\,dt = t^{1/2} + C = \sqrt{1-x^2} + C$. Hence,
$\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2} + C$.

7. Let $u = x^2$, $dv = \cos 3x\,dx \Rightarrow du = 2x\,dx$, $v = \frac{1}{3}\sin 3x$.
Then $I = \int x^2\cos 3x\,dx = \frac{1}{3}x^2\sin 3x - \frac{2}{3}\int x\sin 3x\,dx$ by Equation 2. Next let
$U = x$, $dV = \sin 3x\,dx \Rightarrow dU = dx$, $V = -\frac{1}{3}\cos 3x$ to get
$\int x\sin 3x\,dx = -\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x\,dx = -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C_1$. Substituting for $\int x\sin 3x\,dx$, we get
$I = \frac{1}{3}x^2\sin 3x - \frac{2}{3}\left(-\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C_1\right) = \frac{1}{3}x^2\sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x + C$, where $C = -\frac{2}{3}C_1$.

8. Let $u = x^2$, $dv = \sin ax\,dx \;\Rightarrow\; du = 2x\,dx$, $v = -\dfrac{1}{a}\cos ax$. Then

$$I = \int x^2 \sin ax\,dx = -\frac{x^2}{a}\cos ax - \int\left(-\frac{1}{a}\right)\cos ax\,(2x\,dx) = -\frac{x^2}{a}\cos ax + \frac{2}{a}\int x\cos ax\,dx$$

by Equation 2. Let $U = x$, $dV = \cos ax\,dx \;\Rightarrow\; dU = dx$, $V = \dfrac{1}{a}\sin ax$. Then

$\displaystyle\int x\cos ax\,dx = \frac{x}{a}\sin ax - \int\frac{1}{a}\sin ax\,dx = \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax + C_1$. So

$\displaystyle I = -\frac{x^2}{a}\cos ax + \frac{2}{a}\left(\frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax + C_1\right) = -\frac{x^2}{a}\cos ax + \frac{2x}{a^2}\sin ax + \frac{2}{a^3}\cos ax + C.$

9. Let $u = (\ln x)^2$, $dv = dx \;\Rightarrow\; du = 2\ln x \cdot \frac{1}{x}\,dx$, $v = x$.
Then $I = \int(\ln x)^2\,dx = x(\ln x)^2 - 2\int \ln x\,dx$. Next let $U = \ln x$, $dV = dx \;\Rightarrow\; dU = 1/x\,dx$, $V = x$ to get $\int \ln x\,dx = x\ln x - \int x\cdot(1/x)\,dx = x\ln x - x + C_1$. Thus, $I = x(\ln x)^2 - 2x\ln x + 2x + C$, where $C = -2C_1$.

10. Let $u = t^3$, $dv = e^t\,dt \;\Rightarrow\; du = 3t^2\,dt$, $v = e^t$. Then $I = \int t^3 e^t\,dt = t^3 e^t - \int 3t^2 e^t\,dt$. Integrate by parts twice more with $dv = e^t\,dt$.

$$\begin{aligned} I &= t^3e^t - \left(3t^2e^t - \textstyle\int 6te^t\,dt\right) = t^3e^t - 3t^2e^t + 6te^t - \textstyle\int 6e^t\,dt = t^3e^t - 3t^2e^t + 6te^t - 6e^t + C \\ &= \left(t^3 - 3t^2 + 6t - 6\right)e^t + C \end{aligned}$$

More generally, if $p(t)$ is a polynomial of degree n in t, then repeated integration by parts shows that $\int p(t)\,e^t\,dt = \left[p(t) - p'(t) + p''(t) - p'''(t) + \cdots + (-1)^n p^{(n)}(t)\right]e^t + C$.

11. First let $u = \sin 3\theta$, $dv = e^{2\theta}\,d\theta \;\Rightarrow\; du = 3\cos 3\theta\,d\theta$, $v = \frac{1}{2}e^{2\theta}$. Then
$I = \int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{2}\int e^{2\theta}\cos 3\theta\,d\theta$. Next let $U = \cos 3\theta$,
$dV = e^{2\theta}\,d\theta \;\Rightarrow\; dU = -3\sin 3\theta\,d\theta$, $V = \frac{1}{2}e^{2\theta}$ to get
$\int e^{2\theta}\cos 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\cos 3\theta + \frac{3}{2}\int e^{2\theta}\sin 3\theta\,d\theta$. Substituting in the previous formula gives
$I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}\int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}I \;\Rightarrow$
$\frac{13}{4}I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta + C_1$. Hence, $I = \frac{1}{13}e^{2\theta}(2\sin 3\theta - 3\cos 3\theta) + C$, where $C = \frac{4}{13}C_1$.

12. Let $u = e^{-\theta}$, $dv = \cos 2\theta\,d\theta \;\Rightarrow\; du = -e^{-\theta}\,d\theta$, $v = \frac{1}{2}\sin 2\theta$. Then

$$I = \int e^{-\theta}\cos 2\theta\,d\theta = \tfrac{1}{2}e^{-\theta}\sin 2\theta - \int \tfrac{1}{2}\sin 2\theta\left(-e^{-\theta}\,d\theta\right) = \tfrac{1}{2}e^{-\theta}\sin 2\theta + \tfrac{1}{2}\int e^{-\theta}\sin 2\theta\,d\theta$$

Let $U = e^{-\theta}$, $dV = \sin 2\theta\,d\theta \;\Rightarrow\; dU = -e^{-\theta}\,d\theta$, $V = -\frac{1}{2}\cos 2\theta$, so
$\int e^{-\theta}\sin 2\theta\,d\theta = -\frac{1}{2}e^{-\theta}\cos 2\theta - \int\left(-\frac{1}{2}\right)\cos 2\theta\left(-e^{-\theta}\,d\theta\right) = -\frac{1}{2}e^{-\theta}\cos 2\theta - \frac{1}{2}\int e^{-\theta}\cos 2\theta\,d\theta$. So
$I = \frac{1}{2}e^{-\theta}\sin 2\theta + \frac{1}{2}\left[\left(-\frac{1}{2}e^{-\theta}\cos 2\theta\right) - \frac{1}{2}I\right] = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta - \frac{1}{4}I \;\Rightarrow$
$\frac{5}{4}I = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1 \;\Rightarrow$
$\int e^{-\theta}\cos 2\theta\,d\theta = I = \frac{4}{5}\left(\frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1\right) = \frac{2}{5}e^{-\theta}\sin 2\theta - \frac{1}{5}e^{-\theta}\cos 2\theta + C$.

13. Let $u = y$, $dv = \sinh y\,dy \;\Rightarrow\; du = dy$, $v = \cosh y$. Then
$\int y\sinh y\,dy = y\cosh y - \int\cosh y\,dy = y\cosh y - \sinh y + C$.

14. Let $u = y$, $dv = \cosh ay\,dy \Rightarrow du = dy$, $v = \dfrac{\sinh ay}{a}$. Then

$$\int y\cosh ay\,dy = \frac{y\sinh ay}{a} - \frac{1}{a}\int \sinh ay\,dy = \frac{y\sinh ay}{a} - \frac{\cosh ay}{a^2} + C.$$

15. Let $u = t$, $dv = e^{-t}\,dt \Rightarrow du = dt$, $v = -e^{-t}$. By Formula 6,

$\int_0^1 te^{-t}\,dt = \left[-te^{-t}\right]_0^1 + \int_0^1 e^{-t}\,dt = -1/e + \left[-e^{-t}\right]_0^1 = -1/e - 1/e + 1 = 1 - 2/e.$

16. Let $u = x^2 + 1$, $dv = e^{-x}\,dx \Rightarrow du = 2x\,dx$, $v = -e^{-x}$. By (6),

$\int_0^1 \left(x^2+1\right)e^{-x}\,dx = \left[-\left(x^2+1\right)e^{-x}\right]_0^1 + \int_0^1 2xe^{-x}\,dx = -2e^{-1} + 1 + 2\int_0^1 xe^{-x}\,dx$. Let

$U = x$, $dV = e^{-x}\,dx \Rightarrow dU = dx$, $V = -e^{-x}$. By (6) again,

$\int_0^1 xe^{-x}\,dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx = -e^{-1} + \left[-e^{-x}\right]_0^1 = -e^{-1} - e^{-1} + 1 = -2e^{-1} + 1$. So

$\int_0^1 \left(x^2+1\right)e^{-x}\,dx = -2e^{-1} + 1 + 2\left(-2e^{-1}+1\right) = -2e^{-1} + 1 - 4e^{-1} + 2 = -6e^{-1} + 3.$

17. Let $u = \ln x$, $dv = x^{-2}\,dx \Rightarrow du = \dfrac{1}{x}\,dx$, $v = -x^{-1}$. By (6),

$$\int_1^2 \frac{\ln x}{x^2}\,dx = \left[-\frac{\ln x}{x}\right]_1^2 + \int_1^2 x^{-2}\,dx = -\tfrac{1}{2}\ln 2 + \ln 1 + \left[-\frac{1}{x}\right]_1^2 = -\tfrac{1}{2}\ln 2 + 0 - \tfrac{1}{2} + 1 = \tfrac{1}{2} - \tfrac{1}{2}\ln 2.$$

18. Let $u = \ln t$, $dv = \sqrt{t}\,dt \Rightarrow du = dt/t$, $v = \frac{2}{3}t^{3/2}$. By Formula 6,

$\int_1^4 \sqrt{t}\ln t\,dt = \left[\frac{2}{3}t^{3/2}\ln t\right]_1^4 - \frac{2}{3}\int_1^4 \sqrt{t}\,dt = \frac{2}{3}\cdot 8\cdot\ln 4 - 0 - \left[\frac{2}{3}\cdot\frac{2}{3}t^{3/2}\right]_1^4 = \frac{16}{3}\ln 4 - \frac{4}{9}(8-1) = \frac{16}{3}\ln 4 - \frac{28}{9}.$

19. $I = \int_1^4 \ln\sqrt{x}\,dx = \frac{1}{2}\int_1^4 \ln x\,dx = \frac{1}{2}\left[x\ln x - x\right]_1^4$ as in Example 2. So

$I = \frac{1}{2}\left[(4\ln 4 - 4) - (0 - 1)\right] = 2\ln 4 - \frac{3}{2}.$

20. Let $u = x$, $dv = \csc^2 x\,dx \Rightarrow du = dx$, $v = -\cot x$. Then

$$\begin{aligned}\int_{\pi/4}^{\pi/2} x\csc^2 x\,dx &= \left[-x\cot x\right]_{\pi/4}^{\pi/2} + \int_{\pi/4}^{\pi/2}\cot x\,dx = -\tfrac{\pi}{2}\cdot 0 + \tfrac{\pi}{4}\cdot 1 + \left[\ln|\sin x|\right]_{\pi/4}^{\pi/2} = \tfrac{\pi}{4} + \ln 1 - \ln\tfrac{1}{\sqrt{2}}\\ &= \tfrac{\pi}{4} + \tfrac{1}{2}\ln 2\end{aligned}$$

21. Let $u = \cos^{-1}x$, $dv = dx \Rightarrow du = -\dfrac{dx}{\sqrt{1-x^2}}$, $v = x$. Then

$$I = \int_0^{1/2}\cos^{-1}x\,dx = \left[x\cos^{-1}x\right]_0^{1/2} + \int_0^{1/2}\frac{x\,dx}{\sqrt{1-x^2}} = \tfrac{1}{2}\cdot\tfrac{\pi}{3} + \int_1^{3/4} t^{-1/2}\left[-\tfrac{1}{2}\,dt\right],\text{ where } t = 1 - x^2 \Rightarrow$$

$dt = -2x\,dx$. Thus, $I = \frac{\pi}{6} + \frac{1}{2}\int_{3/4}^1 t^{-1/2}\,dt = \left[\sqrt{t}\right]_{3/4}^1 = \frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2} = \frac{1}{6}\left(\pi + 6 - 3\sqrt{3}\right).$

22. Let $u = x$, $dv = 5^x\,dx \Rightarrow du = dx$, $v = (5^x/\ln 5)\,dx$. Then

$$\begin{aligned}\int_0^1 x5^x\,dx &= \left[\frac{x5^x}{\ln 5}\right]_0^1 - \int_0^1 \frac{5^x}{\ln 5}\,dx = \frac{5}{\ln 5} - 0 - \frac{1}{\ln 5}\left[\frac{5^x}{\ln 5}\right]_0^1 = \frac{5}{\ln 5} - \frac{5}{(\ln 5)^2} + \frac{1}{(\ln 5)^2}\\ &= \frac{5}{\ln 5} - \frac{4}{(\ln 5)^2}\end{aligned}$$

23. Let $u = \ln(\sin x)$, $dv = \cos x\,dx \Rightarrow du = \dfrac{\cos x}{\sin x}\,dx$, $v = \sin x$. Then

$I = \int \cos x\ln(\sin x)\,dx = \sin x\ln(\sin x) - \int\cos x\,dx = \sin x\ln(\sin x) - \sin x + C.$

Another Method: Substitute $t = \sin x$, so $dt = \cos x\,dx$. Then $I = \int \ln t\,dt = t\ln t - t + C$ (see Example 2) and so $I = \sin x\,(\ln\sin x - 1) + C$.

24. Let $u = \tan^{-1} x$, $dv = x\,dx \;\Rightarrow\; du = dx/\left(1+x^2\right)$, $v = \frac{1}{2}x^2$.
Then $\int x \tan^{-1} x\,dx = \frac{1}{2}x^2 \tan^{-1} x - \dfrac{1}{2}\displaystyle\int \frac{x^2}{1+x^2}\,dx$. But
$\displaystyle\int \frac{x^2}{1+x^2}\,dx = \int \frac{\left(1+x^2\right)-1}{1+x^2}\,dx = \int 1\,dx - \int \frac{1}{1+x^2}\,dx = x - \tan^{-1} x + C_1$, so
$\int x \tan^{-1} x\,dx = \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\left(x - \tan^{-1} x + C_1\right) = \frac{1}{2}\left(x^2 \tan^{-1} x + \tan^{-1} x - x\right) + C$.

25. Let $w = \ln x \;\Leftrightarrow\; dw = dx/x$. Then $x = e^w$ and $dx = e^w\,dw$, so

$$\begin{aligned}\int \cos(\ln x)\,dx &= \int e^w \cos w\,dw = \tfrac{1}{2}e^w(\sin w + \cos w) + C \quad \text{(by the method of Example 4)}\\ &= \tfrac{1}{2}x\,[\sin(\ln x) + \cos(\ln x)] + C\end{aligned}$$

26. Let $u = r^2$, $dv = \dfrac{r}{\sqrt{4+r^2}}\,dr \;\Rightarrow\; du = 2r\,dr$, $v = \sqrt{4+r^2}$. By (6),

$$\begin{aligned}\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr &= \left[r^2\sqrt{4+r^2}\right]_0^1 - 2\int_0^1 r\sqrt{4+r^2}\,dr = \sqrt{5} - \tfrac{2}{3}\left[\left(4+r^2\right)^{3/2}\right]_0^1\\ &= \sqrt{5} - \tfrac{2}{3}(5)^{3/2} + \tfrac{2}{3}(8) = \sqrt{5}\left(1 - \tfrac{10}{3}\right) + \tfrac{16}{3} = \tfrac{16}{3} - \tfrac{7}{3}\sqrt{5}\end{aligned}$$

27. Let $u = (\ln x)^2$, $dv = x^4\,dx \;\Rightarrow\; du = 2\dfrac{\ln x}{x}\,dx$, $v = \dfrac{x^5}{5}$. By (6),
$\displaystyle\int_1^2 x^4 (\ln x)^2\,dx = \left[\frac{x^5}{5}(\ln x)^2\right]_1^2 - 2\int_1^2 \frac{x^4}{5}\ln x\,dx = \tfrac{32}{5}(\ln 2)^2 - 0 - 2\int_1^2 \frac{x^4}{5}\ln x\,dx.$
Let $U = \ln x$, $dV = \dfrac{x^4}{5}\,dx \;\Rightarrow\; dU = \dfrac{1}{x}\,dx$, $V = \dfrac{x^5}{25}$. So
$\displaystyle 2\int_1^2 \frac{x^4}{5}\ln x\,dx = \left[\frac{x^5}{25}\ln x\right]_1^2 - \int_1^2 \frac{x^4}{25}\,dx = \tfrac{32}{25}\ln 2 - 0 - \left[\frac{x^5}{125}\right]_1^2 = \tfrac{32}{25}\ln 2 - \left(\tfrac{32}{125} - \tfrac{1}{125}\right)$. So
$\int_1^2 x^4 (\ln x)^2\,dx = \frac{32}{5}(\ln 2)^2 - 2\left(\frac{32}{25}\ln 2 - \frac{31}{125}\right) = \frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln 2 + \frac{62}{125}$.

28. Let $u = \sin(t-s)$, $dv = e^s\,ds \;\Rightarrow\; du = -\cos(t-s)\,ds$, $v = e^s$. Then
$I = \int_0^t e^s \sin(t-s)\,ds = \left[e^s \sin(t-s)\right]_0^t + \int_0^t e^s \cos(t-s)\,ds = e^t \sin 0 - e^0 \sin t + I_1$. For
I_1, let $U = \cos(t-s)$, $dV = e^s\,ds \;\Rightarrow\; dU = \sin(t-s)\,ds$, $V = e^s$. So
$I_1 = \left[e^s \cos(t-s)\right]_0^t - \int_0^t e^s \sin(t-s)\,ds = e^t \cos 0 - e^0 \cos t - I$. Thus, $I = -\sin t + e^t - \cos t - I \;\Rightarrow$
$2I = e^t - \cos t - \sin t \;\Rightarrow\; I = \frac{1}{2}\left(e^t - \cos t - \sin t\right)$.

29. Let $w = \sqrt{x}$, so that $x = w^2$ and $dx = 2w\,dw$. Thus, $\int \sin\sqrt{x}\,dx = \int 2w \sin w\,dw$.
Now use parts with $u = 2w$, $dv = \sin w\,dw$, $du = 2\,dw$, $v = -\cos w$ to get
$\int 2w \sin w\,dw = -2w\cos w + \int 2\cos w\,dw = -2w\cos w + 2\sin w + C = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$.

30. Let $w = \sqrt{x}$, so that $x = w^2$ and $dx = 2w\,dw$. Thus, $\int_1^4 e^{\sqrt{x}}\,dx = \int_1^2 e^w 2w\,dw$. Now use parts with $u = 2w$,
$dv = e^w\,dw$, $du = 2\,dw$, $v = e^w$ to get $\int_1^2 e^w 2w\,dw = \left[2we^w\right]_1^2 - 2\int_1^2 e^w\,dw = 4e^2 - 2e - 2\left(e^2 - e\right) = 2e^2$.

31. Let $x = \theta^2$, so $dx = 2\theta\,d\theta$. So $\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos\left(\theta^2\right)d\theta = \frac{1}{2}\int_{\pi/2}^{\pi} x\cos x\,dx$. Let $u = x$, $dv = \cos x\,dx \;\Rightarrow$
$du = dx$, $v = \sin x$. So

$$\begin{aligned}\tfrac{1}{2}\int_{\pi/2}^{\pi} x\cos x\,dx &= \tfrac{1}{2}\left(\left[x\sin x\right]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} \sin x\,dx\right) = \tfrac{1}{2}\left[x\sin x + \cos x\right]_{\pi/2}^{\pi}\\ &= \tfrac{1}{2}(\pi\sin\pi + \cos\pi) - \tfrac{1}{2}\left(\tfrac{\pi}{2}\sin\tfrac{\pi}{2} + \cos\tfrac{\pi}{2}\right) = \tfrac{1}{2}(\pi\cdot 0 - 1) - \tfrac{1}{2}\left(\tfrac{\pi}{2}\cdot 1 + 0\right) = -\tfrac{1}{2} - \tfrac{\pi}{4}\end{aligned}$$

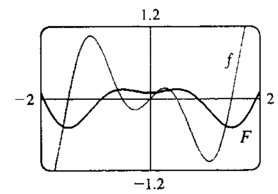
32. $\int x^5 e^{x^2}\,dx = \int (x^2)^2 e^{x^2} x\,dx = \int t^2 e^t \tfrac{1}{2}\,dt$ (where $t = x^2 \Rightarrow \tfrac{1}{2}\,dt = x\,dx$)
$= \tfrac{1}{2}\left(t^2 - 2t + 2\right)e^t + C$ (by Example 3) $= \tfrac{1}{2}\left(x^4 - 2x^2 + 2\right)e^{x^2} + C$

In Exercises 33–36, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

33. Let $u = x$, $dv = \cos\pi x\,dx \Rightarrow du = dx$, $v = (\sin\pi x)/x$. Then

$$\int x\cos\pi x\,dx = x\cdot\frac{\sin\pi x}{\pi} - \int\frac{\sin\pi x}{\pi}\,dx = \frac{x\sin\pi x}{\pi} + \frac{\cos\pi x}{\pi^2} + C.$$

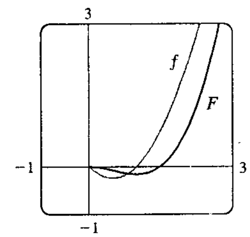
We see from the graph that this is reasonable, since F has extreme values where f is 0.



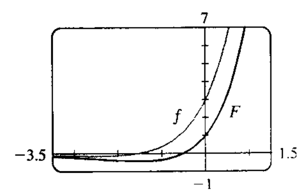
34. Let $u = \ln x$, $dv = x^{3/2}\,dx \Rightarrow du = \dfrac{1}{x}\,dx$, $v = \tfrac{2}{5}x^{5/2}$. So

$$\int x^{3/2}\ln x\,dx = \tfrac{2}{5}x^{5/2}\ln x - \tfrac{2}{5}\int x^{3/2}\,dx = \tfrac{2}{5}x^{5/2}\ln x - \left(\tfrac{2}{5}\right)^2 x^{5/2} + C$$
$$= \tfrac{2}{5}x^{5/2}\ln x - \tfrac{4}{25}x^{5/2} + C$$

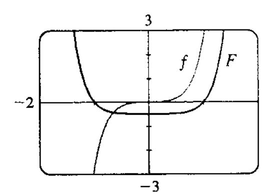
We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive.



35. Let $u = 2x + 3$, $dv = e^x\,dx \Rightarrow du = 2\,dx$, $v = e^x$. Then $\int(2x+3)e^x\,dx = (2x+3)e^x - 2\int e^x\,dx = (2x+3)e^x - 2e^x + C = (2x+1)e^x + C$. We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive.



36. $\int x^3 e^{x^2}\,dx = \int x^2\cdot xe^{x^2}\,dx = I$. Let $u = x^2$, $dv = xe^{x^2}\,dx \Rightarrow du = 2x\,dx$, $v = \tfrac{1}{2}e^{x^2}$. Then
$I = \tfrac{1}{2}x^2 e^{x^2} - \int xe^{x^2}\,dx = \tfrac{1}{2}x^2 e^{x^2} - \tfrac{1}{2}e^{x^2} + C = \tfrac{1}{2}e^{x^2}\left(x^2 - 1\right) + C$.
We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive.



37. (a) Take $n = 2$ in Example 6 to get $\int\sin^2 x\,dx = -\tfrac{1}{2}\cos x\sin x + \tfrac{1}{2}\int 1\,dx = \dfrac{x}{2} - \dfrac{\sin 2x}{4} + C$.

(b) $\int\sin^4 x\,dx = -\tfrac{1}{4}\cos x\sin^3 x + \tfrac{3}{4}\int\sin^2 x\,dx = -\tfrac{1}{4}\cos x\sin^3 x + \tfrac{3}{8}x - \tfrac{3}{16}\sin 2x + C$.

38. (a) Let $u = \cos^{n-1} x$, $dv = \cos x\, dx \;\Rightarrow\; du = -(n-1)\cos^{n-2} x \sin x\, dx$, $v = \sin x$ in (2):

$$\begin{aligned}\int \cos^n x\, dx &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\, dx \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\left(1-\cos^2 x\right) dx \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx - (n-1)\int \cos^n x\, dx\end{aligned}$$

Rearranging terms gives $n\int \cos^n x\, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx$ or

$$\int \cos^n x\, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\, dx.$$

(b) Take $n = 2$ in (a) to get $\int \cos^2 x\, dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}\int 1\, dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$.

(c) $\int \cos^4 x\, dx = \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\int \cos^2 x\, dx = \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}x + \frac{3}{16}\sin 2x + C$

39. (a) $\displaystyle\int_0^{\pi/2} \sin^n x\, dx = \left[-\frac{\cos x \sin^{n-1} x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$

(b) $\int_0^{\pi/2} \sin^3 x\, dx = \frac{2}{3}\int_0^{\pi/2} \sin x\, dx = \left[-\frac{2}{3}\cos x\right]_0^{\pi/2} = \frac{2}{3}$; $\int_0^{\pi/2} \sin^5 x\, dx = \frac{4}{5}\int_0^{\pi/2} \sin^3 x\, dx = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15}$

(c) The formula holds for $n = 1$ (that is, $2n+1 = 3$) by (b). Assume it holds for some $k \geq 1$. Then $\displaystyle\int_0^{\pi/2} \sin^{2k+1} x\, dx = \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)}$. By Example 6, $\displaystyle\int_0^{\pi/2} \sin^{2k+3} x\, dx = \frac{2k+2}{2k+3}\int_0^{\pi/2} \sin^{2k+1} x\, dx = \frac{2\cdot 4\cdot 6\cdot\cdots\cdot[2(k+1)]}{2\cdot 4\cdot 6\cdot\cdots\cdot[2(k+1)+1]}$ as desired. By induction, the formula holds for all $n \geq 1$.

40. Using Exercise 39(a), we see that the formula holds for $n = 1$, because $\int_0^{\pi/2} \sin^2 x\, dx = \frac{1}{2}\int_0^{\pi/2} 1\, dx = \frac{1}{2}[x]_0^{\pi/2} = \frac{1}{2}\cdot\frac{\pi}{2}$.

Now assume it holds for some $k \geq 1$. Then $\displaystyle\int_0^{\pi/2} \sin^{2k} x\, dx = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}\frac{\pi}{2}$. By Exercise 39(a),

$\displaystyle\int_0^{\pi/2} \sin^{2(k+1)} x\, dx = \frac{2k+1}{2k+2}\int_0^{\pi/2} \sin^{2k} x\, dx = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k+1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k+2)}\cdot\frac{\pi}{2}$, so the formula holds for $n = k+1$.

By induction, the formula holds for all $n \geq 1$.

41. Let $u = (\ln x)^n$, $dv = dx \;\Rightarrow\; du = n(\ln x)^{n-1}(dx/x)$, $v = x$. By Equation 2, $\int (\ln x)^n\, dx = x(\ln x)^n - n\int (\ln x)^{n-1}\, dx$.

42. Let $u = x^n$, $dv = e^x\, dx \;\Rightarrow\; du = nx^{n-1}\, dx$, $v = e^x$. By Equation 2, $\int x^n e^x\, dx = x^n e^x - n\int x^{n-1} e^x\, dx$.

43. Let $u = \left(x^2+a^2\right)^n$, $dv = dx \;\Rightarrow\; du = n\left(x^2+a^2\right)^{n-1} 2x\, dx$, $v = x$. Then

$$\begin{aligned}\int \left(x^2+a^2\right)^n dx &= x\left(x^2+a^2\right)^n - 2n\int x^2\left(x^2+a^2\right)^{n-1} dx \\ &= x\left(x^2+a^2\right)^n - 2n\left[\int \left(x^2+a^2\right)^n dx - a^2\int \left(x^2+a^2\right)^{n-1} dx\right] \quad \left[\text{since } x^2 = \left(x^2+a^2\right) - a^2\right]\end{aligned}$$

$\Rightarrow\; (2n+1)\int \left(x^2+a^2\right)^n dx = x\left(x^2+a^2\right)^n + 2na^2\int \left(x^2+a^2\right)^{n-1} dx$, and

$$\int \left(x^2+a^2\right)^n dx = \frac{x\left(x^2+a^2\right)^n}{2n+1} + \frac{2na^2}{2n+1}\int \left(x^2+a^2\right)^{n-1} dx \quad (\text{provided } 2n+1 \neq 0).$$

44. Let $u = \sec^{n-2} x$, $dv = \sec^2 x\,dx \Rightarrow du = (n-2)\sec^{n-3} x \sec x \tan x\,dx$, $v = \tan x$. Then by Equation 2,

$$\begin{aligned}\int \sec^n x\,dx &= \tan x \sec^{n-2} x - (n-2)\int \sec^{n-2} x \tan^2 x\,dx \\ &= \tan x \sec^{n-2} x - (n-2)\int \sec^{n-2} x\left(\sec^2 x - 1\right)dx \\ &= \tan x \sec^{n-2} x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx\end{aligned}$$

so $(n-1)\int \sec^n x\,dx = \tan x \sec^{n-2} x + (n-2)\int \sec^{n-2} x\,dx$. If $n - 1 \neq 0$, then

$$\int \sec^n x\,dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx.$$

45. Take $n = 3$ in Exercise 41 to get

$\int (\ln x)^3\,dx = x(\ln x)^3 - 3\int (\ln x)^2\,dx = x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C$ (by Exercise 9).

Or: Instead of using Exercise 9, apply Exercise 41 again with $n = 2$.

46. Take $n = 4$ in Exercise 42 to get

$$\begin{aligned}\int x^4 e^x\,dx &= x^4 e^x - 4\int x^3 e^x\,dx = x^4 e^x - 4\left(x^3 - 3x^2 + 6x - 6\right)e^x + C \quad \text{(by Exercise 10)} \\ &= e^x\left(x^4 - 4x^3 + 12x^2 - 24x + 24\right) + C\end{aligned}$$

Or: Instead of using Exercise 10, apply Exercise 42 with $n = 3$, then $n = 2$, then $n = 1$.

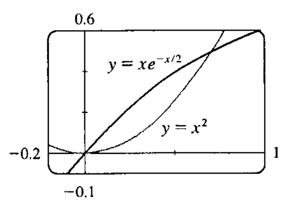
47. Let $u = \sin^{-1} x$, $dv = dx \Rightarrow du = \dfrac{dx}{\sqrt{1-x^2}}$, $v = x$. Then

$$\begin{aligned}\text{area} &= \int_0^{1/2} \sin^{-1} x\,dx = \left[x\sin^{-1} x\right]_0^{1/2} - \int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx \\ &= \tfrac{1}{2}\left(\tfrac{\pi}{6}\right) + \left[\sqrt{1-x^2}\right]_0^{1/2} = \tfrac{\pi}{12} + \tfrac{\sqrt{3}}{2} - 1 = \tfrac{1}{12}\left(\pi + 6\sqrt{3} - 12\right)\end{aligned}$$

48. The curves intersect when $(x-5)\ln x = 0$; that is, when $x = 1$ or $x = 5$. For $1 < x < 5$, we have $5\ln x > x\ln x$ since $\ln x > 0$. Thus, area $= \int_1^5 (5\ln x - x\ln x)\,dx$. Let $u = \ln x$, $dv = (5-x)\,dx \Rightarrow du = dx/x$, $v = 5x - \frac{1}{2}x^2$. Then

$$\begin{aligned}\text{area} &= (\ln x)\left[5x - \tfrac{1}{2}x^2\right]_1^5 - \int_1^5 \left(5x - \tfrac{1}{2}x^2\right)\tfrac{1}{x}\,dx = (\ln 5)\left(\tfrac{25}{2}\right) - 0 - \int_1^5 \left[5 - \tfrac{1}{2}x\right]dx \\ &= \tfrac{25}{2}\ln 5 - \left[5x - \tfrac{1}{4}x^2\right]_1^5 = \tfrac{25}{2}\ln 5 - \left[\left(25 - \tfrac{25}{4}\right) - \left(5 - \tfrac{1}{4}\right)\right] = \tfrac{25}{2}\ln 5 - 14\end{aligned}$$

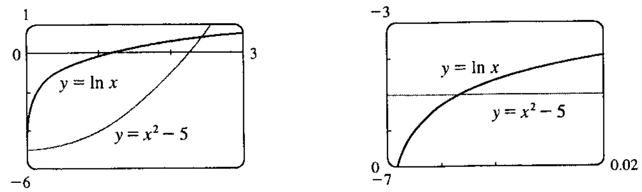
49.



From the graph, we see that the curves intersect at approximately $x = 0$ and $x = 0.70$, with $xe^{-x/2} > x^2$ on $(0, 0.70)$. So the area bounded by the curves is approximately $A = \int_0^{0.70}\left(xe^{-x/2} - x^2\right)dx$. We separate this into two integrals, and evaluate the first one by parts with $u = x$, $dv = e^{-x/2}\,dx \Rightarrow du = dx$, $v = -2e^{-x/2}$:

$$\begin{aligned}A &= \left[-2xe^{-x/2}\right]_0^{0.70} - \int_0^{0.70}\left(-2e^{-x/2}\right)dx - \left[\tfrac{1}{3}x^3\right]_0^{0.70} \\ &= \left[-2(0.70)e^{-0.35} - 0\right] - \left[4e^{-x/2}\right]_0^{0.70} - \tfrac{1}{3}\left[0.70^3 - 0\right] \approx 0.080\end{aligned}$$

50.



From the graphs, we see that the curves intersect at approximately $x = 0.0067$ and $x = 2.43$, with $\ln x > x^2 - 5$ on $(0.0067, 2.43)$. So the area bounded by the curves is about

$$A = \int_{0.0067}^{2.43} \left[\ln x - \left(x^2 - 5\right)\right] dx = \int_{0.0067}^{2.43} \left(\ln x - x^2 + 5\right) dx$$
$$= \left[(x \ln x - x) - \tfrac{1}{3}x^3 + 5x\right]_{0.0067}^{2.43} \quad \text{(see Example 2)} \quad \approx 7.10$$

51. Volume $= \int_{2\pi}^{3\pi} 2\pi x \sin x \, dx$. Let $u = x$, $dv = \sin x \, dx \;\Rightarrow\; du = dx$, $v = -\cos x \;\Rightarrow$
$V = 2\pi \left[-x \cos x + \sin x\right]_{2\pi}^{3\pi} = 2\pi \left[(3\pi + 0) - (-2\pi + 0)\right] = 2\pi\,(5\pi) = 10\pi^2$.

52. Volume $= \int_0^1 2\pi x \left(e^x - e^{-x}\right) dx = 2\pi \int_0^1 \left(xe^x - xe^{-x}\right) dx$
$= 2\pi \left[\int_0^1 xe^x \, dx - \int_0^1 xe^{-x} \, dx\right]$ (both integrals by parts)
$= 2\pi \left[\left(xe^x - e^x\right) - \left(-xe^{-x} - e^{-x}\right)\right]_0^1 = 2\pi\,[2/e - 0] = 4\pi/e$

53. Volume $= \int_{-1}^{0} 2\pi\,(1 - x)\,e^{-x} \, dx$. Let $u = 1 - x$, $dv = e^{-x} \, dx \;\Rightarrow\; du = -dx$, $v = -e^{-x} \;\Rightarrow$
$V = 2\pi \left[xe^{-x}\right]_{-1}^{0} = 2\pi\,(0 + e) = 2\pi e$

54. Volume $= \int_1^{\pi} 2\pi y \cdot \ln y \, dy = 2\pi \left[\tfrac{1}{2}y^2 \ln y - \tfrac{1}{4}y^2\right]_1^{\pi} = 2\pi \left[\tfrac{1}{4}y^2\,(2\ln y - 1)\right]_1^{\pi}$ (by parts)

$$= 2\pi \left[\frac{\pi^2\,(2\ln \pi - 1)}{4} - \frac{(0 - 1)}{4}\right] = \pi^3 \ln \pi - \frac{\pi^3}{2} + \frac{\pi}{2}$$

55. Let $u = x$, $dv = \cos 2x \, dx \;\Rightarrow\; du = dx$, $v = \tfrac{1}{2}\sin 2x \, dx$. Then
$\int_0^{\pi/2} x \cos 2x \, dx = \left[\tfrac{1}{2}x \sin 2x\right]_0^{\pi/2} - \tfrac{1}{2}\int_0^{\pi/2} \sin 2x \, dx = 0 + \left[\tfrac{1}{4}\cos 2x\right]_0^{\pi/2} = \tfrac{1}{4}(-1 - 1) = -\tfrac{1}{2}$. Hence, the average value of f is $\dfrac{-1/2}{\pi/2 - 0} = -\dfrac{1}{\pi}$.

56. The rocket will have height $H = \int_0^{60} v\,(t) \, dt$ after 60 seconds.

$$H = \int_0^{60} \left[-gt - v_e \ln\left(\frac{m - rt}{m}\right)\right] dt = -g\left[\tfrac{1}{2}t^2\right]_0^{60} - v_e\left[\int_0^{60} \ln(m - rt)\,dt - \int_0^{60} \ln m \, dt\right]$$
$$= -g\,(1800) + v_e\,(\ln m)\,(60) - v_e \int_0^{60} \ln(m - rt)\,dt$$

Let $u = \ln(m - rt)$, $dv = dt \Rightarrow du = \dfrac{1}{m - rt}(-r)\,dt$, $v = t$. Then

$$\int_0^{60} \ln(m - rt)\,dt = [t\ln(m - rt)]_0^{60} + \int_0^{60} \frac{rt}{m - rt}\,dt = 60\ln(m - 60r) + \int_0^{60}\left(-1 + \frac{m}{m - rt}\right)dt$$
$$= 60\ln(m - 60r) + \left[-t - \frac{m}{r}\ln(m - rt)\right]_0^{60}$$
$$= 60\ln(m - 60r) - 60 - \frac{m}{r}\ln(m - 60r) + \frac{m}{r}\ln m$$

So $H = -1800g + 60v_e\ln m - 60v_e\ln(m - 60r) + 60v_e + \dfrac{m}{r}v_e\ln(m - 60r) - \dfrac{m}{r}v_e\ln m$. Substituting $g = 9.8$, $m = 30{,}000$, $r = 160$, and $v_e = 3000$ gives us $H \approx 14{,}844$ m.

57. Since $v(t) > 0$ for all t, the desired distance $s(t) = \int_0^t v(w)\,dw = \int_0^t w^2e^{-w}\,dw$. Let $u = w^2$, $dv = e^{-w}\,dw$ $\Rightarrow du = 2w\,dw$, $v = -e^{-w}$. Then $s(t) = \left[-w^2e^{-w}\right]_0^t + 2\int_0^t we^{-w}\,dw$. Now let $U = w$, $dV = e^{-w}\,dw \Rightarrow$ $dU = dw$, $V = -e^{-w}$. Then

$$s(t) = -t^2e^{-t} + 2\left(\left[-we^{-w}\right]_0^t + \int_0^t e^{-w}\,dw\right) = -t^2e^{-t} - 2te^{-t} - 2e^{-t} + 2$$
$$= 2 - e^{-t}\left(t^2 + 2t + 2\right) \text{ meters}$$

58. Suppose $f(0) = g(0) = 0$ and let $u = f(x)$, $dv = g''(x)\,dx \Rightarrow du = f'(x)\,dx$, $v = g'(x)$. Then $\int_0^a f(x)g''(x)\,dx = \left[f(x)g'(x)\right]_0^a - \int_0^a f'(x)g'(x)\,dx = f(a)g'(a) - \int_0^a f'(x)g'(x)\,dx$. Now let $U = f'(x)$, $dV = g'(x)\,dx \Rightarrow dU = f''(x)\,dx$ and $V = g(x)$, so $\int_0^a f'(x)g'(x)\,dx = \left[f'(x)g(x)\right]_0^a - \int_0^a f''(x)g(x)\,dx = f'(a)g(a) - \int_0^a f''(x)g(x)\,dx$. Combining the two results, we get $\int_0^a f(x)g''(x)\,dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)\,dx$.

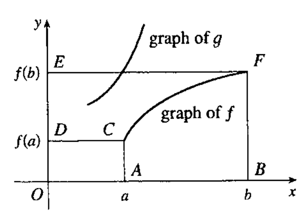
59. Take $g(x) = x$ and $g'(x) = 1$ in Equation 1.

60. By Exercise 59, $\int_a^b f(x)\,dx = bf(b) - af(a) - \int_a^b x f'(x)\,dx$. Now let $y = f(x)$, so that $x = g(y)$ and $dy = f'(x)\,dx$. Then $\int_a^b x f'(x)\,dx = \int_{f(a)}^{f(b)} g(y)\,dy$. The result follows.

61. By Exercise 60, $\int_1^e \ln x\,dx = e\ln e - 1\ln 1 - \int_{\ln 1}^{\ln e} e^y\,dy = e - \int_0^1 e^y\,dy = e - [e^y]_0^1 = e - (e - 1) = 1$.

62. Exercise 60 says that the area of region $ABFC$ is

(area of rectangle $OBFE$) $-$ (area of rectangle $OACD$) $-$ (area of region $DCFE$)



63. Using the formula for volumes of rotation and the figure, we see that

Volume $= \int_0^d \pi b^2\,dy - \int_0^c \pi a^2\,dy - \int_c^d \pi\,[g(y)]^2\,dy = \pi b^2 d - \pi a^2 c - \int_c^d \pi\,[g(y)]^2\,dy$. Let $y = f(x)$, which gives $dy = f'(x)\,dx$ and $g(y) = x$, so that $V = \pi b^2 d - \pi a^2 c - \pi \int_a^b x^2 f'(x)\,dx$. Now integrate by parts with $u = x^2$, and $dv = f'(x)\,dx \Rightarrow du = 2x\,dx$, $v = f(x)$, and

$\int_a^b x^2 f'(x)\,dx = \left[x^2 f(x)\right]_a^b - \int_a^b 2x\,f(x)\,dx = b^2 f(b) - a^2 f(a) - \int_a^b 2x\,f(x)\,dx$, but $f(a) = c$ and $f(b) = d \Rightarrow V = \pi b^2 d - \pi a^2 c - \pi\left[b^2 d - a^2 c - \int_a^b 2x f(x)\,dx\right] = \int_a^b 2\pi x f(x)\,dx$.

64. (a) We note that for $0 \le x \le \frac{\pi}{2}$, $0 \le \sin x \le 1$, so $\sin^{2n+2} x \le \sin^{2n+1} x \le \sin^{2n} x$. So by the second Comparison Property of the Integral, $I_{2n+2} \le I_{2n+1} \le I_{2n}$.

(b) Substituting directly into the result from Exercise 39, we get

$$\frac{I_{2n+2}}{I_{2n}} = \frac{\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot[2(n+1)-1]}{2\cdot 4\cdot 6\cdot\cdots\cdot[2(n+1)]}\dfrac{\pi}{2}}{\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}\dfrac{\pi}{2}} = \frac{2(n+1)-1}{2(n+1)} = \frac{2n+1}{2n+2}$$

(c) We divide the result from part (a) by I_{2n}. The inequalities are preserved since I_{2n} is positive:

$\dfrac{I_{2n+2}}{I_{2n}} \le \dfrac{I_{2n+1}}{I_{2n}} \le \dfrac{I_{2n}}{I_{2n}}$. Now from part (b), the left term is equal to $\dfrac{2n+1}{2n+2}$, so the expression becomes

$\dfrac{2n+1}{2n+2} \le \dfrac{I_{2n+1}}{I_{2n}} \le 1$. Now $\lim\limits_{n\to\infty}\dfrac{2n+1}{2n+2} = \lim\limits_{n\to\infty} 1 = 1$, so by the Squeeze Theorem, $\lim\limits_{n\to\infty}\dfrac{I_{2n+1}}{I_{2n}} = 1$.

(d) We substitute the results from Exercises 39 and 40 into the result from part (c):

$$\begin{aligned} 1 = \lim_{n\to\infty}\frac{I_{2n+1}}{I_{2n}} &= \lim_{n\to\infty}\frac{\dfrac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}}{\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}\dfrac{\pi}{2}} \\ &= \lim_{n\to\infty}\left[\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}\right]\left[\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}\left(\frac{2}{\pi}\right)\right] \end{aligned}$$

Rearranging the terms and multiplying by $\dfrac{\pi}{2}$, we get $\lim\limits_{n\to\infty}\dfrac{2}{1}\cdot\dfrac{2}{3}\cdot\dfrac{4}{3}\cdot\dfrac{4}{5}\cdot\dfrac{6}{5}\cdot\dfrac{6}{7}\cdot\cdots\cdot\dfrac{2n}{2n-1}\cdot\dfrac{2n}{2n+1} = \dfrac{\pi}{2}$, as required.

(e) The area of the kth rectangle is k. At the $2n$th step, the area is increased from $2n-1$ to $2n$ by multiplying the width by $\dfrac{2n}{2n-1}$, and at the $(2n+1)$th step, the area is increased from $2n$ to $2n+1$ by multiplying the height by $\dfrac{2n+1}{2n}$. These two steps multiply the ratio of width to height by $\dfrac{2n}{2n-1}$ and $\dfrac{1}{(2n+1)/(2n)} = \dfrac{2n}{2n+1}$ respectively. So, by part (d), the limiting ratio is $\dfrac{2}{1}\cdot\dfrac{2}{3}\cdot\dfrac{4}{3}\cdot\dfrac{4}{5}\cdot\dfrac{6}{5}\cdot\dfrac{6}{7}\cdot\cdots = \dfrac{\pi}{2}$.

8.2 Trigonometric Integrals

1. $\int \sin^3 x \cos^2 x\,dx = \int \sin^2 x \cos^2 x \sin x\,dx = \int (1-\cos^2 x)\cos^2 x \sin x\,dx \overset{c}{=} \int (1-u^2)\,u^2\,(-du)$
$= \int (u^2-1)\,u^2\,du = \int (u^4-u^2)\,du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$

2. $\int \sin^6 x \cos^3 x\,dx = \int \sin^6 x \cos^2 x \cos x\,dx = \int \sin^6 x\,(1-\sin^2 x)\cos x\,dx \overset{s}{=} \int u^6\,(1-u^2)\,du$
$= \int (u^6-u^8)\,du = \frac{1}{7}u^7 - \frac{1}{9}u^9 + C = \frac{1}{7}\sin^7 x - \frac{1}{9}\sin^9 x + C$

3. $\int_{\pi/2}^{3\pi/4} \sin^5 x \cos^3 x\,dx = \int_{\pi/2}^{3\pi/4} \sin^5 x \cos^2 x \cos x\,dx = \int_{\pi/2}^{3\pi/4} \sin^5 x\,(1-\sin^2 x)\cos x\,dx \overset{s}{=} \int_1^{\sqrt{2}/2} u^5\,(1-u^2)\,du$
$= \int_1^{\sqrt{2}/2} (u^5-u^7)\,du = \left[\frac{1}{6}u^6 - \frac{1}{8}u^8\right]_1^{\sqrt{2}/2} = \left(\frac{1/8}{6} - \frac{1/16}{8}\right) - \left(\frac{1}{6}-\frac{1}{8}\right) = -\frac{11}{384}$

4. $\int_0^{\pi/2} \cos^5 x\,dx = \int_0^{\pi/2} (\cos^2 x)^2 \cos x\,dx = \int_0^{\pi/2} (1-\sin^2 x)^2 \cos x\,dx \overset{s}{=} \int_0^1 (1-u^2)^2\,du$
$= \int_0^1 (1-2u^2+u^4)\,du = \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_0^1 = \left(1 - \frac{2}{3} + \frac{1}{5}\right) - 0 = \frac{8}{15}$

5. $\int \cos^5 x \sin^4 x\,dx = \int \cos^4 x \sin^4 x \cos x\,dx = \int (1-\sin^2 x)^2 \sin^4 x \cos x\,dx \overset{s}{=} \int (1-u^2)^2 u^4\,du$
$= \int (1-2u^2+u^4)\,u^4\,du = \int (u^4-2u^6+u^8)\,du = \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C$
$= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$

6. $\int \sin^3 mx\,dx = \int (1-\cos^2 mx)\sin mx\,dx = -\frac{1}{m}\int (1-u^2)\,du$ $[u = \cos mx,\ du = -m\sin mx\,dx]$
$= -\frac{1}{m}\left(u - \frac{1}{3}u^3\right) + C = -\frac{1}{m}\left(\cos mx - \frac{1}{3}\cos^3 mx\right) + C = \frac{1}{3m}\cos^3 mx - \frac{1}{m}\cos mx + C$

7. $\int_0^{\pi/2} \sin^2 3x\,dx = \int_0^{\pi/2} \frac{1}{2}(1-\cos 6x)\,dx = \left[\frac{1}{2}x - \frac{1}{12}\sin 6x\right]_0^{\pi/2} = \frac{\pi}{4}$

8. $\int_0^{\pi/2} \cos^2 x\,dx = \int_0^{\pi/2} \frac{1}{2}(1+\cos 2x)\,dx = \left[\frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\pi/2} = \frac{\pi}{4}$

9. $\int \cos^4 t\,dt = \int \left[\frac{1}{2}(1+\cos 2t)\right]^2 dt = \frac{1}{4}\int (1+2\cos 2t+\cos^2 2t)\,dt$
$= \frac{1}{4}t + \frac{1}{4}\sin 2t + \frac{1}{4}\int \frac{1}{2}(1+\cos 4t)\,dt = \frac{1}{4}\left[t + \sin 2t + \frac{1}{2}t + \frac{1}{8}\sin 4t\right] + C$
$= \frac{3}{8}t + \frac{1}{4}\sin 2t + \frac{1}{32}\sin 4t + C$

10. $\int \sin^6 \pi x\,dx = \int (\sin^2 \pi x)^3\,dx = \int \left[\frac{1}{2}(1-\cos 2\pi x)\right]^3 dx$
$= \frac{1}{8}\int (1 - 3\cos 2\pi x + 3\cos^2 2\pi x - \cos^3 2\pi x)\,dx$
$= \frac{1}{8}\int \left[1 - 3\cos 2\pi x + \frac{3}{2}(1+\cos 4\pi x) - (1-\sin^2 2\pi x)\cos 2\pi x\right] dx$
$= \frac{1}{8}\int \left(\frac{5}{2} - 4\cos 2\pi x + \frac{3}{2}\cos 4\pi x + \sin^2 2\pi x \cos 2\pi x\right) dx$
$= \frac{1}{8}\left[\frac{5}{2}x - \frac{4}{2\pi}\sin 2\pi x + \frac{3}{8\pi}\sin 4\pi x + \frac{1}{3\cdot 2\pi}\sin^3 2\pi x\right] + C$
$= \frac{5}{16}x - \frac{1}{4\pi}\sin 2\pi x + \frac{3}{64\pi}\sin 4\pi x + \frac{1}{48\pi}\sin^3 2\pi x + C$

11. $\int (1-\sin 2x)^2\,dx = \int (1 - 2\sin 2x + \sin^2 2x)\,dx = \int \left[1 - 2\sin 2x + \frac{1}{2}(1-\cos 4x)\right] dx$
$= \int \left[\frac{3}{2} - 2\sin 2x - \frac{1}{2}\cos 4x\right] dx = \frac{3}{2}x + \cos 2x - \frac{1}{8}\sin 4x + C$

12. $\int \sin\left(\theta + \frac{\pi}{6}\right)\cos\theta\,d\theta = \int \left(\sin\theta \cdot \frac{\sqrt{3}}{2} + \cos\theta \cdot \frac{1}{2}\right)\cos\theta\,d\theta$

$= \frac{\sqrt{3}}{4}\int \sin 2\theta\,d\theta + \frac{1}{4}\int (1 + \cos 2\theta)\,d\theta = -\frac{\sqrt{3}}{8}\cos 2\theta + \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta + C$

13. $\int_0^{\pi/4} \sin^4 x \cos^2 x\,dx = \int_0^{\pi/4} \sin^2 x\,(\sin x \cos x)^2\,dx = \int_0^{\pi/4} \frac{1}{2}(1 - \cos 2x)\left(\frac{1}{2}\sin 2x\right)^2 dx$

$= \frac{1}{8}\int_0^{\pi/4} (1 - \cos 2x)\sin^2 2x\,dx = \frac{1}{8}\int_0^{\pi/4} \sin^2 2x\,dx - \frac{1}{8}\int_0^{\pi/4} \sin^2 2x \cos 2x\,dx$

$= \frac{1}{16}\int_0^{\pi/4} (1 - \cos 4x)\,dx - \frac{1}{16}\left[\frac{1}{3}\sin^3 2x\right]_0^{\pi/4} = \frac{1}{16}\left[x - \frac{1}{4}\sin 4x - \frac{1}{3}\sin^3 2x\right]_0^{\pi/4}$

$= \frac{1}{16}\left(\frac{\pi}{4} - 0 - \frac{1}{3}\right) = \frac{1}{192}(3\pi - 4)$

14. $\int_0^{\pi/2} \sin^2 x \cos^2 x\,dx = \frac{1}{4}\int_0^{\pi/2} \sin^2 2x\,dx = \frac{1}{8}\int_0^{\pi/2} (1 - \cos 4x)\,dx = \frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi/2} = \frac{1}{8}\left(\frac{\pi}{2}\right) = \frac{\pi}{16}$

15. $\int \sin^3 x \sqrt{\cos x}\,dx = \int (1 - \cos^2 x)\sqrt{\cos x}\,\sin x\,dx \overset{c}{=} \int (1 - u^2)\,u^{1/2}\,(-du) = \int (u^{5/2} - u^{1/2})\,du$

$= \frac{2}{7}u^{7/2} - \frac{2}{3}u^{3/2} + C = \frac{2}{7}(\cos x)^{7/2} - \frac{2}{3}(\cos x)^{3/2} + C = \left[\frac{2}{7}\cos^3 x - \frac{2}{3}\cos x\right]\sqrt{\cos x} + C$

16. Let $u = x^2 \quad\Rightarrow\quad du = 2x\,dx$. Then

$\int x \sin^3(x^2)\,dx = \int \sin^3 u \cdot \frac{1}{2}du = \frac{1}{2}\left(-\cos u + \frac{1}{3}\cos^3 u\right) + C$ (by Exercise 6 with $m = 1$)

$= -\frac{1}{2}\cos\left(x^2\right) + \frac{1}{6}\cos^3\left(x^2\right) + C$

17. $\int \cos^2 x \tan^3 x\,dx = \int \frac{\sin^3 x}{\cos x}dx \overset{c}{=} \int \frac{(1 - u^2)(-du)}{u} = \int \left[\frac{-1}{u} + u\right] du$

$= -\ln|u| + \frac{1}{2}u^2 + C = \frac{1}{2}\cos^2 x - \ln|\cos x| + C$

18. $\int \cot^5\theta \sin^4\theta\,d\theta = \int \frac{\cos^5\theta}{\sin^5\theta}\sin^4\theta\,d\theta = \int \frac{\cos^5\theta}{\sin\theta}\,d\theta = \int \frac{\cos^4\theta}{\sin\theta}\cos\theta\,d\theta = \int \frac{(1 - \sin^2\theta)^2}{\sin\theta}\cos\theta\,d\theta$

$\overset{s}{=} \int \frac{(1 - u^2)^2}{u}\,du = \int \frac{1 - 2u^2 + u^4}{u}\,du = \int \left(\frac{1}{u} - 2u + u^3\right) du$

$= \ln|u| - u^2 + \frac{1}{4}u^4 + C = \ln|\sin\theta| - \sin^2\theta + \frac{1}{4}\sin^4\theta + C$

19. $\int \frac{1 - \sin x}{\cos x}\,dx = \int (\sec x - \tan x)\,dx = \ln|\sec x + \tan x| - \ln|\sec x| + C \quad \left[\begin{array}{c}\text{by (1) and the boxed}\\ \text{formula above it}\end{array}\right]$

$= \ln|(\sec x + \tan x)\cos x| + C = \ln|1 + \sin x| + C$

$= \ln(1 + \sin x) + C$ since $1 + \sin x \geq 0$

Or: $\int \frac{1 - \sin x}{\cos x}\,dx = \int \frac{1 - \sin x}{\cos x} \cdot \frac{1 + \sin x}{1 + \sin x}\,dx = \int \frac{(1 - \sin^2 x)\,dx}{\cos x\,(1 + \sin x)} = \int \frac{\cos x\,dx}{1 + \sin x}$

$= \int \frac{dw}{w}$ (where $w = 1 + \sin x$, $dw = \cos x\,dx$)

$= \ln|w| + C = \ln|1 + \sin x| + C = \ln(1 + \sin x) + C$

20. $\int \frac{dx}{\cos x - 1} = \int \frac{1}{\cos x - 1} \cdot \frac{\cos x + 1}{\cos x + 1}\,dx = \int \frac{\cos x + 1}{\cos^2 x - 1}\,dx = \int \frac{\cos x + 1}{-\sin^2 x}\,dx$

$= \int (-\cot x \csc x - \csc^2 x)\,dx = \csc x + \cot x + C$

21. $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C$

22. $\int \tan^4 x\,dx = \int \tan^2 x\,(\sec^2 x - 1)\,dx = \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx = \frac{1}{3}\tan^3 x - \tan x + x + C$

(Set $u = \tan x$ in the first integral and use Exercise 21 for the second.)

23. $\int \sec^4 x\,dx = \int (\tan^2 x + 1)\sec^2 x\,dx = \int \tan^2 x \sec^2 x\,dx + \int \sec^2 x\,dx = \frac{1}{3}\tan^3 x + \tan x + C$

24. $\int \sec^6 x\,dx = \int (\tan^2 x + 1)^2 \sec^2 x\,dx = \int \tan^4 x \sec^2 x\,dx + 2\int \tan^2 x \sec^2 x\,dx + \int \sec^2 x\,dx$

$= \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C$ (Set $u = \tan x$ in the first two integrals.)

Thus, $\int_0^{\pi/4} \sec^6 x\,dx = \left[\frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x\right]_0^{\pi/4} = \frac{1}{5} + \frac{2}{3} + 1 = \frac{28}{15}$.

25. Let $u = \tan t \quad \Rightarrow \quad du = \sec^2 t\,dt$. Then $\int_0^{\pi/4} \tan^4 t \sec^2 t\,dt = \int_0^1 u^4\,du = \left[\frac{1}{5}u^5\right]_0^1 = \frac{1}{5}$.

26. Let $u = \tan x \quad \Rightarrow \quad du = \sec^2 x\,dx$. Then

$\int_0^{\pi/4} \tan^2 x \sec^4 x\,dx = \int_0^1 u^2 (u^2 + 1)\,du = \int_0^1 (u^4 + u^2)\,du = \left[\frac{1}{5}u^5 + \frac{1}{3}u^3\right]_0^1 = \frac{1}{5} + \frac{1}{3} = \frac{8}{15}$.

27. $\int \tan^3 x \sec x\,dx = \int \tan^2 x \sec x \tan x\,dx = \int (\sec^2 x - 1)\sec x \tan x\,dx$

$= \int (u^2 - 1)\,du \quad [u = \sec x,\ du = \sec x \tan x\,dx] \quad = \frac{1}{3}u^3 - u + C = \frac{1}{3}\sec^3 x - \sec x + C$

28. Let $u = \sec x \quad \Rightarrow \quad du = \sec x \tan x\,dx$. Then

$$\int \tan^3 x \sec^3 x\,dx = \int \sec^2 x \tan^2 x \sec x \tan x\,dx = \int u^2 (u^2 - 1)\,du = \int (u^4 - u^2)\,du$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$$

29. Let $u = \sec x \quad \Rightarrow \quad du = \sec x \tan x\,dx$. Then

$$\int_0^{\pi/3} \tan^5 x \sec x\,dx = \int_0^{\pi/3} (\sec^2 x - 1)^2 \sec x \tan x\,dx = \int_1^2 (u^2 - 1)^2\,du = \int_1^2 (u^4 - 2u^2 + 1)\,du$$

$$= \left[\frac{1}{5}u^5 - \frac{2}{3}u^3 + u\right]_1^2 = \left(\frac{32}{5} - \frac{16}{3} + 2\right) - \left(\frac{1}{5} - \frac{2}{3} + 1\right) = \frac{38}{15}$$

30. $\int_0^{\pi/3} \tan^5 x \sec^6 x\,dx = \int_0^{\pi/3} \tan^5 x \sec^4 x \sec^2 x\,dx = \int_0^{\pi/3} \tan^5 x\,(1 + \tan^2 x)^2 \sec^2 x\,dx$

$= \int_0^{\sqrt{3}} u^5 (1 + u^2)^2\,du \quad [u = \tan x,\ du = \sec^2 x\,dx] \quad = \int_0^{\sqrt{3}} u^5 (1 + 2u^2 + u^4)\,du$

$= \int_0^{\sqrt{3}} (u^5 + 2u^7 + u^9)\,du = \left[\frac{1}{6}u^6 + \frac{1}{4}u^8 + \frac{1}{10}u^{10}\right]_0^{\sqrt{3}} = \frac{27}{6} + \frac{81}{4} + \frac{243}{10} = \frac{981}{20}$

31. $\int \tan^5 x\,dx = \int (\sec^2 x - 1)^2 \tan x\,dx = \int \sec^4 x \tan x\,dx - 2\int \sec^2 x \tan x\,dx + \int \tan x\,dx$

$= \int \sec^3 x \sec x \tan x\,dx - 2\int \tan x \sec^2 x\,dx + \int \tan x\,dx$

$= \frac{1}{4}\sec^4 x - \tan^2 x + \ln|\sec x| + C$ [or $\frac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C$]

32. $\int \tan^6 ay\,dy = \int \tan^4 ay\,(\sec^2 ay - 1)\,dy = \int \tan^4 ay \sec^2 ay\,dy - \int \tan^4 ay\,dy$

$= \frac{1}{5a}\tan^5 ay - \int \tan^2 ay\,(\sec^2 ay - 1)\,dy$

$= \frac{1}{5a}\tan^5 ay - \int \tan^2 ay \sec^2 ay\,dy + \int (\sec^2 ay - 1)\,dy$

$= \frac{1}{5a}\tan^5 ay - \frac{1}{3a}\tan^3 ay + \frac{1}{a}\tan ay - y + C$

33. Let $u = \tan x \quad \Rightarrow \quad du = \sec^2 x\,dx$. Then

$$\int \frac{\sec^2 x}{\cot x}\,dx = \int \tan x \sec^2 x\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2 x + C.$$

34. $\int \tan^2 x \sec x\,dx = \int (\sec^2 x - 1)\sec x\,dx = \int \sec^3 x\,dx - \int \sec x\,dx$

$= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) - \ln|\sec x + \tan x| + C$ [by Example 8 and (1)]

$= \frac{1}{2}(\sec x \tan x - \ln|\sec x + \tan x|) + C$

35. $\int_{\pi/6}^{\pi/2} \cot^2 x\,dx = \int_{\pi/6}^{\pi/2} (\csc^2 x - 1)\,dx = [-\cot x - x]_{\pi/6}^{\pi/2} = \left(0 - \frac{\pi}{2}\right) - \left(-\sqrt{3} - \frac{\pi}{6}\right) = \sqrt{3} - \frac{\pi}{3}$

36. $\int_{\pi/4}^{\pi/2} \cot^3 x\,dx = \int_{\pi/4}^{\pi/2} \cot x\left(\csc^2 x - 1\right)dx = \int_{\pi/4}^{\pi/2} \cot x \csc^2 x\,dx - \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x}\,dx$

$$= \left[-\tfrac{1}{2}\cot^2 x - \ln|\sin x|\right]_{\pi/4}^{\pi/2} = (0 - \ln 1) - \left[-\tfrac{1}{2} - \ln\tfrac{1}{\sqrt{2}}\right] = \tfrac{1}{2} + \ln\tfrac{1}{\sqrt{2}} = \tfrac{1}{2}(1 - \ln 2)$$

37. $\int \cot^2 w \csc^4 w\,dw = \int \cot^2 w \csc^2 w \csc^2 w\,dw = \int \cot^2 w\left(1 + \cot^2 w\right)\csc^2 w\,dw$

$$= \int u^2\left(1 + u^2\right)(-du) \quad [u = \cot w,\ du = -\csc^2 w\,dw] \quad = -\int\left(u^2 + u^4\right)du$$

$$= -\tfrac{1}{3}u^3 - \tfrac{1}{5}u^5 + C = -\tfrac{1}{3}\cot^3 w - \tfrac{1}{5}\cot^5 w + C$$

38. Let $u = \cot x \quad\Rightarrow\quad du = -\csc^2 x\,dx$. Then

$$\int \cot^3 x \csc^4 x\,dx = \int \cot^3 x\left(\cot^2 x + 1\right)\csc^2 x\,dx = \int u^3\left(u^2 + 1\right)(-du)$$

$$= -\tfrac{1}{6}u^6 - \tfrac{1}{4}u^4 + C = -\tfrac{1}{6}\cot^6 x - \tfrac{1}{4}\cot^4 x + C$$

39. $I = \displaystyle\int \csc x\,dx = \int \frac{\csc x\,(\csc x - \cot x)}{\csc x - \cot x}\,dx = \int \frac{-\csc x\cot x + \csc^2 x}{\csc x - \cot x}\,dx$. Let $u = \csc x - \cot x \quad\Rightarrow$ $du = \left(-\csc x\cot x + \csc^2 x\right)dx$. Then $I = \int du/u = \ln|u| = \ln|\csc x - \cot x| + C$.

40. Let $u = \csc x$, $dv = \csc^2 x\,dx$. Then $du = -\csc x\cot x\,dx$, $v = -\cot x \quad\Rightarrow$

$$\int \csc^3 x\,dx = -\csc x\cot x - \int \csc x\cot^2 x\,dx = -\csc x\cot x - \int \csc x\left(\csc^2 x - 1\right)dx$$

$$= -\csc x\cot x + \int \csc x\,dx - \int \csc^3 x\,dx$$

Solving for $\int \csc^3 x\,dx$ and using Exercise 39, we get
$\int \csc^3 x\,dx = -\frac{1}{2}\csc x\cot x + \frac{1}{2}\int \csc x\,dx = -\frac{1}{2}\csc x\cot x + \frac{1}{2}\ln|\csc x - \cot x| + C$. Thus,

$$\int_{\pi/6}^{\pi/3} \csc^3 x\,dx = \left[-\tfrac{1}{2}\csc x\cot x + \tfrac{1}{2}\ln|\csc x - \cot x|\right]_{\pi/6}^{\pi/3}$$

$$= -\tfrac{1}{2}\cdot\tfrac{2}{\sqrt{3}}\cdot\tfrac{1}{\sqrt{3}} + \tfrac{1}{2}\ln\left|\tfrac{2}{\sqrt{3}} - \tfrac{1}{\sqrt{3}}\right| + \tfrac{1}{2}\cdot 2\cdot\sqrt{3} - \tfrac{1}{2}\ln\left|2 - \sqrt{3}\right|$$

$$= -\tfrac{1}{3} + \sqrt{3} + \tfrac{1}{2}\ln\tfrac{1}{\sqrt{3}} - \tfrac{1}{2}\ln\left(2 - \sqrt{3}\right) \approx 1.7825$$

41. $\int \sin 5x \sin 2x\,dx = \int \frac{1}{2}\left[\cos(5x - 2x) - \cos(5x + 2x)\right]dx = \frac{1}{2}\int(\cos 3x - \cos 7x)\,dx$

$$= \tfrac{1}{6}\sin 3x - \tfrac{1}{14}\sin 7x + C$$

42. $\int \sin 3x \cos x\,dx = \int \frac{1}{2}\left[\sin(3x + x) + \sin(3x - x)\right]dx = \frac{1}{2}\int(\sin 4x + \sin 2x)\,dx$

$$= -\tfrac{1}{8}\cos 4x - \tfrac{1}{4}\cos 2x + C$$

43. $\int \cos 7\theta \cos 5\theta\,d\theta = \int \frac{1}{2}\left[\cos(7\theta - 5\theta) + \cos(7\theta + 5\theta)\right]d\theta = \frac{1}{2}\int(\cos 2\theta + \cos 12\theta)\,d\theta$

$$= \tfrac{1}{2}\left(\tfrac{1}{2}\sin 2\theta + \tfrac{1}{12}\sin 12\theta\right) + C = \tfrac{1}{4}\sin 2\theta + \tfrac{1}{24}\sin 12\theta + C$$

44. $\displaystyle\int \frac{\cos x + \sin x}{\sin 2x}\,dx = \frac{1}{2}\int \frac{\cos x + \sin x}{\sin x\cos x}\,dx$

$$= \tfrac{1}{2}\int(\csc x + \sec x)\,dx = \tfrac{1}{2}\left(\ln|\csc x - \cot x| + \ln|\sec x + \tan x|\right) + C$$

45. $\displaystyle\int \frac{1 - \tan^2 x}{\sec^2 x}\,dx = \int\left(\cos^2 x - \sin^2 x\right)dx = \int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$

46. $\int e^x \cos^7(e^x)\,dx = \int \cos^7 u\,du \quad [u = e^x,\ du = e^x\,dx] \quad = \int\left(\cos^2 u\right)^3 \cos u\,du = \int\left(1 - \sin^2 u\right)^3 \cos u\,du$

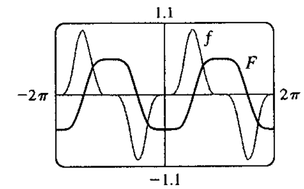
$$= \int\left(1 - v^2\right)^3 dv \quad [v = \sin u,\ dv = \cos u\,du] \quad = \int\left(1 - 3v^2 + 3v^4 - v^6\right)dv$$

$$= v - v^3 + \tfrac{3}{5}v^5 - \tfrac{1}{7}v^7 + C = \sin u - \sin^3 u + \tfrac{3}{5}\sin^5 u - \tfrac{1}{7}\sin^7 u + C$$

$$= \sin e^x - \sin^3 e^x + \tfrac{3}{5}\sin^5 e^x - \tfrac{1}{7}\sin^7 e^x + C$$

47. Let $u = \cos x \quad \Rightarrow \quad du = -\sin x\,dx$. Then

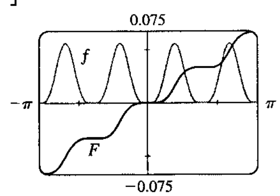
$$\begin{aligned}\int \sin^5 x\,dx &= \int \left(1-\cos^2 x\right)^2 \sin x\,dx = \int \left(1-u^2\right)^2 (-du)\\ &= \int \left(-1+2u^2-u^4\right) du = -\tfrac{1}{5}u^5+\tfrac{2}{3}u^3-u+C\\ &= -\tfrac{1}{5}\cos^5 x+\tfrac{2}{3}\cos^3 x-\cos x+C\end{aligned}$$



Notice that F is increasing when $f(x) > 0$, so the graphs serve as a check on our work.

48. $\int \sin^4 x \cos^4 x\,dx = \int \left(\frac{1}{2}\sin 2x\right)^4 dx = \frac{1}{16}\int \sin^4 2x\,dx = \frac{1}{16}\int \left[\frac{1}{2}(1-\cos 4x)\right]^2 dx$

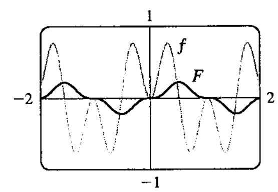
$$\begin{aligned}&= \tfrac{1}{64}\int \left(1-2\cos 4x+\cos^2 4x\right) dx\\ &= \tfrac{1}{64}\left(x-\tfrac{1}{2}\sin 4x\right)+\tfrac{1}{128}\int (1+\cos 8x)\,dx\\ &= \tfrac{1}{64}\left(x-\tfrac{1}{2}\sin 4x\right)+\tfrac{1}{128}\left(x+\tfrac{1}{8}\sin 8x\right)+C\\ &= \tfrac{3}{128}x-\tfrac{1}{128}\sin 4x+\tfrac{1}{1024}\sin 8x+C\end{aligned}$$



Notice that $f(x) = 0$ whenever F has a horizontal tangent.

49. $\int \sin 3x \sin 6x\,dx = \int \frac{1}{2}\left[\cos(3x-6x)-\cos(3x+6x)\right] dx$

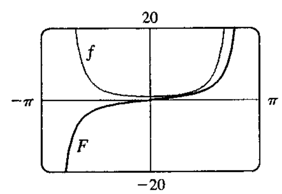
$$\begin{aligned}&= \tfrac{1}{2}\int (\cos 3x-\cos 9x)\,dx\\ &= \tfrac{1}{6}\sin 3x-\tfrac{1}{18}\sin 9x+C\end{aligned}$$



Notice that $f(x) = 0$ whenever F has a horizontal tangent.

50. $\int \sec^4 \frac{x}{2}\,dx = \int \left(\tan^2 \frac{x}{2}+1\right)\sec^2 \frac{x}{2}\,dx$

$$\begin{aligned}&= \int \left(u^2+1\right) 2\,du \quad \left[u=\tan\tfrac{x}{2},\ du=\tfrac{1}{2}\sec^2\tfrac{x}{2}\,dx\right]\\ &= \tfrac{2}{3}u^3+2u+C = \tfrac{2}{3}\tan^3\tfrac{x}{2}+2\tan\tfrac{x}{2}+C\end{aligned}$$



Notice that F is increasing and f is positive on the intervals on which they are defined. Also, F has no horizontal tangent and f is never zero.

51. $f_{\text{ave}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 x \cos^3 x\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 x\left(1-\sin^2 x\right)\cos x\,dx = \frac{1}{2\pi}\int_0^0 u^2\left(1-u^2\right) du$ (where $u = \sin x$) $= 0$

52. (a) Let $u = \cos x$. Then $du = -\sin x\,dx \quad \Rightarrow \quad \int \sin x \cos x\,dx = \int u\,(-du) = -\frac{1}{2}u^2+C = -\frac{1}{2}\cos^2 x+C_1$.

(b) Let $u = \sin x$. Then $du = \cos x\,dx \quad \Rightarrow \quad \int \sin x \cos x\,dx = \int u\,du = \frac{1}{2}u^2+C = \frac{1}{2}\sin^2 x+C_2$.

(c) $\int \sin x \cos x\,dx = \int \frac{1}{2}\sin 2x\,dx = -\frac{1}{4}\cos 2x+C_3$

(d) Let $u = \sin x$, $dv = \cos x\,dx$. Then $du = \cos x\,dx$, $v = \sin x$, so $\int \sin x \cos x\,dx = \sin^2 x - \int \sin x \cos x\,dx$, by Equation 2, so $\int \sin x \cos x\,dx = \frac{1}{2}\sin^2 x+C_4$.

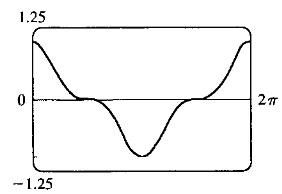
The answers differ from one another by constants. Since $\cos 2x = 1-2\sin^2 x = 2\cos^2 x-1$, we find that $-\frac{1}{4}\cos 2x = \frac{1}{2}\sin^2 x-\frac{1}{4} = -\frac{1}{2}\cos^2 x+\frac{1}{4}$.

53. For $0 < x < \frac{\pi}{2}$, we have $0 < \sin x < 1$, so $\sin^3 x < \sin x$. Hence the area is $\int_0^{\pi/2} \left(\sin x - \sin^3 x\right) dx = \int_0^{\pi/2} \sin x \left(1 - \sin^2 x\right) dx = \int_0^{\pi/2} \cos^2 x \sin x \, dx$. Now let $u = \cos x \Rightarrow du = -\sin x \, dx$. Then area $= \int_1^0 u^2 \, (-du) = \int_0^1 u^2 du = \left[\frac{1}{3}u^3\right]_0^1 = \frac{1}{3}$.

54. $\sin x > 0$ for $0 < x < \frac{\pi}{2}$, so the sign of $2\sin^2 x - \sin x$ [which equals $2 \sin x \left(\sin x - \frac{1}{2}\right)$] is the same as that of $\sin x - \frac{1}{2}$. Thus $2\sin^2 x - \sin x$ is positive on $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ and negative on $\left(0, \frac{\pi}{6}\right)$. The desired area is

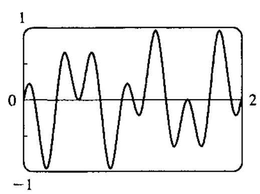
$$\begin{aligned}
\int_0^{\pi/6} \left(\sin x - 2\sin^2 x\right) dx + \int_{\pi/6}^{\pi/2} \left(2\sin^2 x - \sin x\right) dx &= \int_0^{\pi/6} (\sin x - 1 + \cos 2x)\, dx \\
&\quad + \int_{\pi/6}^{\pi/2} (1 - \cos 2x - \sin x)\, dx \\
&= \left[-\cos x - x + \tfrac{1}{2}\sin 2x\right]_0^{\pi/6} + \left[x - \tfrac{1}{2}\sin 2x + \cos x\right]_{\pi/6}^{\pi/2} \\
&= -\tfrac{\sqrt{3}}{2} - \tfrac{\pi}{6} + \tfrac{\sqrt{3}}{4} - (-1) + \tfrac{\pi}{2} - \left(\tfrac{\pi}{6} - \tfrac{\sqrt{3}}{4} + \tfrac{\sqrt{3}}{2}\right) = 1 + \tfrac{\pi}{6} - \tfrac{\sqrt{3}}{2}
\end{aligned}$$

55.



It seems from the graph that $\int_0^{2\pi} \cos^3 x \, dx = 0$, since the area below the x-axis and above the graph looks about equal to the area above the axis and below the graph. By Example 1, the integral is $\left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{2\pi} = 0$. Note that due to symmetry, the integral of any odd power of $\sin x$ or $\cos x$ between limits which differ by $2n\pi$ (n any integer) is 0.

56.



It seems from the graph that $\int_0^2 \sin 2\pi x \cos 5\pi x \, dx = 0$, since each bulge above the x-axis seems to have a corresponding depression below the x-axis. To evaluate the integral, we use a trigonometric identity:

$$\begin{aligned}
\int_0^1 \sin 2\pi x \cos 5\pi x \, dx &= \tfrac{1}{2}\int_0^2 \left[\sin(2\pi x - 5\pi x) + \sin(2\pi x + 5\pi x)\right] dx \\
&= \tfrac{1}{2}\int_0^2 \left[\sin(-3\pi x) + \sin 7\pi x\right] dx \\
&= \tfrac{1}{2}\left[\tfrac{1}{3\pi}\cos(-3\pi x) - \tfrac{1}{7\pi}\cos 7\pi x\right]_0^2 \\
&= \tfrac{1}{2}\left[\tfrac{1}{3\pi}(1 - 1) - \tfrac{1}{8\pi}(1 - 1)\right] = 0
\end{aligned}$$

57. $V = \int_{\pi/2}^{\pi} \pi \sin^2 x \, dx = \pi \int_{\pi/2}^{\pi} \frac{1}{2}(1 - \cos 2x)\, dx = \pi \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_{\pi/2}^{\pi} = \pi \left(\frac{\pi}{2} - 0 - \frac{\pi}{4} + 0\right) = \frac{\pi^2}{4}$

58. Volume $= \int_0^{\pi/4} \pi \left(\tan^2 x\right)^2 dx = \pi \int_0^{\pi/4} \tan^2 x \left(\sec^2 x - 1\right) dx = \pi \int_0^{\pi/4} \tan^2 x \sec^2 x \, dx - \pi \int_0^{\pi/4} \tan^2 x \, dx$

$= \pi \int_0^{\pi/4} u^2 du - \pi \int_0^{\pi/4} \left(\sec^2 x - 1\right) dx$ [where $u = \tan x$ and $du = \sec^2 x \, dx$]

$= \pi \left[\frac{1}{3}u^3\right]_0^{\pi/4} - \pi \left[\tan x - x\right]_0^{\pi/4} = \pi \left[\frac{1}{3}\tan^3 x - \tan x + x\right]_0^{\pi/4} = \pi \left[\frac{1}{3} - 1 + \frac{\pi}{4}\right] = \pi \left(\frac{\pi}{4} - \frac{2}{3}\right)$

59. Volume $= \pi \int_0^{\pi/2} \left[(1 + \cos x)^2 - 1^2\right] dx = \pi \int_0^{\pi/2} \left(2\cos x + \cos^2 x\right) dx$

$= \pi \left[2\sin x + \frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\pi/2} = \pi \left(2 + \frac{\pi}{4}\right) = 2\pi + \frac{\pi^2}{4}$

60. Volume $= \pi \int_0^{\pi/2} \left[1^2 - (1-\cos x)^2\right] dx = \pi \int_0^{\pi/2} \left(2\cos x - \cos^2 x\right) dx$

$$= \pi \left[2\sin x - \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_0^{\pi/2} = \pi \left[\left(2 - \tfrac{\pi}{4} - 0\right) - 0\right] = 2\pi - \tfrac{\pi^2}{4}$$

61. $s = f(t) = \int_0^t \sin\omega u \cos^2 \omega u\, du$. Let $y = \cos\omega u \;\Rightarrow\; dy = -\omega \sin\omega u\, du$. Then

$$s = -\tfrac{1}{\omega}\int_1^{\cos\omega t} y^2 dy = -\tfrac{1}{\omega}\left[\tfrac{1}{3}y^3\right]_1^{\cos\omega t} = \tfrac{1}{3\omega}\left(1 - \cos^3\omega t\right).$$

62. (a) We want to calculate the square root of the average value of $[E(t)]^2 = [155\sin(120\pi t)]^2 = 155^2\sin^2(120\pi t)$. First, we calculate the average value itself, by integrating $[E(t)]^2$ over one cycle (between $t = 0$ and $t = \frac{1}{60}$, since there are 60 cycles per second) and dividing by $\left(\frac{1}{60} - 0\right)$:

$$[E(t)]^2_{\text{ave}} = \tfrac{1}{1/60}\int_0^{1/60}\left[155^2\sin^2(120\pi t)\right] dt = 60 \cdot 155^2 \int_0^{1/60} \tfrac{1}{2}\left[1 - 2\cos(240\pi t)\right] dt$$

$$= 60 \cdot 155^2\left(\tfrac{1}{2}\right)\left[t - 2\cdot\tfrac{1}{240\pi}\sin(240\pi t)\right]_0^{1/60} = 60\cdot 155^2\left(\tfrac{1}{2}\right)\left[\left(\tfrac{1}{60} - 0\right) - (0 - 0)\right] = \tfrac{155^2}{2}$$

The RMS value is just the square root of this quantity, which is $\frac{155}{\sqrt{2}} \approx 110$ V.

(b) $220 = \sqrt{[E(t)]^2_{\text{ave}}} \;\Rightarrow$

$$220^2 = [E(t)]^2_{\text{ave}} = \tfrac{1}{1/60}\int_0^{1/60} A^2\sin^2(120\pi t)\, dt = 60A^2\int_0^{1/60}\tfrac{1}{2}\left[1 - 2\cos(240\pi t)\right] dt$$

$$= 30A^2\left[t - \tfrac{2}{240\pi}\sin(240\pi t)\right]_0^{1/60} = 30A^2\left[\left(\tfrac{1}{60} - 0\right) - (0 - 0)\right] = \tfrac{1}{2}A^2$$

Thus, $220^2 = \frac{1}{2}A^2 \;\Rightarrow\; A = 220\sqrt{2} \approx 311$ V.

63. Just note that the integrand is odd $[f(-x) = -f(x)]$.

Or: If $m \neq n$, calculate

$$\int_{-\pi}^{\pi} \sin mx \cos nx\, dx = \int_{-\pi}^{\pi} \tfrac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right] dx$$

$$= \frac{1}{2}\left[-\frac{\cos(m-n)x}{m-n} - \frac{\cos(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$$

If $m = n$, then the first term in each set of brackets is zero.

64. $\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right] dx$. If $m \neq n$, this is equal to $\dfrac{1}{2}\left[\dfrac{\sin(m-n)x}{m-n} - \dfrac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$. If $m = n$, we get

$$\int_{-\pi}^{\pi} \tfrac{1}{2}\left[1 - \cos(m+n)x\right] dx = \left[\tfrac{1}{2}x\right]_{-\pi}^{\pi} - \left[\frac{\sin(m+n)x}{2(m+n)}\right]_{-\pi}^{\pi} = \pi - 0 = \pi.$$

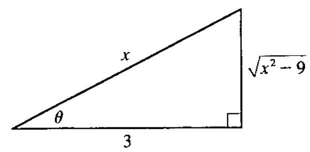
65. $\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right] dx$. If $m \neq n$, this is equal to $\dfrac{1}{2}\left[\dfrac{\sin(m-n)x}{m-n} + \dfrac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$. If $m = n$, we get

$$\int_{-\pi}^{\pi} \tfrac{1}{2}\left[1 + \cos(m+n)x\right] dx = \left[\tfrac{1}{2}x\right]_{-\pi}^{\pi} + \left[\frac{\sin(m+n)x}{2(m+n)}\right]_{-\pi}^{\pi} = \pi + 0 = \pi.$$

66. $\displaystyle \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\, dx = \sum_{n=1}^{m} \frac{a_n}{\pi}\int_{-\pi}^{\pi} \sin mx \sin nx\, dx$. By Exercise 64, every term is zero except the mth one, and that term is $\dfrac{a_m}{\pi}\cdot\pi = a_m$.

8.3 Trigonometric Substitution

1. Let $x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = 3\sec\theta\tan\theta\,d\theta$ and

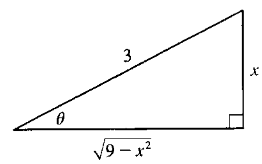


$$\sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9} = \sqrt{9\left(\sec^2\theta - 1\right)} = \sqrt{9\tan^2\theta}$$
$$= 3\,|\tan\theta| = 3\tan\theta \text{ for the relevant values of } \theta.$$

$$\int \frac{1}{x^2\sqrt{x^2-9}}\,dx = \int \frac{1}{9\sec^2\theta \cdot 3\tan\theta} 3\sec\theta\tan\theta\,d\theta = \tfrac{1}{9}\int \cos\theta\,d\theta = \tfrac{1}{9}\sin\theta + C = \frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C$$

Note that $-\sec(\theta+\pi) = \sec\theta$, so the figure is sufficient for the case $\pi \le \theta < \frac{3\pi}{2}$.

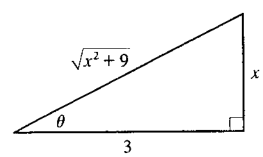
2. Let $x = 3\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 3\cos\theta\,d\theta$ and



$$\sqrt{9-x^2} = \sqrt{9 - 9\sin^2\theta} = \sqrt{9\left(1-\sin^2\theta\right)} = \sqrt{9\cos^2\theta}$$
$$= 3\,|\cos\theta| = 3\cos\theta \text{ for the relevant values of } \theta.$$

$$\begin{aligned}
\int x^3\sqrt{9-x^2}\,dx &= \int 3^3\sin^3\theta \cdot 3\cos\theta \cdot 3\cos\theta\,d\theta = 3^5\int \sin^3\theta\cos^2\theta\,d\theta \\
&= 3^5\int \sin^2\theta\cos^2\theta\sin\theta\,d\theta = 3^5\int\left(1-\cos^2\theta\right)\cos^2\theta\sin\theta\,d\theta \\
&= 3^5\int\left(1-u^2\right)u^2\,(-du) \quad [u = \cos\theta,\ du = -\sin\theta\,d\theta] \\
&= 3^5\int\left(u^4-u^2\right)du = 3^5\left(\tfrac{1}{5}u^5 - \tfrac{1}{3}u^3\right) + C = 3^5\left(\tfrac{1}{5}\cos^5\theta - \tfrac{1}{3}\cos^3\theta\right) + C \\
&= 3^5\left[\frac{1}{5}\frac{\left(9-x^2\right)^{5/2}}{3^5} - \frac{1}{3}\frac{\left(9-x^2\right)^{3/2}}{3^3}\right] + C \\
&= \tfrac{1}{5}\left(9-x^2\right)^{5/2} - 3\left(9-x^2\right)^{3/2} + C \text{ or } -\tfrac{1}{5}\left(x^2+6\right)(9-x)^{3/2} + C
\end{aligned}$$

3. Let $x = 3\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 3\sec^2\theta\,d\theta$ and



$$\sqrt{x^2+9} = \sqrt{9\tan^2\theta + 9} = \sqrt{9\left(\tan^2\theta + 1\right)} = \sqrt{9\sec^2\theta}$$
$$= 3\,|\sec\theta| = 3\sec\theta \text{ for the relevant values of } \theta.$$

$$\begin{aligned}
\int \frac{x^3}{\sqrt{x^2+9}}\,dx &= \int \frac{3^3\tan^3\theta}{3\sec\theta} 3\sec^2\theta\,d\theta = 3^3\int \tan^3\theta\sec\theta d\theta = 3^3\int \tan^2\theta\tan\theta\sec\theta\,d\theta \\
&= 3^3\int\left(\sec^2\theta - 1\right)\tan\theta\sec\theta\,d\theta = 3^3\int\left(u^2-1\right)du \quad [u = \sec\theta,\ du = \sec\theta\tan\theta\,d\theta] \\
&= 3^3\left(\tfrac{1}{3}u^3 - u\right) + C = 3^3\left(\tfrac{1}{3}\sec^3\theta - \sec\theta\right) + C = 3^3\left[\frac{1}{3}\frac{\left(x^2+9\right)^{3/2}}{3^3} - \frac{\sqrt{x^2+9}}{3}\right] + C \\
&= \tfrac{1}{3}\left(x^2+9\right)^{3/2} - 9\sqrt{x^2+9} + C \text{ or } \tfrac{1}{3}\left(x^2-18\right)\sqrt{x^2+9} + C
\end{aligned}$$

4. Let $x = 4\sin\theta$. Then $dx = 4\cos\theta\, d\theta$, so

$$\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx = \int_0^{\pi/3} \frac{4^3\sin^3\theta}{4\cos\theta} 4\cos\theta\,d\theta = 4^3 \int_0^{\pi/3} \sin^3\theta\,d\theta = 4^3 \int_0^{\pi/3} \left(1-\cos^2\theta\right)\sin\theta\,d\theta$$

$$= -4^3 \int_1^{1/2} \left(1-u^2\right) du \quad [u = \cos\theta,\ du = -\sin\theta\,d\theta] \quad = -64\left[u - \tfrac{1}{3}u^3\right]_1^{1/2}$$

$$= -64\left[\left(\tfrac{1}{2} - \tfrac{1}{24}\right) - \left(1 - \tfrac{1}{3}\right)\right] = -64\left(-\tfrac{5}{24}\right) = \tfrac{40}{3}$$

Or: Let $u = 16 - x^2$, $x^2 = 16 - u$, $du = -2x\,dx$.

5. Let $t = \sec\theta$, so $dt = \sec\theta\tan\theta\,d\theta$, $t = \sqrt{2} \Rightarrow \theta = \frac{\pi}{4}$, and $t = 2 \Rightarrow \theta = \frac{\pi}{3}$. Then

$$\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\,dt = \int_{\pi/4}^{\pi/3} \frac{1}{\sec^3\theta\tan\theta}\sec\theta\tan\theta\,d\theta = \int_{\pi/4}^{\pi/3} \cos^2\theta\,d\theta = \int_{\pi/4}^{\pi/3} \tfrac{1}{2}\left(1+\cos 2\theta\right)d\theta$$

$$= \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/3} = \tfrac{1}{2}\left[\left(\tfrac{\pi}{3} + \tfrac{1}{2}\tfrac{\sqrt{3}}{2}\right) - \left(\tfrac{\pi}{4} + \tfrac{1}{2}\cdot 1\right)\right]$$

$$= \tfrac{1}{2}\left(\tfrac{\pi}{12} + \tfrac{\sqrt{3}}{4} - \tfrac{1}{2}\right) = \tfrac{\pi}{24} + \tfrac{\sqrt{3}}{8} - \tfrac{1}{4}$$

6. Let $x = 2\tan\theta$, so $dx = 2\sec^2\theta\,d\theta$, $x = 0 \Rightarrow \theta = 0$, and $x = 2 \Rightarrow \theta = \frac{\pi}{4}$. Then

$$\int_0^2 x^3\sqrt{x^2+4}\,dx = \int_0^{\pi/4} 2^3\tan^3\theta\cdot 2\sec\theta\cdot 2\sec^2\theta\,d\theta$$

$$= 2^5\int_0^{\pi/4} \frac{\sin^3\theta}{\cos^6\theta}\,d\theta = 2^5\int_0^{\pi/4} \frac{\sin^2\theta}{\cos^6\theta}\sin\theta\,d\theta = 2^5\int_0^{\pi/4} \frac{1-\cos^2\theta}{\cos^6\theta}\sin\theta\,d\theta$$

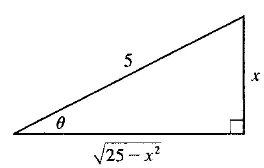
$$= -2^5\int_1^{1/\sqrt{2}} \frac{1-u^2}{u^6}\,du \quad [u = \cos\theta, du = -\sin\theta\,d\theta] \quad = 2^5\int_{1/\sqrt{2}}^{1} \left(u^{-6} - u^{-4}\right)du$$

$$= 2^5\left[-\frac{1}{5u^5} + \frac{1}{3u^3}\right]_{1/\sqrt{2}}^{1} = 2^5\left[\left(-\tfrac{1}{5} + \tfrac{1}{3}\right) - \left(-\tfrac{4\sqrt{2}}{5} + \tfrac{2\sqrt{2}}{3}\right)\right]$$

$$= 2^5\left(\tfrac{2}{15} + \tfrac{2\sqrt{2}}{15}\right) = \tfrac{64}{15}\left(\sqrt{2}+1\right)$$

Or: Let $u = x^2 + 4$, $x^2 = u - 4$, $du = 2x\,dx$.

7. Let $x = 5\sin\theta$, so $dx = 5\cos\theta\,d\theta$. Then



$$\int \frac{1}{x^2\sqrt{25-x^2}}\,dx = \int \frac{1}{5^2\sin^2\theta\cdot 5\cos\theta} 5\cos\theta\,d\theta$$

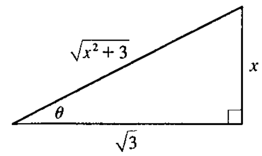
$$= \tfrac{1}{25}\int \csc^2\theta\,d\theta = -\tfrac{1}{25}\cot\theta + C$$

$$= -\frac{1}{25}\frac{\sqrt{25-x^2}}{x} + C$$

8. Let $u = x^2 + 4 \Rightarrow du = 2x\,dx$. Then

$$\int \frac{x\,dx}{\left(x^2+4\right)^{5/2}} = \frac{1}{2}\int u^{-5/2}du = \frac{1}{2}\left(-\frac{2}{3}\right)u^{-3/2} + C = \left(-\frac{1}{3}\right)u^{-3/2} + C = \left(-\frac{1}{3}\right)\left(x^2+4\right)^{-3/2} + C.$$

9. Let $x = \sqrt{3}\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then

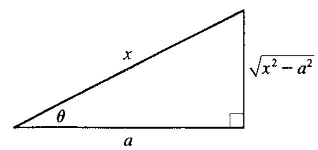


$$\int \frac{dx}{x\sqrt{x^2+3}} = \int \frac{\sqrt{3}\sec^2\theta d\theta}{\sqrt{3}\tan\theta\sqrt{3}\sec\theta}$$

$$= \tfrac{1}{\sqrt{3}}\int \csc\theta d\theta = \tfrac{1}{\sqrt{3}}\ln|\csc\theta - \cot\theta| + C$$

$$= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{x^2+3}-\sqrt{3}}{x}\right| + C$$

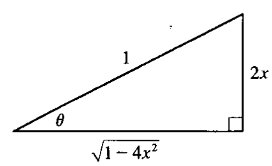
10. Let $x = a\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = a\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-a^2} = a\tan\theta$, so



$$\int \frac{\sqrt{x^2-a^2}}{x^4}dx = \int \frac{a\tan\theta}{a^4\sec^4\theta}a\sec\theta\tan\theta\,d\theta = \frac{1}{a^2}\int \sin^2\theta\cos\theta\,d\theta$$

$$= \frac{1}{3a^2}\sin^3\theta + C = \frac{\left(x^2-a^2\right)^{3/2}}{3a^2x^3} + C$$

11. Let $2x = \sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $x = \frac{1}{2}\sin\theta$, $dx = \frac{1}{2}\cos\theta\,d\theta$, and $\sqrt{1-4x^2} = \sqrt{1-(2x)^2} = \cos\theta$.



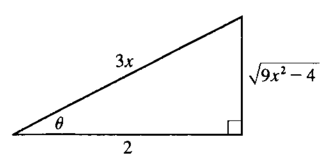
$$\int \sqrt{1-4x^2}dx = \int \cos\theta\left(\tfrac{1}{2}\cos\theta\right)d\theta = \tfrac{1}{4}\int(1+\cos 2\theta)\,d\theta$$

$$= \tfrac{1}{4}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{1}{4}(\theta + \sin\theta\cos\theta) + C$$

$$= \tfrac{1}{4}\left[\sin^{-1}(2x) + 2x\sqrt{1-4x^2}\right] + C$$

12. Let $u = 25 + x^2$, so $du = 2x\,dx$. Then $\int x\sqrt{25+x^2}\,dx = \frac{1}{2}\int\sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{3}\left(x^2+25\right)^{3/2} + C$.

13. $9x^2 - 4 = (3x)^2 - 4$, so let $3x = 2\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = \frac{2}{3}\sec\theta\tan\theta\,d\theta$ and $\sqrt{9x^2-4} = 2\tan\theta$.

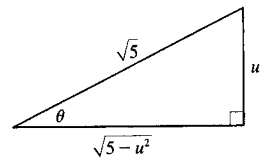


$$\int \frac{\sqrt{9x^2-4}}{x}dx = \int \frac{2\tan\theta}{\frac{2}{3}\sec\theta}\cdot\frac{2}{3}\sec\theta\tan\theta d\theta$$

$$= 2\int \tan^2\theta d\theta = 2\int\left(\sec^2\theta - 1\right)d\theta = 2(\tan\theta - \theta) + C$$

$$= \sqrt{9x^2-4} - 2\sec^{-1}\left(\frac{3x}{2}\right) + C$$

14. Let $u = \sqrt{5}\sin\theta$, so $du = \sqrt{5}\cos\theta\, d\theta$. Then



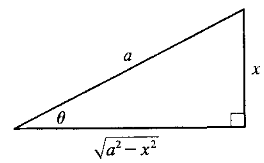
$$\int \frac{du}{u\sqrt{5-u^2}} = \int \frac{1}{\sqrt{5}\sin\theta \cdot \sqrt{5}\cos\theta}\sqrt{5}\cos\theta\, d\theta = \frac{1}{\sqrt{5}}\int \csc\theta\, d\theta$$

$$= \tfrac{1}{\sqrt{5}}\ln|\csc\theta - \cot\theta| + C \quad \text{(by Exercise 8.2.39)}$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}}{u} - \frac{\sqrt{5-u^2}}{u}\right| + C$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}-\sqrt{5-u^2}}{u}\right| + C$$

15. Let $x = a\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = a\cos\theta\, d\theta$ and

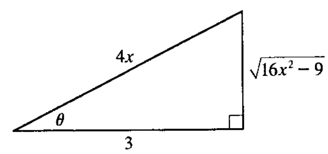


$$\int \frac{x^2\,dx}{(a^2-x^2)^{3/2}} = \int \frac{a^2\sin^2\theta a\cos\theta\, d\theta}{a^3\cos^3\theta} = \int \tan^2\theta\, d\theta$$

$$= \int \left(\sec^2\theta - 1\right) d\theta = \tan\theta - \theta + C$$

$$= \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a} + C$$

16. Let $4x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = \frac{3}{4}\sec\theta\tan\theta\, d\theta$ and $\sqrt{16x^2-9} = 3\tan\theta$, so



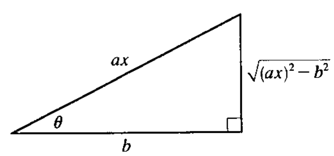
$$\int \frac{dx}{x^2\sqrt{16x^2-9}} = \int \frac{\frac{3}{4}\sec\theta\tan\theta d\theta}{\left(\frac{3}{4}\right)^2\sec^2\theta\, 3\tan\theta}$$

$$= \tfrac{4}{9}\int \cos\theta\, d\theta = \tfrac{4}{9}\sin\theta + C$$

$$= \frac{4}{9}\frac{\sqrt{16x^2-9}}{4x} + C = \frac{\sqrt{16x^2-9}}{9x} + C$$

17. Let $u = x^2 - 7$, so $du = 2x\,dx$. Then $\displaystyle\int \frac{x}{\sqrt{x^2-7}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \tfrac{1}{2}\cdot 2\sqrt{u} + C = \sqrt{x^2-7} + C$.

18. Let $ax = b\sec\theta$, so $(ax)^2 = b^2\sec^2\theta \;\Rightarrow$ $(ax)^2 - b^2 = b^2\sec^2\theta - b^2 = b^2\left(\sec^2\theta - 1\right) = b^2\tan^2\theta$. So $\sqrt{(ax^2) - b^2} = b\tan\theta$, $dx = \dfrac{b}{a}\sec\theta\tan\theta\, d\theta$, and



$$\int \frac{dx}{\left[(ax)^2 - b^2\right]^{3/2}} = \int \frac{\frac{b}{a}\sec\theta\tan\theta}{b^3\tan^3\theta}\,d\theta = \frac{1}{ab^2}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{ab^2}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{ab^2}\int \csc\theta\cot\theta\, d\theta$$

$$= -\frac{1}{ab^2}\csc\theta + C = -\frac{1}{ab^2}\frac{ax}{\sqrt{(ax)^2-b^2}} + C = -\frac{x}{b^2\sqrt{(ax)^2-b^2}} + C$$

19. Let $x = 3\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 3\sec^2\theta\,d\theta$ and $\sqrt{9+x^2} = 3\sec\theta$.

$$\int_0^3 \frac{dx}{\sqrt{9+x^2}} = \int_0^{\pi/4} \frac{3\sec^2\theta\,d\theta}{3\sec\theta} = \int_0^{\pi/4} \sec\theta\,d\theta = [\ln|\sec\theta + \tan\theta|]_0^{\pi/4}$$
$$= \ln\left(\sqrt{2}+1\right) - \ln 1 = \ln\left(\sqrt{2}+1\right)$$

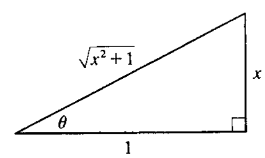
20. Let $u = 9 - x^2$, so $du = -2x\,dx$. Then $\int_0^3 x\sqrt{9-x^2}\,dx = -\frac{1}{2}\int_9^0 \sqrt{u}\,du = -\frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_9^0 = -\frac{1}{3}(0-27) = 9$.

21. Let $u = 4 - 9x^2 \Rightarrow du = -18x\,dx$. Then $x^2 = \frac{1}{9}(4-u)$ and

$$\int_0^{2/3} x^3\sqrt{4-9x^2}\,dx = \int_4^0 \tfrac{1}{9}(4-u)\,u^{1/2}\left(-\tfrac{1}{18}\right)du = \tfrac{1}{162}\int_0^4 \left(4u^{1/2} - u^{3/2}\right)du$$
$$= \tfrac{1}{162}\left[\tfrac{8}{3}u^{3/2} - \tfrac{2}{5}u^{5/2}\right]_0^4 = \tfrac{1}{162}\left[\tfrac{64}{3} - \tfrac{64}{5}\right] = \tfrac{64}{1215}$$

Or: Let $3x = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

22. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$, $\sqrt{x^2+1} = \sec\theta$ and $x = 0 \Rightarrow \theta = 0, x = 1 \Rightarrow \theta = \frac{\pi}{4}$, so

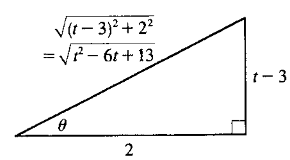


$$\int_0^1 \sqrt{x^2+1}\,dx = \int_0^{\pi/4} \sec\theta\sec^2\theta\,d\theta = \int_0^{\pi/4} \sec^3\theta\,d\theta$$
$$= \tfrac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]_0^{\pi/4} \quad \text{(by Example 8.2.8)}$$
$$= \tfrac{1}{2}\left[\sqrt{2}\cdot 1 + \ln\left(1+\sqrt{2}\right) - 0 - \ln(1+0)\right] = \tfrac{1}{2}\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$$

23. $2x - x^2 = -\left(x^2 - 2x + 1\right) + 1 = 1 - (x-1)^2$. Let $u = x - 1$. Then $du = dx$ and

$$\int\sqrt{2x-x^2}\,dx = \int\sqrt{1-u^2}\,du = \int\cos^2\theta\,d\theta \quad \left(\text{where } u = \sin\theta, -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}\right)$$
$$= \tfrac{1}{2}\int(1+\cos 2\theta)\,d\theta = \tfrac{1}{2}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{1}{2}\left(\sin^{-1}u + u\sqrt{1-u^2}\right) + C$$
$$= \tfrac{1}{2}\left[\sin^{-1}(x-1) + (x-1)\sqrt{2x-x^2}\right] + C$$

24. $t^2 - 6t + 13 = \left(t^2 - 6t + 9\right) + 4 = (t-3)^2 + 2^2$. Let $t - 3 = 2\tan\theta$, so $dt = 2\sec^2\theta\,d\theta$. Then



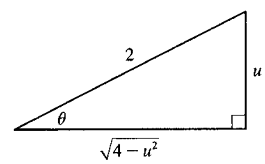
$$\int\frac{dt}{\sqrt{t^2-6t+13}} = \int\frac{1}{\sqrt{(2\tan\theta)^2+2^2}}\,2\sec^2\theta\,d\theta = \int\frac{2\sec^2\theta}{2\sec\theta}\,d\theta$$
$$= \int\sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C_1 \quad \text{(by Formula 8.2.1)}$$
$$= \ln\left|\frac{\sqrt{t^2-6t+13}}{2} + \frac{t-3}{2}\right| + C_1 = \ln\left|\sqrt{t^2-6t+13} + t - 3\right| + C \quad \text{where } C = C_1 - \ln 2$$

25. $9x^2 + 6x - 8 = (3x+1)^2 - 9$, so let $u = 3x+1$, $du = 3dx$. Then $\displaystyle\int \frac{dx}{\sqrt{9x^2+6x-8}} = \int \frac{\frac{1}{3}\,du}{\sqrt{u^2-9}}$. Now let $u = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $du = 3\sec\theta\tan\theta\,d\theta$ and $\sqrt{u^2-9} = 3\tan\theta$, so

$$\int \frac{\frac{1}{3}\,du}{\sqrt{u^2-9}} = \int \frac{\sec\theta\tan\theta\,d\theta}{3\tan\theta} = \tfrac{1}{3}\int \sec\theta d\theta = \tfrac{1}{3}\ln|\sec\theta + \tan\theta| + C_1 = \tfrac{1}{3}\ln\left|\tfrac{u+\sqrt{u^2-9}}{3}\right| + C_1$$
$$= \tfrac{1}{3}\ln\left|u + \sqrt{u^2-9}\right| + C = \tfrac{1}{3}\ln\left|3x+1+\sqrt{9x^2+6x-8}\right| + C$$

26. $4x - x^2 = -\left(x^2 - 4x + 4\right) + 4 = 4 - (x-2)^2$, so let $u = x - 2$. Then $x = u + 2$ and $dx = du$, so

$$\int \frac{x^2\,dx}{\sqrt{4x-x^2}} = \int \frac{(u+2)^2\,du}{\sqrt{4-u^2}} = \int \frac{(2\sin\theta+2)^2}{2\cos\theta} 2\cos\theta\,d\theta \quad (\text{Put } u = 2\sin\theta)$$
$$= 4\int \left(\sin^2\theta + 2\sin\theta + 1\right) d\theta = 2\int (1-\cos 2\theta)\,d\theta + 8\int \sin\theta d\theta + 4\int d\theta$$
$$= 2\theta - \sin 2\theta - 8\cos\theta + 4\theta + C = 6\theta - 8\cos\theta - 2\sin\theta\cos\theta + C$$
$$= 6\sin^{-1}\left(\tfrac{1}{2}u\right) - 4\sqrt{4-u^2} - \tfrac{1}{2}u\sqrt{4-u^2} + C$$
$$= 6\sin^{-1}\left(\frac{x-2}{2}\right) - 4\sqrt{4x-x^2} - \left(\frac{x-2}{2}\right)\sqrt{4x-x^2} + C$$



27. $x^2 + 2x + 2 = (x+1)^2 + 1$. Let $u = x+1$, $du = dx$. Then

$$\int \frac{dx}{\left(x^2+2x+2\right)^2} = \int \frac{du}{\left(u^2+1\right)^2} = \int \frac{\sec^2\theta d\theta}{\sec^4\theta} \quad \begin{pmatrix}\text{where } u = \tan\theta,\ du = \sec^2\theta\,d\theta,\\ \text{and } u^2+1 = \sec^2\theta\end{pmatrix}$$
$$= \int \cos^2\theta\,d\theta = \tfrac{1}{2}\int (1+\cos 2\theta)\,d\theta = \tfrac{1}{2}(\theta + \sin\theta\cos\theta) + C$$
$$= \frac{1}{2}\left[\tan^{-1}u + \frac{u}{1+u^2}\right] + C = \frac{1}{2}\left[\tan^{-1}(x+1) + \frac{x+1}{x^2+2x+2}\right] + C$$

28. $5 - 4x - x^2 = -\left(x^2 + 4x + 4\right) + 9 = 9 - (x+2)^2$. Let $u = x + 2 \;\Rightarrow\; du = dx$. Then

$$\int \frac{dx}{\left(5-4x-x^2\right)^{5/2}} = \int \frac{du}{\left(9-u^2\right)^{5/2}} = \int \frac{3\cos\theta\,d\theta}{(3\cos\theta)^5} \quad \begin{pmatrix}\text{where } u = 3\sin\theta,\ du = 3\cos\theta\,d\theta,\\ \text{and } \sqrt{9-u^2} = 3\cos\theta\end{pmatrix}$$
$$= \tfrac{1}{81}\int \sec^4\theta\,d\theta = \tfrac{1}{81}\int \left(\tan^2\theta + 1\right)\sec^2\theta\,d\theta = \tfrac{1}{81}\left[\tfrac{1}{3}\tan^3\theta + \tan\theta\right] + C$$
$$= \frac{1}{243}\left[\frac{u^3}{\left(9-u^2\right)^{3/2}} + \frac{3u}{\sqrt{9-u^2}}\right] + C = \frac{1}{243}\left[\frac{(x+2)^3}{\left(5-4x-x^2\right)^{3/2}} + \frac{3(x+2)}{\sqrt{5-4x-x^2}}\right] + C$$

29. Let $u = e^t \;\Rightarrow\; du = e^t dt$. Then

$$\int e^t\sqrt{9-e^{2t}}\,dt = \int \sqrt{9-u^2}\,du = \int (3\cos\theta)\,3\cos\theta\,d\theta \quad \left(\text{where } u = 3\sin\theta,\ -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}\right)$$
$$= 9\int \cos^2\theta\,d\theta = \tfrac{9}{2}\int (1+\cos 2\theta)\,d\theta = \tfrac{9}{2}(\theta + \sin\theta\cos\theta) + C$$
$$= \frac{9}{2}\left[\sin^{-1}\left(\frac{u}{3}\right) + \frac{u}{3}\cdot\frac{\sqrt{9-u^2}}{3}\right] + C = \tfrac{9}{2}\sin^{-1}\left(\tfrac{1}{3}e^t\right) + \tfrac{1}{2}e^t\sqrt{9-e^{2t}} + C$$

30. Let $u = e^t$. Then $t = \ln u$ and $dt = du/u$. Hence $I = \int \sqrt{e^{2t} - 9}\,dt = \int \left(\sqrt{u^2 - 9}/u\right) du$. Now let $u = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $\sqrt{u^2 - 9} = 3\tan\theta$ and $du = 3\sec\theta\tan\theta\,d\theta$, so

$$\begin{aligned} I &= \int \frac{3\tan\theta}{3\sec\theta} 3\sec\theta\tan\theta\,d\theta = 3\int \tan^2\theta\,d\theta = 3\int \left(\sec^2\theta - 1\right) d\theta = 3\left(\tan\theta - \theta\right) + C \\ &= 3\left[\tfrac{1}{3}\sqrt{u^2 - 9} - \sec^{-1}\left(\tfrac{1}{3}u\right)\right] + C = \sqrt{e^{2t} - 9} - 3\sec^{-1}\left(\tfrac{1}{3}e^t\right) + C \end{aligned}$$

31. (a) Let $x = a\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $\sqrt{x^2 + a^2} = a\sec\theta$ and

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2 + a^2}} &= \int \frac{a\sec^2\theta\,d\theta}{a\sec\theta} = \int \sec\theta\,d\theta = \ln\left|\sec\theta + \tan\theta\right| + C_1 = \ln\left|\frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a}\right| + C_1 \\ &= \ln\left(x + \sqrt{x^2 + a^2}\right) + C \quad \text{where } C = C_1 - \ln|a| \end{aligned}$$

(b) Let $x = a\sinh t$, so that $dx = a\cosh t\,dt$ and $\sqrt{x^2 + a^2} = a\cosh t$. Then

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{a\cosh t\,dt}{a\cosh t} = t + C = \sinh^{-1}\frac{x}{a} + C.$$

32. (a) Let $x = a\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then

$$\begin{aligned} I &= \int \frac{x^2}{\left(x^2 + a^2\right)^{3/2}}\,dx = \int \frac{a^2\tan^2\theta}{a^3\sec^3\theta} a\sec^2\theta\,d\theta = \int \frac{\tan^2\theta}{\sec\theta}\,d\theta = \int \frac{\sec^2\theta - 1}{\sec\theta}\,d\theta \\ &= \int \left(\sec\theta - \cos\theta\right) d\theta = \ln\left|\sec\theta + \tan\theta\right| - \sin\theta + C \\ &= \ln\left|\frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a}\right| - \frac{x}{\sqrt{x^2 + a^2}} + C = \ln\left(x + \sqrt{x^2 + a^2}\right) - \frac{x}{\sqrt{x^2 + a^2}} + C_1 \end{aligned}$$

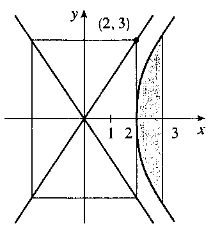
(b) Let $x = a\sinh t$. Then

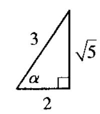
$$\begin{aligned} I &= \int \frac{a^2\sinh^2 t}{a^3\cosh^3 t} a\cosh t\,dt = \int \tanh^2 t\,dt = \int \left(1 - \operatorname{sech}^2 t\right) dt = t - \tanh t + C \\ &= \sinh^{-1}\frac{x}{a} - \frac{x}{\sqrt{a^2 + x^2}} + C \end{aligned}$$

33. $f(x) = \left(4 - x^2\right)^{3/2}$ on $[0, 2]$ $\Rightarrow$ $f_{\text{ave}} = \frac{1}{2-0}\int_0^2 \left(4 - x^2\right)^{3/2} dx$. Let $x = 2\sin\theta$ $\Rightarrow$ $dx = 2\cos\theta\,d\theta$ and $\left(4 - x^2\right)^{3/2} = (2\cos\theta)^3$. So

$$\begin{aligned} f_{\text{ave}} &= \tfrac{1}{2}\int_0^{\pi/2} (2\cos\theta)^3\, 2\cos\theta\,d\theta = 8\int_0^{\pi/2} \cos^4\theta\,d\theta \\ &= 8\left[\tfrac{3}{8}\theta + \tfrac{1}{4}\sin 2\theta + \tfrac{1}{32}\sin 4\theta\right]_0^{\pi/2} \quad \text{[by Exercise 8.2.9]} \quad = 8\left[\left(\tfrac{3\pi}{16} + 0 + 0\right) - (0 + 0 + 0)\right] = \tfrac{3\pi}{2} \end{aligned}$$

34. $9x^2 - 4y^2 = 36 \quad \Rightarrow \quad y = \pm\frac{3}{2}\sqrt{x^2-4} \quad \Rightarrow$





$$\begin{aligned}
\text{area} &= 2\int_2^3 \tfrac{3}{2}\sqrt{x^2-4}\,dx = 3\int_2^3 \sqrt{x^2-4}\,dx \\
&= 3\int_0^\alpha 2\tan\theta\, 2\sec\theta\tan\theta\,d\theta \qquad \begin{bmatrix} \text{where } x = 2\sec\theta, \\ dx = 2\sec\theta\tan\theta\,d\theta, \\ \alpha = \sec^{-1}\frac{3}{2} \end{bmatrix} \\
&= 12\int_0^\alpha \left(\sec^2\theta - 1\right)\sec\theta\,d\theta = 12\int_0^\alpha \left(\sec^3\theta - \sec\theta\right)d\theta \\
&= 12\left[\tfrac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right) - \ln|\sec\theta+\tan\theta|\right]_0^\alpha \\
&= 6\left[\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\right]_0^\alpha \\
&= 6\left[\tfrac{3\sqrt{5}}{4} - \ln\left(\tfrac{3}{2}+\tfrac{\sqrt{5}}{2}\right)\right] = \tfrac{9\sqrt{5}}{2} - 6\ln\left(\tfrac{3+\sqrt{5}}{2}\right)
\end{aligned}$$

35. Area of $\triangle POQ = \frac{1}{2}(r\cos\theta)(r\sin\theta) = \frac{1}{2}r^2\sin\theta\cos\theta$. Area of region $PQR = \int_{r\cos\theta}^{r}\sqrt{r^2-x^2}\,dx$. Let $x = r\cos u \quad \Rightarrow \quad dx = -r\sin u\,du$ for $\theta \le u \le \frac{\pi}{2}$. Then we obtain

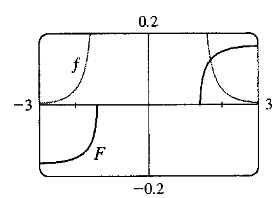
$$\begin{aligned}
\int\sqrt{r^2-x^2}\,dx &= \int r\sin u\,(-r\sin u)\,du = -r^2\int\sin^2 u\,du = -\tfrac{1}{2}r^2\,(u - \sin u\cos u) + C \\
&= -\tfrac{1}{2}r^2\cos^{-1}(x/r) + \tfrac{1}{2}x\sqrt{r^2-x^2} + C
\end{aligned}$$

so

$$\begin{aligned}
\text{area of region } PQR &= \tfrac{1}{2}\left[-r^2\cos^{-1}(x/r) + x\sqrt{r^2-x^2}\right]_{r\cos\theta}^{r} = \tfrac{1}{2}\left[0 - \left(-r^2\theta + r\cos\theta\, r\sin\theta\right)\right] \\
&= \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta\cos\theta
\end{aligned}$$

and thus, (area of sector POR) = (area of $\triangle POQ$) + (area of region PQR) $= \frac{1}{2}r^2\theta$.

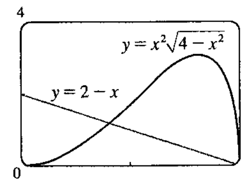
36. Let $x = \sqrt{2}\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$, so $dx = \sqrt{2}\sec\theta\tan\theta\,d\theta$. Then



$$\begin{aligned}
\int\frac{dx}{x^4\sqrt{x^2-2}} &= \int\frac{\sqrt{2}\sec\theta\tan\theta\,d\theta}{4\sec^4\theta\sqrt{2}\tan\theta} \\
&= \tfrac{1}{4}\int\cos^3\theta\,d\theta = \tfrac{1}{4}\int\left(1-\sin^2\theta\right)\cos\theta\,d\theta \\
&= \tfrac{1}{4}\left[\sin\theta - \tfrac{1}{3}\sin^3\theta\right] + C \quad (\text{substitute } u = \sin\theta) \\
&= \frac{1}{4}\left[\frac{\sqrt{x^2-2}}{x} - \frac{\left(x^2-2\right)^{3/2}}{3x^3}\right] + C
\end{aligned}$$

From the graph, it appears that our answer is reasonable. [Notice that $f(x)$ is large when F increases rapidly and small when F levels out.]

37. From the graph, it appears that the curve $y = x^2\sqrt{4-x^2}$ and the line $y = 2 - x$ intersect at about $x = 0.81$ and $x = 2$, with $x^2\sqrt{4-x^2} > 2 - x$ on $(0.81, 2)$. So the area bounded by the curve and the line is



$$A \approx \int_{0.81}^{2}\left[x^2\sqrt{4-x^2} - (2-x)\right]dx = \int_{0.81}^{2} x^2\sqrt{4-x^2}\,dx - \left[2x - \tfrac{1}{2}x^2\right]_{0.81}^{2}.$$

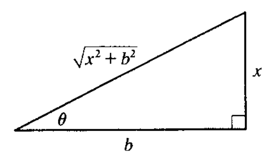
To evaluate the integral, we put $x = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 2\cos\theta\,d\theta$, $x = 2 \Rightarrow \theta = \sin^{-1} 1 = \frac{\pi}{2}$, and $x = 0.81 \Rightarrow \theta = \sin^{-1} 0.405 \approx 0.417$. So

$$\int_{0.81}^{2} x^2\sqrt{4-x^2}\,dx \approx \int_{0.417}^{\pi/2} 4\sin^2\theta\,(2\cos\theta)\,(2\cos\theta\,d\theta) = 4\int_{0.417}^{\pi/2}\sin^2 2\theta\,d\theta = 4\int_{0.417}^{\pi/2}\tfrac{1}{2}(1-\cos 4\theta)\,d\theta$$

$$= 2\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_{0.417}^{\pi/2} = 2\left[\left(\tfrac{\pi}{2} - 0\right) - \left(0.417 - \tfrac{1}{4}(0.995)\right)\right] \approx 2.81$$

Thus, $A \approx 2.81 - \left[\left(2\cdot 2 - \frac{1}{2}\cdot 2^2\right) - \left(2\cdot 0.81 - \frac{1}{2}\cdot 0.81^2\right)\right] \approx 2.10.$

38. Let $x = b\tan\theta$, so that $dx = b\sec^2\theta\,d\theta$ and $\sqrt{x^2+b^2} = b\sec\theta$.



$$E(P) = \int_{-a}^{L-a}\frac{\lambda b}{4\pi\varepsilon_0\left(x^2+b^2\right)^{3/2}}\,dx = \frac{\lambda b}{4\pi\varepsilon_0}\int_{\theta_1}^{\theta_2}\frac{1}{(b\sec\theta)^3}\,b\sec^2\theta\,d\theta$$

$$= \frac{\lambda}{4\pi\varepsilon_0 b}\int_{\theta_1}^{\theta_2}\frac{1}{\sec\theta}\,d\theta = \frac{\lambda}{4\pi\varepsilon_0 b}\int_{\theta_1}^{\theta_2}\cos\theta\,d\theta = \frac{\lambda}{4\pi\varepsilon_0 b}\left[\sin\theta\right]_{\theta_1}^{\theta_2}$$

$$= \frac{\lambda}{4\pi\varepsilon_0 b}\left[\frac{x}{\sqrt{x^2+b^2}}\right]_{-a}^{L-a} = \frac{\lambda}{4\pi\varepsilon_0 b}\left(\frac{L-a}{\sqrt{(L-a)^2+b^2}} + \frac{a}{\sqrt{a^2+b^2}}\right)$$

39. Let the equation of the large circle be $x^2 + y^2 = R^2$. Then the equation of the small circle is $x^2 + (y-b)^2 = r^2$, where $b = \sqrt{R^2-r^2}$ is the distance between the centers of the circles.The desired area is

$$A = \int_{-r}^{r}\left[\left(b + \sqrt{r^2-x^2}\right) - \sqrt{R^2-x^2}\right]dx = 2\int_0^r\left(b + \sqrt{r^2-x^2} - \sqrt{R^2-x^2}\right)dx$$

$$= 2\int_0^r b\,dx + 2\int_0^r\sqrt{r^2-x^2}\,dx - 2\int_0^r\sqrt{R^2-x^2}\,dx$$

The first integral is just $2br = 2r\sqrt{R^2-r^2}$. To evaluate the other two integrals, note that

$$\int\sqrt{a^2-x^2}\,dx = \int a^2\cos^2\theta\,d\theta\ \ (x = a\sin\theta,\ dx = a\cos\theta\,d\theta)\ = \tfrac{1}{2}\left(\tfrac{1}{2}a^2\right)\int(1+\cos 2\theta)$$

$$= \tfrac{1}{2}a^2\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{1}{2}a^2(\theta + \sin\theta\cos\theta) + C$$

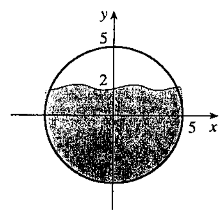
$$= \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{a^2}{2}\left(\frac{x}{a}\right)\frac{\sqrt{a^2-x^2}}{a} + C = \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + C$$

so the desired area is

$$A = 2r\sqrt{R^2-r^2} + \left[r^2\arcsin(x/r) + x\sqrt{r^2-x^2}\right]_0^r - \left[R^2\arcsin(x/R) + x\sqrt{R^2-x^2}\right]_0^r$$

$$= 2r\sqrt{R^2-r^2} + r^2\left(\tfrac{\pi}{2}\right) - \left[R^2\arcsin(r/R) + r\sqrt{R^2-r^2}\right] = r\sqrt{R^2-r^2} + \tfrac{\pi}{2}r^2 - R^2\arcsin(r/R)$$

40. Note that the circular cross-sections of the tank are the same everywhere, so the percentage of the total capacity that is being used is equal to the percentage of any cross-section that is under water. The underwater area is



$$A = 2\int_{-5}^{2}\sqrt{25-y^2}\,dy$$

$$= \left[25\arcsin(y/5) + y\sqrt{25-y^2}\right]_{-5}^{2} \quad (\text{substitute } y = 5\sin\theta)$$

$$= 25\arcsin\tfrac{2}{5} + 2\sqrt{21} + \tfrac{25}{2}\pi \approx 58.72 \text{ ft}^2$$

so the fraction of the total capacity in use is $\dfrac{A}{\pi(5)^2} \approx \dfrac{58.72}{25\pi} \approx 0.748$ or 74.8%.

41. We use cylindrical shells and assume that $R > r$. $x^2 = r^2 - (y-R)^2 \quad \Rightarrow \quad x = \pm\sqrt{r^2-(y-R)^2}$, so $g(y) = 2\sqrt{r^2-(y-R)^2}$ and

$$V = \int_{R-r}^{R+r} 2\pi y \cdot 2\sqrt{r^2-(y-R)^2}\,dy = \int_{-r}^{r} 4\pi(u+R)\sqrt{r^2-u^2}\,du \quad (\text{where } u = y - R)$$

$$= 4\pi\int_{-r}^{r} u\sqrt{r^2-u^2}\,du + 4\pi R\int_{-r}^{r}\sqrt{r^2-u^2}\,du \quad \left(\begin{array}{c}\text{where } u = r\sin\theta,\ du = r\cos\theta\,d\theta \\ \text{in the second integral}\end{array}\right)$$

$$= 4\pi\left[-\tfrac{1}{3}\left(r^2-u^2\right)^{3/2}\right]_{-r}^{r} + 4\pi R\int_{-\pi/2}^{\pi/2} r^2\cos^2\theta\,d\theta = -\tfrac{4\pi}{3}(0-0) + 4\pi Rr^2\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta$$

$$= 2\pi Rr^2\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta = 2\pi Rr^2\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = 2\pi^2 Rr^2$$

Another Method: Use washers instead of shells, so $V = 8\pi R\int_0^r \sqrt{r^2-y^2}\,dy$ as in Exercise 6.2.59(a), but evaluate the integral using $y = r\sin\theta$.

8.4 Integration of Rational Functions by Partial Fractions

1. $\dfrac{3}{(2x+3)(x-1)} = \dfrac{A}{2x+3} + \dfrac{B}{x-1}$

2. $\dfrac{5}{2x^2-3x-2} = \dfrac{5}{(2x+1)(x-2)} = \dfrac{A}{2x+1} + \dfrac{B}{x-2}$

3. $\dfrac{x^2+9x-12}{(3x-1)(x+6)^2} = \dfrac{A}{3x-1} + \dfrac{B}{x+6} + \dfrac{C}{(x+6)^2}$

4. $\dfrac{z^2-4z}{(3z+5)^3(z+2)} = \dfrac{A}{3z+5} + \dfrac{B}{(3z+5)^2} + \dfrac{C}{(3z+5)^3} + \dfrac{D}{z+2}$

5. $\dfrac{1}{x^4-x^3} = \dfrac{1}{x^3(x-1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{D}{x-1}$

6. $\dfrac{x^4+x^3-x^2-x+1}{x^3-x} = x+1+\dfrac{1}{x(x+1)(x-1)} = x+1+\dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1}$

7. $\dfrac{x^2+1}{x^2-1} = 1 + \dfrac{2}{(x-1)(x+1)} = 1 + \dfrac{A}{x-1} + \dfrac{B}{x+1}$

8. $\dfrac{x^3-4x^2+2}{(x^2+1)(x^2+2)}=\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{x^2+2}$

9. $\dfrac{t^4+t^2+1}{(t^2+1)(t^2+4)^2}=\dfrac{At+B}{t^2+1}+\dfrac{Ct+D}{t^2+4}+\dfrac{Et+F}{(t^2+4)^2}$

10. $\dfrac{3-11x}{(x-2)^3(x^2+1)(2x^2+5x+7)^2}=\dfrac{A}{x-2}+\dfrac{B}{(x-2)^2}+\dfrac{C}{(x-2)^3}+\dfrac{Dx+E}{x^2+1}$
$$+\frac{Fx+G}{2x^2+5x+7}+\frac{Hx+I}{(2x^2+5x+7)^2}$$

11. $\dfrac{x^4}{(x^2+9)^3}=\dfrac{Ax+B}{x^2+9}+\dfrac{Cx+D}{(x^2+9)^2}+\dfrac{Ex+F}{(x^2+9)^3}$

12. $\dfrac{1}{x^6-x^3}=\dfrac{1}{x^3(x^3-1)}=\dfrac{1}{x^3(x-1)(x^2+x+1)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x^3}+\dfrac{D}{x-1}+\dfrac{Ex+F}{x^2+x+1}$

13. $\displaystyle\int\frac{x^2}{x+1}dx=\int\left(x-1+\frac{1}{x+1}\right)dx=\tfrac{1}{2}x^2-x+\ln|x+1|+C$

14. $\displaystyle\int\frac{y}{y+2}dy=\int\left(1-\frac{2}{y+2}\right)dy=y-2\ln|y+2|+C$

15. $\dfrac{x-9}{(x+5)(x-2)}=\dfrac{A}{x+5}+\dfrac{B}{x-2}$. Multiply both sides by $(x+5)(x-2)$ to get $x-9=A(x-2)+B(x+5)$. Substituting 2 for x gives $-7=7B \Leftrightarrow B=-1$. Substituting -5 for x gives $-14=-7A \Leftrightarrow A=2$. Thus,
$$\int\frac{x-9}{(x+5)(x-2)}dx=\int\left(\frac{2}{x+5}+\frac{-1}{x-2}\right)dx=2\ln|x+5|-\ln|x-2|+C$$

16. $\dfrac{1}{(t+4)(t-1)}=\dfrac{A}{t+4}+\dfrac{B}{t-1} \Rightarrow 1=A(t-1)+B(t+4)$. $t=1 \Rightarrow 1=5B \Rightarrow B=\tfrac{1}{5}$. $t=-4$ $\Rightarrow 1=-5A \Rightarrow A=-\tfrac{1}{5}$. Thus,
$$\int\frac{1}{(t+4)(t-1)}dt=\int\left(\frac{-1/5}{t+4}+\frac{1/5}{t-1}\right)dt=-\tfrac{1}{5}\ln|t+4|+\tfrac{1}{5}\ln|t-1|+C \text{ or } \frac{1}{5}\ln\left|\frac{t-1}{t+4}\right|+C$$

17. $\dfrac{x^2+1}{x^2-x}=1+\dfrac{x+1}{x(x-1)}=1-\dfrac{1}{x}+\dfrac{2}{x-1}$, so
$$\int\frac{x^2+1}{x^2-x}dx=x-\ln|x|+2\ln|x-1|+C=x+\ln\frac{(x-1)^2}{|x|}+C$$

18. If $a\neq b$, $\dfrac{1}{(x+a)(x+b)}=\dfrac{1}{b-a}\left(\dfrac{1}{x+a}-\dfrac{1}{x+b}\right)$, so if $a\neq b$, then
$$\int\frac{dx}{(x+a)(x+b)}=\frac{1}{b-a}(\ln|x+a|-\ln|x+b|)+C=\frac{1}{b-a}\ln\left|\frac{x+a}{x+b}\right|+C$$

If $a=b$, then $\displaystyle\int\frac{dx}{(x+a)^2}=-\frac{1}{x+a}+C$.

19. $\dfrac{2x+3}{(x+1)^2} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} \Rightarrow 2x+3 = A(x+1)+B$. Take $x=-1$ to get $B=1$, and equate coefficients of x to get $A=2$. Now

$$\int_0^1 \frac{2x+3}{(x+1)^2}dx = \int_0^1 \left[\frac{2}{x+1} + \frac{1}{(x+1)^2}\right] dx = \left[2\ln(x+1) - \frac{1}{x+1}\right]_0^1$$
$$= 2\ln 2 - \tfrac{1}{2} - (2\ln 1 - 1) = 2\ln 2 + \tfrac{1}{2}$$

20. $\dfrac{x^3+x^2-12x+1}{x^2+x-12} = x + \dfrac{1}{x^2+x-12} = x + \dfrac{1}{(x-3)(x+4)} = x + \dfrac{1}{7}\left(\dfrac{1}{x-3} - \dfrac{1}{x+4}\right)$. So

$$\int_0^2 \frac{x^3+x^2-12x+1}{x^2+x-12}dx = \left[\tfrac{1}{2}x^2 + \tfrac{1}{7}(\ln|x-3| - \ln|x+4|)\right]_0^2 = 2 + \tfrac{1}{7}\ln\tfrac{2}{9}$$

21. $\dfrac{4y^2-7y-12}{y(y+2)(y-3)} = \dfrac{A}{y} + \dfrac{B}{y+2} + \dfrac{C}{y-3} \Rightarrow 4y^2-7y-12 = A(y+2)(y-3) + By(y-3) + Cy(y+2)$. Setting $y=0$ gives $-12=-6A$, so $A=2$. Setting $y=-2$ gives $18=10B$, so $B=\frac{9}{5}$. Setting $y=3$ gives $3=15C$, so $C=\frac{1}{5}$. Now

$$\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}dy = \int_1^2\left(\frac{2}{y} + \frac{9/5}{y+2} + \frac{1/5}{y-3}\right)dy = \left[2\ln|y| + \tfrac{9}{5}\ln|y+2| + \tfrac{1}{5}\ln|y-3|\right]_1^2$$
$$= 2\ln 2 + \tfrac{9}{5}\ln 4 + \tfrac{1}{5}\ln 1 - 2\ln 1 - \tfrac{9}{5}\ln 3 - \tfrac{1}{5}\ln 2$$
$$= 2\ln 2 + \tfrac{18}{5}\ln 2 - \tfrac{1}{5}\ln 2 - \tfrac{9}{5}\ln 3 = \tfrac{27}{5}\ln 2 - \tfrac{9}{5}\ln 3 = \tfrac{9}{5}(3\ln 2 - \ln 3) = \tfrac{9}{5}\ln\tfrac{8}{3}$$

22. $\dfrac{1}{x^3+x^2-2x} = \dfrac{1}{x(x+2)(x-1)} = \dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{x-1} \Rightarrow$
$1 = A(x+2)(x-1) + Bx(x-1) + Cx(x+2)$. Setting $x=0$ gives $1=-2A$, so $A=-\frac{1}{2}$. Setting $x=-2$ gives $1=6B$, so $B=\frac{1}{6}$. Setting $x=1$ gives $1=3C$, so $C=\frac{1}{3}$. Now

$$\int_2^3 \frac{1}{x^3+x^2-2x}dx = \int_2^3\left(\frac{-1/2}{x} + \frac{1/6}{x+2} + \frac{1/3}{x-1}\right)dx = \left[-\tfrac{1}{2}\ln|x| + \tfrac{1}{6}\ln|x+2| + \tfrac{1}{3}\ln|x-1|\right]_2^3$$
$$= -\tfrac{1}{2}\ln 3 + \tfrac{1}{6}\ln 5 + \tfrac{1}{3}\ln 2 + \tfrac{1}{2}\ln 2 - \tfrac{1}{6}\ln 4 - \tfrac{1}{3}\ln 1 = \tfrac{1}{2}\ln 2 - \tfrac{1}{2}\ln 3 + \tfrac{1}{6}\ln 5$$

23. $\dfrac{1}{(x+5)^2(x-1)} = \dfrac{A}{x+5} + \dfrac{B}{(x+5)^2} + \dfrac{C}{x-1} \Rightarrow 1 = A(x+5)(x-1) + B(x-1) + C(x+5)^2$. Setting $x=-5$ gives $1=-6B$, so $B=-\frac{1}{6}$. Setting $x=1$ gives $1=36C$, so $C=\frac{1}{36}$. Setting $x=-2$ gives $1 = A(3)(-3) + B(-3) + C(3^2) = -9A - 3B + 9C = -9A + \frac{1}{2} + \frac{1}{4} = -9A + \frac{3}{4}$, so $9A = -\frac{1}{4}$ and $A = -\frac{1}{36}$. Now

$$\int \frac{1}{(x+5)^2(x-1)}dx = \int\left[\frac{-1/36}{x+5} - \frac{1/6}{(x+5)^2} + \frac{1/36}{x-1}\right]dx$$
$$= -\tfrac{1}{36}\ln|x+5| + \frac{1}{6(x+5)} + \tfrac{1}{36}\ln|x-1| + C$$

24. $\dfrac{x^2}{(x-3)(x+2)^2} = \dfrac{A}{x-3} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2} \Rightarrow x^2 = A(x+2)^2 + B(x-3)(x+2) + C(x-3)$. Setting $x = 3$ gives $A = \frac{9}{25}$. Take $x = -2$ to get $C = -\frac{4}{5}$, and equate the coefficients of x^2 to get $1 = A + B \Rightarrow B = \frac{16}{25}$. Then

$$\int \frac{x^2}{(x-3)(x+2)^2}\,dx = \int \left[\frac{9/25}{x-3} + \frac{16/25}{x+2} - \frac{4/5}{(x+2)^2}\right] dx = \tfrac{9}{25}\ln|x-3| + \tfrac{16}{25}\ln|x+2| + \frac{4}{5(x+2)} + C$$

25. $\dfrac{5x^2+3x-2}{x^3+2x^2} = \dfrac{5x^2+3x-2}{x^2(x+2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+2}$. Multiply by $x^2(x+2)$ to get $5x^2 + 3x - 2 = Ax(x+2) + B(x+2) + Cx^2$. Set $x = -2$ to get $C = 3$, and take $x = 0$ to get $B = -1$. Equating the coefficients of x^2 gives $5 = A + C \Rightarrow A = 2$. So

$$\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx = \int \left(\frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2}\right) dx = 2\ln|x| + \frac{1}{x} + 3\ln|x+2| + C.$$

26. $\dfrac{1}{s^2(s-1)^2} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s-1} + \dfrac{D}{(s-1)^2} \Rightarrow 1 = As(s-1)^2 + B(s-1)^2 + Cs^2(s-1) + Ds^2$. Set $s = 0$, giving $B = 1$. Then set $s = 1$ to get $D = 1$. Equate the coefficients of s^3 to get $0 = A + C$ or $A = -C$, and finally set $s = 2$ to get $1 = 2A + 1 - 4A + 4$ or $A = 2$. Now

$$\int \frac{ds}{s^2(s-1)^2} = \int \left[\frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{1}{(s-1)^2}\right] ds = 2\ln|s| - \frac{1}{s} - 2\ln|s-1| - \frac{1}{s-1} + C.$$

27. $\dfrac{x^2}{(x+1)^3} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)^3}$. Multiply by $(x+1)^3$ to get $x^2 = A(x+1)^2 + B(x+1) + C$. Setting $x = -1$ gives $C = 1$. Equating the coefficients of x^2 gives $A = 1$, and setting $x = 0$ gives $B = -2$. Now

$$\int \frac{x^2\,dx}{(x+1)^3} = \int \left[\frac{1}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{(x+1)^3}\right] dx = \ln|x+1| + \frac{2}{x+1} - \frac{1}{2(x+1)^2} + C.$$

28. $\dfrac{x}{x+1} = \dfrac{(x+1)-1}{x+1} = 1 - \dfrac{1}{x+1}$, so $\dfrac{x^3}{(x+1)^3} = \left[1 - \dfrac{1}{x+1}\right]^3 = 1 - \dfrac{3}{x+1} + \dfrac{3}{(x+1)^2} - \dfrac{1}{(x+1)^3}$. Thus,

$$\int \frac{x^3}{(x+1)^3}\,dx = \int \left[1 - \frac{3}{x+1} + \frac{3}{(x+1)^2} - \frac{1}{(x+1)^3}\right] dx = x - 3\ln|x+1| - \frac{3}{x+1} + \frac{1}{2(x+1)^2} + C.$$

29. $\dfrac{x^3}{x^2+1} = \dfrac{(x^3+x)-x}{x^2+1} = x - \dfrac{x}{x^2+1}$, so

$$\int_0^1 \frac{x^3}{x^2+1}\,dx = \int_0^1 x\,dx - \int_0^1 \frac{x\,dx}{x^2+1} = \left[\tfrac{1}{2}x^2\right]_0^1 - \tfrac{1}{2}\int_1^2 \frac{1}{u}\,du \quad (\text{where } u = x^2+1,\ du = 2x\,dx)$$

$$= \tfrac{1}{2} - \left[\tfrac{1}{2}\ln u\right]_1^2 = \tfrac{1}{2} - \tfrac{1}{2}\ln 2 = \tfrac{1}{2}(1 - \ln 2)$$

30. $\dfrac{x^2+3}{x^3+2x} = \dfrac{x^2+3}{x(x^2+2)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+2} \Rightarrow x^2 + 3 = A(x^2+2) + (Bx+C)x = (A+B)x^2 + Cx + 2A$ $\Rightarrow A + B = 1, C = 0$, and $2A = 3 \Rightarrow A = \frac{3}{2}, B = -\frac{1}{2}$, and $C = 0$. Now

$$\int_1^2 \frac{x^2+3}{x^3+2x}\,dx = \int_1^2 \left(\frac{3/2}{x} - \frac{x/2}{x^2+2}\right) dx = \left[\tfrac{3}{2}\ln x - \tfrac{1}{4}\ln\left(x^2+2\right)\right]_1^2 = \tfrac{3}{2}\ln 2 - \tfrac{1}{4}\ln 6 - \tfrac{3}{2}\ln 1 + \tfrac{1}{4}\ln 3$$

$$= \tfrac{3}{2}\ln 2 - \tfrac{1}{4}\ln 2 - \tfrac{1}{4}\ln 3 - 0 + \tfrac{1}{4}\ln 3 = \left(\tfrac{3}{2} - \tfrac{1}{4}\right)\ln 2 = \tfrac{5}{4}\ln 2$$

31. $\dfrac{3x^2-4x+5}{(x-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1} \quad\Rightarrow\quad 3x^2-4x+5 = A(x^2+1) + (Bx+C)(x-1)$. Take $x=1$ to get $4=2A$ or $A=2$. Now $(Bx+C)(x-1) = 3x^2-4x+5-2(x^2+1) = x^2-4x+3$. Equating coefficients of x^2 and then comparing the constant terms, we get $B=1$ and $C=-3$. Hence,

$$\begin{aligned}\int \frac{3x^2-4x+5}{(x-1)(x^2+1)}\,dx &= \int\left[\frac{2}{x-1}+\frac{x-3}{x^2+1}\right]dx = 2\ln|x-1| + \int\frac{x\,dx}{x^2+1} - 3\int\frac{dx}{x^2+1}\\ &= 2\ln|x-1| + \tfrac{1}{2}\ln\left(x^2+1\right) - 3\tan^{-1}x + C\\ &= \ln(x-1)^2 + \ln\sqrt{x^2+1} - 3\tan^{-1}x + C\end{aligned}$$

32. $\dfrac{x^2-2x-1}{(x-1)^2(x^2+1)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{Cx+D}{x^2+1} \quad\Rightarrow$

$x^2-2x-1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$. Setting $x=1$ gives $B=-1$. Equating the coefficients of x^3 gives $A=-C$. Equating the constant terms gives $-1=-A-1+D$, so $D=A$, and setting $x=2$ gives $-1=5A-5-2A+A$ or $A=1$. We have

$$\begin{aligned}\int\frac{x^2-2x-1}{(x-1)^2(x^2+1)}\,dx &= \int\left[\frac{1}{x-1}-\frac{1}{(x-1)^2}-\frac{x-1}{x^2+1}\right]dx\\ &= \ln|x-1| + \frac{1}{x-1} - \tfrac{1}{2}\ln\left(x^2+1\right) + \tan^{-1}x + C\end{aligned}$$

33. $\dfrac{2t^3-t^2+3t-1}{(t^2+1)(t^2+2)} = \dfrac{At+B}{t^2+1} + \dfrac{Ct+D}{t^2+2} \quad\Rightarrow$

$2t^3-t^2+3t-1 = (At+B)(t^2+2) + (Ct+D)(t^2+1) = (A+C)t^3 + (B+D)t^2 + (2A+C)t + (2B+D)$

$\Rightarrow\quad A+C=2,\ B+D=-1,\ 2A+C=3$, and $2B+D=-1 \quad\Rightarrow\quad A=1,\ C=1,\ B=0$, and $D=-1$. Now

$$\begin{aligned}\int\frac{2t^3-t^2+3t-1}{(t^2+1)(t^2+2)}\,dt &= \int\left(\frac{t}{t^2+1}+\frac{t-1}{t^2+2}\right)dt = \frac{1}{2}\int\frac{2t\,dt}{t^2+1} + \frac{1}{2}\int\frac{2t\,dt}{t^2+2} - \int\frac{dt}{t^2+2}\\ &= \tfrac{1}{2}\ln\left(t^2+1\right) + \tfrac{1}{2}\ln\left(t^2+2\right) - \tfrac{1}{\sqrt{2}}\tan^{-1}\left(\tfrac{1}{\sqrt{2}}t\right) + C\\ &\text{or } \tfrac{1}{2}\ln\left(\left(t^2+1\right)\left(t^2+2\right)\right) - \tfrac{\sqrt{2}}{2}\tan^{-1}\left(\tfrac{1}{\sqrt{2}}t\right) + C\end{aligned}$$

34. $\dfrac{x^3-2x^2+x+1}{x^4+5x^2+4} = \dfrac{x^3-2x^2+x+1}{(x^2+1)(x^2+4)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+4} \quad\Rightarrow$

$x^3-2x^2+x+1 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$. Equating coefficients gives $A+C=1$, $B+D=-2$, $4A+C=1$, $4B+D=1 \quad\Rightarrow\quad A=0,\ C=1,\ B=1,\ D=-3$. Now

$\displaystyle\int\frac{x^3-2x^2+x+1}{x^4+5x^2+4}\,dx = \int\frac{dx}{x^2+1} + \int\frac{x-3}{x^2+4}\,dx = \tan^{-1}x + \tfrac{1}{2}\ln\left(x^2+4\right) - \tfrac{3}{2}\tan^{-1}(x/2) + C.$

35. $\dfrac{1}{x^3-1} = \dfrac{1}{(x-1)(x^2+x+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+1} \Rightarrow 1 = A(x^2+x+1) + (Bx+C)(x-1)$. Take $x=1$ to get $A = \frac{1}{3}$. Equating coefficients of x^2 and then comparing the constant terms, we get $0 = \frac{1}{3} + B$, $1 = \frac{1}{3} - C$, so $B = -\frac{1}{3}$, $C = -\frac{2}{3} \Rightarrow$

$$\begin{aligned}\int \frac{dx}{x^3-1} &= \int \frac{\frac{1}{3}}{x-1}\,dx + \int \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2+x+1}\,dx = \tfrac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+2}{x^2+x+1}dx \\ &= \tfrac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+1/2}{x^2+x+1}dx - \frac{1}{3}\int \frac{(3/2)\,dx}{(x+1/2)^2+3/4} \\ &= \tfrac{1}{3}\ln|x-1| - \tfrac{1}{6}\ln\left(x^2+x+1\right) - \tfrac{1}{2}\left(\tfrac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right) + K \\ &= \tfrac{1}{3}\ln|x-1| - \tfrac{1}{6}\ln\left(x^2+x+1\right) - \tfrac{1}{\sqrt{3}}\tan^{-1}\left(\tfrac{1}{\sqrt{3}}(2x+1)\right) + K\end{aligned}$$

36. $\dfrac{x^3}{x^3+1} = \dfrac{(x^3+1)-1}{x^3+1} = 1 - \dfrac{1}{x^3+1} = 1 - \left(\dfrac{A}{x+1} + \dfrac{Bx+C}{x^2-x+1}\right) \Rightarrow$ $1 = A(x^2-x+1) + (Bx+C)(x+1)$. Equate the terms of degree 2, 1 and 0 to get $0 = A+B$, $0 = -A+B+C$, $1 = A+C$. Solve the three equations to get $A = \frac{1}{3}$, $B = -\frac{1}{3}$, and $C = \frac{2}{3}$. So

$$\begin{aligned}\int \frac{x^3}{x^3+1}\,dx &= \int \left[1 - \frac{\frac{1}{3}}{x+1} + \frac{\frac{1}{3}x - \frac{2}{3}}{x^2-x+1}\right]dx \\ &= x - \tfrac{1}{3}\ln|x+1| + \frac{1}{6}\int \frac{2x-1}{x^2-x+1}\,dx - \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}} \\ &= x - \tfrac{1}{3}\ln|x+1| + \tfrac{1}{6}\ln\left(x^2-x+1\right) - \tfrac{1}{\sqrt{3}}\tan^{-1}\left(\tfrac{1}{\sqrt{3}}(2x-1)\right) + K\end{aligned}$$

37. Let $u = x^3 + 3x^2 + 4$. Then $du = 3\left(x^2+2x\right)dx \Rightarrow$
$\displaystyle\int_2^5 \frac{x^2+2x}{x^3+3x^2+4}\,dx = \frac{1}{3}\int_{24}^{204} \frac{du}{u} = \tfrac{1}{3}[\ln u]_{24}^{204} = \tfrac{1}{3}(\ln 204 - \ln 24) = \tfrac{1}{3}\ln\tfrac{204}{24} = \tfrac{1}{3}\ln\tfrac{17}{2}$

38. Let $u = x^4 + 5x^2 + 4 \Rightarrow du = \left(4x^3 + 10x\right)dx = 2\left(2x^3+5x\right)dx$, so
$\displaystyle\int_0^1 \frac{2x^3+5x}{x^4+5x^2+4}\,dx = \frac{1}{2}\int_4^{10} \frac{du}{u} = \tfrac{1}{2}[\ln|u|]_4^{10} = \tfrac{1}{2}(\ln 10 - \ln 4) = \tfrac{1}{2}\ln\tfrac{5}{2}$.

39. $\dfrac{1}{x^4-x^2} = \dfrac{1}{x^2(x-1)(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1} + \dfrac{D}{x+1}$. Multiply by $x^2(x-1)(x+1)$ to get $1 = Ax(x-1)(x+1) + B(x-1)(x+1) + Cx^2(x+1) + Dx^2(x-1)$. Setting $x=1$ gives $C = \frac{1}{2}$, taking $x=-1$ gives $D = -\frac{1}{2}$. Equating the coefficients of x^3 gives $0 = A + C + D = A$. Finally, setting $x=0$ yields $B = -1$. Now $\displaystyle\int \frac{dx}{x^4-x^2} = \int \left[\frac{-1}{x^2} + \frac{1/2}{x-1} - \frac{1/2}{x+1}\right]dx = \frac{1}{x} + \tfrac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$.

40. $\dfrac{x^4}{x^4-1} = 1 + \dfrac{1}{x^4-1}$ and $\dfrac{1}{x^4-1} = \dfrac{1}{(x-1)(x+1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+1} \Rightarrow$
$1 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1)$. Set
$x = 1$ to get $A = \frac{1}{4}$, and set $x = -1$ to get $B = -\frac{1}{4}$. Now take $x = 0$ to get
$1 = A - B - D = -D + \frac{1}{2}$, so that $D = -\frac{1}{2}$. Finally, equate the coefficients of x^3 to get $C = 0$. Now

$$\int \frac{x^4\,dx}{x^4-1} = \int \left[1 + \frac{1/4}{x-1} - \frac{1/4}{x+1} - \frac{1/2}{x^2+1}\right] dx = x + \tfrac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \tfrac{1}{2}\tan^{-1}x + C.$$

41. $\displaystyle\int \frac{x-3}{(x^2+2x+4)^2}\,dx = \int \frac{x-3}{\left[(x+1)^2+3\right]^2}\,dx = \int \frac{u-4}{(u^2+3)^2}\,du$ (with $u = x+1$)

$$= \int \frac{u\,du}{(u^2+3)^2} - 4\int \frac{du}{(u^2+3)^2} = \frac{1}{2}\int \frac{dv}{v^2} - 4\int \frac{\sqrt{3}\sec^2\theta\,d\theta}{9\sec^4\theta} \quad \begin{bmatrix} v = u^2+3 \text{ in the first integral;} \\ u = \sqrt{3}\tan\theta \text{ in the second} \end{bmatrix}$$

$$= \frac{-1}{(2v)} - \frac{4\sqrt{3}}{9}\int \cos^2\theta\,d\theta = \frac{-1}{2(u^2+3)} - \frac{2\sqrt{3}}{9}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{-1}{2(x^2+2x+4)} - \frac{2\sqrt{3}}{9}\left[\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + \frac{\sqrt{3}(x+1)}{x^2+2x+4}\right] + C$$

$$= \frac{-1}{2(x^2+2x+4)} - \frac{2\sqrt{3}}{9}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) - \frac{2(x+1)}{3(x^2+2x+4)} + C$$

42. $\dfrac{x^4+1}{x(x^2+1)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2} \Rightarrow x^4+1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x.$
Setting $x = 0$ gives $A = 1$, and equating the coefficients of x^4 gives $1 = A + B$, so $B = 0$. Now
$\dfrac{C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2} = \dfrac{x^4+1}{x(x^2+1)^2} - \dfrac{1}{x} = \dfrac{1}{x}\left[\dfrac{x^4+1-(x^4+2x^2+1)}{(x^2+1)^2}\right] = \dfrac{-2x}{(x^2+1)^2}$, so we can take $C = 0$,
$D = -2$, and $E = 0$. Hence, $\displaystyle\int \frac{x^4+1}{x(x^2+1)^2}dx = \int \left[\frac{1}{x} - \frac{2x}{(x^2+1)^2}\right] dx = \ln|x| + \frac{1}{x^2+1} + C.$

43. Let $u = \sqrt{x+1}$. Then $x = u^2 - 1$, $dx = 2u\,du \Rightarrow$

$$\int \frac{dx}{x\sqrt{x+1}} = \int \frac{2u\,du}{(u^2-1)u} = 2\int \frac{du}{u^2-1} = \ln\left|\frac{u-1}{u+1}\right| + C = \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C.$$

44. Let $u = \sqrt{x+2}$. Then $x = u^2 - 2$, $dx = 2u\,du \Rightarrow I = \displaystyle\int \frac{dx}{x - \sqrt{x+2}} = \int \frac{2u\,du}{u^2-2-u} = 2\int \frac{u\,du}{u^2-u-2}$
and $\dfrac{u}{u^2-u-2} = \dfrac{A}{u-2} + \dfrac{B}{u+1} \Rightarrow u = A(u+1) + B(u-2) \Rightarrow A = \frac{2}{3}$, $B = \frac{1}{3}$, so

$$I = \frac{2}{3}\int \left[\frac{2}{u-2} + \frac{1}{u+1}\right] du = \tfrac{2}{3}(2\ln|u-2| + \ln|u+1|) + C$$

$$= \tfrac{2}{3}\left[2\ln\left|\sqrt{x+2}-2\right| + \ln\left(\sqrt{x+2}+1\right)\right] + C$$

45. Let $u = \sqrt{x}$, so $u^2 = x$ and $dx = 2u\,du$. Thus,

$$\int_9^{16} \frac{\sqrt{x}}{x-4}\,dx = \int_3^4 \frac{u}{u^2-4}2u\,du = 2\int_3^4 \frac{u^2}{u^2-4}\,du = 2\int_3^4 \left(1+\frac{4}{u^2-4}\right)du = 2+8\int_3^4 \frac{du}{(u+2)(u-2)}$$

$$= 2+8\int_3^4 \left(\frac{-1/4}{u+2}+\frac{1/4}{u-2}\right) \text{(by partial fractions)} = 2+8\left[-\tfrac{1}{4}\ln|u+2|+\tfrac{1}{4}\ln|u-2|\right]_3^4$$

$$= 2+[2\ln|u-2|-2\ln|u+2|]_3^4 = 2+2\left[\ln\left|\frac{u-2}{u+2}\right|\right]_3^4 = 2+2\left(\ln\tfrac{2}{6}-\ln\tfrac{1}{5}\right)$$

$$= 2+2\ln\tfrac{5}{3} \text{ or } 2+\ln\tfrac{25}{9}$$

46. Let $u = \sqrt[3]{x}$. Then $x = u^3$, $dx = 3u^2\,du \Rightarrow$

$$\int_0^1 \frac{x}{1+\sqrt[3]{x}}\,dx = \int_0^1 \frac{3u^2\,du}{1+u} = \int_0^1 \left(3u-3+\frac{3}{1+u}\right)du = \left[\tfrac{3}{2}u^2-3u+3\ln(1+u)\right]_0^1$$

$$= 3\left(\ln 2-\tfrac{1}{2}\right)$$

47. Let $u = \sqrt[3]{x^2+1}$. Then $x^2 = u^3-1$, $2x\,dx = 3u^2\,du \Rightarrow$

$$\int \frac{x^3dx}{\sqrt[3]{x^2+1}} = \int \frac{(u^3-1)\,\frac{3}{2}u^2\,du}{u} = \tfrac{3}{2}\int (u^4-u)\,du = \tfrac{3}{10}u^5-\tfrac{3}{4}u^2+C$$

$$= \tfrac{3}{10}\left(x^2+1\right)^{5/3}-\tfrac{3}{4}\left(x^2+1\right)^{2/3}+C$$

48. Let $u = \sqrt{x}$. Then $x = u^2$, $dx = 2u\,du \Rightarrow$

$$\int_{1/3}^3 \frac{\sqrt{x}}{x^2+x}\,dx = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{u\cdot 2u\,du}{u^4+u^2} = 2\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{du}{u^2+1} = 2\left[\tan^{-1}u\right]_{1/\sqrt{3}}^{\sqrt{3}} = 2\left(\tfrac{\pi}{3}-\tfrac{\pi}{6}\right) = \tfrac{\pi}{3}$$

49. If we were to substitute $u = \sqrt{x}$, then the square root would disappear but a cube root would remain. On the other hand, the substitution $u = \sqrt[3]{x}$ would eliminate the cube root but leave a square root. We can eliminate both roots by means of the substitution $u = \sqrt[6]{x}$. (Note that 6 is the least common multiple of 2 and 3.)

Let $u = \sqrt[6]{x}$. Then $x = u^6$, so $dx = 6u^5\,du$ and $\sqrt{x} = u^3$, $\sqrt[3]{x} = u^2$. Thus,

$$\int \frac{dx}{\sqrt{x}-\sqrt[3]{x}} = \int \frac{6u^5\,du}{u^3-u^2} = 6\int \frac{u^5}{u^2(u-1)}\,du = 6\int \frac{u^3}{u-1}\,du$$

$$= 6\int \left(u^2+u+1+\frac{1}{u-1}\right)du \text{ (by long division)}$$

$$= 6\left(\tfrac{1}{3}u^3+\tfrac{1}{2}u^2+u+\ln|u-1|\right)+C = 2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}+6\ln\left|\sqrt[6]{x}-1\right|+C$$

50. Let $u = \sqrt[12]{x}$. Then $x = u^{12}$, $dx = 12u^{11}\,du \Rightarrow$

$$\int \frac{dx}{\sqrt[3]{x}+\sqrt[4]{x}} = \int \frac{12u^{11}du}{u^4+u^3} = 12\int \frac{u^8du}{u+1} = 12\int \left(u^7-u^6+u^5-u^4+u^3-u^2+u-1+\frac{1}{u+1}\right)du$$

$$= \tfrac{3}{2}u^8-\tfrac{12}{7}u^7+2u^6-\tfrac{12}{5}u^5+3u^4-4u^3+6u^2-12u+12\ln|u+1|+C$$

$$= \tfrac{3}{2}x^{2/3}-\tfrac{12}{7}x^{7/12}+2\sqrt{x}-\tfrac{12}{5}x^{5/12}+3\sqrt[3]{x}-4\sqrt[4]{x}+6\sqrt[6]{x}-12\sqrt[12]{x}+12\ln\left(\sqrt[12]{x}+1\right)+C$$

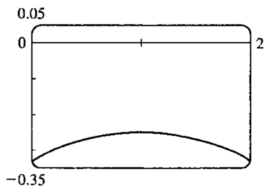
51. Let $u = e^x$. Then $x = \ln u$, $dx = \dfrac{du}{u}$ $\Rightarrow$

$$\int \frac{e^{2x}\,dx}{e^{2x}+3e^x+2} = \int \frac{u^2\,(du/u)}{u^2+3u+2} = \int \frac{u\,du}{(u+1)\,(u+2)} = \int \left[\frac{-1}{u+1} + \frac{2}{u+2}\right] du$$
$$= 2\ln|u+2| - \ln|u+1| + C = \ln\left[(e^x+2)^2 / (e^x+1)\right] + C$$

52. Let $u = \sin x$. Then $du = \cos x\,dx$ $\Rightarrow$

$$\int \frac{\cos x\,dx}{\sin^2 x + \sin x} = \int \frac{du}{u^2+u} = \int \frac{du}{u\,(u+1)} = \int \left[\frac{1}{u} - \frac{1}{u+1}\right] du = \ln\left|\frac{u}{u+1}\right| + C = \ln\left|\frac{\sin x}{1+\sin x}\right| + C.$$

53. From the graph, we see that the integral will be negative, and we guess that the area is about the same as that of a rectangle with width 2 and height 0.3, so we estimate the integral to be $-(2 \cdot 0.3) = -0.6$. Now

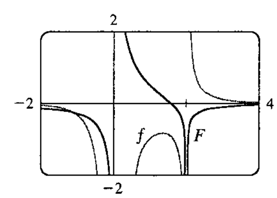


$$\frac{1}{x^2-2x-3} = \frac{1}{(x-3)\,(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} \quad \Leftrightarrow$$

$1 = (A+B)\,x + A - 3B$, so $A = -B$ and $A - 3B = 1$ $\Leftrightarrow$ $A = \frac{1}{4}$ and $B = -\frac{1}{4}$, so the integral becomes

$$\int_0^2 \frac{dx}{x^2-2x-3} = \frac{1}{4}\int_0^2 \frac{dx}{x-3} - \frac{1}{4}\int_0^2 \frac{dx}{x+1} = \tfrac{1}{4}\left[\ln|x-3| - \ln|x+1|\right]_0^2$$
$$= \frac{1}{4}\left[\ln\left|\frac{x-3}{x+1}\right|\right]_0^2 = \tfrac{1}{4}\left(\ln\tfrac{1}{3} - \ln 3\right) = -\tfrac{1}{2}\ln 3 \approx -0.55$$

54. $\dfrac{1}{x^3-2x^2} = \dfrac{1}{x^2\,(x-2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-2}$ $\Rightarrow$



$1 = (A+C)\,x^2 + (B-2A)\,x - 2B$, so $A + C = B - 2A = 0$ and $-2B = 1$ $\Rightarrow$ $B = -\frac{1}{2}$, $A = -\frac{1}{4}$, and $C = \frac{1}{4}$. So the general antiderivative of $f(x) = \dfrac{1}{x^3-2x^2}$ is

$$F(x) = \int \frac{dx}{x^3-2x^2} = -\frac{1}{4}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x^2} + \frac{1}{4}\int \frac{dx}{x-2} = -\tfrac{1}{4}\ln|x| - \tfrac{1}{2}(-1/x) + \tfrac{1}{4}\ln|x-2| + C$$
$$= \tfrac{1}{4}\ln\left|\frac{x-2}{x}\right| + \frac{1}{2x} + C$$

We plot this function with $C = 0$ on the same screen as $y = \dfrac{1}{x^3-2x^2}$.

55. $\displaystyle\int \frac{dx}{x^2-2x} = \int \frac{dx}{(x-1)^2-1} = \int \frac{du}{u^2-1}$ (put $u = x - 1$)

$$= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C \text{ (by Equation 6)} = \frac{1}{2}\ln\left|\frac{x-2}{x}\right| + C$$

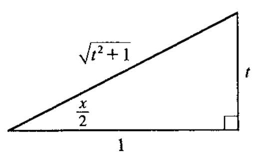
56. $\displaystyle\int \frac{(2x+1)\,dx}{4x^2+12x-7} = \frac{1}{4}\int \frac{(8x+12)\,dx}{4x^2+12x-7} - \int \frac{2\,dx}{(2x+3)^2-16}$

$\displaystyle = \tfrac{1}{4}\ln\left|4x^2+12x-7\right| - \int \frac{du}{u^2-16}$ [put $u = 2x+3$]

$\displaystyle = \tfrac{1}{4}\ln\left|4x^2+12x-7\right| - \tfrac{1}{8}\ln\left|(u-4)\,/\,(u+4)\right| + C$ (by Equation 6)

$\displaystyle = \tfrac{1}{4}\ln\left|4x^2+12x-7\right| - \tfrac{1}{8}\ln\left|(2x-1)\,/\,(2x+7)\right| + C$

57. (a) If $t = \tan\left(\frac{x}{2}\right)$, then $\frac{x}{2} = \tan^{-1} t$. The figure gives $\cos\left(\frac{x}{2}\right) = \dfrac{1}{\sqrt{1+t^2}}$ and $\sin\left(\frac{x}{2}\right) = \dfrac{t}{\sqrt{1+t^2}}$.



(b) $\displaystyle \cos x = \cos\left(2\cdot\frac{x}{2}\right) = 2\cos^2\left(\frac{x}{2}\right) - 1$

$\displaystyle = 2\left(\frac{1}{\sqrt{1+t^2}}\right)^2 - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$

$\displaystyle \sin x = \sin\left(2\cdot\frac{x}{2}\right) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$

$\displaystyle = 2\frac{t}{\sqrt{1+t^2}}\frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$

(c) $\displaystyle \frac{x}{2} = \arctan t \;\Rightarrow\; x = 2\arctan t \;\Rightarrow\; dx = \frac{2}{1+t^2}\,dt$

58. Let $t = \tan\left(\frac{x}{2}\right)$. Then, using Exercise 57, $dx = \dfrac{2}{1+t^2}\,dt$, $\sin x = \dfrac{2t}{1+t^2}$ $\Rightarrow$

$$\int \frac{dx}{3-5\sin x} = \int \frac{2\,dt/\left(1+t^2\right)}{3-10t/\left(1+t^2\right)} = \int \frac{2\,dt}{3\left(1+t^2\right)-10t} = 2\int \frac{dt}{3t^2-10t+3}$$

$$= \frac{1}{4}\int\left[\frac{1}{t-3} - \frac{3}{3t-1}\right]dt = \tfrac{1}{4}\left(\ln|t-3| - \ln|3t-1|\right) + C = \frac{1}{4}\ln\left|\frac{\tan(x/2)-3}{3\tan(x-2)-1}\right| + C$$

59. Let $t = \tan(x/2)$. Then, using the expressions in Exercise 57, we have

$$\int \frac{1}{3\sin x - 4\cos x}\,dx = \int \frac{1}{3\left(\dfrac{2t}{1+t^2}\right) - 4\left(\dfrac{1-t^2}{1+t^2}\right)}\frac{2\,dt}{1+t^2} = 2\int \frac{dt}{3\,(2t)-4\left(1-t^2\right)} = \int \frac{dt}{2t^2+3t-2}$$

$$= \int \frac{dt}{(2t-1)\,(t+2)} = \int\left[\frac{2}{5}\frac{1}{2t-1} - \frac{1}{5}\frac{1}{t+2}\right]dt \text{ (using partial fractions)}$$

$$= \tfrac{1}{5}\left[\ln|2t-1| - \ln|t+2|\right] + C = \frac{1}{5}\ln\left|\frac{2t-1}{t+2}\right| + C = \frac{1}{5}\ln\left|\frac{2\tan(x/2)-1}{\tan(x/2)+2}\right| + C$$

60. Let $t = \tan\left(\frac{x}{2}\right)$. Then, by Exercise 57,

$$\int_{\pi/3}^{\pi/2} \frac{dx}{1+\sin x-\cos x} = \int_{1/\sqrt{3}}^{1} \frac{2dt/\left(1+t^2\right)}{1+2t/\left(1+t^2\right) - \left(1-t^2\right)/\left(1+t^2\right)} = \int_{1/\sqrt{3}}^{1} \frac{2dt}{1+t^2+2t-1+t^2}$$

$$= \int_{1/\sqrt{3}}^{1}\left[\frac{1}{t} - \frac{1}{t+1}\right]dt = \left[\ln t - \ln(t+1)\right]_{1/\sqrt{3}}^{1} = \ln\frac{1}{2} - \ln\frac{1}{\sqrt{3}+1} = \ln\frac{\sqrt{3}+1}{2}$$

61. Let $t = \tan\left(\frac{x}{2}\right)$. Then, by Exercise 57,

$$\begin{aligned}\int \frac{dx}{2\sin x + \sin 2x} &= \frac{1}{2}\int \frac{dx}{\sin x + \sin x \cos x} = \frac{1}{2}\int \frac{2\,dt/\left(1+t^2\right)}{2t/\left(1+t^2\right) + 2t\left(1-t^2\right)/\left(1+t^2\right)^2} \\ &= \frac{1}{2}\int \frac{\left(1+t^2\right)dt}{t\left(1+t^2\right) + t\left(1-t^2\right)} = \frac{1}{4}\int \frac{\left(1+t^2\right)dt}{t} = \frac{1}{4}\int \left(\frac{1}{t} + t\right)dt \\ &= \tfrac{1}{4}\ln|t| + \tfrac{1}{8}t^2 + C = \tfrac{1}{4}\ln\left|\tan\left(\tfrac{1}{2}x\right)\right| + \tfrac{1}{8}\tan^2\left(\tfrac{1}{2}x\right) + C\end{aligned}$$

62. $x^2 - 6x + 8 = (x-3)^2 - 1$ is positive for $5 \le x \le 10$, so

$$\begin{aligned}\text{area} &= \int_5^{10} \frac{dx}{(x-3)^2 - 1} = \int_2^7 \frac{du}{u^2 - 1} \ (\text{put } u = x - 3) \ = \left[\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|\right]_2^7 \\ &= \tfrac{1}{2}\ln\tfrac{3}{4} - \tfrac{1}{2}\ln\tfrac{1}{3} = \tfrac{1}{2}(\ln 3 - 2\ln 2 + \ln 3) = \ln 3 - \ln 2 = \ln\tfrac{3}{2}\end{aligned}$$

63. $\dfrac{x+1}{x-1} = 1 + \dfrac{2}{x-1} > 0$ for $2 \le x \le 3$, so

$$\text{area} = \int_2^3 \left[1 + \frac{2}{x-1}\right]dx = [x + 2\ln|x-1|]_2^3 = (3 + 2\ln 2) - (2 + 2\ln 1) = 1 + 2\ln 2.$$

64. (a) We use disks, so the volume is $V = \pi\int_0^1 \left[\dfrac{1}{x^2+3x+2}\right]^2 dx = \pi\int_0^1 \dfrac{dx}{(x+1)^2(x+2)^2}$. To evaluate the integral, we use partial fractions: $\dfrac{1}{(x+1)^2(x+2)^2} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x+2} + \dfrac{D}{(x+2)^2} \Rightarrow$ $1 = A(x+1)(x+2)^2 + B(x+2)^2 + C(x+1)^2(x+2) + D(x+1)^2$. We set $x = -1$, giving $B = 1$, then set $x = -2$, giving $D = 1$. Now equating coefficients of x^3 gives $A = -C$, and then equating constants gives $1 = 4A + 4 + 2(-A) + 1 \Rightarrow A = -2 \Rightarrow C = 2$. So the expression becomes

$$\begin{aligned}V &= \pi\int_0^1 \left[\frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{2}{(x+2)} + \frac{1}{(x+2)^2}\right]dx = \pi\left[2\ln\left|\frac{x+2}{x+1}\right| - \frac{1}{x+1} - \frac{1}{x+2}\right]_0^1 \\ &= \pi\left[\left(2\ln\tfrac{3}{2} - \tfrac{1}{2} - \tfrac{1}{3}\right) - \left(2\ln 2 - 1 - \tfrac{1}{2}\right)\right] = \pi\left(2\ln\tfrac{3/2}{2} + \tfrac{2}{3}\right) = \pi\left(\tfrac{2}{3} + \ln\tfrac{9}{16}\right)\end{aligned}$$

(b) In this case, we use cylindrical shells, so the volume is $V = 2\pi\displaystyle\int_0^1 \frac{x\,dx}{x^2+3x+2} = 2\pi\int_0^1 \frac{x\,dx}{(x+1)(x+2)}$.

We use partial fractions to simplify the integrand: $\dfrac{x}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2} \Rightarrow$ $x = (A+B)x + 2A + B$. So $A + B = 1$ and $2A + B = 0 \Rightarrow A = -1$ and $B = 2$. So the volume is

$$\begin{aligned}2\pi\int_0^1 \left[\frac{-1}{x+1} + \frac{2}{x+2}\right]dx &= 2\pi\left[-\ln|x+1| + 2\ln|x+2|\right]_0^1 \\ &= 2\pi(-\ln 2 + 2\ln 3 + \ln 1 - 2\ln 2) = 2\pi(2\ln 3 - 3\ln 2) = 2\pi\ln\tfrac{9}{8}\end{aligned}$$

65. $\dfrac{P+S}{P\,[(r-1)\,P-S]}=\dfrac{A}{P}+\dfrac{B}{(r-1)\,P-S} \;\Rightarrow\; P+S=A\,[(r-1)\,P-S]+BP=[(r-1)\,A+B]\,P-AS$
$\Rightarrow\; (r-1)\,A+B=1,\ -A=1 \;\Rightarrow\; A=-1,\ B=r$. Now

$$t=\int\frac{P+S}{P\,[(r-1)\,P-S]}\,dP=\int\left[\frac{-1}{P}+\frac{r}{(r-1)\,P-S}\right]dP=-\int\frac{dP}{P}+\frac{r}{r-1}\int\frac{r-1}{(r-1)\,P-S}\,dP$$

so $t=-\ln P+\dfrac{r}{r-1}\ln|(r-1)\,P-S|+C$. Here $r=0.10$ and $S=900$, so

$$t=-\ln P+\tfrac{0.1}{-0.9}\ln|-0.9P-900|+C=-\ln P-\tfrac{1}{9}\ln\left(|-1|\,|0.9P+900|\right)=-\ln P-\tfrac{1}{9}\ln(0.9P+900)+C$$

When $t=0$, $P=10{,}000$, so $0=-\ln 10{,}000-\frac{1}{9}\ln(9900)+C$. Thus, $C=\ln 10{,}000+\frac{1}{9}\ln 9900$ [≈ 10.2326], so our equation becomes

$$t=\ln 10,000-\ln P+\tfrac{1}{9}\ln 9900-\tfrac{1}{9}\ln(0.9P+900)=\ln\frac{10{,}000}{P}+\frac{1}{9}\ln\frac{9900}{0.9P+900}$$
$$=\ln\frac{10{,}000}{P}+\frac{1}{9}\ln\frac{1100}{0.1P+100}=\ln\frac{10{,}000}{P}+\frac{1}{9}\ln\frac{11{,}000}{P+1000}$$

66. If we subtract and add $2x^2$, we get

$$x^4+1=x^4+2x^2+1-2x^2=\left(x^2+1\right)^2-2x^2=\left(x^2+1\right)^2-\left(\sqrt{2}x\right)^2$$
$$=\left[\left(x^2+1\right)-\sqrt{2}x\right]\left[\left(x^2+1\right)+\sqrt{2}x\right]=\left(x^2-\sqrt{2}x+1\right)\left(x^2+\sqrt{2}x+1\right)$$

So we can decompose $\dfrac{1}{x^4+1}=\dfrac{Ax+B}{x^2+\sqrt{2}x+1}+\dfrac{Cx+D}{x^2-\sqrt{2}x+1} \;\Rightarrow$
$1=(Ax+B)\left(x^2-\sqrt{2}x+1\right)+(Cx+D)\left(x^2+\sqrt{2}x+1\right)$. Setting the constant terms equal gives $B+D=1$, then from the coefficients of x^3 we get $A+C=0$. Now from the coefficients of x we get $A+C+(B-D)\sqrt{2}=0 \;\Leftrightarrow\; [(1-D)-D]\sqrt{2}=0 \;\Rightarrow\; D=\frac{1}{2} \;\Rightarrow\; B=\frac{1}{2}$, and finally, from the coefficients of x^2 we get $\sqrt{2}\,(C-A)+B+D=0 \;\Rightarrow\; C-A=-\frac{1}{\sqrt{2}} \;\Rightarrow\; C=-\frac{\sqrt{2}}{4}$ and $A=\frac{\sqrt{2}}{4}$. So we rewrite the integrand, splitting the terms into forms which we know how to integrate:

$$\frac{1}{x^4+1}=\frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2+\sqrt{2}x+1}+\frac{-\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2-\sqrt{2}x+1}=\frac{1}{4\sqrt{2}}\left[\frac{2x+2\sqrt{2}}{x^2+\sqrt{2}x+1}-\frac{2x-2\sqrt{2}}{x^2-\sqrt{2}x+1}\right]$$
$$=\frac{\sqrt{2}}{8}\left[\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}-\frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right]+\frac{1}{4}\left[\frac{1}{\left(x+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}+\frac{1}{\left(x-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}\right]$$

Now we integrate: $\displaystyle\int\frac{dx}{x^4+1}=\frac{\sqrt{2}}{8}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)+\frac{\sqrt{2}}{4}\left[\tan^{-1}\left(\sqrt{2}x+1\right)+\tan^{-1}\left(\sqrt{2}x-1\right)\right]+C.$

67. (a) In Maple, we define $f(x)$, and then use `convert(f,parfrac,x);` to obtain

$$f(x) = \frac{24{,}110/4879}{5x+2} - \frac{668/323}{2x+1} - \frac{9438/80{,}155}{3x-7} + \frac{(22{,}098x + 48{,}935)/260{,}015}{x^2+x+5}.$$

In Mathematica, we use the command `Apart`, and in Derive, we use `Expand`.

(b) $$\int f(x)\,dx = \tfrac{24{,}110}{4879}\cdot\tfrac{1}{5}\ln|5x+2| - \tfrac{668}{323}\cdot\tfrac{1}{2}\ln|2x+1| - \tfrac{9438}{80{,}155}\cdot\tfrac{1}{3}\ln|3x-7|$$

$$+\frac{1}{260{,}015}\int\frac{22{,}098\left(x+\frac{1}{2}\right)+37{,}886}{\left(x+\frac{1}{2}\right)^2+\frac{19}{4}}\,dx + C$$

$$= \tfrac{24{,}110}{4879}\cdot\tfrac{1}{5}\ln|5x+2| - \tfrac{668}{323}\cdot\tfrac{1}{2}\ln|2x+1| - \tfrac{9438}{80{,}155}\cdot\tfrac{1}{3}\ln|3x-7|$$

$$+\tfrac{1}{260{,}015}\left[22{,}098\cdot\tfrac{1}{2}\ln\left(x^2+x+5\right) + 37{,}886\cdot\sqrt{\tfrac{4}{19}}\tan^{-1}\left(\tfrac{1}{\sqrt{19/4}}\left(x+\tfrac{1}{2}\right)\right)\right]+C$$

$$= \tfrac{4822}{4879}\ln|5x+2| - \tfrac{334}{323}\ln|2x+1| - \tfrac{3146}{80{,}155}\ln|3x-7| + \tfrac{11{,}049}{260{,}015}\ln\left(x^2+x+5\right)$$

$$+\tfrac{75{,}772}{260{,}015\sqrt{19}}\tan^{-1}\left[\tfrac{1}{\sqrt{19}}(2x+1)\right]+C.$$

Using a CAS, we get

$$\frac{4822\ln(5x+2)}{4879} - \frac{334\ln(2x+1)}{323} - \frac{3146\ln(3x-7)}{80{,}155}$$

$$+\frac{11{,}049\ln\left(x^2+x+5\right)}{260{,}015} + \frac{3988\sqrt{19}}{260{,}115}\tan^{-1}\left[\frac{\sqrt{19}}{19}(2x+1)\right]$$

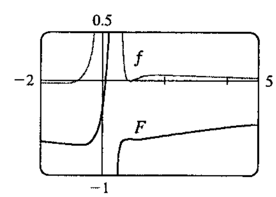
The main difference in this answer is that the absolute value signs and the constant of integration have been omitted. Also, the fractions have been reduced and the denominators rationalized.

68. (a) In Maple, we define $f(x)$, and then use `convert(f,parfrac,x);` to get

$$f(x) = \frac{5828/1815}{(5x-2)^2} - \frac{59{,}096/19{,}965}{5x-2} + \frac{2(2843x+816)/3993}{2x^2+1} + \frac{(313x-251)/363}{\left(2x^2+1\right)^2}.$$

In Mathematica, we use the command `Apart`, and in Derive, we use `Expand`.

(b) As we saw in Exercise 67, computer algebra systems omit the absolute value signs in $\int (1/y)\,dy = \ln|y|$. So we use the CAS to integrate the expression in part (a) and add the necessary absolute value signs and constant of integration to get



$$\int f(x)\,dx = -\frac{5828}{9075(5x-2)} - \frac{59{,}096\ln|5x-2|}{99{,}825} + \frac{2843\ln\left(2x^2+1\right)}{7986}$$

$$+\frac{503}{15{,}972}\sqrt{2}\tan^{-1}\left(\sqrt{2}x\right) - \frac{1}{2904}\frac{1004x+626}{2x^2+1}+C$$

(c) From the graph, we see that f goes from negative to positive at $x \approx -0.78$, then back to negative at $x \approx 0.8$, and finally back to positive at $x = 1$. Also, $\lim_{x\to 0.4} f(x) = \infty$. So we see (by the First Derivative Test) that $\int f(x)\,dx$ has minima at $x \approx -0.78$ and $x = 1$, and a maximum at $x \approx 0.80$, and that $\int f(x)\,dx$ is unbounded as $x \to 0.4$. Note also that just to the right of $x = 0.4$, f has large values, so $\int f(x)\,dx$ increases rapidly, but slows down as f drops toward 0. $\int f(x)\,dx$ decreases from about 0.8 to 1, then increases slowly since f stays small and positive.

69. There are only finitely many values of x where $Q(x) = 0$ (assuming that Q is not the zero polynomial). At all other values of x, $F(x)/Q(x) = G(x)/Q(x)$, so $F(x) = G(x)$. In other words, the values of F and G agree at all except perhaps finitely many values of x. By continuity of F and G, the polynomials F and G must agree at those values of x too.

More explicitly: if a is a value of x such that $Q(a) = 0$, then $Q(x) \neq 0$ for all x sufficiently close to a. Thus,

$$\begin{aligned} F(a) &= \lim_{x\to a} F(x) \quad \text{(by continuity of } F) \quad = \lim_{x\to a} G(x) \quad [\text{whenever } Q(x) \neq 0] \\ &= G(a) \quad \text{(by continuity of } G). \end{aligned}$$

70. Let $f(x) = ax^2 + bx + c$. We calculate the partial fraction decomposition of $\dfrac{f(x)}{x^2(x+1)^3}$. Since $f(0) = 1$, we must have $c = 1$, so $\dfrac{f(x)}{x^2(x+1)^3} = \dfrac{ax^2+bx+1}{x^2(x+1)^3} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1} + \dfrac{D}{(x+1)^2} + \dfrac{E}{(x+1)^3}$. Now in order for the integral not to contain any logarithms (that is, in order for it to be a rational function), we must have $A = C = 0$, so $ax^2 + bx + 1 = B(x+1)^3 + Dx^2(x+1) + Ex^2$. Equating constant terms gives $B = 1$, then equating coefficients of x gives $3B = b \Rightarrow b = 3$. This is the quantity we are looking for, since $f'(0) = b$.

8.5 Strategy for Integration

1. Let $u = \sin x$. Then $\displaystyle\int \frac{\cos x\,dx}{1+\sin^2 x} = \int \frac{du}{1+u^2} = \tan^{-1} u + C = \tan^{-1}(\sin x) + C$.

2. $\displaystyle\int \frac{1+\cos x}{\sin x}\,dx = \int (\csc x + \cot x)\,dx = \ln|\csc x - \cot x| + \ln|\sin x| + C = \ln|1-\cos x| + C$.

Or: $\displaystyle\int \frac{1+\cos x}{\sin x}\,dx = \int \frac{1-\cos^2 x}{\sin x\,(1-\cos x)}\,dx = \int \frac{\sin x\,dx}{1-\cos x} = \ln|1-\cos x| + C$.

3. Let $u = \arctan y$. Then $du = \dfrac{dy}{1+y^2} \Rightarrow \displaystyle\int_{-1}^{1} \frac{e^{\arctan y}}{1+y^2}\,dy = \int_{-\pi/4}^{\pi/4} e^u\,du = \left[e^u\right]_{-\pi/4}^{\pi/4} = e^{\pi/4} - e^{-\pi/4}$.

4. Integrate by parts with $u = \ln x$, $dv = x^3\,dx \Rightarrow du = dx/x$, $v = x^4/4$:

$\int_1^2 x^3 \ln x\,dx = \left[\frac14 x^4 \ln x\right]_1^2 - \frac14 \int_1^2 x^3\,dx = 4\ln 2 - \frac{1}{16}\left[x^4\right]_1^2 = 4\ln 2 - \frac{15}{16}$.

5. $$\begin{aligned} \int \sin^2 x \cos^3 x\,dx &= \int \sin^2 x\,(1-\sin^2 x)\cos x\,dx = \int u^2(1-u^2)\,du \quad (\text{put } u = \sin x) \\ &= \int (u^2 - u^4)\,du = \tfrac13 u^3 - \tfrac15 u^5 + C = \tfrac13 \sin^3 x - \tfrac15 \sin^5 x + C \end{aligned}$$

6. Let $u = \cos x$. Then $du = -\sin x\,dx \Rightarrow$

$\int \sin x \cos(\cos x)\,dx = -\int \cos u\,du = -\sin u + C = -\sin(\cos x) + C$.

7. Let $u = \sqrt{9-x^2}$. Then $u^2 = 9 - x^2$, $u\,du = -x\,dx \Rightarrow$

$$\begin{aligned} \int \frac{\sqrt{9-x^2}}{x}\,dx &= \int \frac{\sqrt{9-x^2}}{x^2}\,x\,dx = \int \frac{u}{9-u^2}(-u)\,du = \int \left[1 - \frac{9}{9-u^2}\right]du \\ &= u + 9\int \frac{du}{u^2-9} = u + \frac{9}{2\cdot 3}\ln\left|\frac{u-3}{u+3}\right| + C = \sqrt{9-x^2} + \frac32 \ln\left|\frac{\sqrt{9-x^2}-3}{\sqrt{9-x^2}+3}\right| + C \\ &= \sqrt{9-x^2} + \frac32 \ln \frac{(\sqrt{9-x^2}-3)^2}{x^2} + C = \sqrt{9-x^2} + 3\ln\left|\frac{3-\sqrt{9-x^2}}{x}\right| + C \end{aligned}$$

Or: Put $x = 3\sin\theta$.

8. Let $u = x^2$. Then $du = 2x\,dx \Rightarrow \displaystyle\int \frac{x\,dx}{\sqrt{3-x^4}} = \frac{1}{2}\int \frac{du}{\sqrt{3-u^2}} = \frac{1}{2}\sin^{-1}\frac{u}{\sqrt{3}} + C = \frac{1}{2}\sin^{-1}\frac{x^2}{\sqrt{3}} + C.$

9. Let $u = 1 - x^2 \Rightarrow du = -2x\,dx$. Then

$$\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_1^{3/4} \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int_{3/4}^{1} u^{-1/2}\,du = \tfrac{1}{2}\left[2u^{1/2}\right]_{3/4}^{1} = \left[\sqrt{u}\right]_{3/4}^{1} = 1 - \tfrac{\sqrt{3}}{2}$$

10. $\displaystyle\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/4} \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta \quad [x = \sin\theta,\ dx = \cos\theta\,d\theta]$

$$= \textstyle\int_0^{\pi/4} \frac{1}{2}(1-\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \frac{1}{2}\left[\left(\frac{\pi}{4} - \frac{1}{2}\right) - (0-0)\right] = \frac{\pi}{8} - \frac{1}{4}$$

11. $\displaystyle\int_0^2 \frac{2t}{(t-3)^2}\,dt = \int_{-3}^{-1} \frac{2(u+3)}{u^2}\,du \quad [u = t-3,\ du = dt] \quad = \int_{-3}^{-1}\left(\frac{2}{u} + \frac{6}{u^2}\right)du = \left[2\ln|u| - \frac{6}{u}\right]_{-3}^{-1}$

$$= (2\ln 1 + 6) - (2\ln 3 + 2) = 4 - 2\ln 3 \text{ or } 4 - \ln 9$$

12. $\dfrac{x-1}{x^2-4x-5} = \dfrac{x-1}{(x-5)(x+1)} = \dfrac{A}{x-5} + \dfrac{B}{x+1} \Rightarrow x - 1 = A(x+1) + B(x-5)$. Setting $x = -1$ gives $-2 = -6B$, so $B = \frac{1}{3}$. Setting $x = 5$ gives $4 = 6A$, so $A = \frac{2}{3}$. Now

$$\int_0^4 \frac{x-1}{x^2-4x-5}\,dx = \int_0^4 \left(\frac{2/3}{x-5} + \frac{1/3}{x+1}\right)dx = \left[\tfrac{2}{3}\ln|x-5| + \tfrac{1}{3}\ln|x+1|\right]_0^4$$

$$= \tfrac{2}{3}\ln 1 + \tfrac{1}{3}\ln 5 - \tfrac{2}{3}\ln 5 - \tfrac{1}{3}\ln 1 = -\tfrac{1}{3}\ln 5$$

13. $\displaystyle\int \frac{x-1}{x^2-4x+5}\,dx = \int \frac{(x-2)+1}{(x-2)^2+1}\,dx = \int\left(\frac{u}{u^2+1} + \frac{1}{u^2+1}\right)du \quad [u = x-2,\ du = dx]$

$$= \tfrac{1}{2}\ln\left(u^2+1\right) + \tan^{-1}u + C = \tfrac{1}{2}\ln\left(x^2-4x+5\right) + \tan^{-1}(x-2) + C$$

14. $\displaystyle\int \frac{x}{x^4+x^2+1}\,dx = \int \frac{\frac{1}{2}\,du}{u^2+u+1} \quad [u = x^2,\ du = 2x\,dx] \quad = \frac{1}{2}\int \frac{du}{\left(u+\frac{1}{2}\right)^2 + \frac{3}{4}}$

$$= \frac{1}{2}\int \frac{\frac{\sqrt{3}}{2}\,dv}{\frac{3}{4}\left(v^2+1\right)} \quad \left[u + \tfrac{1}{2} = \tfrac{\sqrt{3}}{2}v,\ du = \tfrac{\sqrt{3}}{2}\,dv\right] \quad = \frac{\sqrt{3}}{4}\cdot\frac{4}{3}\int \frac{dv}{v^2+1}$$

$$= \tfrac{1}{\sqrt{3}}\tan^{-1}v + C = \tfrac{1}{\sqrt{3}}\tan^{-1}\left(\tfrac{2}{\sqrt{3}}\left(x^2+\tfrac{1}{2}\right)\right) + C$$

15. Let $u = e^x$. Then $\int e^{x+e^x}\,dx = \int e^{e^x}e^x\,dx = \int e^u\,du = e^u + C = e^{e^x} + C$.

16. Let $u = \sqrt[3]{x}$. Then $x = u^3 \Rightarrow \int e^{\sqrt[3]{x}}\,dx = \int e^u \cdot 3u^2\,du$. Now use parts: let $w = u^2$, $dv = e^u\,du \Rightarrow$ $dw = 2u\,du$, $v = e^u \Rightarrow 3\int e^u u^2\,du = 3\left(u^2e^u - 2\int ue^u\,du\right)$. Now use parts again with $w = u$, $dv = e^u\,du$ to get $\int e^u 3u^2\,du = e^u\left(3u^2 - 6u + 6\right) + C = 3e^{\sqrt[3]{x}}\left(x^{2/3} - 2\sqrt[3]{x} + 2\right) + C$.

17. Use integration by parts: $u = \ln\left(1+x^2\right)$, $dv = dx \Rightarrow du = \dfrac{2x}{1+x^2}\,dx$, $v = x$, so

$$\int \ln\left(1+x^2\right)dx = x\ln\left(1+x^2\right) - \int x\cdot\frac{2x\,dx}{1+x^2} = x\ln\left(1+x^2\right) - 2\int\left[1 - \frac{1}{1+x^2}\right]dx$$

$$= x\ln\left(1+x^2\right) - 2x + 2\tan^{-1}x + C$$

18. Let $u = \ln x$. Then $du = dx/x \Rightarrow$

$$\int \frac{\sqrt{1+\ln x}}{x\ln x}\,dx = \int \frac{\sqrt{1+u}}{u}\,du = \int \frac{v}{v^2-1}\,2v\,dv \quad [\text{put } v = \sqrt{1+u},\ u = v^2-1,\ du = 2v\,dv]$$
$$= 2\int\left(1+\frac{1}{v^2-1}\right)dv = 2v + \ln\left|\frac{v-1}{v+1}\right| + C = 2\sqrt{1+\ln x} + \ln\left(\frac{\sqrt{1+\ln x}-1}{\sqrt{1+\ln x}+1}\right) + C$$

19. Integrate by parts three times, first with $u = t^3$, $dv = e^{-2t}\,dt$:

$$\int t^3e^{-2t}\,dt = -\tfrac{1}{2}t^3e^{-2t} + \tfrac{1}{2}\int 3t^2e^{-2t}\,dt = -\tfrac{1}{2}t^3e^{-2t} - \tfrac{3}{4}t^2e^{-2t} + \tfrac{1}{2}\int 3te^{-2t}\,dt$$
$$= -e^{-2t}\left[\tfrac{1}{2}t^3 + \tfrac{3}{4}t^2\right] - \tfrac{3}{4}te^{-2t} + \tfrac{3}{4}\int e^{-2t}\,dt = -e^{-2t}\left[\tfrac{1}{2}t^3 + \tfrac{3}{4}t^2 + \tfrac{3}{4}t + \tfrac{3}{8}\right] + C$$
$$= -\tfrac{1}{8}e^{-2t}\left(4t^3 + 6t^2 + 6t + 3\right) + C$$

20. Integrate by parts: $u = \sin^{-1}x$, $dv = x\,dx \Rightarrow du = \left(1\big/\sqrt{1-x^2}\right)dx$, $v = \frac{1}{2}x^2$, so

$$\int x\sin^{-1}x\,dx = \tfrac{1}{2}x^2\sin^{-1}x - \frac{1}{2}\int\frac{x^2\,dx}{\sqrt{1-x^2}} = \tfrac{1}{2}x^2\sin^{-1}x - \frac{1}{2}\int\frac{\sin^2\theta\cos\theta\,d\theta}{\cos\theta} \quad \begin{bmatrix}\text{where } x = \sin\theta \\ \text{for } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\end{bmatrix}$$
$$= \tfrac{1}{2}x^2\sin^{-1}x - \tfrac{1}{4}\int(1-\cos 2\theta)\,d\theta = \tfrac{1}{2}x^2\sin^{-1}x - \tfrac{1}{4}(\theta - \sin\theta\cos\theta) + C$$
$$= \tfrac{1}{2}x^2\sin^{-1}x - \tfrac{1}{4}\left[\sin^{-1}x - x\sqrt{1-x^2}\right] + C = \tfrac{1}{4}\left[\left(2x^2-1\right)\sin^{-1}x + x\sqrt{1-x^2}\right] + C$$

21. Let $u = 1+\sqrt{x}$. Then $x = (u-1)^2$, $dx = 2(u-1)\,du \Rightarrow$

$$\int_0^1\left(1+\sqrt{x}\right)^8 dx = \int_1^2 u^8\cdot 2(u-1)\,du = 2\int_1^2\left(u^9-u^8\right)du = \left[\tfrac{1}{5}u^{10} - 2\cdot\tfrac{1}{9}u^9\right]_1^2$$
$$= \tfrac{1024}{5} - \tfrac{1024}{9} - \tfrac{1}{5} + \tfrac{2}{9} = \tfrac{4097}{45}$$

22. $\int_0^1 \sqrt{z}\left(z+\sqrt[3]{z}\right)dz = \int_0^1 u^3\left(u^6+u^2\right)6u^5\,du$ $[u = \sqrt[6]{z},\ u^6 = z,\ dz = 6u^5\,du]$ $= 6\int_0^1\left(u^{14}+u^{10}\right)du$

$$= 6\left[\tfrac{1}{15}u^{15} + \tfrac{1}{11}u^{11}\right]_0^1 = 6\left(\tfrac{1}{15}+\tfrac{1}{11}\right) = 6\cdot\tfrac{26}{165} = \tfrac{52}{55}$$

(Note that this integral can also be evaluated without substitution.)

23. $\dfrac{3x^2-2}{x^2-2x-8} = 3 + \dfrac{6x+22}{(x-4)(x+2)} = 3 + \dfrac{A}{x-4} + \dfrac{B}{x+2} \Rightarrow 6x+22 = A(x+2) + B(x-4)$. Setting $x = 4$ gives $46 = 6A$, so $A = \frac{23}{3}$. Setting $x = -2$ gives $10 = -6B$, so $B = -\frac{5}{3}$. Now

$$\int\frac{3x^2-2}{x^2-2x-8}\,dx = \int\left(3 + \frac{23/3}{x-4} - \frac{5/3}{x+2}\right)dx = 3x + \tfrac{23}{3}\ln|x-4| - \tfrac{5}{3}\ln|x+2| + C.$$

24. $\displaystyle\int\frac{3x^2-2}{x^3-2x-8}\,dx = \int\frac{du}{u}$ $[u = x^3-2x-8,\ du = (3x^2-2)\,dx]$ $= \ln|u| + C = \ln\left|x^3-2x-8\right| + C$

25. Let $u = \ln(\sin x)$. Then $du = \cot x\,dx \Rightarrow \int\cot x\ln(\sin x)\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}[\ln(\sin x)]^2 + C$.

26. $\int\sin\sqrt{at}\,dt = \int\sin u\cdot\frac{2}{a}u\,du$ $[u = \sqrt{at},\ 2u\,du = a\,dt,\ u^2 = at]$ $= \frac{2}{a}\int u\sin u\,du$

$$= \tfrac{2}{a}\left[-u\cos u + \sin u\right] + C \ \ [\text{integration by parts}] \ = -\tfrac{2}{a}\sqrt{at}\cos\sqrt{at} + \tfrac{2}{a}\sin\sqrt{at} + C$$
$$= -2\sqrt{\tfrac{t}{a}}\cos\sqrt{at} + \tfrac{2}{a}\sin\sqrt{at} + C$$

27. $\int_{-3}^{3} |x^3 + x^2 - 2x|\,dx = \int_{-3}^{3} |(x+2)x(x-1)|\,dx$

$= -\int_{-3}^{-2} (x^3 + x^2 - 2x)\,dx + \int_{-2}^{0} (x^3 + x^2 - 2x)\,dx - \int_{0}^{1} (x^3 + x^2 - 2x)\,dx + \int_{1}^{3} (x^3 + x^2 - 2x)\,dx$

Let $f(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 - x^2$. Then $f'(x) = x^3 + x^2 - 2x$, so

$$\int_{-3}^{3} |x^3 + x^2 - 2x|\,dx = -f(-2) + f(-3) + f(0) - f(-2) - f(1) + f(0) + f(3) - f(1)$$
$$= f(-3) - 2f(-2) + 2f(0) - 2f(1) + f(3) = \tfrac{9}{4} - 2\left(-\tfrac{8}{3}\right) + 2 \cdot 0 - 2\left(-\tfrac{5}{12}\right) + \tfrac{81}{4} = \tfrac{86}{3}$$

28. Let $x - \frac{1}{2} = \frac{\sqrt{5}}{2}u$. Then $dx = \frac{\sqrt{5}}{2}\,du$, $u = \frac{1}{\sqrt{5}}(2x-1) \Rightarrow$

$$\int \sqrt{1 + x - x^2}\,dx = \int \sqrt{\tfrac{5}{4} - \left(x - \tfrac{1}{2}\right)^2}\,dx = \tfrac{5}{4}\int \sqrt{1-u^2}\,du = \tfrac{5}{4}\int \cos\theta\cos\theta\,d\theta$$
$$= \tfrac{5}{8}(\theta + \sin\theta\cos\theta) + C = \frac{5}{8}\left[\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + \frac{2x-1}{\sqrt{5}}\frac{2}{\sqrt{5}}\sqrt{1+x-x^2}\right] + C$$
$$= \tfrac{5}{8}\sin^{-1}\left(\tfrac{1}{\sqrt{5}}(2x-1)\right) + \tfrac{1}{4}(2x-1)\sqrt{1+x-x^2} + C$$

29. As in Example 5,

$$\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x\,dx}{\sqrt{1-x^2}}$$
$$= \sin^{-1}x - \sqrt{1-x^2} + C$$

Another Method: Substitute $u = \sqrt{(1+x)/(1-x)}$.

30. $\displaystyle\int \frac{\sqrt{2x-1}}{2x+3}\,dx = \int \frac{u \cdot u\,du}{u^2+4} \begin{bmatrix} u = \sqrt{2x-1},\ 2x+3 = u^2+4, \\ u^2 = 2x-1,\ u\,du = dx \end{bmatrix} = \int \left(1 - \frac{4}{u^2+4}\right)du$

$= u - 4 \cdot \frac{1}{2}\tan^{-1}\left(\frac{1}{2}u\right) + C = \sqrt{2x-1} - 2\tan^{-1}\left(\frac{1}{2}\sqrt{2x-1}\right) + C$

31. $\displaystyle\int_0^5 \frac{3w-1}{w+2}\,dw = \int_0^5 \left(3 - \frac{7}{w+2}\right)dw = [3w - 7\ln|w+2|]_0^5 = 15 - 7\ln 7 + 7\ln 2 = 15 - 7\ln\tfrac{7}{2}$

32. $\displaystyle\int \frac{dx}{x^3-8} = \int \left[\frac{1/12}{x-2} - \frac{x/12 + 1/3}{x^2+2x+4}\right]dx = \frac{1}{12}\int \left[\frac{1}{x-2} - \frac{x+4}{x^2+2x+4}\right]dx$

$\displaystyle= \tfrac{1}{12}\ln|x-2| - \frac{1}{24}\int \frac{2x+2}{x^2+2x+4}\,dx - \frac{1}{4}\int \frac{dx}{(x+1)^2+3}$

$= \frac{1}{12}\ln|x-2| - \frac{1}{24}\ln\left(x^2+2x+4\right) - \frac{1}{4\sqrt{3}}\tan^{-1}\left[\frac{1}{\sqrt{3}}(x+1)\right] + C$

33. $I = \int e^{2x}\sin 3x\,dx = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int e^{2x}\cos 3x\,dx \quad \begin{bmatrix} u = e^{2x},\ dv = \sin 3x\,dx, \\ du = 2e^{2x},\ v = -\frac{1}{3}\cos 3x \end{bmatrix}$

$= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\left(\frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\int e^{2x}\sin 3x\,dx\right) \quad \begin{bmatrix} U = e^{2x},\ dV = \cos 3x\,dx, \\ dU = 2e^{2x},\ V = \frac{1}{3}\sin 3x \end{bmatrix}$

So $I = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x - \frac{4}{9}I \Rightarrow \frac{13}{9}I = \frac{1}{9}e^{2x}(2\sin 3x - 3\cos 3x) + C_1$ and $I = \frac{1}{13}e^{2x}(2\sin 3x - 3\cos 3x) + C$, where $C = \frac{9}{13}C_1$.

34. Let $u = \cos^2 x \Rightarrow du = -2\cos x\sin x\,dx = -\sin 2x\,dx \Rightarrow$

$\displaystyle\int \frac{\sin 2x}{\sqrt{9-\cos^4 x}}\,dx = \int \frac{-du}{\sqrt{9-u^2}} = -\sin^{-1}\left(\tfrac{1}{3}u\right) + C = -\sin^{-1}\left(\tfrac{1}{3}\cos^2 x\right) + C.$

35. Because $f(x) = x^8\sin x$ is the product of an even function and an odd function, it is odd. Therefore, $\int_{-1}^{1} x^8\sin x\,dx = 0$ [by (5.5.6)(b)].

36. $\sin 4x \cos 3x = \frac{1}{2}(\sin x + \sin 7x)$ by Formula 8.2.2(a), so

$\int \sin 4x \cos 3x\, dx = \frac{1}{2}\int (\sin x + \sin 7x)\, dx = \frac{1}{2}\left[-\cos x - \frac{1}{7}\cos 7x\right] + C = -\frac{1}{2}\cos x - \frac{1}{14}\cos 7x + C.$

37. $\int_0^{\pi/4} \cos^2\theta \tan^2\theta\, d\theta = \int_0^{\pi/4} \sin^2\theta\, d\theta = \int_0^{\pi/4} \frac{1}{2}(1 - \cos 2\theta)\, d\theta = \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/4}$

$$= \left(\frac{\pi}{8} - \frac{1}{4}\right) - (0 - 0) = \frac{\pi}{8} - \frac{1}{4}$$

38. Let $u = \tan x$. Then

$$\int_0^{\pi/4} \tan^3 x \sec^4 x\, dx = \int_0^{\pi/4} \tan^3 x\left(\tan^2 x + 1\right)\sec^2 x\, dx = \int_0^1 u^3\left(u^2 + 1\right) du = \int_0^1 \left(u^5 + u^3\right) du$$

$$= \left[\frac{1}{6}u^6 + \frac{1}{4}u^4\right]_0^1 = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}$$

39. Let $u = 1 - x^2$. Then $du = -2x\, dx \Rightarrow$

$$\int \frac{x\, dx}{1 - x^2 + \sqrt{1 - x^2}} = -\frac{1}{2}\int \frac{du}{u + \sqrt{u}} = -\int \frac{v\, dv}{v^2 + v} \quad \left(v = \sqrt{u},\, u = v^2,\, du = 2v\, dv\right)$$

$$= -\int \frac{dv}{v + 1} = -\ln|v + 1| + C = -\ln\left(\sqrt{1 - x^2} + 1\right) + C$$

40. $4y^2 - 4y - 3 = (2y - 1)^2 - 2^2$, so let $u = 2y - 1 \Rightarrow du = 2\, dy$. Thus,

$$\int \frac{dy}{\sqrt{4y^2 - 4y - 3}} = \int \frac{dy}{\sqrt{(2y - 1)^2 - 2^2}} = \frac{1}{2}\int \frac{du}{\sqrt{u^2 - 2^2}}$$

$$= \frac{1}{2}\ln\left|u + \sqrt{u^2 - 2^2}\right| \quad \text{(by Formula 20 in the table in this section)}$$

$$= \frac{1}{2}\ln\left|2y - 1 + \sqrt{4y^2 - 4y - 3}\right| + C$$

41. Let $u = \theta$, $dv = \tan^2\theta\, d\theta = \left(\sec^2\theta - 1\right) d\theta \Rightarrow du = d\theta$ and $v = \tan\theta - \theta$. So

$$\int \theta \tan^2\theta\, d\theta = \theta(\tan\theta - \theta) - \int (\tan\theta - \theta)\, d\theta = \theta\tan\theta - \theta^2 - \ln|\sec\theta| + \frac{1}{2}\theta^2 + C$$

$$= \theta\tan\theta - \frac{1}{2}\theta^2 - \ln|\sec\theta| + C$$

42. $\int \tan^2 4x\, dx = \int \left(\sec^2 4x - 1\right) dx = \frac{1}{4}\tan 4x - x + C$

43. Let $t = x^3$. Then $dt = 3x^2\, dx \Rightarrow I = \int x^5 e^{-x^3}\, dx = \frac{1}{3}\int te^{-t}\, dt$. Now integrate by parts with $u = t$, $dv = e^{-t}\, dt$: $I = -\frac{1}{3}te^{-t} + \frac{1}{3}\int e^{-t}\, dt = -\frac{1}{3}te^{-t} - \frac{1}{3}e^{-t} + C = -\frac{1}{3}e^{-x^3}\left(x^3 + 1\right) + C$.

44. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \Rightarrow$

$$\int \frac{1 + e^x}{1 - e^x}\, dx = \int \frac{(1 + u)\, du}{(1 - u)\, u} = -\int \frac{(u + 1)\, du}{(u - 1)\, u} = -\int \left(\frac{2}{u - 1} - \frac{1}{u}\right) du$$

$$= \ln|u| - 2\ln|u - 1| + C = \ln e^x - 2\ln\left|e^x - 1\right| + C = x - 2\ln\left|e^x - 1\right| + C$$

45. $\displaystyle\int \frac{x + a}{x^2 + a^2}\, dx = \frac{1}{2}\int \frac{2x\, dx}{x^2 + a^2} + a\int \frac{dx}{x^2 + a^2} = \frac{1}{2}\ln\left(x^2 + a^2\right) + a \cdot \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

$$= \ln\sqrt{x^2 + a^2} + \tan^{-1}(x/a) + C$$

46. Let $u = x^2$. Then $du = 2x\, dx \Rightarrow$

$$\int \frac{x\, dx}{x^4 - a^4} = \int \frac{\frac{1}{2}\, du}{u^2 - \left(a^2\right)^2} = \frac{1}{4a^2}\ln\left|\frac{u - a^2}{u + a^2}\right| + C = \frac{1}{4a^2}\ln\left|\frac{x^2 - a^2}{x^2 + a^2}\right| + C.$$

47. $\int \sin^2 \pi x \cos^4 \pi x\, dx = \int \frac{1}{2}(1 - \cos 2\pi x)\left[\frac{1}{2}(1 + \cos 2\pi x)\right]^2 dx$

$= \frac{1}{8}\int (1 - \cos 2\pi x)\left(1 + 2\cos 2\pi x + \cos^2 2\pi x\right) dx$

$= \frac{1}{8}\int \left(1 + 2\cos 2\pi x + \cos^2 2\pi x - \cos 2\pi x - 2\cos^2 2\pi x - \cos^3 2\pi x\right) dx$

$= \frac{1}{8}\int \left(1 + \cos 2\pi x - \cos^2 2\pi x - \cos^3 2\pi x\right) dx$

$= \frac{1}{8}\int \left[1 + \cos 2\pi x - \frac{1}{2}(1 + \cos 4\pi x)\right] dx - \frac{1}{8}\int \cos^3 2\pi x\, dx$

$= \frac{1}{8}\int \left(\frac{1}{2} + \cos 2\pi x - \frac{1}{2}\cos 4\pi x\right) dx - \frac{1}{8}\int \left(1 - \sin^2 2\pi x\right)\cos 2\pi x\, dx$

$= \frac{1}{8}\left(\frac{1}{2}x + \frac{1}{2\pi}\sin 2\pi x - \frac{1}{8\pi}\sin 4\pi x\right) - \frac{1}{8}\int \left(1 - u^2\right)\frac{1}{2\pi}\, du$ $[u = \sin 2\pi x,\ du = 2\pi \cos 2\pi x\, dx]$

$= \frac{1}{16}x + \frac{1}{16\pi}\sin 2\pi x - \frac{1}{64\pi}\sin 4\pi x - \frac{1}{16\pi}\left(u - \frac{1}{3}u^3\right) + C$

$= \frac{1}{16}x + \frac{1}{16\pi}\sin 2\pi x - \frac{1}{64\pi}\sin 4\pi x - \frac{1}{16\pi}\sin 2\pi x + \frac{1}{48\pi}\sin^3 2\pi x + C$

$= \frac{1}{16}x - \frac{1}{64\pi}\sin 4\pi x + \frac{1}{48\pi}\sin^3 2\pi x + C$

48. Integrate by parts with $u = \tan^{-1} x$, $dv = x^2\, dx \Rightarrow du = dx/\left(1 + x^2\right)$, $v = \frac{1}{3}x^3$:

$$\int x^2 \tan^{-1} x\, dx = \tfrac{1}{3}x^3 \tan^{-1} x - \int \frac{x^3}{3}\frac{dx}{1 + x^2} = \tfrac{1}{3}x^3 \tan^{-1} x - \frac{1}{3}\int \left[x - \frac{x}{x^2 + 1}\right] dx$$

$$= \tfrac{1}{3}x^3 \tan^{-1} x - \tfrac{1}{6}x^2 + \tfrac{1}{6}\ln\left(x^2 + 1\right) + C$$

49. Let $u = \sqrt{4x + 1} \Rightarrow u^2 = 4x + 1 \Rightarrow 2u\, du = 4\, dx \Rightarrow dx = \frac{1}{2}u\, du$. So

$$\int \frac{1}{x\sqrt{4x + 1}}\, dx = \int \frac{\frac{1}{2}u\, du}{\frac{1}{4}\left(u^2 - 1\right)u} = 2\int \frac{du}{u^2 - 1} = 2\left(\tfrac{1}{2}\right)\ln\left|\frac{u - 1}{u + 1}\right| + C \quad \text{(by Formula 19)}$$

$$= \ln\left|\frac{\sqrt{4x + 1} - 1}{\sqrt{4x + 1} + 1}\right| + C$$

50. As in Exercise 49, let $u = \sqrt{4x + 1}$. Then $\displaystyle\int \frac{dx}{x^2\sqrt{4x + 1}} = \int \frac{\frac{1}{2}u\, du}{\left[\frac{1}{4}\left(u^2 - 1\right)\right]^2 u} = 8\int \frac{du}{\left(u^2 - 1\right)^2}$. Now

$$\frac{1}{\left(u^2 - 1\right)^2} = \frac{1}{(u + 1)^2 (u - 1)^2} = \frac{A}{u + 1} + \frac{B}{(u + 1)^2} + \frac{C}{u - 1} + \frac{D}{(u - 1)^2} \Rightarrow$$

$1 = A(u + 1)(u - 1)^2 + B(u - 1)^2 + C(u - 1)(u + 1)^2 + D(u + 1)^2$. $u = 1 \Rightarrow D = \frac{1}{4}$, $u = -1 \Rightarrow$ $B = \frac{1}{4}$. Equating coefficients of u^3 gives $A + C = 0$, and equating coefficients of 1 gives $1 = A + B - C + D$ $\Rightarrow 1 = A + \frac{1}{4} - C + \frac{1}{4} \Rightarrow \frac{1}{2} = A - C$. So $A = \frac{1}{4}$ and $C = -\frac{1}{4}$. Therefore,

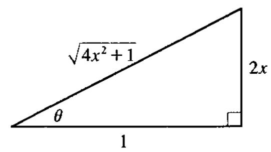
$$\int \frac{dx}{x^2\sqrt{4x + 1}} = 8\int \left[\frac{1/4}{u + 1} + \frac{1/4}{(u + 1)^2} + \frac{-1/4}{u - 1} + \frac{1/4}{(u - 1)^2}\right] du$$

$$= \int \left[\frac{2}{u + 1} + 2(u + 1)^{-2} - \frac{2}{u - 1} + 2(u - 1)^{-2}\right] du$$

$$= 2\ln|u + 1| - \frac{2}{u + 1} - 2\ln|u - 1| - \frac{2}{u - 1} + C$$

$$= 2\ln\left(\sqrt{4x + 1} + 1\right) - \frac{2}{\sqrt{4x + 1} + 1} - 2\ln\left|\sqrt{4x + 1} - 1\right| - \frac{2}{\sqrt{4x + 1} - 1} + C$$

51. Let $2x = \tan\theta \quad\Rightarrow\quad x = \frac{1}{2}\tan\theta,\ dx = \frac{1}{2}\sec^2\theta\,d\theta,\ \sqrt{4x^2+1} = \sec\theta$, so



$$\int \frac{dx}{x\sqrt{4x^2+1}} = \int \frac{\frac{1}{2}\sec^2\theta\,d\theta}{\frac{1}{2}\tan\theta\sec\theta} = \int \frac{\sec\theta}{\tan\theta}\,d\theta = \int \csc\theta\,d\theta$$

$$= -\ln|\csc\theta + \cot\theta| + C \quad [\text{or } \ln|\csc\theta - \cot\theta| + C]$$

$$= -\ln\left|\frac{\sqrt{4x^2+1}}{2x} + \frac{1}{2x}\right| + C \quad \left[\text{or } \ln\left|\frac{\sqrt{4x^2+1}}{2x} - \frac{1}{2x}\right| + C\right]$$

52. Let $u = x^2$. Then $du = 2x\,dx \quad\Rightarrow$

$$\int \frac{dx}{x\left(x^4+1\right)} = \int \frac{x\,dx}{x^2\left(x^4+1\right)} = \frac{1}{2}\int \frac{du}{u\left(u^2+1\right)} = \frac{1}{2}\int \left[\frac{1}{u} - \frac{u}{u^2+1}\right]du = \tfrac{1}{2}\ln|u| - \tfrac{1}{4}\ln\left(u^2+1\right) + C$$

$$= \tfrac{1}{2}\ln\left(x^2\right) - \tfrac{1}{4}\ln\left(x^4+1\right) + C = \tfrac{1}{4}\left[\ln\left(x^4\right) - \ln\left(x^4+1\right)\right] + C = \frac{1}{4}\ln\left(\frac{x^4}{x^4+1}\right) + C$$

Or: Write $I = \displaystyle\int \frac{x^3\,dx}{x^4\left(x^4+1\right)}$ and let $u = x^4$.

53. $\int x^2 \sinh(mx)\,dx = \frac{1}{m}x^2\cosh(mx) - \frac{2}{m}\int x\cosh(mx)\,dx \quad \begin{bmatrix} u = x^2,\ dv = \sinh(mx)\,dx, \\ du = 2x\,dx,\ v = \frac{1}{m}\cosh(mx) \end{bmatrix}$

$$= \tfrac{1}{m}x^2\cosh(mx) - \tfrac{2}{m}\left(\tfrac{1}{m}x\sinh(mx) - \tfrac{1}{m}\textstyle\int \sinh(mx)\,dx\right) \quad \begin{bmatrix} U = x,\ dV = \cosh(mx)\,dx, \\ dU = dx,\ V = \frac{1}{m}\sinh(mx) \end{bmatrix}$$

$$= \tfrac{1}{m}x^2\cosh(mx) - \tfrac{2}{m^2}x\sinh(mx) + \tfrac{2}{m^3}\cosh(mx) + C$$

54. $\int (x+\sin x)^2\,dx = \int \left(x^2 + 2x\sin x + \sin^2 x\right)dx = \frac{1}{3}x^3 + 2(\sin x - x\cos x) + \frac{1}{2}(x - \sin x\cos x) + C$

$$= \tfrac{1}{3}x^3 + \tfrac{1}{2}x + 2\sin x - \tfrac{1}{2}\sin x\cos x - 2x\cos x + C$$

55. Let $u = \sqrt{x+1}$. Then $x = u^2 - 1 \quad\Rightarrow$

$$\int \frac{dx}{x+4+4\sqrt{x+1}} = \int \frac{2u\,du}{u^2+3+4u} = \int \left[\frac{-1}{u+1} + \frac{3}{u+3}\right]du$$

$$= 3\ln|u+3| - \ln|u+1| + C = 3\ln\left(\sqrt{x+1}+3\right) - \ln\left(\sqrt{x+1}+1\right) + C$$

56. Let $t = \sqrt{x^2-1}$. Then $dt = \left(x\big/\sqrt{x^2-1}\right)dx,\ x^2 - 1 = t^2,\ x = \sqrt{t^2+1}$, so

$$I = \int \frac{x\,\ln x}{\sqrt{x^2-1}}\,dx = \int \ln\sqrt{t^2+1}\,dt = \frac{1}{2}\int \ln\left(t^2+1\right)dt.$$ Now use parts with $u = \ln\left(t^2+1\right)$, $dv = dt$:

$$I = \tfrac{1}{2}t\ln\left(t^2+1\right) - \int \frac{t^2}{t^2+1}\,dt = \tfrac{1}{2}t\ln\left(t^2+1\right) - \int\left[1 - \frac{1}{t^2+1}\right]dt$$

$$= \tfrac{1}{2}t\ln\left(t^2+1\right) - t + \tan^{-1}t + C = \sqrt{x^2-1}\,\ln x - \sqrt{x^2-1} + \tan^{-1}\sqrt{x^2-1} + C$$

Another Method: First integrate by parts with $u = \ln x$, $dv = \left(x\big/\sqrt{x^2-1}\right)dx$ and then use substitution $\left(x = \sec\theta \text{ or } u = \sqrt{x^2-1}\right)$.

57. Let $u = \sqrt[3]{x+c}$. Then $x = u^3 - c \quad\Rightarrow$

$$\int x\sqrt[3]{x+c}\,dx = \int \left(u^3 - c\right)u\cdot 3u^2\,du = 3\int\left(u^6 - cu^3\right)du = \tfrac{3}{7}u^7 - \tfrac{3}{4}cu^4 + C$$

$$= \tfrac{3}{7}(x+c)^{7/3} - \tfrac{3}{4}c(x+c)^{4/3} + C$$

58. Integrate by parts with $u = \ln(1+x)$, $dv = x^2\,dx \Rightarrow du = dx/(1+x)$, $v = \frac{1}{3}x^3$:

$$\int x^2 \ln(1+x)\,dx = \tfrac{1}{3}x^3 \ln(1+x) - \int \frac{x^3\,dx}{3(1+x)} = \tfrac{1}{3}x^3\ln(1+x) - \frac{1}{3}\int\left(x^2 - x + 1 - \frac{1}{x+1}\right)dx$$

$$= \tfrac{1}{3}x^3\ln(1+x) - \tfrac{1}{9}x^3 + \tfrac{1}{6}x^2 - \tfrac{1}{3}x + \tfrac{1}{3}\ln(1+x) + C$$

59. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \Rightarrow$

$$\int \frac{dx}{e^{3x} - e^x} = \int \frac{du/u}{u^3 - u} = \int \frac{du}{(u-1)u^2(u+1)} = \int\left[\frac{1/2}{u-1} - \frac{1}{u^2} - \frac{1/2}{u+1}\right]du$$

$$= \frac{1}{u} + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C = e^{-x} + \frac{1}{2}\ln\left|\frac{e^x - 1}{e^x + 1}\right| + C$$

60. Let $u = \sqrt[3]{x}$. Then $x = u^3$, $dx = 3u^2\,du \Rightarrow$

$$\int \frac{dx}{x + \sqrt[3]{x}} = \int \frac{3u^2\,du}{u^3 + u} = \frac{3}{2}\int\frac{2u\,du}{u^2+1} = \tfrac{3}{2}\ln\left(u^2+1\right) + C = \tfrac{3}{2}\ln\left(x^{2/3}+1\right) + C.$$

61. Let $u = x^5$. Then $du = 5x^4\,dx \Rightarrow$

$$\int \frac{x^4\,dx}{x^{10}+16} = \int \frac{\frac{1}{5}\,du}{u^2+16} = \tfrac{1}{5}\cdot\tfrac{1}{4}\tan^{-1}\left(\tfrac{1}{4}u\right) + C = \tfrac{1}{20}\tan^{-1}\left(\frac{1}{4}x^5\right) + C.$$

62. Let $u = x + 1$. Then $du = dx \Rightarrow$

$$\int \frac{x^3}{(x+1)^{10}}\,dx = \int \frac{(u-1)^3}{u^{10}}\,du = \int\left(u^{-7} - 3u^{-8} + 3u^{-9} - u^{-10}\right)du$$

$$= -\tfrac{1}{6}u^{-6} + \tfrac{3}{7}u^{-7} - \tfrac{3}{8}u^{-8} + \tfrac{1}{9}u^{-9} + C$$

$$= (x+1)^{-9}\left[-\tfrac{1}{6}(x+1)^3 + \tfrac{3}{7}(x+1)^2 - \tfrac{3}{8}(x+1) + \tfrac{1}{9}\right] + C.$$

63. Let $u = \csc 2x$. Then $du = -2\cot 2x\,\csc 2x\,dx \Rightarrow$

$$\int \cot^3 2x\,\csc^3 2x\,dx = \int \csc^2 2x\left(\csc^2 2x - 1\right)\cot 2x\,\csc 2x\,dx = \int u^2\left(u^2 - 1\right)\left(-\tfrac{1}{2}\,du\right)$$

$$= -\tfrac{1}{2}\int\left(u^4 - u^2\right)du = -\tfrac{1}{2}\left[\tfrac{1}{5}u^5 - \tfrac{1}{3}u^3\right] + C = \tfrac{1}{6}\csc^3 2x - \tfrac{1}{10}\csc^5 2x + C$$

64. Let $u = \tan x$. Then

$$\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)\,dx}{\sin x\,\cos x} = \int_{\pi/4}^{\pi/3}\frac{\ln(\tan x)}{\tan x}\sec^2 x\,dx = \int_1^{\sqrt{3}}\frac{\ln u}{u}\,du$$

$$= \left[\tfrac{1}{2}(\ln u)^2\right]_1^{\sqrt{3}} = \tfrac{1}{2}\left(\ln\sqrt{3}\right)^2 = \tfrac{1}{8}(\ln 3)^2$$

65. $\displaystyle\int \frac{dx}{\sqrt{x+1}+\sqrt{x}} = \int\left(\sqrt{x+1} - \sqrt{x}\right)dx = \frac{2}{3}\left[(x+1)^{3/2} - x^{3/2}\right] + C$

66. $\displaystyle\int \frac{u^3+1}{u^3-u^2}\,du = \int\left[1 + \frac{u^2+1}{(u-1)u^2}\right]du = u + \int\left[\frac{2}{u-1} - \frac{1}{u} - \frac{1}{u^2}\right]du = u + 2\ln|u-1| - \ln|u| + \frac{1}{u} + C.$

Thus,

$$\int_2^3 \frac{u^3+1}{u^3-u^2}\,du = \left[u + 2\ln(u-1) - \ln u + \frac{1}{u}\right]_2^3 = \left(3 + 2\ln 2 - \ln 3 + \tfrac{1}{3}\right) - \left(2 + 2\ln 1 - \ln 2 + \tfrac{1}{2}\right)$$

$$= 1 + 3\ln 2 - \ln 3 - \tfrac{1}{6} = \tfrac{5}{6} + \ln\tfrac{8}{3}$$

67. Let $u = \sqrt{t}$. Then $du = dt/\left(2\sqrt{t}\right) \Rightarrow$

$$\int \frac{\arctan\sqrt{t}}{\sqrt{t}}\,dt = \int \tan^{-1} u\, 2\,du = 2u\tan^{-1} u - \int \frac{2u\,du}{1+u^2} \quad \text{(by parts)}$$
$$= 2u\tan^{-1} u - \ln\left(1+u^2\right) + C = 2\sqrt{t}\tan^{-1}\sqrt{t} - \ln(1+t) + C$$

68. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \Rightarrow$

$$\int \frac{dx}{1+2e^x - e^{-x}} = \int \frac{du/u}{1+2u-1/u} = \int \frac{du}{2u^2+u-1} = \int \left[\frac{2/3}{2u-1} - \frac{1/3}{u+1}\right] du$$
$$= \tfrac{1}{3}\ln|2u-1| - \tfrac{1}{3}\ln|u+1| + C = \tfrac{1}{3}\ln\left|(2e^x-1)/\left(e^x+1\right)\right| + C$$

69. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \Rightarrow$

$$\int \frac{e^{2x}}{1+e^x}\,dx = \int \frac{u^2}{1+u}\frac{du}{u} = \int \frac{u}{1+u}\,du = \int \left(1 - \frac{1}{1+u}\right) du$$
$$= u - \ln|1+u| + C = e^x - \ln\left(1+e^x\right) + C$$

70. Use parts with $u = \ln(x+1)$, $dv = dx/x^2$:

$$\int \frac{\ln(x+1)}{x^2}\,dx = -\frac{1}{x}\ln(x+1) + \int \frac{dx}{x(x+1)} = -\frac{1}{x}\ln(x+1) + \int \left[\frac{1}{x} - \frac{1}{x+1}\right] dx$$
$$= -\tfrac{1}{x}\ln(x+1) + \ln|x| - \ln(x+1) + C = -\left(1+\tfrac{1}{x}\right)\ln(x+1) + \ln|x| + C$$

71. $$\frac{x}{x^4+4x^2+3} = \frac{x}{\left(x^2+3\right)\left(x^2+1\right)} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{x^2+1} \Rightarrow$$

$$x = (Ax+B)\left(x^2+1\right) + (Cx+D)\left(x^2+3\right) = \left(Ax^3+Bx^2+Ax+B\right) + \left(Cx^3+Dx^2+3Cx+3D\right)$$
$$= (A+C)x^3 + (B+D)x^2 + (A+3C)x + (B+3D) \Rightarrow$$

$A+C=0$, $B+D=0$, $A+3C=1$, $B+3D=0 \Rightarrow A = -\frac{1}{2}$, $C = \frac{1}{2}$, $B=0$, $D=0$. Thus,

$$\int \frac{x}{x^4+4x^2+3}\,dx = \int \left(\frac{-\frac{1}{2}x}{x^2+3} + \frac{\frac{1}{2}x}{x^2+1}\right) dx$$
$$= -\tfrac{1}{4}\ln\left(x^2+3\right) + \tfrac{1}{4}\ln\left(x^2+1\right) + C \text{ or } \frac{1}{4}\ln\left(\frac{x^2+1}{x^2+3}\right) + C$$

72. Let $u = \sqrt[6]{t}$. Then $t = u^6$, $dt = 6u^5\,du \Rightarrow$

$$\int \frac{\sqrt{t}\,dt}{1+\sqrt[3]{t}} = \int \frac{u^3 \cdot 6u^5\,du}{1+u^2} = 6\int \frac{u^8}{u^2+1}\,du = 6\int \left(u^6 - u^4 + u^2 - 1 + \frac{1}{u^2+1}\right) du$$
$$= 6\left(\tfrac{1}{7}u^7 - \tfrac{1}{5}u^5 + \tfrac{1}{3}u^3 - u + \tan^{-1} u\right) + C = 6\left(\tfrac{1}{7}t^{7/6} - \tfrac{1}{5}t^{5/6} + \tfrac{1}{3}t^{1/2} - t^{1/6} + \tan^{-1} t^{1/6}\right) + C$$

73. $\dfrac{1}{(x-2)(x^2+4)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+4} \quad \Rightarrow$

$1 = A(x^2+4) + (Bx+C)(x-2) = (A+B)x^2 + (C-2B)x + (4A-2C)$. So $0 = A + B = C - 2B$, $1 = 4A - 2C$. Setting $x = 2$ gives $A = \frac{1}{8} \Rightarrow B = -\frac{1}{8}$ and $C = -\frac{1}{4}$. So

$$\int \frac{1}{(x-2)(x^2+4)}\,dx = \int\left(\frac{\frac{1}{8}}{x-2} + \frac{-\frac{1}{8}x - \frac{1}{4}}{x^2+4}\right)dx = \frac{1}{8}\int\frac{dx}{x-2} - \frac{1}{16}\int\frac{2x\,dx}{x^2+4} - \frac{1}{4}\int\frac{dx}{x^2+4}$$
$$= \tfrac{1}{8}\ln|x-2| - \tfrac{1}{16}\ln\left(x^2+4\right) - \tfrac{1}{8}\tan^{-1}(x/2) + C$$

74. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \Rightarrow$

$$\int\frac{dx}{e^x - e^{-x}} = \int\frac{e^x\,dx}{e^{2x}-1} = \int\frac{u}{u^2-1}\frac{du}{u} = \int\frac{du}{u^2-1} = \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C = \frac{1}{2}\ln\left(\frac{|e^x-1|}{e^x+1}\right) + C.$$

75. $\int \sin x \sin 2x \sin 3x\,dx = \int \sin x \cdot \frac{1}{2}[\cos(2x-3x) - \cos(2x+3x)]\,dx = \frac{1}{2}\int(\sin x\cos x - \sin x\cos 5x)\,dx$

$$= \tfrac{1}{4}\int \sin 2x\,dx - \tfrac{1}{2}\int\tfrac{1}{2}[\sin(x+5x) + \sin(x-5x)]\,dx$$
$$= -\tfrac{1}{8}\cos 2x - \tfrac{1}{4}\int(\sin 6x - \sin 4x)\,dx = -\tfrac{1}{8}\cos 2x + \tfrac{1}{24}\cos 6x - \tfrac{1}{16}\cos 4x + C$$

76. $\int(x^2 - bx)\sin 2x\,dx = -\frac{1}{2}(x^2-bx)\cos 2x + \frac{1}{2}\int(2x-b)\cos 2x\,dx$

$$\left[u = x^2 - bx,\ dv = \sin 2x\,dx,\ du = (2x-b)\,dx,\ v = -\tfrac{1}{2}\cos 2x\right]$$
$$= -\tfrac{1}{2}(x^2-bx)\cos 2x + \tfrac{1}{2}\left[\tfrac{1}{2}(2x-b)\sin 2x - \int\sin 2x\,dx\right]$$
$$\left[U = 2x - b,\ dV = \cos 2x\,dx,\ dU = 2\,dx,\ V = \tfrac{1}{2}\sin 2x\right]$$
$$= -\tfrac{1}{2}(x^2-bx)\cos 2x + \tfrac{1}{4}(2x-b)\sin 2x + \tfrac{1}{4}\cos 2x + C$$

8.6 Integration Using Tables and Computer Algebra Systems

Keep in mind that there are several ways to approach many of these exercises, and different methods can lead to different forms of the answer.

1. We could make the substitution $u = \sqrt{2}x$ to obtain the radical $\sqrt{7-u^2}$ and then use Formula 33 with $a = \sqrt{7}$. Alternatively, we will factor $\sqrt{2}$ out of the radical and use $a = \sqrt{\frac{7}{2}}$.

$$\int\frac{\sqrt{7-2x^2}}{x^2}\,dx = \sqrt{2}\int\frac{\sqrt{\frac{7}{2}-x^2}}{x^2}\,dx \overset{33}{=} \sqrt{2}\left[-\frac{1}{x}\sqrt{\tfrac{7}{2}-x^2} - \sin^{-1}\frac{x}{\sqrt{\frac{7}{2}}}\right] + C$$
$$= -\frac{1}{x}\sqrt{7-2x^2} - \sqrt{2}\sin^{-1}\left(\sqrt{\tfrac{2}{7}}x\right) + C$$

2. $$\int\frac{3x}{\sqrt{3-2x}}\,dx = 3\int\frac{x}{\sqrt{3+(-2)x}}\,dx \overset{55}{=} 3\left[\frac{2}{3(-2)^2}(-2x - 2\cdot 3)\sqrt{3+(-2)x}\right] + C$$
$$= \tfrac{1}{2}(-2x-6)\sqrt{3-2x} + C = -(x+3)\sqrt{3-2x} + C$$

3. Let $u = \pi x \Rightarrow du = \pi\, dx$, so

$$\begin{aligned}\int \sec^3(\pi x)\,dx &= \tfrac{1}{\pi}\int \sec^3 u\,du \overset{71}{=} \tfrac{1}{\pi}\left(\tfrac{1}{2}\sec u\tan u + \tfrac{1}{2}\ln|\sec u + \tan u|\right) + C \\ &= \tfrac{1}{2\pi}\sec \pi x\tan \pi x + \tfrac{1}{2\pi}\ln|\sec \pi x + \tan \pi x| + C\end{aligned}$$

4. $\displaystyle\int e^{2\theta}\sin 3\theta\,d\theta \overset{98}{=} \frac{e^{2\theta}}{2^2+3^2}(2\sin 3\theta - 3\cos 3\theta) + C = \tfrac{2}{13}e^{2\theta}\sin 3\theta - \tfrac{3}{13}e^{2\theta}\cos 3\theta + C$

5. $\displaystyle\int_0^1 2x\cos^{-1}x\,dx \overset{91}{=} 2\left[\frac{2x^2-1}{4}\cos^{-1}x - \frac{x\sqrt{1-x^2}}{4}\right]_0^1 = 2\left[\left(\tfrac{1}{4}\cdot 0 - 0\right) - \left(-\tfrac{1}{4}\cdot\tfrac{\pi}{2} - 0\right)\right] = 2\left(\tfrac{\pi}{8}\right) = \tfrac{\pi}{4}$

6. $$\begin{aligned}\int_2^3 \frac{1}{x^2\sqrt{4x^2-7}}\,dx &= 2\int_4^6 \frac{du}{u^2\sqrt{u^2-7}}\quad [u = 2x,\ du = 2\,dx] \overset{45}{=} 2\left[\frac{\sqrt{u^2-7}}{7u}\right]_4^6 \\ &= 2\left(\tfrac{\sqrt{29}}{42} - \tfrac{3}{28}\right) = \tfrac{\sqrt{29}}{21} - \tfrac{3}{14}\end{aligned}$$

7. By Formula 99 with $a = -3$ and $b = 4$,

$$\int e^{-3x}\cos 4x\,dx = \frac{e^{-3x}}{(-3)^2+4^2}(-3\cos 4x + 4\sin 4x) + C = \frac{e^{-3x}}{25}(-3\cos 4x + 4\sin 4x) + C.$$

8. Let $u = x/2$, so $dx = 2\,du$, and we use Formula 72:

$$\begin{aligned}\int \csc^3(x/2)dx &= 2\int \csc^3 u\,du = -\csc u\cot u + \ln|\csc u - \cot u| + C \\ &= -\csc(x/2)\cot(x/2) + \ln|\csc(x/2) - \cot(x/2)| + C\end{aligned}$$

9. Let $u = x^2$. Then $du = 2x\,dx$, so

$$\begin{aligned}\int x\sin^{-1}(x^2)\,dx &= \tfrac{1}{2}\int \sin^{-1}u\,du \overset{87}{=} \tfrac{1}{2}\left(u\sin^{-1}u + \sqrt{1-u^2}\right) + C \\ &= \tfrac{1}{2}\left(x^2\sin^{-1}\left(x^2\right) + \sqrt{1-x^4}\right) + C\end{aligned}$$

10. Let $u = x^2$. Then $du = 2x\,dx$, so

$$\begin{aligned}\int x^3\sin^{-1}(x^2)\,dx &= \tfrac{1}{2}\int u\sin^{-1}u\,du \overset{90}{=} \frac{2u^2-1}{8}\sin^{-1}u + \frac{u\sqrt{1-u^2}}{8} + C \\ &= \frac{2x^4-1}{8}\sin^{-1}\left(x^2\right) + \frac{x^2\sqrt{1-x^4}}{8} + C\end{aligned}$$

11. $$\begin{aligned}\int_{-1}^0 t^2e^{-t}\,dt &\overset{97}{=} \left[\frac{1}{-1}t^2e^{-t}\right]_{-1}^0 - \frac{2}{-1}\int_{-1}^0 te^{-t}\,dt = e + 2\int_{-1}^0 te^{-t}\,dt \overset{96}{=} e + 2\left[\frac{1}{(-1)^2}(-t-1)e^{-t}\right]_{-1}^0 \\ &= e + 2\left[-e^0 + 0\right] = e - 2\end{aligned}$$

12. Let $u = 3x$. Then $du = 3\,dx$, so

$$\begin{aligned}\int x^2\cos 3x\,dx &= \tfrac{1}{27}\int u^2\cos u\,du \overset{85}{=} \tfrac{1}{27}\left(u^2\sin u - 2\int u\sin u\,du\right) \\ &\overset{82}{=} \tfrac{1}{3}x^2\sin 3x - \tfrac{2}{27}(\sin 3x - 3x\cos 3x) + C \\ &= \tfrac{1}{27}\left[\left(9x^2-2\right)\sin 3x + 6x\cos 3x\right] + C\end{aligned}$$

Thus, $\int_0^\pi x^2\cos 3x\,dx = \tfrac{1}{27}\left[(9x^2-2)\sin 3x + 6x\cos 3x\right]_0^\pi = \tfrac{1}{27}\left[(0 + 6\pi(-1)) - (0+0)\right] = -\tfrac{6\pi}{27} = -\tfrac{2\pi}{9}$.

13. Let $u = 3x$. Then $du = 3\,dx$, so

$$\int \frac{\sqrt{9x^2-1}}{x^2}\,dx = \int \frac{\sqrt{u^2-1}}{u^2/9}\frac{du}{3} = 3\int \frac{\sqrt{u^2-1}}{u^2}\,du \overset{42}{=} -\frac{3\sqrt{u^2-1}}{u} + 3\ln\left|u+\sqrt{u^2-1}\right| + C$$
$$= -\frac{\sqrt{9x^2-1}}{x} + 3\ln\left|3x+\sqrt{9x^2-1}\right| + C$$

14. By Formula 32,

$$\int \frac{\sqrt{4-3x^2}}{x}\,dx = \int \frac{\sqrt{4-u^2}}{u/\sqrt{3}}\frac{du}{\sqrt{3}} \quad (u=\sqrt{3}x,\, du=\sqrt{3}\,dx) \quad = \int \frac{\sqrt{4-u^2}}{u}\,du$$
$$= \sqrt{4-u^2} - 2\ln\left|\frac{2+\sqrt{4-u^2}}{u}\right| + C_1 = \sqrt{4-3x^2} - 2\ln\left|\frac{2+\sqrt{4-3x^2}}{\sqrt{3}x}\right| + C_1$$
$$= \sqrt{4-3x^2} - 2\ln\left|\frac{2+\sqrt{4-3x^2}}{x}\right| + C$$

15. Let $u = e^x$. Then $du = e^x\,dx$, so

$$\int e^x \operatorname{sech}(e^x)\,dx = \int \operatorname{sech} u\,du \overset{107}{=} \tan^{-1}|\sinh u| + C = \tan^{-1}[\sinh(e^x)] + C$$

16. $\displaystyle\int \frac{\sin\theta}{1+2\cos\theta}\,d\theta = -\frac{1}{2}\int \frac{1}{u}\,du\ [u = 1+2\cos\theta,\, du = -2\sin\theta\,d\theta] = -\tfrac{1}{2}\ln|u| + C = -\tfrac{1}{2}\ln|1+2\cos\theta| + C$

17. $\int_{-2}^{1}\sqrt{5-4x-x^2}\,dx = \int_{-2}^{1}\sqrt{5-(x^2+4x)}\,dx = \int_{-2}^{1}\sqrt{5+4-(x^2+4x+4)}\,dx$

$$= \int_{-2}^{1}\sqrt{9-(x+2)^2}\,dx = \int_0^3\sqrt{3^2-u^2}\,du \quad [u = x+2,\, du = dx]$$
$$\overset{30}{=} \left[\tfrac{u}{2}\sqrt{9-u^2} + \tfrac{9}{2}\sin^{-1}\tfrac{u}{3}\right]_0^3 = \left[\left(0+\tfrac{9}{2}\cdot\tfrac{\pi}{2}\right) - (0+0)\right] = \tfrac{9\pi}{4}$$

18. Let $u = x^2$. Then $du = 2x\,dx$, so by Formula 48,

$$\int \frac{x^5\,dx}{x^2+\sqrt{2}} = \frac{1}{2}\int \frac{u^2}{u+\sqrt{2}}\,du = \tfrac{1}{2}\cdot\tfrac{1}{2}\left[(u+\sqrt{2})^2 - 4\sqrt{2}(u+\sqrt{2}) + 4\ln\left|u+\sqrt{2}\right|\right] + C$$
$$= \tfrac{1}{4}\left[(x^2+\sqrt{2})^2 - 4\sqrt{2}(x^2+\sqrt{2}) + 4\ln(x^2+\sqrt{2})\right] + C = \tfrac{1}{4}x^4 - \tfrac{1}{\sqrt{2}}x^2 + \ln(x^2+\sqrt{2}) + K$$

Or: Let $u = x^2 + \sqrt{2}$.

19. Let $u = \sin x$. Then $du = \cos x\,dx$, so

$\int \sin^2 x\cos x\ln(\sin x)\,dx = \int u^2\ln u\,du \overset{101}{=} \tfrac{1}{9}u^3(3\ln u - 1) + C = \tfrac{1}{9}\sin^3 x\,[3\ln(\sin x) - 1] + C.$

20. Let $u = e^x$. Then $x = \ln u$, $dx = du/u$, so

$$\int \frac{dx}{e^x(1+2e^x)} = \int \frac{du/u}{u(1+2u)} = \int \frac{du}{u^2(1+2u)} \overset{50}{=} -\frac{1}{u} + 2\ln\left|\frac{1+2u}{u}\right| + C = -e^{-x} + 2\ln\left(e^{-x}+2\right) + C.$$

21. $\displaystyle\int \sqrt{2+3\cos x}\tan x\,dx = -\int \frac{\sqrt{2+3\cos x}}{\cos x}(-\sin x\,dx) = -\int \frac{\sqrt{2+3u}}{u}\,du \quad (u = \cos x,\, du = -\sin x\,dx)$

$$\overset{58}{=} -2\sqrt{2+3u} - 2\int \frac{du}{u\sqrt{2+3u}} \overset{57}{=} -2\sqrt{2+3u} - 2\cdot\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2+3u}-\sqrt{2}}{\sqrt{2+3u}+\sqrt{2}}\right| + C$$
$$= -2\sqrt{2+3\cos x} - \sqrt{2}\ln\left|\frac{\sqrt{2+3\cos x}-\sqrt{2}}{\sqrt{2+3\cos x}+\sqrt{2}}\right| + C$$

22. Let $u = x - 2$. Then

$$\int \frac{x\,dx}{\sqrt{x^2-4x}} = \int \frac{(x-2)+2}{\sqrt{(x-2)^2-4}}\,dx = \int \frac{u\,du}{\sqrt{u^2-4}} + 2\int \frac{du}{\sqrt{u^2-4}}$$
$$= \frac{1}{2}\int v^{-1/2}\,dv + 2\int \frac{du}{\sqrt{u^2-4}} \quad (v = u^2-4,\ dv = 2u\,du)$$
$$\overset{2,43}{=} v^{1/2} + 2\ln\left|u+\sqrt{u^2-4}\right| + C = \sqrt{x^2-4x} + 2\ln\left|x-2+\sqrt{x^2-4x}\right| + C$$

23. $\int \sec^5 x\,dx \overset{77}{=} \frac{1}{4}\tan x\sec^3 x + \frac{3}{4}\int \sec^3 x\,dx \overset{77}{=} \frac{1}{4}\tan x\sec^3 x + \frac{3}{4}\left(\frac{1}{2}\tan x\sec x + \frac{1}{2}\int \sec x\,dx\right)$

$\overset{14}{=} \frac{1}{4}\tan x\sec^3 x + \frac{3}{8}\tan x\sec x + \frac{3}{8}\ln|\sec x + \tan x| + C$

24. Let $u = 2x$. Then $du = 2\,dx$, so

$$\int \sin^6 2x\,dx = \frac{1}{2}\int \sin^6 u\,du \overset{73}{=} \frac{1}{2}\left(-\frac{1}{6}\sin^5 u\cos u + \frac{5}{6}\int \sin^4 u\,du\right)$$
$$\overset{73}{=} -\frac{1}{12}\sin^5 u\cos u + \frac{5}{12}\left(-\frac{1}{4}\sin^3 u\cos u + \frac{3}{4}\int \sin^2 u\,du\right)$$
$$\overset{63}{=} -\frac{1}{12}\sin^5 u\cos u - \frac{5}{48}\sin^3 u\cos u + \frac{5}{16}\left(\frac{1}{2}u - \frac{1}{4}\sin 2u\right) + C$$
$$= -\frac{1}{12}\sin^5 2x\cos 2x - \frac{5}{48}\sin^3 2x\cos 2x - \frac{5}{64}\sin 4x + \frac{5}{16}x + C$$

25. $\int_0^{\pi/2} \cos^5 x\,dx \overset{74}{=} \frac{1}{5}\left[\cos^4 x\sin x\right]_0^{\pi/2} + \frac{4}{5}\int_0^{\pi/2}\cos^3 x\,dx \overset{68}{=} 0 + \frac{4}{5}\left[\frac{1}{3}\left(2+\cos^2 x\right)\sin x\right]_0^{\pi/2} = \frac{4}{15}(2-0) = \frac{8}{15}$

26. Since

$$\int x^4 e^{-x}\,dx \overset{97}{=} -x^4e^{-x} + 4\int x^3e^{-x}\,dx \overset{97}{=} -x^4e^{-x} + 4\left(-x^3e^{-x} + 3\int x^2e^{-x}\,dx\right)$$
$$\overset{97}{=} -\left(x^4+4x^3\right)e^{-x} + 12\left(-x^2e^{-x} + 2\int xe^{-x}\,dx\right)$$
$$\overset{96}{=} -\left(x^4+4x^3+12x^2\right)e^{-x} + 24\left[(-x-1)e^{-x}\right] + C$$
$$= -\left(x^4+4x^3+12x^2+24x+24\right)e^{-x} + C$$

we find that $\int_0^1 x^4e^{-x}\,dx = -(1+4+12+24+24)e^{-1} + 24e^0 = 24 - 65e^{-1}$.

27. Let $u = e^x \Rightarrow \ln u = x \Rightarrow dx = \dfrac{du}{u}$. Then

$$\int \sqrt{e^{2x}-1}\,dx = \int \frac{\sqrt{u^2-1}}{u}\,du \overset{41}{=} \sqrt{u^2-1} - \cos^{-1}(1/u) + C = \sqrt{e^{2x}-1} - \cos^{-1}\left(e^{-x}\right) + C.$$

28. Let $u = \alpha t - 3$ and assume that $\alpha \neq 0$. Then $du = \alpha\,dt$ and

$$\int e^t \sin(\alpha t - 3)\,dt = \frac{1}{\alpha}\int e^{(u+3)/\alpha}\sin u\,du = \frac{1}{\alpha}e^{3/\alpha}\int e^{(1/\alpha)u}\sin u\,du$$
$$\overset{98}{=} \frac{1}{\alpha}e^{3/\alpha}\frac{e^{(1/\alpha)u}}{(1/\alpha)^2+1^2}\left(\frac{1}{\alpha}\sin u - \cos u\right) + C$$
$$= \frac{1}{\alpha}e^{3/\alpha}e^{(1/\alpha)u}\frac{\alpha^2}{1+\alpha^2}\left(\frac{1}{\alpha}\sin u - \cos u\right) + C = \frac{1}{1+\alpha^2}e^{(u+3)/\alpha}(\sin u - \alpha\cos u) + C$$
$$= \frac{1}{1+\alpha^2}e^t\left[\sin(\alpha t - 3) - \alpha\cos(\alpha t - 3)\right] + C$$

29. Let $u = x^5$, $du = 5x^4\,dx$. Then

$$\int \frac{x^4\,dx}{\sqrt{x^{10}-2}} = \frac{1}{5}\int \frac{du}{\sqrt{u^2-2}} \overset{43}{=} \tfrac{1}{5}\ln\left|u+\sqrt{u^2-2}\right| + C = \tfrac{1}{5}\ln\left|x^5+\sqrt{x^{10}-2}\right| + C.$$

30. Using Formula 95 with $n = 2$,

$$\begin{aligned}\int x^2\tan^{-1}x\,dx &= \frac{1}{3}\left[x^3\tan^{-1}x - \int \frac{x^3\,dx}{1+x^2}\right] = \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int\left(x - \frac{x}{x^2+1}\right)dx \\ &= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\frac{x^2}{2} + \frac{1}{6}\int\frac{du}{u} \quad [u = x^2+1, \text{ so } du = 2x\,dx] \\ &= \tfrac{1}{3}x^3\tan^{-1}x - \tfrac{1}{6}x^2 + \tfrac{1}{6}\ln\left(1+x^2\right) + C\end{aligned}$$

31. $$\text{Volume} = \int_0^1 \frac{2\pi x}{(1+5x)^2}\,dx \overset{51}{=} 2\pi\left[\frac{1}{25(1+5x)} + \tfrac{1}{25}\ln|1+5x|\right]_0^1 = \tfrac{2\pi}{25}\left(\tfrac{1}{6} + \ln 6 - 1 - \ln 1\right)$$
$$= \tfrac{2\pi}{25}\left(\ln 6 - \tfrac{5}{6}\right)$$

32. $$\text{Volume} = \int_0^{\pi/4}\pi\tan^4x\,dx \overset{75}{=} \pi\left(\left[\tfrac{1}{3}\tan^3x\right]_0^{\pi/4} - \int_0^{\pi/4}\tan^2x\,dx\right) \overset{65}{=} \pi\left[\tfrac{1}{3}\tan^3x - \tan x + x\right]_0^{\pi/4}$$
$$= \pi\left(\tfrac{1}{3} - 1 + \tfrac{\pi}{4}\right) = \pi\left(\tfrac{\pi}{4} - \tfrac{2}{3}\right)$$

33. (a) $$\frac{d}{du}\left[\frac{1}{b^3}\left(a + bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C\right] = \frac{1}{b^3}\left[b + \frac{ba^2}{(a+bu)^2} - \frac{2ab}{(a+bu)}\right]$$
$$= \frac{1}{b^3}\left[\frac{b(a+bu)^2 + ba^2 - (a+bu)\,2ab}{(a+bu)^2}\right] = \frac{1}{b^3}\left[\frac{b^3u^2}{(a+bu)^2}\right] = \frac{u^2}{(a+bu)^2}$$

(b) Let $t = a + bu \;\Rightarrow\; dt = b\,du$.

$$\begin{aligned}\int\frac{u^2\,du}{(a+bu)^2} &= \frac{1}{b^3}\int\frac{(t-a)^2}{t^2}\,dt = \frac{1}{b^3}\int\left(1 - \frac{2a}{t} + \frac{a^2}{t^2}\right)dt = \frac{1}{b^3}\left(t - 2a\ln|t| - \frac{a^2}{t}\right) + C \\ &= \frac{1}{b^3}\left(a + bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C\end{aligned}$$

34. (a) $$\frac{d}{du}\left[\frac{u}{8}\left(2u^2-a^2\right)\sqrt{a^2-u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C\right]$$
$$\begin{aligned}&= \sqrt{a^2-u^2}\left[\tfrac{1}{8}\left(2u^2-a^2\right) + \frac{u}{8}(4u)\right] + \frac{u}{8}\left(2u^2-a^2\right)\frac{-u}{\sqrt{a^2-u^2}} + \frac{a^4}{8}\frac{1/a}{\sqrt{1-u^2/a^2}} \\ &= \sqrt{a^2-u^2}\left[\frac{2u^2-a^2}{8} + \frac{u^2}{2}\right] - \frac{u^2\left(2u^2-a^2\right)}{8\sqrt{a^2-u^2}} + \frac{a^4}{8\sqrt{a^2-u^2}} \\ &= \tfrac{1}{2}\left(a^2-u^2\right)^{-1/2}\left[\tfrac{1}{4}\left(a^2-u^2\right)\left(2u^2-a^2\right) + u^2\left(a^2-u^2\right) - \frac{u^2}{4}\left(2u^2-a^2\right) + \frac{a^4}{4}\right] \\ &= \tfrac{1}{2}\left(a^2-u^2\right)^{-1/2}\left[2u^2a^2 - 2u^4\right] = \frac{u^2\left(a^2-u^2\right)}{\sqrt{a^2-u^2}} = u^2\sqrt{a^2-u^2}\end{aligned}$$

(b) Let $u = a\sin\theta \;\Rightarrow\; du = a\cos\theta d\theta$. Then

$$\begin{aligned}
\int u^2\sqrt{a^2-u^2}\,du &= \int a^2\sin^2\theta\, a\sqrt{1-\sin^2\theta}\,a\cos\theta\,d\theta = a^4\int \sin^2\theta\cos^2\theta\,d\theta \\
&= a^4\int \tfrac{1}{2}(1+\cos 2\theta)\,\tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{4}a^4\int\left(1-\cos^2 2\theta\right)d\theta \\
&= \tfrac{1}{4}a^4\int\left[1-\tfrac{1}{2}(1+\cos 4\theta)\right]d\theta = \tfrac{1}{4}a^4\left(\tfrac{1}{2}\theta-\tfrac{1}{8}\sin 4\theta\right)+C \\
&= \tfrac{1}{4}a^4\left(\tfrac{1}{2}\theta-\tfrac{1}{8}2\sin 2\theta\cos 2\theta\right)+C = \tfrac{1}{4}a^4\left[\tfrac{1}{2}\theta-\tfrac{1}{2}\sin\theta\cos\theta\left(1-2\sin^2\theta\right)\right]+C \\
&= \frac{a^4}{8}\left[\sin^{-1}\frac{u}{a}-\frac{u}{a}\frac{\sqrt{a^2-u^2}}{a}\left(1-\frac{2u^2}{a^2}\right)\right]+C \\
&= \tfrac{1}{8}a^4\sin^{-1}(u/a)+\tfrac{1}{8}u\sqrt{a^2-u^2}\left(2u^2-a^2\right)+C
\end{aligned}$$

35. Maple, Mathematica and Derive all give $\int x^2\sqrt{5-x^2}\,dx = -\frac{1}{4}x\left(5-x^2\right)^{3/2}+\frac{5}{8}x\sqrt{5-x^2}+\frac{25}{8}\sin^{-1}\left(\frac{1}{\sqrt{5}}x\right)$. Using Formula 31, we get $\int x^2\sqrt{5-x^2}\,dx = \frac{1}{8}x\left(2x^2-5\right)\sqrt{5-x^2}+\frac{1}{8}\left(5^2\right)\sin^{-1}\left(\frac{1}{\sqrt{5}}x\right)+C$. But $-\frac{1}{4}x\left(5-x^2\right)^{3/2}+\frac{5}{8}x\sqrt{5-x^2} = \frac{1}{8}x\sqrt{5-x^2}\left[5-2\left(5-x^2\right)\right] = \frac{1}{8}x\left(2x^2-5\right)\sqrt{5-x^2}$, and the $\sin^{-1}$ terms are the same in each expression, so the answers are equivalent.

36. Maple and Mathematica both give $\int x^2\left(1+x^3\right)^4 dx = \frac{1}{15}x^{15}+\frac{1}{3}x^{12}+\frac{2}{3}x^9+\frac{2}{3}x^6+\frac{1}{3}x^3$, while Derive gives $\int x^2\left(1+x^3\right)^4 dx = \frac{1}{15}\left(x^3+1\right)^5$. Using the substitution $u = 1+x^3 \;\Rightarrow\; du = 3x^2\,dx$, we get $\int x^2\left(1+x^3\right)^4 dx = \int u^4\left(\frac{1}{3}du\right) = \frac{1}{15}u^5+C = \frac{1}{15}\left(1+x^3\right)^5+C$. We can use the Binomial Theorem or a CAS to expand this expression, and we get $\frac{1}{15}\left(1+x^3\right)^5+C = \frac{1}{15}+\frac{1}{3}x^3+\frac{2}{3}x^6+\frac{2}{3}x^9+\frac{1}{3}x^{12}+\frac{1}{15}x^{15}+C$.

37. Maple and Derive both give $\int \sin^3 x\cos^2 x\,dx = -\frac{1}{5}\sin^2 x\cos^3 x-\frac{2}{15}\cos^3 x$ (although Derive factors the expression), and Mathematica gives $\int \sin^3 x\cos^2 x\,dx = -\frac{1}{8}\cos x-\frac{1}{48}\cos 3x+\frac{1}{80}\cos 5x$. We can use a CAS to show that both of these expressions are equal to $-\frac{1}{3}\cos^3 x+\frac{1}{5}\cos^5 x$. Using Formula 86, we write

$$\begin{aligned}
\int \sin^3 x\cos^2 x\,dx &= -\tfrac{1}{5}\sin^2 x\cos^3 x+\tfrac{2}{5}\int \sin x\cos^2 x\,dx = -\tfrac{1}{5}\sin^2 x\cos^3 x+\tfrac{2}{5}\left(-\tfrac{1}{3}\cos^3 x\right)+C \\
&= -\tfrac{1}{5}\sin^2 x\cos^3 x-\tfrac{2}{15}\cos^3 x+C
\end{aligned}$$

38. Maple gives $\int \tan^2 x\sec^4 dx = \dfrac{1}{5}\dfrac{\sin^3 x}{\cos^5 x}+\dfrac{2}{15}\dfrac{\sin^3 x}{\cos^3 x}$,
Mathematica gives $\int \tan^2 x\sec^4 dx = -\frac{1}{120}\sec^5 x\left(-20\sin x+5\sin 3x+\sin 5x\right)$, and
Derive gives $\int \tan^2 x\sec^4 dx = -\frac{2}{15}\tan x-\dfrac{\sin x}{15\cos^3 x}+\dfrac{\sin x}{5\cos^5 x}$. All of these expressions can be "simplified" to $-\dfrac{1}{15}\dfrac{\sin x\left(\cos^2 x-2\cos^4 x-3\right)}{\cos^5 x}$ using Maple. Using the identity $1+\tan^2 x = \sec^2 x$, we write $\int \tan^2 x\sec^4 x\,dx = \int \tan^2 x\left(1+\tan^2 x\right)\sec^2 x\,dx = \int\left(\tan^2 x+\tan^4 x\right)\sec^2 x\,dx$. Now we substitute $u = \tan x$ $\Rightarrow\; du = \sec^2 x\,dx$, and the integral becomes $\int\left(u^2+u^4\right)du = \frac{1}{3}u^3+\frac{1}{5}u^5+C = \frac{1}{3}\tan^3 x+\frac{1}{5}\tan^5 x+C$. If we write $\sin^5 x = \sin^3 x\left(1-\cos^2 x\right)$ and substitute into the numerator of the $\tan^5 x$ term, this becomes $\dfrac{1}{3}\dfrac{\sin^3 x}{\cos^3 x}+\dfrac{1}{5}\dfrac{\sin^3 x\left(1-\cos^2 x\right)}{\cos^5 x}+C = \dfrac{1}{5}\dfrac{\sin^3 x}{\cos^5 x}+\left(\dfrac{1}{3}-\dfrac{1}{5}\right)\dfrac{\sin^3 x}{\cos^3 x}+C = \dfrac{1}{5}\dfrac{\sin^3 x}{\cos^5 x}+\dfrac{2}{15}\dfrac{\sin^3 x}{\cos^3 x}+C$, which is the same as Maple's expression.

39. Maple gives $\int x\sqrt{1+2x}\,dx = \frac{1}{10}(1+2x)^{5/2} - \frac{1}{6}(1+2x)^{3/2}$, Mathematica gives $\sqrt{1+2x}\left(\frac{2}{5}x^2 + \frac{1}{15}x - \frac{1}{15}\right)$, and Derive gives $\frac{1}{15}(1+2x)^{3/2}(3x-1)$. The first two expressions can be simplified to Derive's result. If we use Formula 54, we get

$$\begin{aligned}\int x\sqrt{1+2x}\,dx &= \tfrac{2}{15(2)^2}(3\cdot 2x - 2\cdot 1)(1+2x)^{3/2} + C = \tfrac{1}{30}(6x-2)(1+2x)^{3/2} + C\\ &= \tfrac{1}{15}(3x-1)(1+2x)^{3/2}\end{aligned}$$

40. Maple and Derive both give $\int \sin^4 x\,dx = -\frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\cos x\sin x + \frac{3}{8}x$, while Mathematica gives $\frac{1}{32}(12x - 8\sin 2x + \sin 4x)$, which can be expanded and simplified to give the other expression. Now

$$\begin{aligned}\int \sin^4 x\,dx &\overset{73}{=} -\tfrac{1}{4}\sin^3 x\cos x + \tfrac{3}{4}\int \sin^2 x\,dx \overset{63}{=} -\tfrac{1}{4}\sin^3 x\cos x + \tfrac{3}{4}\left(\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right) + C\\ &= -\tfrac{1}{4}\sin^3 x\cos x - \tfrac{3}{8}\sin x\cos x + \tfrac{3}{8}x + C \text{ since } \sin 2x = 2\sin x\cos x\end{aligned}$$

41. Maple gives $\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \frac{1}{2}\ln(1+\tan^2 x)$, Mathematica gives $\int \tan^5 x\,dx = \frac{1}{4}[-1 - 2\cos(2x)]\sec^4 x - \ln(\cos x)$, and Derive gives $\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln(\cos x)$. These expressions are equivalent, and none includes absolute value bars or a constant of integration. Note that Mathematica's and Derive's expressions suggest that the integral is undefined where $\cos x < 0$, which is not the case.
Using Formula 75, $\int \tan^5 x\,dx = \frac{1}{5-1}\tan^{5-1}x - \int \tan^{5-2}x\,dx = \frac{1}{4}\tan^4 x - \int \tan^3 x\,dx$. Using Formula 69, $\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x + \ln|\cos x| + C$, so $\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C$.

42. Maple gives $\int x^5\sqrt{x^2+1}\,dx = \frac{1}{35}x^4\sqrt{1+x^2} - \frac{4}{105}x^2\sqrt{1+x^2} + \frac{8}{105}\sqrt{1+x^2} + \frac{1}{7}x^6\sqrt{1+x^2}$. When we use the `factor` command on this expression, it becomes $\frac{1}{105}(1+x^2)^{3/2}(15x^4 - 12x^2 + 8)$. Mathematica gives $\sqrt{1+x^2}\left(\frac{8}{105} - \frac{4}{105}x^2 + \frac{1}{35}x^4 + \frac{1}{7}x^6\right)$, which again factors to give the above expression, and Derive gives the factored form immediately. If we substitute $u = \sqrt{x^2+1} \Rightarrow x^4 = (u^2-1)^2$, $x\,dx = u\,du$, then the integral becomes

$$\begin{aligned}\int (u^2-1)^2 u\,(u\,du) &= \int (u^4 - 2u^2 + 1)\,u^2\,du = \tfrac{1}{7}u^7 - \tfrac{2}{5}u^5 + \tfrac{1}{3}u^3 + C\\ &= \left(x^2+1\right)^{3/2}\left[\tfrac{1}{7}\left(x^2+1\right)^2 - \tfrac{2}{5}\left(x^2+1\right) + \tfrac{1}{3}\right] + C\\ &= \tfrac{1}{105}\left(x^2+1\right)^{3/2}\left[15\left(x^2+1\right)^2 - 42\left(x^2+1\right) + 35\right] + C\\ &= \tfrac{1}{105}\left(x^2+1\right)^{3/2}\left(15x^4 - 12x^2 + 8\right) + C\end{aligned}$$

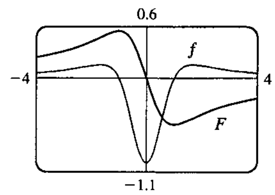
43. Derive gives $I = \int 2^x\sqrt{4^x - 1}\,dx = \dfrac{2^{x-1}\sqrt{2^{2x}-1}}{\ln 2} - \dfrac{\ln\left(\sqrt{2^{2x}-1} + 2^x\right)}{2\ln 2}$ immediately. Neither Maple nor Mathematica is able to evaluate I in its given form. However, if we instead write I as $\int 2^x\sqrt{(2^x)^2 - 1}\,dx$, both systems give the same answer as Derive (after minor simplification). Our trick works because the CAS now recognizes 2^x as a promising substitution.

44. None of Maple, Mathematica and Derive is able to evaluate $\int (1+\ln x)\sqrt{1+(x\ln x)^2}\,dx$. However, if we let $u = x\ln x$, then $du = (1+\ln x)\,dx$ and the integral is simply $\int \sqrt{1+u^2}\,du$, which any CAS can evaluate. The antiderivative is $\frac{1}{2}\ln\left(x\ln x + \sqrt{1+(x\ln x)^2}\right) + \frac{1}{2}x\ln x\sqrt{1+(x\ln x)^2} + C$.

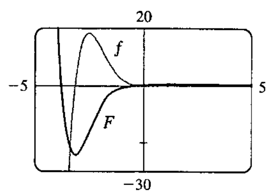
45. Maple gives the antiderivative

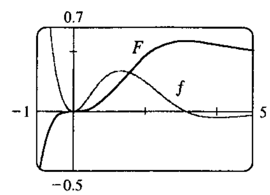
$$F(x) = \int \frac{x^2-1}{x^4+x^2+1}\,dx = -\tfrac{1}{2}\ln\left(x^2+x+1\right) + \tfrac{1}{2}\ln\left(x^2-x+1\right).$$

We can see that at 0, this antiderivative is 0. From the graphs, it appears that F has a maximum at $x=-1$ and a minimum at $x=1$ [since $F'(x) = f(x)$ changes sign at these x-values], and that F has inflection points at $x \approx -1.7$, $x=0$ and $x \approx 1.7$ [since $f(x)$ has extrema at these x-values].



46. Maple gives the antiderivative which, after we use the `simplify` command, becomes $\int xe^{-x}\sin x\,dx = -\frac{1}{2}e^{-x}(\cos x + x\cos x + x\sin x)$. At $x=0$, this antiderivative has the value $-\frac{1}{2}$, so we use $F(x) = -\frac{1}{2}e^{-x}(\cos x + x\cos x + x\sin x) + \frac{1}{2}$ to make $F(0)=0$.



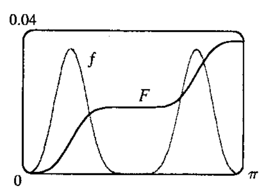


From the graphs, it appears that F has a minimum at $x \approx -3.1$ and a maximum at $x \approx 3.1$ [note that $f(x)=0$ at $x=\pm\pi$], and that F has inflection points where f' changes sign, at $x \approx -2.5$, $x=0$, $x\approx 1.3$ and $x \approx 4.1$.

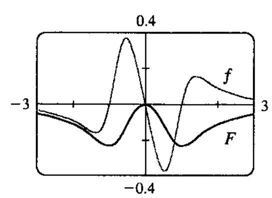
47. Since f is everywhere positive, we know that its antiderivative F is increasing. Maple gives

$$\int \sin^4 x\cos^6 x\,dx = -\tfrac{1}{10}\sin^3 x\cos^7 x - \tfrac{3}{80}\sin x\cos^7 x + \tfrac{1}{160}\cos^5 x\sin x + \tfrac{1}{128}\cos^3 x\sin x + \tfrac{3}{256}\cos x\sin x + \tfrac{3}{256}x$$

and this is 0 at $x=0$.



48. From the graph of f, we can see that F has a maximum at $x=0$, and minima at $x \approx \pm 1$. The antiderivative given by Maple is $F(x) = -\frac{1}{3}\ln\left(x^2+1\right) + \frac{1}{6}\ln\left(x^4-x^2+1\right)$, and $F(0)=0$. Note that f is odd, and its antiderivative F is even.



Discovery Project □ Patterns in Integrals

1. (a) The CAS results are listed. Note that the absolute value symbols are missing, as is the familiar " $+C$".

(i) $\displaystyle\int \frac{1}{(x+2)(x+3)}\,dx = \ln(x+2) - \ln(x+3)$

(ii) $\displaystyle\int \frac{1}{(x+1)(x+5)}\,dx = \frac{\ln(x+1)}{4} - \frac{\ln(x+5)}{4}$

(iii) $\displaystyle\int \frac{1}{(x+2)(x-5)}\,dx = \frac{\ln(x-5)}{7} - \frac{\ln(x+2)}{7}$

(iv) $\displaystyle\int \frac{1}{(x+2)^2}\,dx = -\frac{1}{x+2}$

(b) If $a \neq b$, it appears that $\ln(x+a)$ is divided by $b-a$ and $\ln(x+b)$ is divided by $a-b$, so we guess that

$$\int \frac{1}{(x+a)(x+b)}\,dx = \frac{\ln(x+a)}{b-a} + \frac{\ln(x+b)}{a-b} + C$$

If $a = b$, as in part (a)(iv), it appears that

$$\int \frac{1}{(x+a)^2}\,dx = -\frac{1}{x+a} + C$$

(c) The CAS verifies our guesses. Now

$$\frac{1}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b} \quad \Rightarrow \quad 1 = A(x+b) + B(x+a)$$

Setting $x = -b$ gives $B = 1/(a-b)$ and setting $x = -a$ gives $A = 1/(b-a)$. So

$$\int \frac{1}{(x+a)(x+b)}\,dx = \int \left[\frac{1/(b-a)}{x+a} + \frac{1/(a-b)}{x+b}\right] dx = \frac{\ln|x+a|}{b-a} + \frac{\ln|x+b|}{a-b} + C$$

and our guess for $a \neq b$ is correct.

If $a = b$, then $\dfrac{1}{(x+a)(x+b)} = \dfrac{1}{(x+a)^2} = (x+a)^{-2}$. Letting $u = x+a \;\Rightarrow\; du = dx$, we have $\int (x+a)^{-2}\,dx = \int u^{-2}\,du = -\dfrac{1}{u} + C = -\dfrac{1}{x+a} + C$, and our guess for $a = b$ is also correct.

2. (a) (i) $\int \sin x \cos 2x\, dx = \dfrac{\cos x}{2} - \dfrac{\cos 3x}{6}$

(ii) $\int \sin 3x \cos 7x\, dx = \dfrac{\cos 4x}{8} - \dfrac{\cos 10x}{20}$

(iii) $\int \sin 8x \cos 3x\, dx = -\dfrac{\cos 11x}{22} - \dfrac{\cos 5x}{10}$

(b) Looking at the sums and differences of a and b in part (a), we guess that

$$\int \sin ax \cos bx\, dx = \frac{\cos((a-b)x)}{2(b-a)} - \frac{\cos(a+b)x}{2(a+b)} + C$$

Note that $\cos((a-b)x) = \cos((b-a)x)$.

(c) The CAS verifies our guess. To integrate directly, we can use Formula 2(a) from Section 8.2.

$$\int \sin ax \cos bx\, dx = \int \left\{\tfrac{1}{2}[\sin(ax-bx) + \sin(ax+bx)]\right\} dx = \tfrac{1}{2}\int [\sin((a-b)x) + \sin((a+b)x)]\, dx$$
$$= \frac{1}{2}\left[-\frac{\cos((a-b)x)}{b-a} - \frac{\cos((a+b)x)}{a+b}\right] + C = \frac{\cos((a-b)x)}{2(b-a)} - \frac{\cos(a+b)x}{2(a+b)} + C$$

Our formula is valid for $a \neq b$.

3. (a) (i) $\int \ln x\, dx = x\ln x - x$

(ii) $\int x\ln x\, dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$

(iii) $\int x^2\ln x\, dx = \frac{1}{3}x^3\ln x - \frac{1}{9}x^3$

(iv) $\int x^3\ln x\, dx = \frac{1}{4}x^4\ln x - \frac{1}{16}x^4$

(v) $\int x^7\ln x\, dx = \frac{1}{8}x^8\ln x - \frac{1}{64}x^8$

(b) We guess that $\int x^n \ln x\, dx = \dfrac{1}{n+1}x^{n+1}\ln x - \dfrac{1}{(n+1)^2}x^{n+1}$.

(c) Let $u = \ln x$, $dv = x^n\, dx \quad\Rightarrow\quad du = \dfrac{dx}{x}$, $v = \dfrac{1}{n+1}x^{n+1}$.

Then

$$\int x^n \ln x\, dx = \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{n+1}\int x^n\, dx$$
$$= \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{n+1}\cdot\frac{1}{n+1}x^{n+1}$$

which verifies our guess. We must have $n+1 \neq 0 \quad\Leftrightarrow\quad n \neq -1$.

4. (a) (i) $\int xe^x\,dx = e^x\,(x-1)$

(ii) $\int x^2e^x\,dx = e^x\,(x^2-2x+2)$

(iii) $\int x^3e^x\,dx = e^x\,(x^3-3x^2+6x-6)$

(iv) $\int x^4e^x\,dx = e^x\,(x^4-4x^3+12x^2-24x+24)$

(v) $\int x^5e^x\,dx = e^x\,(x^5-5x^4+20x^3-60x^2+120x-120)$

(b) Notice from part (a) that we can write

$$\int x^4e^x\,dx = e^x\,(x^4-4x^3+4\cdot 3x^2-4\cdot 3\cdot 2x+4\cdot 3\cdot 2\cdot 1)$$

and

$$\int x^5e^x\,dx = e^x\,(x^5-5x^4+5\cdot 4x^3-5\cdot 4\cdot 3x^2+5\cdot 4\cdot 3\cdot 2x-5\cdot 4\cdot 3\cdot 2\cdot 1)$$

So we guess that

$$\begin{aligned}\int x^6e^x\,dx &= e^x\left(x^6-6x^5+6\cdot 5x^4-6\cdot 5\cdot 4x^3+6\cdot 5\cdot 4\cdot 3x^2-6\cdot 5\cdot 4\cdot 3\cdot 2x+6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\right)\\ &= e^x\left(x^6-6x^5+30x^4-120x^3+360x^2-720x+720\right)\end{aligned}$$

The CAS verifies our guess.

(c) From the results in part (a), as well as our prediction in part (b), we speculate that

$$\begin{aligned}\int x^ne^x\,dx &= e^x\left[x^n-nx^{n-1}+n\,(n-1)\,x^{n-2}-n\,(n-1)\,(n-2)\,x^{n-3}+\cdots\pm n!x\mp n!\right]\\ &= e^x\sum_{i=0}^{n}(-1)^{n-i}\,\frac{n!}{i!}x^i\end{aligned}$$

(We have reversed the order of the polynomial's terms.)

(d) Let S_n be the statement that $\int x^ne^x\,dx = e^x\displaystyle\sum_{i=0}^{n}(-1)^{n-i}\,\frac{n!}{i!}x^i$.

S_1 is true by part (a)(i). Suppose S_k is true for some k, and consider S_{k+1}. Integrating by parts with $u = x^{k+1}$, $dv = e^x\,dx \;\Rightarrow\; du = (k+1)\,x^k\,dx$, $v = e^x$, we get

$$\begin{aligned}\int x^{k+1}e^x\,dx &= x^{k+1}e^x-(k+1)\int x^ke^x\,dx\\ &= x^{k+1}e^x-(k+1)\left[e^x\sum_{i=0}^{k}(-1)^{k-i}\,\frac{k!}{i!}x^i\right]\\ &= e^x\left[x^{k+1}-(k+1)\sum_{i=0}^{k}(-1)^{k-i}\,\frac{k!}{i!}x^i\right]\\ &= e^x\left[x^{k+1}+\sum_{i=0}^{k}(-1)^{k-i+1}\,\frac{(k+1)\,k!}{i!}x^i\right]\\ &= e^x\sum_{i=0}^{k+1}(-1)^{(k+1)-i}\,\frac{(k+1)!}{i!}x^i\end{aligned}$$

This verifies S_n for $n = k+1$. Thus, by mathematical induction, S_n is true for all n, where n is a positive integer.

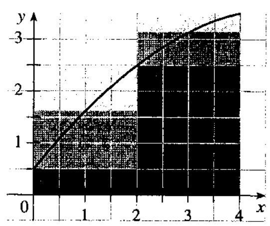
8.7 Approximate Integration

1. (a) $L_2 = \sum_{i=1}^{2} f(x_{i-1})\,\Delta x = f(x_0)\cdot 2 + f(x_1)\cdot 2 = 2\,[f(0)+f(2)] = 2\,(0.5+2.5) = 6$

$R_2 = \sum_{i=1}^{2} f(x_i)\,\Delta x = f(x_1)\cdot 2 + f(x_2)\cdot 2 = 2\,[f(2)+f(4)] = 2\,(2.5+3.5) = 12$

$M_2 = \sum_{i=1}^{2} f(\overline{x}_i)\,\Delta x = f(\overline{x}_1)\cdot 2 + f(\overline{x}_2)\cdot 2 = 2\,[f(1)+f(3)] \approx 2\,(1.7+3.2) = 9.8$

(b)



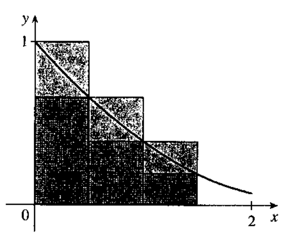
L_2 is an underestimate, since the area under the small rectangles is less than the area under the curve, and R_2 is an overestimate, since the area under the large rectangles is greater than the area under the curve. It appears that M_2 is an overestimate, though it is fairly close to I. See the solution to Exercise 43 for a proof of the fact that if f is concave down on (a, b), then the Midpoint Rule is an overestimate of $\int_a^b f(x)\,dx$.

(c) $T_2 = \left(\frac{1}{2}\Delta x\right)[f(x_0)+2f(x_1)+f(x_2)] = \frac{2}{2}\,[f(0)+2f(2)+f(4)] = 0.5+2\,(2.5)+3.5 = 9.$

This approximation is an underestimate, since the graph is concave down. See the solution to Exercise 43 for a general proof of this conclusion.

(d) For any n, we will have $L_n < T_n < I < M_n < R_n$.

2.



The diagram shows that $L_4 > T_4 > \int_0^2 f(x)\,dx > R_4$, and it appears that M_4 is a bit less than $\int_0^2 f(x)\,dx$. In fact, for any function that is concave upward, it can be shown that

$L_n > T_n > \int_0^2 f(x)\,dx > M_n > R_n$.

(a) Since $0.9540 > 0.8675 > 0.8632 > 0.7811$, it follows that $L_n = 0.9540$, $T_n = 0.8675$, $M_n = 0.8632$, and $R_n = 0.7811$.

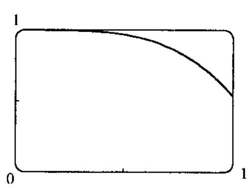
(b) Since $M_n < \int_0^2 f(x)\,dx < T_n$, we have

$0.8632 < \int_0^2 f(x)\,dx < 0.8675$.

3. $f(x) = \cos\left(x^2\right)$, $\Delta x = \frac{1-0}{4} = \frac{1}{4}$

(a) $T_4 = \frac{1}{4\cdot 2}\left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{2}{4}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right] \approx 0.895759$

(b) $M_4 = \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right] \approx 0.908907$



The graph shows that f is concave down on $[0, 1]$. So T_4 is an underestimate and M_4 is an overestimate. We can conclude that $0.895759 < \int_0^1 \cos\left(x^2\right)dx < 0.908907$.

4. 

(a) Since f is increasing on $[0, 1]$, L_2 will underestimate I (since the area under the small rectangles is less than the area under the curve), and R_2 will overestimate I. Since f is concave upward on $[0, 1]$, M_2 will underestimate I and T_2 will overestimate I (the area under the straight line segments is greater than the area under the curve).

(b) For any n, we will have $L_n < M_n < I < T_n < R_n$.

(c) $L_5 = \sum\limits_{i=1}^{5} f(x_{i-1})\,\Delta x = \frac{1}{5}\left[f(0.0) + f(0.2) + f(0.4) + f(0.6) + f(0.8)\right] \approx 0.1187$

$R_5 = \sum\limits_{i=1}^{5} f(x_i)\,\Delta x = \frac{1}{5}\left[f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1)\right] \approx 0.2146$

$M_5 = \sum\limits_{i=1}^{5} f(\overline{x}_i)\,\Delta x = \frac{1}{5}\left[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)\right] \approx 0.1622$

$T_5 = \left(\frac{1}{2}\Delta x\right)\left[f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1)\right] \approx 0.1666$

From the graph, it appears that the Midpoint Rule gives the best approximation. (This is in fact the case, since $I \approx 0.16371405$.)

5. $f(x) = x^2 \sin x$, $\Delta x = \dfrac{b-a}{n} = \dfrac{\pi - 0}{8} = \dfrac{\pi}{8}$

(a) $M_8 = \frac{\pi}{8}\left[f\left(\frac{\pi}{16}\right) + f\left(\frac{3\pi}{16}\right) + f\left(\frac{5\pi}{16}\right) + \cdots + f\left(\frac{15\pi}{16}\right)\right] \approx 5.932957$

(b) $S_8 = \frac{\pi}{8\cdot 3}\left[f(0) + 4f\left(\frac{\pi}{8}\right) + 2f\left(\frac{2\pi}{8}\right) + 4f\left(\frac{3\pi}{8}\right) + 2f\left(\frac{4\pi}{8}\right) + 4f\left(\frac{5\pi}{8}\right) + 2f\left(\frac{6\pi}{8}\right) + 4f\left(\frac{7\pi}{8}\right) + f(\pi)\right]$
≈ 5.869247

Actual: $\int_0^\pi x^2 \sin x\,dx \overset{84}{=} \left[-x^2 \cos x\right]_0^\pi + 2\int_0^\pi x \cos x\,dx \overset{83}{=} \left[-\pi^2(-1) - 0\right] + 2\left[\cos x + x \sin x\right]_0^\pi$
$= \pi^2 + 2\left[(-1 + 0) - (1 + 0)\right] = \pi^2 - 4 \approx 5.869604$

Errors: $E_M = \text{actual} - M_8 = \int_0^\pi x^2 \sin x\,dx - M_8 \approx -0.063353$

$E_S = \text{actual} - S_8 = \int_0^\pi x^2 \sin x\,dx - S_8 \approx 0.000357$

6. $f(x) = e^{-\sqrt{x}}$, $\Delta x = \dfrac{b-a}{n} = \dfrac{1-0}{6} = \dfrac{1}{6}$

(a) $M_6 = \frac{1}{6}\left[f\left(\frac{1}{12}\right) + f\left(\frac{3}{12}\right) + f\left(\frac{5}{12}\right) + f\left(\frac{7}{12}\right) + f\left(\frac{9}{12}\right) + f\left(\frac{11}{12}\right)\right] \approx 0.525100$

(b) $S_6 = \frac{1}{6\cdot 3}\left[f(0) + 4f\left(\frac{1}{6}\right) + 2f\left(\frac{2}{6}\right) + 4f\left(\frac{3}{6}\right) + 2f\left(\frac{4}{6}\right) + 4f\left(\frac{5}{6}\right) + f(1)\right] \approx 0.533979$

Actual: $\int_0^1 e^{-\sqrt{x}}\,dx = \int_0^{-1} e^u 2u\,du \quad \left[u = -\sqrt{x},\, u^2 = x,\, 2u\,du = dx\right] \overset{96}{=} 2\left[(u-1)e^u\right]_0^{-1}$
$= 2\left[-2e^{-1} - (-1e^0)\right] = 2 - 4e^{-1} \approx 0.528482$

Errors: $E_M = \text{actual} - M_6 = \int_0^1 e^{-\sqrt{x}}\,dx - M_6 \approx 0.003382$

$E_S = \text{actual} - S_6 = \int_0^1 e^{-\sqrt{x}}\,dx - S_6 \approx -0.005497$

7. $f(x) = \sqrt{1+x^3}$, $\Delta x = \dfrac{1-(-1)}{8} = \dfrac{1}{4}$

(a) $T_8 = \frac{0.25}{2}\left[f(-1) + 2f\left(-\frac{3}{4}\right) + 2f\left(-\frac{1}{2}\right) + \cdots + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right] \approx 1.913972$

(b) $S_8 = \frac{0.25}{3}\left[f(-1) + 4f\left(-\frac{3}{4}\right) + 2f\left(-\frac{1}{2}\right) + 4f\left(-\frac{1}{4}\right) + 2f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1)\right]$
≈ 1.934766

8. $f(x) = \sin(x^2)$, $\Delta x = \dfrac{1/2-0}{4} = \dfrac{1}{8}$

(a) $T_4 = \frac{1}{8\cdot 2}\left[f(0) + 2f\left(\frac{1}{8}\right) + 2f\left(\frac{2}{8}\right) + 2f\left(\frac{3}{8}\right) + f\left(\frac{1}{2}\right)\right] \approx 0.042743$

(b) $S_4 = \frac{1}{8\cdot 3}\left[f(0) + 4f\left(\frac{1}{8}\right) + 2f\left(\frac{2}{8}\right) + 4f\left(\frac{3}{8}\right) + f\left(\frac{1}{2}\right)\right] \approx 0.041478$

9. $f(x) = \dfrac{\sin x}{x}$, $\Delta x = \dfrac{\pi - \pi/2}{6} = \dfrac{\pi}{12}$

(a) $T_6 = \frac{\pi}{24}\left[f\left(\frac{\pi}{2}\right) + 2f\left(\frac{7\pi}{12}\right) + 2f\left(\frac{2\pi}{3}\right) + 2f\left(\frac{3\pi}{4}\right) + 2f\left(\frac{5\pi}{6}\right) + 2f\left(\frac{11\pi}{12}\right) + f(\pi)\right] \approx 0.481672$

(b) $S_6 = \frac{\pi}{36}\left[f\left(\frac{\pi}{2}\right) + 4f\left(\frac{7\pi}{12}\right) + 2f\left(\frac{2\pi}{3}\right) + 4f\left(\frac{3\pi}{4}\right) + 2f\left(\frac{5\pi}{6}\right) + 4f\left(\frac{11\pi}{12}\right) + f(\pi)\right] \approx 0.481172$

10. $f(x) = x\tan x$, $\Delta x = \dfrac{\pi/4 - 0}{6} = \dfrac{\pi}{24}$

(a) $T_6 = \frac{\pi}{48}\left[f(0) + 2f\left(\frac{\pi}{24}\right) + 2f\left(\frac{\pi}{12}\right) + \cdots + 2f\left(\frac{5\pi}{24}\right) + f\left(\frac{\pi}{4}\right)\right] \approx 0.189445$

(b) $S_6 = \frac{\pi}{72}\left[f(0) + 4f\left(\frac{\pi}{24}\right) + 2f\left(\frac{\pi}{12}\right) + 4f\left(\frac{\pi}{8}\right) + 2f\left(\frac{\pi}{6}\right) + 4f\left(\frac{5\pi}{24}\right) + f\left(\frac{\pi}{4}\right)\right] \approx 0.185822$

11. $f(x) = e^{-x^2}$, $\Delta x = \dfrac{1-0}{10} = \dfrac{1}{10}$

(a) $T_{10} = \frac{1}{10\cdot 2}\left[f(0) + 2f(0.1) + 2f(0.2) + \cdots + 2f(0.8) + 2f(0.9) + f(1)\right] \approx 0.746211$

(b) $M_{10} = \frac{1}{10}\left[f(0.05) + f(0.15) + f(0.25) + \cdots + f(0.75) + f(0.85) + f(0.95)\right] \approx 0.747131$

(c) $S_{10} = \frac{1}{10\cdot 3}[f(0) + 4f(0.1) + 2f(0.2) + 4f(0.3) + 2f(0.4) + 4f(0.5)$
$+ 2f(0.6) + 4f(0.7) + 2f(0.8) + 4f(0.9) + f(1)] \approx 0.746825$

12. $f(x) = \dfrac{1}{\sqrt{1+x^3}}$, $\Delta x = \dfrac{2-0}{10} = \dfrac{1}{5}$

(a) $T_{10} = \frac{1}{5\cdot 2}\left[f(0) + 2f(0.2) + 2f(0.4) + \cdots + 2f(1.6) + 2f(1.8) + f(2)\right] \approx 1.401435$

(b) $M_{10} = \frac{1}{5}\left[f(0.1) + f(0.3) + f(0.5) + \cdots + f(1.7) + f(1.9)\right] \approx 1.402558$

(c) $S_{10} = \frac{1}{5\cdot 3}[f(0) + 4f(0.2) + 2f(0.4) + 4f(0.6) + 2f(0.8) + 4f(1)$
$+ 2f(1.2) + 4f(1.4) + 2f(1.6) + 4f(1.8) + f(2)] \approx 1.402206$

13. $f(t) = \sin(e^{t/2})$, $\Delta t = \dfrac{1/2-0}{8} = \dfrac{1}{16}$

(a) $T_8 = \frac{1}{16\cdot 2}\left[f(0) + 2f\left(\frac{1}{16}\right) + 2f\left(\frac{2}{16}\right) + \cdots + 2f\left(\frac{7}{16}\right) + f\left(\frac{1}{2}\right)\right] \approx 0.451948$

(b) $M_8 = \frac{1}{16}\left[f\left(\frac{1}{32}\right) + f\left(\frac{3}{32}\right) + f\left(\frac{5}{32}\right) + \cdots + f\left(\frac{13}{32}\right) + f\left(\frac{15}{32}\right)\right] \approx 0.451991$

(c) $S_8 = \frac{1}{16\cdot 3}\left[f(0) + 4f\left(\frac{1}{16}\right) + 2f\left(\frac{2}{16}\right) + \cdots + 4f\left(\frac{7}{16}\right) + f\left(\frac{1}{2}\right)\right] \approx 0.451976$

14. $f(x) = \dfrac{1}{\ln x}$, $\Delta x = \dfrac{3-2}{10} = \dfrac{1}{10}$

(a) $T_{10} = \frac{1}{10 \cdot 2}[f(2) + 2f(2.1) + 2f(2.2) + \cdots + 2f(2.9) + f(3)] \approx 1.119061$

(b) $M_{10} = \frac{1}{10}[f(2.05) + f(2.15) + f(2.25) + \cdots + f(2.85) + f(2.95)] \approx 1.118107$

(c) $S_{10} = \frac{1}{10 \cdot 3}[f(2) + 4f(2.1) + 2f(2.2) + 4f(2.3) + 2f(2.4) + 4f(2.5) + 2f(2.6) + 4f(2.7) + 2f(2.8) + 4f(2.9) + f(3)] \approx 1.118428$

15. $f(x) = e^{1/x}$, $\Delta x = \dfrac{2-1}{4} = \dfrac{1}{4}$

(a) $T_4 = \frac{1}{4 \cdot 2}[f(1) + 2f(1.25) + 2f(1.5) + 2f(1.75) + f(2)] \approx 2.031893$

(b) $M_4 = \frac{1}{4}[f(1.125) + f(1.375) + f(1.625) + f(1.875)] \approx 2.014207$

(c) $S_4 = \frac{1}{4 \cdot 3}[f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2)] \approx 2.020651$

16. $f(x) = \ln(1 + e^x)$, $\Delta x = \dfrac{1-0}{8} = \dfrac{1}{8}$

(a) $T_8 = \frac{1}{8 \cdot 2}\left[f(0) + 2f\left(\frac{1}{8}\right) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{3}{8}\right) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{5}{8}\right) + 2f\left(\frac{3}{4}\right) + 2f\left(\frac{7}{8}\right) + f(1)\right] \approx 0.984120$

(b) $M_8 = \frac{1}{8}\left[f\left(\frac{1}{16}\right) + f\left(\frac{3}{16}\right) + f\left(\frac{5}{16}\right) + f\left(\frac{7}{16}\right) + \cdots + f\left(\frac{15}{16}\right)\right] \approx 0.983669$

(c) $S_8 = \frac{1}{8 \cdot 3}\left[f(0) + 4f\left(\frac{1}{8}\right) + 2f\left(\frac{1}{4}\right) + 4f\left(\frac{3}{8}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{5}{8}\right) + 2f\left(\frac{3}{4}\right) + 4f\left(\frac{7}{8}\right) + f(1)\right] \approx 0.983819$

17. $f(x) = x^5 e^x$, $\Delta x = \dfrac{1-0}{10} = \dfrac{1}{10}$

(a) $T_{10} = \frac{1}{10 \cdot 2}[f(0) + 2f(0.1) + 2f(0.2) + \cdots + 2f(0.9) + f(1)] \approx 0.409140$

(b) $M_{10} = \frac{1}{10}[f(0.05) + f(0.15) + f(0.25) + \cdots + f(0.95)] \approx 0.388849$

(c) $S_{10} = \frac{1}{10 \cdot 3}[f(0) + 4f(0.1) + 2f(0.2) + 4f(0.3) + 2f(0.4) + 4f(0.5) + 2f(0.6) + 4f(0.7) + 2f(0.8) + 4f(0.9) + f(1)] \approx 0.395802$

18. $f(x) = \sqrt{x}\sin x$, $\Delta x = \dfrac{4-0}{8} = \dfrac{1}{2}$

(a) $T_8 = \frac{1}{2 \cdot 2}\left\{f(0) + 2\left[f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2) + f\left(\frac{5}{2}\right) + f(3) + f\left(\frac{7}{2}\right)\right] + f(4)\right\} \approx 1.732865$

(b) $M_8 = \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + \cdots + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right)\right] \approx 1.787427$

(c) $S_8 = \frac{1}{2 \cdot 3}\left[f(0) + 4f\left(\frac{1}{2}\right) + 2f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + f(4)\right] \approx 1.772142$

19. $f(y) = \dfrac{1}{1+y^5}$, $\Delta y = \dfrac{3-0}{6} = \dfrac{1}{2}$

(a) $T_6 = \frac{1}{2 \cdot 2}\left[f(0) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{2}{2}\right) + 2f\left(\frac{3}{2}\right) + 2f\left(\frac{4}{2}\right) + 2f\left(\frac{5}{2}\right) + f(3)\right] \approx 1.064275$

(b) $M_6 = \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right)\right] \approx 1.067416$

(c) $S_6 = \frac{1}{2 \cdot 3}\left[f(0) + 4f\left(\frac{1}{2}\right) + 2f\left(\frac{2}{2}\right) + 4f\left(\frac{3}{2}\right) + 2f\left(\frac{4}{2}\right) + 4f\left(\frac{5}{2}\right) + f(3)\right] \approx 1.074915$

20. $f(x) = \frac{e^x}{x}$, $\Delta x = \frac{4-2}{10} = \frac{1}{5}$

(a) $T_{10} = \frac{1}{5 \cdot 2}\{f(2) + 2[f(2.2) + f(2.4) + f(2.6) + \cdots + f(3.8)] + f(4)\} \approx 14.704592$

(b) $M_{10} = \frac{1}{5}[f(2.1) + f(2.3) + f(2.5) + f(2.7) + \cdots + f(3.7) + f(3.9)] \approx 14.662669$

(c) $S_{10} = \frac{1}{5 \cdot 3}[f(2) + 4f(2.2) + 2f(2.4) + 4f(2.6) + \cdots + 2f(3.6) + 4f(3.8) + f(4)] \approx 14.676696$

21. $f(x) = e^{-x^2}$, $\Delta x = \frac{2-0}{10} = \frac{1}{5}$

(a) $T_{10} = \frac{1}{5 \cdot 2}\{f(0) + 2[f(0.2) + f(0.4) + \cdots + f(1.8)] + f(2)\} \approx 0.881839$
$M_{10} = \frac{1}{5}[f(0.1) + f(0.3) + f(0.5) + \cdots + f(1.7) + f(1.9)] \approx 0.882202$

(b) $f(x) = e^{-x^2}$, $f'(x) = -2xe^{-x^2}$, $f''(x) = (4x^2 - 2)e^{-x^2}$, $f'''(x) = 4x(3 - 2x^2)e^{-x^2}$. $f'''(x) = 0 \Leftrightarrow$ $x = 0$ or $x = \pm\sqrt{\frac{3}{2}}$. So to find the maximum value of $|f''(x)|$ on $[0, 2]$, we need only consider its values at $x = 0$, $x = 2$, and $x = \sqrt{\frac{3}{2}}$. $|f''(0)| = 2$, $|f''(2)| \approx 0.2564$ and $\left|f''\left(\sqrt{\frac{3}{2}}\right)\right| \approx 0.8925$. Thus, taking $K = 2$, $a = 0$, $b = 2$, and $n = 10$ in Theorem 3, we get $|E_T| \le 2 \cdot 2^3/(12 \cdot 10^2) = \frac{1}{75} = 0.01\overline{3}$, and $|E_M| \le |E_T|/2 \le 0.00\overline{6}$.

(c) Take $K = 2$ [as in part (b)] in Theorem 3. $|E_T| \le \frac{K(b-a)^3}{12n^2} \le 10^{-5} \Leftrightarrow \frac{2(2-0)^3}{12n^2} \le 10^{-5} \Leftrightarrow$ $\frac{3}{4}n^2 \ge 10^5 \Leftrightarrow n \ge 365.1\ldots \Leftrightarrow n \ge 366$. Take $n = 366$ for T_n. For E_M, again take $K = 2$ in Theorem 3 to get $|E_M| \le 10^{-5} \Leftrightarrow \frac{3}{2}n^2 \ge 10^5 \Leftrightarrow n \ge 258.2 \Rightarrow n \ge 259$. Take $n = 259$ for M_n.

22. (a) $T_8 = \frac{1}{8 \cdot 2}\left\{f(0) + 2\left[f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{7}{8}\right)\right] + f(1)\right\} \approx 0.902333$
$M_8 = \frac{1}{8}\left[f\left(\frac{1}{16}\right) + f\left(\frac{3}{16}\right) + f\left(\frac{5}{16}\right) + \cdots + f\left(\frac{15}{16}\right)\right] = 0.905620$

(b) $f(x) = \cos(x^2)$, $f'(x) = -2x\sin(x^2)$, $f''(x) = -2\sin(x^2) - 4x^2\cos(x^2)$. For $0 \le x \le 1$, sin and cos are positive, so $|f''(x)| = 2\sin(x^2) + 4x^2\cos(x^2) \le 2 \cdot 1 + 4 \cdot 1 \cdot 1 = 6$ since $\sin(x^2) \le 1$ and $\cos(x^2) \le 1$ for all x, and $x^2 \le 1$ for $0 \le x \le 1$. So for $n = 8$, we take $K = 6$, $a = 0$, and $b = 1$ in Theorem 3, to get $|E_T| \le 6 \cdot 1^3/(12 \cdot 8^2) = \frac{1}{128} = 0.0078125$ and $|E_M| \le \frac{1}{256} = 0.00390625$. [A better estimate is obtained by noting from a graph of f'' that $|f''(x)| \le 4$ for $0 \le x \le 1$.]

(c) Using $K = 6$ as in part (b), we have $|E_T| \le 6 \cdot 1^3/(12n^2) = 1/(2n^2) \le 10^{-5} \Rightarrow 2n^2 \ge 10^5 \Rightarrow$ $n \ge \sqrt{\frac{1}{2} \cdot 10^5}$ or $n \ge 224$. To guarantee that $|E_M| \le 0.00001$, we need $6 \cdot 1^3/(24n^2) \le 10^{-5} \Rightarrow$ $4n^2 \ge 10^5 \Rightarrow n \ge \sqrt{\frac{1}{4} \cdot 10^5}$ or $n \ge 159$.

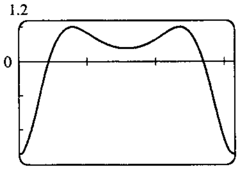
23. (a) $T_{10} = \frac{1}{10 \cdot 2}\{f(0) + 2[f(0.1) + f(0.2) + \cdots + f(0.9)] + f(1)\} \approx 1.71971349$
$S_{10} = \frac{1}{10 \cdot 3}[f(0) + 4f(0.1) + 2f(0.2) + 4f(0.3) + \cdots + 4f(0.9) + f(1)] \approx 1.71828278$
Since $I = \int_0^1 e^x\,dx = [e^x]_0^1 = e - 1 \approx 1.71828183$, $E_T = I - T_{10} \approx -0.00143166$ and $E_S = I - S_{10} \approx -0.00000095$.

(b) $f(x) = e^x \Rightarrow f''(x) = e^x \le e$ for $0 \le x \le 1$. Taking $K = e$, $a = 0$, $b = 1$, and $n = 10$ in Theorem 3, we get $|E_T| \le e(1)^3/(12 \cdot 10^2) \approx 0.002265 > 0.00143166$ [actual $|E_T|$ from (a)]. $f^{(4)}(x) = e^x < e$ for $0 \le x \le 1$. Using Theorem 4, we have $|E_S| \le e(1)^5/(180 \cdot 10^4) \approx 0.0000015 > 0.00000095$ [actual $|E_S|$ from (a)]. We see that the actual errors are about two-thirds the size of the error estimates.

(c) From part (b), we take $K = e$ to get $|E_T| \leq \dfrac{K\,(b-a)^3}{12n^2} \leq 0.00001 \;\Rightarrow\; n^2 \geq \dfrac{e\,(1^3)}{12\,(0.00001)} \;\Rightarrow$ $n \geq 150.5$. Take $n = 151$ for T_n. Now $|E_M| \leq \dfrac{K\,(b-a)^3}{24n^2} \leq 0.00001 \;\Rightarrow\; n \geq 106.4$. Take $n = 107$ for M_n. Finally, $|E_S| \leq \dfrac{K\,(b-a)^5}{180n^4} \leq 0.00001 \;\Rightarrow\; n^4 \geq \dfrac{e\,(1^5)}{180\,(0.00001)} \;\Rightarrow\; n \geq 6.23$. Take $n = 8$ for S_n (since n has to be even for Simpson's Rule).

24. From Example 7(b), we take $K = 76e$ to get $|E_S| \leq 76e\,(1)^5/(180n^4) \leq 0.00001 \;\Rightarrow$ $n^4 \geq 76e\,/[180\,(0.00001)] \;\Rightarrow\; n \geq 18.4$. Take $n = 20$ (since n must be even.)

25. (a) Using the CAS, we differentiate $f(x) = e^{\cos x}$ twice, and find that $f''(x) = e^{\cos x}\left(\sin^2 x - \cos x\right)$. From the graph, we see that the maximum value of $|f''(x)|$ occurs at the endpoints of the interval $[0, 2\pi]$. Since $f''(0) = -e$, we can use $K = e$ or $K = 2.8$.



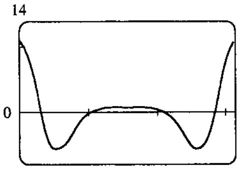
(b) A CAS gives $M_{10} \approx 7.954926518$. (In Maple, use `student[middlesum]`.)

(c) Using Theorem 3 for the Midpoint Rule, with $K = e$, we get $|E_M| \leq \dfrac{e\,(2\pi - 0)^3}{24 \cdot 10^2} \approx 0.280945995$. With $K = 2.8$, we get $|E_M| \leq \dfrac{2.8\,(2\pi - 0)^3}{24 \cdot 10^2} = 0.289391916$.

(d) A CAS gives $I \approx 7.954926521$.

(e) The actual error is only about 3×10^{-9}, much less than the estimate in part (c).

(f) We use the CAS to differentiate twice more, and then graph $f^{(4)}(x) = e^{\cos x}\left(\sin^4 x - 6\sin^2 x\cos x + 3 - 7\sin^2 x + \cos x\right)$. From the graph, we see that the maximum value of $|f^{(4)}(x)|$ occurs at the endpoints of the interval $[0, 2\pi]$. Since $f^{(4)}(0) = 4e$, we can use $K = 4e$ or $K = 10.9$.



(g) A CAS gives $S_{10} \approx 7.953789422$. (In Maple, use `student[simpson]`.)

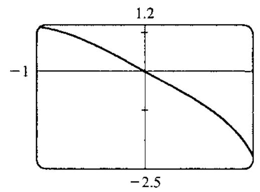
(h) Using Theorem 4 with $K = 4e$, we get $|E_S| \leq \dfrac{4e\,(2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059153618$. With $K = 10.9$, we get $|E_S| \leq \dfrac{10.9\,(2\pi - 0)^5}{180 \cdot 10^4} = 0.059299814$.

(i) The actual error is about $7.954926521 - 7.953789422 \approx 0.00114$. This is quite a bit smaller than the estimate in part (h), though the difference is not nearly as great as it was in the case of the Midpoint Rule.

(j) To ensure that $|E_S| \le 0.0001$, we use Theorem 4: $|E_S| \le \dfrac{4e\,(2\pi)^5}{180 \cdot n^4} \le 0.0001 \quad \Rightarrow \quad \dfrac{4e\,(2\pi)^5}{180 \cdot 0.0001} \le n^4 \quad \Rightarrow$ $n^4 \ge 5{,}915{,}362 \quad \Leftrightarrow \quad n \ge 49.3$. So we must take $n \ge 50$ to ensure that $|I - S_n| \le 0.0001$. ($K = 10.9$ leads to the same value of n.)

26. (a) Using the CAS, we differentiate $f(x) = \sqrt{4 - x^3}$ twice, and find that

$$f''(x) = -\frac{9x^4}{4\left(4 - x^3\right)^{3/2}} - \frac{3x}{\left(4 - x^3\right)^{1/2}}.$$



From the graph, we see that $|f''(x)| < 2.2$ on $[-1, 1]$.

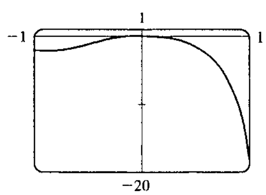
(b) A CAS gives $M_{10} \approx 3.995804152$. (In Maple, use `student[middlesum]`.)

(c) Using Theorem 3 for the Midpoint Rule, with $K = 2.2$, we get $|E_M| \le \dfrac{2.2\,[1 - (-1)]^3}{24 \cdot 10^2} \approx 0.00733$.

(d) A CAS gives $I \approx 3.995487677$.

(e) The actual error is about -0.0003165, much less than the estimate in part (c).

(f) We use the CAS to differentiate twice more, and then graph $f^{(4)}(x) = \dfrac{9}{16}\dfrac{x^2\left(x^6 - 224x^3 - 1280\right)}{\left(4 - x^3\right)^{7/2}}$.



From the graph, we see that $\left|f^{(4)}(x)\right| < 18.1$ on $[-1, 1]$.

(g) A CAS gives $S_{10} \approx 3.995449790$. (In Maple, use `student[simpson]`.)

(h) Using Theorem 4 with $K = 18.1$, we get $|E_S| \le \dfrac{18.1\,[1 - (-1)]^5}{180 \cdot 10^4} \approx 0.000322$.

(i) The actual error is about $3.995487677 - 3.995449790 \approx 0.0000379$. This is quite a bit smaller than the estimate in part (h).

(j) To ensure that $|E_S| \le 0.0001$,we use Theorem 4: $|E_S| \le \dfrac{18.1\,(2)^5}{180 \cdot n^4} \le 0.0001 \quad \Rightarrow \quad \dfrac{18.1\,(2)^5}{180 \cdot 0.0001} \le n^4 \quad \Rightarrow$ $n^4 \ge 32{,}178 \quad \Rightarrow n \ge 13.4$. So we must take $n \ge 14$ to ensure that $|I - S_n| \le 0.0001$.

27. $I = \int_0^1 x^3 dx = \left[\frac{1}{4}x^4\right]_0^1 = 0.25$. $f(x) = x^3$.

$n = 4$: $L_4 = \frac{1}{4}\left[0^3 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{3}{4}\right)^3\right] = 0.140625$

$R_4 = \frac{1}{4}\left[\left(\frac{1}{4}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{3}{4}\right)^3 + 1^3\right] = 0.390625$

$T_4 = \frac{1}{4 \cdot 2}\left[0^3 + 2\left(\frac{1}{4}\right)^3 + 2\left(\frac{1}{2}\right)^3 + 2\left(\frac{3}{4}\right)^3 + 1^3\right] = 0.265625,$

$M_4 = \frac{1}{4}\left[\left(\frac{1}{8}\right)^3 + \left(\frac{3}{8}\right)^3 + \left(\frac{5}{8}\right)^3 + \left(\frac{7}{8}\right)^3\right] = 0.2421875,$

$E_L = I - L_4 = \frac{1}{4} - 0.140625 = 0.109375$, $E_R = \frac{1}{4} - 0.390625 = -0.140625$,

$E_T = \frac{1}{4} - 0.265625 = -0.015625$, $E_M = \frac{1}{4} - 0.2421875 = 0.0078125$

$n = 8$: $L_8 = \frac{1}{8}\left[f(0) + f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{7}{8}\right)\right] \approx 0.191406$

$R_8 = \frac{1}{8}\left[f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{7}{8}\right) + f(1)\right] \approx 0.316406$

$T_8 = \frac{1}{8 \cdot 2}\left\{f(0) + 2\left[f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{7}{8}\right)\right] + f(1)\right\} \approx 0.253906$

$M_8 = \frac{1}{8}\left[f\left(\frac{1}{16}\right) + f\left(\frac{3}{16}\right) + \cdots + f\left(\frac{13}{16}\right) + f\left(\frac{15}{16}\right)\right] = 0.248047$

$E_L \approx \frac{1}{4} - 0.191406 \approx 0.058594$, $E_R \approx \frac{1}{4} - 0.316406 \approx -0.066406$,

$E_T \approx \frac{1}{4} - 0.253906 \approx -0.003906$, $E_M \approx \frac{1}{4} - 0.248047 \approx 0.001953$.

$n = 16$: $L_{16} = \frac{1}{16}\left[f(0) + f\left(\frac{1}{16}\right) + f\left(\frac{2}{16}\right) + \cdots + f\left(\frac{15}{16}\right)\right] \approx 0.219727$

$R_{16} = \frac{1}{16}\left[f\left(\frac{1}{16}\right) + f\left(\frac{2}{16}\right) + \cdots + f\left(\frac{15}{16}\right) + f(1)\right] \approx 0.282227$

$T_{16} = \frac{1}{16 \cdot 2}\left\{f(0) + 2\left[f\left(\frac{1}{16}\right) + f\left(\frac{2}{16}\right) + \cdots + f\left(\frac{15}{16}\right)\right] + f(1)\right\} \approx 0.250977$

$M_{16} = \frac{1}{16}\left[f\left(\frac{1}{32}\right) + f\left(\frac{3}{32}\right) + \cdots + f\left(\frac{31}{32}\right)\right] \approx 0.249512$

$E_L \approx \frac{1}{4} - 0.219727 \approx 0.030273$, $E_R \approx \frac{1}{4} - 0.282227 \approx -0.032227$,

$E_T \approx \frac{1}{4} - 0.250977 \approx -0.000977$, $E_M \approx \frac{1}{4} - 0.249512 \approx 0.000488$.

n	L_n	R_n	T_n	M_n
4	0.140625	0.390625	0.265625	0.242188
8	0.191406	0.316406	0.253906	0.248047
16	0.219727	0.282227	0.250977	0.249512

n	E_L	E_R	E_T	E_M
4	0.109375	−0.140625	−0.015625	0.007813
8	0.058594	−0.066406	−0.003906	0.001953
16	0.030273	−0.032227	−0.000977	0.000488

Observations:

1. E_L and E_R are always opposite in sign, as are E_T and E_M.
2. As n is doubled, E_L and E_R are decreased by about a factor of 2, and E_T and E_M are decreased by a factor of about 4.
3. The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
4. All the approximations become more accurate as the value of n increases.
5. The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

28. $\int_0^2 e^x\,dx = [e^x]_0^2 = e^2 - 1 \approx 6.389056.$ $f(x) = e^x$

$n = 4$: $\Delta x = (2-0)/4 = \frac{1}{2}$

$L_4 = \frac{1}{2}\left[e^0 + e^{1/2} + e^1 + e^{3/2}\right] \approx 4.924346$

$R_4 = \frac{1}{2}\left[e^{1/2} + e^1 + e^{3/2} + e^2\right] \approx 8.118874$

$T_4 = \frac{1}{2\cdot 2}\left[e^0 + 2e^{1/2} + 2e^1 + 2e^{3/2} + e^2\right] \approx 6.521610$

$M_4 = \frac{1}{2}\left[e^{1/4} + e^{3/4} + e^{5/4} + e^{7/4}\right] \approx 6.322986.$

$E_L \approx 6.389056 - 4.924346 \approx 1.464710$, $E_R \approx 6.389056 - 8.118874 = -1.729818$,

$E_T \approx 6.389056 - 6.521610 \approx -0.132554$, $E_M \approx 6.389056 - 6.322986 = 0.0660706.$

$n = 8$: $\Delta x = (2-0)/8 = \frac{1}{4}$

$L_8 = \frac{1}{4}\left[e^0 + e^{1/4} + e^{1/2} + e^{3/4} + e^1 + e^{5/4} + e^{3/2} + e^{7/4}\right] \approx 5.623666$

$R_8 = \frac{1}{4}\left[e^{1/4} + e^{1/2} + e^{3/4} + e^1 + e^{5/4} + e^{3/2} + e^{7/4} + e^2\right] \approx 7.220930$

$T_8 = \frac{1}{4\cdot 2}\left[e^0 + 2e^{1/4} + 2e^{1/2} + 2e^{3/4} + 2e^1 + 2e^{5/4} + 2e^{3/2} + 2e^{7/4} + e^2\right] \approx 6.422298$

$M_8 = \frac{1}{4}\left[e^{1/8} + e^{3/8} + e^{5/8} + e^{7/8} + e^{9/8} + e^{11/8} + e^{13/8} + e^{15/8}\right] \approx 6.372448$

$E_L \approx 6.389056 - 5.623666 \approx 0.765390$, $E_R \approx 6.389056 - 7.220930 \approx -0.831874$,

$E_T \approx 6.389056 - 6.422298 \approx -0.033242$, $E_M \approx 6.389056 - 6.372448 \approx 0.016608.$

$n = 16$: $\Delta x = (2-0)/16 = \frac{1}{8}$

$L_{16} = \frac{1}{8}\left[f(0) + f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{14}{8}\right) + f\left(\frac{15}{8}\right)\right] \approx 5.998057$

$R_{16} = \frac{1}{8}\left[f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + f\left(\frac{3}{8}\right) + \cdots + f\left(\frac{15}{8}\right) + f(2)\right] \approx 6.796689$

$T_{16} = \frac{1}{8\cdot 2}\left(f(0) + 2\left[f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + f\left(\frac{3}{8}\right) + \cdots + f\left(\frac{15}{8}\right)\right] + f(2)\right) \approx 6.397373$

$M_{16} = \frac{1}{8}\left[f\left(\frac{1}{16}\right) + f\left(\frac{3}{16}\right) + f\left(\frac{5}{16}\right) + \cdots + f\left(\frac{29}{16}\right) + f\left(\frac{31}{16}\right)\right] \approx 6.384899$

$E_L \approx 6.389056 - 5.998057 \approx 0.390999$, $E_R \approx 6.389056 - 6.796689 \approx -0.407633$,

$E_T \approx 6.389056 - 6.397373 \approx -0.008317$, $E_M \approx 6.389056 - 6.384899 \approx 0.004158.$

n	E_L	E_R	E_T	E_M
4	1.464710	-1.729818	-0.132554	0.066071
8	0.765390	-0.831874	-0.033242	0.016608
16	0.390999	-0.407633	-0.008317	0.004158

Observations:

1. E_L and E_R are always opposite in sign, as are E_T and E_M.
2. As n is doubled, E_L and E_R are decreased by a factor of about 2, and E_T and E_M are decreased by a factor of about 4.
3. The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
4. All the approximations become more accurate as the value of n increases.
5. The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

29. $\int_1^4 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_1^4 = \frac{2}{3}(8-1) = \frac{14}{3} \approx 4.666667$

$n = 6$: $\Delta x = (4-1)/6 = \frac{1}{2}$

$T_6 = \frac{1}{2\cdot 2}\left[\sqrt{1} + 2\sqrt{1.5} + 2\sqrt{2} + 2\sqrt{2.5} + 2\sqrt{3} + 2\sqrt{3.5} + \sqrt{4}\right] \approx 4.661488$

$M_6 = \frac{1}{2}\left[\sqrt{1.25} + \sqrt{1.75} + \sqrt{2.25} + \sqrt{2.75} + \sqrt{3.25} + \sqrt{3.75}\right] \approx 4.669245$

$S_6 = \frac{1}{2\cdot 3}\left[\sqrt{1} + 4\sqrt{1.5} + 2\sqrt{2} + 4\sqrt{2.5} + 2\sqrt{3} + 4\sqrt{3.5} + \sqrt{4}\right] \approx 4.666563$

$E_T \approx \frac{14}{3} - 4.661488 \approx 0.005178$, $E_M \approx \frac{14}{3} - 4.669245 \approx -0.002578$,

$E_S \approx \frac{14}{3} - 4.666563 \approx 0.000104$.

$n = 12$: $\Delta x = (4-1)/12 = \frac{1}{4}$

$T_{12} = \frac{1}{4\cdot 2}\left(f(1) + 2[f(1.25) + f(1.5) + \cdots + f(3.5) + f(3.75)] + f(4)\right) \approx 4.665367$

$M_{12} = \frac{1}{4}[f(1.125) + f(1.375) + f(1.625) + \cdots + f(3.875)] \approx 4.667316$

$S_{12} = \frac{1}{4\cdot 3}[f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + \cdots + 4f(3.75) + f(4)] \approx 4.666659$

$E_T \approx \frac{14}{3} - 4.665367 \approx 0.001300$, $E_M \approx \frac{14}{3} - 4.667316 \approx -0.000649$,

$E_S \approx \frac{14}{3} - 4.666659 \approx 0.000007$.

Note: These errors were computed more precisely and then rounded to six places. That is, they were not computed by comparing the rounded values of T_n, M_n, and S_n with the rounded value of the actual definite integral.

n	T_n	M_n	S_n
6	4.661488	4.669245	4.666563
12	4.665367	4.667316	4.666659

n	E_T	E_M	E_S
6	0.005178	−0.002578	0.000104
12	0.001300	−0.000649	0.000007

Observations:

1. E_T and E_M are opposite in sign and decrease by a factor of about 4 as n is doubled.

2. The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and seems to decrease by a factor of about 16 as n is doubled.

30. $I = \int_{-1}^2 xe^x\,dx = [xe^x - e^x]_{-1}^2 = e^2 + 2/e \approx 8.124815$. $f(x) = xe^x$.

$n = 6$: $\Delta x = [2-(-1)]/6 = \frac{1}{2}$

$T_6 = \frac{1}{2\cdot 2}\{f(-1) + 2[f(-0.5) + f(0) + \cdots + f(1.5)] + f(2)\} \approx 8.583514$

$M_6 = \frac{1}{2}[f(-0.75) + f(-0.25) + \cdots + f(1.75)] \approx 7.896632$

$S_6 = \frac{1}{2\cdot 3}[f(-1) + 4f(-0.5) + 2f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(2)] \approx 8.136885$

$E_T \approx I - 8.583514 \approx -0.458699$, $E_M \approx I - 7.896632 \approx 0.228183$,

$E_S \approx I - 8.136885 \approx -0.012070$.

$n = 12$: $\Delta x = [2 - (-1)]/12 = \frac{1}{4}$

$$T_{12} = \tfrac{1}{4 \cdot 2}\{f(-1) + 2[f(-0.75) + f(-0.5) + \cdots + f(1.75)] + f(2)\} \approx 8.240073$$

$$M_{12} = \tfrac{1}{4}\left[f\left(-\tfrac{7}{8}\right) + f\left(-\tfrac{5}{8}\right) + \cdots + f\left(\tfrac{13}{8}\right) + f\left(\tfrac{15}{8}\right)\right] \approx 8.067259$$

$$S_{12} = \tfrac{1}{4 \cdot 3}[f(-1) + 4f(-0.75) + 2f(-0.5) + \cdots + 2f(1.5) + 4f(1.75) + f(2)] \approx 8.125593$$

$E_T \approx I - 8.240073 \approx -0.115258$, $E_M \approx I - 8.067259 \approx 0.057556$,

$E_S \approx I - 8.125593 \approx -0.000778$

n	T_n	M_n	S_n
6	8.583514	7.896632	8.136885
12	8.240073	8.067259	8.125593

n	E_T	E_M	E_S
6	−0.458699	0.228183	−0.012070
12	−0.115258	0.057556	−0.000778

Observations:

1. E_T and E_M are opposite in sign and decrease by a factor of about 4 as n is doubled.
2. The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and seems to decrease by a factor of about 16 as n is doubled.

31. $\Delta x = (4 - 0)/4 = 1$

(a) $T_4 = \frac{1}{2}[f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)] \approx \frac{1}{2}[0 + 2(3) + 2(5) + 2(3) + 1] = 11.5$

(b) $M_4 = 1 \cdot [f(0.5) + f(1.5) + f(2.5) + f(3.5)] \approx 1 + 4.5 + 4.5 + 2 = 12$

(c) $S_4 = \frac{1}{3}[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)] \approx \frac{1}{3}[0 + 4(3) + 2(5) + 4(3) + 1] = 11.\overline{6}$

32. If x = distance from left end of pool and $w = w(x)$ = width at x, then Simpson's Rule with $n = 8$ and $\Delta x = 2$ gives Area $= \int_0^{16} w\,dx \approx \frac{2}{3}[0 + 4(6.2) + 2(7.2) + 4(6.8) + 2(5.6) + 4(5.0) + 2(4.8) + 4(4.8) + 0] \approx 84\text{ m}^2$.

33. (a) $\int_1^{3.2} y\,dx \approx \frac{0.2}{2}[4.9 + 2(5.4) + 2(5.8) + 2(6.2) + 2(6.7) + 2(7.0)$
$+ 2(7.3) + 2(7.5) + 2(8.0) + 2(8.2) + 2(8.3) + 8.3] = 15.4$

(b) $-1 \le f''(x) \le 3 \;\Rightarrow\; |f''(x)| \le 3$, so use $K = 3$, $a = 1$, $b = 3.2$, and $n = 11$ in Theorem 3. So

$$|E_T| \le \frac{3(3.2 - 1)^3}{12(11)^2} = 0.022.$$

34. We use Simpson's Rule with $n = 10$ and $\Delta x = \frac{1}{2}$:

$$\begin{aligned} \text{distance} &= \int_0^5 v(t)\,dt \approx S_{10} = \tfrac{1}{2 \cdot 3}[f(0) + 4f(0.5) + 2f(1) + \cdots + 4f(4.5) + f(5)] \\ &= \tfrac{1}{6}[0 + 4(4.67) + 2(7.34) + 4(8.86) + 2(9.73) + 4(10.22) + 2(10.51) \\ &\quad + 4(10.67) + 2(10.76) + 4(10.81) + 10.81] = \tfrac{1}{6}(268.41) = 44.735\text{ m} \end{aligned}$$

35. By the Total Change Theorem, the increase in velocity is equal to $\int_0^6 a(t)\,dt$. We use Simpson's Rule with $n = 6$ and $\Delta t = 1$ to estimate this integral:

$$\begin{aligned} \int_0^6 a(t)\,dt &\approx S_6 = \tfrac{1}{1 \cdot 3}[a(0) + 4a(1) + 2a(2) + 4a(3) + 2a(4) + 4a(5) + a(6)] \\ &\approx \tfrac{1}{3}[0 + 4(0.5) + 2(4.1) + 4(9.8) + 2(12.9) + 4(9.5) + 0] = \tfrac{1}{3}(113.2) = 37.7\overline{3}\text{ ft/s} \end{aligned}$$

36. By the Total Change Theorem, the amount of water leaked is equal to $\int_0^4 r(t)\,dt$. We use Simpson's Rule with $n = 4$ and $\Delta t = 1$ to estimate this integral:

$$\begin{aligned}\int_0^4 r(t)\,dt \approx S_4 &= \tfrac{1}{1\cdot 3}\left[r(0) + 4r(1) + 2r(2) + 4r(3) + r(4)\right]\\ &\approx \tfrac{1}{3}\left[6 + 4(5.7) + 2(5.1) + 4(4.1) + 3\right] = \tfrac{1}{3}(58.4) = 19.4\overline{6}\text{ L}\end{aligned}$$

37. By the Total Change Theorem, the total percentage increase is equal to $\int_{1987}^{1997} r(t)\,dt$. We use Simpson's Rule with $n = 10$ and $\Delta t = 1$ to estimate this integral:

$$\begin{aligned}\int_{1987}^{1997} r(t)\,dt \approx S_{10} &= \tfrac{1}{3}\left[r(1987) + 4r(1988) + 2r(1989) + \cdots + 4r(1996) + r(1997)\right]\\ &\approx \tfrac{1}{3}\left[4.0 + 4(4.1) + 2(5.7) + 4(5.8) + 2(3.6) + 4(1.4) + 2(2.1) + 4(2.3) + 2(2.8) + 4(3.2) + 2.6\right]\\ &= \tfrac{1}{3}(102.2) = 34.0\overline{6}\% \approx 34.1\%\end{aligned}$$

38. By the Total Change Theorem, the energy used is equal to $\int_0^{12} P(t)\,dt$. We use Simpson's Rule with $n = 12$ and $\Delta t = 1$ to estimate this integral:

$$\begin{aligned}\int_0^{12} P(t)\,dt \approx S_{12} &= \tfrac{1}{1\cdot 3}\left[P(0) + 4P(1) + 2P(2) + \cdots + 2P(10) + 4P(11) + P(12)\right]\\ &\approx \tfrac{1}{3}\left[4182 + 4(3856) + 2(3640) + 4(3558) + 2(3547) + 4(3679) + 2(4112)\right.\\ &\qquad\left. + 4(4699) + 2(5151) + 4(5514) + 2(5751) + 4(6044) + 6206\right]\\ &= \tfrac{1}{3}(164{,}190) = 54{,}730\text{ megawatt-hours}\end{aligned}$$

39. Volume $= \pi \int_0^2 \left(\sqrt[3]{1+x^3}\right)^2 dx = \pi \int_0^2 \left(1+x^3\right)^{2/3} dx$. $V \approx \pi \cdot S_{10}$ where $f(x) = \left(1+x^3\right)^{2/3}$ and $\Delta x = (2-0)/10 = \frac{1}{5}$. Therefore,

$$\begin{aligned}V \approx \pi \cdot S_{10} = \pi \frac{1}{5\cdot 3}&\left[f(0) + 4f(0.2) + 2f(0.4) + 4f(0.6) + 2f(0.8) + 4f(1)\right.\\ &\left. + 2f(1.2) + 4f(1.4) + 2f(1.6) + 4f(1.8) + f(2)\right] \approx 12.325078\end{aligned}$$

40. Using Simpson's Rule with $n = 10$, $\Delta x = \frac{\pi/2}{10}$, $L = 1$, $\theta_0 = \frac{42\pi}{180}$ radians, $g = 9.8\text{ m/s}^2$, $k^2 = \sin^2\left(\frac{1}{2}\theta_0\right)$, and $f(x) = 1\Big/\sqrt{1 - k^2\sin^2 x}$, we get

$$\begin{aligned}T &= 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-k^2\sin^2 x}} \approx 4\sqrt{\frac{L}{g}}S_{10}\\ &= 4\sqrt{\tfrac{1}{9.8}}\left(\tfrac{\pi/2}{10\cdot 3}\right)\left[f(0) + 4f\left(\tfrac{\pi}{20}\right) + 2f\left(\tfrac{2\pi}{20}\right) + \cdots + 4f\left(\tfrac{9\pi}{20}\right) + f\left(\tfrac{\pi}{2}\right)\right] \approx 2.07665\end{aligned}$$

41. $I(\theta) = \dfrac{N^2\sin^2 k}{k^2}$, where $k = \dfrac{\pi N d\sin\theta}{\lambda}$, $N = 10{,}000$, $d = 10^{-4}$, and $\lambda = 632.8 \times 10^{-9}$. So $I(\theta) = \dfrac{\left(10^4\right)^2\sin^2 k}{k^2}$, where $k = \dfrac{\pi\left(10^4\right)\left(10^{-4}\right)\sin\theta}{632.8\times 10^{-9}}$. Now $n = 10$ and $\Delta\theta = \dfrac{10^{-6} - \left(-10^{-6}\right)}{10} = 2\times 10^{-7}$, so $M_{10} = 2\times 10^{-7}\left[I(-0.0000009) + I(-0.0000007) + \cdots + I(0.0000009)\right] \approx 59.4$.

42. $f(x) = \cos(\pi x)$, $\Delta x = \frac{20-0}{10} = 2 \Rightarrow$

$$\begin{aligned} T_{10} &= \tfrac{2}{2}\{f(0) + 2[f(2) + f(4) + \cdots + f(18)] + f(20)\} \\ &= 1[\cos 0 + 2(\cos 2\pi + \cos 4\pi + \cdots + \cos 18\pi) + \cos 20\pi] \\ &= 1 + 2(1+1+1+1+1+1+1+1+1) + 1 = 20 \end{aligned}$$

The actual value is $\int_0^{20} \cos(\pi x)\, dx = \frac{1}{\pi}[\sin \pi x]_0^{20} = \frac{1}{\pi}(\sin 20\pi - \sin 0) = 0$. The discrepancy is due to the fact that the function is sampled only at points of the form $2n$, where its value is $f(2n) = \cos(2n\pi) = 1$.

43. Since the Trapezoidal and Midpoint approximations on the interval $[a, b]$ are the sums of the Trapezoidal and Midpoint approximations on the subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, we can focus our attention on one such interval. The condition $f''(x) < 0$ for $a \le x \le b$ means that the graph of f is concave down as in Figure 5. In that figure, T_n is the area of the trapezoid $AQRD$, $\int_a^b f(x)\, dx$ is the area of the region $AQPRD$, and M_n is the area of the trapezoid $ABCD$, so $T_n < \int_a^b f(x)\, dx < M_n$. In general, the condition $f'' < 0$ implies that the graph of f on $[a, b]$ lies above the chord joining the points $(a, f(a))$ and $(b, f(b))$. Thus, $\int_a^b f(x)\, dx > T_n$. Since M_n is the area under a tangent to the graph, and since $f'' < 0$ implies that the tangent lies above the graph, we also have $M_n > \int_a^b f(x)\, dx$. Thus, $T_n < \int_a^b f(x)\, dx < M_n$.

44. Let f be a polynomial of degree ≤ 3; say $f(x) = Ax^3 + Bx^2 + Cx + D$. It will suffice to show that Simpson's estimate is exact when there are two subintervals ($n = 2$), because for a larger even number of subintervals the sum of exact estimates is exact. As in the derivation of Simpson's Rule, we can assume that $x_0 = -h$, $x_1 = 0$, and $x_2 = h$. Then Simpson's approximation is

$$\begin{aligned} \int_{-h}^{h} f(x)\, dx &\approx \tfrac{1}{3}h[f(-h) + 4f(0) + f(h)] \\ &= \tfrac{1}{3}h\left[\left(-Ah^3 + Bh^2 - Ch + D\right) + 4D + \left(Ah^3 + Bh^2 + Ch + D\right)\right] \\ &= \tfrac{1}{3}h\left[2Bh^2 + 6D\right] = \tfrac{2}{3}Bh^3 + 2Dh \end{aligned}$$

The exact value of the integral is

$$\int_{-h}^{h} (Ax^3 + Bx^2 + Cx + D)\, dx = 2\int_0^h (Bx^2 + D)\, dx = 2\left[\tfrac{1}{3}Bx^3 + Dx\right]_0^h = \tfrac{2}{3}Bh^3 + 2Dh$$

Thus, Simpson's Rule is exact.

45. $T_n = \frac{1}{2}\Delta x\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$ and $M_n = \Delta x\left[f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_{n-1}) + f(\overline{x}_n)\right]$, where $\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$. Now

$$\begin{aligned} T_{2n} = \tfrac{1}{2}\left(\tfrac{1}{2}\Delta x\right)[f(x_0) &+ 2f(\overline{x}_1) + 2f(x_1) + 2f(\overline{x}_2) + 2f(x_2) + \cdots \\ &+ 2f(\overline{x}_{n-1}) + 2f(x_{n-1}) + 2f(\overline{x}_n) + f(x_n)] \end{aligned}$$

so

$$\begin{aligned} \tfrac{1}{2}(T_n + M_n) &= \tfrac{1}{2}T_n + \tfrac{1}{2}M_n \\ &= \tfrac{1}{4}\Delta x\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right] \\ &\qquad + \tfrac{1}{4}\Delta x\left[2f(\overline{x}_1) + 2f(\overline{x}_2) + \cdots + 2f(\overline{x}_{n-1}) + 2f(\overline{x}_n)\right] \\ &= T_{2n} \end{aligned}$$

46. $T_n = \dfrac{\Delta x}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$ and $M_n = \Delta x \sum_{i=1}^{n} f\left(x_i - \dfrac{\Delta x}{2}\right)$, so

$$\frac{1}{3}T_n + \frac{2}{3}M_n = \frac{1}{3}(T_n + 2M_n) = \frac{\Delta x}{3 \cdot 2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) + 4\sum_{i=1}^{n} f\left(x_i - \frac{\Delta x}{2}\right)\right]$$

where $\Delta x = \dfrac{b-a}{n}$. Let $\delta x = \dfrac{b-a}{2n}$. Then $\Delta x = 2\delta x$, so

$$\begin{aligned} \tfrac{1}{3}T_n + \tfrac{2}{3}M_n &= \frac{\delta x}{3}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) + 4\sum_{i=1}^{n} f(x_i - \delta x)\right] \\ &= \tfrac{1}{3}\delta x\,[f(x_0) + 4f(x_1 - \delta x) + 2f(x_1) + 4f(x_2 - \delta x) \\ &\qquad + 2f(x_2) + \cdots + 2f(x_{n-1}) + 4f(x_n - \delta x) + f(x_n)] \end{aligned}$$

Since $x_0, x_1 - \delta x, x_1, x_2 - \delta x, x_2, \ldots, x_{n-1}, x_n - \delta x, x_n$ are the subinterval endpoints for S_{2n}, and since $\delta x = \dfrac{b-a}{2n}$ is the width of the subintervals for S_{2n}, the last expression for $\frac{1}{3}T_n + \frac{2}{3}M_n$ is the usual expression for S_{2n}. Therefore, $\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}$.

8.8 Improper Integrals

1. (a) Since $\int_1^\infty x^4 e^{-x^4}\,dx$ has an infinite interval of integration, this is an improper integral of Type I.

(b) Since $y = \sec x$ has an infinite discontinuity at $x = \frac{\pi}{2}$, $\int_0^{\pi/2} \sec x\,dx$ is a Type II improper integral.

(c) Since $y = \dfrac{x}{(x-2)(x-3)}$ has an infinite discontinuity at $x = 2$, $\displaystyle\int_0^2 \frac{x}{x^2 - 5x + 6}\,dx$ is a Type II improper integral.

(d) Since $\displaystyle\int_{-\infty}^0 \frac{1}{x^2+25}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

2. (a) Since $y = 1/(2x-1)$ is defined and continuous on $[1, 2]$, the integral is proper.

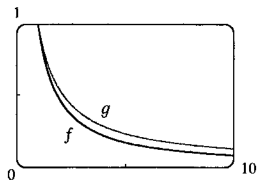
(b) Since $y = \dfrac{1}{2x-1}$ has an infinite discontinuity at $x = \frac{1}{2}$, $\displaystyle\int_0^1 \frac{1}{2x-1}\,dx$ is a Type II improper integral.

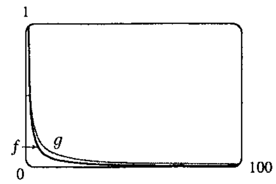
(c) Since $\displaystyle\int_{-\infty}^\infty \frac{\sin x}{1+x^2}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

(d) Since $y = \ln(x-1)$ has an infinite discontinuity at $x = 1$, $\int_1^2 \ln(x-1)\,dx$ is a Type II improper integral.

3. The area under the graph of $y = 1/x^3 = x^{-3}$ between $x = 1$ and $x = t$ is $A(t) = \int_1^t x^{-3}\,dx = \left[-\frac{1}{2}x^{-2}\right]_1^t = \frac{1}{2} - 1/(2t^2)$. So the area for $1 \le x \le 10$ is $A(10) = 0.5 - 0.005 = 0.495$, the area for $1 \le x \le 100$ is $A(100) = 0.5 - 0.00005 = 0.49995$, and the area for $1 \le x \le 1000$ is $A(1000) = 0.5 - 0.0000005 = 0.4999995$. The total area under the curve for $x \ge 1$ is $\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left[\frac{1}{2} - 1/(2t^2)\right] = \frac{1}{2}$.

4. (a)





(b) The area under the graph of f from $x = 1$ to $x = t$ is

$$F(t) = \int_1^t f(x)\,dx = \int_1^t x^{-1.1}dx = \left[-\tfrac{1}{0.1}x^{-0.1}\right]_1^t$$
$$= -10\left(t^{-0.1} - 1\right) = 10\left(1 - t^{-0.1}\right)$$

and the area under the graph of g is

$$G(t) = \int_1^t g(x)\,dx = \int_1^t x^{-0.9}dx = \left[\tfrac{1}{0.1}x^{0.1}\right]_1^t$$
$$= 10\left(t^{0.1} - 1\right)$$

t	$F(t)$	$G(t)$
10	2.06	2.59
100	3.69	5.85
10^4	6.02	15.12
10^6	7.49	29.81
10^{10}	9	90
10^{20}	9.9	990

(c) The total area under the graph of f is $\lim_{t\to\infty} F(t) = \lim_{t\to\infty} 10\left(1 - t^{-0.1}\right) = 10$.

The total area under the graph of g does not exist, since $\lim_{t\to\infty} G(t) = \lim_{t\to\infty} 10\left(t^{0.1} - 1\right) = \infty$.

5. $\displaystyle\int_1^\infty \frac{1}{(3x+1)^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{(3x+1)^2}\,dx = \lim_{t\to\infty}\int_4^{3t+1}\frac{1}{u^2}\cdot\frac{1}{3}\,du$ $[u = 3x+1,\ du = 3\,dx]$

$$= \frac{1}{3}\lim_{t\to\infty}\left[-\frac{1}{u}\right]_4^{3t+1} = \frac{1}{3}\lim_{t\to\infty}\left[-\frac{1}{3t+1} + \frac{1}{4}\right] = \tfrac{1}{3}\left(0 + \tfrac{1}{4}\right) = \tfrac{1}{12}.$$ Convergent

6. $\displaystyle\int_{-\infty}^0 \frac{1}{2x-5}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{1}{2x-5}\,dx = \lim_{t\to-\infty}\left[\tfrac{1}{2}\ln|2x-5|\right]_t^0 = \lim_{t\to\infty}\left[\tfrac{1}{2}\ln 5 - \tfrac{1}{2}\ln|2t-5|\right] = -\infty.$
Divergent

7. $\displaystyle\int_{-\infty}^{-1}\frac{1}{\sqrt{2-w}}\,dw = \lim_{t\to-\infty}\int_t^{-1}\frac{1}{\sqrt{2-w}}\,dw = \lim_{t\to-\infty}\left[-2\sqrt{2-w}\right]_t^{-1}$ $[u = 2-w,\ du = -dw]$

$$= \lim_{t\to-\infty}\left[-2\sqrt{3} + 2\sqrt{2-t}\right] = \infty.$$ Divergent

8. $\displaystyle\int_2^\infty \frac{dx}{(x+3)^{3/2}} = \lim_{t\to\infty}\int_2^t \frac{dx}{(x+3)^{3/2}} = \lim_{t\to\infty}\left[\frac{-2}{\sqrt{x+3}}\right]_2^t = \lim_{t\to\infty}\left[\frac{-2}{\sqrt{t+3}} + \frac{2}{\sqrt{5}}\right] = \frac{2}{\sqrt{5}}$

9. $\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_0^t = \lim_{t\to\infty}\left(-e^{-t} + 1\right) = 1$

10. $\int_{-\infty}^{-1} e^{-2t}\,dt = \lim_{b\to-\infty}\int_b^{-1} e^{-2t}\,dt = \lim_{b\to-\infty}\left[-\tfrac{1}{2}e^{-2t}\right]_b^{-1} = \lim_{b\to-\infty}\left[-\tfrac{1}{2}e^2 + \tfrac{1}{2}e^{-2b}\right] = \infty$. Divergent

11. $\int_{-\infty}^\infty x^3\,dx = \int_{-\infty}^0 x^3\,dx + \int_0^\infty x^3\,dx$, but $\int_{-\infty}^0 x^3\,dx = \lim_{t\to-\infty}\left[\tfrac{1}{4}x^4\right]_t^0 = \lim_{t\to-\infty}\left(-\tfrac{1}{4}t^4\right) = -\infty$. Divergent

12. $I = \int_{-\infty}^{\infty} \frac{1}{\sqrt[3]{w-5}}\,dw = \lim_{a\to-\infty}\int_a^0 \frac{dw}{\sqrt[3]{w-5}} + \lim_{b\to 5^-}\int_0^b \frac{dw}{\sqrt[3]{w-5}} + \lim_{c\to 5^+}\int_c^{10} \frac{dw}{\sqrt[3]{w-5}} + \lim_{d\to\infty}\int_{10}^d \frac{dw}{\sqrt[3]{w-5}}.$
(The values 0 and 10 could be any pair of values surrounding 5.) If any one of these four integrals diverges, then I diverges, and (for example) $\lim_{d\to\infty}\int_{10}^d \frac{dw}{\sqrt[3]{w-5}} = \lim_{d\to\infty}\left[\frac{3}{2}(w-5)^{2/3}\right]_{10}^d = \lim_{d\to\infty}\left[\frac{3}{2}(d-5)^{2/3} - \frac{3}{2}(5)^{2/3}\right] = \infty.$
Thus, I is divergent.

13. $\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^0 xe^{-x^2}\,dx + \int_0^{\infty} xe^{-x^2}\,dx.$ $\int_{-\infty}^0 xe^{-x^2}\,dx = \lim_{t\to-\infty}\left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_t^0$
$= \lim_{t\to-\infty}\left(-\frac{1}{2}\right)\left(1-e^{-t^2}\right) = -\frac{1}{2}$, and
$\int_0^{\infty} xe^{-x^2}\,dx = \lim_{t\to\infty}\left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_0^t = \lim_{t\to\infty}\left(-\frac{1}{2}\right)\left(e^{-t^2}-1\right) = \frac{1}{2}$. Therefore, $\int_{-\infty}^{\infty} xe^{-x^2}\,dx = -\frac{1}{2}+\frac{1}{2} = 0.$

14. $\int_{-\infty}^{\infty} x^2e^{-x^3}\,dx = \int_{-\infty}^0 x^2e^{-x^3}\,dx + \int_0^{\infty} x^2e^{-x^3}\,dx$, and
$\int_{-\infty}^0 x^2e^{-x^3}\,dx = \lim_{t\to-\infty}\left[-\frac{1}{3}e^{-x^3}\right]_t^0 = -\frac{1}{3}+\frac{1}{3}\left(\lim_{t\to-\infty} e^{-t^3}\right) = \infty$. Divergent

15. $\int_0^{\infty} \frac{dx}{(x+2)(x+3)} = \lim_{t\to\infty}\int_0^t \left[\frac{1}{x+2} - \frac{1}{x+3}\right]dx = \lim_{t\to\infty}\left[\ln\left(\frac{x+2}{x+3}\right)\right]_0^t = \lim_{t\to\infty}\left[\ln\left(\frac{t+2}{t+3}\right) - \ln\frac{2}{3}\right]$
$= \ln 1 - \ln\frac{2}{3} = -\ln\frac{2}{3}$

16. $\int_0^{\infty} \frac{x\,dx}{(x+2)(x+3)} = \lim_{t\to\infty}\int_0^t \left[\frac{-2}{x+2} + \frac{3}{x+3}\right]dx = \lim_{t\to\infty}\left[3\ln(x+3) - 2\ln(x+2)\right]_0^t =$
$\lim_{t\to\infty}\left[\ln\frac{(t+3)^3}{(t+2)^2} - \ln\frac{27}{4}\right] = \infty$. Divergent

17. $\int_0^{\infty} \cos x\,dx = \lim_{t\to\infty}\left[\sin x\right]_0^t = \lim_{t\to\infty}\sin t$, which does not exist. Divergent

18. $\int_{-\infty}^{\pi/2} \sin 2\theta\,d\theta = \lim_{t\to-\infty}\int_t^{\pi/2} \sin 2\theta\,d\theta = \lim_{t\to-\infty}\left[-\frac{1}{2}\cos 2\theta\right]_t^{\pi/2} = \lim_{t\to-\infty}\left(\frac{1}{2}+\frac{1}{2}\cos 2t\right)$. This limit does not exist, so the integral is divergent.

19. $\int_{-\infty}^1 xe^{2x}\,dx = \lim_{t\to-\infty}\int_t^1 xe^{2x}\,dx = \lim_{t\to-\infty}\left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right]_t^1$ (by parts)
$= \lim_{t\to-\infty}\left[\frac{1}{2}e^2 - \frac{1}{4}e^2 - \frac{1}{2}te^{2t} + \frac{1}{4}e^{2t}\right] = \frac{1}{4}e^2 - 0 + 0 = \frac{1}{4}e^2$
since $\lim_{t\to-\infty} te^{2t} = \lim_{t\to-\infty}\frac{t}{e^{-2t}} \overset{\text{H}}{=} \lim_{t\to-\infty}\frac{1}{-2e^{-2t}} = \lim_{t\to-\infty} -\frac{1}{2}e^{2t} = 0.$

20. $\int_0^{\infty} xe^{-x}\,dx = \lim_{t\to\infty}\left[-xe^{-x} - e^{-x}\right]_0^t = \lim_{t\to\infty}\left[1-(t+1)e^{-t}\right] = 1 - \lim_{t\to\infty}\frac{t+1}{e^t} \overset{\text{H}}{=} 1 - \lim_{t\to\infty}\frac{1}{e^t} = 1-0 = 1$

21. $\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t = \lim_{t\to\infty}\frac{(\ln t)^2}{2} = \infty$. Divergent

22. $\int_{-\infty}^{\infty} e^{-|x|}\,dx = \int_{-\infty}^0 e^x\,dx + \int_0^{\infty} e^{-x}\,dx$, $\int_{-\infty}^0 e^x\,dx = \lim_{t\to-\infty}\left[e^x\right]_t^0 = \lim_{t\to-\infty}\left(1-e^t\right) = 1$, and
$\int_0^{\infty} e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_0^t = \lim_{t\to\infty}\left(1-e^{-t}\right) = 1$. Therefore, $\int_{-\infty}^{\infty} e^{-|x|}\,dx = 1+1 = 2.$

23. $\int_{-\infty}^{\infty} \frac{x\,dx}{1+x^2} = \int_{-\infty}^0 \frac{x\,dx}{1+x^2} + \int_0^{\infty} \frac{x\,dx}{1+x^2}$ and
$\int_{-\infty}^0 \frac{x\,dx}{1+x^2} = \lim_{t\to-\infty}\left[\frac{1}{2}\ln\left(1+x^2\right)\right]_t^0 = \lim_{t\to-\infty}\left[0 - \frac{1}{2}\ln\left(1+t^2\right)\right] = -\infty$. Divergent

24. Since $f(r) = \dfrac{1}{r^2+4}$ is even,

$$I = \int_{-\infty}^{\infty} f(r)\,dr = 2\int_0^{\infty} f(r)\,dr = 2\lim_{t\to\infty}\int_0^t \frac{1}{r^2+4}\,dr = 2\lim_{t\to\infty}\left[\frac{1}{2}\arctan\frac{r}{2}\right]_0^t$$

$$= \lim_{t\to\infty}\left(\arctan\frac{t}{2} - 0\right) = \frac{\pi}{2}$$

25. Integrate by parts with $u = \ln x$, $dv = dx/x^2 \Rightarrow du = dx/x$, $v = -1/x$.

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left(-\frac{\ln t}{t} - \frac{1}{t} + 0 + 1\right)$$

$$= -0 - 0 + 0 + 1 = 1$$

since $\lim\limits_{t\to\infty}\dfrac{\ln t}{t} \overset{\text{H}}{=} \lim\limits_{t\to\infty}\dfrac{1/t}{1} = 0$.

26. Integrate by parts with $u = \ln x$, $dv = dx/x^3$, $du = dx/x$, $v = -1/(2x^2)$:

$$\int_1^{\infty} \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty}\left(\left[-\frac{1}{2x^2}\ln x\right]_1^t + \frac{1}{2}\int_1^t \frac{1}{x^3}\,dx\right)$$

$$= \lim_{t\to\infty}\left(-\frac{1}{2}\frac{\ln t}{t^2} + 0 - \frac{1}{4t^2} + \frac{1}{4}\right) = \frac{1}{4}\,.$$

since, by l'Hospital's Rule, $\lim\limits_{t\to\infty}\dfrac{\ln t}{t^2} = \lim\limits_{t\to\infty}\dfrac{1/t}{2t} = \lim\limits_{t\to\infty}\dfrac{1}{2t^2} = 0$.

27. $\displaystyle\int_0^3 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+}\int_t^3 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+}\left[2\sqrt{x}\right]_t^3 = \lim_{t\to 0^+}\left(2\sqrt{3} - 2\sqrt{t}\right) = 2\sqrt{3}$

28. $\displaystyle\int_0^3 \frac{dx}{x\sqrt{x}} = \lim_{t\to 0^+}\int_t^3 \frac{dx}{x^{3/2}} = \lim_{t\to 0^+}\left[\frac{-2}{\sqrt{x}}\right]_t^3 = \frac{-2}{\sqrt{3}} + \lim_{t\to 0^+}\frac{2}{\sqrt{t}} = \infty$. Divergent

29. $\displaystyle\int_{-1}^0 \frac{dx}{x^2} = \lim_{t\to 0^-}\int_{-1}^t \frac{dx}{x^2} = \lim_{t\to 0^-}\left[\frac{-1}{x}\right]_{-1}^t = \lim_{t\to 0^-}\left[-\frac{1}{t} + \frac{1}{-1}\right] = \infty$. Divergent

30. $\displaystyle\int_1^9 \frac{dx}{\sqrt[3]{x-9}} = \lim_{t\to 9^-}\int_1^t \frac{dx}{\sqrt[3]{x-9}} = \lim_{t\to 9^-}\left[\tfrac{3}{2}(x-9)^{2/3}\right]_1^t = \lim_{t\to 9^-}\left[\frac{3}{2}(t-9)^{2/3} - 6\right] = -6$

31. $\int_0^{\pi/4} \csc^2 t\,dt = \lim\limits_{s\to 0^+}\int_s^{\pi/4}\csc^2 t\,dt = \lim\limits_{s\to 0^+}\left[-\cot t\right]_s^{\pi/4} = \lim\limits_{s\to 0^+}\left[-\cot\frac{\pi}{4} + \cot s\right] = \infty$. Divergent

32. $\displaystyle\int_0^{\pi/4} \frac{\cos x\,dx}{\sqrt{\sin x}} = \lim_{t\to 0^+}\int_t^{\pi/4} \frac{\cos x\,dx}{\sqrt{\sin x}} = \lim_{t\to 0^+}\left[2\sqrt{\sin x}\right]_t^{\pi/4} = \lim_{t\to 0^+}\left(2\sqrt{\tfrac{1}{\sqrt{2}}} - 2\sqrt{\sin t}\right)$

$$= 2\sqrt{\tfrac{1}{\sqrt{2}}} = \tfrac{2}{2^{1/4}} = 2^{3/4}$$

33. $\displaystyle\int_{-2}^3 \frac{dx}{x^4} = \int_{-2}^0 \frac{dx}{x^4} + \int_0^3 \frac{dx}{x^4}$, but $\displaystyle\int_{-2}^0 \frac{dx}{x^4} = \lim_{t\to 0^-}\left[-\frac{x^{-3}}{3}\right]_{-2}^t = \lim_{t\to 0^-}\left[-\frac{1}{3t^3} - \frac{1}{24}\right] = \infty$. Divergent

34. $f(y) = 1/(4y-1)$ has an infinite discontinuity at $y = \frac{1}{4}$.

$$\int_{1/4}^1 \frac{1}{4y-1}\,dy = \lim_{t\to(1/4)^+}\int_t^1 \frac{1}{4y-1}\,dy = \lim_{t\to(1/4)^+}\left[\tfrac{1}{4}\ln|4y-1|\right]_t^1$$

$$= \lim_{t\to(1/4)^+}\left[\tfrac{1}{4}\ln 3 - \tfrac{1}{4}\ln(4t-1)\right] = \infty$$

so $\displaystyle\int_{1/4}^1 \frac{1}{4y-1}\,dy$ diverges, and hence, $\displaystyle\int_0^1 \frac{1}{4y-1}\,dy$ diverges.

35. $\int_0^\pi \sec x\,dx = \int_0^{\pi/2}\sec x\,dx + \int_{\pi/2}^{\pi}\sec x\,dx$. $\int_0^{\pi/2}\sec x\,dx = \lim\limits_{t\to\pi/2^-}\int_0^t \sec x\,dx$

$$= \lim_{t\to\pi/2^-}\left[\ln|\sec x+\tan x|\right]_0^t = \lim_{t\to\pi/2^-}\ln|\sec t+\tan t| = \infty. \text{ Divergent}$$

36. $\displaystyle\int_0^4 \frac{dx}{x^2+x-6} = \int_0^4 \frac{dx}{(x+3)(x-2)} = \int_0^2 \frac{dx}{(x-2)(x+3)} + \int_2^4 \frac{dx}{(x-2)(x+3)}$, and

$$\int_0^2 \frac{dx}{(x-2)(x+3)} = \lim_{t\to 2^-}\int_0^t\left[\frac{1/5}{x-2}-\frac{1/5}{x+3}\right]dx = \lim_{t\to 2^-}\left[\frac{1}{5}\ln\left|\frac{x-2}{x+3}\right|\right]_0^t$$
$$= \lim_{t\to 2^-}\frac{1}{5}\left[\ln\left|\frac{t-2}{t+3}\right| - \ln\frac{2}{3}\right] = -\infty. \text{ Divergent}$$

37. $\displaystyle\int_{-2}^2 \frac{dx}{x^2-1} = \int_{-2}^{-1}\frac{dx}{x^2-1} + \int_{-1}^{0}\frac{dx}{x^2-1} + \int_0^1\frac{dx}{x^2-1} + \int_1^2\frac{dx}{x^2-1}$,

and $\displaystyle\int\frac{dx}{x^2-1} = \int\frac{dx}{(x-1)(x+1)} = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$, so

$\displaystyle\int_0^1\frac{dx}{x^2-1} = \lim_{t\to 1^-}\left[\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\right]_0^t = \lim_{t\to 1^-}\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| = -\infty$. Divergent

38. $\displaystyle\int_0^2\frac{x-3}{2x-3}\,dx = \int_0^{3/2}\frac{x-3}{2x-3}\,dx + \int_{3/2}^2\frac{x-3}{2x-3}\,dx$ and

$\displaystyle\int\frac{x-3}{2x-3}\,dx = \frac{1}{2}\int\frac{2x-6}{2x-3}\,dx = \frac{1}{2}\int\left[1-\frac{3}{2x-3}\right]dx = \tfrac{1}{2}x - \tfrac{3}{4}\ln|2x-3| + C$, so

$\displaystyle\int_0^{3/2}\frac{x-3}{2x-3}\,dx = \lim_{t\to 3/2^-}\tfrac{1}{4}\left[2x-3\ln|2x-3|\right]_0^t = \infty$. Divergent

39. $I = \int_0^2 z^2\ln z\,dz = \lim\limits_{t\to 0^+}\int_t^2 z^2\ln z\,dz \overset{101}{=} \lim\limits_{t\to 0^+}\left[\dfrac{z^3}{3^2}(3\ln z - 1)\right]_t^2$

$= \lim\limits_{t\to 0^+}\left[\tfrac{8}{9}(3\ln 2 - 1) - \tfrac{1}{9}t^3(3\ln t - 1)\right] = \tfrac{8}{3}\ln 2 - \tfrac{8}{9} - \tfrac{1}{9}\lim\limits_{t\to 0^+}\left[t^3(3\ln t - 1)\right] = \tfrac{8}{3}\ln 2 - \tfrac{8}{9} - \tfrac{1}{9}L$

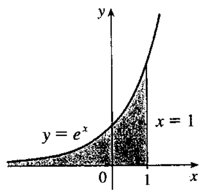
Now $L = \lim\limits_{t\to 0^+}\left[t^3(3\ln t - 1)\right] = \lim\limits_{t\to 0^+}\dfrac{3\ln t - 1}{t^{-3}} \overset{H}{=} \lim\limits_{t\to 0^+}\dfrac{3/t}{-3/t^4} = \lim\limits_{t\to 0^+}\left(-t^3\right) = 0$. Thus, $L = 0$ and $I = \tfrac{8}{3}\ln 2 - \tfrac{8}{9}$.

40. Integrate by parts with $u = \ln x$, $dv = dx\,/\sqrt{x}$, $du = dx/x$, $v = 2\sqrt{x}$:

$$\int_0^1\frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1\frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\left(\left[2\sqrt{x}\ln x\right]_t^1 - 2\int_t^1\frac{dx}{\sqrt{x}}\right) = \lim_{t\to 0^+}\left(-2\sqrt{t}\ln t - 4\left[\sqrt{x}\right]_t^1\right)$$
$$= \lim_{t\to 0^+}\left(-2\sqrt{t}\ln t - 4 + 4\sqrt{t}\right) = -4$$

since, by l'Hospital's Rule, $\lim\limits_{t\to 0^+}\sqrt{t}\ln t = \lim\limits_{t\to 0^+}\dfrac{\ln t}{t^{-1/2}} = \lim\limits_{t\to 0^+}\dfrac{1/t}{-t^{-3/2}/2} = \lim\limits_{t\to 0^+}\left(-2\sqrt{t}\right) = 0$.

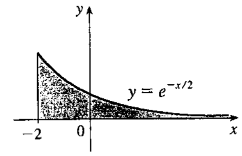
41.



Area $= \int_{-\infty}^{1} e^x\,dx = \lim_{t\to-\infty} [e^x]_t^1$

$= e - \lim_{t\to-\infty} e^t = e$

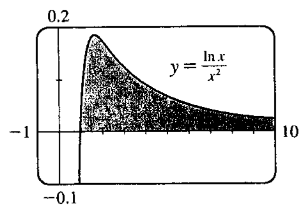
42.



Area $= \int_{-2}^{\infty} e^{-x/2}\,dx = -2\lim_{t\to\infty}\left[e^{-x/2}\right]_{-2}^{t}$

$= -2\lim_{t\to\infty} e^{-t/2} + 2e = 2e$

43.

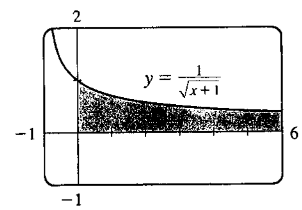


We integrate by parts with $u = \ln x$, $dv = (1/x^2)\,dx$, $du = dx/x$, $v = -1/x$:

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left(\left[-\frac{1}{x}\ln x\right]_1^t + \int_1^t \frac{1}{x^2}\,dx\right)$$

$$= \lim_{t\to\infty}\left(-\frac{\ln t}{t} - \frac{1}{t} + 1\right) = 1$$

44.

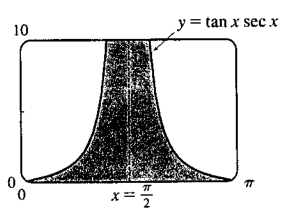


$$\text{Area} = \int_0^{\infty} \frac{dx}{\sqrt{x+1}} = \lim_{t\to\infty}\left[2\sqrt{x+1}\right]_0^t$$

$= \infty$

so the area is infinite.

45.



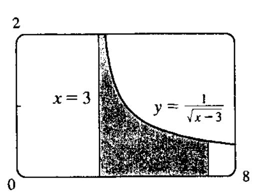
$\int_0^{\pi} \tan x \sec x\,dx = \int_0^{\pi/2} \tan x \sec x\,dx + \int_{\pi/2}^{\pi} \tan x \sec x\,dx.$

$\int_0^{\pi/2} \tan x \sec x\,dx = \lim_{t\to(\pi/2)^-} \int_0^t \tan x \sec x\,dx$

$= \lim_{t\to(\pi/2)^-} [\sec x]_0^t = \lim_{t\to(\pi/2)^-} (\sec t - 1)$

$= \infty$. Divergent

46.



$$\text{Area} = \int_3^7 \frac{dx}{\sqrt{x-3}} = \lim_{t\to3^+}\left[2\sqrt{x-3}\right]_t^7$$

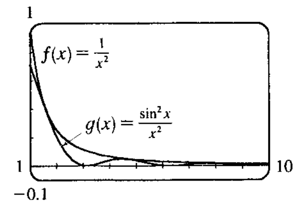
$= 4 - \lim_{t\to3^+} 2\sqrt{t-3} = 4 - 0 = 4$

47. (a)

t	$\int_1^t g(x)\,dx$
2	0.447453
5	0.577101
10	0.621306
100	0.668479
1000	0.672957
10,000	0.673407

$g(x) = \dfrac{\sin^2 x}{x^2}$. It appears that the integral is convergent.

(c)



Since $\int_1^\infty f(x)\,dx$ is finite and the area under $g(x)$ is less than the area under $f(x)$ on any interval $[1, t]$, $\int_1^\infty g(x)\,dx$ must be finite; that is, the integral is convergent.

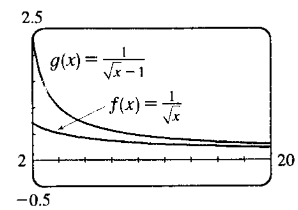
(b) $-1 \le \sin x \le 1 \;\Rightarrow\; 0 \le \sin^2 x \le 1 \;\Rightarrow\; 0 \le \dfrac{\sin^2 x}{x^2} \le \dfrac{1}{x^2}$. Since $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent (Equation 2 with $p = 2 > 1$), $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent by the Comparison Theorem.

48. (a)

t	$\int_2^t g(x)\,dx$
5	3.830327
10	6.801200
100	23.328769
1000	69.023361
10,000	208.124560

$g(x) = \dfrac{1}{\sqrt{x} - 1}$. It appears that the integral is divergent.

(c)



Since $\int_2^\infty f(x)\,dx$ is infinite and the area under $g(x)$ is greater than the area under $f(x)$ on any interval $[2, t]$, $\int_2^\infty g(x)\,dx$ must be infinite; that is, the integral is divergent.

(b) For $x \ge 2$, $\sqrt{x} > \sqrt{x} - 1 \;\Rightarrow\; \dfrac{1}{\sqrt{x}} < \dfrac{1}{\sqrt{x} - 1}$. Since $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}}\,dx$ is divergent (Equation 2 with $p = \frac{1}{2} \le 1$), $\displaystyle\int_2^\infty \frac{1}{\sqrt{x} - 1}\,dx$ is divergent by the Comparison Theorem.

49. For $x \ge 1$, $\dfrac{\cos^2 x}{1 + x^2} \le \dfrac{1}{1 + x^2} < \dfrac{1}{x^2}$. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by Equation 2 with $p = 2 > 1$, so $\displaystyle\int_1^\infty \frac{\cos^2 x}{1 + x^2}\,dx$ is convergent by the Comparison Theorem.

50. $\dfrac{1}{\sqrt{x^3 + 1}} \le \dfrac{1}{x^{3/2}}$ on $[1, \infty)$. $\displaystyle\int_1^\infty \frac{dx}{x^{3/2}}$ converges by Equation 2 with $p = \frac{3}{2} > 1$, so $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^3 + 1}}$ converges by the Comparison Theorem.

51. For $x \geq 1$, $x + e^{2x} > e^{2x} > 0 \Rightarrow \dfrac{1}{x + e^{2x}} \leq \dfrac{1}{e^{2x}} = e^{-2x}$ on $[1, \infty)$.

$\int_1^\infty e^{-2x}\,dx = \lim\limits_{t\to\infty}\left[-\frac{1}{2}e^{-2x}\right]_1^t = \lim\limits_{t\to\infty}\left[-\frac{1}{2}e^{-2t} + \frac{1}{2}e^{-2}\right] = \frac{1}{2}e^{-2}$. Therefore, $\int_1^\infty e^{-2x}\,dx$ is convergent, and by the Comparison Theorem, $\displaystyle\int_1^\infty \frac{dx}{x + e^{2x}}$ is also convergent.

52. $\dfrac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} > \dfrac{1}{\sqrt{x}}$ on $[1, \infty)$. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}}$ is divergent by Equation 2 with $p = \frac{1}{2} \leq 1$, so $\displaystyle\int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$ is divergent by the Comparison Theorem.

53. $\dfrac{1}{x\sin x} \geq \dfrac{1}{x}$ on $\left(0, \frac{\pi}{2}\right]$ since $0 \leq \sin x \leq 1$. $\displaystyle\int_0^{\pi/2} \frac{dx}{x} = \lim_{t\to 0^+}\int_t^{\pi/2}\frac{dx}{x} = \lim_{t\to 0^+}\left[\ln x\right]_t^{\pi/2}$.

But $\ln t \to -\infty$ as $t \to 0^+$, so $\displaystyle\int_0^{\pi/2}\frac{dx}{x}$ is divergent, and by the Comparison Theorem, $\displaystyle\int_0^{\pi/2}\frac{dx}{x\sin x}$ is also divergent.

54. $e^{-x} \leq 1 \Rightarrow \dfrac{e^{-x}}{\sqrt{x}} \leq \dfrac{1}{\sqrt{x}}$ for $0 \leq x \leq 1$, and $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\left(2 - 2\sqrt{t}\right) = 2$ is convergent. Therefore, $\displaystyle\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$ is convergent by the Comparison Theorem.

55.
$$\begin{aligned}
\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} &= \int_0^1 \frac{dx}{\sqrt{x}\,(1+x)} + \int_1^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{\sqrt{x}\,(1+x)} + \lim_{t\to\infty}\int_1^t \frac{dx}{\sqrt{x}\,(1+x)} \\
&= \lim_{t\to 0^+}\int_{\sqrt{t}}^1 \frac{2\,du}{1+u^2} + \lim_{t\to\infty}\int_1^{\sqrt{t}}\frac{2\,du}{1+u^2} \quad (u = \sqrt{x}, x = u^2) \\
&= \lim_{t\to 0^+}\left[2\tan^{-1}u\right]_{\sqrt{t}}^1 + \lim_{t\to\infty}\left[2\tan^{-1}u\right]_1^{\sqrt{t}} \\
&= \lim_{t\to 0^+}\left[2\left(\tfrac{\pi}{4}\right) - 2\tan^{-1}\sqrt{t}\right] + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{t} - 2\left(\tfrac{\pi}{4}\right)\right] = \tfrac{\pi}{2} - 0 + 2\left(\tfrac{\pi}{2}\right) - \tfrac{\pi}{2} = \pi
\end{aligned}$$

56. $\displaystyle\int_2^\infty \frac{dx}{x\sqrt{x^2-4}} = \int_2^3 \frac{dx}{x\sqrt{x^2-4}} + \int_3^\infty \frac{dx}{x\sqrt{x^2-4}} = \lim_{t\to 2^+}\int_t^3 \frac{dx}{x\sqrt{x^2-4}} + \lim_{t\to\infty}\int_3^t \frac{dx}{x\sqrt{x^2-4}}$, and

$\displaystyle\int \frac{dx}{x\sqrt{x^2-4}} = \int \frac{2\sec\theta\tan\theta\,d\theta}{2\sec\theta\,2\tan\theta} \quad (x = 2\sec\theta) \quad = \tfrac{1}{2}\theta + C = \tfrac{1}{2}\sec^{-1}\left(\tfrac{1}{2}x\right) + C$, so

$$\begin{aligned}
\int_2^\infty \frac{dx}{x\sqrt{x^2-4}} &= \lim_{t\to 2^+}\left[\tfrac{1}{2}\sec^{-1}\left(\tfrac{1}{2}x\right)\right]_t^3 + \lim_{t\to\infty}\left[\tfrac{1}{2}\sec^{-1}\left(\tfrac{1}{2}x\right)\right]_3^t \\
&= \tfrac{1}{2}\sec^{-1}\left(\tfrac{3}{2}\right) - 0 + \tfrac{1}{2}\left(\tfrac{\pi}{2}\right) - \tfrac{1}{2}\sec^{-1}\left(\tfrac{3}{2}\right) = \tfrac{\pi}{4}
\end{aligned}$$

57. If $p = 1$, then $\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+} [\ln x]_t^1 = \infty$. Divergent. If $p \neq 1$, then

$$\begin{aligned}\int_0^1 \frac{dx}{x^p} &= \lim_{t\to 0^+} \int_t^1 \frac{dx}{x^p} \quad \text{(note that the integral is not improper if } p < 0\text{)}\\ &= \lim_{t\to 0^+} \left[\frac{x^{-p+1}}{-p+1}\right]_t^1 = \lim_{t\to 0^+} \frac{1}{1-p}\left[1 - \frac{1}{t^{p-1}}\right]\end{aligned}$$

If $p > 1$, then $p - 1 > 0$, so $\frac{1}{t^{p-1}} \to \infty$ as $t \to 0^+$, and the integral diverges. Finally, if $p < 1$, then $\int_0^1 \frac{dx}{x^p} = \frac{1}{1-p}\left[\lim_{t\to 0^+}\left(1 - t^{1-p}\right)\right] = \frac{1}{1-p}$. Thus, the integral converges if and only if $p < 1$, and in that case its value is $\frac{1}{1-p}$.

58. Let $u = \ln x$. Then $du = dx/x \Rightarrow \int_e^\infty \frac{dx}{x(\ln x)^p} = \int_1^\infty \frac{du}{u^p}$. By Example 4, this converges to $\frac{1}{p-1}$ if $p > 1$ and diverges otherwise.

59. First suppose $p = -1$. Then

$\int_0^1 x^p \ln x\, dx = \int_0^1 \frac{\ln x}{x}\, dx = \lim_{t\to 0^+} \int_t^1 \frac{\ln x}{x}\, dx = \lim_{t\to 0^+} \left[\frac{1}{2}(\ln x)^2\right]_t^1 = -\frac{1}{2}\lim_{t\to 0^+} (\ln t)^2 = -\infty$, so the integral diverges. Now suppose $p \neq -1$. Then integration by parts gives

$\int x^p \ln x\, dx = \frac{x^{p+1}}{p+1}\ln x - \int \frac{x^p}{p+1}\, dx = \frac{x^{p+1}}{p+1}\ln x - \frac{x^{p+1}}{(p+1)^2} + C$. If $p < -1$, then $p + 1 < 0$, so

$$\int_0^1 x^p \ln x\, dx = \lim_{t\to 0^+}\left[\frac{x^{p+1}}{p+1}\ln x - \frac{x^{p+1}}{(p+1)^2}\right]_t^1 = \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\left[t^{p+1}\left(\ln t - \frac{1}{p+1}\right)\right] = \infty.$$

If $p > -1$, then $p + 1 > 0$ and

$$\begin{aligned}\int_0^1 x^p \ln x\, dx &= \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\frac{\ln t - 1/(p+1)}{t^{-(p+1)}} \overset{\text{H}}{=} \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\frac{1/t}{-(p+1)t^{-(p+2)}}\\ &= \frac{-1}{(p+1)^2} + \frac{1}{(p+1)^2}\lim_{t\to 0^+} t^{p+1} = \frac{-1}{(p+1)^2}\end{aligned}$$

Thus, the integral converges to $-\frac{1}{(p+1)^2}$ if $p > -1$ and diverges otherwise.

60. For n a nonnegative integer, integration by parts with $u = x^{n+1}$, $dv = e^{-x}\,dx$, gives $\int x^{n+1}e^{-x}\,dx = -x^{n+1}e^{-x} + (n+1)\int x^n e^{-x}\,dx$, so

$$\int_0^\infty x^{n+1}e^{-x}\,dx = \lim_{t\to\infty}\int_0^t x^{n+1}e^{-x}\,dx = \lim_{t\to\infty}\left[-x^{n+1}e^{-x}\right]_0^t + (n+1)\int_0^\infty x^n e^{-x}\,dx$$

$$= \lim_{t\to\infty}\frac{-t^{n+1}}{e^t} + (n+1)\int_0^\infty x^n e^{-x}\,dx = (n+1)\int_0^\infty x^n e^{-x}\,dx$$

Now $\int_0^\infty x^0 e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_0^t = 1$, so $\int_0^\infty x^1 e^{-x}\,dx = 1\cdot 1 = 1$, $\int_0^\infty x^2 e^{-x}\,dx = 2\cdot 1 = 2$, and $\int_0^\infty x^3 e^{-x}\,dx = 3\cdot 2 = 6$. In general, we guess that $\int_0^\infty x^n e^{-x}\,dx = n! = 1\cdot 2\cdot 3\cdot\cdots\cdot n$, when n is a positive integer. (Since $0! = 1$, our guess holds for $n = 0$ too.) Our guess works for $n \le 3$. Suppose that $\int_0^\infty x^k e^{-x}\,dx = k!$ for some positive integer k. Then $\int_0^\infty x^{k+1}e^{-x}\,dx = (k+1)\int_0^\infty x^k e^{-x}\,dx = (k+1)\,k! = (k+1)!$, so the formula holds for $k+1$. By induction, the formula holds for all integers $n \ge 0$.

61. (a) $\int_{-\infty}^\infty x\,dx = \int_{-\infty}^0 x\,dx + \int_0^\infty x\,dx$, and $\int_0^\infty x\,dx = \lim_{t\to\infty}\int_0^t x\,dx = \lim_{t\to\infty}\left[\frac12 t^2 - \frac12(0^2)\right] = \infty$, so the integral is divergent.

(b) $\int_{-t}^t x\,dx = \left[\frac12 x^2\right]_{-t}^t = \frac12 t^2 - \frac12 t^2 = 0$, so $\lim_{t\to\infty}\int_{-t}^t x\,dx = 0$. Therefore, $\int_{-\infty}^\infty x\,dx \ne \lim_{t\to\infty}\int_{-t}^t x\,dx$.

62. Let $k = \dfrac{M}{2RT}$ so that $\bar{v} = \dfrac{4}{\sqrt{\pi}}k^{3/2}\displaystyle\int_0^\infty v^3 e^{-kv^2}\,dv$. Let I denote the integral and use parts to integrate I. Let $\alpha = v^2$, $d\beta = ve^{-kv^2}\,dv$, $d\alpha = 2v\,dv$, $\beta = -\dfrac{1}{2k}e^{-kv^2}$:

$$I = \lim_{t\to\infty}\left[-\frac{1}{2k}v^2 e^{-kv^2}\right]_0^t + \frac1k\int_0^\infty ve^{-kv^2}\,dv = -\frac{1}{2k}\lim_{t\to\infty}\left(t^2 e^{-kt^2}\right) + \frac1k\lim_{t\to\infty}\left[-\frac{1}{2k}e^{-kv^2}\right]_0^t$$

$$\overset{\text{H}}{=} -\frac{1}{2k}\cdot 0 - \frac{1}{2k^2}(0-1) = \frac{1}{2k^2}$$

Thus, $\bar{v} = \dfrac{4}{\sqrt{\pi}}k^{3/2}\cdot\dfrac{1}{2k^2} = \dfrac{2}{(k\pi)^{1/2}} = \dfrac{2}{[\pi M/(2RT)]^{1/2}} = \dfrac{2\sqrt{2}\sqrt{RT}}{\sqrt{\pi M}} = \sqrt{\dfrac{8RT}{\pi M}}$.

63. $$\text{Volume} = \int_1^\infty \pi\left(\frac1x\right)^2 dx = \pi\lim_{t\to\infty}\int_1^t\frac{dx}{x^2} = \pi\lim_{t\to\infty}\left[-\frac1x\right]_1^t = \pi\lim_{t\to\infty}\left(1-\frac1t\right) = \pi < \infty.$$

64. $$\text{Work} = \int_R^\infty\frac{GMm}{r^2}\,dr = \lim_{t\to\infty}\int_R^t\frac{GMm}{r^2}\,dr = \lim_{t\to\infty}GMm\left[\frac{-1}{r}\right]_R^t = GMm\lim_{t\to\infty}\left(\frac{-1}{t}+\frac1R\right) = \frac{GMm}{R},$$

where M = mass of earth $= 5.98\times 10^{24}$ kg, m = mass of satellite $= 10^3$ kg, R = radius of earth $= 6.37\times 10^6$ m, and G = gravitational constant $= 6.67\times 10^{-11}$ N·m²/kg.

Therefore, $\text{Work} = \dfrac{6.67\times 10^{-11}\cdot 5.98\times 10^{24}\cdot 10^3}{6.37\times 10^6} \approx 6.26\times 10^{10}$ J.

65. $\text{Work} = \displaystyle\int_R^\infty F\,dr = \lim_{t\to\infty}\int_R^t\frac{GmM}{r^2}\,dr = \lim_{t\to\infty}GmM\left(\frac1R-\frac1t\right) = \frac{GmM}{R}$. The initial kinetic energy provides the work, so $\frac12 mv_0^2 = \dfrac{GmM}{R} \Rightarrow v_0 = \sqrt{\dfrac{2GM}{R}}$.

66. $y(s) = \int_s^R \frac{2r}{\sqrt{r^2 - s^2}} x(r)\, dr$ and $x(r) = \frac{1}{2}(R - r)^2 \Rightarrow$

$$\begin{aligned} y(s) &= \lim_{t \to s^+} \int_t^R \frac{r(R-r)^2}{\sqrt{r^2 - s^2}}\, dr = \lim_{t \to s^+} \int_t^R \frac{r^3 - 2Rr^2 + R^2 r}{\sqrt{r^2 - s^2}}\, dr \\ &= \lim_{t \to s^+} \left[\int_t^R \frac{r^3\, dr}{\sqrt{r^2 - s^2}} - 2R \int_t^R \frac{r^2\, dr}{\sqrt{r^2 - s^2}} + R^2 \int_t^R \frac{r\, dr}{\sqrt{r^2 - s^2}} \right] \\ &= \lim_{t \to s^+} \left(I_1 - 2RI_2 + R^2 I_3 \right) = L \end{aligned}$$

For I_1: Let $u = \sqrt{r^2 - s^2} \Rightarrow u^2 = r^2 - s^2,\ r^2 = u^2 + s^2,\ 2r\, dr = 2u\, du$, so, omitting limits and constant of integration,

$$\begin{aligned} I_1 &= \int \frac{(u^2 + s^2)\, u}{u}\, du = \int \left(u^2 + s^2 \right) du = \tfrac{1}{3}u^3 + s^2 u = \tfrac{1}{3}u\left(u^2 + 3s^2 \right) \\ &= \tfrac{1}{3}\sqrt{r^2 - s^2}\left(r^2 - s^2 + 3s^2 \right) = \tfrac{1}{3}\sqrt{r^2 - s^2}\left(r^2 + 2s^2 \right) \end{aligned}$$

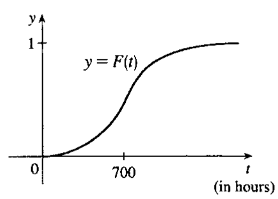
For I_2: Using Formula 44, $I_2 = \frac{r}{2}\sqrt{r^2 - s^2} + \frac{s^2}{2} \ln\left| r + \sqrt{r^2 - s^2} \right|$.

For I_3: Let $u = r^2 - s^2 \Rightarrow du = 2r\, dr$. Then $I_3 = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2} \cdot 2\sqrt{u} = \sqrt{r^2 - s^2}$.

Thus,

$$\begin{aligned} L &= \lim_{t \to s^+} \left[\tfrac{1}{3}\sqrt{r^2 - s^2}\left(r^2 + 2s^2 \right) - 2R\left(\frac{r}{2}\sqrt{r^2 - s^2} + \frac{s^2}{2} \ln\left| r + \sqrt{r^2 - s^2} \right| \right) + R^2\sqrt{r^2 - s^2} \right]_t^R \\ &= \lim_{t \to s^+} \left[\tfrac{1}{3}\sqrt{R^2 - s^2}\left(R^2 + 2s^2 \right) - 2R\left(\frac{R}{2}\sqrt{R^2 - s^2} + \frac{s^2}{2} \ln\left| R + \sqrt{R^2 - s^2} \right| \right) + R^2\sqrt{R^2 - s^2} \right] \\ &\qquad - \lim_{t \to s^+} \left[\tfrac{1}{3}\sqrt{t^2 - s^2}\left(t^2 + 2s^2 \right) - 2R\left(\frac{t}{2}\sqrt{t^2 - s^2} + \frac{s^2}{2} \ln\left| t + \sqrt{t^2 - s^2} \right| \right) + R^2\sqrt{t^2 - s^2} \right] \\ &= \left[\tfrac{1}{3}\sqrt{R^2 - s^2}\left(R^2 + 2s^2 \right) - Rs^2 \ln\left| R + \sqrt{R^2 - s^2} \right| \right] - \left[-Rs^2 \ln|s| \right] \\ &= \tfrac{1}{3}\sqrt{R^2 - s^2}\left(R^2 + 2s^2 \right) - Rs^2 \ln\left(\frac{R + \sqrt{R^2 - s^2}}{s} \right) \end{aligned}$$

67. (a)



(b) $r(t) = F'(t)$ is the rate at which the fraction $F(t)$ of burnt-out bulbs increases as t increases. This could be interpreted as a fractional burnout rate.

(c) $\int_0^\infty r(t)\, dt = \lim_{x \to \infty} F(x) = 1$, since all of the bulbs will eventually burn out.

68. $I = \int_0^\infty te^{kt}\, dt = \lim_{s \to \infty} \left[\frac{1}{k^2}(kt - 1)e^{kt} \right]_0^s$ (Formula 96, or parts) $= \lim_{s \to \infty} \left[\left(\frac{1}{k}se^{ks} - \frac{1}{k^2}e^{ks} \right) - \left(-\frac{1}{k^2} \right) \right]$.
Since $k < 0$ the first two terms approach 0 (you can verify that the first term does so with l'Hospital's Rule) so the whole expression is equal to $1/k^2$. Thus, $M = -kI = -k\left(1/k^2 \right) = -1/k = -1/(-0.000121) \approx 8264.5$ years.

69. $I = \int_a^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_a^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \left[\tan^{-1} x\right]_a^t = \lim_{t\to\infty} \left(\tan^{-1} t - \tan^{-1} a\right) = \frac{\pi}{2} - \tan^{-1} a.$

$I < 0.001 \quad\Rightarrow\quad \frac{\pi}{2} - \tan^{-1} a < 0.01 \quad\Rightarrow\quad \tan^{-1} a > \frac{\pi}{2} - 0.001 \quad\Rightarrow\quad a > \tan\left(\frac{\pi}{2} - 0.001\right) \approx 1000.$

70. $f(x) = e^{-x^2}$ and $\Delta x = \frac{4-0}{8} = \frac{1}{2}$.

$$\begin{aligned}\textstyle\int_0^4 f(x)\,dx \approx S_8 &= \tfrac{1}{2\cdot 3}\left[f(0) + 4f(0.5) + 2f(1) + \cdots + 2f(3) + 4f(3.5) + f(4)\right]\\ &\approx \tfrac{1}{6}(5.31717808) \approx 0.8862\end{aligned}$$

Now $x > 4 \quad\Rightarrow\quad -x \cdot x < -x \cdot 4 \quad\Rightarrow\quad e^{-x^2} < e^{-4x} \quad\Rightarrow\quad \int_4^\infty e^{-x^2}\,dx < \int_4^\infty e^{-4x}\,dx.$

$\int_4^\infty e^{-4x}\,dx = \lim_{t\to\infty}\left[-\frac{1}{4}e^{-4x}\right]_4^t = -\frac{1}{4}\left(0 - e^{-16}\right) = 1/\left(4e^{16}\right) \approx 0.0000000281 < 0.0000001$, as desired.

71. (a) $F(s) = \int_0^\infty f(t)\,e^{-st}\,dt = \int_0^\infty e^{-st}\,dt = \lim_{n\to\infty}\left[-\frac{e^{-st}}{s}\right]_0^n = \lim_{n\to\infty}\left(\frac{e^{-sn}}{-s} + \frac{1}{s}\right)$. This converges to $\frac{1}{s}$ only if $s > 0$. Therefore $F(s) = \frac{1}{s}$ with domain $\{s \mid s > 0\}$.

(b) $F(s) = \int_0^\infty f(t)\,e^{-st}\,dt = \int_0^\infty e^t e^{-st}\,dt = \lim_{n\to\infty}\int_0^n e^{t(1-s)}\,dt = \lim_{n\to\infty}\left[\frac{1}{1-s}e^{t(1-s)}\right]_0^n$

$= \lim_{n\to\infty}\left(\frac{e^{(1-s)n}}{1-s} - \frac{1}{1-s}\right)$

This converges only if $1 - s < 0 \quad\Rightarrow\quad s > 1$, in which case $F(s) = \frac{1}{s-1}$ with domain $\{s \mid s > 1\}$.

(c) $F(s) = \int_0^\infty f(t)\,e^{-st}\,dt = \lim_{n\to\infty}\int_0^n te^{-st}\,dt$. Use integration by parts: let $u = t$, $dv = e^{-st}\,dt \quad\Rightarrow$ $du = dt$, $v = -\frac{e^{-st}}{s}$. Then $F(s) = \lim_{n\to\infty}\left[-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right]_0^n = \lim_{n\to\infty}\left(\frac{-n}{se^{sn}} - \frac{1}{s^2e^{sn}} + 0 + \frac{1}{s^2}\right) = \frac{1}{s^2}$ only if $s > 0$. Therefore, $F(s) = \frac{1}{s^2}$ and the domain of F is $\{s \mid s > 0\}$.

72. $0 \le f(t) \le Me^{at} \quad\Rightarrow\quad 0 \le f(t)\,e^{-st} \le Me^{at}e^{-st}$ for $t \ge 0$. Now use the Comparison Theorem:

$$\int_0^\infty Me^{at}e^{-st}\,dt = \lim_{n\to\infty} M\int_0^n e^{t(a-s)}\,dt = M \cdot \lim_{n\to\infty}\left[\frac{1}{a-s}e^{t(a-s)}\right]_0^n = M \cdot \lim_{n\to\infty}\frac{1}{a-s}\left[e^{n(a-s)} - 1\right]$$

This is convergent only when $a - s < 0 \quad\Rightarrow\quad s > a$. Therefore, by the Comparison Theorem, $F(s) = \int_0^\infty f(t)\,e^{-st}\,dt$ is also convergent for $s > a$.

73. $G(s) = \int_0^\infty f'(t)\,e^{-st}\,dt$. Integrate by parts with $u = e^{-st}$, $dv = f'(t)\,dt \quad\Rightarrow\quad du = -se^{-st}$, $v = f(t)$:

$$G(s) = \lim_{n\to\infty}\left[f(t)\,e^{-st}\right]_0^n + s\int_0^\infty f(t)\,e^{-st}\,dt = \lim_{n\to\infty} f(n)\,e^{-sn} - f(0) + sF(s)$$

But $0 \le f(t) \le Me^{at} \quad\Rightarrow\quad 0 \le f(t)\,e^{-st} \le Me^{at}e^{-st}$ and $\lim_{t\to\infty} Me^{t(a-s)} = 0$ for $s > a$. So by the Squeeze Theorem, $\lim_{t\to\infty} f(t)\,e^{-st} = 0$ for $s > a \quad\Rightarrow\quad G(s) = 0 - f(0) + sF(s) = sF(s) - f(0)$ for $s > a$.

74. Assume without loss of generality that $a < b$. Then

$$\begin{aligned}
\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx &= \lim_{t\to-\infty}\int_{t}^{a} f(x)\,dx + \lim_{u\to\infty}\int_{a}^{u} f(x)\,dx \\
&= \lim_{t\to-\infty}\int_{t}^{a} f(x)\,dx + \lim_{u\to\infty}\left[\int_{a}^{b} f(x)\,dx + \int_{b}^{u} f(x)\,dx\right] \\
&= \lim_{t\to-\infty}\int_{t}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx + \lim_{u\to\infty}\int_{b}^{u} f(x)\,dx \\
&= \lim_{t\to-\infty}\left[\int_{t}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx\right] + \int_{b}^{\infty} f(x)\,dx \\
&= \lim_{t\to-\infty}\int_{t}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx
\end{aligned}$$

75. We use integration by parts: let $u = x$, $dv = xe^{-x^2}\,dx$, $du = dx$, $v = -\frac{1}{2}e^{-x^2}$. So

$$\int_0^\infty x^2 e^{-x^2}\,dx = \lim_{t\to\infty}\left[-\tfrac{1}{2}xe^{-x^2}\right]_0^t + \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx = \lim_{t\to\infty} -t\Big/\left(2e^{t^2}\right) + \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx = \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx$$

(The limit is 0 by l'Hospital's Rule.)

76. $\int_0^\infty e^{-x^2}\,dx$ is the area under the curve $y = e^{-x^2}$ for $0 \le x < \infty$ and $0 < y \le 1$. Solving $y = e^{-x^2}$ for x, we get $y = e^{-x^2} \Rightarrow \ln y = -x^2 \Rightarrow -\ln y = x^2 \Rightarrow x = \pm\sqrt{-\ln y}$. Since x is positive, choose $x = \sqrt{-\ln y}$, and the area is represented by $\int_0^1 \sqrt{-\ln y}\,dy$. Therefore, each integral represents the same area, so the integrals are equal.

77. For the first part of the integral, let $x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta\,d\theta$.
$\displaystyle\int \frac{1}{\sqrt{x^2+4}}\,dx = \int \sec\theta = \ln|\sec\theta + \tan\theta|$. But $\tan\theta = \frac{1}{2}x$, and
$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\frac{1}{4}x^2} = \frac{1}{2}\sqrt{x^2+4}$. So

$$\begin{aligned}
\int_0^\infty \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2}\right)dx &= \lim_{t\to\infty}\left[\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| - C\ln|x+2|\right]_0^t \\
&= \lim_{t\to\infty}\ln\left(\frac{\sqrt{t^2+4}+t}{2\,(t+2)^C}\right) - (\ln 1 - C\ln 2) \\
&= \ln\left(\lim_{t\to\infty}\frac{t+\sqrt{t^2+4}}{(t+2)^C}\right) + \ln 2^{C-1}
\end{aligned}$$

By l'Hospital's Rule, $\displaystyle\lim_{t\to\infty}\frac{t+\sqrt{t^2+4}}{(t+2)^C} = \lim_{t\to\infty}\frac{1+t/\sqrt{t^2+4}}{C\,(t+2)^{C-1}} = \frac{2}{C\lim\limits_{t\to\infty}(t+2)^{C-1}}$.

If $C < 1$, we get ∞ and the interval diverges. If $C = 1$, we get 2, so the original integral converges to $\ln 2 + \ln 2^0 = \ln 2$. If $C > 1$, we get 0, so the original integral diverges to $-\infty$.

78. $\displaystyle\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1}\right) dx = \lim_{t\to\infty}\left[\tfrac{1}{2}\ln(x^2+1) - \tfrac{1}{3}C\ln(3x+1)\right]_0^t$

$$= \lim_{t\to\infty}\left[\ln\left(t^2+1\right)^{1/2} - \ln(3t+1)^{C/3}\right]$$

$$= \lim_{t\to\infty}\left(\ln\frac{\left(t^2+1\right)^{1/2}}{(3t+1)^{C/3}}\right) = \ln\left(\lim_{t\to\infty}\frac{\sqrt{t^2+1}}{(3t+1)^{C/3}}\right)$$

Clearly the integral diverges for $C \le 0$. For $C > 0$, we use l'Hospital's Rule and get

$$\ln\left(\lim_{t\to\infty}\frac{t\Big/\sqrt{t^2+1}}{C\,(3t+1)^{(C/3)-1}}\right) = \ln\left(\frac{1}{C}\lim_{t\to\infty}\frac{1}{(3t+1)^{(C/3)-1}}\right)$$

For $C/3 < 1 \iff C < 3$, the integral diverges. For $C = 3$, $\ln\left(\frac{1}{3}\lim_{t\to\infty}\frac{1}{1}\right) = \ln\frac{1}{3}$. For $C > 3$, the limit is 0, so the integral diverges to $-\infty$.

8 Review

CONCEPT CHECK

1. See Formula 8.1.1 or 8.1.2. We try to choose $u = f(x)$ to be a function that becomes simpler when differentiated (or at least not more complicated) as long as $dv = g'(x)\,dx$ can be readily integrated to give v.
2. See the Strategy for Evaluating $\int \sin^m x \cos^n x\,dx$ on page 514.
3. If $\sqrt{a^2-x^2}$ occurs, try $x = a\sin\theta$; if $\sqrt{a^2+x^2}$ occurs, try $x = a\tan\theta$, and if $\sqrt{x^2-a^2}$ occurs, try $x = a\sec\theta$. See the Table of Trigonometric Substitutions on page 518.
4. See Equation 2 and Expressions 7, 9, and 11 in Section 8.4.
5. See the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule, as well as their associated error bounds, all in Section 8.7. We would expect the best estimate to be given by Simpson's Rule.
6. See Definitions 1(a), (b), and (c) in Section 8.8.
7. See Definitions 3(b), (a), and (c) in Section 8.8.
8. See the Comparison Theorem on page 564.

TRUE-FALSE QUIZ

1. False. Since the numerator has a higher degree than the denominator,
$$\frac{x\left(x^2+4\right)}{x^2-4} = x + \frac{8x}{x^2-4} = x + \frac{A}{x+2} + \frac{B}{x-2}.$$
2. True. In fact, $A = -1$, $B = C = 1$.
3. False. It can be put in the form $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-4}$.
4. False. The form is $\dfrac{A}{x} + \dfrac{Bx+C}{x^2+4}$.

5. False. This is an improper integral, since the denominator vanishes at $x = 1$.

$$\int_0^4 \frac{x}{x^2-1}\,dx = \int_0^1 \frac{x}{x^2-1}\,dx + \int_1^4 \frac{x}{x^2-1}\,dx \text{ and}$$

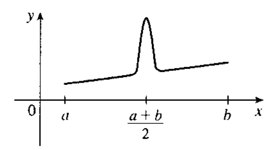
$$\int_0^1 \frac{x}{x^2-1} = \lim_{t\to 1^-}\int_0^t \frac{x}{x^2-1}\,dx = \lim_{t\to 1^-}\left[\tfrac{1}{2}\ln\left|x^2-1\right|\right]_0^t = \lim_{t\to 1^-}\tfrac{1}{2}\ln\left|t^2-1\right| = \infty$$

So the integral diverges.

6. True by Theorem 8.8.2 with $p = \sqrt{2} > 1$.

7. False. See Exercise 61 in Section 8.8.

8. False. For example, with $n = 1$ the Trapezoidal Rule is much more accurate than the Midpoint Rule for the function in the diagram.



9. False. Examples include the functions $f(x) = e^{x^2}$, $g(x) = \sin(x^2)$, and $h(x) = \dfrac{\sin x}{x}$.

10. False. $\int_0^\infty f(x)\,dx$ could converge or diverge. For example, if $g(x) = 1$, then $\int_0^\infty f(x)\,dx$ diverges if $f(x) = 1$ and converges if $f(x) = 0$.

EXERCISES

1. $\displaystyle\int \frac{6x+1}{3x+2}\,dx = \int\left(2 - \frac{3}{3x+2}\right)dx = 2x - 3\cdot\tfrac{1}{3}\ln|3x+2| + C = 2x - \ln|3x+2| + C$

2. Let $u = x$, $dv = \cos 3x\,dx \Rightarrow du = dx$, $v = \frac{1}{3}\sin 3x$. Then

$$\int x\cos 3x\,dx = \tfrac{1}{3}x\sin 3x - \tfrac{1}{3}\int \sin 3x\,dx = \tfrac{1}{3}x\sin 3x + \tfrac{1}{9}\cos 3x + C$$

3. $\int \cot^2 x\,dx = \int(\csc^2 x - 1)\,dx = -\cot x - x + C$

4. Let $u = \tan\theta$. Then $\displaystyle\int \frac{\sec^2\theta\,d\theta}{1-\tan\theta} = \int \frac{du}{1-u} = -\ln|1-u| + C = -\ln|1-\tan\theta| + C.$

5. Use integration by parts with $u = \ln x$, $dv = x^4\,dx \Rightarrow du = dx/x$, $v = x^5/5$:

$$\int x^4\ln x\,dx = \tfrac{1}{5}x^5\ln x - \tfrac{1}{5}\int x^4\,dx = \tfrac{1}{5}x^5\ln x - \tfrac{1}{25}x^5 + C = \tfrac{1}{25}x^5(5\ln x - 1) + C$$

6. $\dfrac{1}{y^2-4y-12} = \dfrac{1}{(y-6)(y+2)} = \dfrac{A}{y-6} + \dfrac{B}{y+2} \Rightarrow 1 = A(y+2) + B(y-6)$. Letting $y = -2 \Rightarrow$ $B = -\frac{1}{8}$ and letting $y = 6 \Rightarrow A = \frac{1}{8}$. So

$$\int \frac{1}{y^2-4y-12}\,dy = \int\left(\frac{1/8}{y-6} + \frac{-1/8}{y+2}\right)dy = \tfrac{1}{8}\ln|y-6| - \tfrac{1}{8}\ln|y+2| + C$$

7. Let $u = \sec x$. Then $du = \sec x\tan x\,dx$, so

$$\begin{aligned}\int \tan^7 x\sec^3 x\,dx &= \int \tan^6 x\sec^2 x\sec x\tan x\,dx = \int (u^2-1)^3 u^2\,du = \int (u^8 - 3u^6 + 3u^4 - u^2)\,du \\ &= \tfrac{1}{9}u^9 - \tfrac{3}{7}u^7 + \tfrac{3}{5}u^5 - \tfrac{1}{3}u^3 + C = \tfrac{1}{9}\sec^9 x - \tfrac{3}{7}\sec^7 x + \tfrac{3}{5}\sec^5 x - \tfrac{1}{3}\sec^3 x + C\end{aligned}$$

8. Let $x = \tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then

$$\int \frac{dx}{x^2\sqrt{1+x^2}} = \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta \sec\theta} = \int \frac{\sec\theta\, d\theta}{\tan^2\theta} = \int \frac{\cos\theta\, d\theta}{\sin^2\theta} = \int \frac{du}{u^2} \quad (\text{put } u = \sin\theta)$$
$$= -\frac{1}{u} + C = -\frac{1}{\sin\theta} + C = -\frac{\sqrt{1+x^2}}{x} + C$$

9. Let $u = x^2$. Then $du = 2x\,dx$, so $\int x \sin\left(x^2\right) dx = \frac{1}{2}\int \sin u\, du = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos\left(x^2\right) + C$.

10. Integrate by parts twice, first with $u = x^2$, $dv = e^{-3x}\,dx \quad \Rightarrow \quad du = 2x\,dx$, $v = -\frac{1}{3}e^{-3x}$:

$$\int x^2 e^{-3x}\,dx = -\tfrac{1}{3}x^2 e^{-3x} + \tfrac{2}{3}\int x e^{-3x}\,dx = -\tfrac{1}{3}x^2 e^{-3x}\,dx + \tfrac{2}{3}\left(-\tfrac{1}{3}xe^{-3x} + \tfrac{1}{3}\int e^{-3x}\,dx\right)$$
$$= -\left(\tfrac{1}{3}x^2 + \tfrac{2}{9}x + \tfrac{2}{27}\right)e^{-3x} + C$$

11. $\displaystyle\int \frac{dx}{x^3+x} = \int \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx = \ln|x| - \tfrac{1}{2}\ln\left(x^2+1\right) + C$

12. $\displaystyle\int \frac{x^2+2}{x+2}\,dx = \int \left(x - 2 + \frac{6}{x+2}\right) dx = \tfrac{1}{2}x^2 - 2x + 6\ln|x+2| + C$

13. $\int \sin^2\theta \cos^5\theta\, d\theta = \int \sin^2\theta \left(\cos^2\theta\right)^2 \cos\theta\, d\theta = \int \sin^2\theta \left(1 - \sin^2\theta\right)^2 \cos\theta\, d\theta$

$$= \int u^2\left(1-u^2\right)^2 du \quad [u = \sin\theta,\ du = \cos\theta\, d\theta] \quad = \int u^2\left(1 - 2u^2 + u^4\right) du$$
$$= \int \left(u^2 - 2u^4 + u^6\right) du = \tfrac{1}{3}u^3 - \tfrac{2}{5}u^5 + \tfrac{1}{7}u^7 + C = \tfrac{1}{3}\sin^3\theta - \tfrac{2}{5}\sin^5\theta + \tfrac{1}{7}\sin^7\theta + C$$

14. Let $u = \cos x$. Then

$$\int \frac{\sin^3 x}{\cos x}\,dx = \int \frac{\left(1-\cos^2 x\right)\sin x}{\cos x}\,dx = -\int \frac{1-u^2}{u}\,du = \int \left(u - \frac{1}{u}\right) du = \frac{u^2}{2} - \ln|u| + C$$
$$= \tfrac{1}{2}\cos^2 x - \ln|\cos x| + C$$

15. Integrate by parts with $u = x$, $dv = \sec x \tan x\, dx \quad \Rightarrow \quad du = dx$, $v = \sec x$:

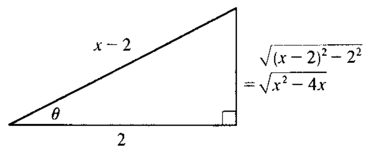
$$\int x \sec x \tan x\, dx = x \sec x - \int \sec x\, dx = x\sec x - \ln|\sec x + \tan x| + C$$

16. $\displaystyle\int \frac{dx}{x^3 - 2x^2 + x} = \int \frac{dx}{(x-1)^2 x} = \int \left[\frac{-1}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{x}\right] dx = -\frac{1}{x-1} + \ln\left|\frac{x}{x-1}\right| + C$

17. Let $u = \arctan x$. Then $du = \dfrac{dx}{1+x^2}$, so $\displaystyle\int \frac{(\arctan x)^5}{1+x^2}\,dx = \int u^5\,du = \tfrac{1}{6}u^6 + C = \tfrac{1}{6}(\arctan x)^6 + C$.

18. $\displaystyle\int \frac{dt}{\sin^2 t + \cos 2t} = \int \frac{dt}{\sin^2 t + \left(\cos^2 t - \sin^2 t\right)} = \int \frac{dt}{\cos^2 t} = \int \sec^2 t\, dt = \tan t + C$

19.



$$\int \frac{dx}{\sqrt{x^2-4x}} = \int \frac{dx}{\sqrt{(x^2-4x+4)-4}} = \int \frac{dx}{\sqrt{(x-2)^2-2^2}}$$

$$= \int \frac{2\sec\theta\tan\theta\,d\theta}{2\tan\theta} \quad \left[\begin{array}{c} x-2=2\sec\theta, \\ dx=2\sec\theta\tan\theta\,d\theta \end{array}\right] = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C_1$$

$$= \ln\left|\frac{x-2}{2} + \frac{\sqrt{x^2-4x}}{2}\right| + C_1 = \ln\left|x-2+\sqrt{x^2-4x}\right| + C, \text{ where } C = C_1 - \ln 2$$

20. Let $u = x + 1$. Then

$$\int \frac{x^3}{(x+1)^{10}}\,dx = \int \frac{(u-1)^3}{u^{10}}\,dx = \int \frac{u^3-3u^2+3u-1}{u^{10}}\,du$$

$$= \int \left(u^{-7} - 3u^{-8} + 3u^{-9} - u^{-10}\right) du = -\tfrac{1}{6}u^{-6} + \tfrac{3}{7}u^{-7} - \tfrac{3}{8}u^{-8} + \tfrac{1}{9}u^{-9} + C$$

$$= \frac{-1}{6(x+1)^6} + \frac{3}{7(x+1)^7} - \frac{3}{8(x+1)^8} + \frac{1}{9(x+1)^9} + C$$

21. Let $u = \cot 4x$. Then $du = -4\csc^2 4x\,dx \quad \Rightarrow$

$$\int \csc^4 4x\,dx = \int \left(\cot^2 4x + 1\right)\csc^2 4x\,dx = \int \left(u^2+1\right)\left(-\tfrac{1}{4}\,du\right)$$

$$= -\tfrac{1}{4}\left(\tfrac{1}{3}u^3 + u\right) + C = -\tfrac{1}{12}\left(\cot^3 4x + 3\cot 4x\right) + C$$

22. Let $u = 2x$. Then

$$\int x\sin^2 x\,dx = \tfrac{1}{2}\int x\,(1-\cos 2x)\,dx = \tfrac{1}{4}x^2 - \tfrac{1}{8}\int 2x\cos 2x 2\,dx = \tfrac{1}{4}x^2 - \tfrac{1}{8}\int u\cos u\,du$$

$$= \tfrac{1}{4}x^2 - \tfrac{1}{8}\left(u\sin u + \cos u\right) + C = \tfrac{1}{4}x^2 - \tfrac{1}{4}x\sin 2x - \tfrac{1}{8}\cos 2x + C$$

23. Let $u = \ln x$. Then $\displaystyle\int \frac{\ln(\ln x)}{x}\,dx = \int \ln u\,du$. Now use parts with $w = \ln u$, $dv = du \quad \Rightarrow \quad dw = du/u$, $v = u \quad \Rightarrow \quad \int \ln u\,du = u\ln u - u + C = (\ln x)\left[\ln(\ln x) - 1\right] + C$.

24. Let $u = \cos x$, $dv = e^x\,dx \quad \Rightarrow \quad du = -\sin x\,dx$, $v = e^x$: $(*)\ I = \int e^x\cos x\,dx = e^x\cos x + \int e^x\sin x\,dx$. To integrate $\int e^x\sin x\,dx$, let $U = \sin x$, $dV = e^x\,dx \quad \Rightarrow \quad dU = \cos x\,dx$, $V = e^x$. Then $\int e^x\sin x\,dx = e^x\sin x - \int e^x\cos x\,dx = e^x\sin x - I$. By substitution in $(*)$, $I = e^x\cos x + e^x\sin x - I \quad \Rightarrow$ $2I = e^x(\cos x + \sin x) \quad \Rightarrow \quad I = \tfrac{1}{2}e^x(\cos x + \sin x) + C$.

25. $\dfrac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+2} \quad\Rightarrow\quad 3x^3 - x^2 + 6x - 4 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1).$

Equating the coefficients gives $A + C = 3$, $B + D = -1$, $2A + C = 6$, and $2B + D = -4 \quad\Rightarrow\quad A = 3, C = 0,$ $B = -3$, and $D = 2$. Now

$$\begin{aligned}\int \frac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)}\,dx &= 3\int \frac{x-1}{x^2+1}\,dx + 2\int \frac{dx}{x^2+2} \\ &= \tfrac{3}{2}\ln\left(x^2+1\right) - 3\tan^{-1}x + \sqrt{2}\tan^{-1}\left(\tfrac{1}{\sqrt{2}}x\right) + C\end{aligned}$$

26. Let $u = e^x$. Then $x = \ln u$, $dx = \dfrac{du}{u}$, so

$$\begin{aligned}\int \frac{dx}{1+e^x} = \int \frac{du/u}{1+u} = \int \left[\frac{1}{u} - \frac{1}{u+1}\right] du &= \ln u - \ln(u+1) + C = \ln e^x - \ln\left(1+e^x\right) + C \\ &= x - \ln\left(1+e^x\right) + C\end{aligned}$$

27. Let $u = e^{2r} \quad\Rightarrow\quad du = 2e^{2r}\,dr$.

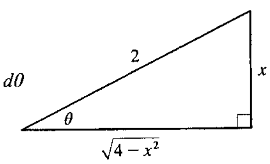
$$\begin{aligned}\int \frac{e^{2r}}{e^{4r}-1}\,dr = \frac{1}{2}\int \frac{du}{u^2-1} &= \frac{1}{2}\int \left[\frac{1}{2(u-1)} - \frac{1}{2(u+1)}\right] du \\ &= \frac{1}{4}\ln\left|\frac{u-1}{u+1}\right| + C = \frac{1}{4}\ln\left|\frac{e^{2r}-1}{e^{2r}+1}\right| + C\end{aligned}$$

28. Let $u = \sqrt[3]{x}$. Then $x = u^3$, $dx = 3u^2 du \quad\Rightarrow$

$$\begin{aligned}\int \frac{\sqrt[3]{x}+1}{\sqrt[3]{x}-1}\,dx = \int \frac{u+1}{u-1}\,3u^2 du &= 3\int \left(u^2 + 2u + 2 + \frac{2}{u-1}\right) du \\ &= u^3 + 3u^2 + 6u + 6\ln|u-1| + C = x + 3x^{2/3} + 6\sqrt[3]{x} + 6\ln\left|\sqrt[3]{x} - 1\right| + C\end{aligned}$$

29. Let $x = 2\sin\theta \quad\Rightarrow\quad \left(4 - x^2\right)^{3/2} = (2\cos\theta)^3$, $dx = 2\cos\theta\,d\theta$, so

$$\begin{aligned}\int \frac{x^2}{\left(4-x^2\right)^{3/2}}\,dx = \int \frac{4\sin^2\theta}{8\cos^3\theta}\,2\cos\theta\,d\theta &= \int \tan^2\theta\,d\theta = \int \left(\sec^2\theta - 1\right) d\theta \\ &= \tan\theta - \theta + C = \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C\end{aligned}$$



30. Integrate by parts twice, first with $u = (\arcsin x)^2$, $dv = dx$:

$$I = \int (\arcsin x)^2\,dx = x(\arcsin x)^2 - \int 2x\arcsin x\left(\frac{dx}{\sqrt{1-x^2}}\right)$$

Now let $U = \arcsin x$, $dV = \dfrac{x}{\sqrt{1-x^2}}\,dx \quad\Rightarrow\quad dU = \dfrac{1}{\sqrt{1-x^2}}\,dx$, $V = -\sqrt{1-x^2}$. So

$$I = x(\arcsin x)^2 - 2\left[\arcsin x\left(-\sqrt{1-x^2}\right) + \int dx\right] = x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C$$

31. $\int (\cos x + \sin x)^2 \cos 2x\,dx = \int \left(\cos^2 x + 2\sin x\cos x + \sin^2 x\right)\cos 2x\,dx$

$= \int (1 + \sin 2x)\cos 2x\,dx = \int \cos 2x\,dx + \frac{1}{2}\int \sin 4x\,dx = \frac{1}{2}\sin 2x - \frac{1}{8}\cos 4x + C$

Or: $\int (\cos x + \sin x)^2 \cos 2x\,dx = \int (\cos x + \sin x)^2\left(\cos^2 x - \sin^2 x\right) dx$

$= \int (\cos x + \sin x)^3(\cos x - \sin x)\,dx = \frac{1}{4}(\cos x + \sin x)^4 + C_1$

32. Let $u = \left(\tan^{-1} x\right)^2$, $dv = x\,dx \;\Rightarrow\; du = 2\left(\tan^{-1} x\right)/\left(1+x^2\right)dx$, $v = \frac{1}{2}x^2$. Then

$$I = \int x\left(\tan^{-1} x\right)^2 dx = \tfrac{1}{2}x^2\left(\tan^{-1} x\right)^2 - \int \frac{x^2 \tan^{-1} x}{1+x^2}\,dx$$

Now let $w = \tan^{-1} x$, $dw = 1/\left(1+x^2\right)dx$, and $x^2 = \tan^2 w$. So

$$\begin{aligned}
I &= \tfrac{1}{2}x^2\left(\tan^{-1} x\right)^2 - \textstyle\int w \tan^2 w\,dw = \tfrac{1}{2}x^2\left(\tan^{-1} x\right)^2 - \textstyle\int w \sec^2 w\,dw + \textstyle\int w\,dw \\
&= \tfrac{1}{2}x^2\left(\tan^{-1} x\right)^2 - \left(x\tan^{-1} x - \ln\sqrt{x^2+1}\right) + \tfrac{1}{2}\left(\tan^{-1} x\right)^2 \quad \text{[parts with } u = w,\ dv = \sec^2 w\,dw\text{]} \\
&= \tfrac{1}{2}\left(x^2+1\right)\left(\tan^{-1} x\right)^2 - x\tan^{-1} x + \ln\sqrt{x^2+1} + C \\
&\quad\text{or } \tfrac{1}{2}\left(x^2+1\right)\left(\tan^{-1} x\right)^2 - x\tan^{-1} x + \tfrac{1}{2}\ln\left(x^2+1\right) + C
\end{aligned}$$

33. $$\begin{aligned}\int_1^\infty \frac{1}{(2x+1)^3}\,dx &= \lim_{t\to\infty}\int_1^t \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\int_1^t \tfrac{1}{2}(2x+1)^{-3}\,2\,dx \\
&= \lim_{t\to\infty}\left[-\frac{1}{4(2x+1)^2}\right]_1^t = -\frac{1}{4}\lim_{t\to\infty}\left[\frac{1}{(2t+1)^2} - \frac{1}{9}\right] = -\frac{1}{4}\left(0 - \frac{1}{9}\right) = \frac{1}{36}\end{aligned}$$

34. $$\begin{aligned}\int_0^\infty \frac{dx}{(x+1)^2(x+2)} &= \lim_{t\to\infty}\int_0^t \left[\frac{1}{x+2} - \frac{1}{x+1} + \frac{1}{(x+1)^2}\right]dx = \lim_{t\to\infty}\left[\ln\left(\frac{x+2}{x+1}\right) - \frac{1}{x+1}\right]_0^t \\
&= 1 - \ln 2\end{aligned}$$

35. $$\begin{aligned}\int_0^4 \frac{\ln x}{\sqrt{x}}\,dx &= \lim_{t\to 0^+}\int_t^4 \frac{\ln x}{\sqrt{x}}\,dx \overset{*}{=} \lim_{t\to 0^+}\left[2\sqrt{x}\ln x - 4\sqrt{x}\right]_t^4 \\
&= \lim_{t\to 0^+}\left[(2\cdot 2\ln 4 - 4\cdot 2) - \left(2\sqrt{t}\ln t - 4\sqrt{t}\right)\right] \overset{**}{=} (4\ln 4 - 8) - (0 - 0) = 4\ln 4 - 8\end{aligned}$$

(∗) Let $u = \ln x$, $dv = \dfrac{1}{\sqrt{x}}\,dx \;\Rightarrow\; du = \dfrac{1}{x}\,dx$, $v = 2\sqrt{x}$. Then

$$\int \frac{\ln x}{\sqrt{x}}\,dx = 2\sqrt{x}\ln x - 2\int \frac{dx}{\sqrt{x}} = 2\sqrt{x}\ln x - 4\sqrt{x} + C$$

(∗∗) $\displaystyle\lim_{t\to 0^+}\left(2\sqrt{t}\ln t\right) = \lim_{t\to 0^+}\frac{2\ln t}{t^{-1/2}} \overset{\text{H}}{=} \lim_{t\to 0^+}\frac{2/t}{-\frac{1}{2}t^{-3/2}} = \lim_{t\to 0^+}\left(-4\sqrt{t}\right) = 0$

36. Let $u = \dfrac{1}{x}$. Then $du = -\dfrac{dx}{x^2}$, so $\displaystyle\int_1^4 \frac{e^{1/x}}{x^2}\,dx = \int_{1/4}^1 e^u\,du = \left[e^u\right]_{1/4}^1 = e - e^{1/4}$.

37. $\displaystyle\int_0^1 \frac{t^2-1}{t^2+1}\,dt = \int_0^1\left(1 - \frac{2}{t^2+1}\right)dt = \left[t - 2\tan^{-1} t\right]_0^1 = \left(1 - 2\cdot\tfrac{\pi}{4}\right) - 0 = 1 - \tfrac{\pi}{2}$

38. $\dfrac{t^2+1}{t^2-1} = \dfrac{(t^2-1)+2}{t^2-1} = 1 + \dfrac{2}{(t+1)(t-1)}$. Now $\dfrac{2}{(t+1)(t-1)} = \dfrac{A}{t+1} + \dfrac{B}{t-1} \;\Rightarrow$ $2 = A(t-1) + B(t+1)$. Letting $t = 1 \;\Rightarrow\; B = 1$ and letting $t = -1 \;\Rightarrow\; A = -1$. So

$$\begin{aligned}\int_0^1 \frac{t^2+1}{t^2-1}\,dt &= \lim_{b\to 1^-}\int_0^b\left(1 + \frac{-1}{t+1} + \frac{1}{t-1}\right)dt = \lim_{b\to 1^-}\left[t - \ln|t+1| + \ln|t-1|\right]_0^b \\
&= \lim_{b\to 1^-}\left[t + \ln\left|\frac{t-1}{t+1}\right|\right]_0^b = \lim_{b\to 1^-}\left[\left(b + \ln\left|\frac{b-1}{b+1}\right|\right) - (0+0)\right] = -\infty. \text{ Divergent}\end{aligned}$$

39. $\int_0^{\pi/2} \cos^3 x \sin 2x\, dx = \int_0^{\pi/2} 2\cos^4 x \sin x\, dx = \left[-\frac{2}{5}\cos^5 x\right]_0^{\pi/2} = \frac{2}{5}$

40. Let $u = \sqrt{y-2}$. Then $y = u^2 + 2$, so

$$\int \frac{y\,dy}{\sqrt{y-2}} = \int \frac{(u^2+2)\,2u\,du}{u} = 2\int \left(u^2+2\right) du = 2\left[\tfrac{1}{3}u^3 + 2u\right] + C$$

Thus

$$\int_2^6 \frac{y\,dy}{\sqrt{y-2}} = \lim_{t\to 2^+} \int_t^6 \frac{y\,dy}{\sqrt{y-2}} = \lim_{t\to 2^+} \left[\tfrac{2}{3}(y-2)^{3/2} + 4\sqrt{y-2}\right]_t^6$$
$$= \lim_{t\to 2^+} \left[\tfrac{16}{3} + 8 - \tfrac{2}{3}(t-2)^{3/2} - 4\sqrt{t-2}\right] = \tfrac{40}{3}$$

41. $\displaystyle\int_0^3 \frac{dx}{x^2-x-2} = \int_0^3 \frac{dx}{(x+1)(x-2)} = \int_0^2 \frac{dx}{(x+1)(x-2)} + \int_2^3 \frac{dx}{(x+1)(x-2)}$, and

$$\int_2^3 \frac{dx}{x^2-x-2} = \lim_{t\to 2^+} \int_t^3 \left[\frac{-1/3}{x+1} + \frac{1/3}{x-2}\right] dx = \lim_{t\to 2^+} \left[\frac{1}{3}\ln\left|\frac{x-2}{x+1}\right|\right]_t^3$$
$$= \lim_{t\to 2^+} \left[\tfrac{1}{3}\ln\tfrac{1}{4} - \frac{1}{3}\ln\left|\frac{t-2}{t+1}\right|\right] = \infty$$

so $\displaystyle\int_0^3 \frac{dx}{x^2-x-2}$ diverges.

42. Note that $f(x) = 1/(2-3x)$ has an infinite discontinuity at $x = \frac{2}{3}$. Now

$$\int_0^{2/3} \frac{1}{2-3x}\,dx = \lim_{t\to (2/3)^-} \int_0^t \frac{1}{2-3x}\,dx = \lim_{t\to (2/3)^-} \left[-\tfrac{1}{3}\ln|2-3x|\right]_0^t$$
$$= -\tfrac{1}{3}\lim_{t\to (2/3)^-} \left[\ln|2-3t| - \ln 2\right] = \infty$$

Since $\displaystyle\int_0^{2/3} \frac{1}{2-3x}\,dx$ diverges, so does $\displaystyle\int_0^1 \frac{1}{2-3x}\,dx$.

43. Let $u = \sqrt{x} + 2$. Then $x = (u-2)^2$, $dx = 2(u-2)\,du$, so

$$\int_1^4 \frac{\sqrt{x}\,dx}{\sqrt{x}+2} = \int_3^4 \frac{2(u-2)^2\,du}{u} = \int_3^4 \left[2u - 8 + \frac{8}{u}\right] du = \left[u^2 - 8u + 8\ln u\right]_3^4$$
$$= (16 - 32 + 8\ln 4) - (9 - 24 + 8\ln 3) = -1 + 8\ln 4 - 8\ln 3 = 8\ln\tfrac{4}{3} - 1$$

44. Let $u = \ln x$. Then $du = \dfrac{dx}{x}$, so $\displaystyle\int_1^e \frac{dx}{x\left[1+(\ln x)^2\right]} = \int_0^1 \frac{du}{1+u^2} = \left[\tan^{-1} u\right]_0^1 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$.

45. Let $x = \sec\theta$. Then

$$\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx = \int_0^{\pi/3} \frac{\tan\theta}{\sec\theta}\sec\theta\tan\theta\,d\theta = \int_0^{\pi/3} \tan^2\theta\,d\theta = \int_0^{\pi/3} \left(\sec^2\theta - 1\right) d\theta$$
$$= [\tan\theta - \theta]_0^{\pi/3} = \sqrt{3} - \tfrac{\pi}{3}$$

46. $\int_{-1}^1 \left(x^{-1/3} + x^{-4/3}\right) dx = \int_{-1}^0 \left(x^{-1/3} + x^{-4/3}\right) dx + \int_0^1 \left(x^{-1/3} + x^{-4/3}\right) dx$. But

$$\int_0^1 \left(x^{-1/3} + x^{-4/3}\right) dx = \lim_{t\to 0^+} \int_t^1 \left(x^{-1/3} + x^{-4/3}\right) dx = \lim_{t\to 0^+} \left[\tfrac{3}{2}x^{2/3} - 3x^{-1/3}\right]_t^1$$
$$= \lim_{t\to 0^+} \left[\tfrac{3}{2} - 3 - \tfrac{3}{2}t^{2/3} + 3t^{-1/3}\right] = \infty. \text{ Divergent}$$

47. $\int_0^{\pi/4} \tan^2\theta \sec^2\theta \, d\theta = \int_0^1 u^2 \, du \;\; [u = \tan\theta, du = \sec^2\theta \, d\theta] \; = \left[\frac{1}{3}u^3\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$

48. $\int_{-3}^{3} x\sqrt{1+x^4} \, dx = 0$, since the integrand is an odd function.

49. Let $u = 2x + 1$. Then

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{4x^2+4x+5} &= \int_{-\infty}^{\infty} \frac{\frac{1}{2}\,du}{u^2+4} = \frac{1}{2}\int_{-\infty}^{0} \frac{du}{u^2+4} + \frac{1}{2}\int_{0}^{\infty} \frac{du}{u^2+4} \\ &= \frac{1}{2}\lim_{t\to-\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{1}{2}u\right)\right]_t^0 + \frac{1}{2}\lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{1}{2}u\right)\right]_0^t \\ &= \frac{1}{4}\left[0 - \left(-\frac{\pi}{2}\right)\right] + \frac{1}{4}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{4}\end{aligned}$$

50. $\displaystyle\int_1^{\infty} \frac{\tan^{-1}x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\tan^{-1}x}{x^2}\,dx$. Integrate by parts:

$$\begin{aligned}\int \frac{\tan^{-1}x}{x^2}\,dx &= \frac{-\tan^{-1}x}{x} + \int \frac{1}{x}\frac{dx}{1+x^2} = \frac{-\tan^{-1}x}{x} + \int\left[\frac{1}{x} - \frac{x}{x^2+1}\right]dx \\ &= \frac{-\tan^{-1}x}{x} + \ln|x| - \frac{1}{2}\ln\left(x^2+1\right) + C = \frac{-\tan^{-1}x}{x} + \frac{1}{2}\ln\frac{x^2}{x^2+1} + C\end{aligned}$$

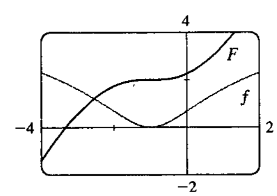
Thus,

$$\begin{aligned}\int_1^{\infty} \frac{\tan^{-1}x}{x^2}\,dx &= \lim_{t\to\infty}\left[-\frac{\tan^{-1}x}{x} + \frac{1}{2}\ln\frac{x^2}{x^2+1}\right]_1^t = \lim_{t\to\infty}\left[-\frac{\tan^{-1}t}{t} + \frac{1}{2}\ln\frac{t^2}{t^2+1} + \frac{\pi}{4} - \frac{1}{2}\ln\frac{1}{2}\right] \\ &= 0 + \frac{1}{2}\ln 1 + \frac{\pi}{4} + \frac{1}{2}\ln 2 = \frac{\pi}{4} + \frac{1}{2}\ln 2\end{aligned}$$

51. We first make the substitution $t = x + 1$, so $\ln\left(x^2+2x+2\right) = \ln\left[(x+1)^2+1\right] = \ln\left(t^2+1\right)$. Then we use parts with $u = \ln\left(t^2+1\right)$, $dv = dt$:

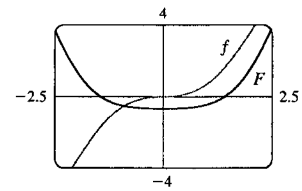
$$\begin{aligned}\int \ln\left(t^2+1\right)dt &= t\ln\left(t^2+1\right) - \int\frac{t\,(2t)\,dt}{t^2+1} = t\ln\left(t^2+1\right) - 2\int\frac{t^2\,dt}{t^2+1} \\ &= t\ln\left(t^2+1\right) - 2\int\left(1 - \frac{1}{t^2+1}\right)dt = t\ln\left(t^2+1\right) - 2t + 2\arctan t + C \\ &= (x+1)\ln\left(x^2+2x+2\right) - 2x + 2\arctan(x+1) + K, \text{ where } K = C - 2\end{aligned}$$

[Alternatively, we could have integrated by parts immediately with $u = \ln\left(x^2+2x+2\right)$.] Notice from the graph that $f = 0$ where F has a horizontal tangent. Also, F is always increasing, and $f \geq 0$.



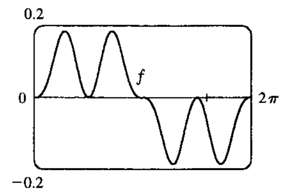
52. $u = x^2 + 1 \quad \Rightarrow \quad x^2 = u - 1$ and $x\,dx = \frac{1}{2}\,du$, so

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+1}}\,dx &= \int \frac{(u-1)}{\sqrt{u}}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int \left(u^{1/2} - u^{-1/2}\right)du \\ &= \tfrac{1}{2}\left(\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right) + C = \tfrac{1}{3}\left(x^2+1\right)^{3/2} - \left(x^2+1\right)^{1/2} + C \\ &= \tfrac{1}{3}\sqrt{x^2+1}\,\left(x^2 - 2\right) + C\end{aligned}$$



53. From the graph, it seems that $\int_0^{2\pi} \cos^2 x \sin^3 x\,dx = 0$. To evaluate the integral, we write the integral as $I = \int_0^{2\pi} \cos^2 x\,(1 - \cos^2 x)\sin x\,dx$ and let $u = \cos x \quad \Rightarrow \quad du = -\sin x\,dx$. Thus,

$$I = \int_1^1 u^2\left(1 - u^2\right)(-du) = 0$$

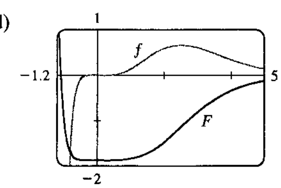


54. (a) To evaluate $\int x^5 e^{-2x}\,dx$ by hand, we would integrate by parts repeatedly, always taking $dv = e^{-2x}$ and starting with $u = x^5$. Each time we would reduce by 1 the degree of the x-factor.

(b) To evaluate the integral using tables, we would use Formula 97 (which is proved using integration by parts) until the exponent of x was reduced to 1, and then we would use Formula 96.

(c) $\int x^5 e^{-2x}\,dx = -\frac{1}{8}e^{-2x}\left(4x^5 + 10x^4 + 20x^3 + 30x^2 + 30x + 15\right) + C$

(d)



55. $u = e^x \quad \Rightarrow \quad du = e^x\,dx$, so

$$\int e^x\sqrt{1 - e^{2x}}\,dx = \int \sqrt{1 - u^2}\,du \overset{30}{=} \tfrac{1}{2}u\sqrt{1-u^2} + \tfrac{1}{2}\sin^{-1} u + C = \tfrac{1}{2}\left[e^x\sqrt{1 - e^{2x}} + \sin^{-1}\left(e^x\right)\right] + C$$

56. $$\begin{aligned}\int \csc^5 t\,dt &\overset{78}{=} -\tfrac{1}{4}\cot t\csc^3 t + \tfrac{3}{4}\int \csc^3 t\,dt \overset{72}{=} -\tfrac{1}{4}\cot t\csc^3 t + \tfrac{3}{4}\left[-\tfrac{1}{2}\csc t\cot t + \tfrac{1}{2}\ln\left|\csc t - \cot t\right|\right] + C \\ &= -\tfrac{1}{4}\cot t\csc^3 t - \tfrac{3}{8}\csc t\cot t + \tfrac{3}{8}\ln\left|\csc t - \cot t\right| + C\end{aligned}$$

57. $u = x + \frac{1}{2} \quad \Rightarrow \quad du = dx$, so

$$\begin{aligned}\int \sqrt{x^2 + x + 1}\,dx &= \int \sqrt{\left(x + \tfrac{1}{2}\right)^2 + \tfrac{3}{4}}\,dx = \int \sqrt{u^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2}\,du \\ &\overset{21}{=} \tfrac{1}{2}u\sqrt{u^2 + \tfrac{3}{4}} + \tfrac{3}{8}\ln\left|u + \sqrt{u^2 + \tfrac{3}{4}}\right| + C \\ &= \frac{2x+1}{4}\sqrt{x^2 + x + 1} + \tfrac{3}{8}\ln\left|x + \tfrac{1}{2} + \sqrt{x^2 + x + 1}\right| + C\end{aligned}$$

58. $u = \sin x \quad \Rightarrow \quad du = \cos x\,dx$, so

$$\int \frac{\cot x\,dx}{\sqrt{1 + 2\sin x}} = \int \frac{du}{u\sqrt{1+2u}} \overset{\substack{57 \text{ with} \\ a=1,\,b=2}}{=} \ln\left|\frac{\sqrt{1+2u} - 1}{\sqrt{1+2u} + 1}\right| + C = \ln\left|\frac{\sqrt{1 + 2\sin x} - 1}{\sqrt{1 + 2\sin x} + 1}\right| + C$$

59. (a) $\frac{d}{du}\left[-\frac{1}{u}\sqrt{a^2-u^2}-\sin^{-1}\left(\frac{u}{a}\right)+C\right]=\frac{1}{u^2}\sqrt{a^2-u^2}+\frac{1}{\sqrt{a^2-u^2}}-\frac{1}{\sqrt{1-u^2/a^2}}\cdot\frac{1}{a}$

$$=\left(a^2-u^2\right)^{-1/2}\left[\frac{1}{u^2}\left(a^2-u^2\right)+1-1\right]=\frac{\sqrt{a^2-u^2}}{u^2}$$

(b) Let $u=a\sin\theta \;\Rightarrow\; du=a\cos\theta\,d\theta,\ a^2-u^2=a^2\left(1-\sin^2\theta\right)=a^2\cos^2\theta.$

$$\int\frac{\sqrt{a^2-u^2}}{u^2}\,du=\int\frac{a^2\cos^2\theta}{a^2\sin^2\theta}\,d\theta=\int\frac{1-\sin^2\theta}{\sin^2\theta}\,d\theta=\int\left(\csc^2\theta-1\right)d\theta=-\cot\theta-\theta+C$$

$$=-\frac{\sqrt{a^2-u^2}}{u}-\sin^{-1}\left(\frac{u}{a}\right)+C$$

60. Work backward, and use integration by parts with $U=u^{-(n-1)}$ and $dV=(a+bu)^{-1/2}\,du \;\Rightarrow$ $dU=\frac{-(n-1)\,du}{u^n}$ and $V=\frac{2}{b}\sqrt{a+bu}$, to get

$$\int\frac{du}{u^{n-1}\sqrt{a+bu}}=\int U\,dV=UV-\int V\,dU=\frac{2\sqrt{a+bu}}{bu^{n-1}}+\frac{2(n-1)}{b}\int\frac{\sqrt{a+bu}}{u^n}\,du$$

$$=\frac{2\sqrt{a+bu}}{bu^{n-1}}+\frac{2(n-1)}{b}\int\frac{a+bu}{u^n\sqrt{a+bu}}\,du$$

$$=\frac{2\sqrt{a+bu}}{bu^{n-1}}+2(n-1)\int\frac{du}{u^{n-1}\sqrt{a+bu}}+\frac{2a(n-1)}{b}\int\frac{du}{u^n\sqrt{a+bu}}$$

Rearranging the equation gives $\frac{2a(n-1)}{b}\int\frac{du}{u^n\sqrt{a+bu}}=-\frac{2\sqrt{a+bu}}{bu^{n-1}}-(2n-3)\int\frac{du}{u^{n-1}\sqrt{a+bu}} \;\Rightarrow$

$$\int\frac{du}{u^n\sqrt{a+bu}}=\frac{-\sqrt{a+bu}}{a(n-1)u^{n-1}}-\frac{b(2n-3)}{2a(n-1)}\int\frac{du}{u^{n-1}\sqrt{a+bu}}$$

61. For $n\geq 0$, $\int_0^\infty x^n\,dx=\lim_{t\to\infty}\left[x^{n+1}/(n+1)\right]_0^t=\infty$. For $n<0$, $\int_0^\infty x^n\,dx=\int_0^1 x^n\,dx+\int_1^\infty x^n\,dx$. Both integrals are improper. By (8.8.2), the second integral diverges if $-1\leq n<0$. By Exercise 8.8.57, the first integral diverges if $n\leq -1$. Thus, $\int_0^\infty x^n\,dx$ is divergent for all values of n.

62. $I=\int_0^\infty e^{ax}\cos x\,dx=\lim_{t\to\infty}\int_0^t e^{ax}\cos x\,dx\overset{\substack{99\text{ with}\\ b=1}}{=}\lim_{t\to\infty}\left[\frac{e^{ax}}{a^2+1}(a\cos x+\sin x)\right]_0^t$

$$=\lim_{t\to\infty}\left[\frac{e^{at}}{a^2+1}(a\cos t+\sin t)-\frac{1}{a^2+1}(a)\right]=\frac{1}{a^2+1}\lim_{t\to\infty}\left[e^{at}(a\cos t+\sin t)-a\right].$$

For $a\geq 0$, the limit does not exist due to oscillation. For $a<0$, $\lim_{t\to\infty}\left[e^{at}(a\cos t+\sin t)\right]=0$ by the Squeeze Theorem, because $\left|e^{at}(a\cos t+\sin t)\right|\leq e^{at}(|a|+1)$, so $I=\frac{1}{a^2+1}(-a)=-\frac{a}{a^2+1}$.

63. $f(x)=\sqrt{1+x^4}, \Delta x=\frac{b-a}{n}=\frac{1-0}{10}=\frac{1}{10}$

(a) $T_{10}=\frac{1}{10\cdot 2}\{f(0)+2[f(0.1)+f(0.2)+\cdots+f(0.9)]+f(1)\}\approx 1.090608$

(b) $M_{10}=\frac{1}{10}\left[f\left(\frac{1}{20}\right)+f\left(\frac{3}{20}\right)+f\left(\frac{5}{20}\right)+\cdots+f\left(\frac{19}{20}\right)\right]\approx 1.088840$

(c) $S_{10}=\frac{1}{10\cdot 3}[f(0)+4f(0.1)+2f(0.2)+\cdots+4f(0.9)+f(1)]\approx 1.089429$

f is concave upward, so the Trapezoidal Rule gives us an overestimate, the Midpoint Rule gives an underestimate, and we cannot tell whether Simpson's Rule gives us an overestimate or an underestimate.

64. $f(x) = \sqrt{\sin x}$, $\Delta x = \dfrac{\frac{\pi}{2} - 0}{10} = \dfrac{\pi}{20}$

(a) $T_{10} = \frac{\pi}{20 \cdot 2} \left\{ f(0) + 2\left[f\left(\frac{\pi}{20}\right) + f\left(\frac{2\pi}{20}\right) + \cdots + f\left(\frac{9\pi}{20}\right) \right] + f\left(\frac{\pi}{2}\right) \right\} \approx 1.185197$

(b) $M_{10} = \frac{\pi}{20} \left[f\left(\frac{\pi}{40}\right) + f\left(\frac{3\pi}{40}\right) + f\left(\frac{5\pi}{40}\right) + \cdots + f\left(\frac{17\pi}{40}\right) + f\left(\frac{19\pi}{40}\right) \right] \approx 1.201932$

(c) $S_{10} = \frac{\pi}{20 \cdot 3} \left[f(0) + 4f\left(\frac{\pi}{20}\right) + 2f\left(\frac{2\pi}{20}\right) + \cdots + 4f\left(\frac{9\pi}{20}\right) + f\left(\frac{\pi}{2}\right) \right] \approx 1.193089$

f is concave downward, so the Trapezoidal Rule gives us an underestimate, the Midpoint Rule gives an overestimate, and we cannot tell whether Simpson's Rule gives us an overestimate or an underestimate.

65. $f(x) = \left(1 + x^4\right)^{1/2}$, $f'(x) = \frac{1}{2}\left(1 + x^4\right)^{-1/2}\left(4x^3\right) = 2x^3\left(1 + x^4\right)^{-1/2}$, $f''(x) = \left(2x^6 + 6x^2\right)\left(1 + x^4\right)^{-3/2}$. A graph of f'' on $[0, 1]$ shows that it has its maximum at $x = 1$, so $|f''(x)| \le f''(1) = \sqrt{8}$ on $[0, 1]$. By taking $K = \sqrt{8}$, we find that the error in Exercise 63(a) is bounded by $\dfrac{K(b - a)^3}{12n^2} = \dfrac{\sqrt{8}}{1200} \approx 0.0024$, and in (b) by about $\frac{1}{2}(0.0024) = 0.0012$.

Note: Another way to estimate K is to let $x = 1$ in the factor $2x^6 + 6x^2$ (maximizing the numerator) and let $x = 0$ in the factor $\left(1 + x^4\right)^{-3/2}$ (minimizing the denominator). Doing so gives us $K = 8$ and errors of 0.0067 and $0.00\overline{3}$.

Using $K = 8$ for the Trapezoidal Rule, we have $|E_T| \le \dfrac{K(b - a)^3}{12n^2} \le 0.00001 \Leftrightarrow \dfrac{8(1 - 0)^3}{12n^2} \le \dfrac{1}{100{,}000} \Leftrightarrow$ $n^2 \ge \dfrac{800{,}000}{12} \Leftrightarrow n \gtrsim 258.2$, so we should take $n = 259$.

For the Midpoint Rule, $|E_M| \le \dfrac{K(b - a)^3}{24n^2} \le 0.00001 \Leftrightarrow n^2 \ge \dfrac{800{,}000}{24} \Leftrightarrow n \gtrsim 182.6$, so we should take $n = 183$.

66. $\int_1^4 \frac{e^x}{x}\,dx \approx S_6 = \frac{(4-1)/6}{3}\left[f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + f(4)\right] \approx 17.739438$

67. $\Delta t = \left(\frac{10}{60} - 0\right) / 10 = \frac{1}{60}$.

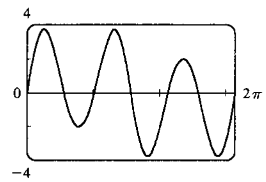
$$\begin{aligned}\text{Distance traveled} = \int_0^{10} v\,dt \approx S_{10} &= \tfrac{1}{60 \cdot 3}\,[40 + 4(42) + 2(45) + 4(49) + 2(52) \\ &\qquad + 4(54) + 2(56) + 4(57) + 2(57) + 4(55) + 56] \\ &= \tfrac{1}{180}(1544) = 8.5\overline{7}\text{ mi}\end{aligned}$$

68. We use Simpson's Rule with $n = 6$ and $\Delta t = \frac{24 - 0}{6} = 4$:

$$\begin{aligned}\text{Increase in bee population} &= \int_0^{24} r(t)\,dt \approx S_6 \\ &= \tfrac{4}{3}\,[r(0) + 4r(4) + 2r(8) + 4r(12) + 2r(16) + 4r(20) + r(24)] \\ &= \tfrac{4}{3}\,[0 + 4(300) + 2(3000) + 4(11{,}000) + 2(4000) + 4(400) + 0] \\ &= \tfrac{4}{3}(60{,}800) \approx 81{,}067\text{ bees}\end{aligned}$$

69. (a) $f(x) = \sin(\sin x)$. A CAS gives

$$f^{(4)}(x) = \sin(\sin x)\left[\cos^4 x + 7\cos^2 x - 3\right] + \cos(\sin x)\left[6\cos^2 x \sin x + \sin x\right]$$



From the graph, we see that $\left|f^{(4)}(x)\right| < 3.8$ for $x \in [0, \pi]$.

(b) We use Simpson's Rule with $f(x) = \sin(\sin x)$ and $\Delta x = \frac{\pi}{10}$:

$$\int_0^\pi f(x)\,dx \approx \tfrac{\pi}{10 \cdot 3}\left[f(0) + 4f\left(\tfrac{\pi}{10}\right) + 2f\left(\tfrac{2\pi}{10}\right) + \cdots + 4f\left(\tfrac{9\pi}{10}\right) + f(\pi)\right] \approx 1.786721$$

From part (a), we know that $f^{(4)}(x) < 3.8$ on $[0, \pi]$, so we use Theorem 8.7.4 with $K = 3.8$, and estimate the error as $|E_S| \le \frac{3.8(\pi - 0)^5}{180(10)^4} \approx 0.000646$.

(c) If we want the error to be less than 0.00001, we must have $|E_S| \le \frac{3.8\pi^5}{180n^4} \le 0.00001$, so $n^4 \ge \frac{3.8\pi^5}{180(0.00001)} \approx 646{,}041.6 \;\Rightarrow\; n \ge 28.35$. Since n must be even for Simpson's Rule, we must have $n \ge 30$ to ensure the desired accuracy.

70. With an x-axis in the normal position, at $x = 7$ we have $C = 2\pi r = 45 \;\Rightarrow\; r(7) = \frac{45}{2\pi}$. Using Simpson's Rule with $n = 4$ and $\Delta x = 7$, we have

$$V = \int_0^{28} \pi\,[r(x)]^2\,dx \approx S_4 = \tfrac{7}{3}\left[0 + 4\pi\left(\tfrac{45}{2\pi}\right)^2 + 2\pi\left(\tfrac{53}{2\pi}\right)^2 + 4\pi\left(\tfrac{45}{2\pi}\right)^2 + 0\right] = \tfrac{7}{3}\left(\tfrac{21{,}818}{4\pi}\right) \approx 4051 \text{ cm}^3$$

71. $\dfrac{x^3}{x^5+2} \le \dfrac{x^3}{x^5} = \dfrac{1}{x^2}$ for x in $[1, \infty)$. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by (8.8.2) with $p = 2 > 1$. Therefore, $\displaystyle\int_1^\infty \frac{x^3}{x^5+2}\,dx$ is convergent by the Comparison Theorem.

72. The line $y = 3$ intersects the hyperbola $y^2 - x^2 = 1$ at two points on its upper branch, namely $\left(-2\sqrt{2}, 3\right)$ and $\left(2\sqrt{2}, 3\right)$. The desired area is

$$A = \int_{-2\sqrt{2}}^{2\sqrt{2}} \left(3 - \sqrt{x^2+1}\right)dx = 2\int_0^{2\sqrt{2}} \left(3 - \sqrt{x^2+1}\right)dx \overset{21}{=} 2\left[3x - \tfrac{1}{2}x\sqrt{x^2+1} - \tfrac{1}{2}\ln\left(x + \sqrt{x^2+1}\right)\right]_0^{2\sqrt{2}}$$

$$= \left[6x - x\sqrt{x^2+1} - \ln\left(x + \sqrt{x^2+1}\right)\right]_0^{2\sqrt{2}} = 12\sqrt{2} - 2\sqrt{2}\cdot 3 - \ln\left(2\sqrt{2} + 3\right) = 6\sqrt{2} - \ln\left(3 + 2\sqrt{2}\right)$$

Another Method: $A = 2\int_1^3 \sqrt{y^2 - 1}\,dy$ and use Formula 39.

73. For x in $\left[0, \frac{\pi}{2}\right]$, $0 \le \cos^2 x \le \cos x$. For x in $\left[\frac{\pi}{2}, \pi\right]$, $\cos x \le 0 \le \cos^2 x$. Thus,

$$\begin{aligned}\text{area} &= \int_0^{\pi/2} \left(\cos x - \cos^2 x\right)dx + \int_{\pi/2}^{\pi} \left(\cos^2 x - \cos x\right)dx \\ &= \left[\sin x - \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_0^{\pi/2} + \left[\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x - \sin x\right]_{\pi/2}^{\pi} \\ &= \left[\left(1 - \tfrac{\pi}{4}\right) - 0\right] + \left[\tfrac{\pi}{2} - \left(\tfrac{\pi}{4} - 1\right)\right] = 2\end{aligned}$$

74. The curves $y = \dfrac{1}{2 \pm \sqrt{x}}$ are defined for $x \geq 0$. For $x > 0$, $\dfrac{1}{2-\sqrt{x}} > \dfrac{1}{2+\sqrt{x}}$. Thus, the required area is

$$\int_0^1 \left(\frac{1}{2-\sqrt{x}} - \frac{1}{2+\sqrt{x}}\right) dx = \int_0^1 \left(\frac{1}{2-u} - \frac{1}{2+u}\right) 2u\,du \text{ (put } u = \sqrt{x}) = 2\int_0^1 \left(-\frac{u}{u-2} - \frac{u}{u+2}\right) du$$

$$= 2\int_0^1 \left(-1 - \frac{2}{u-2} - 1 + \frac{2}{u+2}\right) du = 2\left[2\ln\left|\frac{u+2}{u-2}\right| - 2u\right]_0^1 = 4\ln 3 - 4$$

75. Using the formula for disks, the volume is

$$V = \int_0^{\pi/2} \pi\,[f(x)]^2\,dx = \pi \int_0^{\pi/2} \left(\cos^2 x\right)^2 dx = \pi \int_0^{\pi/2} \left[\tfrac{1}{2}(1+\cos 2x)\right]^2 dx$$

$$= \tfrac{\pi}{4} \int_0^{\pi/2} \left(1 + \cos^2 2x + 2\cos 2x\right) dx = \tfrac{\pi}{4} \int_0^{\pi/2} \left[1 + \tfrac{1}{2}(1+\cos 4x) + 2\cos 2x\right] dx$$

$$= \tfrac{\pi}{4}\left[\tfrac{3}{2}x + \tfrac{1}{2}\left(\tfrac{1}{4}\sin 4x\right) + 2\left(\tfrac{1}{2}\sin 2x\right)\right]_0^{\pi/2} = \tfrac{\pi}{4}\left[\left(\tfrac{3\pi}{4} + \tfrac{1}{8}\cdot 0 + 0\right) - 0\right] = \tfrac{3\pi^2}{16}$$

76. Using the formula for cylindrical shells, the volume is

$$V = \int_0^{\pi/2} 2\pi x f(x)\,dx = 2\pi \int_0^{\pi/2} x\cos^2 x\,dx = 2\pi \int_0^{\pi/2} x\left[\tfrac{1}{2}(1+\cos 2x)\right] dx = 2\left(\tfrac{1}{2}\right)\pi \int_0^{\pi/2} (x + x\cos 2x)\,dx$$

$$= \pi\left(\left[\tfrac{1}{2}x^2\right]_0^{\pi/2} + \left[x\left(\tfrac{1}{2}\sin 2x\right)\right]_0^{\pi/2} - \int_0^{\pi/2} \tfrac{1}{2}\sin 2x\,dx\right) \quad \text{(parts with } u = x,\ dv = \cos 2x\,dx)$$

$$= \pi\left[\tfrac{1}{2}\left(\tfrac{\pi}{2}\right)^2 + 0 - \tfrac{1}{2}\left[-\tfrac{1}{2}\cos 2x\right]_0^{\pi/2}\right] = \tfrac{\pi^3}{8} + \tfrac{\pi}{4}(-1-1) = \tfrac{1}{8}\left(\pi^3 - 4\pi\right)$$

77. By the Fundamental Theorem of Calculus,

$$\int_0^\infty f'(x)\,dx = \lim_{t\to\infty} \int_0^t f'(x)\,dx = \lim_{t\to\infty} [f(t) - f(0)] = \lim_{t\to\infty} f(t) - f(0) = 0 - f(0) = -f(0)$$

78. (a) $\left(\tan^{-1} x\right)_{\text{ave}} = \displaystyle\lim_{t\to\infty} \frac{1}{t-0}\int_0^t \tan^{-1} x\,dx \overset{89}{=} \lim_{t\to\infty}\left\{\frac{1}{t}\left[x\tan^{-1}x - \tfrac{1}{2}\ln\left(1+x^2\right)\right]_0^t\right\}$

$$= \lim_{t\to\infty}\left[\frac{1}{t}\left(t\tan^{-1}t - \tfrac{1}{2}\ln\left(1+t^2\right)\right)\right] = \lim_{t\to\infty}\left[\tan^{-1}t - \frac{\ln\left(1+t^2\right)}{2t}\right]$$

$$\overset{\text{H}}{=} \frac{\pi}{2} - \lim_{t\to\infty} \frac{2t/\left(1+t^2\right)}{2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

(b) $f(x) \geq 0$ and $\int_a^\infty f(x)\,dx$ is divergent $\Rightarrow$ $\displaystyle\lim_{t\to\infty}\int_a^t f(x)\,dx = \infty$.

$$f_{\text{ave}} = \lim_{t\to\infty} \frac{\int_a^t f(x)\,dx}{t-a}\,dx \overset{\text{H}}{=} \lim_{t\to\infty} \frac{f(t)}{1} \text{ (by FTC1)} = \lim_{x\to\infty} f(x), \text{ if this limit exists.}$$

(c) Suppose $\int_a^\infty f(x)\,dx$ converges; that is, $\displaystyle\lim_{t\to\infty}\int_a^t f(x)\,dx = L < \infty$. Then

$$f_{\text{ave}} = \lim_{t\to\infty}\left[\frac{1}{t-a}\int_a^t f(x)\,dx\right] = \lim_{t\to\infty}\frac{1}{t-a}\cdot\lim_{t\to\infty}\int_a^t f(x)\,dx = 0\cdot L = 0$$

(d) $(\sin x)_{\text{ave}} = \displaystyle\lim_{t\to\infty}\frac{1}{t}\int_0^t \sin x\,dx = \lim_{t\to\infty}\left(\frac{1}{t}[-\cos x]_0^t\right) = \lim_{t\to\infty}\left(-\frac{\cos t}{t} + \frac{1}{t}\right) = \lim_{t\to\infty}\frac{1-\cos t}{t} = 0$

79. Let $u = 1/x \;\Rightarrow\; x = 1/u \;\Rightarrow\; dx = -\left(1/u^2\right) du$.

$$\int_0^\infty \frac{\ln x}{1+x^2}\,dx = \int_\infty^0 \frac{\ln(1/u)}{1+1/u^2}\left(-\frac{du}{u^2}\right) = \int_\infty^0 \frac{-\ln u}{u^2+1}\,(-du) = \int_\infty^0 \frac{\ln u}{1+u^2}\,du = -\int_0^\infty \frac{\ln u}{1+u^2}\,du$$

Therefore $\displaystyle\int_0^\infty \frac{\ln x}{1+x^2}\,dx = -\int_0^\infty \frac{\ln x}{1+x^2}\,dx = 0$.

80. If the distance between P and the point charge is d, then the potential V at P is

$$V = W = \int_\infty^d F\,dr = \int_\infty^d \frac{q}{4\pi\varepsilon_0 r^2}\,dr = \lim_{t\to\infty} \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_t^d = \frac{q}{4\pi\varepsilon_0}\lim_{t\to\infty}\left(-\frac{1}{d}+\frac{1}{t}\right) = -\frac{q}{4\pi\varepsilon_0 d}$$

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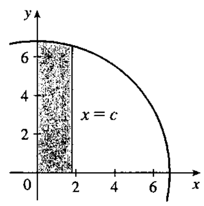
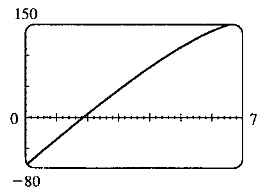
Problems Plus

1.





By symmetry, the problem can be reduced to finding the line $x = c$ such that the shaded area is one-third of the area of the quarter-circle. The equation of the circle is $y = \sqrt{49 - x^2}$, so we require that

$\int_0^c \sqrt{49 - x^2}dx = \frac{1}{3} \cdot \frac{1}{4}\pi\,(7)^2 \quad \Leftrightarrow$

$\left[\frac{1}{2}x\sqrt{49 - x^2} + \frac{49}{2}\sin^{-1}(x/7)\right]_0^c = \frac{49}{12}\pi$ (by Formula 30) $\quad \Leftrightarrow$

$\frac{1}{2}c\sqrt{49 - c^2} + \frac{49}{2}\sin^{-1}(c/7) = \frac{49}{12}\pi$.

This equation would be difficult to solve exactly, so we plot the left-hand side as a function of c, and find that the equation holds for $c \approx 1.85$. So the cuts should be made at distances of about 1.85 inches from the center of the pizza.

2. $\displaystyle\int \frac{1}{x^7 - x}\,dx = \int \frac{dx}{x\,(x^6 - 1)} = \int \frac{x^5}{x^6\,(x^6 - 1)}\,dx = \frac{1}{6}\int \frac{du}{u\,(u - 1)}\,du \quad [u = x^6]$

$$= \frac{1}{6}\int\left(\frac{1}{u - 1} - \frac{1}{u}\right)du = \frac{1}{6}\left(\ln|u - 1| - \ln|u|\right) + C = \frac{1}{6}\ln\left|\frac{u - 1}{u}\right| + C$$

$$= \frac{1}{6}\ln\left|\frac{x^6 - 1}{x^6}\right| + C$$

Alternate Method: $\displaystyle\int \frac{1}{x^7 - x}\,dx = \int \frac{x^{-7}}{1 - x^{-6}}\,dx \quad [u = 1 - x^{-6},\ du = 6x^{-7}\,dx]$

$$= \frac{1}{6}\int \frac{du}{u} = \tfrac{1}{6}\ln|u| + C = \tfrac{1}{6}\ln\left|1 - x^{-6}\right| + C$$

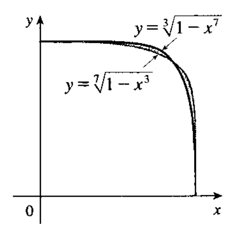
Other Methods: Substitute $u = x^3$ or $x^3 = \sec\theta$.

3. The given integral represents the difference of the shaded areas, which appears to be 0. It can be calculated by integrating with respect to either x or y, so we find x in terms of y for each curve: $y = \sqrt[3]{1 - x^7} \;\Rightarrow\; x = \sqrt[7]{1 - y^3}$ and $y = \sqrt[7]{1 - x^3} \;\Rightarrow\; x = \sqrt[3]{1 - y^7}$, so

$$\int_0^1 \left(\sqrt[3]{1 - y^7} - \sqrt[7]{1 - y^3}\right)dy = \int_0^1 \left(\sqrt[7]{1 - x^3} - \sqrt[3]{1 - x^7}\right)dx$$

But this equation is of the form $z = -z$. So

$$\int_0^1 \left(\sqrt[3]{1 - x^7} - \sqrt[7]{1 - x^3}\right)dx = 0$$



4. (a) The tangent to the curve $y = f(x)$ at $x = x_0$ has the equation $y - f(x_0) = f'(x_0)(x - x_0)$. The y-intercept of this tangent line is $f(x_0) - f'(x_0)x_0$. Thus, L is the distance from the point $(0, f(x_0) - f'(x_0)x_0)$ to the point $(x_0, f'(x_0))$. That is, $L^2 = x_0^2 + [f'(x_0)]^2 x_0^2$, so $[f'(x_0)]^2 = \dfrac{L^2 - x_0^2}{x_0^2}$ and $f'(x_0) = -\dfrac{\sqrt{L^2 - x_0^2}}{x_0}$ for each $0 < x_0 < L$.

(b) $\dfrac{dy}{dx} = -\dfrac{\sqrt{L^2 - x^2}}{x} \quad \Rightarrow$

$$y = \int\left(-\frac{\sqrt{L^2-x^2}}{x}\right)dx = \int \frac{-L\cos\theta\, L\cos\theta\, d\theta}{L\sin\theta} \quad (\text{where } x = L\sin\theta)$$

$$= L\int \frac{\sin^2\theta - 1}{\sin\theta}\,d\theta = L\int(\sin\theta - \csc\theta)\,d\theta = -L\cos\theta + L\ln|\csc\theta + \cot\theta| + C$$

$$= -\sqrt{L^2 - x^2} + L\ln\left(\frac{L}{x} + \frac{\sqrt{L^2-x^2}}{x}\right) + C$$

When $x = L$, $0 = y = -0 + L\ln(1+0) + C$, so $C = 0$. Therefore,

$$y = -\sqrt{L^2 - x^2} + L\ln\left(\frac{L + \sqrt{L^2 - x^2}}{x}\right)$$

5. Recall that $\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$. So

$$f(x) = \int_0^\pi \cos t\cos(x-t)\,dt = \tfrac{1}{2}\int_0^\pi [\cos(t + x - t) + \cos(t - x + t)]\,dt$$

$$= \tfrac{1}{2}\int_0^\pi [\cos x + \cos(2t - x)]\,dt = \tfrac{1}{2}\left[t\cos x + \tfrac{1}{2}\sin(2t-x)\right]_0^\pi$$

$$= \tfrac{\pi}{2}\cos x + \tfrac{1}{4}\sin(2\pi - x) - \tfrac{1}{4}\sin(-x) = \tfrac{\pi}{2}\cos x + \tfrac{1}{4}\sin(-x) - \tfrac{1}{4}\sin(-x)$$

$$= \tfrac{\pi}{2}\cos x$$

The minimum of $\cos x$ on this domain is -1, so the minimum value of $f(x)$ is $f(\pi) = -\frac{\pi}{2}$.

6. n is a positive integer, so

$$\int(\ln x)^n\,dx = x(\ln x)^n - \int x\cdot n(\ln x)^{n-1}(dx/x) \quad (\text{by parts}) \quad = x(\ln x)^n - n\int(\ln x)^{n-1}\,dx$$

Thus,

$$\int_0^1(\ln x)^n\,dx = \lim_{t\to0^+}\int_t^1(\ln x)^n\,dx = \lim_{t\to0^+}\left[x(\ln x)^n\right]_t^1 - n\lim_{t\to0^+}\int_t^1(\ln x)^{n-1}\,dx$$

$$= -\lim_{t\to0^+}\frac{(\ln t)^n}{1/t} - n\int_0^1(\ln x)^{n-1}\,dx = -n\int_0^1(\ln x)^{n-1}\,dx$$

by repeated application of l'Hospital's Rule. We want to prove that $\int_0^1(\ln x)^n\,dx = (-1)^n n!$ for every positive integer n. For $n = 1$, we have

$$\int_0^1(\ln x)^1\,dx = (-1)\int_0^1(\ln x)^0\,dx = -\int_0^1 dx = -1 \left(\text{or } \int_0^1 \ln x\,dx = \lim_{t\to0^+}[x\ln x - x]_t^1 = -1\right)$$

Assuming that the formula holds for n, we find that

$$\int_0^1(\ln x)^{n+1}\,dx = -(n+1)\int_0^1(\ln x)^n\,dx = -(n+1)(-1)^n n! = (-1)^{n+1}(n+1)!$$

This is the formula for $n+1$. Thus, the formula holds for all positive integers n by induction.

7. In accordance with the hint, we let $I_k = \int_0^1 \left(1 - x^2\right)^k \, dx$, and we find an expression for I_{k+1} in terms of I_k. We integrate I_{k+1} by parts with $u = \left(1 - x^2\right)^{k+1} \Rightarrow du = (k+1)\left(1 - x^2\right)^k (-2x)$, $dv = dx \Rightarrow v = x$, and then split the remaining integral into identifiable quantities:

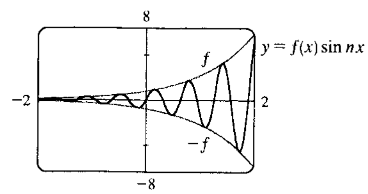
$$I_{k+1} = x\left(1 - x^2\right)^{k+1}\Big|_0^1 + 2(k+1)\int_0^1 x^2\left(1 - x^2\right)^k dx = (2k+2)\int_0^1 \left(1 - x^2\right)^k \left[1 - \left(1 - x^2\right)\right] dx$$
$$= (2k+2)\left(I_k - I_{k+1}\right)$$

So $I_{k+1}\left[1 + (2k+2)\right] = (2k+2)\, I_k \Rightarrow I_{k+1} = \frac{2k+2}{2k+3} I_k$. Now to complete the proof, we use induction:
$I_0 = 1 = \dfrac{2^0 (0!)^2}{1!}$, so the formula holds for $n = 0$. Now suppose it holds for $n = k$. Then

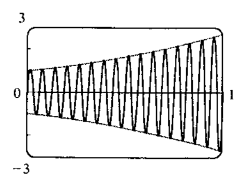
$$I_{k+1} = \frac{2k+2}{2k+3} I_k = \frac{2k+2}{2k+3}\left[\frac{2^{2k}(k!)^2}{(2k+1)!}\right] = \frac{2(k+1)\,2^{2k}(k!)^2}{(2k+3)(2k+1)!} = \frac{2(k+1)}{2k+2}\cdot\frac{2(k+1)\,2^{2k}(k!)^2}{(2k+3)(2k+1)!}$$
$$= \frac{[2(k+1)]^2\, 2^{2k}(k!)^2}{(2k+3)(2k+2)(2k+1)!} = \frac{2^{2(k+1)}\left[(k+1)!\right]^2}{[2(k+1)+1]!}$$

So by induction, the formula holds for all integers $n \geq 0$.

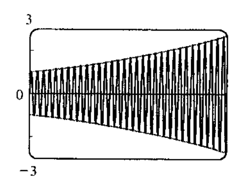
8. (a) Since $-1 \leq \sin \leq 1$, we have $-f(x) \leq f(x)\sin nx \leq f(x)$, and the graph of $y = f(x)\sin nx$ oscillates between $f(x)$ and $-f(x)$. (The diagram shows the case $f(x) = e^x$ and $n = 10$.) As $n \to \infty$, the graph oscillates more and more frequently; see the graphs in part (b).



(b) From the graphs of the integrand, it seems that $\lim_{n\to\infty} \int_0^1 f(x)\sin nx\, dx = 0$, since as n increases, the integrand oscillates more and more rapidly, and thus (since f' is continuous) it makes sense that the areas above the x-axis and below it during each oscillation approach equality.



$n = 100$



$n = 200$

(c) We integrate by parts with $u = f(x) \Rightarrow du = f'(x)\,dx$, $dv = \sin nx\, dx \Rightarrow v = -\dfrac{\cos nx}{n}$:

$$\int_0^1 f(x)\sin nx\, dx = \left[-\frac{f(x)\cos nx}{n}\right]_0^1 + \int_0^1 \frac{\cos nx}{n} f'(x)\, dx$$
$$= \tfrac{1}{n}\left(\int_0^1 \cos nx f'(x)\, dx - \left[f(x)\cos nx\right]_0^1\right)$$
$$= \tfrac{1}{n}\left[\int_0^1 \cos nx f'(x)\, dx + f(0) - f(1)\cos n\right]$$

Taking absolute values of the first and last terms in this equality, and using the facts that $|\alpha \pm \beta| \leq |\alpha| + |\beta|$, $\int_0^1 f(x)\,dx \leq \int_0^1 |f(x)|\,dx$, $|f(0)| = f(0)$ (f is positive), $|f'(x)| \leq M$ for $0 \leq x \leq 1$, and $|\cos nx| \leq 1$,

$$\left|\int_0^1 f(x)\sin nx\, dx\right| \leq \tfrac{1}{n}\left[\left|\int_0^1 M\,dx\right| + |f(0)| + |f(1)|\right] = \tfrac{1}{n}\left[M + |f(0)| + |f(1)|\right]$$

which approaches 0 as $n \to \infty$. The result follows by the Squeeze Theorem.

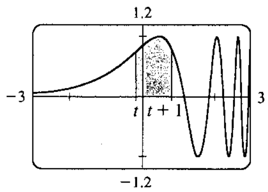
9. $0 < a < b$. Now

$$\int_0^1 [bx + a(1-x)]^t\,dx = \int_a^b \frac{u^t}{(b-a)}\,du \;[\text{put } u = bx + a(1-x)] \;= \left[\frac{u^{t+1}}{(t+1)(b-a)}\right]_a^b = \frac{b^{t+1} - a^{t+1}}{(t+1)(b-a)}.$$

Now let $y = \lim\limits_{t\to 0}\left[\dfrac{b^{t+1} - a^{t+1}}{(t+1)(b-a)}\right]^{1/t}$. Then $\ln y = \lim\limits_{t\to 0}\left[\dfrac{1}{t}\ln\dfrac{b^{t+1} - a^{t+1}}{(t+1)(b-a)}\right]$. This limit is of the form $0/0$, so we can apply l'Hospital's Rule to get

$$\ln y = \lim_{t\to 0}\left[\frac{b^{t+1}\ln b - a^{t+1}\ln a}{b^{t+1} - a^{t+1}} - \frac{1}{t+1}\right] = \frac{b\ln b - a\ln a}{b-a} - 1 = \frac{b\ln b}{b-a} - \frac{a\ln a}{b-a} - \ln e = \ln\frac{b^{b/(b-a)}}{ea^{a/(b-a)}}.$$

Therefore, $y = e^{-1}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}$.

10. 

From the graph, it appears that the area under the graph of $f(x) = \sin(e^x)$ on the interval $[t, t+1]$ is greatest when $t \approx -0.2$. To find the exact value, we write the integral as $I = \int_t^{t+1} f(x)\,dx = \int_0^{t+1} f(x)\,dx - \int_0^t f(x)\,dx$, and use FTC1 to find $dI/dt = f(t+1) - f(t) = \sin\left(e^{t+1}\right) - \sin\left(e^t\right) = 0$ when $\sin\left(e^{t+1}\right) = \sin\left(e^t\right)$.

Now we have $\sin x = \sin y$ whenever $x - y = 2k\pi$ and also whenever x and y are the same distance from $\left(k + \frac{1}{2}\right)\pi$, k any integer, since $\sin x$ is symmetric about the line $x = \left(k + \frac{1}{2}\right)\pi$. The first possibility is the more obvious one, but if we calculate $e^{t+1} - e^t = 2k\pi$, we get $t = \ln(2k\pi/(e-1))$, which is about 1.3 for $k = 1$ (the least possible value of k). From the graph, this looks unlikely to give the maximum we are looking for. So instead we set $e^{t+1} - \left(k + \frac{1}{2}\right)\pi = \left(k + \frac{1}{2}\right)\pi - e^t \;\Leftrightarrow\; e^{t+1} + e^t = (2k+1)\pi \;\Leftrightarrow\; e^t(e+1) = (2k+1)\pi \;\Leftrightarrow\; t = \ln((2k+1)\pi/(e+1))$. Now $k = 0 \;\Rightarrow\; t = \ln(\pi/(e+1)) \approx -0.16853$, which does give the maximum value, as we have seen from the graph of f.

11. We integrate by parts with $u = \dfrac{1}{\ln(1+x+t)}$, $dv = \sin t\,dt$, so $du = \dfrac{-1}{(1+x+t)[\ln(1+x+t)]^2}$ and $v = -\cos t$. The integral becomes

$$I = \int_0^\infty \frac{\sin t\,dt}{\ln(1+x+t)} = \lim_{b\to\infty}\left(\left[\frac{-\cos t}{\ln(1+x+t)}\right]_0^b - \int_0^b \frac{\cos t\,dt}{(1+x+t)[\ln(1+x+t)]^2}\right)$$

$$= \lim_{b\to\infty}\frac{-\cos b}{\ln(1+x+b)} + \frac{1}{\ln(1+x)} + \int_0^\infty \frac{-\cos t\,dt}{(1+x+t)[\ln(1+x+t)]^2} = \frac{1}{\ln(1+x)} + J$$

where $J = \displaystyle\int_0^\infty \frac{-\cos t\,dt}{(1+x+t)[\ln(1+x+t)]^2}$.

Now $-1 \le -\cos t \le 1$ for all t; in fact, the inequality is strict except at isolated points. So

$$-\int_0^\infty \frac{dt}{(1+x+t)\,[\ln(1+x+t)]^2} < J < \int_0^\infty \frac{dt}{(1+x+t)\,[\ln(1+x+t)]^2} \quad\Leftrightarrow$$

$$-\frac{1}{\ln(1+x)} < J < \frac{1}{\ln(1+x)} \quad\Leftrightarrow\quad 0 < I < \frac{2}{\ln(1+x)}.$$

12. (a) $T_n(x) = \cos(n \arccos x)$. The domain of arccos is $[-1, 1]$, and the domain of cos is $\mathbb{R}$, so the domain of $T_n(x)$ is $[-1, 1]$. As for the range, $T_0(x) = \cos 0 = 1$, so the range of $T_0(x)$ is $\{1\}$. But since the range of $n \arccos x$ is at least $[0, \pi]$ for $n > 0$, and since $\cos y$ takes on all values in $[-1, 1]$ for $y \in [0, \pi]$, the range of $T_n(x)$ is $[-1, 1]$ for $n > 0$.

(b) Using the usual trigonometric identities, $T_2(x) = \cos(2 \arccos x) = 2[\cos(\arccos x)]^2 - 1 = 2x^2 - 1$, and

$$\begin{aligned}
T_3(x) &= \cos(3 \arccos x) = \cos(\arccos x + 2 \arccos x) \\
&= \cos(\arccos x)\cos(2 \arccos x) - \sin(\arccos x)\sin(2 \arccos x) \\
&= x\left(2x^2 - 1\right) - \sin(\arccos x)\,[2\sin(\arccos x)\cos(\arccos x)] \\
&= 2x^3 - x - 2\left[\sin^2(\arccos x)\right]x = 2x^3 - x - 2x\left[1 - \cos^2(\arccos x)\right] \\
&= 2x^3 - x - 2x\left(1 - x^2\right) = 4x^3 - 3x
\end{aligned}$$

(c) Let $y = \arccos x$. Then

$$\begin{aligned}
T_{n+1}(x) &= \cos[(n+1)y] = \cos(y + ny) = \cos y \cos ny - \sin y \sin ny \\
&= 2\cos y \cos ny - (\cos y \cos ny + \sin y \sin ny) = 2x T_n(x) - \cos(ny - y) \\
&= 2x T_n(x) - T_{n-1}(x)
\end{aligned}$$

(d) Here we use induction. $T_0(x) = 1$, a polynomial of degree 0. Now assume that $T_k(x)$ is a polynomial of degree k. Then $T_{k+1}(x) = 2x T_k(x) - T_{k-1}(x)$. By assumption, the leading term of T_k is $a_k x^k$, say, so the leading term of T_{k+1} is $2x a_k x^k = 2a_k x^{k+1}$, and so T_{k+1} has degree $k+1$.

(e) $T_4(x) = 2x T_3(x) - T_2(x) = 2x\left(4x^3 - 3x\right) - \left(2x^2 - 1\right) = 8x^4 - 8x^2 + 1$,

$T_5(x) = 2x T_4(x) - T_3(x) = 2x\left(8x^4 - 8x^2 + 1\right) - \left(4x^3 - 3x\right) = 16x^5 - 20x^3 + 5x$,

$T_6(x) = 2x T_5(x) - T_4(x) = 2x\left(16x^5 - 20x^3 + 5x\right) - \left(8x^4 - 8x^2 + 1\right) = 32x^6 - 48x^4 + 18x^2 - 1$,

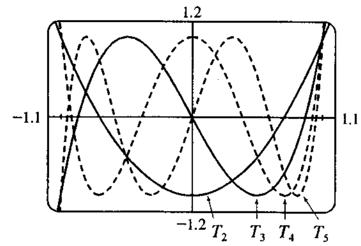
$T_7(x) = 2x T_6(x) - T_5(x) = 2x\left(32x^6 - 48x^4 + 18x^2 - 1\right) - \left(16x^5 - 20x^3 + 5x\right)$

$= 64x^7 - 112x^5 + 56x^3 - 7x$

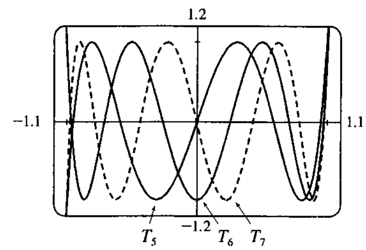
(f) The zeros of $T_n(x) = \cos(n \arccos x)$ occur where $n \arccos x = k\pi + \frac{\pi}{2}$ for some integer k, since then $\cos(n \arccos x) = \cos\left(k\pi + \frac{\pi}{2}\right) = 0$. Note that there will be restrictions on k, since $0 \le \arccos x \le \pi$. We continue: $n \arccos x = k\pi + \frac{\pi}{2} \quad\Leftrightarrow\quad \arccos x = \dfrac{k\pi + \frac{\pi}{2}}{n}$. This only has solutions for $0 \le \dfrac{k\pi + \frac{\pi}{2}}{n} \le \pi$ $\Leftrightarrow\quad 0 < k\pi + \frac{\pi}{2} < n\pi \quad\Leftrightarrow\quad 0 \le k < n$. [This makes sense, because then $T_n(x)$ has n zeros, and it is a polynomial of degree n.] So, taking cosines of both sides of the last equation, we find that the zeros of $T_n(x)$ occur at $x = \cos \dfrac{k\pi + \frac{\pi}{2}}{n}$, k an integer with $0 \le k < n$. To find the values of x at which $T_n(x)$ has local extrema, we set $0 = T_n'(x) = -\sin(n \arccos x)\dfrac{-n}{\sqrt{1-x^2}} = \dfrac{n \sin(n \arccos x)}{\sqrt{1-x^2}} \quad\Leftrightarrow\quad \sin(n \arccos x) = 0 \quad\Leftrightarrow$ $n \arccos x = k\pi$, k some integer $\quad\Leftrightarrow\quad \arccos x = k\pi/n$. This has solutions for $0 \le k \le n$, but we disallow the cases $k = 0$ and $k = n$, since these give $x = 1$ and $x = -1$ respectively. So the local extrema of $T_n(x)$

occur at $x = \cos(k\pi/n)$, k an integer with $0 < k < n$. [Again, this seems reasonable, since a polynomial of degree n has at most $(n-1)$ extrema.] By the First Derivative Test, the cases where k is even give maxima of $T_n(x)$, since then $n \arccos[\cos(k\pi/n)] = k\pi$ is an even multiple of π, so $\sin(n \arccos x)$ goes from negative to positive at $x = \cos(k\pi/n)$. Similarly, the cases where k is odd represent minima of $T_n(x)$.

(g)



(h)



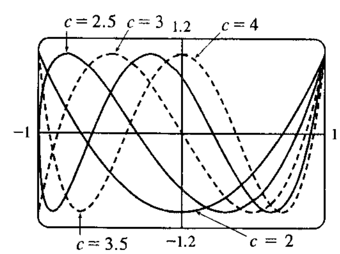
(i) From the graphs, it seems that the zeros of T_n and T_{n+1} alternate; that is, between two adjacent zeros of T_n, there is a zero of T_{n+1}, and vice versa. The same is true of the x-coordinates of the extrema of T_n and T_{n+1}: between the x-coordinates of any two adjacent extrema of one, there is the x-coordinate of an extremum of the other.

(j) When n is odd, the function $T_n(x)$ is odd, since all of its terms have odd degree, and so $\int_{-1}^{1} T_n(x)\,dx = 0$. When n is even, $T_n(x)$ is even, and it appears that the integral is negative, but decreases in absolute value as n gets larger.

(k) $\int_{-1}^{1} T_n(x)\,dx = \int_{-1}^{1} \cos(n \arccos x)\,dx$. We substitute $u = \arccos x \Rightarrow x = \cos u \Rightarrow dx = -\sin u\,du$, $x = -1 \Rightarrow u = \pi$, and $x = 1 \Rightarrow u = 0$. So the integral becomes

$$\begin{aligned}
\int_0^{\pi} \cos(nu)\sin u\,du &= \int_0^{\pi} \tfrac{1}{2}[\sin(u - nu) + \sin(u + nu)]\,du \\
&= \frac{1}{2}\left[\frac{\cos[(1-n)u]}{n-1} - \frac{\cos[(1+n)u]}{n+1}\right]_0^{\pi} \\
&= \begin{cases} \dfrac{1}{2}\left[\left(\dfrac{-1}{n-1} - \dfrac{-1}{n+1}\right) - \left(\dfrac{1}{n-1} - \dfrac{1}{n+1}\right)\right] & \text{if } n \text{ is even} \\[2ex] \dfrac{1}{2}\left[\left(\dfrac{1}{n-1} - \dfrac{1}{n+1}\right) - \left(\dfrac{1}{n-1} - \dfrac{1}{n+1}\right)\right] & \text{if } n \text{ is odd} \end{cases} = \begin{cases} -\dfrac{2}{n^2-1} & \text{if } n \text{ is even} \\[2ex] 0 & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

(l) From the graph, we see that as c increases through an integer, the graph of f gains a local extremum, which starts at $x = -1$ and moves rightward, compressing the graph of f as c continues to increase.



9 Further Applications of Integration

9.1 Arc Length

1. $y = 2 - 3x \quad\Rightarrow\quad L = \int_{-2}^{1} \sqrt{1 + (dy/dx)^2}\,dx = \int_{-2}^{1} \sqrt{1 + (-3)^2}\,dx = \sqrt{10}\,[1 - (-2)] = 3\sqrt{10}.$
The arc length can be calculated using the distance formula, since the curve is a line segment, so

$$L = [\text{distance from } (-2, 8) \text{ to } (1, -1)] = \sqrt{[1 - (-2)]^2 + [(-1) - 8]^2} = \sqrt{90} = 3\sqrt{10}$$

2. Using the arc length formula with $y = \sqrt{4 - x^2} \quad\Rightarrow\quad \dfrac{dy}{dx} = -\dfrac{x}{\sqrt{4 - x^2}}$, we get

$$L = \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^2 \sqrt{1 + \frac{x^2}{4 - x^2}}\,dx = \int_0^2 \frac{2\,dx}{\sqrt{4 - x^2}} = 2 \lim_{t \to 0^+} \int_t^2 \frac{dx}{\sqrt{2^2 - x^2}}$$

$$= 2 \lim_{t \to 0^+} \left[\sin^{-1}(x/2)\right]_t^2 = 2 \lim_{t \to 0^+} \left[\sin^{-1} 1 - \sin^{-1}(t/2)\right] = 2\left(\tfrac{\pi}{2} - 0\right) = \pi$$

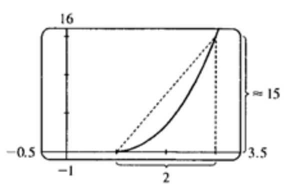
The curve is a quarter of a circle with radius 2, so the length of the arc is $\frac{1}{4}(2\pi \cdot 2) = \pi$, as above.

3. $y^2 = (x - 1)^3,\ y = (x - 1)^{3/2} \quad\Rightarrow\quad \dfrac{dy}{dx} = \frac{3}{2}(x - 1)^{1/2} \quad\Rightarrow\quad 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \frac{9}{4}(x - 1)$. So

$$L = \int_1^2 \sqrt{1 + \tfrac{9}{4}(x - 1)}\,dx = \int_1^2 \sqrt{\tfrac{9}{4}x - \tfrac{5}{4}}\,dx = \left[\tfrac{4}{9} \cdot \tfrac{2}{3}\left(\tfrac{9}{4}x - \tfrac{5}{4}\right)^{3/2}\right]_1^2 = \tfrac{13\sqrt{13} - 8}{27}$$

4. $12xy = 4y^4 + 3,\ x = \dfrac{y^3}{3} + \dfrac{y^{-1}}{4} \quad\Rightarrow\quad \dfrac{dx}{dy} = y^2 - \dfrac{y^{-2}}{4}$, so $\left(\dfrac{dx}{dy}\right)^2 = y^4 - \dfrac{1}{2} + \dfrac{y^{-4}}{16} \quad\Rightarrow$
$1 + \left(\dfrac{dx}{dy}\right)^2 = y^4 + \dfrac{1}{2} + \dfrac{y^{-4}}{16} \quad\Rightarrow\quad \sqrt{1 + \left(\dfrac{dx}{dy}\right)^2} = y^2 + \dfrac{y^{-2}}{4}$. So
$L = \displaystyle\int_1^2 \left(y^2 + \frac{y^{-2}}{4}\right) dy = \left[\frac{y^3}{3} - \frac{1}{4y}\right]_1^2 = \left(\frac{8}{3} - \frac{1}{8}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{59}{24}.$

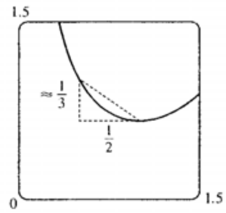
5.



From the figure, the length of the curve is slightly larger than the hypotenuse of the triangle formed by the points $(1, 0)$, $(3, 0)$, and $(3, f(3)) \approx (3, 15)$, where $y = f(x) = \frac{2}{3}(x^2 - 1)^{3/2}$. This length is about $\sqrt{15^2 + 2^2} \approx 15$, so we might estimate the length to be 15.5.
$y = \frac{2}{3}(x^2 - 1)^{3/2} \quad\Rightarrow\quad y' = (x^2 - 1)^{1/2}(2x) \quad\Rightarrow$
$1 + (y')^2 = 1 + 4x^2(x^2 - 1) = 4x^4 - 4x^2 + 1 = (2x^2 - 1)^2$, so, using the fact that $2x^2 - 1 > 0$ for $1 \le x \le 3$,

$$L = \int_1^3 \sqrt{(2x^2 - 1)^2}\,dx = \int_1^3 \left|2x^2 - 1\right| dx = \int_1^3 (2x^2 - 1)\,dx = \left[\tfrac{2}{3}x^3 - x\right]_1^3 = (18 - 3) - \left(\tfrac{2}{3} - 1\right) = \tfrac{46}{3} = 15.\overline{3}$$

6.



From the figure, the length of the curve is slightly larger than the hypotenuse, which is about $\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2} \approx 0.6$, so we might estimate the length to be about 0.7. $y = \dfrac{x^3}{6} + \dfrac{1}{2x} \Rightarrow y' = \dfrac{x^2}{2} - \dfrac{x^{-2}}{2} \Rightarrow$ $1 + (y')^2 = \dfrac{x^4}{4} + \dfrac{1}{2} + \dfrac{x^{-4}}{4} = \left(\dfrac{x^2}{2} + \dfrac{x^{-2}}{2}\right)^2$, so, using the fact that the parenthetical expression is positive,

$$L = \int_{1/2}^{1} \sqrt{\left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right)^2}\, dx = \int_{1/2}^{1} \left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right) dx = \left[\frac{x^3}{6} - \frac{1}{2x}\right]_{1/2}^{1}$$
$$= \left(\tfrac{1}{6} - \tfrac{1}{2}\right) - \left(\tfrac{1}{48} - 1\right) = \tfrac{31}{48} \approx 0.646$$

7. $y = \frac{1}{3}\left(x^2+2\right)^{3/2} \Rightarrow dy/dx = \frac{1}{2}\left(x^2+2\right)^{1/2}(2x) = x\sqrt{x^2+2} \Rightarrow$
$1 + (dy/dx)^2 = 1 + x^2\left(x^2+2\right) = \left(x^2+1\right)^2$. So $L = \int_0^1 \left(x^2+1\right) dx = \left[\frac{1}{3}x^3 + x\right]_0^1 = \frac{4}{3}$.

8. $y = \dfrac{x^2}{2} - \dfrac{\ln x}{4} \Rightarrow \dfrac{dy}{dx} = x - \dfrac{1}{4x} \Rightarrow 1 + \left(\dfrac{dy}{dx}\right)^2 = x^2 + \dfrac{1}{2} + \dfrac{1}{16x^2}$. So

$$L = \int_2^4 \left(x + \frac{1}{4x}\right) dx = \left[\frac{x^2}{2} + \frac{\ln x}{4}\right]_2^4 = \left(8 + \frac{2\ln 2}{4}\right) - \left(2 + \frac{\ln 2}{4}\right) = 6 + \frac{\ln 2}{4}$$

9. $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2} \Rightarrow \dfrac{dy}{dx} = x^3 - \dfrac{1}{4x^3} \Rightarrow 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + x^6 - \dfrac{1}{2} + \dfrac{1}{16x^6} = x^6 + \dfrac{1}{2} + \dfrac{1}{16x^6}$. So
$L = \int_1^3 \left(x^3 + \frac{1}{4}x^{-3}\right) dx = \left[\frac{1}{4}x^4 - \frac{1}{8}x^{-2}\right]_1^3 = \left(\frac{81}{4} - \frac{1}{72}\right) - \left(\frac{1}{4} - \frac{1}{8}\right) = \frac{181}{9}$.

10. $x = \frac{1}{3}\sqrt{y}\,(y-3) = \frac{1}{3}y^{3/2} - y^{1/2} \Rightarrow dx/dy = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2} \Rightarrow$
$1 + (dx/dy)^2 = 1 + \frac{1}{4}\left(y - 2 + y^{-1}\right) = \frac{1}{4}\left(y + 2 + y^{-1}\right) = \left[\frac{1}{2}\left(y^{1/2} + y^{-1/2}\right)\right]^2$, so

$$L = \int_0^9 \sqrt{1 + (dx/dy)^2}\, dy = \int_0^9 \tfrac{1}{2}\left(y^{1/2} + y^{-1/2}\right) dy = \lim_{t\to 0^+} \int_t^9 \tfrac{1}{2}\left(y^{1/2} + y^{-1/2}\right) dy$$
$$= \lim_{t\to 0^+} \left[\tfrac{1}{3}y^{3/2} + y^{1/2}\right]_t^9 = \lim_{t\to 0^+} \left[(9+3) - \left(\tfrac{1}{3}t^{3/2} + t^{1/2}\right)\right] = 12 - 0 = 12$$

11. $y = \ln(\sec x) \Rightarrow \dfrac{dy}{dx} = \dfrac{\sec x \tan x}{\sec x} = \tan x \Rightarrow 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x$, so
$L = \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx = \int_0^{\pi/4} \sec x\, dx = \left[\ln(\sec x + \tan x)\right]_0^{\pi/4} = \ln\left(\sqrt{2}+1\right) - \ln(1+0) = \ln\left(\sqrt{2}+1\right)$

12. $y = \ln(\sin x) \Rightarrow \dfrac{dy}{dx} = \dfrac{\cos x}{\sin x} = \cot x \Rightarrow 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \cot^2 x = \csc^2 x$. So

$$L = \int_{\pi/6}^{\pi/3} \csc x\, dx = \left[\ln(\csc x - \cot x)\right]_{\pi/6}^{\pi/3} = \ln\left(\tfrac{2}{\sqrt{3}} - \tfrac{1}{\sqrt{3}}\right) - \ln\left(2 - \sqrt{3}\right) = \ln \frac{1}{\sqrt{3}\left(2-\sqrt{3}\right)}$$
$$= \ln \tfrac{2+\sqrt{3}}{\sqrt{3}} = \ln\left(1 + \tfrac{2}{\sqrt{3}}\right)$$

13. $y = \ln(1 - x^2) \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{-2x}{1 - x^2} \;\Rightarrow\; 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{4x^2}{(1 - x^2)^2} = \dfrac{(1 + x^2)^2}{(1 - x^2)^2}$. So

$$L = \int_0^{1/2} \frac{1 + x^2}{1 - x^2}\,dx = \int_0^{1/2} \left[-1 + \frac{2}{(1 - x)(1 + x)}\right] dx = \int_0^{1/2} \left[-1 + \frac{1}{1 + x} + \frac{1}{1 - x}\right] dx$$
$$= [-x + \ln(1 + x) - \ln(1 - x)]_0^{1/2} = -\tfrac{1}{2} + \ln\tfrac{3}{2} - \ln\tfrac{1}{2} - 0 = \ln 3 - \tfrac{1}{2}$$

14. $y = \ln x \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{1}{x} \;\Rightarrow\; \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \sqrt{1 + \left(\dfrac{1}{x}\right)^2} = \dfrac{\sqrt{1 + x^2}}{x}$. So $L = \displaystyle\int_1^{\sqrt{3}} \frac{\sqrt{1 + x^2}}{x}\,dx$. Now let $v = \sqrt{1 + x^2}$, so $v^2 = 1 + x^2$ and $v\,dv = x\,dx$. Thus

$$L = \int_{\sqrt{2}}^2 \frac{v}{v^2 - 1} v\,dv = \int_{\sqrt{2}}^2 \left(1 + \frac{1/2}{v - 1} - \frac{1/2}{v + 1}\right) dv = \left[v + \tfrac{1}{2}\ln|v - 1| - \tfrac{1}{2}\ln|v + 1|\right]_{\sqrt{2}}^2$$
$$= \left[v - \frac{1}{2}\ln\left|\frac{v + 1}{v - 1}\right|\right]_{\sqrt{2}}^2 = 2 - \sqrt{2} - \tfrac{1}{2}\ln 3 + \frac{1}{2}\ln\left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right) = 2 - \sqrt{2} + \ln\left(\sqrt{2} + 1\right) - \tfrac{1}{2}\ln 3$$

Or: Use Formula 23 in the table of integrals.

15. $y = \cosh x \;\Rightarrow\; y' = \sinh x \;\Rightarrow\; 1 + (y')^2 = 1 + \sinh^2 x = \cosh^2 x$. So

$$L = \textstyle\int_0^1 \cosh x\,dx = [\sinh x]_0^1 = \sinh 1 = \frac{1}{2}(e - 1/e)$$

16. $y^2 = 4x$, $x = \frac{1}{4}y^2 \;\Rightarrow\; \frac{dx}{dy} = \frac{1}{2}y \;\Rightarrow\; 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{4}y^2$. So

$$L = \textstyle\int_0^2 \sqrt{1 + \frac{1}{4}y^2}\,dy = \int_0^1 \sqrt{1 + u^2} \cdot 2\,du \quad (u = \frac{1}{2}y,\ dy = 2\,du)$$
$$\overset{21}{=} \left[u\sqrt{1 + u^2} + \ln\left|u + \sqrt{1 + u^2}\right|\right]_0^1 = \sqrt{2} + \ln\left(1 + \sqrt{2}\right)$$

17. $y = e^x \;\Rightarrow\; y' = e^x \;\Rightarrow\; 1 + (y')^2 = 1 + e^{2x}$. So

$$L = \int_0^1 \sqrt{1 + e^{2x}}\,dx = \int_1^e \sqrt{1 + u^2}\frac{du}{u} \quad [u = e^x,\text{ so } x = \ln u,\ dx = du/u]$$
$$= \int_1^e \frac{\sqrt{1 + u^2}}{u^2} u\,du = \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{v}{v^2 - 1} v\,dv \quad \left[v = \sqrt{1 + u^2},\text{ so } v^2 = 1 + u^2,\ v\,dv = u\,du\right]$$
$$= \int_{\sqrt{2}}^{\sqrt{1+e^2}} \left(1 + \frac{1/2}{v - 1} - \frac{1/2}{v + 1}\right) dv = \left[v + \frac{1}{2}\ln\frac{v - 1}{v + 1}\right]_{\sqrt{2}}^{\sqrt{1+e^2}}$$
$$= \sqrt{1 + e^2} - \sqrt{2} + \frac{1}{2}\ln\frac{\sqrt{1 + e^2} - 1}{\sqrt{1 + e^2} + 1} - \frac{1}{2}\ln\frac{\sqrt{2} - 1}{\sqrt{2} + 1}$$
$$= \sqrt{1 + e^2} - \sqrt{2} + \ln\left(\sqrt{1 + e^2} - 1\right) - 1 - \ln\left(\sqrt{2} - 1\right)$$

Or: Use Formula 23 for $\int\left(\sqrt{1 + u^2}/u\right)du$, or substitute $u = \tan\theta$.

18. $y = \ln\left(\frac{e^x+1}{e^x-1}\right) = \ln(e^x+1) - \ln(e^x-1) \Rightarrow y' = \frac{e^x}{e^x+1} - \frac{e^x}{e^x-1} = \frac{-2e^x}{e^{2x}-1} \Rightarrow$

$1 + (y')^2 = 1 + \frac{4e^{2x}}{(e^{2x}-1)^2} = \frac{(e^{2x}+1)^2}{(e^{2x}-1)^2} \Rightarrow \sqrt{1+(y')^2} = \frac{e^{2x}+1}{e^{2x}-1} = \frac{e^x+e^{-x}}{e^x-e^{-x}} = \frac{\cosh x}{\sinh x}$. So

$$L = \int_a^b \frac{\cosh x}{\sinh x}\,dx = \ln\sinh x\big|_a^b = \ln\left(\frac{\sinh b}{\sinh a}\right) = \ln\left(\frac{e^b - e^{-b}}{e^a - e^{-a}}\right)$$

19. $y = x^3 \Rightarrow y' = 3x^2 \Rightarrow 1 + (y')^2 = 1 + 9x^4$. So $L = \int_0^1 \sqrt{1+9x^4}\,dx$.

20. $y = 2^x \Rightarrow dy/dx = (2^x)\ln 2 \Rightarrow L = \int_0^3 \sqrt{1 + (\ln 2)^2\, 2^{2x}}\,dx$

21. $y = e^x \cos x \Rightarrow y' = e^x(\cos x - \sin x) \Rightarrow$

$$1 + (y')^2 = 1 + e^{2x}\left(\cos^2 x - 2\cos x \sin x + \sin^2 x\right) = 1 + e^{2x}(1 - \sin 2x)$$

So $L = \int_0^{\pi/2} \sqrt{1 + e^{2x}(1 - \sin 2x)}\,dx$.

22. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $y = \pm b\sqrt{1 - x^2/a^2} = \pm\frac{b}{a}\sqrt{a^2-x^2}$. $y = \frac{b}{a}\sqrt{a^2-x^2} \Rightarrow \frac{dy}{dx} = \frac{-bx}{a\sqrt{a^2-x^2}} \Rightarrow$

$\left(\frac{dy}{dx}\right)^2 = \frac{b^2x^2}{a^2(a^2-x^2)}$. So $L = 2\int_{-a}^{a}\left[1 + \frac{b^2x^2}{a^2(a^2-x^2)}\right]^{1/2} dx = \frac{4}{a}\int_0^a \left[\frac{(b^2-a^2)x^2 + a^4}{a^2-x^2}\right]^{1/2} dx.$

23. $y = x^3 \Rightarrow 1 + (y')^2 = 1 + (3x^2)^2 = 1 + 9x^4 \Rightarrow L = \int_0^1 \sqrt{1+9x^4}\,dx$.
Let $f(x) = \sqrt{1+9x^4}$. Then by Simpson's Rule with $n = 10$,
$L \approx \frac{1/10}{3}[f(0) + 4f(0.1) + 2f(0.2) + 4f(0.3) + \cdots + 2f(0.8) + 4f(0.9) + f(1)] \approx 1.548.$

24. $y = \frac{1}{x} \Rightarrow \frac{dy}{dx} = -\frac{1}{x^2} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{x^4}$. Therefore, with $f(x) = \sqrt{1 + \frac{1}{x^4}}$,

$$\begin{aligned} L = \int_1^2 \sqrt{1 + 1/x^4}\,dx &\approx S_{10} \\ &= \tfrac{2-1}{10\cdot 3}[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + 2f(1.4) + 4f(1.5) + 2f(1.6) \\ &\quad + 4f(1.7) + 2f(1.8) + 4f(1.9) + f(2)] \approx 1.132104 \end{aligned}$$

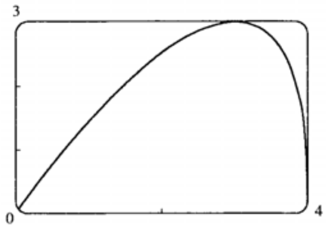
25. $y = \sin x$, $1 + (dy/dx)^2 = 1 + \cos^2 x$, $L = \int_0^\pi \sqrt{1+\cos^2 x}\,dx$. Let $g(x) = \sqrt{1+\cos^2 x}$. Then

$$\begin{aligned} L \approx \tfrac{\pi/10}{3}\Big[&g(0) + 4g\left(\tfrac{\pi}{10}\right) + 2g\left(\tfrac{\pi}{5}\right) + 4g\left(\tfrac{3\pi}{10}\right) + 2g\left(\tfrac{2\pi}{5}\right) + 4g\left(\tfrac{\pi}{2}\right) \\ &+ 2g\left(\tfrac{3\pi}{5}\right) + 4g\left(\tfrac{7\pi}{10}\right) + 2g\left(\tfrac{4\pi}{5}\right) + 4g\left(\tfrac{9\pi}{10}\right) + g(\pi)\Big] \approx 3.820 \end{aligned}$$

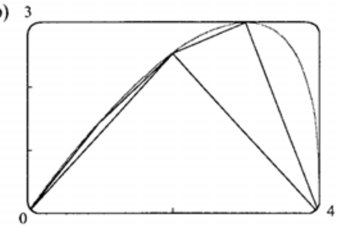
26. $y = \tan x \Rightarrow 1 + (y')^2 = 1 + \sec^4 x$. So $L = \int_0^{\pi/4} \sqrt{1+\sec^4 x}\,dx$. Let $g(x) = \sqrt{1+\sec^4 x}$. Then

$$\begin{aligned} L \approx \tfrac{\pi/40}{3}\Big[&g(0) + 4g\left(\tfrac{\pi}{40}\right) + 2g\left(\tfrac{2\pi}{40}\right) + 4g\left(\tfrac{3\pi}{40}\right) + 2g\left(\tfrac{4\pi}{40}\right) + 4g\left(\tfrac{5\pi}{40}\right) \\ &+ 2g\left(\tfrac{6\pi}{40}\right) + 4g\left(\tfrac{7\pi}{40}\right) + 2g\left(\tfrac{8\pi}{40}\right) + 4g\left(\tfrac{9\pi}{40}\right) + g\left(\tfrac{\pi}{4}\right)\Big] \approx 1.278 \end{aligned}$$

27. (a)



(b)



Let $f(x) = y = x\sqrt[3]{4-x}$. The polygon with one side is just the line segment joining the points $(0, f(0)) = (0, 0)$ and $(4, f(4)) = (4, 0)$, and its length is 4. The polygon with two sides joins the points $(0, 0)$, $(2, f(2)) = \left(2, 2\sqrt[3]{2}\right)$ and $(4, 0)$. Its length is

$$\sqrt{(2-0)^2 + \left(2\sqrt[3]{2} - 0\right)^2} + \sqrt{(4-2)^2 + \left(0 - 2\sqrt[3]{2}\right)^2} = 2\sqrt{4 + 2^{8/3}} \approx 6.43$$

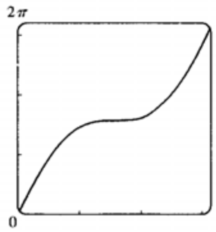
Similarly, the inscribed polygon with four sides joins the points $(0, 0)$, $\left(1, \sqrt[3]{3}\right)$, $\left(2, 2\sqrt[3]{2}\right)$, $(3, 3)$, and $(4, 0)$, so its length is

$$\sqrt{1 + \left(\sqrt[3]{3}\right)^2} + \sqrt{1 + \left(2\sqrt[3]{2} - \sqrt[3]{3}\right)^2} + \sqrt{1 + \left(3 - 2\sqrt[3]{2}\right)^2} + \sqrt{1+9} \approx 7.50$$

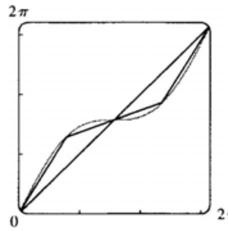
(c) Using the arc length formula with $\dfrac{dy}{dx} = x\left[\frac{1}{3}(4-x)^{-2/3}(-1)\right] + \sqrt[3]{4-x} = \dfrac{12-4x}{3(4-x)^{2/3}}$, the length of the curve is $L = \displaystyle\int_0^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^4 \sqrt{1 + \left[\frac{12-4x}{3(4-x)^{2/3}}\right]^2}\,dx.$

(d) According to a CAS, the length of the curve is $L \approx 7.7988$. The actual value is larger than any of the approximations in part (b). This is always true, since any approximating straight line between two points on the curve is shorter than the length of the curve between the two points.

28. (a)



(b)



Let $f(x) = y = x + \sin x$. The polygon with one side is just the line segment joining the points $(0, f(0)) = (0, 0)$ and $(2\pi, f(2\pi)) = (2\pi, 2\pi)$, and its length is $\sqrt{(2\pi - 0)^2 + (2\pi - 0)^2} = 2\sqrt{2}\pi \approx 8.9$. The polygon with two sides joins the points $(0, 0)$, $(\pi, f(\pi)) = (\pi, \pi)$, and $(2\pi, 2\pi)$. Its length is

$$\sqrt{(\pi - 0)^2 + (\pi - 0)^2} + \sqrt{(2\pi - \pi)^2 + (2\pi - \pi)^2} = \sqrt{2}\pi + \sqrt{2}\pi = 2\sqrt{2}\pi \approx 8.9$$

Note from the diagram that the two approximations are the same because the sides of the 2-sided polygon are in fact on the same line, since $f(\pi) = \pi = \frac{1}{2} f(2\pi)$.

The four-sided polygon joins the points $(0, 0)$, $\left(\frac{\pi}{2}, \frac{\pi}{2} + 1\right)$, (π, π), $\left(\frac{3\pi}{2}, \frac{3\pi}{2} - 1\right)$, and $(2\pi, 2\pi)$, so its length is

$$\sqrt{\left(\tfrac{\pi}{2}\right)^2 + \left(\tfrac{\pi}{2} + 1\right)^2} + \sqrt{\left(\tfrac{\pi}{2}\right)^2 + \left(\tfrac{\pi}{2} - 1\right)^2} + \sqrt{\left(\tfrac{\pi}{2}\right)^2 + \left(\tfrac{\pi}{2} - 1\right)^2} + \sqrt{\left(\tfrac{\pi}{2}\right)^2 + \left(\tfrac{\pi}{2} + 1\right)^2} \approx 9.4$$

(c) Using the arc length formula with $dy/dx = 1 + \cos x$, the length of the curve is

$$L = \int_0^{2\pi} \sqrt{1 + (1 + \cos x)^2}\, dx = \int_0^{2\pi} \sqrt{2 + 2\cos x + \cos^2 x}\, dx$$

(d) The CAS approximates the integral as 9.5076. The actual length is larger than the approximations in part (b).

29. $x = \ln\left(1 - y^2\right) \;\Rightarrow\; \dfrac{dx}{dy} = \dfrac{-2y}{1 - y^2} \;\Rightarrow\; 1 + \left(\dfrac{dx}{dy}\right)^2 = 1 + \dfrac{4y^2}{\left(1 - y^2\right)^2} = \dfrac{\left(1 + y^2\right)^2}{\left(1 - y^2\right)^2}$. So

$$L = \int_0^{1/2} \sqrt{\frac{\left(1 + y^2\right)^2}{\left(1 - y^2\right)^2}}\, dy = \int_0^{1/2} \frac{1 + y^2}{1 - y^2}\, dy = \ln 3 - \tfrac{1}{2} \text{ [from a CAS] } \approx 0.599$$

30. $y = x^{4/3} \;\Rightarrow\; dy/dx = \frac{4}{3}x^{1/3} \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + \frac{16}{9}x^{2/3} \;\Rightarrow$

$$L = \int_0^1 \sqrt{1 + \tfrac{16}{9}x^{2/3}}\, dx = \int_0^{4/3} \sqrt{1 + u^2}\, \tfrac{81}{64} u^2\, du \quad \left[\begin{array}{c} u = \frac{4}{3}x^{1/3},\; du = \frac{4}{9}x^{-2/3}\, dx, \\ dx = \frac{9}{4}x^{2/3}\, du = \frac{9}{4} \cdot \frac{9}{16}u^2\, du = \frac{81}{64}u^2\, du \end{array}\right]$$

$$\stackrel{22}{=} \tfrac{81}{64}\left[\tfrac{1}{8}u\left(1 + 2u^2\right)\sqrt{1 + u^2} - \tfrac{1}{8}\ln\left(u + \sqrt{1 + u^2}\right)\right]_0^{4/3}$$

$$= \tfrac{81}{64}\left[\tfrac{1}{6}\left(1 + \tfrac{32}{9}\right)\sqrt{\tfrac{25}{9}} - \tfrac{1}{8}\ln\left(\tfrac{4}{3} + \sqrt{\tfrac{25}{9}}\right)\right] = \tfrac{81}{64}\left(\tfrac{1}{6} \cdot \tfrac{41}{9} \cdot \tfrac{5}{3} - \tfrac{1}{8}\ln 3\right)$$

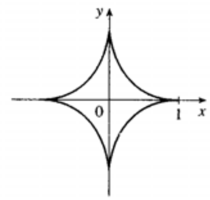
$$= \tfrac{205}{128} - \tfrac{81}{512}\ln 3 \approx 1.4277586$$

31. $y^{2/3} = 1 - x^{2/3} \Rightarrow y = \left(1 - x^{2/3}\right)^{3/2} \Rightarrow$

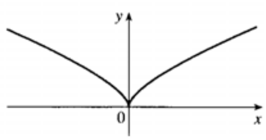
$\dfrac{dy}{dx} = \frac{3}{2}\left(1 - x^{2/3}\right)^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = -x^{-1/3}\left(1 - x^{2/3}\right)^{1/2} \Rightarrow$

$\left(\dfrac{dy}{dx}\right)^2 = x^{-2/3}\left(1 - x^{2/3}\right) = x^{-2/3} - 1$. Thus

$L = 4\int_0^1 \sqrt{1 + \left(x^{-2/3} - 1\right)}\,dx = 4\int_0^1 x^{-1/3}\,dx = 4\lim\limits_{t\to 0^+}\left[\frac{3}{2}x^{2/3}\right]_t^1 = 6$



32. (a)



(b) $y = x^{2/3} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{2}{3}x^{-1/3}\right)^2 = 1 + \frac{4}{9}x^{-2/3}$. So $L = \int_0^1 \sqrt{1 + \frac{4}{9}x^{-2/3}}\,dx$ (an improper integral). $x = y^{3/2} \Rightarrow$ $1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{3}{2}y^{1/2}\right)^2 = 1 + \frac{9}{4}y$.

So $L = \int_0^1 \sqrt{1 + \frac{9}{4}y}\,dy$. The second integral equals $\frac{4}{9}\cdot\frac{2}{3}\left[\left(1 + \frac{9}{4}y\right)^{3/2}\right]_0^1 = \frac{8}{27}\left(\frac{13\sqrt{13}}{8} - 1\right) = \frac{13\sqrt{13}-8}{27}$.

The first integral can be evaluated as follows:

$$\int_0^1 \sqrt{1 + \tfrac{4}{9}x^{-2/3}}\,dx = \int_0^1 \frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}\,dx = \int_0^9 \frac{\sqrt{u+4}}{18}\,du \quad \left(u = 9x^{2/3},\ du = 6x^{-1/3}\,dx\right)$$

$$= \lim_{t\to 0^+}\frac{1}{18}\cdot\left[\tfrac{2}{3}(u+4)^{3/2}\right]_t^9 = \frac{13\sqrt{13}-8}{27}$$

(c) L = length of the arc of this curve from $(-1, 1)$ to $(8, 4)$

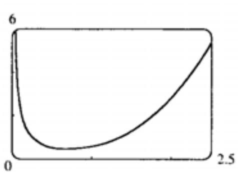
$= \int_0^1 \sqrt{1 + \frac{9}{4}y}\,dy + \int_0^4 \sqrt{1 + \frac{9}{4}y}\,dy = \frac{13\sqrt{13}-8}{27} + \frac{8}{27}\left[\left(1 + \frac{9}{4}y\right)^{3/2}\right]_0^4$ [from part (b)]

$= \frac{13\sqrt{13}-8}{27} + \frac{8}{27}\left(10\sqrt{10} - 1\right) = \frac{13\sqrt{13}+80\sqrt{10}-16}{27}$

33. $y = 2x^{3/2} \Rightarrow y' = 3x^{1/2} \Rightarrow 1 + (y')^2 = 1 + 9x$. The arc length function with starting point $P_0(1, 2)$ is

$$s(x) = \int_1^x \sqrt{1+9t}\,dt = \left[\tfrac{2}{27}(1+9t)^{3/2}\right]_1^x = \tfrac{2}{27}\left[(1+9x)^{3/2} - 10\sqrt{10}\right]$$

34. (a)



(b) $1 + \left(\dfrac{dy}{dx}\right)^2 = x^4 + \dfrac{1}{2} + \dfrac{1}{16x^4}$,

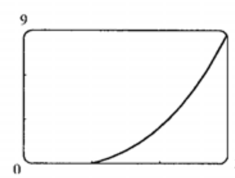
$s(x) = \int_1^x \left[t^2 + 1/\left(4t^2\right)\right]dt$

$= \left[\frac{1}{3}t^3 - 1/(4t)\right]_1^x$

$= \frac{1}{3}x^3 - 1/(4x) - \left(\frac{1}{3} - \frac{1}{4}\right)$

$= \frac{1}{3}x^3 - 1/(4x) - \frac{1}{12}$ for $x \geq 1$

(c)



35. The prey hits the ground when $y = 0 \;\Leftrightarrow\; 180 - \frac{1}{45}x^2 = 0 \;\Leftrightarrow\; x^2 = 45 \cdot 180 \;\Rightarrow\; x = \sqrt{8100} = 90$, since x must be positive. $y' = -\frac{2}{45}x \;\Rightarrow\; 1 + (y')^2 = 1 + \frac{4}{45^2}x^2$, so the distance traveled by the prey is

$$\begin{aligned} L &= \int_0^{90} \sqrt{1 + \tfrac{4}{45^2}x^2}\,dx = \int_0^4 \sqrt{1+u^2}\left(\tfrac{45}{2}\,du\right) \quad [u = \tfrac{2}{45}x,\; du = \tfrac{2}{45}\,dx] \\ &\overset{21}{=} \tfrac{45}{2}\left[\tfrac{1}{2}u\sqrt{1+u^2} + \tfrac{1}{2}\ln\left(u + \sqrt{1+u^2}\right)\right]_0^4 = \tfrac{45}{2}\left[2\sqrt{17} + \tfrac{1}{2}\ln\left(4 + \sqrt{17}\right)\right] \\ &= 45\sqrt{17} + \tfrac{45}{4}\ln\left(4+\sqrt{17}\right) \approx 209.1 \text{ m} \end{aligned}$$

36. $y = 150 - \frac{1}{40}(x-50)^2 \;\Rightarrow\; y' = -\frac{1}{20}(x-50) \;\Rightarrow\; 1 + (y')^2 = 1 + \frac{1}{20^2}(x-50)^2$, so the distance traveled by the kite is

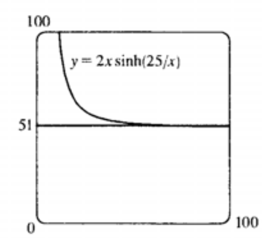
$$\begin{aligned} L &= \int_0^{80} \sqrt{1 + \tfrac{1}{20^2}(x-50)^2}\,dx = \int_{-5/2}^{3/2} \sqrt{1+u^2}\,(20\,du) \quad [u = \tfrac{1}{20}(x-50),\, du = \tfrac{1}{20}\,dx] \\ &\overset{21}{=} 20\left[\tfrac{1}{2}u\sqrt{1+u^2} + \tfrac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\right]_{-5/2}^{3/2} \\ &= 10\left[\tfrac{3}{2}\sqrt{\tfrac{13}{4}} + \ln\left(\tfrac{3}{2} + \sqrt{\tfrac{13}{4}}\right) + \tfrac{5}{2}\sqrt{\tfrac{29}{4}} - \ln\left(-\tfrac{5}{2} + \sqrt{\tfrac{29}{4}}\right)\right] \\ &= \tfrac{15}{2}\sqrt{13} + \tfrac{25}{2}\sqrt{29} + 10\ln\left(\tfrac{3+\sqrt{13}}{-5+\sqrt{29}}\right) \approx 122.8 \text{ ft} \end{aligned}$$

37. The sine wave has amplitude 1 and period 14, since it goes through two periods in a distance of 28 in., so its equation is $y = 1\sin\left(\frac{2\pi}{14}x\right) = \sin\left(\frac{\pi}{7}x\right)$. The width w of the flat metal sheet needed to make the panel is the arc length of the sine curve from $x = 0$ to $x = 28$. We set up the integral to evaluate w using the arc length formula with $\frac{dy}{dx} = \frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)$: $L = \int_0^{28}\sqrt{1 + \left[\frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)\right]^2}\,dx = 2\int_0^{14}\sqrt{1 + \left[\frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)\right]^2}\,dx$. This integral would be very difficult to evaluate exactly, so we use a CAS, and find that $L \approx 29.36$ inches.

38. (a) $y = c + a\cosh\left(\frac{x}{a}\right) \;\Rightarrow\; y' = \sinh\left(\frac{x}{a}\right) \;\Rightarrow\; 1 + (y')^2 = 1 + \sinh^2\left(\frac{x}{a}\right) = \cosh^2\left(\frac{x}{a}\right)$. So

$$L = \int_{-b}^{b}\sqrt{\cosh^2\left(\tfrac{x}{a}\right)}\,dx = 2\int_0^b \cosh\left(\tfrac{x}{a}\right)dx = 2\left[a\sinh\left(\tfrac{x}{a}\right)\right]_0^b = 2a\sinh\left(\tfrac{b}{a}\right)$$

(b) At $x = 0$, $y = c + a$, so $c + a = 20$. The poles are 50 ft apart, so $b = 25$, and $L = 51 \;\Rightarrow\; 51 = 2a\sinh(b/a)$ [from part (a)]. From the figure, we see that $y = 51$ intersects $y = 2x\sinh(25/x)$ at $x \approx 72.3843$ for $x > 0$. So $a \approx 72.3843$ and the wire should be attached at a distance of $y = c + a\cosh(25/a) = 20 - a + a\cosh(25/a) \approx 24.36$ ft above the ground.



39. $y = \int_1^x \sqrt{t^3 - 1}\,dt \;\Rightarrow\; \frac{dy}{dx} = \sqrt{x^3 - 1}$ (by FTC1) $\;\Rightarrow\; 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\sqrt{x^3-1}\right)^2 = x^3 \;\Rightarrow$

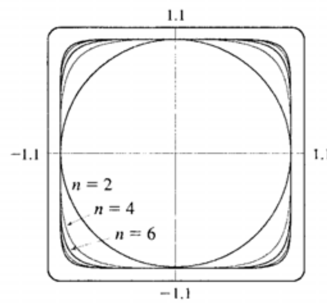
$$L = \int_1^4 \sqrt{x^3}\,dx = \int_1^4 x^{3/2}\,dx = \tfrac{2}{5}\left[x^{5/2}\right]_1^4 = \tfrac{2}{5}(32-1) = \tfrac{62}{5} = 12.4$$

40. By symmetry, the length of the curve in each quadrant is the same, so we find the length in the first quadrant and multiply by 4.

$x^{2k} + y^{2k} = 1 \quad \Rightarrow \quad y^{2k} = 1 - x^{2k} \quad \Rightarrow \quad y = \left(1 - x^{2k}\right)^{1/(2k)}$

(in the first quadrant), so we use the arc length formula with

$$\frac{dy}{dx} = \frac{1}{2k}\left(1 - x^{2k}\right)^{1/(2k)-1}\left(-2kx^{2k-1}\right) = -x^{2k-1}\left(1 - x^{2k}\right)^{1/(2k)-1}$$



The total length is therefore

$$L_{2k} = 4\int_0^1 \sqrt{1 + \left[-x^{2k-1}\left(1 - x^{2k}\right)^{1/(2k)-1}\right]^2}\,dx = 4\int_0^1 \sqrt{1 + x^{2(2k-1)}\left(1 - x^{2k}\right)^{1/k-2}}\,dx$$

Now from the graph, we see that as k increases, the "corners" of these fat circles get closer to the points $(\pm 1, \pm 1)$, and the "edges" of the fat circles approach the lines joining these four points. It seems plausible that as $k \to \infty$, the total length of the fat circle with $n = 2k$ will approach the length of the perimeter of the square with sides of length 2. This is supported by taking the limit as $k \to \infty$ of the equation of the fat circle in the first quadrant: $\lim\limits_{k\to\infty}\left(1 - x^{2k}\right)^{1/(2k)} = 1$ for $0 \le x < 1$. So we guess that $\lim\limits_{k\to\infty} L_{2k} = 4 \cdot 2 = 8$.

9.2 Area of a Surface of Revolution

1. $y = \ln x \quad \Rightarrow \quad ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + (1/x)^2}\,dx \quad \Rightarrow \quad S = \int_1^3 2\pi\,(\ln x)\sqrt{1 + (1/x)^2}\,dx$ [by (7)]

2. $y = \sin^2 x \quad \Rightarrow \quad ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + (2\sin x\cos x)^2}\,dx \quad \Rightarrow$
$S = \int_0^{\pi/2} 2\pi\sin^2 x\sqrt{1 + (2\sin x\cos x)^2}\,dx$ [by (7)]

3. $y = \sec x \quad \Rightarrow \quad ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + (\sec x\tan x)^2}\,dx \quad \Rightarrow$
$S = \int_0^{\pi/4} 2\pi x\sqrt{1 + (\sec x\tan x)^2}\,dx$ [by (8)]

4. $y = e^x \quad \Rightarrow \quad ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + e^{2x}}\,dx \quad \Rightarrow$
$S = \int_0^{\ln 2} 2\pi x\sqrt{1 + e^{2x}}\,dx$ [by (8)] or $\int_1^2 2\pi\,(\ln y)\sqrt{1 + (1/y)^2}\,dy$ [by (6)]

5. $y = x^3 \quad \Rightarrow \quad y' = 3x^2$. So

$$S = \int_0^2 2\pi y\sqrt{1 + (y')^2}\,dx = 2\pi\int_0^2 x^3\sqrt{1 + 9x^4}\,dx \quad (u = 1 + 9x^4,\ du = 36x^3\,dx)$$
$$= \tfrac{2\pi}{36}\int_1^{145}\sqrt{u}\,du = \tfrac{\pi}{18}\left[\tfrac{2}{3}u^{3/2}\right]_1^{145} = \tfrac{\pi}{27}\left(145\sqrt{145} - 1\right)$$

6. The curve $y^2 = 4x + 4$ is symmetric about the x-axis, which is the axis of rotation, so we need only consider the upper half of the curve, given by $y = \sqrt{4x+4} = 2\sqrt{x+1}$. Then

$\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x+1}} \quad\Rightarrow\quad \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \sqrt{1 + \dfrac{1}{x+1}}$. So

$$S = 2\pi \int_0^8 2\sqrt{x+1}\sqrt{1 + \frac{1}{x+1}}\, dx = 4\pi \int_0^8 \sqrt{x+2}\, dx = 4\pi \left[\tfrac{2}{3}(x+2)^{3/2}\right]_0^8 = \tfrac{8\pi}{3}\left(10\sqrt{10} - 2\sqrt{2}\right)$$

Another Method: Use $S = \int_2^6 2\pi y \sqrt{1 + (dx/dy)^2}\, dy$, where $x = \frac{1}{4}y^2 - 1$.

7. $y = \sqrt{x} \quad\Rightarrow\quad 1 + (dy/dx)^2 = 1 + \left[1/\left(2\sqrt{x}\right)\right]^2 = 1 + 1/(4x)$. So

$$\begin{aligned} S &= \int_4^9 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_4^9 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}}\, dx = 2\pi \int_4^9 \sqrt{x + \tfrac{1}{4}}\, dx \\ &= 2\pi \left[\tfrac{2}{3}\left(x + \tfrac{1}{4}\right)^{3/2}\right]_4^9 = \tfrac{4\pi}{3}\left[\tfrac{1}{8}(4x+1)^{3/2}\right]_4^9 = \tfrac{\pi}{6}\left(37\sqrt{37} - 17\sqrt{17}\right) \end{aligned}$$

8. $y = \dfrac{x^2}{4} - \dfrac{\ln x}{2} \quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{x}{2} - \dfrac{1}{2x} \quad\Rightarrow\quad 1 + \left(\dfrac{dy}{dx}\right)^2 = \dfrac{x^2}{4} + \dfrac{1}{2} + \dfrac{1}{4x^2}$. So

$$\begin{aligned} S &= 2\pi \int_1^4 \left(\frac{x^2}{4} - \frac{\ln x}{2}\right)\left(\frac{x}{2} + \frac{1}{2x}\right) dx = \frac{\pi}{2}\int_1^4 \left(\frac{x^2}{2} - \ln x\right)\left(x + \frac{1}{x}\right) dx \\ &= \frac{\pi}{2}\int_1^4 \left(\frac{x^3}{2} + \frac{x}{2} - x\ln x - \frac{\ln x}{x}\right) dx = \frac{\pi}{2}\left[\frac{x^4}{8} + \frac{x^2}{4} - \frac{x^2}{2}\ln x + \frac{x^2}{4} - \tfrac{1}{2}(\ln x)^2\right]_1^4 \\ &= \tfrac{\pi}{2}\left[\left(32 + 4 - 8\ln 4 + 4 - \tfrac{1}{2}(\ln 4)^2\right) - \left(\tfrac{1}{8} + \tfrac{1}{4} - 0 + \tfrac{1}{4} - 0\right)\right] = \pi\left[\tfrac{315}{16} - 8\ln 2 - (\ln 2)^2\right] \end{aligned}$$

9. $y = \sin x \quad\Rightarrow\quad 1 + (dy/dx)^2 = 1 + \cos^2 x$. So

$$\begin{aligned} S &= 2\pi \textstyle\int_0^\pi \sin x\sqrt{1 + \cos^2 x}\, dx = 2\pi \int_{-1}^1 \sqrt{1+u^2}\, du \quad (u = -\cos x,\ du = \sin x\, dx) \\ &= 4\pi \textstyle\int_0^1 \sqrt{1+u^2}\, du = 4\pi \int_0^{\pi/4} \sec^3\theta\, d\theta \quad (u = \tan\theta,\ du = \sec^2\theta\, d\theta) \\ &= 2\pi\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]_0^{\pi/4} = 2\pi\left[\sqrt{2} + \ln\left(\sqrt{2}+1\right)\right] \end{aligned}$$

10. $y = \cos 2x \quad\Rightarrow\quad ds = \sqrt{1 + (dy/dx)^2}\, dx = \sqrt{1 + (-2\sin 2x)^2}\, dx \quad\Rightarrow$

$$S = \textstyle\int_0^{\pi/6} 2\pi\cos 2x\sqrt{1 + 4\sin^2 2x}\, dx = 2\pi\int_0^{\sqrt{3}} \sqrt{1+u^2}\left(\tfrac{1}{4}\, du\right) \quad [u = 2\sin 2x,\ du = 4\cos 2x\, dx]$$

$$\overset{21}{=} \tfrac{\pi}{2}\left[\tfrac{1}{2}u\sqrt{1+u^2} + \tfrac{1}{2}\ln\left(u + \sqrt{1+u^2}\right)\right]_0^{\sqrt{3}} = \tfrac{\pi}{2}\left[\tfrac{\sqrt{3}}{2}\cdot 2 + \tfrac{1}{2}\ln\left(\sqrt{3}+2\right)\right] = \tfrac{\pi\sqrt{3}}{2} + \tfrac{\pi}{4}\ln\left(2+\sqrt{3}\right)$$

11. $y = \cosh x \quad\Rightarrow\quad 1 + (dy/dx)^2 = 1 + \sinh^2 x = \cosh^2 x$. So

$$\begin{aligned} S &= 2\pi \textstyle\int_0^1 \cosh x\cosh x\, dx = 2\pi\int_0^1 \tfrac{1}{2}(1 + \cosh 2x)\, dx = \pi\left[x + \tfrac{1}{2}\sinh 2x\right]_0^1 \\ &= \pi\left(1 + \tfrac{1}{2}\sinh 2\right) \text{ or } \pi\left[1 + \tfrac{1}{4}\left(e^2 - e^{-2}\right)\right] \end{aligned}$$

12. $2y = 3x^{2/3}$, $y = \frac{3}{2}x^{2/3} \Rightarrow dy/dx = x^{-1/3} \Rightarrow 1 + (dy/dx)^2 = 1 + x^{-2/3}$. So

$$S = 2\pi \int_1^8 \tfrac{3}{2}x^{2/3}\sqrt{1 + x^{-2/3}}\,dx = 3\pi \int_1^2 u^2\sqrt{1 + 1/u^2}\,3u^2\,du \quad (u = x^{1/3},\ x = u^3,\ dx = 3u^2\,du)$$
$$= 9\pi \int_1^2 u^3\sqrt{u^2+1}\,du = \tfrac{9\pi}{2}\int_1^2 u^2\sqrt{u^2+1}\,2u\,du = \tfrac{9\pi}{2}\int_2^5 (y-1)\sqrt{y}\,dy \quad (y = u^2+1,\ dy = 2u\,du)$$
$$= \tfrac{9\pi}{2}\int_2^5 \left(y^{3/2} - y^{1/2}\right)dy = \tfrac{9\pi}{2}\left[\tfrac{2}{5}y^{5/2} - \tfrac{2}{3}y^{3/2}\right]_2^5 = 9\pi\left[\left(\tfrac{1}{5}\cdot 5^{5/2} - \tfrac{1}{3}\cdot 5^{3/2}\right) - \left(\tfrac{1}{5}\cdot 2^{5/2} - \tfrac{1}{3}\cdot 2^{3/2}\right)\right]$$
$$= 9\pi\left[5\sqrt{5} - \tfrac{5\sqrt{5}}{3} - \tfrac{4\sqrt{2}}{5} + \tfrac{2\sqrt{2}}{3}\right] = \tfrac{3\pi}{5}\left(50\sqrt{5} - 2\sqrt{2}\right)$$

13. $x = \frac{1}{3}\left(y^2+2\right)^{3/2} \Rightarrow dx/dy = \frac{1}{2}\left(y^2+2\right)^{1/2}(2y) = y\sqrt{y^2+2} \Rightarrow$
$1 + (dx/dy)^2 = 1 + y^2\left(y^2+2\right) = \left(y^2+1\right)^2$. So

$$S = 2\pi \int_1^2 y\left(y^2+1\right)dy = 2\pi\left[\tfrac{1}{4}y^4 + \tfrac{1}{2}y^2\right]_1^2 = 2\pi\left(4 + 2 - \tfrac{1}{4} - \tfrac{1}{2}\right) = \tfrac{21\pi}{2}$$

14. $x = 1 + 2y^2 \Rightarrow 1 + (dx/dy)^2 = 1 + (4y)^2 = 1 + 16y^2$. So

$$S = 2\pi \int_1^2 y\sqrt{1+16y^2}\,dy = \tfrac{\pi}{16}\int_1^2 \left(16y^2+1\right)^{1/2} 32y\,dy = \tfrac{\pi}{16}\left[\tfrac{2}{3}\left(16y^2+1\right)^{3/2}\right]_1^2$$
$$= \tfrac{\pi}{24}\left(65\sqrt{65} - 17\sqrt{17}\right)$$

15. $y = \sqrt[3]{x} \Rightarrow x = y^3 \Rightarrow 1 + (dx/dy)^2 = 1 + 9y^4$. So

$$S = 2\pi \int_1^2 x\sqrt{1 + (dx/dy)^2}\,dy = 2\pi \int_1^2 y^3\sqrt{1+9y^4}\,dy = \tfrac{2\pi}{36}\int_1^2 \sqrt{1+9y^4}\,36y^3\,dy$$
$$= \tfrac{\pi}{18}\left[\tfrac{2}{3}\left(1+9y^4\right)^{3/2}\right]_1^2 = \tfrac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right)$$

16. $y = 1 - x^2 \Rightarrow 1 + (dy/dx)^2 = 1 + 4x^2 \Rightarrow$

$$S = 2\pi \int_0^1 x\sqrt{1+4x^2}\,dx = \tfrac{\pi}{4}\int_0^1 8x\sqrt{4x^2+1}\,dx = \tfrac{\pi}{4}\left[\tfrac{2}{3}\left(4x^2+1\right)^{3/2}\right]_0^1 = \tfrac{\pi}{6}\left(5\sqrt{5} - 1\right)$$

17. $x = e^{2y} \Rightarrow 1 + (dx/dy)^2 = 1 + 4e^{4y}$. So

$$S = 2\pi \int_0^{1/2} e^{2y}\sqrt{1 + \left(2e^{2y}\right)^2}\,dy = 2\pi \int_2^{2e} \sqrt{1+u^2}\,\frac{1}{4}\,du \quad (u = 2e^{2y},\ du = 4e^{2y}\,dy)$$
$$= \tfrac{\pi}{2}\int_2^{2e} \sqrt{1+u^2}\,du = \tfrac{\pi}{2}\left[\tfrac{1}{2}u\sqrt{1+u^2} + \tfrac{1}{2}\ln\left|u + \sqrt{1+u^2}\right|\right]_2^{2e} \quad (u = \tan\theta \text{ or use Formula 21})$$
$$= \tfrac{\pi}{2}\left[e\sqrt{1+4e^2} + \tfrac{1}{2}\ln\left(2e + \sqrt{1+4e^2}\right) - \sqrt{5} - \tfrac{1}{2}\ln\left(2+\sqrt{5}\right)\right]$$
$$= \frac{\pi}{4}\left[2e\sqrt{1+4e^2} - 2\sqrt{5} + \ln\left(\frac{2e + \sqrt{1+4e^2}}{2+\sqrt{5}}\right)\right]$$

18. $x = \sqrt{2y - y^2} \Rightarrow \dfrac{dx}{dy} = \dfrac{1-y}{\sqrt{2y-y^2}} \Rightarrow 1 + \left(\dfrac{dx}{dy}\right)^2 = 1 + \dfrac{1 - 2y + y^2}{2y - y^2} = \dfrac{1}{2y - y^2}$. So

$$S = 2\pi \int_0^1 \sqrt{2y-y^2}\left(\frac{1}{\sqrt{2y-y^2}}\right)dy = 2\pi \int_0^1 dy = 2\pi$$

19. $x = \frac{1}{2\sqrt{2}}(y^2 - \ln y) \Rightarrow \frac{dx}{dy} = \frac{1}{2\sqrt{2}}\left(2y - \frac{1}{y}\right) \Rightarrow$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{8}\left(2y - \frac{1}{y}\right)^2 = 1 + \frac{1}{8}\left(4y^2 - 4 + \frac{1}{y^2}\right) = \frac{1}{8}\left(4y^2 + 4 + \frac{1}{y^2}\right) = \left[\frac{1}{2\sqrt{2}}\left(2y + \frac{1}{y}\right)\right]^2$$

So

$$\begin{aligned} S &= 2\pi \int_1^2 \frac{1}{2\sqrt{2}}\left(y^2 - \ln y\right)\frac{1}{2\sqrt{2}}\left(2y + \frac{1}{y}\right)dy = \frac{\pi}{4}\int_1^2 \left(2y^3 + y - 2y\ln y - \frac{\ln y}{y}\right)dy \\ &= \frac{\pi}{4}\left[\frac{1}{2}y^4 + \frac{1}{2}y^2 - y^2\ln y + \frac{1}{2}y^2 - \frac{1}{2}(\ln y)^2\right]_1^2 = \frac{\pi}{8}\left[y^4 + 2y^2 - 2y^2\ln y - (\ln y)^2\right]_1^2 \\ &= \frac{\pi}{8}\left[16 + 8 - 8\ln 2 - (\ln 2)^2 - 1 - 2\right] = \frac{\pi}{8}\left[21 - 8\ln 2 - (\ln 2)^2\right] \end{aligned}$$

20. $x = a\cosh(y/a) \Rightarrow 1 + (dx/dy)^2 = 1 + \sinh^2(y/a) = \cosh^2(y/a)$. So

$$\begin{aligned} S &= 2\pi\int_{-a}^{a} a\cosh\left(\frac{y}{a}\right)\cosh\left(\frac{y}{a}\right)dy = 4\pi a\int_0^a \cosh^2\left(\frac{y}{a}\right)dy = 2\pi a\int_0^a\left[1 + \cosh\left(\frac{2y}{a}\right)\right]dy \\ &= 2\pi a\left[y + \frac{a}{2}\sinh\left(\frac{2y}{a}\right)\right]_0^a = 2\pi a\left[a + \frac{a}{2}\sinh 2\right] = 2\pi a^2\left[1 + \frac{1}{2}\sinh 2\right] \text{ or } \frac{\pi a^2\left(e^2 + 4 - e^{-2}\right)}{2} \end{aligned}$$

21. With $f(x) = x^4\sqrt{16x^6 + 1}$,

$$\begin{aligned} S &= 2\pi\int_0^1 x^4\sqrt{1 + (4x^3)^2}\,dx = 2\pi\int_0^1 x^4\sqrt{16x^6 + 1}\,dx \\ &\approx 2\pi\,\tfrac{1/10}{3}\left[f(0) + 4f(0.1) + 2f(0.2) + 4f(0.3) + 2f(0.4)\right. \\ &\qquad \left. + 4f(0.5) + 2f(0.6) + 4f(0.7) + 2f(0.8) + 4f(0.9) + f(1)\right] \approx 3.44 \end{aligned}$$

22. With $f(x) = \tan x\sqrt{1 + \sec^4 x}$,

$$\begin{aligned} S &= 2\pi\int_0^{\pi/4}\tan x\sqrt{1 + (\sec^2 x)^2}\,dx \\ &\approx 2\pi\,\tfrac{\pi/40}{3}\left[f(0) + 4f\left(\tfrac{\pi}{40}\right) + 2f\left(\tfrac{\pi}{20}\right) + 4f\left(\tfrac{3\pi}{40}\right) + 2f\left(\tfrac{\pi}{10}\right)\right. \\ &\qquad \left. + 4f\left(\tfrac{\pi}{8}\right) + 2f\left(\tfrac{3\pi}{20}\right) + 4f\left(\tfrac{7\pi}{40}\right) + 2f\left(\tfrac{\pi}{5}\right) + 4f\left(\tfrac{9\pi}{40}\right) + f\left(\tfrac{\pi}{4}\right)\right] \approx 3.84 \end{aligned}$$

23. $y = 1/x \Rightarrow ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + (-1/x^2)^2}\,dx = \sqrt{1 + 1/x^4}\,dx \Rightarrow$

$$\begin{aligned} S &= \int_1^2 2\pi\cdot\frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx = 2\pi\int_1^2\frac{\sqrt{x^4 + 1}}{x^3}\,dx = 2\pi\int_1^4\frac{\sqrt{u^2 + 1}}{u^2}\left(\tfrac{1}{2}\,du\right)\quad [u = x^2, du = 2x\,dx] \\ &= \pi\int_1^4\frac{\sqrt{1 + u^2}}{u^2}\,du \overset{24}{=} \pi\left[-\frac{\sqrt{1 + u^2}}{u} + \ln\left(u + \sqrt{1 + u^2}\right)\right]_1^4 \\ &= \pi\left[-\tfrac{\sqrt{17}}{4} + \ln\left(4 + \sqrt{17}\right) + \tfrac{\sqrt{2}}{1} - \ln\left(1 + \sqrt{2}\right)\right] = \pi\left[\sqrt{2} - \tfrac{\sqrt{17}}{4} + \ln\left(\tfrac{4 + \sqrt{17}}{1 + \sqrt{2}}\right)\right] \end{aligned}$$

24. $y = \sqrt{x^2+1} \quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2+1}} \quad\Rightarrow\quad ds = \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\,dx = \sqrt{1+\dfrac{x^2}{x^2+1}}\,dx \quad\Rightarrow$

$$S = \int_0^3 2\pi\sqrt{x^2+1}\sqrt{1+\frac{x^2}{x^2+1}}\,dx = 2\pi\int_0^3 \sqrt{2x^2+1}\,dx = 2\sqrt{2}\pi\int_0^3\sqrt{x^2+\left(\tfrac{1}{\sqrt{2}}\right)^2}\,dx$$

$$\stackrel{21}{=} 2\sqrt{2}\pi\left[\tfrac{1}{2}x\sqrt{x^2+\tfrac{1}{2}}+\tfrac{1}{4}\ln\left(x+\sqrt{x^2+\tfrac{1}{2}}\right)\right]_0^3$$

$$= 2\sqrt{2}\pi\left[\tfrac{3}{2}\sqrt{9+\tfrac{1}{2}}+\tfrac{1}{4}\ln\left(3+\sqrt{9+\tfrac{1}{2}}\right)-\tfrac{1}{4}\ln\tfrac{1}{\sqrt{2}}\right] = 2\sqrt{2}\pi\left[\tfrac{3}{2}\sqrt{\tfrac{19}{2}}+\tfrac{1}{4}\ln\left(3+\sqrt{\tfrac{19}{2}}\right)+\tfrac{1}{4}\ln\sqrt{2}\right]$$

$$= 2\sqrt{2}\pi\left[\tfrac{3}{2}\tfrac{\sqrt{19}}{\sqrt{2}}+\tfrac{1}{4}\ln\left(3\sqrt{2}+\sqrt{19}\right)\right] = 3\sqrt{19}\pi+\tfrac{\pi}{\sqrt{2}}\ln\left(3\sqrt{2}+\sqrt{19}\right)$$

25. $y = x^3$ and $0 \le y \le 1 \quad\Rightarrow\quad y' = 3x^2$ and $0 \le x \le 1$.

$$S = \textstyle\int_0^1 2\pi x\sqrt{1+(3x^2)^2}\,dx = 2\pi\int_0^3\sqrt{1+u^2}\,\tfrac{1}{6}\,du \quad [u = 3x^2,\ du = 6x\,dx]$$

$$= \tfrac{\pi}{3}\textstyle\int_0^3\sqrt{1+u^2}\,du \stackrel{21}{=} \tfrac{\pi}{3}\left[\tfrac{1}{2}u\sqrt{1+u^2}+\tfrac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\right]_0^3$$

$$= \tfrac{\pi}{3}\left[\tfrac{3}{2}\sqrt{10}+\tfrac{1}{2}\ln\left(3+\sqrt{10}\right)\right] = \tfrac{\pi}{6}\left[3\sqrt{10}+\ln\left(3+\sqrt{10}\right)\right]$$

26. $y = \ln(x+1)$, $0 \le x \le 1$. $ds = \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\,dx = \sqrt{1+\left(\dfrac{1}{x+1}\right)^2}\,dx$, so

$$S = \int_0^1 2\pi x\sqrt{1+\frac{1}{(x+1)^2}}\,dx = \int_1^2 2\pi\,(u-1)\sqrt{1+\frac{1}{u^2}}\,du \quad [u = x+1,\ du = dx]$$

$$= 2\pi\int_1^2 u\frac{\sqrt{1+u^2}}{u}\,du - 2\pi\int_1^2\frac{\sqrt{1+u^2}}{u}\,du = 2\pi\int_1^2\sqrt{1+u^2}\,du - 2\pi\int_1^2\frac{\sqrt{1+u^2}}{u}\,du$$

$$\stackrel{21,\,23}{=} 2\pi\left[\tfrac{1}{2}u\sqrt{1+u^2}+\tfrac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\right]_1^2 - 2\pi\left[\sqrt{1+u^2}-\ln\left(\frac{1+\sqrt{1+u^2}}{u}\right)\right]_1^2$$

$$= 2\pi\left[\sqrt{5}+\tfrac{1}{2}\ln\left(2+\sqrt{5}\right)-\tfrac{1}{2}\sqrt{2}-\tfrac{1}{2}\ln\left(1+\sqrt{2}\right)\right] - 2\pi\left[\sqrt{5}-\ln\left(\tfrac{1+\sqrt{5}}{2}\right)-\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right]$$

$$= 2\pi\left[\tfrac{1}{2}\ln\left(2+\sqrt{5}\right)+\ln\left(\tfrac{1+\sqrt{5}}{2}\right)+\tfrac{\sqrt{2}}{2}-\tfrac{3}{2}\ln\left(1+\sqrt{2}\right)\right] \approx 3.694990$$

27. $S = 2\pi\displaystyle\int_1^\infty y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_1^\infty\frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx = 2\pi\int_1^\infty\frac{\sqrt{x^4+1}}{x^3}\,dx$. Rather than trying to evaluate this integral, note that $\sqrt{x^4+1} > \sqrt{x^4} = x^2$ for $x > 0$. Thus, if the area is finite,

$$S = 2\pi\int_1^\infty\frac{\sqrt{x^4+1}}{x^3}\,dx > 2\pi\int_1^\infty\frac{x^2}{x^3}\,dx = 2\pi\int_1^\infty\frac{1}{x}\,dx$$

But we know that this integral diverges, so the area S is infinite.

28. $S = 2\pi \int_0^\infty y\sqrt{1 + (dy/dx)^2}\, dx = 2\pi \int_0^\infty e^{-x}\sqrt{1 + e^{-2x}}\, dx = 2\pi \int_0^1 \sqrt{1 + u^2}\, du \quad \begin{bmatrix} u = e^{-x}, \\ du = -e^{-x}\, dx \end{bmatrix}$

$\overset{21}{=} 2\pi \left[\frac{1}{2}u\sqrt{1 + u^2} + \frac{1}{2}\ln\left|u + \sqrt{1 + u^2}\right|\right]_0^1 = 2\pi\left[\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln\left(1 + \sqrt{2}\right)\right] = \pi\left[\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right]$

29. The curve $8y^2 = x^2\left(1 - x^2\right)$ actually consists of two loops in the region described by the inequalities $|x| \le 1$, $|y| \le \frac{\sqrt{2}}{8}$. (The maximum value of $|y|$ is attained when $|x| = \frac{1}{\sqrt{2}}$.) If we consider the loop in the region $x \ge 0$, the surface area S it generates when rotated about the x-axis is calculated as follows: $16y\dfrac{dy}{dx} = 2x - 4x^3$, so $\left(\dfrac{dy}{dx}\right)^2 = \left(\dfrac{x - 2x^3}{8y}\right)^2 = \dfrac{x^2\left(1 - 2x^2\right)^2}{64y^2} = \dfrac{x^2\left(1 - 2x^2\right)^2}{8x^2\left(1 - x^2\right)} = \dfrac{\left(1 - 2x^2\right)^2}{8\left(1 - x^2\right)}$ for $x \ne 0, \pm 1$. The formula also holds for $x = 0$ by continuity. $1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{\left(1 - 2x^2\right)^2}{8\left(1 - x^2\right)} = \dfrac{9 - 12x^2 + 4x^4}{8\left(1 - x^2\right)} = \dfrac{\left(3 - 2x^2\right)^2}{8\left(1 - x^2\right)}$. So

$$\begin{aligned} S &= 2\pi \int_0^1 \frac{\sqrt{x^2\left(1 - x^2\right)}}{2\sqrt{2}} \cdot \frac{3 - 2x^2}{2\sqrt{2}\sqrt{1 - x^2}}\, dx \\ &= \tfrac{\pi}{4}\int_0^1 x\left(3 - 2x^2\right) dx = \tfrac{\pi}{4}\left[\tfrac{3}{2}x^2 - \tfrac{1}{2}x^4\right]_0^1 = \tfrac{\pi}{4}\left(\tfrac{3}{2} - \tfrac{1}{2}\right) = \tfrac{\pi}{4} \end{aligned}$$

30. In general, if the parabola $y = ax^2$, $-c \le x \le c$, is rotated about the y-axis, the surface area it generates is

$$\begin{aligned} 2\pi \int_0^c x\sqrt{1 + (2ax)^2}\, dx &= 2\pi \int_0^{2ac} \frac{u}{2a}\sqrt{1 + u^2}\,\frac{1}{2a}\, du \quad \begin{pmatrix} u = 2ax, \\ du = 2a\, dx \end{pmatrix} \quad = \frac{\pi}{4a^2}\int_0^{2ac}\left(1 + u^2\right)^{1/2} 2u\, du \\ &= \frac{\pi}{4a^2}\left[\tfrac{2}{3}\left(1 + u^2\right)^{3/2}\right]_0^{2ac} = \frac{\pi}{6a^2}\left[\left(1 + 4a^2c^2\right)^{3/2} - 1\right] \end{aligned}$$

Here $2c = 10$ ft and $ac^2 = 2$ ft, so $c = 5$ and $a = \frac{2}{25}$. Thus, the surface area is

$$\begin{aligned} S &= \tfrac{\pi}{6}\tfrac{625}{4}\left[\left(1 + 4 \cdot \tfrac{4}{625} \cdot 25\right)^{3/2} - 1\right] = \tfrac{625\pi}{24}\left[\left(1 + \tfrac{16}{25}\right)^{3/2} - 1\right] = \tfrac{625\pi}{24}\left(\tfrac{41\sqrt{41}}{125} - 1\right) \\ &= \tfrac{5\pi}{24}\left(41\sqrt{41} - 125\right) \approx 90.01\ \text{ft}^2 \end{aligned}$$

31. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad \Rightarrow \quad \dfrac{y\,(dy/dx)}{b^2} = -\dfrac{x}{a^2} \quad \Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{b^2 x}{a^2 y} \quad \Rightarrow$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{b^4x^2}{a^4y^2} = \frac{b^4x^2 + a^4y^2}{a^4y^2} = \frac{b^4x^2 + a^4b^2\left(1 - x^2/a^2\right)}{a^4b^2\left(1 - x^2/a^2\right)} = \frac{a^4b^2 + b^4x^2 - a^2b^2x^2}{a^4b^2 - a^2b^2x^2}$$
$$= \frac{a^4 + b^2x^2 - a^2x^2}{a^4 - a^2x^2} = \frac{a^4 - \left(a^2 - b^2\right)x^2}{a^2\left(a^2 - x^2\right)}$$

The ellipsoid's surface area is twice the area generated by rotating the first quadrant portion of the ellipse about the x-axis. Thus,

$$S = 2\int_0^a 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 4\pi\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,\frac{\sqrt{a^4 - \left(a^2 - b^2\right)x^2}}{a\sqrt{a^2 - x^2}}\,dx$$
$$= \frac{4\pi b}{a^2}\int_0^a \sqrt{a^4 - \left(a^2 - b^2\right)x^2}\,dx = \frac{4\pi b}{a^2}\int_0^{a\sqrt{a^2-b^2}} \sqrt{a^4 - u^2}\,\frac{du}{\sqrt{a^2 - b^2}} \quad \left(u = \sqrt{a^2 - b^2}\,x\right)$$
$$\overset{30}{=} \frac{4\pi b}{a^2\sqrt{a^2 - b^2}}\left[\frac{u}{2}\sqrt{a^4 - u^2} + \frac{a^4}{2}\sin^{-1}\frac{u}{a^2}\right]_0^{a\sqrt{a^2-b^2}}$$
$$= \frac{4\pi b}{a^2\sqrt{a^2 - b^2}}\left[\frac{a\sqrt{a^2 - b^2}}{2}\sqrt{a^4 - a^2\left(a^2 - b^2\right)} + \frac{a^4}{2}\sin^{-1}\frac{\sqrt{a^2 - b^2}}{a}\right] = 2\pi\left[b^2 + \frac{a^2 b\sin^{-1}\dfrac{\sqrt{a^2 - b^2}}{a}}{\sqrt{a^2 - b^2}}\right]$$

32. The upper half of the torus is generated by rotating the curve $(x - R)^2 + y^2 = r^2$, $y > 0$, about the y-axis.

$y\dfrac{dy}{dx} = -(x - R) \quad \Rightarrow \quad 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{(x - R)^2}{y^2} = \dfrac{y^2 + (x - R)^2}{y^2} = \dfrac{r^2}{r^2 - (x - R)^2}$. Thus,

$$S = 2\int_{R-r}^{R+r} 2\pi x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 4\pi\int_{R-r}^{R+r} \frac{rx}{\sqrt{r^2 - (x - R)^2}}\,dx = 4\pi r\int_{-r}^{r} \frac{u + R}{\sqrt{r^2 - u^2}}\,du \quad (u = x - R)$$
$$= 4\pi r\int_{-r}^{r} \frac{u\,du}{\sqrt{r^2 - u^2}} + 4\pi Rr\int_{-r}^{r} \frac{du}{\sqrt{r^2 - u^2}}$$
$$= 4\pi r \cdot 0 + 8\pi Rr\int_0^r \frac{du}{\sqrt{r^2 - u^2}} \quad \text{(since the first integrand is odd and the second is even)}$$
$$= 8\pi Rr\left[\sin^{-1}(u/r)\right]_0^r = 8\pi Rr\left(\tfrac{\pi}{2}\right) = 4\pi^2 Rr$$

33. The analogue of $f\left(x_i^*\right)$ in the derivation of (4) is now $c - f\left(x_i^*\right)$, so

$$S = \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi\left[c - f\left(x_i^*\right)\right]\sqrt{1 + \left[f'\left(x_i^*\right)\right]^2}\,\Delta x = \int_a^b 2\pi\left[c - f(x)\right]\sqrt{1 + [f'(x)]^2}\,dx$$

34. $y = x^{1/2} \Rightarrow y' = \frac{1}{2}x^{-1/2} \Rightarrow 1 + (y')^2 = 1 + 1/4x$, so by Exercise 33,
$S = \int_0^4 2\pi\,(4 - \sqrt{x})\,\sqrt{1 + 1/(4x)}\,dx$. Using a CAS, we get

$$S = 2\pi \ln\left(\sqrt{17} + 4\right) + \tfrac{\pi}{6}\left(31\sqrt{17} + 1\right) \approx 80.6095$$

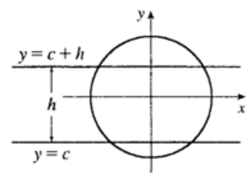
35. For the upper semicircle, $f(x) = \sqrt{r^2 - x^2}$, $f'(x) = -x/\sqrt{r^2 - x^2}$. The surface area generated is

$$S_1 = \int_{-r}^{r} 2\pi\left(r - \sqrt{r^2 - x^2}\right)\sqrt{1 + \frac{x^2}{r^2 - x^2}}\,dx = 4\pi\int_0^r \left(r - \sqrt{r^2 - x^2}\right)\frac{r}{\sqrt{r^2 - x^2}}\,dx$$
$$= 4\pi\int_0^r \left(\frac{r^2}{\sqrt{r^2 - x^2}} - r\right)dx$$

For the lower semicircle, $f(x) = -\sqrt{r^2 - x^2}$ and $f'(x) = \dfrac{x}{\sqrt{r^2 - x^2}}$, so $S_2 = 4\pi\displaystyle\int_0^r \left(\frac{r^2}{\sqrt{r^2 - x^2}} + r\right)dx$.

Thus, the total area is $S = S_1 + S_2 = 8\pi\displaystyle\int_0^r \left(\frac{r^2}{\sqrt{r^2 - x^2}}\right)dx = 8\pi\left[r^2 \sin^{-1}\left(\frac{x}{r}\right)\right]_0^r = 8\pi r^2\left(\frac{\pi}{2}\right) = 4\pi^2 r^2$.

36. Take the sphere $x^2 + y^2 + z^2 = \frac{1}{4}d^2$ and let the intersecting planes be $y = c$ and $y = c + h$, where $-\frac{1}{2}d \le c \le \frac{1}{2}d - h$. The sphere intersects the xy-plane in the circle $x^2 + y^2 = \frac{1}{4}d^2$. From this equation, we get $x\dfrac{dx}{dy} + y = 0$, so $\dfrac{dx}{dy} = -\dfrac{y}{x}$. The desired surface area is



$$S = 2\pi\int x\,ds = 2\pi\int_c^{c+h} x\sqrt{1 + (dx/dy)^2}\,dy = 2\pi\int_c^{c+h} x\sqrt{1 + y^2/x^2}\,dy = 2\pi\int_c^{c+h}\sqrt{x^2 + y^2}\,dy$$
$$= 2\pi\int_c^{c+h} \tfrac{1}{2}d\,dy = \pi d\int_c^{c+h} dy = \pi d h$$

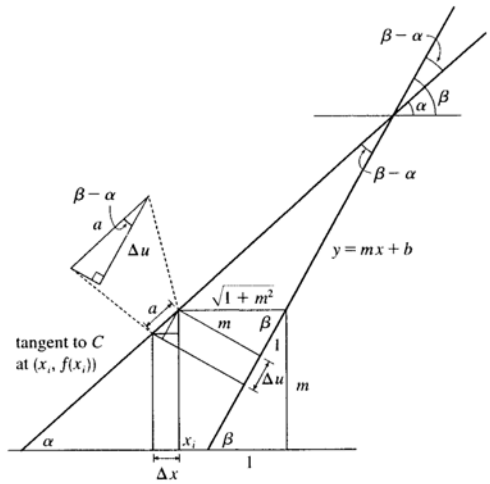
37. In the derivation of (4), we computed a typical contribution to the surface area to be $2\pi\dfrac{y_{i-1} + y_i}{2}\,|P_{i-1}P_i|$, the area of a frustum of a cone. When $f(x)$ is not necessarily positive, the approximations $y_i = f(x_i) \approx f(x_i^*)$ and $y_{i-1} = f(x_{i-1}) \approx f(x_i^*)$ must be replaced by $y_i = |f(x_i)| \approx |f(x_i^*)|$ and $y_{i-1} = |f(x_{i-1})| \approx |f(x_i^*)|$. Thus, $2\pi\dfrac{y_{i-1} + y_i}{2}\,|P_{i-1}P_i| \approx 2\pi\,|f(x_i^*)|\sqrt{1 + [f'(x_i^*)]^2}\,\Delta x$. Continuing with the rest of the derivation as before, we obtain $S = \int_a^b 2\pi\,|f(x)|\sqrt{1 + [f'(x)]^2}\,dx$.

38. Since $g(x) = f(x) + c$, we have $g'(x) = f'(x)$. Thus,

$$S_g = \int_a^b 2\pi g(x)\sqrt{1 + [g'(x)]^2}\,dx = \int_a^b 2\pi\,[f(x) + c]\sqrt{1 + [f'(x)]^2}\,dx$$
$$= \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx + 2\pi c\int_a^b \sqrt{1 + [f'(x)]^2}\,dx = S_f + 2\pi c L$$

Discovery Project □ Rotating on a Slant

1.



In the figure, the segment a lying above the interval $[x_i - \Delta x, x_i]$ along the tangent to C has length $\Delta x \sec \alpha = \Delta x \sqrt{1 + \tan^2 \alpha} = \sqrt{1 + [f'(x_i)]^2}\,\Delta x$. The segment from $(x_i, f(x_i))$ drawn perpendicular to the line $y = mx + b$ has length

$$g(x_i) = [f(x_i) - mx_i - b]\cos\beta = \frac{f(x_i) - mx_i - b}{\sec\beta} = \frac{f(x_i) - mx_i - b}{\sqrt{1 + \tan^2\beta}} = \frac{f(x_i) - mx_i - b}{\sqrt{1 + m^2}}$$

Also, $\cos(\beta - \alpha) = \dfrac{\Delta u}{\Delta x \sec\alpha} \quad \Rightarrow$

$$\Delta u = \Delta x \sec\alpha \cos(\beta - \alpha) = \Delta x\,\frac{\cos\beta\cos\alpha + \sin\beta\sin\alpha}{\cos\alpha} = \Delta x\,(\cos\beta + \sin\beta\tan\alpha)$$

$$= \Delta x\left[\frac{1}{\sqrt{1+m^2}} + \frac{m}{\sqrt{1+m^2}}\,f'(x_i)\right] = \frac{1 + mf'(x_i)}{\sqrt{1+m^2}}\,\Delta x$$

Thus,

$$\text{Area}(\mathscr{R}) = \lim_{n\to\infty}\sum_{i=1}^{n} g(x_i)\,\Delta u = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{f(x_i) - mx_i - b}{\sqrt{1+m^2}}\cdot\frac{1 + mf'(x_i)}{\sqrt{1+m^2}}\,\Delta x$$

$$= \frac{1}{1+m^2}\int_p^q [f(x) - mx - b]\left[1 + mf'(x)\right]dx$$

2. From Problem 1,

$$\text{Area} = \frac{1}{1+1^2}\int_0^{2\pi}[x+\sin x-(x-2)][1+1(1+\cos x)]\,dx = \tfrac{1}{2}\int_0^{2\pi}(\sin x+2)(2+\cos x)\,dx$$
$$= \tfrac{1}{2}\int_0^{2\pi}(2\sin x+\sin x\cos x+4+2\cos x)\,dx = \tfrac{1}{2}\left[-2\cos x+\tfrac{1}{2}\sin^2 x+4x+2\sin x\right]_0^{2\pi}$$
$$= \tfrac{1}{2}[(-2+0+8\pi+0)-(-2+0+0+0)] = \tfrac{1}{2}(8\pi) = 4\pi$$

3. $$V = \lim_{n\to\infty}\sum_{i=1}^{n}\pi[g(x_i)]^2\,\Delta u = \lim_{n\to\infty}\sum_{i=1}^{n}\pi\left[\frac{f(x_i)-mx_i-b}{\sqrt{1+m^2}}\right]^2\frac{1+mf'(x_i)}{\sqrt{1+m^2}}\,\Delta x$$
$$= \frac{\pi}{(1+m^2)^{3/2}}\int_p^q [f(x)-mx-b]^2[1+mf'(x)]\,dx$$

4. $$V = \frac{\pi}{(1+1^2)^{3/2}}\int_0^{2\pi}(x+\sin x-x+2)^2(1+1+\cos x)\,dx = \frac{\pi}{2\sqrt{2}}\int_0^{2\pi}(\sin x+2)^2(\cos x+2)\,dx$$
$$= \tfrac{\pi}{2\sqrt{2}}\int_0^{2\pi}\left(\sin^2 x+4\sin x+4\right)(\cos x+2)\,dx$$
$$= \tfrac{\pi}{2\sqrt{2}}\int_0^{2\pi}\left(\sin^2 x\cos x+4\sin x\cos x+4\cos x+2\sin^2 x+8\sin x+8\right)dx$$
$$= \tfrac{\pi}{2\sqrt{2}}\left[\tfrac{1}{3}\sin^3 x+2\sin^2 x+4\sin x+x-\tfrac{1}{2}\sin 2x-8\cos x+8x\right]_0^{2\pi} \quad [\text{since } 2\sin^2 x = 1-\cos 2x]$$
$$= \tfrac{\pi}{2\sqrt{2}}[(2\pi-8+16\pi)-(-8)] = \tfrac{9\sqrt{2}}{2}\pi^2$$

5. $$S = \int_p^q 2\pi g(x)\sqrt{1+[f'(x)]^2}\,dx = \frac{2\pi}{\sqrt{1+m^2}}\int_p^q [f(x)-mx-b]\sqrt{1+[f'(x)]^2}\,dx$$

6. From Problem 5 with $f(x)=\sqrt{x}$, $p=0$, $q=4$, $m=\frac{1}{2}$, and $b=0$,

$$S = \frac{2\pi}{\sqrt{1+\left(\frac{1}{2}\right)^2}}\int_0^4\left(\sqrt{x}-\tfrac{1}{2}x\right)\sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\,dx = \frac{\pi}{\sqrt{5}}\left[\frac{\ln\left(\sqrt{17}+4\right)}{32}+\frac{37\sqrt{17}}{24}-\frac{1}{3}\right] \quad \text{(from CAS)}$$
$$\approx 8.554$$

9.3 Applications to Physics and Engineering

1. The weight density of water is $\delta = 62.5\ \text{lb/ft}^3$.

(a) $P = \delta d \approx \left(62.5\ \text{lb/ft}^3\right)(3\ \text{ft}) = 187.5\ \text{lb/ft}^2$

(b) $F = PA \approx \left(187.5\ \text{lb/ft}^2\right)(5\ \text{ft})(2\ \text{ft}) = 1875\ \text{lb}$. ($A$ is the area of the bottom of the tank.)

(c) As in Example 1, the area of the ith strip is $2(\Delta x)$ and the pressure is $\delta d = \delta x_i$. Thus,

$$F = \int_0^3 \delta x\cdot 2\,dx \approx (62.5)(2)\int_0^3 x\,dx = 125\left[\tfrac{1}{2}x^2\right]_0^3 = 125\left(\tfrac{9}{2}\right) = 562.5\ \text{lb}$$

2. (a) $P = \rho g d = 1030(9.8)(2.5) = 25{,}235 \approx 2.52\times 10^4\ \text{Pa} = 25.2\ \text{kPa}$

(b) $F = PA \approx \left(2.52\times 10^4\ \text{N/m}^2\right)\left(50\ \text{m}^2\right) = 1.26\times 10^6\ \text{N}$

(c) $F = \int_0^{2.5}\rho g x\cdot 5\,dx = (1030)(9.8)(5)\int_0^{2.5}x\,dx \approx 2.52\times 10^4\left[x^2\right]_0^{2.5} \approx 1.58\times 10^5\ \text{N}$

In Exercises 3–9, n is the number of subintervals of length Δx and x_i^* is a sample point in the ith subinterval $[x_{i-1}, x_i]$.

3. In the middle of the figure in the text, draw a vertical x-axis that increases in the downward direction. The area of the ith rectangular strip is $2\sqrt{100-(x_i^*)^2}\,\Delta x$ and the pressure on the strip is $\rho g x_i^*$ [$\rho = 1000\text{ kg/m}^3$ and $g = 9.8\text{ m/s}^2$]. Thus, the hydrostatic force on the ith strip is the product $\rho g x_i^* 2\sqrt{100-(x_i^*)^2}\,\Delta x$.

$$F = \lim_{n\to\infty}\sum_{i=1}^{n}\rho g x_i^* 2\sqrt{100-(x_i^*)^2}\,\Delta x = \int_0^{10}\rho g x\cdot 2\sqrt{100-x^2}\,dx = 9.8\times 10^3\int_0^{10}\sqrt{100-x^2}\,2x\,dx$$
$$= 9.8\times 10^3\int_{100}^{0}u^{1/2}\,(-du)\quad (u = 100-x^2)\quad = 9.8\times 10^3\int_0^{100}u^{1/2}\,du$$
$$= 9.8\times 10^3\left[\tfrac{2}{3}u^{3/2}\right]_0^{100} = \tfrac{2}{3}\cdot 9.8\times 10^6 \approx 6.5\times 10^6\text{ N}$$

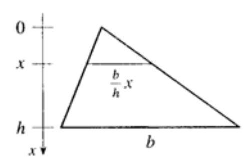
4. This is like Exercise 3, except that the pressure on the strip is $\rho g\,(x_i^* - 5)$.

$$F = \lim_{n\to\infty}\sum_{i=1}^{n}\rho g\,(x_i^*-5)\,2\sqrt{100-(x_i^*)^2}\,\Delta x = \int_5^{10}\rho g\,(x-5)\cdot 2\sqrt{100-x^2}\,dx$$
$$= \rho g\int_5^{10}2x\sqrt{100-x^2}\,dx - 10\rho g\int_5^{10}\sqrt{100-x^2}\,dx$$
$$= -\rho g\left[\tfrac{2}{3}\left(100-x^2\right)^{3/2}\right]_5^{10} - 10\rho g\left[\tfrac{1}{2}x\sqrt{100-x^2} + 50\sin^{-1}(x/10)\right]_5^{10}$$
$$= \tfrac{2}{3}\rho g\,(75)^{3/2} - 10\rho g\left[50\left(\tfrac{\pi}{2}\right) - \tfrac{5}{2}\sqrt{75} - 50\left(\tfrac{\pi}{6}\right)\right] = 250\rho g\left(\tfrac{3\sqrt{3}}{2} - \tfrac{2\pi}{3}\right) \approx 1.23\times 10^6\text{ N}$$

5. Place an x-axis as in Exercise 3. Using similar triangles, $\dfrac{4\text{ ft wide}}{6\text{ ft high}} = \dfrac{w\text{ ft wide}}{x_i^*\text{ ft high}}$, so $w = \frac{4}{6}x_i^*$ and the area of the ith rectangular strip is $\frac{4}{6}x_i^*\,\Delta x$. The pressure on the strip is $\delta\,(x_i^*-2)$ [$\delta = \rho g = 62.5\text{ lb/ft}^3$] and the hydrostatic force is $\delta\,(x_i^*-2)\,\frac{4}{6}x_i^*\,\Delta x$.

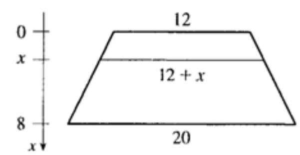
$$F = \lim_{n\to\infty}\sum_{i=1}^{n}\delta\,(x_i^*-2)\,\tfrac{4}{6}x_i^*\,\Delta x = \int_2^6\delta\,(x-2)\,\tfrac{2}{3}x\,dx = \tfrac{2}{3}\delta\int_2^6\left(x^2-2x\right)dx = \tfrac{2}{3}\delta\left[\tfrac{1}{3}x^3 - x^2\right]_2^6$$
$$= \tfrac{2}{3}\delta\left[36 - \left(-\tfrac{4}{3}\right)\right] = \tfrac{224}{9}\delta \approx 1.56\times 10^3\text{ lb}$$

6. This is like Exercise 5, except that the area of the ith rectangular strip is $\dfrac{b}{h}x_i^*\,\Delta x$ and the pressure on the strip is δx_i^*.



$$F = \lim_{n\to\infty}\sum_{i=1}^{n}\rho g x_i^*\frac{b}{h}x_i^*\,\Delta x = \int_0^h\rho g x\cdot\frac{b}{h}x\,dx = \left[\frac{\rho g b}{3h}x^3\right]_0^h$$
$$= \tfrac{1}{3}\rho g b h^2$$

7. Using similar triangles, $\dfrac{4 \text{ ft wide}}{8 \text{ ft high}} = \dfrac{a \text{ ft wide}}{x_i^* \text{ ft high}}$, so $a = \frac{1}{2}x_i^*$ and the width of the ith rectangular strip is $12 + 2a = 12 + x_i^*$. The area of the strip is $(12 + x_i^*)\,\Delta x$. The pressure on the strip is δx_i^*.

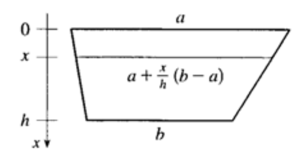


$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \delta x_i^* \left(12 + x_i^*\right) \Delta x = \int_0^8 \delta x \cdot (12 + x)\, dx$$

$$= \delta \int_0^8 \left(12x + x^2\right) dx = \delta \left[6x^2 + \tfrac{x^3}{3}\right]_0^8 = \delta \left(384 + \tfrac{512}{3}\right)$$

$$= (62.5)\,\tfrac{1664}{3} \approx 3.47 \times 10^4 \text{ lb}$$

8. In the figure, deleting a $b \times h$ rectangle leaves a triangle with base $a - b$ and height h. By similar triangles, $\dfrac{(a-b) \text{ ft wide}}{h \text{ ft high}} = \dfrac{d \text{ ft wide}}{(h - x_i^*) \text{ ft high}}$, so the width of the triangle is



$$d = \frac{h - x_i^*}{h}(a - b) = \left(1 - \frac{x_i^*}{h}\right)(a - b) = a - b + \frac{x_i^*}{h}(a - b)$$

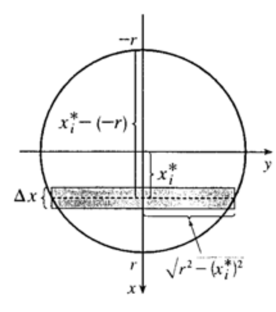
and the width of the trapezoid is $b + d = a + \dfrac{x_i^*}{h}(a - b)$. The area of the ith rectangular strip is $\left[a + \dfrac{x_i^*}{h}(a - b)\right]\Delta x$ and the pressure on it is $\rho g x_i^*$.

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho g x_i^* \left[a + \frac{x_i^*}{h}(a - b)\right] \Delta x = \int_0^h \rho g x \left[a + \frac{x}{h}(b - a)\right] dx$$

$$= \rho g a \int_0^h x\, dx + \frac{\rho g (b - a)}{h} \int_0^h x^2\, dx = \rho g a \frac{h^2}{2} + \rho g \frac{b - a}{h} \frac{h^3}{3}$$

$$= \rho g h^2 \left(\frac{a}{2} + \frac{b - a}{3}\right) = \rho g h^2 \frac{a + 2b}{6} \approx \frac{500}{3} g h^2 (a + 2b) \text{ N}$$

9. From the figure, the area of the ith rectangular strip is $2\sqrt{r^2 - (x_i^*)^2}\,\Delta x$ and the pressure on it is $\rho g \left(x_i^* + r\right)$.



$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho g \left(x_i^* + r\right) 2\sqrt{r^2 - (x_i^*)^2}\,\Delta x$$

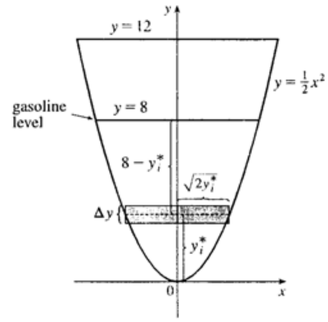
$$= \int_{-r}^{r} \rho g (x + r) \cdot 2\sqrt{r^2 - x^2}\, dx$$

$$= \rho g \int_{-r}^{r} \sqrt{r^2 - x^2}\, 2x\, dx + 2\rho g r \int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$$

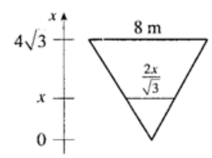
The first integral is 0 because the integrand is an odd function. The second integral can be interpreted as the area of a semicircular disk with radius r, or we could make the trigonometric substitution $x = r \sin \theta$. Continuing: $F = \rho g \cdot 0 + 2\rho g r \cdot \frac{1}{2}\pi r^2 = \rho g \pi r^3 = 1000 g \pi r^3$ N (SI units assumed).

10. The area of the ith rectangular strip is $2\sqrt{2y_i^*}\,\Delta y$ and the pressure on it is $\delta d_i = \delta\,(8 - y_i^*)$.

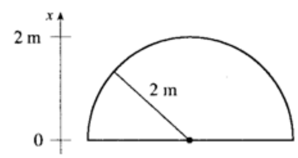
$$F = \int_0^8 \delta\,(8-y)\,2\sqrt{2y}\,dy = 42\cdot 2\cdot \sqrt{2}\int_0^8 (8-y)\,y^{1/2}\,dy$$
$$= 84\sqrt{2}\int_0^8 \left(8y^{1/2} - y^{3/2}\right)dy = 84\sqrt{2}\left[8\cdot\tfrac{2}{3}y^{3/2} - \tfrac{2}{5}y^{5/2}\right]_0^8$$
$$= 84\sqrt{2}\left[8\cdot\tfrac{2}{3}\cdot 16\sqrt{2} - \tfrac{2}{5}\cdot 128\sqrt{2}\right]$$
$$= 84\sqrt{2}\cdot 256\sqrt{2}\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = 43{,}008\cdot\tfrac{2}{15} = 5734.4 \text{ lb}$$



11. $$F = \int_0^{4\sqrt{3}} \rho g\left(4\sqrt{3} - x\right)\frac{2x}{\sqrt{3}}\,dx = 8\rho g\int_0^{4\sqrt{3}} x\,dx - \frac{2\rho g}{\sqrt{3}}\int_0^{4\sqrt{3}} x^2\,dx$$
$$= 4\rho g\left[x^2\right]_0^{4\sqrt{3}} - \frac{2\rho g}{3\sqrt{3}}\left[x^3\right]_0^{4\sqrt{3}} = 192\rho g - \frac{2\rho g}{3\sqrt{3}}64\cdot 3\sqrt{3}$$
$$= 192\rho g - 128\rho g = 64\rho g \approx 64\,(840)\,(9.8) \approx 5.27\times 10^5 \text{ N}$$



12. $$F = \int_0^2 \rho g\,(10 - x)\,2\sqrt{4 - x^2}\,dx$$
$$= 20\rho g\int_0^2 \sqrt{4 - x^2}\,dx - \rho g\int_0^2 \sqrt{4 - x^2}\,2x\,dx$$
$$= 20\rho g\tfrac{1}{4}\pi\left(2^2\right) - \rho g\int_0^4 u^{1/2}\,du \quad \left(u = 4 - x^2,\ du = -2x\,dx\right)$$
$$= 20\pi\rho g - \tfrac{2}{3}\rho g\left[u^{3/2}\right]_0^4 = 20\pi\rho g - \tfrac{16}{3}\rho g = \rho g\left(20\pi - \tfrac{16}{3}\right)$$
$$\approx 5.63\times 10^5 \text{ N}$$

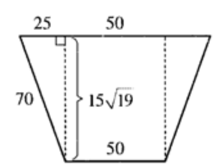


13. (a) $F = \rho g d A \approx (1000)\,(9.8)\,(0.8)\,(0.2)^2 \approx 314$ N

(b) $F = \int_{0.8}^1 \rho g x\,(0.2)\,dx = 0.2\rho g\left[\frac{1}{2}x^2\right]_{0.8}^1 = (0.2\rho g)\,(0.18) = 0.036\rho g \approx 353$ N

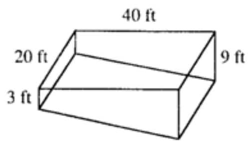
14. The height of the dam is $h = 15\sqrt{19}\left(\frac{\sqrt{3}}{2}\right)$, so

$$F = \int_0^h \delta x\left(100 - \frac{50x}{h}\right)\frac{2}{\sqrt{3}}\,dx = \frac{200\delta}{\sqrt{3}}\int_0^h x\,dx - \frac{100\delta}{h\sqrt{3}}\int_0^h x^2\,dx$$
$$= \frac{200\delta}{\sqrt{3}}\frac{h^2}{2} - \frac{100\delta}{h\sqrt{3}}\frac{h^3}{3} = \frac{200\delta h^2}{3\sqrt{3}} = \frac{200\,(62.5)}{3\sqrt{3}}\cdot\frac{12{,}825}{4}$$
$$\approx 7.71\times 10^6 \text{ lb}$$



15. Assume that the pool is filled with water.

(a) $F = \int_0^3 \delta x 20\,dx = 20\delta\left[\frac{1}{2}x^2\right]_0^3 = 20\delta \cdot \frac{9}{2} = 90\delta \approx 5625 \text{ lb} \approx 5.63 \times 10^3 \text{ lb}$



(b) $F = \int_0^9 \delta x 20\,dx = 20\delta\left[\frac{1}{2}x^2\right]_0^9 = 810\delta \approx 50{,}625 \text{ lb} \approx 5.06 \times 10^4 \text{ lb}.$

(c) For the first 3 ft, the length of the side is constant at 40 ft. For $3 < x \le 9$, we can use similar triangles to find the length a: $\dfrac{a}{40} = \dfrac{9-x}{6} \Rightarrow a = 40 \cdot \dfrac{9-x}{6}$.

$$F = \int_0^3 \delta x 40\,dx + \int_3^9 \delta x\,(40)\,\tfrac{9-x}{6}\,dx = 40\delta\left[\tfrac{1}{2}x^2\right]_0^3 + \tfrac{20}{3}\delta\int_3^9 (9x - x^2)\,dx$$

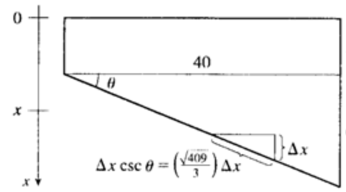
$$= 180\delta + \tfrac{20}{3}\delta\left[\tfrac{9}{2}x^2 - \tfrac{1}{3}x^3\right]_3^9 = 180\delta + \tfrac{20}{3}\delta\left[\left(\tfrac{729}{2} - 243\right) - \left(\tfrac{81}{2} - 9\right)\right]$$

$$= 780\delta \approx 48{,}750 \text{ lb} \approx 4.88 \times 10^4 \text{ lb}$$

(d) For any right triangle with hypotenuse on the bottom,

$\csc\theta = \dfrac{\Delta x}{\text{hypotenuse}} \Rightarrow$

$\text{hypotenuse} = \Delta x \csc\theta = \Delta x \dfrac{\sqrt{40^2 + 6^2}}{6} = \dfrac{\sqrt{409}}{3}\Delta x.$



$$F = \int_3^9 \delta x 20 \tfrac{\sqrt{409}}{3}\,dx = \tfrac{1}{3}\left(20\sqrt{409}\right)\delta\left[\tfrac{1}{2}x^2\right]_3^9$$

$$= \tfrac{1}{3} \cdot 10\sqrt{409}\delta\,(81 - 9)$$

$$\approx 303{,}356 \text{ lb} \approx 3.03 \times 10^5 \text{ lb}$$

16. Partition the interval $[a, b]$ by points x_i as usual and choose $x_i^* \in [x_{i-1}, x_i]$ for each i. The ith horizontal strip of the immersed plate is approximated by a rectangle of height Δx_i and width $w(x_i^*)$, so its area is $A_i \approx w(x_i^*)\,\Delta x_i$. For small Δx_i, the pressure P_i on the ith strip is almost constant and $P_i \approx \rho g x_i^*$ by Equation 1. The hydrostatic force F_i acting on the ith strip is $F_i = P_i A_i \approx \rho g x_i^* w(x_i^*)\,\Delta x_i$. Adding these forces and taking the limit as $n \to \infty$, we obtain the hydrostatic force on the immersed plate:

$$F = \lim_{n\to\infty}\sum_{i=1}^{n} F_i = \lim_{n\to\infty}\sum_{i=1}^{n} \rho g x_i^* w(x_i^*)\,\Delta x_i = \int_a^b \rho g x w(x)\,dx$$

17. $\overline{x} = A^{-1}\int_a^b x w(x)\,dx$ (Equation 8) $\Rightarrow$ $A\overline{x} = \int_a^b x w(x)\,dx$ $\Rightarrow$ $(\rho g \overline{x})\,A = \int_a^b \rho g x w(x)\,dx = F$ by Exercise 16.

18. $F = (\rho g \overline{x})\,A = (\rho g r)\,\pi r^2 = \rho g \pi r^3$. Note that $\overline{x} = r$ because the centroid of a circle is its center, which in this case is at a depth of r meters.

19. $m_1 = 4$, $m_2 = 8$; $P_1(-1, 2)$, $P_2(2, 4)$. $m = \sum_{i=1}^{2} m_i = m_1 + m_2 = 12$. $M_x = \sum_{i=1}^{2} m_i y_i = 4 \cdot 2 + 8 \cdot 4 = 40$;

$M_y = \sum_{i=1}^{2} m_i x_i = 4 \cdot (-1) + 8 \cdot 2 = 12$; $\overline{x} = M_y/m = 1$ and $\overline{y} = M_x/m = \frac{10}{3}$, so the center of mass of the system is $(\overline{x}, \overline{y}) = \left(1, \frac{10}{3}\right)$.

20. $M_x = \sum_{i=1}^{4} m_i y_i = 6(-2) + 5(4) + 1(-7) + 4(-1) = -3$, $M_y = \sum_{i=1}^{4} m_i x_i = 6(1) + 5(3) + 1(-3) + 4(6) = 42$,

and $m = \sum_{i=1}^{4} m_i = 16$, so $\overline{x} = M_y/m = \frac{42}{16} = \frac{21}{8}$ and $\overline{y} = M_x/m = -\frac{3}{16}$; the center of mass is

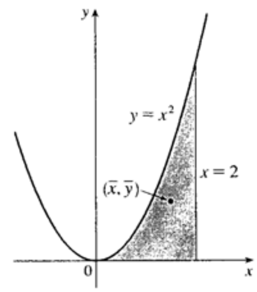
$(\overline{x}, \overline{y}) = \left(\frac{21}{8}, -\frac{3}{16}\right)$.

21. $A = \int_0^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^2 = \frac{8}{3}$,

$\overline{x} = A^{-1} \int_0^2 x \cdot x^2\,dx = \frac{3}{8}\left[\frac{1}{4}x^4\right]_0^2 = \frac{3}{8} \cdot 4 = \frac{3}{2}$,

$\overline{y} = A^{-1} \int_0^2 \frac{1}{2}\left(x^2\right)^2 dx = \frac{3}{8} \cdot \frac{1}{2}\left[\frac{1}{5}x^5\right]_0^2 = \frac{3}{16} \cdot \frac{32}{5} = \frac{6}{5}$.

Centroid $(\overline{x}, \overline{y}) = \left(\frac{3}{2}, \frac{6}{5}\right) = (1.5, 1.2)$.

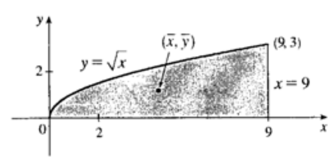


22. $A = \int_0^9 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_0^9 = \frac{2}{3} \cdot 27 = 18$,

$\overline{x} = \frac{1}{A} \int_0^9 x\sqrt{x}\,dx = \frac{1}{18}\left[\frac{2}{5}x^{5/2}\right]_0^9 = \frac{1}{18} \cdot \frac{2}{5} \cdot 243 = \frac{27}{5}$,

$\overline{y} = \frac{1}{A} \int_0^9 \frac{1}{2}\left(\sqrt{x}\right)^2 dx = \frac{1}{18} \cdot \frac{1}{2}\left[\frac{1}{2}x^2\right]_0^9 = \frac{81}{72} = \frac{9}{8}$.

Centroid $(\overline{x}, \overline{y}) = \left(\frac{27}{5}, \frac{9}{8}\right) = (5.4, 1.125)$



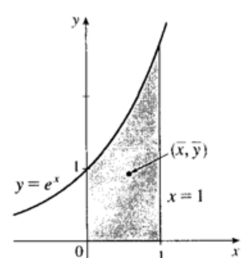
23. $A = \int_0^1 e^x\,dx = [e^x]_0^1 = e - 1$,

$\overline{x} = \frac{1}{A} \int_0^1 xe^x\,dx = \frac{1}{e-1}\left[xe^x - e^x\right]_0^1$ (by parts)

$= \frac{1}{e-1}[0 - (-1)] = \frac{1}{e-1}$,

$\overline{y} = \frac{1}{A} \int_0^1 \frac{1}{2}(e^x)^2\,dx = \frac{1}{e-1} \cdot \frac{1}{4}\left[e^{2x}\right]_0^1 = \frac{1}{4(e-1)}\left(e^2 - 1\right) = \frac{e+1}{4}$.

Centroid $(\overline{x}, \overline{y}) = \left(\frac{1}{e-1}, \frac{e+1}{4}\right) \approx (0.58, 0.93)$.

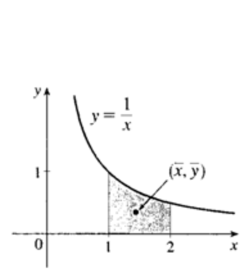


24. $A = \int_1^2 \frac{1}{x}\,dx = [\ln x]_1^2 = \ln 2$, $\overline{x} = \frac{1}{A}\int_1^2 x \cdot \frac{1}{x}\,dx = \frac{1}{A}[x]_1^2 = \frac{1}{A} = \frac{1}{\ln 2}$,

$\overline{y} = \frac{1}{A}\int_1^2 \frac{1}{2}\left(\frac{1}{x}\right)^2 dx = \frac{1}{2A}\int_1^2 x^{-2}\,dx = \frac{1}{2A}\left[-\frac{1}{x}\right]_1^2$

$= \frac{1}{2\ln 2}\left(-\frac{1}{2}+1\right) = \frac{1}{4\ln 2}$.

Centroid $(\overline{x},\overline{y}) = \left(\frac{1}{\ln 2}, \frac{1}{4\ln 2}\right) \approx (1.44, 0.36)$



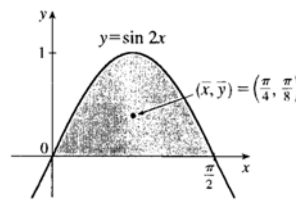
25. From the figure we see that $\overline{x} = \frac{\pi}{4}$ (halfway from $x = 0$ to $\frac{\pi}{2}$). Now

$A = \int_0^{\pi/2} \sin 2x\,dx = -\frac{1}{2}[\cos 2x]_0^{\pi/2} = -\frac{1}{2}(-1-1) = 1$, so

$\overline{y} = \frac{1}{A}\int_0^{\pi/2} \frac{1}{2}\sin^2 2x\,dx = \frac{1}{1}\int_0^{\pi/2} \frac{1}{2}\cdot\frac{1}{2}(1-\cos 4x)\,dx$

$= \frac{1}{4}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi/2} = \frac{\pi}{8}$.

Centroid $(\overline{x},\overline{y}) = \left(\frac{\pi}{4},\frac{\pi}{8}\right)$.



26. $A = \int_1^e \ln x\,dx = [x\ln x - x]_1^e = 0-(-1) = 1$,

$$\overline{x} = \frac{1}{A}\int_1^e x\ln x\,dx = \left[\tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2\right]_1^e = \left(\tfrac{1}{2}e^2 - \tfrac{1}{4}e^2\right) - \left(-\tfrac{1}{4}\right) = \frac{e^2+1}{4},$$

$\overline{y} = \dfrac{1}{A}\displaystyle\int_1^e \frac{(\ln x)^2}{2}\,dx = \frac{1}{2}\int_1^e (\ln x)^2\,dx$. To evaluate $\int(\ln x)^2\,dx$, take $u = \ln x$ and $dv = \ln x\,dx$, so that $du = 1/x\,dx$ and $v = x\ln x - x$. Then

$$\begin{aligned}\int(\ln x)^2\,dx &= x(\ln x)^2 - x(\ln x) - \int(x\ln x - x)\tfrac{1}{x}\,dx = x(\ln x)^2 - x(\ln x) - \int(\ln x - 1)\,dx\\ &= x(\ln x)^2 - x\ln x - x\ln x + x + x + C = x(\ln x)^2 - 2x\ln x + 2x + C\end{aligned}$$

Thus, $\overline{y} = \frac{1}{2}\left[x(\ln x)^2 - 2x\ln x + 2x\right]_1^e = \frac{1}{2}[(e - 2e + 2e) - (0 - 0 + 2)] = \frac{e-2}{2}$. So

$(\overline{x},\overline{y}) = \left(\dfrac{e^2+1}{4}, \dfrac{e-2}{2}\right)$.

27. $A = \int_0^{\pi/4}(\cos x - \sin x)\,dx = [\sin x + \cos x]_0^{\pi/4} = \sqrt{2}-1$,

$\overline{x} = A^{-1}\int_0^{\pi/4} x(\cos x - \sin x)\,dx = A^{-1}[x(\sin x + \cos x) + \cos x - \sin x]_0^{\pi/4}$ [integration by parts]

$= A^{-1}\left(\frac{\pi}{4}\sqrt{2}-1\right) = \dfrac{\frac{1}{4}\pi\sqrt{2}-1}{\sqrt{2}-1}$

$\overline{y} = A^{-1}\int_0^{\pi/4}\frac{1}{2}(\cos^2 x - \sin^2 x)\,dx = \frac{1}{2A}\int_0^{\pi/4}\cos 2x\,dx = \frac{1}{4A}[\sin 2x]_0^{\pi/4} = \frac{1}{4A} = \frac{1}{4\left(\sqrt{2}-1\right)}$.

$$(\overline{x},\overline{y}) = \left(\frac{\pi\sqrt{2}-4}{4\left(\sqrt{2}-1\right)}, \frac{1}{4\left(\sqrt{2}-1\right)}\right).$$

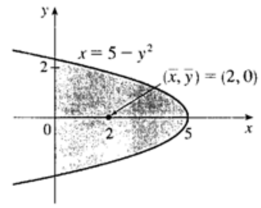
28. $A = \int_0^1 x\,dx + \int_1^2 \frac{1}{x}\,dx = \left[\frac{1}{2}x^2\right]_0^1 + [\ln x]_1^2 = \frac{1}{2} + \ln 2$,

$$\bar{x} = \frac{1}{A}\left[\int_0^1 x^2\,dx + \int_1^2 1\,dx\right] = \frac{1}{A}\left(\left[\tfrac{1}{3}x^3\right]_0^1 + [x]_1^2\right) = \frac{1}{A}\left(\frac{1}{3}+1\right) = \frac{2}{1+2\ln 2}\cdot\frac{4}{3} = \frac{8}{3\,(1+2\ln 2)},$$

$$\bar{y} = \frac{1}{A}\left[\int_0^1 \tfrac{1}{2}x^2\,dx + \int_1^2 \frac{1}{2x^2}\,dx\right] = \frac{1}{2A}\left(\left[\tfrac{1}{3}x^3\right]_0^1 + \left[-\frac{1}{x}\right]_1^2\right) = \frac{1}{2A}\left(\frac{1}{3}+\frac{1}{2}\right) = \frac{5}{12A} = \frac{5}{6+12\ln 2}.$$

$(\bar{x},\bar{y}) = \left(\dfrac{8}{3\,(1+2\ln 2)}, \dfrac{5}{6\,(1+2\ln 2)}\right)$. The principle used in this problem is stated after Example 3: the moment of the union of two non-overlapping regions is the sum of the moments of the individual regions.

29. From the figure we see that $\bar{y} = 0$. Now



$$A = \int_0^5 2\sqrt{5-x}\,dx = 2\left[-\tfrac{2}{3}(5-x)^{3/2}\right]_0^5$$
$$= 2\left(0+\tfrac{2}{3}\cdot 5^{3/2}\right) = \tfrac{20}{3}\sqrt{5}$$

so

$$\bar{x} = \tfrac{1}{A}\int_0^5 x\left[\sqrt{5-x} - \left(-\sqrt{5-x}\right)\right]dx = \tfrac{1}{A}\int_0^5 2x\sqrt{5-x}\,dx$$
$$= \tfrac{1}{A}\int_{\sqrt{5}}^0 2\left(5-u^2\right)u\,(-2u)\,du \quad \left(u = \sqrt{5-x},\ x = 5-u^2,\ u^2 = 5-x,\ dx = -2u\,du\right)$$
$$= \tfrac{4}{A}\int_0^{\sqrt{5}} u^2\left(5-u^2\right)du = \tfrac{4}{A}\left[\tfrac{5}{3}u^3 - \tfrac{1}{5}u^5\right]_0^{\sqrt{5}} = \tfrac{3}{5\sqrt{5}}\left(\tfrac{25}{3}\sqrt{5} - 5\sqrt{5}\right) = 5-3 = 2$$

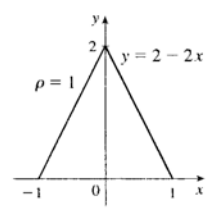
Thus, the centroid is $(\bar{x},\bar{y}) = (2,0)$.

30. By symmetry, $M_y = 0$ and $\bar{x} = 0$; $A = \frac{1}{2}\pi\cdot 1^2 + 4$, so $m = \rho A = 5\left(\frac{\pi}{2}+4\right) = \frac{5}{2}(\pi+8)$;

$$M_x = \rho\cdot 2\int_0^1 \tfrac{1}{2}\left[\left(\sqrt{1-x^2}\right)^2 - (-2)^2\right]dx = 5\int_0^1\left(-x^2-3\right)dx = -5\left[\tfrac{1}{3}x^3+3x\right]_0^1 = -5\cdot\tfrac{10}{3} = -\tfrac{50}{3};$$

$\bar{y} = \frac{1}{m}M_x = \frac{2}{5(\pi+8)}\cdot\frac{-50}{3} = -\frac{20}{3(\pi+8)}$. $(\bar{x},\bar{y}) = \left(0, \dfrac{-20}{3\,(\pi+8)}\right)$.

31. By symmetry, $M_y = 0$ and $\bar{x} = 0$. $A = \frac{1}{2}bh = \frac{1}{2}\cdot 2\cdot 2 = 2$.



$$M_x = 2\rho\int_0^1 \tfrac{1}{2}(2-2x)^2\,dx = \left(2\cdot 1\cdot\tfrac{1}{2}\cdot 2^2\right)\int_0^1 (1-x)^2\,dx$$
$$= 4\left[-\tfrac{1}{3}(1-x)^3\right]_0^1 = 4\cdot\tfrac{1}{3} = \tfrac{4}{3}$$

$\bar{y} = \frac{1}{\rho A}M_x = \frac{1}{1\cdot 2}\cdot\frac{4}{3} = \frac{2}{3}$. $(\bar{x},\bar{y}) = \left(0,\frac{2}{3}\right)$.

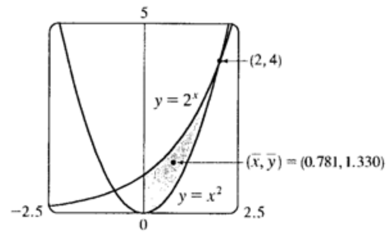
32. By symmetry about the line $y = x$, we expect that $\bar{x} = \bar{y}$. $A = \frac{1}{4}\pi r^2$, so $m = \rho A = 2A = \frac{1}{2}\pi r^2$.

$$M_x = \rho\int_0^r \tfrac{1}{2}\left(\sqrt{r^2-x^2}\right)^2 dx = \int_0^r\left(r^2-x^2\right)dx = \left[r^2x - \tfrac{1}{3}x^3\right]_0^r = \tfrac{2}{3}r^3.$$

$$M_y = \rho\int_0^r x\sqrt{r^2-x^2}\,dx = \int_0^r\left(r^2-x^2\right)^{1/2}2x\,dx = \int_0^{r^2}u^{1/2}\,du \quad \left(u = r^2-x^2\right) = \left[\tfrac{2}{3}u^{3/2}\right]_0^{r^2} = \tfrac{2}{3}r^3.$$

$\bar{x} = \dfrac{1}{m}M_y = \dfrac{2}{\pi r^2}\left(\frac{2}{3}r^3\right) = \frac{4}{3\pi}r$; $\bar{y} = \dfrac{1}{m}M_x = \dfrac{2}{\pi r^2}\left(\frac{2}{3}r^3\right) = \frac{4}{3\pi}r$. $(\bar{x},\bar{y}) = \left(\frac{4}{3\pi}r, \frac{4}{3\pi}r\right)$.

33.

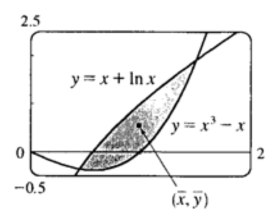


$$A = \int_0^2 \left(2^x - x^2\right) dx = \left[\frac{2^x}{\ln 2} - \frac{x^3}{3}\right]_0^2 = \left(\frac{4}{\ln 2} - \frac{8}{3}\right) - \frac{1}{\ln 2} = \frac{3}{\ln 2} - \frac{8}{3} \approx 1.661418.$$

$$\begin{aligned}
\bar{x} &= \tfrac{1}{A}\textstyle\int_0^2 x\left(2^x - x^2\right) dx = \tfrac{1}{A}\int_0^2 \left(x2^x - x^3\right) dx = \tfrac{1}{A}\int_0^2 \left[xe^{(\ln 2)x} - x^3\right] dx \\
&\overset{96}{=} \frac{1}{A}\left[\frac{1}{(\ln 2)^2}(x \ln 2 - 1)\, e^{(\ln 2)x} - \tfrac{1}{4}x^4\right]_0^2 \quad \text{(or use parts)} \\
&= \frac{1}{A}\left[\left(\frac{x}{\ln 2} - \frac{1}{(\ln 2)^2}\right)2^x - \tfrac{1}{4}x^4\right]_0^2 = \frac{1}{A}\left[\frac{x2^x}{\ln 2} - \frac{2^x}{(\ln 2)^2} - \frac{x^4}{4}\right]_0^2 \\
&= \frac{1}{A}\left[\frac{8}{\ln 2} - \frac{4}{(\ln 2)^2} - 4 + \frac{1}{(\ln 2)^2}\right] = \frac{1}{A}\left[\frac{8}{\ln 2} - \frac{3}{(\ln 2)^2} - 4\right] \\
&\approx \tfrac{1}{A}(1.297453) \approx 0.781
\end{aligned}$$

$$\begin{aligned}
\bar{y} &= \tfrac{1}{A}\textstyle\int_0^2 \tfrac{1}{2}\left[(2^x)^2 - \left(x^2\right)^2\right] dx = \tfrac{1}{A}\int_0^2 \tfrac{1}{2}\left(2^{2x} - x^4\right) dx = \frac{1}{A}\cdot\frac{1}{2}\left[\frac{2^{2x}}{2\ln 2} - \frac{x^5}{5}\right]_0^2 \\
&= \frac{1}{A}\cdot\frac{1}{2}\left(\frac{16}{2\ln 2} - \frac{32}{5} - \frac{1}{2\ln 2}\right) = \frac{1}{A}\left(\frac{15}{4\ln 2} - \frac{16}{5}\right) \approx \frac{1}{A}(2.210106) \approx 1.330
\end{aligned}$$

34. The curves $y = x + \ln x$ and $y = x^3 - x$ intersect at $(a, c) \approx (0.447141, -0.357742)$ and $(b, d) \approx (1.507397, 1.917782)$.

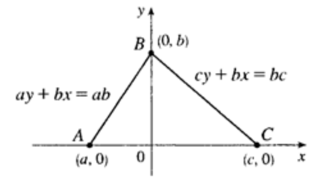


$$\begin{aligned}
A &= \textstyle\int_a^b \left(x + \ln x - x^3 + x\right) dx = \int_a^b \left(2x + \ln x - x^3\right) dx \\
&\overset{100}{=} \left[x^2 + x\ln x - x - \tfrac{1}{4}x^4\right]_a^b \approx 0.709781
\end{aligned}$$

$$\begin{aligned}
\bar{x} &= \tfrac{1}{A}\textstyle\int_a^b x\left(2x + \ln x - x^3\right) dx = \tfrac{1}{A}\int_a^b \left(2x^2 + x\ln x - x^4\right) dx \\
&\overset{101}{=} \tfrac{1}{A}\left[\tfrac{2}{3}x^3 + \tfrac{1}{4}x^2(2\ln x - 1) - \tfrac{1}{5}x^5\right]_a^b \approx \tfrac{1}{A}(0.699489) \approx 0.985501 \\
\bar{y} &= \tfrac{1}{A}\textstyle\int_a^b \tfrac{1}{2}\left[(x + \ln x)^2 - \left(x^3 - x\right)^2\right] dx = \tfrac{1}{2A}\int_a^b \left[2x\ln x + (\ln x)^2 - x^6 + 2x^4\right] dx \\
&\overset{\text{101 and parts}}{=} \tfrac{1}{2A}\left[x^2\ln x - \tfrac{1}{2}x^2 + x(\ln x)^2 - 2x\ln x + 2x - \tfrac{1}{7}x^7 + \tfrac{2}{5}x^5\right]_a^b \approx \tfrac{1}{2A}(0.765092) \approx 0.538964
\end{aligned}$$

So $(\bar{x}, \bar{y}) \approx (0.986, 0.539)$.

35. Choose x- and y-axes so that the base (one side of the triangle) lies along the x-axis with the other vertex along the positive y-axis as shown. From geometry, we know the medians intersect at a point $\frac{2}{3}$ of the way from each vertex (along the median) to the opposite side. The median from B goes to the midpoint $\left(\frac{1}{2}(a+c), 0\right)$ of side AC, so the point of intersection of the medians is $\left(\frac{2}{3}\cdot\frac{1}{2}(a+c), \frac{1}{3}b\right) = \left(\frac{1}{3}(a+c), \frac{1}{3}b\right)$. This can also be verified by finding the equations of two medians, and solving them simultaneously to find their point of intersection. Now let us compute the location of the centroid of the triangle. The area is $A = \frac{1}{2}(c-a)\,b$.



$$\begin{aligned}\overline{x} &= \tfrac{1}{A}\left[\int_a^0 x\cdot\tfrac{b}{a}(a-x)\,dx + \int_0^c x\cdot\tfrac{b}{c}(c-x)\,dx\right] = \tfrac{1}{A}\left[\tfrac{b}{a}\int_a^0 (ax-x^2)\,dx + \tfrac{b}{c}\int_0^c (cx-x^2)\,dx\right]\\ &= \tfrac{b}{Aa}\left[\tfrac{1}{2}ax^2-\tfrac{1}{3}x^3\right]_a^0 + \tfrac{b}{Ac}\left[\tfrac{1}{2}cx^2-\tfrac{1}{3}x^3\right]_0^c = \tfrac{b}{Aa}\left[-\tfrac{1}{2}a^3+\tfrac{1}{3}a^3\right] + \tfrac{b}{Ac}\left[\tfrac{1}{2}c^3-\tfrac{1}{3}c^3\right]\\ &= \tfrac{2}{a(c-a)}\cdot\tfrac{-a^3}{6} + \tfrac{2}{c(c-a)}\cdot\tfrac{c^3}{6} = \tfrac{1}{3(c-a)}\left(c^2-a^2\right) = \tfrac{a+c}{3}\end{aligned}$$

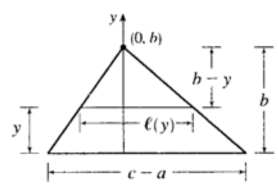
and

$$\begin{aligned}\overline{y} &= \tfrac{1}{A}\left[\int_a^0 \tfrac{1}{2}\left(\tfrac{b}{a}(a-x)\right)^2 dx + \int_0^c \tfrac{1}{2}\left(\tfrac{b}{c}(c-x)\right)^2 dx\right]\\ &= \tfrac{1}{A}\left[\tfrac{b^2}{2a^2}\int_a^0 \left(a^2-2ax+x^2\right)dx + \tfrac{b^2}{2c^2}\int_0^c \left(c^2-2cx+x^2\right)dx\right]\\ &= \tfrac{1}{A}\left[\tfrac{b^2}{2a^2}\left[a^2x-ax^2+\tfrac{1}{3}x^3\right]_a^0 + \tfrac{b^2}{2c^2}\left[c^2x-cx^2+\tfrac{1}{3}x^3\right]_0^c\right]\\ &= \tfrac{1}{A}\left[\tfrac{b^2}{2a^2}\left(-a^3+a^3-\tfrac{1}{3}a^3\right) + \tfrac{b^2}{2c^2}\left(c^3-c^3+\tfrac{1}{3}c^3\right)\right] = \tfrac{1}{A}\left[\tfrac{b^2}{6}(-a+c)\right] = \tfrac{2}{(c-a)b}\cdot\tfrac{(c-a)b^2}{6} = \tfrac{b}{3}\end{aligned}$$

Thus, $(\overline{x},\overline{y}) = \left(\dfrac{a+c}{3}, \dfrac{b}{3}\right)$ as claimed.

Remarks: Actually the computation of $\overline{y}$ is all that is needed. By considering each side of the triangle in turn to be the base, we see that the centroid is $\frac{1}{3}$ of the way from each side to the opposite vertex and must therefore be the intersection of the medians.

The computation of $\overline{y}$ in this problem (and many others) can be simplified by using horizontal rather than vertical approximating rectangles. If the length of a thin rectangle at coordinate y is $\ell(y)$, then its area is $\ell(y)\,\Delta y$, its mass is $\rho\ell(y)\,\Delta y$, and its moment about the x-axis is $\Delta M_x = \rho y\ell(y)\,\Delta y$. Thus,



$$M_x = \int \rho y\ell(y)\,dy \quad\text{and}\quad \overline{y} = \frac{\int \rho y\ell(y)\,dy}{\rho A} = \tfrac{1}{A}\int y\ell(y)\,dy$$

In this problem, $\ell(y) = \dfrac{c-a}{b}(b-y)$ by similar triangles, so

$$\overline{y} = \frac{1}{A}\int_0^b \frac{c-a}{b}y(b-y)\,dy = \frac{2}{b^2}\int_0^b \left(by-y^2\right)dy = \frac{2}{b^2}\left[\tfrac{1}{2}by^2-\tfrac{1}{3}y^3\right]_0^b = \frac{2}{b^2}\cdot\frac{b^3}{6} = \frac{b}{3}$$

Notice that only one integral is needed when this method is used.

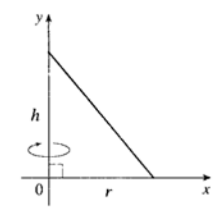
Since the position of a centroid is independent of density when the density is constant, we will assume for convenience that $\rho = 1$ in Exercises 36 and 37.

36. Divide the lamina into three rectangles with masses 2, 2 and 6, with centroids $\left(-\frac{3}{2}, 1\right)$, $\left(0, \frac{1}{2}\right)$ and $\left(2, \frac{3}{2}\right)$ respectively. The total mass of the lamina is 10. So, using Formulas 5, 6, and 7, we have $\overline{x} = \dfrac{\sum m_i x_i}{m} = \frac{2}{10}\left(-\frac{3}{2}\right) + \frac{2}{10}(0) + \frac{6}{10}(2) = \frac{9}{10}$, and $\overline{y} = \dfrac{\sum m_i y_i}{m} = \frac{2}{10}(1) + \frac{2}{10}\left(\frac{1}{2}\right) + \frac{6}{10}\left(\frac{3}{2}\right) = \frac{6}{5}$. Therefore $(\overline{x}, \overline{y}) = \left(\frac{9}{10}, \frac{6}{5}\right)$.

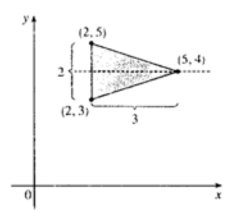
37. Divide the lamina into two triangles and one rectangle with respective masses of 2, 2 and 4, so that the total mass is 8. Using the result of Exercise 35, the triangles have centroids $\left(-1, \frac{2}{3}\right)$ and $\left(1, \frac{2}{3}\right)$. The centroid of the rectangle (its center) is $\left(0, -\frac{1}{2}\right)$. So, using Formulas 5 and 7, we have $\overline{y} = \dfrac{\sum m_i y_i}{m} = \frac{2}{8}\left(\frac{2}{3}\right) + \frac{2}{8}\left(\frac{2}{3}\right) + \frac{4}{8}\left(-\frac{1}{2}\right) = \frac{1}{12}$, and $\overline{x} = 0$, since the lamina is symmetric about the line $x = 0$. Therefore $(\overline{x}, \overline{y}) = \left(0, \frac{1}{12}\right)$.

38. A sphere can be generated by rotating a semicircle about its diameter. By Example 4, the center of mass travels a distance $2\pi\overline{y} = 2\pi\left(\frac{4r}{3\pi}\right) = \frac{8r}{3}$, so by the Theorem of Pappus, the volume of the sphere is $V = Ad = \dfrac{\pi r^2}{2} \cdot \dfrac{8r}{3} = \frac{4}{3}\pi r^3$.

39. A cone of height h and radius r can be generated by rotating a right triangle about one of its legs as shown. By Exercise 35, $\overline{x} = \frac{1}{3}r$, so by the Theorem of Pappus, the volume of the cone is $V = Ad = \frac{1}{2}rh \cdot 2\pi\left(\frac{1}{3}r\right) = \frac{1}{3}\pi r^2 h$.



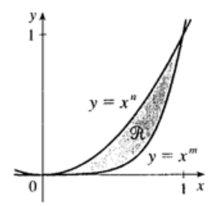
40. From the symmetry in the figure, $\overline{y} = 4$. So the distance traveled by the centroid when rotating the triangle about the x-axis is $d = 2\pi \cdot 4 = 8\pi$. The area of the triangle is $A = \frac{1}{2}bh = \frac{1}{2}(2)(3) = 3$. By the Theorem of Pappus, the volume of the resulting solid is $Ad = 3(8\pi) = 24\pi$.



41. Suppose the region lies between two curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$, as illustrated in Figure 13. Choose points x_i with $a = x_0 < x_1 < \cdots < x_n = b$ and choose x_i^* to be the midpoint of the ith subinterval; that is, $x_i^* = \overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$. Then the centroid of the ith approximating rectangle R_i is its center $C_i = \left(\overline{x}_i, \frac{1}{2}[f(\overline{x}_i) + g(\overline{x}_i)]\right)$. Its area is $[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x$, so its mass is $\rho\,[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x$.
Thus, $M_y(R_i) = \rho\,[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x \cdot \overline{x}_i = \rho\overline{x}_i\,[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x$ and
$M_x(R_i) = \rho\,[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x \cdot \frac{1}{2}[f(\overline{x}_i) + g(\overline{x}_i)] = \rho \cdot \frac{1}{2}\left[f(\overline{x}_i)^2 - g(\overline{x}_i)^2\right]\Delta x$. Summing over i and taking the limit as $n \to \infty$, we get $M_y = \lim\limits_{n\to\infty} \sum_i \rho\overline{x}_i\,[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x = \rho\int_a^b x\,[f(x) - g(x)]\,dx$ and
$M_x = \lim\limits_{n\to\infty} \sum_i \rho \cdot \frac{1}{2}\left[f(\overline{x}_i)^2 - g(\overline{x}_i)^2\right]\Delta x = \rho\int_a^b \frac{1}{2}\left[f(x)^2 - g(x)^2\right]dx$. Thus,

$$\overline{x} = \frac{M_y}{m} = \frac{M_y}{\rho A} = \frac{1}{A}\int_a^b x\,[f(x) - g(x)]\,dx \quad\text{and}\quad \overline{y} = \frac{M_x}{m} = \frac{M_x}{\rho A} = \frac{1}{A}\int_a^b \tfrac{1}{2}\left[f(x)^2 - g(x)^2\right]dx$$

42. (a)



(b) Using Formula 9 and the fact that the area of $\mathfrak{R}$ is

$$A = \int_0^1 (x^n - x^m)\,dx = \frac{1}{n+1} - \frac{1}{m+1} = \frac{m-n}{(n+1)(m+1)}, \text{ we get}$$

$$\begin{aligned}\overline{x} &= \frac{(n+1)(m+1)}{m-n}\int_0^1 x\left[x^n - x^m\right]dx = \frac{(n+1)(m+1)}{m-n}\left[\frac{1}{n+2} - \frac{1}{m+2}\right] \\ &= \frac{(n+1)(m+1)}{(n+2)(m+2)}\end{aligned}$$

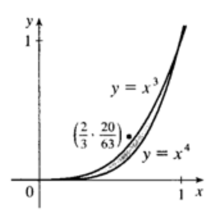
and

$$\begin{aligned}\overline{y} &= \frac{(n+1)(m+1)}{m-n}\int_0^1 \frac{1}{2}\left[(x^n)^2 - (x^m)^2\right]dx = \frac{(n+1)(m+1)}{2(m-n)}\left[\frac{1}{2n+1} - \frac{1}{2m+1}\right] \\ &= \frac{(n+1)(m+1)}{(2n+1)(2m+1)}\end{aligned}$$

(c) If we take $n = 3$ and $m = 4$, then

$$(\overline{x}, \overline{y}) = \left(\frac{4\cdot 5}{5\cdot 6}, \frac{4\cdot 5}{7\cdot 9}\right) = \left(\frac{2}{3}, \frac{20}{63}\right)$$

which lies outside $\mathfrak{R}$ since $\left(\frac{2}{3}\right)^3 = \frac{8}{27} < \frac{20}{63}$. This is the simplest of many possibilities.



9.4 Applications to Economics and Biology

1. $C(2000) = C(0) + \int_0^{2000} C'(x)\,dx = 1{,}500{,}000 + \int_0^{2000} \left(0.006x^2 - 1.5x + 8\right) dx$
$= 1{,}500{,}000 + \left[0.002x^3 - 0.75x^2 + 8x\right]_0^{2000} = \$14{,}516{,}000$

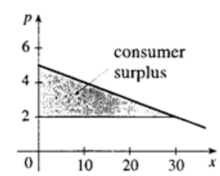
2. $R'(x) = 90 - 0.02x$ and $R(100) = \$8800$, so

$$R(200) = R(100) + \int_{100}^{200} R'(x)\,dx = 8800 + \int_{100}^{200} (90 - 0.02x)\,dx = 8800 + \left[90x - 0.01x^2\right]_{100}^{200}$$
$$= 8800 + (18{,}000 - 400) - (9000 - 100) = \$17{,}500$$

3. $C'(x) = 74 + 1.1x - 0.002x^2 + 0.00004x^3$, so the increase in cost is

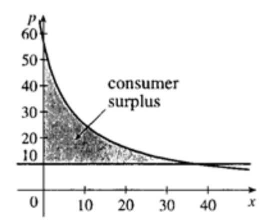
$$\begin{aligned} C(1600) - C(1200) &= \int_{1200}^{1600} \left(74 + 1.1x - 0.002x^2 + 0.00004x^3\right) dx \\ &= \left[74x + 0.55x^2 - \tfrac{0.002}{3}x^3 + 0.00001x^4\right]_{1200}^{1600} \\ &= 64{,}331{,}733.33 - 20{,}464{,}800 = \$43{,}866{,}933.33 \end{aligned}$$

4. Consumer surplus $= \int_0^{30} [p(x) - p(30)]\,dx$
$= \int_0^{30} \left[5 - \frac{1}{10}x - \left(5 - \frac{30}{10}\right)\right] dx$
$= \left[3x - \frac{1}{20}x^2\right]_0^{30} = 90 - 45 = \$45$



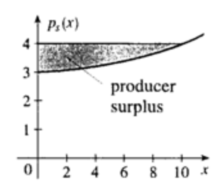
5. $p(x) = 10 = \dfrac{450}{x+8} \;\Rightarrow\; x + 8 = 45 \;\Rightarrow\; x = 37.$

Consumer surplus $= \displaystyle\int_0^{37} [p(x) - 10]\,dx = \int_0^{37} \left(\frac{450}{x+8} - 10\right) dx$
$= [450 \ln(x+8) - 10x]_0^{37}$
$= 450 \ln\left(\frac{45}{8}\right) - 370 \approx \$407.25$



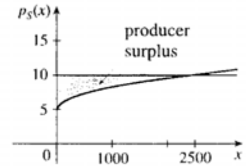
6. $p_S(x) = 3 + 0.01x^2$. $P = p_S(10) = 3 + 1 = 4$.

Producer surplus $= \int_0^{10} [P - p_S(x)]\,dx$
$= \int_0^{10} \left[4 - 3 - 0.01x^2\right] dx = \left[x - \frac{0.01}{3}x^3\right]_0^{10}$
$\approx 10 - 3.33 = \$6.67$



7. $P = p_S(x) = 10 = 5 + \frac{1}{10}\sqrt{x} \quad\Rightarrow\quad 50 = \sqrt{x} \quad\Rightarrow\quad x = 2500.$

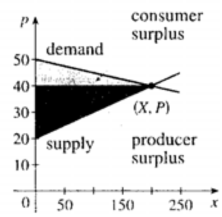
$$\begin{aligned}\text{Producer surplus} &= \int_0^{2500} [P - p_S(x)]\,dx \\ &= \int_0^{2500} \left(10 - 5 - \tfrac{1}{10}\sqrt{x}\right) dx \\ &= \left[5x - \tfrac{1}{15}x^{3/2}\right]_0^{2500} \approx \$4166.67\end{aligned}$$



8. $p = 50 - \frac{1}{20}x$ and $p = 20 + \frac{1}{10}x$ intersect at $p = 40$ and $x = 200$.

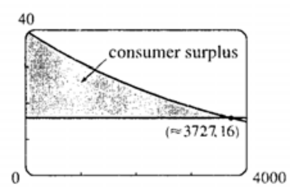
$$\begin{aligned}\text{Consumer surplus} &= \int_0^{200} \left(50 - \tfrac{1}{20}x - 40\right) dx \\ &= \left[10x - \tfrac{1}{40}x^2\right]_0^{200} = \$1000\end{aligned}$$

$$\begin{aligned}\text{Producer surplus} &= \int_0^{200} \left(40 - 20 - \tfrac{1}{10}x\right) dx \\ &= \left[20x - \tfrac{1}{20}x^2\right]_0^{200} = \$2000\end{aligned}$$



9. $p(x) = \dfrac{800{,}000e^{-x/5000}}{x + 20{,}000} = 16 \quad\Rightarrow\quad x = x_1 \approx 3727.04.$

$$\text{Consumer surplus} = \int_0^{x_1} [p(x) - 16]\,dx \approx \$37{,}753.01$$



10. The demand function is linear with slope $\frac{-0.5}{35} = -\frac{1}{70}$ and $p(400) = 7.5$, so an equation is $p - 7.5 = -\frac{1}{70}(x - 400)$ or $p = -\frac{1}{70}x + \frac{185}{14}$. A selling price of \$6 implies that $6 = -\frac{1}{70}x + \frac{185}{14} \quad\Rightarrow$ $\frac{1}{70}x = \frac{185}{14} - \frac{84}{14} = \frac{101}{14} \quad\Rightarrow\quad x = 505.$

$$\text{Consumer surplus} = \int_0^{505} \left(-\tfrac{1}{70}x + \tfrac{185}{14} - 6\right) dx = \left[-\tfrac{1}{140}x^2 + \tfrac{101}{14}x\right]_0^{505} \approx \$1821.61$$

11. $f(8) - f(4) = \int_4^8 f'(t)\,dt = \int_4^8 \sqrt{t}\,dt = \left[\frac{2}{3}t^{3/2}\right]_4^8 = \frac{2}{3}\left(16\sqrt{2} - 8\right) \approx \$9.75$ million

12. $n(9) - n(5) = \displaystyle\int_5^9 \left(2200 + 10e^{0.8t}\right) dt = \left[2200t + \frac{10e^{0.8t}}{0.8}\right]_5^9 = [2200t]_5^9 + \frac{25}{2}\left[e^{0.8t}\right]_5^9$

$= 2200(9 - 5) + 12.5\left(e^{7.2} - e^4\right) \approx 24{,}860$

13. $F = \dfrac{\pi P R^4}{8\eta\ell} = \dfrac{\pi (4000)(0.008)^4}{8(0.027)(2)} \approx 1.19 \times 10^{-4}\ \text{cm}^3/\text{s}$

14. If the flux remains constant, then $\dfrac{\pi P_0 R_0^4}{8\eta\ell} = \dfrac{\pi P R^4}{8\eta\ell} \quad\Rightarrow\quad P_0 R_0^4 = P R^4 \quad\Rightarrow\quad \dfrac{P}{P_0} = \left(\dfrac{R_0}{R}\right)^4.\ R = \frac{3}{4}R_0 \quad\Rightarrow$

$P = P_0\left(\frac{4}{3}\right)^4 \approx 3.1605 > 3P_0.$

15. $\int_0^{12} c(t)\,dt = \int_0^{12} \frac{1}{4}t(12-t)\,dt = \left[\frac{3}{2}t^2 - \frac{1}{12}t^3\right]_0^{12} = (216-144) = 72$ mg · s/L. Therefore, $F = A/72 = \frac{8}{72} = \frac{1}{9}$ L/s $= \frac{60}{9}$ L/min.

16. As in Example 3, we will estimate the cardiac output using Simpson's Rule with $\Delta t = 2$.

$$\begin{aligned}\int_0^{20} c(t)\,dt &\approx \tfrac{2}{3}[1(0) + 4(2.4) + 2(5.1) + 4(7.8) + 2(7.6) \\ &\qquad + 4(5.4) + 2(3.9) + 4(2.3) + 2(1.6) + 4(0.7) + 1(0)] \\ &= \tfrac{2}{3}(110.8) \approx 73.87 \text{ mg}\cdot\text{s/L}\end{aligned}$$

Therefore, $F \approx \dfrac{A}{73.87} = \dfrac{8}{73.87} \approx 0.1083$ L/s or 6.498 L/min.

9.5 Probability

1. (a) $\int_{100}^{200} f(t)\,dt$ is the probability that a randomly chosen battery will have a lifetime of between 100 and 200 hours.

(b) $\int_{200}^{\infty} f(t)\,dt$ is the probability that a randomly chosen battery will have a lifetime of at least 200 hours.

2. (a) $P(180 \le X \le 240) = \int_{180}^{240} f(x)\,dx$

(b) $P(0 \le X \le 200) = \int_0^{200} f(x)\,dx$

3. (a) In general, we must satisfy the two conditions that are mentioned before Example 1 — namely, (1) $f(x) \ge 0$ for all x, and (2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$. Clearly, condition (1) is satisfied. For condition (2), we see that

$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{10} 0.1\,dx = \left[\frac{1}{10}x\right]_0^{10} = 1$. Thus, $f(x)$ is a probability density function.

(b) Since all the numbers between 0 and 10 are equally likely to be selected, we expect the mean to be halfway between the endpoints of the interval; that is, $x = 5$.

$\mu = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^{10} x(0.1)\,dx = \left[\frac{1}{20}x^2\right]_0^{10} = \frac{100}{20} = 5$, as expected.

4. (a) As in the preceding exercise, (1) $f(x) \ge 0$ and

(2) $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{10} f(x)\,dx = \frac{1}{2}(10)(0.2)$ [area of a triangle] $= 1$. So $f(x)$ is a probability density function.

(b) (i) $P(X < 3) = \int_0^3 f(x)\,dx = \frac{1}{2}(3)(0.1) = \frac{3}{20} = 0.15$

(ii) We first compute $P(X > 8)$ and then subtract that value and our answer in (i) from 1 (the total probability).

$P(X > 8) = \int_8^{10} f(x)\,dx = \frac{1}{2}(2)(0.1) = \frac{2}{20} = 0.10$. So $P(3 \le X \le 8) = 1 - 0.15 - 0.10 = 0.75$

(c) We find equations of the lines from $(0, 0)$ to $(6, 0.2)$ and from $(6, 0.2)$ to $(10, 0)$, and find that

$$f(x) = \begin{cases} \frac{1}{30}x & \text{if } 0 \le x < 6 \\ -\frac{1}{20}x + \frac{1}{2} & \text{if } 6 \le x < 10 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}\mu &= \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^6 x\left(\tfrac{1}{30}x\right)dx + \int_6^{10} x\left(-\tfrac{1}{20}x + \tfrac{1}{2}\right)dx = \left[\tfrac{1}{90}x^3\right]_0^6 + \left[-\tfrac{1}{60}x^3 + \tfrac{1}{4}x^2\right]_6^{10} \\ &= \tfrac{216}{90} + \left(-\tfrac{1000}{60} + \tfrac{100}{4}\right) - \left(-\tfrac{216}{60} + \tfrac{36}{4}\right) = \tfrac{16}{3} = 5.\overline{3}\end{aligned}$$

5. We need to find m so that $\int_m^\infty f(t)\,dt = \frac{1}{2} \Rightarrow \lim_{x\to\infty} \int_m^x \frac{1}{5}e^{-t/5}dt = \frac{1}{2} \Rightarrow \lim_{x\to\infty} \left[\frac{1}{5}(-5)\,e^{-t/5}\right]_m^x = \frac{1}{2} \Rightarrow$ $(-1)\left(0 - e^{-m/5}\right) = \frac{1}{2} \Rightarrow e^{-m/5} = \frac{1}{2} \Rightarrow -m/5 = \ln\frac{1}{2} \Rightarrow m = -5\ln\frac{1}{2} = 5\ln 2 \approx 3.47$ min.

6. (a) $\mu = 1000 \Rightarrow f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1000}e^{-t/1000} & \text{if } t \geq 0 \end{cases}$

(i) $P(0 \leq X \leq 200) = \int_0^{200} \frac{1}{1000}e^{-t/1000}\,dt = \left[-e^{-t/1000}\right]_0^{200} = -e^{-1/5} + 1 \approx 0.181$

(ii) $P(X > 800) = \int_{800}^\infty \frac{1}{1000}e^{-t/1000}\,dt = \lim_{x\to\infty}\left[-e^{-t/1000}\right]_{800}^x = 0 + e^{-4/5} \approx 0.449$

(b) We need to find m so that $\int_m^\infty f(t)\,dt = \frac{1}{2} \Rightarrow \lim_{x\to\infty}\int_m^x \frac{1}{1000}e^{-t/1000}\,dt = \frac{1}{2} \Rightarrow \lim_{x\to\infty}\left[-e^{-t/1000}\right]_m^x = \frac{1}{2}$ $\Rightarrow 0 + e^{-m/1000} = \frac{1}{2} \Rightarrow -m/1000 = \ln\frac{1}{2} \Rightarrow m = -1000\ln\frac{1}{2} = 1000\ln 2 \approx 693.1$ h.

7. We use an exponential density function with $\mu = 2.5$ min.

(a) $P(X > 4) = \int_4^\infty f(t)\,dt = \lim_{x\to\infty}\int_4^x \frac{1}{2.5}e^{-t/2.5}\,dt = \lim_{x\to\infty}\left[-e^{-t/2.5}\right]_4^x = 0 + e^{-4/2.5} \approx 0.202$

(b) $P(0 \leq X \leq 2) = \int_0^2 f(t)\,dt = \left[-e^{-t/2.5}\right]_0^2 = -e^{-2/2.5} + 1 \approx 0.551$

(c) We need to find a value a so that $P(X \geq a) = 0.02$, or, equivalently, $P(0 \leq X \leq a) = 0.98 \Leftrightarrow$ $\int_0^a f(t)\,dt = 0.98 \Leftrightarrow \left[-e^{-t/2.5}\right]_0^a = 0.98 \Leftrightarrow -e^{-a/2.5} + 1 = 0.98 \Leftrightarrow e^{-a/2.5} = 0.02 \Leftrightarrow$ $-a/2.5 = \ln 0.02 \Leftrightarrow a = -2.5\ln\frac{1}{50} = 2.5\ln 50 \approx 9.78 \text{ min} \approx 10$ min. The ad should say that if you aren't served within 10 minutes, you get a free hamburger.

8. (a) $P(65 \leq X \leq 73) = \displaystyle\int_{65}^{73} \frac{1}{2.8\sqrt{2\pi}} \exp\left(-\frac{(x-69)^2}{2\cdot 2.8^2}\right)dx \approx 0.847$ (using a calculator or computer to estimate the integral).

(b) $P(X > 72) = 1 - P(0 \leq X \leq 72) \approx 1 - 0.858 = 0.142$, so 14.2% of the adult male population is more than 6 feet tall.

9. $P(X \geq 10) = \displaystyle\int_{10}^\infty \frac{1}{4.2\sqrt{2\pi}} \exp\left(-\frac{(x-9.4)^2}{2\cdot 4.2^2}\right)dx$. To avoid the improper integral we approximate it by the integral from 10 to 100. Thus, $P(X \geq 10) \approx \displaystyle\int_{10}^{100} \frac{1}{4.2\sqrt{2\pi}} \exp\left(-\frac{(x-9.4)^2}{2\cdot 4.2^2}\right)dx \approx 0.443$ (using a calculator or computer to estimate the integral), so 44.3% of the households throw out at least 10 lb of paper a week.

10. (a) $P(0 \leq X \leq 480) = \displaystyle\int_0^{480} \frac{1}{12\sqrt{2\pi}} \exp\left(-\frac{(x-500)^2}{2\cdot 12^2}\right)dx \approx 0.0478$ (using a calculator or computer to estimate the integral), so there is about a 4.78% chance that a particular box contains less than 480 g of cereal.

(b) We need to find μ so that $P(0 \leq X < 500) = 0.05$. Using our calculator or computer to find $P(0 \leq X \leq 500)$ for various values of μ, we find that if $\mu = 519.73$, $P = 0.05007$; and if $\mu = 519.74$, $P = 0.04998$. So a good target weight is at least 519.74 g.

11. $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = \displaystyle\int_{\mu-2\sigma}^{\mu+2\sigma} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx$. Substituting $t = \dfrac{x-\mu}{\sigma}$ and $dt = \dfrac{1}{\sigma}\,dx$ gives us

$$\int_{-2}^{2} \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2}(\sigma\,dt) = \frac{1}{\sqrt{2\pi}}\int_{-2}^{2} e^{-t^2/2}\,dt \approx 0.9545$$

12. Let $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ ce^{-cx} & \text{if } x \geq 0 \end{cases}$ where $c = 1/\mu$. By using parts, tables, or a CAS, we find that

(1): $\int xe^{bx}\,dx = \left(e^{bx}/b^2\right)(bx-1)$

(2): $\int x^2e^{bx}\,dx = \left(e^{bx}/b^3\right)\left(b^2x^2 - 2bx + 2\right)$

Now

$$\begin{aligned}\sigma^2 &= \textstyle\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \int_{-\infty}^{0} (x-\mu)^2 f(x)\,dx + \int_0^{\infty} (x-\mu)^2 f(x)\,dx \\ &= 0 + \lim_{t\to\infty} c \textstyle\int_0^t (x-\mu)^2 e^{-cx}\,dx = c \cdot \lim_{t\to\infty} \int_0^t \left(x^2e^{-cx} - 2x\mu e^{-cx} + \mu^2 e^{-cx}\right)dx\end{aligned}$$

Next we use (2) and (1) with $b = -c$ to get

$$\sigma^2 = c \lim_{t\to\infty}\left[-\frac{e^{-cx}}{c^3}\left(c^2x^2 + 2cx + 2\right) - 2\mu\frac{e^{-cx}}{c^2}(-cx-1) + \mu^2\frac{e^{-cx}}{-c}\right]_0^t$$

Using l'Hospital's Rule several times, along with the fact that $\mu = 1/c$, we get

$$\sigma^2 = c\left[0 - \left(-\frac{2}{c^3} + \frac{2}{c}\cdot\frac{1}{c^2} + \frac{1}{c^2}\cdot\frac{1}{-c}\right)\right] = c\left(\frac{1}{c^3}\right) = \frac{1}{c^2} \quad\Rightarrow\quad \sigma = \frac{1}{c} = \mu$$

13. (a) First $p(r) = \dfrac{4}{a_0^3}r^2e^{-2r/a_0} \geq 0$ for $r \geq 0$. Next,

$$\int_{-\infty}^{\infty} p(r)\,dr = \int_0^{\infty} \frac{4}{a_0^3}r^2e^{-2r/a_0}\,dr = \frac{4}{a_0^3}\lim_{t\to\infty}\int_0^t r^2e^{-2r/a_0}\,dr$$

As in Exercise 12, we use (2) from that solution (with $b = -2/a_0$) and l'Hospital's Rule to get

$\dfrac{4}{a_0^3}\left[\dfrac{a_0^3}{-8}(-2)\right] = 1$. This satisfies the second condition for a function to be a probability density function.

(b) Using l'Hospital's Rule, $\dfrac{4}{a_0^3}\lim\limits_{r\to\infty}\dfrac{r^2}{e^{2r/a_0}} = \dfrac{4}{a_0^3}\lim\limits_{r\to\infty}\dfrac{2r}{(2/a_0)\,e^{2r/a_0}} = \dfrac{2}{a_0^2}\lim\limits_{r\to\infty}\dfrac{2}{(2/a_0)\,e^{2r/a_0}} = 0.$

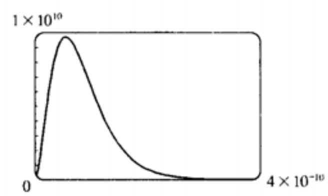
To find the maximum of p, we differentiate:

$$p'(r) = \frac{4}{a_0^3}\left[r^2e^{-2r/a_0}\left(-\frac{2}{a_0}\right) + e^{-2r/a_0}(2r)\right] = \frac{4}{a_0^3}e^{-2r/a_0}(2r)\left(-\frac{r}{a_0}+1\right)$$

$p'(r) = 0 \quad\Leftrightarrow\quad r = 0$ or $1 = \dfrac{r}{a_0} \quad\Leftrightarrow\quad r = a_0$. $p'(r)$ changes from positive to negative at $r = a_0$, so $p(r)$ has its maximum value at $r = a_0$.

(c) It is fairly difficult to find a viewing rectangle, but knowing the maximum value from part (b) helps.

$$p(a_0) = \frac{4}{a_0^3}a_0^2e^{-2a_0/a_0} = \frac{4}{a_0}e^{-2} \approx 9{,}684{,}098{,}979$$



With a maximum of nearly 10 billion and a total area under the curve of 1, we know that the "hump" in the graph must be extremely narrow.

(d) $P(r) = \int_0^r \frac{4}{a_0^3} s^2 e^{-2s/a_0}\, ds \;\Rightarrow\; P(4a_0) = \int_0^{4a_0} \frac{4}{a_0^3} s^2 e^{-2s/a_0}\, ds$. Using (2) from the solution to Exercise 12 (with $b = -2/a_0$),

$$P(4a_0) = \frac{4}{a_0^3}\left[\frac{e^{-2s/a_0}}{-8/a_0^3}\left(\frac{4}{a_0^2}s^2 + \frac{4}{a_0}s + 2\right)\right]_0^{4a_0} = \frac{4}{a_0^3}\left(\frac{a_0^3}{-8}\right)\left[e^{-8}(64+16+2) - 1(2)\right]$$

$$= -\tfrac{1}{2}\left(82e^{-8} - 2\right) = 1 - 41e^{-8} \approx 0.986$$

(e) $\mu = \int_{-\infty}^{\infty} r p(r)\, dr = \frac{4}{a_0^3} \lim_{t\to\infty} \int_0^t r^3 e^{-2r/a_0} dr$. Integrating by parts three times or using a CAS, we find that

$\int x^3 e^{bx} dx = \frac{e^{bx}}{b^4}\left(b^3x^3 - 3b^2x^2 + 6bx - 6\right)$. So with $b = -\frac{2}{a_0}$, we use l'Hospital's Rule, and get

$\mu = \frac{4}{a_0^3}\left[\frac{a_0^4}{16}(-6)\right] = \frac{3}{2}a_0$.

9 Review

CONCEPT CHECK

1. (a) The length of a curve is defined to be the limit of the lengths of the inscribed polygons, as described near Figure 3 in Section 9.1.
   (b) See Equation 9.1.2.
   (c) See Equation 9.1.4.
2. (a) $S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$
   (b) If $x = g(y)$, $c \le y \le d$, then $S = \int_c^d 2\pi y \sqrt{1 + [g'(y)]^2}\, dy$.
   (c) $S = \int_a^b 2\pi x \sqrt{1 + [f'(x)]^2}\, dx$ or $S = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\, dy$
3. Let $c(x)$ be the cross-sectional length of the wall (measured parallel to the surface of the fluid) at depth x. Then the hydrostatic force against the wall is given by $F = \int_a^b \delta x c(x)\, dx$, where a and b are the lower and upper limits for x at points of the wall and δ is the weight density of the fluid.
4. (a) The center of mass is the point at which the plate balances horizontally.
   (b) See Equations 9.3.8.
5. If a plane region $\mathcal{R}$ that lies entirely on one side of a line ℓ in its plane is rotated about ℓ, then the volume of the resulting solid is the product of the area of $\mathcal{R}$ and the distance traveled by the centroid of $\mathcal{R}$.
6. See Figure 3 in Section 9.4, and the discussion which precedes it.
7. (a) See the definition before Figure 6 in Section 9.4.
   (b) See the discussion after Figure 6 in Section 9.4.
8. A probability density function f is a function on the domain of a continuous random variable X such that $\int_a^b f(x)\, dx$ measures the probability that X lies between a and b. Such a function f has nonnegative values and satisfies the relation $\int_D f(x)\, dx = 1$, where D is the domain of the corresponding random variable X. If $D = \mathbb{R}$, or if we define $f(x) = 0$ for real numbers $x \notin D$, then $\int_{-\infty}^{\infty} f(x) = 1$. (Of course, to work with f in this way, we must assume that the integrals of f exist.)

9. (a) $\int_0^{100} f(x)\,dx$ represents the probability that the weight of a randomly chosen female college student is less than 100 pounds.

(b) $\mu = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^{\infty} xf(x)\,dx$

(c) The median of f is the number m such that $\int_m^{\infty} f(x)\,dx = \frac{1}{2}$.

EXERCISES

1. $3x = 2(y-1)^{3/2}$, $2 \le y \le 5$. $x = \frac{2}{3}(y-1)^{3/2}$, so $dx/dy = (y-1)^{1/2}$ and the arc length formula gives

$$L = \int_2^5 \sqrt{1+(dx/dy)^2}\,dy = \int_2^5 \sqrt{1+(y-1)}\,dy = \int_2^5 \sqrt{y}\,dy = \left[\tfrac{2}{3}y^{3/2}\right]_2^5 = \tfrac{2}{3}\left(5\sqrt{5}-2\sqrt{2}\right)$$

2. $y = \ln x - \dfrac{x^2}{8} \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{1}{x} - \dfrac{x}{4} \;\Rightarrow\; 1+\left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{1}{x^2} - \dfrac{1}{2} + \dfrac{x^2}{16} = \dfrac{1}{x^2} + \dfrac{1}{2} + \dfrac{x^2}{16} = \left(\dfrac{1}{x} + \dfrac{x}{4}\right)^2 \;\Rightarrow$

$$L = \int_1^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^4 \left(\frac{1}{x}+\frac{x}{4}\right)dx = \left[\ln x + \frac{x^2}{8}\right]_1^4 = (\ln 4 + 2) - \left(\ln 1 + \tfrac{1}{8}\right) = 2\ln 2 + \tfrac{15}{8}$$

3. (a) $y = \frac{1}{6}x^3 + \dfrac{1}{2x}$, $1 \le x \le 2 \;\Rightarrow\; y' = \dfrac{1}{2}\left(x^2 - \dfrac{1}{x^2}\right) \;\Rightarrow\; (y')^2 = \dfrac{1}{4}\left(x^4 - 2 + \dfrac{1}{x^4}\right) \;\Rightarrow$

$$1+(y')^2 = \frac{1}{4}\left(x^4+2+\frac{1}{x^4}\right) = \frac{1}{4}\left(x^2+\frac{1}{x^2}\right)^2 \;\Rightarrow$$

$$L = \int_1^2 \sqrt{1+(y')^2}\,dy = \frac{1}{2}\int_1^2 \left(x^2+\frac{1}{x^2}\right)dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{1}{x}\right]_1^2 = \frac{1}{2}\left(\frac{17}{6}\right) = \frac{17}{12}$$

(b) $S = \displaystyle\int_1^2 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_1^2 \left(\frac{x^3}{6}+\frac{1}{2x}\right)\frac{1}{2}\left(x^2+\frac{1}{x^2}\right)dx = \pi\int_1^2 \left(\tfrac{1}{6}x^5 + \tfrac{2}{3}x + \tfrac{1}{2}x^{-3}\right)dx$

$= \pi\left[\frac{1}{36}x^6 + \frac{1}{3}x^2 - \frac{1}{4}x^{-2}\right]_1^2 = \pi\left[\left(\frac{64}{36}+\frac{4}{3}-\frac{1}{16}\right) - \left(\frac{1}{36}+\frac{1}{3}-\frac{1}{4}\right)\right] = \frac{47\pi}{16}$

4. (a) $y = x^2 \;\Rightarrow\; 1+(y')^2 = 1+4x^2 \;\Rightarrow$

$$S = \int_0^1 2\pi x\sqrt{1+4x^2}\,dx = \int_1^5 \tfrac{\pi}{4}\sqrt{u}\,du \;\;(u = 1+4x^2) \;= \tfrac{\pi}{6}\left[u^{3/2}\right]_1^5 = \tfrac{\pi}{6}\left(5^{3/2}-1\right)$$

(b) $y = x^2 \;\Rightarrow\; 1+(y')^2 = 1+4x^2$. So

$$S = 2\pi\int_0^1 x^2\sqrt{1+4x^2}\,dx = 2\pi\int_0^2 \tfrac{1}{4}u^2\sqrt{1+u^2}\,\tfrac{1}{2}\,du \;\;(u=2x) \;= \tfrac{\pi}{4}\int_0^2 u^2\sqrt{1+u^2}\,du$$

$$= \tfrac{\pi}{4}\left[\tfrac{1}{8}u\left(1+2u^2\right)\sqrt{1+u^2} - \tfrac{1}{8}\ln\left|u+\sqrt{1+u^2}\right|\right]_0^2 \;\;(u = \tan\theta \text{ or use Formula 22})$$

$$= \tfrac{\pi}{4}\left[\tfrac{1}{4}(9)\sqrt{5} - \tfrac{1}{8}\ln\left(2+\sqrt{5}\right) - 0\right] = \tfrac{\pi}{32}\left[18\sqrt{5} - \ln\left(2+\sqrt{5}\right)\right]$$

5. $y = \sqrt[3]{x} = x^{1/3} \;\Rightarrow\; dy/dx = \frac{1}{3}x^{-2/3} \;\Rightarrow\; \sqrt{1+(dy/dx)^2} = \sqrt{1+1/\left(9x^{4/3}\right)}$. Call this $f(x)$. Then

$L = \int_1^8 f(x)\,dx$

$\approx S_{14} = \frac{8-1}{14\cdot 3}\left[f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + \cdots + 2f(7) + 4f(7.5) + f(8)\right] \approx 7.082581$

6. Set $g(x) = x^{1/3}\sqrt{1 + 1/\left(9x^{4/3}\right)} = x^{1/3} f(x)$. (See the solution to Exercise 5.) Then

$$\begin{aligned} S &= 2\pi \textstyle\int_1^8 g(x)\,dx \\ &\approx 2\pi S_{14} = 2\pi \tfrac{8-1}{14\cdot 3}\left[g(1) + 4g(1.5) + 2g(2) + 4g(2.5) + \cdots + 2g(7) + 4g(7.5) + g(8)\right] \\ &\approx 2\pi\,(11.364436) \approx 71.404857 \end{aligned}$$

7. The loop lies between $x = 0$ and $x = 3a$ and is symmetric about the x-axis. We can assume without loss of generality that $a > 0$, since if $a = 0$, the graph is the parallel lines $x = 0$ and $x = 3a$, so there is no loop. The upper half of the loop is given by $y = \dfrac{1}{3\sqrt{a}}\sqrt{x}\,(3a - x) = \sqrt{a}\,x^{1/2} - \dfrac{x^{3/2}}{3\sqrt{a}}$, $0 \le x \le 3a$. The desired surface area is twice the area generated by the upper half of the loop, that is, $S = 2\,(2\pi)\displaystyle\int_0^{3a} x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$.

$\dfrac{dy}{dx} = \dfrac{\sqrt{a}}{2}x^{-1/2} - \dfrac{x^{1/2}}{2\sqrt{a}} \quad\Rightarrow\quad 1 + \left(\dfrac{dy}{dx}\right)^2 = \dfrac{a}{4x} + \dfrac{1}{2} + \dfrac{x}{4a}$. Therefore

$$\begin{aligned} S &= 2\,(2\pi)\int_0^{3a} x\left(\frac{\sqrt{a}}{2}x^{-1/2} + \frac{x^{1/2}}{2\sqrt{a}}\right)dx = 2\pi\int_0^{3a}\left(\sqrt{a}\,x^{1/2} + \frac{x^{3/2}}{\sqrt{a}}\right)dx \\ &= 2\pi\left[\frac{2\sqrt{a}}{3}x^{3/2} + \frac{2}{5\sqrt{a}}x^{5/2}\right]_0^{3a} = 2\pi\left[\frac{2\sqrt{a}}{3}3a\sqrt{3a} + \frac{2}{5\sqrt{a}}9a^2\sqrt{3a}\right] = \frac{56\sqrt{3}\pi a^2}{5} \end{aligned}$$

8. $y^2 = ax - \frac{2}{3}x^2 + \dfrac{x^3}{9a} = \left(\sqrt{ax} - \dfrac{x^{3/2}}{3\sqrt{a}}\right)^2$, $y = \sqrt{ax} - \dfrac{x^{3/2}}{3\sqrt{a}} \quad\Rightarrow$

$1 + \left(\dfrac{dy}{dx}\right)^2 = \left(\dfrac{\sqrt{a}}{2\sqrt{x}} - \dfrac{\sqrt{x}}{2\sqrt{a}}\right)^2 + 1 = \left(\dfrac{\sqrt{a}}{2\sqrt{x}} + \dfrac{\sqrt{x}}{2\sqrt{a}}\right)^2$. So

$$\begin{aligned} S &= 2\pi\int_0^{3a}\left(\sqrt{ax} - \frac{x^{3/2}}{3\sqrt{a}}\right)\left(\frac{\sqrt{a}}{2\sqrt{x}} + \frac{\sqrt{x}}{2\sqrt{a}}\right)dx = 2\pi\int_0^{3a}\left(\frac{a}{2} + \frac{x}{2} - \frac{x}{6} - \frac{x^2}{6a}\right)dx \\ &= 2\pi\left[\frac{a}{2}x + \frac{x^2}{6} - \frac{x^3}{18a}\right]_0^{3a} = 2\pi\left(\frac{3a^2}{2} + \frac{9a^2}{6} - \frac{27a^3}{18a}\right) = 3\pi a^2 \end{aligned}$$

9. As in Example 1 of Section 9.3,

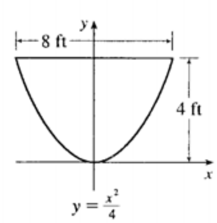
$$F = \textstyle\int_0^2 \rho g x\,(5 - x)\,dx = \rho g\left[\frac{5}{2}x^2 - \frac{1}{3}x^3\right]_0^2 = \rho g\left(10 - \frac{8}{3}\right) = \frac{22}{3}\delta \approx \frac{22}{3}\cdot 62.5 \approx 458.3 \text{ lb}$$

10. $F = \int_0^4 \delta\,(4 - y)\,2\left(2\sqrt{y}\right)dy = 4\delta\int_0^4\left(4y^{1/2} - y^{3/2}\right)dy$

$= 4\delta\left[\frac{8}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^4 = 4\delta\left(\frac{64}{3} - \frac{64}{5}\right)$

$= 256\delta\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{512}{15}\delta$

≈ 2133.3 lb $\left(\delta \approx 62.5 \text{ lb/ft}^3\right)$



11. $A = \int_{-2}^{1}\left[(4-x^2)-(x+2)\right]dx = \int_{-2}^{1}(2-x-x^2)\,dx = \left[2x-\frac{1}{2}x^2-\frac{1}{3}x^3\right]_{-2}^{1}$

$= \left(2-\frac{1}{2}-\frac{1}{3}\right)-\left(-4-2+\frac{8}{3}\right) = \frac{9}{2} \quad\Rightarrow$

$$\overline{x} = A^{-1}\int_{-2}^{1}x\left(2-x-x^2\right)dx = \frac{2}{9}\int_{-2}^{1}\left(2x-x^2-x^3\right)dx = \frac{2}{9}\left[x^2-\frac{1}{3}x^3-\frac{1}{4}x^4\right]_{-2}^{1}$$

$$= \frac{2}{9}\left[\left(1-\frac{1}{3}-\frac{1}{4}\right)-\left(4+\frac{8}{3}-4\right)\right] = -\frac{1}{2}$$

and

$$\overline{y} = A^{-1}\int_{-2}^{1}\frac{1}{2}\left[\left(4-x^2\right)^2-(x+2)^2\right]dx = \frac{1}{9}\int_{-2}^{1}\left(x^4-9x^2-4x+12\right)dx$$

$$= \frac{1}{9}\left[\frac{1}{5}x^5-3x^3-2x^2+12x\right]_{-2}^{1} = \frac{1}{9}\left[\left(\frac{1}{5}-3-2+12\right)-\left(-\frac{32}{5}+24-8-24\right)\right] = \frac{12}{5}$$

So $(\overline{x},\overline{y}) = \left(-\frac{1}{2},\frac{12}{5}\right)$.

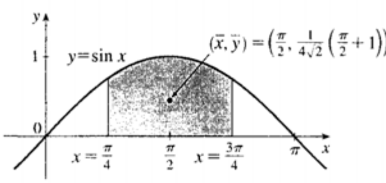
12. From the symmetry of the region, $\overline{x} = \frac{\pi}{2}$. $A = \int_{\pi/4}^{3\pi/4}\sin x\,dx = \left[-\cos x\right]_{\pi/4}^{3\pi/4} = \frac{1}{\sqrt{2}}-\left(-\frac{1}{\sqrt{2}}\right) = \sqrt{2}$

$$\overline{y} = \frac{1}{A}\int_{\pi/4}^{3\pi/4}\frac{1}{2}\sin^2 x\,dx = \frac{1}{A}\int_{\pi/4}^{3\pi/4}\frac{1}{4}(1-\cos 2x)\,dx$$

$$= \frac{1}{4\sqrt{2}}\left[x-\frac{1}{2}\sin 2x\right]_{\pi/4}^{3\pi/4}$$

$$= \frac{1}{4\sqrt{2}}\left[\frac{3\pi}{4}-\frac{1}{2}(-1)-\frac{\pi}{4}+\frac{1}{2}\cdot 1\right]$$

$$= \frac{1}{4\sqrt{2}}\left(\frac{\pi}{2}+1\right)$$



So $(\overline{x},\overline{y}) = \left(\frac{\pi}{2},\frac{1}{4\sqrt{2}}\left(\frac{\pi}{2}+1\right)\right) \approx (1.57, 0.45)$.

13. The equation of the line passing through $(0,0)$ and $(3,2)$ is $y = \frac{2}{3}x$. $A = \frac{1}{2}\cdot 3\cdot 2 = 3$. Therefore, using Equations 9.3.8, $\overline{x} = \frac{1}{3}\int_0^3 x\left(\frac{2}{3}x\right)dx = \frac{2}{27}\left[x^3\right]_0^3 = 2$, and $\overline{y} = \frac{1}{3}\int_0^3\frac{1}{2}\left(\frac{2}{3}x\right)^2dx = \frac{2}{81}\left[x^3\right]_0^3 = \frac{2}{3}$. Thus, $(\overline{x},\overline{y}) = \left(2,\frac{2}{3}\right)$.

14. Suppose first that the large rectangle were complete, so that its mass would be $6\cdot 3 = 18$. Its centroid would be $\left(1,\frac{3}{2}\right)$. The mass removed from this object to create the one being studied is 3. The centroid of the cut-out piece is $\left(\frac{3}{2},\frac{3}{2}\right)$. Therefore, for the actual lamina, whose mass is 15, $\overline{x} = \frac{18}{15}(1)-\frac{3}{15}\left(\frac{3}{2}\right) = \frac{9}{10}$, and $\overline{y} = \frac{3}{2}$, since the lamina is symmetric about the line $y = \frac{3}{2}$. Therefore $(\overline{x},\overline{y}) = \left(\frac{9}{10},\frac{3}{2}\right)$.

15. The centroid of this circle, $(1,0)$, travels a distance $2\pi\,(1)$ when the lamina is rotated about the y-axis. The area of the circle is $\pi\,(1)^2$. So by the Theorem of Pappus, $V = A2\pi\overline{x} = \pi\,(1)^2\,2\pi\,(1) = 2\pi^2$.

16. The semicircular region has an area of $\frac{1}{2}\pi r^2$, and sweeps out a sphere of radius r when rotated about the x-axis. $\overline{x} = 0$ because of symmetry about the line $x = 0$. And by the Theorem of Pappus, Volume $= \frac{4}{3}\pi r^3 = \frac{1}{2}\pi r^2\,(2\pi\,\overline{y})$ $\Rightarrow\quad \overline{y} = \frac{4}{3\pi}r$. Therefore, $(\overline{x},\overline{y}) = \left(0,\frac{4}{3\pi}r\right)$.

17. $x = 100 \quad \Rightarrow \quad P = 2000 - 0.1\,(100) - 0.01\,(100)^2 = 1890$

$$\text{Consumer surplus} = \int_0^{100} [p\,(x) - P]\,dx = \int_0^{100} \left(2000 - 0.1x - 0.01x^2 - 1890\right) dx$$
$$= \left[110x - 0.05x^2 - \tfrac{0.01}{3}x^3\right]_0^{100} = 11,000 - 500 - \tfrac{10,000}{3} \approx \$7166.67$$

18. $\int_0^{24} c\,(t)\,dt \approx S_{12} = \frac{24-0}{12 \cdot 3}\,[1\,(0) + 4\,(1.9) + 2\,(3.3) + 4\,(5.1) + 2\,(7.6) + 4\,(7.1) + 2\,(5.8)$
$+ 4\,(4.7) + 2\,(3.3) + 4\,(2.1) + 2\,(1.1) + 4\,(0.5) + 1\,(0)]$
$= \frac{2}{3}\,(127.8) = 85.2\ \text{mg} \cdot \text{s/L}$

Therefore, $F \approx A/85.2 = 6/85.2 \approx 0.0704$ L/s or 4.225 L/min.

19. $f\,(x) = \begin{cases} \frac{\pi}{20}\sin\left(\frac{\pi}{10}x\right) & \text{if } 0 \le x \le 10 \\ 0 & \text{if } x < 0 \text{ or } x > 10 \end{cases}$

(a) $f\,(x) \ge 0$ for all real numbers x and

$\int_{-\infty}^{\infty} f\,(x)\,dx = \int_0^{10} \frac{\pi}{20}\sin\left(\frac{\pi}{10}x\right) dx = \frac{\pi}{20} \cdot \frac{10}{\pi}\left[-\cos\left(\frac{\pi}{10}x\right)\right]_0^{10} = \frac{1}{2}\,(-\cos\pi + \cos 0) = \frac{1}{2}\,(1+1) = 1.$

(b) $P\,(X < 4) = \int_{-\infty}^{4} f\,(x)\,dx = \int_0^4 \frac{\pi}{20}\sin\left(\frac{\pi}{10}x\right) dx = \frac{1}{2}\left[-\cos\left(\frac{\pi}{10}x\right)\right]_0^4 = \frac{1}{2}\left(-\cos\frac{2\pi}{5} + \cos 0\right)$

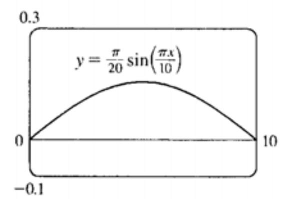
$\approx \frac{1}{2}\,(-0.309017 + 1) \approx 0.345492$

(c) $\mu = \int_{-\infty}^{\infty} x f\,(x)\,dx = \int_0^{10} \frac{\pi}{20}x\sin\left(\frac{\pi}{10}x\right) dx$

$= \int_0^{\pi} \frac{\pi}{20} \cdot \frac{10}{\pi}u\,(\sin u)\left(\frac{10}{\pi}\right) du \quad \left[u = \frac{\pi}{10}x,\ du = \frac{\pi}{10}\,dx\right]$

$= \frac{5}{\pi}\int_0^{\pi} u \sin u\,du \overset{82}{=} \frac{5}{\pi}\,[\sin u - u\cos u]_0^{\pi} = \frac{5}{\pi}\,[0 - \pi\,(-1)] = 5$

This answer is expected because the graph of f is symmetric about the line $x = 5$.



20. $P\,(250 \le X \le 280) = \displaystyle\int_{250}^{280} \frac{1}{15\sqrt{2\pi}} \exp\left(\frac{-(x-268)^2}{2 \cdot 15^2}\right) dx \approx 0.673$. Thus, the percentage of pregnancies that last between 250 and 280 days is about 67.3%.

21. (a) The probability density function is $f\,(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{8}e^{-t/8} & \text{if } t \ge 0 \end{cases}$

$P\,(0 \le X \le 3) = \int_0^3 \frac{1}{8}e^{-t/8}\,dt = \left[-e^{-t/8}\right]_0^3 = -e^{-3/8} + 1 \approx 0.3127$

(b) $P\,(X > 10) = \int_{10}^{\infty} \frac{1}{8}e^{-t/8}\,dt = \lim\limits_{x\to\infty}\left[-e^{-t/8}\right]_{10}^{x} = \lim\limits_{x\to\infty}\left(-e^{-x/8} + e^{-10/8}\right) = 0 + e^{-5/4} \approx 0.2865$

(c) We need to find m such that $P\,(X \ge m) = \frac{1}{2} \quad \Rightarrow \quad \int_m^{\infty} \frac{1}{8}e^{-t/8}\,dt = \frac{1}{2} \quad \Rightarrow \quad \lim\limits_{x\to\infty}\left[-e^{-t/8}\right]_m^x = \frac{1}{2} \quad \Rightarrow$

$\lim\limits_{x\to\infty}\left(-e^{-x/8} + e^{-m/8}\right) = \frac{1}{2} \quad \Rightarrow \quad e^{-m/8} = \frac{1}{2} \quad \Rightarrow \quad -m/8 = \ln\frac{1}{2} \quad \Rightarrow$

$m = -8\ln\frac{1}{2} = 8\ln 2 \approx 5.55$ minutes.

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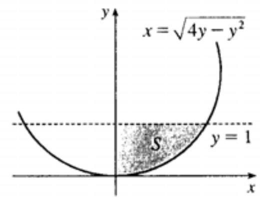
Problems Plus

1. $x^2 + y^2 \leq 4y \quad \Leftrightarrow \quad x^2 + (y-2)^2 \leq 4$, so S is part of a circle, as shown in the diagram. The area of S is



$$\int_0^1 \sqrt{4y - y^2}\,dy = \int_{-2}^{-1} \sqrt{4 - v^2}\,dv \qquad (v = y - 2)$$

$$\overset{30}{=} \left[\tfrac{1}{2} v\sqrt{4 - v^2} + \tfrac{1}{2}(4)\sin^{-1}\left(\tfrac{1}{2}v\right)\right]_{-2}^{-1} = \tfrac{2\pi}{3} - \tfrac{\sqrt{3}}{2}$$

Another Method (without calculus): Note that $\theta = \angle ABC = \frac{\pi}{3}$, so the area is

$$(\text{area of sector } AOC) - (\text{area of } \triangle ABC) = \tfrac{1}{2}\left(2^2\right)\tfrac{\pi}{3} - \tfrac{1}{2}(1)\sqrt{3} = \tfrac{2\pi}{3} - \tfrac{\sqrt{3}}{2}$$

2. $y = \pm\sqrt{x^3 - x^4} \quad \Rightarrow$ The loop of the curve is symmetric about $y = 0$, and therefore $\overline{y} = 0$. At each point x where $0 \leq x \leq 1$, the lamina has a vertical length of $\sqrt{x^3 - x^4} - \left(-\sqrt{x^3 - x^4}\right) = 2\sqrt{x^3 - x^4}$. Therefore,

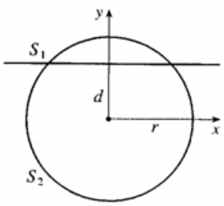
$$\overline{x} = \frac{\int_0^1 x \cdot 2\sqrt{x^3 - x^4}\,dx}{\int_0^1 2\sqrt{x^3 - x^4}\,dx} = \frac{\int_0^1 x\sqrt{x^3 - x^4}\,dx}{\int_0^1 \sqrt{x^3 - x^4}\,dx}.$$ We evaluate the integrals separately:

$$\begin{aligned}
\int_0^1 x\sqrt{x^3 - x^4}\,dx &= \int_0^1 x^{5/2}\sqrt{1 - x}\,dx \\
&= \int_0^{\pi/2} 2\sin^6\theta\cos\theta\sqrt{1 - \sin^2\theta}\,d\theta \quad \begin{bmatrix} \sin\theta = \sqrt{x},\ \cos\theta\,d\theta = dx/\left(2\sqrt{x}\right), \\ 2\sin\theta\cos\theta\,d\theta = dx \end{bmatrix} \\
&= \int_0^{\pi/2} 2\sin^6\theta\cos^2\theta\,d\theta = \int_0^{\pi/2} 2\left[\tfrac{1}{2}(1 - \cos 2\theta)\right]^3 \tfrac{1}{2}(1 + \cos 2\theta)\,d\theta \\
&= \int_0^{\pi/2} \tfrac{1}{8}\left(1 - 2\cos 2\theta + 2\cos^3 2\theta - \cos^4 2\theta\right)d\theta \\
&= \int_0^{\pi/2} \tfrac{1}{8}\left[1 - 2\cos 2\theta + 2\cos 2\theta\left(1 - \sin^2 2\theta\right) - \tfrac{1}{4}(1 + \cos 4\theta)^2\right]d\theta \\
&= \tfrac{1}{8}\left[\theta - \tfrac{1}{3}\sin^3 2\theta\right]_0^{\pi/2} - \tfrac{1}{32}\int_0^{\pi/2}\left(1 + 2\cos 4\theta + \cos^2 4\theta\right)d\theta \\
&= \tfrac{\pi}{16} - \tfrac{1}{32}\left[\theta + \tfrac{1}{2}\sin 4\theta\right]_0^{\pi/2} - \tfrac{1}{64}\int_0^{\pi/2}(1 + \cos 8\theta)\,d\theta \\
&= \tfrac{3\pi}{64} - \tfrac{1}{64}\left[\theta + \tfrac{1}{8}\sin 8\theta\right]_0^{\pi/2} = \tfrac{5\pi}{128}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \sqrt{x^3 - x^4}\,dx &= \int_0^1 x^{3/2}\sqrt{1 - x}\,dx = \int_0^{\pi/2} 2\sin^4\theta\cos\theta\sqrt{1 - \sin^2\theta}\,d\theta \quad \left(\sin\theta = \sqrt{x}\right) \\
&= \int_0^{\pi/2} 2\sin^4\theta\cos^2\theta\,d\theta = \int_0^{\pi/2} 2 \cdot \tfrac{1}{4}(1 - \cos 2\theta)^2 \cdot \tfrac{1}{2}(1 + \cos 2\theta)\,d\theta \\
&= \int_0^{\pi/2} \tfrac{1}{4}\left(1 - \cos 2\theta - \cos^2 2\theta + \cos^3 2\theta\right)d\theta \\
&= \int_0^{\pi/2} \tfrac{1}{4}\left[1 - \cos 2\theta - \tfrac{1}{2}(1 + \cos 4\theta) + \cos 2\theta\left(1 - \sin^2 2\theta\right)\right]d\theta \\
&= \tfrac{1}{4}\left[\tfrac{\theta}{2} - \tfrac{1}{8}\sin 4\theta - \tfrac{1}{6}\sin^3 2\theta\right]_0^{\pi/2} = \tfrac{\pi}{16}
\end{aligned}$$

Therefore, $\overline{x} = \dfrac{5\pi/128}{\pi/16} = \dfrac{5}{8}$, and $(\overline{x}, \overline{y}) = \left(\dfrac{5}{8}, 0\right)$.

3. (a) The two spherical zones, whose surface areas we will call S_1 and S_2, are generated by rotation about the y-axis of circular arcs, as indicated in the figure. The arcs are the upper and lower portions of the circle $x^2 + y^2 = r^2$ that are obtained when the circle is cut with the line $y = d$. The portion of the upper arc in the first quadrant is sufficient to generate the upper spherical zone. That portion of the arc can be described



by the relation $x = \sqrt{r^2 - y^2}$ for $d \le y \le r$. Thus, $dy/dx = -y \Big/ \sqrt{r^2 - y^2}$ and

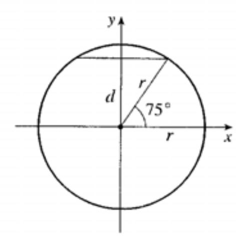
$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)}\, dy = \sqrt{1 + \frac{y^2}{r^2 - y^2}}\, dy = \sqrt{\frac{r^2}{r^2 - y^2}}\, dy = \frac{r\,dy}{\sqrt{r^2 - y^2}}$$

From Formula 9.2.8 we have

$$S_1 = \int_d^r 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_d^r 2\pi \sqrt{r^2 - y^2} \frac{r\,dy}{\sqrt{r^2 - y^2}} = \int_d^r 2\pi r\, dy = 2\pi r\,(r - d)$$

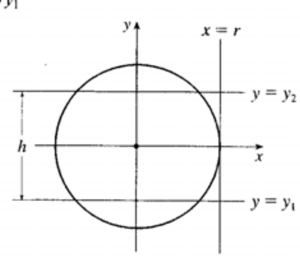
Similarly, we can compute $S_2 = \int_{-r}^{d} 2\pi x \sqrt{1 + (dx/dy)^2}\, dy = \int_{-r}^{d} 2\pi r\, dy = 2\pi r\,(r + d)$. Note that $S_1 + S_2 = 4\pi r^2$, the surface area of the entire sphere.

(b) $r = 3960$ mi and $d = r\,(\sin 75°) \approx 3825$ mi, so the surface area of the Arctic Ocean is about $2\pi r\,(r - d) \approx 2\pi\,(3960)\,(135) \approx 3.36 \times 10^6\ \text{mi}^2$.



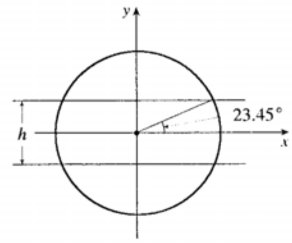
(c) The area on the sphere lies between planes $y = y_1$ and $y = y_2$, where $y_2 - y_1 = h$. Thus, we compute the surface area on the sphere to be $S = \displaystyle\int_{y_1}^{y_2} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_{y_1}^{y_2} 2\pi r\, dy = 2\pi r\,(y_2 - y_1) = 2\pi r h$.

This equals the lateral area of a cylinder of radius r and height h, since such a cylinder is obtained by rotating the line $x = r$ about the y-axis, so the surface area of the cylinder between the planes $y = y_1$ and $y = y_2$ is

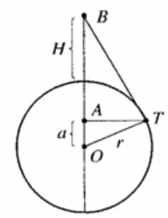


$$A = \int_{y_1}^{y_2} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_{y_1}^{y_2} 2\pi r \sqrt{1 + 0^2}\, dy$$
$$= 2\pi r y\Big|_{y=y_1}^{y_2} = 2\pi r\,(y_2 - y_1) = 2\pi r h$$

(d) $h = 2r \sin 23.45° \approx 3152$ mi, so the surface area of the Torrid Zone is $2\pi r h \approx 2\pi\,(3960)\,(3152) \approx 7.84 \times 10^7$ mi^2.



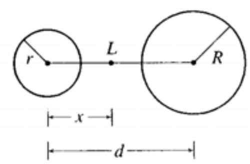
4. (a) Since the right triangles OAT and OBT are similar, we have $\dfrac{H+r}{r} = \dfrac{r}{a} \;\Rightarrow\; a = \dfrac{r^2}{H+r}$. The surface area visible from B is $S = \int_a^r 2\pi x\sqrt{1 + (dx/dy)^2}\,dy$. From $x^2 + y^2 = r^2$, we get $\dfrac{dy}{dx} = -\dfrac{y}{x}$ and $1 + \left(\dfrac{dx}{dy}\right)^2 = \dfrac{x^2 + y^2}{x^2} = \dfrac{r^2}{x^2}$. Thus



$$S = \int_a^r 2\pi x \cdot \frac{r}{x}\,dy = 2\pi r\,(r - a) = 2\pi r\left(r - \frac{r^2}{H+r}\right)$$

$$= 2\pi r^2\left(1 - \frac{r}{H+r}\right) = 2\pi r^2 \cdot \frac{H}{H+R} = \frac{2\pi r^2 H}{r+H}$$

(b) If a light is placed at point L, at a distance x from the center of the sphere of radius r, then from (a) we find that the total illuminated area on the two spheres is $A = \dfrac{2\pi r^2\,(x-R)}{x} + \dfrac{2\pi\,R^2\,(d-x-R)}{d-x}$.



$\dfrac{A}{2\pi} = r^2\left(1 - \dfrac{r}{x}\right) + R^2\left(1 - \dfrac{R}{d-x}\right) \;\Rightarrow\; \dfrac{dA}{dx} = 0 \;\Leftrightarrow$

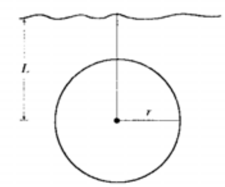
$0 = r^2 \cdot \dfrac{r}{x^2} + R^2 \cdot \dfrac{-R}{(d-x)^2} \;\Leftrightarrow\; 0 = \dfrac{r^3}{x^2} - \dfrac{R^3}{(d-x)^2} \;\Leftrightarrow\; R^3x^2 = r^3\left[d^2 - (2d)\,x + x^2\right] \;\Leftrightarrow$

$\left(R^3 - r^3\right)x^2 + \left(2r^3d\right)x - r^3d^2 = 0 \;\Leftrightarrow\; \left[(R/r)^3 - 1\right]x^2 + (2d)\,x - d^2 = 0$. Assume, without loss of generality, that $R = \lambda r$, where $\lambda \geq 1$. Then $dA/dx = 0 \;\Leftrightarrow\; \left(\lambda^3 - 1\right)x^2 + (2d)\,x - d^2 = 0 \;\Leftrightarrow$

$x = \dfrac{-2d \pm 2d\sqrt{\lambda^3}}{2\left(\lambda^3 - 1\right)} \;\Leftrightarrow\; x = \dfrac{\lambda^{3/2} - 1}{\lambda^3 - 1}d$ (since $x > 0$) $\;\Leftrightarrow\; x = \dfrac{d}{\lambda^{3/2} - 1}$.

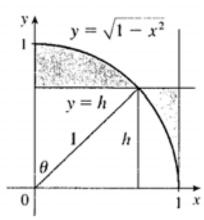
5. (a) Choose a vertical x-axis pointing downward with its origin at the surface. In order to calculate the pressure at depth z, consider n subintervals of the interval $[0, z]$ by points x_i and choose a point $x_i^* \in [x_{i-1}, x_i]$ for each i. The thin layer of water lying between depth x_{i-1} and depth x_i has a density of approximately $\rho(x_i^*)$, so the weight of a piece of that layer with unit cross-sectional area is $\rho(x_i^*)\, g\,\Delta x$. The total weight of a column of water extending from the surface to depth z (with unit cross-sectional area) would be approximately $\sum_{i=1}^{n} \rho(x_i^*)\, g\,\Delta x$. The estimate becomes exact if we take the limit as $n \to \infty$; weight (or force) per unit area at depth z is $W = \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*)\, g\,\Delta x$. In other words, $P(z) = \int_0^z \rho(x)\, g\, dx$. More generally, if we make no assumptions about the location of the origin, then $P(z) = P_0 + \int_0^z \rho(x)\, g\, dx$, where P_0 is the pressure at $x = 0$. Differentiating, we get $dP/dz = \rho(z)\, g$.

(b)



$$
\begin{aligned}
F &= \int_{-r}^{r} P(L + x) \cdot 2\sqrt{r^2 - x^2}\, dx \\
&= \int_{-r}^{r} \left(P_0 + \int_0^{L+x} \rho_0 e^{z/H} g\, dz\right) \cdot 2\sqrt{r^2 - x^2}\, dx \\
&= P_0 \int_{-r}^{r} 2\sqrt{r^2 - x^2}\, dx + \rho_0 g H \int_{-r}^{r} \left(e^{(L+x)/H} - 1\right) \cdot 2\sqrt{r^2 - x^2}\, dx \\
&= (P_0 - \rho_0 g H) \int_{-r}^{r} 2\sqrt{r^2 - x^2}\, dx + \rho_0 g H \int_{-r}^{r} e^{(L+x)/H} \cdot 2\sqrt{r^2 - x^2}\, dx \\
&= (P_0 - \rho_0 g H)(\pi r^2) + \rho_0 g H e^{L/H} \int_{-r}^{r} e^{x/H} \cdot 2\sqrt{r^2 - x^2}\, dx
\end{aligned}
$$

6. The problem can be reduced to finding the line which minimizes the shaded area in the diagram. The equation of the circle in the first quadrant is $y = \sqrt{1 - x^2}$, so if the equation of the line is $y = h$, then the circle and the line intersect where $h = \sqrt{1 - x^2} \;\Rightarrow\; x = \sqrt{1 - h^2}$. So the shaded area is



$$
\begin{aligned}
A &= \int_0^{\sqrt{1-h^2}} \left(\sqrt{1 - x^2} - h\right) dx + \int_{\sqrt{1-h^2}}^{1} \left(h - \sqrt{1 - x^2}\right) \\
&\overset{\bigstar}{=} [-hx]_0^{\sqrt{1-h^2}} + [hx]_{\sqrt{1-h^2}}^{1} + \int_0^{\sqrt{1-h^2}} \sqrt{1 - x^2}\, dx + \int_1^{\sqrt{1-h^2}} \sqrt{1 - x^2}\, dx \\
&= -h\sqrt{1 - h^2} + h - h\sqrt{1 - h^2} + \int_0^{\sqrt{1-h^2}} \sqrt{1 - x^2}\, dx + \int_1^{\sqrt{1-h^2}} \sqrt{1 - x^2}\, dx \\
&= h\left(1 - 2\sqrt{1 - h^2}\right) + \int_0^{\sqrt{1-h^2}} \sqrt{1 - x^2}\, dx + \int_1^{\sqrt{1-h^2}} \sqrt{1 - x^2}\, dx
\end{aligned}
$$

Note that at ($\bigstar$), we reversed the limits of integration and changed the sign in the last integral. We are interested in the minimum of $A(h) = h\left(1 - 2\sqrt{1 - h^2}\right) + \int_0^{\sqrt{1-h^2}} \sqrt{1 - x^2}\, dx + \int_1^{\sqrt{1-h^2}} \sqrt{1 - x^2}\, dx$, so we find dA/dh using FTC1 and the Chain Rule:

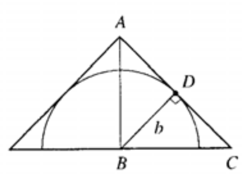
$$
\begin{aligned}
\frac{dA}{dh} &= h\left(-2\frac{-h}{\sqrt{1 - h^2}}\right) + \left(1 - 2\sqrt{1 - h^2}\right) + 2\left[\sqrt{1 - \left(\sqrt{1 - h^2}\right)^2}\right] \frac{d}{dh}\left(\sqrt{1 - h^2}\right) \\
&= \frac{1}{\sqrt{1 - h^2}}\left[2h^2 + \sqrt{1 - h^2} - 2\left(1 - h^2\right)\right] + 2h\frac{-h}{\sqrt{1 - h^2}}
\end{aligned}
$$

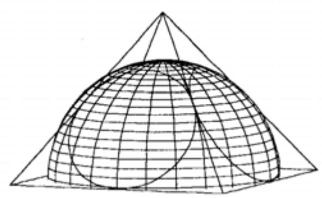
This is 0 when $\sqrt{1-h^2} - 2\left(1-h^2\right) = 0 \;\Leftrightarrow\; u - 2u^2 = 0$ (where $u = \sqrt{1-h^2}$) $\Leftrightarrow\; u = 0$ or $\frac{1}{2} \;\Leftrightarrow$ $h = 1$ or $\frac{\sqrt{3}}{2}$. By the First Derivative Test, $h = \frac{\sqrt{3}}{2}$ represents a minimum for $A(h)$, since $A'(h) = 1 - \dfrac{2}{\sqrt{1-h^2}}$ goes from negative to positive at $h = \frac{\sqrt{3}}{2}$.

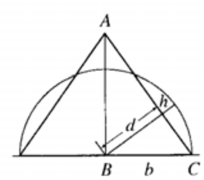
Another Method: Use FTC2 to evaluate all of the integrals before differentiating.

Note: Another strategy is to use the angle θ as the variable (see diagram above) and show that $A = \theta + \cos\theta - \frac{\pi}{4} - \frac{1}{2}\sin 2\theta$, which is minimized when $\theta = \frac{\pi}{3}$.

7. To find the height of the pyramid, we use similar triangles. The first figure shows a cross-section of the pyramid passing through the top and through two opposite corners of the square base. Now $|BD| = b$, since it is a radius of the sphere, which has diameter $2b$ since it is tangent to the opposite sides of the square base. Also, $|AD| = b$ since $\triangle ADB$ is isosceles. So the height is $|AB| = \sqrt{b^2 + b^2} = \sqrt{2}b$.



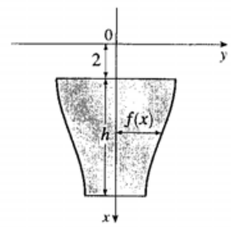




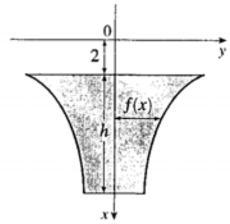
We observe that the shared volume is equal to half the volume of the sphere, minus the sum of the four equal volumes (caps of the sphere) cut off by the triangular faces of the pyramid. See Exercise 6.2.47 for a derivation of the formula for the volume of a cap of a sphere. To use the formula, we need to find the perpendicular distance h of each triangular face from the surface of the sphere. We first find the distance d from the center of the sphere to one of the triangular faces. The third figure shows a cross-section of the pyramid through the top and through the midpoints of opposite sides of the square base. From similar triangles we find that $\dfrac{d}{b} = \dfrac{|AB|}{|AC|} = \dfrac{\sqrt{2}b}{\sqrt{b^2 + \left(\sqrt{2}b\right)^2}}$

$\Rightarrow\quad d = \dfrac{\sqrt{2}b^2}{\sqrt{3b^2}} = \frac{\sqrt{6}}{3}b$. So $h = b - \frac{\sqrt{6}}{3}b = \frac{3-\sqrt{6}}{3}b$. So, using the formula from Exercise 6.2.47 with $r = b$, we find that the volume of each of the caps is $\pi\left(\frac{3-\sqrt{6}}{3}b\right)^2\left(b - \frac{3-\sqrt{6}}{3\cdot 3}b\right) = \frac{15-6\sqrt{6}}{9}\cdot\frac{6+\sqrt{6}}{9}\pi b^3 = \left(\frac{2}{3} - \frac{7}{27}\sqrt{6}\right)\pi b^3$. So, using our first observation, the shared volume is $V = \frac{1}{2}\left(\frac{4}{3}\pi b^3\right) - 4\left(\frac{2}{3} - \frac{7}{27}\sqrt{6}\right)\pi b^3 = \left(\frac{28}{27}\sqrt{6} - 2\right)\pi b^3$.

8. Orient the positive x-axis as in the figure. Suppose that the plate has height h and is symmetric about the x-axis. At depth x below the water $(2 \le x \le 2+h)$, let the width of the plate be $2f(x)$. Now each of the n horizontal strips has height h/n and the ith strip $(1 \le i \le n)$ goes from $x = 2 + \left(\frac{i-1}{n}\right)h$ to $x = 2 + \left(\frac{i}{n}\right)h$. The hydrostatic force on the ith strip is $F(i) = \int_{2+[(i-1)/n]h}^{2+(i/n)h} 62.5x\,[2f(x)]\,dx$. If we now let $x\,[2f(x)] = k$ (a constant) so that $f(x) = k/(2x)$, then



$$F(i) = \int_{2+[(i-1)/n]h}^{2+(i/n)h} 62.5k\,dx = 62.5k\,[x]_{2+[(i-1)/n]h}^{2+(i/n)h}$$
$$= 62.5k\left[\left(2+\tfrac{i}{n}h\right) - \left(2+\tfrac{i-1}{n}h\right)\right] = 62.5k\left(\tfrac{h}{n}\right)$$



So the hydrostatic force on the ith strip is independent of i, that is, the force on each strip is the same. So the plate can be shaped as shown in the figure. (In fact, the required condition is satisfied whenever the plate has width C/x at depth x, for some constant C. Many shapes are possible.)

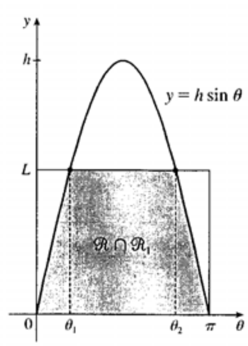
9. If $h = L$, then

$$P = \frac{\text{area under } y = L\sin\theta}{\text{area of rectangle}} = \frac{\int_0^\pi L\sin\theta\,d\theta}{\pi L} = \frac{[-\cos\theta]_0^\pi}{\pi} = \frac{-(-1)+1}{\pi} = \frac{2}{\pi}$$

If $h = L/2$, then

$$P = \frac{\text{area under } y = \frac{1}{2}L\sin\theta}{\text{area of rectangle}} = \frac{\int_0^\pi \frac{1}{2}L\sin\theta\,d\theta}{\pi L} = \frac{[-\cos\theta]_0^\pi}{2\pi} = \frac{2}{2\pi} = \frac{1}{\pi}$$

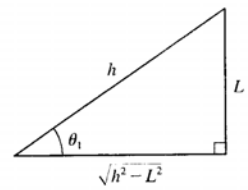
10. (a) The total set of possibilities can be identified with the rectangular region $\mathcal{R} = \{(\theta, y) \mid 0 \le y < L, 0 \le \theta < \pi\}$. Even when $h > L$, the needle intersects at least one line if and only if $y \le h\sin\theta$. Let $\mathcal{R}_1 = \{(\theta, y) \mid 0 \le y \le h\sin\theta, 0 \le \theta < \pi\}$. When $h \le L$, $\mathcal{R}_1$ is contained in $\mathcal{R}$, but that is no longer true when $h > L$. Thus, the probability that the needle intersects a line becomes



$$P = \frac{\text{area}(\mathcal{R}\cap\mathcal{R}_1)}{\text{area}(\mathcal{R})} = \frac{\text{area}(\mathcal{R}\cap\mathcal{R}_1)}{\pi L}$$

When $h > L$, the curve $y = h\sin\theta$ intersects the line $y = L$ twice — at $\left(\sin^{-1}(L/h), L\right)$ and at $\left(\pi - \sin^{-1}(L/h), L\right)$. Set $\theta_1 = \sin^{-1}(L/h)$ and $\theta_2 = \pi - \theta_1$.

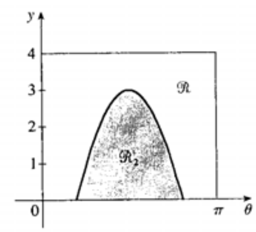
Then



$$\begin{aligned}\text{area}\,(\mathcal{R}\cap\mathcal{R}_1) &= \int_0^{\theta_1} h\sin\theta\,d\theta + \int_{\theta_1}^{\theta_2} L\,d\theta + \int_{\theta_2}^{\pi} h\sin\theta\,d\theta \\ &= 2\int_0^{\theta_1} h\sin\theta\,d\theta + L\,(\theta_2-\theta_1) \\ &= 2h\,[-\cos\theta]_0^{\theta_1} + L\,(\pi-2\theta_1) \\ &= 2h\,(1-\cos\theta_1) + L\,(\pi-2\theta_1) \\ &= 2h\left(1-\frac{\sqrt{h^2-L^2}}{h}\right) + L\left[\pi - 2\sin^{-1}\left(\frac{L}{h}\right)\right] \\ &= 2h - 2\sqrt{h^2-L^2} + \pi L - 2L\sin^{-1}\left(\frac{L}{h}\right)\end{aligned}$$

We are told that $L=4$ and $h=7$, so $\text{area}(\mathcal{R}\cap\mathcal{R}_1) = 14 - 2\sqrt{33} + 4\pi - 8\sin^{-1}\left(\frac{4}{7}\right) \approx 10.21128$ and $P = \frac{1}{4\pi}\text{area}(\mathcal{R}\cap\mathcal{R}_1) \approx 0.812588$. (By comparison, $P = \frac{2}{\pi} \approx 0.636620$ when $h = L$, as shown in the solution to Problem 9.)

(b) The needle intersects at least two lines when $y + L \le h\sin\theta$; that is, when $y \le h\sin\theta - L$. Set $\mathcal{R}_2 = \{(\theta, y) \mid 0 \le y \le h\sin\theta - L,\ 0 \le \theta < \pi\}$. Then the probability that the needle intersects at least two lines is $P_2 = \dfrac{\text{area}\,(\mathcal{R}\cap\mathcal{R}_2)}{\text{area}\,(\mathcal{R})} = \dfrac{\text{area}\,(\mathcal{R}\cap\mathcal{R}_2)}{\pi L}$. When $L=4$ and $h=7$, $\mathcal{R}_2$ is contained in $\mathcal{R}$ (see the figure).



Thus,

$$\begin{aligned}P_2 &= \tfrac{1}{4\pi}\,\text{area}\,(\mathcal{R}_2) = \tfrac{1}{4\pi}\int_{\sin^{-1}(4/7)}^{\pi-\sin^{-1}(4/7)} (7\sin\theta - 4)\,d\theta = \tfrac{1}{4\pi}\cdot 2\int_{\sin^{-1}(4/7)}^{\pi/2} (7\sin\theta-4)\,d\theta \\ &= \tfrac{1}{2\pi}\,[-7\cos\theta - 4\theta]_{\sin^{-1}(4/7)}^{\pi/2} = \tfrac{1}{2\pi}\left[0 - 2\pi + 7\tfrac{\sqrt{33}}{7} + 4\sin^{-1}(4/7)\right] \\ &= \frac{\sqrt{33} + 4\sin^{-1}(4/7) - 2\pi}{2\pi} \approx 0.301497\end{aligned}$$

(c) The needle intersects at least three lines when $y + 2L \le h\sin\theta$: that is, when $y \le h\sin\theta - 2L$. Set $\mathcal{R}_3 = \{(\theta, y) \mid 0 \le y \le h\sin\theta - 2L,\ 0 \le \theta < \pi\}$. Then the probability that the needle intersects at least three lines is $P_3 = \dfrac{\text{area}\,(\mathcal{R}\cap\mathcal{R}_3)}{\text{area}\,(\mathcal{R})} = \dfrac{\text{area}\,(\mathcal{R}\cap\mathcal{R}_3)}{\pi L}$. (At this point, the generalization to P_n, n any positive integer, should be clear.) Under the given assumption,

$$\begin{aligned}P_3 &= \tfrac{1}{\pi L}\,\text{area}\,(\mathcal{R}_3) = \tfrac{1}{\pi L}\int_{\sin^{-1}(2L/h)}^{\pi-\sin^{-1}(2L/h)} (h\sin\theta - 2L)\,d\theta = \tfrac{2}{\pi L}\int_{\sin^{-1}(2L/h)}^{\pi/2} (h\sin\theta - 2L)\,d\theta \\ &= \tfrac{2}{\pi L}\,[-h\cos\theta - 2L\theta]_{\sin^{-1}(2L/h)}^{\pi/2} = \tfrac{2}{\pi L}\left[-\pi L + \sqrt{h^2-4L^2} + 2L\sin^{-1}(2L/h)\right]\end{aligned}$$

Note that the probability that a needle touches exactly one line is $P_1 - P_2$, the probability that it touches exactly two lines is $P_2 - P_3$, and so on.

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10 Differential Equations

10.1 Modeling with Differential Equations

1. $y = 2 + e^{-x^3} \Rightarrow y' = -3x^2e^{-x^3}$.

$$\text{LHS} = y' + 3x^2y = -3x^2e^{-x^3} + 3x^2\left(2 + e^{-x^3}\right) = -3x^2e^{-x^3} + 6x^2 + 3x^2e^{-x^3} = 6x^2 = \text{RHS}$$

2. $y = \dfrac{2 + \ln x}{x} \Rightarrow y' = \dfrac{x(1/x) - (2 + \ln x)(1)}{(x)^2} = \dfrac{-1 - \ln x}{x^2}$ and $y(1) = \dfrac{2 + \ln 1}{1} = 2$.

$$\text{LHS} = x^2y' + xy = x^2\left(\frac{-1 - \ln x}{x^2}\right) + x\left(\frac{2 + \ln x}{x}\right) = (-1 - \ln x) + (2 + \ln x) = 1 = \text{RHS}$$

3. (a) $y = \sin kt \Rightarrow y' = k\cos kt \Rightarrow y'' = -k^2\sin kt$. $y'' + 9y = 0 \Rightarrow -k^2\sin kt + 9\sin kt = 0 \Rightarrow$ $\left(9 - k^2\right)\sin kt = 0$ (for all t) $\Rightarrow k = \pm 3$

(b) $y = A\sin kt + B\cos kt \Rightarrow y' = Ak\cos kt - Bk\sin kt \Rightarrow y'' = -Ak^2\sin kt - Bk^2\cos kt \Rightarrow$ $y'' + 9y = -Ak^2\sin kt - Bk^2\cos kt + 9(A\sin kt + B\cos kt) = \left(9 - k^2\right)A\sin kt + \left(9 - k^2\right)B\cos kt = 0$. The last equation is true for all values of A and B if $k = \pm 3$.

4. $y = e^{rt} \Rightarrow y' = re^{rt} \Rightarrow y'' = r^2e^{rt}$. $y'' + y' - 6y = 0 \Rightarrow r^2e^{rt} + re^{rt} - 6e^{rt} = 0 \Rightarrow$ $\left(r^2 + r - 6\right)e^{rt} = 0 \Rightarrow (r + 3)(r - 2) = 0 \Rightarrow r = -3$ or 2

5. (a) $y = e^t \Rightarrow y' = e^t \Rightarrow y'' = e^t$. $\text{LHS} = y'' + 2y' + y = e^t + 2e^t + e^t = 4e^t \neq 0$, so $y = e^t$ is not a solution of the differential equation.

(b) $y = e^{-t} \Rightarrow y' = -e^{-t} \Rightarrow y'' = e^{-t}$. $\text{LHS} = y'' + 2y' + y = e^{-t} - 2e^{-t} + e^{-t} = 0 = \text{RHS}$, so $y = e^{-t}$ is a solution.

(c) $y = te^{-t} \Rightarrow y' = e^{-t}(1 - t) \Rightarrow y'' = e^{-t}(t - 2)$.

$$\begin{aligned}\text{LHS} = y'' + 2y' + y &= e^{-t}(t - 2) + 2e^{-t}(1 - t) + te^{-t} = e^{-t}[(t - 2) + 2(1 - t) + t] \\ &= e^{-t}(0) = 0 = \text{RHS}\end{aligned}$$

so $y = te^{-t}$ is a solution.

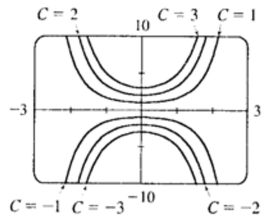
(d) $y = t^2e^{-t} \Rightarrow y' = te^{-t}(2 - t) \Rightarrow y'' = e^{-t}\left(t^2 - 4t + 2\right)$.

$$\begin{aligned}\text{LHS} = y'' + 2y' + y &= e^{-t}\left(t^2 - 4t + 2\right) + 2te^{-t}(2 - t) + t^2e^{-t} \\ &= e^{-t}\left[\left(t^2 - 4t + 2\right) + 2t(2 - t) + t^2\right] = e^{-t}(2) \neq 0\end{aligned}$$

so $y = t^2e^{-t}$ is not a solution.

6. (a) $y = Ce^{x^2/2} \Rightarrow y' = Ce^{x^2/2}(2x/2) = xCe^{x^2/2} = xy$.

(b)



(c) $y(0) = 5 \Rightarrow Ce^0 = 5 \Rightarrow C = 5$, so the solution is $y = 5e^{x^2/2}$.

(d) $y(1) = 2 \Rightarrow Ce^{1/2} = 2 \Rightarrow C = 2e^{-1/2}$, so the solution is

$y = 2e^{-1/2}e^{x^2/2} = 2e^{\left(x^2-1\right)/2}$.

7. (a) Since the derivative y' is always negative (or 0), the function y must be decreasing (or have a horizontal tangent) on any interval on which it is defined.

(b) $y = \dfrac{1}{x+C} \Rightarrow y' = -\dfrac{1}{(x+C)^2}$. LHS $= y' = -\dfrac{1}{(x+C)^2} = -\left(\dfrac{1}{x+C}\right)^2 = -y^2 =$ RHS

(c) $y = 0$ is a solution of $y' = -y^2$.

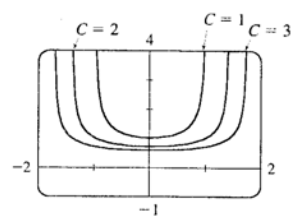
(d) $y(0) = \dfrac{1}{0+C} = \dfrac{1}{C}$ and $y(0) = 0.5 \Rightarrow C = 2$, so $y = \dfrac{1}{x+2}$.

8. (a) If x is close to 0, then xy^3 is close to 0, and hence, y' is close to 0. Thus, the graph of y must have a tangent line that is nearly horizontal. If x is large, then xy^3 is large, and the graph of y must have a tangent line that is nearly vertical. (In both cases, we assume reasonable values for y.)

(b) $y = \left(C - x^2\right)^{-1/2} \Rightarrow y' = x\left(C - x^2\right)^{-3/2}$.

$$\text{RHS} = xy^3 = x\left[\left(C - x^2\right)^{-1/2}\right]^3$$
$$= x\left(C - x^2\right)^{-3/2} = y' = \text{LHS}$$

(c)



When x is close to 0, y' is also close to 0. As x gets larger, so does $|y'|$.

(d) $y(0) = (C - 0)^{-1/2} = 1/\sqrt{C}$ and $y(0) = 2 \Rightarrow \sqrt{C} = \frac{1}{2}$

$\Rightarrow C = \frac{1}{4}$, so $y = \left(\frac{1}{4} - x^2\right)^{-1/2}$.

9. (a) $\dfrac{dP}{dt} = 1.2P\left(1 - \dfrac{P}{4200}\right)$. $\dfrac{dP}{dt} > 0 \Rightarrow 1 - \dfrac{P}{4200} > 0 \Rightarrow P < 4200 \Rightarrow$ the population is increasing for $P < 4200$ (assuming that $P \geq 0$).

(b) $\dfrac{dP}{dt} < 0 \Rightarrow P > 4200$

(c) $\dfrac{dP}{dt} = 0 \Rightarrow P = 4200$ or $P = 0$

10. (a) $y = k \Rightarrow y' = 0$, so $\dfrac{dy}{dt} = y^4 - 6y^3 + 5y^2 \Rightarrow 0 = k^4 - 6k^3 + 5k^2 \Rightarrow k^2\left(k^2 - 6k + 5\right) = 0 \Rightarrow$ $k^2(k-1)(k-5) = 0 \Rightarrow k = 0, 1$, or 5

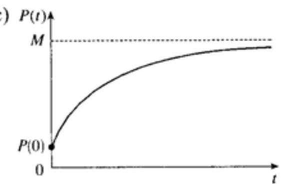
(b) y is increasing $\Leftrightarrow \dfrac{dy}{dt} > 0 \Leftrightarrow y^2(y-1)(y-5) > 0 \Leftrightarrow y \in (-\infty, 0) \cup (0, 1) \cup (5, \infty)$

(c) y is decreasing $\Leftrightarrow \dfrac{dy}{dt} < 0 \Leftrightarrow y \in (1, 5)$

11. (a) P increases most rapidly at the beginning, since there are usually many simple, easily-learned sub-skills associated with learning a skill. As t increases, we would expect dP/dt to remain positive, but decrease. This is because as time progresses, the only points left to learn are the more difficult ones.

(b) dP/dt is always positive, so the level of performance is increasing. As P gets close to M, dP/dt gets close to 0, that is, the performance levels off, as explained in part (a).

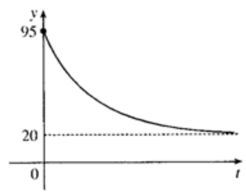
(c)



12. (a) The coffee cools most quickly as soon as it is removed from the heat source. The rate of cooling decreases toward 0 since the coffee approaches room temperature.

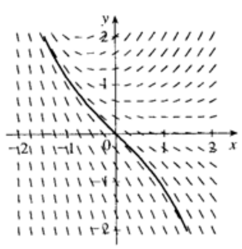
(b) $\dfrac{dy}{dt} = k\,(y - R)$, where k is a proportionality constant, y is the temperature of the coffee, and R is the room temperature. The initial condition is $y\,(0) = 95°\text{C}$. The answer and the model support each other because as y approaches R, dy/dt approaches 0, so the model seems appropriate.

(c)

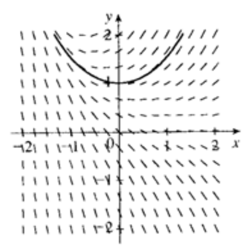


10.2 Direction Fields and Euler's Method

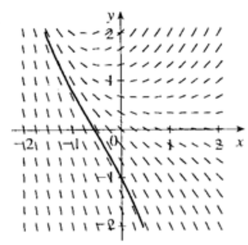
1. (a)



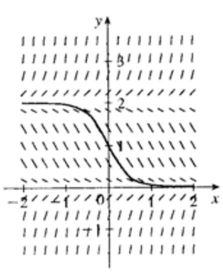
(b)



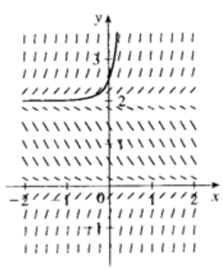
(c)



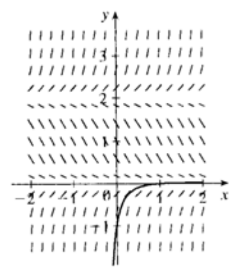
2. (a) (i)



(ii)



(iii)



(b) For $c \le 2$, $\lim\limits_{t\to\infty} y\,(t)$ is finite. In fact, if $c = 2$ then $\lim\limits_{t\to\infty} y\,(t) = 2$ and if $c < 2$ then $\lim\limits_{t\to\infty} y\,(t) = 0$. The equilibrium solutions are $y = 0$ and $y = 2$.

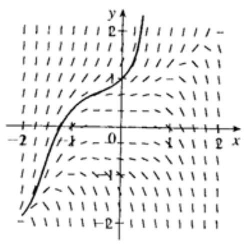
3. $y' = y - 1$. The slopes at each point are independent of x, so the slopes are the same along each line parallel to the x-axis. Thus, IV is the direction field for this equation. Note that for $y = 1$, $y' = 0$.

4. $y' = y - x = 0$ on the line $y = x$, when $x = 0$ the slope is y, and when $y = 0$ the slope is $-x$. Direction field II satisfies these conditions. [Looking at the slope at the point $(0, 2)$, II looks more like it has a slope of 2 than does direction field I.]

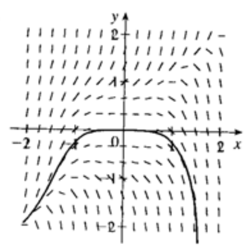
5. $y' = y^2 - x^2 = 0 \quad \Rightarrow \quad y = \pm x$. There are horizontal tangents on these lines only in graph III, so this equation corresponds to direction field III.

6. $y' = y^3 - x^3 = 0$ on the line $y = x$, when $x = 0$ the slope is y^3, and when $y = 0$ the slope is $-x^3$. The graph is similar to the graph for Exercise 4, but the segments must get steeper very rapidly as they move away from the origin, because x and y are raised to the third power. This is the case in direction field I.

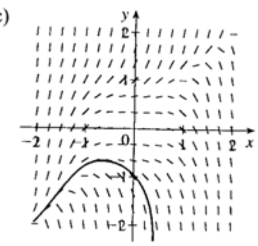
7. (a)



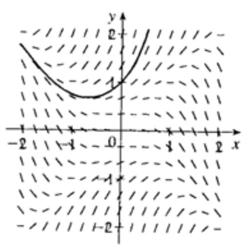
(b)



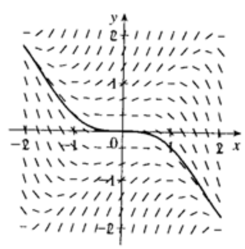
(c)



8. (a)



(b)



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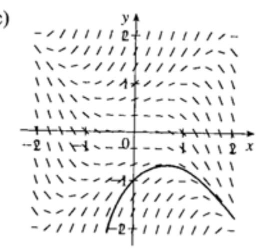
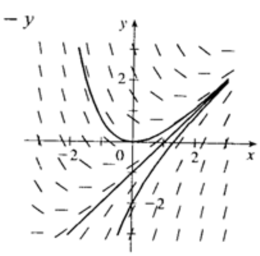
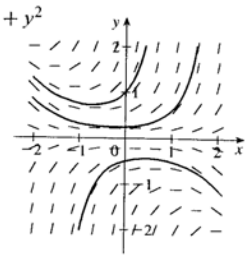


9. $y' = x - y$

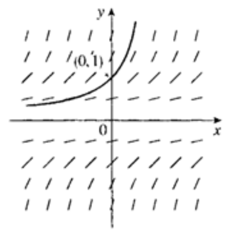


10. $y' = xy + y^2$



11.

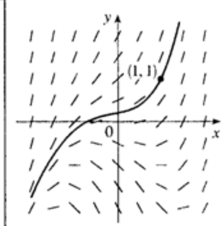
x	y	$y' = y^2$
0	0	0
0	1	1
0	-1	1
1	0	0
-1	0	0
1	-1	1
1	1	1
1	2	4
1	-2	4
-1	2	4
-1	-2	4



The solution curve through (0, 1)

12.

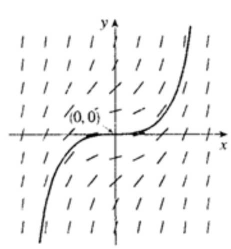
x	y	$y' = x^2 + y$
0	0	0
0	1	1
0	-1	-1
1	0	1
-1	0	1
1	1	2
-1	1	2
1	-1	0
-1	-1	0
2	0	4
2	1	5
2	-1	3



The solution curve through (1, 1)

13.

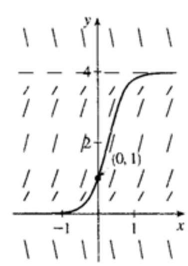
x	y	$y' = x^2 + y^2$
0	0	0
0	1	1
1	0	1
1	1	2
-1	1	2
0	2	4
2	0	4
2	2	8
2	1	5
-2	-1	5
1	2	5



The solution curve through (0, 0)

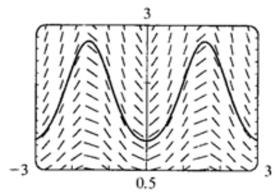
14.

x	y	$y' = y\,(4 - y)$
0	0	0
0	1	3
0	-1	-5
0	2	4
0	-2	-12
0	0.5	1.75
0	-0.5	-2.25
1	0	0
1	1	3
1	2	4
1	-1	-5



Note: The solution curve is asymptotic to $y = 0$ and $y = 4$.

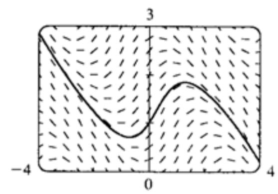
15. In Maple, we can use either `directionfield` (in Maple's share library) or `plots[fieldplot]` to plot the direction field. To plot the solution, we can either use the initial-value option in `directionfield`, or actually solve the equation. In Mathematica, we use `PlotVectorField` for the direction field, and the `Plot[Evaluate[...]]` construction to plot the solution, which is $y = e^{(1-\cos 2x)/2}$.



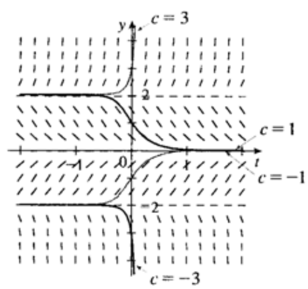
In Derive, use `Direction_Field` (in utility file `ODE_APPR`) to plot the direction field. Then use `DSOLVE1(-y*SIN(2*x),1,x,y,0,1)` (in utility file `ODE1`) to solve the equation. Simplify each result.

16. See Exercise 15 for specific CAS directions. The exact solution is

$$y = -x - 2\arctan \frac{2 + x - \dfrac{2}{1+\tan(1/2)}}{x - \dfrac{2}{1+\tan(1/2)}}$$

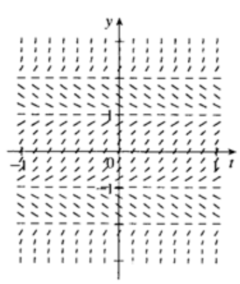


17.



$L = \lim_{t\to\infty} y(t)$ exists for $-2 \le c \le 2$; $L = \pm 2$ for $c = \pm 2$ and $L = 0$ for $-2 < c < 2$. For other values of c, L does not exist.

18.



Note that when $f(y) = 0$, we have $y' = f(y) = 0$; so we get horizontal segments at $y = \pm 1, \pm 2$. We get segments with negative slopes only for $1 < |y| < 2$. All other segments have positive slope. For the limiting behavior of solutions:

- If $y(0) > 2$, then $\lim_{t\to\infty} y = \infty$ and $\lim_{t\to-\infty} y = 2$.
- If $1 < y(0) < 2$, then $\lim_{t\to\infty} y = 1$ and $\lim_{t\to-\infty} y = 2$.
- If $-1 < y(0) < 1$, then $\lim_{t\to\infty} y = 1$ and $\lim_{t\to-\infty} y = -1$.
- If $-2 < y(0) < -1$, then $\lim_{t\to\infty} y = -2$ and $\lim_{t\to-\infty} y = -1$.
- If $y < -2$, then $\lim_{t\to\infty} y = -2$ and $\lim_{t\to-\infty} y = -\infty$.

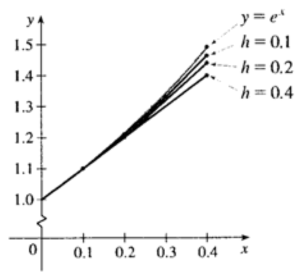
19. (a) $y' = F(x, y) = y$ and $y(0) = 1 \Rightarrow x_0 = 0,\ y_0 = 1$.

(i) $h = 0.4$ and $y_1 = y_0 + hF(x_0, y_0) \Rightarrow y_1 = 1 + 0.4 \cdot 1 = 1.4$. $x_1 = x_0 + h = 0 + 0.4 = 0.4$, so $y_1 = y(0.4) = 1.4$.

(ii) $h = 0.2 \Rightarrow x_1 = 0.2$ and $x_2 = 0.4$, so we need to find y_2.
$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.2y_0 = 1 + 0.2 \cdot 1 = 1.2$,
$y_2 = y_1 + hF(x_1, y_1) = 1.2 + 0.2y_1 = 1.2 + 0.2 \cdot 1.2 = 1.44$.

(iii) $h = 0.1 \Rightarrow x_4 = 0.4$, so we need to find y_4. $y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1y_0 = 1 + 0.1 \cdot 1 = 1.1$,
$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1y_1 = 1.1 + 0.1 \cdot 1.1 = 1.21$,
$y_3 = y_2 + hF(x_2, y_2) = 1.21 + 0.1y_2 = 1.21 + 0.1 \cdot 1.21 = 1.331$,
$y_4 = y_3 + hF(x_3, y_3) = 1.331 + 0.1y_3 = 1.331 + 0.1 \cdot 1.331 = 1.4641$.

(b)



We see that the estimates are underestimates since they are all below the graph of $y = e^x$.

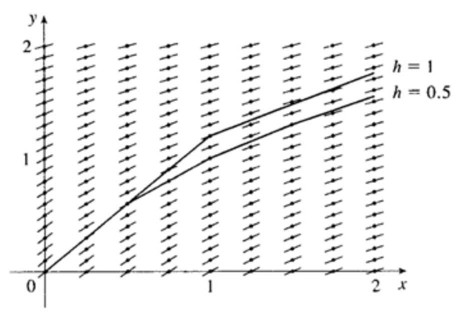
(c) (i) For $h = 0.4$:
(exact value) − (approximate value)
$= e^{0.4} - 1.4 \approx 0.0918$

(ii) For $h = 0.2$:
(exact value) − (approximate value)
$= e^{0.4} - 1.44 \approx 0.0518$

(iii) For $h = 0.1$:
(exact value) − (approximate value)
$= e^{0.4} - 1.4641 \approx 0.0277$

Each time the step size is halved, the error estimate also appears to be halved (approximately).

20.



As x increases, the slopes decrease and all of the estimates are above the true values. Thus, all of the estimates are overestimates.

21. $h = 0.5$, $x_0 = 1$, $y_0 = 2$, and $F(x, y) = 1 + 3x - 2y$. So
$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}) = y_{n-1} + 0.5(1 + 3x_{n-1} - 2y_{n-1}) = 0.5 + 1.5x_{n-1}$. Thus, $y_1 = 0.5 + 1.5 \cdot 1 = 2$,
$y_2 = 0.5 + 1.5 \cdot 1.5 = 2.75$, $y_3 = 0.5 + 1.5 \cdot 2 = 3.5$, $y_4 = 0.5 + 1.5 \cdot 2.5 = 4.25$.

22. $h = 0.2$, $x_0 = 0$, $y_0 = 0$, and $F(x, y) = x + y^2$. We need to find y_5, because $x_5 = 1$. So
$y_n = y_{n-1} + 0.2\left(x_{n-1} + y_{n-1}^2\right)$. $y_1 = 0 + 0.2(0 + 0) = 0$, $y_2 = 0 + 0.2\left(0.2 + 0^2\right) = 0.04$,
$y_3 = 0.04 + 0.2\left(0.4 + 0.04^2\right) = 0.12032$, $y_4 = 0.12032 + 0.2\left(0.6 + 0.12032^2\right) \approx 0.24322$,
$y_5 = 0.24322 + 0.2\left(0.8 + 0.24322^2\right) \approx 0.4150 \approx y(1)$.

23. $h = 0.1$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = x^2 + y^2$. We need to find y_5, because $x_5 = 0.5$. So $y_n = y_{n-1} + 0.1\left(x_{n-1}^2 + y_{n-1}^2\right)$. $y_1 = 1 + 0.1\left(0^2 + 1^2\right) = 1.1$, $y_2 = 1.1 + 0.1\left(0.1^2 + 1.1^2\right) = 1.222$, $y_3 = 1.222 + 0.1\left(0.2^2 + 1.222^2\right) \approx 1.37533$, $y_4 = 1.37533 + 0.1\left(0.3^2 + 1.37533^2\right) \approx 1.57348$, $y_5 = 1.57348 + 0.1\left(0.4^2 + 1.57348^2\right) \approx 1.8371 \approx y(0.5)$.

24. (a) $h = 0.2$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = 2xy^2$. We need to find y_2, because $x_2 = 0.4$. $y_1 = 1 + 0.2\left(2 \cdot 0 \cdot 1^2\right) = 1$, $y_2 = 1 + 0.2\left(2 \cdot 0.2 \cdot 1^2\right) = 1.08 \approx y(0.4)$.

(b) $h = 0.1$ now, so we need to find y_4. $y_1 = 1 + 0.1\left(2 \cdot 0 \cdot 1^2\right) = 1$, $y_2 = 1 + 0.1\left(2 \cdot 0.1 \cdot 1^2\right) = 1.02$, $y_3 = 1.02 + 0.1\left(2 \cdot 0.2 \cdot 1.02^2\right) \approx 1.06162$, $y_4 = 1.06162 + 0.1\left(2 \cdot 0.3 \cdot 1.06162^2\right) \approx 1.1292 \approx y(0.4)$.

25. (a) $dy/dx + 3x^2y = 6x^2 \Rightarrow y' = 6x^2 - 3x^2y$. Store this expression in Y_1 and use the following simple program to evaluate $y(1)$ for each part, using $\mathrm{H} = h = 1$ and $\mathrm{N} = 1$ for part (i), $\mathrm{H} = 0.1$ and $\mathrm{N} = 10$ for part (ii), and so forth.

$h \to$ H: $0 \to$ X: $3 \to$ Y:
For(I, 1, N): Y + HY$_1$ $\to$ Y: X + H $\to$ X:
End(loop):
Display Y.

(i) $\mathrm{H} = 1, \mathrm{N} = 1 \Rightarrow y(1) = 3$

(ii) $\mathrm{H} = 0.1, \mathrm{N} = 10 \Rightarrow y(1) \approx 2.3928$

(iii) $\mathrm{H} = 0.01, \mathrm{N} = 100 \Rightarrow y(1) \approx 2.3701$

(iv) $\mathrm{H} = 0.001, \mathrm{N} = 1000 \Rightarrow y(1) \approx 2.3681$

(b) $y = 2 + e^{-x^3} \Rightarrow y' = -3x^2e^{-x^3}$

$$\mathrm{LHS} = y' + 3x^2y = -3x^2e^{-x^3} + 3x^2\left(2 + e^{-x^3}\right) = -3x^2e^{-x^3} + 6x^2 + 3x^2e^{-x^3} = 6x^2 = \mathrm{RHS}$$

$y(0) = 2 + e^{-0} = 2 + 1 = 3$

(c) (i) For $h = 1$: (exact value) $-$ (approximate value) $= 2 + e^{-1} - 3 \approx -0.6321$

(ii) For $h = 0.1$: (exact value) $-$ (approximate value) $= 2 + e^{-1} - 2.3928 \approx -0.0249$

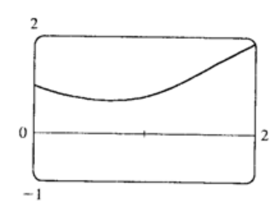
(iii) For $h = 0.01$: (exact value) $-$ (approximate value) $= 2 + e^{-1} - 2.3701 \approx -0.0022$

(iv) For $h = 0.001$: (exact value) $-$ (approximate value) $= 2 + e^{-1} - 2.3681 \approx -0.0002$

In (ii)–(iv), it seems that when the step size is divided by 10, the error estimate is also divided by 10 (approximately).

26. (a) We use the program from the solution to Exercise 25 with $\mathrm{Y}_1 = x^2 - y^2$, $\mathrm{H} = 0.01$, and $\mathrm{N} = 200$. With $(x_0, y_0) = (0, 1)$, we get $y(2) \approx 1.9000$.

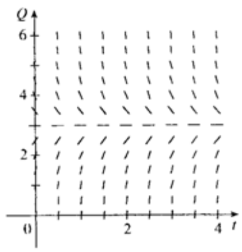
(b)



Notice from the graph that $y(2) \approx 1.9$, which serves as a check on our calculation in part (a).

27. (a) $R\dfrac{dQ}{dt} + \dfrac{1}{C}Q = E(t)$ becomes

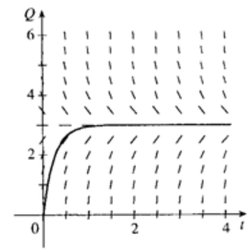
$5Q' + \frac{1}{0.05}Q = 60$ or $Q' + 4Q = 12$.



(b) From the graph, it appears that the limiting value of the charge Q is about 3.

(c) If $Q' = 0$, then $4Q = 12 \Rightarrow Q = 3$ is an equilibrium solution.

(d)



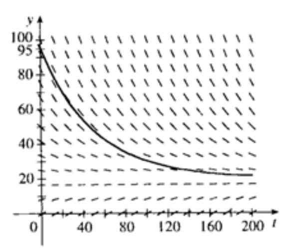
(e) $Q' + 4Q = 12 \Rightarrow Q' = 12 - 4Q$. $Q(0) = 0$, so $t_0 = 0$ and $Q_0 = 0$.

$$Q_1 = Q_0 + hF(t_0, Q_0) = 0 + 0.1(12 - 4 \cdot 0) = 1.2$$
$$Q_2 = Q_1 + hF(t_1, Q_1) = 1.2 + 0.1(12 - 4 \cdot 1.2) = 1.92$$
$$Q_3 = Q_2 + hF(t_2, Q_2) = 1.92 + 0.1(12 - 4 \cdot 1.92) = 2.352$$
$$Q_4 = Q_3 + hF(t_3, Q_3) = 2.352 + 0.1(12 - 4 \cdot 2.352) = 2.6112$$
$$Q_5 = Q_4 + hF(t_4, Q_4) = 2.6112 + 0.1(12 - 4 \cdot 2.6112) = 2.76672$$

Thus, $Q_5 = Q(0.5) \approx 2.77$ C.

28. (a) From Exercise 10.1.12, we have $dy/dt = k(y - R)$. We are given that $R = 20°\text{C}$ and $dy/dt = -1°\text{C/min}$ when $y = 70°\text{C}$. Thus, $-1 = k(70 - 20) \Rightarrow k = -\frac{1}{50}$ and the differential equation becomes $dy/dt = -\frac{1}{50}(y - 20)$.

(b)



The limiting value of the temperature is 20°C, that is, the temperature of the room.

(c) From part (a), $dy/dt = -\frac{1}{50}(y - 20)$. With $t_0 = 0$, $y_0 = 95$, and $h = 2$ min, we get

$$y_1 = y_0 + hF(t_0, y_0) = 95 + 2\left[-\tfrac{1}{50}(95 - 20)\right] = 92$$
$$y_2 = y_1 + hF(t_1, y_1) = 92 + 2\left[-\tfrac{1}{50}(92 - 20)\right] = 89.12$$
$$y_3 = y_2 + hF(t_2, y_2) = 89.12 + 2\left[-\tfrac{1}{50}(89.12 - 20)\right] = 86.3552$$
$$y_4 = y_3 + hF(t_3, y_3) = 86.3552 + 2\left[-\tfrac{1}{50}(86.3552 - 20)\right] = 83.700992$$
$$y_5 = y_4 + hF(t_4, y_4) = 83.700992 + 2\left[-\tfrac{1}{50}(83.700992 - 20)\right] = 81.15295232$$

Thus, $y(10) \approx 81.15$ °C.

10.3 Separable Equations

1. $\dfrac{dy}{dx} = y^2 \;\Rightarrow\; \dfrac{dy}{y^2} = dx \; (y \neq 0) \;\Rightarrow\; \displaystyle\int \frac{dy}{y^2} = \int dx \;\Rightarrow\; -\frac{1}{y} = x + C \;\Rightarrow\; -y = \frac{1}{x+C} \;\Rightarrow$
$y = \dfrac{-1}{x+C}$, and $y = 0$ is also a solution.

2. $\dfrac{dy}{dx} = \dfrac{e^{2x}}{4y^3} \;\Rightarrow\; 4y^3\,dy = e^{2x}\,dx \;\Rightarrow\; \int 4y^3\,dy = \int e^{2x}\,dx \;\Rightarrow\; y^4 = \frac{1}{2}e^{2x} + C \;\Rightarrow\; y = \pm\sqrt[4]{\frac{1}{2}e^{2x} + C}$

3. $yy' = x \;\Rightarrow\; \int y\,dy = \int x\,dx \;\Rightarrow\; \frac{1}{2}y^2 = \frac{1}{2}x^2 + C_1 \;\Rightarrow\; y^2 = x^2 + 2C_1 \;\Rightarrow$
$x^2 - y^2 = C$ (where $C = -2C_1$). This represents a family of hyperbolas.

4. $y' = xy \;\Rightarrow\; \displaystyle\int \frac{dy}{y} = \int x\,dx \quad (y \neq 0) \;\Rightarrow\; \ln|y| = \frac{x^2}{2} + C \;\Rightarrow\; |y| = e^C e^{x^2/2} \;\Rightarrow\; y = Ke^{x^2/2}$, where $K = \pm e^C$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y = 0$ by allowing K to be zero.)

5. $\dfrac{dy}{dt} = \dfrac{te^t}{y\sqrt{1+y^2}} \;\Rightarrow\; y\sqrt{1+y^2}\,dy = te^t\,dt \;\Rightarrow\; \int y\sqrt{1+y^2}\,dy = \int te^t\,dt \;\Rightarrow$
$\frac{1}{3}\left(1+y^2\right)^{3/2} = te^t - e^t + C$ (where the first integral is evaluated by substitution and the second by parts) $\Rightarrow$
$1 + y^2 = \left[3\left(te^t - e^t + C\right)\right]^{2/3} \;\Rightarrow\; y = \pm\sqrt{\left[3\left(te^t - e^t + C\right)\right]^{2/3} - 1}$

6. $y' = \dfrac{xy}{2\ln y} \;\Rightarrow\; \dfrac{2\ln y}{y}\,dy = x\,dx \;\Rightarrow\; \displaystyle\int \frac{2\ln y}{y}\,dy = \int x\,dx \;\Rightarrow\; (\ln y)^2 = \frac{x^2}{2} + C \;\Rightarrow$
$\ln y = \pm\sqrt{x^2/2 + C} \;\Rightarrow\; y = e^{\pm\sqrt{x^2/2+C}}$

7. $\dfrac{du}{dt} = 2 + 2u + t + tu \;\Rightarrow\; \dfrac{du}{dt} = (1+u)(2+t) \;\Rightarrow\; \displaystyle\int \frac{du}{1+u} = \int (2+t)\,dt \;\; (u \neq -1) \;\Rightarrow$
$\ln|1+u| = \frac{1}{2}t^2 + 2t + C \;\Rightarrow\; |1+u| = e^{t^2/2+2t+C} = Ke^{t^2/2+2t}$, where $K = e^C \;\Rightarrow\; 1 + u = \pm Ke^{t^2/2+2t}$
$\Rightarrow\; u = -1 \pm Ke^{t^2/2+2t}$ where $K > 0$. $u = -1$ is also a solution, so $u = -1 + ke^{t^2/2+2t}$, where k is an arbitrary constant.

8. $\dfrac{dz}{dt} + e^{t+z} = 0 \;\Rightarrow\; \dfrac{dz}{dt} = -e^t e^z \;\Rightarrow\; \int e^{-z}\,dz = -\int e^t\,dt \;\Rightarrow\; -e^{-z} = -e^t + C \;\Rightarrow\; e^{-z} = e^t - C$
$\Rightarrow\; \dfrac{1}{e^z} = e^t - C \;\Rightarrow\; e^z = \dfrac{1}{e^t - C} \;\Rightarrow\; z = -\ln\left(e^t - C\right)$

9. $\dfrac{dy}{dx} = y^2 + 1,\; y(1) = 0.\; \displaystyle\int \frac{dy}{y^2+1} = \int dx \;\Leftrightarrow\; \tan^{-1} y = x + C$. $y = 0$ when $x = 1$, so $1 + C = \tan^{-1} 0 = 0$
$\Rightarrow\; C = -1$. Thus, $\tan^{-1} y = x - 1$ and $y = \tan(x-1)$.

10. $\dfrac{dy}{dx} = \dfrac{1+x}{xy},\; x > 0,\; y(1) = -4.\; \displaystyle\int y\,dy = \int \frac{1+x}{x}\,dx = \int \left(\frac{1}{x} + 1\right)dx \;\Rightarrow$
$\frac{1}{2}y^2 = \ln|x| + x + C = \ln x + x + C$ (since $x > 0$). $y(1) = -4 \;\Rightarrow\; \frac{(-4)^2}{2} = \ln 1 + 1 + C \;\Rightarrow$
$8 = 0 + 1 + C \;\Rightarrow\; C = 7$, so $y^2 = 2\ln x + 2x + 14$.

11. $xe^{-t}\dfrac{dx}{dt} = t,\; x(0) = 1.\; \int x\,dx = \int te^t\,dt \;\Rightarrow\; \frac{1}{2}x^2 = (t-1)e^t + C$. $x(0) = 1$, so $\frac{1}{2} = (0-1)e^0 + C$ and
$C = \frac{3}{2}$. Thus, $x^2 = 2(t-1)e^t + 3 \;\Rightarrow\; x = \sqrt{2(t-1)e^t + 3}$.

12. $x + 2y\sqrt{x^2+1}\,\dfrac{dy}{dx} = 0 \quad\Rightarrow\quad x\,dx + 2y\sqrt{x^2+1}\,dy = 0,\ y(0) = 1.\ \displaystyle\int 2y\,dy = -\int \frac{x\,dx}{\sqrt{x^2+1}} \quad\Rightarrow$
$y^2 = -\sqrt{x^2+1} + C.\ y(0) = 1 \quad\Rightarrow\quad 1 = -1 + C \quad\Rightarrow\quad C = 2$, so $y^2 = 2 - \sqrt{x^2+1}$.

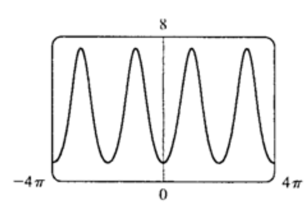
13. $\dfrac{du}{dt} = \dfrac{2t + \sec^2 t}{2u},\ u(0) = -5.\ \int 2u\,du = \int \left(2t + \sec^2 t\right) dt \quad\Rightarrow\quad u^2 = t^2 + \tan t + C$, where
$[u(0)]^2 = 0^2 + \tan 0 + C \quad\Rightarrow\quad C = 25$. Therefore, $u^2 = t^2 + \tan t + 25$, so $u = \pm\sqrt{t^2 + \tan t + 25}$. Since
$u(0) = -5$, we must have $u = -\sqrt{t^2 + \tan t + 25}$.

14. $\dfrac{dy}{dt} = te^y,\ y(1) = 0.\ \int e^{-y}\,dy = \int t\,dt \quad\Rightarrow\quad -e^{-y} = \frac{1}{2}t^2 + C$. Since $y(1) = 0$, $-e^0 = \frac{1}{2}\cdot 1^2 + C$. Therefore,
$C = -1 - \frac{1}{2} = -\frac{3}{2}$ and $-e^{-y} = \frac{1}{2}t^2 - \frac{3}{2}$. $e^{-y} = \frac{3}{2} - \frac{1}{2}t^2 = \dfrac{3 - t^2}{2} \quad\Rightarrow\quad e^y = \dfrac{2}{3 - t^2} \quad\Rightarrow$
$y = \ln 2 - \ln\left(3 - t^2\right)$ for $|t| < \sqrt{3}$.

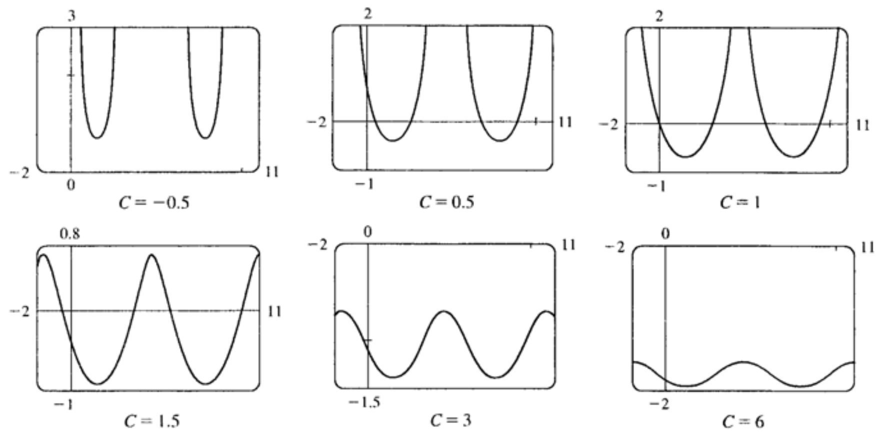
15. $\dfrac{dy}{dx} = 4x^3 y,\ y(0) = 7.\ \dfrac{dy}{y} = 4x^3\,dx$ (if $y \neq 0$) $\quad\Rightarrow\quad \displaystyle\int \frac{dy}{y} = \int 4x^3\,dx \quad\Rightarrow\quad \ln|y| = x^4 + C \quad\Rightarrow$
$e^{\ln|y|} = e^{x^4 + C} \quad\Rightarrow\quad |y| = e^{x^4}e^C \quad\Rightarrow\quad y = Ae^{x^4};\ y(0) = 7 \quad\Rightarrow\quad A = 7 \quad\Rightarrow\quad y = 7e^{x^4}$.

16. $\dfrac{dy}{dx} = \dfrac{y^2}{x^3},\ y(1) = 1.\ \displaystyle\int \frac{dy}{y^2} = \int \frac{dx}{x^3} \quad\Rightarrow\quad -\frac{1}{y} = -\frac{1}{2x^2} + C.\ y(1) = 1 \quad\Rightarrow\quad -1 = -\frac{1}{2} + C \quad\Rightarrow\quad C = -\frac{1}{2}$.
So $\dfrac{1}{y} = \dfrac{1}{2x^2} + \dfrac{1}{2} = \dfrac{2 + 2x^2}{2\cdot 2x^2} \quad\Rightarrow\quad y = \dfrac{2x^2}{x^2+1}$.

17. $y' = y\sin x,\ y(0) = 1.\ \displaystyle\int \frac{dy}{y} = \int \sin x\,dx \quad\Leftrightarrow\quad \ln|y| = -\cos x + C$
$\Rightarrow\quad |y| = e^{-\cos x + C} \quad\Rightarrow\quad y(x) = Ae^{-\cos x}.\ y(0) = Ae^{-1} = 1 \quad\Leftrightarrow$
$A = e^1$, so $y = e\cdot e^{-\cos x} = e^{1 - \cos x}$.

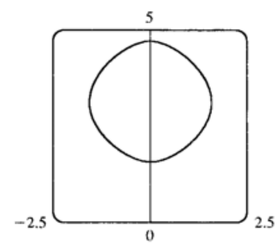


18. $e^{-y}y' + \cos x = 0 \quad \Leftrightarrow \quad \int e^{-y}\,dy = -\int \cos x\,dx \quad \Leftrightarrow \quad -e^{-y} = -\sin x + C_1 \quad \Leftrightarrow \quad y = -\ln(\sin x + C)$. The solution is periodic, with period 2π. Note that for $C > 1$, the domain of the solution is $\mathbb{R}$, but for $-1 < C \leq 1$ it is only defined on the intervals where $\sin x + C > 0$, and it is meaningless for $C \leq -1$, since then $\sin x + C \leq 0$, and the logarithm is undefined.

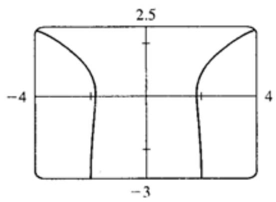


For $-1 < C < 1$, the solution curve consists of concave-up pieces separated by intervals on which the solution is not defined (where $\sin x + C \leq 0$). For $C = 1$, the solution curve consists of concave-up pieces separated by vertical asymptotes at the points where $\sin x + C = 0 \quad \Leftrightarrow \quad \sin x = -1$. For $C > 1$, the curve is continuous, and as C increases, the graph moves downward, and the amplitude of the oscillations decreases.

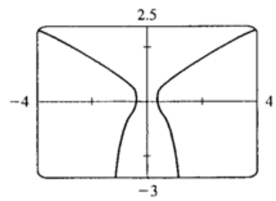
19. $\dfrac{dy}{dx} = \dfrac{\sin x}{\sin y}$, $y(0) = \frac{\pi}{2}$. So $\int \sin y\,dy = \int \sin x\,dx \quad \Leftrightarrow$ $-\cos y = -\cos x + C \quad \Leftrightarrow \quad \cos y = \cos x - C$. From the initial condition, we need $\cos\frac{\pi}{2} = \cos 0 - C \quad \Rightarrow \quad 0 = 1 - C \quad \Rightarrow \quad C = 1$, so the solution is $\cos y = \cos x - 1$. Note that we cannot take $\cos^{-1}$ of both sides, since that would unnecessarily restrict the solution to the case where $-1 \leq \cos x - 1 \quad \Leftrightarrow \quad 0 \leq \cos x$, as $\cos^{-1}$ is defined only on $[-1, 1]$. Instead we plot the graph using Maple's `plots[implicitplot]` or Mathematica's `Plot[Evaluate[···]]`.



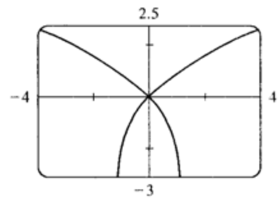
20. $\dfrac{dy}{dx} = \dfrac{x\sqrt{x^2+1}}{ye^y} \quad \Leftrightarrow \quad \int ye^y\,dy = \int x\sqrt{x^2+1}\,dx$. We use parts on the LHS with $u = y$, $dv = e^y\,dy$, and on the RHS we use the substitution $z = x^2 + 1$, so $dz = 2x\,dx$. The equation becomes $ye^y - \int e^y\,dy = \frac{1}{2}\int \sqrt{z}\,dz$ $\Leftrightarrow \quad e^y(y-1) = \frac{1}{3}(x^2+1)^{3/2} + C$, so we see that the curves are symmetric about the y-axis. Every point (x, y) in the plane lies on one of the curves, namely the one for which $C = (y-1)e^y - \frac{1}{3}(x^2+1)^{3/2}$. For example, along the y-axis, $C = (y-1)e^y - \frac{1}{3}$, so the origin lies on the curve with $C = -\frac{4}{3}$. We use Maple's `plots[implicitplot]` command or `Plot[Evaluate[···]]` in Mathematica to plot the solution curves for various values of C.



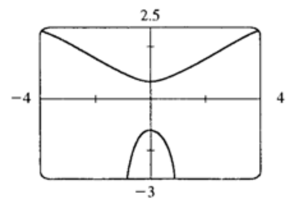
$C = -4$



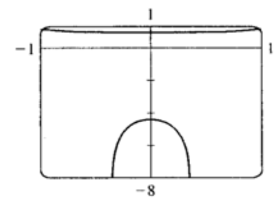
$C = -1.4$



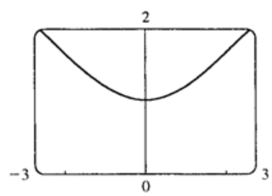
$C = -4/3$



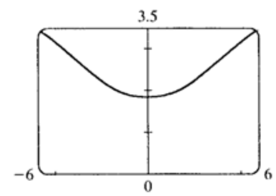
$C = -1$



$C = -0.4$



$C = -1/3$

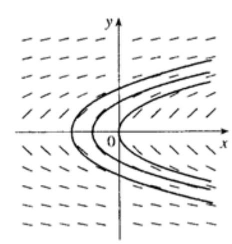


$C = 5$

It seems that the transitional values of C are $-\frac{4}{3}$ and $-\frac{1}{3}$. For $C < -\frac{4}{3}$, the graph consists of left and right branches. At $C = -\frac{4}{3}$, the two branches become connected at the origin, and as C increases, the graph splits into top and bottom branches. At $C = -\frac{1}{3}$, the bottom half disappears. As C increases further, the graph moves upward, but doesn't change shape much.

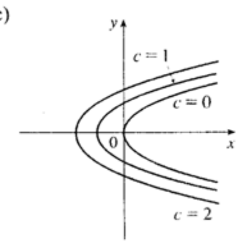
21. (a)

x	y	$y' = 1/y$
0	0.5	2
0	−0.5	−2
0	1	1
0	−1	−1
0	2	0.5
0	−2	−0.5
0	4	0.25
0	3	$0.\overline{3}$
0	0.25	4
0	$0.\overline{3}$	3



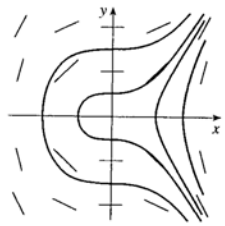
(b) $y\,dy = dx$, so $\frac{1}{2}y^2 = x + c$ or $y = \pm\sqrt{2(x+c)}$.

(c)



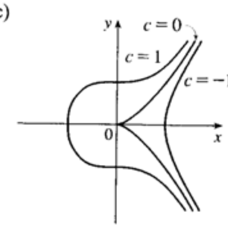
22. (a)

x	y	$y' = x^2/y$
1	1	1
−1	1	1
−1	−1	−1
1	−1	−1
1	2	0.5
2	1	4
2	2	2
1	0.5	2
0.5	1	0.25
2	0.5	8

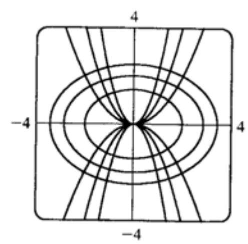


(b) $y\,dy = x^2\,dx$, so $\frac{1}{2}y^2 = \frac{1}{3}x^3 + c_1$, or $y = \pm\left(\frac{2}{3}x^3 + c\right)^{1/2}$.

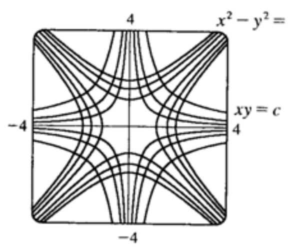
(c)



23. The curves $y = kx^2$ form a family of parabolas with axis the y-axis. Differentiating gives $y' = 2kx$, but $k = y/x^2$, so $y' = 2y/x$. Thus, the slope of the tangent line at any point (x, y) on one of the parabolas is $y' = 2y/x$, so the orthogonal trajectories must satisfy $y' = -x\,/(2y) \quad \Leftrightarrow \quad 2y\,dy = -x\,dx \quad \Leftrightarrow \quad y^2 = -x^2/\,2 + c_1 \quad \Leftrightarrow \quad x^2 + 2y^2 = c$. This is a family of ellipses.



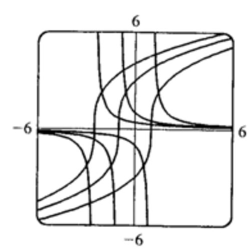
24. The curves $x^2 - y^2 = k$ form a family of hyperbolas. Differentiating gives $2x - 2y\,(dy/dx) = 0$ or $y' = x/y$, the slope of the tangent line at (x, y) on one of the hyperbolas. Thus, the orthogonal trajectories must satisfy $y' = -y/x \quad \Leftrightarrow \quad dy/y = -dx/x \quad \Leftrightarrow \quad \ln|y| = -\ln|x| + c_1 \quad \Leftrightarrow \quad \ln|x| + \ln|y| = c_1 \quad \Leftrightarrow \quad \ln|xy| = c_1 \quad \Leftrightarrow \quad xy = c$. This is a family of hyperbolas.



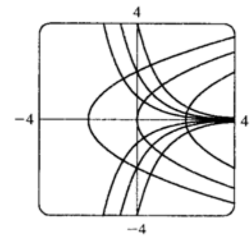
25. Differentiating $y = (x + k)^{-1}$ gives $y' = -\dfrac{1}{(x+k)^2}$, but $k = \dfrac{1}{y} - x$, so $y' = -\dfrac{1}{(1/y)^2} = -y^2$. Thus, the orthogonal trajectories must satisfy

$$y' = -\frac{1}{-y^2} = \frac{1}{y^2} \quad \Leftrightarrow \quad y^2\,dy = dx \quad \Leftrightarrow \quad \frac{y^3}{3} = x + c \text{ or}$$

$y = [3\,(x + c)]^{1/3}$



26. Differentiating $y = ke^{-x}$ gives $y' = -ke^{-x}$, but $k = ye^x$, so $y' = -y$. Thus, the orthogonal trajectories must satisfy $y' = -1/(-y) = 1/y \Leftrightarrow y\,dy = dx \Leftrightarrow \frac{1}{2}y^2 = x + c \Leftrightarrow y = \pm[2(c+x)]^{1/2}$. This is a family of parabolas with axis the x-axis.



27. From Exercise 10.2.27, $\frac{dQ}{dt} = 12 - 4Q \Leftrightarrow \int \frac{dQ}{12 - 4Q} = \int dt \Leftrightarrow -\frac{1}{4}\ln|12 - 4Q| = t + C \Leftrightarrow$
$\ln|12 - 4Q| = -4t - 4C \Leftrightarrow |12 - 4Q| = e^{-4t-4C} \Leftrightarrow 12 - 4Q = Ke^{-4t}\ (K = \pm e^{-4C}) \Leftrightarrow$
$4Q = 12 - Ke^{-4t} \Leftrightarrow Q = 3 - Ae^{-4t}\ (A = K/3)$. $Q(0) = 0 \Leftrightarrow 0 = 3 - A \Leftrightarrow A = 3 \Leftrightarrow$
$Q(t) = 3 - 3e^{-4t}$. As $t \to \infty$, $Q(t) \to 3 - 0 = 3$ (the limiting value).

28. From Exercise 10.2.28, $\frac{dy}{dt} = -\frac{1}{50}(y - 20) \Leftrightarrow \int \frac{dy}{y - 20} = \int \left(-\frac{1}{50}\right) dt \Leftrightarrow \ln|y - 20| = -\frac{1}{50}t + C$
$\Leftrightarrow y - 20 = Ke^{-t/50} \Leftrightarrow y(t) = Ke^{-t/50} + 20$. $y(0) = 95 \Leftrightarrow 95 = K + 20 \Leftrightarrow K = 75 \Leftrightarrow$
$y(t) = 75e^{-t/50} + 20$.

29. $\frac{dP}{dt} = k(M - P) \Leftrightarrow \int \frac{dP}{P - M} = \int (-k)\,dt \Leftrightarrow \ln|P - M| = -kt + C \Leftrightarrow |P - M| = e^{-kt+C} \Leftrightarrow$
$P - M = Ae^{-kt}\ (A = \pm e^C) \Leftrightarrow P = M + Ae^{-kt}$. If we assume that performance is at level 0 when $t = 0$, then $P(0) = 0 \Leftrightarrow 0 = M + A \Leftrightarrow A = -M \Leftrightarrow P(t) = M - Me^{-kt}$. $\lim_{t\to\infty} P(t) = M - M \cdot 0 = M$.

30. (a) $\frac{dx}{dt} = k(a - x)(b - x)$, $a \neq b$. $\int \frac{dx}{(a - x)(b - x)} = \int k\,dt \Rightarrow$

$\frac{1}{b - a}(-\ln|a - x| + \ln|b - x|) = kt + C$ (from CAS) $\Rightarrow \ln\left|\frac{b - x}{a - x}\right| = (b - a)(kt + C)$. The concentrations $[A] = a - x$ and $[B] = b - x$ cannot be negative, so $\frac{b - x}{a - x} \geq 0$ and $\left|\frac{b - x}{a - x}\right| = \frac{b - x}{a - x}$. We now have $\ln\left(\frac{b - x}{a - x}\right) = (b - a)(kt + C)$. Since $x(0) = 0$, we get $\ln\left(\frac{b}{a}\right) = (b - a)C$.

Hence, $\ln\left(\frac{b - x}{a - x}\right) = (b - a)kt + \ln\left(\frac{b}{a}\right) \Rightarrow \frac{b - x}{a - x} = \frac{b}{a}e^{(b-a)kt} \Rightarrow$

$$x = \frac{b\left[e^{(b-a)kt} - 1\right]}{be^{(b-a)kt}/a - 1} = \frac{ab\left[e^{(b-a)kt} - 1\right]}{be^{(b-a)kt} - a}\ \text{moles/L}.$$

(b) If $b = a$, then $\frac{dx}{dt} = k(a - x)^2$, so $\int \frac{dx}{(a - x)^2} = \int k\,dt$ and $\frac{1}{a - x} = kt + C$. Since $x(0) = 0$, we get $C = \frac{1}{a}$. Thus, $a - x = \frac{1}{kt + 1/a}$ and $x = a - \frac{a}{akt + 1} = \frac{a^2kt}{akt + 1}\,\frac{\text{moles}}{\text{L}}$.

Suppose $x = [C] = a/2$ when $t = 20$. Then $x(20) = a/2 \Rightarrow \frac{a}{2} = \frac{20a^2k}{20ak + 1} \Rightarrow 40a^2k = 20a^2k + a$
$\Rightarrow 20a^2k = a \Rightarrow k = \frac{1}{20a}$, so $x = \frac{a^2t/(20a)}{1 + at/(20a)} = \frac{at/20}{1 + t/20} = \frac{at}{t + 20}\,\frac{\text{moles}}{\text{L}}$.

31. (a) $\dfrac{dC}{dt} = r - kC \quad\Rightarrow\quad \dfrac{dC}{dt} = -(kC - r) \quad\Rightarrow\quad \displaystyle\int \frac{dC}{kC - r} = \int -dt \quad\Rightarrow\quad (1/k)\ln|kC - r| = -t + M_1$

$\Rightarrow\quad \ln|kC - r| = -kt + M_2 \quad\Rightarrow\quad |kC - r| = e^{-kt+M_2} \quad\Rightarrow\quad kC - r = M_3e^{-kt} \quad\Rightarrow\quad kC = M_3e^{-kt} + r$

$\Rightarrow\quad C(t) = M_4e^{-kt} + r/k$. $C(0) = C_0 \quad\Rightarrow\quad C_0 = M_4 + r/k \quad\Rightarrow\quad M_4 = C_0 - r/k \quad\Rightarrow$

$C(t) = (C_0 - r/k)\,e^{-kt} + r/k$.

(b) If $C_0 < r/k$, then $C_0 - r/k < 0$ and the formula for $C(t)$ shows that $C(t)$ increases and $\lim\limits_{t\to\infty} C(t) = r/k$. As t increases, the formula for $C(t)$ shows how the role of C_0 steadily diminishes as that of r/k increases.

32. (a) Use 1 billion dollars as the x-unit and 1 day as the t-unit. Initially, there is \$10 billion of old currency in circulation, so all of the \$50 million returned to the banks is old. At time t, the amount of new currency is $x(t)$ billion dollars, so $10 - x(t)$ billion dollars of currency is old. The fraction of circulating money that is old is $[10 - x(t)]/10$, and the amount of old currency being returned to the banks each day is $\dfrac{10 - x(t)}{10}0.05$ billion dollars. This amount of new currency per day is introduced into circulation, so $\dfrac{dx}{dt} = \dfrac{10 - x}{10}\cdot 0.05 = 0.005\,(10 - x)$ billion dollars per day.

(b) $\dfrac{dx}{10 - x} = 0.005\,dt \quad\Rightarrow\quad \dfrac{-dx}{10 - x} = -0.005\,dt \quad\Rightarrow\quad \ln(10 - x) = -0.005t + c \quad\Rightarrow\quad 10 - x = Ce^{-0.005t}$, where $C = e^c \quad\Rightarrow\quad x(t) = 10 - Ce^{-0.005t}$. From $x(0) = 0$, we get $C = 10$, so $x(t) = 10\left(1 - e^{-0.005t}\right)$.

(c) The new bills make up 90% of the circulating currency when $x(t) = 0.9 \cdot 10 = 9$ billion dollars.

$9 = 10\left(1 - e^{-0.005t}\right) \quad\Rightarrow\quad 0.9 = 1 - e^{-0.005t} \quad\Rightarrow\quad e^{-0.005t} = 0.1 \quad\Rightarrow\quad -0.005t = -\ln 10 \quad\Rightarrow$

$t = 200\ln 10 \approx 460.517$ days ≈ 1.26 years.

33. (a) Let $y(t)$ be the amount of salt (in kg) after t minutes. Then $y(0) = 15$. The amount of liquid in the tank is 1000 L at all times, so the concentration at time t (in minutes) is $y(t)/1000$ kg/L and

$\dfrac{dy}{dt} = -\left[\dfrac{y(t)}{1000}\,\dfrac{\text{kg}}{\text{L}}\right]\left(10\,\dfrac{\text{L}}{\text{min}}\right) = -\dfrac{y(t)}{100}\,\dfrac{\text{kg}}{\text{min}}$. $\displaystyle\int \frac{dy}{y} = -\frac{1}{100}\int dt \quad\Rightarrow\quad \ln y = -\frac{t}{100} + C$, and

$y(0) = 15 \quad\Rightarrow\quad \ln 15 = C$, so $\ln y = \ln 15 - \dfrac{t}{100}$. It follows that $\ln\left(\dfrac{y}{15}\right) = -\dfrac{t}{100}$ and $\dfrac{y}{15} = e^{-t/100}$, so

$y = 15e^{-t/100}$ kg.

(b) After 20 minutes, $y = 15e^{-20/100} = 15e^{-0.2} \approx 12.3$ kg.

34. (a) If $y(t)$ is the amount of salt (in kg) after t minutes, then $y(0) = 0$ and the total amount of liquid in the tank remains constant at 1000 L.

$$\begin{aligned}\frac{dy}{dt} &= \left(0.05\,\frac{\text{kg}}{\text{L}}\right)\left(5\,\frac{\text{L}}{\text{min}}\right) + \left(0.04\,\frac{\text{kg}}{\text{L}}\right)\left(10\,\frac{\text{L}}{\text{min}}\right) - \left(\frac{y(t)}{1000}\,\frac{\text{kg}}{\text{L}}\right)\left(15\,\frac{\text{L}}{\text{min}}\right)\\ &= 0.25 + 0.40 - 0.015y = 0.65 - 0.015y = \frac{130 - 3y}{200}\,\frac{\text{kg}}{\text{min}}\end{aligned}$$

so $\displaystyle\int \frac{dy}{130 - 3y} = \int \frac{dt}{200}$ and $-\frac{1}{3}\ln|130 - 3y| = \frac{1}{200}t + C$; since $y(0) = 0$, we have $-\frac{1}{3}\ln 130 = C$, so $-\frac{1}{3}\ln|130 - 3y| = \frac{1}{200}t - \frac{1}{3}\ln 130 \quad\Rightarrow\quad \ln|130 - 3y| = -\frac{3}{200}t + \ln 130 = \ln\left(130e^{-3t/200}\right)$, and $|130 - 3y| = 130e^{-3t/200}$. Since y is continuous, $y(0) = 0$, and the right-hand side is never zero, we deduce that $130 - 3y$ is always positive. Thus, $130 - 3y = 130e^{-3t/200}$ and $y = \frac{130}{3}\left(1 - e^{-3t/200}\right)$ kg.

(b) After an hour, $y = \frac{130}{3}\left(1 - e^{-180/200}\right) = \frac{130}{3}\left(1 - e^{-0.9}\right) \approx 25.7$ kg.

Note: As $t \to \infty$, $y(t) \to \frac{130}{3} = 43\frac{1}{3}$ kg.

35. Assume that the raindrop begins at rest, so that $v(0) = 0$. $dm/dt = km$ and $(mv)' = gm \Rightarrow m'v + mv' = gm$ $\Rightarrow (km)v + mv' = gm \Rightarrow v' = g - kv \Rightarrow \int \frac{dv}{g - kv} = \int dt \Rightarrow -(1/k)\ln|g - kv| = t + C \Rightarrow$ $g - kv = Ae^{-kt}$. $v(0) = 0 \Rightarrow A = g$. So $v = (g/k)\left(1 - e^{-kt}\right)$. Since $k > 0$, as $t \to \infty$, $e^{-kt} \to 0$ and therefore, $\lim\limits_{t\to\infty} v(t) = g/k$.

36. (a) $m\frac{dv}{dt} = -kv \Rightarrow \frac{dv}{v} = -\frac{k}{m}\,dt \Rightarrow \ln|v| = -\frac{k}{m}t + C$. Since $v(0) = v_0$, $\ln|v_0| = C$. Therefore, $\ln\left|\frac{v}{v_0}\right| = -\frac{k}{m}t \Rightarrow \left|\frac{v}{v_0}\right| = e^{-kt/m} \Rightarrow v(t) = \pm v_0 e^{-kt/m}$. The sign is $+$ when $t = 0$, and we assume v is continuous, so that the sign is $+$ for all t. Thus, $v(t) = v_0e^{-kt/m}$. $ds/dt = v_0e^{-kt/m} \Rightarrow$ $s(t) = -\frac{mv_0}{k}e^{-kt/m} + C'$. From $s(0) = s_0$, we get $s_0 = -\frac{mv_0}{k} + C'$, so $C' = s_0 + \frac{mv_0}{k}$ and $s(t) = s_0 + \frac{mv_0}{k}\left(1 - e^{-kt/m}\right)$. The distance traveled from time 0 to time t is $s(t) - s_0$, so the total distance traveled is $\lim\limits_{t\to\infty}[s(t) - s_0] = \frac{mv_0}{k}$.

Note: In finding the limit, we use the fact that $k > 0$ to conclude that $\lim\limits_{t\to\infty} e^{-kt/m} = 0$.

(b) $m\frac{dv}{dt} = -kv^2 \Rightarrow \frac{dv}{v^2} = -\frac{k}{m}\,dt \Rightarrow \frac{-1}{v} = -\frac{kt}{m} + C \Rightarrow \frac{1}{v} = \frac{kt}{m} - C$. Since $v(0) = v_0$, $C = -\frac{1}{v_0}$ and $\frac{1}{v} = \frac{kt}{m} + \frac{1}{v_0}$. Therefore, $v(t) = \frac{1}{kt/m + 1/v_0} = \frac{mv_0}{kv_0t + m}$. $\frac{ds}{dt} = \frac{mv_0}{kv_0t + m} \Rightarrow$ $s(t) = \frac{m}{k}\int \frac{kv_0\,dt}{kv_0t + m} = \frac{m}{k}\ln|kv_0t + m| + C'$. Since $s(0) = s_0$, we get $s_0 = \frac{m}{k}\ln m + C' \Rightarrow$ $C' = s_0 - \frac{m}{k}\ln m \Rightarrow s(t) = s_0 + \frac{m}{k}\left(\ln|kv_0t + m| - \ln m\right) = s_0 + \frac{m}{k}\ln\left|\frac{kv_0t + m}{m}\right|$. We can rewrite the formulas for $v(t)$ and $s(t)$ as $v(t) = \frac{v_0}{1 + (kv_0/m)t}$ and $s(t) = s_0 + \frac{m}{k}\ln\left|1 + \frac{kv_0}{m}t\right|$.

Remarks: This model of horizontal motion through a resistive medium was designed to handle the case in which $v_0 > 0$. Then the term $-kv^2$ representing the resisting force causes the object to decelerate. The absolute value in the expression for $s(t)$ is unnecessary (since k, v_0, and m are all positive), and $\lim\limits_{t\to\infty} s(t) = \infty$. In other words, the object travels infinitely far. However, $\lim\limits_{t\to\infty} v(t) = 0$. When $v_0 < 0$, the term $-kv^2$ increases the magnitude of the object's negative velocity. According to the formula for $s(t)$, the position of the object approaches $-\infty$ as t approaches $m/k(-v_0)$: $\lim\limits_{t\to -m/(kv_0)} s(t) = -\infty$. Again the object travels infinitely far, but this time the feat is accomplished in a finite amount of time. Notice also that $\lim\limits_{t\to -m/(kv_0)} v(t) = -\infty$ when $v_0 < 0$, showing that the speed of the object increases without limit.

37. (a) The rate of growth of the area is jointly proportional to $\sqrt{A(t)}$ and $M - A(t)$; that is, the rate is proportional to the product of those two quantities. So for some constant k, $dA/dt = k\sqrt{A}(M - A)$. We are interested in the maximum of the function dA/dt (when the tissue grows the fastest), so we differentiate, using the Chain Rule and then substituting for dA/dt from the differential equation:

$$\frac{d}{dt}\left(\frac{dA}{dt}\right) = k\left[\tfrac{1}{2}A^{-1/2}(M - A)\frac{dA}{dt} + \sqrt{A}(-1)\frac{dA}{dt}\right] = \tfrac{1}{2}kA^{-1/2}\frac{dA}{dt}\left[(M - A) - 2A\right]$$
$$= \tfrac{1}{2}k^2(M - A)(M - 3A)$$

This is 0 when $M - A = 0$ [this situation never actually occurs, since the graph of $A(t)$ is asymptotic to the line $y = M$, as in the logistic model] and when $M - 3A = 0 \Leftrightarrow A(t) = M/3$. This represents a maximum by the First Derivative Test, since $\dfrac{d}{dt}\left(\dfrac{dA}{dt}\right)$ goes from positive to negative when $A(t) = M/3$.

(b) From the CAS, we get $A(t) = M\left(\dfrac{Ce^{\sqrt{M}kt} - 1}{Ce^{\sqrt{M}kt} + 1}\right)^2$. To get C in terms of the initial area A_0 and the maximum area M, we substitute $t = 0$ and $A = A_0$: $A_0 = M\left(\dfrac{C - 1}{C + 1}\right)^2 \Leftrightarrow (C + 1)\sqrt{A_0} = (C - 1)\sqrt{M} \Leftrightarrow$ $C\sqrt{A_0} + \sqrt{A_0} = C\sqrt{M} - \sqrt{M} \Leftrightarrow \sqrt{A_0} + \sqrt{M} = C\sqrt{M} - C\sqrt{A_0} \Leftrightarrow C = \dfrac{\sqrt{M} + \sqrt{A_0}}{\sqrt{M} - \sqrt{A_0}}$. (Notice that if $A_0 = 0$, then $C = 1$.)

38. (a) According to the hint we use the Chain Rule: $m\dfrac{dv}{dt} = m\dfrac{dv}{dx}\cdot\dfrac{dx}{dt} = mv\dfrac{dv}{dx} = -\dfrac{mgR^2}{(x + R)^2} \Rightarrow$ $\displaystyle\int v\,dv = \int\frac{-gR^2\,dx}{(x + R)^2} \Rightarrow \frac{v^2}{2} = \frac{gR^2}{x + R} + C$. When $x = 0$, $v = v_0$, so $\dfrac{v_0^2}{2} = \dfrac{gR^2}{0 + R} + C \Rightarrow$ $C = \frac{1}{2}v_0^2 - gR \Rightarrow \frac{1}{2}v^2 - \frac{1}{2}v_0^2 = \dfrac{gR^2}{x + R} - gR$. Now at the top of its flight, the rocket's velocity will be 0, and its height will be $x = h$. Solving for v_0: $-\frac{1}{2}v_0^2 = \dfrac{gR^2}{h + R} - gR \Rightarrow$

$$\frac{v_0^2}{2} = g\left[-\frac{R^2}{R + h} + \frac{R(R + h)}{R + h}\right] = \frac{gRh}{R + h} \Rightarrow v_0 = \sqrt{\frac{2gRh}{R + h}}.$$

(b) $v_e = \lim\limits_{h\to\infty} v_0 = \lim\limits_{h\to\infty}\sqrt{\dfrac{2gRh}{R + h}} = \lim\limits_{h\to\infty}\sqrt{\dfrac{2gR}{(R/h) + 1}} = \sqrt{2gR}$

(c) $v_e = \sqrt{2\cdot 32\text{ ft/s}^2\cdot 3960\text{ mi}\cdot 5280\text{ ft/mi}} \approx 36{,}581\text{ ft/s} \approx 6.93\text{ mi/s}$

39. (a) We have $V(t) = \pi r^2 y(t) \Rightarrow \dfrac{dV}{dy} = \pi r^2 = 4\pi$ where $\dfrac{dV}{dt} = \dfrac{dV}{dy}\dfrac{dy}{dt}$. Thus, $\dfrac{dV}{dt} = -a\sqrt{2gy} \Rightarrow$ $\dfrac{dV}{dy}\dfrac{dy}{dt} = -\pi\left(\frac{1}{12}\right)^2\sqrt{2\cdot 32y} \Rightarrow 4\pi\dfrac{dy}{dt} = -\pi\frac{8}{144}\sqrt{y} \Rightarrow \dfrac{dy}{dt} = -\frac{1}{72}\sqrt{y}$.

(b) $\dfrac{dy}{dt} = -\frac{1}{72}\sqrt{y} \Rightarrow y^{-1/2}\,dy = -\frac{1}{72}\,dt \Rightarrow 2\sqrt{y} = -\frac{1}{72}t + C$. $y(0) = 6 \Rightarrow 2\sqrt{6} = 0 + C \Rightarrow$ $C = 2\sqrt{6} \Rightarrow y = \left(-\frac{1}{144}t + \sqrt{6}\right)^2$.

(c) We want to find t when $y = 0$, so we set $y = 0 = \left(-\frac{1}{144}t + \sqrt{6}\right)^2 \Rightarrow t = 144\sqrt{6} \approx 5$ min 53 s.

40. (a) If the radius of the circular cross-section at height y is r, then the Pythagorean Theorem gives $r^2 = 2^2 - (2-y)^2$ since the radius of the tank is 2 m. So $A(y) = \pi r^2 = \pi\left[4-(2-y)^2\right] = \pi\left(4y-y^2\right)$. Thus, $A(y)\dfrac{dy}{dt} = -a\sqrt{2gy} \;\Rightarrow\; \pi\left(4y-y^2\right)\dfrac{dy}{dt} = -\pi(0.01)^2\sqrt{2\cdot 10y} \;\Rightarrow$ $\left(4y-y^2\right)\dfrac{dy}{dt} = -(0.0001)\sqrt{20y}$.

(b) From part (a) we have $\left(4y^{1/2}-y^{3/2}\right)dy = -\left(0.0001\sqrt{20}\right)dt \;\Rightarrow$ $\frac{8}{3}y^{3/2} - \frac{2}{5}y^{5/2} = -\left(0.0001\sqrt{20}\right)t + C$. $y(0)=2 \;\Rightarrow\; \frac{8}{3}(2)^{3/2} - \frac{2}{5}(2)^{5/2} = C \;\Rightarrow\; C = \frac{56}{15}\sqrt{2}$. To find out how long it will take to drain all the water we evaluate t when $y=0$: $0 = -\left(0.0001\sqrt{20}\right)t + C \;\Rightarrow$

$$t = \frac{C}{0.0001\sqrt{20}} = \frac{56\sqrt{2}\big/15}{0.0001\sqrt{20}} \approx 11,806 \text{ s} \approx 3 \text{ h } 17 \text{ min}.$$

Applied Project □ Which is Faster, Going Up or Coming Down?

1. $mv' = -pv - mg \;\Rightarrow\; m\dfrac{dv}{dt} = -(pv+mg) \;\Rightarrow\; \displaystyle\int \frac{dv}{pv+mg} = \int -\frac{1}{m}\,dt \;\Rightarrow$ $\dfrac{1}{p}\ln(pv+mg) = -\dfrac{1}{m}t + C$ $(pv+mg>0)$. At $t=0$, $v=v_0$, so $C = \dfrac{1}{p}\ln(pv_0+mg)$. Thus, $\dfrac{1}{p}\ln(pv+mg) = -\dfrac{1}{m}t + \dfrac{1}{p}\ln(pv_0+mg) \;\Rightarrow\; \ln(pv+mg) = -\dfrac{p}{m}t + \ln(pv_0+mg) \;\Rightarrow$ $pv+mg = e^{-pt/m}(pv_0+mg) \;\Rightarrow\; pv = (pv_0+mg)e^{-pt/m} - mg \;\Rightarrow$ $v(t) = \left(v_0 + \dfrac{mg}{p}\right)e^{-pt/m} - \dfrac{mg}{p}$.

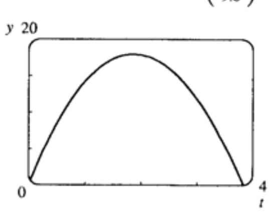
2. $y(t) = \int v(t)\,dt = \displaystyle\int\left[\left(v_0+\frac{mg}{p}\right)e^{-pt/m} - \frac{mg}{p}\right]dt = \left(v_0+\frac{mg}{p}\right)e^{-pt/m}\left(-\frac{m}{p}\right) - \frac{mg}{p}t + C$

At $t=0$, $y=0$, so $C = \left(v_0+\dfrac{mg}{p}\right)\dfrac{m}{p}$. Thus,

$$y(t) = \left(v_0+\frac{mg}{p}\right)\frac{m}{p} - \left(v_0+\frac{mg}{p}\right)\frac{m}{p}e^{-pt/m} - \frac{mgt}{p} = \left(v_0+\frac{mg}{p}\right)\frac{m}{p}\left(1-e^{-pt/m}\right) - \frac{mgt}{p}$$

3. $y'(t) = \left(v_0+\dfrac{mg}{p}\right)\dfrac{m}{p}\left(\dfrac{p}{m}e^{-pt/m}\right) - \dfrac{mg}{p}$, so $y'(t)=0 \;\Rightarrow\; \dfrac{mg}{p} = \left(v_0+\dfrac{mg}{p}\right)e^{-pt/m} \;\Rightarrow$ $e^{pt/m} = \dfrac{pv_0}{mg} + 1 \;\Rightarrow\; \dfrac{pt}{m} = \ln\left(\dfrac{pv_0}{mg}+1\right) \;\Rightarrow\; t_1 = \dfrac{m}{p}\ln\left(\dfrac{mg+pv_0}{mg}\right)$. With $m=1$, $v_0=20$, $p=\frac{1}{10}$, and $g=9.8$, we have $t_1 = 10\ln\left(\frac{11.8}{9.8}\right) \approx 1.86$ s.

4.



The figure shows the graph of $y = 1180\left(1-e^{-0.1t}\right) - 98t$. The zeros are at $t=0$ and $t_2 \approx 3.84$. Thus, $t_1 - 0 \approx 1.86$ and $t_2 - t_1 \approx 1.98$. So the time it takes to come down is about 0.12 s longer than the time it takes to go up; hence, going up is faster.

5. $y(2t_1) = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}\left(1 - e^{-2pt_1/m}\right) - \frac{mg}{p} \cdot 2t_1$

$= \left(\frac{pv_0 + mg}{p}\right)\frac{m}{p}\left[1 - \left(e^{pt_1/m}\right)^{-2}\right] - \frac{mg}{p} \cdot 2\frac{m}{p}\ln\left(\frac{pv_0 + mg}{mg}\right)$

Substituting $x = e^{pt_1/m} = \frac{pv_0}{mg} + 1 = \frac{pv_0 + mg}{mg}$ (from Problem 3), we get

$y(2t_1) = \left(x \cdot \frac{mg}{p}\right)\frac{m}{p}\left(1 - x^{-2}\right) - \frac{m^2 g}{p^2} \cdot 2\ln x = \frac{m^2 g}{p^2}\left(x - \frac{1}{x} - 2\ln x\right)$. Now $p > 0$, $m > 0$, $t_1 > 0 \Rightarrow$ $x = e^{pt_1/m} > e^0 = 1$. $f(x) = x - \frac{1}{x} - 2\ln x \Rightarrow f'(x) = 1 + \frac{1}{x^2} - \frac{2}{x} = \frac{x^2 - 2x + 1}{x^2} = \frac{(x-1)^2}{x^2} > 0$ for $x > 1 \Rightarrow f(x)$ is increasing for $x > 1$. Since $f(1) = 0$, it follows that $f(x) > 0$ for every $x > 1$. Therefore, $y(2t_1) = \frac{m^2 g}{p^2} f(x)$ is positive, which means that the ball has not yet reached the ground at time $2t_1$. This tells us that the time spent going up is always less than the time spent coming down, so *ascent is faster*.

10.4 Exponential Growth and Decay

1. The relative growth rate is $\frac{1}{P}\frac{dP}{dt} = 0.7944$, so $\frac{dP}{dt} = 0.7944P$ and, by Theorem 2, $P(t) = P(0)e^{0.7944t} = 2e^{0.7944t}$. Thus, $P(6) = 2e^{0.7944(6)} \approx 234.99$ or about 235 members.

2. (a) By Theorem 2, $P(t) = P(0)e^{kt} = 60e^{kt}$. In 20 minutes ($\frac{1}{3}$ hour), there are 120 cells, so $P\left(\frac{1}{3}\right) = 60e^{k/3} = 120 \Rightarrow e^{k/3} = 2 \Rightarrow k/3 = \ln 2 \Rightarrow k = 3\ln 2 = \ln\left(2^3\right) = \ln 8$.

(b) $P(t) = 60e^{(\ln 8)t} = 60 \cdot 8^t$

(c) $P(8) = 60 \cdot 8^8 = 60 \cdot 2^{24} = 1{,}006{,}632{,}960$

(d) $dP/dt = kP \Rightarrow P'(8) = kP(8) = (\ln 8)\,P(8) \approx 2.093$ billion cells/h

(e) $P(t) = 20{,}000 \Rightarrow 60 \cdot 8^t = 20{,}000 \Rightarrow 8^t = 1000/3 \Rightarrow t\ln 8 = \ln(1000/3) \Rightarrow$ $t = \frac{\ln(1000/3)}{\ln 8} \approx 2.79$ h

3. (a) By Theorem 2, $y(t) = y(0)e^{kt} = 500e^{kt}$. $y(3) = 500e^{3k} = 8000 \Rightarrow e^{3k} = 16 \Rightarrow 3k = \ln 16 \Rightarrow$ $k = (\ln 16)/3$. So $y(t) = 500e^{(\ln 16)t/3} = 500 \cdot 16^{t/3}$

(b) $y(4) = 500 \cdot 16^{4/3} \approx 20{,}159$

(c) $dy/dt = ky \Rightarrow y'(4) = ky(4) = \frac{1}{3}\ln 16\left(500 \cdot 16^{4/3}\right)$ [from part (a)] $\approx 18{,}631$ cells/h

(d) $y(t) = 500 \cdot 16^{t/3} = 30{,}000 \Rightarrow 16^{t/3} = 60 \Rightarrow \frac{1}{3}t\ln 16 = \ln 60 \Rightarrow t = 3(\ln 60)/(\ln 16) \approx 4.4$ h

4. (a) $y(t) = y(0)e^{kt} \Rightarrow y(2) = y(0)e^{2k} = 400$, $y(6) = y(0)e^{6k} = 25{,}600$. Dividing these equations, we get $e^{6k}/e^{2k} = 25{,}600/400 \Rightarrow e^{4k} = 64 \Rightarrow 4k = \ln 64 = 6\ln 2 \Rightarrow k = \frac{3}{2}\ln 2 = \frac{1}{2}\ln 8$. Thus, $y(0) = 400/e^{2k} = 400/e^{\ln 8} = \frac{400}{8} = 50$.

(b) $y(t) = y(0)e^{kt} = 50e^{(\ln 8)t/2}$ or $y = 50 \cdot 8^{t/2}$

(c) $y(t) = 50e^{(3\ln 2)t/2} = 100 \Leftrightarrow e^{(3\ln 2)t/2} = 2 \Leftrightarrow (3\ln 2)t/2 = \ln 2 \Leftrightarrow t = 2/3 \text{ h} = 40 \text{ min}$

(d) $50e^{(\ln 8)t/2} = 100{,}000 \Leftrightarrow e^{(\ln 8)t/2} = 2000 \Leftrightarrow (\ln 8)t/2 = \ln 2000 \Leftrightarrow t = (2\ln 2000)/\ln 8 \approx 7.3$ h.

5. (a) Let the population (in millions) in the year t be $P(t)$. Since the initial time is the year 1750, we substitute $t - 1750$ for t in Theorem 2, so the exponential model gives $P(t) = P(1750)\, e^{k(t-1750)}$. Then $P(1800) = 906 = 728e^{k(1800-1750)} \quad \Rightarrow \quad \ln \frac{906}{728} = k(50) \quad \Rightarrow \quad k = \frac{1}{50} \ln \frac{906}{728} \approx 0.0043748$. So with this model, we have $P(1900) = 728e^{150(0.0043748)} \approx 1403$ million, and $P(1950) \approx 728e^{200(0.0043748)} \approx 1746$ million. Both of these estimates are much too low.

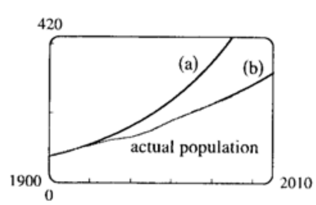
(b) In this case, the exponential model gives $P(t) = P(1850)\, e^{k(t-1850)} \quad \Rightarrow$ $P(1900) = 1608 = 1171e^{k(1900-1850)} \quad \Rightarrow \quad \ln \frac{1608}{1171} = k(50) \quad \Rightarrow \quad k = \frac{1}{50} \ln \frac{1608}{1171} \approx 0.006343$. So with this model, we estimate $P(1950) \approx 1171e^{100(0.006343)} \approx 2208$ million. This is still too low, but closer than the estimate of $P(1950)$ in part (a).

(c) The exponential model gives $P(t) = P(1900)\, e^{k(t-1900)} \quad \Rightarrow \quad P(1950) = 2517 = 1608e^{k(1950-1900)} \quad \Rightarrow$ $\ln \frac{2517}{1608} = k(50) \quad \Rightarrow \quad k = \frac{1}{50} \ln \frac{2517}{1608} \approx 0.008962$. With this model, we estimate $P(1992) \approx 1608e^{0.008962(1992-1900)} \approx 3667$ million. This is much too low. The discrepancy is explained by the fact that the world birth rate (average yearly number of births per person) is about the same as always, whereas the mortality rate (especially the infant mortality rate) is much lower, owing mostly to advances in medical science and to the wars in the first part of the twentieth century. The exponential model assumes, among other things, that the birth and mortality rates will remain constant.

6. (a) Let $P(t)$ be the population (in millions) in the year t. Since the initial time is the year 1900, we substitute $t - 1900$ for t in Theorem 2, and find that the exponential model gives $P(t) = P(1900)\, e^{k(t-1900)} \quad \Rightarrow$ $P(1910) = 92 = 76e^{k(1910-1900)} \quad \Rightarrow \quad k = \frac{1}{10} \ln \frac{92}{76} \approx 0.0191$. With this model, we estimate $P(1990) = 76e^{0.0191(1990-1900)} \approx 424$ million. This estimate is much too high. The discrepancy is explained by the fact that, between the years 1900 and 1910, an enormous number of immigrants (compared to the total population) came to the United States. Since that time, immigration (as a proportion of total population) has been much lower. Also, the birth rate in the United States has declined since the turn of the century. So our calculation of the constant k was based partly on factors which no longer exist.

(b) Substituting $t - 1970$ for t in Theorem 2, we find that the exponential model gives $P(t) = P(1970)\, e^{k(t-1970)}$ $\Rightarrow \quad P(1980) = 227 = 203e^{k(1990-1980)} \quad \Rightarrow \quad k = \frac{1}{10} \ln \frac{227}{203} \approx 0.01117$. With this model, we estimate $P(1990) \approx 203e^{0.01117(1990-1970)} \approx 254$ million. This is quite accurate. The further estimates are $P(2000) = 203e^{30k} \approx 284$ million and $P(2010) = 203e^{40k} \approx 317$ million.

(c)



The model in part (a) is quite inaccurate after 1910 (off by 5 million in 1920 and 12 million in 1930). The model in part (b) is more accurate (which is not surprising, since it is based on more recent information).

7. (a) If $y = [N_2O_5]$ then by Theorem 2, $\dfrac{dy}{dt} = -0.0005y \quad \Rightarrow \quad y(t) = y(0)\, e^{-0.0005t} = Ce^{-0.0005t}$.

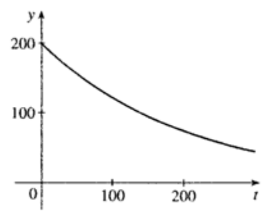
(b) $y(t) = Ce^{-0.0005t} = 0.9C \quad \Rightarrow \quad e^{-0.0005t} = 0.9 \quad \Rightarrow \quad -0.0005t = \ln 0.9 \quad \Rightarrow \quad t = -2000 \ln 0.9 \approx 211$ s

8. (a) The mass remaining after t days is $y(t) = y(0)\,e^{kt} = 200e^{kt}$. Since the half-life is 140 days, $y(140) = 200e^{140k} = 100 \Rightarrow e^{140k} = \frac{1}{2}$ $\Rightarrow 140k = \ln\frac{1}{2} \Rightarrow k = -(\ln 2)/140$, so $y(t) = 200e^{-(\ln 2)t/140} = 200 \cdot 2^{-t/140}$.

(b) $y(100) = 200 \cdot 2^{-100/140} \approx 121.9$ mg

(c) $200e^{-(\ln 2)t/140} = 10 \Leftrightarrow -\ln 2\frac{t}{140} = \ln\frac{1}{20} = -\ln 20 \Leftrightarrow$ $t = 140\frac{\ln 20}{\ln 2} \approx 605$ days

(d)



9. (a) If $y(t)$ is the mass remaining after t days, then $y(t) = y(0)\,e^{kt} = 50e^{kt}$. $y(0.00014) = 50e^{0.00014k} = 25 \Rightarrow$ $e^{0.00014k} = \frac{1}{2} \Rightarrow k = -(\ln 2)/0.00014 \Rightarrow y(t) = 50e^{-(\ln 2)t/0.00014} = 50 \cdot 2^{-t/0.00014}$

(b) $y(0.01) = 50 \cdot 2^{-0.01/0.00014} \approx 1.57 \times 10^{-20}$ mg

(c) $50e^{-(\ln 2)t/0.00014} = 40 \Rightarrow -(\ln 2)\,t/0.00014 = \ln 0.8 \Rightarrow t = -0.00014\dfrac{\ln 0.8}{\ln 2} \approx 4.5 \times 10^{-5}$ s

10. (a) If $y(t)$ is the mass after t days and $y(0) = A$, then $y(t) = Ae^{kt}$. $y(3) = Ae^{3k} = 0.58A \Rightarrow e^{3k} = 0.58$ $\Rightarrow 3k = \ln 0.58 \Rightarrow k = \frac{1}{3}\ln 0.58$. Then $Ae^{\ln(0.58)t/3} = \dfrac{A}{2} \Leftrightarrow \dfrac{\ln(0.58)\,t}{3} = \ln\frac{1}{2}$, so the half-life is $t = -\dfrac{3\ln 2}{\ln 0.58} \approx 3.82$ days.

(b) $Ae^{\ln(0.58)t/3} = \dfrac{A}{10} \Rightarrow \dfrac{\ln(0.58)\,t}{3} = \ln\frac{1}{10} \Leftrightarrow t = -\dfrac{3\ln 10}{\ln 0.58} \approx 12.68$ days

11. Let $y(t)$ be the level of radioactivity. Thus, $y(t) = y(0)\,e^{-kt}$ and k is determined by using the half-life: $y(5730) = \frac{1}{2}y(0) \Rightarrow \frac{1}{2} = e^{-5730k} \Rightarrow k = -\dfrac{\ln\frac{1}{2}}{5730} = \frac{\ln 2}{5730}$. If 74% of the ^{14}C remains, then we know that $y(t) = 0.74y(0) \Rightarrow 0.74 = e^{-t(\ln 2)/5730} \Rightarrow \ln 0.74 = -\dfrac{t\ln 2}{5730} \Rightarrow$ $t = -\dfrac{5730(\ln 0.74)}{\ln 2} \approx 2489 \approx 2500$ years.

12. From the information given, we know that $\dfrac{dy}{dx} = 2y \Rightarrow y = Ce^{2x}$ by Theorem 2. To calculate C we use the point $(0, 5)$: $5 = Ce^{2(0)} \Rightarrow C = 5$. Thus, the equation of the curve is $y = 5e^{2x}$.

13. (a) If $y = u - 75$, $u(0) = 185 \Rightarrow y(0) = 185 - 75 = 110$, and the initial-value problem is $dy/dt = ky$ with $y(0) = 110$. So the solution is $y(t) = 110e^{kt}$.

(b) $y(30) = 110e^{30k} = 150 - 75 \Rightarrow e^{30k} = \frac{75}{110} = \frac{15}{22} \Rightarrow k = \frac{1}{30}\ln\frac{15}{22}$, so $y(t) = 110e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)}$ and $y(45) = 110e^{\frac{45}{30}\ln\left(\frac{15}{22}\right)} \approx 62°$F. Thus, $u(45) \approx 62 + 75 = 137°$F.

(c) $u(t) = 100 \Rightarrow y(t) = 25$. $y(t) = 110e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)} = 25 \Rightarrow e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)} = \frac{25}{110} \Rightarrow \frac{1}{30}t\ln\frac{15}{22} = \ln\frac{25}{110}$ $\Rightarrow t = \dfrac{30\ln\frac{25}{110}}{\ln\frac{15}{22}} \approx 116$ min.

14. (a) Let $y(t)$ = temperature after t minutes. Newton's Law of Cooling implies that $\dfrac{dy}{dt} = k(y-5)$. Let $u(t) = y(t) - 5$. Then $\dfrac{du}{dt} = ku$, so $u(t) = u(0)\,e^{kt} = 15e^{kt} \Rightarrow y(t) = 5 + 15e^{kt} \Rightarrow$ $y(1) = 5 + 15e^{k} = 12 \Rightarrow e^{k} = \frac{7}{15} \Rightarrow k = \ln\frac{7}{15}$, so $y(t) = 5 + 15e^{\ln(7/15)t}$ and $y(2) = 5 + 15e^{2\ln(7/15)} \approx 8.3\ ^\circ\text{C}$.

(b) $5 + 15e^{\ln(7/15)t} = 6$ when $e^{\ln(7/15)t} = \frac{1}{15} \Rightarrow \ln\left(\frac{7}{15}\right)t = \ln\frac{1}{15} \Rightarrow t = \dfrac{\ln\frac{1}{15}}{\ln\frac{7}{15}} \approx 3.6$ min.

15. (a) Let $P(h)$ be the pressure at altitude h. Then $dP/dh = kP \Rightarrow P(h) = P(0)\,e^{kh} = 101.3e^{kh}$. $P(1000) = 101.3e^{1000k} = 87.14 \Rightarrow 1000k = \ln\left(\frac{87.14}{101.3}\right) \Rightarrow P(h) = 101.3\,e^{\frac{1}{1000}h\ln\left(\frac{87.14}{101.3}\right)}$, so $P(3000) = 101.3e^{3\ln\left(\frac{87.14}{101.3}\right)} \approx 64.5$ kPa.

(b) $P(6187) = 101.3\,e^{\frac{6187}{1000}\ln\left(\frac{87.14}{101.3}\right)} \approx 39.9$ kPa

16. (a) Using $A = A_0\left(1 + \dfrac{r}{n}\right)^{nt}$ with $A_0 = 500$, $r = 0.14$, and $t = 2$, we have:

(i) Annually: $n = 1$; $A = 500\,(1.14)^2 = \$649.80$

(ii) Quarterly: $n = 4$; $A = 500\left(1 + \frac{0.14}{4}\right)^{8} = \$658.40$

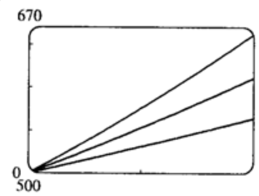
(iii) Monthly: $n = 12$; $A = 500\left(1 + \frac{0.14}{12}\right)^{24} = \$660.49$

(iv) Daily: $n = 365$; $A = 500\left(1 + \frac{0.14}{365}\right)^{2\cdot 365} = \$661.53$

(v) Hourly: $n = 365\cdot 24$; $A = 500\left(1 + \frac{0.14}{365\cdot 24}\right)^{2\cdot 365\cdot 24} = \$661.56$

(vi) Continuously: $A = 500e^{(0.14)2} = \$661.56$

(b)



$A_{0.14}(2) = \$661.56$,
$A_{0.10}(2) = \$610.70$, and
$A_{0.06}(2) = \$563.75$.

17. (a) Using $A = A_0\left(1 + \dfrac{r}{n}\right)^{nt}$ with $A_0 = 3000$, $r = 0.05$, and $t = 5$, we have:

(i) Annually: $n = 1$; $A = 3000\,(1.05)^5 = \$3828.84$

(ii) Semiannually: $n = 2$; $A = 3000\left(1 + \frac{0.05}{2}\right)^{10} = \$3840.25$

(iii) Monthly: $n = 12$; $A = 3000\left(1 + \frac{0.05}{12}\right)^{60} = \$3850.08$

(iv) Weekly: $n = 52$; $A = 3000\left(1 + \frac{0.05}{52}\right)^{5\cdot 52} = \$3851.61$

(v) Daily: $n = 365$; $A = 3000\left(1 + \frac{0.05}{365}\right)^{5\cdot 365} = \$3852.01$

(vi) Continuously: $A = 3000e^{(0.05)5} = \$3852.08$

(b) $dA/dt = 0.05A$ and $A(0) = 3000$.

18. $A_0e^{0.06t} = 2A_0 \Leftrightarrow e^{0.06t} = 2 \Leftrightarrow 0.06t = \ln 2 \Leftrightarrow t = \frac{50}{3}\ln 2 \approx 11.55$, so the investment will double in about 11.55 years.

19. (a) $\frac{dP}{dt} = kP - m = k\left(P - \frac{m}{k}\right)$. Let $y = P - \frac{m}{k}$, so the equation becomes $\frac{dy}{dt} = ky$. The solution is $y = y_0 e^{kt}$ $\Rightarrow$ $P - \frac{m}{k} = \left(P_0 - \frac{m}{k}\right)e^{kt}$ $\Rightarrow$ $P(t) = \frac{m}{k} + \left(P_0 - \frac{m}{k}\right)e^{kt}$.

(b) There will be an exponential expansion $\Leftrightarrow$ $P_0 - \frac{m}{k} > 0$ $\Leftrightarrow$ $m < kP_0$.

(c) The population will be constant if $P_0 - \frac{m}{k} = 0$ $\Leftrightarrow$ $m = kP_0$. It will decline if $P_0 - \frac{m}{k} < 0$ $\Leftrightarrow$ $m > kP_0$.

(d) $P_0 = 8{,}000{,}000$, $k = \alpha - \beta = 0.016$, $m = 210{,}000$ $\Rightarrow$ $m > kP_0$ ($= 128{,}000$), so by part (c), the population was declining.

20. (a) $\frac{dy}{dt} = ky^{1+c}$ $\Rightarrow$ $y^{-1-c}\,dy = k\,dt$ $\Rightarrow$ $\frac{y^{-c}}{-c} = kt + C$. Since $y(0) = y_0$, we have $C = \frac{y_0^{-c}}{-c}$. Thus, $\frac{y^{-c}}{-c} = kt + \frac{y_0^{-c}}{-c}$, or $y^{-c} = y_0^{-c} - ckt$. So $y^c = \frac{1}{y_0^{-c} - ckt} = \frac{y_0^c}{1 - cy_0^c kt}$ and $y(t) = \frac{y_0}{\left(1 - cy_0^c kt\right)^{1/c}}$.

(b) $y(t) \to \infty$ as $1 - cy_0^c kt \to 0$, that is, as $t \to \frac{1}{cy_0^c k}$. Define $T = \frac{1}{cy_0^c k}$. Then $\lim\limits_{t \to T^-} y(t) = \infty$.

(c) According to the data given, we have $c = 0.01$, $y(0) = 2$, and $y(3) = 16$, where the time t is given in months. Thus, $y_0 = 2$ and $16 = y(3) = \frac{y_0}{\left(1 - cy_0^c k \cdot 3\right)^{1/c}}$. Since $T = \frac{1}{cy_0^c k}$, we will solve for $cy_0^c k$.

$16 = \frac{2}{\left(1 - 3cy_0^c k\right)^{100}}$ $\Rightarrow$ $1 - 3cy_0^c k = \left(\frac{1}{8}\right)^{0.01} = 8^{-0.01}$ $\Rightarrow$ $cy_0^c k = \frac{1}{3}\left(1 - 8^{-0.01}\right)$. Thus, doomsday occurs when $t = T = \frac{1}{cy_0^c k} = \frac{3}{1 - 8^{-0.01}} \approx 145.77$ months or 12.15 years.

Applied Project □ Calculus and Baseball

1. (a) $F = ma = m\frac{dv}{dt}$, so by the Substitution Rule we have

$$\int_{t_0}^{t_1} F(t)\,dt = \int_{t_0}^{t_1} m\left(\frac{dv}{dt}\right)dt = m\int_{v_0}^{v_1} dv = [mv]_{v_0}^{v_1} = mv_1 - mv_0 = p(t_1) - p(t_0)$$

(b) (i) We have $v_1 = 110$ mi/h $= \frac{110(5280)}{3600}$ ft/s $= 161.\overline{3}$ ft/s, $v_0 = -90$ mi/h $= -132$ ft/s, and the mass of the baseball is $m = \frac{w}{g} = \frac{5/16}{32} = \frac{5}{512}$. So the change in momentum is

$p(t_1) - p(t_0) = mv_1 - mv_0 = \frac{5}{512}\left[161.\overline{3} - (-132)\right] \approx 2.86$ slug-ft/s.

(ii) From part (a) and part (b)(i), we have $\int_0^{0.001} F(t)\,dt = p(0.001) - p(0) \approx 2.86$, so the average force over the interval $[0, 0.001]$ is $\frac{1}{0.001}\int_0^{0.001} F(t)\,dt \approx \frac{1}{0.001}(2.86) = 2860$ lb.

2. (a) $W = \int_{s_0}^{s_1} F(s)\,ds$, where $F(s) = m\frac{dv}{dt} = m\frac{dv}{ds}\frac{ds}{dt} = mv\frac{dv}{ds}$ and so, by the Substitution Rule,

$$W = \int_{s_0}^{s_1} F(s)\,ds = \int_{s_0}^{s_1} mv\frac{dv}{ds}\,ds = \int_{v(s_0)}^{v(s_1)} mv\,dv = \left[\tfrac{1}{2}mv^2\right]_{v_0}^{v_1} = \tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_0^2$$

(b) From part (b)(i), 90 mi/h = 132 ft/s. Assume $v_0 = v(s_0) = 0$ and $v_1 = v(s_1) = 132$ ft/s (note that s_1 is the point of release of the baseball). $m = \frac{5}{512}$, so the work done is

$$W = \tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_0^2 = \tfrac{1}{2} \cdot \tfrac{5}{512} \cdot (132)^2 \approx 85 \text{ ft-lb}$$

3. (a) Here we have a differential equation of the form $dv/dt = kv$, so by Theorem 10.4.2, the solution is $v(t) = v(0)\,e^{kt}$. In this case $k = -\frac{1}{10}$ and $v(0) = 100$ ft/s, so $v(t) = 100e^{-t/10}$. We are interested in the time t that the ball takes to travel 280 ft, so we find the distance function

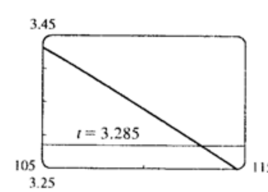
$$\begin{aligned} s(t) &= \textstyle\int_0^t v(x)\,dx = \int_0^t 100e^{-x/10}\,dx = 100\left[-10e^{-x/10}\right]_0^t = -1000\left(e^{-t/10} - 1\right) \\ &= 1000\left(1 - e^{-t/10}\right) \end{aligned}$$

Now we set $s(t) = 280$ and solve for t: $280 = 1000\left(1 - e^{-t/10}\right) \;\Rightarrow\; 1 - e^{-t/10} = \frac{7}{25} \;\Rightarrow$ $-\frac{1}{10}t = \ln\left(1 - \frac{7}{25}\right) \;\Rightarrow\; t \approx 3.285$ seconds.

(b) Let x be the distance of the shortstop from home plate. We calculate the time for the ball to reach home plate as a function of x, then differentiate with respect to x to find the value of x which corresponds to the minimum time. The total time that it takes the ball to reach home is the sum of the times of the two throws, plus the relay time ($\frac{1}{2}$ s). The distance from the fielder to the shortstop is $280 - x$, so to find the time t_1 taken by the first throw, we solve the equation $s_1(t_1) = 280 - x \;\Leftrightarrow\; 1 - e^{-t_1/10} = \dfrac{280 - x}{1000} \;\Leftrightarrow\; t_1 = -10\ln\dfrac{720 + x}{1000}$. We find the time t_2 taken by the second throw if the shortstop throws with velocity w, since we see that this velocity varies in the rest of the problem. We use $v = we^{-t/10}$ and isolate t_2 in the equation $s(t_2) = 10w\left(1 - e^{-t_2/10}\right) = x \;\Leftrightarrow\; e^{-t_2/10} = 1 - \dfrac{x}{10w} \;\Leftrightarrow\; t_2 = -10\ln\dfrac{10w - x}{10w}$, so the total time is $t_w(x) = \dfrac{1}{2} - 10\left[\ln\dfrac{720 + x}{1000} + \ln\dfrac{10w - x}{10w}\right]$. To find the minimum, we differentiate:
$\dfrac{dt_w}{dx} = -10\left[\dfrac{1}{720 + x} - \dfrac{1}{10w - x}\right]$, which changes from negative to positive when $720 + x = 10w - x \;\Leftrightarrow$ $x = 5w - 360$. By the First Derivative Test, t_w has a minimum at this distance from the shortstop to home plate. So if the shortstop throws at $w = 105$ ft/s from a point $x = 5(105) - 360 = 165$ ft from home plate, the minimum time is $t_{105}(165) = \frac{1}{2} - 10\left(\ln\frac{720 + 165}{1000} + \ln\frac{1050 - 165}{1050}\right) \approx 3.431$ seconds. This is longer than the time taken in part (a), so in this case the manager should encourage a direct throw.
If $w = 115$ ft/s, then $x = 215$ ft from home, and the minimum time is
$t_{115}(215) = \frac{1}{2} - 10\left(\ln\frac{720 + 215}{1000} + \ln\frac{1150 - 215}{1150}\right) \approx 3.242$ seconds. This is less than the time taken in part (a), so in this case, the manager should encourage a relayed throw.

(c) In general, the minimum time is

$$\begin{aligned} t_w(5w - 360) &= \frac{1}{2} - 10\left[\ln\frac{360 + 5w}{1000} + \ln\frac{360 + 5w}{10w}\right] \\ &= \frac{1}{2} - 10\ln\frac{(w + 72)^2}{400w} \end{aligned}$$



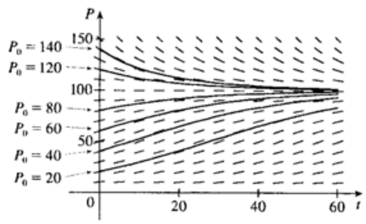
We want to find out when this is about 3.285 seconds, the same time as the direct throw. From the graph, we estimate that this is the case for $w \approx 112.8$ ft/s. So if the shortstop can throw the ball with this velocity, then a relayed throw takes the same time as a direct throw.

10.5 The Logistic Equation

1. (a) $dP/dt = 0.05P - 0.0005P^2 = 0.05P\,(1 - 0.01P) = 0.05P\,(1 - P/100)$. Comparing to Equation 1, $dP/dt = kP\,(1 - P/K)$, we see that the carrying capacity is $K = 100$ and the value of k is 0.05.

(b) The slopes close to 0 occur where P is near 100. The largest slopes appear to be on the line $P = 50$. The solutions are increasing for $0 < P_0 < 100$ and decreasing for $P_0 > 100$.

(c)

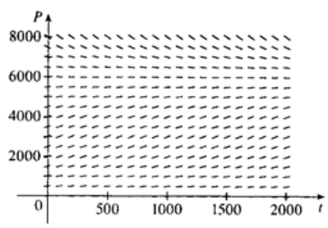


All of the solutions approach $P = 100$ as t increases. As in part (b), the solutions differ since for $0 < P_0 < 100$ they are increasing, and for $P_0 > 100$ they are decreasing. Also, some have an IP and some don't. It appears that the solutions which have $P_0 = 20$ and $P_0 = 40$ have inflection points at $P = 50$.

(d) The equilibrium solutions are $P = 0$ (trivial solution) and $P = 100$. The increasing solutions move away from $P = 0$ and all nonzero solutions approach $P = 100$ as $t \to \infty$.

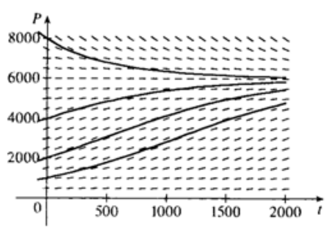
2. (a) $K = 6000$ and $k = 0.0015 \quad \Rightarrow \quad dP/dt = 0.0015P\,(1 - P/6000)$.

(b)



All of the solution curves approach 6000 as $t \to \infty$.

(c)

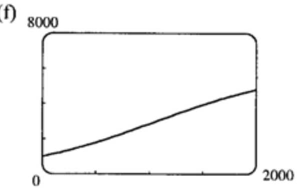


The curves with $P_0 = 1000$ and $P_0 = 2000$ appear to be concave upward at first and then concave downward. The curve with $P_0 = 4000$ appears to be concave downward everywhere. The curve with $P_0 = 8000$ appears to be concave upward everywhere. The inflection points are where the population grows the fastest.

(d) See the solution to Exercise 10.2.25 for a possible program to calculate $P\,(50)$. [In this case, we use $\mathrm{X} = 0$, $\mathrm{H} = 1$, $\mathrm{N} = 50$, $\mathrm{Y}_1 = 0.0015y\,(1 - y/6000)$, and $\mathrm{Y} = 1000$.] We find that $P\,(50) \approx 1064$.

(e) Using Equation 4 with $K = 6000$, $k = 0.0015$, and $P_0 = 1000$, we have $P\,(t) = \dfrac{K}{1 + Ae^{-kt}} = \dfrac{6000}{1 + Ae^{-0.0015t}}$, where $A = \dfrac{K - P_0}{P_0} = \dfrac{6000 - 1000}{1000} = 5$. Thus, $P\,(50) = \dfrac{6000}{1 + 5e^{-0.0015(50)}} \approx 1064.1$, which is extremely close to the estimate obtained in part (d).

(f)

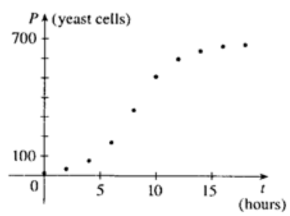


The curves are very similar.

3. (a) $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{K}\right) \Rightarrow y(t) = \dfrac{K}{1 + Ae^{-kt}}$ with $A = \dfrac{K - y(0)}{y(0)}$. With $K = 8 \times 10^7$, $k = 0.71$, and $y(0) = 2 \times 10^7$, we get the model $y(t) = \dfrac{8 \times 10^7}{1 + 3e^{-0.71t}}$, so $y(1) = \dfrac{8 \times 10^7}{1 + 3e^{-0.71}} \approx 3.23 \times 10^7$ kg.

(b) $y(t) = 4 \times 10^7 \Rightarrow \dfrac{8 \times 10^7}{1 + 3e^{-0.71t}} = 4 \times 10^7 \Rightarrow 2 = 1 + 3e^{-0.71t} \Rightarrow e^{-0.71t} = \frac{1}{3} \Rightarrow$
$-0.71t = \ln\frac{1}{3} \Rightarrow t = \dfrac{\ln 3}{0.71} \approx 1.55$ years

4. (a)



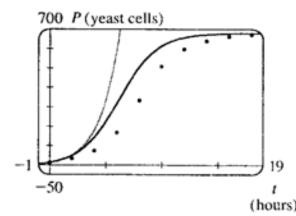
From the graph, we estimate the carrying capacity K for the yeast population to be 680.

(b) An estimate of the initial relative growth rate is
$\dfrac{1}{P_0}\dfrac{dP}{dt} = \dfrac{1}{18} \cdot \dfrac{39 - 18}{2 - 0} = \dfrac{7}{12} = 0.58\overline{3}$.

(c) An exponential model is $P(t) = 18e^{7t/12}$. A logistic model is $P(t) = \dfrac{680}{1 + Ae^{-7t/12}}$, where $A = \frac{680 - 18}{18} = \frac{331}{9}$.

(d)

Time in Hours	Observed Values	Exponential Model	Logistic Model
0	18	18	18
2	39	58	55
4	80	186	149
6	171	596	322
8	336	1914	505
10	509	6147	614
12	597	19,739	658
14	640	63,389	673
16	664	203,558	678
18	672	653,679	679



The exponential model is a poor fit for anything beyond the first two observed values.
The logistic model varies more for the middle values than it does for the values at either end, but provides a good general fit, as shown in the figure.

(e) $P(7) = \dfrac{680}{1 + \frac{331}{9}e^{-7(7/12)}} \approx 420$ yeast cells

5. (a) We will assume that the difference in the birth and death rates is 20 million/y. Let $t = 0$ correspond to the year 1990 and use a unit of 1 million for all calculations. $k \approx \frac{1}{P}\frac{dP}{dt} = \frac{1}{5300}(20) = \frac{1}{265}$, so

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) = \frac{1}{265}P\left(1 - \frac{P}{100{,}000}\right)$$

(b) $A = \frac{K - P_0}{P_0} = \frac{100{,}000 - 5300}{5300} = \frac{947}{53} \approx 17.8679$. $P(t) = \frac{K}{1 + Ae^{-kt}} = \frac{100{,}000}{1 + \frac{947}{53}e^{-(1/265)t}}$, so $P(10) \approx 5492.6$ (or 5.5 billion), $P(110) \approx 7813.8$, and $P(510) \approx 27{,}718.3$.

(c) If $K = 50{,}000$, then $P(t) = \frac{50{,}000}{1 + \frac{447}{53}e^{-(1/265)t}}$. So $P(10) \approx 5481.5$, $P(110) \approx 7611.8$, and $P(510) \approx 22{,}412.6$

6. (a) If we assume that the carrying capacity for the world population is 100 billion, it would seem reasonable that the carrying capacity for the U.S. is 3–5 billion by using current populations and simple proportions. We will use $K = 4$ billion or 4000 million. With $t = 0$ corresponding to 1980, we have

$$P(t) = \frac{4000}{1 + \left(\frac{4000 - 228}{228}\right)e^{-kt}} = \frac{4000}{1 + \frac{943}{57}e^{-kt}}$$

(b) $P(10) = 250 \;\Rightarrow\; \frac{4000}{1 + \frac{943}{57}e^{-10k}} = 250 \;\Rightarrow\; 1 + \frac{943}{57}e^{-10k} = 16 \;\Rightarrow\; e^{-10k} = \frac{855}{943} \;\Rightarrow\; -10k = \ln\frac{855}{943} \;\Rightarrow\; k = -\frac{1}{10}\ln\frac{855}{943} \approx 0.0097965$.

(c) $2100 - 1980 = 120$ and $P(120) \approx 655$ million.
$2200 - 1980 = 220$ and $P(220) \approx 1371$ million, or about 1.4 billion.

(d) $P(t) = 300 \;\Rightarrow\; \frac{4000}{1 + \frac{943}{57}e^{-kt}} = 300 \;\Rightarrow\; 1 + \frac{943}{57}e^{-kt} = \frac{40}{3} \;\Rightarrow\; e^{-kt} = \frac{37}{3}\cdot\frac{57}{943} \;\Rightarrow\; -kt = \ln\frac{703}{943}$ $\Rightarrow\; t = 10\frac{\ln\frac{703}{943}}{\ln\frac{855}{943}} \approx 29.98 \approx 30$. So we predict that the U.S. population will exceed 300 million in the year $1980 + 30 = 2010$.

7. (a) Our assumption is that $\frac{dy}{dt} = ky(1 - y)$, where y is the fraction of the population that has heard the rumor.

(b) Using the logistic equation (1), $\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$, we substitute $y = \frac{P}{K}$, $P = Ky$, and $\frac{dP}{dt} = K\frac{dy}{dt}$, to obtain $K\frac{dy}{dt} = k(Ky)(1 - y) \;\Leftrightarrow\; \frac{dy}{dt} = ky(1 - y)$, our equation in part (a). Now the solution to (1) is $P(t) = \frac{K}{1 + Ae^{-kt}}$, where $A = \frac{K - P_0}{P_0}$. We use the same substitution to obtain $Ky = \frac{K}{1 + \frac{K - Ky_0}{Ky_0}e^{-kt}}$ $\Rightarrow\; y = \frac{y_0}{y_0 + (1 - y_0)e^{-kt}}$.
Alternatively, we could use the same steps as outlined in "The Analytic Solution", following Example 2.

(c) Let t be the number of hours since 8 A.M. Then $y_0 = y(0) = \frac{80}{1000} = 0.08$ and $y(4) = \frac{1}{2}$, so

$\frac{1}{2} = y(4) = \dfrac{0.08}{0.08 + 0.92e^{-4k}}$. Thus, $0.08 + 0.92e^{-4k} = 0.16$, $e^{-4k} = \frac{0.08}{0.92} = \frac{2}{23}$, and $e^{-k} = \left(\frac{2}{23}\right)^{1/4}$, so

$y = \dfrac{0.08}{0.08 + 0.92\,(2/23)^{t/4}} = \dfrac{2}{2 + 23\,(2/23)^{t/4}}$. Solving this equation for t, we get

$2y + 23y\left(\dfrac{2}{23}\right)^{t/4} = \dfrac{2-2y}{23y} \quad\Rightarrow\quad \left(\dfrac{2}{23}\right)^{t/4} = \dfrac{2}{23}\cdot\dfrac{1-y}{y} \quad\Rightarrow\quad \left(\dfrac{2}{23}\right)^{t/4-1} = \dfrac{1-y}{y}$. It follows that

$\dfrac{t}{4} - 1 = \dfrac{\ln[(1-y)/y]}{\ln\frac{2}{23}}$, so $t = 4\left[1 + \dfrac{\ln((1-y)/y)}{\ln\frac{2}{23}}\right]$. When $y = 0.9$, $\dfrac{1-y}{y} = \frac{1}{9}$, so

$t = 4\left(1 - \dfrac{\ln 9}{\ln\frac{2}{23}}\right) \approx 7.6$ h or 7 h 36 min. Thus, 90% of the population will have heard the rumor by 3:36 P.M.

8. (a) $P(0) = P_0 = 400$, $P(1) = 1200$ and $K = 10{,}000$. From the solution to the logistic differential equation

$P(t) = \dfrac{P_0K}{P_0 + (K - P_0)\,e^{-kt}}$, we get $P = \dfrac{400\,(10{,}000)}{400 + (9600)\,e^{-kt}} = \dfrac{10{,}000}{1 + 24e^{-kt}}$. $P(1) = 1200 \quad\Rightarrow$

$1 + 24e^{-k} = \frac{100}{12} \quad\Rightarrow\quad e^k = \frac{288}{88} \quad\Rightarrow\quad k = \ln\frac{36}{11}$. So $P = \dfrac{10{,}000}{1 + 24e^{-t\ln(36/11)}} = \dfrac{10{,}000}{1 + 24\cdot(11/36)^t}$.

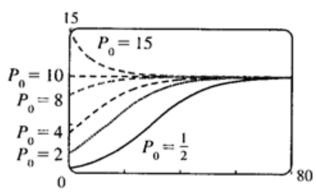
(b) $5000 = \dfrac{10{,}000}{1 + 24\,(11/36)^t} \quad\Rightarrow\quad 24\left(\frac{11}{36}\right)^t = 1 \quad\Rightarrow\quad t\ln\frac{11}{36} = \ln\frac{1}{24} \quad\Rightarrow\quad t \approx 2.68$ years.

9. (a) $\dfrac{dP}{dt} = k(P)\left(1 - \dfrac{P}{K}\right) \quad\Rightarrow$

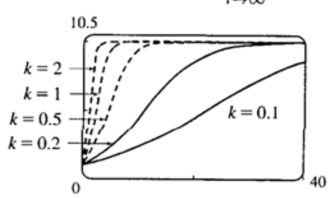
$$\begin{aligned}\frac{d^2P}{dt^2} &= k\left[P\left(-\frac{1}{K}\frac{dP}{dt}\right) + \left(1 - \frac{P}{K}\right)\frac{dP}{dt}\right] = k\frac{dP}{dt}\left(-\frac{P}{K} + 1 - \frac{P}{K}\right)\\ &= k\left[kP\left(1 - \frac{P}{K}\right)\right]\left[1 - \frac{2P}{K}\right] = k^2P\left(1 - \frac{P}{K}\right)\left(1 - \frac{2P}{K}\right)\end{aligned}$$

(b) P grows fastest when P' has a maximum, that is, when $P'' = 0$. From part (a), $P'' = 0 \;\Leftrightarrow\; P = 0$, $P = K$, or $P = K/2$. Since $0 < P < K$, we see that $P'' = 0 \;\Leftrightarrow\; P = K/2$.

10.



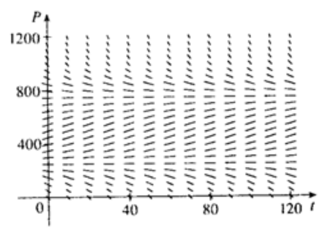
First we keep k constant (at 0.1, say) and change P_0 in the function $P = \dfrac{10P_0}{P_0 + (10 - P_0)\,e^{-0.1t}}$. (Notice that P_0 is the P-intercept.) If $P_0 = 0$, the function is 0 everywhere. For $0 < P_0 < 5$, the curve has an inflection point, which moves to the right as P_0 decreases. If $5 < P_0 < 10$, the graph is concave down everywhere. (We are considering only $t \geq 0$.) If $P_0 = 10$, the function is the constant function $P = 10$, and if $P_0 > 10$, the function decreases. For all $P_0 \neq 0$, $\lim_{t\to\infty} P = 10$.



Now we instead keep P_0 constant (at $P_0 = 1$) and change k in the function $P = \dfrac{10}{1 + 9e^{-kt}}$. It seems that as k increases, the graph approaches the line $P = 10$ more and more quickly. (Note that the only difference in the shape of the curves is in the horizontal scaling; if we choose suitable x-scales, the graphs all look the same.)

11. (a) The term -15 represents a harvesting of fish at a constant rate — in this case, 15 fish/week. This is the rate at which fish are caught.

(b)



(c) From the graph in part (b), it appears that $P(t) = 250$ and $P(t) = 750$ are the equilibrium solutions. We confirm this analytically by solving the equation $dP/dt = 0$ as follows:

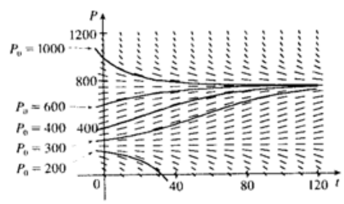
$0.08P\,(1 - P/1000) - 15 = 0 \quad \Rightarrow$

$0.08P - 0.00008P^2 - 15 = 0 \quad \Rightarrow$

$-0.00008\left(P^2 - 1000P + 187{,}500\right) = 0 \quad \Rightarrow$

$(P - 250)(P - 750) = 0 \quad \Rightarrow \quad P = 250$ or 750.

(d)



For $0 < P_0 < 250$, $P(t)$ decreases to 0. For $P_0 = 250$, $P(t)$ remains constant. For $250 < P_0 < 750$, $P(t)$ increases and approaches 750. For $P_0 = 750$, $P(t)$ remains constant. For $P_0 > 750$, $P(t)$ decreases and approaches 750.

(e) $\dfrac{dP}{dt} = 0.08P\left(1 - \dfrac{P}{1000}\right) - 15 \quad \Leftrightarrow \quad -\dfrac{100{,}000}{8} \cdot \dfrac{dP}{dt} = \left(0.08P - 0.00008P^2 - 15\right) \cdot \left(-\dfrac{100{,}000}{8}\right) \quad \Leftrightarrow$

$-12{,}500\dfrac{dP}{dt} = P^2 - 1000P + 187{,}500 \quad \Leftrightarrow \quad \dfrac{dP}{(P-250)(P-750)} = -\frac{1}{12{,}500}\,dt \quad \Leftrightarrow$

$\displaystyle\int \left(\frac{-1/500}{P-250} + \frac{1/500}{P-750}\right) dP = -\tfrac{1}{12{,}500}\,dt \quad \Leftrightarrow \quad \int \left(\frac{1}{P-250} - \frac{1}{P-750}\right) dP = \tfrac{1}{25}\,dt \quad \Leftrightarrow$

$\ln|P - 250| - \ln|P - 750| = \frac{1}{25}t + C \quad \Leftrightarrow \quad \ln\left|\dfrac{P-250}{P-750}\right| = \frac{1}{25}t + C \quad \Leftrightarrow \quad \left|\dfrac{P-250}{P-750}\right| = e^{t/25 + C} = ke^{t/25}$

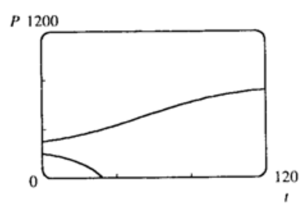
$\Leftrightarrow \quad \dfrac{P-250}{P-750} = ke^{t/25} \quad \Leftrightarrow \quad P - 250 = Pke^{t/25} - 750ke^{t/25} \quad \Leftrightarrow \quad P - Pke^{t/25} = 250 - 750ke^{t/25} \quad \Leftrightarrow$

$P(t) = \dfrac{250 - 750ke^{t/25}}{1 - ke^{t/25}}$. If $t = 0$ and $P = 200$, then $200 = \dfrac{250 - 750k}{1-k}$

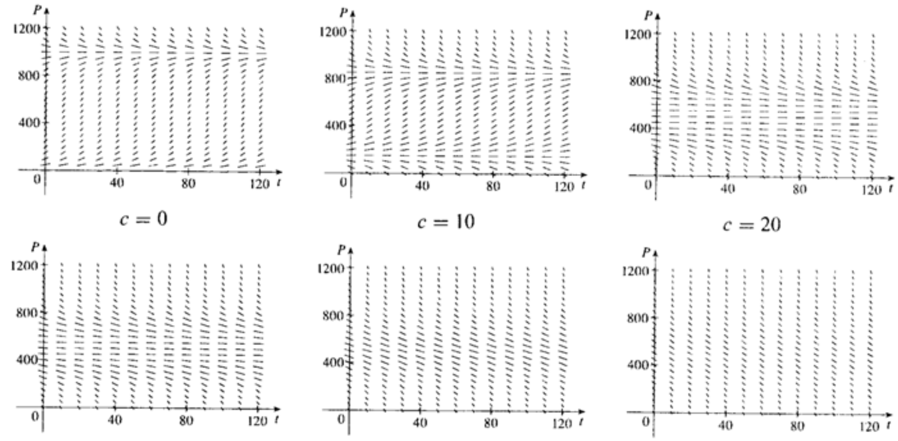
$\Leftrightarrow \quad 200 - 200k = 250 - 750k \quad \Leftrightarrow \quad 550k = 50 \quad \Leftrightarrow \quad k = \frac{1}{11}$.

Similarly, if $t = 0$ and $P = 300$, then $k = -\frac{1}{9}$. Simplifying P with these two values of k gives us

$P(t) = \dfrac{250\left(3e^{t/25} - 11\right)}{e^{t/25} - 11}$ and $P(t) = \dfrac{750\left(e^{t/25} + 3\right)}{e^{t/25} + 9}$.



12. (a)



$c = 0$ $c = 10$ $c = 20$ $c = 21$ $c = 25$ $c = 30$

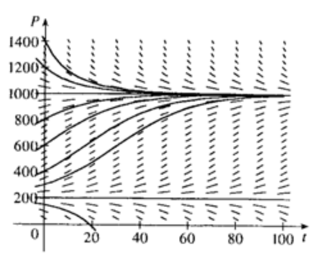
(b) For $0 \le c \le 20$, there is at least one equilibrium solution. For $c > 20$, the population always dies out.

(c) $\dfrac{dP}{dt} = 0.08P - 0.00008P^2 - c$. $\dfrac{dP}{dt} = 0 \iff P = \dfrac{-0.08 \pm \sqrt{(0.08)^2 - 4(-0.00008)(-c)}}{2(-0.00008)}$, which has at least one solution when the discriminant is nonnegative $\Rightarrow$ $0.0064 - 0.00032c \ge 0 \iff c \le 20$. For $0 \le c \le 20$, there is at least one value of P such that $dP/dt = 0$ and hence, at least one equilibrium solution. For $c > 20$, $dP/dt < 0$ and the population always dies out.

(d) The weekly catch should be less than 20 fish per week.

13. (a) $\dfrac{dP}{dt} = (kP)\left(1 - \dfrac{P}{K}\right)\left(1 - \dfrac{m}{P}\right)$. If $m < P < K$, then $dP/dt = (+)(+)(+) = + \Rightarrow P$ is increasing. If $0 < P < m$, then $dP/dt = (+)(+)(-) = - \Rightarrow P$ is decreasing.

(b)



$k = 0.08$, $K = 1000$, and $m = 200 \Rightarrow$

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)\left(1 - \frac{200}{P}\right)$$

For $0 < P_0 < 200$, the population dies out. For $P_0 = 200$, the population is steady. For $200 < P_0 < 1000$, the population increases and approaches 1000. For $P_0 > 1000$, the population decreases and approaches 1000.

The equilibrium solutions are $P(t) = 200$ and $P(t) = 1000$.

(c) $\dfrac{dP}{dt} = kP\left(1-\dfrac{P}{K}\right)\left(1-\dfrac{m}{P}\right) = kP\left(\dfrac{K-P}{K}\right)\left(\dfrac{P-m}{P}\right) = \dfrac{k}{K}(K-P)(P-m) \quad\Leftrightarrow$

$\displaystyle\int \frac{dP}{(K-P)(P-m)} = \int \frac{k}{K}\,dt.$

By partial fractions, $\dfrac{1}{(K-P)(P-m)} = \dfrac{A}{K-P} + \dfrac{B}{P-m}$, so $A(P-m) + B(K-P) = 1$.

If $P = m$, $B = \dfrac{1}{K-m}$; if $P = K$, $A = \dfrac{1}{K-m}$, so $\dfrac{1}{K-m}\displaystyle\int\left(\frac{1}{K-P} + \frac{1}{P-m}\right)dP = \int \frac{k}{K}\,dt \quad\Rightarrow$

$\dfrac{1}{K-m}\left(-\ln|K-P| + \ln|P-m|\right) = \dfrac{k}{K}t + M.$

But $m < P < K$, so $\dfrac{1}{K-m}\ln\dfrac{P-m}{K-P} = \dfrac{k}{K}t + M \quad\Rightarrow\quad \ln\dfrac{P-m}{K-P} = (K-m)\dfrac{k}{K}t + M_1 \quad\Leftrightarrow$

$\dfrac{P-m}{K-P} = De^{(K-m)(k/K)t} \quad (D = e^{M_1})$. Let $t = 0$: $\dfrac{P_0-m}{K-P_0} = D$. So $\dfrac{P-m}{K-P} = \dfrac{P_0-m}{K-P_0}e^{(K-m)(k/K)t}$.

Solving for P, we get $P(t) = \dfrac{m(K-P_0) + K(P_0-m)e^{(K-m)(k/K)t}}{K-P_0+(P_0-m)e^{(K-m)(k/K)t}}$.

(d) If $P_0 < m$, then $P_0 - m < 0$. Let $N(t)$ be the numerator of the expression for $P(t)$ in part (c). Then $N(0) = P_0(K-m) > 0$, and $P_0 - m < 0 \quad\Leftrightarrow\quad \lim\limits_{t\to\infty} K(P_0-m)e^{(K-m)(k/K)t} = -\infty \quad\Rightarrow$ $\lim\limits_{t\to\infty} N(t) = -\infty$. Since N is continuous, there is a number t such that $N(t) = 0$ and thus $P(t) = 0$. So the species will become extinct.

14. (a) $\dfrac{dP}{dt} = c\ln\left(\dfrac{K}{P}\right)P \quad\Rightarrow\quad \displaystyle\int \frac{dP}{P\ln(K/P)} = \int c\,dt$. Let $u = \ln\left(\dfrac{K}{P}\right) = \ln K - \ln P \quad\Rightarrow\quad du = -\dfrac{dP}{P} \quad\Rightarrow$

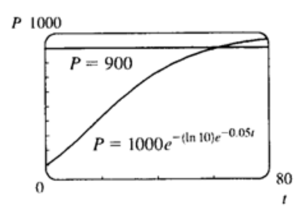
$\displaystyle\int -\frac{du}{u} = ct + D \quad\Rightarrow\quad \ln|u| = -ct - D \quad\Rightarrow\quad |u| = e^{-(ct+D)} \quad\Rightarrow\quad |\ln(K/P)| = e^{-(ct+D)} \quad\Rightarrow$

$\ln(K/P) = e^{-(ct+D)}$ [we know that $\ln(K/P) \geq 0$ since $K \geq P$]. Letting $t = 0$, we get $\ln(K/P_0) = e^{-D}$, so $\ln(K/P) = e^{-ct-D} = e^{-ct}e^{-D} = \ln(K/P_0)e^{-ct} \quad\Rightarrow\quad K/P = e^{\ln(K/P_0)e^{-ct}} \quad\Rightarrow$

$P(t) = Ke^{-\ln(K/P_0)e^{-ct}}$, $c \neq 0$.

(b) $\lim\limits_{t\to\infty} P(t) = \lim\limits_{t\to\infty} Ke^{-\ln(K/P_0)e^{-ct}} = Ke^{-\ln(K/P_0)\cdot 0} = Ke^0 = K$

(c)



The graphs look very similar. For the Gompertz function, $P(40) \approx 732$, nearly the same as the logistic function. The Gompertz function reaches $P = 900$ at $t \approx 61.7$ and its value at $t = 80$ is about 959, so it doesn't increase quite as fast as the logistic curve.

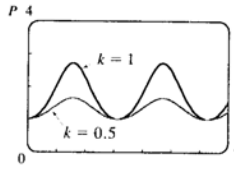
(d) $\dfrac{dP}{dt} = c\ln\left(\dfrac{K}{P}\right)P = cP(\ln K - \ln P) \quad\Rightarrow$

$$\begin{aligned}\frac{d^2P}{dt^2} &= c\left[P\left(-\frac{1}{P}\frac{dP}{dt}\right) + (\ln K - \ln P)\frac{dP}{dt}\right] = c\frac{dP}{dt}\left[-1 + \ln\left(\frac{K}{P}\right)\right] \\ &= c\left[c\ln(K/P)\,P\right]\left[\ln(K/P) - 1\right] = c^2P\ln(K/P)\left[\ln(K/P) - 1\right]\end{aligned}$$

Since $0 < P < K$, $P'' = 0 \quad\Leftrightarrow\quad \ln(K/P) = 1 \quad\Leftrightarrow\quad K/P = e \quad\Leftrightarrow\quad P = K/e$. $P'' > 0$ for $0 < P < K/e$ and $P'' < 0$ for $K/e < P < K$, so P' is a maximum (and P grows fastest) when $P = K/e$.

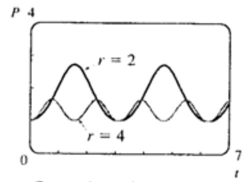
15. (a) $dP/dt = kP\cos(rt-\phi) \Rightarrow (dP)/P = k\cos(rt-\phi)\,dt \Rightarrow \int (dP)/P = k\int \cos(rt-\phi)\,dt \Rightarrow$ $\ln P = (k/r)\sin(rt-\phi) + C$. (Since this is a growth model, $P > 0$ and we can write $\ln P$ instead of $\ln|P|$.) Since $P(0) = P_0$, we obtain $\ln P_0 = (k/r)\sin(-\phi) + C = -(k/r)\sin\phi + C \Rightarrow C = \ln P_0 + (k/r)\sin\phi$. Thus, $\ln P = (k/r)\sin(rt-\phi) + \ln P_0 + (k/r)\sin\phi$, which we can rewrite as $\ln(P/P_0) = (k/r)[\sin(rt-\phi) + \sin\phi]$ or, after exponentiation, $P(t) = P_0 e^{(k/r)[\sin(rt-\phi)+\sin\phi]}$.

(b) As k increases, the amplitude increases, but the minimum value stays the same.



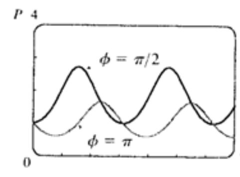
Comparing values of k with $P_0 = 1$, $r = 2$, and $\phi = \pi/2$

As r increases, the amplitude and the period decrease.



Comparing values of r with $P_0 = 1$, $k = 1$, and $\phi = \pi/2$

A change in ϕ produces slight adjustments in the phase shift and amplitude.



Comparing values of ϕ with $P_0 = 1$, $k = 1$, and $r = 2$

$P(t)$ oscillates between $P_0 e^{(k/r)(1+\sin\phi)}$ and $P_0 e^{(k/r)(-1+\sin\phi)}$ (the extreme values are attained when $rt - \phi$ is an odd multiple of $\frac{\pi}{2}$), so $\lim\limits_{t\to\infty} P(t)$ does not exist.

16. (a) $dP/dt = kP\cos^2(rt-\phi) \Rightarrow (dP)/P = k\cos^2(rt-\phi)\,dt \Rightarrow \int (dP)/P = k\int \cos^2(rt-\phi)\,dt$

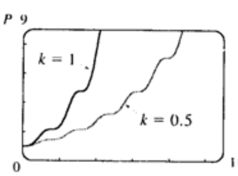
$\Rightarrow \ln P = k\int \dfrac{1+\cos(2(rt-\phi))}{2}\,dt = \dfrac{k}{2}t + \dfrac{k}{4r}\sin(2(rt-\phi)) + C$. From $P(0) = P_0$, we get

$\ln P_0 = \dfrac{k}{4r}\sin(-2\phi) + C = C - \dfrac{k}{4r}\sin 2\phi$, so $C = \ln P_0 + \dfrac{k}{4r}\sin 2\phi$ and

$\ln P = \dfrac{k}{2}t + \dfrac{k}{4r}\sin(2(rt-\phi)) + \ln P_0 + \dfrac{k}{4r}\sin 2\phi$. Simplifying, we get

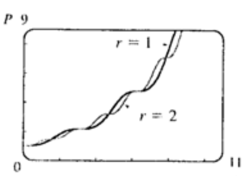
$\ln\dfrac{P}{P_0} = \dfrac{k}{2}t + \dfrac{k}{4r}[\sin(2(rt-\phi)) + \sin 2\phi] = f(t)$, or $P(t) = P_0 e^{f(t)}$.

(b) An increase in k stretches the graph of P vertically while maintaining $P(0) = P_0$.



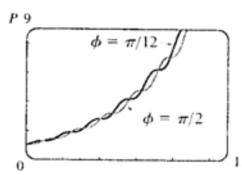
Comparing values of k with $P_0 = 1$, $r = 2$, and $\phi = \pi/2$

An increase in r compresses the graph of P horizontally — similar to changing the period in Exercise 15.



Comparing values of r with $P_0 = 1$, $k = 0.5$, and $\phi = \pi/2$

As in Exercise 15, a change in ϕ only makes slight adjustments in the growth of P, as shown in the figure.



Comparing values of ϕ with $P_0 = 1$, $k = 0.5$, and $r = 2$

$f'(t) = k/2 + [k/(4r)][2r\cos(2(rt-\phi))] = (k/2)[1+\cos(2(rt-\phi))] \ge 0$. Since $P(t) = P_0 e^{f(t)}$, we have $P'(t) = P_0 f'(t)\,e^{f(t)} \ge 0$, with equality only when $\cos(2(rt-\phi)) = -1$, that is, when $rt - \phi$ is an odd multiple of $\frac{\pi}{2}$. Therefore, $P(t)$ is an increasing function on $(0,\infty)$. P can also be written as $P(t) = P_0 e^{kt/2} e^{(k/4r)[\sin(2(rt-\phi))+\sin 2\phi]}$. The second exponential oscillates between $e^{(k/4r)(1+\sin 2\phi)}$ and $e^{(k/4r)(-1+\sin 2\phi)}$, while the first one, $e^{kt/2}$, grows without bound. So $\lim\limits_{t\to\infty} P(t) = \infty$.

10.6 Linear Equations

1. $y' + e^x y = x^2 y^2$ is not linear since it cannot be put into the standard linear form (1).

2. $y + \sin x = x^3 y' \quad \Rightarrow \quad x^3 y' - y = \sin x \quad \Rightarrow \quad y' + \left(-\dfrac{1}{x^3}\right) y = \dfrac{\sin x}{x^3}$. This equation is in the standard linear form (1), so it is linear.

3. $xy' + \ln x - x^2 y = 0 \quad \Rightarrow \quad xy' - x^2 y = -\ln x \quad \Rightarrow \quad y' + (-x)\, y = -\dfrac{\ln x}{x}$, which is in the standard linear form (1), so this equation is linear.

4. $yy' = \sin x$ is not linear since it cannot be put into the standard linear form (1).

5. Comparing the given equation, $y' + 2y = 2e^x$, with the general form, $y' + P(x)\, y = Q(x)$, we see that $P(x) = 2$ and the integrating factor is $I(x) = e^{\int P(x)dx} = e^{\int 2\,dx} = e^{2x}$. Multiplying the differential equation by $I(x)$ gives $e^{2x} y' + 2e^{2x} y = 2e^{3x} \quad \Rightarrow \quad (e^{2x} y)' = 2e^{3x} \quad \Rightarrow \quad e^{2x} y = \int 2e^{3x}\, dx \quad \Rightarrow \quad e^{2x} y = \frac{2}{3} e^{3x} + C \quad \Rightarrow$
$y = \frac{2}{3} e^x + Ce^{-2x}$.

6. $y' = x + 5y \quad \Rightarrow \quad y' - 5y = x$. $I(x) = e^{\int P(x)dx} = e^{\int (-5)dx} = e^{-5x}$. Multiplying the differential equation by $I(x)$ gives $e^{-5x} y' - 5e^{-5x} y = xe^{-5x} \quad \Rightarrow \quad (e^{-5x} y)' = xe^{-5x} \quad \Rightarrow$
$e^{-5x} y = \int xe^{-5x}\, dx \overset{96}{=} -\frac{1}{5} xe^{-5x} - \frac{1}{25} e^{-5x} + C \quad \Rightarrow \quad y = -\frac{1}{5} x - \frac{1}{25} + Ce^{5x}$

7. $I(x) = e^{\int (-2x)dx} = e^{-x^2}$. Multiplying the differential equation by $I(x)$ gives $e^{-x^2}(y' - 2xy) = xe^{-x^2} \quad \Rightarrow$
$\left(e^{-x^2} y\right)' = xe^{-x^2} \quad \Rightarrow \quad y = e^{x^2}\left(\int xe^{-x^2}\, dx + C\right) = Ce^{x^2} - \frac{1}{2}$.

8. $y' + \dfrac{2}{x} y = \dfrac{1}{x} e^{x^2}$ $(x \neq 0)$, so $I(x) = e^{\int 2x^{-1}\, dx} = e^{\ln x^2} = x^2$. Multiplying the differential equation by $I(x)$ gives $x^2 y' + 2xy = xe^{x^2} \quad \Rightarrow \quad (x^2 y)' = xe^{x^2}$
$\Rightarrow \quad y = x^{-2}\left[\int \left(xe^{x^2}\right) dx + C_1\right] = x^{-2}\left[\left(\frac{1}{2} e^{x^2}\right) + C_1\right] = \dfrac{e^{x^2} + C}{2x^2}$.

9. $y' - y \tan x = \dfrac{\sin 2x}{\cos x} = 2 \sin x$, so $I(x) = e^{\int -\tan x\, dx} = e^{\ln|\cos x|} = \cos x$ (since $-\frac{\pi}{2} < x < \frac{\pi}{2}$). Multiplying the differential equation by $I(x)$ gives $(y' - y \tan x) \cos x = \dfrac{\sin 2x}{\cos x} \cos x \quad \Rightarrow \quad (y \cos x)' = \sin 2x \quad \Rightarrow$
$y \cos x = \int \sin 2x\, dx + C = -\frac{1}{2} \cos 2x + C = \frac{1}{2} - \cos^2 x + C \quad \Rightarrow \quad y = \dfrac{\sec x}{2} - \cos x + C \sec x$.

10. $y' - y = 1/x$ $(x \neq 0)$, so $I(x) = e^{\int (-1)dx} = e^{-x}$. Multiplying the differential equation by $I(x)$ gives $e^{-x} y' - e^{-x} y = e^{-x}/x \quad \Rightarrow \quad (e^{-x} y)' = e^{-x}/x \quad \Rightarrow \quad y = e^x \left[\int (e^{-x}/x)\, dx + C\right]$.

11. $I(x) = e^{\int 2x\, dx} = e^{x^2}$. Multiplying the differential equation by $I(x)$ gives $e^{x^2} y' + 2xe^{x^2} y = x^2 e^{x^2} \quad \Rightarrow \quad \left(e^{x^2} y\right)' = x^2 e^{x^2}$. Thus
$y = e^{-x^2}\left[\int x^2 e^{x^2} dx + C\right] = e^{-x^2}\left[\frac{1}{2} xe^{x^2} - \int \frac{1}{2} e^{x^2}\, dx + C\right] = \frac{1}{2} x + Ce^{-x^2} - e^{-x^2} \int \frac{1}{2} e^{x^2}\, dx$.

12. $I(x) = e^{\int -\tan x\,dx} = e^{\ln|\cos x|} = \cos x$ (since $-\frac{\pi}{2} < x < \frac{\pi}{2}$). Multiplying the differential equation by $I(x)$ gives $y'\cos x - y\tan x\cos x = x\cos x\sin 2x \;\Rightarrow\; (y\cos x)' = x\cos x\sin 2x$. So

$$y = \frac{1}{\cos x}\left[\int x\cos x\sin 2x\,dx + C\right] = \frac{1}{\cos x}\left[\int 2x\cos^2 x\sin x\,dx + C\right]$$
$$= \frac{1}{\cos x}\left[\frac{-2x\cos^3 x}{3} + \frac{2}{3}\left(\sin x - \frac{\sin^3 x}{3}\right) + C\right] = \frac{-2x\cos^2 x}{3} + \frac{C}{\cos x} + 2\tan x\,\frac{3-\sin^2 x}{9}$$

13. $(1+t)\dfrac{du}{dt} + u = 1+t,\ t>0 \;\Rightarrow\; \dfrac{d}{dt}[(1+t)u] = 1+t \;\Rightarrow\; (1+t)u = \int(1+t)\,dt = t + \frac{1}{2}t^2 + C \;\Rightarrow$ $u = \dfrac{t+\frac{1}{2}t^2+C}{1+t}$ or $u = \dfrac{t^2+2t+2C}{2(t+1)}$.

14. $y' + \left(1+\dfrac{1}{x}\right)y = \dfrac{e^{-x}}{x}$, so $I(x) = e^{\int(1+1/x)dx} = xe^x$. Multiplying the differential equation by $I(x)$ gives $xe^xy' + (xe^x+e^x)y = 1 \;\Rightarrow\; (xe^xy)' = 1 \;\Rightarrow\; xe^xy = \int 1\,dx \;\Rightarrow\; xe^xy = x + C \;\Rightarrow$ $y = e^{-x}(1 + C/x)$.

15. $I(x) = e^{\int dx} = e^x$. Multiplying the differential equation by $I(x)$ gives $e^xy' + e^xy = e^x(x+e^x) \;\Rightarrow$ $(e^xy)' = e^x(x+e^x)$. Thus $y = e^{-x}\left[\int e^x(x+e^x)\,dx + C\right] = e^{-x}\left[xe^x - e^x + \dfrac{e^{2x}}{2} + C\right] = x - 1 + \dfrac{e^x}{2} + \dfrac{C}{e^x}$. But $0 = y(0) = -1 + \frac{1}{2} + C$, so $C = \frac{1}{2}$, and the solution to the initial-value problem is $y = x - 1 + \frac{1}{2}e^x + \frac{1}{2}e^{-x} = x - 1 + \cosh x$.

16. $t\dfrac{dy}{dt} + 2y = t^3,\ t>0,\ y(1) = 0$. Divide by t to get $\dfrac{dy}{dt} + \dfrac{2}{t}y = t^2$, which is linear. $I(t) = e^{\int(2/t)dt} = e^{2\ln t} = t^2$. Multiplying by t^2 gives $t^2\dfrac{dy}{dt} + 2ty = t^4 \;\Rightarrow\; (t^2y)' = t^4 \;\Rightarrow\; t^2y = \frac{1}{5}t^5 + C \;\Rightarrow\; y = \dfrac{t^3}{5} + \dfrac{C}{t^2}$. $0 = y(1) = \frac{1}{5} + C \;\Rightarrow\; C = -\frac{1}{5}$, so $y = \dfrac{t^3}{5} - \dfrac{1}{5t^2}$.

17. $\dfrac{dv}{dt} - 2tv = 3t^2e^{t^2},\ v(0) = 5$. $I(t) = e^{\int(-2t)dt} = e^{-t^2}$. Multiply the differential equation by $I(t)$ to get $e^{-t^2}\dfrac{dv}{dt} - 2te^{-t^2}v = 3t^2 \;\Rightarrow\; \left(e^{-t^2}v\right)' = 3t^2 \;\Rightarrow\; e^{-t^2}v = \int 3t^2\,dt = t^3 + C \;\Rightarrow\; v = t^3e^{t^2} + Ce^{t^2}$. $5 = v(0) = 0\cdot 1 + C\cdot 1 = C$, so $v = t^3e^{t^2} + 5e^{t^2}$.

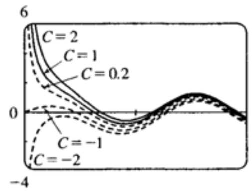
18. $y' + \left(\dfrac{2x}{1+x^2}\right)y = \dfrac{3\sqrt{x}}{1+x^2}$, so $I(x) = e^{\int\left[2x/\left(1+x^2\right)\right]dx} = e^{\ln\left(1+x^2\right)} = 1 + x^2$. Multiplying the differential equation by $I(x)$ gives $\left(1+x^2\right)y' + 2xy = 3\sqrt{x} \;\Rightarrow\; \left(\left(1+x^2\right)y\right)' = 3\sqrt{x} \;\Rightarrow$ $y = \left(1+x^2\right)^{-1}\left(\int 3\sqrt{x}\,dx + C\right) = \dfrac{2x^{3/2}+C}{1+x^2}$. But $2 = y(0) = C$, so the solution to the initial-value problem is $y = \dfrac{2x^{3/2}+2}{1+x^2}$.

19. $y' + 2\dfrac{y}{x} = \dfrac{\cos x}{x^2}$ $(x \neq 0)$, so $I(x) = e^{\int(2/x)dx} = x^2$. Multiplying the differential equation by $I(x)$ gives $x^2y' + 2xy = \cos x \;\Rightarrow\; \left(x^2y\right)' = \cos x \;\Rightarrow\; y = x^{-2}\left[\int\cos x\,dx + C\right] = x^{-2}(\sin x + C)$ $(x \neq 0)$. But $0 = y(\pi) = C$, so the solution to the initial-value problem is $y = (\sin x)/x^2$.

20. $y' - \dfrac{y}{x(x+1)} = 1$ $(x > 0)$, so $I(x) = e^{-\int 1/[x(x+1)]dx} = e^{-(\ln|x| - \ln|x+1|)} = \dfrac{x+1}{x}$. Multiplying the differential equation by $I(x)$ gives $\dfrac{x+1}{x}y' - \dfrac{y}{x(x+1)}\dfrac{x+1}{x} = \dfrac{x+1}{x} \Rightarrow \left(\dfrac{x+1}{x}y\right)' = \dfrac{x+1}{x}$. Then $y = \dfrac{x}{x+1}\left[\int\left(1+\dfrac{1}{x}\right)dx + C\right] = \dfrac{x}{x+1}(x + \ln x + C)$. But $0 = y(1) = \frac{1}{2}[1+C]$ so $C = -1$ and the solution to the initial-value problem is $y = \dfrac{x}{x+1}(x - 1 + \ln x)$.

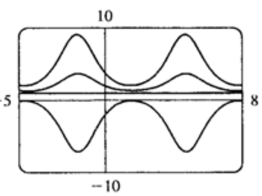
21. $y' + \dfrac{1}{x}y = \cos x$ $(x \neq 0)$, so $I(x) = e^{\int(1/x)dx} = e^{\ln|x|} = x$ (for $x > 0$). Multiplying the differential equation by $I(x)$ gives $xy' + y = x\cos x \Rightarrow (xy)' = x\cos x$. Thus,

$$y = \frac{1}{x}\left[\int x\cos x\,dx + C\right] = \frac{1}{x}[x\sin x + \cos x + C]$$
$$= \sin x + \frac{\cos x}{x} + \frac{C}{x}$$



The solutions are asymptotic to the y-axis (except for $C = -1$). In fact, for $C > -1$, $y \to \infty$ as $x \to 0^+$, whereas for $C < -1$, $y \to -\infty$ as $x \to 0^+$. As x gets larger, the solutions approximate $y = \sin x$ more closely. The graphs for larger C lie above those for smaller C. The distance between the graphs lessens as x increases.

22. $I(x) = e^{\int\cos x\,dx} = e^{\sin x}$. Multiplying the differential equation by $I(x)$ gives $e^{\sin x}y' + \cos x e^{\sin x}y = \cos x e^{\sin x} \Rightarrow$ $(e^{\sin x}y)' = \cos x e^{\sin x} \Rightarrow$ $y = e^{-\sin x}\left[\int \cos x e^{\sin x}\,dx + C\right] = 1 + Ce^{-\sin x}$. The graphs for $C = -3, 0, 1$, and 3 are shown. As the values of C get further from zero the graph is stretched away from the line $y = 1$, which is the value for $C = 0$. The graphs are all periodic in x, with a period of 2π.



23. Setting $u = y^{1-n}$, $\dfrac{du}{dx} = (1-n)y^{-n}\dfrac{dy}{dx}$ or $\dfrac{dy}{dx} = \dfrac{y^n}{1-n}\dfrac{du}{dx} = \dfrac{u^{n/(1-n)}}{1-n}\dfrac{du}{dx}$. Then the Bernoulli differential equation becomes $\dfrac{u^{n/(1-n)}}{1-n}\dfrac{du}{dx} + P(x)u^{1/(1-n)} = Q(x)u^{n/(1-n)}$ or $\dfrac{du}{dx} + (1-n)P(x)u = Q(x)(1-n)$.

24. Here $y' + \dfrac{y}{x} = -y^2$, so $n = 2$, $P(x) = \dfrac{1}{x}$ and $Q(x) = -1$. Setting $u = y^{-1}$, u satisfies $u' - \dfrac{1}{x}u = 1$. Then $I(x) = e^{\int(-1/x)dx} = \dfrac{1}{x}$ (for $x > 0$) and $u = x\left(\int\dfrac{1}{x}dx + C\right) = x(\ln|x| + C)$. Thus, $y = \dfrac{1}{x(C + \ln|x|)}$.

25. Here $n = 3$, $P(x) = \dfrac{2}{x}$, $Q(x) = \dfrac{1}{x^2}$ and setting $u = y^{-2}$, u satisfies $u' - \dfrac{4u}{x} = -\dfrac{2}{x^2}$. Then $I(x) = e^{\int(-4/x)dx} = x^{-4}$ and $u = x^4\left(\int -\dfrac{2}{x^6}dx + C\right) = x^4\left(\dfrac{2}{5x^5} + C\right) = Cx^4 + \dfrac{2}{5x}$. Thus, $y = \pm\left(Cx^4 + \dfrac{2}{5x}\right)^{-1/2}$.

26. Here $n=3$, $P(x)=1$, $Q(x)=x$ and setting $u=y^{-2}$, u satisfies $u'-2u=-2x$. Then $I(x)=e^{\int(-2)dx}=e^{-2x}$ and $u=e^{2x}\left[\int -2xe^{-2x}\,dx+C\right]=e^{2x}\left(xe^{-2x}+\frac{1}{2}e^{-2x}+C\right)=x+\frac{1}{2}+Ce^{2x}$. So $y=\left[x+\frac{1}{2}+Ce^{2x}\right]^{-1/2}$.

27. (a) $2\dfrac{dI}{dt}+10I=40$ or $\dfrac{dI}{dt}+5I=20$. Then the integrating factor is $e^{\int 5\,dt}=e^{5t}$. Multiplying the differential equation by the integrating factor gives $e^{5t}\dfrac{dI}{dt}+5Ie^{5t}=20e^{5t}\quad\Rightarrow\quad\left(e^{5t}I\right)'=20e^{5t}\quad\Rightarrow$
$I(t)=e^{-5t}\left[\int 20e^{5t}\,dt+C\right]=4+Ce^{-5t}$. But $0=I(0)=4+C$, so $I(t)=4-4e^{-5t}$.

(b) $I(0.1)=4-4e^{-0.5}\approx 1.57$ A

28. (a) $\dfrac{dI}{dt}+20I=40\sin 60t$, so the integrating factor is e^{20t}. Multiplying the differential equation by the integrating factor gives $e^{20t}\dfrac{dI}{dt}+20Ie^{20t}=40\sin 60te^{20t}\quad\Rightarrow\quad\left(e^{20t}I\right)'=40\sin 60te^{20t}\quad\Rightarrow$

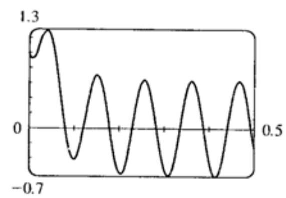
$$\begin{aligned}I(t)&=e^{-20t}\left[\int 40e^{20t}\sin 60t\,dt+C\right]\\&=e^{-20t}\left[40e^{20t}\left(\tfrac{1}{4000}\right)(20\sin 60t-60\cos 60t)\right]+Ce^{-20t}\\&=\frac{\sin 60t-3\cos 60t}{5}+Ce^{-20t}\end{aligned}$$

But $1=I(0)=-\frac{3}{5}+C$, so

$$I(t)=\frac{\sin 60t-3\cos 60t+8e^{-20t}}{5}.$$

(b) $I(0.1)=\dfrac{\sin 6-3\cos 6+8e^{-2}}{5}\approx -0.42$ A

(c)



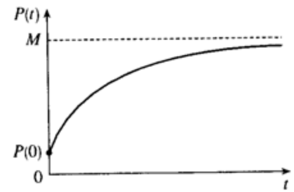
29. $5\dfrac{dQ}{dt}+20Q=60$ with $Q(0)=0$C. Then the integrating factor is $e^{\int 4\,dt}=e^{4t}$, and multiplying the differential equation by the integrating factor gives $e^{4t}\dfrac{dQ}{dt}+4e^{4t}Q=12e^{4t}\quad\Rightarrow\quad\left(e^{4t}Q\right)'=12e^{4t}\quad\Rightarrow$
$Q(t)=e^{-4t}\left[\int 12e^{4t}\,dt+C\right]=3+Ce^{-4t}$. But $0=Q(0)=3+C$ so $Q(t)=3\left(1-e^{-4t}\right)$ is the charge at time t and $I=dQ/dt=12e^{-4t}$ is the current at time t.

30. $2\dfrac{dQ}{dt}+100Q=10\sin 60t$ or $\dfrac{dQ}{dt}+50Q=5\sin 60t$. Then the integrating factor is $e^{\int 50\,dt}=e^{50t}$, and multiplying the differential equation by the integrating factor gives $e^{50t}\dfrac{dQ}{dt}+50e^{50t}Q=5e^{50t}\sin 60t\quad\Rightarrow$
$\left(e^{50t}Q\right)'=5e^{50t}\sin 60t\quad\Rightarrow$

$$\begin{aligned}Q(t)&=e^{-50t}\left[\int 5e^{50t}\sin 60t\,dt+C\right]=e^{-50t}\left[5e^{50t}\left(\tfrac{1}{6100}\right)(50\sin 60t-60\cos 60t)\right]+Ce^{-50t}\\&=\tfrac{1}{122}(5\sin 60t-6\cos 60t)+Ce^{-50t}\end{aligned}$$

But $0=Q(0)=-\frac{6}{122}+C$ so $C=\frac{3}{61}$ and $Q(t)=\dfrac{5\sin 60t-6\cos 60t}{122}+\dfrac{3e^{-50t}}{61}$ is the charge at time t, while the current is $I(t)=\dfrac{dQ}{dt}=\dfrac{150\cos 60t+180\sin 60t-150e^{-50t}}{61}$.

31. $\dfrac{dP}{dt} + kP = kM$, so $I(t) = e^{\int k\,dt} = e^{kt}$. Multiplying the differential equation by $I(t)$ gives $e^{kt}\dfrac{dP}{dt} + kPe^{kt} = kMe^{kt} \Rightarrow \left(e^{kt}P\right)' = kMe^{kt}$ $\Rightarrow P(t) = e^{-kt}\left(\int kMe^{kt}dt + C\right) = M + Ce^{-kt}$, $k > 0$. Furthermore, it is reasonable to assume that $0 \le P(0) \le M$, so $-M \le C \le 0$.



32. Since $P(0) = 0$, we have $P(t) = M\left(1 - e^{-kt}\right)$. If $P_1(t)$ is Jim's learning curve, then $P_1(1) = 25$ and $P_1(2) = 45$. Hence, $25 = M_1\left(1 - e^{-k}\right)$ and $45 = M_1\left(1 - e^{-2k}\right)$, so $1 - 25/M_1 = e^{-k}$ or $k = -\ln\left(1 - \dfrac{25}{M_1}\right) = \ln\left(\dfrac{M_1}{M_1 - 25}\right)$. But $45 = M_1\left(1 - e^{-2k}\right)$ so $45 = M_1\left[1 - \left(\dfrac{M_1 - 25}{M_1}\right)^2\right]$ or $45 = \dfrac{50M_1 - 625}{M_1}$. Thus, $M_1 = 125$ is the maximum number of units per hour Jim is capable of processing. Similarly, if $P_2(t)$ is Mark's learning curve, then $P_2(1) = 35$ and $P_2(2) = 50$. So $k = \ln\left(\dfrac{M_2}{M_2 - 35}\right)$ and $50 = M_2\left[1 - \left(\dfrac{M_2 - 35}{M_2}\right)^2\right]$ or $M_2 = 61.25$. Hence the maximum number of units per hour for Mark is approximately 61.

33. $y(0) = 0$ kg. Salt is added at a rate of $\left(0.4\,\dfrac{\text{kg}}{\text{L}}\right)\left(5\,\dfrac{\text{L}}{\text{min}}\right) = 2\,\dfrac{\text{kg}}{\text{min}}$. Since solution is drained from the tank at a rate of 3 L/min, but salt solution is added at a rate of 5 L/min, the tank, which starts out with 100 L of water, contains $(100 + 2t)$ L of liquid after t min. Thus, the salt concentration at time t is $\dfrac{y(t)}{100 + 2t}\,\dfrac{\text{kg}}{\text{L}}$. Salt therefore leaves the tank at a rate of $\left(\dfrac{y(t)}{100 + 2t}\,\dfrac{\text{kg}}{\text{L}}\right)\left(3\,\dfrac{\text{L}}{\text{min}}\right) = \dfrac{3y}{100 + 2t}\,\dfrac{\text{kg}}{\text{min}}$. Combining the rates at which salt enters and leaves the tank, we get $\dfrac{dy}{dt} = 2 - \dfrac{3y}{100 + 2t}$. Rewriting this equation as $\dfrac{dy}{dt} + \left(\dfrac{3}{100 + 2t}\right)y = 2$, we see that it is linear. $I(t) = \exp\left(\displaystyle\int \dfrac{3\,dt}{100 + 2t}\right) = \exp\left(\tfrac{3}{2}\ln(100 + 2t)\right) = (100 + 2t)^{3/2}$. Multiplying the differential equation by $I(t)$ gives $(100 + 2t)^{3/2}\dfrac{dy}{dt} + 3(100 + 2t)^{1/2}y = 2(100 + 2t)^{3/2} \Rightarrow$
$\left[(100 + 2t)^{3/2}y\right]' = 2(100 + 2t)^{3/2} \Rightarrow (100 + 2t)^{3/2}y = \tfrac{2}{5}(100 + 2t)^{5/2} + C \Rightarrow$
$y = \tfrac{2}{5}(100 + 2t) + C(100 + 2t)^{-3/2}$. Now $0 = y(0) = \tfrac{2}{5}(100) + C \cdot 100^{-3/2} = 40 + \tfrac{1}{1000}C \Rightarrow$
$C = -40{,}000$, so $y = \left[\tfrac{2}{5}(100 + 2t) - 40{,}000(100 + 2t)^{-3/2}\right]$ kg. From this solution (no pun intended), we calculate the salt concentration at time t to be $C(t) = \dfrac{y(t)}{100 + 2t} = \left[\dfrac{-40{,}000}{(100 + 2t)^{5/2}} + \dfrac{2}{5}\right]\dfrac{\text{kg}}{\text{L}}$. In particular, $C(20) = \dfrac{-40{,}000}{140^{5/2}} + \dfrac{2}{5} \approx 0.2275\,\dfrac{\text{kg}}{\text{L}}$ and $y(20) = \tfrac{2}{5}(140) - 40{,}000(140)^{-3/2} \approx 31.85$ kg.

34. Let $y(t)$ denote the amount of chlorine in the tank at time t (in seconds). $y(0) = (0.05\text{ g/L})(400\text{ L}) = 20$ g. The amount of liquid in the tank at time t is $(400 - 6t)$ L since 4 L of water enters the tank each second and 10 L of liquid leaves the tank each second. Thus, the concentration of chlorine at time t is $\dfrac{y(t)}{400-6t}\,\dfrac{\text{g}}{\text{L}}$. Chlorine doesn't enter the tank, but it leaves at a rate of $\left[\dfrac{y(t)}{400-6t}\,\dfrac{\text{g}}{\text{L}}\right]\left[10\,\dfrac{\text{L}}{\text{s}}\right] = \dfrac{10\,y(t)}{400-6t}\,\dfrac{\text{g}}{\text{s}} = \dfrac{5\,y(t)}{200-3t}\,\dfrac{\text{g}}{\text{s}}$. Therefore,

$\dfrac{dy}{dt} = -\dfrac{5y}{200-3t} \quad\Rightarrow\quad \displaystyle\int \frac{dy}{y} = \int \frac{-5\,dt}{200-3t} \quad\Rightarrow\quad \ln y = \tfrac{5}{3}\ln(200-3t) + C \quad\Rightarrow$

$y = \exp\left(\tfrac{5}{3}\ln(200-3t) + C\right) = e^C(200-3t)^{5/3}$. Now $20 = y(0) = e^C \cdot 200^{5/3} \quad\Rightarrow\quad e^C = \dfrac{20}{200^{5/3}}$, so

$y(t) = 20\dfrac{(200-3t)^{5/3}}{200^{5/3}} = 20(1-0.015t)^{5/3}$ g for $0 \le t \le 66\frac{2}{3}$ s, at which time the tank is empty.

35. (a) $\dfrac{dv}{dt} + \dfrac{c}{m}v = g$ and $I(t) = e^{\int (c/m)\,dt} = e^{(c/m)t}$, and multiplying the differential equation by $I(t)$ gives

$e^{(c/m)t}\dfrac{dv}{dt} + \dfrac{vce^{(c/m)t}}{m} = ge^{(c/m)t} \quad\Rightarrow\quad \left[e^{(c/m)t}v\right]' = ge^{(c/m)t}$. Hence,

$v(t) = e^{-(c/m)t}\left[\int ge^{(c/m)t}\,dt + K\right] = mg/c + Ke^{-(c/m)t}$. But the object is dropped from rest, so $v(0) = 0$ and $K = -mg/c$. Thus, the velocity at time t is $v(t) = (mg/c)\left[1 - e^{-(c/m)t}\right]$.

(b) $\lim\limits_{t\to\infty} v(t) = mg/c$.

(c) $s(t) = \int v(t)\,dt = (mg/c)\left[t + (m/c)e^{-(c/m)t}\right] + c_1$ where $c_1 = s(0) - m^2g/c^2$, $s(0)$ is the initial position, so $s(0) = 0$ and $s(t) = (mg/c)\left[t + (m/c)e^{-(c/m)t}\right] - m^2g/c^2$.

36. $v = (mg/c)\left(1 - e^{-ct/m}\right) \quad\Rightarrow$

$\dfrac{dv}{dm} = \dfrac{mg}{c}\left(0 - e^{-ct/m}\cdot\dfrac{ct}{m^2}\right) + \dfrac{g}{c}\left(1 - e^{-ct/m}\right)\cdot 1 = -\dfrac{gt}{m}e^{-ct/m} + \dfrac{g}{c} - \dfrac{g}{c}e^{-ct/m} = \dfrac{g}{c}\left(1 - e^{-ct/m} - \dfrac{ct}{m}e^{-ct/m}\right)$

$\Rightarrow\quad \dfrac{c}{g}\dfrac{dv}{dm} = 1 - \left(1 + \dfrac{ct}{m}\right)e^{-ct/m} = 1 - \dfrac{1 + ct/m}{e^{ct/m}} = 1 - \dfrac{1+Q}{e^Q}$, where $Q = \dfrac{ct}{m} \ge 0$. Since $e^Q > 1 + Q$ for all $Q > 0$ [see Exercise 7.2.85(a) or 7.3*.85(a)], it follows that $dv/dm > 0$ for $t > 0$. In other words, for all $t > 0$, v increases as m increases.

10.7 Predator-Prey Systems

1. (a) $dx/dt = -0.05x + 0.0001xy$. If $y = 0$, we have $dx/dt = -0.05x$, which indicates that in the absence of y, x declines at a rate proportional to itself. So x represents the predator population and y represents the prey population. The growth of the prey population, $0.1y$ (from $dy/dt = 0.1y - 0.005xy$), is restricted only by encounters with predators (the term $-0.005xy$). The predator population increases only through the term $0.0001xy$, that is, by encounters with the prey and not through additional food sources.

(b) $dy/dt = -0.015y + 0.00008xy$. If $x = 0$, we have $dy/dt = -0.015y$, which indicates that in the absence of x, y would decline at a rate proportional to itself. So y represents the predator population and x represents the prey population. The growth of the prey population, $0.2x$ (from $dx/dt = 0.2x - 0.0002x^2 - 0.006xy = 0.2x(1 - 0.001x) - 0.006xy$), is restricted by a carrying capacity of 1000 [from the term $1 - 0.001x = 1 - x/1000$] and by encounters with predators (the term $-0.006xy$). The predator population increases only through the term $0.00008xy$, that is, by encounters with the prey and not through additional food sources.

2. (a) $dx/dt = 0.12x - 0.0006x^2 + 0.00001xy$. $dy/dt = 0.08y + 0.00004xy$.

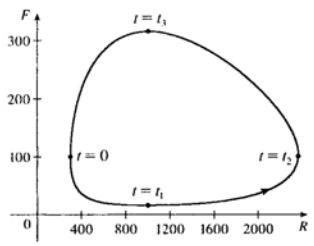
The xy terms represent encounters between the two species x and y. An increase in y makes dx/dt (the growth rate of x) larger due to the positive term $0.00001xy$. An increase in x makes dy/dt (the growth rate of y) larger due to the positive term $0.00004xy$. Hence, the system describes a cooperation model.

(b) $dx/dt = 0.15x - 0.0002x^2 - 0.0006xy = 0.15x\,(1 - x/750) - 0.0006xy$.

$dy/dt = 0.2y - 0.00008y^2 - 0.0002xy = 0.2y\,(1 - y/2500) - 0.0002xy$.

The system shows that x and y have carrying capacities of 750 and 2500. An increase in x reduces the growth rate of y due to the negative term $-0.0002xy$. An increase in y reduces the growth rate of x due to the negative term $-0.0006xy$. Hence, the system describes a competition model.

3. (a) At $t = 0$, there are about 300 rabbits and 100 foxes. At $t = t_1$, the number of foxes reaches a minimum of about 20 while the number of rabbits is about 1000. At $t = t_2$, the number of rabbits reaches a maximum of about 2400, while the number of foxes rebounds to 100. At $t = t_3$, the number of rabbits decreases to about 1000 and the number of foxes reaches a maximum of about 315. As t increases, the number of foxes decreases greatly to 100, and the number of rabbits decreases to 300 (the initial populations), and the cycle starts again.



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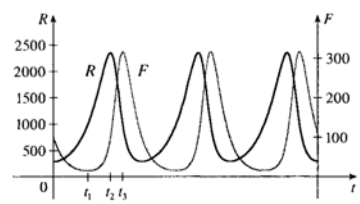
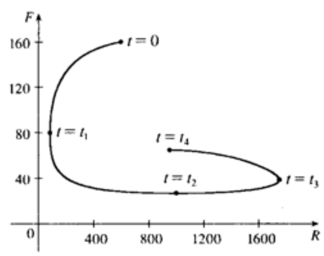


4. (a) At $t = 0$, there are about 600 rabbits and 160 foxes. At $t = t_1$, the number of rabbits reaches a minimum of about 80 and the number of foxes is also 80. At $t = t_2$, the number of foxes reaches a minimum of about 25 while the number of rabbits rebounds to 1000. At $t = t_3$, the number of foxes has increased to 40 and the rabbit population has reached a maximum of about 1750. The curve ends at $t = t_4$, where the number of foxes has increased to 65 and the number of rabbits has decreased to about 950.



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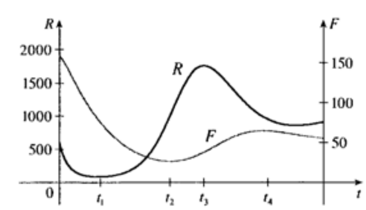

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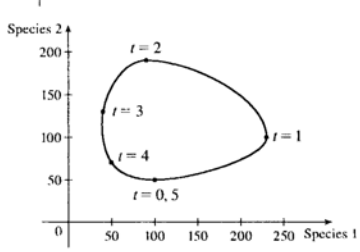

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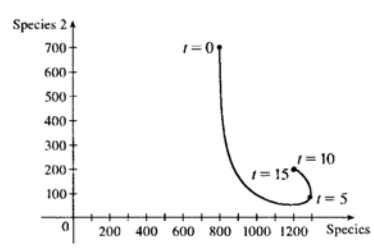


7. $\dfrac{dW}{dR} = \dfrac{-0.02W + 0.00002RW}{0.08R - 0.001RW} \iff (0.08 - 0.001W)\,R\,dW = (-0.02 + 0.00002R)\,W\,dR \iff$
$\dfrac{0.08 - 0.001W}{W}\,dW = \dfrac{-0.02 + 0.00002R}{R}\,dR \iff \displaystyle\int\left(\frac{0.08}{W} - 0.001\right)dW = \int\left(-\frac{0.02}{R} + 0.00002\right)dR$
$\iff 0.08\ln|W| - 0.001W = -0.02\ln|R| + 0.00002R + K \iff$
$0.08\ln W + 0.02\ln R = 0.001W + 0.00002R + K \iff \ln\left(W^{0.08}R^{0.02}\right) = 0.00002R + 0.001W + K \iff$
$W^{0.08}R^{0.02} = e^{0.00002R + 0.001W + K} \iff R^{0.02}W^{0.08} = Ce^{0.00002R}e^{0.001W} \iff \dfrac{R^{0.02}W^{0.08}}{e^{0.00002R}e^{0.001W}} = C.$

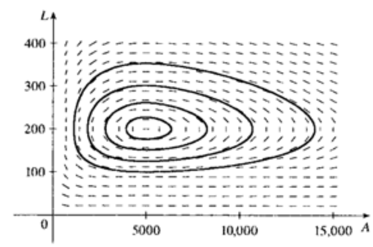
In general, if $\dfrac{dy}{dx} = \dfrac{-ry + bxy}{kx - axy}$, then $C = \dfrac{x^r y^k}{e^{bx}e^{ay}}$.

8. (a) A and L are constant $\Rightarrow A' = 0$ and $L' = 0 \Rightarrow \begin{Bmatrix} 0 = 2A - 0.01AL \\ 0 = -0.5L + 0.0001AL \end{Bmatrix} \Rightarrow \begin{cases} 0 = A(2 - 0.01L) \\ 0 = L(-0.5 + 0.0001A) \end{cases}$

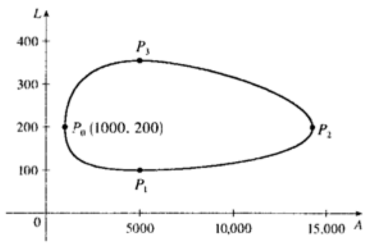
So either $A = L = 0$ or $L = \frac{2}{0.01} = 200$ and $A = \frac{0.5}{0.0001} = 5000$. The trivial solution $A = L = 0$ just says that if there aren't any aphids or ladybugs, then the populations will not change. The non-trivial solution, $L = 200$ and $A = 5000$, indicates the population sizes needed so that there are no changes in either the number of aphids or the number of ladybugs.

(b) $\dfrac{dL}{dA} = \dfrac{dL/dt}{dA/dt} = \dfrac{-0.5L + 0.0001AL}{2A - 0.01AL}$

(c) The solution curves (phase trajectories) are all closed curves that have the equilibrium point (5000, 200) inside them.

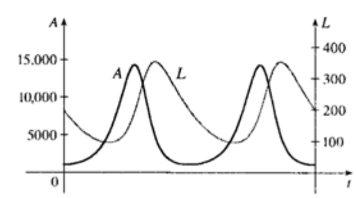


(d)



At P_0 (1000, 200), $dA/dt = 0$ and $dL/dt = -80 < 0$, so the number of ladybugs is decreasing and hence, we are proceeding in a counterclockwise direction. At P_0, there aren't enough aphids to support the ladybug population, so the number of ladybugs decreases and the number of aphids begins to increase. The ladybug population reaches a minimum at P_1 (5000, 100) while the aphid population increases in a dramatic way, reaching its maximum at P_2 (14,250, 200). Meanwhile, the ladybug population is increasing from P_1 to P_3 (5000, 355), and as we pass through P_2, the increasing number of ladybugs starts to deplete the aphid population. At P_3 the ladybugs reach a maximum population, and start to decrease due to the reduced aphid population. Both populations then decrease until P_0, where the cycle starts over again.

(e) Both graphs have the same period and the graph of L peaks about a quarter of a cycle after the graph of A.



9. (a) Letting $W = 0$ gives us $dR/dt = 0.08R\,(1 - 0.0002R)$. $dR/dt = 0 \quad \Leftrightarrow \quad R = 0$ or 5000. Since $dR/dt > 0$ for $0 < R < 5000$, we would expect the rabbit population to *increase* to 5000 for these values of R. Since $dR/dt < 0$ for $R > 5000$, we would expect the rabbit population to *decrease* to 5000 for these values of R. Hence, in the absence of wolves, we would expect the rabbit population to stabilize at 5000.

(b) R and W are constant $\Rightarrow$ $R' = 0$ and $W' = 0$ $\Rightarrow$ $\left\{\begin{matrix} 0 = 0.08R\,(1 - 0.0002R) - 0.001\,RW \\ 0 = -0.02W + 0.00002RW \end{matrix}\right\}$ $\Rightarrow$

$\begin{cases} 0 = R\,[0.08\,(1 - 0.0002R) - 0.001W] \\ 0 = W\,(-0.02 + 0.00002R) \end{cases}$

The second equation is true if $W = 0$ or $R = \frac{0.02}{0.00002} = 1000$. If $W = 0$ in the first equation, then either $R = 0$ or $R = \frac{1}{0.0002} = 5000$ [as in part (a)]. If $R = 1000$, then $0 = 1000\,[0.08\,(1 - 0.0002 \cdot 1000) - 0.001W] \quad \Leftrightarrow$ $0 = 80\,(1 - 0.2) - W \quad \Leftrightarrow \quad W = 64$.

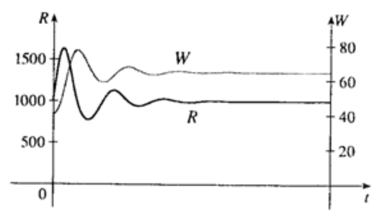
Case (i): $W = 0$, $R = 0$: both populations are zero

Case (ii): $W = 0$, $R = 5000$: see part (a)

Case (iii): $R = 1000$, $W = 64$: the predator/prey interaction balances and the populations are stable.

(c) The populations of wolves and rabbits fluctuate around 64 and 1000, respectively, and eventually stabilize at those values.

(d)



10. (a) If $L = 0$, $dA/dt = 2A\,(1 - 0.0001A)$, so $dA/dt = 0 \quad \Leftrightarrow \quad A = 0$ or $A = \frac{1}{0.0001} = 10{,}000$. Since $dA/dt > 0$ for $0 < A < 10{,}000$, we expect the aphid population to *increase* to 10,000 for these values of A. Since $dA/dt < 0$ for $A > 10{,}000$, we expect the aphid population to *decrease* to 10,000 for these values of A. Hence, in the absence of ladybugs we expect the aphid population to stabilize at 10,000.

(b) A and L are constant $\Rightarrow$ $A' = 0$ and $L' = 0$ $\Rightarrow$

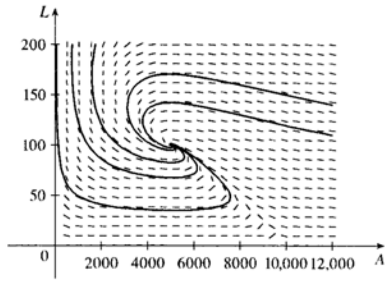
$\left\{\begin{matrix} 0 = 2A\,(1 - 0.0001A) - 0.01AL \\ 0 = -0.5L + 0.0001AL \end{matrix}\right\} \Rightarrow \begin{cases} 0 = A\,[2\,(1 - 0.0001A) - 0.01L] \\ 0 = L\,(-0.5 + 0.0001A) \end{cases}$

The second equation is true if $L = 0$ or $A = \frac{0.5}{0.0001} = 5000$. If $L = 0$ in the first equation, then either $A = 0$ or $A = \frac{1}{0.0001} = 10{,}000$. If $A = 5000$, then $0 = 5000\,[2\,(1 - 0.0001 \cdot 5000) - 0.01L] \quad \Leftrightarrow$ $0 = 10{,}000\,(1 - 0.5) - 50L \quad \Leftrightarrow \quad 50L = 5000 \quad \Leftrightarrow \quad L = 100$. The equilibrium solutions are:

(i) $L = 0$, $A = 0$ (ii) $L = 0$, $A = 10{,}000$ (iii) $A = 5000$, $L = 100$

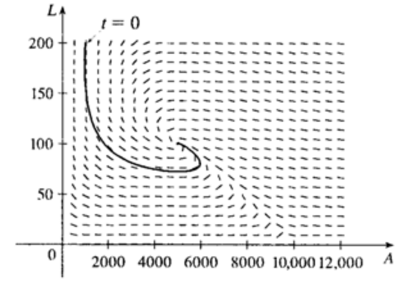
(c) $\dfrac{dL}{dA} = \dfrac{dL/dt}{dA/dt} = \dfrac{-0.5L + 0.0001AL}{2A\,(1 - 0.0001A) - 0.01AL}$

(d)



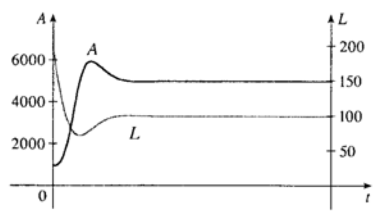
All of the phase trajectories spiral tightly around the equilibrium solution (5000, 100).

(e)



At $t = 0$, the ladybug population decreases rapidly and the aphid population decreases slightly before beginning to increase. As the aphid population continues to increase, the ladybug population reaches a minimum at about (5000, 75). The ladybug population starts to increase and quickly stabilizes at 100, while the aphid population stabilizes at 5000.

(f)



The graph of A peaks just after the graph of L has a minimum.

10 Review

CONCEPT CHECK

1. (a) A differential equation is an equation that contains an unknown function and one or more of its derivatives.

(b) The order of a differential equation is the order of the highest derivative that occurs in the equation.

(c) An initial condition is a condition of the form $y(t_0) = y_0$.

2. $y' = x^2 + y^2 \geq 0$ for all x and y. $y' = 0$ only at the origin, so there is a horizontal tangent at $(0, 0)$, but nowhere else. The graph of the solution is increasing on every interval.

3. See the paragraph preceding Example 1 in Section 10.2.

4. See the paragraph after Figure 14 in Section 10.2.

5. A separable equation is a first-order differential equation in which the expression for dy/dx can be factored as a function of x times a function of y, that is, $dy/dx = g(x)\,f(y)$. We can solve the equation by integrating both sides of the equation $dy/f(y) = g(x)\,dx$ and solving for y.

6. A first-order linear differential equation is a differential equation that can be put in the form $\dfrac{dy}{dx} + P(x)\,y = Q(x)$, where P and Q are continuous functions on a given interval. To solve such an equation, multiply it by the integrating factor $I(x) = e^{\int P(x)dx}$ to put it in the form $[I(x)\,y]' = I(x)\,Q(x)$ and then integrate both sides to get $I(x)\,y = \int I(x)\,Q(x)\,dx$, that is, $e^{\int P(x)dx}y = \int e^{\int P(x)dx}Q(x)\,dx$. Solving for y gives us $y = e^{-\int P(x)dx}\int e^{\int P(x)dx}Q(x)\,dx$.

7. (a) $dy/dt = ky$

(b) The equation in part (a) is an appropriate model for population growth, assuming that there is enough room and nutrition to support the growth.

(c) If $y(0) = y_0$, then the solution is $y(t) = y_0 e^{kt}$.

8. (a) $dP/dt = kP(1 - P/K)$, where K is the carrying capacity.

(b) The equation in part (a) is an appropriate model for population growth, assuming that the population grows at a rate proportional to the size of the population in the beginning, but eventually levels off and approaches its carrying capacity because of limited resources.

9. (a) $dF/dt = kF - aFS$ and $dS/dt = -rS + bFS$.

(b) In the absence of sharks, an ample food supply would support exponential growth of the fish population, that is, $dF/dt = kF$, where k is a positive constant. In the absence of fish, we assume that the shark population would decline at a rate proportional to itself, that is, $dS/dt = -rS$, where r is a positive constant.

TRUE-FALSE QUIZ

1. False. $y = 0$ is a solution of $y' = -y^4$, but $y = 0$ is not a decreasing function. (All non-trivial solutions are decreasing, however.)

2. True. $y = \dfrac{\ln x}{x} \Rightarrow y' = \dfrac{1 - \ln x}{x^2}$.

LHS $= x^2 y' + xy = x^2 \cdot \dfrac{1 - \ln x}{x^2} + x \cdot \dfrac{\ln x}{x} = (1 - \ln x) + \ln x = 1 =$ RHS, so $y = \dfrac{\ln x}{x}$ is a solution of $x^2 y' + xy = 1$.

3. False. $x + y$ cannot be written in the form $g(x) f(y)$.

4. True. $y' = 3y - 2x + 6xy - 1 = 6xy - 2x + 3y - 1 = 2x(3y - 1) + 1(3y - 1) = (2x + 1)(3y - 1)$, so y' can be written in the form $g(x) f(y)$.

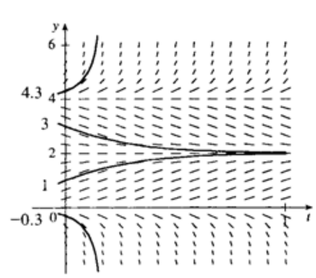
5. True. $e^x y' = y \Rightarrow y' = e^{-x} y \Rightarrow y' + (-e^{-x}) y = 0$, which is of the form $y' + P(x) y = Q(x)$.

6. False. $y' + xy = e^y$ cannot be put in the form $y' + P(x) y = Q(x)$, so it is not linear.

7. True. By comparing $\dfrac{dy}{dt} = 2y\left(1 - \dfrac{y}{5}\right)$ with the logistic differential equation (10.5.1), we see that the carrying capacity is 5, that is, $\lim_{t\to\infty} y = 5$.

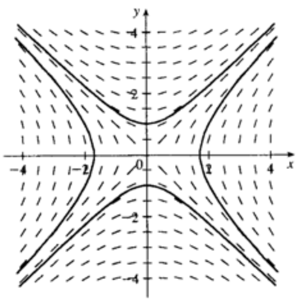
EXERCISES

1. (a)



(b) $\lim_{t\to\infty} y(t)$ appears to be finite for $0 \le c \le 4$. In fact $\lim_{t\to\infty} y(t) = 4$ for $c = 4$, $\lim_{t\to\infty} y(t) = 2$ for $0 < c < 4$, and $\lim_{t\to\infty} y(t) = 0$ for $c = 0$. The equilibrium solutions are $y(t) = 0$, $y(t) = 2$, and $y(t) = 4$.

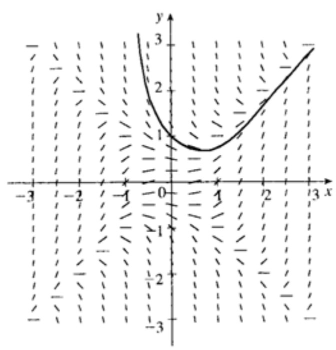
2. (a)



We sketch the direction field and four solution curves, as shown. Note that the slope $y' = x/y$ is not defined on the line $y = 0$.

(b) $y' = x/y \quad \Leftrightarrow \quad y\,dy = x\,dx \quad \Leftrightarrow y^2 = x^2 + C$. For $C = 0$, this is the pair of lines $y = \pm x$. For $C \neq 0$, it is the hyperbola $x^2 - y^2 = -C$.

3. (a)



We estimate that when $x = 0.3$, $y = 0.8$, so $y(0.3) \approx 0.8$.

(b) $h = 0.1$, $x_0 = 0$, $y_0 = 1$ and $F(x, y) = x^2 - y^2$. So $y_n = y_{n-1} + 0.1\left(x_{n-1}^2 - y_{n-1}^2\right)$. Thus, $y_1 = 1 + 0.1\left(0^2 - 1^2\right) = 0.9$, $y_2 = 0.9 + 0.1\left(0.1^2 - 0.9^2\right) = 0.82$, $y_3 = 0.82 + 0.1\left(0.2^2 - 0.82^2\right) = 0.75676 \approx y(0.3)$.

(c) The centers of the horizontal line segments of the direction field are located on the lines $y = x$ and $y = -x$. When a solution curve crosses one of these lines, it has a local maximum or minimum.

4. (a) $h = 0.2$, $x_0 = 0$, $y_0 = 1$ and $F(x, y) = 2xy^2$. We need y_2. $y_1 = 1 + 0.2\left(2 \cdot 0 \cdot 1^2\right) = 1$, $y_2 = 1 + 0.2\left(2 \cdot 0.2 \cdot 1^2\right) = 1.08 \approx y(0.4)$.

(b) $h = 0.1$ now, so $y_1 = 1 + 0.1\left(2 \cdot 0 \cdot 1^2\right) = 1$, $y_2 = 1 + 0.1\left(2 \cdot 0.1 \cdot 1^2\right) = 1.02$, $y_3 = 1.02 + 0.1\left(2 \cdot 0.2 \cdot 1.02^2\right) \approx 1.06162$, $y_4 = 1.06162 + 0.1\left(2 \cdot 0.3 \cdot 1.06162^2\right) \approx 1.1292 \approx y(0.4)$.

(c) The equation is separable, so we write $\dfrac{dy}{y^2} = 2x\,dx \quad \Rightarrow \quad \displaystyle\int \frac{dy}{y^2} = \int 2x\,dx \quad \Leftrightarrow \quad -\frac{1}{y} = x^2 + C$, but $y(0) = 1$, so $C = -1$ and $y(x) = \dfrac{1}{1 - x^2} \quad \Leftrightarrow \quad y(0.4) = \dfrac{1}{1 - 0.16} \approx 1.1905$. From this we see that the approximation was greatly improved by increasing the number of steps, but the approximations were still far off.

5. $y^2 \dfrac{dy}{dx} = x + \sin x \quad \Rightarrow \quad \int y^2\,dy = \int (x + \sin x)\,dx \quad \Rightarrow \quad \dfrac{y^3}{3} = \dfrac{x^2}{2} - \cos x + C \quad \Rightarrow$
$y^3 = \frac{3}{2}x^2 - 3\cos x + K$ (where $K = 3C$) $\quad \Rightarrow \quad y = \sqrt[3]{\frac{3}{2}x^2 - 3\cos x + K}$

6. Since it's linear, $I(x) = e^{\int 2x\,dx} = e^{x^2}$ and multiplying the differential equation by $I(x)$ gives $e^{x^2}y' + 2xe^{x^2}y = 2x^3e^{x^2} \quad \Rightarrow \quad \left(e^{x^2}y\right)' = 2x^3e^{x^2} \quad \Rightarrow$
$y(x) = e^{-x^2}\left(\int 2x^3e^{x^2}\,dx + C\right) = e^{-x^2}\left(x^2e^{x^2} - e^{x^2} + C\right) = x^2 - 1 + Ce^{-x^2}$.

7. Since it's linear, $I(x) = e^{\int -2/x\,dx} = x^{-2}$ and multiplying by $I(x)$ gives $x^{-2}y' - 2y^{-3} = 1 \Rightarrow (x^{-2}y)' = 1$ $\Rightarrow y(x) = x^2\left(\int 1\,dx + C\right) = x^2[x + C] = Cx^2 + x^3$.

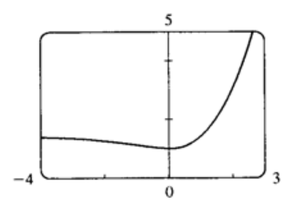
8. $y' = (2+y)\left(1+x^2\right) \Rightarrow \dfrac{dy}{2+y} = \left(1+x^2\right)dx \Rightarrow \ln|2+y| = x + \dfrac{x^3}{3} + c_1 \Rightarrow 2+y = ke^{x+x^3/3}$ and the solution is $y(x) = ke^{x+x^3/3} - 2$.

9. $xyy' = \ln x \Rightarrow y\,dy = \dfrac{\ln x}{x}\,dx \Rightarrow \displaystyle\int y\,dy = \int \frac{\ln x}{x}\,dx$ (Make the substitution $u = \ln x$; then $du = dx/x$.) So $\int y\,dy = \int u\,du \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}u^2 + C \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}(\ln x)^2 + C$. $y(1) = 2 \Rightarrow \frac{1}{2}2^2 = \frac{1}{2}(\ln 1)^2 + C = C$ $\Leftrightarrow C = 2$. Therefore, $\frac{1}{2}y^2 = \frac{1}{2}(\ln x)^2 + 2$, or $y = \sqrt{(\ln x)^2 + 4}$. The negative square root is inadmissible, since $y(1) > 0$.

10. $1 + x = 2xyy' \Rightarrow y' = \dfrac{1+x}{2xy} \Leftrightarrow y\,dy = \dfrac{1+x}{2x}\,dx \Rightarrow \dfrac{y^2}{2} = \dfrac{\ln|x|}{2} + \dfrac{x}{2} + c_1$. But $x > 0$, so $y^2 = \ln x + x + c \Leftrightarrow y(x) = \pm\sqrt{c + x + \ln x}$. But $-2 = y(1)$ so choose the negative square root and $-2 = -\sqrt{c+1}$ so $c = 3$. Thus, the solution is $y(x) = -\sqrt{3 + x + \ln x}$.

11. Since the equation is linear, let $I(x) = e^{\int dx} = e^x$. Then multiplying by $I(x)$ gives $e^x y' + e^x y = \sqrt{x} \Rightarrow$ $(e^x y)' = \sqrt{x} \Rightarrow y(x) = e^{-x}\left(\int \sqrt{x}\,dx + c\right) = e^{-x}\left(\frac{2}{3}x^{3/2} + c\right)$. But $3 = y(0) = c$, so the solution to the initial-value problem is $y(x) = e^{-x}\left(\frac{2}{3}x^{3/2} + 3\right)$.

12. $2yy' = xe^x \Rightarrow \int 2y\,dy = \int xe^x\,dx \Rightarrow$ $y^2 = xe^x - \int e^x\,dx$ (by parts) $= (x-1)e^x + C$. We substitute the initial condition: $1^2 = (0-1)e^0 + C \Rightarrow C = 2$. So the solution is $y = \sqrt{(x-1)e^x + 2}$. The negative square root is inadmissible due to the initial condition.



13. The curves $kx^2 + y^2 = 1$ form a family of ellipses for $k > 0$, a family of hyperbolas for $k < 0$, and two parallel lines $y = \pm 1$ for $k = 0$. Solving $kx^2 + y^2 = 1$ for k gives $k = \dfrac{1-y^2}{x^2}$. Differentiating gives $2kx + 2yy' = 0 \Leftrightarrow$ $y' = -\dfrac{kx}{y} = -\left(1-y^2\right)\dfrac{x}{yx^2} = \dfrac{y^2-1}{xy}$. Thus, for $k \neq 0$ the orthogonal trajectories must satisfy $y' = -\dfrac{xy}{y^2-1}$ $\Rightarrow \dfrac{y^2-1}{y}\,dy = -x\,dx \Rightarrow \dfrac{y^2}{2} - \ln|y| = \dfrac{-x^2}{2} + c$. For $k = 0$, the orthogonal trajectories are given by $x = c_1$ for c_1 an arbitrary constant.

14. Differentiating both sides of $y = \dfrac{k}{1+x^2}$ gives $y' = -\dfrac{2kx}{\left(1+x^2\right)^2} = -2xy\dfrac{1+x^2}{\left(1+x^2\right)^2} = -\dfrac{2xy}{1+x^2}$. Thus, for $k \neq 0$ the orthogonal trajectories must satisfy $y' = \dfrac{1+x^2}{2xy} \Rightarrow 2y\,dy = \left(\dfrac{1}{x} + x\right)dx \Rightarrow y^2 = \dfrac{x^2}{2} + \ln|x| + c$. For $k = 0$, the orthogonal trajectories are given by $x = c_2$ for c_2 an arbitrary constant.

15. (a) $y(t) = y(0)\,e^{kt} = 1000e^{kt} \quad\Rightarrow\quad y(2) = 1000e^{2k} = 9000 \quad\Rightarrow\quad e^{2k} = 9 \quad\Rightarrow\quad 2k = \ln 9 \quad\Rightarrow$
$k = \frac{1}{2}\ln 9 = \ln 3 \quad\Rightarrow\quad y(t) = 1000e^{(\ln 3)t} = 1000 \cdot 3^t$

(b) $y(3) = 1000 \cdot 3^3 = 27{,}000$

(c) $dy/dt = 1000 \cdot 3^t \cdot \ln 3$; $t = 3 \quad\Rightarrow\quad dy/dt = 27{,}000 \ln 3 \approx 29{,}663$

(d) $1000 \cdot 3^t = 2000 \quad\Rightarrow\quad 3^t = 2 \quad\Rightarrow\quad t\ln 3 = \ln 2 \quad\Rightarrow\quad t = (\ln 2)/(\ln 3) \approx 0.63$ h

16. (a) If $y(t)$ is the mass remaining after t years, then $y(t) = y(0)\,e^{kt} = 18e^{kt}$. $y(25) = 18e^{25k} = 9 \quad\Rightarrow\quad e^{25k} = \frac{1}{2}$
$\Rightarrow\quad 25k = -\ln 2 \quad\Rightarrow\quad k = -\frac{1}{25}\ln 2 \quad\Rightarrow\quad y(t) = 18\,e^{-(\ln 2)t/25} = 18 \cdot 2^{-t/25}$.

(b) $18 \cdot 2^{-t/25} = 2 \quad\Rightarrow\quad 2^{-t/25} = \frac{1}{9} \quad\Rightarrow\quad -\frac{1}{25}t\ln 2 = -\ln 9 \quad\Rightarrow\quad t = 25\frac{\ln 9}{\ln 2} \approx 79$ years

17. (a) $C'(t) = -kC(t) \quad\Rightarrow\quad C(t) = C(0)\,e^{-kt}$ by Theorem 10.4.2. But $C(0) = C_0$, so $C(t) = C_0e^{-kt}$.

(b) $C(30) = \frac{1}{2}C_0$ since the concentration is reduced by half. Thus, $\frac{1}{2}C_0 = C_0e^{-30k} \quad\Rightarrow \ln\frac{1}{2} = -30k \quad\Rightarrow$
$k = -\frac{1}{30}\ln\frac{1}{2} = \frac{1}{30}\ln 2$. Since 10% of the original concentration remains if 90% is eliminated, we want the value of t such that $C(t) = \frac{1}{10}C_0$. Therefore, $\frac{1}{10}C_0 = C_0e^{-t(\ln 2)/30} \quad\Rightarrow\quad \ln 0.1 = -t(\ln 2)/30 \quad\Rightarrow$
$t = -\frac{30}{\ln 2}\ln 0.1 \approx 100$ h.

18. (a) Let $t = 0$ correspond to 1980 so that $P = 4.45e^{kt}$ is a starting point for the model. When $t = 10$, $P = 5.3$. So
$5.3 = 4.45e^{10k} \quad\Rightarrow\quad 10k = \ln\frac{5.3}{4.45} \quad\Rightarrow\quad k = \frac{1}{10}\ln\frac{5.3}{4.45} \approx 0.01748$, and the model is $P(t) = 4.45e^{0.01748t}$.
For the year 2020, $t = 40$, and $P(40) = 4.45e^{40k} \approx 8.95$ billion.

(b) $P = 10 \quad\Rightarrow\quad 4.45e^{kt} = 10 \quad\Rightarrow\quad \frac{10}{4.45} = e^{kt} \quad\Rightarrow\quad kt = \ln\frac{10}{4.45} \quad\Rightarrow\quad t = 10\dfrac{\ln\frac{10}{4.45}}{\ln\frac{5.3}{4.45}} \approx 46.32$ years, that is, in 2026.

(c) $P(t) = \dfrac{K}{1 + Ae^{-kt}} = \dfrac{100}{1 + Ae^{-kt}}$, where $A = \dfrac{100 - 4.45}{4.45} \approx 21.47$. So a model is
$P(t) = \dfrac{100}{1 + 21.47e^{-0.01748t}}$ and $P(40) \approx 8.57$ billion, slightly lower than our estimate in part (a).

(d) $P = 10 \quad\Rightarrow\quad 1 + Ae^{-kt} = \frac{100}{10} \quad\Rightarrow\quad Ae^{-kt} = 9 \quad\Rightarrow\quad e^{-kt} = 9/A \quad\Rightarrow\quad -kt = \ln(9/A) \quad\Rightarrow$
$t = -\dfrac{1}{k}\ln\dfrac{9}{A} \approx 49.74$ years (that is, in 2029), which is later than the prediction in part (b).

19. (a) $\dfrac{dL}{dt} \propto L_\infty - L \quad\Rightarrow\quad \dfrac{dL}{dt} = k(L_\infty - L) \quad\Rightarrow\quad \displaystyle\int \frac{dL}{L_\infty - L} = \int k\,dt \quad\Rightarrow\quad -\ln|L_\infty - L| = kt + C \quad\Rightarrow$
$\ln|L_\infty - L| = -kt - C \quad\Rightarrow\quad |L_\infty - L| = e^{-kt-C} \quad\Rightarrow\quad L_\infty - L = Ae^{-kt} \quad\Rightarrow\quad L = L_\infty - Ae^{-kt}$. At
$t = 0$, $L = L(0) = L_\infty - A \quad\Rightarrow\quad A = L_\infty - L(0) \quad\Rightarrow\quad L(t) = L_\infty - [L_\infty - L(0)]\,e^{-kt}$

(b) $L_\infty = 53$ cm, $L(0) = 10$ cm, and $k = 0.2 \quad\Rightarrow\quad L(t) = 53 - (53 - 10)\,e^{-0.2t} = 53 - 43e^{-0.2t}$.

20. Denote the amount of salt in the tank (in kg) by y. $y(0) = 0$ since initially there is only water in the tank. The rate at which y increases is equal to the rate at which salt flows into the tank minus the rate at which it flows out. That rate is $\dfrac{dy}{dt} = 0.1\dfrac{\text{kg}}{\text{L}} \times 10\dfrac{\text{L}}{\text{min}} - \frac{y}{100}\dfrac{\text{kg}}{\text{L}} \times 10\dfrac{\text{L}}{\text{min}} = 1 - \dfrac{y}{10}\dfrac{\text{kg}}{\text{min}} \quad\Rightarrow\quad \displaystyle\int \frac{dy}{10 - y} = \int \frac{1}{10}\,dt \quad\Rightarrow$
$-\ln|10 - y| = \frac{1}{10}t + C \quad\Rightarrow\quad 10 - y = Ae^{-t/10}$. $y(0) = 0 \quad\Rightarrow\quad 10 = A \quad\Rightarrow\quad y = 10\left(1 - e^{-t/10}\right)$. At $t = 6$ minutes, $y = 10\left(1 - e^{-6/10}\right) \approx 4.512$ kg.

21. Let P be the population and I be the number of infected people. The rate of spread dI/dt is jointly proportional to I and to $P - I$, so for some constant k, $dI/dt = kI\,(P - I)$ $\Rightarrow$ $I = \dfrac{I_0 P}{I_0 + (P - I_0)\,e^{-kPt}}$ (from the discussion of logistic growth in Section 10.5).

Now, measuring t in days, we substitute $t = 7$, $P = 5000$, $I_0 = 160$ and $I\,(7) = 1200$ to find k:

$1200 = \dfrac{160 \cdot 5000}{160 + (5000 - 160)\,e^{-5000 \cdot 7 \cdot k}}$ $\Leftrightarrow$ $k \approx 0.00006448$. So, putting $I = 5000 \times 80\% = 4000$, we solve for t: $4000 = \dfrac{160 \cdot 5000}{160 + (5000 - 160)\,e^{-0.00006448 \cdot 5000 \cdot t}}$ $\Leftrightarrow$ $160 + 4840e^{-0.3224t} = 200$ $\Leftrightarrow$

$-0.3224t = \ln \frac{40}{4840}$ $\Leftrightarrow$ $t \approx 14.9$. So it takes about 15 days for 80% of the population to be infected.

22. $\dfrac{1}{R}\dfrac{dR}{dt} = \dfrac{k}{S}\dfrac{dS}{dt}$ $\Rightarrow$ $\dfrac{d}{dt}(\ln R) = \dfrac{d}{dt}(k \ln S)$ $\Rightarrow$ $\ln R = k \ln S + C$ $\Rightarrow$ $R = e^{k \ln S + C} = e^C \left(e^{\ln S}\right)^k$ $\Rightarrow$ $R = AS^k$, where $A = e^C$ is a positive constant.

23. $\dfrac{dh}{dt} = -\dfrac{R}{V}\left(\dfrac{h}{k+h}\right)$ $\Rightarrow$ $\displaystyle\int \frac{k+h}{h}\,dh = \int \left(-\frac{R}{V}\right) dt$ $\Rightarrow$ $\displaystyle\int \left(1 + \frac{k}{h}\right) dh = -\frac{R}{V}\int 1\,dt$ $\Rightarrow$

$h + k \ln h = -\dfrac{R}{V}t + C$. This equation gives a relationship between h and t, but it is not possible to isolate h and express it in terms of t.

24. $dx/dt = 0.4x - 0.002xy$, $dy/dt = -0.2y + 0.000008xy$

(a) The xy terms represent encounters between the birds and the insects. Since the y-population increases from these terms and the x-population decreases, we expect y to represent the birds and x the insects.

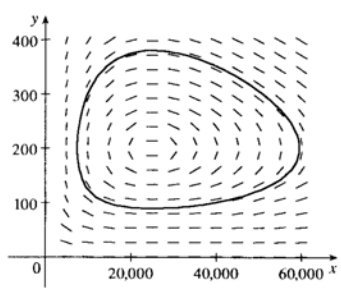
(b) x and y are constant $\Rightarrow$ $x' = 0$ and $y' = 0$ $\Rightarrow$

$$\left\{\begin{array}{l} 0 = 0.4x - 0.002xy \\ 0 = -0.2y + 0.000008xy \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 0 = 0.4x\,(1 - 0.005y) \\ 0 = -0.2y\,(1 - 0.00004x) \end{array}\right. \Rightarrow y = 0 \text{ and } x = 0 \text{ (zero populations)}$$

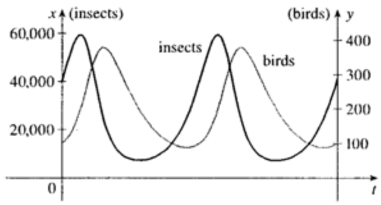
or $y = \frac{1}{0.005} = 200$ and $x = \frac{1}{0.00004} = 25{,}000$. The non-trivial solution represents the population sizes needed so that there are no changes in either the number of birds or the number of insects.

(c) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-0.2y + 0.000008xy}{0.4x - 0.002xy}$

(d)



At $(x, y) = (40{,}000,\ 100)$, $dx/dt = 8000 > 0$, so as t increases we are proceeding in a counterclockwise direction. The populations increase to approximately (59,646, 200), at which point the insect population starts to decrease. The birds attain a maximum population of about 380 when the insect population is 25,000. The populations decrease to about (7370, 200), at which point the insect population starts to increase. The birds attain a minimum population of about 88 when the insect population is 25,000, and then the cycle repeats.

(e) 

Both graphs have the same period and the bird population peaks about a quarter-cycle after the insect population.

25. (a) $dx/dt = 0.4x\,(1 - 0.000005x) - 0.002xy$, $dy/dt = -0.2y + 0.000008xy$. If $y = 0$, then $dx/dt = 0.4x\,(1 - 0.000005x)$, so $dx/dt = 0 \iff x = 0$ or $x = 200{,}000$, which shows that the insect population increases logistically with a carrying capacity of 200,000. Since $dx/dt > 0$ for $0 < x < 200{,}000$ and $dx/dt < 0$ for $x > 200{,}000$, we expect the insect population to stabilize at 200,000.

(b) x and y are constant $\Rightarrow$ $x' = 0$ and $y' = 0$ $\Rightarrow$

$$\begin{Bmatrix} 0 = 0.4x\,(1 - 0.000005x) - 0.002xy \\ 0 = -0.2y + 0.000008xy \end{Bmatrix} \Rightarrow \begin{cases} 0 = 0.4x\,[(1 - 0.000005x) - 0.005y] \\ 0 = y\,(-0.2 + 0.000008x) \end{cases}$$

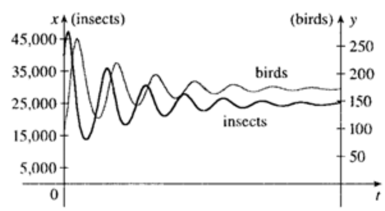
The second equation is true if $y = 0$ or $x = \frac{0.2}{0.000008} = 25{,}000$. If $y = 0$ in the first equation, then either $x = 0$ or $x = \frac{1}{0.000005} = 200{,}000$. If $x = 25{,}000$, then $0 = 0.4\,(25{,}000)\,[(1 - 0.000005 \cdot 25{,}000) - 0.005y]$ $\Rightarrow$ $0 = 10{,}000\,[(1 - 0.125) - 0.005y]$ $\Rightarrow$ $0 = 8750 - 50y$ $\Rightarrow$ $y = 175$.

Case (i): $y = 0$, $x = 0$: zero populations

Case (ii): $y = 0$, $x = 200{,}000$: in the absence of birds, the insect population is constantly 200,000.

Case (iii): $x = 25{,}000$, $y = 175$: the predator/prey interaction balances and the populations are stable.

(c) The populations of the birds and insects fluctuate around 175 and 25,000, respectively, and eventually stabilize at those values.

(d) 

26. First note that, in this question, "weighs" is used in the informal sense, so what we really require is Barbara's mass m in kg as a function of t. Barbara's net intake of calories per day at time t (measured in days) is $c\,(t) = 1600 - 850 - 15m\,(t) = 750 - 15m\,(t)$, where $m\,(t)$ is her mass at time t. We are given that $m\,(0) = 60$ kg and $\dfrac{dm}{dt} = \dfrac{c\,(t)}{10{,}000}$, so $\dfrac{dm}{dt} = \dfrac{750 - 15m}{10{,}000} = \dfrac{150 - 3m}{2000} = \dfrac{-3\,(m - 50)}{2000}$ with $m\,(0) = 60$. From $\displaystyle\int \frac{dm}{m - 50} = \int \frac{-3\,dt}{2000}$, we get $\ln|m - 50| = -\frac{3}{2000}t + C$. Since $m\,(0) = 60$, $C = \ln 10$. Now $\ln\dfrac{|m - 50|}{10} = -\dfrac{3t}{2000}$, so $|m - 50| = 10e^{-3t/2000}$. The quantity $m - 50$ is continuous and initially positive; the right-hand side is never zero. Thus, $m - 50$ is positive for all t, and $m\,(t) = 50 + 10e^{-3t/2000}$ kg. As $t \to \infty$, $m\,(t) \to 50$ kg. Thus, Barbara's mass gradually settles down to 50 kg.

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Problems Plus

1. We use the Fundamental Theorem of Calculus to differentiate the given equation:

$[f(x)]^2 = 100 + \int_0^x \left\{[f(t)]^2 + [f'(t)]^2\right\} dt \quad \Rightarrow \quad 2f(x)\,f'(x) = [f(x)]^2 + [f'(x)]^2 \quad \Rightarrow$

$[f(x)]^2 + [f'(x)]^2 - 2f(x)\,f'(x) = 0 \quad \Rightarrow \quad [f(x) - f'(x)]^2 = 0 \quad \Leftrightarrow \quad f(x) = f'(x)$. We can solve this as a separable equation, or else use Theorem 10.4.2 with $k = 1$, which says that the solutions are $f(x) = Ce^x$. Now $[f(0)]^2 = 100$, so $f(0) = C = \pm 10$, and hence $f(x) = \pm 10e^x$ are the only functions satisfying the given equation.

2. $(fg)' = f'g'$, where $f(x) = e^{x^2} \quad \Rightarrow \quad \left(e^{x^2}g\right)' = 2xe^{x^2}g'$. Since the student's mistake did not affect the answer, $\left(e^{x^2}g\right)' = e^{x^2}g' + 2xe^{x^2}g = 2xe^{x^2}g'$. So $(2x-1)\,g' = 2xg$, or $\dfrac{g'}{g} = \dfrac{2x}{2x-1} = 1 + \dfrac{1}{2x-1} \quad \Rightarrow$

$\ln|g(x)| = x + \frac{1}{2}\ln(2x-1) + C \quad \Rightarrow \quad g(x) = Ae^x\sqrt{2x-1}$.

3. $f'(x) = \displaystyle\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{f(x)\,[f(h) - 1]}{h}$ [since $f(x+h) = f(x)\,f(h)$]

$\displaystyle = f(x)\lim_{h\to 0}\frac{f(h) - 1}{h} = f(x)\lim_{h\to 0}\frac{f(h) - f(0)}{h - 0} = f(x)\,f'(0) = f(x)$

Therefore, $f'(x) = f(x)$ for all x and from Theorem 10.4.2 we get $f(x) = Ae^x$. Now $f(0) = 1 \quad \Rightarrow \quad A = 1$ $\Rightarrow \quad f(x) = e^x$.

4. $\displaystyle\left[\int f(x)\,dx\right]\left[\int \frac{dx}{f(x)}\right] = -1 \Rightarrow \int \frac{dx}{f(x)} = \frac{-1}{\int f(x)\,dx} \Rightarrow \frac{1}{f(x)} = \frac{f(x)}{\left[\int f(x)\,dx\right]^2}$ (after differentiating)

$\Rightarrow \quad \int f(x)\,dx = \pm f(x)$ (after taking square roots) $\quad \Rightarrow \quad f(x) = \pm f'(x)$ (after differentiating again) $\quad \Rightarrow$ $y = Ae^x$ or $y = Ae^{-x}$ by (10.4.2). Therefore, $f(x) = Ae^x$ or $f(x) = Ae^{-x}$, for all non-zero constants A, are the functions satisfying the original equation.

5. (a) We are given that $V = \frac{1}{3}\pi r^2 h$, $dV/dt = 60{,}000\pi$ ft^3/h, and $r = 1.5h = \frac{3}{2}h$. So $V = \frac{1}{3}\pi\left(\frac{3}{2}h\right)^2 h = \frac{3}{4}\pi h^3$

$\Rightarrow \quad \dfrac{dV}{dt} = \frac{3}{4}\pi \cdot 3h^2\dfrac{dh}{dt} = \frac{9}{4}\pi h^2\dfrac{dh}{dt}$. Therefore, $\dfrac{dh}{dt} = \dfrac{4\,(dV/dt)}{9\pi h^2} = \dfrac{240{,}000\pi}{9\pi h^2} = \dfrac{80{,}000}{3h^2}$ $(\bigstar)$ $\quad \Rightarrow$

$\int 3h^2\,dh = \int 80{,}000\,dt \quad \Rightarrow \quad h^3 = 80{,}000t + C$. When $t = 0$, $h = 60$. Thus, $C = 60^3 = 216{,}000$, so $h^3 = 80{,}000t + 216{,}000$. Let $h = 100$. Then $100^3 = 1{,}000{,}000 = 80{,}000t + 216{,}000 \quad \Rightarrow$ $80{,}000t = 784{,}000 \quad \Rightarrow \quad t = 9.8$, so the time required is 9.8 hours.

(b) The floor area of the silo is $F = \pi \cdot 200^2 = 40{,}000\pi$ ft^2, and the area of the base of the pile is $A = \pi r^2 = \pi\left(\frac{3}{2}h\right)^2 = \frac{9\pi}{4}h^2$. So the area of the floor which is not covered when $h = 60$ is $F - A = 40{,}000\pi - 8100\pi = 31{,}900\pi \approx 100{,}217$ ft^2. Now $A = \frac{9\pi}{4}h^2 \quad \Rightarrow \quad dA/dt = \frac{9\pi}{4}\cdot 2h\,(dh/dt)$, and from $(\bigstar)$ in part (a) we know that when $h = 60$, $dh/dt = \frac{80{,}000}{3(60)^2} = \frac{200}{27}$ ft/h. Therefore,

$dA/dt = \frac{9\pi}{4}\,(2)\,(60)\left(\frac{200}{27}\right) = 2000\pi \approx 6283$ ft^2/h.

(c) At $h = 90$ ft, $dV/dt = 60{,}000\pi - 20{,}000\pi = 40{,}000\pi$ ft^3/h. From $(\bigstar)$ in part (a),

$\dfrac{dh}{dt} = \dfrac{4\,(dV/dt)}{9\pi h^2} = \dfrac{4\,(40{,}000\pi)}{9\pi h^2} = \dfrac{160{,}000}{9h^2} \quad \Rightarrow \quad \int 9h^2\,dh = \int 160{,}000\,dt \quad \Rightarrow \quad 3h^3 = 160{,}000t + C$.

When $t = 0$, $h = 90$; therefore, $C = 3 \cdot 729{,}000 = 2{,}187{,}000$. So $3h^3 = 160{,}000t + 2{,}187{,}000$. At the top, $h = 100 \Rightarrow 3\,(100)^3 = 160{,}000t + 2{,}187{,}000 \Rightarrow t = \frac{813{,}000}{160{,}000} \approx 5.1$. The pile reaches the top after about 5.1 h.

6. Let b be the number of hours before noon that it began to snow, t the time measured in hours after noon, and $x = x(t) =$ distance traveled by the plow at time t. Then $dx/dt =$ speed of plow. Since the snow falls steadily, the height at time t is $h(t) = k(t+b)$, where k is a constant. We are given that the rate of removal is constant, say R (in m^3/h). If the width of the path is w, then $R = \text{height} \times \text{width} \times \text{speed} = h(t) \times w \times \dfrac{dx}{dt} = k(t+b)\,w\dfrac{dx}{dt}$.

Thus, $\dfrac{dx}{dt} = \dfrac{C}{t+b}$, where $C = \dfrac{R}{kw}$ is a constant. This is a separable equation. $\int dx = C\int \dfrac{dt}{t+b} \quad \Rightarrow$

$x(t) = C\ln(t+b) + K$.

Put $t = 0$: $0 = C\ln b + K \quad \Rightarrow \quad K = -C\ln b$, so $x(t) = C\ln(t+b) - C\ln b = C\ln(1 + t/b)$.

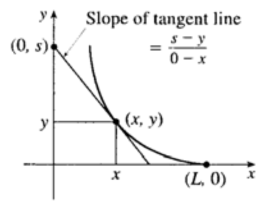
Put $t = 1$: $6000 = C\ln(1 + 1/b)$ $[x = 6 \text{ km}]$.

Put $t = 2$: $9000 = C\ln(1 + 2/b)$ $[x = (6+3) \text{ km}]$.

Solve for b: $\dfrac{\ln(1+1/b)}{6000} = \dfrac{\ln(1+2/b)}{9000} \quad \Rightarrow \quad 3\ln\left(1+\dfrac{1}{b}\right) = 2\ln\left(1+\dfrac{2}{b}\right) \quad \Rightarrow \quad \left(1+\dfrac{1}{b}\right)^3 = \left(1+\dfrac{2}{b}\right)^2$

$\Rightarrow \quad 1 + \dfrac{3}{b} + \dfrac{3}{b^2} + \dfrac{1}{b^3} = 1 + \dfrac{4}{b} + \dfrac{4}{b^2} \quad \Rightarrow \quad \dfrac{1}{b} + \dfrac{1}{b^2} - \dfrac{1}{b^3} = 0 \quad \Rightarrow \quad b^2 + b - 1 = 0 \quad \Rightarrow \quad b = \frac{-1 \pm \sqrt{5}}{2}$. But $b > 0$, so $b = \frac{-1+\sqrt{5}}{2} \approx 0.618 \text{ h} \approx 37$ min. The snow began to fall $\frac{\sqrt{5}-1}{2}$ hours before noon, that is, at about 11:23 A.M.

7. (a) While running from $(L, 0)$ to (x, y), the dog travels a distance



$s = \displaystyle\int_x^L \sqrt{1 + (dy/dx)^2}\,dx = -\int_L^x \sqrt{1 + (dy/dx)^2}\,dx$, so

$\dfrac{ds}{dx} = -\sqrt{1 + (dy/dx)^2}$. The dog and rabbit run at the same speed, so the rabbit's position when the dog has traveled a distance s is $(0, s)$. Since the dog runs straight for the rabbit, $\dfrac{dy}{dx} = \dfrac{s-y}{0-x}$ (see the figure).

Thus, $s = y - x\dfrac{dy}{dx} \quad \Rightarrow \quad \dfrac{ds}{dx} = \dfrac{dy}{dx} - \left(x\dfrac{d^2y}{dx^2} + 1\dfrac{dy}{dx}\right) = -x\dfrac{d^2y}{dx^2}$. Equating the two expressions for $\dfrac{ds}{dx}$ gives us $x\dfrac{d^2y}{dx^2} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$, as claimed.

(b) Letting $z = \dfrac{dy}{dx}$, we obtain the differential equation $x\dfrac{dz}{dx} = \sqrt{1+z^2}$, or $\dfrac{dz}{\sqrt{1+z^2}} = \dfrac{dx}{x}$. Integrating:

$\ln x = \displaystyle\int \frac{dz}{\sqrt{1+z^2}} \overset{25}{=} \ln\left|z + \sqrt{1+z^2}\right| + C$. When $x = L$, $z = dy/dx = 0$, so $\ln L = \ln 1 + C$. Therefore,

$C = \ln L$, so $\ln x = \ln\left(\sqrt{1+z^2} + z\right) + \ln L = \ln\left(L\left(\sqrt{1+z^2} + z\right)\right) \quad \Rightarrow \quad x = L\left(\sqrt{1+z^2} + z\right) \quad \Rightarrow$

$\sqrt{1+z^2} = \dfrac{x}{L} - z \quad \Rightarrow \quad 1 + z^2 = \left(\dfrac{x}{L}\right)^2 - \dfrac{2xz}{L} + z^2 \quad \Rightarrow \quad \left(\dfrac{x}{L}\right)^2 - 2z\left(\dfrac{x}{L}\right) - 1 = 0 \quad \Rightarrow$

$z = \dfrac{(x/L)^2 - 1}{2(x/L)} = \dfrac{x^2 - L^2}{2Lx} = \dfrac{x}{2L} - \dfrac{L}{2}\dfrac{1}{x}$ (for $x > 0$). Since $z = \dfrac{dy}{dx}$, $y = \dfrac{x^2}{4L} - \dfrac{L}{2}\ln x + C_1$. Since

$y = 0$ when $x = L$, $0 = \dfrac{L}{4} - \dfrac{L}{2}\ln L + C_1 \quad \Rightarrow \quad C_1 = \dfrac{L}{2}\ln L - \dfrac{L}{4}$. Thus,

$y = \dfrac{x^2}{4L} - \dfrac{L}{2}\ln x + \dfrac{L}{2}\ln L - \dfrac{L}{4} = \dfrac{x^2 - L^2}{4L} - \dfrac{L}{2}\ln\left(\dfrac{x}{L}\right)$.

(c) As $x \to 0^+$, $y \to \infty$, so the dog never catches the rabbit.

8. (a) If the dog runs twice as fast as the rabbit, then the rabbit's position when the dog has traveled a distance s is $(0, s/2)$. Since the dog runs straight toward the rabbit, the tangent line to the dog's path has slope $\dfrac{dy}{dx} = \dfrac{s/2 - y}{0 - x}$. Thus, $s = 2y - 2x\dfrac{dy}{dx} \;\Rightarrow\; \dfrac{ds}{dx} = 2\dfrac{dy}{dx} - \left(2x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx}\right) = -2x\dfrac{d^2y}{dx^2}$. From Problem 7(a), $\dfrac{ds}{dx} = -\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$, so $2x\dfrac{d^2y}{dx^2} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$. Letting $z = \dfrac{dy}{dx}$, we obtain the differential equation $2x\dfrac{dz}{dx} = \sqrt{1 + z^2}$, or $\dfrac{2dz}{\sqrt{1 + z^2}} = \dfrac{dx}{x}$. Integrating, we get $\ln x = \displaystyle\int \frac{2dz}{\sqrt{1 + z^2}} = 2\ln\left|\sqrt{1 + z^2} + z\right| + C$. [See Problem 7(b).] When $x = L$, $z = dy/dx = 0$, so $\ln L = 2\ln 1 + C = C$. Thus, $\ln x = 2\ln\left(\sqrt{1 + z^2} + z\right) + \ln L = \ln\left(L\left(\sqrt{1 + z^2} + z\right)^2\right) \;\Rightarrow$

$$x = L\left(\sqrt{1 + z^2} + z\right)^2 \;\Rightarrow\; \sqrt{1 + z^2} = \sqrt{\frac{x}{L}} - z \;\Rightarrow\; 1 + z^2 = \frac{x}{L} - 2\sqrt{\frac{x}{L}}\,z + z^2 \;\Rightarrow\; 2\sqrt{\frac{x}{L}}\,z = \frac{x}{L} - 1$$

$\Rightarrow\; \dfrac{dy}{dx} = z = \dfrac{1}{2}\sqrt{\dfrac{x}{L}} - \dfrac{1}{2\sqrt{x/L}} = \dfrac{1}{2\sqrt{L}}x^{1/2} - \dfrac{\sqrt{L}}{2}x^{-1/2} \;\Rightarrow\; y = \dfrac{1}{3\sqrt{L}}x^{3/2} - \sqrt{L}x^{1/2} + C_1$. When $x = L$, $y = 0$, so $0 = \dfrac{1}{3\sqrt{L}}L^{3/2} - \sqrt{L}L^{1/2} + C_1 = \dfrac{L}{3} - L + C_1 = C_1 - \frac{2}{3}L$. Therefore, $C_1 = \frac{2}{3}L$ and $y = \dfrac{x^{3/2}}{3\sqrt{L}} - \sqrt{L}x^{1/2} + \frac{2}{3}L$. As $x \to 0$, $y \to \frac{2}{3}L$, so the dog catches the rabbit when the rabbit is at $\left(0, \frac{2}{3}L\right)$. (At that point, the dog has traveled a distance of $\frac{4}{3}L$, twice as far as the rabbit has run.)

(b) As in the solutions to part (a) and Problem 7, we get $z = \dfrac{dy}{dx} = \dfrac{x^2}{2L^2} - \dfrac{L^2}{2x^2}$ and hence $y = \dfrac{x^3}{6L^2} + \dfrac{L^2}{2x} - \frac{2}{3}L$. We want to minimize the distance D from the dog at (x, y) to the rabbit at $(0, 2s)$. Now $s = \frac{1}{2}y - \frac{1}{2}x\dfrac{dy}{dx} \;\Rightarrow$

$2s = y - xz = \dfrac{L^2}{x} - \dfrac{x^3}{3L^2} - \dfrac{2L}{3}$, so

$$\begin{aligned} D &= \sqrt{(x - 0)^2 + \left[\left(\frac{x^3}{6L^2} + \frac{L^2}{2x} - \frac{2L}{3}\right) - \left(\frac{L^2}{x} - \frac{x^3}{3L^2} - \frac{2L}{3}\right)\right]^2} \\ &= \sqrt{x^2 + \left(\frac{L^2}{2x} - \frac{x^3}{2L^2}\right)^2} = \sqrt{\frac{x^6}{4L^4} + \frac{x^2}{2} + \frac{L^4}{4x^2}} = \sqrt{\left(\frac{x^3}{2L^2} + \frac{L^2}{2x}\right)^2} = \frac{x^3}{2L^2} + \frac{L^2}{2x} \end{aligned}$$

$$D' = 0 \;\Leftrightarrow\; \frac{3x^2}{2L^2} - \frac{L^2}{2x^2} = 0 \;\Leftrightarrow\; \frac{3x^2}{2L^2} = \frac{L^2}{2x^2} \;\Leftrightarrow\; x^4 = \frac{L^4}{3} \;\Leftrightarrow\; x = \frac{L}{\sqrt[4]{3}},\; x > 0,\; L > 0.$$

Since $D''(x) = \dfrac{3x}{L^2} + \dfrac{L^2}{x^3} > 0$ for all $x > 0$, we know that $D\left(\dfrac{L}{\sqrt[4]{3}}\right) = \dfrac{\left(L \cdot 3^{-1/4}\right)^3}{2L^2} + \dfrac{L^2}{2\left(L \cdot 3^{-1/4}\right)} = \dfrac{2L}{3^{3/4}}$ is the minimum value of D, that is, the closest the dog gets to the rabbit. The positions at this distance are

Dog: $(x, y) = \left(\dfrac{L}{\sqrt[4]{3}}, \left(\dfrac{5}{3^{7/4}} - \dfrac{2}{3}\right)L\right) = \left(\dfrac{L}{\sqrt[4]{3}}, \dfrac{5\sqrt[4]{3} - 6}{9}L\right)$

Rabbit: $(0, 2s) = \left(0, \dfrac{8\sqrt[4]{3}L}{9} - \dfrac{2L}{3}\right) = \left(0, \dfrac{8\sqrt[4]{3} - 6}{9}L\right)$

9. (a) $\dfrac{d^2y}{dx^2} = k\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$. Setting $z = \dfrac{dy}{dx}$, we get $\dfrac{dz}{dx} = k\sqrt{1+z^2} \Rightarrow \dfrac{dz}{\sqrt{1+z^2}} = k\,dx$. Using Formula 25 gives $\ln\left(z+\sqrt{1+z^2}\right) = kx + c \Rightarrow z + \sqrt{1+z^2} = Ce^{kx}$ (where $C = e^c$) $\Rightarrow \sqrt{1+z^2} = Ce^{kx} - z$ $\Rightarrow 1 + z^2 = C^2e^{2kx} - 2Ce^{kx}z + z^2 \Rightarrow 2Ce^{kx}z = C^2e^{2kx} - 1 \Rightarrow z = \dfrac{C}{2}e^{kx} - \dfrac{1}{2C}e^{-kx}$. Now $\dfrac{dy}{dx} = \dfrac{C}{2}e^{kx} - \dfrac{1}{2C}e^{-kx} \Rightarrow y = \dfrac{C}{2k}e^{kx} + \dfrac{1}{2Ck}e^{-kx} + C'$. From the diagram in the text, we see that $y(0) = a$ and $y(\pm b) = h$. $a = y(0) = \dfrac{C}{2k} + \dfrac{1}{2Ck} + C' \Rightarrow C' = a - \dfrac{C}{2k} - \dfrac{1}{2Ck}$ $\Rightarrow y = \dfrac{C}{2k}\left(e^{kx}-1\right) + \dfrac{1}{2Ck}\left(e^{-kx}-1\right) + a$. From $h = y(\pm b)$, we find $h = \dfrac{C}{2k}\left(e^{kb}-1\right) + \dfrac{1}{2Ck}\left(e^{-kb}-1\right) + a$ and $h = \dfrac{C}{2k}\left(e^{-kb}-1\right) + \dfrac{1}{2Ck}\left(e^{kb}-1\right) + a$. Subtracting the second equation from the first, we get $0 = \dfrac{C}{k}\dfrac{e^{kb}-e^{-kb}}{2} - \dfrac{1}{Ck}\dfrac{e^{kb}-e^{-kb}}{2} = \dfrac{1}{k}\left(C - \dfrac{1}{C}\right)\sinh(kb)$. Now $k > 0$ and $b > 0$, so $\sinh(kb) > 0$ and $C = \pm 1$. If $C = 1$, then $y = \dfrac{1}{2k}\left(e^{kx}-1\right) + \dfrac{1}{2k}\left(e^{-kx}-1\right) + a = \dfrac{1}{k}\dfrac{e^{kx}+e^{-kx}}{2} - \dfrac{1}{k} + a = a + \dfrac{1}{k}(\cosh kx - 1)$. If $C = -1$, then $y = -\dfrac{1}{2k}\left(e^{kx}-1\right) - \dfrac{1}{2k}\left(e^{-kx}-1\right) + a = \dfrac{-1}{k}\dfrac{e^{kx}+e^{-kx}}{2} + \dfrac{1}{k} + a = a - \dfrac{1}{k}(\cosh kx - 1)$. Since $k > 0$, $\cosh kx \ge 1$, and $y \ge a$, we conclude that $C = 1$ and $y = a + \dfrac{1}{k}(\cosh kx - 1)$, where $h = y(b) = a + \dfrac{1}{k}(\cosh kb - 1)$. Since $\cosh(kb) = \cosh(-kb)$, there is no further information to extract from the condition that $y(b) = y(-b)$. However, we could replace a with the expression $h - \dfrac{1}{k}(\cosh kb - 1)$, obtaining $y = h + \dfrac{1}{k}(\cosh kx - \cosh kb)$. It would be better still to keep a in the expression for y, and use the expression for h to solve for k in terms of a, b, and h. That would enable us to express y in terms of x and the given parameters a, b, and h. Sadly, it is not possible to solve for k in closed form. That would have to be done by numerical methods when specific parameter values are given.

(b) The length of the cable is

$$L = \int_{-b}^{b}\sqrt{1+(dy/dx)^2}\,dx = \int_{-b}^{b}\sqrt{1+\sinh^2 kx}\,dx = \int_{-b}^{b}\cosh kx\,dx = \left[\tfrac{1}{k}\sinh kx\right]_{-b}^{b}$$
$$= (1/k)\left[\sinh(kb) - \sinh(-kb)\right] = (2/k)\sinh(kb)$$

10. Suppose C is a curve with the required property and let $P = (x_0, y_0)$ be a point on C. The equation of the normal line to C at P is $y - y_0 = -\dfrac{1}{y_0'}(x - x_0)$, where y_0' is the value of $\dfrac{dy}{dx}$ at $x = x_0$. This equation makes sense only if $y_0' \neq 0$. If $y_0' = 0$, then the normal line at P is $x = x_0$, which does not intersect the y-axis at all unless $x_0 = 0$.

So let's assume that $y_0' \neq 0$. Then the normal line to C at P intersects the x-axis at $\left(x_0 + y_0y_0', 0\right)$, and it intersects the y-axis at $\left(0, y_0 + x_0/y_0'\right)$. The condition on C implies that

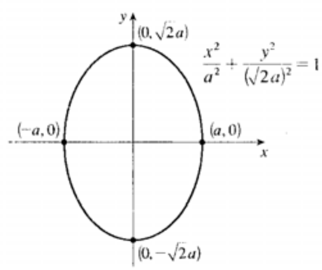
$$\left[\text{distance from } P(x_0, y_0) \text{ to } \left(0, y_0 + x_0/y_0'\right)\right] = \left[\text{distance from } \left(0, y_0 + x_0/y_0'\right) \text{ to } \left(x_0 + y_0y_0', 0\right)\right]$$
$$\sqrt{(0 - x_0)^2 + \left(y_0 + x_0/y_0' - y_0\right)^2} = \sqrt{\left(x_0 + y_0y_0' - 0\right)^2 + \left[0 - \left(y_0 + x_0/y_0'\right)\right]^2}$$

Squaring both sides, we get $x_0^2 + x_0^2/(y_0')^2 = (x_0 + y_0 y_0')^2 + (y_0 + x_0/y_0')^2$ or

$x_0^2 + \dfrac{x_0^2}{(y_0')^2} = x_0^2 + 2x_0 y_0 y_0' + y_0^2 (y_0')^2 + y_0^2 + 2\dfrac{x_0 y_0}{y_0'} + \dfrac{x_0^2}{(y_0')^2}$. Subtracting $x_0^2 + \dfrac{x_0^2}{(y_0')^2}$ from both sides and multiplying by y_0', we get

$$0 = y_0^2 y_0' + y_0^2 (y_0')^3 + 2x_0 y_0 \left[1 + (y_0')^2\right] = y_0 \left\{y_0 y_0' + y_0 (y_0')^3 + 2x_0 \left[1 + (y_0')^2\right]\right\}$$
$$= y_0 \left\{y_0 y_0' \left[1 + (y_0')^2\right] + 2x_0 \left[1 + (y_0')^2\right]\right\} = y_0 (y_0 y_0' + 2x_0) \left[1 + (y_0')^2\right]$$

Since $1 + (y_0')^2 \geq 1 > 0$, we conclude that $y_0 (y_0 y_0' + 2x_0) = 0$. Now P is an arbitrary point on C for which $y_0' \neq 0$. Thus, we have shown that $y(yy' + 2x) = 0$ for points (x, y) along C where $y' \neq 0$. One solution of this equation is $y = 0$, but that curve (the x-axis) doesn't satisfy the condition required of C, since its normal lines at points for $x \neq 0$ don't intersect the y-axis. Thus, we can focus our attention on points of C where $y \neq 0$, and conclude that $yy' + 2x = 0$ at points of C where $y \neq 0$ and $y' \neq 0$. Integrating both sides of $yy' + 2x = 0$, we get $\frac{1}{2}y^2 + x^2 = c$. Clearly $c > 0$ (since $y \neq 0$), so we can write $c = a^2$, where $a = \sqrt{c} > 0$. Thus, $\frac{1}{2}y^2 + x^2 = a^2$ and $x^2/a^2 + y^2/(\sqrt{2}a)^2 = 1$.



This shows that C is (part of) the ellipse centered at $(0, 0)$ with semimajor axis $\sqrt{2}a$ in the y-direction and semiminor axis a in the x-direction. The points of C where $y = 0$ or $y' = 0$ are the vertices $\left(0, \pm\sqrt{2}a\right)$ and $(\pm a, 0)$. At these points, the condition on C is satisfied in a degenerate way. [When $P = (\pm a, 0)$, the normal line at P *is* the x-axis, so *all* the points of the normal line can be viewed as points of intersection with the x-axis. The intersection with the y-axis at $(0, 0)$ is midway between $(a, 0)$ and $(-a, 0)$; one of these points is P, and the other can be regarded as an intersection of the normal line with the x-axis. Similarly, when $P = \left(0, \pm\sqrt{2}a\right)$, the normal line is the y-axis, and the point $\left(0, \pm\sqrt{2}a/2\right)$, which can be regarded as an intersection of the normal line with the y-axis, is midway between P and $(0, 0)$, the intersection with the x-axis.]

Conversely, if C is part of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{2a^2} = 1$ for some $a > 0$, then the normal line at a point (x_0, y_0) of C (other than the four vertices) has equation $y - y_0 = \dfrac{y_0}{2x_0}(x - x_0)$. Its intersections with the coordinate axes are $\left(0, \dfrac{y_0}{2}\right)$ and $(-x_0, 0)$. $\left[\text{distance from } (x_0, y_0) \text{ to } \left(0, \dfrac{y_0}{2}\right)\right]^2 = x_0^2 + \dfrac{y_0^2}{4}$ and $\left[\text{distance from } \left(0, \dfrac{y_0}{2}\right) \text{ to } (-x_0, 0)\right]^2 = x_0^2 + \dfrac{y_0^2}{4}$, so the required condition is met at points other than the four vertices. As we have explained, if we are willing to interpret the condition broadly, then it can be viewed as holding even at the four vertices.

Another Method: Let $P(x_0, y_0)$ be a point on the curve. Since the midpoint of the line segment determined by the normal line from (x_0, y_0) to its intersection with the x-axis has x-coordinate 0, the x-coordinate of the point of intersection with the x-axis must be $-x_0$. Hence, the normal line has slope $\dfrac{y_0 - 0}{x_0 - (-x_0)} = \dfrac{y_0}{2x_0}$. So the tangent line has slope $-\dfrac{2x_0}{y_0}$. This gives the differential equation $y' = -\dfrac{2x}{y} \;\Rightarrow\; y\,dy = -2x\,dx \;\Rightarrow$ $\int y\,dy = \int(-2x)\,dx \;\Rightarrow\; \frac{1}{2}y^2 = -x^2 + C \;\Rightarrow\; x^2 + \frac{1}{2}y^2 = C \;\; (C > 0)$.

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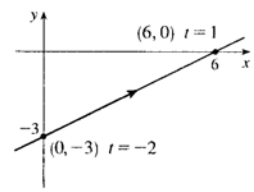
11 Parametric Equations and Polar Coordinates

11.1 Curves Defined by Parametric Equations

1. (a) $x = 2t + 4$, $y = t - 1$

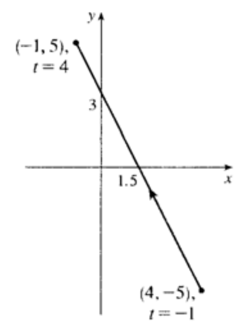
t	-3	-2	-1	0	1	2
x	-2	0	2	4	6	8
y	-4	-3	-2	-1	0	1



(b) $x = 2t + 4$, $y = t - 1 \quad\Rightarrow\quad x = 2(y + 1) + 4 = 2y + 6$ or $y = \frac{1}{2}x - 3$

2. (a) $x = 3 - t$, $y = 2t - 3$, $-1 \le t \le 4$

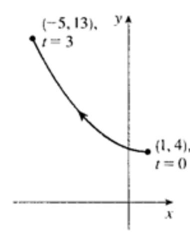
t	-1	0	1	2	3	4
x	4	3	2	1	0	-1
y	-5	-3	-1	1	3	5



(b) $x = 3 - t \quad\Rightarrow\quad t = 3 - x \quad\Rightarrow\quad y = 2t - 3 = 2(3 - x) - 3 \quad\Rightarrow$ $y = 3 - 2x$

3. (a) $x = 1 - 2t$, $y = t^2 + 4$, $0 \le t \le 3$

t	0	1	2	3
x	1	-1	-3	-5
y	4	5	8	13



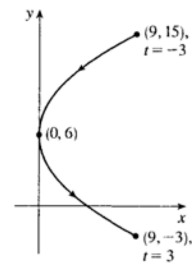
(b) $x = 1 - 2t \quad\Rightarrow\quad 2t = 1 - x \quad\Rightarrow\quad t = \dfrac{1 - x}{2} \quad\Rightarrow$

$y = t^2 + 4 = \left(\dfrac{1 - x}{2}\right)^2 + 4 = \frac{1}{4}(x - 1)^2 + 4$ or

$y = \frac{1}{4}x^2 - \frac{1}{2}x + \frac{17}{4}$

4. (a) $x = t^2$, $y = 6 - 3t$

t	-3	-2	-1	0	1	2	3
x	9	4	1	0	1	4	9
y	15	12	9	6	3	0	-3

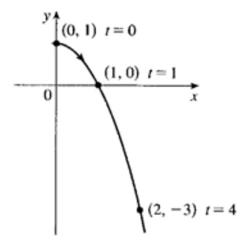


(b) $y = 6 - 3t \;\Rightarrow\; 3t = 6 - y \;\Rightarrow\; t = \dfrac{6-y}{3} \;\Rightarrow$

$x = t^2 = \left(\dfrac{6-y}{3}\right)^2 = \frac{1}{9}(y-6)^2$

5. (a) $x = \sqrt{t}$, $y = 1 - t$

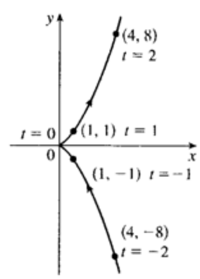
t	0	1	2	3	4
x	0	1	1.414	1.732	2
y	1	0	-1	-2	-3



(b) $x = \sqrt{t} \;\Rightarrow\; t = x^2$. $y = 1 - t = 1 - x^2$. Since $t \geq 0$, $x \geq 0$.

6. (a) $x = t^2$, $y = t^3$

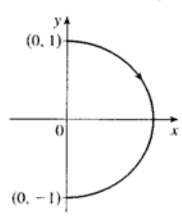
t	-2	-1	0	1	2
x	4	1	0	1	4
y	-8	-1	0	1	8



(b) $y = t^3 \;\Rightarrow\; t = \sqrt[3]{y}$. $x = t^2 = \left(\sqrt[3]{y}\right)^2 = y^{2/3}$. $t \in \mathbb{R}$, $y \in \mathbb{R}$, $x \geq 0$.

7. (a) $x = \sin\theta$, $y = \cos\theta$, $0 \leq \theta \leq \pi$.
$x^2 + y^2 = \sin^2\theta + \cos^2\theta = 1$, $0 \leq x \leq 1$.

(b)

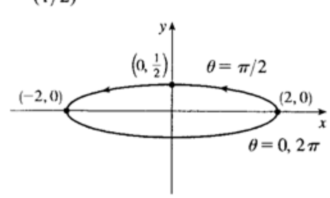


8. (a) $x = 2\cos\theta$, $y = \frac{1}{2}\sin\theta$, $0 \leq \theta \leq 2\pi$.

$1 = \cos^2\theta + \sin^2\theta = \left(\dfrac{x}{2}\right)^2 + \left(\dfrac{y}{1/2}\right)^2$, so

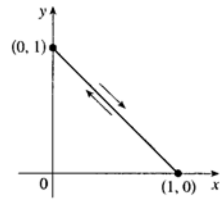
$\dfrac{x^2}{2^2} + \dfrac{y^2}{(1/2)^2} = 1$.

(b)



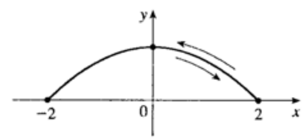
9. (a) $x = \sin^2\theta$, $y = \cos^2\theta$.
$x + y = \sin^2\theta + \cos^2\theta = 1$, $0 \le x \le 1$.
Note that the curve is at $(0, 1)$ whenever $\theta = \pi n$ and is at $(1, 0)$ whenever $\theta = \frac{\pi}{2}n$ for every integer n.

(b)



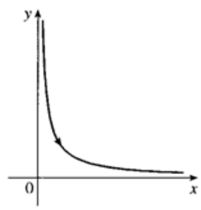
10. (a) $x = 2\cos\theta$, $y = \sin^2\theta$.
$1 = \cos^2\theta + \sin^2\theta = \left(\frac{x}{2}\right)^2 + y$, so
$y = 1 - \frac{x^2}{4}$, $-2 \le x \le 2$. The curve is at $(2, 0)$ whenever $\theta = 2\pi n$.

(b)



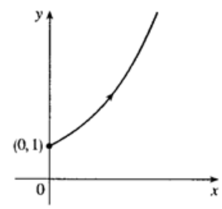
11. (a) $x = e^t$, $y = e^{-t}$, $y = 1/x$, $x > 0$

(b)



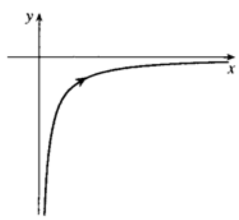
12. (a) $x = \ln t$, $y = \sqrt{t}$, $t \ge 1$. $x = \ln t \Rightarrow t = e^x \Rightarrow y = \sqrt{t} = e^{x/2}$, $x \ge 0$.

(b)



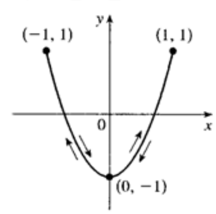
13. (a) $x = \tan\theta + \sec\theta$, $y = \tan\theta - \sec\theta$,
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. $xy = \tan^2\theta - \sec^2\theta = -1$
$\Rightarrow y = -1/x$, $x > 0$.

(b)



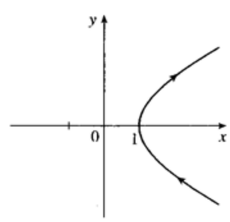
14. (a) $x = \cos t$, $y = \cos 2t$.
$y = \cos 2t = 2\cos^2 t - 1 = 2x^2 - 1$, so
$y + 1 = 2x^2$, $-1 \le x \le 1$.

(b)



15. (a) $x = \cosh t$, $y = \sinh t$, $x^2 - y^2 = \cosh^2 t - \sinh^2 t = 1$, $x \ge 1$

(b)



16. $x = 4 - 4t$, $y = 2t + 5$, $0 \le t \le 2$. $x = 4 - 2(2t) = 4 - 2(y - 5) = -2y + 14$, so the particle moves along the line $y = -\frac{1}{2}x + 7$ from $(4, 5)$ to $(-4, 9)$.

17. $x^2 + y^2 = \cos^2 \pi t + \sin^2 \pi t = 1$, $1 \le t \le 2$, so the particle moves counterclockwise along the circle $x^2 + y^2 = 1$ from $(-1, 0)$ to $(1, 0)$, along the lower half of the circle.

18. $(x - 2)^2 + (y - 3)^2 = \cos^2 t + \sin^2 t = 1$, so the motion takes place on a unit circle centered at $(2, 3)$. As t goes from 0 to 2π, the particle makes one complete counterclockwise rotation around the circle, starting and ending at $(3, 3)$.

19. $\left(\frac{1}{2}x\right)^2 + \left(\frac{1}{3}y\right)^2 = \sin^2 t + \cos^2 t = 1$, so the particle moves once clockwise along the ellipse $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$, starting and ending at $(0, 3)$.

20. $x = \cos^2 t = y^2$, so the particle moves along the parabola $x = y^2$. As t goes from 0 to 4π, the particle moves from $(1, 1)$ down to $(1, -1)$ (at $t = \pi$), back up to $(1, 1)$ again (at $t = 2\pi$), and then repeats this entire cycle between $t = 2\pi$ and $t = 4\pi$.

21. $x = \tan t$, $y = \cot t$, $\frac{\pi}{6} \le t \le \frac{\pi}{3}$. $y = 1/x$ for $\frac{1}{\sqrt{3}} \le x \le \sqrt{3}$. The particle moves along the first quadrant branch of the hyperbola $y = 1/x$ from $\left(\frac{1}{\sqrt{3}}, \sqrt{3}\right)$ to $\left(\sqrt{3}, \frac{1}{\sqrt{3}}\right)$.

22. (a) Note that as $t \to -\infty$, we have $x \to -\infty$ and $y \to \infty$, whereas when $t \to \infty$, both x and $y \to \infty$. This description fits only IV. [But also note that $x(t)$ increases, then decreases, then increases again.]

(b) Note that as $t \to \pm\infty$, $y \to -\infty$. This is only the case with VI.

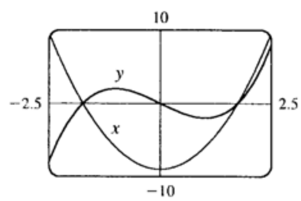
(c) If $t = 0$, then $(x, y) = (\sin 0, \sin 0) = (0, 0)$. Also, $|x| = |\sin 3t| \le 1$ for all t, and $|y| = |\sin 4t| \le 1$ for all t. The only graph which includes the point $(0, 0)$ and which has $|x| \le 1$ and $|y| \le 1$, is V.

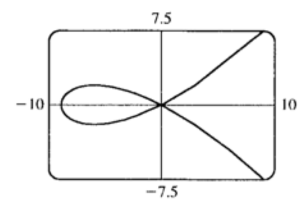
(d) Note that as $t \to -\infty$, both x and $y \to -\infty$, and as $t \to \infty$, both x and $y \to \infty$. This description fits only III. (Also note that, since $\sin 2t$ and $\sin 3t$ lie between -1 and 1, the curve never strays very far from the line $y = x$.)

(e) Note that both $x(t)$ and $y(t)$ are periodic with period 2π and satisfy $|x| \le 1$ and $|y| \le 1$. Now the only y-intercepts occur when $x = \sin(t + \sin t) = 0 \quad \Leftrightarrow \quad t = 0$ or π. So there should be two y-intercepts: $y(0) = \cos 1 \approx 0.54$ and $y(\pi) = \cos(\pi - 1) \approx -0.54$. Similarly, there should be two x-intercepts: $x\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2} + 1\right) \approx 0.54$ and $x\left(\frac{3\pi}{2}\right) = \sin\left(\frac{3\pi}{2} - 1\right) \approx -0.54$. The only curve with these x- and y-intercepts is I.

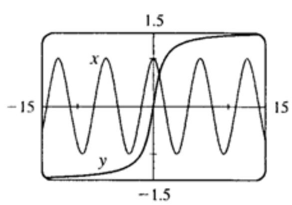
(f) Note that $x(t)$ is periodic with period 2π, so the only y-intercepts occur when $x = \cos t = 0 \quad \Leftrightarrow \quad t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Also, the graph is symmetric about the x-axis, since $y(-t) = \sin(-t + \sin 5(-t)) = \sin(-t - \sin 5t) = -\sin(t + \sin 5t) = -y(t)$, and $x(-t) = \cos(-t) = \cos t = x(t)$. The only graph which has only two y-intercepts, and is symmetric about the x-axis, is II.

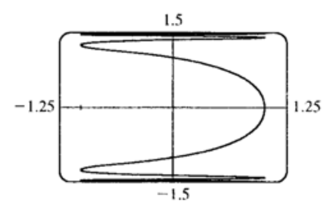
23. From the graphs, it seems that as $t \to -\infty$, $x \to \infty$ and $y \to -\infty$. So the point $(x(t), y(t))$ will move from far out in the fourth quadrant as t increases. At $t = -\sqrt{3}$, both x and y are 0, so the graph passes through the origin. After that the graph passes through the second quadrant (x is negative, y is positive), then intersects the x-axis at $x = -9$ when $t = 0$. After this, the graph passes through the third quadrant, going through the origin again at $t = \sqrt{3}$, and then as $t \to \infty$, $x \to \infty$ and $y \to \infty$. Note that for every point $(x(t), y(t)) = \left(3\left(t^2 - 3\right), t^3 - 3t\right)$, we can substitute $-t$ to get the corresponding point $(x(-t), y(-t)) = \left(3\left[(-t)^2 - 3\right], (-t)^3 - 3(-t)\right) = (x(t), -y(t))$, and so the graph is symmetric about the x-axis. The first figure was obtained using $x_1 = t$, $y_1 = 3\left(t^2 - 3\right)$; $x_2 = t$, $y_2 = t^3 - 3t$; and $-2\pi \le t \le 2\pi$.



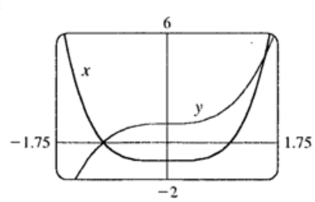


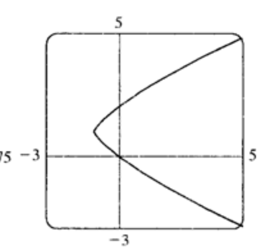
24. As $t \to -\infty$, $y \to -\frac{\pi}{2}$ and x oscillates between 1 and -1. Then, as t increases through 0, y increases while x continues to oscillate, and the graph passes through the origin. Then, as $t \to \infty$, $y \to \frac{\pi}{2}$ as x oscillates.



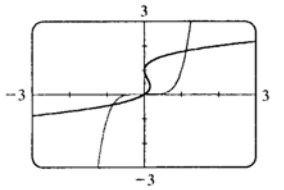


25. As $t \to -\infty$, $x \to \infty$ and $y \to -\infty$. The graph passes through the origin at $t = -1$, and then goes through the second quadrant (x negative, y positive), passing through the point $(-1, 1)$ at $t = 0$. As t increases, the graph passes through the point $(0, 2)$ at $t = 1$, and then as $t \to \infty$, both x and y approach ∞. The first figure was obtained using $x_1 = t$, $y_1 = t^4 - 1$; $x_2 = t$, $y_2 = t^3 + 1$; and $-2\pi \le t \le 2\pi$.



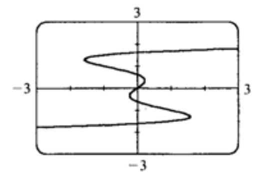


26. We use $x_1 = t$, $y_1 = t^5$ and $x_2 = t\,(t-1)^2$, $y_2 = t$ with $-2\pi \le t \le 2\pi$.



There are 3 points of intersection; $(0, 0)$ is fairly obvious. The point in quadrant III is approximately $(-0.8, -0.4)$ and the point in quadrant I is approximately $(1.1, 1.8)$.

27. As in Example 4, we let $y = t$ and $x = t - 3t^3 + t^5$ and use a t-interval of $[-2\pi, 2\pi]$.



28. (a) Clearly the curve passes through (x_1, y_1) when $t = 0$ and through (x_2, y_2) when $t = 1$. For $0 < t < 1$, x is strictly between x_1 and x_2 and y is strictly between y_1 and y_2. For every value of t, x and y satisfy the relation $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, which is the equation of the line through (x_1, y_1) and (x_2, y_2).

Finally, any point (x, y) on that line satisfies $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$; if we call that common value t, then the given parametric equations yield the point (x, y); and any (x, y) on the line between (x_1, y_1) and (x_2, y_2) yields a value of t in $[0, 1]$. So the given parametric equations exactly specify the line segment from (x_1, y_1) to (x_2, y_2).

(b) $x = -2 + [3 - (-2)]\,t = -2 + 5t$ and $y = 7 + (-1 - 7)\,t = 7 - 8t$ for $0 \le t \le 1$.

29. The circle $x^2 + y^2 = 4$ can be represented parametrically by $x = 2\cos t$, $y = 2\sin t$; $0 \le t \le 2\pi$. The circle $x^2 + (y-1)^2 = 4$ can be represented by $x = 2\cos t$, $y = 1 + 2\sin t$; $0 \le t \le 2\pi$. This representation gives us the circle with a counterclockwise orientation starting at $(2, 1)$.

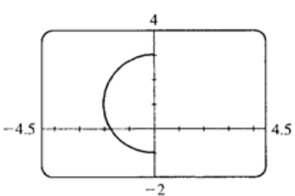
(a) To get a clockwise orientation, we could change the equations to $x = 2\cos t$, $y = 1 - 2\sin t$.

(b) To get three times around in the counterclockwise direction, we use the original equations $x = 2\cos t$, $y = 1 + 2\sin t$ with the domain expanded to $0 \le t \le 6\pi$.

(c) To start at $(0, 3)$ using the original equations, we must have $x_1 = 0$; that is, $2\cos t = 0$. Hence, $t = \frac{\pi}{2}$. So we use $x = 2\cos t$, $y = 1 + 2\sin t$; $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$.

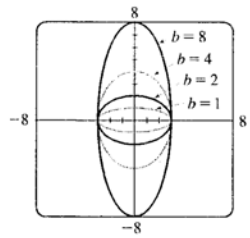
Alternatively, if we want t to start at 0, we could change the equations of the curve. For example, we could use $x = -2\sin t$, $y = 1 + 2\cos t$, $0 \le t \le \pi$.

30.



31. (a) Let $x^2/a^2 = \sin^2 t$ and $y^2/b^2 = \cos^2 t$ to obtain $x = a\sin t$ and $y = b\cos t$ with $0 \le t \le 2\pi$ as possible parametric equations for the ellipse $x^2/a^2 + y^2/b^2 = 1$.

(b) The equations are $x = 3\sin t$ and $y = b\cos t$ for $b \in \{1, 2, 4, 8\}$.



(c) As b increases, the ellipse is stretched vertically.

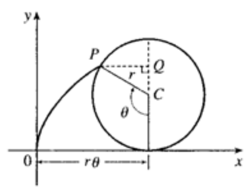
32. The possible parametrizations of the curve $y = x^3$ include

(1) $x = t,\ y = t^3,\ t \in \mathbb{R}$

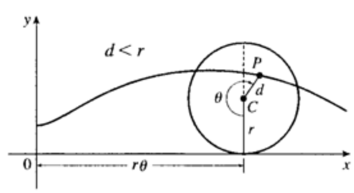
(2) $x = -t,\ y = -t^3,\ t \in \mathbb{R}$

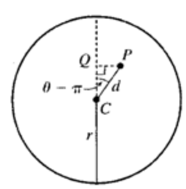
(3) $x = t + 1,\ y = (t+1)^3,\ t \in \mathbb{R}$

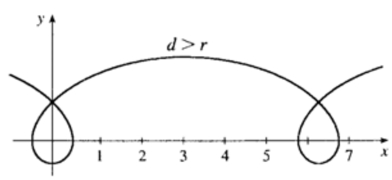
33. The case $\frac{\pi}{2} < \theta < \pi$ is illustrated. C has coordinates $(r\theta, r)$ as before, and Q has coordinates $(r\theta, r + r\cos(\pi - \theta)) = (r\theta, r(1 - \cos\theta))$ [since $\cos(\pi - \alpha) = \cos\pi\cos\alpha + \sin\pi\sin\alpha = -\cos\alpha$], so P has coordinates $(r\theta - r\sin(\pi - \theta), r(1 - \cos\theta)) = (r(\theta - \sin\theta), r(1 - \cos\theta))$ [since $\sin(\pi - \alpha) = \sin\pi\cos\alpha - \cos\pi\sin\alpha = \sin\alpha$]. Again we have the parametric equations $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.



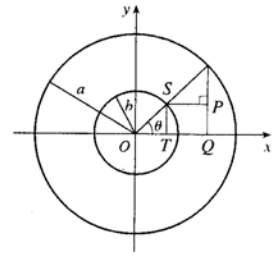
34. The first two diagrams depict the case $\pi < \theta < \frac{3\pi}{2}$, $d < r$. As in Exercise 33, C has coordinates $(r\theta, r)$. Now Q (in the second diagram) has coordinates $(r\theta, r + d\cos(\theta - \pi)) = (r\theta, r - d\cos\theta)$, so a typical point P of the trochoid has coordinates $(r\theta + d\sin(\theta - \pi), r - d\cos\theta)$. That is, P has coordinates (x, y), where $x = r\theta - d\sin\theta$ and $y = r - d\cos\theta$. When $d = r$, these equations agree with those of the cycloid.





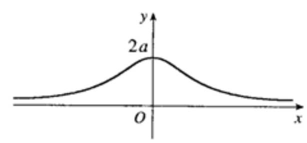


35. It is apparent that $x = |OQ|$ and $y = |QP| = |ST|$. From the diagram, $x = |OQ| = a\cos\theta$ and $y = |ST| = b\sin\theta$. Thus, the parametric equations are $x = a\cos\theta$ and $y = b\sin\theta$. To eliminate θ we rearrange: $\sin\theta = y/b \;\Rightarrow\; \sin^2\theta = (y/b)^2$ and $\cos\theta = x/a$ $\Rightarrow\; \cos^2\theta = (x/a)^2$. Adding the two equations: $\sin^2\theta + \cos^2\theta = 1 = x^2/a^2 + y^2/b^2$. Thus, we have an ellipse.



36. A has coordinates $(a\cos\theta, a\sin\theta)$. Since OA is perpendicular to AB, ΔOAB is a right triangle and B has coordinates $(a\sec\theta, 0)$. It follows that P has coordinates $(a\sec\theta, b\sin\theta)$. Thus, the parametric equations are $x = a\sec\theta$, $y = b\sin\theta$.

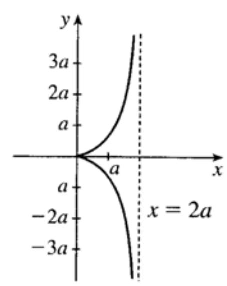
37. $C = (2a\cot\theta, 2a)$, so the x-coordinate of P is $x = 2a\cot\theta$. Let $B = (0, 2a)$. Then $\angle OAB$ is a right angle and $\angle OBA = \theta$, so $|OA| = 2a\sin\theta$ and $A = (2a\sin\theta\cos\theta, 2a\sin^2\theta)$. Thus, the y-coordinate of P is $y = 2a\sin^2\theta$.



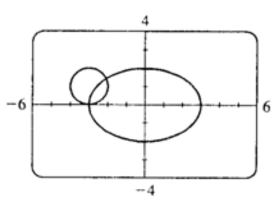
38. Let θ be the angle of inclination of segment OP. Then $|OB| = \dfrac{2a}{\cos\theta}$. Let $C = (2a, 0)$. Then by use of right triangle OAC we see that $|OA| = 2a\cos\theta$. Now

$$|OP| = |AB| = |OB| - |OA| = 2a\left(\frac{1}{\cos\theta} - \cos\theta\right)$$
$$= 2a\frac{1-\cos^2\theta}{\cos\theta} = 2a\frac{\sin^2\theta}{\cos\theta} = 2a\sin\theta\tan\theta$$

So P has coordinates $x = 2a\sin\theta\tan\theta\cdot\cos\theta = 2a\sin^2\theta$ and $y = 2a\sin\theta\tan\theta\cdot\sin\theta = 2a\sin^2\theta\tan\theta$.



39. (a)



There are 2 points of intersection: $(-3, 0)$ and approximately $(-2.1, 1.4)$.

(b) As an aid in finding collision points, set your graphing utility to graph both curves simultaneously and closely observe the drawing of the graphs. In this case, we have one collision point: both particles are at $(-3, 0)$ when $t = \frac{3\pi}{2}$. [Notice that the first curve passes through $(-2.1, 1.4)$ when $t \approx 5.5$, but the second curve passes through $(-2.1, 1.4)$ when $t \approx 0.4$.]

(c) The circle is centered at $(3, 1)$ instead of $(-3, 1)$. There are still 2 intersection points: $(3, 0)$ and $(2.1, 1.4)$, but there are no collision points.

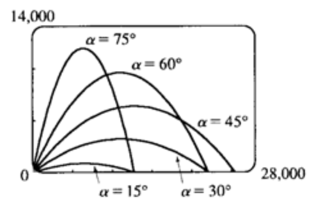
40. (a) If $\alpha = 30°$ and $v_0 = 500$ m/s, then the equations become $x = (500\cos 30°)\,t = 250\sqrt{3}t$ and $y = (500\sin 30°)\,t - \frac{1}{2}(9.8)\,t^2 = 250t - 4.9t^2$. $y = 0$ when $t = 0$ (when the gun is fired) and again when $t = \frac{250}{4.9} \approx 51$ s. Then $x = \left(250\sqrt{3}\right)\left(\frac{250}{4.9}\right) \approx 22{,}092$ m, so the bullet hits the ground about 22 km from the gun.

The formula for y is quadratic in t. To find the maximum y-value, we will complete the square:

$$y = -4.9\left(t^2 - \frac{250}{4.9}t\right) = -4.9\left[t^2 - \frac{250}{4.9}t + \left(\frac{125}{4.9}\right)^2\right] + \frac{125^2}{4.9} = -4.9\left(t - \frac{125}{4.9}\right)^2 + \frac{125^2}{4.9} \le \frac{125^2}{4.9}$$

with equality when $t = \frac{125}{4.9}$ s, so the maximum height attained is $\frac{125^2}{4.9} \approx 3189$ m.

(b)

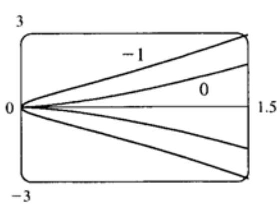


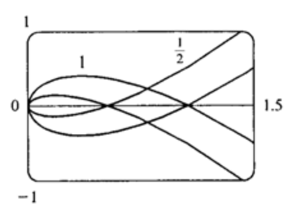
As α $(0° < \alpha < 90°)$ increases up to 45°, the projectile attains a greater height and a greater range. As α increases past 45°, the projectile attains a greater height, but its range decreases.

(c) $x = (v_0\cos\alpha)\,t \quad\Rightarrow\quad t = \dfrac{x}{v_0\cos\alpha}$.

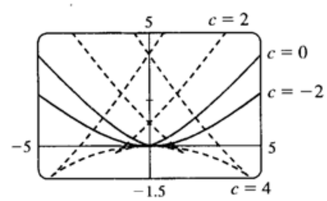
$$y = (v_0\sin\alpha)\,\frac{x}{v_0\cos\alpha} - \frac{g}{2}\left(\frac{x}{v_0\cos\alpha}\right)^2 = (\tan\alpha)\,x - \left(\frac{g}{2v_0^2\cos^2\alpha}\right)x^2,$$ which is the equation of a parabola (quadratic in x).

41. $x = t^2$, $y = t^3 - ct$. We use a graphing device to produce the graphs for various values of c with $-\pi \le t \le \pi$. Note that all the members of the family are symmetric about the x-axis. For $c < 0$, the graph does not cross itself, but for $c = 0$ it has a cusp at $(0, 0)$ and for $c > 0$ the graph crosses itself at $x = c$, so the loop grows larger as c increases.

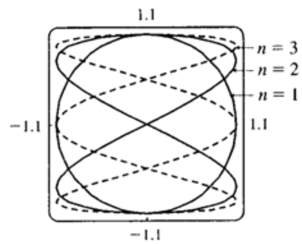




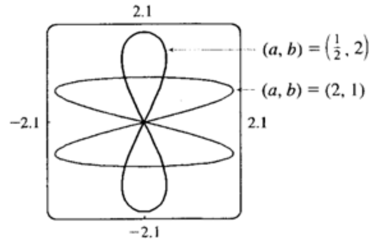
42. $x = 2ct - 4t^3$, $y = -ct^2 + 3t^4$. We use a graphing device to produce the graphs for various values of c with $-\pi \le t \le \pi$. Note that all the members of the family are symmetric about the y-axis. When $c < 0$, the graph resembles that of a polynomial of even degree, but when $c = 0$ there is a corner at the origin, and when $c > 0$, the graph crosses itself at the origin, and has two cusps below the x-axis. The size of the "swallowtail" increases as c increases.



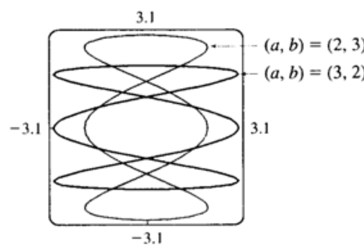
43. Note that all the Lissajous figures are symmetric about the x-axis. The parameters a and b simply stretch the graph in the x- and y-directions respectively. For $a = b = n = 1$ the graph is simply a circle with radius 1. For $n = 2$ the graph crosses itself at the origin and there are loops above and below the x-axis. In general, the figures have $n - 1$ points of intersection, all of which are on the y-axis, and a total of n closed loops.



$a = b = 1$

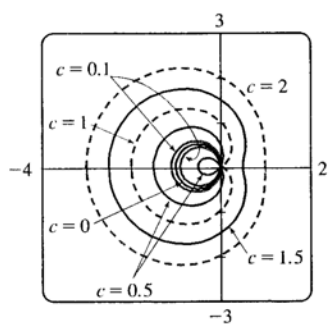


$n = 2$



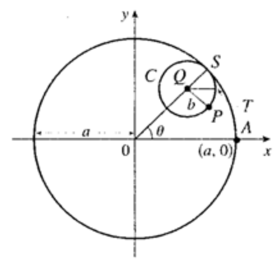
$n = 3$

44. We use $-\pi \leq t \leq \pi$ in the viewing rectangle $[-4, 2] \times [-3, 3]$. We first observe that for $c = 0$, we obtain a circle with center $\left(-\frac{1}{2}, 0\right)$ and radius $\frac{1}{2}$. As the value of c increases, there is a larger outer loop and a smaller inner loop until $c = 1$, when we obtain a curve with a dent (called a **cardioid**). As c increases, we get curve with a dimple (called a **limaçon**) until $c = 2$. For $c > 2$, we have convex limaçons. For negative values of c, we obtain the same graphs as for positive c, but with different values of t corresponding to the points on the curve.



Laboratory Project □ Families of Hypocycloids

1. The center Q of the smaller circle has coordinates $((a-b)\cos\theta, (a-b)\sin\theta)$. Arc PS on circle C has length $a\theta$ since it is equal in length to arc AS (the smaller circle rolls without slipping against the larger.) Thus, $\angle PQS = \frac{a}{b}\theta$ and $\angle PQT = \frac{a}{b}\theta - \theta$, so P has coordinates

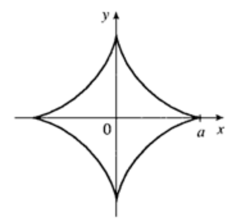


$$x = (a-b)\cos\theta + b\cos(\angle PQT) = (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\theta\right)$$

and

$$y = (a-b)\sin\theta - b\sin(\angle PQT) = (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\theta\right)$$

2. With $b = 1$ and a a positive integer greater than 2, we obtain a hypocycloid of a cusps. Shown in the figure is the graph for $a = 4$. Let $a = 4$ and $b = 1$. Using the sum identities to expand $\cos 3\theta$ and $\sin 3\theta$, we obtain



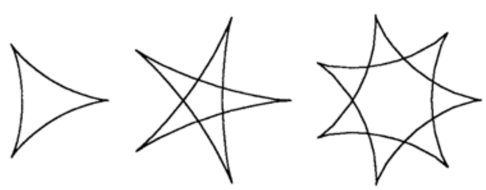
$$x = 3\cos\theta + \cos 3\theta = 3\cos\theta + \left(4\cos^3\theta - 3\cos\theta\right) = 4\cos^3\theta$$

and

$$y = 3\sin\theta - \sin 3\theta = 3\sin\theta - \left(3\sin\theta - 4\sin^3\theta\right) = 4\sin^3\theta$$

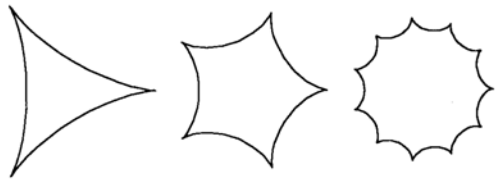
3. The following graphs are obtained with $b = 1$ and $a = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, and $\frac{1}{10}$ with $-2\pi \leq \theta \leq 2\pi$. We conclude that as the denominator d increases, the graph gets smaller, but maintains the basic shape shown.

Letting $d = 2$ and $n = 3, 5$, and 7 with $-2\pi \leq \theta \leq 2\pi$ gives us the following:



[continued]

So if d is held constant and n varies, we get a graph with n cusps (assuming n/d is in lowest form). When $n = d + 1$, we obtain a hypocycloid of n cusps. As n increases, we must expand the range of θ in order to get a closed curve. The following graphs have $a = \frac{3}{2}$, $\frac{5}{4}$, and $\frac{11}{10}$.

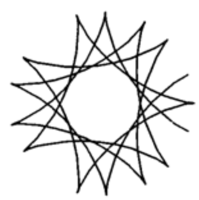


If $b = 1$, the equations for the hypocycloid are

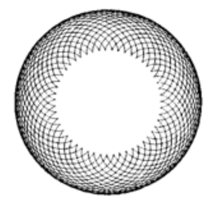
$$x = (a-1)\cos\theta + \cos((a-1)\theta),\ y = (a-1)\sin\theta - \sin((a-1)\theta)$$

which is a hypocycloid of a cusps (from Problem 2). If $n = d + 1$, then $a = (d+1)/d$, and the equations become $x = \frac{1}{d}\cos\theta + \cos\frac{\theta}{d}$, $y = \frac{1}{d}\sin\theta - \sin\frac{\theta}{d}$. Now letting $\varphi = -\theta/d$ and multiplying by d (from the hint) gives us $X = \cos(-d\varphi) + d\cos(-\varphi)$, $Y = \sin(-d\varphi) - d\sin(-\varphi)$ or, equivalently, $X = d\cos\varphi + \cos d\varphi$, $Y = d\sin\varphi - \sin d\varphi$. We recognize these equations as those of a hypocycloid with $(d+1)$ cusps.

4. In general, if $a > 1$, we get a figure with cusps on the "outside ring" and if $a < 1$, the cusps are on the "inside ring". In any case, as the values of θ get larger, we get a figure that looks more and more like a washer. If we were to graph the hypocycloid for all values of θ, every point on the washer would eventually be arbitrarily close to a point on the curve.

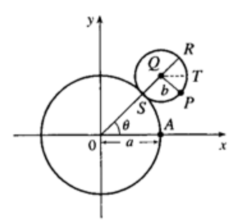


$a = \sqrt{2}$, $-10\pi \le \theta \le 10\pi$



$a = e - 2$, $0 \le \theta \le 446$

5. The center Q of the smaller circle has coordinates $((a+b)\cos\theta, (a+b)\sin\theta)$. Arc PS has length $a\theta$ (as in Problem 1), so that $\angle PQS = \frac{a\theta}{b}$, $\angle PQR = \pi - \frac{a\theta}{b}$, and $\angle PQT = \pi - \frac{a\theta}{b} - \theta = \pi - \left(\frac{a+b}{b}\right)\theta$ since $\angle RQT = \theta$.



Thus, the coordinates of P are

$$x = (a+b)\cos\theta + b\cos\left(\pi - \frac{a+b}{b}\theta\right) = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right)$$

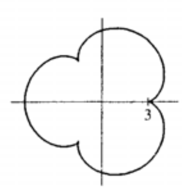
and

$$y = (a+b)\sin\theta - b\sin\left(\pi - \frac{a+b}{b}\theta\right) = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\theta\right).$$

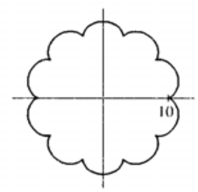
6. Let $b = 1$ and the equations become

$$x = (a + 1)\cos\theta - \cos((a + 1)\theta) \qquad\qquad y = (a + 1)\sin\theta - \sin((a + 1)\theta)$$

If $a = 1$, we have a cardioid. If a is a positive integer greater than 1, we get the graph of an "a-leafed clover", with cusps that are a units from the origin. (Some of the pairs of figures are not to scale.)

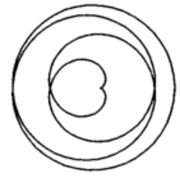


$a = 3, -2\pi \le \theta \le 2\pi$



$a = 10, -2\pi \le \theta \le 2\pi$

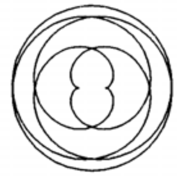
If $a = n/d$ with $n = 1$, we obtain a figure that does not increase in size and requires $-d\pi \le \theta \le d\pi$ to be a closed curve traced exactly once.



$a = \frac{1}{4}, -4\pi \le \theta \le 4\pi$

$a = \frac{1}{7}, -7\pi \le \theta \le 7\pi$

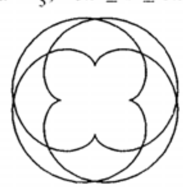
Next, we keep d constant and let n vary. As n increases, so does the size of the figure. There is an n-pointed star in the middle.



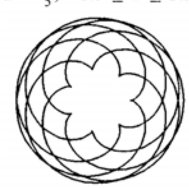
$a = \frac{2}{5}, -5\pi \le \theta \le 5\pi$

$a = \frac{7}{5}, -5\pi \le \theta \le 5\pi$

Now if $n = d + 1$ we obtain figures similar to the previous ones, but the size of the figure does not increase.

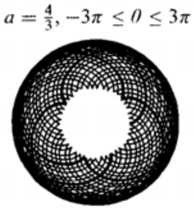


$a = \frac{4}{3}, -3\pi \le \theta \le 3\pi$

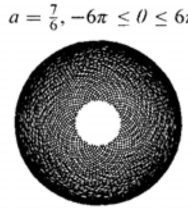


$a = \frac{7}{6}, -6\pi \le \theta \le 6\pi$

If a is irrational, we get washers that increase in size as a increases.



$a = \sqrt{2}, 0 \le \theta \le 200$



$a = e - 2, 0 \le \theta \le 446$

7. The equations for the epicycloid are

$$x = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right) \qquad y = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\theta\right)$$

For the first part of the problem, we set $a = n$ and $b = 1$, so these equations become

$$x = (n+1)\cos\theta - \cos((n+1)\theta) \qquad y = (n+1)\sin\theta - \sin((n+1)\theta)$$

If we substitute $\varphi = (n+1)\theta$ and rearrange the terms, these become

$$x = -\cos\varphi + (n+1)\cos\frac{\varphi}{n+1} \qquad y = -\sin\varphi + (n+1)\sin\frac{\varphi}{n+1}$$

so

$$X = \frac{1}{n+1}x = -\frac{\cos\varphi}{n+1} + \cos\frac{\varphi}{n+1} \qquad Y = \frac{1}{n+1}y = -\frac{\sin\varphi}{n+1} + \sin\frac{\varphi}{n+1}$$

whereas the equations of a hypocycloid with $a = \dfrac{n}{n+1}$ and $b = 1$ are

$$x = \left(\frac{n}{n+1} - 1\right)\cos\theta + \cos\left(\left(\frac{n}{n+1} - 1\right)\theta\right) = -\frac{\cos\theta}{n+1} + \cos\frac{\theta}{n+1}$$

$$y = \left(\frac{n}{n+1} - 1\right)\sin\theta - \sin\left(\left(\frac{n}{n+1} - 1\right)\theta\right) = -\frac{\sin\theta}{n+1} + \sin\frac{\theta}{n+1}$$

For the second part of the problem, we set $a = \dfrac{1}{n}$ in the equations for the epicycloid:

$$x = \frac{n+1}{n}\cos\theta - \cos\left(\frac{n+1}{n}\theta\right) \qquad y = \frac{n+1}{n}\sin\theta - \sin\left(\frac{n+1}{n}\theta\right)$$

Multiplying by $\dfrac{n}{n+1}$, substituting $\varphi = \dfrac{n+1}{n}\theta$ and rearranging the terms, we get

$$X = -\frac{n}{n+1}\cos\varphi + \cos\left(\frac{n}{n+1}\varphi\right) \qquad Y = -\frac{n}{n+1}\sin\varphi + \sin\left(\frac{n}{n+1}\varphi\right)$$

But these are exactly the equations of a hypocycloid $\{(X, Y)\}$ with $a = \dfrac{1}{n+1}$.

11.2 Tangents and Areas

1. $x = t - t^3$, $y = 2 - 5t \Rightarrow \dfrac{dy}{dt} = -5$, $\dfrac{dx}{dt} = 1 - 3t^2$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-5}{1-3t^2}$ or $\dfrac{5}{3t^2-1}$.

2. $x = \sqrt{t} - t$, $y = t^3 - t \Rightarrow \dfrac{dy}{dt} = 3t^2 - 1$, $\dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t}} - 1$, and

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-1}{1/\left(2\sqrt{t}\right) - 1} = \frac{\left(3t^2-1\right)\left(2\sqrt{t}\right)}{1 - 2\sqrt{t}}$$

3. $x = t\ln t$, $y = \sin^2 t \Rightarrow \dfrac{dy}{dt} = 2\sin t\cos t$, $\dfrac{dx}{dt} = t\left(\dfrac{1}{t}\right) + (\ln t)\cdot 1 = 1 + \ln t$, and

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\sin t\cos t}{1 + \ln t}$$

4. $x = te^t, y = t + e^t \Rightarrow \frac{dy}{dt} = 1 + e^t, \frac{dx}{dt} = te^t + e^t$, and $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 + e^t}{te^t + e^t}$.

5. $x = t^2 + t, y = t^2 - t; t = 0$. $\frac{dy}{dt} = 2t - 1, \frac{dx}{dt} = 2t + 1$, so $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t - 1}{2t + 1}$. When $t = 0$, $x = y = 0$ and $\frac{dy}{dx} = -1$. An equation of the tangent is $y - 0 = (-1)(x - 0)$ or $y = -x$.

6. $x = 2t^2 + 1, y = \frac{1}{3}t^3 - t; t = 3$. $\frac{dy}{dt} = t^2 - 1, \frac{dx}{dt} = 4t$, and $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t^2 - 1}{4t}$. When $t = 3$, $(x, y) = (19, 6)$ and $dy/dx = \frac{8}{12} = \frac{2}{3}$, so an equation of the tangent line is $y - 6 = \frac{2}{3}(x - 19)$ or $y = \frac{2}{3}x - \frac{20}{3}$.

7. $x = e^{\sqrt{t}}, y = t - \ln t^2; t = 1$. $\frac{dy}{dt} = 1 - \frac{2t}{t^2} = 1 - \frac{2}{t}, \frac{dx}{dt} = \frac{e^{\sqrt{t}}}{2\sqrt{t}}$, and

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - 2/t}{e^{\sqrt{t}}/(2\sqrt{t})} \cdot \frac{2t}{2t} = \frac{2t - 4}{\sqrt{t}e^{\sqrt{t}}}$. When $t = 1$, $(x, y) = (e, 1)$ and $\frac{dy}{dx} = -\frac{2}{e}$, so an equation of the tangent line is $y - 1 = -\frac{2}{e}(x - e)$ or $y = -\frac{2}{e}x + 3$.

8. $x = t\sin t, y = t\cos t; t = \pi$. $\frac{dy}{dt} = \cos t - t\sin t, \frac{dx}{dt} = \sin t + t\cos t$, and $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t - t\sin t}{\sin t + t\cos t}$.

When $t = \pi$, $(x, y) = (0, -\pi)$ and $\frac{dy}{dx} = \frac{-1}{-\pi} = \frac{1}{\pi}$, so an equation of the tangent is $y + \pi = \frac{1}{\pi}(x - 0)$ or $y = \frac{1}{\pi}x - \pi$.

9. (a) $x = e^t, y = (t - 1)^2; (1, 1)$. $\frac{dy}{dt} = 2(t - 1), \frac{dx}{dt} = e^t$, and $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2(t - 1)}{e^t}$.

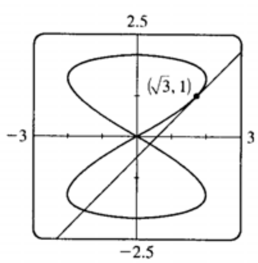
At $(1, 1)$, $t = 0$ and $\frac{dy}{dx} = -2$, so an equation of the tangent is $y - 1 = -2(x - 1)$ or $y = -2x + 3$.

(b) $x = e^t \Rightarrow t = \ln x$, so $y = (t - 1)^2 = (\ln x - 1)^2$ and $\frac{dy}{dx} = 2(\ln x - 1)\left(\frac{1}{x}\right)$. When $x = 1$, $\frac{dy}{dx} = 2(-1)(1) = -2$, so an equation of the tangent is $y = -2x + 3$, as in part (a).

10. (a) $x = 5\cos t, y = 5\sin t; (3, 4)$. $\frac{dy}{dt} = 5\cos t, \frac{dx}{dt} = -5\sin t, \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\cot t$. At $(3, 4)$, $t = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{4}{3}$, so $\frac{dy}{dx} = -\frac{3}{4}$, and an equation of the tangent is $y - 4 = -\frac{3}{4}(x - 3)$, or $y = -\frac{3}{4}x + \frac{25}{4}$.

(b) $x^2 + y^2 = 25$, so $2x + 2y\frac{dy}{dx} = 0$, or $\frac{dy}{dx} = -\frac{x}{y}$. At $(3, 4)$, $\frac{dy}{dx} = -\frac{3}{4}$, and as in part (a), an equation of the tangent is $y = -\frac{3}{4}x + \frac{25}{4}$.

11. $x = 2\sin 2t, y = 2\sin t; \left(\sqrt{3}, 1\right)$.



$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{2 \cdot 2\cos 2t} = \frac{\cos t}{2\cos 2t}$. The point $\left(\sqrt{3}, 1\right)$ corresponds to $t = \frac{\pi}{6}$, so the slope of the tangent at that point is

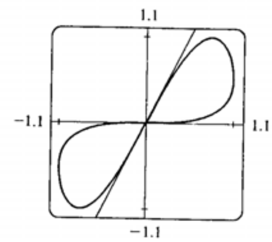
$\frac{\cos\frac{\pi}{6}}{2\cos\frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{2 \cdot \frac{1}{2}} = \frac{\sqrt{3}}{2}$. An equation of the tangent is therefore

$(y - 1) = \frac{\sqrt{3}}{2}\left(x - \sqrt{3}\right)$ or $y = \frac{\sqrt{3}}{2}x - \frac{1}{2}$.

12. $x = \sin t$, $y = \sin(t + \sin t)$; $(0, 0)$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos(t + \sin t)(1 + \cos t)}{\cos t} = (\sec t + 1)\cos(t + \sin t)$$

Note that there are two tangents at the point $(0, 0)$, since both $t = 0$ and $t = \pi$ correspond to the origin. The tangent corresponding to $t = 0$ has slope $(\sec 0 + 1)\cos(0 + \sin 0) = 2\cos 0 = 2$, and its equation is $y = 2x$. The tangent corresponding to $t = \pi$ has slope $(\sec \pi + 1)\cos(\pi + \sin \pi) = 0$, so it is the x-axis.



13. $x = t^4 - 1$, $y = t - t^2 \;\Rightarrow\; \dfrac{dy}{dt} = 1 - 2t$, $\dfrac{dx}{dt} = 4t^3$, $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 - 2t}{4t^3} = \frac{1}{4}t^{-3} - \frac{1}{2}t^{-2}$;

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = -\tfrac{3}{4}t^{-4} + t^{-3},\ \frac{d^2y}{dx^2} = \frac{d\,(dy/dx)/dt}{dx/dt} = \frac{-\frac{3}{4}t^{-4} + t^{-3}}{4t^3} \cdot \frac{4t^4}{4t^4} = \frac{-3 + 4t}{16t^7}.$$

14. $x = t^3 + t^2 + 1$, $y = 1 - t^2$. $\dfrac{dy}{dt} = -2t$, $\dfrac{dx}{dt} = 3t^2 + 2t$; $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-2t}{3t^2 + 2t} = -\dfrac{2}{3t + 2}$;

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{6}{(3t + 2)^2};\ \frac{d^2y}{dx^2} = \frac{d\,(dy/dx)/dt}{dx/dt} = \frac{6}{t\,(3t + 2)^3}.$$

15. $x = \sin \pi t$, $y = \cos \pi t$. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-\pi \sin \pi t}{\pi \cos \pi t} = -\tan \pi t$;

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d\,(dy/dx)/dt}{dx/dt} = \frac{-\pi \sec^2 \pi t}{\pi \cos \pi t} = -\sec^3 \pi t.$$

16. $x = 1 + \tan t$, $y = \cos 2t \;\Rightarrow\; \dfrac{dy}{dt} = -2\sin 2t$, $\dfrac{dx}{dt} = \sec^2 t$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2\sin 2t}{\sec^2 t} = -4\sin t \cos t \cdot \cos^2 t = -4\sin t \cos^3 t;$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = -4\sin t\,(3\cos^2 t)(-\sin t) - 4\cos^4 t = 12\sin^2 t\cos^2 t - 4\cos^4 t,$$

$$\frac{d^2y}{dx^2} = \frac{d\,(dy/dx)/dt}{dx/dt} = \frac{4\cos^2 t\,(3\sin^2 t - \cos^2 t)}{\sec^2 t} = 4\cos^4 t\,(3\sin^2 t - \cos^2 t).$$

17. $x = e^{-t}$, $y = te^{2t}$. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{(2t+1)\,e^{2t}}{-e^{-t}} = -\,(2t+1)\,e^{3t}$;

$\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) = -3\,(2t+1)\,e^{3t} - 2e^{3t} = -\,(6t+5)\,e^{3t}$;

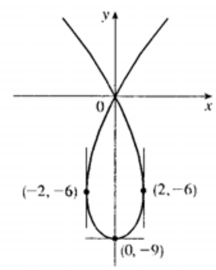
$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d\,(dy/dx)\,/dt}{dx/dt} = \dfrac{-\,(6t+5)\,e^{3t}}{-e^{-t}} = (6t+5)\,e^{4t}$.

18. $x = 1 + t^2$, $y = t\ln t$. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1+\ln t}{2t}$; $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) = \dfrac{2t\,(1/t) - (1+\ln t)\,2}{(2t)^2} = -\dfrac{\ln t}{2t^2}$;

$\dfrac{d^2y}{dx^2} = \dfrac{d\,(dy/dx)\,/dt}{dx/dt} = -\dfrac{\ln t}{4t^3}$.

19. $x = t\left(t^2 - 3\right) = t^3 - 3t$, $y = 3\left(t^2 - 3\right)$. $\dfrac{dx}{dt} = 3t^2 - 3 = 3\,(t-1)\,(t+1)$; $\dfrac{dy}{dt} = 6t$. $\dfrac{dy}{dt} = 0 \;\Leftrightarrow\; t = 0 \;\Leftrightarrow$ $(x,y) = (0,-9)$. $\dfrac{dx}{dt} = 0 \;\Leftrightarrow\; t = \pm 1 \;\Leftrightarrow\; (x,y) = (-2,-6)$ or $(2,-6)$. So there is a horizontal tangent at $(0,-9)$ and there are vertical tangents at $(-2,-6)$ and $(2,-6)$.

	$t < -1$	$-1 < t < 0$	$0 < t < 1$	$t > 1$
dx/dt	$+$	$-$	$-$	$+$
dy/dt	$-$	$-$	$+$	$+$
x	$\rightarrow$	$\leftarrow$	$\leftarrow$	$\rightarrow$
y	$\downarrow$	$\downarrow$	$\uparrow$	$\uparrow$
curve	$\searrow$	$\swarrow$	$\nwarrow$	$\nearrow$

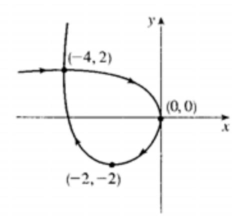


20. $x = t^3 - 3t^2$, $y = t^3 - 3t$. $\dfrac{dx}{dt} = 3t^2 - 6t = 3t\,(t-2)$,

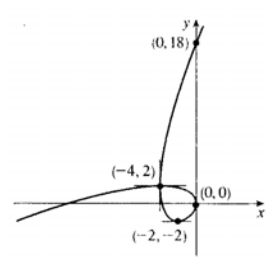
$\dfrac{dy}{dt} = 3t^2 - 3 = 3\,(t-1)\,(t+1)$. $\dfrac{dy}{dt} = 0 \;\Leftrightarrow\; t = +1$ or $-1 \;\Leftrightarrow$

$(x,y) = (-2,-2)$ or $(-4,2)$. $\dfrac{dx}{dt} = 0 \;\Leftrightarrow\; t = 0$ or $2 \;\Leftrightarrow$

$(x,y) = (0,0)$ or $(-4,2)$. So the tangent is horizontal at $(-2,-2)$ and vertical at $(0,0)$. At $(-4,2)$ the curve crosses itself and there are two tangents, one horizontal and one vertical.



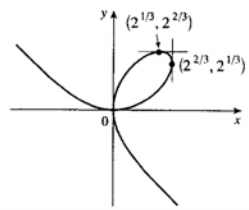
	$t < -1$	$-1 < t < 0$	$0 < t < 1$	$1 < t < 2$	$t > 2$
dx/dt	$+$	$+$	$-$	$-$	$+$
dy/dt	$+$	$-$	$-$	$+$	$+$
x	$\rightarrow$	$\rightarrow$	$\leftarrow$	$\leftarrow$	$\rightarrow$
y	$\uparrow$	$\downarrow$	$\downarrow$	$\uparrow$	$\uparrow$
curve	$\nearrow$	$\searrow$	$\swarrow$	$\nwarrow$	$\nearrow$



21. $x = \dfrac{3t}{1+t^3}$, $y = \dfrac{3t^2}{1+t^3}$. $\dfrac{dx}{dt} = \dfrac{(1+t^3)\,3 - 3t\,(3t^2)}{(1+t^3)^2} = \dfrac{3-6t^3}{(1+t^3)^2}$,

$\dfrac{dy}{dt} = \dfrac{(1+t^3)\,(6t) - 3t^2\,(3t^2)}{(1+t^3)^2} = \dfrac{6t-3t^4}{(1+t^3)^2} = \dfrac{3t\,(2-t^3)}{(1+t^3)^2}$. $\dfrac{dy}{dt} = 0 \iff t = 0$ or $\sqrt[3]{2} \iff$

$(x,y) = (0,0)$ or $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$. $\dfrac{dx}{dt} = 0 \iff t^3 = \frac{1}{2} \iff t = 2^{-1/3}$



$\iff (x,y) = \left(\sqrt[3]{4}, \sqrt[3]{2}\right)$. There are horizontal tangents at $(0,0)$ and $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$, and there are vertical tangents at $\left(\sqrt[3]{4}, \sqrt[3]{2}\right)$ and $(0,0)$. [The vertical tangent at $(0,0)$ is undetectable by the methods of this section because that tangent corresponds to the limiting position of the point (x,y) as $t \to \pm\infty$.] In the following table, $a = \sqrt[3]{2}$.

	$t < -1$	$-1 < t < 0$	$0 < t < 1/a$	$1/a < t < a$	$t > a$
dx/dt	+	+	+	−	−
dy/dt	−	−	+	+	−
x	→	→	→	←	←
y	↓	↓	↑	↑	↓
curve	↘	↘	↗	↖	↙

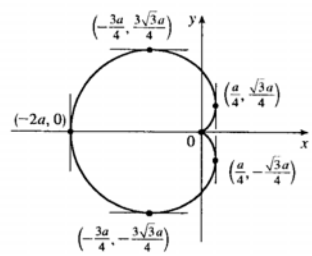
22. $x = a\left(\cos\theta - \cos^2\theta\right)$, $y = a\left(\sin\theta - \sin\theta\cos\theta\right)$. $\dfrac{dx}{d\theta} = a\left(-\sin\theta + 2\cos\theta\sin\theta\right)$,

$\dfrac{dy}{d\theta} = a\left(\cos\theta + \sin^2\theta - \cos^2\theta\right) = a\left(\cos\theta + 1 - 2\cos^2\theta\right)$, $\dfrac{dy}{d\theta} = 0 \iff$

$0 = 2\cos^2\theta - \cos\theta - 1 = (2\cos\theta + 1)(\cos\theta - 1) \iff \cos\theta = -\frac{1}{2}$ or $1 \iff (x,y) = \left(-\frac{3}{4}a, \pm\frac{3\sqrt{3}}{4}a\right)$ or

$(0,0)$. $\dfrac{dx}{d\theta} = 0 \iff (2\cos\theta - 1)\sin\theta = 0 \iff \cos\theta = \frac{1}{2}$ or $\sin\theta = 0 \iff (x,y) = (0,0)$ or

$\left(\frac{1}{4}a, \pm\frac{\sqrt{3}}{4}a\right)$ or $(-2a, 0)$. The curve has horizontal tangents at $\left(-\frac{3}{4}a, \pm\frac{3\sqrt{3}}{4}a\right)$ and vertical tangents at $(-2a, 0)$ and $\left(\frac{1}{4}a, \pm\frac{\sqrt{3}}{4}a\right)$.

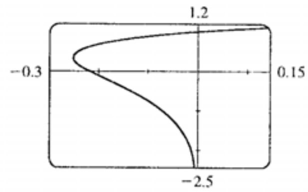


Since $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{(2\cos\theta + 1)(1 - \cos\theta)}{(2\cos\theta - 1)\sin\theta}$, we see that

$\displaystyle\lim_{\theta\to 0}\frac{dy}{dx} = \lim_{\theta\to 0}\frac{2\cos\theta + 1}{2\cos\theta - 1}\cdot\lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta} = 3\cdot 0 = 0$ (using l'Hospital's Rule). Thus, the curve has a horizontal tangent at $(0,0)$, where both $dx/d\theta$ and $dy/d\theta$ are 0.

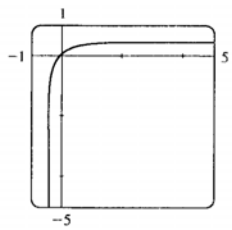
	$0 < t < \frac{\pi}{3}$	$\frac{\pi}{3} < t < \frac{2\pi}{3}$	$\frac{2\pi}{3} < t < \pi$	$\pi < t < \frac{4\pi}{3}$	$\frac{4\pi}{3} < t < \frac{5\pi}{3}$	$\frac{5\pi}{3} < t < 2\pi$
dx/dt	+	−	−	+	+	−
dy/dt	+	+	−	−	+	+
x	→	←	←	→	→	←
y	↑	↑	↓	↓	↑	↑
curve	↗	↖	↙	↘	↗	↖

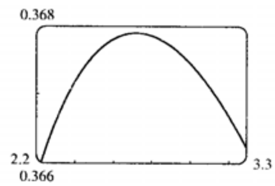
23. From the graph, it appears that the leftmost point on the curve $x = t^4 - t^2$, $y = t + \ln t$ is about $(-0.25, 0.36)$. To find the exact coordinates, we find the value of t for which the graph has a vertical tangent, that is, $0 = dx/dt = 4t^3 - 2t \quad \Leftrightarrow \quad 2t\left(2t^2 - 1\right) = 0 \quad \Leftrightarrow$ $2t\left(\sqrt{2}t + 1\right)\left(\sqrt{2}t - 1\right) = 0 \quad \Leftrightarrow \quad t = 0 \text{ or } \pm\frac{1}{\sqrt{2}}$. The negative and 0 roots are inadmissible since $y(t)$ is only defined for $t > 0$, so the leftmost point must be



$$\left(x\left(\tfrac{1}{\sqrt{2}}\right), y\left(\tfrac{1}{\sqrt{2}}\right)\right) = \left(\left(\tfrac{1}{\sqrt{2}}\right)^4 - \left(\tfrac{1}{\sqrt{2}}\right)^2, \tfrac{1}{\sqrt{2}} + \ln\tfrac{1}{\sqrt{2}}\right) = \left(-\tfrac{1}{4}, \tfrac{1}{\sqrt{2}} - \tfrac{1}{2}\ln 2\right)$$

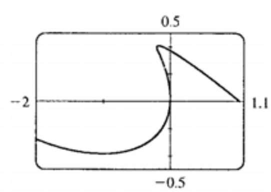
24. The curve is symmetric about the line $y = -x$, so if we can find the highest point (x_h, y_h), then the leftmost point is $(x_l, y_l) = (-y_h, -x_h)$. After carefully zooming in, we estimate that the highest point on the curve $x = te^t$, $y = te^{-t}$ is about $(2.7, 0.37)$.

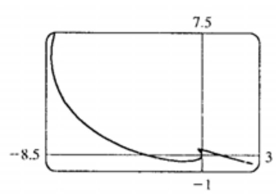




To find the exact coordinates of the highest point, we find the value of t for which the curve has a horizontal tangent, that is, $dy/dt = 0 \quad \Leftrightarrow \quad t\left(-e^{-t}\right) + e^{-t} = 0 \quad \Leftrightarrow \quad (1 - t)e^{-t} = 0 \quad \Leftrightarrow \quad t = 1$. This corresponds to the point $(x(1), y(1)) = (e, 1/e)$. To find the leftmost point, we find the value of t for which $0 = dx/dt = te^t + e^t \quad \Leftrightarrow \quad (1 + t)e^t = 0 \quad \Leftrightarrow \quad t = -1$. This corresponds to the point $(x(-1), y(-1)) = (-1/e, -e)$. As $t \to -\infty$, $x(t) = te^t \to 0^-$ by l'Hospital's Rule and $y(t) = te^{-t} \to -\infty$, so the y-axis is an asymptote. As $t \to \infty$, $x(t) \to \infty$ and $y(t) \to 0^+$, so the x-axis is the other asymptote. The asymptotes can also be determined from the graph, if we use a larger t-interval.

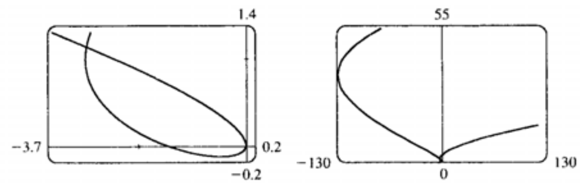
25. We graph the curve $x = t^4 - 2t^3 - 2t^2$, $y = t^3 - t$ in the viewing rectangle $[-2, 1.1]$ by $[-0.5, 0.5]$. This rectangle corresponds approximately to $t \in [-1, 0.8]$. We estimate that the curve has horizontal tangents at about $(-1, -0.4)$ and $(-0.17, 0.39)$ and vertical tangents at about $(0, 0)$ and $(-0.19, 0.37)$. We calculate





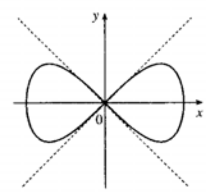
$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2 - 1}{4t^3 - 6t^2 - 4t}$. The horizontal tangents occur when $dy/dt = 3t^2 - 1 = 0 \quad \Leftrightarrow \quad t = \pm\frac{1}{\sqrt{3}}$, so both horizontal tangents are shown in our graph. The vertical tangents occur when $dx/dt = 2t\left(2t^2 - 3t - 2\right) = 0$ $\Leftrightarrow \quad 2t(2t + 1)(t - 2) = 0 \quad \Leftrightarrow \quad t = 0, -\frac{1}{2}$ or 2. It seems that we have missed one vertical tangent, and indeed if we plot the curve on the t-interval $[-1.2, 2.2]$ we see that there is another vertical tangent at $(-8, 6)$.

26. We graph the curve $x = t^4 + 4t^3 - 8t^2$, $y = 2t^2 - t$ in the viewing rectangle $[-3.7, 0.2]$ by $[-0.2, 1.4]$. It appears that there is a horizontal tangent at about $(-0.4, -0.1)$, and vertical tangents at about $(-3, 1)$ and $(0, 0)$.

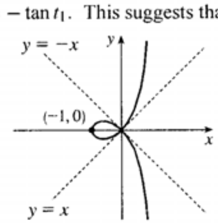


We calculate $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4t - 1}{4t^3 + 12t^2 - 16t}$, so there is a horizontal tangent where $dy/dt = 4t - 1 = 0 \Leftrightarrow t = \frac{1}{4}$. This point (the lowest point) is shown in the first graph. There are vertical tangents where $dx/dt = 4t^3 + 12t^2 - 16t = 0 \Leftrightarrow 4t\left(t^2 + 3t - 4\right) = 0 \Leftrightarrow 4t\,(t + 4)\,(t - 1) = 0$. We have missed one vertical tangent corresponding to $t = -4$, and if we plot the graph for $t \in [-5, 3]$, we see that the curve has another vertical tangent line at approximately $(-128, 36)$.

27. $x = \cos t$, $y = \sin t \cos t$. $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = -\sin^2 t + \cos^2 t = \cos 2t$. $(x, y) = (0, 0) \Leftrightarrow \cos t = 0 \Leftrightarrow t$ is an odd multiple of $\frac{\pi}{2}$. When $t = \frac{\pi}{2}$, $\frac{dx}{dt} = -1$ and $\frac{dy}{dt} = -1$, so $\frac{dy}{dx} = 1$. When $t = \frac{3\pi}{2}$, $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = -1$. So $\frac{dy}{dx} = -1$. Thus, $y = x$ and $y = -x$ are both tangent to the curve at $(0, 0)$.



28. $x = 1 - 2\cos^2 t = -\cos 2t$, $y = (\tan t)\left(1 - 2\cos^2 t\right) = -(\tan t)\cos 2t$. To find a point where the curve crosses itself, we look for two values of t that give the same point (x, y). Call these values t_1 and t_2. Then $\cos^2 t_1 = \cos^2 t_2$ (from the equation for x) and either $\tan t_1 = \tan t_2$ or $\cos^2 t_1 = \cos^2 t_2 = \frac{1}{2}$ (from the equation for y). We can satisfy $\cos^2 t_1 = \cos^2 t_2$ and $\tan t_1 = \tan t_2$ by choosing t_1 arbitrarily and taking $t_2 = t_1 + \pi$, so evidently the whole curve is retraced every time t traverses an interval of length π. Thus, we can restrict our attention to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. If $t_2 = -t_1$, then $\cos^2 t_2 = \cos^2 t_1$, but $\tan t_2 = -\tan t_1$. This suggests that we try to satisfy the condition $\cos^2 t_1 = \cos^2 t_2 = \frac{1}{2}$. Taking $t_1 = \frac{\pi}{4}$ and $t_2 = -\frac{\pi}{4}$ gives $(x, y) = (0, 0)$ for both values of t. $\frac{dx}{dt} = 2\sin 2t$,and $\frac{dy}{dt} = 2\sin 2t \tan t - \cos 2t \sec^2 t$. When $t = \frac{\pi}{4}$, $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 2$, so $\frac{dy}{dx} = 1$. When $t = -\frac{\pi}{4}$, $\frac{dx}{dt} = -2$ and $\frac{dy}{dt} = 2$, so $\frac{dy}{dx} = -1$. Thus, the equations of the two tangents at $(0, 0)$ are $y = x$ and $y = -x$.



29. (a) $x = r\theta - d\sin\theta$, $y = r - d\cos\theta$; $\dfrac{dx}{d\theta} = r - d\cos\theta$, $\dfrac{dy}{d\theta} = d\sin\theta$. So $\dfrac{dy}{dx} = \dfrac{d\sin\theta}{r - d\cos\theta}$.

(b) If $0 < d < r$, then $|d\cos\theta| \le d < r$, so $r - d\cos\theta \ge r - d > 0$. This shows that $dx/d\theta$ never vanishes, so the trochoid can have no vertical tangent if $d < r$.

30. $x = a\cos^3\theta$, $y = a\sin^3\theta$.

(a) $\dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$, $\dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$, so $\dfrac{dy}{dx} = -\dfrac{\sin\theta}{\cos\theta} = -\tan\theta$.

(b) The tangent is horizontal $\Leftrightarrow dy/dx = 0 \Leftrightarrow \tan\theta = 0 \Leftrightarrow \theta = n\pi \Leftrightarrow (x, y) = (\pm a, 0)$. The tangent is vertical $\Leftrightarrow \cos\theta = 0 \Leftrightarrow \theta$ is an odd multiple of $\frac{\pi}{2} \Leftrightarrow (x, y) = (0, \pm a)$.

(c) $dy/dx = \pm 1 \Leftrightarrow \tan\theta = \pm 1 \Leftrightarrow \theta$ is an odd multiple of $\frac{\pi}{4} \Leftrightarrow (x, y) = \left(\pm\frac{\sqrt{2}}{4}a, \pm\frac{\sqrt{2}}{4}a\right)$ (All sign choices are valid.)

31. The line with parametric equations $x = -7t$, $y = 12t - 5$ is $y = 12\left(-\frac{1}{7}x\right) - 5$, which has slope $-\frac{12}{7}$. The curve $x = t^3 + 4t$, $y = 6t^2$ has slope $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{12t}{3t^2+4}$. This equals $-\frac{12}{7} \iff 3t^2 + 4 = -7t \iff (3t+4)(t+1) = 0 \iff t = -1$ or $t = -\frac{4}{3} \iff (x, y) = (-5, 6)$ or $\left(-\frac{208}{27}, \frac{32}{3}\right)$.

32. $x = 3t^2 + 1$, $y = 2t^3 + 1$, $\dfrac{dx}{dt} = 6t$, $\dfrac{dy}{dt} = 6t^2$, so $\dfrac{dy}{dx} = \dfrac{6t^2}{6t} = t$ (even where $t = 0$).
So at the point corresponding to parameter value t, an equation of the tangent line is $y - \left(2t^3 + 1\right) = t\left[x - \left(3t^2 + 1\right)\right]$. If this line is to pass through $(4, 3)$, we must have $3 - \left(2t^3 + 1\right) = t\left[4 - \left(3t^2 + 1\right)\right] \iff 2t^3 - 2 = 3t^3 - 3t \iff t^3 - 3t + 2 = 0 \iff (t-1)^2(t+2) = 0 \iff t = 1$ or -2. Hence, the desired equations are $y - 3 = x - 4$, or $y = x - 1$, tangent to the curve at $(4, 3)$, and $y - (-15) = -2(x - 13)$, or $y = -2x + 11$, tangent to the curve at $(13, -15)$.

33. $A = \int_0^1 (y-1)\,dx = \int_{\pi/2}^0 \left(e^t - 1\right)(-\sin t)\,dt = \int_0^{\pi/2} \left(e^t \sin t - \sin t\right) dt \overset{98}{=} \left[\frac{1}{2}e^t(\sin t - \cos t) + \cos t\right]_0^{\pi/2}$
$= \frac{1}{2}\left(e^{\pi/2} - 1\right)$

34. $t + 1/t = 2.5 \iff t = \frac{1}{2}$ or 2, and for $\frac{1}{2} < t < 2$, we have $t + 1/t < 2.5$. $x = -\frac{3}{2}$ when $t = \frac{1}{2}$ and $x = \frac{3}{2}$ when $t = 2$.

$$A = \int_{-3/2}^{3/2} (2.5 - y)\,dx = \int_{1/2}^{2} \left(\tfrac{5}{2} - t - 1/t\right)\left(1 + 1/t^2\right) dt \quad \left[x = t - 1/t,\ dx = \left(1 + 1/t^2\right) dt\right]$$
$$= \int_{1/2}^{2} \left(-t + \tfrac{5}{2} - 2t^{-1} + \tfrac{5}{2}t^{-2} - t^{-3}\right) dt = \left[\frac{-t^2}{2} + \frac{5t}{2} - 2\ln|t| - \frac{5}{2t} + \frac{1}{2t^2}\right]_{1/2}^{2}$$
$$= \left(-2 + 5 - 2\ln 2 - \tfrac{5}{4} + \tfrac{1}{8}\right) - \left(-\tfrac{1}{8} + \tfrac{5}{4} + 2\ln 2 - 5 + 2\right) = \tfrac{15}{4} - 4\ln 2$$

35. By symmetry of the ellipse about the x- and y-axes,

$$A = 4\int_0^a y\,dx = 4\int_{\pi/2}^0 b\sin\theta\,(-a\sin\theta)\,d\theta = 4ab\int_0^{\pi/2} \sin^2\theta\,d\theta = 4ab\int_0^{\pi/2} \tfrac{1}{2}(1 - \cos 2\theta)\,d\theta$$
$$= 2ab\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 2ab\left(\tfrac{\pi}{2}\right) = \pi ab$$

36. By symmetry, $A = 4\int_0^a y\,dx = 4\int_{\pi/2}^0 a\sin^3\theta\left(-3a\cos^2\theta\sin\theta\right) d\theta = 12a^2\int_0^{\pi/2} \sin^4\theta\cos^2\theta\,d\theta$. Now

$$\int \sin^4\theta\cos^2\theta\,d\theta = \int \sin^2\theta\left(\tfrac{1}{4}\sin^2 2\theta\right) d\theta = \tfrac{1}{8}\int (1 - \cos 2\theta)\sin^2 2\theta\,d\theta$$
$$= \tfrac{1}{8}\int \left[\tfrac{1}{2}(1 - \cos 4\theta) - \sin^2 2\theta\cos 2\theta\right] d\theta = \tfrac{1}{16}\theta - \tfrac{1}{64}\sin 4\theta - \tfrac{1}{48}\sin^3 2\theta + C$$

so $\int_0^{\pi/2} \sin^4\theta\cos^2\theta\,d\theta = \left[\frac{1}{16}\theta - \frac{1}{64}\sin 4\theta - \frac{1}{48}\sin^3 2\theta\right]_0^{\pi/2} = \frac{\pi}{32}$. Thus, $A = 12a^2\left(\frac{\pi}{32}\right) = \frac{3}{8}\pi a^2$.

37. $A = \int_0^{2\pi r} y\,dx = \int_0^{2\pi} (r - d\cos\theta)(r - d\cos\theta)\,d\theta = \int_0^{2\pi} \left(r^2 - 2dr\cos\theta + d^2\cos^2\theta\right) d\theta$
$= \left[r^2\theta - 2dr\sin\theta + \frac{1}{2}d^2\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} = 2\pi r^2 + \pi d^2$

38. (a) By symmetry, the area of $\mathcal{R}$ is twice the area inside $\mathcal{R}$ above the x-axis. The top half of the loop is described by $x = t^2$, $y = t^3 - 3t$, $-\sqrt{3} \le t \le 0$, so, using the Substitution Rule with $y = t^3 - 3t$ and $dx = 2t\,dt$, we find that

$$\text{area} = 2\int_0^3 y\,dx = 2\int_0^{-\sqrt{3}} (t^3 - 3t)\,2t\,dt = 2\int_0^{-\sqrt{3}} (2t^4 - 6t^2)\,dt = 2\left[\tfrac{2}{5}t^5 - 2t^3\right]_0^{-\sqrt{3}}$$
$$= 2\left[\tfrac{2}{5}\left(-3^{1/2}\right)^5 - 2\left(-3^{1/2}\right)^3\right] = 2\left[\tfrac{2}{5}\left(-9\sqrt{3}\right) - 2\left(-3\sqrt{3}\right)\right] = \tfrac{24}{5}\sqrt{3}$$

(b) Here we use the formula for disks and use the Substitution Rule as in part (a):

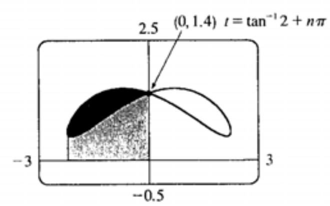
$$\text{volume} = \pi\int_0^3 y^2\,dx = \pi\int_0^{-\sqrt{3}} (t^3 - 3t)^2\,2t\,dt = 2\pi\int_0^{-\sqrt{3}} (t^6 - 6t^4 + 9t^2)\,t\,dt$$
$$= 2\pi\left[\tfrac{1}{8}t^8 - t^6 + \tfrac{9}{4}t^4\right]_0^{-\sqrt{3}} = 2\pi\left[\tfrac{1}{8}\left(-3^{1/2}\right)^8 - \left(-3^{1/2}\right)^6 + \tfrac{9}{4}\left(-3^{1/2}\right)^4\right]$$
$$= 2\pi\left[\tfrac{81}{8} - 27 + \tfrac{81}{4}\right] = \tfrac{27}{4}\pi$$

(c) By symmetry, the y-coordinate of the centroid is 0. To find the x-coordinate, we note that it is the same as the x-coordinate of the centroid of the top half of $\mathcal{R}$, the area of which is $\frac{1}{2}\cdot\frac{24}{5}\sqrt{3} = \frac{12}{5}\sqrt{3}$. So, using Formula 9.3.8 with $A = \frac{12}{5}\sqrt{3}$, we get

$$\overline{x} = \frac{5}{12\sqrt{3}}\int_0^3 xy\,dx = \frac{5}{12\sqrt{3}}\int_0^{-\sqrt{3}} t^2\,(t^3 - 3t)\,2t\,dt = \frac{5}{6\sqrt{3}}\left[\tfrac{1}{7}t^7 - \tfrac{3}{5}t^5\right]_0^{-\sqrt{3}}$$
$$= \frac{5}{6\sqrt{3}}\left[\tfrac{1}{7}\left(-3^{1/2}\right)^7 - \tfrac{3}{5}\left(-3^{1/2}\right)^5\right] = \frac{5}{6\sqrt{3}}\left[-\tfrac{27}{7}\sqrt{3} + \tfrac{27}{5}\sqrt{3}\right] = \tfrac{9}{7}$$

So the coordinates of the centroid of $\mathcal{R}$ are $(x, y) = \left(\frac{9}{7}, 0\right)$.

39. The graph of $x = \sin t - 2\cos t$, $y = 1 + \sin t \cos t$ is symmetric about the y-axis. The graph intersects the y-axis when $x = 0 \Rightarrow \sin t - 2\cos t = 0 \Rightarrow \sin t = 2\cos t \Rightarrow \tan t = 2 \Rightarrow t = \tan^{-1} 2 + n\pi$. The left loop is traced in a clockwise direction from $t = \tan^{-1} 2 - \pi$ to $t = \tan^{-1} 2$, so the area of the loop is given (as in Example 4) by

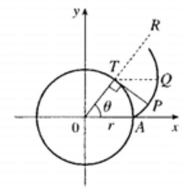


$$A = \int_{\tan^{-1}2-\pi}^{\tan^{-1}2} y\,dx \approx \int_{-2.0344}^{1.1071} (1 + \sin t\cos t)(\cos t + 2\sin t)\,dt \approx 0.8944$$

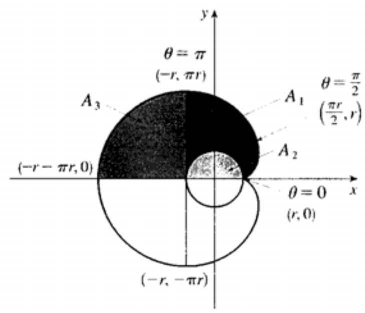
This integral can be evaluated exactly; its value is $\frac{2}{5}\sqrt{5}$.

40. If f' is continuous and $f'(t) \ne 0$ for $a \le t \le b$, then either $f'(t) > 0$ for all t in $[a, b]$ or $f'(t) < 0$ for all t in $[a, b]$. Thus, f is monotonic (in fact, strictly increasing or strictly decreasing) on $[a, b]$. It follows that f has an inverse. Set $F = g \circ f^{-1}$, that is, define F by $F(x) = g\left(f^{-1}(x)\right)$. Then $x = f(t) \Rightarrow f^{-1}(x) = t$, so $y = g(t) = g\left(f^{-1}(x)\right) = F(x)$.

41. The coordinates of T are $(r\cos\theta, r\sin\theta)$. Since TP was unwound from arc TA, TP has length $r\theta$. Also $\angle PTQ = \angle PTR - \angle QTR = \frac{1}{2}\pi - \theta$, so P has coordinates $x = r\cos\theta + r\theta\cos\left(\frac{1}{2}\pi - \theta\right) = r(\cos\theta + \theta\sin\theta)$, $y = r\sin\theta - r\theta\sin\left(\frac{1}{2}\pi - \theta\right) = r(\sin\theta - \theta\cos\theta)$.



42. If the cow walks with the rope taut, it traces out the portion of the involute in Exercise 41 corresponding to the range $0 \le \theta \le \pi$, arriving at the point $(-r, \pi r)$ when $\theta = \pi$. With the rope now fully extended, the cow walks in a semicircle of radius πr, arriving at $(-r, -\pi r)$. Finally, the cow traces out another portion of the involute, namely the reflection about the x-axis of the initial involute path. (This corresponds to the range $-\pi \le \theta \le 0$.) Referring to the figure, we see that the total grazing area is $2(A_1 + A_3)$. A_3 is one-quarter of the area of a circle of radius πr, so $A_3 = \frac{1}{4}\pi(\pi r)^2 = \frac{1}{4}\pi^3 r^2$. We will compute $A_1 + A_2$ and then subtract $A_2 = \frac{1}{2}\pi r^2$ to obtain A_1.



To find $A_1 + A_2$, first note that the rightmost point of the involute is $\left(\frac{\pi}{2}r, r\right)$. [To see this, note that $dx/d\theta = 0$ when $\theta = 0$ or $\frac{\pi}{2}$. $\theta = 0$ corresponds to the cusp at $(r, 0)$ and $\theta = \frac{\pi}{2}$ corresponds to $\left(\frac{\pi}{2}r, r\right)$.] The leftmost point of the involute is $(-r, \pi r)$. Thus, $A_1 + A_2 = \int_{\theta=\pi}^{\pi/2} y\,dx - \int_{\theta=0}^{\pi/2} y\,dx = \int_{\theta=\pi}^{0} y\,dx$. Now $y\,dx = r(\sin\theta - \theta\cos\theta)\,r\theta\cos\theta\,d\theta = r^2\left(\theta\sin\theta\cos\theta - \theta^2\cos^2\theta\right)d\theta$. Integrate: $\left(1/r^2\right)\int y\,dx = -\theta\cos^2\theta - \frac{1}{2}\left(\theta^2 - 1\right)\sin\theta\cos\theta - \frac{1}{6}\theta^3 + \frac{1}{2}\theta + C$. This enables us to compute

$$\begin{aligned} A_1 + A_2 &= r^2\left[-\theta\cos^2\theta - \frac{1}{2}\left(\theta^2 - 1\right)\sin\theta\cos\theta - \frac{1}{6}\theta^3 + \frac{1}{2}\theta\right]_\pi^0 = r^2\left[0 - \left(-\pi - \frac{\pi^3}{6} + \frac{\pi}{2}\right)\right] \\ &= r^2\left(\frac{\pi}{2} + \frac{\pi^3}{6}\right) \end{aligned}$$

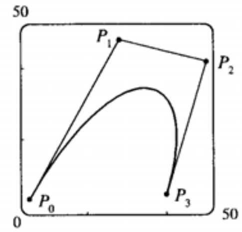
Therefore, $A_1 = (A_1 + A_2) - A_2 = \frac{1}{6}\pi^3 r^2$, so the grazing area is $2(A_1 + A_3) = 2\left(\frac{1}{6}\pi^3 r^2 + \frac{1}{4}\pi^3 r^2\right) = \frac{5}{6}\pi^3 r^2$.

Laboratory Project □ Bézier Curves

1. We are given the points $P_0\,(x_0, y_0) = (4, 1)$, $P_1\,(x_1, y_1) = (28, 48)$, $P_2\,(x_2, y_2) = (50, 42)$, and $P_3\,(x_3, y_3) = (40, 5)$. The curve is then given by

$$x\,(t) = 4\,(1-t)^3 + 3 \cdot 28t\,(1-t)^2 + 3 \cdot 50t^2\,(1-t) + 40t^3$$

$$y\,(t) = 1\,(1-t)^3 + 3 \cdot 48t\,(1-t)^2 + 3 \cdot 42t^2\,(1-t) + 5t^3$$



where $0 \le t \le 1$. The line segments are of the form $x = x_0 + (x_1 - x_0)\,t$, $y = y_0 + (y_1 - y_0)\,t$:

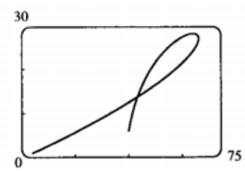
P_0P_1	$x = 4 + 24t,$	$y = 1 + 47t$
P_1P_2	$x = 28 + 22t,$	$y = 48 - 6t$
P_2P_3	$x = 50 - 10t,$	$y = 42 - 37t$

2. It suffices to show that the slope of the tangent at P_0 is the same as that of line segment P_0P_1, namely $\dfrac{y_1 - y_0}{x_1 - x_0}$. We calculate the slope of the tangent to the Bézier curve:

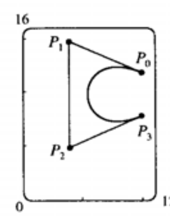
$$\frac{dy/dt}{dx/dt} = \frac{-3y_0\,(1-t)^2 + 3y_1\left[-2t\,(1-t) + (1-t)^2\right] + 3y_2\left[-t^2 + (2t)\,(1-t)\right] + 3y_3t^2}{-3x_0\,(1-t)^2 + 3x_1\left[-2t\,(1-t) + (1-t)^2\right] + 3x_2\left[-t^2 + (2t)\,(1-t)\right] + 3x_3t^2}$$

At point P_0, $t = 0$, so the slope of the tangent is $\dfrac{-3y_0 + 3y_1}{-3x_0 + 3x_1} = \dfrac{y_1 - y_0}{x_1 - x_0}$. So the tangent to the curve at P_0 passes through P_1. Similarly, the slope of the tangent at point P_3 (where $t = 1$) is $\dfrac{-3y_2 + 3y_3}{-3x_2 + 3x_3} = \dfrac{y_3 - y_2}{x_3 - x_2}$, which is also the slope of line P_2P_3.

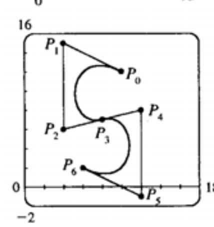
3. It seems that if P_1 were to the right of P_2, a loop would appear. We try setting $P_1 = (110, 30)$, and the resulting curve does indeed have a loop.



4. Based on the behavior of the Bézier curve in Problems 1–3, we suspect that the four control points should be in an exaggerated C shape. We try $P_0\,(10, 12)$, $P_1\,(4, 15)$, $P_2\,(4, 5)$, and $P_3\,(10, 8)$, and these produce a decent C. If you are using a CAS, it may be necessary to instruct it to make the x- and y-scales the same so as not to distort the figure (this is called a "constrained projection" in Maple.)



5. We use the same P_0 and P_1 as in part (a), and use part of our C as the top of an S. To prevent the center line from slanting up too much, we move P_2 up to $(4, 6)$ and P_3 down and to the left, to $(8, 7)$. In order to have a smooth joint between the top and bottom halves of the S (and a symmetric S), we determine points P_4, P_5, and P_6 by rotating points P_2, P_1, and P_0 about the center of the letter (point P_3). The points are therefore $P_4\,(12, 8)$, $P_5\,(12, -1)$, and $P_6\,(6, 2)$.



11.3 Arc Length and Surface Area

1. $x = t - t^2$, $y = \frac{4}{3}t^{3/2}$, $1 \le t \le 2$. $dx/dt = 1 - 2t$ and $dy/dt = 2t^{1/2}$, so
$(dx/dt)^2 + (dy/dt)^2 = (1-2t)^2 + \left(2t^{1/2}\right)^2 = 1 - 4t + 4t^2 + 4t = 1 + 4t^2$ and
$L = \int_\alpha^\beta \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_1^2 \sqrt{1 + 4t^2}\,dt$.

2. $x = 1 + e^t$, $y = t^2$, $-3 \le t \le 3$. $dx/dt = e^t$ and $dy/dt = 2t$, so $(dx/dt)^2 + (dy/dt)^2 = e^{2t} + 4t^2$ and
$L = \int_{-3}^3 \sqrt{e^{2t} + 4t^2}\,dt$.

3. $x = t\sin t$, $y = t\cos t$, $0 \le t \le \frac{\pi}{2}$. $dx/dt = t\cos t + \sin t$ and $dy/dt = t(-\sin t) + \cos t$, so

$$\begin{aligned}(dx/dt)^2 + (dy/dt)^2 &= (t\cos t + \sin t)^2 + (\cos t - t\sin t)^2 \\ &= t^2\cos^2 t + 2t\sin t\cos t + \sin^2 t + \cos^2 t - 2t\sin t\cos t + t^2\sin^2 t \\ &= t^2\left(\cos^2 t + \sin^2 t\right) + \sin^2 t + \cos^2 t = t^2 + 1\end{aligned}$$

and $L = \int_0^{\pi/2} \sqrt{t^2 + 1}\,dt$.

4. $x = \ln t$, $y = \sqrt{t+1}$, $1 \le t \le 5$. $\dfrac{dx}{dt} = \dfrac{1}{t}$ and $\dfrac{dy}{dt} = \dfrac{1}{2\sqrt{t+1}}$, so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{1}{t^2} + \frac{1}{4(t+1)} = \frac{t^2 + 4t + 4}{4t^2(t+1)} \text{ and } L = \int_1^5 \sqrt{\frac{t^2+4t+4}{4t^2(t+1)}}\,dt = \int_1^5 \frac{t+2}{2t\sqrt{t+1}}\,dt.$$

5. $x = t^3$, $y = t^2$, $0 \le t \le 4$. $(dx/dt)^2 + (dy/dt)^2 = \left(3t^2\right)^2 + (2t)^2 = 9t^4 + 4t^2$.
$L = \int_0^4 \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^4 \sqrt{9t^4 + 4t^2}\,dt = \int_0^4 t\sqrt{9t^2+4}\,dt = \frac{1}{18}\int_4^{148}\sqrt{u}\,du$ (where $u = 9t^2 + 4$).
So $L = \frac{1}{18}\left(\frac{2}{3}\right)\left[u^{3/2}\right]_4^{148} = \frac{1}{27}\left(148^{3/2} - 4^{3/2}\right) = \frac{8}{27}\left(37^{3/2} - 1\right)$.

6. $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$, $0 \le \theta \le \pi$.

$$\begin{aligned}(dx/d\theta)^2 + (dy/d\theta)^2 &= a^2\left[(-\sin\theta + \sin\theta + \theta\cos\theta)^2 + (\cos\theta - \cos\theta + \theta\sin\theta)^2\right] \\ &= a^2\theta^2\left(\cos^2\theta + \sin^2\theta\right) = (a\theta)^2\end{aligned}$$

$L = \int_0^\pi a\theta\,d\theta = \frac{1}{2}\pi^2 a$

7. $x = \dfrac{t}{1+t}$, $y = \ln(1+t)$, $0 \le t \le 2$. $\dfrac{dx}{dt} = \dfrac{(1+t)\cdot 1 - t\cdot 1}{(1+t)^2} = \dfrac{1}{(1+t)^2}$ and $\dfrac{dy}{dt} = \dfrac{1}{1+t}$, so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{1}{(1+t)^4} + \frac{1}{(1+t)^2} = \frac{1}{(1+t)^4}\left[1 + (1+t)^2\right] = \frac{t^2+2t+2}{(1+t)^4} \text{ and}$$

$$L = \int_0^2 \frac{\sqrt{t^2+2t+2}}{(1+t)^2}\,dt = \int_1^3 \frac{\sqrt{u^2+1}}{u^2}\,du \quad [u = t+1,\ du = dt] \quad \stackrel{24}{=} \left[-\frac{\sqrt{u^2+1}}{u} + \ln\left(u + \sqrt{u^2+1}\right)\right]_1^3$$

$$= -\tfrac{\sqrt{10}}{3} + \ln\left(3 + \sqrt{10}\right) + \sqrt{2} - \ln\left(1 + \sqrt{2}\right)$$

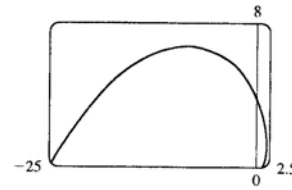
8. $x = e^t + e^{-t}$, $y = 5 - 2t$, $0 \le t \le 3$. $dx/dt = e^t - e^{-t}$ and $dy/dt = -2$, so
$(dx/dt)^2 + (dy/dt)^2 = e^{2t} - 2 + e^{-2t} + 4 = e^{2t} + 2 + e^{-2t} = \left(e^t + e^{-t}\right)^2$ and
$L = \int_0^3 \left(e^t + e^{-t}\right)dt = \left[e^t - e^{-t}\right]_0^3 = e^3 - e^{-3} - (1 - 1) = e^3 - e^{-3}$.

9. $x = e^t \cos t$, $y = e^t \sin t$, $0 \le t \le \pi$.

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left[e^t (\cos t - \sin t)\right]^2 + \left[e^t (\sin t + \cos t)\right]^2$$

$$= e^{2t}\left(2\cos^2 t + 2\sin^2 t\right) = 2e^{2t}$$

$$\Rightarrow \quad L = \int_0^\pi \sqrt{2}e^t \, dt = \sqrt{2}\,(e^\pi - 1)$$

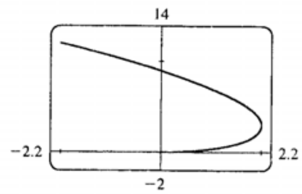


10. $x = 3t - t^3$, $y = 3t^2$, $0 \le t \le 2$.

$$(dx/dt)^2 + (dy/dt)^2 = \left(3 - 3t^2\right)^2 + (6t)^2$$

$$= 9\left(1 + 2t^2 + t^4\right) = \left[3\left(1 + t^2\right)\right]^2$$

$$L = \int_0^2 3\left(1 + t^2\right) dt = \left[3t + t^3\right]_0^2 = 14$$



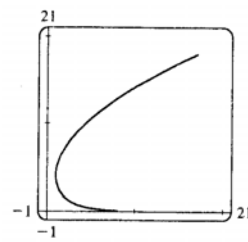
11. $x = e^t - t$, $y = 4e^{t/2}$, $-8 \le t \le 3$.

$$(dx/dt)^2 + (dy/dt)^2 = \left(e^t - 1\right)^2 + \left(2e^{t/2}\right)^2 = e^{2t} - 2e^t + 1 + 4e^t$$

$$= e^{2t} + 2e^t + 1 = \left(e^t + 1\right)^2$$

$$L = \int_{-8}^3 \sqrt{\left(e^t + 1\right)^2}\, dt = \int_{-8}^3 \left(e^t + 1\right) dt = \left[e^t + t\right]_{-8}^3$$

$$= \left(e^3 + 3\right) - \left(e^{-8} - 8\right) = e^3 - e^{-8} + 11$$



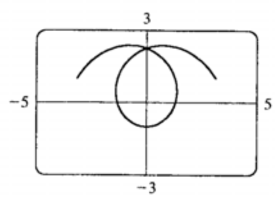
12. $x = t\cos t + \sin t$, $y = t\sin t - \cos t$, $-\pi \le t \le \pi$. $dx/dt = -t\sin t + 2\cos t$ and $dy/dt = t\cos t + 2\sin t$, so $(dx/dt)^2 + (dy/dt)^2 = t^2\sin^2 t - 4t\sin t\cos t + 4\cos^2 t + t^2\cos^2 t + 4t\sin t\cos t + 4\sin^2 t = t^2 + 4$ and

$$L = \int_{-\pi}^{\pi} \sqrt{t^2 + 4}\, dt = 2\int_0^\pi \sqrt{t^2 + 4}\, dt$$

$$\overset{21}{=} 2\left[\tfrac{1}{2}t\sqrt{t^2 + 4} + 2\ln\left(t + \sqrt{t^2 + 4}\right)\right]_0^\pi$$

$$= 2\left[\tfrac{\pi}{2}\sqrt{\pi^2 + 4} + 2\ln\left(\pi + \sqrt{\pi^2 + 4}\right) - 2\ln 2\right]$$

$$= \pi\sqrt{\pi^2 + 4} + 4\ln\left(\pi + \sqrt{\pi^2 + 4}\right) - 4\ln 2 \approx 16.633506$$



13. $x = \ln t$ and $y = e^{-t}$ $\Rightarrow$ $\dfrac{dx}{dt} = \dfrac{1}{t}$ and $\dfrac{dy}{dt} = -e^{-t}$ $\Rightarrow$ $L = \int_1^2 \sqrt{t^{-2} + e^{-2t}}\, dt$. Using Simpson's Rule with $n = 10$, $\Delta x = (2-1)/10 = 0.1$ and $f(t) = \sqrt{t^{-2} + e^{-2t}}$ we get $L \approx \frac{0.1}{3}\left[f(1.0) + 4f(1.1) + 2f(1.2) + \cdots + 2f(1.8) + 4f(1.9) + f(2.0)\right] \approx 0.7314$.

14. $x = 2a\cot\theta$ $\Rightarrow$ $dx/dt = -2a\csc^2\theta$ and $y = 2a\sin^2\theta$ $\Rightarrow$ $dy/dt = 4a\sin\theta\cos\theta = 2a\sin 2\theta$. So $L = \int_{\pi/4}^{\pi/2} \sqrt{4a^2\csc^4\theta + 4a^2\sin^2 2\theta}\, d\theta = 2a\int_{\pi/4}^{\pi/2} \sqrt{\csc^4\theta + \sin^2 2\theta}\, d\theta$. Using Simpson's Rule with $n = 4$, $\Delta x = \frac{\pi/4}{4} = \frac{\pi}{16}$ and $f(\theta) = \sqrt{\csc^4\theta + \sin^2 2\theta}$, we get $L \approx 2a \cdot S_4 = (2a)\,\frac{\pi}{16 \cdot 3}\left[f\left(\frac{\pi}{4}\right) + 4f\left(\frac{5\pi}{16}\right) + 2f\left(\frac{3\pi}{8}\right) + 4f\left(\frac{7\pi}{16}\right) + f\left(\frac{\pi}{2}\right)\right] \approx 2.2605a$.

15. $x = \sin^2\theta,\ y = \cos^2\theta,\ 0 \le \theta \le 3\pi.$

$(dx/d\theta)^2 + (dy/d\theta)^2 = (2\sin\theta\cos\theta)^2 + (-2\cos\theta\sin\theta)^2 = 8\sin^2\theta\cos^2\theta = 2\sin^2 2\theta \quad \Rightarrow$

$$\text{Distance} = \int_0^{3\pi} \sqrt{2}\,|\sin 2\theta|\,d\theta = 6\sqrt{2}\int_0^{\pi/2} \sin 2\theta\,d\theta \text{ (by symmetry) } = \Big[-3\sqrt{2}\cos 2\theta\Big]_0^{\pi/2}$$

$$= -3\sqrt{2}\,(-1-1) = 6\sqrt{2}$$

The full curve is traversed as θ goes from 0 to $\frac{\pi}{2}$, because the curve is the segment of $x + y = 1$ that lies in the first quadrant (since $x, y \ge 0$), and this segment is completely traversed as θ goes from 0 to $\frac{\pi}{2}$.

Thus $L = \int_0^{\pi/2} \sin 2\theta\,d\theta = \sqrt{2}$, as above.

16. $x = \cos^2 t,\ y = \cos t,\ 0 \le t \le 4\pi.\ \left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = (-2\cos t\sin t)^2 + (-\sin t)^2 = \sin^2 t\left(4\cos^2 t + 1\right)$

$$\text{Distance } = \int_0^{4\pi} |\sin t|\,\sqrt{4\cos^2 t + 1}\,dt = 4\int_0^{\pi} \sin t\sqrt{4\cos^2 t + 1}\,dt$$

$$= -4\int_1^{-1}\sqrt{4u^2+1}\,du\ \ [u = \cos t,\ du = -\sin t\,dt] = 4\int_{-1}^{1}\sqrt{4u^2+1}\,du = 8\int_0^1\sqrt{4u^2+1}\,du$$

$$= 8\int_0^{\tan^{-1}2} \sec\theta\,\tfrac{1}{2}\sec^2\theta\,d\theta = 4\int_0^{\tan^{-1}2}\sec^3\theta\,d\theta \stackrel{21}{=} \left[2\sec\theta\tan\theta + 2\ln|\sec\theta + \tan\theta|\right]_0^{\tan^{-1}2}$$

$$= 4\sqrt{5} + 2\ln\left(\sqrt{5}+2\right)$$

$L = \int_0^{\pi} |\sin t|\,\sqrt{4\cos^2 t + 1}\,dt = \sqrt{5} + \frac{1}{2}\ln\left(\sqrt{5}+2\right)$

17. $x = a\sin\theta,\ y = b\cos\theta,\ 0 \le \theta \le 2\pi.$

$$\left(\tfrac{dx}{d\theta}\right)^2 + \left(\tfrac{dy}{d\theta}\right)^2 = (a\cos\theta)^2 + (-b\sin\theta)^2 = a^2\cos^2\theta + b^2\sin^2\theta = a^2\left(1-\sin^2\theta\right) + b^2\sin^2\theta$$

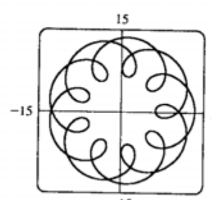
$$= a^2 - \left(a^2 - b^2\right)\sin^2\theta = a^2 - c^2\sin^2\theta = a^2\left(1 - \frac{c^2}{a^2}\sin^2\theta\right) = a^2\left(1 - e^2\sin^2\theta\right)$$

So $L = 4\int_0^{\pi/2}\sqrt{a^2\left(1-e^2\sin^2\theta\right)}\,d\theta$ (by symmetry) $= 4a\int_0^{\pi/2}\sqrt{1-e^2\sin^2\theta}\,d\theta$

18. $x = a\cos^3\theta,\ y = a\sin^3\theta.\ (dx/d\theta)^2 + (dy/d\theta)^2 = \left(-3a\cos^2\theta\sin\theta\right)^2 + \left(3a\sin^2\theta\cos\theta\right)^2 = 9a^2\sin^2\theta\cos^2\theta.$

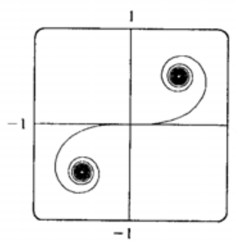
$L = 4\int_0^{\pi/2} 3a\sin\theta\cos\theta\,d\theta = \left[12a\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2} = 6a.$

19. (a) Notice that $0 \le t \le 2\pi$ does not give the complete curve because $x(0) \ne x(2\pi)$. In fact, we must take $t \in [0, 4\pi]$ in order to obtain the complete curve, since the first term in each of the parametric equations has period 2π and the second has period $\frac{4\pi}{11}$, and the least common integer multiple of these two numbers is 4π.



(b) We use the CAS to find the derivatives dx/dt and dy/dt, and then use Theorem 4 to find the arc length. Recent versions of Maple express the integral $\int_0^{4\pi}\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$ as $88E\left(2\sqrt{2}i\right)$, where $E(x)$ is the elliptic integral $\displaystyle\int_0^1 \frac{\sqrt{1-x^2t^2}}{\sqrt{1-t^2}}\,dt$ and i is the imaginary number $\sqrt{-1}$. Some earlier versions of Maple (as well as Mathematica) cannot do the integral exactly, so we use the command `evalf(Int(sqrt(diff(x,t)^2+diff(y,t)^2),t=0..4*Pi));` to estimate the length, and find that the arc length is approximately 294.03.

20. (a) It appears that as $t \to \infty$, $(x, y) \to \left(\frac{1}{2}, \frac{1}{2}\right)$, and as $t \to -\infty$, $(x, y) \to \left(-\frac{1}{2}, -\frac{1}{2}\right)$.



(b) By the Fundamental Theorem of Calculus, $dx/dt = \cos\left(\frac{\pi}{2}t^2\right)$ and $dy/dt = \sin\left(\frac{\pi}{2}t^2\right)$, so by Theorem 4, the length of the curve from the origin to the point with parameter value t is

$$L = \int_0^t \sqrt{(dx/du)^2 + (dy/du)^2}\, du = \int_0^t \sqrt{\cos^2\left(\tfrac{\pi}{2}u^2\right) + \sin^2\left(\tfrac{\pi}{2}u^2\right)}\, du = \int_0^t 1\, du = t \ \text{ (or } -t \text{ if } t < 0)$$

We have used u as the dummy variable so as not to confuse it with the upper limit of integration.

21. $x = t^3$ and $y = t^4 \Rightarrow dx/dt = 3t^2$ and $dy/dt = 4t^3$. So
$S = \int_0^1 2\pi t^4 \sqrt{9t^4 + 16t^6}\, dt = \int_0^1 2\pi t^6 \sqrt{9 + 16t^2}\, dt$.

22. $x = \sin^2 t$, $y = \sin 3t$, $0 \le t \le \frac{\pi}{3}$. $dx/dt = 2\sin t \cos t = \sin 2t$ and $dy/dt = 3\cos 3t$, so
$(dx/dt)^2 + (dy/dt)^2 = \sin^2 2t + 9\cos^2 3t$ and $S = \int 2\pi y\, ds = \int_0^{\pi/3} 2\pi \sin 3t \sqrt{\sin^2 2t + 9\cos^2 3t}\, dt$.

23. $x = t^3$, $y = t^2$, $0 \le t \le 1$. $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = \left(3t^2\right)^2 + (2t)^2 = 9t^4 + 4t^2$.

$$S = \int_0^1 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 2\pi t^2 \sqrt{9t^4 + 4t^2}\, dt$$
$$= 2\pi \int_4^{13} \frac{u-4}{9} \sqrt{u} \left(\tfrac{1}{18} du\right) \ \text{(where } u = 9t^2 + 4) \ = \tfrac{\pi}{81}\left[\tfrac{2}{5}u^{5/2} - \tfrac{8}{3}u^{3/2}\right]_4^{13} = \tfrac{2\pi}{1215}\left(247\sqrt{13} + 64\right)$$

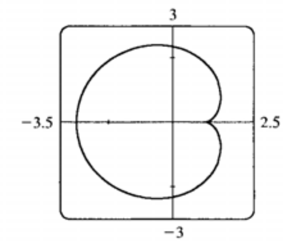
24. $x = 3t - t^3$, $y = 3t^2$, $0 \le t \le 1$. $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = \left(3 - 3t^2\right)^2 + (6t)^2 = 9\left(1 + 2t^2 + t^4\right) = \left[3\left(1 + t^2\right)\right]^2$.
$S = \int_0^1 2\pi\, 3t^2 3\left(1 + t^2\right) dt = 18\pi \int_0^1 \left(t^2 + t^4\right) dt = 18\pi \left[\frac{1}{3}t^3 + \frac{1}{5}t^5\right]_0^1 = \frac{48}{5}\pi$

25. $x = a\cos^3\theta$, $y = a\sin^3\theta$, $0 \le \theta \le \frac{\pi}{2}$.
$\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2 = \left(-3a\cos^2\theta \sin\theta\right)^2 + \left(3a\sin^2\theta\cos\theta\right)^2 = 9a^2\sin^2\theta\cos^2\theta$.
$S = \int_0^{\pi/2} 2\pi a \sin^3\theta\, 3a\sin\theta\cos\theta\, d\theta = 6\pi a^2 \int_0^{\pi/2} \sin^4\theta\cos\theta\, d\theta = \frac{6}{5}\pi a^2 \left[\sin^5\theta\right]_0^{\pi/2} = \frac{6}{5}\pi a^2$

26. $\left(\dfrac{dx}{d\theta}\right)^2+\left(\dfrac{dy}{d\theta}\right)^2=(-2\sin\theta+2\sin 2\theta)^2+(2\cos\theta-2\cos 2\theta)^2$

$$=4\left[\left(\sin^2\theta-2\sin\theta\sin 2\theta+\sin^2 2\theta\right)+\left(\cos^2\theta-2\cos\theta\cos 2\theta+\cos^2 2\theta\right)\right]$$
$$=4\left[1+1-2\left(\cos 2\theta\cos\theta+\sin 2\theta\sin\theta\right)\right]=8\left[1-\cos\left(2\theta-\theta\right)\right]=8\left(1-\cos\theta\right)$$

We plot the graph with parameter interval $[0,2\pi]$, and see that we should only integrate between 0 and π. (If the interval $[0,2\pi]$ were taken, the surface of revolution would be generated twice.) Also note that $y=2\sin\theta-\sin 2\theta=2\sin\theta\,(1-\cos\theta)$. So



$S=\int_0^\pi 2\pi 2\sin\theta\,(1-\cos\theta)\,2\sqrt{2}\sqrt{1-\cos\theta}\,d\theta$

$$=8\sqrt{2}\pi\int_0^\pi (1-\cos\theta)^{3/2}\sin\theta\,d\theta=8\sqrt{2}\pi\int_0^2\sqrt{u^3}\,du$$

$$[\text{where } u=1-\cos\theta,\ du=\sin\theta\,d\theta]\quad=\left[8\sqrt{2}\pi\left(\tfrac{2}{5}\right)u^{5/2}\right]_0^2=\tfrac{128}{5}\pi$$

27. $x=t+t^3,\ y=t-\dfrac{1}{t^2},\ 1\le t\le 2.\ \dfrac{dx}{dt}=1+3t^2$ and $\dfrac{dy}{dt}=1+\dfrac{2}{t^3}$, so

$\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2=\left(1+3t^2\right)^2+\left(1+\dfrac{2}{t^3}\right)^2$ and

$$S=\int 2\pi y\,ds=\int_1^2 2\pi\left(t-\frac{1}{t^2}\right)\sqrt{\left(1+3t^2\right)^2+\left(1+\frac{2}{t^3}\right)^2}\,dt\approx 59.101.$$

28. $S=\int_{\pi/4}^{\pi/2}2\pi\cdot 2a\sin^2\theta\sqrt{\csc^4\theta+\sin^2 2\theta}\,dt=4\pi a\int_{\pi/4}^{\pi/2}\sin^2\theta\sqrt{\csc^4\theta+\sin^2 2\theta}\,d\theta$. Using Simpson's Rule with $n=4$, $\Delta x=\frac{\pi}{16}$ and $f(\theta)=\sin^2\theta\sqrt{\csc^4\theta+\sin^2 2\theta}$, we get

$S\approx(4\pi a)\frac{\pi}{16\cdot 3}\left[f\left(\frac{\pi}{4}\right)+4f\left(\frac{5\pi}{16}\right)+2f\left(\frac{3\pi}{8}\right)+4f\left(\frac{7\pi}{16}\right)+f\left(\frac{\pi}{2}\right)\right]\approx 11.0893a.$

29. $\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2=(6t)^2+\left(6t^2\right)^2=36t^2\left(1+t^2\right)\quad\Rightarrow$

$$S=\int_0^5 2\pi x\sqrt{(dx/dt)^2+(dy/dt)^2}\,dt=\int_0^5 2\pi\left(3t^2\right)6t\sqrt{1+t^2}\,dt=18\pi\int_0^5 t^2\sqrt{1+t^2}\,2t\,dt$$
$$=18\pi\int_1^{26}(u-1)\sqrt{u}\,du\ \ (\text{where } u=1+t^2)\ =18\pi\int_1^{26}\left(u^{3/2}-u^{1/2}\right)du=18\pi\left[\tfrac{2}{5}u^{5/2}-\tfrac{2}{3}u^{3/2}\right]_1^{26}$$
$$=18\pi\left[\left(\tfrac{2}{5}\cdot 676\sqrt{26}-\tfrac{2}{3}\cdot 26\sqrt{26}\right)-\left(\tfrac{2}{5}-\tfrac{2}{3}\right)\right]=\tfrac{24}{5}\pi\left(949\sqrt{26}+1\right)$$

30. $x=e^t-t,\ y=4e^{t/2},\ 0\le t\le 1.\ \left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2=\left(e^t-1\right)^2+\left(2e^{t/2}\right)^2=e^{2t}+2e^t+1=\left(e^t+1\right)^2.$

$$S=\int_0^1 2\pi\left(e^t-t\right)\sqrt{\left(e^t-1\right)^2+\left(2e^{t/2}\right)^2}\,dt=\int_0^1 2\pi\left(e^t-t\right)\left(e^t+1\right)dt$$
$$=2\pi\left[\tfrac{1}{2}e^{2t}+e^t-(t-1)\,e^t-\tfrac{1}{2}t^2\right]_0^1=\pi\left(e^2+2e-6\right)$$

31. $x = a\cos\theta$, $y = b\sin\theta$, $0 \le \theta \le 2\pi$.

$$(dx/d\theta)^2 + (dy/d\theta)^2 = (-a\sin\theta)^2 + (b\cos\theta)^2 = a^2\sin^2\theta + b^2\cos^2\theta = a^2\left(1-\cos^2\theta\right) + b^2\cos^2\theta$$
$$= a^2 - \left(a^2 - b^2\right)\cos^2\theta = a^2 - c^2\cos^2\theta = a^2\left(1 - \frac{c^2}{a^2}\cos^2\theta\right) = a^2\left(1 - e^2\cos^2\theta\right)$$

(a) $S = \int_0^\pi 2\pi b\sin\theta a\sqrt{1-e^2\cos^2\theta}\,d\theta = 2\pi ab\int_{-e}^{e}\sqrt{1-u^2}\left(\frac{1}{e}\right)du$ (where $u = -e\cos\theta$, $du = e\sin\theta\,d\theta$)

$= \frac{4\pi ab}{e}\int_0^e\left(1-u^2\right)^{1/2}du = \frac{4\pi ab}{e}\int_0^{\sin^{-1}e}\cos^2 v\,dv$ (where $u = \sin v$) $= \frac{2\pi ab}{e}\int_0^{\sin^{-1}e}(1+\cos 2v)\,dv$

$= \frac{2\pi ab}{e}\left[v + \frac{1}{2}\sin 2v\right]_0^{\sin^{-1}e} = \frac{2\pi ab}{e}\left[v + \sin v\cos v\right]_0^{\sin^{-1}e} = \frac{2\pi ab}{e}\left(\sin^{-1}e + e\sqrt{1-e^2}\right)$

But $\sqrt{1-e^2} = \sqrt{1 - \dfrac{c^2}{a^2}} = \sqrt{\dfrac{a^2-c^2}{a^2}} = \sqrt{\dfrac{b^2}{a^2}} = \dfrac{b}{a}$, so $S = \dfrac{2\pi ab}{e}\sin^{-1}e + 2\pi b^2$.

(b) $S = \int_{-\pi/2}^{\pi/2} 2\pi a\cos\theta a\sqrt{1-e^2\cos^2\theta}\,d\theta = 4\pi a^2\int_0^{\pi/2}\cos\theta\sqrt{(1-e^2)+e^2\sin^2\theta}\,d\theta$

$= \dfrac{4\pi a^2\left(1-e^2\right)}{e}\displaystyle\int_0^{\pi/2}\frac{e}{\sqrt{1-e^2}}\cos\theta\sqrt{1+\left(\frac{e\sin\theta}{\sqrt{1-e^2}}\right)^2}\,d\theta$

$= \dfrac{4\pi a^2\left(1-e^2\right)}{e}\displaystyle\int_0^{e/\sqrt{1-e^2}}\sqrt{1+u^2}\,du\left(\text{where } u = \frac{e\sin\theta}{\sqrt{1-e^2}}\right)$

$= \dfrac{4\pi a^2\left(1-e^2\right)}{e}\displaystyle\int_0^{\sin^{-1}e}\sec^3 v\,dv$ (where $u = \tan v$, $du = \sec^2 v\,dv$)

$= \dfrac{2\pi a^2\left(1-e^2\right)}{e}\left[\sec v\tan v + \ln|\sec v + \tan v|\right]_0^{\sin^{-1}e}$

$= \dfrac{2\pi a^2\left(1-e^2\right)}{e}\left[\dfrac{1}{\sqrt{1-e^2}}\dfrac{e}{\sqrt{1-e^2}} + \ln\left|\dfrac{1}{\sqrt{1-e^2}} + \dfrac{e}{\sqrt{1-e^2}}\right|\right]$

$= 2\pi a^2 + \dfrac{2\pi a^2\left(1-e^2\right)}{e}\ln\sqrt{\dfrac{1+e}{1-e}} = 2\pi a^2 + \dfrac{2\pi b^2}{e}\dfrac{1}{2}\ln\left(\dfrac{1+e}{1-e}\right)$ $\left(\text{since } 1-e^2 = \dfrac{b^2}{a^2}\right)$

$= 2\pi\left[a^2 + \dfrac{b^2}{2e}\ln\dfrac{1+e}{1-e}\right]$

32. By Formula 11.3.5, $S = \int_a^b 2\pi F(x)\sqrt{1+F'(x)^2}\,dx$. Now

$1 + F'(x)^2 = 1 + \left(\dfrac{dy/dt}{dx/dt}\right)^2 = \dfrac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}$. Using the Substitution Rule with $x = x(t)$ $\Rightarrow$

$dx = \dfrac{dx}{dt}\,dt$, we have $S = \displaystyle\int_\alpha^\beta 2\pi y\sqrt{\frac{(dx/dt)^2+(dy/dt)^2}{(dx/dt)^2}}\frac{dx}{dt}\,dt = \int_\alpha^\beta 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.

33. (a) $\phi = \tan^{-1}\left(\frac{dy}{dx}\right) \Rightarrow \frac{d\phi}{dt} = \frac{d}{dt}\tan^{-1}\left(\frac{dy}{dx}\right) = \frac{1}{1+(dy/dx)^2}\left[\frac{d}{dt}\left(\frac{dy}{dx}\right)\right]$. But $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}} \Rightarrow$

$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right) = \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2} \Rightarrow \frac{d\phi}{dt} = \frac{1}{1+(\dot{y}/\dot{x})^2}\left(\frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2}\right) = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}$.

Using the Chain Rule, and the fact that $s = \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \Rightarrow$

$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \left(\dot{x}^2 + \dot{y}^2\right)^{1/2}$, we have that

$\frac{d\phi}{ds} = \frac{d\phi/dt}{ds/dt} = \left(\frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}\right)\frac{1}{\left(\dot{x}^2 + \dot{y}^2\right)^{1/2}} = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}}$. So

$\kappa = \left|\frac{d\phi}{ds}\right| = \left|\frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}}\right| = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}}$.

(b) $x = x$ and $y = f(x) \Rightarrow \dot{x} = 1, \ddot{x} = 0$ and $\dot{y} = \frac{dy}{dx}, \ddot{y} = \frac{d^2y}{dx^2}$.

So $\kappa = \frac{\left|1 \cdot (d^2y/dx^2) - 0 \cdot (dy/dx)\right|}{\left[1 + (dy/dx)^2\right]^{3/2}} = \frac{\left|d^2y/dx^2\right|}{\left[1 + (dy/dx)^2\right]^{3/2}}$.

34. (a) $y = x^2 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow \frac{d^2y}{dx^2} = 2$. So $\kappa = \frac{\left|d^2y/dx^2\right|}{\left[1 + (dy/dx)^2\right]^{3/2}} = \frac{2}{\left(1 + 4x^2\right)^{3/2}}$, and at $(1, 1)$,

$\kappa = \frac{2}{5^{3/2}} = \frac{2}{5\sqrt{5}}$.

(b) $\kappa' = \frac{d\kappa}{dx} = -3\left(1 + 4x^2\right)^{-5/2}(8x) = 0 \Leftrightarrow x = 0 \Rightarrow y = 0$. This is a maximum since $\kappa' > 0$ for $x < 0$ and $\kappa' < 0$ for $x > 0$. So the parabola $y = x^2$ has maximum curvature at the origin.

35. $x = \theta - \sin\theta \Rightarrow \dot{x} = 1 - \cos\theta \Rightarrow \ddot{x} = \sin\theta$, and $y = 1 - \cos\theta \Rightarrow \dot{y} = \sin\theta \Rightarrow \ddot{y} = \cos\theta$.

Therefore, $\kappa = \frac{\left|\cos\theta - \cos^2\theta - \sin^2\theta\right|}{\left[(1-\cos\theta)^2 + \sin^2\theta\right]^{3/2}} = \frac{\left|\cos\theta - \left(\cos^2\theta + \sin^2\theta\right)\right|}{\left(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta\right)^{3/2}} = \frac{|\cos\theta - 1|}{(2 - 2\cos\theta)^{3/2}}$. The top of the arch is characterized by a horizontal tangent, and from Example 1 in Section 11.2, the tangent is horizontal when $\theta = (2n-1)\pi$, so take $n = 1$ and substitute $\theta = \pi$ into the expression for κ:

$\kappa = \frac{|\cos\pi - 1|}{(2 - 2\cos\pi)^{3/2}} = \frac{|-1-1|}{[2 - 2(-1)]^{3/2}} = \frac{1}{4}$.

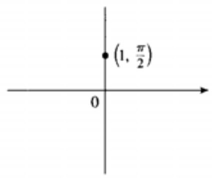
36. (a) Every straight line has parametrizations of the form $x = a + vt$, $y = b + wt$, where a, b are arbitrary and v, $w \neq 0$. For example, a straight line passing through distinct points (a, b) and (c, d) can be described as the parametrized curve $x = a + (c-a)t$, $y = b + (d-b)t$. Starting with $x = a + vt$, $y = b + wt$, we compute $\dot{x} = v$, $\dot{y} = w$, $\ddot{x} = \ddot{y} = 0$, and $\kappa = \frac{|v \cdot 0 - w \cdot 0|}{\left(v^2 + w^2\right)^{3/2}} = 0$.

(b) Parametric equations for a circle of radius r are $x = r\cos\theta$ and $y = r\sin\theta$. We can take the center to be the origin. So $\dot{x} = -r\sin\theta \Rightarrow \ddot{x} = -r\cos\theta$ and $\dot{y} = r\cos\theta \Rightarrow \ddot{y} = -r\sin\theta$. Therefore,

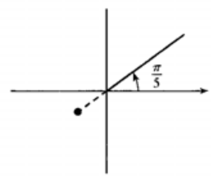
$\kappa = \frac{\left|r^2\sin^2\theta + r^2\cos^2\theta\right|}{\left(r^2\sin^2\theta + r^2\cos^2\theta\right)^{3/2}} = \frac{r^2}{r^3} = \frac{1}{r}$. And so for any θ (and thus any point), $\kappa = \frac{1}{r}$.

11.4 Polar Coordinates

1. (a) By adding 2π to $\frac{\pi}{2}$, we obtain the point $\left(1, \frac{5\pi}{2}\right)$. The direction opposite $\frac{\pi}{2}$ is $\frac{3\pi}{2}$, so $\left(-1, \frac{3\pi}{2}\right)$ is a point that satisfies the $r < 0$ requirement.

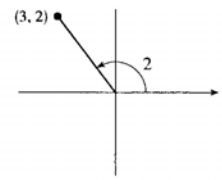


(b) $\left(-2, \frac{\pi}{4}\right)$



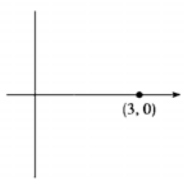
$\left(2, \frac{5\pi}{4}\right), \left(-2, \frac{9\pi}{4}\right)$

(c) $(3, 2)$



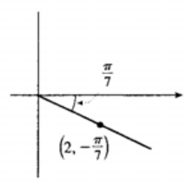
$(3, 2 + 2\pi), (-3, 2 + \pi)$

2. (a) $(3, 0)$



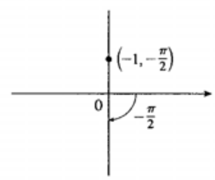
$(3, 2\pi), (-3, \pi)$

(b) $\left(2, -\frac{\pi}{7}\right)$



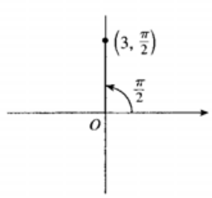
$\left(2, \frac{13\pi}{7}\right), \left(-2, \frac{6\pi}{7}\right)$

(c) $\left(-1, -\frac{\pi}{2}\right)$



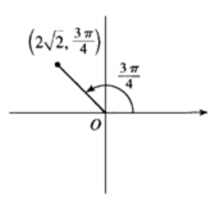
$\left(1, \frac{\pi}{2}\right), \left(-1, \frac{3\pi}{2}\right)$

3. (a)



$x = 3\cos\frac{\pi}{2} = 3\,(0) = 0$ and $y = 3\sin\frac{\pi}{2} = 3\,(1) = 3$ give us $(0, 3)$.

(b)



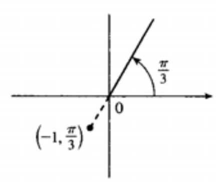
$x = 2\sqrt{2}\cos\frac{3\pi}{4}$

$= 2\sqrt{2}\left(-\frac{1}{\sqrt{2}}\right) = -2$ and

$y = 2\sqrt{2}\sin\frac{3\pi}{4} = 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 2$

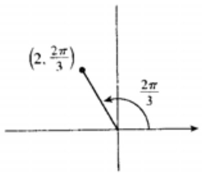
give us $(-2, 2)$.

(c)



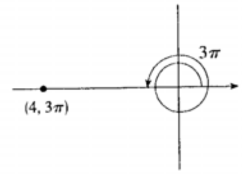
$x = -1\cos\frac{\pi}{3} = -\frac{1}{2}$ and $y = -1\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$ give us $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

4. (a)



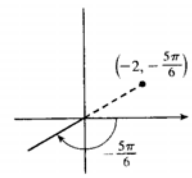
$x = 2\cos\frac{2\pi}{3} = -1$,
$y = 2\sin\frac{2\pi}{3} = \sqrt{3}$

(b)



$x = 4\cos 3\pi = -4$, $y = 4\sin 3\pi = 0$

(c)



$x = -2\cos\left(-\frac{5\pi}{6}\right) = \sqrt{3}$,
$y = -2\sin\left(-\frac{5\pi}{6}\right) = 1$

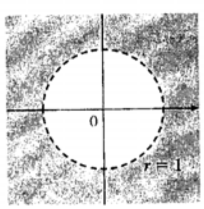
5. (a) $x = 1$ and $y = 1 \Rightarrow r = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$. Since $(1, 1)$ is in the first quadrant, the polar coordinates are (i) $\left(\sqrt{2}, \frac{\pi}{4}\right)$ and (ii) $\left(-\sqrt{2}, \frac{5\pi}{4}\right)$.

(b) $x = 2\sqrt{3}$ and $y = -2 \Rightarrow r = \sqrt{\left(2\sqrt{3}\right)^2 + (-2)^2} = 4$ and $\theta = \tan^{-1}\left(-\frac{2}{2\sqrt{3}}\right) = -\frac{\pi}{6}$. Since $\left(2\sqrt{3}, -2\right)$ is in the fourth quadrant and $0 \le \theta \le 2\pi$, the polar coordinates are (i) $\left(4, \frac{11\pi}{6}\right)$ and (ii) $\left(-4, \frac{5\pi}{6}\right)$.

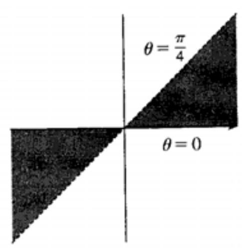
6. (a) $(x, y) = \left(-1, -\sqrt{3}\right)$, $r = \sqrt{1 + 3} = 2$, $\tan\theta = y/x = \sqrt{3}$ and (x, y) is in the third quadrant, so $\theta = \frac{4\pi}{3}$. The polar coordinates are (i) $\left(2, \frac{4\pi}{3}\right)$ and (ii) $\left(-2, \frac{\pi}{3}\right)$.

(b) $(x, y) = (-2, 3)$, $r = \sqrt{4 + 9} = \sqrt{13}$, $\tan\theta = y/x = -\frac{3}{2}$ and (x, y) is in the second quadrant, so $\theta = \tan^{-1}\left(-\frac{3}{2}\right) + \pi$. The polar coordinates are (i) $\left(\sqrt{13}, \theta\right)$ and (ii) $\left(-\sqrt{13}, \theta + \pi\right)$.

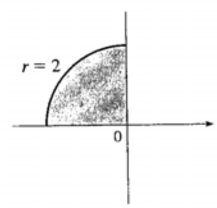
7. $r > 1$



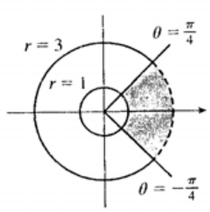
8. $0 \le \theta < \frac{\pi}{4}$



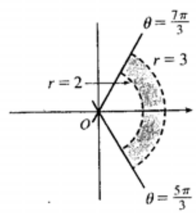
9. $0 \le r \le 2$, $\frac{\pi}{2} \le \theta \le \pi$



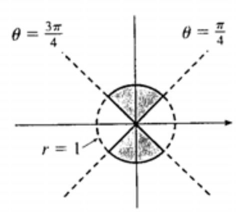
10. $1 \le r < 3$, $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$



11. $2 < r < 3$, $\frac{5\pi}{3} \le \theta \le \frac{7\pi}{3}$



12. $-1 \le r \le 1$, $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$



13. $\left(1, \frac{\pi}{6}\right)$ is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ Cartesian and $\left(3, \frac{3\pi}{4}\right)$ is $\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ Cartesian. The square of the distance between them is $\left(\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}-\frac{3}{\sqrt{2}}\right)^2=\frac{1}{4}\left(40+6\sqrt{6}-6\sqrt{2}\right)$, so the distance is $\frac{1}{2}\sqrt{40+6\sqrt{6}-6\sqrt{2}}$.

14. The points in Cartesian coordinates are $(r_1\cos\theta_1, r_1\sin\theta_1)$ and $(r_2\cos\theta_2, r_2\sin\theta_2)$ respectively. So the square of the distance between them is $(r_2\cos\theta_2-r_1\cos\theta_1)^2+(r_2\sin\theta_2-r_1\sin\theta_1)^2=r_1^2-2r_1r_2\cos(\theta_1-\theta_2)+r_2^2$, and the distance is $\sqrt{r_1^2-2r_1r_2\cos(\theta_1-\theta_2)+r_2^2}$.

15. $r=2 \quad\Leftrightarrow\quad \sqrt{x^2+y^2}=2 \quad\Leftrightarrow\quad x^2+y^2=4$, a circle of radius 2 centered at the origin.

16. $r\cos\theta=1 \quad\Leftrightarrow\quad x=1$, a vertical line.

17. $r=3\sin\theta \quad\Rightarrow\quad r^2=3r\sin\theta \quad\Leftrightarrow\quad x^2+y^2=3y \quad\Leftrightarrow\quad x^2+\left(y-\frac{3}{2}\right)^2=\left(\frac{3}{2}\right)^2$, a circle of radius $\frac{3}{2}$ centered at $\left(0, \frac{3}{2}\right)$. The first two equations are actually equivalent since $r^2=3r\sin\theta \quad\Rightarrow\quad r(r-3\sin\theta)=0 \quad\Rightarrow$ $r=0$ or $r=3\sin\theta$. But $r=3\sin\theta$ gives the point $r=0$ (the pole) when $\theta=0$. Thus, the single equation $r=3\sin\theta$ is equivalent to the compound condition ($r=0$ or $r=3\sin\theta$).

18. $r=\dfrac{1}{1+2\sin\theta} \quad\Rightarrow\quad r+2r\sin\theta=1 \quad\Leftrightarrow\quad r=1-2r\sin\theta \quad\Leftrightarrow\quad \sqrt{x^2+y^2}=1-2y \quad\Rightarrow$

$x^2+y^2=1-4y+4y^2 \quad\Leftrightarrow\quad 3y^2-4y-x^2=-1 \quad\Leftrightarrow\quad 3\left(y^2-\frac{4}{3}y+\frac{4}{9}\right)-x^2=\frac{4}{3}-1 \quad\Leftrightarrow$

$3\left(y-\frac{2}{3}\right)^2-x^2=\frac{1}{3} \quad\Leftrightarrow\quad 9\left(y-\frac{2}{3}\right)^2-3x^2=1 \quad\Leftrightarrow\quad \dfrac{\left(y-\frac{2}{3}\right)^2}{\left(\frac{1}{3}\right)^2}-\dfrac{x^2}{\left(\frac{1}{\sqrt{3}}\right)^2}=1$. This is a hyperbola opening up and down and centered at $\left(0, \frac{2}{3}\right)$.

19. $r^2=\sin 2\theta=2\sin\theta\cos\theta \quad\Leftrightarrow\quad r^4=2r\sin\theta\, r\cos\theta \quad\Leftrightarrow\quad (x^2+y^2)^2=2yx$

20. $r^2=\theta \quad\Rightarrow\quad \tan(r^2)=\tan\theta \quad\Rightarrow\quad \tan(x^2+y^2)=y/x$

21. $y=5 \quad\Leftrightarrow\quad r\sin\theta=5$

22. $y=2x-1 \quad\Leftrightarrow\quad r\sin\theta=2r\cos\theta-1 \quad\Leftrightarrow\quad r(2\cos\theta-\sin\theta)=1 \quad\Leftrightarrow\quad r=\dfrac{1}{2\cos\theta-\sin\theta}$. (We can divide by $2\cos\theta-\sin\theta$ because it must be nonzero in order that its product with r equal 1.)

23. $x^2+y^2=25 \quad\Leftrightarrow\quad r^2=25 \quad\Rightarrow\quad r=5$

24. $x^2=4y \quad\Leftrightarrow\quad r^2\cos^2\theta=4r\sin\theta \quad\Leftrightarrow\quad r\cos^2\theta=4\sin\theta \quad\Leftrightarrow\quad r=4\tan\theta\sec\theta$

25. $2xy=1 \quad\Leftrightarrow\quad 2r\cos\theta\, r\sin\theta=1 \quad\Leftrightarrow\quad r^2\sin 2\theta=1 \quad\Leftrightarrow\quad r^2=\csc 2\theta$

26. $x^2-y^2=1 \quad\Leftrightarrow\quad r^2\left(\cos^2\theta-\sin^2\theta\right)=1 \quad\Leftrightarrow\quad r^2\cos 2\theta=1 \quad\Rightarrow\quad r^2=\sec 2\theta$

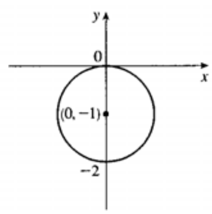
27. (a) The description leads immediately to the polar equation $\theta=\frac{\pi}{6}$, and the Cartesian equation $y=\tan\left(\frac{\pi}{6}\right)x=\frac{1}{\sqrt{3}}x$ is slightly more difficult to derive.

(b) The easier description here is the Cartesian equation $x=3$.

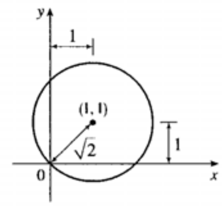
28. (a) Because its center is not at the origin, it is more easily described by its Cartesian equation, $(x-2)^2+(y-3)^2=5^2$.

(b) This circle is more easily given in polar coordinates: $r=4$. The Cartesian equation is also simple: $x^2+y^2=16$.

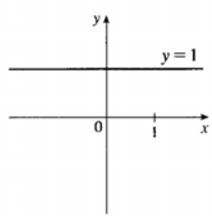
29. $r = -2\sin\theta \iff r^2 = -2r\sin\theta$ (since the possibility $r = 0$ is covered by the equation $r = -2\sin\theta$) $\iff x^2 + y^2 = -2y \iff x^2 + y^2 + 2y + 1 = 1 \iff x^2 + (y+1)^2 = 1$.



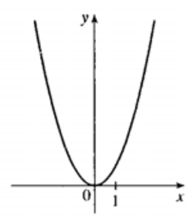
30. $r = 2\sin\theta + 2\cos\theta \iff r^2 = 2r\sin\theta + 2r\cos\theta, x^2 + y^2 = 2y + 2x \iff (x-1)^2 + (y-1)^2 = 2$



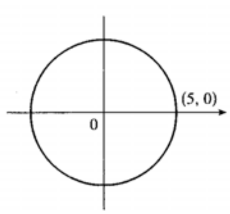
31. $r = \csc\theta = \dfrac{1}{\sin\theta} \iff r\sin\theta = 1$. (The right-hand equation implies that $\sin\theta \neq 0$, so we can divide by $\sin\theta$ to get the left-hand equation) $\iff y = 1$.



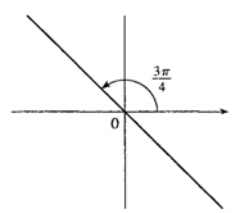
32. $r = \tan\theta\sec\theta \Rightarrow r = \dfrac{\sin\theta}{\cos\theta}\cdot\dfrac{1}{\cos\theta} \Rightarrow r\cos^2\theta = \sin\theta \Rightarrow r^2\cos^2\theta = r\sin\theta \Rightarrow x^2 = y$



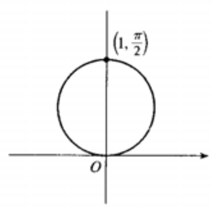
33. As in Example 4, $r = 5$ represents the circle with center O and radius 5.



34. $\theta = \frac{3\pi}{4}$ is a line through the origin.

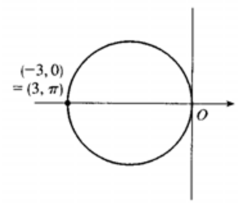


35. $r = \sin\theta \iff r^2 = r\sin\theta \iff x^2 + y^2 = y \iff x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$. The reasoning here is the same as in Exercise 29. This is a circle of radius $\frac{1}{2}$ centered at $\left(0, \frac{1}{2}\right)$.

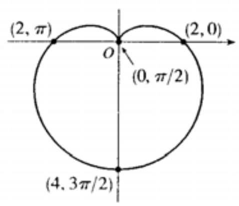


36. $r = -3\cos\theta \iff r^2 = -3r\cos\theta \iff x^2 + y^2 = -3x \iff \left(x + \frac{3}{2}\right)^2 + y^2 = \left(\frac{3}{2}\right)^2$.

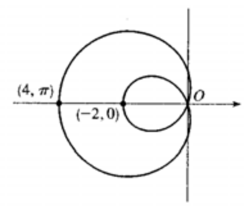
This curve is a circle of radius $\frac{3}{2}$.



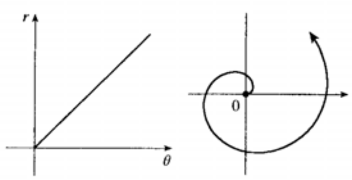
37. $r = 2\,(1 - \sin\theta)$. This curve is a cardioid.



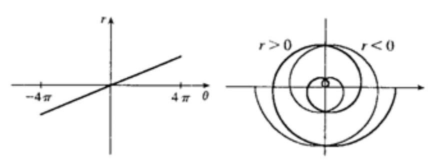
38. $r = 1 - 3\cos\theta$. This is a limaçon.



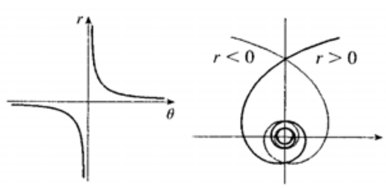
39. $r = \theta, \theta \geq 0$



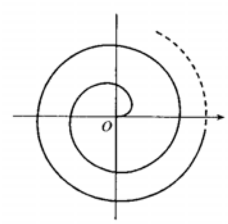
40. $r = \theta/2, -4\pi \leq \theta \leq 4\pi$



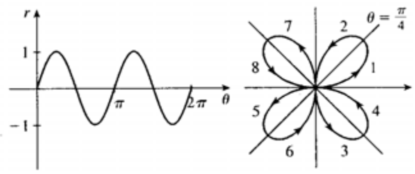
41. $r = 1/\theta$



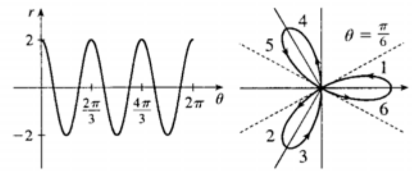
42. $r = \sqrt{\theta}$. This curve is a spiral.



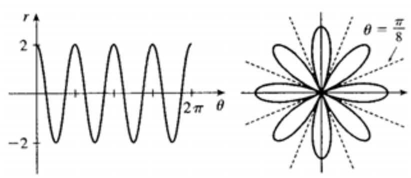
43. $r = \sin 2\theta$



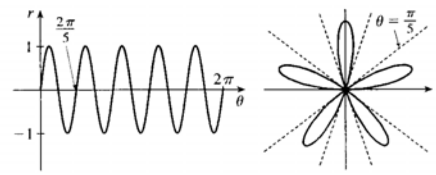
44. $r = 2\cos 3\theta$



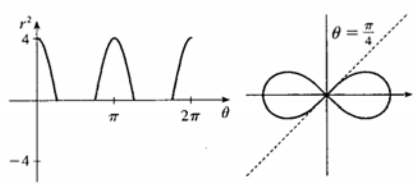
45. $r = 2\cos 4\theta$



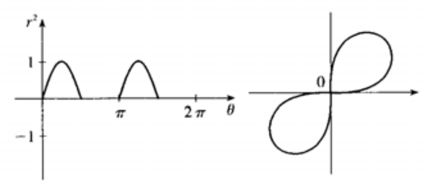
46. $r = \sin 5\theta$



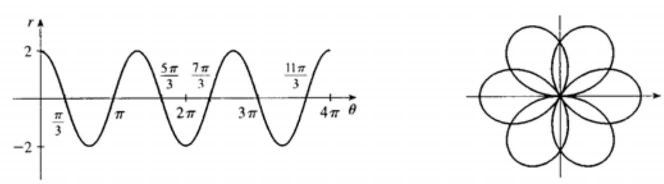
47. $r^2 = 4\cos 2\theta$



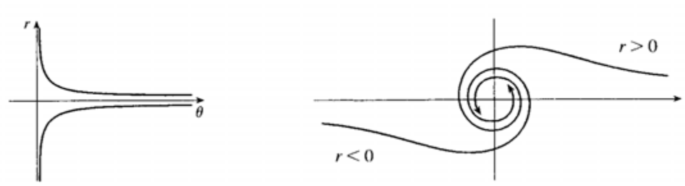
48. $r^2 = \sin 2\theta$

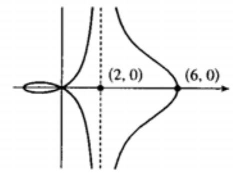


49. $r = 2\cos\left(\frac{3}{2}\theta\right)$

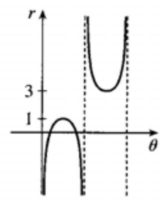


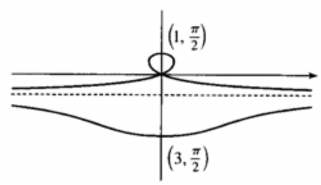
50. $r^2\theta = 1 \quad \Leftrightarrow \quad r = \pm 1/\sqrt{\theta}$ for $\theta > 0$



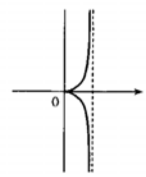


51. $x = (r)\cos\theta = (4 + 2\sec\theta)\cos\theta = 4\cos\theta + 2$. Now, $r \to \infty \Rightarrow (4 + 2\sec\theta) \to \infty \Rightarrow \theta \to \left(\frac{\pi}{2}\right)^-$ or $\theta \to \left(\frac{3\pi}{2}\right)^+$ (since we need only consider $0 \le \theta < 2\pi$), so $\lim\limits_{r\to\infty} x = \lim\limits_{\theta\to\pi/2^-} (4\cos\theta + 2) = 2$. Also, $r \to -\infty \Rightarrow (4 + 2\sec\theta) \to -\infty \Rightarrow \theta \to \left(\frac{\pi}{2}\right)^+$ or $\theta \to \left(\frac{3\pi}{2}\right)^-$, so $\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi/2^+} (4\cos\theta + 2) = 2$. Therefore, $\lim\limits_{r\to\pm\infty} x = 2 \Rightarrow x = 2$ is a vertical asymptote.





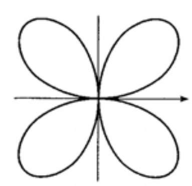
52. $y = r\sin\theta = 2\sin\theta - \csc\theta\sin\theta = 2\sin\theta - 1$. $r \to \infty \Rightarrow (2 - \csc\theta) \to \infty \Rightarrow \csc\theta \to -\infty \Rightarrow \theta \to \pi^+$ (since we need only consider $0 \le \theta < 2\pi$) and so $\lim\limits_{r\to\infty} y = \lim\limits_{\theta\to\pi^+} 2\sin\theta - 1 = -1$. Also $r \to -\infty \Rightarrow (2 - \csc\theta) \to -\infty \Rightarrow \csc\theta \to \infty \Rightarrow \theta \to \pi^-$ and so $\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi^-} 2\sin\theta - 1 = -1$. Therefore $\lim\limits_{r\to\pm\infty} y = -1 \Rightarrow y = -1$ is a horizontal asymptote.



53. To show that $x = 1$ is an asymptote we must prove $\lim\limits_{r\to\pm\infty} x = 1$.

$x = (r)\cos\theta = (\sin\theta\tan\theta)\cos\theta = \sin^2\theta$. Now, $r \to \infty \Rightarrow \sin\theta\tan\theta \to \infty \Rightarrow \theta \to \left(\frac{\pi}{2}\right)^-$, so $\lim\limits_{r\to\infty} x = \lim\limits_{\theta\to\pi/2^-} \sin^2\theta = 1$. Also, $r \to -\infty \Rightarrow \sin\theta\tan\theta \to -\infty \Rightarrow \theta \to \left(\frac{\pi}{2}\right)^+$, so $\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi/2^+} \sin^2\theta = 1$.

Therefore, $\lim\limits_{r\to\pm\infty} x = 1 \Rightarrow x = 1$ is a vertical asymptote. Also notice that $x = \sin^2\theta \ge 0$ for all θ, and $x = \sin^2\theta \le 1$ for all θ. And $x \ne 1$, since the curve is not defined at odd multiples of $\frac{\pi}{2}$. Therefore, the curve lies entirely within the vertical strip $0 \le x < 1$.



54. The equation is $\left(x^2 + y^2\right)^3 = 4x^2y^2$, but using polar coordinates we know that $x^2 + y^2 = r^2$ and $x = r\cos\theta$ and $y = r\sin\theta$. Substituting into the given equation: $r^6 = 4r^2\cos^2\theta\, r^2\sin^2\theta \Rightarrow r^2 = 4\cos^2\theta\sin^2\theta \Rightarrow r = \pm 2\cos\theta\sin\theta = \pm\sin 2\theta$. $r = \pm\sin 2\theta$ is sketched at right.

55. (a) We see that the curve crosses itself at the origin, where $r = 0$ (in fact the inner loop corresponds to negative r-values,) so we solve the equation of the limaçon for $r = 0 \Leftrightarrow c\sin\theta = -1 \Leftrightarrow \sin\theta = -1/c$. Now if $|c| < 1$, then this equation has no solution and hence there is no inner loop. But if $c < -1$, then on the interval $(0, 2\pi)$ the equation has the two solutions $\theta = \sin^{-1}(-1/c)$ and $\theta = \pi - \sin^{-1}(-1/c)$, and if $c > 1$, the solutions are $\theta = \pi + \sin^{-1}(1/c)$ and $\theta = 2\pi - \sin^{-1}(1/c)$. In each case, $r < 0$ for θ between the two solutions, indicating a loop.

(b) For $0 < c < 1$, the dimple (if it exists) is characterized by the fact that y has a local maximum at $\theta = \frac{3\pi}{2}$. So we determine for what c-values $\dfrac{d^2y}{d\theta^2}$ is negative at $\theta = \frac{3\pi}{2}$, since by the Second Derivative Test this indicates a maximum: $y = r\sin\theta = \sin\theta + c\sin^2\theta \;\Rightarrow\; \dfrac{dy}{d\theta} = \cos\theta + 2c\sin\theta\cos\theta = \cos\theta + c\sin 2\theta \;\Rightarrow$ $\dfrac{d^2y}{d\theta^2} = -\sin\theta + 2c\cos 2\theta$. At $\theta = \frac{3\pi}{2}$, this is equal to $-(-1) + 2c(-1) = 1 - 2c$, which is negative only for $c > \frac{1}{2}$. A similar argument shows that for $-1 < c < 0$, y only has a local minimum at $\theta = \frac{\pi}{2}$ (indicating a dimple) for $c < -\frac{1}{2}$.

56. (a) $r = \sin(\theta/2)$. This equation must correspond to one of II, III or VI, since these are the only graphs which are bounded. In fact it must be VI, since this is the only graph which is completed after a rotation of exactly 4π.

(b) $r = \sin(\theta/4)$. This equation must correspond to III, since this is the only graph which is completed after a rotation of exactly 8π.

(c) $r = \sec(3\theta)$. This must correspond to IV, since the graph is unbounded at $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{2\pi}{3}$, and so on.

(d) $r = \theta\sin\theta$. This must correspond to V. Note that $r = 0$ whenever θ is a multiple of π. This graph is unbounded, and each time θ moves through an interval of 2π, the same basic shape is repeated (because of the periodic $\sin\theta$ factor) but it gets larger each time (since θ increases each time we go around.)

(e) $r = 1 + 4\cos 5\theta$. This corresponds to II, since it is bounded, has fivefold rotational symmetry, and takes only one takes only one rotation through 2π to be complete.

(f) $r = 1\big/\sqrt{\theta}$. This corresponds to I, since it is unbounded at $\theta = 0$, and r decreases as θ increases; in fact $r \to 0$ as $\theta \to \infty$.

57. Using Equation 3 with $r = 3\cos\theta$, we have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(dr/d\theta)(\sin\theta) + r\cos\theta}{(dr/d\theta)(\cos\theta) - r\sin\theta} = \frac{-3\sin\theta\sin\theta + 3\cos\theta\cos\theta}{-3\sin\theta\cos\theta - 3\cos\theta\sin\theta} = \frac{3\left(\cos^2\theta - \sin^2\theta\right)}{-3\,(2\sin\theta\cos\theta)}$$
$$= -\frac{\cos 2\theta}{\sin 2\theta} = -\cot 2\theta = \frac{1}{\sqrt{3}} \text{ when } \theta = \frac{\pi}{3}$$

Another Solution: $r = 3\cos\theta \;\Rightarrow\; x = r\cos\theta = 3\cos^2\theta,\ y = r\sin\theta = 3\sin\theta\cos\theta \;\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-3\sin^2\theta + 3\cos^2\theta}{-6\cos\theta\sin\theta} = \frac{\cos 2\theta}{-\sin 2\theta} = -\cot 2\theta = \frac{1}{\sqrt{3}} \text{ when } \theta = \frac{\pi}{3}$$

58. Using Equation 3 with $r = \cos\theta + \sin\theta$, we have

$$\frac{dy}{dx} = \frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta} = \frac{(-\sin\theta + \cos\theta)\sin\theta + (\cos\theta + \sin\theta)\cos\theta}{(-\sin\theta + \cos\theta)\cos\theta - (\cos\theta + \sin\theta)\sin\theta} = -1 \text{ when } \theta = \frac{\pi}{4}$$

Another Solution: $r = \cos\theta + \sin\theta \;\Rightarrow\; x = r\cos\theta = (\cos\theta + \sin\theta)\cos\theta,\ y = r\sin\theta = (\cos\theta + \sin\theta)\sin\theta \;\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta\,(-\sin\theta + \cos\theta) + (\cos\theta + \sin\theta)\cos\theta}{\cos\theta\,(-\sin\theta + \cos\theta) - (\cos\theta + \sin\theta)\sin\theta} = -1 \text{ when } \theta = \frac{\pi}{4}$$

59. $r = 1/\theta \;\Rightarrow\; x = r\cos\theta = (\cos\theta)/\theta,\ y = r\sin\theta = (\sin\theta)/\theta \;\Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta\left(-1/\theta^2\right) + (1/\theta)\cos\theta}{\cos\theta\left(-1/\theta^2\right) - (1/\theta)\sin\theta} \cdot \frac{\theta^2}{\theta^2} = \frac{-\sin\theta + \theta\cos\theta}{-\cos\theta - \theta\sin\theta} = -\pi \text{ when } \theta = \pi$$

60. $r = \ln\theta \Rightarrow x = r\cos\theta = \ln\theta\cos\theta,\ y = r\sin\theta = \ln\theta\sin\theta \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta\,(1/\theta) + \ln\theta\cos\theta}{\cos\theta\,(1/\theta) - \ln\theta\sin\theta} = \frac{\sin e + e\cos e}{\cos e - e\sin e} \text{ when } \theta = e$$

61. $r = 1 + \cos\theta \Rightarrow x = r\cos\theta = \cos\theta + \cos^2\theta,\ y = r\sin\theta = \sin\theta + \sin\theta\cos\theta \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta - 2\cos\theta\sin\theta} = \frac{\cos\theta + \cos 2\theta}{-\sin\theta - \sin 2\theta} = -1 \text{ when } \theta = \frac{\pi}{6}$$

62. $r = \sin 3\theta \Rightarrow x = r\cos\theta = \sin 3\theta\cos\theta,\ y = r\sin\theta = \sin 3\theta\sin\theta \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos 3\theta\sin\theta + \sin 3\theta\cos\theta}{3\cos 3\theta\cos\theta - \sin 3\theta\sin\theta} = -\sqrt{3} \text{ when } \theta = \frac{\pi}{6}$$

63. $r = 3\cos\theta \Rightarrow x = r\cos\theta = 3\cos\theta\cos\theta,\ y = r\sin\theta = 3\cos\theta\sin\theta \Rightarrow$
$dy/d\theta = -3\sin^2\theta + 3\cos^2\theta = 3\cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2} \Leftrightarrow \theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$. So the tangent is horizontal at $\left(\frac{3}{\sqrt{2}}, \frac{\pi}{4}\right)$ and $\left(-\frac{3}{\sqrt{2}}, \frac{3\pi}{4}\right)$ $\left[\text{same as } \left(\frac{3}{\sqrt{2}}, -\frac{\pi}{4}\right)\right]$. $dx/d\theta = -6\sin\theta\cos\theta = -3\sin 2\theta = 0 \Rightarrow$ $2\theta = 0$ or $\pi \Leftrightarrow \theta = 0$ or $\frac{\pi}{2}$. So the tangent is vertical at $(3, 0)$ and $\left(0, \frac{\pi}{2}\right)$.

64. $y = r\sin\theta = \cos\theta\sin\theta + \sin^2\theta = \frac{1}{2}\sin 2\theta + \sin^2\theta \Rightarrow dy/d\theta = \cos 2\theta + \sin 2\theta = 0 \Rightarrow \tan 2\theta = -1$ $\Rightarrow 2\theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4} \Leftrightarrow \theta = \frac{3\pi}{8}$ or $\frac{7\pi}{8} \Rightarrow$ horizontal tangents at $\left(\cos\frac{3\pi}{8} + \sin\frac{3\pi}{8}, \frac{3\pi}{8}\right)$ and $\left(\cos\frac{7\pi}{8} + \sin\frac{7\pi}{8}, \frac{7\pi}{8}\right)$. $x = r\cos\theta = \cos^2\theta + \cos\theta\sin\theta \Rightarrow dx/d\theta = -\sin 2\theta + \cos 2\theta = 0 \Rightarrow$ $\tan 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{4}$ or $\frac{5\pi}{4} \Leftrightarrow \theta = \frac{\pi}{8}$ or $\frac{5\pi}{8} \Rightarrow$ vertical tangents at $\left(\cos\frac{\pi}{8} + \sin\frac{\pi}{8}, \frac{\pi}{8}\right)$ and $\left(\cos\frac{5\pi}{8} + \sin\frac{5\pi}{8}, \frac{5\pi}{8}\right)$.
Note: These expressions can be simplified using trigonometric identities. For example,
$\cos\frac{\pi}{8} + \sin\frac{\pi}{8} = \frac{1}{2}\sqrt{4 + 2\sqrt{2}}$.

65. $r = 1 + \cos\theta \Rightarrow x = r\cos\theta = \cos\theta\,(1 + \cos\theta),\ y = r\sin\theta = \sin\theta\,(1 + \cos\theta) \Rightarrow$
$dy/d\theta = (1 + \cos\theta)\cos\theta - \sin^2\theta = 2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1) = 0 \Rightarrow \cos\theta = \frac{1}{2}$ or -1 $\Rightarrow \theta = \frac{\pi}{3}, \pi$, or $\frac{5\pi}{3} \Rightarrow$ horizontal tangent at $\left(\frac{3}{2}, \frac{\pi}{3}\right)$, $(0, \pi)$, and $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$.
$dx/d\theta = -(1 + \cos\theta)\sin\theta - \cos\theta\sin\theta = -\sin\theta\,(1 + 2\cos\theta) = 0 \Rightarrow \sin\theta = 0$ or $\cos\theta = -\frac{1}{2} \Rightarrow$ $\theta = 0, \pi, \frac{2\pi}{3}$, or $\frac{4\pi}{3} \Rightarrow$ vertical tangent at $(2, 0)$, $\left(\frac{1}{2}, \frac{2\pi}{3}\right)$, and $\left(\frac{1}{2}, \frac{4\pi}{3}\right)$. Note that the tangent is horizontal, not vertical when $\theta = \pi$, since $\displaystyle\lim_{\theta\to\pi} \frac{dy/d\theta}{dx/d\theta} = 0$.

66. $\dfrac{dy}{d\theta} = e^\theta\sin\theta + e^\theta\cos\theta = e^\theta(\sin\theta + \cos\theta) = 0 \Rightarrow \sin\theta = -\cos\theta \Rightarrow \tan\theta = -1 \Rightarrow$
$\theta = -\frac{1}{4}\pi + n\pi$ (n any integer) $\Rightarrow$ horizontal tangents at $\left(e^{\pi(n-1/4)}, \pi\left(n - \frac{1}{4}\right)\right)$.
$\dfrac{dx}{d\theta} = e^\theta\cos\theta - e^\theta\sin\theta = e^\theta(\cos\theta - \sin\theta) = 0 \Rightarrow \sin\theta = \cos\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{1}{4}\pi + n\pi$ (n any integer) $\Rightarrow$ vertical tangents at $\left(e^{\pi(n+1/4)}, \pi\left(n + \frac{1}{4}\right)\right)$.

67. $r = \cos 2\theta \Rightarrow x = r\cos\theta = \cos 2\theta\cos\theta,\ y = r\sin\theta = \cos 2\theta\sin\theta \Rightarrow$

$$\begin{aligned} dy/d\theta &= -2\sin 2\theta\sin\theta + \cos 2\theta\cos\theta = -4\sin^2\theta\cos\theta + \left(\cos^3\theta - \sin^2\theta\cos\theta\right) \\ &= \cos\theta\left(\cos^2\theta - 5\sin^2\theta\right) = \cos\theta\left(1 - 6\sin^2\theta\right) = 0 \Rightarrow \end{aligned}$$

$\cos\theta = 0$ or $\sin\theta = \pm\frac{1}{\sqrt{6}} \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \alpha, \pi - \alpha, \pi + \alpha$, or $2\pi - \alpha$ (where $\alpha = \sin^{-1}\frac{1}{\sqrt{6}}$).

So the tangent is horizontal at $\left(-1, \frac{\pi}{2}\right)$, $\left(-1, \frac{3\pi}{2}\right)$, $\left(\frac{2}{3}, \alpha\right)$, $\left(\frac{2}{3}, \pi - \alpha\right)$, $\left(\frac{2}{3}, \pi + \alpha\right)$, and $\left(\frac{2}{3}, 2\pi - \alpha\right)$.

$$dx/d\theta = -2\sin 2\theta\cos\theta - \cos 2\theta\sin\theta = -4\sin\theta\cos^2\theta - \left(2\cos^2\theta - 1\right)\sin\theta$$

$$= \sin\theta\left(1 - 6\cos^2\theta\right) = 0 \quad \Rightarrow$$

$\sin\theta = 0$ or $\cos\theta = \pm\frac{1}{\sqrt{6}} \Rightarrow \theta = 0, \pi, \frac{\pi}{2} - \alpha, \frac{\pi}{2} + \alpha, \frac{3\pi}{2} - \alpha$, or $\frac{3\pi}{2} + \alpha$ (where $\alpha = \cos^{-1}\frac{1}{\sqrt{6}}$).

So the tangent is vertical at $(1, 0)$, $(1, \pi)$, $\left(\frac{2}{3}, \frac{3\pi}{2} - \alpha\right)$, $\left(\frac{2}{3}, \frac{3\pi}{2} + \alpha\right)$, $\left(\frac{2}{3}, \frac{\pi}{2} - \alpha\right)$, and $\left(\frac{2}{3}, \frac{\pi}{2} + \alpha\right)$.

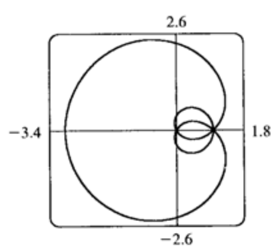
68. $dr/d\theta = (1/r)\cos 2\theta$ (by differentiating implicitly), so
$\dfrac{dy}{d\theta} = \dfrac{1}{r}\cos 2\theta\sin\theta + r\cos\theta = \dfrac{1}{r}\left(\cos 2\theta\sin\theta + r^2\cos\theta\right) = \dfrac{1}{r}\left(\cos 2\theta\sin\theta + \sin 2\theta\cos\theta\right) = \dfrac{1}{r}\sin 3\theta$. This is 0 when $\sin 3\theta = 0 \Rightarrow \theta = 0, \frac{\pi}{3}$ or $\frac{4\pi}{3}$ (restricting θ to the domain of the lemniscate), so there are horizontal tangents at $\left(\sqrt[4]{\frac{3}{4}}, \frac{\pi}{3}\right)$, $\left(\sqrt[4]{\frac{3}{4}}, \frac{4\pi}{3}\right)$ and $(0, 0)$. Similarly, $dx/d\theta = (1/r)\cos 3\theta = 0$ when $\theta = \frac{\pi}{6}$ or $\frac{7\pi}{6}$, so there are vertical tangents at $\left(\sqrt[4]{\frac{3}{4}}, \frac{\pi}{6}\right)$ and $\left(\sqrt[4]{\frac{3}{4}}, \frac{7\pi}{6}\right)$ [and $(0, 0)$]. See the sketch in Exercise 48.

69. $r = a\sin\theta + b\cos\theta \Rightarrow r^2 = ar\sin\theta + br\cos\theta \Rightarrow x^2 + y^2 = ay + bx \Rightarrow$
$\left(x - \frac{1}{2}b\right)^2 + \left(y - \frac{1}{2}a\right)^2 = \frac{1}{4}\left(a^2 + b^2\right)$, and this is a circle with center $\left(\frac{1}{2}b, \frac{1}{2}a\right)$ and radius $\frac{1}{2}\sqrt{a^2 + b^2}$.

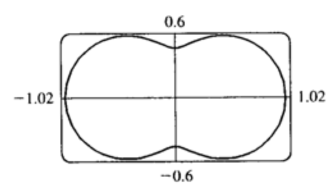
70. These curves are circles which intersect at the origin and at $\left(\frac{1}{\sqrt{2}}a, \frac{\pi}{4}\right)$. At the origin, the first circle has a horizontal tangent and the second a vertical one, so the tangents are perpendicular here. For the first circle $(r = a\sin\theta)$, $dy/d\theta = a\cos\theta\sin\theta + a\sin\theta\cos\theta = a\sin 2\theta = a$ at $\theta = \frac{\pi}{4}$ and $dx/d\theta = a\cos^2\theta - a\sin^2\theta = a\cos 2\theta = 0$ at $\theta = \frac{\pi}{4}$, so the tangent here is vertical. Similarly, for the second circle $(r = a\cos\theta)$, $dy/d\theta = a\cos 2\theta = 0$ and $dx/d\theta = -a\sin 2\theta = -a$ at $\theta = \frac{\pi}{4}$, so the tangent is horizontal, and again the tangents are perpendicular.

Note for Exercises 71–76: Maple is able to plot polar curves using the `polarplot` command, or using the `coords=polar` option in a regular `plot` command. In Mathematica, use `PolarPlot`. In Derive, change to `Polar` under `Options State`. If your graphing device cannot plot polar equations, you must convert to parametric equations. For example, in Exercise 71, $x = r\cos\theta = [1 + 2\sin(\theta/2)]\cos\theta$, $y = r\sin\theta = [1 + 2\sin(\theta/2)]\sin\theta$.

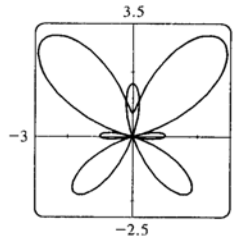
71. $r = 1 + 2\sin(\theta/2)$. The parameter interval is $[0, 4\pi]$.



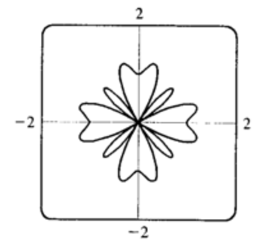
72. $r = \sqrt{1 - 0.8\sin^2\theta}$. The parameter interval is $[0, 2\pi]$.



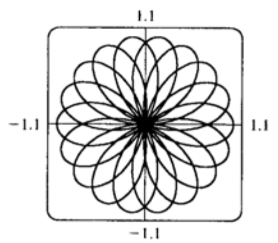
73. $r = e^{\sin\theta} - 2\cos(4\theta)$. The parameter interval is $[0, 2\pi]$.



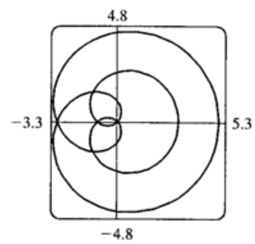
74. $r = \sin^2(4\theta) + \cos(4\theta)$. The parameter interval is $[0, 2\pi]$.



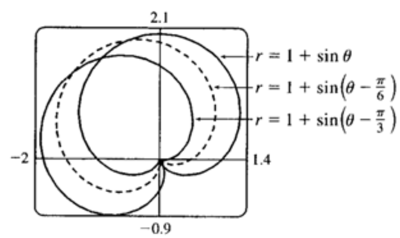
75. $r = \sin(9\theta/4)$. The parameter interval is $[0, 8\pi]$.



76. $r = 1 + 4\cos(\theta/3)$. The parameter interval is $[0, 6\pi]$.



77.



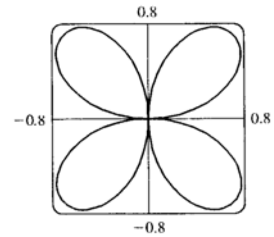
It appears that the graph of $r = 1 + \sin\left(\theta - \frac{\pi}{6}\right)$ is the same shape as the graph of $r = 1 + \sin\theta$, but rotated counterclockwise about the origin by $\frac{\pi}{6}$. Similarly, the graph of $r = 1 + \sin\left(\theta - \frac{\pi}{3}\right)$ is rotated by $\frac{\pi}{3}$. In general, the graph of $r = f(\theta - \alpha)$ is the same shape as that of $r = f(\theta)$, but rotated counterclockwise through α about the origin. That is, for any point (r_0, θ_0) on the curve $r = f(\theta)$, the point $(r_0, \theta_0 + \alpha)$ is on the curve $r = f(\theta - \alpha)$, since $r_0 = f(\theta_0) = f((\theta_0 + \alpha) - \alpha)$.

78. From the graph, the highest points seem to have $y \approx 0.77$.

To find the exact value, we solve $dy/d\theta = 0$.

$y = r \sin\theta = \sin\theta \sin 2\theta \quad \Rightarrow$

$$\begin{aligned} dy/d\theta &= 2\sin\theta\cos 2\theta + \cos\theta\sin 2\theta \\ &= 2\sin\theta\left(2\cos^2\theta - 1\right) + \cos\theta\,(2\sin\theta\cos\theta) \\ &= 2\sin\theta\left(3\cos^2\theta - 1\right) \end{aligned}$$



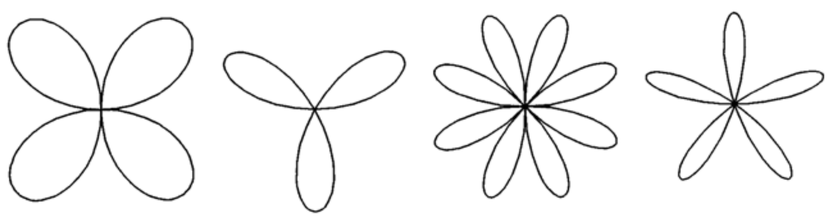
In the first quadrant, this is 0 when $\cos\theta = \frac{1}{\sqrt{3}} \quad \Leftrightarrow \quad \sin\theta = \sqrt{\frac{2}{3}} \quad \Leftrightarrow$

$y = 2\sin^2\theta\cos\theta = 2 \cdot \frac{2}{3} \cdot \frac{1}{\sqrt{3}} = \frac{4\sqrt{3}}{9} \approx 0.77.$

79. (a) $r = \sin n\theta$. From the graphs, it seems that when n is even, the number of loops in the curve (called a rose) is $2n$, and when n is odd, the number of loops is simply n.

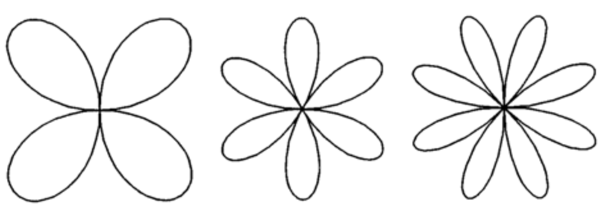
This is because in the case of n odd, every point on the graph is traversed twice, due to the fact that

$$r\,(\theta + \pi) = \sin\left[n\,(\theta + \pi)\right] = \sin n\theta\cos n\pi + \cos n\theta\sin n\pi = \begin{cases} \sin n\theta & \text{if } n \text{ is even} \\ -\sin n\theta & \text{if } n \text{ is odd} \end{cases}$$

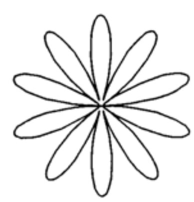


$n = 2$ $\qquad$ $n = 3$ $\qquad$ $n = 4$ $\qquad$ $n = 5$

(b) The graph of $r = |\sin n\theta|$ has $2n$ loops whether n is odd or even, since $r\,(\theta + \pi) = r\,(\theta)$.



$n = 2$ $\qquad$ $n = 3$ $\qquad$ $n = 4$



$n = 5$

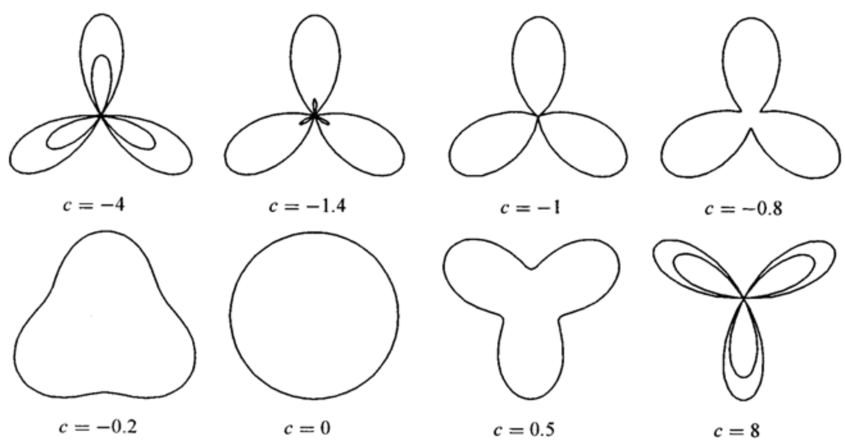
80. $r = 1 + c\sin n\theta$. We vary n while keeping c constant at 2. As n changes, the curves change in the same way as those in Exercise 79: the number of loops increases. Note that if n is even, the smaller loops are outside the larger ones; if n is odd, they are inside.

$c = 2$

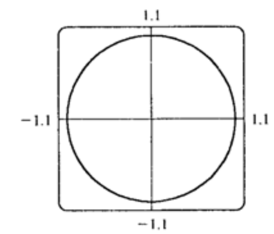
$n = 2$ $\quad$ $n = 3$ $\quad$ $n = 4$ $\quad$ $n = 5$

Now we vary c while keeping $n = 3$. As c increases toward 0, the entire graph gets smaller (the graphs below are not to scale) and the smaller loops shrink in relation to the large ones. At $c = -1$, the small loops disappear entirely, and for $-1 < c < 1$, the graph is a simple, closed curve (at $c = 0$ it is a circle). As c continues to increase, the same changes are seen, but in reverse order, since $1 + (-c)\sin n\theta = 1 + c\sin n(\theta + \pi)$, so the graph for $c = c_0$ is the same as that for $c = -c_0$, with a rotation through π. As $c \to \infty$, the smaller loops get relatively closer in size to the large ones. Note that the distance between the outermost points of corresponding inner and outer loops is always 2. Maple's `animate` command (or Mathematica's `Animate`) is very useful for seeing the changes that occur as c varies.

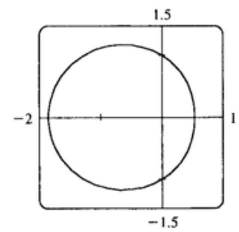
$n = 3$



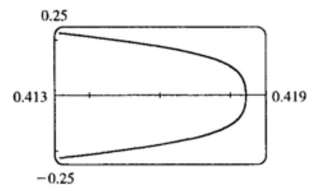
81. $r = \dfrac{1 - a\cos\theta}{1 + a\cos\theta}$. We start with $a = 0$, since in this case the curve is simply the circle $r = 1$. As a increases, the graph moves to the left, and its right side becomes flattened. As a increases through about 0.4, the right side seems to grow a dimple, which upon closer investigation (with narrower θ-ranges) seems to appear at $a \approx 0.42$ (the actual value is $\sqrt{2} - 1$). As $a \to 1$, this dimple becomes more pronounced, and the curve begins to stretch out horizontally, until at $a = 1$ the denominator vanishes at $\theta = \pi$, and the dimple becomes an actual cusp. For $a > 1$ we must choose our parameter interval carefully, since $r \to \infty$ as $1 + a\cos\theta \to 0 \quad \Leftrightarrow$ $\theta \to \pm\cos^{-1}(-1/a)$. As a increases from 1, the curve splits into two parts. The left part has a loop, which grows larger as a increases, and the right part grows broader vertically, and its left tip develops a dimple when $a \approx 2.42$ (actually, $\sqrt{2} + 1$). As a increases, the dimple grows more and more pronounced.
If $a < 0$, we get the same graph as we do for the corresponding positive a-value, but with a rotation through π about the pole, as happened when c was replaced with $-c$ in Exercise 80.



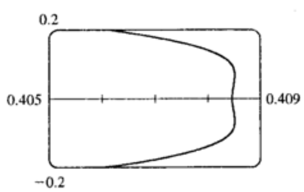
$a = 0$



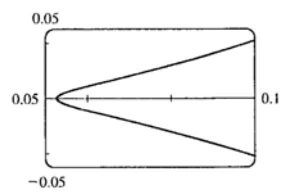
$a = 0.3$



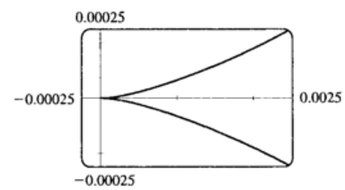
$a = 0.41, |\theta| \le 0.5$



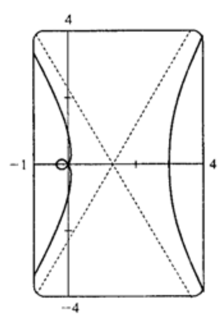
$a = 0.42, |\theta| \le 0.5$



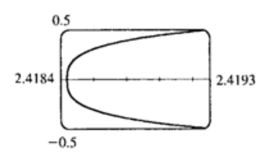
$a = 0.9, |\theta| \le 0.5$



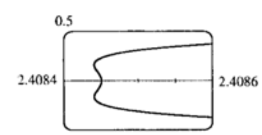
$a = 1, |\theta| \le 0.1$



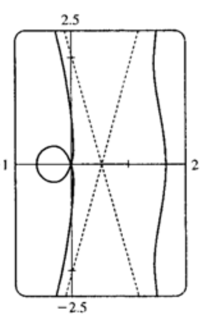
$a = 2$



$a = 2.41, |\theta - \pi| \le 0.2$



$a = 2.42, |\theta - \pi| \le 0.2$



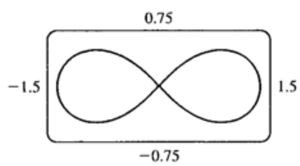
$a = 4$

82. Most graphing devices cannot plot implicit polar equations, so we must first find an explicit expression (or expressions) for r in terms of θ, a, and c. We note that the given equation is a quadratic in r^2, so we use the quadratic formula and find that

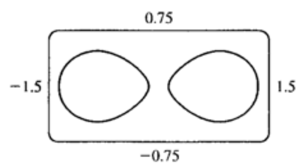
$$r^2 = \frac{2c^2\cos 2\theta \pm \sqrt{4c^4\cos^2 2\theta - 4\left(c^4 - a^4\right)}}{2} = c^2\cos 2\theta \pm \sqrt{a^4 - c^4\sin^2 2\theta}$$

so $r = \pm\sqrt{c^2\cos 2\theta \pm \sqrt{a^4 - c^4\sin^2 2\theta}}$. So for each graph, we must plot four curves to be sure of plotting all the points which satisfy the given equation. Note that all four functions have period π.

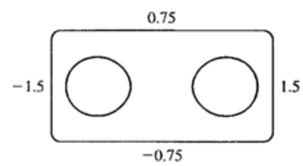
We start with the case $a = c = 1$, and the resulting curve resembles the symbol for infinity. If we let a decrease, the curve splits into two symmetric parts, and as a decreases further, the parts become smaller, further apart, and rounder. If instead we let a increase from 1, the two lobes of the curve join together, and as a increases further they continue to merge, until at $a \approx 1.4$, the graph no longer has dimples, and has an oval shape. As $a \to \infty$, the oval becomes larger and rounder, since the c^2 and c^4 terms lose their significance. Note that the shape of the graph seems to depend only on the ratio c/a, while the size of the graph varies as c and a jointly increase.



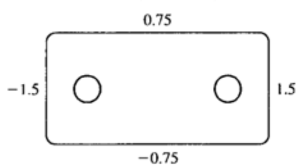
$(a, c) = (1, 1)$



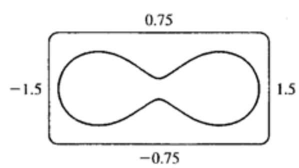
$(a, c) = (0.99, 1)$



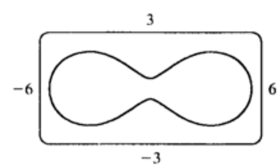
$(a, c) = (0.9, 1)$



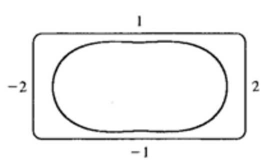
$(a, c) = (0.6, 1)$



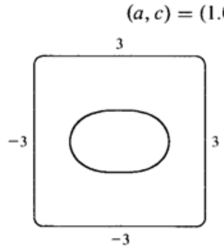
$(a, c) = (1.01, 1)$



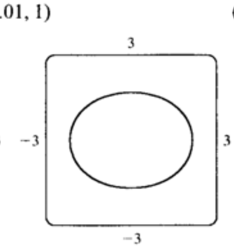
$(a, c) = (4.04, 4)$



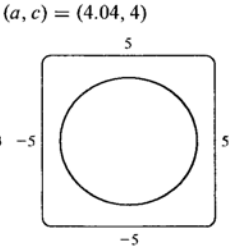
$(a, c) = (1.3, 1)$



$(a, c) = (1.5, 1)$



$(a, c) = (2, 1)$



$(a, c) = (4, 1)$

83. $\tan\psi = \tan(\phi-\theta) = \dfrac{\tan\phi-\tan\theta}{1+\tan\phi\tan\theta} = \dfrac{\dfrac{dy}{dx}-\tan\theta}{1+\dfrac{dy}{dx}\tan\theta} = \dfrac{\dfrac{dy/d\theta}{dx/d\theta}-\tan\theta}{1+\dfrac{dy/d\theta}{dx/d\theta}\tan\theta}$

$$= \frac{\dfrac{dy}{d\theta}-\dfrac{dx}{d\theta}\tan\theta}{\dfrac{dx}{d\theta}+\dfrac{dy}{d\theta}\tan\theta} = \frac{\left(\dfrac{dr}{d\theta}\sin\theta + r\cos\theta\right)-\tan\theta\left(\dfrac{dr}{d\theta}\cos\theta - r\sin\theta\right)}{\left(\dfrac{dr}{d\theta}\cos\theta - r\sin\theta\right)+\tan\theta\left(\dfrac{dr}{d\theta}\sin\theta + r\cos\theta\right)}$$

$$= \frac{r\cos\theta + r\cdot\dfrac{\sin^2\theta}{\cos\theta}}{\dfrac{dr}{d\theta}\cos\theta+\dfrac{dr}{d\theta}\cdot\dfrac{\sin^2\theta}{\cos\theta}} = \frac{r\cos^2\theta + r\sin^2\theta}{\dfrac{dr}{d\theta}\cos^2\theta+\dfrac{dr}{d\theta}\sin^2\theta} = \frac{r}{dr/d\theta}$$

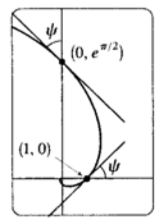
84. (a) $r = e^\theta \Rightarrow dr/d\theta = e^\theta$, so by Exercise 83,

$\tan\psi = r/e^\theta = 1 \Rightarrow \psi = \arctan 1 = \frac{\pi}{4}$.

(c) Let a be the tangent of the angle between the tangent and radial lines, that is, $a = \tan\psi$. Then, by Exercise 83, $a = \dfrac{r}{dr/d\theta} \Rightarrow \dfrac{dr}{d\theta} = \dfrac{1}{a}r$

$\Rightarrow r = Ce^{\theta/a}$ (by Theorem 10.4.2).

(b) The Cartesian equation of the tangent line at $(1,0)$ is $y = x-1$, and that of the tangent line at $(0, e^{\pi/2})$ is $y = e^{\pi/2} - x$.



11.5 Areas and Lengths in Polar Coordinates

1. $r = \sqrt{\theta}$, $0 \le \theta \le \frac{\pi}{4}$. $A = \int_0^{\pi/4} \frac{1}{2}r^2\,d\theta = \int_0^{\pi/4}\frac{1}{2}\left(\sqrt{\theta}\right)^2 d\theta = \int_0^{\pi/4}\frac{1}{2}\theta\,d\theta = \left[\frac{1}{4}\theta^2\right]_0^{\pi/4} = \frac{1}{64}\pi^2$

2. $r = e^{\theta/2}$, $\pi \le \theta \le 2\pi$. $A = \int_\pi^{2\pi}\frac{1}{2}\left(e^{\theta/2}\right)^2 d\theta = \int_\pi^{2\pi}\frac{1}{2}e^\theta\,d\theta = \frac{1}{2}\left[e^\theta\right]_\pi^{2\pi} = \frac{1}{2}\left(e^{2\pi}-e^\pi\right)$

3. $r = \sin\theta$, $\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$.

$$A = \int_{\pi/3}^{2\pi/3}\tfrac{1}{2}\sin^2\theta\,d\theta = \tfrac{1}{4}\int_{\pi/3}^{2\pi/3}(1-\cos 2\theta)\,d\theta = \tfrac{1}{4}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_{\pi/3}^{2\pi/3}$$

$$= \tfrac{1}{4}\left[\tfrac{2\pi}{3} - \tfrac{1}{2}\sin\tfrac{4\pi}{3} - \tfrac{\pi}{3} + \tfrac{1}{2}\sin\tfrac{2\pi}{3}\right] = \tfrac{1}{4}\left[\tfrac{2\pi}{3} - \tfrac{1}{2}\left(-\tfrac{\sqrt{3}}{2}\right) - \tfrac{\pi}{3} + \tfrac{1}{2}\left(\tfrac{\sqrt{3}}{2}\right)\right] = \tfrac{1}{4}\left(\tfrac{\pi}{3}+\tfrac{\sqrt{3}}{2}\right) = \tfrac{\pi}{12}+\tfrac{\sqrt{3}}{8}$$

4. $r = \sqrt{\sin\theta}$, $0 \le \theta \le \pi$. $A = \int_0^\pi \frac{1}{2}\left(\sqrt{\sin\theta}\right)^2 d\theta = \int_0^\pi \frac{1}{2}\sin\theta\,d\theta = \left[-\frac{1}{2}\cos\theta\right]_0^\pi = \frac{1}{2}+\frac{1}{2} = 1$

5. $r = \theta$, $0 \le \theta \le \pi$. $A = \int_0^\pi \frac{1}{2}\theta^2\,d\theta = \left[\frac{1}{6}\theta^3\right]_0^\pi = \frac{1}{6}\pi^3$

6. $r = 1+\sin\theta$, $\frac{\pi}{2} \le \theta \le \pi$.

$$A = \int_{\pi/2}^{\pi}\tfrac{1}{2}(1+\sin\theta)^2\,d\theta = \tfrac{1}{2}\int_{\pi/2}^{\pi}\left(1+2\sin\theta+\sin^2\theta\right)d\theta = \tfrac{1}{2}\int_{\pi/2}^{\pi}\left[1+2\sin\theta+\tfrac{1}{2}(1-\cos 2\theta)\right]d\theta$$

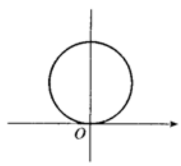
$$= \tfrac{1}{2}\left[\theta - 2\cos\theta + \tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_{\pi/2}^{\pi} = \tfrac{1}{2}\left[\pi + 2 + \tfrac{\pi}{2} - 0 - \left(\tfrac{\pi}{2} - 0 + \tfrac{\pi}{4} - 0\right)\right] = \tfrac{1}{2}\left(\tfrac{3\pi}{4}+2\right) = \tfrac{3\pi}{8}+1$$

7. $r = 4 + 3\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

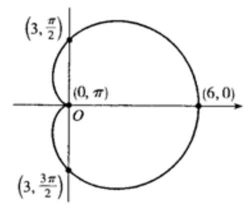
$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/2} \tfrac{1}{2}\,(4 + 3\sin\theta)^2\, d\theta = \tfrac{1}{2}\int_{-\pi/2}^{\pi/2} \left(16 + 24\sin\theta + 9\sin^2\theta\right) d\theta \\ &= \tfrac{1}{2}\int_{-\pi/2}^{\pi/2} \left(16 + 9\sin^2\theta\right) d\theta \quad \text{[by Theorem 5.5.6(b)]} \\ &= \tfrac{1}{2}\cdot 2\int_0^{\pi/2} \left[16 + 9\cdot\tfrac{1}{2}\,(1 - \cos 2\theta)\right] d\theta \quad \text{[by Theorem 5.5.6(a)]} \\ &= \int_0^{\pi/2} \left(\tfrac{41}{2} - \tfrac{9}{2}\cos 2\theta\right) d\theta = \left[\tfrac{41}{2}\theta - \tfrac{9}{4}\sin 2\theta\right]_0^{\pi/2} = \left(\tfrac{41\pi}{4} - 0\right) - (0 - 0) = \tfrac{41\pi}{4} \end{aligned}$$

8. $r = \sin 4\theta,\ 0 \le \theta \le \frac{\pi}{4}$. $A = \int_0^{\pi/4} \frac{1}{2}\sin^2 4\theta\, d\theta = \int_0^{\pi/4} \frac{1}{4}\,(1 - \cos 8\theta)\, d\theta = \left[\frac{1}{4}\theta - \frac{1}{32}\sin 8\theta\right]_0^{\pi/4} = \frac{\pi}{16}$

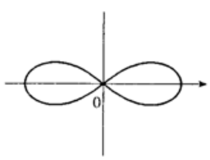
9.
$$\begin{aligned} A &= \int_0^{\pi} \tfrac{1}{2}\,(5\sin\theta)^2\, d\theta \\ &= \tfrac{25}{4}\int_0^{\pi} (1 - \cos 2\theta)\, d\theta \\ &= \tfrac{25}{4}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi} = \tfrac{25}{4}\pi \end{aligned}$$



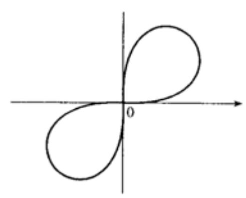
10.
$$\begin{aligned} A &= \int_0^{2\pi} \tfrac{1}{2}r^2\, d\theta = \int_0^{2\pi} \tfrac{1}{2}\,[3\,(1 + \cos\theta)]^2\, d\theta \\ &= \tfrac{9}{2}\int_0^{2\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta \\ &= \tfrac{9}{2}\int_0^{2\pi} \left[1 + 2\cos\theta + \tfrac{1}{2}\,(1 + \cos 2\theta)\right] d\theta \\ &= \tfrac{9}{2}\left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = \tfrac{27}{2}\pi \end{aligned}$$



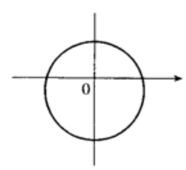
11.
$$\begin{aligned} A &= 4\int_0^{\pi/4} \tfrac{1}{2}r^2 d\theta = 2\int_0^{\pi/4} (4\cos 2\theta)\, d\theta \\ &= 8\int_0^{\pi/4} \cos 2\theta d\theta = 4\,[\sin 2\theta]_0^{\pi/4} = 4 \end{aligned}$$



12.
$$\begin{aligned} A &= 4\int_0^{\pi/4} \tfrac{1}{2}r^2\, d\theta = 2\int_0^{\pi/4} \sin 2\theta\, d\theta \\ &= [-\cos 2\theta]_0^{\pi/4} = 1 \end{aligned}$$



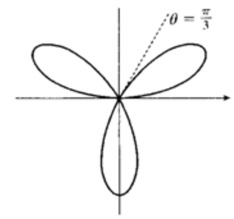
13.
$$\begin{aligned} A &= 2\int_{-\pi/2}^{\pi/2} \tfrac{1}{2}\,(4 - \sin\theta)^2\, d\theta = \int_{-\pi/2}^{\pi/2} \left(16 - 8\sin\theta + \sin^2\theta\right) d\theta \\ &= \int_{-\pi/2}^{\pi/2} \left(16 + \sin^2\theta\right) d\theta \quad \text{[by Theorem 5.5.6(b)]} \\ &= 2\int_0^{\pi/2} \left(16 + \sin^2\theta\right) d\theta \quad \text{[by Theorem 5.5.6(a)]} \\ &= 2\int_0^{\pi/2} \left[16 + \tfrac{1}{2}\,(1 - \cos 2\theta)\right] d\theta = 2\left[\tfrac{33}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2} \\ &= \tfrac{33\pi}{2} \end{aligned}$$



14. $A = 6\int_0^{\pi/6} \frac{1}{2}\sin^2 3\theta\, d\theta = 3\int_0^{\pi/6} \frac{1}{2}(1-\cos 6\theta)\, d\theta$

$= \frac{3}{2}\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/6}$

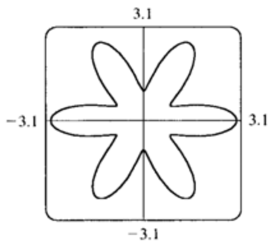
$= \frac{\pi}{4}$



15. By symmetry, the total area is twice the area enclosed above the polar axis, so

$$A = 2\int_0^{\pi} \tfrac{1}{2}r^2 d\theta = \int_0^{\pi} [2+\cos 6\theta]^2\, d\theta = \int_0^{\pi} \left(4 + 4\cos 6\theta + \cos^2 6\theta\right) d\theta$$

$$= \left[4\theta + 4\left(\tfrac{1}{6}\sin 6\theta\right) + \left(\tfrac{1}{24}\sin 12\theta + \tfrac{1}{2}\theta\right)\right]_0^{\pi} = 4\pi + \tfrac{\pi}{2} = \tfrac{9\pi}{2}$$

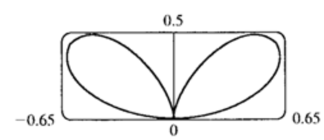


16. Note that the entire curve $r = 2\sin\theta\cos^2\theta$ is generated by $\theta \in [0, \pi]$. The radius is positive on this interval, so the area enclosed is

$$A = \int_0^{\pi} \tfrac{1}{2}r^2 d\theta = \int_0^{\pi} \tfrac{1}{2}\left(2\sin\theta\cos^2\theta\right)^2 d\theta = 2\int_0^{\pi} \sin^2\theta\cos^4\theta d\theta = 2\int_0^{\pi} (\sin\theta\cos\theta)^2\cos^2\theta d\theta$$

$$= 2\int_0^{\pi} \left(\tfrac{1}{2}\sin 2\theta\right)^2 \cos^2\theta d\theta = \tfrac{1}{4}\int_0^{\pi} \sin^2 2\theta\,(\cos 2\theta + 1)\, d\theta = \tfrac{1}{4}\left[\int_0^{\pi} \sin^2 2\theta\cos 2\theta d\theta + \int_0^{\pi} \sin^2 2\theta d\theta\right]$$

$$= \tfrac{1}{4}\left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi} \quad \text{(the first integral vanishes)} \quad = \tfrac{\pi}{8}$$



17. $A = \int_0^{\pi/2} \frac{1}{2}\sin^2 2\theta\, d\theta = \int_0^{\pi/2} \frac{1}{4}(1-\cos 4\theta)\, d\theta = \frac{1}{4}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{\pi}{8}$

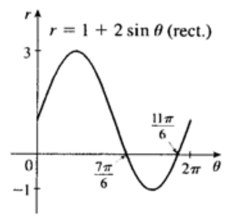
18. $A = \int_0^{\pi/3} \frac{1}{2}(4\sin 3\theta)^2\, d\theta = 8\int_0^{\pi/3} \sin^2 3\theta\, d\theta = 4\int_0^{\pi/3}(1-\cos 6\theta)\, d\theta = 4\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \frac{4\pi}{3}$

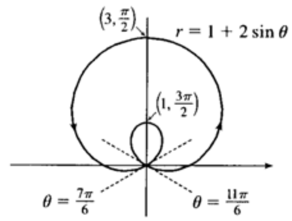
19. $r = 0 \;\Rightarrow\; 3\cos 5\theta = 0 \;\Rightarrow\; 5\theta = \frac{\pi}{2} \;\Rightarrow\; \theta = \frac{\pi}{10}$.

$A = \int_{-\pi/10}^{\pi/10} \frac{1}{2}(3\cos 5\theta)^2\, d\theta = \int_0^{\pi/10} 9\cos^2 5\theta\, d\theta = \frac{9}{2}\int_0^{\pi/10}(1+\cos 10\theta)\, d\theta = \frac{9}{2}\left[\theta + \frac{1}{10}\sin 10\theta\right]_0^{\pi/10} = \frac{9\pi}{20}$

20. $A = 2\int_0^{\pi/8} \frac{1}{2}(2\cos 4\theta)^2\, d\theta = 2\int_0^{\pi/8}(1+\cos 8\theta)\, d\theta = 2\left[\theta + \frac{1}{8}\sin 8\theta\right]_0^{\pi/8} = \frac{\pi}{4}$

21.



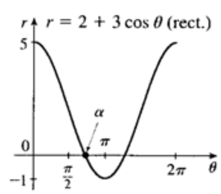


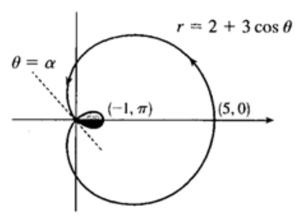
This is a limaçon, with inner loop traced out between $\theta = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$ [found by solving $r = 0$].

$$A = 2\int_{7\pi/6}^{3\pi/2} \tfrac{1}{2}(1 + 2\sin\theta)^2\,d\theta = \int_{7\pi/6}^{3\pi/2}\left(1 + 4\sin\theta + 4\sin^2\theta\right)d\theta = [\theta - 4\cos\theta + 2\theta - \sin 2\theta]_{7\pi/6}^{3\pi/2}$$
$$= \left(\tfrac{9\pi}{2}\right) - \left(\tfrac{7\pi}{2} + 2\sqrt{3} - \tfrac{\sqrt{3}}{2}\right) = \pi - \tfrac{3\sqrt{3}}{2}$$

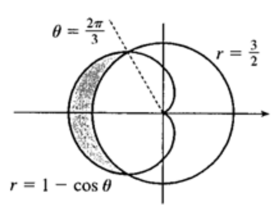
22. $2 + 3\cos\theta = 0 \;\Rightarrow\; \cos\theta = -\frac{2}{3} \;\Rightarrow\; \theta = \cos^{-1}\left(-\frac{2}{3}\right)$ $(= \alpha)$ or $2\pi - \cos^{-1}\left(-\frac{2}{3}\right) \;\Rightarrow$

$$A = 2\int_\alpha^\pi \tfrac{1}{2}(2 + 3\cos\theta)^2\,d\theta = \int_\alpha^\pi\left(4 + 12\cos\theta + 9\cos^2\theta\right)d\theta = \int_\alpha^\pi\left(\tfrac{17}{2} + 12\cos\theta + \tfrac{9}{2}\cos 2\theta\right)d\theta$$
$$= \left[\tfrac{17}{2}\theta + 12\sin\theta + \tfrac{9}{4}\sin 2\theta\right]_\alpha^\pi = \tfrac{17}{2}(\pi - \alpha) - 12\sin\alpha - \tfrac{9}{2}\sin\alpha\cos\alpha$$
$$= \tfrac{17}{2}\left[\pi - \cos^{-1}\left(-\tfrac{2}{3}\right)\right] - 12\left(\tfrac{\sqrt{5}}{3}\right) - \tfrac{9}{2}\left(\tfrac{\sqrt{5}}{3}\right)\left(-\tfrac{2}{3}\right) = \tfrac{17}{2}\cos^{-1}\tfrac{2}{3} - 3\sqrt{5}$$





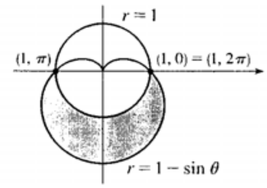
23. $1 - \cos\theta = \frac{3}{2} \;\Rightarrow\; \cos\theta = -\frac{1}{2} \;\Rightarrow\; \theta = \frac{2\pi}{3}$ or $\frac{4\pi}{3} \;\Rightarrow$



$$A = 2\int_{2\pi/3}^{\pi} \tfrac{1}{2}\left[(1 - \cos\theta)^2 - \left(\tfrac{3}{2}\right)^2\right]d\theta = \int_{2\pi/3}^{\pi}\left(-\tfrac{5}{4} - 2\cos\theta + \cos^2\theta\right)d\theta$$
$$= \left[-\tfrac{5}{4}\theta - 2\sin\theta\right]_{2\pi/3}^{\pi} + \tfrac{1}{2}\int_{2\pi/3}^{\pi}(1 + \cos 2\theta)\,d\theta$$
$$= -\tfrac{5}{12}\pi + \sqrt{3} + \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{2\pi/3}^{\pi}$$
$$= -\tfrac{5}{12}\pi + \sqrt{3} + \tfrac{1}{6}\pi + \tfrac{\sqrt{3}}{8} = \tfrac{9\sqrt{3}}{8} - \tfrac{1}{4}\pi$$

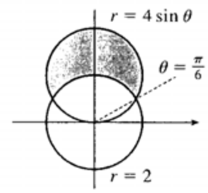
24. $1-\sin\theta=1 \quad\Rightarrow\quad \sin\theta=0 \quad\Rightarrow\quad \theta=0 \text{ or } \pi \quad\Rightarrow$

$$\begin{aligned} A &= \int_\pi^{2\pi} \tfrac{1}{2}\left[(1-\sin\theta)^2-1\right]d\theta = \tfrac{1}{2}\int_\pi^{2\pi}\left(\sin^2\theta-2\sin\theta\right)d\theta \\ &= \tfrac{1}{4}\int_\pi^{2\pi}(1-\cos 2\theta-4\sin\theta)\,d\theta = \tfrac{1}{4}\left[\theta-\tfrac{1}{2}\sin 2\theta+4\cos\theta\right]_\pi^{2\pi} \\ &= \tfrac{1}{4}\pi+2 \end{aligned}$$



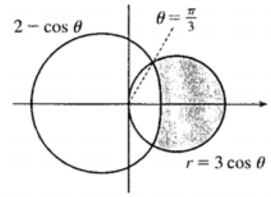
25. $4\sin\theta=2 \quad\Leftrightarrow\quad \sin\theta=\frac{1}{2} \quad\Leftrightarrow\quad \theta=\frac{\pi}{6} \text{ or } \frac{5\pi}{6} \quad\Leftrightarrow$

$$\begin{aligned} A &= 2\int_{\pi/6}^{\pi/2}\tfrac{1}{2}\left[(4\sin\theta)^2-2^2\right]d\theta = \int_{\pi/6}^{\pi/2}\left(16\sin^2\theta-4\right)d\theta \\ &= \int_{\pi/6}^{\pi/2}\left[8\,(1-\cos 2\theta)-4\right]d\theta = \left[4\theta-4\sin 2\theta\right]_{\pi/6}^{\pi/2} \\ &= \tfrac{4}{3}\pi+2\sqrt{3} \end{aligned}$$



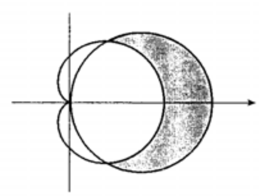
26. $3\cos\theta=2-\cos\theta \quad\Rightarrow\quad \cos\theta=\frac{1}{2} \quad\Rightarrow\quad \theta=\pm\frac{\pi}{3} \quad\Rightarrow$

$$\begin{aligned} A &= 2\int_0^{\pi/3}\tfrac{1}{2}\left[(3\cos\theta)^2-(2-\cos\theta)^2\right]d\theta \\ &= \int_0^{\pi/3}\left(8\cos^2\theta+4\cos\theta-4\right)d\theta \\ &= \int_0^{\pi/3}(4\cos 2\theta+4\cos\theta)\,d\theta = \left[2\sin 2\theta+4\sin\theta\right]_0^{\pi/3} = 3\sqrt{3} \end{aligned}$$

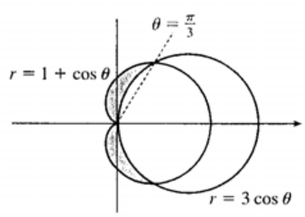


27. $3\cos\theta=1+\cos\theta \quad\Leftrightarrow\quad \cos\theta=\frac{1}{2} \quad\Rightarrow\quad \theta=\frac{\pi}{3} \text{ or } -\frac{\pi}{3}.$

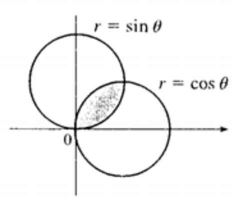
$$\begin{aligned} A &= 2\int_0^{\pi/3}\tfrac{1}{2}\left[(3\cos\theta)^2-(1+\cos\theta)^2\right]d\theta \\ &= \int_0^{\pi/3}\left(8\cos^2\theta-2\cos\theta-1\right)d\theta = \int_0^{\pi/3}\left[4\,(1+\cos 2\theta)-2\cos\theta-1\right]d\theta \\ &= \left[3\theta+2\sin 2\theta-2\sin\theta\right]_0^{\pi/3} = \pi+\sqrt{3}-\sqrt{3} = \pi \end{aligned}$$



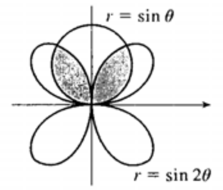
28.
$$\begin{aligned} A &= 2\int_{\pi/3}^{\pi}\tfrac{1}{2}(1+\cos\theta)^2\,d\theta - 2\int_{\pi/3}^{\pi/2}\tfrac{1}{2}(3\cos\theta)^2\,d\theta \\ &= \left[\theta+2\sin\theta+\tfrac{1}{2}\left(\theta+\tfrac{1}{2}\sin 2\theta\right)\right]_{\pi/3}^{\pi} - \tfrac{9}{2}\left[\theta+\tfrac{1}{2}\sin 2\theta\right]_{\pi/3}^{\pi/2} \\ &= \left(\pi-\tfrac{9}{8}\sqrt{3}\right)-\tfrac{9}{2}\left(\tfrac{\pi}{6}-\tfrac{1}{4}\sqrt{3}\right) = \tfrac{\pi}{4} \end{aligned}$$



29.
$$\begin{aligned} A &= 2\int_0^{\pi/4}\tfrac{1}{2}\sin^2\theta\,d\theta = \int_0^{\pi/4}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta \\ &= \left[\tfrac{1}{2}\theta-\tfrac{1}{4}\sin 2\theta\right]_0^{\pi/4} \\ &= \tfrac{1}{8}\pi-\tfrac{1}{4} \end{aligned}$$

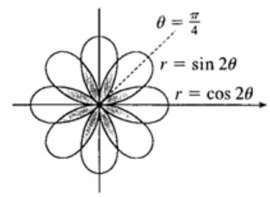


30. $\sin\theta = \pm\sin 2\theta = \pm 2\sin\theta\cos\theta \;\Rightarrow\; \sin\theta\,(1 \pm 2\cos\theta) = 0$. From the figure we can see that the intersections occur where $\cos\theta = \pm\frac{1}{2}$, or $\theta = \frac{\pi}{3}$ and $\frac{2\pi}{3}$.



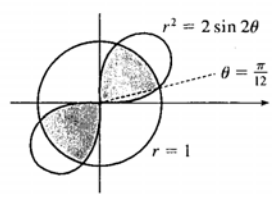
$$A = 2\left[\int_0^{\pi/3} \tfrac{1}{2}\sin^2\theta\,d\theta + \int_{\pi/3}^{\pi/2} \tfrac{1}{2}\sin^2 2\theta\,d\theta\right]$$
$$= \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/3} + \tfrac{1}{2}\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_{\pi/3}^{\pi/2} = \frac{4\pi - 3\sqrt{3}}{16}$$

31. $\sin 2\theta = \cos 2\theta \;\Rightarrow\; \tan 2\theta = 1 \;\Rightarrow\; 2\theta = \frac{\pi}{4} \;\Rightarrow\; \theta = \frac{\pi}{8} \;\Rightarrow$



$$A = 16\int_0^{\pi/8} \tfrac{1}{2}\sin^2 2\theta\,d\theta = 4\int_0^{\pi/8} (1 - \cos 4\theta)\,d\theta$$
$$= 4\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/8} = \tfrac{1}{2}\pi - 1$$

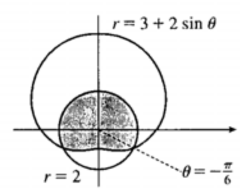
32. $2\sin 2\theta = 1^2 \;\Rightarrow\; \sin 2\theta = \frac{1}{2} \;\Rightarrow\; 2\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6} \;\Rightarrow\; \theta = \frac{\pi}{12}$ or $\frac{5\pi}{12}$.



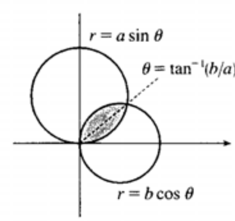
$$A = 4\left[\int_0^{\pi/12} \tfrac{1}{2}\cdot 2\sin 2\theta\,d\theta + \int_{\pi/12}^{\pi/4} \tfrac{1}{2}\left(1^2\right)d\theta\right]$$
$$= \left[-2\cos 2\theta\right]_0^{\pi/12} + 2\left(\tfrac{1}{4}\pi - \tfrac{1}{12}\pi\right) = 2 - \sqrt{3} + \tfrac{\pi}{3}$$

33. $A = 2\left[\int_{-\pi/2}^{-\pi/6} \tfrac{1}{2}(3 + 2\sin\theta)^2\,d\theta + \int_{-\pi/6}^{\pi/2} \tfrac{1}{2}2^2\,d\theta\right]$

$$= \int_{-\pi/2}^{-\pi/6} \left(9 + 12\sin\theta + 4\sin^2\theta\right)d\theta + \left[4\theta\right]_{-\pi/6}^{\pi/2}$$
$$= \left[9\theta - 12\cos\theta + 2\theta - \sin 2\theta\right]_{-\pi/2}^{-\pi/6} + \tfrac{8\pi}{3} = \tfrac{19\pi}{3} - \tfrac{11\sqrt{3}}{2}$$



34. Let $\alpha = \tan^{-1}(b/a)$. Then



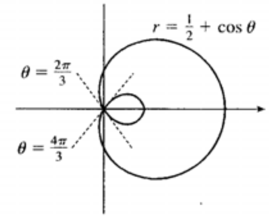
$$A = \int_0^{\alpha} \tfrac{1}{2}(a\sin\theta)^2\,d\theta + \int_{\alpha}^{\pi/2} \tfrac{1}{2}(b\cos\theta)^2\,d\theta$$
$$= \tfrac{1}{4}a^2\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\alpha} + \tfrac{1}{4}b^2\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{\alpha}^{\pi/2}$$
$$= \tfrac{1}{4}\alpha\left(a^2 - b^2\right) + \tfrac{1}{8}\pi b^2 - \tfrac{1}{4}\left(a^2 + b^2\right)(\sin\alpha\cos\alpha)$$
$$= \tfrac{1}{4}\left(a^2 - b^2\right)\tan^{-1}(b/a) + \tfrac{1}{8}\pi b^2 - \tfrac{1}{4}ab$$

35. $A = 2\left[\int_0^{2\pi/3} \frac{1}{2}\left(\frac{1}{2} + \cos\theta\right)^2 d\theta - \int_{2\pi/3}^{\pi} \frac{1}{2}\left(\frac{1}{2} + \cos\theta\right)^2 d\theta\right]$

$= \int_0^{2\pi/3}\left(\frac{1}{4} + \cos\theta + \cos^2\theta\right) d\theta - \int_{2\pi/3}^{\pi}\left(\frac{1}{4} + \cos\theta + \cos^2\theta\right) d\theta$

$= \left[\frac{\theta}{4} + \sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi/3} - \left[\frac{\theta}{4} + \sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{2\pi/3}^{\pi}$

$= \left(\frac{\pi}{2} + \frac{3\sqrt{3}}{8}\right) - \left(\frac{3\pi}{4}\right) + \left(\frac{\pi}{2} + \frac{3\sqrt{3}}{8}\right) = \frac{1}{4}\left(\pi + 3\sqrt{3}\right)$



36. The points of intersection occur where $\sqrt{1 - 0.8\sin^2\theta} = \sin\theta \quad \Leftrightarrow$

$1.8\sin^2\theta = 1 \quad \Leftrightarrow \quad \theta = \arcsin\sqrt{\frac{5}{9}}$ ($= \alpha$, so $\cos\alpha = \frac{2}{3}$). So the area is

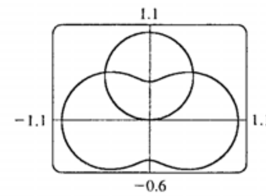
$$A = 2\int_0^{\alpha} \frac{1}{2}\sin^2\theta d\theta + 2\int_{\alpha}^{\pi/2} \frac{1}{2}\left(\sqrt{1 - 0.8\sin^2\theta}\right)^2 d\theta$$

$$= \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\alpha} + \left[\theta - 0.8\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\right]_{\alpha}^{\pi/2}$$

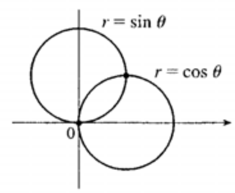
$$= \frac{1}{2}\alpha - \frac{1}{4}(2\sin\alpha\cos\alpha) + 0.6\cdot\frac{\pi}{2} - [0.6\alpha + 0.2(2\sin\alpha\cos\alpha)]$$

$$= \frac{1}{2}\arcsin\frac{\sqrt{5}}{3} - \frac{1}{2}\frac{\sqrt{5}}{3}\frac{2}{3} + 0.3\pi - 0.6\arcsin\frac{\sqrt{5}}{3} - 0.4\cdot\frac{\sqrt{5}}{3}\frac{2}{3}$$

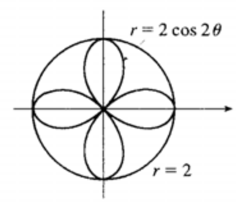
$$= \frac{3}{10}\pi - \frac{1}{10}\arcsin\frac{\sqrt{5}}{3} - \frac{1}{5}\sqrt{5} \approx 0.411$$



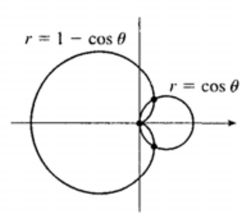
37. The two circles intersect at the pole since $(0, 0)$ satisfies the first equation and $\left(0, \frac{\pi}{2}\right)$ the second. The other intersection point $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$ occurs where $\sin\theta = \cos\theta$.



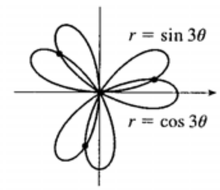
38. $2\cos 2\theta = \pm 2 \quad \Rightarrow \quad \cos 2\theta = \pm 1 \quad \Rightarrow \quad \theta = 0, \frac{\pi}{2}, \pi$, or $\frac{3\pi}{2}$, so the points are $(2, 0)$, $\left(2, \frac{\pi}{2}\right)$, $(2, \pi)$, and $\left(2, \frac{3\pi}{2}\right)$.



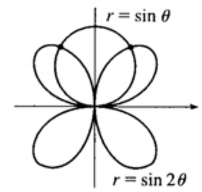
39. The curves intersect at the pole since $\left(0, \frac{\pi}{2}\right)$ satisfies $r = \cos\theta$ and $(0, 0)$ satisfies $r = 1 - \cos\theta$. $\cos\theta = 1 - \cos\theta \Longrightarrow \cos\theta = \frac{1}{2} \Longrightarrow$ $\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3} \Longrightarrow$ the other intersection points are $\left(\frac{1}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{1}{2}, \frac{5\pi}{3}\right)$.



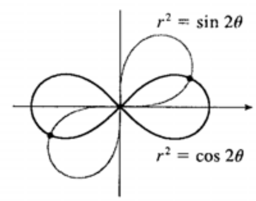
40. Clearly the pole lies on both curves. $\sin 3\theta = \cos 3\theta \Rightarrow \tan 3\theta = 1 \Rightarrow 3\theta = \frac{1}{4}\pi + n\pi$ (n any integer) $\Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}$, or $\frac{3\pi}{4}$, so the three remaining intersection points are $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{12}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{5\pi}{12}\right)$, and $\left(\frac{1}{\sqrt{2}}, \frac{3\pi}{4}\right)$.



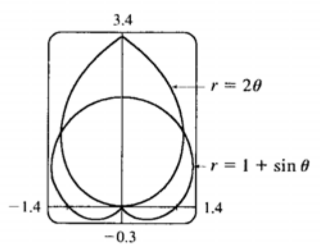
41. The pole is a point of intersection. $\sin\theta = \sin 2\theta = 2\sin\theta\cos\theta \Leftrightarrow \sin\theta\,(1 - 2\cos\theta) = 0 \Leftrightarrow \sin\theta = 0$ or $\cos\theta = \frac{1}{2} \Rightarrow \theta = 0, \pi, \frac{\pi}{3}, -\frac{\pi}{3} \Rightarrow \left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)$ (by symmetry) are the other intersection points.

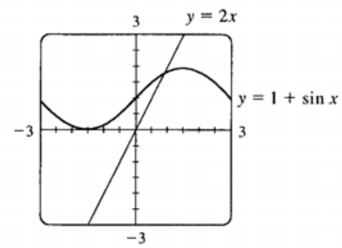


42. Clearly the pole is a point of intersection. $\sin 2\theta = \cos 2\theta \Rightarrow \tan 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{4} + 2n\pi$ (since $\sin 2\theta$ and $\cos 2\theta$ must be positive in the equations) $\Rightarrow \theta = \frac{\pi}{8} + n\pi \Rightarrow \theta = \frac{\pi}{8}$ or $\frac{9\pi}{8}$. So the curves also intersect at $\left(\frac{1}{\sqrt[4]{2}}, \frac{\pi}{8}\right)$ and $\left(\frac{1}{\sqrt[4]{2}}, \frac{9\pi}{8}\right)$.



43.



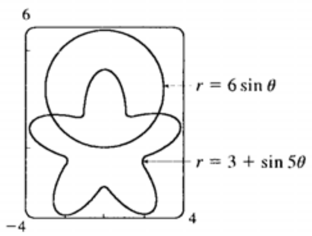


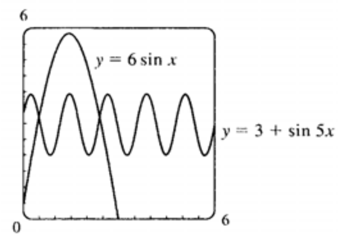
From the first graph, we see that the pole is one point of intersection. By zooming in or using the cursor, we estimate the θ-values of the intersection points to be about 0.89 and $\pi - 0.89 \approx 2.25$. (The first of these values may be more easily estimated by plotting $y = 1 + \sin x$ and $y = 2x$ in rectangular coordinates; see the second graph.)

By symmetry, the total area contained is twice the area contained in the first quadrant, that is,

$$\begin{aligned} A &\approx 2\int_0^{0.89} \tfrac{1}{2}(2\theta)^2\,d\theta + 2\int_{0.89}^{\pi/2} \tfrac{1}{2}(1+\sin\theta)^2\,d\theta \\ &= \left[\tfrac{4}{3}\theta^3\right]_0^{0.89} + \left[\theta - 2\cos\theta + \left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right)\right]_{0.89}^{\pi/2} \approx 3.46 \end{aligned}$$

44.





From the first graph, it appears that the θ-values of the points of intersection are about 0.58 and 2.57. (These values may be more easily estimated by plotting $y = 3 + \sin 5x$ and $y = 6\sin x$ in rectangular coordinates; see the second graph.) By symmetry, the total area enclosed in both curves is

$$A \approx 2\int_0^{0.58} \tfrac{1}{2}(6\sin\theta)^2\,d\theta + 2\int_{0.58}^{\pi/2} \tfrac{1}{2}(3+\sin 5\theta)^2\,d\theta = \int_0^{0.58} 36\sin^2\theta\,d\theta + \int_{0.58}^{\pi/2}(9 + 6\sin 5\theta + \sin^2 5\theta)\,d\theta$$

$$= \left[36\left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right)\right]_0^{0.58} + \left[9\theta - \tfrac{6}{5}\cos 5\theta + \tfrac{1}{5}\left(\tfrac{5}{2}\theta - \tfrac{1}{4}\sin 10\theta\right)\right]_{0.58}^{\pi/2} \approx 10.41$$

45. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_0^{3\pi/4} \sqrt{(5\cos\theta)^2 + (-5\sin\theta)^2}\,d\theta = 5\int_0^{3\pi/4} \sqrt{\cos^2\theta + \sin^2\theta}\,d\theta$
$= 5\int_0^{3\pi/4} d\theta = \frac{15}{4}\pi$

46. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_0^{2\pi} \sqrt{(e^{2\theta})^2 + (2e^{2\theta})^2}\,d\theta = \int_0^{2\pi} \sqrt{e^{4\theta} + 4e^{4\theta}}\,d\theta = \int_0^{2\pi} \sqrt{5e^{4\theta}}\,d\theta$
$= \sqrt{5}\int_0^{2\pi} e^{2\theta}\,d\theta = \frac{\sqrt{5}}{2}\left[e^{2\theta}\right]_0^{2\pi} = \frac{\sqrt{5}}{2}\left(e^{4\pi} - 1\right)$

47. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_0^{2\pi} \sqrt{(2^\theta)^2 + \left[(\ln 2)\,2^\theta\right]^2}\,d\theta = \int_0^{2\pi} 2^\theta\sqrt{1 + \ln^2 2}\,d\theta$
$= \left[\sqrt{1+\ln^2 2}\left(\dfrac{2^\theta}{\ln 2}\right)\right]_0^{2\pi} = \dfrac{\sqrt{1+\ln^2 2}\,(2^{2\pi} - 1)}{\ln 2}$

48. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_0^{2\pi} \sqrt{\theta^2 + 1}\,d\theta \overset{21}{=} \left[\frac{\theta}{2}\sqrt{\theta^2+1} + \frac{1}{2}\ln\left(\theta + \sqrt{\theta^2+1}\right)\right]_0^{2\pi}$
$= \pi\sqrt{4\pi^2 + 1} + \frac{1}{2}\ln\left(2\pi + \sqrt{4\pi^2+1}\right)$

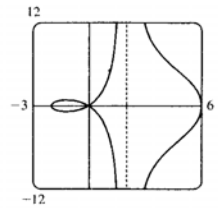
49. $L = \int_0^{2\pi} \sqrt{(\theta^2)^2 + (2\theta)^2}\,d\theta = \int_0^{2\pi} \theta\sqrt{\theta^2 + 4}\,d\theta = \frac{1}{2}\cdot\frac{2}{3}\left[(\theta^2+4)^{3/2}\right]_0^{2\pi} = \frac{8}{3}\left[(\pi^2+1)^{3/2} - 1\right]$

50. $L = 2\int_0^{\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\,d\theta = 2\sqrt{2}\int_0^{\pi} \sqrt{1+\cos\theta}\,d\theta = 2\sqrt{2}\int_0^{\pi}\sqrt{2\cos^2(\theta/2)}\,d\theta$
$= \left[8\sin(\theta/2)\right]_0^{\pi} = 8$

51. From Figure 4 in Example 1,

$$L = \int_{-\pi/4}^{\pi/4} \sqrt{r^2 + (r')^2}\,d\theta = 2\int_0^{\pi/4} \sqrt{\cos^2 2\theta + 4\sin^2 2\theta}\,d\theta \approx 2\,(1.211056) \approx 2.4221$$

52. $4+2\sec\theta=0 \Rightarrow \sec\theta=-2 \Rightarrow \cos\theta=-\frac{1}{2} \Rightarrow \theta=\frac{2\pi}{3}$, $\frac{4\pi}{3}$. $L=\int_{2\pi/3}^{4\pi/3}\sqrt{(4+2\sec\theta)^2+(2\sec\theta\tan\theta)^2}\,d\theta \approx 5.8128$



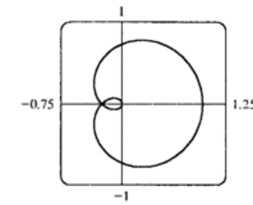
53. $L=2\int_0^{2\pi}\sqrt{\cos^8(\theta/4)+\cos^6(\theta/4)\sin^2(\theta/4)}\,d\theta$

$=2\int_0^{2\pi}\left|\cos^3(\theta/4)\right|\sqrt{\cos^2(\theta/4)+\sin^2(\theta/4)}\,d\theta$

$=2\int_0^{2\pi}\left|\cos^3(\theta/4)\right|d\theta=8\int_0^{\pi/2}\cos^3 u\,du \quad (\text{where } u=\frac{1}{4}\theta)$

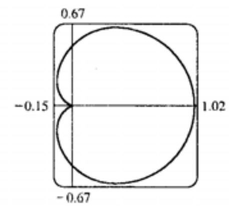
$=8\left[\sin u-\frac{1}{3}\sin^3 u\right]_0^{\pi/2}=\frac{16}{3}$

Note that the curve is retraced after every interval of length 4π.



54. $L=2\int_0^{\pi}\sqrt{\left[\cos^2\left(\frac{1}{2}\theta\right)\right]^2+\left[-\cos\left(\frac{1}{2}\theta\right)\sin\left(\frac{1}{2}\theta\right)\right]^2}\,d\theta$

$=2\int_0^{\pi}\cos\left(\frac{1}{2}\theta\right)d\theta=4\left[\sin\left(\frac{1}{2}\theta\right)\right]_0^{\pi}=4$



55. (a) From (11.3.5),

$S=\int_a^b 2\pi y\sqrt{(dx/d\theta)^2+(dy/d\theta)^2}\,d\theta$

$=\int_a^b 2\pi y\sqrt{r^2+(dr/d\theta)^2}\,d\theta$ (see the derivation of Equation 5) $=\int_a^b 2\pi r\sin\theta\sqrt{r^2+(dr/d\theta)^2}\,d\theta$

(b) $r^2=\cos 2\theta \Rightarrow 2r\dfrac{dr}{d\theta}=-2\sin 2\theta \Rightarrow \left(\dfrac{dr}{d\theta}\right)^2=\dfrac{\sin^2 2\theta}{r^2}=\dfrac{\sin^2 2\theta}{\cos 2\theta}$.

$S=2\int_0^{\pi/4}2\pi\sqrt{\cos 2\theta}\sin\theta\sqrt{\cos 2\theta+\left(\sin^2 2\theta\right)/\cos 2\theta}\,d\theta=4\pi\int_0^{\pi/4}\sin\theta\,d\theta$

$=\left[-4\pi\cos\theta\right]_0^{\pi/4}=-4\pi\left(\frac{1}{\sqrt{2}}-1\right)=2\pi\left(2-\sqrt{2}\right)$

56. (a) Rotation around $\theta=\frac{\pi}{2}$ is the same as rotation around the y-axis, that is, $S=\int_a^b 2\pi x\,ds$ where $ds=\sqrt{(dx/dt)^2+(dy/dt)^2}\,dt$ for a parametric equation, and for the special case of a polar equation, $x=r\cos\theta$ and $ds=\sqrt{(dx/d\theta)^2+(dy/d\theta)^2}\,d\theta=\sqrt{r^2+(dr/d\theta)^2}\,d\theta$ (see the derivation of Equation 5.) Therefore, for a polar equation, rotated around $\theta=\frac{\pi}{2}$, $S=\int_a^b 2\pi r\cos\theta\sqrt{r^2+(dr/d\theta)^2}\,d\theta$.

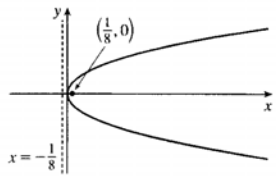
(b) In the case of the lemniscate we are concerned with $-\frac{\pi}{4}\le\theta\le\frac{\pi}{4}$ and $r^2=\cos 2\theta \Rightarrow$ $2r\,dr/d\theta=-2\sin 2\theta \Rightarrow (dr/d\theta)^2=\left(\sin^2 2\theta\right)/r^2=\left(\sin^2 2\theta\right)/\cos 2\theta$. Therefore

$S=\int_{-\pi/4}^{\pi/4}2\pi\sqrt{\cos 2\theta}\cos\theta\sqrt{\cos 2\theta+\left(\sin^2 2\theta\right)/\cos 2\theta}\,d\theta$

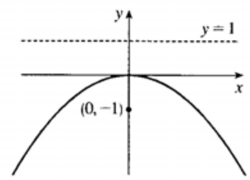
$=4\pi\int_0^{\pi/4}\cos\theta\sqrt{\cos 2\theta}\sqrt{1/\cos 2\theta}\,d\theta=4\pi\int_0^{\pi/4}\cos\theta\,d\theta=2\sqrt{2}\pi$

11.6 Conic Sections

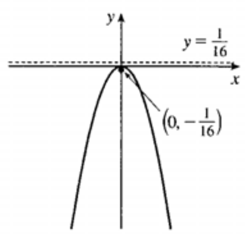
1. $x = 2y^2 \;\Rightarrow\; y^2 = \frac{1}{2}x$. $4p = \frac{1}{2}$, so $p = \frac{1}{8}$. The vertex is $(0, 0)$, the focus is $\left(\frac{1}{8}, 0\right)$, and the directrix is $x = -\frac{1}{8}$.



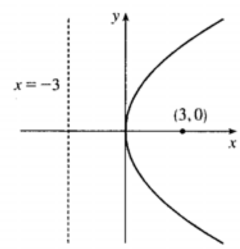
2. $4y + x^2 = 0 \;\Rightarrow\; x^2 = -4y$. $4p = -4$, so $p = -1$. The vertex is $(0, 0)$, the focus is $(0, -1)$, and the directrix is $y = 1$.



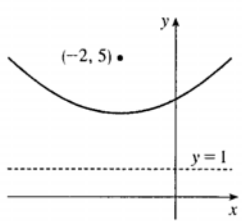
3. $4x^2 = -y \;\Rightarrow\; x^2 = -\frac{1}{4}y$. $4p = -\frac{1}{4}$, so $p = -\frac{1}{16}$. The vertex is $(0, 0)$, the focus is $\left(0, -\frac{1}{16}\right)$, and the directrix is $y = \frac{1}{16}$.



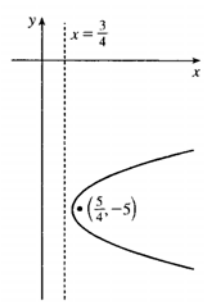
4. $y^2 = 12x$. $4p = 12$, so $p = 3$. The vertex is $(0, 0)$, the focus is $(3, 0)$, and the directrix is $x = -3$.



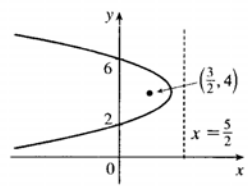
5. $(x + 2)^2 = 8(y - 3)$. $4p = 8$, so $p = 2$. The vertex is $(-2, 3)$, the focus is $(-2, 5)$, and the directrix is $y = 1$.



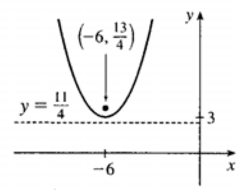
6. $x - 1 = (y + 5)^2$. $4p = 1$, so $p = \frac{1}{4}$. The vertex is $(1, -5)$, the focus is $\left(\frac{5}{4}, -5\right)$, and the directrix is $x = \frac{3}{4}$.



7. $2x + y^2 - 8y + 12 = 0 \Rightarrow$
$(y-4)^2 = -2(x-2) \Rightarrow p = -\frac{1}{2} \Rightarrow$
vertex $(2, 4)$, focus $\left(\frac{3}{2}, 4\right)$, directrix $x = \frac{5}{2}$



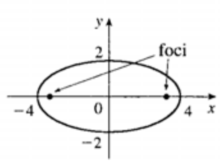
8. $x^2 + 12x - y + 39 = 0 \Leftrightarrow (x+6)^2 = y - 3$
$\Rightarrow p = \frac{1}{4} \Rightarrow$ vertex $(-6, 3)$, focus $\left(-6, \frac{13}{4}\right)$,
directrix $y = \frac{11}{4}$



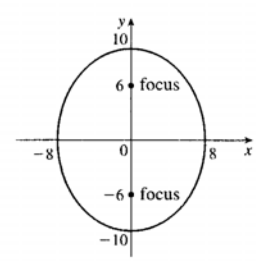
9. The equation has the form $y^2 = 4px$, where $p < 0$. Since the parabola passes through $(-1, 1)$, we have $1^2 = 4p(-1)$, so $4p = -1$ and an equation is $y^2 = -x$ or $x = -y^2$. $4p = -1$, so $p = -\frac{1}{4}$ and the focus is $\left(-\frac{1}{4}, 0\right)$ while the directrix is $x = \frac{1}{4}$.

10. The vertex is $(2, -2)$, so the equation is of the form $(x-2)^2 = 4p(y+2)$, where $p > 0$. The point $(0, 0)$ is on the parabola, so $4 = 4p(2)$ and $4p = 2$. Thus, an equation is $(x-2)^2 = 2(y+2)$. $4p = 2$, so $p = \frac{1}{2}$ and the focus is $\left(2, -\frac{3}{2}\right)$ while the directrix is $y = -\frac{5}{2}$.

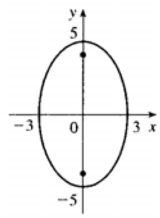
11. $x^2/16 + y^2/4 = 1 \Rightarrow a = 4, b = 2,$
$c = \sqrt{16-4} = 2\sqrt{3} \Rightarrow$ center $(0, 0)$, vertices $(\pm 4, 0)$, foci $\left(\pm 2\sqrt{3}, 0\right)$



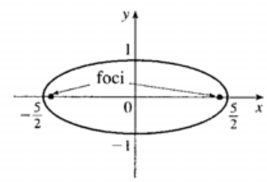
12. $\dfrac{x^2}{64} + \dfrac{y^2}{100} = 1 \Rightarrow a = 10, b = 8,$
$c = \sqrt{a^2 - b^2} = 6$. The ellipse is centered at $(0, 0)$, with vertices at $(0, \pm 10)$. The foci are $(0, \pm 6)$.



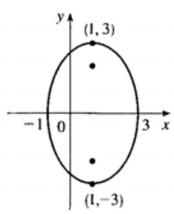
13. $25x^2 + 9y^2 = 225 \quad \Leftrightarrow \quad \frac{1}{9}x^2 + \frac{1}{25}y^2 = 1 \quad \Rightarrow$ $a = 5, b = 3, c = 4 \quad \Rightarrow \quad$ center $(0, 0)$, vertices $(0, \pm 5)$, foci $(0, \pm 4)$



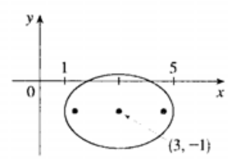
14. $4x^2 + 25y^2 = 25 \quad \Rightarrow \quad \dfrac{x^2}{25/4} + \dfrac{y^2}{1} = 1 \quad \Rightarrow$ $a = \frac{5}{2}, b = 1, c = \sqrt{a^2 - b^2} = \sqrt{\frac{21}{4}} = \frac{\sqrt{21}}{2}$. The ellipse is centered at $(0, 0)$, with vertices at $\left(\pm\frac{5}{2}, 0\right)$. The foci are $\left(\pm\frac{\sqrt{21}}{2}, 0\right)$.



15. $9x^2 - 18x + 4y^2 = 27 \quad \Leftrightarrow \quad \dfrac{(x-1)^2}{4} + \dfrac{y^2}{9} = 1$ $\Rightarrow \quad a = 3, b = 2, c = \sqrt{5} \quad \Rightarrow \quad$ center $(1, 0)$, vertices $(1, \pm 3)$, foci $\left(1, \pm\sqrt{5}\right)$



16. $x^2 - 6x + 2y^2 + 4y = -7 \quad \Leftrightarrow$ $\dfrac{(x-3)^2}{4} + \dfrac{(y+1)^2}{2} = 1 \quad \Rightarrow \quad a = 2,$ $b = \sqrt{2} = c \quad \Rightarrow \quad$ center $(3, -1)$, vertices $(1, -1)$ and $(5, -1)$, foci $\left(3 \pm \sqrt{2}, -1\right)$

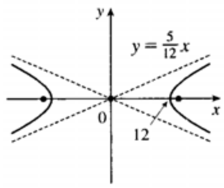


17. The center is $(0, 0)$, $a = 3$, and $b = 2$, so an equation is $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$. $c = \sqrt{a^2 - b^2} = \sqrt{5}$, so the foci are $\left(0, \pm\sqrt{5}\right)$.

18. The ellipse is centered at $(2, 1)$, with $a = 3$ and $b = 2$. An equation is $\dfrac{(x-2)^2}{9} + \dfrac{(y-1)^2}{4} = 1$. $c = \sqrt{a^2 - b^2} = \sqrt{5}$, so the foci are $\left(2 \pm \sqrt{5}, 1\right)$.

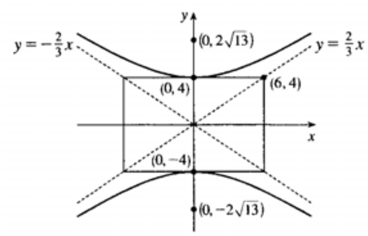
19. $\dfrac{x^2}{144} - \dfrac{y^2}{25} = 1 \quad\Rightarrow\quad a = 12, b = 5,$

$c = \sqrt{144 + 25} = 13 \quad\Rightarrow\quad$ center $(0, 0)$, vertices $(\pm 12, 0)$, foci $(\pm 13, 0)$, asymptotes $y = \pm\frac{5}{12}x$



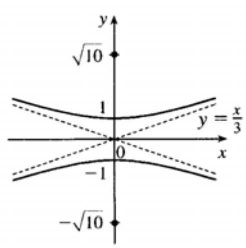
20. $\dfrac{y^2}{16} - \dfrac{x^2}{36} = 1 \quad\Rightarrow\quad a = 4, b = 6,$

$c = \sqrt{a^2 + b^2} = \sqrt{16 + 36} = \sqrt{52} = 2\sqrt{13}$. The center is $(0, 0)$, the vertices are $(0, \pm 4)$, the foci are $\left(0, \pm 2\sqrt{13}\right)$, and the asymptotes are the lines $y = \pm\frac{a}{b}x = \pm\frac{2}{3}x$.



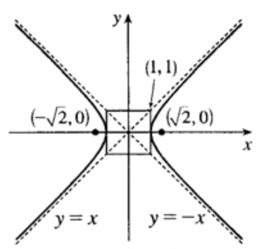
21. $9y^2 - x^2 = 9 \quad\Rightarrow\quad y^2 - \frac{1}{9}x^2 = 1 \quad\Rightarrow\quad a = 1,$

$b = 3, c = \sqrt{10} \quad\Rightarrow\quad$ center $(0, 0)$, vertices $(0, \pm 1)$, foci $\left(0, \pm\sqrt{10}\right)$, asymptotes $y = \pm\frac{1}{3}x$



22. $x^2 - y^2 = 1 \quad\Rightarrow\quad a = b = 1, c = \sqrt{2} \quad\Rightarrow$

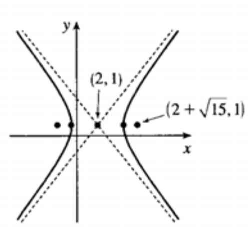
center $(0, 0)$, vertices $(\pm 1, 0)$, foci $\left(\pm\sqrt{2}, 0\right)$, asymptotes $y = \pm x$



23. $2y^2 - 4y - 3x^2 + 12x = -8 \quad\Leftrightarrow$

$\dfrac{(x-2)^2}{6} - \dfrac{(y-1)^2}{9} = 1 \quad\Rightarrow\quad a = \sqrt{6}, b = 3,$

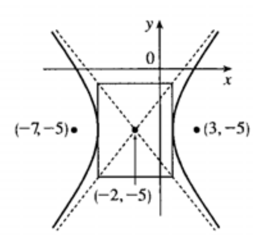
$c = \sqrt{15} \quad\Rightarrow\quad$ center $(2, 1)$, vertices $\left(2 \pm \sqrt{6}, 1\right)$, foci $\left(2 \pm \sqrt{15}, 1\right)$, asymptotes $y - 1 = \pm\frac{3}{\sqrt{6}}(x - 2)$



24. $16x^2 + 64x - 9y^2 - 90y = 305 \quad\Leftrightarrow$

$\dfrac{(x+2)^2}{9} - \dfrac{(y+5)^2}{16} = 1 \quad\Rightarrow\quad a = 3, b = 4,$

$c = 5 \quad\Rightarrow\quad$ center $(-2, -5)$, vertices $(-5, -5)$ and $(1, -5)$, foci $(-7, -5)$ and $(3, -5)$, asymptotes $y + 5 = \pm\frac{4}{3}(x + 2)$



25. The parabola with vertex $(0,0)$ and focus $(0,-2)$ opens downward and has $p=-2$, so its equation is $x^2=4py=-8y$.

26. The parabola with vertex $(1,0)$ and directrix $x=-5$ opens to the right and has $p=6$, so its equation is $y^2=4p(x-1)=24(x-1)$.

27. Vertex at $(2,0)$, $p=1$, opens to right $\Rightarrow$ $y^2=4p(x-2)=4(x-2)$

28. Vertex $(1,2)$, parabola opens down $\Rightarrow$ $p=-3$ $\Rightarrow$ $(x-1)^2=4p(y-2)=-12(y-2)$ $\Leftrightarrow$ $x^2-2x+12y-23=0$

29. The parabola must have equation $y^2=4px$, so $(-4)^2=4p(1)$ $\Rightarrow$ $p=4$ $\Rightarrow$ $y^2=16x$.

30. Vertical axis $\Rightarrow$ $(x-h)^2=4p(y-k)$. Substituting $(-2,3)$ and $(0,3)$ gives $(-2-h)^2=4p(3-k)$ and $(-h)^2=4p(3-k)$ $\Rightarrow$ $(-2-h)^2=(-h)^2$ $\Rightarrow$ $4+4h+h^2=h^2$ $\Rightarrow$ $h=-1$ $\Rightarrow$ $1=4p(3-k)$. Substituting $(1,9)$ gives $[1-(-1)]^2=4p(9-k)$ $\Rightarrow$ $4=4p(9-k)$. Solving for p from these equations gives $p=\dfrac{1}{4(3-k)}=\dfrac{1}{9-k}$ $\Rightarrow$ $4(3-k)=9-k$ $\Rightarrow$ $k=1$ $\Rightarrow$ $p=\frac{1}{8}$ $\Rightarrow$ $(x+1)^2=\frac{1}{2}(y-1)$ $\Rightarrow$ $2x^2+4x-y+3=0$.

31. The ellipse with foci $(\pm2,0)$ and vertices $(\pm5,0)$ has center $(0,0)$ and a horizontal major axis, with $a=5$ and $c=2$, so $b=\sqrt{a^2-c^2}=\sqrt{21}$. An equation is $\dfrac{x^2}{25}+\dfrac{y^2}{21}=1$.

32. The ellipse with foci $(0,\pm5)$ and vertices $(0,\pm13)$ has center $(0,0)$ and a vertical major axis, with $c=5$ and $a=13$, so $b=\sqrt{a^2-c^2}=12$. An equation is $\dfrac{x^2}{144}+\dfrac{y^2}{169}=1$.

33. Center $(3,0)$, $c=1$, $a=3$ $\Rightarrow$ $b=\sqrt{8}=2\sqrt{2}$ $\Rightarrow$ $\frac{1}{8}(x-3)^2+\frac{1}{9}y^2=1$

34. Center $(0,2)$, $c=1$, $a=3$, major axis horizontal $\Rightarrow$ $b=2\sqrt{2}$ and $\frac{1}{9}x^2+\frac{1}{8}(y-2)^2=1$

35. Center $(2,2)$, $c=2$, $a=3$ $\Rightarrow$ $b=\sqrt{5}$ $\Rightarrow$ $\frac{1}{9}(x-2)^2+\frac{1}{5}(y-2)^2=1$

36. Center $(0,0)$, $c=2$, major axis horizontal $\Rightarrow$ $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and $b^2=a^2-c^2=a^2-4$. Since the ellipse passes through $(2,1)$, we have $2a=|PF_1|+|PF_2|=\sqrt{17}+1$ $\Rightarrow$ $a^2=\frac{9+\sqrt{17}}{2}$ and $b^2=\frac{1+\sqrt{17}}{2}$, so the ellipse has equation $\dfrac{2x^2}{9+\sqrt{17}}+\dfrac{2y^2}{1+\sqrt{17}}=1$.

37. Center $(0,0)$, vertical axis, $c=3$, $a=1$ $\Rightarrow$ $b=\sqrt{8}=2\sqrt{2}$ $\Rightarrow$ $y^2-\frac{1}{8}x^2=1$

38. Center $(0,0)$, horizontal axis, $c=6$, $a=4$ $\Rightarrow$ $b=2\sqrt{5}$ $\Rightarrow$ $\frac{1}{16}x^2-\frac{1}{20}y^2=1$

39. Center $(4,3)$, horizontal axis, $c=3$, $a=2$ $\Rightarrow$ $b=\sqrt{5}$ $\Rightarrow$ $\frac{1}{4}(x-4)^2-\frac{1}{5}(y-3)^2=1$

40. Center $(2,3)$, vertical axis, $c=5$, $a=3$ $\Rightarrow$ $b=4$ $\Rightarrow$ $\frac{1}{9}(y-3)^2-\frac{1}{16}(x-2)^2=1$

41. Center $(0,0)$, horizontal axis, $a=3$, $\frac{b}{a}=2$ $\Rightarrow$ $b=6$ $\Rightarrow$ $\frac{1}{9}x^2-\frac{1}{36}y^2=1$

42. Center $(4,2)$, horizontal axis, asymptotes $y-2=\pm(x-4)$ $\Rightarrow$ $c=2$, $b/a=1$ $\Rightarrow$ $a=b$ $\Rightarrow$ $c^2=4=a^2+b^2=2a^2$ $\Rightarrow$ $a^2=2$ $\Rightarrow$ $\frac{1}{2}(x-4)^2-\frac{1}{2}(y-2)^2=1$

43. In Figure 8, we see that the point on the ellipse closest to a focus is the closer vertex (which is a distance $a-c$ from it) while the farthest point is the other vertex (at a distance of $a+c$). So for this lunar orbit, $(a-c)+(a+c)=2a=(1728+110)+(1728+314)$, or $a=1940$; and $(a+c)-(a-c)=2c=314-110$, or $c=102$. Thus, $b^2=a^2-c^2=3{,}753{,}196$, and the equation is $\dfrac{x^2}{3{,}763{,}600}+\dfrac{y^2}{3{,}753{,}196}=1$.

44. (a) Choose V to be the origin, with x-axis through V and F. Then F is $(p,0)$, A is $(p,5)$, so substituting A into the equation $y^2=4px$ gives $25=4p^2$ so $p=\frac{5}{2}$ and $y^2=10x$.

(b) $x=11 \quad\Rightarrow\quad y=\sqrt{110} \quad\Rightarrow\quad |CD|=2\sqrt{110}$

45. (a) Set up the coordinate system so that A is $(-200,0)$ and B is $(200,0)$.
$|PA|-|PB|=(1200)(980)=1{,}176{,}000\text{ ft}=\frac{2450}{11}\text{ mi}=2a \quad\Rightarrow\quad a=\frac{1225}{11}$, and $c=200$ so $b^2=c^2-a^2=\frac{3{,}339{,}375}{121} \quad\Rightarrow\quad \dfrac{121x^2}{1{,}500{,}625}-\dfrac{121y^2}{3{,}339{,}375}=1$.

(b) Due north of $B \quad\Rightarrow\quad x=200 \quad\Rightarrow\quad \dfrac{(121)(200)^2}{1{,}500{,}625}-\dfrac{121y^2}{3{,}339{,}375}=1 \quad\Rightarrow\quad y=\dfrac{133{,}575}{539}\approx 248$ mi

46. $|PF_1|-|PF_2|=\pm 2a \quad\Leftrightarrow\quad \sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=\pm 2a \quad\Leftrightarrow$

$\sqrt{(x+c)^2+y^2}=\sqrt{(x-c)^2+y^2}\pm 2a \quad\Leftrightarrow\quad (x+c)^2+y^2=(x-c)^2+y^2+4a^2\pm 4a\sqrt{(x-c)^2+y^2} \quad\Leftrightarrow$

$4cx-4a^2=\pm 4a\sqrt{(x-c)^2+y^2} \quad\Leftrightarrow\quad c^2x^2-2a^2cx+a^4=a^2\left(x^2-2cx+c^2+y^2\right) \quad\Leftrightarrow$

$\left(c^2-a^2\right)x^2-a^2y^2=a^2\left(c^2-a^2\right) \quad\Leftrightarrow\quad b^2x^2-a^2y^2=a^2b^2$ (where $b^2=c^2-a^2$) $\quad\Leftrightarrow\quad \dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

47. The function whose graph is the upper branch of this hyperbola is concave upward. The function is $y=f(x)=a\sqrt{1+\dfrac{x^2}{b^2}}=\dfrac{a}{b}\sqrt{b^2+x^2}$, so $y'=\dfrac{a}{b}x\left(b^2+x^2\right)^{-1/2}$ and

$y''=\dfrac{a}{b}\left[\left(b^2+x^2\right)^{-1/2}-x^2\left(b^2+x^2\right)^{-3/2}\right]=ab\left(b^2+x^2\right)^{-3/2}>0$ for all x, and so f is concave upward.

48. We can follow exactly the same sequence of steps as in the derivation of Formula 4, except we use the points $(1,1)$ and $(-1,-1)$ in the distance formula (first equation of that derivation) so $\sqrt{(x-1)^2+(y-1)^2}+\sqrt{(x+1)^2+(y+1)^2}=4$ will lead to $3x^2-2xy+3y^2=8$.

49. (a) ellipse

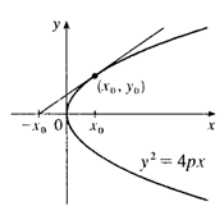
(b) hyperbola

(c) empty graph (no curve)

(d) In case (a), $a^2=k$, $b^2=k-16$, and $c^2=a^2-b^2=16$, so the foci are at $(\pm 4,0)$. In case (b), $k-16<0$, so $a^2=k$, $b^2=16-k$, and $c^2=a^2+b^2=16$, and so again the foci are at $(\pm 4,0)$.

50. (a) $y^2=4px \quad\Rightarrow\quad 2yy'=4p \quad\Rightarrow\quad y'=\dfrac{2p}{y}$, so the tangent line is

$y-y_0=\dfrac{2p}{y_0}(x-x_0) \quad\Rightarrow\quad yy_0-y_0^2=2p(x-x_0) \quad\Leftrightarrow$
$yy_0-4px_0=2px-2px_0 \quad\Rightarrow\quad yy_0=2p(x+x_0)$.



(b) The x-intercept is $-x_0$.

51. Use the parametrization $x = 2\cos t$, $y = \sin t$, $0 \le t \le 2\pi$ to get

$$L = 4\int_0^{\pi/2} \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = 4\int_0^{\pi/2} \sqrt{4\sin^2 t + \cos^2 t}\,dt = 4\int_0^{\pi/2} \sqrt{3\sin^2 t + 1}\,dt$$

Using Simpson's Rule with $n = 10$,

$$L \approx \tfrac{4}{3}\left(\tfrac{\pi}{20}\right)\left[f(0) + 4f\left(\tfrac{\pi}{20}\right) + 2f\left(\tfrac{\pi}{10}\right) + \cdots + 2f\left(\tfrac{2\pi}{5}\right) + 4f\left(\tfrac{9\pi}{20}\right) + f\left(\tfrac{\pi}{2}\right)\right]$$

with $f(t) = \sqrt{3\sin^2 t + 1}$, so $L \approx 9.69$.

52. The length of the major axis is $2a$, so $a = \frac{1}{2}\left(1.18 \times 10^{10}\right) = 5.9 \times 10^9$. The length of the minor axis is $2b$, so $b = \frac{1}{2}\left(1.14 \times 10^{10}\right) = 5.7 \times 10^9$. Therefore the equation of the ellipse is $\dfrac{x^2}{(5.9 \times 10^9)^2} + \dfrac{y^2}{(5.7 \times 10^9)^2} = 1$.

Converting into parametric equations, $x = 5.9 \times 10^9 \cos\theta$ and $y = 5.7 \times 10^9 \sin\theta$. So

$$\begin{aligned} L &= 4\int_0^{\pi/2} \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\,d\theta = 4\int_0^{\pi/2} \sqrt{3.48 \times 10^{19}\sin^2\theta + 3.249 \times 10^{19}\cos^2\theta}\,d\theta \\ &= 4\sqrt{3.249 \times 10^{19}}\int_0^{\pi/2} \sqrt{1.0714\sin^2\theta + \cos^2\theta}\,d\theta \end{aligned}$$

Using Simpson's Rule with $n = 10$, $\Delta x = \frac{\pi/2}{10} = \frac{\pi}{20}$ and $f(\theta) = \sqrt{1.0714\sin^2\theta + \cos^2\theta}$ we get

$$\begin{aligned} L &\approx 4\left(5.7 \times 10^9\right) \cdot S_{10} \\ &= 4\left(5.7 \times 10^9\right)\tfrac{\pi}{20 \cdot 3}\left[f(0) + 4f\left(\tfrac{\pi}{20}\right) + 2f\left(\tfrac{\pi}{10}\right) + \cdots + 2f\left(\tfrac{2\pi}{5}\right) + 4f\left(\tfrac{9\pi}{20}\right) + f\left(\tfrac{\pi}{2}\right)\right] \\ &\approx \tfrac{\pi}{15}\left(5.7 \times 10^9\right)(30.529) \approx 3.64 \times 10^{10}\text{ km} \end{aligned}$$

53. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \;\Rightarrow\; \dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0 \;\Rightarrow\; y' = -\dfrac{b^2 x}{a^2 y}$ $(y \neq 0)$. Thus, the slope of the tangent line at P is $-\dfrac{b^2 x_1}{a^2 y_1}$. The slope of $F_1 P$ is $\dfrac{y_1}{x_1 + c}$ and of $F_2 P$ is $\dfrac{y_1}{x_1 - c}$. By the formula from Problems Plus, we have

$$\begin{aligned} \tan\alpha &= \frac{\dfrac{y_1}{x_1 + c} + \dfrac{b^2 x_1}{a^2 y_1}}{1 - \dfrac{b^2 x_1 y_1}{a^2 y_1 (x_1 + c)}} = \frac{a^2 y_1^2 + b^2 x_1 (x_1 + c)}{a^2 y_1 (x_1 + c) - b^2 x_1 y_1} = \frac{a^2 b^2 + b^2 c x_1}{c^2 x_1 y_1 + a^2 c y_1} \quad \left(\begin{array}{c}\text{using } b^2 x_1^2 + a^2 y_1^2 = a^2 b^2 \\ \text{and } a^2 - b^2 = c^2\end{array}\right) \\ &= \frac{b^2 (c x_1 + a^2)}{c y_1 (c x_1 + a^2)} = \frac{b^2}{c y_1} \end{aligned}$$

and

$$\tan\beta = \frac{-\dfrac{y_1}{x_1 - c} - \dfrac{b^2 x_1}{a^2 y_1}}{1 - \dfrac{b^2 x_1 y_1}{a^2 y_1 (x_1 - c)}} = \frac{-a^2 y_1^2 - b^2 x_1 (x_1 - c)}{a^2 y_1 (x_1 - c) - b^2 x_1 y_1} = \frac{-a^2 b^2 + b^2 c x_1}{c^2 x_1 y_1 - a^2 c y_1} = \frac{b^2 (c x_1 - a^2)}{c y_1 (c x_1 - a^2)} = \frac{b^2}{c y_1}$$

So $\alpha = \beta$.

54. The slopes of the line segments F_1P and F_2P are $\dfrac{y_1}{x_1+c}$ and $\dfrac{y_1}{x_1-c}$, where P is (x_1, y_1). Differentiating implicitly, $\dfrac{2x}{a^2} - \dfrac{2yy'}{b^2} = 0 \Rightarrow y' = \dfrac{b^2x}{a^2y} \Rightarrow$ the slope of the tangent at P is $\dfrac{b^2x_1}{a^2y_1}$, so by the formula from Problems Plus,

$$\tan\alpha = \frac{\dfrac{b^2x_1}{a^2y_1} - \dfrac{y_1}{x_1+c}}{1 + \dfrac{b^2x_1y_1}{a^2y_1(x_1+c)}} = \frac{b^2x_1(x_1+c) - a^2y_1^2}{a^2y_1(x_1+c) + b^2x_1y_1}$$

$$= \frac{b^2(cx_1+a^2)}{cy_1(cx_1+a^2)} \left(\begin{array}{c}\text{using } x_1^2/a^2 - y_1^2/b^2 = 1 \\ \text{and } a^2+b^2=c^2\end{array}\right) = \frac{b^2}{cy_1}$$

and

$$\tan\beta = \frac{-\dfrac{b^2x_1}{a^2y_1} + \dfrac{y_1}{x_1-c}}{1 + \dfrac{b^2x_1y_1}{a^2y_1(x_1-c)}} = \frac{-b^2x_1(x_1-c) + a^2y_1^2}{a^2y_1(x_1-c) + b^2x_1y_1} = \frac{b^2(cx_1-a^2)}{cy_1(cx_1-a^2)} = \frac{b^2}{cy_1}$$

So $\alpha = \beta$.

11.7 Conic Sections in Polar Coordinates

1. The directrix $y = 6$ is above the focus at the origin, so we use the form with "$+\, e\sin\theta$" in the denominator.
$r = \dfrac{ed}{1+e\sin\theta} = \dfrac{\frac{7}{4}\cdot 6}{1+\frac{7}{4}\sin\theta} = \dfrac{42}{4+7\sin\theta}$

2. The directrix $x = 4$ is to the right of the focus at the origin, so we use the form with "$+\, e\cos\theta$" in the denominator. $e = 1$ for a parabola, so an equation is $r = \dfrac{ed}{1+e\cos\theta} = \dfrac{1\cdot 4}{1+1\cos\theta} = \dfrac{4}{1+\cos\theta}$

3. The directrix $x = -5$ is to the left of the focus at the origin, so we use the form with "$-\, e\cos\theta$" in the denominator. $r = \dfrac{ed}{1-e\cos\theta} = \dfrac{\frac{3}{4}\cdot 5}{1-\frac{3}{4}\cos\theta} = \dfrac{15}{4-3\cos\theta}$

4. The directrix $y = -2$ is below the focus at the origin, so we use the form with "$-\, e\sin\theta$" in the denominator.
$r = \dfrac{ed}{1-e\sin\theta} = \dfrac{2\cdot 2}{1-2\sin\theta} = \dfrac{4}{1-2\sin\theta}$

5. $r = 5\sec\theta \Leftrightarrow x = r\cos\theta = 5$, so $r = \dfrac{ed}{1+e\cos\theta} = \dfrac{4\cdot 5}{1+4\cos\theta} = \dfrac{20}{1+4\cos\theta}$

6. $r = 2\csc\theta \Leftrightarrow y = r\sin\theta = 2$, so $r = \dfrac{ed}{1+e\sin\theta} = \dfrac{\frac{3}{5}\cdot 2}{1+\frac{3}{5}\sin\theta} = \dfrac{6}{5+3\sin\theta}$

7. Focus $(0,0)$, vertex $\left(5, \frac{\pi}{2}\right) \Rightarrow$ directrix $y = 10 \Rightarrow r = \dfrac{ed}{1+e\sin\theta} = \dfrac{10}{1+\sin\theta}$

8. The directrix is $x = 4$, so $r = \dfrac{ed}{1+e\cos\theta} = \dfrac{\frac{2}{5}\cdot 4}{1+\frac{2}{5}\cos\theta} = \dfrac{8}{5+2\cos\theta}$

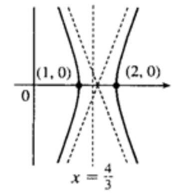
9. $r = \dfrac{4}{1 + 3\cos\theta}$

(a) $e = 3$

(b) Since $e = 3 > 1$, the conic is a hyperbola.

(c) $ed = 4 \quad \Rightarrow \quad d = \frac{4}{3} \quad \Rightarrow \quad$ directrix $x = \frac{4}{3}$

(d) The vertices are $(1, 0)$ and $(-2, \pi) = (2, 0)$; the center is $\left(\frac{3}{2}, 0\right)$; the asymptotes are parallel to $\theta = \pm\cos^{-1}\left(-\frac{1}{3}\right)$



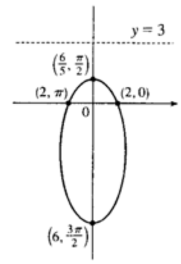
10. $r = \dfrac{6}{3 + 2\sin\theta} = \dfrac{2}{1 + \frac{2}{3}\sin\theta} = \dfrac{\frac{2}{3}\cdot 3}{1 + \frac{2}{3}\sin\theta}$

(a) $e = \frac{2}{3}$

(b) Ellipse

(c) $y = 3$

(d) Vertices $\left(\frac{6}{5}, \frac{\pi}{2}\right)$ and $\left(6, \frac{3\pi}{2}\right)$; center $\left(\frac{12}{5}, \frac{3\pi}{2}\right)$



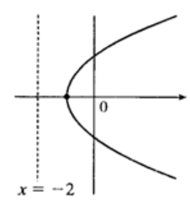
11. $r = \dfrac{2}{1 - \cos\theta}$

(a) $e = 1$

(b) Parabola

(c) $ed = 2 \quad \Rightarrow \quad d = 2 \quad \Rightarrow \quad$ directrix $x = -2$

(d) Vertex $(-1, 0) = (1, \pi)$



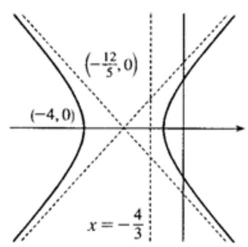
12. $r = \dfrac{8}{4 - 6\cos\theta} = \dfrac{2}{1 - \frac{3}{2}\cos\theta} = \dfrac{\frac{3}{2}\cdot\frac{4}{3}}{1 - \frac{3}{2}\cos\theta}$

(a) $e = \frac{3}{2}$

(b) Hyperbola

(c) $x = -\frac{4}{3}$

(d) The vertices are $(-4, 0)$ and $\left(\frac{4}{5}, \pi\right) = \left(-\frac{4}{5}, 0\right)$, so the center is $\left(-\frac{12}{5}, 0\right)$. The asymptotes are parallel to $\theta = \pm\cos^{-1}\frac{2}{3}$. [Their slopes are $\pm\tan\left(\cos^{-1}\frac{2}{3}\right) = \pm\frac{\sqrt{5}}{2}$.]



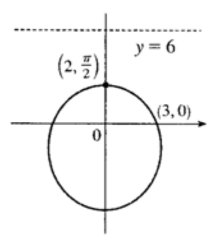
13. $r = \dfrac{3}{1 + \frac{1}{2}\sin\theta}$

(a) $e = \frac{1}{2}$

(b) Ellipse

(c) $ed = 3 \quad \Rightarrow \quad d = 6 \quad \Rightarrow \quad$ directrix $y = 6$

(d) Vertices $\left(2, \frac{\pi}{2}\right)$ and $\left(6, \frac{3\pi}{2}\right)$; center $\left(2, \frac{3\pi}{2}\right)$



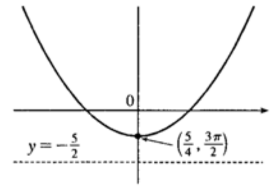
14. $r = \dfrac{5}{2 - 2\sin\theta} = \dfrac{\frac{5}{2}}{1 - \sin\theta}$

(a) $e = 1$

(b) Parabola

(c) $y = -\frac{5}{2}$

(d) The focus is $(0, 0)$, so the vertex is $\left(\frac{5}{4}, \frac{3\pi}{2}\right)$ and the parabola opens up.



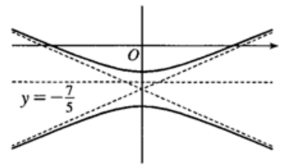
15. $r = \dfrac{7/2}{1 - \frac{5}{2}\sin\theta}$

(a) $e = \frac{5}{2}$

(b) Hyperbola

(c) $ed = \frac{7}{2} \;\Rightarrow\; d = \frac{7}{5} \;\Rightarrow\;$ directrix $y = -\frac{7}{5}$

(d) Center $\left(\frac{5}{3}, \frac{3\pi}{2}\right)$; vertices $\left(-\frac{7}{3}, \frac{\pi}{2}\right) = \left(\frac{7}{3}, \frac{3\pi}{2}\right)$ and $\left(1, \frac{3\pi}{2}\right)$



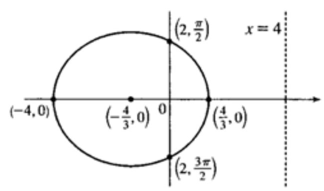
16. $r = \dfrac{4}{2 + \cos\theta} = \dfrac{2}{1 + \frac{1}{2}\cos\theta} = \dfrac{\frac{1}{2} \cdot 4}{1 + \frac{1}{2}\cos\theta}$

(a) $e = \frac{1}{2}$

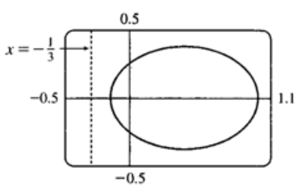
(b) Ellipse

(c) $x = 4$

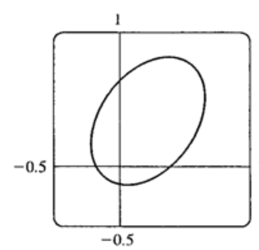
(d) The vertices are $\left(\frac{4}{3}, 0\right)$ and $(4, \pi) = (-4, 0)$, so the center is $\left(-\frac{4}{3}, 0\right)$.



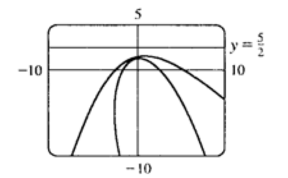
17. (a) The equation is $r = \dfrac{1}{4 - 3\cos\theta} = \dfrac{1/4}{1 - \frac{3}{4}\cos\theta}$, so $e = \frac{3}{4}$ and $ed = \frac{1}{4} \;\Rightarrow\; d = \frac{1}{3}$. The conic is an ellipse, and the equation of its directrix is $x = r\cos\theta = -\frac{1}{3} \;\Rightarrow\; r = -\dfrac{1}{3\cos\theta}$. We must be careful in our choice of parameter values in this equation ($-1 \le \theta \le 1$ works well).



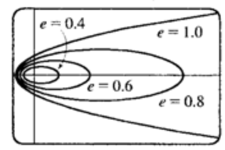
(b) The equation is obtained by replacing θ with $\theta - \frac{\pi}{3}$ in the equation of the original conic (see Example 4), so $r = \dfrac{1}{4 - 3\cos\left(\theta - \frac{\pi}{3}\right)}$.



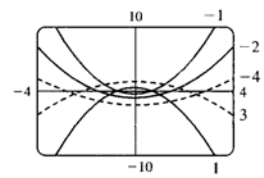
18. $r = \dfrac{5}{2 + 2\sin\theta} = \dfrac{5/2}{1 + \sin\theta}$, so $e = 1$ and $d = \frac{5}{2}$. The equation of the directrix is $y = r\sin\theta = \frac{5}{2} \Rightarrow r = \dfrac{5}{2\sin\theta}$. If the parabola is rotated about its focus (the origin) through $\frac{\pi}{6}$, its equation is the same as that of the original, with θ replaced by $\theta - \frac{\pi}{6}$ (see Example 4), so $r = \dfrac{5}{2 + 2\sin(\theta - \pi/6)}$. In graphing each of these curves, we must be careful to select parameter ranges which prevent the denominator from vanishing while still showing enough of the curve.



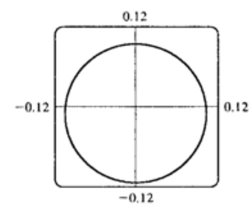
19. For $e < 1$ the curve is an ellipse. It is nearly circular when e is close to 0. As e increases, the graph is stretched out to the right, and grows larger (that is, its right-hand focus moves to the right while its left-hand focus remains at the origin.) At $e = 1$, the curve becomes a parabola with focus at the origin.



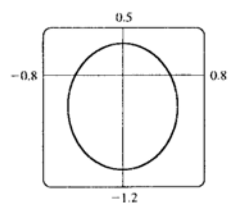
20. (a) The value of d does not seem to affect the shape of the conic (a parabola) at all, just its size, position, and orientation (for $d < 0$ it opens upward, for $d > 0$ it opens downward).



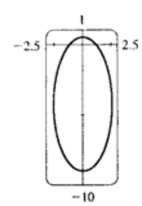
(b) We consider only positive values of e. When $0 < e < 1$, the conic is an ellipse. As $e \to 0^+$, the graph approaches perfect roundness and zero size. As e increases, the ellipse becomes more elongated, until at $e = 1$ it turns into a parabola. For $e > 1$, the conic is a hyperbola, which moves downward and gets broader as e continues to increase.



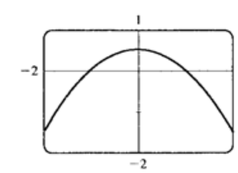
$e = 0.1$



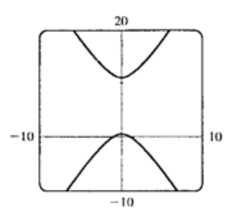
$e = 0.5$



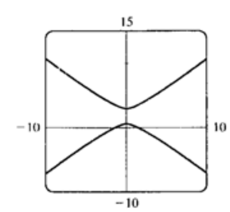
$e = 0.9$



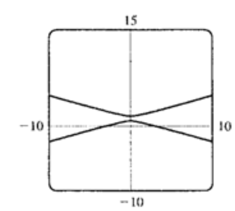
$e = 1$



$e = 1.1$



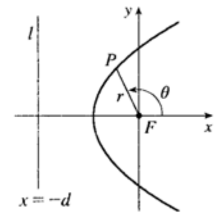
$e = 1.5$



$e = 10$

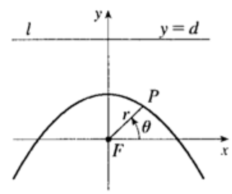
21. $|PF| = e\,|Pl| \;\Rightarrow\; r = e\,[d - r\cos(\pi - \theta)] = e\,(d + r\cos\theta) \;\Rightarrow$

$r\,(1 - e\cos\theta) = ed \;\Rightarrow\; r = \dfrac{ed}{1 - e\cos\theta}$



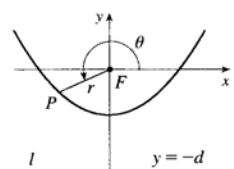
22. $|PF| = e\,|Pl| \;\Rightarrow\; r = e\,[d - r\sin\theta] \;\Rightarrow\; r\,(1 + e\sin\theta) = ed \;\Rightarrow$

$r = \dfrac{ed}{1 + e\sin\theta}$



23. $|PF| = e\,|Pl| \;\Rightarrow\; r = e\,[d - r\sin(\theta - \pi)] = e\,(d + r\sin\theta) \;\Rightarrow$

$r\,(1 - e\sin\theta) = ed \;\Rightarrow\; r = \dfrac{ed}{1 - e\sin\theta}$



24. The parabolas intersect at the two points where $\dfrac{c}{1+\cos\theta} = \dfrac{d}{1-\cos\theta} \;\Rightarrow\; \cos\theta = \dfrac{c-d}{c+d} \;\Rightarrow\; r = \dfrac{c+d}{2}$.

For the first parabola, $\dfrac{dr}{d\theta} = \dfrac{c\sin\theta}{(1+\cos\theta)^2}$, so

$$\frac{dy}{dx} = \frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta} = \frac{c\sin^2\theta + c\cos\theta\,(1+\cos\theta)}{c\sin\theta\cos\theta - c\sin\theta\,(1+\cos\theta)} = \frac{1+\cos\theta}{-\sin\theta}$$

and similarly for the second, $\dfrac{dy}{dx} = \dfrac{1-\cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1+\cos\theta}$. Since the product of these slopes is -1, the parabolas intersect at right angles.

25. (a) If the directrix is $x = -d$, then $r = \dfrac{ed}{1 - e\cos\theta}$ [see Figure 2(b)], and, from (4), $a^2 = \dfrac{e^2d^2}{\left(1-e^2\right)^2} \;\Rightarrow$

$ed = a\left(1 - e^2\right)$. Therefore, $r = \dfrac{a\left(1-e^2\right)}{1 - e\cos\theta}$.

(b) $e = 0.017$ and the major axis $= 2a = 2.99 \times 10^8 \;\Rightarrow\; a = 1.495 \times 10^8$.

Therefore $r = \dfrac{1.495 \times 10^8\left[1 - (0.017)^2\right]}{1 - 0.017\cos\theta} \approx \dfrac{1.49 \times 10^8}{1 - 0.017\cos\theta}$.

26. (a) At perihelion, $\theta = \pi$, so $r = \dfrac{a\left(1-e^2\right)}{1 - e\cos\pi} = \dfrac{a\left(1-e^2\right)}{1 - e\,(-1)} = \dfrac{a\,(1-e)\,(1+e)}{1+e} = a\,(1-e)$.

At aphelion, $\theta = 0$, so $r = \dfrac{a\left(1-e^2\right)}{1 - e\cos 0} = \dfrac{a\,(1-e)\,(1+e)}{1-e} = a\,(1+e)$.

(b) At perihelion, $r = a\,(1-e) \approx \left(1.495 \times 10^8\right)(1 - 0.017) \approx 1.47 \times 10^8$ km. At aphelion,

$r = a\,(1+e) \approx \left(1.495 \times 10^8\right)(1 + 0.017) \approx 1.52 \times 10^8$ km.

27. Here $2a =$ length of major axis $= 36.18$ AU $\Rightarrow$ $a = 18.09$ AU and $e = 0.97$. By Exercise 25(a), the equation of the orbit is $r = \dfrac{18.09\left[1-(0.97)^2\right]}{1-0.97\cos\theta} \approx \dfrac{1.07}{1-0.97\cos\theta}$. By Exercise 26(a), the maximum distance from the comet to the sun is $18.09\,(1+0.97) \approx 35.64$ AU or about 3.314 billion miles.

28. Here $2a =$ length of major axis $= 356.5$ AU $\Rightarrow$ $a = 178.25$ AU and $e = 0.9951$. By Exercise 25(a), the equation of the orbit is $r = \dfrac{178.25\left[1-(0.9951)^2\right]}{1-0.9951\cos\theta} \approx \dfrac{1.7426}{1-0.9951\cos\theta}$. By Exercise 26(a), the minimum distance from the comet to the sun is $178.25\,(1-0.9951) \approx 0.8734$ AU or about 81 million miles.

29. The minimum distance is at perihelion where $4.6\times 10^7 = r = a\,(1-e) = a\,(1-0.206) = a\,(0.794)$ $\Rightarrow$ $a = 4.6\times 10^7/0.794$. So the maximum distance, which is at aphelion, is $r = a\,(1+e) = \left(4.6\times 10^7/0.794\right)\times 10^7\,(1.206) \approx 7.0\times 10^7$ km.

30. At perihelion, $r = a\,(1-e) = 4.43\times 10^9$, and at aphelion, $r = a\,(1+e) = 7.37\times 10^9$. Adding, we get $2a = 11.80\times 10^9$, so $a = 5.90\times 10^9$ km. Therefore $1+e = a\,(1+e)\,/e = \frac{7.37}{5.90} \approx 1.249$ and $e \approx 0.249$.

31. From Exercise 29, we have $e = 0.206$ and $a\,(1-e) = 4.6\times 10^7$ km. Thus, $a = 4.6\times 10^7/0.794$. From Exercise 25, we can write the equation of Mercury's orbit as $r = a\dfrac{1-e^2}{1-e\cos\theta}$. So since

$$\frac{dr}{d\theta} = \frac{-a\left(1-e^2\right)e\sin\theta}{(1-e\cos\theta)^2} \quad\Rightarrow$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \frac{a^2\left(1-e^2\right)^2}{(1-e\cos\theta)^2} + \frac{a^2\left(1-e^2\right)^2 e^2\sin^2\theta}{(1-e\cos\theta)^4} = \frac{a^2\left(1-e^2\right)^2}{(1-e\cos\theta)^4}\left(1-2e\cos\theta+e^2\right)$$

the length of the orbit is

$$L = \int_0^{2\pi}\sqrt{r^2+(dr/d\theta)^2}\,d\theta = a\left(1-e^2\right)\int_0^{2\pi}\frac{\sqrt{1+e^2-2e\cos\theta}}{(1-e\cos\theta)^2}\,d\theta \approx 3.6\times 10^8\text{ km}$$

This seems reasonable, since Mercury's orbit is nearly circular, and the circumference of a circle of radius a is $2\pi a \approx 3.6\times 10^8$ km.

11 Review

CONCEPT CHECK

1. (a) A parametric curve is a set of points of the form $(x, y) = (f(t), g(t))$, where f and g are continuous functions of a variable t.

(b) Sketching a parametric curve, like sketching the graph of a function, is difficult to do in general. We can plot points on the curve by finding $f(t)$ and $g(t)$ for various values of t, either by hand or with a calculator or computer. Sometimes, when f and g are given by formulas, we can eliminate t from the equations $x = f(t)$ and $y = g(t)$ to get a Cartesian equation relating x and y. It may be easier to graph that equation than to work with the original formulas for x and y in terms of t.

2. (a) You can find $\dfrac{dy}{dx}$ as a function of t by calculating $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ (if $dx/dt \neq 0$).

(b) Calculate the area as $\int_a^b y\,dx = \int_\alpha^\beta g(t)\,f'(t)\,dt$ [or $\int_\beta^\alpha g(t)\,f'(t)\,dt$ if the leftmost point is $(f(\beta), g(\beta))$ rather than $(f(\alpha), g(\alpha))$].

3. (a) $L = \int_\alpha^\beta \sqrt{(dx/dt)^2+(dy/dt)^2}\,dt = \int_\alpha^\beta \sqrt{[f'(t)]^2+[g'(t)]^2}\,dt$

(b) $S = \int_\alpha^\beta 2\pi y\sqrt{(dx/dt)^2+(dy/dt)^2}\,dt = \int_\alpha^\beta 2\pi g(t)\sqrt{[f'(t)]^2+[g'(t)]^2}\,dt$

4. (a) See Figure 5 in Section 11.4.

(b) $x = r\cos\theta$, $y = r\sin\theta$

(c) To find a polar representation (r, θ) with $r \geq 0$ and $0 \leq \theta < 2\pi$, first calculate $r = \sqrt{x^2 + y^2}$. Then θ is specified by $\cos\theta = x/r$ and $\sin\theta = y/r$.

5. (a) Calculate $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{(d/d\theta)(r\sin\theta)}{(d/d\theta)(r\cos\theta)} = \dfrac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta}$, where $r = f(\theta)$.

(b) Calculate $A = \int_a^b \frac{1}{2}r^2\,d\theta = \int_a^b \frac{1}{2}[f(\theta)]^2\,d\theta$

(c) $L = \int_a^b \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\,d\theta = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_a^b \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta$

6. (a) A parabola is a set of points in a plane whose distances from a fixed point F (the focus) and a fixed line ℓ (the directrix) are equal.

(b) $x^2 = 4py$; $y^2 = 4px$

7. (a) An ellipse is a set of points in a plane the sum of whose distances from two fixed points (the foci) is a constant.

(b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - c^2} = 1$.

8. (a) A hyperbola is a set of points in a plane the difference of whose distances from two fixed points (the foci) is a constant.

(b) $\dfrac{x^2}{a^2} - \dfrac{y^2}{c^2 - a^2} = 1$

(c) $y = \pm\dfrac{\sqrt{c^2 - a^2}}{a}x$

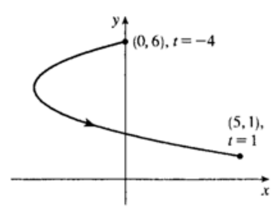
9. (a) If a conic section has focus F and corresponding directrix ℓ, then the eccentricity e is the fixed ratio $|PF|/|P\ell|$ for points P of the conic section.

(b) $e < 1$ for an ellipse; $e > 1$ for a hyperbola; $e = 1$ for a parabola.

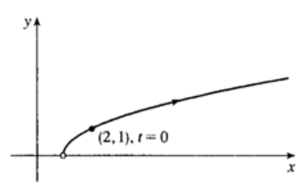
(c) $x = d$: $r = \dfrac{ed}{1 + e\cos\theta}$. $x = -d$: $r = \dfrac{ed}{1 - e\cos\theta}$. $y = d$: $r = \dfrac{ed}{1 + e\sin\theta}$. $y = -d$: $r = \dfrac{ed}{1 - e\sin\theta}$.

EXERCISES

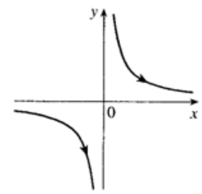
1. $x = t^2 + 4t$, $y = 2 - t$, $-4 \leq t \leq 1$. $t = 2 - y$, so $x = (2 - y)^2 + 4(2 - y) = 4 - 4y + y^2 + 8 - 4y = y^2 - 8y + 12$. This is part of a parabola with vertex $(-4, 4)$, opening to the right.



2. $x = 1 + e^{2t}$, $y = e^t$. $x = 1 + y^2$, $y > 0$.

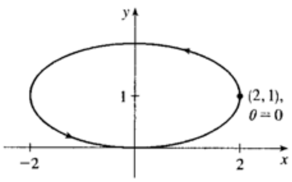


3. $x = \tan\theta$, $y = \cot\theta$. $y = 1/\tan\theta = 1/x$. The whole curve is traced out as θ ranges over the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ [or any open interval of the form $\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$, where n is an integer].

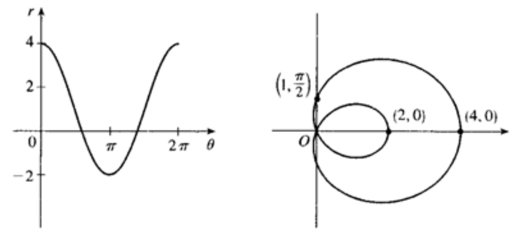


4. $x = 2\cos\theta$, $y = 1 + \sin\theta$, $\cos^2\theta + \sin^2\theta = 1 \quad \Rightarrow$

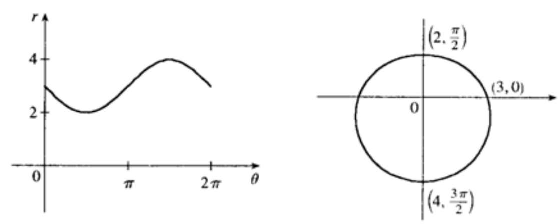
$\left(\frac{x}{2}\right)^2 + (y-1)^2 = 1 \quad \Rightarrow \quad \frac{x^2}{4} + (y-1)^2 = 1$. This is an ellipse, centered at $(0, 1)$, with semimajor axis of length 2 and semiminor axis of length 1.



5. $r = 1 + 3\cos\theta$



6. $r = 3 - \sin\theta$

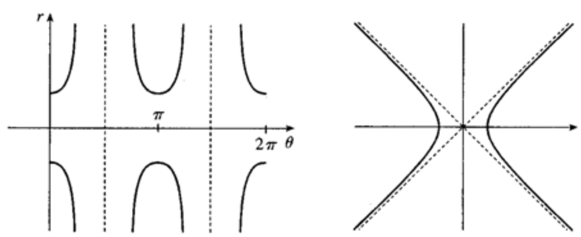


7. $r^2 = \sec 2\theta \quad \Rightarrow \quad r^2\cos 2\theta = 1$

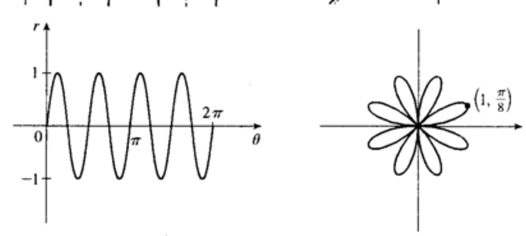
$\Rightarrow \quad r^2\left(\cos^2\theta - \sin^2\theta\right) = 1$

$\Rightarrow \quad r^2\cos^2\theta - r^2\sin^2\theta = 1$

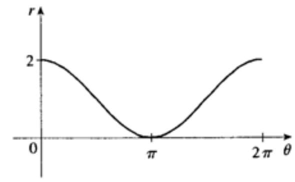
$\Rightarrow \quad x^2 - y^2 = 1$, a hyperbola

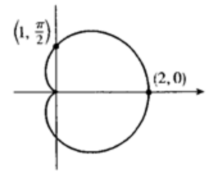


8. $r = \sin 4\theta$. This is an eight-leafed rose.

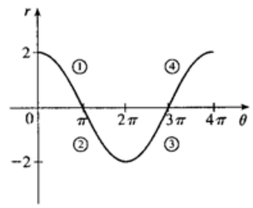


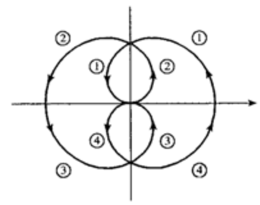
9. $r = 2\cos^2(\theta/2) = 1 + \cos\theta$



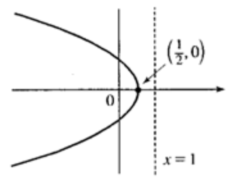


10. $r = 2\cos(\theta/2)$. The curve is symmetric about the pole and both the horizontal and vertical axes.

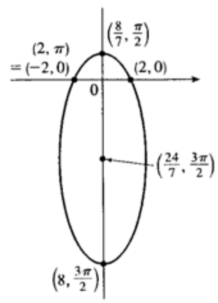




11. $r = \dfrac{1}{1+\cos\theta} \;\Rightarrow\; e = 1 \;\Rightarrow\;$ parabola; $d = 1 \;\Rightarrow\;$ directrix $x = 1$ and vertex $\left(\frac{1}{2}, 0\right)$; y-intercepts are $\left(1, \frac{\pi}{2}\right)$ and $\left(1, \frac{3\pi}{2}\right)$.



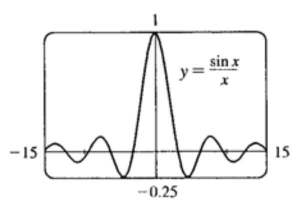
12. $r = \dfrac{8}{4+3\sin\theta} = \dfrac{\frac{3}{4}\cdot\frac{8}{3}}{1+\frac{3}{4}\sin\theta}$. This is an ellipse with focus at the pole, eccentricity $\frac{3}{4}$, and directrix $y = \frac{8}{3}$. The center is $\left(\frac{24}{7}, \frac{3\pi}{2}\right)$.

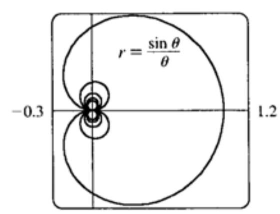


13. $x + y = 2 \;\Leftrightarrow\; r\cos\theta + r\sin\theta = 2 \;\Leftrightarrow\; r(\cos\theta + \sin\theta) = 2 \;\Leftrightarrow\; r = \dfrac{2}{\cos\theta + \sin\theta}$

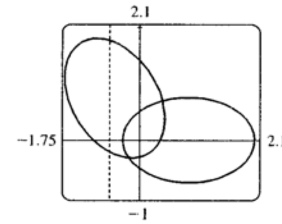
14. $x^2 + y^2 = 2 \;\Rightarrow\; r^2 = 2 \;\Rightarrow\; r = \sqrt{2}$. ($r = -\sqrt{2}$ gives the same curve.)

15. $r = (\sin\theta)/\theta$. As $\theta \to \pm\infty$, $r \to 0$.





16. $r = \dfrac{2}{4 - 3\cos\theta} = \dfrac{1/2}{1 - \frac{3}{4}\cos\theta} \Rightarrow e = \frac{3}{4}$ and $d = \frac{2}{3}$. The equation of the directrix is $x = r\cos\theta = -\frac{2}{3} \Rightarrow r = -2/(3\cos\theta)$. To obtain the equation of the rotated ellipse, we replace θ in the original equation with $\theta - \frac{2\pi}{3}$, and get $r = \dfrac{2}{4 - 3\cos\left(\theta - \frac{2\pi}{3}\right)}$.



17. $x = \ln t$, $y = 1 + t^2$; $t = 1$. $\dfrac{dy}{dt} = 2t$ and $\dfrac{dx}{dt} = \dfrac{1}{t}$, so $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t}{1/t} = 2t^2$. When $t = 1$, $\dfrac{dy}{dx} = 2$.

18. $x = te^t$, $y = 1 + \sqrt{1+t}$. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1/\left(2\sqrt{1+t}\right)}{(1+t)\,e^t} = \dfrac{1}{2}$ when $t = 0$.

19. $\dfrac{dy}{dx} = \dfrac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta} = \dfrac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta} = \dfrac{\frac{1}{\sqrt{2}} + \frac{\pi}{4}\cdot\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}} - \frac{\pi}{4}\cdot\frac{1}{\sqrt{2}}} = \dfrac{4+\pi}{4-\pi}$ when $\theta = \frac{\pi}{4}$.

20. $\dfrac{dy}{dx} = \dfrac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta} = \dfrac{-2\cos\theta\sin\theta + (3 - 2\sin\theta)\cos\theta}{-2\cos^2\theta - (3 - 2\sin\theta)\sin\theta} = \dfrac{3\cos\theta - 2\sin 2\theta}{-3\sin\theta - 2\cos 2\theta} = 0$ when $\theta = \frac{\pi}{2}$.

21. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{t\cos t + \sin t}{-t\sin t + \cos t}$. $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$, where

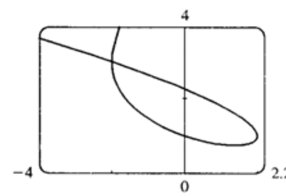
$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{(-t\sin t + \cos t)(-t\sin t + 2\cos t) - (t\cos t + \sin t)(-t\cos t - 2\sin t)}{(-t\sin t + \cos t)^2} = \frac{t^2+2}{(-t\sin t + \cos t)^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{t^2+2}{(-t\sin t + \cos t)^3}.$$

22. $x = 1 + t^2$, $y = t - t^3$. $\dfrac{dy}{dt} = 1 - 3t^2$ and $\dfrac{dx}{dt} = 2t$, so $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 - 3t^2}{2t} = \frac{1}{2}t^{-1} - \frac{3}{2}t$.

$$\frac{d^2y}{dx^2} = \frac{d\,(dy/dx)/dt}{dx/dt} = \frac{-\frac{1}{2}t^{-2} - \frac{3}{2}}{2t} = -\tfrac{1}{4}t^{-3} - \tfrac{3}{4}t^{-1} = -\frac{1}{4t^3}\left(1 + 3t^2\right) = -\frac{3t^2+1}{4t^3}.$$

23. We graph the curve for $-2.2 \le t \le 1.2$. By zooming in or using a cursor, we find that the lowest point is about $(1.4, 0.75)$. To find the exact values, we find the t-value at which $dy/dt = 2t + 1 = 0 \Leftrightarrow t = -\frac{1}{2} \Leftrightarrow$ $(x, y) = \left(\frac{11}{8}, \frac{3}{4}\right)$.



24. We estimate the coordinates of the point of intersection to be $(-2, 3)$. In fact this is exact, since both $t = -2$ and $t = 1$ give the point $(-2, 3)$. So the area enclosed by the loop is

$$\int_{t=-2}^{t=1} y\,dx = \int_{-2}^{1}\left(t^2 + t + 1\right)\left(3t^2 - 3\right)dt = \int_{-2}^{1}\left(3t^4 + 3t^3 - 3t - 3\right)dt$$

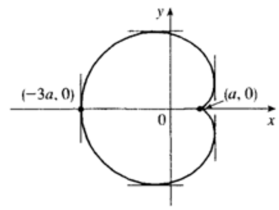
$$= \left[\tfrac{3}{5}t^5 + \tfrac{3}{4}t^4 - \tfrac{3}{2}t^2 - 3t\right]_{-2}^{1} = \left(\tfrac{3}{5} + \tfrac{3}{4} - \tfrac{3}{2} - 3\right) - \left[-\tfrac{96}{5} + 12 - 6 - (-6)\right] = \tfrac{81}{20}$$

25. $\dfrac{dx}{dt} = -2a\sin t + 2a\sin 2t = 2a\sin t\,(2\cos t - 1) = 0 \Leftrightarrow \sin t = 0$ or $\cos t = \frac{1}{2} \Rightarrow t = 0, \frac{\pi}{3}, \pi$, or $\frac{5\pi}{3}$.

$\dfrac{dy}{dt} = 2a\cos t - 2a\cos 2t = 2a\left(1 + \cos t - 2\cos^2 t\right) = 2a\,(1 - \cos t)\,(1 + 2\cos t) = 0 \Rightarrow t = 0, \frac{2\pi}{3}$, or $\frac{4\pi}{3}$.

Thus the graph has vertical tangents where $t = \frac{\pi}{3}, \pi$ and $\frac{5\pi}{3}$, and horizontal tangents where $t = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$. To determine what the slope is where $t = 0$, we use l'Hospital's Rule to evaluate $\lim\limits_{t\to 0} \dfrac{dy/dt}{dx/dt} = 0$, so there is a horizontal tangent there.

t	x	y
0	a	0
$\frac{\pi}{3}$	$\frac{3}{2}a$	$\frac{\sqrt{3}}{2}a$
$\frac{2\pi}{3}$	$-\frac{1}{2}a$	$\frac{3\sqrt{3}}{2}a$
π	$-3a$	0
$\frac{4\pi}{3}$	$-\frac{1}{2}a$	$-\frac{3\sqrt{3}}{2}a$
$\frac{5\pi}{3}$	$\frac{3}{2}a$	$-\frac{\sqrt{3}}{2}a$



26. From Exercise 25, $x = 2a\cos t - a\cos 2t$, $y = 2a\sin t - a\sin 2t \Rightarrow$

$$A = 2\int_\pi^0 (2a\sin t - a\sin 2t)\,(-2a\sin t + 2a\sin 2t)\,dt = 4a^2\int_0^\pi \left(2\sin^2 t + \sin^2 2t - 3\sin t\sin 2t\right)dt$$

$$= 4a^2\int_0^\pi \left[(1 - \cos 2t) + \tfrac{1}{2}(1 - \cos 4t) - 6\sin^2 t\cos t\right]dt = 4a^2\left[t - \tfrac{1}{2}\sin 2t + \tfrac{1}{2}t - \tfrac{1}{8}\sin 4t - 2\sin^3 t\right]_0^\pi$$

$$= 4a^2\left(\frac{3}{2}\right)\pi = 6\pi a^2$$

27. This curve has 10 "petals". For instance, for $-\frac{\pi}{10} \le \theta \le \frac{\pi}{10}$, there are two petals, one with $r > 0$ and one with $r < 0$.

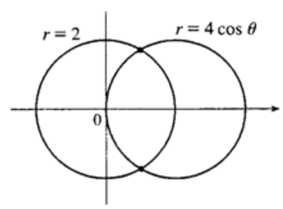
$$A = 10\int_{-\pi/10}^{\pi/10} \tfrac{1}{2}r^2\,d\theta = 5\int_{-\pi/10}^{\pi/10} 9\cos 5\theta\,d\theta = 90\int_0^{\pi/10}\cos 5\theta\,d\theta = [18\sin 5\theta]_0^{\pi/10} = 18$$

28. $r = 1 - 3\sin\theta$. The inner loop is traced out as θ goes from $\alpha = \sin^{-1}\frac{1}{3}$ to $\pi - \alpha$, so

$$A = \int_\alpha^{\pi-\alpha} \tfrac{1}{2}r^2\,d\theta = \int_\alpha^{\pi/2} (1 - 3\sin\theta)^2\,d\theta = \int_\alpha^{\pi/2}\left[1 - 6\sin\theta + \tfrac{9}{2}(1 - \cos 2\theta)\right]d\theta$$

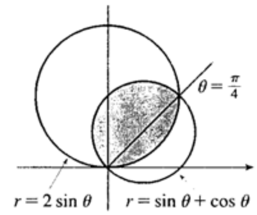
$$= \left[\tfrac{11}{2}\theta + 6\cos\theta - \tfrac{9}{4}\sin 2\theta\right]_\alpha^{\pi/2} = \tfrac{11}{4}\pi - \tfrac{11}{2}\sin^{-1}\tfrac{1}{3} - 3\sqrt{2}$$

29. The curves intersect where $4\cos\theta = 2$; that is, at $\left(2, \frac{\pi}{3}\right)$ and $\left(2, -\frac{\pi}{3}\right)$.



30. The two curves clearly both contain the pole. For other points of intersection, $\cot\theta = 2\cos(\theta + 2n\pi)$ or $-2\cos(\theta + \pi + 2n\pi)$, both of which reduce to $\cot\theta = 2\cos\theta \Leftrightarrow \cos\theta = 2\sin\theta\cos\theta \Leftrightarrow$ $\cos\theta\,(1 - 2\sin\theta) = 0 \Rightarrow \cos\theta = 0$ or $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ or $\frac{3\pi}{2} \Rightarrow$ intersection points are $\left(0, \frac{\pi}{2}\right)$, $\left(\sqrt{3}, \frac{\pi}{6}\right)$, and $\left(\sqrt{3}, \frac{11\pi}{6}\right)$.

31. The curves intersect where $2\sin\theta = \sin\theta + \cos\theta \quad\Rightarrow$ $\sin\theta = \cos\theta \quad\Rightarrow \theta = \frac{\pi}{4}$, and also at the origin (at which $\theta = \frac{3\pi}{4}$ on the second curve).



$$A = \int_0^{\pi/4} \tfrac{1}{2}\,(2\sin\theta)^2\,d\theta + \int_{\pi/4}^{3\pi/4} \tfrac{1}{2}\,(\sin\theta + \cos\theta)^2\,d\theta$$

$$= \int_0^{\pi/4} (1 - \cos 2\theta)\,d\theta + \tfrac{1}{2}\int_{\pi/4}^{3\pi/4} (1 + \sin 2\theta)\,d\theta$$

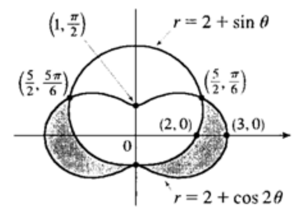
$$= \Big[\theta - \tfrac{1}{2}\sin 2\theta\Big]_0^{\pi/4} + \Big[\tfrac{1}{2}\theta - \tfrac{1}{4}\cos 2\theta\Big]_{\pi/4}^{3\pi/4} = \tfrac{1}{2}\,(\pi - 1)$$

32. $A = 2\int_{-\pi/2}^{\pi/6} \frac{1}{2}\left[(2+\cos 2\theta)^2 - (2+\sin\theta)^2\right]d\theta$

$$= \int_{-\pi/2}^{\pi/6} \left[4\cos 2\theta + \cos^2 2\theta - 4\sin\theta - \sin^2\theta\right]d\theta$$

$$= \Big[2\sin 2\theta + \tfrac{1}{2}\theta + \tfrac{1}{8}\sin 4\theta + 4\cos\theta - \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\Big]_{-\pi/2}^{\pi/6}$$

$$= \tfrac{51}{16}\sqrt{3}$$



33. $x = 3t^2$, $y = 2t^3$.

$$L = \int_0^2 \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^2 \sqrt{(6t)^2 + \left(6t^2\right)^2}\,dt = 6\int_0^2 t\sqrt{1+t^2}\,dt$$

$$= \left[2\left(1+t^2\right)^{3/2}\right]_0^2 = 2\left(5\sqrt{5} - 1\right)$$

34. $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = \left[-\sin t + \dfrac{\frac{1}{2}\sec^2(t/2)}{\tan(t/2)}\right]^2 + \cos^2 t = \left[-\sin t + \dfrac{1}{2\sin(t/2)\cos(t/2)}\right]^2 + \cos^2 t$

$$= \left(-\sin t + \frac{1}{\sin t}\right)^2 + \cos^2 t = \csc^2 t - 1 = \cot^2 t \quad\Rightarrow$$

$L = \int_{\pi/2}^{3\pi/4} |\cot t|\,dt = -\int_{\pi/2}^{3\pi/4} \cot t\,dt = [-\ln|\sin t|]_{\pi/2}^{3\pi/4} = \ln\sqrt{2}$

35. $L = \int_{\pi}^{2\pi} \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_{\pi}^{2\pi} \sqrt{(1/\theta)^2 + \left(-1/\theta^2\right)^2}\,d\theta$

$$\overset{24}{=} \int_{\pi}^{2\pi} \frac{\sqrt{\theta^2+1}}{\theta^2}\,d\theta = \left[-\frac{\sqrt{\theta^2+1}}{\theta} + \ln\left|\theta + \sqrt{\theta^2+1}\right|\right]_{\pi}^{2\pi}$$

$$= \frac{\sqrt{\pi^2+1}}{\pi} - \frac{\sqrt{4\pi^2+1}}{2\pi} + \ln\left|\frac{2\pi + \sqrt{4\pi^2+1}}{\pi + \sqrt{\pi^2+1}}\right|$$

36. $L = \int_0^{\pi} \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_0^{\pi} \sqrt{\sin^6\left(\frac{1}{3}\theta\right) + \sin^4\left(\frac{1}{3}\theta\right)\cos^2\left(\frac{1}{3}\theta\right)}\,d\theta$

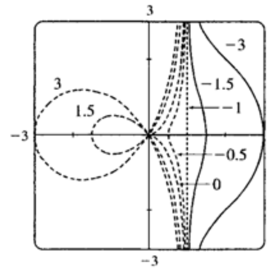
$$= \int_0^{\pi} \sin^2\left(\tfrac{1}{3}\theta\right)d\theta = \left[\tfrac{1}{2}\left(\theta - \tfrac{3}{2}\sin\left(\tfrac{2}{3}\theta\right)\right)\right]_0^{\pi} = \tfrac{1}{2}\pi - \tfrac{3}{8}\sqrt{3}$$

37. $S = \int_1^4 2\pi y\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_1^4 2\pi\left(\frac{1}{3}t^3 + \frac{1}{2}t^{-2}\right)\sqrt{\left(2/\sqrt{t}\right)^2 + \left(t^2 - t^{-3}\right)^2}\,dt$

$$= 2\pi\int_1^4 \left(\tfrac{1}{3}t^3 + \tfrac{1}{2}t^{-2}\right)\sqrt{\left(t^2 + t^{-3}\right)^2}\,dt = 2\pi\int_1^4 \left(\tfrac{1}{3}t^5 + \tfrac{5}{6} + \tfrac{1}{2}t^{-5}\right)dt = 2\pi\left[\tfrac{1}{18}t^6 + \tfrac{5}{6}t - \tfrac{1}{8}t^{-4}\right]_1^4 = \tfrac{471{,}295}{1024}\pi$$

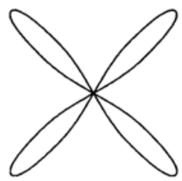
38. From Exercise 34, we find that $S = \int_{\pi/2}^{3\pi/4} 2\pi \sin t\, |\cot t|\, dt = -2\pi \int_{\pi/2}^{3\pi/4} \cos t\, dt = \pi\left(2 - \sqrt{2}\right)$.

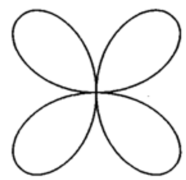
39. For all c except -1, the curve is asymptotic to the line $x = 1$. For $c < -1$, the curve bulges to the right near $y = 0$. As c increases, the bulge becomes smaller, until at $c = -1$ the curve is the straight line $x = 1$. As c continues to increase, the curve bulges to the left, until at $c = 0$ there is a cusp at the origin. For $c > 0$, there is a loop to the left of the origin, whose size and roundness increase as c increases.Note that the x-intercept of the curve is always $-c$.

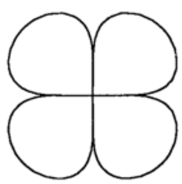


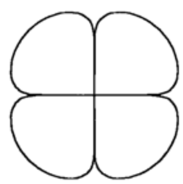
40. For a close to 0, the graph consists of four thin petals. As a increases, the petals get fatter, until as $a \to \infty$, each petal occupies almost its entire quarter-circle.

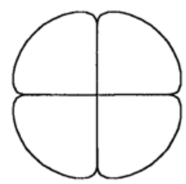

$a = 0.01$


$a = 0.1$

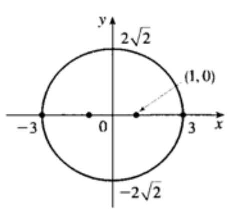

$a = 1$


$a = 5$

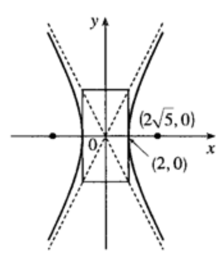

$a = 10$


$a = 25$

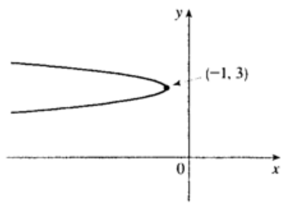
41. Ellipse, center $(0, 0)$, $a = 3$, $b = 2\sqrt{2}$, $c = 1$ $\Rightarrow$ foci $(\pm 1, 0)$, vertices $(\pm 3, 0)$.



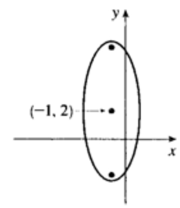
42. $x^2/4 - y^2/16 = 1$ is a hyperbola with center $(0, 0)$, vertices $(\pm 2, 0)$, $a = 2$, $b = 4$, $c = \sqrt{16 + 4} = 2\sqrt{5}$, foci $\left(\pm 2\sqrt{5}, 0\right)$ and asymptotes $y = \pm 2x$.



43. $6\left(y^2 - 6y + 9\right) = -(x+1) \quad \Leftrightarrow$
$(y-3)^2 = -\frac{1}{6}(x+1)$, a parabola with vertex $(-1, 3)$, opening to the left, $p = -\frac{1}{24} \quad \Rightarrow \quad$ focus $\left(-\frac{25}{24}, 3\right)$ and directrix $x = -\frac{23}{24}$.



44. $25(x+1)^2 + 4(y-2)^2 = 100 \quad \Leftrightarrow$
$\frac{1}{4}(x+1)^2 + \frac{1}{25}(y-2)^2 = 1$ is an ellipse centered at $(-1, 2)$ with foci on the line $x = -1$, vertices $(-1, 7)$ and $(-1, -3)$; $a = 5$,
$b = 2 \quad \Rightarrow c = \sqrt{21} \quad \Rightarrow \quad$ foci $\left(-1, 2 \pm \sqrt{21}\right)$.



45. The parabola opens upward with vertex $(0, 4)$ and $p = 2$, so its equation is $(x-0)^2 = 4 \cdot 2(y-4) \quad \Leftrightarrow$ $x^2 = 8(y-4)$.

46. Center is $(0, 0)$, and $c = 5$, $a = 2 \Rightarrow b = \sqrt{21}$; foci on y-axis $\quad \Rightarrow \quad$ equation of the hyperbola is $\dfrac{y^2}{4} - \dfrac{x^2}{21} = 1$.

47. The hyperbola has center $(0, 0)$ and foci on the x-axis. $c = 3$ and $b/a = \frac{1}{2}$ (from the asymptotes) $\Rightarrow \quad 9 = c^2 = a^2 + b^2 = (2b)^2 + b^2 = 5b^2 \quad \Rightarrow \quad b = \frac{3}{\sqrt{5}} \quad \Rightarrow \quad a = \frac{6}{\sqrt{5}} \quad \Rightarrow \quad$ the equation is $\dfrac{x^2}{36/5} - \dfrac{y^2}{9/5} = 1 \quad \Leftrightarrow \quad 5x^2 - 20y^2 = 36$.

48. Center is $(3, 0)$, and $a = \frac{8}{2} = 4$, $c = 2 \quad \Leftrightarrow \quad b = \sqrt{4^2 - 2^2} = 2\sqrt{3} \quad \Rightarrow \quad$ the equation of the ellipse is $\dfrac{(x-3)^2}{12} + \dfrac{y^2}{16} = 1$.

49. $x^2 = -y + 100$ has its vertex at $(0, 100)$, so one of the vertices of the ellipse is $(0, 100)$. Another form of the equation of a parabola is $x^2 = 4p(y - 100)$ so $4p(y-100) = -y + 100 \quad \Rightarrow \quad 4py - 4p(100) = 100 - y \quad \Rightarrow$ $4p = \dfrac{100 - y}{y - 100} \quad \Rightarrow \quad p = -\frac{1}{4}$. Therefore the shared focus is found at $\left(0, \frac{399}{4}\right)$ so $2c = \frac{399}{4} - 0 \quad \Rightarrow \quad c = \frac{399}{8}$ and the center of the ellipse is $\left(0, \frac{399}{8}\right)$. So $a = 100 - \frac{399}{8} = \frac{401}{8}$ and $b^2 = a^2 - c^2 = \frac{401^2 - 399^2}{8^2} = 25$. So the equation of the ellipse is $\dfrac{x^2}{b^2} + \dfrac{\left(y - \frac{399}{8}\right)^2}{a^2} = 1 \quad \Rightarrow \quad \dfrac{x^2}{25} + \dfrac{\left(y - \frac{399}{8}\right)^2}{\left(\frac{401}{8}\right)^2} = 1$ or $\dfrac{x^2}{25} + \dfrac{(8y - 399)^2}{160{,}801} = 1$.

50. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}$. Therefore $\frac{dy}{dx} = m \Leftrightarrow y = -\frac{b^2}{a^2}\frac{x}{m}$. Combining this condition with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we find that $x = \pm\frac{a^2 m}{\sqrt{a^2m^2 + b^2}}$. In other words, the two points on the ellipse where the tangent has slope m are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$. The tangent lines at these points have the equations $y \pm \frac{b^2}{\sqrt{a^2m^2+b^2}} = m\left(x \mp \frac{a^2m}{\sqrt{a^2m^2+b^2}}\right)$ or
$y = mx \mp \frac{a^2m^2}{\sqrt{a^2m^2+b^2}} \mp \frac{b^2}{\sqrt{a^2m^2+b^2}} = mx \pm \sqrt{a^2m^2+b^2}$.

51. Directrix $x = 4 \Rightarrow d = 4$, so $e = \frac{1}{3} \Rightarrow r = \frac{ed}{1 + e\cos\theta} = \frac{4}{3 + \cos\theta}$.

52. See the end of the proof of Theorem 11.7.1. If $e > 1$, then $1 - e^2 < 0$ and Equations 11.7.4 become $a^2 = \frac{e^2d^2}{(e^2-1)^2}$ and $b^2 = \frac{e^2d^2}{e^2-1}$, so $\frac{b^2}{a^2} = e^2 - 1$. The asymptotes $y = \pm\frac{b}{a}x$ have slopes $\pm\frac{b}{a} = \pm\sqrt{e^2-1}$, so the angles they make with the polar axis are $\pm\tan^{-1}\left[\sqrt{e^2-1}\right] = \cos^{-1}(\pm 1/e)$.

53. In polar coordinates, an equation for the circle is $r = 2a\sin\theta$. Thus, the coordinates of Q are $x = r\cos\theta = 2a\sin\theta\cos\theta$ and $y = r\sin\theta = 2a\sin^2\theta$. The coordinates of R are $x = 2a\cot\theta$ and $y = 2a$. Since P is the midpoint of QR, we use the midpoint formula to get $x = a(\sin\theta\cos\theta + \cot\theta)$ and $y = a(1 + \sin^2\theta)$.

Problems Plus

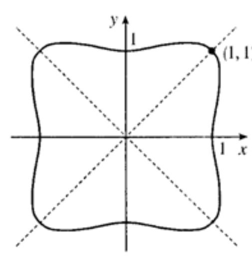
1. $x = \int_1^t \frac{\cos u}{u}\,du$, $y = \int_1^t \frac{\sin u}{u}\,du$, so by FTC1, we have $\frac{dx}{dt} = \frac{\cos t}{t}$ and $\frac{dy}{dt} = \frac{\sin t}{t}$. Vertical tangent lines occur when $\frac{dx}{dt} = 0 \;\Leftrightarrow\; \cos t = 0$. The parameter value corresponding to $(x, y) = (0, 0)$ is $t = 1$, so the nearest vertical tangent occurs when $t = \frac{\pi}{2}$. Therefore, the arc length between these points is

$$L = \int_1^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_1^{\pi/2} \sqrt{\frac{\cos^2 t}{t^2} + \frac{\sin^2 t}{t^2}}\,dt = \int_1^{\pi/2} \frac{dt}{t} = [\ln t]_1^{\pi/2} = \ln \tfrac{\pi}{2}$$

2. (a) The curve $x^4 + y^4 = x^2 + y^2$ is symmetric about both axes and about the line $y = x$ (since interchanging x and y does not change the equation) so we need only consider $y \geq x \geq 0$ to begin with. Implicit differentiation gives $4x^3 + 4y^3 y' = 2x + 2yy' \;\Rightarrow\; y' = \frac{x(1 - 2x^2)}{y(2y^2 - 1)} \;\Rightarrow\; y' = 0$ when $x = 0$ and when $x = \pm\frac{1}{\sqrt{2}}$. If $x = 0$, then $y^4 = y^2 \;\Rightarrow\; y^2(y^2 - 1) = 0 \;\Rightarrow\; y = 0$ or ± 1. The point $(0, 0)$ can't be a highest or lowest point because it is isolated. [If $-1 < x < 1$ and $-1 < y < 1$, then $x^4 < x^2$ and $y^4 < y^2 \;\Rightarrow\; x^4 + y^4 < x^2 + y^2$, except for $(0, 0)$.] If $x = \frac{1}{\sqrt{2}}$, then $x^2 = \frac{1}{2}$, $x^4 = \frac{1}{4}$, so $\frac{1}{4} + y^4 = \frac{1}{2} + y^2 \;\Rightarrow\; 4y^4 - 4y^2 - 1 = 0 \;\Rightarrow\; y^2 = \frac{4 \pm \sqrt{16 + 16}}{8} = \frac{1 \pm \sqrt{2}}{2}$. But $y^2 > 0$, so $y^2 = \frac{1 + \sqrt{2}}{2} \;\Rightarrow\; y = \pm\sqrt{\frac{1}{2}\left(1 + \sqrt{2}\right)}$.

Near the point $(0, 1)$, the denominator of y' is positive and the numerator changes from negative to positive as x increases through 0, so $(0, 1)$ is a local minimum point. At $\left(\frac{1}{\sqrt{2}}, \sqrt{\frac{1+\sqrt{2}}{2}}\right)$, y' changes from positive to negative, so that point gives a maximum. By symmetry, the highest points on the curve are $\left(\pm\frac{1}{\sqrt{2}}, \sqrt{\frac{1+\sqrt{2}}{2}}\right)$ and the lowest points are $\left(\pm\frac{1}{\sqrt{2}}, -\sqrt{\frac{1+\sqrt{2}}{2}}\right)$.

(b) We use the information from part (a), together with symmetry with respect to the axes and the lines $y = \pm x$, to sketch the curve.



(c) In polar coordinates, $x^4 + y^4 = x^2 + y^2$ becomes $r^4 \cos^4\theta + r^4 \sin^4\theta = r^2$ or $r^2 = 1/\left(\cos^4\theta + \sin^4\theta\right)$. By the symmetry shown in part (b), the area enclosed by the curve is $A = 8 \cdot \frac{1}{2}\int_0^{\pi/4} r^2\,d\theta = 4\int_0^{\pi/4} \frac{d\theta}{\cos^4\theta + \sin^4\theta}$. (If we have a CAS, this can be evaluated to give $\sqrt{2}\pi$).

The usual Weierstrass substitution $t = \tan(\theta/2)$ leads to a complicated integrand, so we first simplify:

$$\begin{aligned}\cos^4\theta + \sin^4\theta &= \left(1 - \sin^2\theta\right)^2 + \sin^4\theta = 1 - 2\sin^2\theta + 2\sin^4\theta = 1 - 2\sin^2\theta\left(1 - \sin^2\theta\right) \\ &= 1 - 2\sin^2\theta\cos^2\theta = 1 - \tfrac{1}{2}\sin^2 2\theta\end{aligned}$$

[continued]

Then we substitute $t = \tan 2\theta$, which gives $\theta = \frac{1}{2}\tan^{-1} t \Rightarrow d\theta = \dfrac{dt}{2\left(1+t^2\right)}$ and $\sin 2\theta = \dfrac{t}{\sqrt{1+t^2}}$.

Also, $\theta \to \frac{\pi}{4} \Rightarrow t \to \infty$, so we get the following improper integral:

$$A = 4\int_0^{\pi/4} \frac{d\theta}{1-\frac{1}{2}\sin^2 2\theta} = 4\int_0^\infty \frac{1}{1-\frac{1}{2}\left[t^2/\left(1+t^2\right)\right]}\frac{dt}{2\left(1+t^2\right)} = 4\int_0^\infty \frac{dt}{t^2+2}$$
$$= \lim_{x\to\infty} 4\left[\tfrac{1}{\sqrt{2}}\tan^{-1}\left(\tfrac{1}{\sqrt{2}}t\right)\right]_0^x = 2\sqrt{2}\lim_{x\to\infty}\tan^{-1}\left(\tfrac{1}{\sqrt{2}}x\right) = 2\sqrt{2}\cdot\tfrac{\pi}{2} = \sqrt{2}\pi$$

3. (a) If $\tan\theta = \sqrt{\dfrac{y}{C-y}}$, then $\tan^2\theta = \dfrac{y}{C-y}$, so $C\tan^2\theta - y\tan^2\theta = y$ and

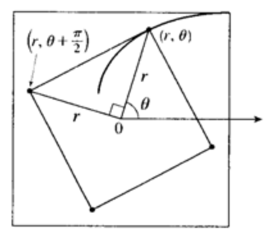
$y = \dfrac{C\tan^2\theta}{1+\tan^2\theta} = \dfrac{C\tan^2\theta}{\sec^2\theta} = C\tan^2\theta\cos^2\theta = C\sin^2\theta = \dfrac{C}{2}(1-\cos 2\theta)$. Now

$$dx = \sqrt{\frac{y}{C-y}}\,dy = \tan\theta\cdot\frac{C}{2}\cdot 2\sin 2\theta\,d\theta = C\tan\theta\cdot 2\sin\theta\cos\theta\,d\theta = 2C\sin^2\theta\,d\theta = C(1-\cos 2\theta)\,d\theta$$

Thus, $x = C\left(\theta - \frac{1}{2}\sin 2\theta\right) + K$ for some constant K. When $\theta = 0$, we have $y = 0$. We require that $x = 0$ when $\theta = 0$ so that the curve passes through the origin when $\theta = 0$. This yields $K = 0$. We now have $x = \frac{1}{2}C(2\theta - \sin 2\theta)$, $y = \frac{1}{2}C(1-\cos 2\theta)$.

(b) Setting $\phi = 2\theta$ and $r = \frac{1}{2}C$, we get $x = r(\phi - \sin\phi)$, $y = r(1-\cos\phi)$. Comparison with Equations 11.1.1 shows that the curve is a cycloid.

4. (a) Let us find the polar equation of the path of the bug that starts in the upper right corner of the square. If the polar coordinates of this bug, at a particular moment, are (r, θ), then the polar coordinates of the bug that it is crawling toward must be $\left(r, \theta + \frac{\pi}{2}\right)$. (The next bug must be the same distance from the origin and the angle between the lines joining the bugs to the pole must be $\frac{\pi}{2}$.) The Cartesian coordinates of the first bug are $(r\cos\theta, r\sin\theta)$ and for the second bug we have



$x = r\cos\left(\theta + \frac{\pi}{2}\right) = -r\sin\theta$, $y = r\sin\left(\theta + \frac{\pi}{2}\right) = r\cos\theta$. So the slope of the line joining the bugs is $\dfrac{r\cos\theta - r\sin\theta}{-r\sin\theta - r\cos\theta} = \dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$. This must be equal to the slope of the tangent line at (r, θ), so by Equation 11.4.3 we have $\dfrac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta} = \dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$. Solving for $\dfrac{dr}{d\theta}$, we get

$$\frac{dr}{d\theta}\sin^2\theta + \frac{dr}{d\theta}\sin\theta\cos\theta + r\sin\theta\cos\theta + r\cos^2\theta = \frac{dr}{d\theta}\sin\theta\cos\theta - \frac{dr}{d\theta}\cos^2\theta - r\sin^2\theta + r\sin\theta\cos\theta$$

$\Rightarrow \dfrac{dr}{d\theta}\left(\sin^2\theta + \cos^2\theta\right) + r\left(\cos^2\theta + \sin^2\theta\right) = 0 \Rightarrow \dfrac{dr}{d\theta} = -r$. Solving this differential equation as a separable equation (as in Section 10.3), or using Theorem 10.4.2 with $k = -1$, we get $r = Ce^{-\theta}$. To determine C we use the fact that, at its starting position, $\theta = \frac{\pi}{4}$ and $r = \frac{1}{\sqrt{2}}a$, so $\frac{1}{\sqrt{2}}a = Ce^{-\pi/4} \Rightarrow C = \frac{1}{\sqrt{2}}ae^{\pi/4}$. Therefore, a polar equation of the bug's path is $r = \frac{1}{\sqrt{2}}ae^{\pi/4}e^{-\theta}$ or $r = \frac{1}{\sqrt{2}}ae^{(\pi/4)-\theta}$.

(b) The distance traveled by this bug is $L = \int_{\pi/4}^{\infty} \sqrt{r^2 + (dr/d\theta)^2}\,d\theta$, where $\dfrac{dr}{d\theta} = \dfrac{a}{\sqrt{2}} e^{\pi/4} \left(-e^{-\theta}\right)$ and so

$$r^2 + (dr/d\theta)^2 = \tfrac{1}{2} a^2 e^{\pi/2} e^{-2\theta} + \tfrac{1}{2} a^2 e^{\pi/2} e^{-2\theta} = a^2 e^{\pi/2} e^{-2\theta}$$

Thus

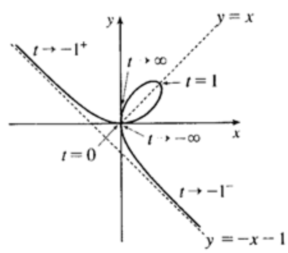
$$L = \int_{\pi/4}^{\infty} a e^{\pi/4} e^{-\theta}\,d\theta = a e^{\pi/4} \lim_{t\to\infty} \int_{\pi/4}^{t} e^{-\theta}\,d\theta = a e^{\pi/4} \lim_{t\to\infty} \left[-e^{-\theta}\right]_{\pi/4}^{t} = a e^{\pi/4} \lim_{t\to\infty} \left[e^{-\pi/4} - e^{-t}\right]$$
$$= a e^{\pi/4} e^{-\pi/4} = a$$

5. (a) If (a, b) lies on the curve, then there is some parameter value t_1 such that $\dfrac{3t_1}{1 + t_1^3} = a$ and $\dfrac{3t_1^2}{1 + t_1^3} = b$. If $t_1 = 0$, the point is $(0, 0)$, which lies on the line $y = x$. If $t_1 \neq 0$, then the point corresponding to $t = \dfrac{1}{t_1}$ is given by $x = \dfrac{3\,(1/t_1)}{1 + (1/t_1)^3} = \dfrac{3t_1^2}{t_1^3 + 1} = b$, $y = \dfrac{3\,(1/t_1)^2}{1 + (1/t_1)^3} = \dfrac{3t_1}{t_1^3 + 1} = a$. So (b, a) also lies on the curve. [Another way to see this is to do part (e) first; the result is immediate.] The curve intersects the line $y = x$ when $\dfrac{3t}{1 + t^3} = \dfrac{3t^2}{1 + t^3} \Rightarrow t = t^2 \Rightarrow t = 0$ or 1, so the points are $(0, 0)$ and $\left(\frac{3}{2}, \frac{3}{2}\right)$.

(b) $\dfrac{dy}{dt} = \dfrac{(1 + t^3)(6t) - 3t^2(3t^2)}{(1 + t^3)^2} = \dfrac{6t - 3t^4}{(1 + t^3)^2} = 0$ when $6t - 3t^4 = 3t\left(2 - t^3\right) = 0 \Rightarrow t = 0$ or $t = \sqrt[3]{2}$, so there are horizontal tangents at $(0, 0)$ and $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$. Using the symmetry from part (a), we see that there are vertical tangents at $(0, 0)$ and $\left(\sqrt[3]{4}, \sqrt[3]{2}\right)$.

(c) Notice that as $t \to -1^+$, we have $x \to -\infty$ and $y \to \infty$. As $t \to -1^-$, we have $x \to \infty$ and $y \to -\infty$. Also $y - (-x - 1) = y + x + 1 = \dfrac{3t + 3t^2 + (1 + t^3)}{1 + t^3} = \dfrac{(t + 1)^3}{1 + t^3} = \dfrac{(t + 1)^2}{t^2 - t + 1} \to 0$ as $t \to -1$. So $y = -x - 1$ is a slant asymptote.

(d) $\dfrac{dx}{dt} = \dfrac{(1 + t^3)(3) - 3t(3t^2)}{(1 + t^3)^2} = \dfrac{3 - 6t^3}{(1 + t^3)^2}$ and from part (b) we have $\dfrac{dy}{dt} = \dfrac{6t - 3t^4}{(1 + t^3)^2}$. So $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{t\,(2 - t^3)}{1 - 2t^3}$. Also $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \dfrac{2\,(1 + t^3)^4}{3\,(1 - 2t^3)^3} > 0 \Leftrightarrow t < \frac{1}{\sqrt[3]{2}}$. So the curve is concave upward there and has a minimum point at $(0, 0)$ and a maximum point at $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$. Using this together with the information from parts (a), (b), and (c), we sketch the curve.



(e) $x^3+y^3=\left(\dfrac{3t}{1+t^3}\right)^3+\left(\dfrac{3t^2}{1+t^3}\right)^3=\dfrac{27t^3+27t^6}{(1+t^3)^3}=\dfrac{27t^3\,(1+t^3)}{(1+t^3)^3}=\dfrac{27t^3}{(1+t^3)^2}$ and

$3xy=3\left(\dfrac{3t}{1+t^3}\right)\left(\dfrac{3t^2}{1+t^3}\right)=\dfrac{27t^3}{(1+t^3)^2}$, so $x^3+y^3=3xy$.

(f) We start with the equation from part (e) and substitute $x=r\cos\theta$, $y=r\sin\theta$. Then $x^3+y^3=3xy \Rightarrow$ $r^3\cos^3\theta+r^3\sin^3\theta=3r^2\cos\theta\sin\theta$. For $r\neq 0$, this gives $r=\dfrac{3\cos\theta\sin\theta}{\cos^3\theta+\sin^3\theta}$. Dividing numerator and denominator by $\cos^3\theta$, we obtain $r=\dfrac{3\left(\dfrac{1}{\cos\theta}\right)\dfrac{\sin\theta}{\cos\theta}}{1+\dfrac{\sin^3\theta}{\cos^3\theta}}=\dfrac{3\sec\theta\tan\theta}{1+\tan^3\theta}$.

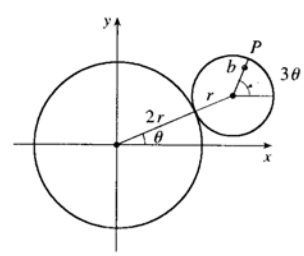
(g) The loop corresponds to $\theta\in\left(0,\frac{\pi}{2}\right)$, so its area is

$$A=\int_0^{\pi/2}\frac{r^2}{2}\,d\theta=\frac{1}{2}\int_0^{\pi/2}\left(\frac{3\sec\theta\tan\theta}{1+\tan^3\theta}\right)^2d\theta=\frac{9}{2}\int_0^{\pi/2}\frac{\sec^2\theta\tan^2\theta}{\left(1+\tan^3\theta\right)^2}\,d\theta$$

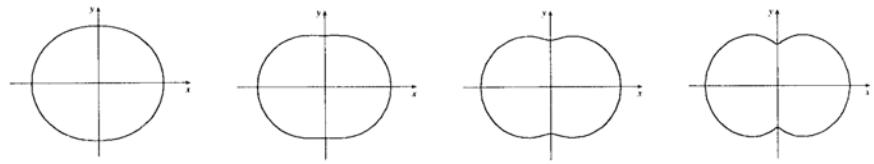
$$=\frac{9}{2}\int_0^{\infty}\frac{u^2du}{(1+u^3)^2}\ \ (\text{put u}=\tan\theta)\ =\lim_{b\to\infty}\tfrac{9}{2}\left[-\tfrac{1}{3}\left(1+u^3\right)^{-1}\right]_0^b=\tfrac{3}{2}$$

(h) By symmetry, the area between the folium and the line $y=-x-1$ is equal to the enclosed area in the third quadrant, plus twice the enclosed area in the fourth quadrant. The area in the third quadrant is $\frac{1}{2}$, and since $y=-x-1 \Rightarrow r\sin\theta=-r\cos\theta-1 \Rightarrow r=-\dfrac{1}{\sin\theta+\cos\theta}$, the area in the fourth quadrant is $\dfrac{1}{2}\displaystyle\int_{-\pi/2}^{-\pi/4}\left[\left(-\frac{1}{\sin\theta+\cos\theta}\right)^2-\left(\frac{3\sec\theta\tan\theta}{1+\tan^3\theta}\right)^2\right]d\theta=\dfrac{1}{2}$ (using a CAS). Therefore, the total area is $\frac{1}{2}+2\left(\frac{1}{2}\right)=\frac{3}{2}$.

6. (a) Since the smaller circle rolls without slipping around C, the amount of arc traversed on C ($2r\theta$ in the figure) must equal the amount of arc of the smaller circle that has been in contact with C. Since the smaller circle has radius r, it must have turned through an angle of $2r\theta/r=2\theta$. In addition to turning through an angle 2θ, the little circle has rolled through an angle θ against C. Thus, P has turned through an angle of 3θ as shown in the figure. (If the little circle had turned through an angle of 2θ with its center pinned to the x-axis, then P would have turned only 2θ instead of 3θ. The movement of the little circle around C adds θ to the angle.) From the figure, we see that the center of the small circle has coordinates $(3r\cos\theta, 3r\sin\theta)$. Thus, P has coordinates (x, y), where $x=3r\cos\theta+b\cos 3\theta$ and $y=3r\sin\theta+b\sin 3\theta$.

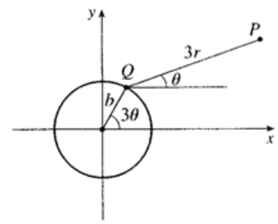


(b)

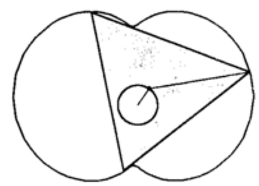


$b = \frac{1}{5}r$ $\quad$ $b = \frac{2}{5}r$ $\quad$ $b = \frac{3}{5}r$ $\quad$ $b = \frac{4}{5}r$

(c) The diagram gives an alternate description of point P on the epitrochoid. Q moves around a circle of radius b, and P rotates one-third as fast with respect to Q at a distance of $3r$. Place an equilateral triangle with sides of length $3\sqrt{3}r$ so that its centroid is at Q and one vertex is at P. (The distance from the centroid to a vertex is $\frac{1}{\sqrt{3}}$ times the length of a side of the equilateral triangle.)



As θ increases by $\frac{2\pi}{3}$, the point Q travels once around the circle of radius b, returning to its original position. At the same time, P (and the rest of the triangle) rotate through an angle of $\frac{2\pi}{3}$ about Q, so P's position is occupied by another vertex. In this way, we see that the epitrochoid traced out by P is simultaneously traced out by the other two vertices as well. The whole equilateral triangle sits inside the epitrochoid (touching it only with its vertices) and each vertex traces out the curve once while the centroid moves around the circle three times.



(d) We view the epitrochoid as being traced out in the same way as in part (c), by a rotor for which the distance from its center to each vertex is $3r$, so it has radius $6r$. To show that the rotor fits inside the epitrochoid, it suffices to show that for any position of the tracing point P, there are no points on the opposite side of the rotor which are outside the epitrochoid. But the most likely case of intersection is when P is on the y-axis, so as long as the diameter of the rotor (which is $3\sqrt{3}r$) is less than the distance between the y-intercepts, the rotor will fit. The y-intercepts occur when $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$ $\Rightarrow$ $y = \pm(3r - b)$, so the distance between the intercepts is $6r - 2b$, and the rotor will fit if $3\sqrt{3}r \leq 6r - 2b$ $\Leftrightarrow$ $b \leq \frac{3\left(2 - \sqrt{3}\right)}{2}r$.

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12 Infinite Sequences and Series

12.1 Sequences

1. (a) A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.

(b) The terms a_n approach 8 as n becomes large. In fact, we can make a_n as close to 8 as we like by taking n sufficiently large.

(c) The terms a_n become large as n becomes large.

2. (a) From Definition 1, a convergent sequence is a sequence for which $\lim_{n\to\infty} a_n$ exists. Examples: $\{1/n\}$, $\{1/2^n\}$

(b) A divergent sequence is a sequence for which $\lim_{n\to\infty} a_n$ *does not* exist. Examples: $\{n\}$, $\{\sin n\}$

3. $a_n = 1 - (0.2)^n$, so the sequence is $\{0.8, 0.96, 0.992, 0.9984, 0.99968, \dots\}$.

4. $a_n = \dfrac{n+1}{3n-1}$, so the sequence is $\left\{\dfrac{2}{2}, \dfrac{3}{5}, \dfrac{4}{8}, \dfrac{5}{11}, \dfrac{6}{14}, \dots\right\} = \left\{1, \dfrac{3}{5}, \dfrac{1}{2}, \dfrac{5}{11}, \dfrac{3}{7}, \dots\right\}$.

5. $a_n = \dfrac{3(-1)^n}{n!}$, so the sequence is $\left\{\dfrac{-3}{1}, \dfrac{3}{2}, \dfrac{-3}{6}, \dfrac{3}{24}, \dfrac{-3}{120}, \dots\right\} = \left\{-3, \dfrac{3}{2}, -\dfrac{1}{2}, \dfrac{1}{8}, -\dfrac{1}{40}, \dots\right\}$.

6. $a_n = 2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)$, so the sequence is
$\{2, 2 \cdot 4, 2 \cdot 4 \cdot 6, 2 \cdot 4 \cdot 6 \cdot 8, 2 \cdot 4 \cdot 6 \cdot 8 \cdot 10, \dots\} = \{2, 8, 48, 384, 3840, \dots\}$.

7. $a_n = \sin \dfrac{n\pi}{2}$, so the sequence is $\{1, 0, -1, 0, 1, \dots\}$.

8. $a_1 = 1$, $a_{n+1} = \dfrac{1}{1+a_n}$, so the sequence is

$$\left\{1, \frac{1}{1+1}, \frac{1}{1+\frac{1}{2}}, \frac{1}{1+\frac{2}{3}}, \frac{1}{1+\frac{3}{5}}, \dots\right\} = \left\{1, \frac{1}{2}, \frac{1}{\frac{3}{2}}, \frac{1}{\frac{5}{3}}, \frac{1}{\frac{8}{5}}\right\} = \left\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \dots\right\}$$

9. The numerators are all 1 and the denominators are powers of 2, so $a_n = \dfrac{1}{2^n}$.

10. The numerators are all 1 and the denominators are multiples of 2, so $a_n = \dfrac{1}{2n}$.

11. $\{2, 7, 12, 17, \dots\}$. Each term is larger than the preceding one by 5, so
$a_n = a_1 + d(n-1) = 2 + 5(n-1) = 5n - 3$.

12. $\left\{-\dfrac{1}{4}, \dfrac{2}{9}, -\dfrac{3}{16}, \dfrac{4}{25}, \dots\right\}$. The numerator of the nth term is n and its denominator is $(n+1)^2$. Including the alternating signs, we get $a_n = (-1)^n \dfrac{n}{(n+1)^2}$.

13. $\left\{1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \dots\right\}$. Each term is $-\frac{2}{3}$ times the preceding one, so $a_n = \left(-\frac{2}{3}\right)^{n-1}$.

14. $\{0, 2, 0, 2, 0, 2, \ldots\}$. One is halfway between 0 and 2, so we can think of alternately subtracting and adding 1 (from 1 and to 1) to obtain the given sequence: $a_n = 1 - (-1)^{n-1}$.

15. $a_n = n(n-1)$. $a_n \to \infty$ as $n \to \infty$, so the sequence diverges.

16. $a_n = \dfrac{n+1}{3n-1} = \dfrac{1+1/n}{3-1/n}$, so $a_n \to \dfrac{1+0}{3-0} = \dfrac{1}{3}$ as $n \to \infty$. Converges

17. $a_n = \dfrac{3+5n^2}{n+n^2} = \dfrac{5+3/n^2}{1+1/n}$, so $a_n \to \dfrac{5+0}{1+0} = 5$ as $n \to \infty$. Converges

18. $a_n = \dfrac{\sqrt{n}}{1+\sqrt{n}} = \dfrac{1}{1/\sqrt{n}+1}$, so $a_n \to \dfrac{1}{0+1} = 1$ as $n \to \infty$. Converges

19. $a_n = \dfrac{2^n}{3^{n+1}} = \dfrac{1}{3}\left(\dfrac{2}{3}\right)^n$, so $\lim\limits_{n\to\infty} a_n = \frac{1}{3}\lim\limits_{n\to\infty}\left(\frac{2}{3}\right)^n = \frac{1}{3}\cdot 0 = 0$ by (7) with $r = \frac{2}{3}$. Converges

20. $a_n = \dfrac{n}{1+\sqrt{n}} = \dfrac{\sqrt{n}}{1/\sqrt{n}+1}$. The numerator approaches ∞ and the denominator approaches $0+1=1$ as $n \to \infty$, so $a_n \to \infty$ as $n \to \infty$ and the sequence diverges.

21. $a_n = \dfrac{(-1)^{n-1}\,n}{n^2+1} = \dfrac{(-1)^{n-1}}{n+1/n}$, so $0 \le |a_n| = \dfrac{1}{n+1/n} \le \dfrac{1}{n} \to 0$ as $n \to \infty$, so $a_n \to 0$ by the Squeeze Theorem and Theorem 5. Converges

22. $\{a_n\} = \{1, 0, -1, 0, 1, 0, -1, \ldots\}$. This sequence oscillates among 1, 0, and -1, so the sequence diverges.

23. $a_n = 2 + \cos n\pi$, so

$$\begin{aligned}\{a_n\} &= \{2+\cos\pi, 2+\cos 2\pi, 2+\cos 3\pi, 2+\cos 4\pi, \ldots\} = \{2-1, 2+1, 2-1, 2+1, \ldots\}\\ &= \{1, 3, 1, 3, \ldots\}\end{aligned}$$

This sequence oscillates between 1 and 3, so it diverges.

24. $2n \to \infty$ as $n \to \infty$, so since $\lim\limits_{x\to\infty} \arctan x = \frac{\pi}{2}$, we have $\lim\limits_{n\to\infty} \arctan 2n = \frac{\pi}{2}$. Convergent

25. $0 < \dfrac{3+(-1)^n}{n^2} \le \dfrac{4}{n^2}$ and $\lim\limits_{n\to\infty}\dfrac{4}{n^2} = 0$, so $\left\{\dfrac{3+(-1)^n}{n^2}\right\}$ converges to 0 by the Squeeze Theorem.

26. $\lim\limits_{n\to\infty}\dfrac{n!}{(n+2)!} = \lim\limits_{n\to\infty}\dfrac{1\cdot 2\cdot 3\cdot\cdots\cdot n}{1\cdot 2\cdot 3\cdot\cdots\cdot n\,(n+1)\,(n+2)} = \lim\limits_{n\to\infty}\dfrac{1}{(n+2)\,(n+1)} = 0$. Convergent

27. $\lim\limits_{x\to\infty}\dfrac{\ln\left(x^2\right)}{x} = \lim\limits_{x\to\infty}\dfrac{2\ln x}{x} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{2/x}{1} = 0$, so by Theorem 2, $\left\{\dfrac{\ln\left(n^2\right)}{n}\right\}$ converges to 0.

28. $\lim\limits_{n\to\infty}\sin\left(\dfrac{1}{n}\right) = \sin 0 = 0$ since $\dfrac{1}{n} \to 0$ as $n \to \infty$, so by Theorem 5, $\left\{(-1)^n \sin\left(\dfrac{1}{n}\right)\right\}$ converges to 0.

29. $b_n = \sqrt{n+2} - \sqrt{n} = \left(\sqrt{n+2}-\sqrt{n}\right)\dfrac{\sqrt{n+2}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n}} = \dfrac{2}{\sqrt{n+2}+\sqrt{n}} < \dfrac{2}{2\sqrt{n}} = \dfrac{1}{\sqrt{n}} \to 0$ as $n \to \infty$. So by the Squeeze Theorem with $a_n = 0$ and $c_n = 1/\sqrt{n}$, $\left\{\sqrt{n+2}-\sqrt{n}\right\}$ converges to 0.

30. $\lim\limits_{x\to\infty}\dfrac{\ln(2+e^x)}{3x} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{e^x/(2+e^x)}{3} = \lim\limits_{x\to\infty}\dfrac{1}{6e^{-x}+3} = \dfrac{1}{3}$, so by Theorem 2, $\lim\limits_{n\to\infty}\dfrac{\ln(2+e^n)}{3n} = \dfrac{1}{3}$. Convergent

31. $\lim\limits_{x\to\infty}\dfrac{x}{2^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{1}{(\ln 2)\,2^x} = 0$, so by Theorem 2, $\left\{n2^{-n}\right\}$ converges to 0.

32. $a_n = \ln(n+1) - \ln n = \ln\left(\dfrac{n+1}{n}\right) = \ln\left(1+\dfrac{1}{n}\right) \to \ln(1) = 0$ as $n \to \infty$. Convergent

33. $0 \le \dfrac{\cos^2 n}{2^n} \le \dfrac{1}{2^n}$ [since $0 \le \cos^2 n \le 1$], so since $\lim\limits_{n\to\infty} \dfrac{1}{2^n} = 0$, $\left\{\dfrac{\cos^2 n}{2^n}\right\}$ converges to 0 by the Squeeze Theorem.

34. $y = (1+3x)^{1/x} \quad\Rightarrow\quad \ln(y) = \dfrac{1}{x}\ln(1+3x) \quad\Rightarrow\quad \lim\limits_{x\to\infty} \ln y = \lim\limits_{x\to\infty} \dfrac{\ln(1+3x)}{x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{3/(1+3x)}{1} = 0 \quad\Rightarrow$ $\lim\limits_{x\to\infty} y = e^0 = 1$, so by Theorem 2, $\left\{(1+3n)^{1/n}\right\}$ converges to 1.

35. The series converges, since

$$a_n = \frac{1+2+3+\cdots+n}{n^2} = \frac{n(n+1)/2}{n^2} \;\text{[sum of the first } n \text{ positive integers]}\; = \frac{n+1}{2n} = \frac{1+1/n}{2} \to \frac{1}{2}$$

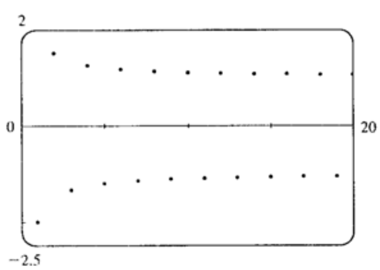
as $n \to \infty$.

36. $0 \le |a_n| = \dfrac{n\,|\cos n|}{n^2+1} \le \dfrac{n}{n^2+1} = \dfrac{1}{n+1/n} \to 0$ as $n \to \infty$, so by the Squeeze Theorem and Theorem 5, $\{a_n\}$ converges to 0.

37. $a_n = \dfrac{1}{2}\cdot\dfrac{2}{2}\cdot\dfrac{3}{2}\cdot\cdots\cdot\dfrac{(n-1)}{2}\cdot\dfrac{n}{2} \ge \dfrac{1}{2}\cdot\dfrac{n}{2} = \dfrac{n}{4} \to \infty$ as $n \to \infty$, so $\{a_n\}$ diverges.

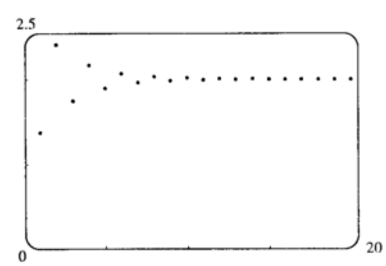
38. $0 < |a_n| = \dfrac{3^n}{n!} = \dfrac{3}{1}\cdot\dfrac{3}{2}\cdot\dfrac{3}{3}\cdot\cdots\cdot\dfrac{3}{(n-1)}\cdot\dfrac{3}{n} \le 3\cdot\dfrac{3}{2}\cdot\dfrac{3}{n} = \dfrac{27}{2n} \to 0$ as $n \to \infty$, so by the Squeeze Theorem and Theorem 5, $\left\{(-3)^n/n\right\}$ converges to 0.

39.



From the graph, we see that the sequence $\left\{(-1)^n \dfrac{n+1}{n}\right\}$ is divergent, since it oscillates between 1 and −1 (approximately).

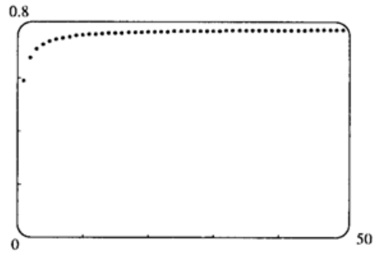
40.



From the graph, it appears that the sequence converges to 2.

$\left\{\left(-\frac{2}{\pi}\right)^n\right\}$ converges to 0 by (7), and hence $\left\{2+\left(-\frac{2}{\pi}\right)^n\right\}$ converges to $2+0=2$.

41.

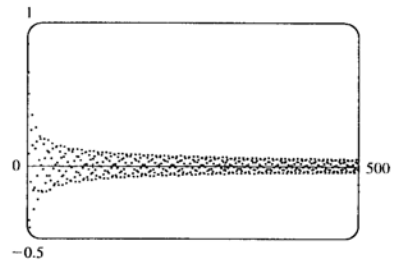


From the graph, it appears that the sequence converges to about 0.78.

$\lim\limits_{n\to\infty} \dfrac{2n}{2n+1} = \lim\limits_{n\to\infty} \dfrac{2}{2+1/n} = 1$, so

$\lim\limits_{n\to\infty} \arctan\left(\dfrac{2n}{2n+1}\right) = \arctan 1 = \dfrac{\pi}{4}$.

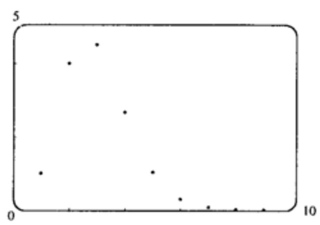
42.



From the graph, it appears that the sequence converges (slowly) to 0.

$0 \le \dfrac{|\sin n|}{\sqrt{n}} \le \dfrac{1}{\sqrt{n}} \to 0$ as $n \to \infty$, so by the Squeeze Theorem and Theorem 5, $\left\{\dfrac{\sin n}{\sqrt{n}}\right\}$ converges to 0.

43.



From the graph, it appears that the sequence converges to 0.

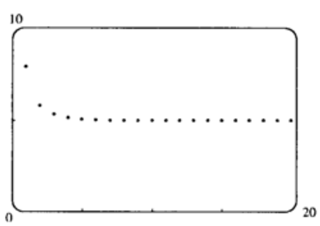
$$0 < a_n = \frac{n^3}{n!} = \frac{n}{n} \cdot \frac{n}{(n-1)} \cdot \frac{n}{(n-2)} \cdot \frac{1}{(n-3)} \cdot \cdots \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{1}$$

$$\le \frac{n^2}{(n-1)(n-2)(n-3)} \quad (\text{for } n \ge 4)$$

$$= \frac{1/n}{(1-1/n)(1-2/n)(1-3/n)} \to 0 \text{ as } n \to \infty$$

So by the Squeeze Theorem, $\{n^3/n!\}$ converges to 0.

44.



From the graph, it appears that the sequence converges to 5.

$$5 = \sqrt[n]{5^n} \le \sqrt[n]{3^n + 5^n} \le \sqrt[n]{5^n + 5^n} = \sqrt[n]{2}\sqrt[n]{5^n}$$

$$= \sqrt[n]{2} \cdot 5 \to 5 \text{ as } n \to \infty$$

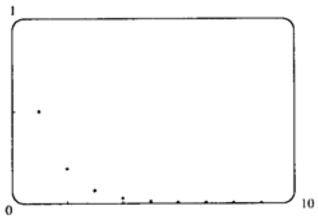
Hence, $a_n \to 5$ by the Squeeze Theorem.

Alternate Solution: Let $y = (3^x + 5^x)^{1/x}$. Then

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(3^x + 5^x)}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{3^x \ln 3 + 5^x \ln 5}{3^x + 5^x}$$

$$= \lim_{x\to\infty} \frac{\left(\frac{3}{5}\right)^x \ln 3 + \ln 5}{\left(\frac{3}{5}\right)^x + 1} = \ln 5$$

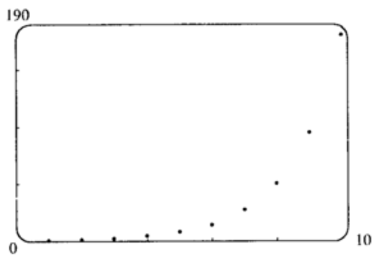
so $\lim\limits_{x\to\infty} y = e^{\ln 5} = 5$, and so $\left\{\sqrt[n]{3^n + 5^n}\right\}$ converges to 5.

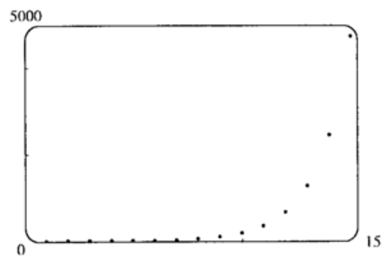
45. 

From the graph, it appears that the sequence approaches 0.

$$0 < a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n)^n} = \frac{1}{2n} \cdot \frac{3}{2n} \cdot \frac{5}{2n} \cdot \cdots \cdot \frac{2n-1}{2n}$$
$$\leq \frac{1}{2n} \cdot (1) \cdot (1) \cdot \cdots \cdot (1) = \frac{1}{2n} \to 0 \text{ as } n \to \infty$$

So by the Squeeze Theorem, $\left\{ \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n)^n} \right\}$ converges to 0.

46. 



From the graphs, it seems that the sequence diverges. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!}$. We first prove by induction that $a_n \geq \left(\dfrac{3}{2}\right)^{n-1}$ for all n. This is clearly true for $n = 1$, so let $P(n)$ be the statement that the above is true for n. We must show it is then true for $n+1$. $a_{n+1} = a_n \cdot \dfrac{2n+1}{n+1} \geq \left(\dfrac{3}{2}\right)^{n-1} \cdot \dfrac{2n+1}{n+1}$ (induction hypothesis). But $\dfrac{2n+1}{n+1} \geq \dfrac{3}{2}$ [since $2(2n+1) \geq 3(n+1) \iff 4n+2 \geq 3n+3 \iff n \geq 1$], and so we get that $a_{n+1} \geq \left(\frac{3}{2}\right)^{n-1} \cdot \frac{3}{2} = \left(\frac{3}{2}\right)^n$ which is $P(n+1)$. Thus, we have proved our first assertion, so since $\left\{\left(\frac{3}{2}\right)^{n-1}\right\}$ diverges (by Equation 7), so does the given sequence $\{a_n\}$.

47. (a) $a_n = 1000(1.06)^n \Rightarrow a_1 = 1060$, $a_2 = 1123.60$, $a_3 = 1191.02$, $a_4 = 1262.48$, and $a_5 = 1338.23$.

(b) $\lim\limits_{n\to\infty} a_n = 1000 \lim\limits_{n\to\infty} (1.06)^n$, so the sequence diverges by (7) with $r = 1.06 > 1$.

48. $a_{n+1} = \begin{cases} \frac{1}{2}a_n & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$ When $a_1 = 11$, the first 40 terms are 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4. When $a_1 = 25$, the first 40 terms are 25, 76, 38, 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4. The famous Collatz conjecture is that this sequence always reaches 1, regardless of the starting point a_1.

49. If $|r| \geq 1$, then $\{r^n\}$ diverges by (7), so $\{nr^n\}$ diverges also, since $|nr^n| = n\,|r^n| \geq |r^n|$. If $|r| < 1$ then $\lim\limits_{x\to\infty} xr^x = \lim\limits_{x\to\infty} \dfrac{x}{r^{-x}} \overset{H}{=} \lim\limits_{x\to\infty} \dfrac{1}{(-\ln r)\,r^{-x}} = \lim\limits_{x\to\infty} \dfrac{r^x}{-\ln r} = 0$, so $\lim\limits_{n\to\infty} nr^n = 0$, and hence $\{nr^n\}$ converges whenever $|r| < 1$.

50. (a) Let $\lim_{n\to\infty} a_n = L$. By Definition 1, this means that for every $\varepsilon > 0$ there is an integer N such that $|a_n - L| < \varepsilon$ whenever $n > N$. Thus, $|a_{n+1} - L| < \varepsilon$ whenever $n + 1 > N \Leftrightarrow n > N - 1$. It follows that $\lim_{n\to\infty} a_{n+1} = L$ and so $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$.

(b) If $L = \lim_{n\to\infty} a_n$ then $\lim_{n\to\infty} a_{n+1} = L$ also, so L must satisfy $L = 1/(1+L) \Rightarrow L^2 + L - 1 = 0 \Rightarrow$ $L = \frac{-1+\sqrt{5}}{2}$ (since L has to be non-negative if it exists).

51. Since $\{a_n\}$ is a decreasing sequence, $a_n > a_{n+1}$ for all $n \geq 1$. Because all of its terms lie between 5 and 8, $\{a_n\}$ is a bounded sequence. By the Monotonic Sequence Theorem, $\{a_n\}$ is convergent, that is, $\{a_n\}$ has a limit L. L must be less than 8 since $\{a_n\}$ is decreasing, so $5 \leq L < 8$.

52. $a_n = 1/5^n$ defines a decreasing geometric sequence since $a_{n+1} = \frac{1}{5}a_n < a_n$ for each $n \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{5}$ for all $n \geq 1$.

53. $a_n = \dfrac{1}{2n+3}$ is decreasing since $a_{n+1} = \dfrac{1}{2(n+1)+3} = \dfrac{1}{2n+5} < \dfrac{1}{2n+3} = a_n$ for each $n \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{5}$ for all $n \geq 1$.

54. $a_n = \dfrac{2n-3}{3n+4}$ defines an increasing sequence since for $f(x) = \dfrac{2x-3}{3x+4}$,

$f'(x) = \dfrac{(3x+4)(2) - (2x-3)(3)}{(3x+4)^2} = \dfrac{17}{(3x+4)^2} > 0$. The sequence is bounded since $a_n \geq a_1 = -\frac{1}{7}$ for $n \geq 1$, and $a_n < \dfrac{2n-3}{3n} < \dfrac{2n}{3n} = \dfrac{2}{3}$ for $n \geq 1$.

55. $a_n = \cos(n\pi/2)$ is not monotonic. The first few terms are $0, -1, 0, 1, 0, -1, 0, 1, \ldots$. In fact, the sequence consists of the terms $0, -1, 0, 1$ repeated over and over again in that order. The sequence is bounded since $|a_n| \leq 1$ for all $n \geq 1$.

56. $a_n = 3 + (-1)^n/n$ defines a sequence that is not monotonic. The first few terms are $2, 3.5, 2.\overline{6}, 3.25$, and 2.8, showing that the sequence is neither increasing nor decreasing. The sequence is bounded since $2 \leq a_n \leq 3.5$ for all $n \geq 1$.

57. $a_n = \dfrac{n}{n^2+1}$ defines a decreasing sequence since for $f(x) = \dfrac{x}{x^2+1}$,

$f'(x) = \dfrac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} \leq 0$ for $x \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{2}$ for all $n \geq 1$.

58. $a_n = \dfrac{\sqrt{n}}{n+2}$ defines a sequence that is neither increasing nor decreasing since $a_1 < a_2$ and $a_2 > a_3$. ($a_1 = \frac{1}{3} = 0.\overline{3}$, $a_2 = \frac{\sqrt{2}}{4} \approx 0.354$, and $a_3 = \frac{\sqrt{3}}{5} \approx 0.346$.) But the sequence $\{a_n \mid n \geq 2\}$ obtained by omitting the first term a_1 *is* decreasing. To see this, note that if $f(x) = \dfrac{\sqrt{x}}{x+2}$ for $x \geq 0$, then

$f'(x) = \dfrac{\frac{x+2}{2\sqrt{x}} - \sqrt{x}}{(x+2)^2} = \dfrac{(x+2) - 2x}{2\sqrt{x}(x+2)^2} = \dfrac{2-x}{2\sqrt{x}(x+2)^2} \leq 0$ for $x \geq 2$. The sequence is bounded since $a_n > 0$ for all $n \geq 1$ and $a_n \leq a_2 = \frac{\sqrt{2}}{4}$ for all $n \geq 1$.

59. $a_1 = 2^{1/2}$, $a_2 = 2^{3/4}$, $a_3 = 2^{7/8}$, ..., so $a_n = 2^{(2^n-1)/2^n} = 2^{1-(1/2^n)}$. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 2^{1-(1/2^n)} = 2^1 = 2$.

Alternate Solution: Let $L = \lim\limits_{n\to\infty} a_n$. (We could show the limit exists by showing that $\{a_n\}$ is bounded and increasing.) So L must satisfy $L = \sqrt{2 \cdot L} \Rightarrow L^2 = 2L \Rightarrow L(L-2) = 0$ ($L \neq 0$ since the sequence increases), so $L = 2$.

60. (a) Let P_n be the statement that $a_{n+1} \geq a_n$ and $a_n \leq 3$. P_1 is obviously true. We will assume that P_n is true and then show that as a consequence P_{n+1} must also be true. $a_{n+2} \geq a_{n+1} \Leftrightarrow \sqrt{2 + a_{n+1}} \geq \sqrt{2 + a_n} \Leftrightarrow 2 + a_{n+1} \geq 2 + a_n \Leftrightarrow a_{n+1} \geq a_n$ which is the induction hypothesis. $a_{n+1} \leq 3 \Leftrightarrow \sqrt{2 + a_n} \leq 3 \Leftrightarrow 2 + a_n \leq 9 \Leftrightarrow a_n \leq 7$, which is certainly true because we are assuming that $a_n \leq 3$. So P_n is true for all n, and so $a_1 \leq a_n \leq 3$ (the sequence is bounded), and hence by the Monotonic Sequence Theorem, $\lim\limits_{n\to\infty} a_n$ exists.

(b) If $L = \lim\limits_{n\to\infty} a_n$, then $\lim\limits_{n\to\infty} a_{n+1} = L$ also, so $L = \sqrt{2 + L} \Rightarrow L^2 - L - 2 = 0 \Rightarrow (L+1)(L-2) = 0 \Rightarrow L = 2$ (since L can't be negative).

61. We show by induction that $\{a_n\}$ is increasing and bounded above by 3.
Let P_n be the proposition that $a_{n+1} > a_n$ and $0 < a_n < 3$. Clearly P_1 is true. Assume that P_n is true. Then

$$a_{n+1} > a_n \Rightarrow \frac{1}{a_{n+1}} < \frac{1}{a_n} \Rightarrow -\frac{1}{a_{n+1}} > -\frac{1}{a_n}.$$

Now $a_{n+2} = 3 - \dfrac{1}{a_{n+1}} > 3 - \dfrac{1}{a_n} = a_{n+1} \Leftrightarrow P_{n+1}$. This proves that $\{a_n\}$ is increasing and bounded above by 3, so $1 = a_1 < a_n < 3$, that is, $\{a_n\}$ is bounded, and hence convergent by the Monotonic Sequence Theorem. If $L = \lim\limits_{n\to\infty} a_n$, then $\lim\limits_{n\to\infty} a_{n+1} = L$ also, so L must satisfy $L = 3 - 1/L \Rightarrow L^2 - 3L + 1 = 0 \Rightarrow L = \frac{3 \pm \sqrt{5}}{2}$. But $L > 1$, so $L = \frac{3+\sqrt{5}}{2}$.

62. We use induction. Let P_n be the statement that $0 < a_{n+1} \leq a_n \leq 2$. Clearly P_1 is true, since $a_2 = 1/(3-2) = 1$. Now assume that P_n is true. Then $a_{n+1} \leq a_n \Rightarrow -a_{n+1} \geq -a_n \Rightarrow 3 - a_{n+1} \geq 3 - a_n \Rightarrow$

$a_{n+2} = \dfrac{1}{3 - a_{n+1}} \leq \dfrac{1}{3 - a_n} = a_{n+1}$. Also $a_{n+2} > 0$ (since $3 - a_{n+1}$ is positive) and $a_{n+1} \leq 2$ by the induction hypothesis, so P_{n+1} is true.

To find the limit, we use the fact that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1} \Rightarrow L = \frac{1}{3-L} \Rightarrow L^2 - 3L + 1 = 0 \Rightarrow L = \frac{3\pm\sqrt{5}}{2}$. But $L \leq 2$, so we must have $L = \frac{3-\sqrt{5}}{2}$.

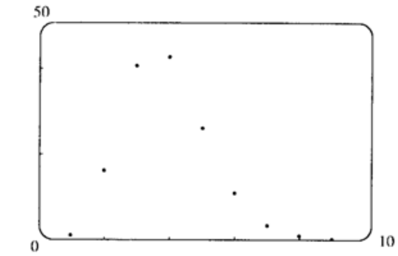
63. (a) Let a_n be the number of rabbit pairs in the nth month. Clearly $a_1 = 1 = a_2$. In the nth month, each pair that is 2 or more months old (that is, a_{n-2} pairs) will produce a new pair to add to the a_{n-1} pairs already present. Thus, $a_n = a_{n-1} + a_{n-2}$, so that $\{a_n\} = \{f_n\}$, the Fibonacci sequence.

(b) $a_n = \dfrac{f_{n+1}}{f_n} \Rightarrow a_{n-1} = \dfrac{f_n}{f_{n-1}} = \dfrac{f_{n-1} + f_{n-2}}{f_{n-1}} = 1 + \dfrac{f_{n-2}}{f_{n-1}} = 1 + \dfrac{1}{f_{n-1}/f_{n-2}} = 1 + \dfrac{1}{a_{n-2}}$. If $L = \lim\limits_{n\to\infty} a_n$, then $L = \lim\limits_{n\to\infty} a_{n-1}$ and $L = \lim\limits_{n\to\infty} a_{n-2}$, so L must satisfy $L = 1 + \dfrac{1}{L} \Rightarrow L^2 - L - 1 = 0 \Rightarrow L = \frac{1+\sqrt{5}}{2}$ (since L must be positive).

64. (a) If f is continuous, then $f(L) = f\left(\lim\limits_{n\to\infty} a_n\right) = \lim\limits_{n\to\infty} f(a_n) = \lim\limits_{n\to\infty} a_{n+1} = L$ by Exercise 50(a).

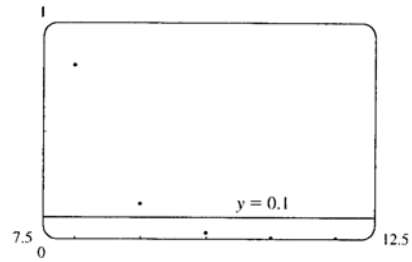
(b) By repeatedly pressing the cosine key on the calculator (that is, taking cosine of the previous answer) until the displayed value stabilizes, we see that $L \approx 0.73909$.

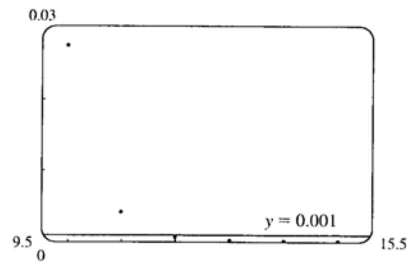
65. (a)



From the graph, it appears that the sequence $\left\{\dfrac{n^5}{n!}\right\}$ converges to 0, that is, $\lim\limits_{n\to\infty}\dfrac{n^5}{n!}=0$.

(b)





From the first graph, it seems that the smallest possible value of N corresponding to $\varepsilon = 0.1$ is 9, since $n^5/n! < 0.1$ whenever $n \geq 10$, but $9^5/9! > 0.1$. From the second graph, it seems that for $\varepsilon = 0.001$, the smallest possible value for N is 11.

66. Let $\varepsilon > 0$ and let N be any positive integer larger than $\ln(\varepsilon)/\ln|r|$. If $n > N$ then $n > \ln(\varepsilon)/\ln|r| \Rightarrow n\ln|r| < \ln\varepsilon$ [since $|r| < 1 \Rightarrow \ln|r| < 0$] $\Rightarrow \ln\left(|r|^n\right) < \ln\varepsilon \Rightarrow |r|^n < \varepsilon \Rightarrow |r^n - 0| < \varepsilon$, and so by Definition 1, $\lim\limits_{n\to\infty} r^n = 0$.

67. If $\lim\limits_{n\to\infty}|a_n| = 0$ then $\lim\limits_{n\to\infty} -|a_n| = 0$, and since $-|a_n| \leq a_n \leq |a_n|$, we have that $\lim\limits_{n\to\infty} a_n = 0$ by the Squeeze Theorem.

68. (a)
$$\begin{aligned}\frac{b^{n+1}-a^{n+1}}{b-a} &= b^n + b^{n-1}a + b^{n-2}a^2 + b^{n-3}a^3 + \cdots + ba^{n-1} + a^n \\ &< b^n + b^{n-1}b + b^{n-2}b^2 + b^{n-3}b^3 + \cdots + bb^{n-1} + b^n = (n+1)\,b^n\end{aligned}$$

(b) Since $b - a > 0$, we have $b^{n+1} - a^{n+1} < (n+1)\,b^n\,(b-a) \Rightarrow b^{n+1} - (n+1)\,b^n\,(b-a) < a^{n+1} \Rightarrow b^n\left[(n+1)\,a - nb\right] < a^{n+1}$.

(c) With this substitution, $(n+1)\,a - nb = 1$, and so $b^n = \left(1+\dfrac{1}{n}\right)^n < a^{n+1} = \left(1+\dfrac{1}{n+1}\right)^{n+1}$.

(d) With this substitution, we get $\left(1+\dfrac{1}{2n}\right)^n\left(\dfrac{1}{2}\right) < 1 \Rightarrow \left(1+\dfrac{1}{2n}\right)^n < 2 \Rightarrow \left(1+\dfrac{1}{2n}\right)^{2n} < 4$.

(e) $a_n < a_{2n}$ since $\{a_n\}$ is increasing, so $a_n < a_{2n} < 4$.

(f) Since $\{a_n\}$ is increasing and bounded above by 4, $a_1 \leq a_n \leq 4$, and so $\{a_n\}$ is bounded and monotonic, and hence has a limit by Theorem 10.

69. (a) First we show that $a > a_1 > b_1 > b$.

$a_1 - b_1 = \frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}\left(a - 2\sqrt{ab} + b\right) = \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2 > 0$ (since $a > b$) $\Rightarrow$ $a_1 > b_1$. Also

$a - a_1 = a - \frac{1}{2}(a+b) = \frac{1}{2}(a-b) > 0$ and $b - b_1 = b - \sqrt{ab} = \sqrt{b}\left(\sqrt{b} - \sqrt{a}\right) < 0$, so $a > a_1 > b_1 > b$.

In the same way we can show that $a_1 > a_2 > b_2 > b_1$ and so the given assertion is true for $n = 1$. Suppose it is true for $n = k$, that is, $a_k > a_{k+1} > b_{k+1} > b_k$. Then

$$a_{k+2} - b_{k+2} = \tfrac{1}{2}(a_{k+1} + b_{k+1}) - \sqrt{a_{k+1}b_{k+1}} = \tfrac{1}{2}\left(a_{k+1} - 2\sqrt{a_{k+1}b_{k+1}} + b_{k+1}\right)$$

$$= \tfrac{1}{2}\left(\sqrt{a_{k+1}} - \sqrt{b_{k+1}}\right)^2 > 0$$

$$a_{k+1} - a_{k+2} = a_{k+1} - \tfrac{1}{2}(a_{k+1} + b_{k+1}) = \tfrac{1}{2}(a_{k+1} - b_{k+1}) > 0$$

and $b_{k+1} - b_{k+2} = b_{k+1} - \sqrt{a_{k+1}b_{k+1}} = \sqrt{b_{k+1}}\left(\sqrt{b_{k+1}} - \sqrt{a_{k+1}}\right) < 0$ $\Rightarrow$ $a_{k+1} > a_{k+2} > b_{k+2} > b_{k+1}$, so the assertion is true for $n = k + 1$. Thus, it is true for all n by mathematical induction.

(b) From part (a) we have $a > a_n > a_{n+1} > b_{n+1} > b_n > b$, which shows that both sequences, $\{a_n\}$ and $\{b_n\}$, are monotonic and bounded. So they are both convergent by the Monotonic Sequence Theorem.

(c) Let $\lim\limits_{n\to\infty} a_n = \alpha$ and $\lim\limits_{n\to\infty} b_n = \beta$. Then $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \dfrac{a_n + b_n}{2}$ $\Rightarrow$ $\alpha = \dfrac{\alpha + \beta}{2}$ $\Rightarrow$ $2\alpha = \alpha + \beta$ $\Rightarrow$ $\alpha = \beta$.

70. (a) Let $\varepsilon > 0$. Since $\lim\limits_{n\to\infty} a_{2n} = L$, there exists N_1 such that $|a_{2n} - L| < \varepsilon$ for $n > N_1$. Since $\lim\limits_{n\to\infty} a_{2n+1} = L$, there exists N_2 such that $|a_{2n+1} - L| < \varepsilon$ for $n > N_2$. Let $N = \max\{2N_1, 2N_2 + 1\}$ and let $n > N$. If n is even, then $n = 2m$ where $m > N_1$, so $|a_n - L| = |a_{2m} - L| < \varepsilon$. If n is odd, then $n = 2m + 1$, where $m > N_2$, so $|a_n - L| = |a_{2m+1} - L| < \varepsilon$. Therefore $\lim\limits_{n\to\infty} a_n = L$.

(b) $a_1 = 1$, $a_2 = 1 + \frac{1}{1+1} = \frac{3}{2} = 1.5$, $a_3 = 1 + \frac{1}{5/2} = \frac{7}{5} = 1.4$, $a_4 = 1 + \frac{1}{12/5} = \frac{17}{12} = 1.41\overline{6}$,

$a_5 = 1 + \frac{1}{29/12} = \frac{41}{29} \approx 1.413793$, $a_6 = 1 + \frac{1}{70/29} = \frac{99}{70} \approx 1.414286$, $a_7 = 1 + \frac{1}{169/70} = \frac{239}{169} \approx 1.414201$,

$a_8 = 1 + \frac{1}{408/169} = \frac{577}{408} \approx 1.414216$. Notice that $a_1 < a_3 < a_5 < a_7$ and $a_2 > a_4 > a_6 > a_8$. It appears that the odd terms are increasing and the even terms are decreasing. Let's prove that $a_{2n-2} > a_{2n}$ and $a_{2n-1} < a_{2n+1}$ by mathematical induction. Suppose that $a_{2k-2} > a_{2k}$. Then $1 + a_{2k-2} > 1 + a_{2k}$ $\Rightarrow$

$\dfrac{1}{1 + a_{2k-2}} < \dfrac{1}{1 + a_{2k}}$ $\Rightarrow$ $1 + \dfrac{1}{1 + a_{2k-2}} < 1 + \dfrac{1}{1 + a_{2k}}$ $\Rightarrow$ $a_{2k-1} < a_{2k+1}$ $\Rightarrow$

$1 + a_{2k-1} < 1 + a_{2k+1}$ $\Rightarrow$ $\dfrac{1}{1 + a_{2k-1}} > \dfrac{1}{1 + a_{2k+1}}$ $\Rightarrow$ $1 + \dfrac{1}{1 + a_{2k-1}} > 1 + \dfrac{1}{1 + a_{2k+1}}$ $\Rightarrow$

$a_{2k} > a_{2k+2}$. We have thus shown, by induction, that the odd terms are increasing and the even terms are decreasing. Also all terms lie between 1 and 2, so both $\{a_n\}$ and $\{b_n\}$ are bounded monotonic sequences and therefore convergent by Theorem 10. Let $\lim\limits_{n\to\infty} a_{2n} = L$. Then $\lim\limits_{n\to\infty} a_{2n+2} = L$ also. We have

$a_{n+2} = 1 + \dfrac{1}{1 + 1 + 1/(1 + a_n)} = 1 + \dfrac{1}{(3 + 2a_n)/(1 + a_n)} = \dfrac{4 + 3a_n}{3 + 2a_n}$, so $a_{2n+2} = \dfrac{4 + 3a_{2n}}{3 + 2a_{2n}}$. Taking limits of both sides, we get $L = \dfrac{4 + 3L}{3 + 2L}$ $\Rightarrow$ $3L + 2L^2 = 4 + 3L$ $\Rightarrow$ $L^2 = 2$ $\Rightarrow$ $L = \sqrt{2}$ (since $L > 0$).

Thus, $\lim\limits_{n\to\infty} a_{2n} = \sqrt{2}$. Similarly we find that $\lim\limits_{n\to\infty} a_{2n+1} = \sqrt{2}$. So, by part (a), $\lim\limits_{n\to\infty} a_n = \sqrt{2}$.

Laboratory Project □ Logistic Sequences

1. To write such a program in Maple it is best to calculate all the points first and then graph them. One possible sequence of commands [taking $p_0 = \frac{1}{2}$ and $k = 1.5$ for the difference equation] is

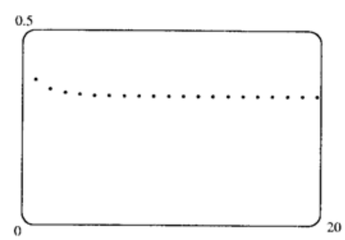
```
p(0):=1/2;k:=1.5;
for j from 1 to 20  do p(j):=k*p(j-1)*(1-p(j-1)) od;
plot({[t,p(t)] $t=0..20},t=0..20,p=0..0.5,style=point);
```

In Mathematica, we can use the following program:

```
p[0]=1/2
k=1.5
p[j_]:=k*p[j-1]*(1-p[j-1])
P=Table[p[t],{t,20}]
ListPlot[P]
```

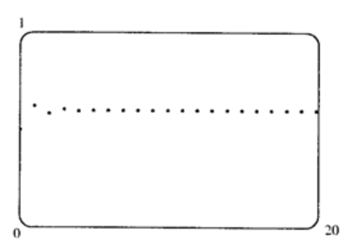
With $p_0 = \frac{1}{2}$ and $k = 1.5$:

n	p_n	n	p_n	n	p_n
0	0.5	7	0.3338465076	14	0.3333373303
1	0.375	8	0.3335895255	15	0.3333353318
2	0.3515625	9	0.3334613309	16	0.3333343326
3	0.3419494629	10	0.3333973076	17	0.3333338329
4	0.3375300416	11	0.3333653143	18	0.3333335831
5	0.3354052689	12	0.3333493223	19	0.3333334582
6	0.3343628617	13	0.3333413274	20	0.3333333958



With $p_0 = \frac{1}{2}$ and $k = 2.5$:

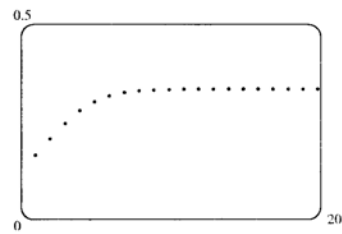
n	p_n	n	p_n	n	p_n
0	0.5	7	0.6004164790	14	0.5999967417
1	0.625	8	0.5997913269	15	0.6000016291
2	0.5859375	9	0.6001042277	16	0.5999991854
3	0.6065368651	10	0.5999478590	17	0.6000004073
4	0.5966247409	11	0.6000260637	18	0.5999997964
5	0.6016591486	12	0.5999869664	19	0.6000001018
6	0.5991635437	13	0.6000065164	20	0.5999999491



Both of these sequences seem to converge (the first to about $\frac{1}{3}$, the second to about 0.60).

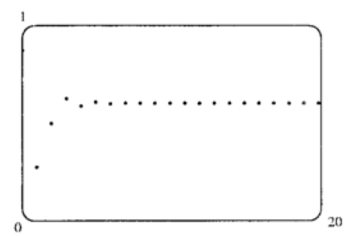
With $p_0 = \frac{7}{8}$ and $k = 1.5$:

n	p_n	n	p_n	n	p_n
0	0.875	7	0.3239166554	14	0.3332554829
1	0.1640625	8	0.3284919837	15	0.3332943990
2	0.2057189941	9	0.3308775005	16	0.3333138639
3	0.2450980344	10	0.3320963702	17	0.3333235980
4	0.2775374819	11	0.3327125567	18	0.3333284655
5	0.3007656421	12	0.3330223670	19	0.3333308994
6	0.3154585059	13	0.3331777051	20	0.3333321164



With $p_0 = \frac{7}{8}$ and $k = 2.5$:

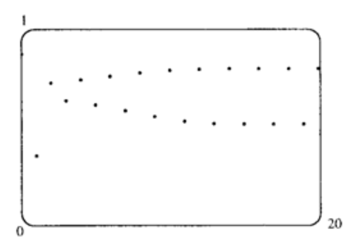
n	p_n	n	p_n	n	p_n
0	0.875	7	0.6016572368	14	0.5999869815
1	0.2734375	8	0.5991645155	15	0.6000065088
2	0.4966735840	9	0.6004159972	16	0.5999967455
3	0.6249723374	10	0.5997915688	17	0.6000016272
4	0.5859547872	11	0.6001041070	18	0.5999991864
5	0.6065294364	12	0.5999479194	19	0.6000004068
6	0.5966286980	13	0.6000260335	20	0.5999997966



The limit of the sequence seems to depend on k, but not on p_0.

2. With $p_0 = \frac{7}{8}$ and $k = 3.2$:

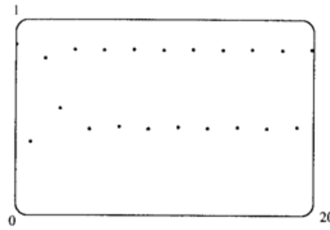
n	p_n	n	p_n	n	p_n
0	0.875	7	0.5830728495	14	0.7990633827
1	0.35	8	0.7779164854	15	0.5137954979
2	0.728	9	0.5528397669	16	0.7993909896
3	0.6336512	10	0.7910654689	17	0.5131681132
4	0.7428395416	11	0.5288988570	18	0.7994451225
5	0.6112926626	12	0.7973275394	19	0.5130643795
6	0.7603646184	13	0.5171082698	20	0.7994538304



It seems that eventually the terms fluctuate between two values (about 0.5 and 0.8 in this case).

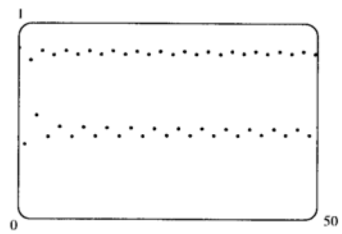
3. With $p_0 = \frac{7}{8}$ and $k = 3.42$:

n	p_n	n	p_n	n	p_n
0	0.875	7	0.4523028596	14	0.8442074951
1	0.3740625	8	0.8472194412	15	0.4498025048
2	0.8007579316	9	0.4426802161	16	0.8463823232
3	0.5456427596	10	0.8437633929	17	0.4446659586
4	0.8478752457	11	0.4508474156	18	0.8445284520
5	0.4411212220	12	0.8467373602	19	0.4490464985
6	0.8431438501	13	0.4438243545	20	0.8461207931

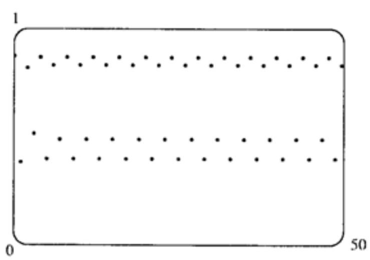


With $p_0 = \frac{7}{8}$ and $k = 3.45$:

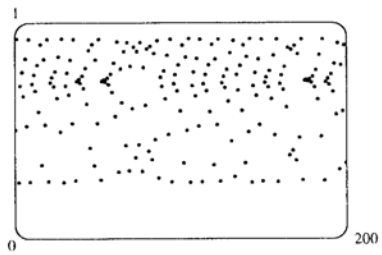
n	p_n	n	p_n	n	p_n
0	0.875	7	0.4670259170	14	0.8403376122
1	0.37734375	8	0.8587488490	15	0.4628875685
2	0.8105962830	9	0.4184824586	16	0.8577482026
3	0.5296783241	10	0.8395743720	17	0.4209559716
4	0.8594612299	11	0.4646778983	18	0.8409445432
5	0.4167173034	12	0.8581956045	19	0.4614610237
6	0.8385707740	13	0.4198508858	20	0.8573758782



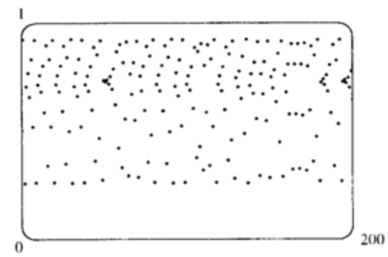
From the graphs above, it seems that for k between 3.4 and 3.5, the terms eventually fluctuate between four values. In the graph below, the pattern followed by the terms is 0.395, 0.832, 0.487, 0.869, 0.395, ... Note that even for $k = 3.42$ (as in the first graph), there are four distinct "branches; even after 1000 terms, the first and third terms in the pattern differ by about 2×10^{-9}, while the first and fifth terms differ by only 2×10^{-10}.
With $p_0 = \frac{7}{8}$ and $k = 3.48$:



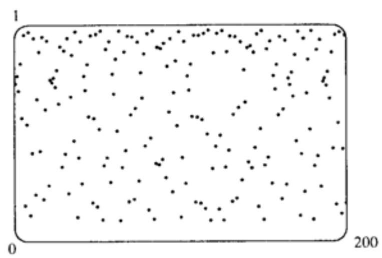
4.



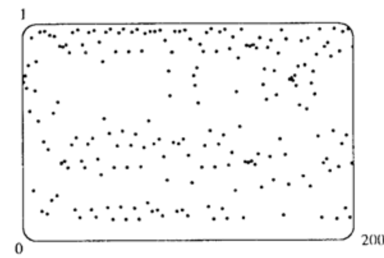
$p_0 = 0.5,\ k = 3.7$



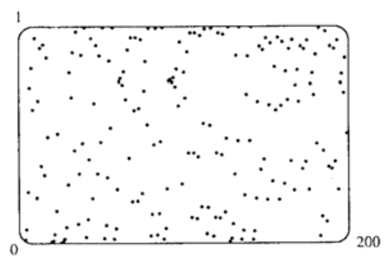
$p_0 = 0.501,\ k = 3.7$



$p_0 = 0.75,\ k = 3.9$



$p_0 = 0.749,\ k = 3.9$



$p_0 = 0.5,\ k = 3.999$

From the graphs, it seems that if p_0 is changed by 0.001, the whole graph changes completely. (Note, however, that this might be partially due to accumulated round-off error in the CAS. These graphs were generated by Maple with 100-digit accuracy, and different degrees of accuracy give different graphs.) There seem to be some some fleeting patterns in these graphs, but on the whole they are certainly very chaotic. As k increases, the graph spreads out vertically, with more extreme values close to 0 or 1.

[redacted]

12.2 Series

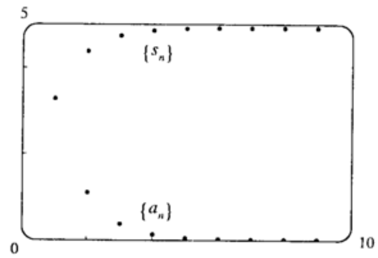
1. (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.

(b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.

2. $\sum_{n=1}^{\infty} a_n = 5$ means that by adding sufficiently many terms of the series we can get as close as we like to the number 5. In other words, it means that $\lim_{n\to\infty} s_n = 5$, where s_n is the nth partial sum, that is, $\sum_{i=1}^{n} a_i$.

3.

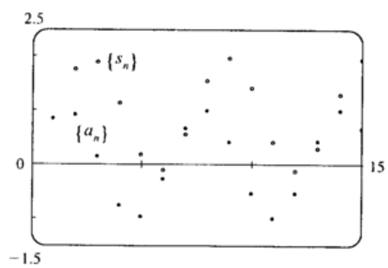
n	s_n
1	3.33333
2	4.44444
3	4.81481
4	4.93827
5	4.97942
6	4.99314
7	4.99771
8	4.99924
9	4.99975
10	4.99992
11	4.99997
12	4.99999



From the graph, it seems that the series converges. In fact, it is a geometric series with $a = \frac{10}{3}$ and $r = \frac{1}{3}$, so its sum is $\sum_{n=1}^{\infty} \frac{10}{3^n} = \frac{10/3}{1 - 1/3} = 5$. Note that the dot corresponding to $n = 1$ is part of both $\{a_n\}$ and $\{s_n\}$.

4.

n	s_n
1	0.8415
2	1.7508
3	1.8919
4	1.1351
5	0.1762
6	−0.1033
7	0.5537
8	1.5431
9	1.9552
10	1.4112
11	0.4112
12	−0.1254

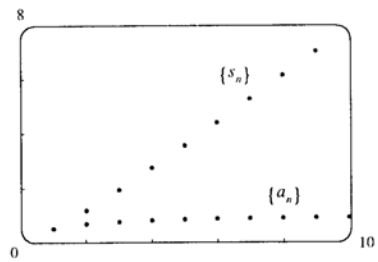


The series diverges,since its terms do not approach 0.

[redacted]

5.

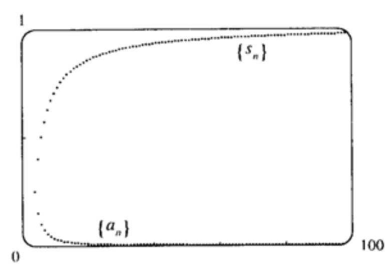
n	s_n
1	0.50000
2	1.16667
3	1.91667
4	2.71667
5	3.55000
6	4.40714
7	5.28214
8	6.17103
9	7.07103
10	7.98012



The series diverges,since its terms do not approach 0.

6.

n	s_n
4	0.25000
5	0.40000
6	0.50000
7	0.57143
8	0.62500
9	0.66667
10	0.70000
11	0.72727
12	0.75000
13	0.76923
...	...
99	0.96970
100	0.97000



From the graph, it seems that the series converges to about 1. To find the sum, we proceed as in Example 6: since $\frac{3}{i\,(i-1)} = \frac{3}{i-1} - \frac{3}{i}$, the partial sums are

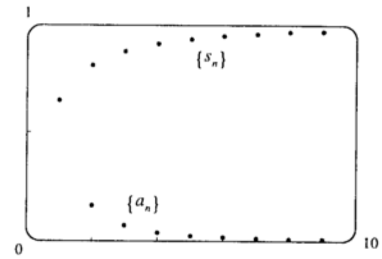
$$s_n = \sum_{i=4}^{n} \left(\frac{3}{i-1} - \frac{3}{i}\right) = \left(\frac{3}{3} - \frac{3}{4}\right) + \left(\frac{3}{4} - \frac{3}{5}\right) + \cdots$$

$$+ \left(\frac{3}{n-2} - \frac{3}{n-1}\right) + \left(\frac{3}{n-1} - \frac{3}{n}\right) = 1 - \frac{3}{n}$$

and so the sum is $\lim_{n\to\infty} s_n = 1$.

7.

n	s_n
1	0.64645
2	0.80755
3	0.87500
4	0.91056
5	0.93196
6	0.94601
7	0.95581
8	0.96296
9	0.96838
10	0.97259



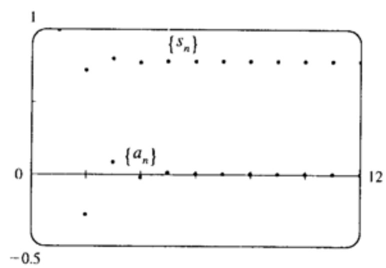
From the graph, it seems that the series converges to 1. To find the sum, we write

$$s_n = \sum_{i=1}^{n} \left(\frac{1}{i^{1.5}} - \frac{1}{(i+1)^{1.5}} \right) = \left(1 - \frac{1}{2^{1.5}}\right) + \left(\frac{1}{2^{1.5}} - \frac{1}{3^{1.5}}\right) + \left(\frac{1}{3^{1.5}} - \frac{1}{4^{1.5}}\right) + \cdots + \left(\frac{1}{n^{1.5}} - \frac{1}{(n+1)^{1.5}}\right) = 1 - \frac{1}{(n+1)^{1.5}}$$

So the sum is $\lim_{n\to\infty} s_n = 1$.

8.

n	s_n
1	1.000000
2	0.714286
3	0.795918
4	0.772595
5	0.779259
6	0.777355
7	0.777899
8	0.777743
9	0.777788
10	0.777775
11	0.777779
12	0.777778



From the graph, it seems that the series converges to about 0.8. In fact, it is a geometric series with $a = 1$ and $r = -\frac{2}{7}$, so its sum is

$$\sum_{n=1}^{\infty} \left(-\frac{2}{7}\right)^{n-1} = \frac{1}{1-(-2/7)} = \frac{7}{9}.$$

9. (a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{2n}{3n+1} = \dfrac{2}{3}$, so the *sequence* $\{a_n\}$ is convergent by (12.1.1).

(b) Since $\lim_{n\to\infty} a_n = \frac{2}{3} \neq 0$, the *series* $\sum_{n=1}^{\infty} a_n$ is divergent by the Test for Divergence (7).

10. (a) Both $\sum\limits_{i=1}^{n} a_i$ and $\sum\limits_{j=1}^{n} a_j$ represent the sum of the first n terms of the sequence $\{a_n\}$, that is, the nth partial sum.

(b) $\sum\limits_{i=1}^{n} a_j = \underbrace{a_j + a_j + \cdots + a_j}_{n \text{ terms}} = na_j$, which, in general, is not the same as $\sum\limits_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$.

11. $4 + \frac{8}{5} + \frac{16}{25} + \frac{32}{125} + \cdots$ is a geometric series with $a = 4$ and $r = \frac{2}{5}$. Since $|r| = \frac{2}{5} < 1$, the series converges to $\frac{4}{1-2/5} = \frac{4}{3/5} = \frac{20}{3}$.

12. $1 - \frac{3}{2} + \frac{9}{4} - \frac{27}{8} + \cdots$ is a geometric series with $a = 1$ and $r = -\frac{3}{2}$. Since $|r| = \frac{3}{2} > 1$, the series diverges.

13. $-2 + \frac{5}{2} - \frac{25}{8} + \frac{125}{32} - \cdots$ is a geometric series with $a = -2$ and $r = \frac{5/2}{-2} = -\frac{5}{4}$. Since $|r| = \frac{5}{4} > 1$, the series diverges by (4).

14. $1 + 0.4 + 0.16 + 0.064 + \cdots$ is a geometric series with ratio 0.4. The series converges to $\frac{a}{1-r} = \frac{1}{1-2/5} = \frac{5}{3}$ since $|r| = \frac{2}{5} < 1$.

15. $\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n-1}$ is a geometric series with $a = 5$ and $r = \frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series converges to $\frac{a}{1-r} = \frac{5}{1-2/3} = \frac{5}{1/3} = 15$.

16. $\sum\limits_{n=1}^{\infty} \frac{(-6)^{n-1}}{5^{n-1}}$ is a geometric series with $a = 1$ and $r = -\frac{6}{5}$. The series diverges since $|r| = \frac{6}{5} > 1$.

17. $\sum\limits_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{1}{4}\sum\limits_{n=1}^{\infty}\left(-\frac{3}{4}\right)^{n-1}$. The latter series is geometric with $a = 1$ and $r = -\frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, it converges to $\frac{1}{1-(-3/4)} = \frac{4}{7}$. Thus, the given series converges to $\left(\frac{1}{4}\right)\left(\frac{4}{7}\right) = \frac{1}{7}$.

18. $\sum\limits_{n=1}^{\infty}\left(\frac{1}{e^2}\right)^n \Rightarrow a = \frac{1}{e^2} = |r| < 1$, so the series converges to $\frac{1/e^2}{1 - 1/e^2} = \frac{1}{e^2 - 1}$.

19. For $\sum_{n=1}^{\infty} 3^{-n}8^{n+1} = \sum_{n=1}^{\infty} 8\left(\frac{8}{3}\right)^n$, $a = \frac{64}{3}$ and $r = \frac{8}{3} > 1$, so the series diverges.

20. $\sum_{n=0}^{\infty} 4\left(\frac{4}{5}\right)^n \Rightarrow a = 4$, $|r| = \frac{4}{5} < 1$, so the series converges to $\frac{4}{1-4/5} = 20$.

21. $\sum\limits_{n=1}^{\infty} \frac{n}{n+5}$ diverges since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n}{n+5} = 1 \neq 0$. [Use (7), the Test for Divergence.]

22. $\sum_{n=1}^{\infty}(3/n) = 3\sum_{n=1}^{\infty}(1/n)$ diverges since each of its partial sums is 3 times the corresponding partial sum of the harmonic series $\sum_{n=1}^{\infty}(1/n)$, which diverges. [If $\sum_{n=1}^{\infty}(3/n)$ were to converge, then $\sum_{n=1}^{\infty}(1/n)$ would also have to converge by Theorem 8(i).] In general, constant multiples of divergent series are divergent.

23. Converges. $s_n = \sum\limits_{i=1}^{n} \frac{1}{i(i+2)} = \sum\limits_{i=1}^{n}\left(\frac{1/2}{i} - \frac{1/2}{i+2}\right)$ (using partial fractions) $= \frac{1}{2}\sum\limits_{i=1}^{n}\left(\frac{1}{i} - \frac{1}{i+2}\right)$. The latter sum is a telescoping series:

$$\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right) = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

Thus, $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{1}{2}\lim\limits_{n\to\infty}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}$.

24. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$ diverges by (7), the Test for Divergence, since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2+2n} = \lim_{n\to\infty}\left(1+\frac{1}{n^2+2n}\right) = 1 \neq 0.$$

25. $\sum_{n=1}^{\infty}\left[2(0.1)^n + (0.2)^n\right] = 2\sum_{n=1}^{\infty}(0.1)^n + \sum_{n=1}^{\infty}(0.2)^n$. These are convergent geometric series and so by Theorem 8, their sum is also convergent. $2\left(\frac{0.1}{1-0.1}\right) + \frac{0.2}{1-0.2} = \frac{2}{9} + \frac{1}{4} = \frac{17}{36}$

26. Converges. $s_n = \displaystyle\sum_{i=1}^{n} \frac{2}{i^2+4i+3} = \sum_{i=1}^{n}\left(\frac{1}{i+1} - \frac{1}{i+3}\right)$ (using partial fractions). The latter sum is

$\left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \left(\frac{1}{4}-\frac{1}{6}\right) + \left(\frac{1}{5}-\frac{1}{7}\right) + \cdots + \left(\frac{1}{n}-\frac{1}{n+2}\right) + \left(\frac{1}{n+1}-\frac{1}{n+3}\right) = \frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}$

(telescoping series). Thus, $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n^2+4n+3} = \lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}\right) = \frac{1}{2}+\frac{1}{3} = \frac{5}{6}.$

27. $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n}{\sqrt{1+n^2}} = \lim_{n\to\infty}\frac{1}{\sqrt{1+1/n^2}} = 1 \neq 0$, so the series diverges by the Test for Divergence.

28. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}}+\frac{2}{3^{n-1}}\right) = \sum_{n=1}^{\infty}\frac{1}{2^{n-1}} + 2\sum_{n=1}^{\infty}\frac{1}{3^{n-1}} = \frac{1}{1-1/2} + 2\left(\frac{1}{1-1/3}\right) = 5$

29. Converges. $\displaystyle\sum_{n=1}^{\infty}\frac{3^n+2^n}{6^n} = \sum_{n=1}^{\infty}\left(\frac{3^n}{6^n}+\frac{2^n}{6^n}\right) = \sum_{n=1}^{\infty}\left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n\right] = \frac{1/2}{1-1/2} + \frac{1/3}{1-1/3} = 1 + \frac{1}{2} = \frac{3}{2}$

30. $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\ln\left(\frac{n}{2n+5}\right) = \lim_{n\to\infty}\left(\frac{1}{2+5/n}\right) = \ln\frac{1}{2} \neq 0$, so the series diverges by the Test for Divergence.

31. $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\arctan n = \frac{\pi}{2} \neq 0$, so the series diverges by the Test for Divergence.

32. $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{5+2^{-n}} = \frac{1}{5} \neq 0$, so the series diverges by the Test for Divergence.

33. $s_n = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + \cdots + [\ln n - \ln(n+1)] = \ln 1 - \ln(n+1) = -\ln(n+1)$ (telescoping series). Thus, $\lim_{n\to\infty} s_n = -\infty$, so the series is divergent.

34. $s_n = \displaystyle\sum_{i=1}^{n}\frac{1}{i(i+1)(i+2)} = \sum_{i=1}^{n}\left(\frac{1/2}{i} - \frac{1}{i+1} + \frac{1/2}{i+2}\right) = \sum_{i=1}^{n}\left(\frac{1/2}{i} - \frac{1/2}{i+1}\right) + \sum_{i=1}^{n}\left(-\frac{1/2}{i+1} + \frac{1/2}{i+2}\right),$

both of which are clearly telescoping sums, so

$$s_n = \left[\frac{1}{2} - \frac{1}{2(n+1)}\right] + \left[-\frac{1}{4} + \frac{1}{2(n+2)}\right] = \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$$

Thus, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)} = \lim_{n\to\infty} s_n = \frac{1}{4}.$

35. $0.\overline{2} = \dfrac{2}{10} + \dfrac{2}{10^2} + \cdots = \dfrac{2/10}{1-1/10} = \dfrac{2}{9}$

36. $0.\overline{73} = \dfrac{73}{10^2} + \dfrac{73}{10^4} + \cdots = \dfrac{73/10^2}{1-1/10^2} = \dfrac{73/100}{99/100} = \dfrac{73}{99}$

37. $3.\overline{417} = 3 + \dfrac{417}{10^3} + \dfrac{417}{10^6} + \cdots = 3 + \dfrac{417/10^3}{1-1/10^3} = 3 + \dfrac{417}{999} = \dfrac{3414}{999} = \dfrac{1138}{333}$

38. $6.2\overline{54} = 6.2 + \dfrac{54}{10^3} + \dfrac{54}{10^5} + \cdots = 6.2 + \dfrac{54/10^3}{1 - 1/10^2} = \dfrac{62}{10} + \dfrac{54}{990} = \dfrac{6192}{990} = \dfrac{344}{55}$

39. $0.123\overline{456} = \dfrac{123}{1000} + \dfrac{0.000456}{1 - 0.001} = \dfrac{123}{1000} + \dfrac{456}{999{,}000} = \dfrac{123{,}333}{999{,}000} = \dfrac{41{,}111}{333{,}000}$

40. $5.\overline{6021} = 5 + \dfrac{6021}{10^4} + \dfrac{6021}{8^4} + \cdots = 5 + \dfrac{6021/10^4}{1 - 1/10^4} = 5 + \dfrac{6021}{9999} = \dfrac{56{,}016}{9999} = \dfrac{6224}{1111}$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{3^n}$ is a geometric series with $r = \dfrac{x}{3}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\dfrac{|x|}{3} < 1$ $\Leftrightarrow$ $|x| < 3$. In that case, the sum of the series is $\dfrac{x/3}{1 - x/3} = \dfrac{x}{3 - x}$.

42. $\sum_{n=1}^{\infty} (x-4)^n$ is a geometric series with $r = x - 4$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $|x - 4| < 1$ $\Leftrightarrow$ $3 < x < 5$. In that case, the sum of the series is $\dfrac{x-4}{1-(x-4)} = \dfrac{x-4}{5-x}$.

43. $\sum_{n=0}^{\infty} 4^n x^n = \sum_{n=0}^{\infty} (4x)^n$ is a geometric series with $r = 4x$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $4\,|x| < 1$ $\Leftrightarrow$ $|x| < \frac{1}{4}$. In that case, the sum of the series is $\dfrac{1}{1-4x}$.

44. $\displaystyle\sum_{n=0}^{\infty} \frac{(x+3)^n}{2^n}$ is a geometric series with $r = \dfrac{x+3}{2}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\dfrac{|x+3|}{2} < 1$ $\Leftrightarrow$ $|x+3| < 2$ $\Leftrightarrow$ $-5 < x < -1$. For these values of x, the sum of the series is $\dfrac{1}{1-(x+3)/2} = \dfrac{2}{2-(x+3)} = -\dfrac{2}{x+1}$.

45. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{x}\right)^n$ is geometric with $r = \dfrac{1}{x}$, so it converges whenever $\left|\dfrac{1}{x}\right| < 1$ $\Leftrightarrow$ $|x| > 1$ $\Leftrightarrow$ $x > 1$ or $x < -1$, and the sum is $\dfrac{1}{1 - 1/x} = \dfrac{x}{x-1}$.

46. $\sum_{n=0}^{\infty} \tan^n x$ is geometric and converges when $|\tan x| < 1$ $\Leftrightarrow$ $-1 < \tan x < 1$ $\Leftrightarrow$ $n\pi - \frac{\pi}{4} < x < n\pi + \frac{\pi}{4}$ (n any integer). On these intervals the sum is $\dfrac{1}{1 - \tan x}$.

47. After defining f, We use `convert(f,parfrac);` in Maple, `Apart` in Mathematica, or `Expand Rational` and `Simplify` in Derive to find that the general term is $\dfrac{1}{(4n+1)(4n-3)} = -\dfrac{1/4}{4n+1} + \dfrac{1/4}{4n-3}$. So the nth partial sum is

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \left(-\frac{1/4}{4k+1} + \frac{1/4}{4k-3} \right) = \frac{1}{4} \left(\frac{1}{4k-3} - \frac{1}{4k+1} \right) \\ &= \frac{1}{4} \left[\left(1 - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{13}\right) + \cdots + \left(\frac{1}{4n-3} - \frac{1}{4n+1}\right) \right] = \frac{1}{4}\left(1 - \frac{1}{4n+1}\right) \end{aligned}$$

The series converges to $\lim\limits_{n\to\infty} s_n = \frac{1}{4}$. This can be confirmed by directly computing the sum using `sum(f,1..infinity);` (in Maple), `Sum[f,{n,1,Infinity}]` (in Mathematica), or `Calculus Sum` (from 1 to ∞) and `Simplify` (in Derive).

48. See Exercise 47 for specific CAS commands. $\dfrac{n^2+3n+1}{\left(n^2+n\right)^2} = \dfrac{1}{n^2} + \dfrac{1}{n} - \dfrac{1}{(n+1)^2} - \dfrac{1}{n+1}$. So the nth partial sum is

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \left(\frac{1}{k^2} + \frac{1}{k} - \frac{1}{(k+1)^2} - \frac{1}{k+1} \right) \\ &= \left(1 + 1 - \frac{1}{2^2} - \frac{1}{2}\right) + \left(\frac{1}{2^2} + \frac{1}{2} - \frac{1}{3^2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n^2} + \frac{1}{n} - \frac{1}{(n+1)^2} - \frac{1}{n+1}\right) \\ &= 1 + 1 - \frac{1}{(n+1)^2} - \frac{1}{n+1} \end{aligned}$$

The series converges to $\lim\limits_{n\to\infty} s_n = 2$.

49. For $n = 1$, $a_1 = 0$ since $s_1 = 0$. For $n > 1$,

$$a_n = s_n - s_{n-1} = \frac{n-1}{n+1} - \frac{(n-1)-1}{(n-1)+1} = \frac{(n-1)\,n - (n+1)\,(n-2)}{(n+1)\,n} = \frac{2}{n\,(n+1)}$$

Also, $\sum\limits_{n=1}^{\infty} a_n = \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \dfrac{1 - 1/n}{1 + 1/n} = 1$.

50. $a_1 = s_1 = \frac{5}{2}$. For $n \neq 1$,

$$a_n = s_n - s_{n-1} = \left(3 - n2^{-n}\right) - \left[3 - (n-1)\,2^{-(n-1)}\right] = -\frac{n}{2^n} + \frac{n-1}{2^{n-1}} \cdot \frac{2}{2} = \frac{2\,(n-1)}{2^n} - \frac{n}{2^n} = \frac{n-2}{2^n}$$

Also, $\sum\limits_{n=1}^{\infty} a_n = \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \left(3 - \dfrac{n}{2^n}\right) = 3$ because $\lim\limits_{x\to\infty} \dfrac{x}{2^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1}{2^x \ln 2} = 0$.

51. (a) The first step in the chain occurs when the local government spends D dollars. The people who receive it spend a fraction c of those D dollars, that is, Dc dollars. Those who receive the Dc dollars spend a fraction c of it, that is, Dc^2 dollars. Continuing in this way, we see that the total spending after n transactions is

$S_n = D + Dc + Dc^2 + \cdots + Dc^{n-1} = \dfrac{D\,(1 - c^n)}{1 - c}$ by (3).

(b) $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \dfrac{D\,(1-c^n)}{1-c} = \dfrac{D}{1-c} \lim\limits_{n\to\infty} (1 - c^n) = \dfrac{D}{1-c}$ (since $0 < c < 1 \Rightarrow \lim\limits_{n\to\infty} c^n = 0$)

$= \dfrac{D}{s}$ (since $c + s = 1$) $= kD$ (since $k = 1/s$)

If $c = 0.8$, then $s = 1 - c = 0.2$ and the multiplier is $k = 1/s = 5$.

52. (a) Initially, the ball falls a distance H, then rebounds a distance rH, falls rH, rebounds r^2H, falls r^2H, etc. The total distance it travels is

$$\begin{aligned} &H + 2rH + 2r^2H + 2r^3H + \cdots = H\left(1 + 2r + 2r^2 + 2r^3 + \cdots\right) \\ &= H\left[1 + 2r\left(1 + r + r^2 + \cdots\right)\right] = H\left[1 + 2r\left(\frac{1}{1-r}\right)\right] = H\left(\frac{1+r}{1-r}\right) \text{ meters} \end{aligned}$$

(b) From Example 3 in Section 2.1, we know that a ball falls $\frac{1}{2}gt^2$ meters in t seconds, where g is the gravitational acceleration. Thus, a ball falls h meters in $t = \sqrt{2h/g}$ seconds. The total travel time in seconds is

$$\begin{aligned} &\sqrt{\frac{2H}{g}} + 2\sqrt{\frac{2H}{g}r} + 2\sqrt{\frac{2H}{g}r^2} + 2\sqrt{\frac{2H}{g}r^3} + \cdots = \sqrt{\frac{2H}{g}}\left[1 + 2\sqrt{r} + 2\sqrt{r}^2 + 2\sqrt{r}^3 + \cdots\right] \\ &= \sqrt{\frac{2H}{g}}\left(1 + 2\sqrt{r}\left[1 + \sqrt{r} + \sqrt{r}^2 + \cdots\right]\right) = \sqrt{\frac{2H}{g}}\left[1 + 2\sqrt{r}\left(\frac{1}{1-\sqrt{r}}\right)\right] = \sqrt{\frac{2H}{g}}\,\frac{1+\sqrt{r}}{1-\sqrt{r}} \end{aligned}$$

(c) It will help to make a chart of the time for each descent and each rebound of the ball, together with the velocity just before and just after each bounce. Recall that the time in seconds needed to fall h meters is $\sqrt{2h/g}$. The ball hits the ground with velocity $-g\sqrt{2h/g} = -\sqrt{2hg}$ (taking the upward direction to be positive) and rebounds with velocity $kg\sqrt{2h/g} = k\sqrt{2hg}$, taking time $k\sqrt{2h/g}$ to reach the top of its bounce, where its velocity is 0. At that point, its height is k^2h. All these results follow from the formulas for vertical motion with gravitational acceleration $-g$: $\dfrac{d^2y}{dt^2} = -g \;\Rightarrow\; v = \dfrac{dy}{dt} = v_0 - gt \;\Rightarrow\; y = y_0 + v_0t - \frac{1}{2}gt^2$.

number of descent	time of descent	speed before bounce	speed after bounce	time of ascent	peak height
1	$\sqrt{2H/g}$	$\sqrt{2Hg}$	$k\sqrt{2Hg}$	$k\sqrt{2H/g}$	k^2H
2	$\sqrt{2k^2H/g}$	$\sqrt{2k^2Hg}$	$k\sqrt{2k^2Hg}$	$k\sqrt{2k^2H/g}$	k^4H
3	$\sqrt{2k^4H/g}$	$\sqrt{2k^4Hg}$	$k\sqrt{2k^4Hg}$	$k\sqrt{2k^4H/g}$	k^6H
...	...	...	...	...	...

The total travel time in seconds is

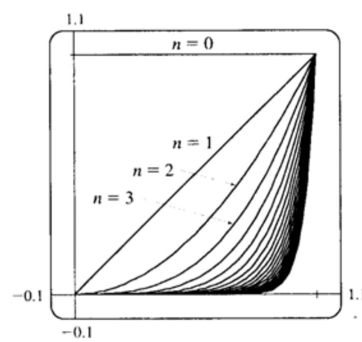
$$\sqrt{\frac{2H}{g}} + k\sqrt{\frac{2H}{g}} + k\sqrt{\frac{2H}{g}} + k^2\sqrt{\frac{2H}{g}} + k^2\sqrt{\frac{2H}{g}} + \cdots = \sqrt{\frac{2H}{g}}\left(1 + 2k + 2k^2 + 2k^3 + \cdots\right)$$
$$= \sqrt{\frac{2H}{g}}\left[1 + 2k\left(1 + k + k^2 + \cdots\right)\right] = \sqrt{\frac{2H}{g}}\left[1 + 2k\left(\frac{1}{1-k}\right)\right] = \sqrt{\frac{2H}{g}}\,\frac{1+k}{1-k}$$

Another Method: We could use part (b). At the top of the bounce, the height is $k^2h = rh$, so $\sqrt{r} = k$ and the result follows from part (b).

53. $\sum_{n=2}^{\infty}(1+c)^{-n}$ is a geometric series with $a = (1+c)^{-2}$ and $r = (1+c)^{-1}$, so the series converges when $\left|(1+c)^{-1}\right| < 1 \;\Leftrightarrow\; |1+c| > 1 \;\Leftrightarrow\; 1+c > 1$ or $1+c < -1 \;\Leftrightarrow\; c > 0$ or $c < -2$. We calculate the sum of the series and set it equal to 2: $\dfrac{(1+c)^{-2}}{1-(1+c)^{-1}} = 2 \;\Leftrightarrow\; \left(\dfrac{1}{1+c}\right)^2 = 2 - 2\left(\dfrac{1}{1+c}\right) \;\Leftrightarrow$ $1 = 2(1+c)^2 - 2(1+c) = 0 \;\Leftrightarrow\; 2c^2 + 2c - 1 = 0 \;\Leftrightarrow\; c = \frac{-2\pm\sqrt{12}}{4} = \frac{\pm\sqrt{3}-1}{2}$. However, the negative root is inadmissible because $-2 < \frac{-\sqrt{3}-1}{2} < 0$. So $c = \frac{\sqrt{3}-1}{2}$.

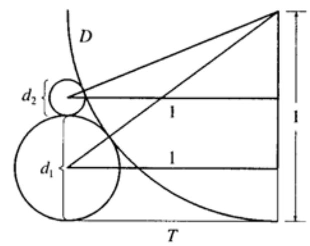
54. The area between $y = x^{n-1}$ and $y = x^n$ for $0 \le x \le 1$ is

$$\int_0^1 \left(x^{n-1} - x^n\right) dx = \left[\frac{x^n}{n} - \frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n} - \frac{1}{n+1}$$
$$= \frac{(n+1)-n}{n(n+1)} = \frac{1}{n(n+1)}$$



We can see from the diagram that as $n \to \infty$, the sum of the areas between the successive curves approaches the area of the unit square, that is, 1. So $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$.

55. Let d_n be the diameter of C_n. We draw lines from the centers of the C_i to the center of D (or C), and using the Pythagorean Theorem, we can write $1^2 + \left(1 - \frac{1}{2}d_1\right)^2 = \left(1 + \frac{1}{2}d_1\right)^2 \iff$ $1 = \left(1 + \frac{1}{2}d_1\right)^2 - \left(1 - \frac{1}{2}d_1\right)^2 = 2d_1$ (difference of squares) $\Rightarrow \quad d_1 = \frac{1}{2}$. Similarly,



$$1 = \left(1 + \tfrac{1}{2}d_2\right)^2 - \left(1 - d_1 - \tfrac{1}{2}d_2\right)^2 = 2d_2 + 2d_1 - d_1^2 - d_1 d_2$$
$$= (2 - d_1)(d_1 + d_2) \quad \iff$$

$d_2 = \dfrac{1}{2 - d_1} - d_1 = \dfrac{(1 - d_1)^2}{2 - d_1}$, $1 = \left(1 + \frac{1}{2}d_3\right)^2 - \left(1 - d_1 - d_2 - \frac{1}{2}d_3\right)^2 \iff d_3 = \dfrac{[1 - (d_1 + d_2)]^2}{2 - (d_1 + d_2)}$, and in general, $d_{n+1} = \dfrac{\left(1 - \sum_{i=1}^n d_i\right)^2}{2 - \sum_{i=1}^n d_i}$. If we actually calculate d_2 and d_3 from the formulas above, we find that they are $\dfrac{1}{6} = \dfrac{1}{2 \cdot 3}$ and $\dfrac{1}{12} = \dfrac{1}{3 \cdot 4}$ respectively, so we suspect that in general, $d_n = \dfrac{1}{n\,(n+1)}$. To prove this, we use induction: assume that for all $k \le n$, $d_k = \dfrac{1}{k\,(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$. Then

$\displaystyle\sum_{i=1}^{n} d_i = 1 - \frac{1}{n+1} = \frac{n}{n+1}$ (telescoping sum). Substituting this into our formula for d_{n+1}, we get

$$d_{n+1} = \frac{\left[1 - \dfrac{n}{n+1}\right]^2}{2 - \left(\dfrac{n}{n+1}\right)} = \frac{\dfrac{1}{(n+1)^2}}{\dfrac{n+2}{n+1}} = \frac{1}{(n+1)\,(n+2)},$$ and the induction is complete.

Now, we observe that the partial sums $\sum_{i=1}^n d_i$ of the diameters of the circles approach 1 as $n \to \infty$; that is, $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n\,(n+1)} = 1$, which is what we wanted to prove.

56. $|CD| = b\sin\theta$, $|DE| = |CD|\sin\theta = b\sin^2\theta$, $|EF| = |DE|\sin\theta = b\sin^3\theta$, ... Therefore, $|CD| + |DE| + |EF| + |FG| + \cdots = b\displaystyle\sum_{n=1}^{\infty}\sin^n\theta = b\left(\frac{\sin\theta}{1 - \sin\theta}\right)$ since this is a geometric series with $r = \sin\theta$ and $|\sin\theta| < 1$ (because $0 < \theta < \frac{\pi}{2}$).

57. The series $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ diverges (geometric series with $r = -1$) so we cannot say that $0 = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$.

58. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$ by Theorem 6, so $\lim_{n\to\infty} \dfrac{1}{a_n} \neq 0$, and so $\sum_{n=1}^{\infty} \dfrac{1}{a_n}$ is divergent by the Test for Divergence.

59. $\sum_{n=1}^{\infty} ca_n = \lim_{n\to\infty} \sum_{i=1}^{n} ca_i = \lim_{n\to\infty} c\sum_{i=1}^{n} a_i = c\lim_{n\to\infty}\sum_{i=1}^{n} a_i = c\sum_{n=1}^{\infty} a_n$, which exists by hypothesis.

60. If $\sum ca_n$ were convergent, then $\sum (1/c)\,(ca_n) = \sum a_n$ would be also, by Theorem 8. But this is not the case, so $\sum ca_n$ must diverge.

61. Suppose on the contrary that $\sum (a_n + b_n)$ converges. Then by Theorem 8(iii), so would $\sum [(a_n + b_n) - a_n] = \sum b_n$, a contradiction.

62. No. For example, take $\sum a_n = \sum n$ and $\sum b_n = \sum (-n)$, which both diverge, yet $\sum (a_n + b_n) = \sum 0$, which converges with sum 0.

63. The partial sums $\{s_n\}$ form an increasing sequence, since $s_n - s_{n-1} = a_n > 0$ for all n. Also, the sequence $\{s_n\}$ is bounded since $s_n \le 1000$ for all n. So by Theorem 12.1.10, the sequence of partial sums converges, that is, the series $\sum a_n$ is convergent.

64. (a) $\text{RHS} = \dfrac{1}{f_{n-1} f_n} - \dfrac{1}{f_n f_{n+1}} = \dfrac{f_n f_{n+1} - f_n f_{n-1}}{f_n^2 f_{n-1} f_{n+1}} = \dfrac{f_{n+1} - f_{n-1}}{f_n f_{n-1} f_{n+1}} = \dfrac{(f_{n-1} + f_n) - f_{n-1}}{f_n f_{n-1} f_{n+1}}$

$= \dfrac{1}{f_{n-1} f_{n+1}} = \text{LHS}$

(b) $$\sum_{n=2}^{\infty} \frac{1}{f_{n-1} f_{n+1}} = \sum_{n=2}^{\infty} \left(\frac{1}{f_{n-1} f_n} - \frac{1}{f_n f_{n+1}} \right) \quad \text{[from part (a)]}$$

$$= \lim_{n\to\infty} \left[\left(\frac{1}{f_1 f_2} - \frac{1}{f_2 f_3} \right) + \left(\frac{1}{f_2 f_3} - \frac{1}{f_3 f_4} \right) + \left(\frac{1}{f_3 f_4} - \frac{1}{f_4 f_5} \right) + \cdots + \left(\frac{1}{f_{n-1} f_n} - \frac{1}{f_n f_{n+1}} \right) \right]$$

$$= \lim_{n\to\infty} \left(\frac{1}{f_1 f_2} - \frac{1}{f_n f_{n+1}} \right) = \frac{1}{f_1 f_2} - 0 = \frac{1}{1 \cdot 1} = 1 \text{ because } f_n \to \infty \text{ as } n \to \infty.$$

(c) $$\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1} f_{n+1}} = \sum_{n=2}^{\infty} \left(\frac{f_n}{f_{n-1} f_n} - \frac{f_n}{f_n f_{n+1}} \right) \quad \text{(as above)} \quad = \sum_{n=2}^{\infty} \left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}} \right)$$

$$= \lim_{n\to\infty} \left[\left(\frac{1}{f_1} - \frac{1}{f_3} \right) + \left(\frac{1}{f_2} - \frac{1}{f_4} \right) + \left(\frac{1}{f_3} - \frac{1}{f_5} \right) + \left(\frac{1}{f_4} - \frac{1}{f_6} \right) + \cdots + \left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}} \right) \right]$$

$$= \lim_{n\to\infty} \left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{1}{f_n} - \frac{1}{f_{n+1}} \right) = 1 + 1 - 0 - 0 = 2 \text{ because } f_n \to \infty \text{ as } n \to \infty.$$

65. (a) At the first step, only the interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ (length $\frac{1}{3}$) is removed. At the second step, we remove the intervals $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$, which have a total length of $2 \cdot \left(\frac{1}{3}\right)^2$. At the third step, we remove 2^2 intervals, each of length $\left(\frac{1}{3}\right)^3$. In general, at the nth step we remove 2^{n-1} intervals, each of length $\left(\frac{1}{3}\right)^n$, for a length of $2^{n-1} \cdot \left(\frac{1}{3}\right)^n = \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$. Thus, the total length of all removed intervals is $\sum\limits_{n=1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} = \frac{1/3}{1-2/3} = 1$ (geometric series with $a = \frac{1}{3}$ and $r = \frac{2}{3}$). Notice that at the nth step, the leftmost interval that is removed is $\left(\left(\frac{1}{3}\right)^n, \left(\frac{2}{3}\right)^n\right)$, so we never remove 0, and 0 is in the Cantor set. Also, the rightmost interval removed is $\left(1 - \left(\frac{2}{3}\right)^n, 1 - \left(\frac{1}{3}\right)^n\right)$, so 1 is never removed. Some other numbers in the Cantor set are $\frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{7}{9}$, and $\frac{8}{9}$.

(b) The area removed at the first step is $\frac{1}{9}$; at the second step, $8 \cdot \left(\frac{1}{9}\right)^2$; at the third step, $(8)^2 \cdot \left(\frac{1}{9}\right)^3$. In general, the area removed at the nth step is $(8)^{n-1}\left(\frac{1}{9}\right)^n = \frac{1}{9}\left(\frac{8}{9}\right)^{n-1}$, so the total area of all removed squares is $\sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{8}{9}\right)^{n-1} = \frac{1/9}{1-8/9} = 1$.

66. (a)

a_1	1	2	4	1	1	1000
a_2	2	3	1	4	1000	1
a_3	1.5	2.5	2.5	2.5	500.5	500.5
a_4	1.75	2.75	1.75	3.25	750.25	250.75
a_5	1.625	2.625	2.125	2.875	625.375	375.625
a_6	1.6875	2.6875	1.9375	3.0625	687.813	313.188
a_7	1.65625	2.65625	2.03125	2.96875	656.594	344.406
a_8	1.67188	2.67188	1.98438	3.01563	672.203	328.797
a_9	1.66406	2.66406	2.00781	2.99219	664.398	336.602
a_{10}	1.66797	2.66797	1.99609	3.00391	668.301	332.699
a_{11}	1.66602	2.66602	2.00195	2.99805	666.350	334.650
a_{12}	1.66699	2.66699	1.99902	3.00098	667.325	333.675

The limits seem to be $\frac{5}{3}, \frac{8}{3}$, 2, 3, 667, and 334. Note that the limits appear to be "weighted" more toward a_2. In general, we guess that the limit is $\dfrac{a_1 + 2a_2}{3}$.

(b) $a_{n+1} - a_n = \frac{1}{2}(a_n + a_{n-1}) - a_n = -\frac{1}{2}(a_n - a_{n-1}) = -\frac{1}{2}\left[\frac{1}{2}(a_{n-1} + a_{n-2}) - a_{n-1}\right]$

$$= -\tfrac{1}{2}\left[-\tfrac{1}{2}(a_{n-1} - a_{n-2})\right] = \cdots = \left(-\tfrac{1}{2}\right)^{n-1}(a_2 - a_1)$$

Note that we have used the formula $a_k = \frac{1}{2}(a_{k-1} + a_{k-2})$ a total of $n - 1$ times in this calculation, once for each k between 3 and $n + 1$. Now we can write

$$a_n = a_1 + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1})$$
$$= a_1 + \textstyle\sum_{k=1}^{n-1}(a_{k+1} - a_k) = a_1 + \sum_{k=1}^{n-1}\left(-\frac{1}{2}\right)^{k-1}(a_2 - a_1)$$

and so

$$\lim_{n\to\infty} a_n = a_1 + (a_2 - a_1)\textstyle\sum_{k=1}^{\infty}\left(-\frac{1}{2}\right)^{k-1} = a_1 + (a_2 - a_1)\left[\frac{1}{1-(-1/2)}\right]$$
$$= a_1 + \tfrac{2}{3}(a_2 - a_1) = \frac{a_1 + 2a_2}{3}$$

67. (a) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{(n+1)!} \Rightarrow s_1 = \frac{1}{1\cdot 2} = \frac{1}{2}, s_2 = \frac{1}{2} + \frac{2}{1\cdot 2\cdot 3} = \frac{5}{6}, s_3 = \frac{5}{6} + \frac{3}{1\cdot 2\cdot 3\cdot 4} = \frac{23}{24},$

$s_4 = \dfrac{23}{24} + \dfrac{4}{1\cdot 2\cdot 3\cdot 4\cdot 5} = \dfrac{119}{120}$. The denominators are $(n+1)!$, so a guess would be $s_n = \dfrac{(n+1)! - 1}{(n+1)!}$.

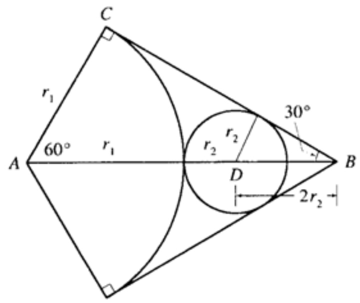
(b) For $n = 1$, $s_1 = \dfrac{1}{2} = \dfrac{2! - 1}{2!}$, so the formula holds for $n = 1$. Assume $s_k = \dfrac{(k+1)! - 1}{(k+1)!}$. Then

$$s_{k+1} = \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+2)!} = \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+1)!\,(k+2)}$$
$$= \frac{(k+2)! - (k+2) + k + 1}{(k+2)!} = \frac{(k+2)! - 1}{(k+2)!}$$

Thus, the formula is true for $n = k + 1$. So by induction, the guess is correct.

(c) $\displaystyle\lim_{n\to\infty} s_n = \lim_{n\to\infty}\frac{(n+1)! - 1}{(n+1)!} = \lim_{n\to\infty}\left[1 - \frac{1}{(n+1)!}\right] = 1$ and so $\displaystyle\sum_{n=0}^{\infty}\frac{n}{(n+1)!} = 1.$

68.



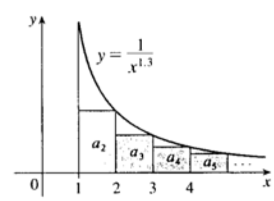
Let r_1 = radius of the large circle, r_2 = radius of next circle, and so on. From the figure we have $\angle BAC = 60°$ and $\cos 60° = r_1/|AB|$, so $|AB| = 2r_1$ and $|DB| = 2r_2$. Therefore, $2r_1 = r_1 + r_2 + 2r_2 = r_1 + 3r_2 \Rightarrow r_1 = 3r_2$. In general, we have $r_{n+1} = \frac{1}{3}r_n$, so the total area is

$$\begin{aligned} A &= \pi r_1^2 + 3\pi r_2^2 + 3\pi r_3^2 + \cdots \\ &= \pi r_1^2 + 3\pi r_2^2\left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \cdots\right) \\ &= \pi r_1^2 + 3\pi r_2^2 \cdot \tfrac{1}{1-1/9} = \pi r_1^2 + \tfrac{27}{8}\pi r_2^2 \end{aligned}$$

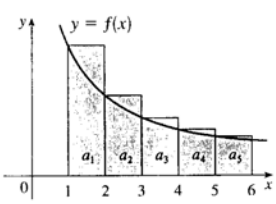
Since the sides of the triangle have length 1, $|BC| = \frac{1}{2}$ and $\tan 30° = \dfrac{r_1}{1/2}$. Thus, $r_1 = \dfrac{\tan 30°}{2} = \dfrac{1}{2\sqrt{3}} \Rightarrow$ $r_2 = \frac{1}{6\sqrt{3}}$, so $A = \pi\left(\frac{1}{2\sqrt{3}}\right)^2 + \frac{27\pi}{8}\left(\frac{1}{6\sqrt{3}}\right)^2 = \frac{\pi}{12} + \frac{\pi}{32} = \frac{11\pi}{96}$. The area of the triangle is $\frac{\sqrt{3}}{4}$, so the circles occupy about 83.1% of the area of the triangle.

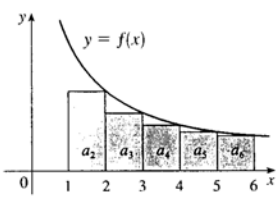
12.3 The Integral Test and Estimates of Sums

1. The picture shows that $a_2 = \dfrac{1}{2^{1.3}} < \displaystyle\int_1^2 \frac{1}{x^{1.3}}\,dx$, $a_3 = \dfrac{1}{3^{1.3}} < \displaystyle\int_2^3 \frac{1}{x^{1.3}}\,dx$, and so on, so $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^{1.3}} < \int_1^{\infty}\frac{1}{x^{1.3}}\,dx$. The integral converges by (8.8.2) with $p = 1.3 > 1$, so the series converges.



2. From the first figure, we see that $\int_1^6 f(x)\,dx < \sum_{i=1}^{5} a_i$. From the second figure, we see that $\sum_{i=2}^{6} a_i < \int_1^6 f(x)\,dx$. Thus, we have $\sum_{i=2}^{6} a_i < \int_1^6 f(x)\,dx < \sum_{i=1}^{5} a_i$.





3. The function $f(x) = 1/x^4$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty}\frac{1}{x^4}\,dx = \lim_{b\to\infty}\int_1^b x^{-4}\,dx = \lim_{b\to\infty}\left[\frac{x^{-3}}{-3}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{3b^3} + \frac{1}{3}\right) = \frac{1}{3}, \text{ so } \sum_{n=1}^{\infty}\frac{1}{n^4} \text{ converges.}$$

4. The function $f(x) = 1/\sqrt[4]{x} = x^{-1/4}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} x^{-1/4}\,dx = \lim_{b\to\infty}\int_1^b x^{-1/4}\,dx = \lim_{b\to\infty}\left[\tfrac{4}{3}x^{3/4}\right]_1^b = \lim_{b\to\infty}\left(\tfrac{4}{3}b^{3/4} - \tfrac{4}{3}\right) = \infty, \text{ so } \sum_{n=1}^{\infty} 1/\sqrt[4]{n} \text{ diverges.}$$

5. The function $f(x) = 1/(3x+1)$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty}\frac{dx}{3x+1} = \lim_{b\to\infty}\int_1^b \frac{dx}{3x+1} = \lim_{b\to\infty}\left[\tfrac{1}{3}\ln(3x+1)\right]_1^b = \lim_{b\to\infty}\left[\tfrac{1}{3}\ln(3b+1) - \tfrac{1}{3}\ln 4\right] = \infty$$

so the improper integral diverges, and so does the series $\sum_{n=1}^{\infty} 1/(3n+1)$.

6. The function $f(x) = e^{-x}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.
$\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx = \lim_{b\to\infty} \left[-e^{-x}\right]_1^b = \lim_{b\to\infty} \left(-e^{-b} + e^{-1}\right) = e^{-1}$, so $\sum_{n=1}^\infty e^{-n}$ converges. *Note:* This is a geometric series, with first term $a = e^{-1}$ and ratio $r = e^{-1}$. Since $|r| < 1$, the series converges to $e^{-1}/\left(1 - e^{-1}\right) = 1/(e-1)$.

7. $f(x) = xe^{-x}$ is continuous and positive on $[1, \infty)$. $f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1-x) < 0$ for $x > 1$, so f is decreasing on $[1, \infty)$. Thus, the Integral Test applies.

$\int_1^\infty xe^{-x}\,dx = \lim_{b\to\infty} \int_1^b xe^{-x}\,dx = \lim_{b\to\infty} \left[-xe^{-x} - e^{-x}\right]_1^b$ (by parts) $= \lim_{b\to\infty} \left[-be^{-b} - e^{-b} + e^{-1} + e^{-1}\right] = 2/e$

since $\lim_{b\to\infty} be^{-b} = \lim_{b\to\infty} \left(b/e^b\right) \overset{\text{H}}{=} \lim_{b\to\infty} \left(1/e^b\right) = 0$ and $\lim_{b\to\infty} e^{-b} = 0$. Thus, $\sum_{n=1}^\infty ne^{-n}$ converges.

8. $\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{1}{15} + \cdots = \sum_{n=1}^\infty \frac{1}{4n-1}$. The function $f(x) = \frac{1}{4x-1}$ is positive, continuous, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^\infty \frac{dx}{4x-1} = \lim_{b\to\infty} \int_1^b \frac{dx}{4x-1} = \lim_{b\to\infty} \left[\tfrac{1}{4} \ln(4x-1)\right]_1^b = \lim_{b\to\infty} \left[\tfrac{1}{4}\ln(4b-1) - \tfrac{1}{4}\ln 3\right] = \infty$$

so the improper integral diverges, and so does the series.

9. $\sum_{n=5}^\infty \left(1/n^{1.0001}\right)$ is a p-series, $p = 1.0001 > 1$, so it converges.

10. $\sum_{n=1}^\infty n^{-0.99} = \sum_{n=1}^\infty \left(1/n^{0.99}\right)$ which diverges since $p = 0.99 < 1$.

11. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots = \sum_{n=1}^\infty \left(1/n^3\right)$. This is a p-series with $p = 3 > 1$, so it converges by (1).

12. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots = \sum_{n=1}^\infty \frac{1}{n\sqrt{n}} = \sum_{n=1}^\infty \frac{1}{n^{3/2}}$. This is a p-series with $p = \frac{3}{2} > 1$, so it converges by (1).

13. $\sum_{n=1}^\infty \frac{5 - 2\sqrt{n}}{n^3} = 5\sum_{n=1}^\infty \frac{1}{n^3} - 2\sum_{n=1}^\infty \frac{1}{n^{5/2}}$ by Theorem 12.2.8, since $\sum_{n=1}^\infty \frac{1}{n^3}$ and $\sum_{n=1}^\infty \frac{1}{n^{5/2}}$ both converge by (1) (with $p = 3$ and $p = \frac{5}{2}$). Thus, $\sum_{n=1}^\infty \frac{5 - 2\sqrt{n}}{n^3}$ converges.

14. $f(x) = \frac{1}{x^2 - 1}$ is positive, continuous, and decreasing on $[2, \infty)$, so applying the Integral Test,

$$\int_2^\infty \frac{dx}{x^2-1} = \int_2^\infty \left(\frac{-1/2}{x+1} + \frac{1/2}{x-1}\right) dx = \lim_{t\to\infty} \left[\ln\left(\frac{x-1}{x+1}\right)^{1/2}\right]_2^t = \ln\sqrt{3} \quad\Rightarrow\quad \sum_{n=2}^\infty \frac{1}{n^2-1} \text{ converges.}$$

15. $f(x) = xe^{-x^2}$ is continuous and positive on $[1, \infty)$, and since $f'(x) = e^{-x^2}\left(1 - 2x^2\right) < 0$ for $x > 1$, f is decreasing as well. Thus, we can use the Integral Test.
$\int_1^\infty xe^{-x^2}\,dx = \lim_{t\to\infty} \left[-\frac{1}{2}e^{-x^2}\right]_1^t = 0 - \left(-\frac{1}{2}e^{-1}\right) = 1/(2e)$. Since the integral converges, the series converges.

16. $f(x) = \frac{x}{2^x}$ is positive and continuous on $[1, \infty)$, and since $f'(x) = \frac{1 - x\ln 2}{2^x} < 0$ when $x > \frac{1}{\ln 2} \approx 1.44$, f is eventually decreasing, so we can apply the Integral Test. Integrating by parts, we get

$$\int_1^\infty \frac{x}{2^x}dx = \lim_{t\to\infty} \left(-\frac{1}{\ln 2}\left[\frac{x}{2^x} + \frac{1}{2^x \ln 2}\right]_1^t\right) = \frac{1}{2\ln 2} + \frac{1}{2(\ln 2)^2}, \text{ since } \lim_{t\to\infty} \frac{t}{2^t} = 0 \text{ by l'Hospital's Rule, and}$$

so $\sum_{n=1}^\infty \frac{n}{2^n}$ converges.

17. $f(x) = \dfrac{x}{x^2+1}$ is continuous and positive on $[1, \infty)$, and since $f'(x) = \dfrac{1-x^2}{(x^2+1)^2} < 0$ for $x > 1$, f is also decreasing. Using the Integral Test, $\displaystyle\int_1^\infty \frac{x}{x^2+1}\,dx = \lim_{t\to\infty}\left[\frac{\ln(x^2+1)}{2}\right]_1^t = \infty$, so the series diverges.

18. The function $f(x) = \dfrac{1}{2x^2+3x+1} = \dfrac{1}{(2x+1)(x+1)}$ is positive, continuous, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\begin{aligned}\int_1^\infty f(x)\,dx &= \lim_{b\to\infty}\int_1^b\left(\frac{2}{2x+1}-\frac{1}{x+1}\right)dx \text{ (partial fractions) } = \lim_{b\to\infty}\left[\ln(2x+1)-\ln(x+1)\right]_1^b \\ &= \lim_{b\to\infty}\left[\ln\left(\frac{2x+1}{x+1}\right)\right]_1^b = \lim_{b\to\infty}\left(\ln\frac{2b+1}{b+1}-\ln\tfrac{3}{2}\right) = \ln 2 - \ln\tfrac{3}{2} = \ln\tfrac{4}{3}\end{aligned}$$

so the series converges.

19. $f(x) = \dfrac{1}{x\ln x}$ is continuous and positive on $[2, \infty)$, and also decreasing since $f'(x) = -\dfrac{1+\ln x}{x^2(\ln x)^2} < 0$ for $x > 2$, so we can use the Integral Test. $\displaystyle\int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{t\to\infty}\left[\ln(\ln x)\right]_2^t = \lim_{t\to\infty}\left[\ln(\ln t) - \ln(\ln 2)\right] = \infty$, so the series diverges.

20. $f(x) = \dfrac{1}{4x^2+1}$ is continuous, positive and decreasing on $[1, \infty)$, so applying the Integral Test,
$\displaystyle\int_1^\infty \frac{dx}{4x^2+1} = \lim_{t\to\infty}\left[\frac{\arctan 2x}{2}\right]_1^t = \frac{\pi}{4} - \frac{\arctan 2}{2} < \infty$, so the series converges.

21. $f(x) = \dfrac{\arctan x}{1+x^2}$ is continuous and positive on $[1, \infty)$. $f'(x) = \dfrac{1-2x\arctan x}{(1+x^2)^2} < 0$ for $x > 1$, since $2x\arctan x \geq \frac{\pi}{2} > 1$ for $x \geq 1$. So f is decreasing and we can use the Integral Test.
$\displaystyle\int_1^\infty \frac{\arctan x}{1+x^2}\,dx = \lim_{t\to\infty}\left[\tfrac{1}{2}\arctan^2 x\right]_1^t = \frac{(\pi/2)^2}{2} - \frac{(\pi/4)^2}{2} = \frac{3\pi^2}{32}$, so the series converges.

22. $f(x) = \dfrac{\ln x}{x^2}$ is continuous and positive for $x \geq 2$, and $f'(x) = \dfrac{1-2\ln x}{x^3} < 0$ for $x \geq 2$ so f is decreasing.
$\displaystyle\int_2^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^t$ (by parts) $= 1$ (by l'Hospital's Rule). Thus, $\displaystyle\sum_{n=1}^\infty \frac{\ln n}{n^2} = \sum_{n=2}^\infty \frac{\ln n}{n^2}$ converges by the Integral Test.

23. $f(x) = \dfrac{1}{x^2+2x+2}$ is continuous and positive on $[1, \infty)$, and $f'(x) = -\dfrac{2x+2}{(x^2+2x+2)^2} < 0$ for $x \geq 1$, so f is decreasing and we can use the Integral Test.
$\displaystyle\int_1^\infty \frac{1}{x^2+2x+2}\,dx = \int_1^\infty \frac{1}{(x+1)^2+1}\,dx = \lim_{t\to\infty}\left[\arctan(x+1)\right]_1^t = \frac{\pi}{2} - \arctan 2$, so the series converges as well.

24. $f(x) = \dfrac{1}{x\ln x\ln(\ln x)}$ is positive and continuous on $[3,\infty)$, and is decreasing since x, $\ln x$, and $\ln(\ln x)$ are all increasing; so we can apply the Integral Test. $\displaystyle\int_3^\infty \frac{dx}{x\ln x\ln(\ln x)} = \lim_{t\to\infty}[\ln(\ln(\ln x))]_3^t$ which diverges, and hence $\displaystyle\sum_{n=3}^\infty \frac{1}{n\ln n\ln(\ln n)}$ diverges.

25. We have already shown (in Exercise 19) that when $p=1$ the series $\displaystyle\sum_{n=2}^\infty \frac{1}{n(\ln n)^p}$ diverges, so assume that $p\neq 1$. $f(x) = \dfrac{1}{x(\ln x)^p}$ is continuous and positive on $[2,\infty)$, and $f'(x) = -\dfrac{p+\ln x}{x^2(\ln x)^{p+1}} < 0$ if $x > e^{-p}$, so that f is eventually decreasing and we can use the Integral Test.

$$\int_2^\infty \frac{1}{x(\ln x)^p}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^{1-p}}{1-p}\right]_2^t \quad (\text{for } p\neq 1) \;= \lim_{t\to\infty}\left[\frac{(\ln t)^{1-p}}{1-p}\right] - \frac{(\ln 2)^{1-p}}{1-p}$$

This limit exists whenever $1-p<0 \Leftrightarrow p>1$, so the series converges for $p>1$.

26. As in Exercise 24, we can apply the Integral Test. $\displaystyle\int_3^\infty \frac{dx}{x\ln x(\ln\ln x)^p} = \lim_{t\to\infty}\left[\frac{(\ln\ln x)^{-p+1}}{-p+1}\right]_3^t$ (for $p\neq 1$; if $p=1$ see Exercise 24) and $\displaystyle\lim_{t\to\infty}\frac{(\ln\ln t)^{-p+1}}{-p+1}$ exists whenever $-p+1<0 \Leftrightarrow p>1$, so the series converges for $p>1$.

27. Clearly the series cannot converge if $p\geq -\frac{1}{2}$, because then $\displaystyle\lim_{n\to\infty} n\left(1+n^2\right)^p \neq 0$. Also, if $p=-1$ the series diverges (see Exercise 17). So assume $p<-\frac{1}{2}$, $p\neq -1$. Then $f(x) = x\left(1+x^2\right)^p$ is continuous, positive, and eventually decreasing on $[1,\infty)$, and we can use the Integral Test.

$\displaystyle\int_1^\infty x\left(1+x^2\right)^p dx = \lim_{t\to\infty}\left[\frac{1}{2}\cdot\frac{\left(1+x^2\right)^{p+1}}{p+1}\right]_1^t = \lim_{t\to\infty}\frac{1}{2}\cdot\frac{\left(1+t^2\right)^{p+1}}{p+1} - \frac{2^p}{p+1}$. This limit exists and is finite $\Leftrightarrow p+1<0 \Leftrightarrow p<-1$, so the series converges whenever $p<-1$.

28. If $p\leq 0$, $\displaystyle\lim_{n\to\infty}\frac{\ln n}{n^p} = \infty$ and the series diverges, so assume $p>0$. $f(x) = \dfrac{\ln x}{x^p}$ is positive and continuous and $f'(x)<0$ for $x>e^{1/p}$, so f is eventually decreasing and we can use the Integral Test. Integration by parts gives

$\displaystyle\int_1^\infty \frac{\ln x}{x^p}dx = \lim_{t\to\infty}\left[\frac{x^{1-p}[(1-p)\ln x - 1]}{(1-p)^2}\right]_1^t$ (for $p\neq 1$) $= \dfrac{1}{(1-p)^2}\left[\lim_{t\to\infty} t^{1-p}[(1-p)\ln t - 1] + 1\right]$ which exists whenever $1-p<0 \Leftrightarrow p>1$. Since we have already done the case $p=1$ in Exercise 25 (set $p=-1$ in that exercise), $\displaystyle\sum_{n=1}^\infty\frac{\ln n}{n^p}$ converges $\Leftrightarrow p>1$.

29. Since this is a p-series with $p=x$, $\zeta(x)$ is defined when $x>1$. Unless specified otherwise, the domain of a function f is the set of numbers x such that the expression for $f(x)$ makes sense and defines a real number. So, in the case of a series, it's the set of numbers x such that the series is convergent.

30. (a) $f(x) = 1/x^4$ is positive and continuous and $f'(x) = -4/x^5$ is negative for $x>1$, and so the Integral Test applies. $\displaystyle\sum_{n=1}^\infty\frac{1}{n^4} \approx s_{10} = \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots+\frac{1}{10^4} \approx 1.082037$.

$R_{10} \leq \displaystyle\int_{10}^\infty \frac{1}{x^4}dx = \lim_{t\to\infty}\left[\frac{1}{-3x^3}\right]_{10}^t = \lim_{t\to\infty}\left(-\frac{1}{3t^3}+\frac{1}{3(10)^3}\right) = \frac{1}{3000}$, so the error is at most $0.000\overline{3}$.

(b) $s_{10} + \int_{11}^{\infty} \frac{1}{x^4}\,dx \le s \le s_{10} + \int_{10}^{\infty} \frac{1}{x^4}\,dx \;\Rightarrow\; s_{10} + \frac{1}{3\,(11)^3} \le s \le s_{10} + \frac{1}{3\,(10)^3} \;\Rightarrow$
$1.082037 + 0.000250 = 1.082287 \le s \le 1.082037 + 0.000333 = 1.082370$, so we get $s \approx 1.08233$ with error ≤ 0.00005.

(c) $R_n \le \int_n^{\infty} \frac{1}{x^4}\,dx = \frac{1}{3n^3}$. So $R_n < 0.00001 \;\Rightarrow\; \frac{1}{3n^3} < \frac{1}{10^5} \;\Rightarrow\; 3n^3 > 10^5 \;\Rightarrow$
$n > \sqrt[3]{(10)^5/3} \approx 32.2$, that is, for $n > 32$.

31. (a) $f(x) = \frac{1}{x^2}$ is positive and continuous and $f'(x) = -\frac{2}{x^3}$ is negative for $x > 1$, and so the Integral Test applies. $\sum_{n=1}^{\infty} \frac{1}{n^2} \approx s_{10} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{10^2} \approx 1.549768$.

$R_{10} \le \int_{10}^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left[\frac{-1}{x}\right]_{10}^{t} = \lim_{t\to\infty}\left(-\frac{1}{t} + \frac{1}{10}\right) = \frac{1}{10}$, so the error is at most 0.1.

(b) $s_{10} + \int_{11}^{\infty} \frac{1}{x^2}\,dx \le s \le s_{10} + \int_{10}^{\infty} \frac{1}{x^2}\,dx \;\Rightarrow\; s_{10} + \frac{1}{11} \le s \le s_{10} + \frac{1}{10} \;\Rightarrow$
$1.549768 + 0.090909 = 1.640677 \le s \le 1.549768 + 0.1 = 1.649768$, so we get $s \approx 1.64522$ (the average of 1.640677 and 1.649768) with error ≤ 0.005 (the maximum of $1.649768 - 1.64522$ and $1.64522 - 1.640677$, rounded up).

(c) $R_n \le \int_n^{\infty} \frac{1}{x^2}\,dx = \frac{1}{n}$. So $R_n < 0.001$ if $\frac{1}{n} < \frac{1}{1000} \;\Leftrightarrow\; n > 1000$.

32. $f(x) = 1/x^5$ is positive and continuous and $f'(x) = -5/x^6$ is negative for $x > 1$, and so the Integral Test applies. Using (2), $R_n \le \int_n^{\infty} x^{-5}\,dx = \lim_{t\to\infty}\left[\frac{-1}{4x^4}\right]_n^t = \frac{1}{4n^4}$. If we take $n = 5$, then $s_5 \approx 1.036662$ and $R_5 \le 0.0004$. So $s \approx s_5 \approx 1.037$.

33. $f(x) = x^{-3/2}$ is positive and continuous and $f'(x) = -\frac{3}{2}x^{-5/2}$ is negative for $x > 1$, so the Integral Test applies. From the end of Example 6, we see that the error is at most half the length of the interval. From (3), the interval is $\left(s_n + \int_{n+1}^{\infty} f(x)\,dx,\, s_n + \int_n^{\infty} f(x)\,dx\right)$, so its length is $\int_n^{\infty} f(x)\,dx - \int_{n+1}^{\infty} f(x)\,dx$. Thus, we need n such that

$$0.01 > \tfrac{1}{2}\left(\int_n^{\infty} x^{-3/2}\,dx - \int_{n+1}^{\infty} x^{-3/2}\,dx\right) = \frac{1}{2}\left(\lim_{t\to\infty}\left[\frac{-2}{\sqrt{x}}\right]_n^t - \lim_{t\to\infty}\left[\frac{-2}{\sqrt{x}}\right]_{n+1}^t\right) = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$$

$\Leftrightarrow\; n > 13.08$. Again from the end of Example 6, we approximate s by the midpoint of this interval. In general, the midpoint is $\frac{1}{2}\left[\left(s_n + \int_{n+1}^{\infty} f(x)\,dx\right) + \left(s_n + \int_n^{\infty} f(x)\,dx\right)\right] = s_n + \frac{1}{2}\left(\int_{n+1}^{\infty} f(x)\,dx + \int_n^{\infty} f(x)\,dx\right)$. So using $n = 14$, we have $s \approx s_{14} + \frac{1}{2}\left(\int_{14}^{\infty} x^{-3/2}\,dx + \int_{15}^{\infty} x^{-3/2}\,dx\right) = 2.0872 + \frac{1}{\sqrt{14}} + \frac{1}{\sqrt{15}} \approx 2.6127$. Any larger value of n will also work. For instance, $s \approx s_{30} + \frac{1}{\sqrt{30}} + \frac{1}{\sqrt{31}} \approx 2.6124$.

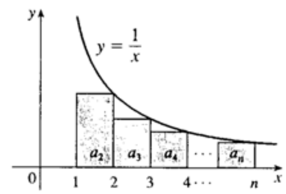
34. $f(x) = \frac{1}{x\,(\ln x)^2}$ is positive and continuous and $f'(x) = -\frac{\ln x + 2}{x^2\,(\ln x)^3}$ is negative for $x > 1$, so the Integral Test applies. Using (2), we need $0.01 > \int_n^{\infty} \frac{dx}{x\,(\ln x)^2} = \lim_{t\to\infty}\left[\frac{-1}{\ln x}\right]_n^t = \frac{1}{\ln n}$. This is true for $n > e^{100}$, so we would have to take this many terms, which would be problematic because $e^{100} \approx 2.7 \times 10^{43}$.

35. (a) From the figure, $a_2 + a_3 + \cdots + a_n \le \int_1^n f(x)\,dx$, so with

$f(x) = \dfrac{1}{x}$,

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \le \int_1^n \frac{1}{x}\,dx = \ln n$$

Thus, $s_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots + \dfrac{1}{n} \le 1 + \ln n$.



(b) By part (a), $s_{10^6} \le 1 + \ln 10^6 \approx 14.82 < 15$ and $s_{10^9} \le 1 + \ln 10^9 \approx 21.72 < 22$.

36. (a) $f(x) = \left(\dfrac{\ln x}{x}\right)^2$ is continuous and positive for $x > 1$, and since $f'(x) = \dfrac{2\ln x\,(1 - \ln x)}{x^3} < 0$ for $x > e$, we can apply the Integral Test. Using a CAS, we get $\displaystyle\int_1^\infty \left(\frac{\ln x}{x}\right)^2 dx = 2$, so the series also converges.

(b) Since the Integral Test applies, the error in $s \approx s_n$ is $R_n \le \displaystyle\int_n^\infty \left(\frac{\ln x}{x}\right)^2 dx = \frac{(\ln n)^2 + 2\ln n + 2}{n}$.

(c) By graphing the functions $y_1 = \dfrac{(\ln x)^2 + 2\ln x + 2}{x}$ and $y_2 = 0.05$, we see that $y_1 < y_2$ for $n \ge 1373$.

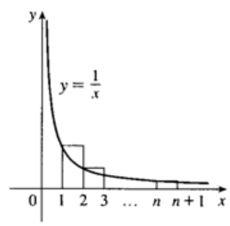
(d) Using the CAS to sum the first 1373 terms, we get $s_{1373} \approx 1.94$.

37. $b^{\ln n} = \left(e^{\ln b}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln b} = n^{\ln b} = \dfrac{1}{n^{-\ln b}}$. This is a p-series, which converges for all b such that $-\ln b > 1$ $\Leftrightarrow$ $\ln b < -1$ $\Leftrightarrow$ $b < e^{-1}$ $\Leftrightarrow$ $b < 1/e$.

38. (a) The sum of the areas of the n rectangles in the graph to the right is $1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}$. Now $\displaystyle\int_1^{n+1} \frac{dx}{x}$ is less than this sum because the rectangles extend above the curve $y = 1/x$, so

$\displaystyle\int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, and since

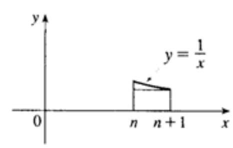
$\ln n < \ln(n+1)$, $0 < 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} - \ln n = t_n$.



(b) The area under $f(x) = 1/x$ between $x = n$ and $x = n + 1$ is

$\displaystyle\int_n^{n+1} \frac{dx}{x} = \ln(n+1) - \ln n$, and this is clearly greater than the area of the inscribed rectangle in the figure to the right $\left[\text{which is } \dfrac{1}{n+1}\right]$,

so $t_n - t_{n+1} = [\ln(n+1) - \ln n] - \dfrac{1}{n+1} > 0$, and so $t_n > t_{n+1}$, so $\{t_n\}$ is a decreasing sequence.



(c) We have shown that $\{t_n\}$ is decreasing and that $t_n > 0$ for all n. Thus, $0 < t_n \le t_1 = 1$, so $\{t_n\}$ is a bounded monotonic sequence, and hence converges by Theorem 12.1.10.

12.4 The Comparison Tests

1. (a) We cannot say anything about $\sum a_n$. If $a_n > b_n$ for all n and $\sum b_n$ is convergent, then $\sum a_n$ could be convergent or divergent. (See Note 2 on page 757.)

(b) If $a_n < b_n$ for all n, then $\sum a_n$ is convergent. [This is part (i) of the Comparison Test.]

2. (a) If $a_n > b_n$ for all n, then $\sum a_n$ is divergent. [This is part (ii) of the Comparison Test.]

(b) We cannot say anything about $\sum a_n$. If $a_n < b_n$ for all n and $\sum b_n$ is divergent, then $\sum a_n$ could be convergent or divergent.

3. $\dfrac{1}{n^2+n+1} < \dfrac{1}{n^2}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+n+1}$ converges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$, which converges because it is a p-series with $p = 2 > 1$.

4. $\dfrac{2}{n^3+4} < \dfrac{2}{n^3}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{2}{n^3+4}$ converges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{2}{n^3} = 2\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3}$, which converges because it is a constant multiple of a convergent p-series ($p = 3 > 1$).

5. $\dfrac{5}{2+3^n} < \dfrac{5}{3^n}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{5}{2+3^n}$ converges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{5}{3^n} = 5\sum\limits_{n=1}^{\infty} \dfrac{1}{3^n}$, which converges because $\sum\limits_{n=1}^{\infty} \dfrac{1}{3^n}$ is a convergent geometric series with $r = \frac{1}{3}$.

6. $\dfrac{1}{n-\sqrt{n}} > \dfrac{1}{n}$ for all $n \geq 2$, so $\sum\limits_{n=2}^{\infty} \dfrac{1}{n-\sqrt{n}}$ diverges by comparison with the divergent (partial) harmonic series $\sum\limits_{n=2}^{\infty} \dfrac{1}{n}$.

7. $\dfrac{n+1}{n^2} > \dfrac{n}{n^2} = \dfrac{1}{n}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n+1}{n^2}$ diverges by comparison with the harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$.

8. $\dfrac{4+3^n}{2^n} > \dfrac{3^n}{2^n} = \left(\dfrac{3}{2}\right)^n$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{4+3^n}{2^n}$ diverges by comparison with the divergent geometric series $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$.

9. $\dfrac{3}{n2^n} \leq \dfrac{3}{2^n}$. $\sum\limits_{n=1}^{\infty} \dfrac{3}{2^n}$ is a geometric series with $|r| = \frac{1}{2} < 1$, and hence converges, so $\sum\limits_{n=1}^{\infty} \dfrac{3}{n2^n}$ converges also, by the Comparison Test.

10. $\dfrac{\sin^2 n}{n\sqrt{n}} \leq \dfrac{1}{n\sqrt{n}} = \dfrac{1}{n^{3/2}}$ and $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ converges ($p = \frac{3}{2} > 1$), so $\sum\limits_{n=1}^{\infty} \dfrac{\sin^2 n}{n\sqrt{n}}$ converges by the Comparison Test.

11. $\dfrac{1}{\sqrt{n(n+1)(n+2)}} < \dfrac{1}{\sqrt{n \cdot n \cdot n}} = \dfrac{1}{n^{3/2}}$ and since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ converges ($p = \frac{3}{2} > 1$), so does $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n(n+1)(n+2)}}$ by the Comparison Test.

12. Use the Limit Comparison Test with $a_n = \dfrac{1}{\sqrt[3]{n(n+1)(n+2)}}$ and $b_n = \dfrac{1}{n}$.

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt[3]{n(n+1)(n+2)}} = \lim_{n\to\infty} \frac{1}{\sqrt[3]{1(1+1/n)(1+2/n)}} = 1 > 0$, so since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n(n+1)(n+2)}}$.

13. If $a_n = \dfrac{n^2+1}{n^3-1}$ and $b_n = \dfrac{1}{n}$, then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3+n}{n^3-1} = \lim_{n\to\infty} \frac{1+1/n^2}{1-1/n^3} = 1$, so $\displaystyle\sum_{n=2}^{\infty} \frac{n^2+1}{n^3-1}$ diverges by the Limit Comparison Test with the divergent (partial) harmonic series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$.

14. $\dfrac{n}{(n+1)2^n} < \dfrac{1}{2^n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series ($|r| = \frac{1}{2} < 1$), so $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)2^n}$ converges by the Comparison Test.

15. $\dfrac{3+\cos n}{3^n} \le \dfrac{4}{3^n}$ since $\cos n \le 1$. $\displaystyle\sum_{n=1}^{\infty} \frac{4}{3^n}$ is a geometric series with $|r| = \frac{1}{3} < 1$ so it converges, and so $\displaystyle\sum_{n=1}^{\infty} \frac{3+\cos n}{3^n}$ converges by the Comparison Test.

16. $\dfrac{5n}{2n^2-5} > \dfrac{5n}{2n^2} = \dfrac{5}{2}\left(\dfrac{1}{n}\right)$ and since $\displaystyle\frac{5}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series) so does $\displaystyle\sum_{n=1}^{\infty} \frac{5n}{2n^2-5}$ by the Comparison Test.

17. $\dfrac{n}{\sqrt{n^5+4}} < \dfrac{n}{\sqrt{n^5}} = \dfrac{1}{n^{3/2}}$. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series ($p = \frac{3}{2} > 1$) so $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5+4}}$ converges by the Comparison Test.

18. $\dfrac{\arctan n}{n^4} < \dfrac{\pi/2}{n^4}$ and $\displaystyle\frac{\pi}{2}\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges ($p = 4 > 1$) so $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^4}$ converges by the Comparison Test.

19. $\dfrac{2^n}{1+3^n} < \dfrac{2^n}{3^n} = \left(\dfrac{2}{3}\right)^n$. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ is a convergent geometric series ($|r| = \frac{2}{3} < 1$), so $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{1+3^n}$ converges by the Comparison Test.

20. Use the Limit Comparison Test with $a_n = \dfrac{1+2^n}{1+3^n}$ and $b_n = \dfrac{2^n}{3^n}$: $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(1/2)^n+1}{(1/3)^n+1} = 1 > 0$. Since $\displaystyle\sum_{n=1}^{\infty} b_n$ converges (geometric series with $|r| = \frac{2}{3} < 1$), $\displaystyle\sum_{n=1}^{\infty} \frac{1+2^n}{1+3^n}$ also converges.

21. Use the Limit Comparison Test with $a_n = \dfrac{1}{1+\sqrt{n}}$ and $b_n = \dfrac{1}{\sqrt{n}}$: $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} = 1 > 0$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series ($p = \frac{1}{2} \le 1$), $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ also diverges.

22. Use the Limit Comparison Test with $a_n = \dfrac{1}{n^2-4}$ and $b_n = \dfrac{1}{n^2}$: $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2-4} = 1 > 0$. Since $\displaystyle\sum_{n=3}^{\infty} b_n$ converges ($p = 2 > 1$), $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2-4}$ also converges.

23. Let $a_n = \dfrac{n^2+1}{n^4+1}$ and $b_n = \dfrac{1}{n^2}$. Then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^4+n^2}{n^4+1} = 1$. Since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ is a convergent p-series $(p = 2 > 1)$, so is $\sum\limits_{n=1}^{\infty} \dfrac{n^2+1}{n^4+1}$ by the Limit Comparison Test.

24. If $a_n = \dfrac{n^2-5n}{n^3+n+1}$ and $b_n = \dfrac{1}{n}$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^3-5n^2}{n^3+n+1} = \lim\limits_{n\to\infty} \dfrac{1-5/n}{1+1/n^2+1/n^3} = 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n^2-5n}{n^3+n+1}$ diverges by the Limit Comparison Test with the divergent harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$. (Note that $a_n > 0$ for $n \geq 6$.)

25. If $a_n = \dfrac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ and $b_n = \dfrac{1}{n}$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n+n^2+n^3}{\sqrt{1+n^2+n^6}} = \lim\limits_{n\to\infty} \dfrac{1/n^2+1/n+1}{\sqrt{1/n^6+1/n^4+1}} = 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ diverges by the Limit Comparison Test with the divergent harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$.

26. If $a_n = \dfrac{n+5}{\sqrt[3]{n^7+n^2}}$ and $b_n = \dfrac{n}{\sqrt[3]{n^7}} = \dfrac{n}{n^{7/3}} = \dfrac{1}{n^{4/3}}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{7/3}+5n^{4/3}}{(n^7+n^2)^{1/3}} \cdot \frac{n^{-7/3}}{n^{-7/3}} = \lim_{n\to\infty} \frac{1+5/n}{\left[(n^7+n^2)/n^7\right]^{1/3}} = \lim_{n\to\infty} \frac{1+5/n}{(1+1/n^5)^{1/3}} = \frac{1+0}{(1+0)^{1/3}} = 1, \text{ so}$$

$\sum\limits_{n=1}^{\infty} \dfrac{n+5}{\sqrt[3]{n^7+n^2}}$ converges by the Limit Comparison Test with the convergent p-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{4/3}}$.

27. Let $a_n = \dfrac{n+1}{n2^n}$ and $b_n = \dfrac{1}{2^n}$. Then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n+1}{n} = 1$. Since $\sum\limits_{n=1}^{\infty} \dfrac{1}{2^n}$ is a convergent geometric series $(|r| = \frac{1}{2} < 1)$, $\sum\limits_{n=1}^{\infty} \dfrac{n+1}{n2^n}$ converges by the Limit Comparison Test.

28. Use the Limit Comparison Test with $a_n = \dfrac{2n^2+7n}{3^n(n^2+5n-1)}$ and $b_n = \dfrac{1}{3^n}$.

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{2n^2+7n}{n^2+5n-1} = 2 > 0$, and since $\sum\limits_{n=1}^{\infty} b_n$ is a convergent geometric series $(|r| = \frac{1}{3} < 1)$, $\sum\limits_{n=1}^{\infty} \dfrac{2n^2+7n}{3^n(n^2+5n-1)}$ converges also.

29. Clearly $n! = n(n-1)(n-2)\cdots(3)(2) \geq 2\cdot 2\cdot 2\cdot\cdots\cdot 2\cdot 2 = 2^{n-1}$, so $\dfrac{1}{n!} \leq \dfrac{1}{2^{n-1}}$. $\sum\limits_{n=1}^{\infty} \dfrac{1}{2^{n-1}}$ is a convergent geometric series $(|r| = \frac{1}{2} < 1)$ so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n!}$ converges by the Comparison Test.

30. $\dfrac{n!}{n^n} = \dfrac{1\cdot 2\cdot 3\cdot\cdots\cdot(n-1)\,n}{n\cdot n\cdot n\cdot\cdots\cdot n\cdot n} \leq \dfrac{1}{n}\cdot\dfrac{2}{n}\cdot 1\cdot 1\cdot\cdots\cdot 1$ for $n \geq 2$, so since $\sum\limits_{n=1}^{\infty} \dfrac{2}{n^2}$ converges $(p = 2 > 1)$, $\sum\limits_{n=1}^{\infty} \dfrac{n!}{n^n}$ converges also by the Comparison Test.

31. Use the Limit Comparison Test with $a_n = \sin\left(\dfrac{1}{n}\right)$ and $b_n = \dfrac{1}{n}$: $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{\sin(1/n)}{1/n} = \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1 > 0$. Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} \sin(1/n)$ also diverges.

32. Use the Limit Comparison Test with $a_n = \dfrac{1}{n^{1+1/n}}$ and $b_n = \dfrac{1}{n}$. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n}{n^{1+1/n}} = \lim\limits_{n\to\infty} \dfrac{1}{n^{1/n}} = 1$

(since $\lim\limits_{x\to\infty} x^{1/x} = 1$ by l'Hospital's Rule), so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges (harmonic series) $\Rightarrow$ $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{1+1/n}}$ diverges.

33. $\sum\limits_{n=1}^{10} \dfrac{1}{n^4+n^2} = \dfrac{1}{2} + \dfrac{1}{20} + \dfrac{1}{90} + \cdots + \dfrac{1}{10{,}100} \approx 0.567975$. Now $\dfrac{1}{n^4+n^2} < \dfrac{1}{n^4}$, so using the reasoning and notation of Example 5, the error is $R_{10} \le T_{10} = \sum\limits_{n=11}^{\infty} \dfrac{1}{n^4} \le \displaystyle\int_{10}^{\infty} \frac{dx}{x^4} = \lim_{t\to\infty}\left[-\frac{x^{-3}}{3}\right]_{10}^{t} = \frac{1}{3000} = 0.000\overline{3}$.

34. $\sum\limits_{n=1}^{10} \dfrac{1+\cos n}{n^5} = 1 + \cos 1 + \dfrac{1+\cos 2}{32} + \dfrac{1+\cos 3}{243} + \cdots + \dfrac{1+\cos 10}{100{,}000} \approx 1.55972$. Now $\dfrac{1+\cos n}{n^5} \le \dfrac{2}{n^5}$, so as in Example 5, $R_{10} \le T_{10} \le \displaystyle\int_{10}^{\infty} \frac{2}{x^5}\,dx = 2\lim_{t\to\infty}\left[-\tfrac{1}{4}x^{-4}\right]_{10}^{t} = 0.00005$.

35. $\sum\limits_{n=1}^{10} \dfrac{1}{1+2^n} = \dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{9} + \cdots + \dfrac{1}{1025} \approx 0.76352$. Now $\dfrac{1}{1+2^n} < \dfrac{1}{2^n}$, so the error is

$R_{10} \le T_{10} = \sum\limits_{n=11}^{\infty} \dfrac{1}{2^n} = \dfrac{1/2^{11}}{1-1/2}$ (geometric series) ≈ 0.00098.

36. $\sum\limits_{n=1}^{10} \dfrac{n}{(n+1)\,3^n} = \dfrac{1}{6} + \dfrac{2}{27} + \dfrac{3}{108} + \cdots + \dfrac{10}{649{,}539} \approx 0.283597$. Now $\dfrac{n}{(n+1)\,3^n} < \dfrac{n}{n\cdot 3^n} = \dfrac{1}{3^n}$, so the error is

$R_{10} \le T_{10} = \sum\limits_{n=11}^{\infty} \dfrac{1}{3^n} = \dfrac{1/3^{11}}{1-1/3} \approx 0.0000085$.

37. Since $\dfrac{d_n}{10^n} \le \dfrac{9}{10^n}$ for each n, and since $\sum\limits_{n=1}^{\infty} \dfrac{9}{10^n}$ is a convergent geometric series ($|r| = \frac{1}{10} < 1$),

$0.d_1d_2d_3\ldots = \sum\limits_{n=1}^{\infty} \dfrac{d_n}{10^n}$ will always converge by the Comparison Test.

38. Clearly, if $p < 0$ then the series diverges, since $\lim\limits_{n\to\infty} \dfrac{1}{n^p \ln n} = \infty$. If $0 \le p \le 1$, then $n^p \ln n \le n \ln n$ $\Rightarrow$ $\dfrac{1}{n^p \ln n} \ge \dfrac{1}{n \ln n}$ and $\sum\limits_{n=2}^{\infty} \dfrac{1}{n \ln n}$ diverges (Exercise 12.3.19), so $\sum\limits_{n=2}^{\infty} \dfrac{1}{n^p \ln n}$ diverges. If $p > 1$, use the Limit Comparison Test with $a_n = \dfrac{1}{n^p \ln n}$ and $b_n = \dfrac{1}{n^p}$. $\sum\limits_{n=2}^{\infty} b_n$ converges, and $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{1}{\ln n} = 0$, so $\sum\limits_{n=2}^{\infty} \dfrac{1}{n^p \ln n}$ also converges. (Or use the Comparison Test, since $n^p \ln n > n^p$ for $n > e$.) In summary, the series converges if and only if $p > 1$.

39. Since $\sum a_n$ converges, $\lim\limits_{n\to\infty} a_n = 0$, so there exists N such that $|a_n - 0| < 1$ for all $n > N$ $\Rightarrow$ $0 \le a_n < 1$ for all $n > N$ $\Rightarrow$ $0 \le a_n^2 \le a_n$. Since $\sum a_n$ converges, so does $\sum a_n^2$ by the Comparison Test.

40. (a) Since $\lim\limits_{n\to\infty} (a_n/b_n) = 0$, there is a number $N > 0$ such that $|a_n/b_n - 0| < 1$ for all $n > N$, and so $a_n < b_n$ since a_n and b_n are positive. Thus, since $\sum b_n$ converges, so does $\sum a_n$ by the Comparison Test.

(b) If $a_n = \dfrac{\ln n}{n^3}$ and $b_n = \dfrac{1}{n^2}$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{\ln n}{n} = \lim\limits_{x\to\infty} \dfrac{\ln x}{x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{1} = 0$, so $\sum\limits_{n=1}^{\infty} \dfrac{\ln n}{n^3}$ converges by part (a).

41. (a) We wish to prove that if $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}=\infty$ and $\sum b_n$ diverges, then so does $\sum a_n$. So suppose on the contrary that $\sum a_n$ converges. Since $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}=\infty$, we have that $\lim\limits_{n\to\infty}\dfrac{b_n}{a_n}=0$, so by the extension of the Limit Comparison Test proved in Exercise 40(a), if $\sum a_n$ converges, so must $\sum b_n$. But this contradicts our hypothesis, so $\sum a_n$ must diverge.

(b) If $a_n=\dfrac{1}{\ln n}$ and $b_n=\dfrac{1}{n}$ for $n\geq 2$, then $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}=\lim\limits_{n\to\infty}\dfrac{n}{\ln n}=\lim\limits_{x\to\infty}\dfrac{x}{\ln x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{1}{1/x}=\lim\limits_{x\to\infty}x=\infty$, so by part (a), $\sum\limits_{n=2}^{\infty}\dfrac{1}{\ln n}$ is divergent.

42. Let $a_n=\dfrac{1}{n^2}$ and $b_n=\dfrac{1}{n}$. Then $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}=\lim\limits_{n\to\infty}\dfrac{1}{n}=0$, but $\sum b_n$ diverges while $\sum a_n$ converges.

43. $\lim\limits_{n\to\infty}na_n=\lim\limits_{n\to\infty}\dfrac{a_n}{1/n}$, so we apply the Limit Comparison Test with $b_n=\dfrac{1}{n}$. Since $\lim\limits_{n\to\infty}na_n>0$ we know that either both series converge or both series diverge, and we also know that $\sum\limits_{n=0}^{\infty}\dfrac{1}{n}$ diverges (p-series with $p=1$). Therefore, $\sum a_n$ must be divergent.

44. First we observe that, by l'Hospital's Rule, $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=\lim\limits_{x\to 0}\dfrac{1}{1+x}=1$. Also, if $\sum a_n$ converges, then $\lim\limits_{n\to\infty}a_n=0$ by Theorem 12.2.6. Therefore, $\lim\limits_{n\to\infty}\dfrac{\ln(1+a_n)}{a_n}=1$. We are given that $\sum a_n$ is convergent and $a_n>0$. Thus, $\sum\ln(1+a_n)$ is convergent by the Limit Comparison Test.

45. Yes. Since $\sum a_n$ converges, its terms approach 0 as $n\to\infty$, so $\lim\limits_{n\to\infty}\dfrac{\sin a_n}{a_n}=1$ by Theorem 3.5.2. Thus, $\sum\sin a_n$ converges by the Limit Comparison Test.

46. Yes. Since $\sum a_n$ converges, its terms approach 0 as $n\to\infty$, so for some integer N, $a_n\leq 1$ for all $n\geq N$. But then $\sum_{n=1}^{\infty}a_nb_n=\sum_{n=1}^{N-1}a_nb_n+\sum_{n=N}^{\infty}a_nb_n\leq\sum_{n=1}^{N-1}a_nb_n+\sum_{n=N}^{\infty}b_n$. The first term is a finite sum, and the second term converges since $\sum_{n=1}^{\infty}b_n$ converges. So $\sum a_nb_n$ converges by the Comparison Test.

12.5 Alternating Series

1. (a) An alternating series is a series whose terms are alternately positive and negative.

(b) An alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges if $0<b_{n+1}\leq b_n$ for all n and $\lim\limits_{n\to\infty}b_n=0$. (This is the Alternating Series Test.)

(c) The error involved in using the partial sum s_n as an approximation to the total sum s is the remainder $R_n=s-s_n$ and the size of the error is smaller than b_{n+1}, that is, $|R_n|\leq b_{n+1}$. (This is the Alternating Series Estimation Theorem.)

2. $-\frac{1}{3}+\frac{2}{4}-\frac{3}{5}+\frac{4}{6}-\frac{5}{7}+\cdots=\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{n}{n+2}$. Here $a_n=(-1)^n\dfrac{n}{n+2}$. Since $\lim\limits_{n\to\infty}a_n\neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

3. $\frac{4}{7}-\frac{4}{8}+\frac{4}{9}-\frac{4}{10}+\frac{4}{11}-\cdots=\sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{4}{n+6}$. $b_n=\dfrac{4}{n+6}>0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty}b_n=0$, so the series converges by the Alternating Series Test.

4. $\sum_{n=2}^{\infty}(-1)^n\frac{1}{\ln n}$. $b_n=\frac{1}{\ln n}$ is positive and $\{b_n\}$ is decreasing; $\lim_{n\to\infty}\frac{1}{\ln n}=0$, so the series converges by the Alternating Series Test.

5. $b_n=\frac{1}{\sqrt{n}}>0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty}b_n=0$, so $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ converges by the Alternating Series Test.

6. $b_n=\frac{1}{3n-1}>0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty}b_n=0$, so the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{3n-1}$ converges by the Alternating Series Test.

7. $a_n=(-1)^n\frac{2n}{4n+1}$, so $|a_n|=\frac{2n}{4n+1}\to\frac{1}{2}$ as $n\to\infty$. Therefore, $\lim_{n\to\infty}a_n\neq 0$ (in fact the limit does not exist) and the series $\sum_{n=1}^{\infty}(-1)^n\frac{2n}{4n+1}$ diverges by the Test for Divergence.

8. $b_n=\frac{2n}{4n^2+1}>0$, $\{b_n\}$ is decreasing [since

$$b_n-b_{n+1}=\frac{2n}{4n^2+1}-\frac{2n+2}{4n^2+8n+5}=\frac{8n^2+8n-2}{(4n^2+1)(4n^2+8n+5)}>0 \text{ for } n\geq 1\text{], and}$$

$\lim_{n\to\infty}b_n=\lim_{n\to\infty}\frac{2/n}{4+1/n^2}=0$, so the series $\sum_{n=1}^{\infty}(-1)^n\frac{2n}{4n^2+1}$ converges by the Alternating Series Test.

9. $b_n=\frac{1}{4n^2+1}>0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty}b_n=0$, so the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2+1}$ converges by the Alternating Series Test.

10. $a_n=(-1)^{n-1}\frac{2n^2}{4n^2+1}$, so $|a_n|=\frac{2n^2}{4n^2+1}\to\frac{1}{2}$ as $n\to\infty$. Therefore, $\lim_{n\to\infty}a_n\neq 0$ (in fact the limit does not exist) and the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n^2}{4n^2+1}$ diverges by the Test for Divergence.

11. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{n}}{n+4}$. $b_n=\frac{\sqrt{n}}{n+4}>0$ for all n. Let $f(x)=\frac{\sqrt{x}}{x+4}$. Then $f'(x)=\frac{4-x}{2\sqrt{x}\,(x+4)^2}<0$ if $x>4$, so $\{b_n\}$ is decreasing after $n=4$. $\lim_{n\to\infty}\frac{\sqrt{n}}{n+4}=\lim_{n\to\infty}\frac{1}{\sqrt{n}+4/\sqrt{n}}=0$. So the series converges by the Alternating Series Test.

12. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2^n}$. $b_n=\frac{n}{2^n}>0$ and $b_n\geq b_{n+1}$ $\Leftrightarrow$ $\frac{n}{2^n}\geq\frac{n+1}{2^{n+1}}$ $\Leftrightarrow$ $2n\geq n+1$ $\Leftrightarrow$ $n\geq 1$ which is certainly true. $\lim_{n\to\infty}(n/2^n)=0$ by l'Hospital's Rule, so the series converges by the Alternating Series Test.

13. $\sum_{n=2}^{\infty}(-1)^n\frac{n}{\ln n}$. $\lim_{n\to\infty}\frac{n}{\ln n}\overset{\text{H}}{=}\lim_{n\to\infty}\frac{1}{1/n}=\infty$, so the series diverges by the Test for Divergence.

14. $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{\ln n}{n}\right)=0+\sum_{n=2}^{\infty}(-1)^{n-1}\left(\frac{\ln n}{n}\right)$. $b_n=\frac{\ln n}{n}>0$ for $n\geq 2$, and if $f(x)=\frac{\ln x}{x}$ then $f'(x)=\frac{1-\ln x}{x^2}<0$ if $x>e$, so $\{b_n\}$ is eventually decreasing. Also, $\lim_{n\to\infty}b_n=\lim_{n\to\infty}\frac{\ln n}{n}\overset{\text{H}}{=}\lim_{n\to\infty}\frac{1/n}{1}=0$, so the series converges by the Alternating Series Test.

15. $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^{3/4}}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{3/4}}$. $b_n=\frac{1}{n^{3/4}}$ is decreasing and positive and $\lim_{n\to\infty}\frac{1}{n^{3/4}}=0$, so the series converges by the Alternating Series Test.

16. $\sin\left(\frac{n\pi}{2}\right) = 0$ if n is even and $(-1)^k$ if $n = 2k+1$, so the series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$. $b_n = \frac{1}{(2n+1)!} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} \frac{1}{(2n+1)!} = 0$, so the series converges by the Alternating Series Test.

17. $\sum_{n=1}^{\infty} (-1)^n \sin\frac{\pi}{n}$. $b_n = \sin\frac{\pi}{n} > 0$ for $n \ge 2$ and $\sin\frac{\pi}{n} \ge \sin\frac{\pi}{n+1}$, and $\lim_{n\to\infty} \sin\frac{\pi}{n} = \sin 0 = 0$, so the series converges by the Alternating Series Test.

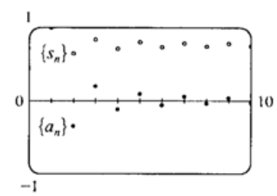
18. $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$. $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1$, so $\lim_{n\to\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$ does not exist and the series diverges by the Test for Divergence.

19. $\frac{n^n}{n!} = \frac{n \cdot n \cdot \cdots \cdot n}{1 \cdot 2 \cdot \cdots \cdot n} \ge n \quad\Rightarrow\quad \lim_{n\to\infty} \frac{n^n}{n!} = \infty \quad\Rightarrow\quad \lim_{n\to\infty} \frac{(-1)^n n^n}{n!}$ does not exist. So the series diverges by the Test for Divergence.

20. $\frac{1}{\sqrt[3]{\ln n}}$ decreases and $\lim_{n\to\infty} \frac{1}{\sqrt[3]{\ln n}} = 0$, so by the Alternating Series Test the series converges.

21.

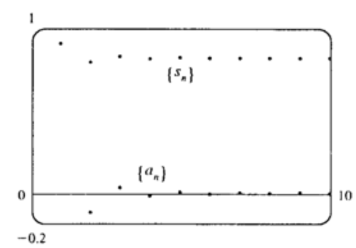
n	a_n	s_n
1	1	1
2	-0.35355	0.64645
3	0.19245	0.83890
4	-0.125	0.71390
5	0.08944	0.80334
6	-0.06804	0.73530
7	0.05399	0.78929
8	-0.04419	0.74510
9	0.03704	0.78214
10	-0.03162	0.75051



By the Alternating Series Estimation Theorem, the error in the approximation $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{3/2}} \approx 0.75051$ is $|s - s_{10}| \le b_{11} = 1/(11)^{3/2} \approx 0.0275$ (to four decimal places, rounded up).

22.

n	a_n	s_n
1	1	1
2	-0.125	0.875
3	0.03704	0.91204
4	-0.01563	0.89641
5	0.008	0.90441
6	-0.00463	0.89978
7	0.00292	0.90270
8	-0.00195	0.90074
9	0.00137	0.90212
10	-0.001	0.90112



By the Alternating Series Estimation Theorem, the error in the approximation $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \approx 0.90112$ is $|s - s_{10}| \le b_{11} = 1/11^3 \approx 0.0007513$.

23. With $b_n = 1/n^2$, $b_{10} = 1/10^2 = 0.01$ and $b_{11} = 1/11^2 = 1/121 \approx 0.008 < 0.01$, so by the Alternating Series Estimation Theorem, $n = 10$.

24. $b_5 = 1/5^4 = 0.0016 > 0.001$ and $b_6 = 1/6^4 \approx 0.00077 < 0.001$, so by the Alternating Series Estimation Theorem, $n = 5$.

25. $b_7 = 2^7/7! \approx 0.025 > 0.01$ and $b_8 = 2^8/8! \approx 0.006 < 0.01$, so by the Alternating Series Estimation Theorem, $n = 7$. (That is, since the 8th term is less than the desired error, we need to add the first 7 terms to get the sum to the desired accuracy.)

26. $b_5 = 5/4^5 \approx 0.0049 > 0.002$ and $b_6 = 6/4^6 \approx 0.0015 < 0.002$, so by the Alternating Series Estimation Theorem, $n = 5$.

27. $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{(2n-1)!}$. $b_5 = \dfrac{1}{(2 \cdot 5 - 1)!} = \dfrac{1}{362{,}880} < 0.00001$, so $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{(2n-1)!} \approx \sum\limits_{n=1}^{4} \dfrac{(-1)^{n-1}}{(2n-1)!} \approx 0.8415$.

28. $b_4 = \dfrac{1}{(2 \cdot 4)!} = \dfrac{1}{40{,}320} \approx 0.000025$ and $s_3 = 1 - \dfrac{1}{2} + \dfrac{1}{24} - \dfrac{1}{720} \approx 0.54028$, so, correct to four decimal places, $\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{(2n)!} \approx 0.5403$.

29. $b_6 = \dfrac{1}{2^6 6!} = \dfrac{1}{46{,}080} \approx 0.000022 < 0.00001$, so $\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{2^n n!} \approx \sum\limits_{n=0}^{5} \dfrac{(-1)^n}{2^n n!} \approx 0.6065$.

30. $b_8 = 1/8^6 \approx 0.0000038 < 0.00001$ and $s_7 = 1 - \frac{1}{64} + \frac{1}{729} - \frac{1}{4096} + \frac{1}{15{,}625} - \frac{1}{46{,}656} + \frac{1}{117{,}649} \approx 0.9855537$, so correct to five decimal places, $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^6} \approx 0.98555$.

31. $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n} = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \cdots + \dfrac{1}{49} - \dfrac{1}{50} + \dfrac{1}{51} - \dfrac{1}{52} + \cdots$. The 50th partial sum of this series is an underestimate, since $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n} = s_{50} + \left(\dfrac{1}{51} - \dfrac{1}{52}\right) + \left(\dfrac{1}{53} - \dfrac{1}{54}\right) + \cdots$, and the terms in parentheses are all positive. The result can be seen geometrically in Figure 1.

32. If $p > 0$, $\dfrac{1}{(n+1)^p} \le \dfrac{1}{n^p}$ and $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$, so the series converges by the Alternating Series Test. If $p \le 0$, $\lim\limits_{n\to\infty} \dfrac{(-1)^{n-1}}{n^p}$ does not exist, so the series diverges by the Test for Divergence. Thus, $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^p}$ converges $\Leftrightarrow$ $p > 0$.

33. Clearly $b_n = \dfrac{1}{n+p}$ is decreasing and eventually positive and $\lim\limits_{n\to\infty} b_n = 0$ for any p. So the series converges (by the Alternating Series Test) for any p for which every b_n is defined, that is, $n + p \neq 0$ for $n \ge 1$, or p is not a negative integer.

34. Let $f(x) = \dfrac{(\ln x)^p}{x}$. Then $f'(x) = \dfrac{(\ln x)^{p-1}(p - \ln x)}{x^2} < 0$ if $x > e^p$ so f is eventually decreasing for every p. Clearly $\lim\limits_{n\to\infty} \dfrac{(\ln n)^p}{n} = 0$ if $p \le 0$, and if $p > 0$ we can apply l'Hospital's Rule $[\![p+1]\!]$ times to get a limit of 0 as well. So the series converges for all p (by the Alternating Series Test).

35. $\sum b_{2n} = \sum 1/(2n)^2$ clearly converges (by comparison with the p-series for $p = 2$). So suppose that $\sum (-1)^{n-1} b_n$ converges. Then by Theorem 12.2.8(ii), so does $\sum \left[(-1)^{n-1} b_n + b_n\right] = 2\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots\right) = 2\sum \frac{1}{2n-1}$. But this diverges by comparison with the harmonic series, a contradiction. Therefore, $\sum (-1)^{n-1} b_n$ must diverge. The Alternating Series Test does not apply since $\{b_n\}$ is not decreasing.

36. (a) We will prove this by induction. Let $P(n)$ be the proposition that $s_{2n} = h_{2n} - h_n$. $P(1)$ is true by an easy calculation. So suppose that $P(n)$ is true. We will show that $P(n+1)$ must be true as a consequence.

$$h_{2n+2} - h_{n+1} = \left(h_{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}\right) - \left(h_n + \frac{1}{n+1}\right) = (h_{2n} - h_n) + \frac{1}{2n+1} - \frac{1}{2n+2}$$
$$= s_{2n} + \frac{1}{2n+1} - \frac{1}{2n+2} = s_{2n+2}$$

which is $P(n+1)$, and proves that $s_{2n} = h_{2n} - h_n$ for all n.

(b) We know that $h_{2n} - \ln 2n \to \gamma$ and $h_n - \ln n \to \gamma$ as $n \to \infty$. So $s_{2n} = h_{2n} - h_n = (h_{2n} - \ln 2n) - (h_n - \ln n) + (\ln 2n - \ln n)$, and $\lim\limits_{n\to\infty} s_{2n} = \gamma - \gamma + \lim\limits_{n\to\infty} [\ln 2n - \ln n] = \lim\limits_{n\to\infty} (\ln 2 + \ln n - \ln n) = \ln 2$.

12.6 Absolute Convergence and the Ratio and Root Tests

1. (a) Since $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 8 > 1$, part (ii) of the Ratio Test tells us that $\sum a_n$ is divergent.

(b) Since $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0.8 < 1$, part (i) of the Ratio Test tells us that $\sum a_n$ is convergent.

(c) Since $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test fails and $\sum a_n$ might converge or it might diverge.

2. The series $\sum\limits_{n=1}^{\infty} \frac{n^2}{2^n}$ has positive terms and $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \left[\frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}\right] = \lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 \cdot \frac{1}{2} = \frac{1}{2} < 1$, so the series is absolutely convergent by the Ratio Test.

3. $\sum\limits_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum\limits_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series ($p = \frac{3}{2} > 1$), so the given series is absolutely convergent.

4. $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}$ converges by the Alternating Series Test, but $\sum\limits_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a divergent p-series ($p = \frac{1}{2} < 1$), so $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}$ converges conditionally.

5. Using the Ratio Test, $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\frac{(-3)^{n+1}/(n+1)^3}{(-3)^n/n^3}\right| = 3 \lim\limits_{n\to\infty} \left(\frac{n}{n+1}\right)^3 = 3 > 1$, so the series diverges.

6. Using the Ratio Test, $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\frac{(-3)^{n+1}/(n+1)!}{(-3)^n/n!}\right| = 3 \lim\limits_{n\to\infty} \frac{1}{n+1} = 0 < 1$, so the series is absolutely convergent.

7. $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{5+n}$ converges by the Alternating Series Test, but $\sum\limits_{n=1}^{\infty} \dfrac{1}{5+n}$ diverges by the Limit Comparison Test with the harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$, so the given series is conditionally convergent.

8. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{n!}{(n+1)!} = \lim\limits_{n\to\infty} \dfrac{1}{n+1} = 0 < 1$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n!}$ is absolutely convergent by the Ratio Test.

9. $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} \dfrac{n}{5+n} = \lim\limits_{n\to\infty} \dfrac{1}{5/n+1} = 1$, so $\lim\limits_{n\to\infty} a_n \neq 0$. Thus, the given series is divergent by the Test for Divergence.

10. $\sum\limits_{n=1}^{\infty} \dfrac{n}{n^2+1}$ diverges by the Limit Comparison Test with the harmonic series:

$\lim\limits_{n\to\infty} \dfrac{n/\left(n^2+1\right)}{1/n} = \lim\limits_{n\to\infty} \dfrac{n^2}{n^2+1} = 1$. But $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{n}{n^2+1}$ converges by the Alternating Series Test:

$\left\{\dfrac{n}{n^2+1}\right\}$ has positive terms, is decreasing since $\left(\dfrac{x}{x^2+1}\right)' = \dfrac{1-x^2}{\left(x^2+1\right)^2} \leq 0$ for $x \geq 1$, and $\lim\limits_{n\to\infty} \dfrac{n}{n^2+1} = 0$.

Thus, $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{n}{n^2+1}$ is conditionally convergent.

11. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{1/(2n+2)!}{1/(2n)!} = \lim\limits_{n\to\infty} \dfrac{(2n)!}{(2n+2)!} = \lim\limits_{n\to\infty} \dfrac{(2n)!}{(2n+2)(2n+1)(2n)!} =$

$\lim\limits_{n\to\infty} \dfrac{1}{(2n+2)(2n+1)} = 0 < 1$, so the series is absolutely convergent by the Ratio Test. Of course, absolute convergence is the same as convergence for this series, since all of its terms are positive.

12. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(n+1)!/e^{n+1}}{n!/e^n}\right| = \dfrac{1}{e} \lim\limits_{n\to\infty} (n+1) = \infty$, so the series diverges by the Ratio Test.

13. $\left|\dfrac{\sin 2n}{n^2}\right| \leq \dfrac{1}{n^2}$ and $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ converges (p-series, $p = 2 > 1$), so $\sum\limits_{n=1}^{\infty} \dfrac{\sin 2n}{n^2}$ converges absolutely by the Comparison Test.

14. $\dfrac{\arctan n}{n^3} < \dfrac{\pi/2}{n^3}$ and $\sum\limits_{n=1}^{\infty} \dfrac{\pi/2}{n^3}$ converges ($p = 3 > 1$), so $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{\arctan n}{n^3}$ converges absolutely by the Comparison Test.

15. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left[\dfrac{(n+1)\,3^{n+1}}{4^n} \cdot \dfrac{4^{n-1}}{n \cdot 3^n}\right] = \lim\limits_{n\to\infty} \left(\dfrac{3}{4} \cdot \dfrac{n+1}{n}\right) = \dfrac{3}{4} < 1$, so the series is absolutely convergent by the Ratio Test.

16. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left[\dfrac{(n+1)^2\,2^{n+1}}{(n+1)!} \cdot \dfrac{n!}{n^2 2^n}\right] = \lim\limits_{n\to\infty} \left[\left(1+\dfrac{1}{n}\right)^2 \cdot \dfrac{2}{n+1}\right] = 0$, so the series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \dfrac{n^2 2^n}{n!}$ is absolutely convergent by the Ratio Test.

17. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left[\dfrac{10^{n+1}}{(n+2)\,4^{2n+3}} \cdot \dfrac{(n+1)\,4^{2n+1}}{10^n}\right] = \lim\limits_{n\to\infty} \left(\dfrac{10}{4^2} \cdot \dfrac{n+1}{n+2}\right) = \dfrac{5}{8} < 1$, so the series is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergence is the same as convergence.

18. $\left|\cos\dfrac{n\pi}{6}\right| \le 1$, so since $\sum_{n=1}^{\infty}\dfrac{1}{n\sqrt{n}}$ converges ($p = \frac{3}{2} > 1$), the given series converges absolutely by the Comparison Test.

19. $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\dfrac{(n+1)!/10^{n+1}}{n!/10^n} = \lim_{n\to\infty}\dfrac{n+1}{10} = \infty$, so the series diverges by the Ratio Test.

20. $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\dfrac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \lim_{n\to\infty}\dfrac{n^n}{(n+1)^n} = \lim_{n\to\infty}\dfrac{1}{(1+1/n)^n} = \dfrac{1}{e} < 1$, so the series converges absolutely by the Ratio Test.

21. $\dfrac{|\cos(n\pi/3)|}{n!} \le \dfrac{1}{n!}$ and $\sum_{n=1}^{\infty}\dfrac{1}{n!}$ converges (Exercise 12.4.29), so the given series converges absolutely by the Comparison Test.

22. $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\dfrac{1}{\ln n} = 0 < 1$ so the series converges absolutely by the Root Test.

23. $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left(\dfrac{n^n}{3^{1+3n}}\right)^{1/n} = \lim_{n\to\infty}\dfrac{n}{\sqrt[n]{3}\cdot 3^3} = \infty$, so the series is divergent by the Root Test.

Or: $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\dfrac{(n+1)^{n+1}}{3^{4+3n}}\cdot\dfrac{3^{1+3n}}{n^n}\right] = \lim_{n\to\infty}\left[\dfrac{1}{3^3}\cdot\left(\dfrac{n+1}{n}\right)^n(n+1)\right]$

$$= \frac{1}{27}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\lim_{n\to\infty}(n+1) = \tfrac{1}{27}e\lim_{n\to\infty}(n+1) = \infty$$

so the series is divergent by the Ratio Test.

24. Since $\left\{\dfrac{1}{n\ln n}\right\}$ is decreasing and $\lim_{n\to\infty}\dfrac{1}{n\ln n} = 0$, the series converges by the Alternating Series Test, but since $\sum_{n=2}^{\infty}\dfrac{1}{n\ln n}$ diverges by the Integral Test (Exercise 12.3.19), the given series converges only conditionally.

25. $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\dfrac{n^2+1}{2n^2+1} = \lim_{n\to\infty}\dfrac{1+1/n^2}{2+1/n^2} = \dfrac{1}{2} < 1$, so the series is absolutely convergent by the Root Test.

26. $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\dfrac{1}{\arctan n} = \dfrac{1}{\pi/2} = \dfrac{2}{\pi} < 1$ so the series converges absolutely by the Root Test.

27. $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\dfrac{(n+1)!/[1\cdot 3\cdot 5\cdot\cdots\cdot(2n+1)]}{n!/[1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)]} = \lim_{n\to\infty}\dfrac{n+1}{2n+1} = \dfrac{1}{2} < 1$, so the series converges absolutely by the Ratio Test.

28. $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\dfrac{\dfrac{1\cdot 4\cdot 7\cdot\cdots\cdot(3n-2)(3n+1)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)(2n+3)}}{\dfrac{1\cdot 4\cdot 7\cdot\cdots\cdot(3n-2)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}}\right| = \lim_{n\to\infty}\dfrac{3n+1}{2n+3} = \dfrac{3}{2} > 1$, so the series diverges by the Ratio Test.

29. $\sum_{n=1}^{\infty}\dfrac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{n!} = \sum_{n=1}^{\infty}\dfrac{2^n n!}{n!} = \sum_{n=1}^{\infty}2^n$ which diverges by the Test for Divergence since $\lim_{n\to\infty}2^n = \infty$.

30. $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\dfrac{\dfrac{2^{n+1}(n+1)!}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+5)}}{\dfrac{2^n n!}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)}}\right| = \lim_{n\to\infty}\dfrac{2(n+1)}{3n+5} = \dfrac{2}{3} < 1$ so the series converges absolutely by the Ratio Test.

31. By the recursive definition, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{5n+1}{4n+3}\right| = \frac{5}{4} > 1$, so the series diverges by the Ratio Test.

32. By the recursive definition, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2+\cos n}{\sqrt{n}}\right| = 0 < 1$, so the series converges absolutely by the Ratio Test.

33. (a) $\lim_{n\to\infty}\left|\frac{1/(n+1)^3}{1/n^3}\right| = \lim_{n\to\infty}\frac{n^3}{(n+1)^3} = \lim_{n\to\infty}\frac{1}{(1+1/n)^3} = 1$. Inconclusive.

(b) $\lim_{n\to\infty}\left|\frac{(n+1)}{2^{n+1}}\cdot\frac{2^n}{n}\right| = \lim_{n\to\infty}\frac{n+1}{2n} = \lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{2n}\right) = \frac{1}{2}$. Conclusive (convergent).

(c) $\lim_{n\to\infty}\left|\frac{(-3)^n}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(-3)^{n-1}}\right| = 3\lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = 3\lim_{n\to\infty}\sqrt{\frac{1}{1+1/n}} = 3$. Conclusive (divergent).

(d) $\lim_{n\to\infty}\left|\frac{\sqrt{n+1}}{1+(n+1)^2}\cdot\frac{1+n^2}{\sqrt{n}}\right| = \lim_{n\to\infty}\left[\sqrt{1+\frac{1}{n}}\cdot\frac{1/n^2+1}{1/n^2+(1+1/n)^2}\right] = 1$. Inconclusive.

34. We use the Ratio Test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{[(n+1)!]^2/[k(n+1)!]}{(n!)^2/(kn)!}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n+1)^2}{[k(n+1)][k(n+1)-1]\cdots[kn+1]}\right|\end{aligned}$$

Now if $k=1$, then this is equal to $\lim_{n\to\infty}\left|\frac{(n+1)^2}{(n+1)}\right| = \infty$, so the series diverges; if $k=2$, the limit is $\lim_{n\to\infty}\left|\frac{(n+1)^2}{(2n+2)(2n+1)}\right| = \frac{1}{4} < 1$, so the series converges, and if $k > 2$, then the highest power of n in the denominator is larger than 2, and so the limit is 0, indicating convergence. So the series converges for $k \geq 2$.

35. (a) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|^{n+1}/(n+1)!}{|x|^n/n!} = |x|\lim_{n\to\infty}\frac{1}{n+1} = 0$, so by the Ratio Test the series converges for all x.

(b) Since the series of part (a) always converges, we must have $\lim_{n\to\infty}\frac{x^n}{n!} = 0$ by Theorem 12.2.6.

36. (a)
$$\begin{aligned} R_n &= a_{n+1}+a_{n+2}+a_{n+3}+a_{n+4}+\cdots = a_{n+1}\left(1+\frac{a_{n+2}}{a_{n+1}}+\frac{a_{n+3}}{a_{n+1}}+\frac{a_{n+4}}{a_{n+1}}+\cdots\right) \\ &= a_{n+1}\left(1+\frac{a_{n+2}}{a_{n+1}}+\frac{a_{n+3}}{a_{n+2}}\frac{a_{n+2}}{a_{n+1}}+\frac{a_{n+4}}{a_{n+3}}\frac{a_{n+3}}{a_{n+2}}\frac{a_{n+2}}{a_{n+1}}+\cdots\right) \\ &= a_{n+1}\left(1+r_{n+1}+r_{n+2}r_{n+1}+r_{n+3}r_{n+2}r_{n+1}+\cdots\right) \quad (\bigstar) \\ &\leq a_{n+1}\left(1+r_{n+1}+r_{n+1}^2+r_{n+1}^3+\cdots\right) \quad [\text{since } \{r_n\} \text{ is decreasing}] \quad = \frac{a_{n+1}}{1-r_{n+1}}\end{aligned}$$

(b) Note that since $\{r_n\}$ is increasing and $r_n \to L$ as $n \to \infty$, we have $r_n < L$ for all n. So, starting with equation $\bigstar$,

$$R_n = a_{n+1}\left(1+r_{n+1}+r_{n+2}r_{n+1}+r_{n+3}r_{n+2}r_{n+1}+\cdots\right) \leq a_{n+1}\left(1+L+L^2+L^3+\cdots\right) = \frac{a_{n+1}}{1-L}$$

37. (a) $s_5 = \sum\limits_{n=1}^{5} \frac{1}{n2^n} = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \frac{1}{160} = \frac{661}{960} \approx 0.68854$. Now the ratios

$r_n = \frac{a_{n+1}}{a_n} = \frac{n2^n}{(n+1)\,2^{n+1}} = \frac{n}{2\,(n+1)}$ form an increasing sequence, since

$r_{n+1} - r_n = \frac{n+1}{2\,(n+2)} - \frac{n}{2\,(n+1)} = \frac{(n+1)^2 - n\,(n+2)}{2\,(n+1)\,(n+2)} = \frac{1}{2\,(n+1)\,(n+2)} > 0$. So by Exercise 36(b),

the error is less than $\frac{a_6}{1 - \lim\limits_{n\to\infty} r_n} = \frac{1/\left(6 \cdot 2^6\right)}{1 - 1/2} = \frac{1}{192} \approx 0.00521$.

(b) The error in using s_n as an approximation to the sum is $R_n = \frac{a_{n+1}}{1 - \frac{1}{2}} = \frac{2}{(n+1)\,2^{n+1}}$. We want $R_n < 0.00005$

$\Leftrightarrow \quad \frac{1}{(n+1)\,2^n} < 0.00005 \quad \Leftrightarrow \quad (n+1)\,2^n > 20{,}000$. To find such an n we can use trial and error or a

graph. We calculate $(11+1)\,2^{11} = 24{,}576$, so $s_{11} = \sum\limits_{n=1}^{11} \frac{1}{n2^n} \approx 0.693109$ is within 0.00005 of the actual

sum.

38. $\sum\limits_{n=1}^{10} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{10}{1024} \approx 1.988$. The ratios $r_n = \frac{a_{n+1}}{a_n} = \frac{n+1}{2n} = \frac{1}{2}\left(1 + \frac{1}{n}\right)$ form a decreasing

sequence, so by Exercise 36(a), using $a_{11} = \frac{11}{2048}$ and $r_{11} = \frac{12}{22} = \frac{6}{11}$, the error in the above approximation is less

than $\frac{a_{11}}{1 - r_{11}} \approx 0.0118$.

39. Summing the inequalities $-|a_i| \le a_i \le |a_i|$ for $i = 1, 2, \ldots, n$, we get $-\sum_{i=1}^{n} |a_i| \le \sum_{i=1}^{n} a_i \le \sum_{i=1}^{n} |a_i| \quad \Rightarrow$
$-\lim\limits_{n\to\infty} \sum_{i=1}^{n} |a_i| \le \lim\limits_{n\to\infty} \sum_{i=1}^{n} a_i \le \lim\limits_{n\to\infty} \sum_{i=1}^{n} |a_i| \quad \Rightarrow \quad -\sum_{n=1}^{\infty} |a_n| \le \sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} |a_n| \quad \Rightarrow$
$\left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty} |a_n|$.

40. (a) Following the hint, we get that $|a_n| < r^n$ for $n \ge N$, and so since the geometric series $\sum_{n=1}^{\infty} r^n$ converges $(0 < r < 1)$, the series $\sum_{n=N}^{\infty} |a_n|$ converges as well by the Comparison Test, and hence so does $\sum_{n=1}^{\infty} |a_n|$, so $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(b) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, then there is an integer N such that $\sqrt[n]{|a_n|} > 1$ for all $n \ge N$, so $|a_n| > 1$ for $n \ge N$. Thus, $\lim\limits_{n\to\infty} a_n \ne 0$, so $\sum_{n=1}^{\infty} a_n$ diverges by the Test for Divergence.

41. (a) Since $\sum a_n$ is absolutely convergent, and since $\left|a_n^+\right| \le |a_n|$ and $\left|a_n^-\right| \le |a_n|$ (because a_n^+ and a_n^- each equal either a_n or 0), we conclude by the Comparison Test that both $\sum a_n^+$ and $\sum a_n^-$ must be absolutely convergent. (Or use Theorem 12.2.8.)

(b) We will show by contradiction that both $\sum a_n^+$ and $\sum a_n^-$ must diverge. For suppose that $\sum a_n^+$ converged. Then so would $\sum \left(a_n^+ - \frac{1}{2}a_n\right)$ by Theorem 12.2.8. But $\sum \left(a_n^+ - \frac{1}{2}a_n\right) = \sum \left[\frac{1}{2}\left(a_n + |a_n|\right) - \frac{1}{2}a_n\right] = \frac{1}{2}\sum |a_n|$, which diverges because $\sum a_n$ is only conditionally convergent. Hence, $\sum a_n^+$ can't converge. Similarly, neither can $\sum a_n^-$.

42. Let $\sum b_n$ be the rearranged series constructed in the hint. [This series can be constructed by virtue of the result of Exercise 41(b).] This series will have partial sums s_n that oscillate in value back and forth across r.
Since $\lim\limits_{n\to\infty} a_n = 0$ (by Theorem 12.2.6), and since the size of the oscillations $|s_n - r|$ is always less than $|a_n|$ because of the way $\sum b_n$ was constructed, we have that $\sum b_n = \lim\limits_{n\to\infty} s_n = r$.

12.7 Strategy for Testing Series

1. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{n^2-1}{n^2+1} = \lim\limits_{n\to\infty} \dfrac{1-1/n^2}{1+1/n} = 1 \neq 0$, so the series diverges by the Test for Divergence.

2. If $a_n = \dfrac{n-1}{n^2+n}$ and $b_n = \dfrac{1}{n}$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^2-n}{n^2+n} = \lim\limits_{n\to\infty} \dfrac{1-1/n}{1+1/n} = 1$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{n-1}{n^2+n}$ diverges by the Limit Comparison Test with the harmonic series.

3. $\dfrac{1}{n^2+n} < \dfrac{1}{n^2}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+n}$ converges by the Comparison Test with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$, a p-series that converges because $p = 2 > 1$.

4. Let $b_n = \dfrac{n-1}{n^2+n}$. Then $b_1 = 0$, and $b_2 = b_3 = \frac{1}{6}$, but $b_n > b_{n+1}$ for $n \geq 3$ since
$\left(\dfrac{x-1}{x^2+x}\right)' = \dfrac{(x^2+x)-(x-1)(2x+1)}{(x^2+x)^2} = \dfrac{-x^2+2x+1}{(x^2+x)^2} < 0$ for $x \geq 3$. (This can be confirmed with a graph.) Thus, $\{b_n \mid n \geq 3\}$ is decreasing and $\lim\limits_{n\to\infty} b_n = 0$, so $\sum\limits_{n=3}^{\infty} (-1)^{n-1} \dfrac{n-1}{n^2+n}$ converges by the Alternating Series Test. Hence, the full series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{n-1}{n^2+n}$ also converges.

5. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left(\dfrac{3^{n+2}}{2^{3n+3}} \cdot \dfrac{2^{3n}}{3^{n+1}}\right) = \lim\limits_{n\to\infty} \dfrac{3}{2^3} = \dfrac{3}{8} < 1$, so the series is absolutely convergent by the Ratio Test.

6. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \dfrac{3n}{1+8n} = \lim\limits_{n\to\infty} \dfrac{3}{1/n+8} = \dfrac{3}{8} < 1$, so $\sum\limits_{n=1}^{\infty} \left(\dfrac{3n}{1+8n}\right)^n$ converges by the Root Test.

7. $\sum\limits_{k=1}^{\infty} k^{-1.7} = \sum\limits_{k=1}^{\infty} \dfrac{1}{k^{1.7}}$ is a convergent p-series ($p = 1.7 > 1$).

8. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{10^{n+1}/(n+1)!}{10^n/n!} = \lim\limits_{n\to\infty} \dfrac{10}{n+1} = 0 < 1$, so the series converges by the Ratio Test.

9. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{(n+1)/e^{n+1}}{n/e^n} = \dfrac{1}{e} \lim\limits_{n\to\infty} \dfrac{n+1}{n} = \dfrac{1}{e} < 1$, so the series converges by the Ratio Test.

10. Let $f(x) = x^2 e^{-x^3}$. Then f is continuous and positive on $[1, \infty)$, and $f'(x) = \dfrac{x(2-3x^3)}{e^{x^3}} < 0$ for $x \geq 1$, so f is decreasing on $[1, \infty)$ as well, and we can apply the Integral Test. $\int_1^\infty x^2 e^{-x^3}\,dx = \lim\limits_{t\to\infty} \left[-\frac{1}{3}e^{-x^3}\right]_1^t = \frac{1}{3e}$, so the series converges.

11. $b_n = \dfrac{1}{n \ln n} > 0$ for $n \geq 2$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^{n+1}}{n \ln n}$ converges by the Alternating Series Test. The series is conditionally convergent since $\sum\limits_{n=2}^{\infty} \dfrac{1}{n \ln n}$ diverges by Exercise 12.3.19.

12. The series $\sum_{n=1}^{\infty} \sin n$ diverges by the Test for Divergence since $\lim\limits_{n\to\infty} \sin n$ does not exist.

13. Let $f(x) = \dfrac{2}{x(\ln x)^3}$. $f(x)$ is clearly positive and decreasing for $x \geq 2$, so we apply the Integral Test.

$\displaystyle\int_2^\infty \frac{2}{x(\ln x)^3}\,dx = \lim_{t\to\infty}\left[\frac{-1}{(\ln x)^2}\right]_2^t = 0 - \frac{-1}{(\ln 2)^2}$, which is finite, so $\displaystyle\sum_{n=2}^\infty \frac{2}{n(\ln n)^3}$ converges.

14. Using the Limit Comparison Test with $a_n = \dfrac{n^2+1}{n^3+1}$ and $b_n = \dfrac{1}{n}$, we have

$\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^3+n}{n^3+1} = \lim_{n\to\infty}\frac{1+1/n^2}{1+1/n^3} = 1 > 0$. Since $\sum_{n=1}^\infty b_n$ is the divergent harmonic series, $\sum_{n=1}^\infty a_n$ is also divergent.

15. $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3^{n+1}(n+1)^2/(n+1)!}{3^n n^2/n!} = 3\lim_{n\to\infty}\frac{n+1}{n^2} = 0$, so the series converges by the Ratio Test.

16. $b_n = \dfrac{1}{\sqrt{n}-1}$ for $n \geq 2$. $\{b_n\}$ is a decreasing sequence of positive numbers and $\lim_{n\to\infty} b_n = 0$, so $\displaystyle\sum_{n=2}^\infty \frac{(-1)^{n-1}}{\sqrt{n}-1}$ converges by the Alternating Series Test.

17. $\dfrac{3^n}{5^n+n} \leq \dfrac{3^n}{5^n} = \left(\dfrac{3}{5}\right)^n$. Since $\displaystyle\sum_{n=1}^\infty \left(\frac{3}{5}\right)^n$ is a convergent geometric series ($|r| = \frac{3}{5} < 1$), $\displaystyle\sum_{n=1}^\infty \frac{3^n}{5^n+n}$ converges by the Comparison Test.

18. $\displaystyle\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{(k+6)/5^{k+1}}{(k+5)/5^k} = \frac{1}{5}\lim_{k\to\infty}\frac{k+6}{k+5} = \frac{1}{5} < 1$, so the series converges by the Ratio Test.

19. $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\dfrac{(n+1)!}{2\cdot5\cdot8\cdot\cdots\cdot(3n+5)}}{\dfrac{n!}{2\cdot5\cdot8\cdot\cdots\cdot(3n+2)}} = \lim_{n\to\infty}\frac{n+1}{3n+5} = \frac{1}{3} < 1$, so the series converges by the Ratio Test.

20. $\displaystyle\lim_{n\to\infty}\frac{n}{(n+1)(n+2)} = \lim_{n\to\infty}\frac{1}{n+3+2/n} = 0$, so since $\{b_n\}$ is a positive, decreasing sequence, $\displaystyle\sum_{n=1}^\infty \frac{(-1)^n n}{(n+1)(n+2)}$ converges by the Alternating Series Test.

21. Use the Limit Comparison Test with $a_i = \dfrac{1}{\sqrt{i(i+1)}}$ and $b_i = \dfrac{1}{i}$.

$\displaystyle\lim_{i\to\infty}\frac{a_i}{b_i} = \lim_{i\to\infty}\frac{i}{\sqrt{i(i+1)}} = \lim_{i\to\infty}\frac{1}{\sqrt{1+1/i}} = 1$. Since $\displaystyle\sum_{i=1}^\infty b_i$ diverges (harmonic series) so does $\displaystyle\sum_{i=1}^\infty \frac{1}{\sqrt{i(i+1)}}$.

22. $\dfrac{\sqrt{n^2-1}}{n^3+2n^2+5} < \dfrac{n}{n^3+2n^2+5} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$ for $n \geq 1$, so $\displaystyle\sum_{n=1}^\infty \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$ converges by the Comparison Test with the convergent p-series $\sum_{n=1}^\infty 1/n^2$.

23. $\lim_{n\to\infty} 2^{1/n} = 2^0 = 1$, so $\lim_{n\to\infty}(-1)^n 2^{1/n}$ does not exist and the series diverges by the Test for Divergence.

24. $\dfrac{|\cos(n/2)|}{n^2+4n} < \dfrac{1}{n^2}$ and since $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2}$ converges ($p = 2 > 1$), $\displaystyle\sum_{n=1}^\infty \frac{\cos(n/2)}{n^2+4n}$ converges absolutely by the Comparison Test.

25. Let $f(x) = \dfrac{\ln x}{\sqrt{x}}$. Then $f'(x) = \dfrac{2 - \ln x}{2x^{3/2}} < 0$ when $\ln x > 2$ or $x > e^2$, so $\dfrac{\ln n}{\sqrt{n}}$ is decreasing for $n > e^2$.

By l'Hospital's Rule, $\lim\limits_{n\to\infty} \dfrac{\ln n}{\sqrt{n}} = \lim\limits_{n\to\infty} \dfrac{1/n}{1/(2\sqrt{n})} = \lim\limits_{n\to\infty} \dfrac{2}{\sqrt{n}} = 0$, so the series converges by the Alternating Series Test.

26. Let $a_n = \dfrac{\tan(1/n)}{n}$ and $b_n = \dfrac{1}{n^2}$. Then

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} n \cdot \tan(1/n) = \lim\limits_{n\to\infty} \dfrac{\tan(1/n)}{1/n} \overset{\text{H}}{=} \lim\limits_{n\to\infty} \dfrac{(-1/n^2)\sec^2(1/n)}{-1/n^2} = \sec^2(0) = 1 > 0$, so since $\sum\limits_{n=1}^{\infty} b_n$ converges ($p = 2 > 1$), $\sum\limits_{n=1}^{\infty} \dfrac{\tan(1/n)}{n}$ converges also, by the Limit Comparison Test.

27. $\sum\limits_{n=1}^{\infty} \dfrac{(-2)^{2n}}{n^n} = \sum\limits_{n=1}^{\infty} \left(\dfrac{4}{n}\right)^n$. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \dfrac{4}{n} = 0$, so the series converges by the Root Test.

28. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \left(\dfrac{n^2 + 2n + 2}{5^{n+1}} \cdot \dfrac{5^n}{n^2 + 1}\right) = \lim\limits_{n\to\infty} \left(\dfrac{1 + 2/n + 2/n^2}{1 + 1/n^2} \cdot \dfrac{1}{5}\right) = \dfrac{1}{5} < 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n^2 + 1}{5^n}$ converges by the Ratio Test.

29. $\displaystyle\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^t$ (using integration by parts) $\overset{\text{H}}{=} 1$. So $\sum\limits_{n=1}^{\infty} \dfrac{\ln n}{n^2}$ converges by the Integral Test, and since $\dfrac{k \ln k}{(k+1)^3} < \dfrac{k \ln k}{k^3} = \dfrac{\ln k}{k^2}$, the given series converges by the Comparison Test.

30. Since $\left\{\dfrac{1}{n}\right\}$ is a decreasing sequence, $e^{1/n} \leq e^{1/1} = e$ for all $n \geq 1$, and $\sum\limits_{n=1}^{\infty} \dfrac{e}{n^2}$ converges ($p = 2 > 1$), so $\sum\limits_{n=1}^{\infty} \dfrac{e^{1/n}}{n^2}$ converges by the Comparison Test. (Or use the Integral Test.)

31. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{2^{n+1}/(2n+3)!}{2^n/(2n+1)!} = 2 \lim\limits_{n\to\infty} \dfrac{1}{(2n+3)(2n+2)} = 0$, so the series converges by the Ratio Test.

32. Let $f(x) = \dfrac{\sqrt{x}}{x+5}$. Then $f(x)$ is continuous and positive on $[1, \infty)$, and since $f'(x) = \dfrac{5 - x}{2\sqrt{x}\,(x+5)^2} < 0$ for $x > 5$, $f(x)$ is eventually decreasing, so we can use the Alternating Series Test.

$\lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{n+5} = \lim\limits_{n\to\infty} \dfrac{1}{n^{1/2} + 5n^{-1/2}} = 0$, so the series converges.

33. $0 < \dfrac{\tan^{-1} n}{n^{3/2}} < \dfrac{\pi/2}{n^{3/2}}$. $\sum\limits_{n=1}^{\infty} \dfrac{\pi/2}{n^{3/2}} = \dfrac{\pi}{2} \sum\limits_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ which is a convergent p-series ($p = \frac{3}{2} > 1$), so $\sum\limits_{n=1}^{\infty} \dfrac{\tan^{-1} n}{n^{3/2}}$ converges by the Comparison Test.

34. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \dfrac{2n}{n^2} = \lim\limits_{n\to\infty} \dfrac{2}{n} = 0$ so the series converges by the Root Test.

35. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \left(\dfrac{n}{n+1}\right)^{n^2/n} = \lim\limits_{n\to\infty} \dfrac{1}{[(n+1)/n]^n} = \dfrac{1}{\lim\limits_{n\to\infty} (1 + 1/n)^n} = \dfrac{1}{e} < 1$ (see Equation 7.4.9), so the series converges by the Root Test.

36. Note that $(\ln n)^{\ln n} = \left(e^{\ln \ln n}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln \ln n} = n^{\ln \ln n}$ and $\ln \ln n \to \infty$ as $n \to \infty$, so $\ln \ln n > 2$ for sufficiently large n. For these n we have $(\ln n)^{\ln n} > n^2$, so $\dfrac{1}{(\ln n)^{\ln n}} < \dfrac{1}{n^2}$. Since $\sum\limits_{n=2}^{\infty} \dfrac{1}{n^2}$ converges ($p = 2 > 1$), so does $\sum\limits_{n=2}^{\infty} \dfrac{1}{(\ln n)^{\ln n}}$ by the Comparison Test.

37. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \left(2^{1/n} - 1\right) = 1 - 1 = 0$, so the series converges by the Root Test.

38. Use the Limit Comparison Test with $a_n = \sqrt[n]{2} - 1$ and $b_n = 1/n$. Then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{2^{1/n} - 1}{1/n} \overset{\text{H}}{=} \ln 2 > 0$. So since $\sum_{n=1}^{\infty} b_n$ diverges (harmonic series), so does $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)$.

Alternate Solution:

$\sqrt[n]{2} - 1 = \dfrac{1}{2^{(n-1)/n} + 2^{(n-2)/n} + 2^{(n-3)/n} + \cdots + 2^{1/n} + 1}$ [rationalize the numerator] $\geq \dfrac{1}{2n}$, and since $\sum\limits_{n=1}^{\infty} \dfrac{1}{2n} = \dfrac{1}{2} \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges (harmonic series), so does $\sum\limits_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)$ by the Comparison Test.

12.8 Power Series

1. A power series is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$, where x is a variable and the c_n's are constants called the coefficients of the series.

More generally, a series of the form $\sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1 (x - a) + c_2 (x - a)^2 + \cdots$ is called a power series in $(x - a)$ or a power series centered at a or a power series about a.

2. (a) Given the power series $\sum_{n=0}^{\infty} c_n (x - a)^n$, the radius of convergence is:

(i) 0 if the series converges only when $x = a$

(ii) ∞ if the series converges for all x, or

(iii) a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

In most cases, R can be found by using the Ratio Test.

(b) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (a), the interval of convergence is: (i) the single point $\{a\}$, (ii) all real numbers, that is, the real number line $(-\infty, \infty)$, or (iii) an interval with endpoints $a - R$ and $a + R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.

3. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x^{n+1}}{\sqrt{n+1}} \cdot \dfrac{\sqrt{n}}{x}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x}{\sqrt{n+1}/\sqrt{n}}\right| = \lim\limits_{n\to\infty} \dfrac{|x|}{\sqrt{1+1/n}} = |x|$. By the Ratio Test, the series converges when $|x| < 1$, so the radius of convergence $R = 1$. When $x = 1$, the series $\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges because it is a p-series with $p = \frac{1}{2} \leq 1$, but when $x = -1$, it converges by the Alternating Series Test. So the interval of convergence is $I = [-1, 1)$.

4. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x^{n+1}}{n+2} \cdot \dfrac{n+1}{x^n}\right| = \lim\limits_{n\to\infty} \dfrac{|x|}{1 + 1/(n+1)} = |x|$. By the Ratio Test, the series $\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^n}{n+1}$ converges when $|x| < 1$, so $R = 1$. When $x = -1$, the series diverges because it is the harmonic series; when $x = 1$, it is the alternating harmonic series, which converges by the Alternating Series Test. Thus, $I = (-1, 1]$.

5. If $a_n = nx^n$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)\,x^{n+1}}{nx^n}\right| = |x|\lim\limits_{n\to\infty}\dfrac{n+1}{n} = |x| < 1$ for convergence (by the Ratio Test). So $R = 1$. When $x = 1$ or -1, $\lim\limits_{n\to\infty} nx^n$ does not exist, so $\sum_{n=0}^{\infty} nx^n$ diverges for $x = \pm 1$. So $I = (-1, 1)$.

6. If $a_n = \dfrac{x^n}{n^2}$ then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = |x|\lim\limits_{n\to\infty}\left(\dfrac{n}{n+1}\right)^2 = |x| < 1$ for convergence (by the Ratio Test), so $R = 1$. If $x = \pm 1$, $\sum\limits_{n=1}^{\infty}|a_n| = \sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$, which converges ($p = 2 > 1$), so $I = [-1, 1]$.

7. If $a_n = \dfrac{x^n}{n!}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}/\,(n+1)!}{x^n/n!}\right| = |x|\lim\limits_{n\to\infty}\dfrac{1}{n+1} = 0 < 1$ for all x. So, by the Ratio Test, $R = \infty$, and $I = (-\infty, \infty)$.

8. Here the Root Test is easier. If $a_n = n^n x^n$ then $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} n\,|x| = \infty$ if $x \neq 0$, so $R = 0$ and $I = \{0\}$.

9. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{(n+1)\,4^{n+1}\,|x|^{n+1}}{n4^n\,|x|^n} = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)4\,|x| = 4\,|x|$. Now $4\,|x| < 1 \;\Leftrightarrow\; |x| < \frac{1}{4}$, so by the Ratio Test, $R = \frac{1}{4}$. When $x = \frac{1}{4}$, we get the divergent series $\sum_{n=1}^{\infty}(-1)^n\,n$, and when $x = -\frac{1}{4}$, we get the divergent series $\sum_{n=1}^{\infty} n$. Thus, $I = \left(-\frac{1}{4}, \frac{1}{4}\right)$.

10. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+1)\,3^{n+1}}\cdot\dfrac{n3^n}{x^n}\right| = \lim\limits_{n\to\infty}\dfrac{|x|}{3\,(1+1/n)} = \dfrac{|x|}{3}$, so by the Ratio Test, the series converges when $\dfrac{|x|}{3} < 1 \;\Leftrightarrow\; |x| < 3$, so $R = 3$. When $x = -3$, the series is an alternating harmonic series, which converges by the Alternating Series Test. When $x = 3$, it is the harmonic series, which diverges. Thus, the interval of convergence is $[-3, 3)$.

11. If $a_n = \dfrac{3^n x^n}{(n+1)^2}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{3^{n+1}x^{n+1}}{(n+2)^2}\cdot\dfrac{(n+1)^2}{3^n x^n}\right| = 3\,|x|\lim\limits_{n\to\infty}\left(\dfrac{n+1}{n+2}\right)^2 = 3\,|x| < 1$ for convergence, so $|x| < \frac{1}{3}$ and $R = \frac{1}{3}$. When $x = \frac{1}{3}$, $\sum\limits_{n=0}^{\infty}\dfrac{3^n x^n}{(n+1)^2} = \sum\limits_{n=0}^{\infty}\dfrac{1}{(n+1)^2} = \sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ which is a convergent p-series ($p = 2 > 1$). When $x = -\frac{1}{3}$, $\sum\limits_{n=0}^{\infty}\dfrac{3^n x^n}{(n+1)^2} = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{(n+1)^2}$ which converges by the Alternating Series Test, so $I = \left[-\frac{1}{3}, \frac{1}{3}\right]$.

12. If $a_n = \dfrac{n^2x^n}{10^n}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{|x|}{10}\lim\limits_{n\to\infty}\left(\dfrac{n+1}{n}\right)^2 = \dfrac{|x|}{10} < 1$ for convergence (by the Ratio Test), so $R = 10$. If $x = \pm 10$, $|a_n| = n^2 \to \infty$ as $n \to \infty$, so $\sum_{n=0}^{\infty} a_n$ diverges (Test for Divergence) and $I = (-10, 10)$.

13. If $a_n = \dfrac{x^n}{\ln n}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{\ln(n+1)}\cdot\dfrac{\ln n}{x^n}\right| = |x|\lim\limits_{n\to\infty}\dfrac{\ln n}{\ln(n+1)} \overset{\text{H}}{=} |x|$, so $R = 1$. When $x = 1$, $\sum\limits_{n=2}^{\infty}\dfrac{x^n}{\ln n} = \sum\limits_{n=2}^{\infty}\dfrac{1}{\ln n}$ which diverges because $\dfrac{1}{\ln n} > \dfrac{1}{n}$ and $\sum\limits_{n=2}^{\infty}\dfrac{1}{n}$ is the divergent harmonic series. When $x = -1$, $\sum\limits_{n=2}^{\infty}\dfrac{x^n}{\ln n} = \sum\limits_{n=2}^{\infty}\dfrac{(-1)^n}{\ln n}$ which converges by the Alternating Series Test. So $I = [-1, 1)$.

14. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3(x-5)^{n+1}}{n^3(x-5)^n}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^3|x-5| = |x-5|$, so by the Ratio Test, the series converges when $|x-5|<1 \Leftrightarrow 4<x<6$. When $x=4$, the series becomes $\sum_{n=0}^{\infty}(-1)^n n^3$, which diverges by the Test for Divergence. When $x=6$, the series becomes $\sum_{n=0}^{\infty} n^3$, which also diverges. Thus, $R=1$ and $I=(4,6)$.

15. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\sqrt{n+1}\,|x-1|^{n+1}}{\sqrt{n}\,|x-1|^n} = \lim_{n\to\infty}\sqrt{1+\frac{1}{n}}\,|x-1| = |x-1|$, so by the Ratio Test, the series converges when $|x-1|<1 \Leftrightarrow -1<x-1<1 \Leftrightarrow 0<x<2$. $R=1$. When $x=2$, the series becomes $\sum_{n=0}^{\infty}\sqrt{n}$, which diverges by the Test for Divergence. When $x=0$, the series becomes $\sum_{n=0}^{\infty}(-1)^n\sqrt{n}$, which also diverges by the Test for Divergence. Thus, $I=(0,2)$.

16. If $a_n = \frac{(-1)^n x^{2n-1}}{(2n-1)!}$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{x^2}{(2n+1)\,2n} = 0 < 1$ for all x. By the Ratio Test the series converges for all x, so $R=\infty$ and $I=(-\infty,\infty)$.

17. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{|x+2|^{n+1}}{(n+1)\,2^{n+1}}\cdot\frac{n2^n}{|x+2|^n}\right] = \lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{|x+2|}{2} = \frac{|x+2|}{2}$, so by the Ratio Test, the series converges when $\frac{|x+2|}{2}<1 \Leftrightarrow |x+2|<2 \Leftrightarrow -2<x+2<2 \Leftrightarrow -4<x<0$. $R=2$. When $x=-4$, the series becomes $\sum_{n=1}^{\infty}(-1)^n\frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{2^n}{n2^n} = \sum_{n=1}^{\infty}\frac{1}{n}$, which is the divergent harmonic series. When $x=0$, the series is $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, the alternating harmonic series, which converges by the Alternating Series Test. Thus, $I=(-4,0]$.

18. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}(x+3)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(-2)^n(x+3)^n}\right| = \lim_{n\to\infty}\frac{2|x+3|}{\sqrt{1+1/n}} = 2|x+3|<1 \Leftrightarrow$ $|x+3|<\frac{1}{2} \Leftrightarrow -\frac{7}{2}<x<-\frac{5}{2}$. Thus, $R=\frac{1}{2}$. When $x=-\frac{5}{2}$, the series becomes $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$, which converges by the Alternating Series Test. When $x=-\frac{7}{2}$, the series becomes $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, which diverges because it is a p-series with $p=\frac{1}{2}\le 1$. Thus, $I=\left(-\frac{7}{2},-\frac{5}{2}\right]$.

19. If $a_n = \frac{(x-2)^n}{n^n}$, then $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{|x-2|}{n} = 0$, so the series converges for all x (by the Root Test). $R=\infty$ and $I=(-\infty,\infty)$.

20. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3x-2)^{n+1}}{(n+1)\,3^{n+1}}\cdot\frac{n3^n}{(3x-2)^n}\right| = \lim_{n\to\infty}\left(\frac{|3x-2|}{3}\cdot\frac{1}{1+1/n}\right) = \frac{|3x-2|}{3} = \left|x-\frac{2}{3}\right|$, so by the Ratio Test, the series converges when $\left|x-\frac{2}{3}\right|<1 \Leftrightarrow -\frac{1}{3}<x<\frac{5}{3}$. $R=1$. When $x=-\frac{1}{3}$, the series is $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, the convergent alternating harmonic series. When $x=\frac{5}{3}$, the series becomes the divergent harmonic series. Thus, $I=\left[-\frac{1}{3},\frac{5}{3}\right)$.

21. If $a_n = \dfrac{2^n (x-3)^n}{n+3}$, then

$\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{2^{n+1}(x-3)^{n+1}}{n+4} \cdot \dfrac{n+3}{2^n (x-3)^n}\right| = 2\,|x-3| \lim\limits_{n\to\infty} \dfrac{n+3}{n+4} = 2\,|x-3| < 1$ for convergence, or

$|x-3| < \frac{1}{2} \quad \Leftrightarrow \quad \frac{5}{2} < x < \frac{7}{2}$, and $R = \frac{1}{2}$. When $x = \frac{5}{2}$, $\sum\limits_{n=0}^{\infty} \dfrac{2^n (x-3)^n}{n+3} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{n+3}$ which converges by the Alternating Series Test. When $x = \frac{7}{2}$, $\sum\limits_{n=0}^{\infty} \dfrac{2^n (x-3)^n}{n+3} = \sum\limits_{n=0}^{\infty} \dfrac{1}{n+3}$, similar to the harmonic series, which diverges. So $I = \left[\frac{5}{2}, \frac{7}{2}\right)$.

22. If $a_n = \dfrac{(x+1)^n}{n\,(n+1)}$, then $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = |x+1| \lim\limits_{n\to\infty} \dfrac{n}{n+2} = |x+1| < 1$ for convergence, or $-2 < x < 0$ and $R = 1$. If $x = -2$ or 0, then $|a_n| = \dfrac{1}{n^2+n} < \dfrac{1}{n^2}$, so $\sum\limits_{n=1}^{\infty} |a_n|$ converges since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ does ($p = 2 > 1$), and $I = [-2, 0]$.

23. If $a_n = n!\,(2x-1)^n$, then $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(n+1)!\,(2x-1)^{n+1}}{n!\,(2x-1)^n}\right| = \lim\limits_{n\to\infty} (n+1)\,|2x-1| \to \infty$ as $n \to \infty$ for all $x \neq \frac{1}{2}$. Since the series diverges for all $x \neq \frac{1}{2}$, $R = 0$ and $I = \left\{\frac{1}{2}\right\}$.

24. If $a_n = \dfrac{nx^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$ then $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = |x| \lim\limits_{n\to\infty} \dfrac{n+1}{n\,(2n+1)} = 0$ for all x. So the series converges for all $x \quad \Rightarrow \quad R = \infty$ and $I = (-\infty, \infty)$.

25. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left[\dfrac{|4x+1|^{n+1}}{(n+1)^2} \cdot \dfrac{n^2}{|4x+1|^n}\right] = \lim\limits_{n\to\infty} \dfrac{|4x+1|}{(1+1/n)^2} = |4x+1|$, so by the Ratio Test, the series converges when $|4x+1| < 1 \quad \Leftrightarrow \quad -1 < 4x+1 < 1 \quad \Leftrightarrow \quad -2 < 4x < 0 \quad \Leftrightarrow \quad -\frac{1}{2} < x < 0$, so $R = \frac{1}{4}$. When $x = -\frac{1}{2}$, the series becomes $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n^2}$, which converges by the Alternating Series Test. When $x = 0$, the series becomes $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$, a convergent p-series ($p = 2 > 1$). $I = \left[-\frac{1}{2}, 0\right]$.

26. If $a_n = \dfrac{(-1)^n (2x+3)^n}{n \ln n}$ then $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = |2x+3| \lim\limits_{n\to\infty} \dfrac{n \ln n}{(n+1)\ln(n+1)} = |2x+3| < 1$ for convergence, so $-2 < x < -1$ and $R = \frac{1}{2}$. When $x = -2$, $\sum\limits_{n=2}^{\infty} a_n = \sum\limits_{n=2}^{\infty} \dfrac{1}{n \ln n}$ which diverges (Integral Test), and when $x = -1$, $\sum\limits_{n=2}^{\infty} a_n = \sum\limits_{n=2}^{\infty} \dfrac{(-1)^n}{n \ln n}$ which converges (Alternating Series Test), so $I = (-2, -1]$.

27. If $a_n = \dfrac{x^n}{(\ln n)^n}$ then $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \dfrac{|x|}{\ln n} = 0 < 1$ for all x, so $R = \infty$ and $I = (-\infty, \infty)$ by the Root Test.

28. If $a_n = \dfrac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)\,x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$ then $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} |x| \left(\dfrac{2n+2}{2n+1}\right) = |x| < 1$ for convergence, so $R = 1$. If $x = \pm 1$, $|a_n| = \dfrac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} > 1$ for all n since each integer in the numerator is larger than the corresponding one in the denominator, so $\sum a_n$ diverges in both cases by the Test for Divergence, and $I = (-1, 1)$.

29. (a) We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x = 4$. So by Theorem 3, it must converge for at least $-4 < x \le 4$. In particular, it converges when $x = -2$, that is, $\sum_{n=0}^{\infty} c_n (-2)^n$ is convergent.

(b) It does not follow that $\sum_{n=0}^{\infty} c_n (-4)^n$ is necessarily convergent. [See the comments after Theorem 3 about convergence at the endpoint of an interval. An example is $c_n = (-1)^n / (n4^n)$.]

30. We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x = -4$ and divergent when $x = 6$. So by Theorem 3 it converges for at least $-4 \le x < 4$ and diverges for at least $x \ge 6$ and $x < -6$. Therefore:

(a) It converges when $x = 1$, that is, $\sum c_n$ is convergent.

(b) It diverges when $x = 8$, that is, $\sum c_n 8^n$ is divergent.

(c) It converges when $x = -3$, that is, $\sum c_n (-3)^n$ is convergent.

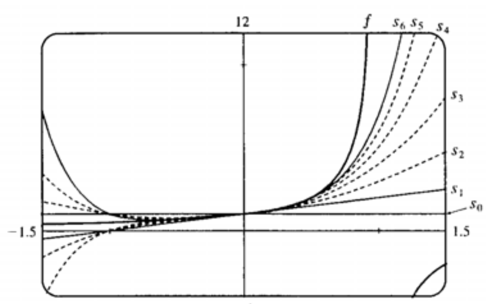
(d) It diverges when $x = -9$, that is, $\sum c_n (-9)^n = \sum (-1)^n c_n 9^n$ is divergent.

31. If $a_n = \dfrac{(n!)^k}{(kn)!} x^n$, then

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{[(n+1)!]^k\,(kn)!}{(n!)^k\,[k(n+1)]!}\,|x| = \lim_{n\to\infty}\frac{(n+1)^k}{(kn+k)(kn+k-1)\cdots(kn+2)(kn+1)}\,|x| \\
&= \lim_{n\to\infty}\left[\frac{(n+1)}{(kn+1)}\frac{(n+1)}{(kn+2)}\cdots\frac{(n+1)}{(kn+k)}\right]|x| \\
&= \lim_{n\to\infty}\left[\frac{n+1}{kn+1}\right]\lim_{n\to\infty}\left[\frac{n+1}{kn+2}\right]\cdots\lim_{n\to\infty}\left[\frac{n+1}{kn+k}\right]|x| = \left(\frac{1}{k}\right)^k |x| < 1 \quad \Leftrightarrow
\end{aligned}$$

$|x| < k^k$ for convergence, and the radius of convergence is $R = k^k$.

32. The partial sums definitely do not converge to $f(x)$ for $x \ge 1$, since f is undefined at $x = 1$ and negative on $(1, \infty)$, while all the partial sums are positive on this interval. The partial sums also fail to converge to f for $x \le -1$, since $0 < f(x) < 1$ on this interval, while the partial sums are either larger than 1 or less than 0. The partial sums seem to converge to f on $(-1, 1)$. This graphical evidence is consistent with what we know about geometric series: convergence for $|x| < 1$, divergence for $|x| \ge 1$ (see Example 12.2.5).

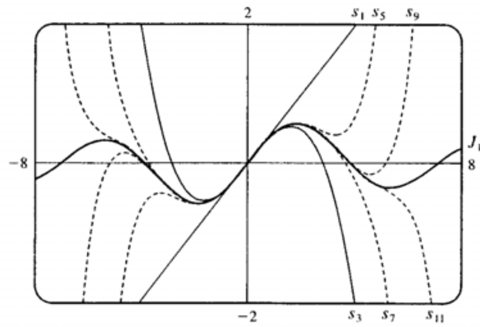


33. (a) If $a_n = \dfrac{(-1)^n x^{2n+1}}{n!\,(n+1)!2^{2n+1}}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \left(\dfrac{x}{2}\right)^2 \lim\limits_{n\to\infty} \dfrac{1}{(n+1)(n+2)} = 0$ for all x. So $J_1(x)$ converges for all x; the domain is $(-\infty, \infty)$.

(b), (c) The initial terms of $J_1(x)$ up to $n = 5$ are

$a_0 = \dfrac{x}{2}$, $a_1 = -\dfrac{x^3}{16}$, $a_2 = \dfrac{x^5}{384}$, $a_3 = -\dfrac{x^7}{18{,}432}$,

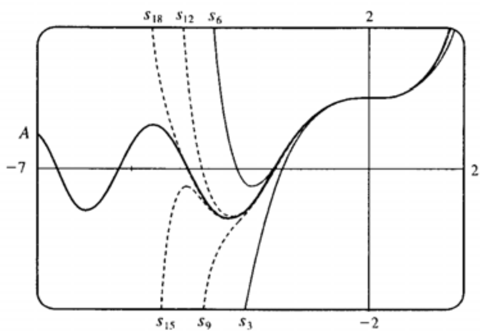
$a_4 = \dfrac{x^9}{1{,}474{,}560}$, and $a_5 = -\dfrac{x^{11}}{176{,}947{,}200}$. The partial sums seem to approximate $J_1(x)$ well near the origin, but as $|x|$ increases, we need to take a large number of terms to get a good approximation.



34. (a) $A(x) = 1 + \sum\limits_{n=1}^{\infty} a_n$, where $a_n = \dfrac{x^{3n}}{2\cdot 3\cdot 5\cdot 6\cdot\cdots\cdot(3n-1)(3n)}$, so

$\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = |x|^3 \lim\limits_{n\to\infty}\dfrac{1}{(3n+2)(3n+3)} = 0$ for all x, so the domain is $\mathbb{R}$.

(b)



$s_0 = 1$ has been omitted from the graph. The partial sums seem to approximate $A(x)$ well near the origin, but as $|x|$ increases, we need to take a large number of terms to get a good approximation.

To plot A, we must first define $A(x)$ for the CAS. Note that for $n \geq 1$, the denominator of a_n is

$2\cdot 3\cdot 5\cdot 6\cdot\cdots\cdot(3n-1)\cdot 3n = \dfrac{(3n)!}{1\cdot 4\cdot 7\cdot\cdots\cdot(3n-2)} = \dfrac{(3n)!}{\prod_{k=1}^{n}(3k-2)}$, so $a_n = 1 + \dfrac{\prod_{k=1}^{n}(3k-2)}{(3n)!}x^{3n}$

and thus $A(x) = 1 + \sum\limits_{n=1}^{\infty}\dfrac{\prod_{k=1}^{n}(3k-2)}{(3n)!}x^{3n}$. Both Maple and Mathematica are able to plot A if we define it this way, and Derive is able to produce a similar graph using a suitable partial sum of $A(x)$.

Derive, Maple and Mathematica all have two initially known Airy functions, called `AI_SERIES(z,m)` and `BI_SERIES(z,m)` from `BESSEL.MTH` in Derive and `AiryAi` and `AiryBi` in Maple and Mathematica (just `Ai` and `Bi` in older versions of Maple). However, it is very difficult to solve for A in terms of the CAS's Airy functions, although in fact $A(x) = \dfrac{\sqrt{3}\texttt{AiryAi}(x) + \texttt{AiryBi}(x)}{\sqrt{3}\texttt{AiryAi}(0) + \texttt{AiryBi}(0)}$.

35. $s_{2n-1} = 1 + 2x + x^2 + 2x^3 + \cdots + x^{2n-2} + 2x^{2n-1} = (1+2x)\left(1 + x^2 + x^4 + \cdots + x^{2n-2}\right)$

$= (1+2x)\dfrac{1-x^{2n}}{1-x^2}$ [by (12.2.3) with $r = x^2$] $\to \dfrac{1+2x}{1-x^2}$ as $n \to \infty$ [by (12.2.4)],

when $|x| < 1$. Also $s_{2n} = s_{2n-1} + x^{2n} \to \dfrac{1+2x}{1-x^2}$ since $x^{2n} \to 0$ for $|x| < 1$. Therefore, $s_n \to \dfrac{1+2x}{1-x^2}$ since s_{2n} and s_{2n-1} both approach $\dfrac{1+2x}{1-x^2}$ as $n \to \infty$. Thus, the interval of convergence is $(-1, 1)$ and $f(x) = \dfrac{1+2x}{1-x^2}$.

36. $s_{4n-1} = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_0 x^4 + c_1 x^5 + c_2 x^6 + c_3 x^7 + \cdots + c_3 x^{4n-1}$

$= \left(c_0 + c_1 x + c_2 x^2 + c_3 x^3\right)\left(1 + x^4 + x^8 + \cdots + x^{4n-4}\right) \to \dfrac{c_0 + c_1 x + c_2 x^2 + c_3 x^3}{1 - x^4}$ as $n \to \infty$

[by (12.2.4) with $r = x^4$] for $|x| < 1$. Also $s_{4n}, s_{4n+1}, s_{4n+2}$ have the same limits (for example, $s_{4n} = s_{4n-1} + c_0 x^{4n}$ and $x^{4n} \to 0$ for $|x| < 1$.) So if at least one of c_0, c_1, c_2, and c_3 is nonzero, then the interval of convergence is $(-1, 1)$ and $f(x) = \dfrac{c_0 + c_1 x + c_2 x^2 + c_3 x^3}{1 - x^4}$.

37. We use the Root Test on the series $\sum c_n x^n$. $\lim\limits_{n\to\infty} \sqrt[n]{|c_n x^n|} = |x| \lim\limits_{n\to\infty} \sqrt[n]{|c_n|} = c\,|x| < 1$ for convergence, or $|x| < 1/c$, so $R = 1/c$.

38. Since $\sum c_n x^n$ converges whenever $|x| < R$, $\sum c_n x^{2n} = \sum c_n \left(x^2\right)^n$ converges whenever $\left|x^2\right| < R \Leftrightarrow$ $|x| < \sqrt{R}$, so the second series has radius of convergence $\sqrt{R}$.

39. For $2 < x < 3$, $\sum c_n x^n$ diverges and $\sum d_n x^n$ converges. By Exercise 12.2.61, $\sum (c_n + d_n) x^n$ diverges. Since both series converge for $|x| < 2$, the radius of convergence of $\sum (c_n + d_n) x^n$ is 2.

12.9 Representations of Functions as Power Series

1. If $f(x) = \sum\limits_{n=0}^{\infty} c_n x^n$ has radius of convergence 10, then $f'(x) = \sum\limits_{n=1}^{\infty} n c_n x^{n-1}$ also has radius of convergence 10 by Theorem 2.

2. If $f(x) = \sum\limits_{n=0}^{\infty} b_n x^n$ converges on $(-2, 2)$, then $\int f(x)\,dx = C + \sum\limits_{n=0}^{\infty} \dfrac{b_n}{n+1} x^{n+1}$ has the same radius of convergence (by Theorem 2), but may not have the same interval of convergence — it may happen that the integrated series converges at an endpoint (or both endpoints).

3. $f(x) = \dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum\limits_{n=0}^{\infty} (-x)^n = \sum\limits_{n=0}^{\infty} (-1)^n x^n$ with $|-x| < 1 \Leftrightarrow |x| < 1$, so $R = 1$ and $I = (-1, 1)$.

4. $f(x) = \dfrac{x}{1-x} = x\left(\dfrac{1}{1-x}\right) = x \sum\limits_{n=0}^{\infty} x^n = \sum\limits_{n=0}^{\infty} x^{n+1} = \sum\limits_{n=1}^{\infty} x^n$ with $R = 1$ and $I = (-1, 1)$.

5. Replacing x with x^3 in (1) gives $f(x) = \dfrac{1}{1-x^3} = \sum\limits_{n=0}^{\infty} \left(x^3\right)^n = \sum\limits_{n=0}^{\infty} x^{3n}$. The series converges when $\left|x^3\right| < 1$; that is, when $|x| < 1$, so $I = (-1, 1)$.

6. $f(x) = \dfrac{1}{1+9x^2} = \dfrac{1}{1-(-9x^2)} = \sum\limits_{n=0}^{\infty} (-9x^2)^n = \sum\limits_{n=0}^{\infty} (-1)^n 3^{2n} x^{2n}$. The series converges when $|-9x^2| < 1$; that is, when $|x| < \frac{1}{3}$, so $I = \left(-\frac{1}{3}, \frac{1}{3}\right)$.

7. $f(x) = \dfrac{1}{4+x^2} = \dfrac{1}{4}\left(\dfrac{1}{1+x^2/4}\right) = \dfrac{1}{4}\sum\limits_{n=0}^{\infty} (-1)^n \left(\dfrac{x^2}{4}\right)^n$ (using Exercise 3) $= \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{4^{n+1}}$, with $\left|\dfrac{x^2}{4}\right| < 1$ $\Leftrightarrow$ $x^2 < 4$ $\Leftrightarrow$ $|x| < 2$, so $R = 2$ and $I = (-2, 2)$.

8. $f(x) = \dfrac{1+x^2}{1-x^2} = 1 + \dfrac{2x^2}{1-x^2} = 1 + 2x^2 \sum\limits_{n=0}^{\infty} (x^2)^n = 1 + \sum\limits_{n=0}^{\infty} 2x^{2n+2} = 1 + \sum\limits_{n=1}^{\infty} 2x^{2n}$, with $|x^2| < 1$ $\Leftrightarrow$ $|x| < 1$, so $R = 1$ and $I = (-1, 1)$.

9. $f(x) = \dfrac{1}{x-5} = -\dfrac{1}{5}\left(\dfrac{1}{1-x/5}\right) = -\dfrac{1}{5}\sum\limits_{n=0}^{\infty}\left(\dfrac{x}{5}\right)^n$. The series converges when $\left|\dfrac{x}{5}\right| < 1$; that is, when $|x| < 5$, so $I = (-5, 5)$.

10. $f(x) = \dfrac{x}{4x+1} = x \cdot \dfrac{1}{1-(-4x)} = x\sum\limits_{n=0}^{\infty} (-4x)^n = \sum\limits_{n=0}^{\infty} (-1)^n 2^{2n} x^{n+1}$. The series converges when $|-4x| < 1$; that is, when $|x| < \frac{1}{4}$, so $I = \left(-\frac{1}{4}, \frac{1}{4}\right)$.

11. $f(x) = \dfrac{3}{x^2+x-2} = \dfrac{3}{(x+2)(x-1)} = \dfrac{A}{x+2} + \dfrac{B}{x-1}$ $\Rightarrow$ $3 = A(x-1) + B(x+2)$. Taking $x = -2$, we get $A = -1$. Taking $x = 1$, we get $B = 1$. Thus,

$$\begin{aligned}\frac{3}{x^2+x-2} &= \frac{1}{x-1} - \frac{1}{x+2} = -\frac{1}{1-x} - \frac{1}{2}\frac{1}{1+x/2} = -\sum_{n=0}^{\infty} x^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n \\ &= \sum_{n=0}^{\infty}\left[-1 - \tfrac{1}{2}\left(-\tfrac{1}{2}\right)^n\right]x^n = \sum_{n=0}^{\infty}\left[-1 + \left(-\tfrac{1}{2}\right)^{n+1}\right]x^n = \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}}{2^{n+1}} - 1\right]x^n\end{aligned}$$

We represented the given function as the sum of two geometric series; the first converges for $x \in (-1, 1)$ and the second converges for $x \in (-2, 2)$. Thus, the sum converges for $x \in (-1, 1) = I$.

12. $f(x) = \dfrac{7x-1}{3x^2+2x-1} = \dfrac{7x-1}{(3x-1)(x+1)} = \dfrac{A}{3x-1} + \dfrac{B}{x+1} = \dfrac{1}{3x-1} + \dfrac{2}{x+1} = 2 \cdot \dfrac{1}{1-(-x)} - \dfrac{1}{1-3x}$
$= 2\sum_{n=0}^{\infty}(-x)^n - \sum_{n=0}^{\infty}(3x)^n = \sum_{n=0}^{\infty}\left[2(-1)^n - 3^n\right]x^n$

The series $\sum(-x)^n$ converges for $x \in (-1, 1)$ and the series $\sum(3x)^n$ converges for $x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$, so their sum converges for $x \in \left(-\frac{1}{3}, \frac{1}{3}\right) = I$.

13. $f(x) = \dfrac{1}{(1+x)^2} = -\dfrac{d}{dx}\left(\dfrac{1}{1+x}\right) = -\dfrac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]$ (from Exercise 3)
$= \sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1}$ [from Theorem 2(a)] $= \sum_{n=0}^{\infty}(-1)^n (n+1) x^n$ with $R = 1$.

14. $f(x) = \dfrac{1}{1+x} = \sum\limits_{n=0}^{\infty}(-1)^n x^n$ (geometric series with $R = 1$), so

$$\begin{aligned}f(x) = \ln(1+x) &= \int\frac{dx}{1+x} = \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1} \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n} \quad [C = 0 \text{ since } f(0) = 0], \text{ with } R = 1\end{aligned}$$

15. $f(x) = \dfrac{1}{(1+x)^3} = -\dfrac{1}{2}\dfrac{d}{dx}\left[\dfrac{1}{(1+x)^2}\right] = -\dfrac{1}{2}\dfrac{d}{dx}\left[\sum\limits_{n=0}^{\infty}(-1)^n(n+1)x^n\right]$ (from Exercise 13)

$= -\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(n+1)nx^{n-1} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ with $R = 1$.

16. $f(x) = x\ln(1+x) = x\left[\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^n}{n}\right]$ (by Exercise 14) $= \sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^{n+1}}{n} = \sum\limits_{n=2}^{\infty}\dfrac{(-1)^n x^n}{n-1}$ with $R = 1$.

17. $f(x) = \ln(5-x) = -\displaystyle\int\frac{dx}{5-x} = -\frac{1}{5}\int\frac{dx}{1-x/5}$

$= -\dfrac{1}{5}\displaystyle\int\left[\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n\right]dx = C - \frac{1}{5}\sum_{n=0}^{\infty}\frac{x^{n+1}}{5^n(n+1)} = C - \sum_{n=1}^{\infty}\frac{x^n}{n5^n}$

Putting $x = 0$, we get $C = \ln 5$. The series converges for $|x/5| < 1 \iff |x| < 5$, so $R = 5$.

18. We know that $\dfrac{1}{1-2x} = \sum\limits_{n=0}^{\infty}(2x)^n$. Differentiating, we get $\dfrac{2}{(1-2x)^2} = \sum\limits_{n=1}^{\infty}2^n nx^{n-1} = \sum\limits_{n=0}^{\infty}2^{n+1}(n+1)x^n$, so

$f(x) = \dfrac{x^2}{(1-2x)^2} = \dfrac{x^2}{2}\cdot\dfrac{2}{(1-2x)^2} = \dfrac{x^2}{2}\sum\limits_{n=0}^{\infty}2^{n+1}(n+1)x^n = \sum\limits_{n=0}^{\infty}2^n(n+1)x^{n+2}$ or $\sum\limits_{n=2}^{\infty}2^{n-2}(n-1)x^n$,

with $R = \frac{1}{2}$.

19. $\dfrac{1}{2-x} = \dfrac{1}{2(1-x/2)} = \dfrac{1}{2}\sum\limits_{n=0}^{\infty}\left(\dfrac{x}{2}\right)^n = \sum\limits_{n=0}^{\infty}\dfrac{1}{2^{n+1}}x^n$ for $\left|\dfrac{x}{2}\right| < 1 \iff |x| < 2$. Now

$\dfrac{1}{(x-2)^2} = \left(\dfrac{1}{2-x}\right)' = \left(\sum\limits_{n=0}^{\infty}\dfrac{1}{2^{n+1}}x^n\right)' = \sum\limits_{n=0}^{\infty}\left(\dfrac{1}{2^{n+1}}x^n\right)' = \sum\limits_{n=1}^{\infty}\dfrac{n}{2^{n+1}}x^{n-1} = \sum\limits_{n=0}^{\infty}\dfrac{n+1}{2^{n+2}}x^n$. So

$f(x) = \dfrac{x^3}{(x-2)^2} = x^3\sum\limits_{n=0}^{\infty}\dfrac{n+1}{2^{n+2}}x^n = \sum\limits_{n=0}^{\infty}\dfrac{n+1}{2^{n+2}}x^{n+3}$ or $\sum\limits_{n=3}^{\infty}\dfrac{n-2}{2^{n-1}}x^n$ for $|x| < 2$. Thus, $R = 2$ and $I = (-2, 2)$.

20. From Example 7, $g(x) = \arctan x = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2n+1}$. Thus,

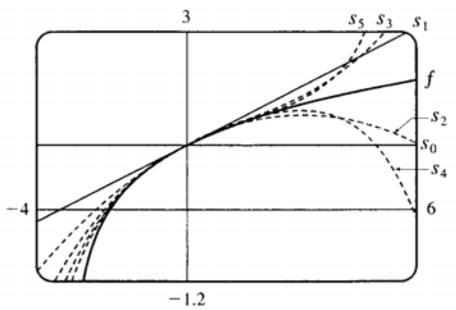
$f(x) = \arctan(x/3) = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{(x/3)^{2n+1}}{2n+1} = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{1}{3^{2n+1}(2n+1)}x^{2n+1}$ for $\left|\dfrac{x}{3}\right| < 1 \iff |x| < 3$, so $R = 3$.

21. $f(x) = \ln(3+x) = \int \frac{dx}{3+x} = \frac{1}{3}\int \frac{dx}{1+x/3} = \frac{1}{3}\int \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{3}\right)^n dx$ (from Exercise 3)

$= C + \frac{1}{3}\sum_{n=0}^{\infty} \frac{(-1/3)^n}{n+1} x^{n+1} = \ln 3 + \frac{1}{3}\sum_{n=1}^{\infty} \frac{(-1/3)^{n-1}}{n} x^n$ $[C = f(0) = \ln 3]$

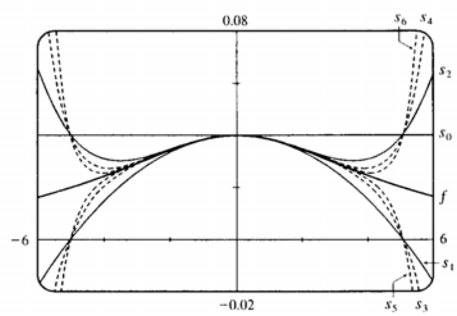
$= \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n3^n} x^n$ with $R = 3$.

The terms of the series are $a_0 = \ln 3$, $a_1 = \frac{x}{3}$, $a_2 = -\frac{x^2}{18}$, $a_3 = \frac{x^3}{81}$, $a_4 = -\frac{x^4}{324}$, $a_5 = \frac{x^5}{1215}$, ...



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-3, 3)$.

22. $f(x) = \frac{1}{x^2+25} = \frac{1/25}{1+(x/5)^2} = \frac{1}{25}\sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{5}\right)^{2n}$ (by Exercise 3) with $R = 5$. The terms of the series are $a_0 = \frac{1}{25}$, $a_1 = -\frac{x^2}{625}$, $a_2 = \frac{x^4}{15{,}625}$, ...



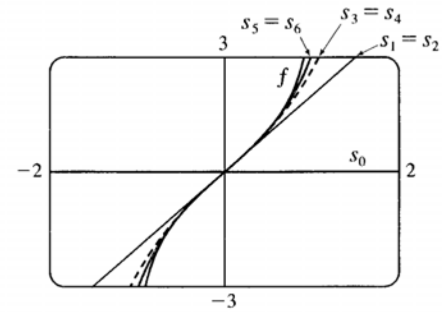
As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-5, 5)$.

23. $f(x) = \ln\left(\dfrac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \displaystyle\int \frac{dx}{1+x} + \int \frac{dx}{1-x}$

$$= \int \left[\sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} x^n\right] dx = \int \sum_{n=0}^{\infty} 2x^{2n}\, dx = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1} + C$$

But $f(0) = \ln\frac{1}{1} = 0$, so $C = 0$ and we have $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$ with $R = 1$. If $x = \pm 1$, then

$f(x) = \pm 2 \displaystyle\sum_{n=0}^{\infty} \frac{1}{2n+1}$, which both diverge by the Limit Comparison Test with $b_n = \dfrac{1}{n}$.

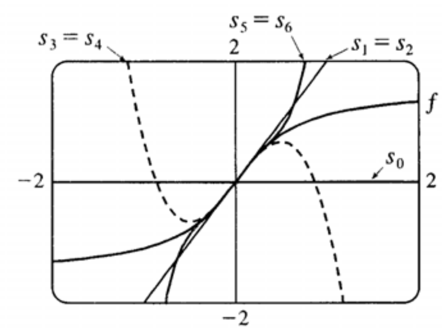


As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-1, 1)$.

24. $f(x) = \tan^{-1} 2x = 2\displaystyle\int \frac{dx}{1+4x^2} = 2\int \sum_{n=0}^{\infty} (-1)^n \left(4x^2\right)^n dx = 2\int \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n}\, dx$

$$= C + 2\sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1} \text{ for } \left|4x^2\right| < 1 \quad \Leftrightarrow \quad |x| < \tfrac{1}{2}, \text{ so } R = \tfrac{1}{2}.$$

If $x = \pm\frac{1}{2}$, then $f(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ and $f(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{2n+1}$, respectively. Both series converge by the Alternating Series Test.



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

25. $\displaystyle\int \frac{dx}{1+x^4} = \int \sum_{n=0}^{\infty} (-1)^n x^{4n}\, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{4n+1}$ with $R = 1$.

26. $\dfrac{1}{1+x^5} = \sum\limits_{n=0}^{\infty} (-1)^n x^{5n} \Rightarrow \dfrac{x}{1+x^5} = \sum\limits_{n=0}^{\infty} (-1)^n x^{5n+1} \Rightarrow \displaystyle\int \frac{x}{1+x^5}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+2}}{5n+2}$ with $R = 1$.

27. By Example 7, $\arctan x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$, so

$\displaystyle\int \frac{\arctan x}{x}\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^2}$ with $R = 1$.

28. By Example 7, $\displaystyle\int \tan^{-1}\left(x^2\right) dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{\left(x^2\right)^{2n+1}}{2n+1}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)(4n+3)}$, with $R = 1$.

29. We use the representation $\displaystyle\int \frac{dx}{1+x^4} = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{4n+1}$ from Exercise 25 with $C = 0$. So

$$\int_0^{0.2} \frac{dx}{1+x^4} = \left[x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13} + \cdots\right]_0^{0.2} = 0.2 - \frac{0.2^5}{5} + \frac{0.2^9}{9} - \frac{0.2^{13}}{13} + \cdots$$

Since the series is alternating, the error in the nth-order approximation is less than the first neglected term, by The Alternating Series Estimation Theorem. If we use only the first two terms of the series, then the error is at most $0.2^9/9 \approx 5.7 \times 10^{-8}$. So, to six decimal places, $\displaystyle\int_0^{0.2} \frac{dx}{1+x^4} \approx 0.2 - \frac{0.2^5}{5} = 0.199936$.

30. We use the representation $\int \tan^{-1}\left(x^2\right) dx = C + \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{4n+3}}{(2n+1)(4n+3)}$ from Exercise 28 with $C = 0$:

$$\int_0^{1/2} \tan^{-1}\left(x^2\right) dx = \left[\frac{x^3}{3} - \frac{x^7}{21} + \frac{x^{11}}{55} - \frac{x^{15}}{105} + \frac{x^{19}}{171} - \cdots\right]_0^{1/2} = \frac{0.5^3}{3} - \frac{0.5^7}{21} + \frac{0.5^{11}}{55} - \frac{0.5^{15}}{105} + \frac{0.5^{19}}{171} - \cdots$$

The series is alternating, so if we use only the first four terms of the series, then the error is at most $0.5^{19}/171 \approx 1.1 \times 10^{-8}$. So, to six decimal places,

$$\textstyle\int_0^{1/2} \tan^{-1}\left(x^2\right) dx \approx \frac{1}{3}(0.5)^3 - \frac{1}{21}(0.5)^7 + \frac{1}{55}(0.5)^{11} - \frac{1}{105}(0.5)^{15} \approx 0.041303$$

31. We substitute x^4 for x in Example 7, and find that

$$\begin{aligned}\int x^2 \tan^{-1}\left(x^4\right) dx &= \int x^2 \sum_{n=0}^{\infty} (-1)^n \frac{\left(x^4\right)^{2n+1}}{2n+1}\,dx \\ &= \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+6}}{2n+1}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+7}}{(2n+1)(8n+7)}\end{aligned}$$

So $\int_0^{1/3} x^2 \tan^{-1}\left(x^4\right) dx = \left[\dfrac{x^7}{7} - \dfrac{x^{15}}{45} + \cdots\right]_0^{1/3} = \dfrac{1}{7 \cdot 3^7} - \dfrac{1}{45 \cdot 3^{15}} + \cdots$. The series is alternating, so if we use only one term, the error is at most $1/\left(45 \cdot 3^{15}\right) \approx 1.5 \times 10^{-9}$. So $\int_0^{1/3} x^2 \tan^{-1}\left(x^4\right) dx \approx 1/\left(7 \cdot 3^7\right) \approx 0.000065$ to six decimal places.

32. $\displaystyle\int_0^{0.5} \frac{dx}{1+x^6} = \int_0^{0.5} \sum_{n=0}^{\infty} (-1)^n x^{6n}\,dx = \sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{6n+1}}{6n+1}\right]_0^{1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+1)\,2^{6n+1}}$

$$= \frac{1}{2} - \frac{1}{7\cdot 2^7} + \frac{1}{13\cdot 2^{13}} - \frac{1}{19\cdot 2^{19}} + \cdots$$

The series is alternating, so if we use only three terms, the error is at most $\frac{1}{19\cdot 2^{19}} \approx 1.0\times 10^{-7}$. So, to six decimal places, $\displaystyle\int_0^{0.5} \frac{dx}{1+x^6} \approx \frac{1}{2} - \frac{1}{7\cdot 2^7} + \frac{1}{13\cdot 2^{13}} \approx 0.498893$.

33. Using the result of Example 6, $\ln(1-x) = -\sum_{n=1}^{\infty} x^n/n$, with $x = -0.1$, we have $\ln 1.1 = \ln[1-(-0.1)] = 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} + \frac{0.00001}{5} - \cdots$. The series is alternating, so if we use only the first four terms, the error is at most $\frac{0.00001}{5} = 0.000002$. So $\ln 1.1 \approx 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} \approx 0.09531$.

34. $\displaystyle f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{(2n)!}$ (the first term disappears), so

$$f''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n)(2n-1)\,x^{2n-2}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2(n-1)}}{[2(n-1)]!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!} \text{ (substituting } n+1 \text{ for } n)$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = -f(x) \Rightarrow f''(x) + f(x) = 0.$$

35. (a) $\displaystyle J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$, $\displaystyle J_0'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2}$, and $\displaystyle J_0''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)\,x^{2n-2}}{2^{2n}(n!)^2}$, so

$$x^2 J_0''(x) + xJ_0'(x) + x^2 J_0(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)\,x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n}}{2^{2n}(n!)^2} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2^{2n}(n!)^2}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)\,x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{2^{2n-2}[(n-1)!]^2}$$

$$= \sum_{n=1}^{\infty} (-1)^n \left[\frac{2n(2n-1) + 2n - 2^2 n^2}{2^{2n}(n!)^2}\right] x^{2n} = \sum_{n=1}^{\infty} (-1)^n \left[\frac{4n^2 - 2n + 2n - 4n^2}{2^{2n}(n!)^2}\right] x^{2n} = 0$$

(b) $\displaystyle\int_0^1 J_0(x)\,dx = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right] dx = \int_0^1 \left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots\right) dx$

$$= \left[x - \frac{x^3}{3\cdot 4} + \frac{x^5}{5\cdot 64} - \frac{x^7}{7\cdot 2304} + \cdots\right]_0^1 = 1 - \frac{1}{12} + \frac{1}{320} - \frac{1}{16{,}128} + \cdots$$

Since $\frac{1}{16{,}128} \approx 0.000062$, it follows from The Alternating Series Estimation Theorem that, correct to three decimal places, $\int_0^1 J_0(x)\,dx \approx 1 - \frac{1}{12} + \frac{1}{320} \approx 0.920$.

36. (a) $J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}}$, $J_1'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{n!\,(n+1)!\,2^{2n+1}}$, and

$$J_1''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(2n) x^{2n-1}}{n!\,(n+1)!\,2^{2n+1}}.$$

$$\begin{aligned}
&x^2 J_1''(x) + x J_1'(x) + \left(x^2 - 1\right) J_1(x) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(2n) x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} \\
&\qquad + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{n!\,(n+1)!\,2^{2n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)(2n) x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} \\
&\qquad - \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(n-1)!\,n!\,2^{2n-1}} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} \qquad \left(\begin{array}{c}\text{Replace } n \text{ with } n-1 \\ \text{in the third term}\end{array}\right) \\
&= \frac{x}{2} - \frac{x}{2} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{(2n+1)(2n) + (2n+1) - (n)(n+1)\,2^2 - 1}{n!\,(n+1)!\,2^{2n+1}}\right] x^{2n+1} = 0
\end{aligned}$$

(b) $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \quad \Rightarrow$

$$\begin{aligned}
J_0'(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{2^{2n}(n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\,2\,(n+1) x^{2n+1}}{2^{2n+2}\,[(n+1)!]^2} \quad \text{(Replace } n \text{ with } n+1\text{)} \\
&= -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}\,(n+1)!\,n!} \quad \text{(cancel 2 and } n+1\text{; take } -1 \text{ outside sum)} \quad = -J_1(x)
\end{aligned}$$

37. (a) $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \Rightarrow \quad f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$

(b) By Theorem 10.4.2, the only solution to the differential equation $df(x)/dx = f(x)$ is $f(x) = Ke^x$, but $f(0) = 1$, so $K = 1$ and $f(x) = e^x$.

Or: We could solve the equation $df(x)/dx = f(x)$ as a separable differential equation.

38. $\frac{|\sin nx|}{n^2} \leq \frac{1}{n^2}$, so $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ converges by the Comparison Test. $\frac{d}{dx}\left(\frac{\sin nx}{n^2}\right) = \frac{\cos nx}{n}$, so when $x = 2k\pi$ (k an integer), $\sum_{n=1}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} \frac{\cos(2kn\pi)}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges (harmonic series). $f_n''(x) = -\sin nx$, so $\sum_{n=1}^{\infty} f_n''(x) = -\sum_{n=1}^{\infty} \sin nx$, which converges only if $\sin nx = 0$, or $x = k\pi$ (k an integer).

39. If $a_n = \dfrac{x^n}{n^2}$, then by the Ratio Test, $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = |x| \lim\limits_{n\to\infty}\left(\dfrac{n}{n+1}\right)^2 = |x| < 1$ for convergence, so $R = 1$. When $x = \pm 1$, $\sum\limits_{n=1}^{\infty}\left|\dfrac{x^n}{n^2}\right| = \sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ which is a convergent p-series ($p = 2 > 1$), so the interval of convergence for f is $[-1, 1]$. By Theorem 2, the radii of convergence of f' and f'' are both 1, so we need only check the endpoints. $f(x) = \sum\limits_{n=1}^{\infty}\dfrac{x^n}{n^2} \Rightarrow f'(x) = \sum\limits_{n=1}^{\infty}\dfrac{nx^{n-1}}{n^2} = \sum\limits_{n=0}^{\infty}\dfrac{x^n}{n+1}$, and this series diverges for $x = 1$ (harmonic series) and converges for $x = -1$ (Alternating Series Test), so the interval of convergence is $[-1, 1)$. $f''(x) = \sum\limits_{n=1}^{\infty}\dfrac{nx^{n-1}}{n+1}$ diverges at both 1 and -1 (Test for Divergence) since $\lim\limits_{n\to\infty}\dfrac{n}{n+1} = 1 \neq 0$, so its interval of convergence is $(-1, 1)$.

40. (a) $\sum\limits_{n=1}^{\infty} nx^{n-1} = \sum\limits_{n=0}^{\infty}\dfrac{d}{dx}x^n = \dfrac{d}{dx}\left[\sum\limits_{n=0}^{\infty} x_n\right] = \dfrac{d}{dx}\left[\dfrac{1}{1-x}\right] = -\dfrac{1}{(1-x)^2}(-1) = \dfrac{1}{(1-x)^2}$, $|x| < 1$.

(b) (i) $\sum\limits_{n=1}^{\infty} nx^n = x\sum\limits_{n=1}^{\infty} nx^{n-1} = x\left[\dfrac{1}{(1-x)^2}\right]$ [from (a)] $= \dfrac{x}{(1-x)^2}$ for $|x| < 1$.

(ii) Put $x = \frac{1}{2}$ in (i): $\sum\limits_{n=1}^{\infty}\dfrac{n}{2^n} = \sum\limits_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \dfrac{1/2}{(1-1/2)^2} = 2$.

(c) (i) $\sum\limits_{n=2}^{\infty} n(n-1)x^n = x^2\sum\limits_{n=2}^{\infty} n(n-1)x^{n-2} = x^2\dfrac{d}{dx}\left[\sum\limits_{n=1}^{\infty} nx^{n-1}\right] = x^2\dfrac{d}{dx}\dfrac{1}{(1-x)^2}$

$$= x^2\frac{2}{(1-x)^3} = \frac{2x^2}{(1-x)^3} \text{ for } |x| < 1.$$

(ii) Put $x = \frac{1}{2}$ in (i): $\sum\limits_{n=2}^{\infty}\dfrac{n^2-n}{2^n} = \sum\limits_{n=2}^{\infty} n(n-1)\left(\frac{1}{2}\right)^n = \dfrac{2(1/2)^2}{(1-1/2)^3} = 4$.

(iii) From (b)(ii) and (c)(ii), we have $\sum\limits_{n=1}^{\infty}\dfrac{n^2}{2^n} = \sum\limits_{n=1}^{\infty}\dfrac{n^2-n}{2^n} + \sum\limits_{n=1}^{\infty}\dfrac{n}{2^n} = 4 + 2 = 6$.

12.10 Taylor and Maclaurin Series

1. Using Theorem 5 with $\sum\limits_{n=0}^{\infty} b_n(x-5)^n$, $b_n = \dfrac{f^{(n)}(a)}{n!}$, so $b_8 = \dfrac{f^{(8)}(5)}{8!}$.

2. Using Formula 6, a power series expansion of f at 1 must have the form $f(1) + f'(1)(x-1) + \cdots$. Comparing to the given series, $0.4 - 0.8(x-1) + \cdots$, we must have $f'(1) = -0.8$. But from the graph, $f'(1)$ is positive. Hence, the given series is *not* the Taylor series of f centered at 1.

3.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos x$	1
1	$-\sin x$	0
2	$-\cos x$	-1
3	$\sin x$	0
4	$\cos x$	1
$\cdots$	$\cdots$	$\cdots$

$$\cos x = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

If $a_n = \dfrac{(-1)^n x^{2n}}{(2n)!}$, then

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = x^2 \lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)} = 0 < 1$ for all x. So $R = \infty$ (Ratio Test).

4.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin 2x$	0
1	$2\cos 2x$	2
2	$-2^2 \sin 2x$	0
3	$-2^3 \cos 2x$	-2^3
4	$2^4 \sin 2x$	0
$\cdots$	$\cdots$	$\cdots$

$f^{(n)}(0) = 0$ if n is even and $f^{(2n+1)}(0) = (-1)^n 2^{2n+1}$, so

$$\sin 2x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!}x^{2n+1}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$$

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^2\,|x|^2}{(2n+3)(2n+2)} = 0 < 1$ for all x, so $R = \infty$ (Ratio Test).

5.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{-3}$	1
1	$-3(1+x)^{-4}$	-3
2	$12(1+x)^{-5}$	12
3	$-60(1+x)^{-6}$	-60
4	$360(1+x)^{-7}$	360
$\cdots$	$\cdots$	$\cdots$

$$(1+x)^{-3} = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$
$$= 1 - 3x + \tfrac{12}{2}x^2 - \tfrac{60}{6}x^3 + \tfrac{360}{24}x^4 + \cdots$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n (n+2)!\,x^n}{2(n!)} = \sum_{n=0}^{\infty}\frac{(-1)^n (n+2)(n+1)\,x^n}{2}$$

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+3)(n+2)\,|x|^{n+1}}{(n+2)(n+1)\,|x|} = |x| < 1$ for convergence, so $R = 1$.

6.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(1+x)$	0
1	$(1+x)^{-1}$	1
2	$-(1+x)^{-2}$	-1
3	$2(1+x)^{-3}$	2
4	$-6(1+x)^{-4}$	-6
5	$24(1+x)^{-5}$	24
$\cdots$	$\cdots$	$\cdots$

$$\ln(1+x) = = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \cdots$$
$$= x - \tfrac{1}{2}x^2 + \tfrac{2}{6}x^3 - \tfrac{6}{24}x^4 + \tfrac{24}{120}x^5 - \cdots$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$$

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{1+1/n} = |x| < 1$ for convergence, so $R = 1$.

7.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sinh x$	0
1	$\cosh x$	1
2	$\sinh x$	0
3	$\cosh x$	1
4	$\sinh x$	0
...	...	...

So $f^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$ and $\sinh x = \sum\limits_{n=0}^{\infty} \dfrac{x^{2n+1}}{(2n+1)!}$. If $a_n = \dfrac{x^{2n+1}}{(2n+1)!}$ then $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = x^2 \lim\limits_{n\to\infty} \dfrac{1}{(2n+3)(2n+2)} = 0 < 1$ for all x, so $R = \infty$.

8.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cosh x$	1
1	$\sinh x$	0
2	$\cosh x$	1
3	$\sinh x$	0
...	...	...

$f^{(n)}(0) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$ so $\cosh x = \sum\limits_{n=0}^{\infty} \dfrac{x^{2n}}{(2n)!}$ with $R = \infty$, by the Ratio Test.

9.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$1 + x + x^2$	7
1	$1 + 2x$	5
2	2	2
3	0	0
4	0	0
...	...	...

$$f(x) = 7 + 5(x-2) + \frac{2}{2!}(x-2)^2 + \sum_{n=3}^{\infty} \frac{0}{n!}(x-2)^n$$
$$= 7 + 5(x-2) + (x-2)^2$$

Since $a_n = 0$ for large n, $R = \infty$.

10.

n	$f^{(n)}(x)$	$f^{(n)}(-1)$
0	x^3	-1
1	$3x^2$	3
2	$6x$	-6
3	6	6
4	0	0
5	0	0
...	...	...

$$f(x) = -1 + 3(x+1) - \tfrac{6}{2}(x+1)^2 + \tfrac{6}{6}(x+1)^3$$
$$= -1 + 3(x+1) - 3(x+1)^2 + (x+1)^3$$

Since $a_n = 0$ for large n, $R = \infty$.

11. Clearly, $f^{(n)}(x) = e^x$, so $f^{(n)}(3) = e^3$ and $e^x = \sum\limits_{n=0}^{\infty} \dfrac{e^3}{n!}(x-3)^n$. If $a_n = \dfrac{e^3}{n!}(x-3)^n$, then

$\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{|x-3|}{n+1} = 0$ for all x, so $R = \infty$.

12.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$\ln x$	$\ln 2$
1	x^{-1}	$\frac{1}{2}$
2	$-x^{-2}$	$-\frac{1}{4}$
3	$2x^{-3}$	$\frac{2}{8}$
4	$-3 \cdot 2x^{-4}$	$-\frac{3 \cdot 2}{16}$
$\cdots$	$\cdots$	$\cdots$

$f^{(n)}(2) = \dfrac{(-1)^{n-1}(n-1)!}{2^n}$ for $n \geq 1$, so $\ln x = \ln 2 + \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-2)^n}{n \cdot 2^n}$.

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-2|}{2}\lim_{n\to\infty}\frac{n}{n+1} = \frac{|x-2|}{2} < 1$ for convergence, so $|x-2| < 2 \quad\Rightarrow\quad R = 2$.

13.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-1}	1
1	$-x^{-2}$	-1
2	$2x^{-3}$	2
3	$-3 \cdot 2x^{-4}$	$-3 \cdot 2$
4	$4 \cdot 3 \cdot 2x^{-5}$	$4 \cdot 3 \cdot 2$
$\cdots$	$\cdots$	$\cdots$

So $f^{(n)}(1) = (-1)^n n!$, and $\dfrac{1}{x} = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n n!}{n!}(x-1)^n = \sum_{n=0}^{\infty}(-1)^n(x-1)^n$. If $a_n = (-1)^n(x-1)^n$ then

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-1| < 1$ for convergence, so $0 < x < 2$ and $R = 1$.

14.

n	$f^{(n)}(x)$	$f^{(n)}(4)$
0	$x^{1/2}$	2
1	$\frac{1}{2}x^{-1/2}$	2^{-2}
2	$-\frac{1}{4}x^{-3/2}$	-2^{-5}
3	$\frac{3}{8}x^{-5/2}$	$3 \cdot 2^{-8}$
4	$-\frac{15}{16}x^{-7/2}$	$-15 \cdot 2^{-11}$
$\cdots$	$\cdots$	$\cdots$

$f^{(n)}(4) = \dfrac{(-1)^{n-1}\, 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^{3n-1}}$ for $n \geq 2$, so

$\sqrt{x} = 2 + \dfrac{x-4}{4} + \displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\, 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^{3n-1} n!}(x-4)^n$.

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-4|}{8}\lim_{n\to\infty}\left(\frac{2n-1}{n+1}\right) = \frac{|x-4|}{4} < 1$ for convergence, so $|x-4| < 4 \quad\Rightarrow\quad R = 4$.

15.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{4}\right)$
0	$\sin x$	$\sqrt{2}/2$
1	$\cos x$	$\sqrt{2}/2$
2	$-\sin x$	$-\sqrt{2}/2$
3	$-\cos x$	$-\sqrt{2}/2$
4	$\sin x$	$\sqrt{2}/2$
$\cdots$	$\cdots$	$\cdots$

$$\begin{aligned}\sin x &= f\left(\tfrac{\pi}{4}\right) + f'\left(\tfrac{\pi}{4}\right)\left(x - \tfrac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x - \tfrac{\pi}{4}\right)^2 \\ &\quad + \frac{f^{(3)}\left(\frac{\pi}{4}\right)}{3!}\left(x - \tfrac{\pi}{4}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{4}\right)}{4!}\left(x - \tfrac{\pi}{4}\right)^4 + \cdots \\ &= \tfrac{\sqrt{2}}{2}\left[1 + \left(x - \tfrac{\pi}{4}\right) - \tfrac{1}{2!}\left(x - \tfrac{\pi}{4}\right)^2 \right. \\ &\quad \left. - \tfrac{1}{3!}\left(x - \tfrac{\pi}{4}\right)^3 + \tfrac{1}{4!}\left(x - \tfrac{\pi}{4}\right)^4 + \cdots\right] \\ &= \tfrac{\sqrt{2}}{2}\left[1 - \tfrac{1}{2!}\left(x - \tfrac{\pi}{4}\right)^2 + \tfrac{1}{4!}\left(x - \tfrac{\pi}{4}\right)^4 - \cdots\right] \\ &\quad + \tfrac{\sqrt{2}}{2}\left[\left(x - \tfrac{\pi}{4}\right) - \tfrac{1}{3!}\left(x - \tfrac{\pi}{4}\right)^3 + \cdots\right] \\ &= \tfrac{\sqrt{2}}{2}\textstyle\sum_{n=0}^{\infty}(-1)^n\left[\tfrac{1}{(2n)!}\left(x - \tfrac{\pi}{4}\right)^{2n} + \tfrac{1}{(2n+1)!}\left(x - \tfrac{\pi}{4}\right)^{2n+1}\right]\end{aligned}$$

The series can also be written in the more elegant form $\sin x = \dfrac{\sqrt{2}}{2}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n(n-1)/2}\left(x - \frac{\pi}{4}\right)^n}{n!}$. If $a_n = \dfrac{(-1)^{n(n-1)/2}\left(x - \frac{\pi}{4}\right)^n}{n!}$, then $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\left|x - \frac{\pi}{4}\right|}{n+1} = 0 < 1$ for all x, so $R = \infty$.

16.

n	$f^{(n)}(x)$	$f^{(n)}\left(-\frac{\pi}{4}\right)$
0	$\cos x$	$\frac{\sqrt{2}}{2}$
1	$-\sin x$	$\frac{\sqrt{2}}{2}$
2	$-\cos x$	$-\frac{\sqrt{2}}{2}$
3	$\sin x$	$-\frac{\sqrt{2}}{2}$
4	$\cos x$	$\frac{\sqrt{2}}{2}$
$\cdots$	$\cdots$	$\cdots$

$f^{(n)}\left(-\frac{\pi}{4}\right) = (-1)^{n(n-1)/2}\,\frac{\sqrt{2}}{2}$, so

$$\cos x = \sum_{n=0}^{\infty}\frac{f^{(n)}\left(-\frac{\pi}{4}\right)}{n!}\left(x + \tfrac{\pi}{4}\right)^n = \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\frac{(-1)^{n(n-1)/2}\left(x + \frac{\pi}{4}\right)^n}{n!}$$

with $R = \infty$ by the Ratio Test (as in Exercise 15).

17. If $f(x) = \cos x$, then by Formula 9 with $a = 0$, $|R_n(x)| \le \dfrac{\left|f^{(n+1)}(x)\right|}{(n+1)!}|x|^{n+1}$. But $f^{(n+1)}(x) = \pm\sin x$ or $\pm\cos x$. In each case, $\left|f^{(n+1)}(x)\right| \le 1$, so $|R_n(x)| \le \dfrac{1}{(n+1)!}|x|^{n+1} \to 0$ as $n \to \infty$ by Equation 10. So $\displaystyle\lim_{n\to\infty} R_n(x) = 0$ and, by Theorem 8, the series in Exercise 3 represents $\cos x$ for all x.

18. If $f(x) = \sin x$, then $|R_n(x)| \le \dfrac{\left|f^{(n+1)}(x)\right|}{(n+1)!}\left|x - \frac{\pi}{4}\right|^{n+1}$. But $f^{(n+1)}(x) = \pm\sin x$ or $\pm\cos x$. In each case, $\left|f^{(n+1)}(x)\right| \le 1$, so $|R_n(x)| \le \dfrac{1}{(n+1)!}\left|x - \frac{\pi}{4}\right|^{n+1} \to 0$ as $n \to \infty$ by Equation 10. So by Theorem 8, the series in Exercise 15 represents $\sin x$ for all x.

19. If $f(x) = \sinh x$, then $R_n(x) = \dfrac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}$, where $0 < |z| < |x|$. But for all n, $\left|f^{(n+1)}(z)\right| \le \cosh z \le \cosh x$ (since all derivatives are either sinh or cosh, $|\sinh z| < |\cosh z|$ for all z, and $|z| < |x| \;\Rightarrow\; \cosh z < \cosh x$), so $|R_n(z)| \le \dfrac{\cosh x}{(n+1)!}x^{n+1} \to 0$ as $n \to \infty$ (by Equation 10). So by Theorem 8, the series represents $\sinh x$ for all x.

20. If $f(x) = \cosh x$, then $R_n(x) = \dfrac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}$, where $0 < |z| < |x|$. But for all n, $\left|f^{(n+1)}(z)\right| \le \cosh z \le \cosh x$ (since all derivatives are either sinh or cosh, $|\sinh z| < |\cosh z|$ for all z, and $|z| < |x| \;\Rightarrow\; \cosh z < \cosh x$), so $|R_n(z)| \le \dfrac{\cosh x}{(n+1)!}x^{n+1} \to 0$ as $n \to \infty$ (by Equation 10). So by Theorem 8, the series represents $\cosh x$ for all x.

21. $\cos x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} \;\Rightarrow\; f(x) = \cos(\pi x) = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n (\pi x)^{2n}}{(2n)!} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n \pi^{2n} x^{2n}}{(2n)!}$, $R = \infty$

22. $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!} \;\Rightarrow\; f(x) = e^{-x/2} = \sum\limits_{n=0}^{\infty} \dfrac{(-x/2)^n}{n!} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{2^n\, n!}x^n$, $R = \infty$

23. $\tan^{-1} x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} \;\Rightarrow\; f(x) = x \tan^{-1} x = x \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+2}}{2n+1}$, $R = 1$

24. $\sin x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} \;\Rightarrow\; f(x) = \sin(x^4) = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{(x^4)^{2n+1}}{(2n+1)!} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!}x^{8n+4}$, $R = \infty$

25. $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!} \;\Rightarrow\; f(x) = x^2 e^{-x} = x^2 \sum\limits_{n=0}^{\infty} \dfrac{(-x)^n}{n!} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{n+2}}{n!}$, $R = \infty$

26. $\cos x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} \;\Rightarrow\; \cos 2x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{(2x)^{2n}}{(2n)!} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n 2^{2n}}{(2n)!}x^{2n} \;\Rightarrow$

$f(x) = x \cos 2x = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n 2^{2n}}{(2n)!}x^{2n+1}$, $R = \infty$

27. $\sin^2 x = \frac{1}{2}[1 - \cos 2x] = \dfrac{1}{2}\left[1 - \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n (2x)^{2n}}{(2n)!}\right] = 2^{-1}\left[1 - 1 - \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n (2x)^{2n}}{(2n)!}\right]$

$= \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!}$, $R = \infty$

28. $\cos^2 x = \frac{1}{2}(1 + \cos 2x) = \dfrac{1}{2}\left[1 + \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n (2x)^{2n}}{(2n)!}\right] = \dfrac{1}{2}\left[1 + 1 + \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n 2^{2n} x^{2n}}{(2n)!}\right]$

$= 1 + \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!}$, $R = \infty$

Another Method: Use $\cos^2 x = 1 - \sin^2 x$ and Exercise 27.

29. $\dfrac{\sin x}{x} = \dfrac{1}{x} \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n+1)!}$ and this series also gives the required value at $x = 0$ (namely 1), so $R = \infty$.

30. $\dfrac{1 - \cos x}{x^2} = x^{-2}\left[1 - \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n)!}\right] = x^{-2}\left[- \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n)!}\right] = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1} x^{2n-2}}{(2n)!} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n+2)!}$, since the series is equal to $\frac{1}{2}$ when $x = 0$; $R = \infty$.

31.

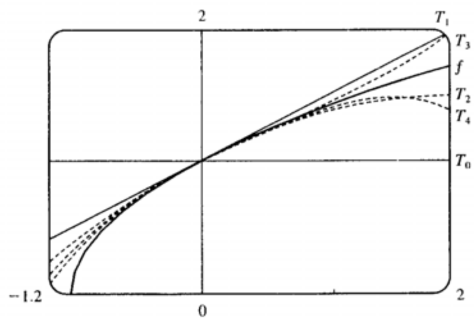
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{1/2}$	1
1	$\frac{1}{2}(1+x)^{-1/2}$	$\frac{1}{2}$
2	$-\frac{1}{4}(1+x)^{-3/2}$	$-\frac{1}{4}$
3	$\frac{3}{8}(1+x)^{-5/2}$	$\frac{3}{8}$
4	$-\frac{15}{16}(1+x)^{-7/2}$	$-\frac{15}{16}$
...	...	...

So $f^{(n)}(0) = \dfrac{(-1)^{n-1}\,1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n}$ for $n \geq 2$, and

$$\sqrt{1+x} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\,1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n n!}x^n.$$

If $a_n = \dfrac{(-1)^{n-1}\,1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n n!}x^n$, then

$\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{|x|}{2}\lim\limits_{n\to\infty}\dfrac{2n-1}{n+1} = |x| < 1$ for convergence, so $R = 1$.



32.

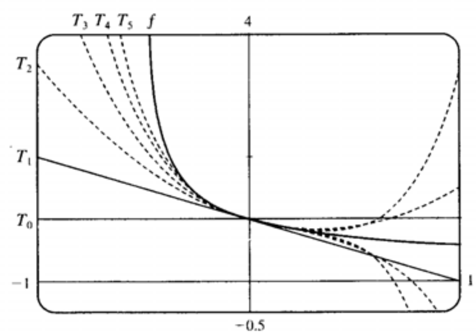
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+2x)^{-1/2}$	1
1	$-\frac{1}{2}(1+2x)^{-3/2}(2)$	-1
2	$\frac{3}{2}(1+2x)^{-5/2}(2)$	3
3	$-3\cdot\frac{5}{2}(1+2x)^{-7/2}(2)$	$-3\cdot 5$
...	...	...

$f^{(n)}(0) = (-1)^n\,1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2n-1)$, so

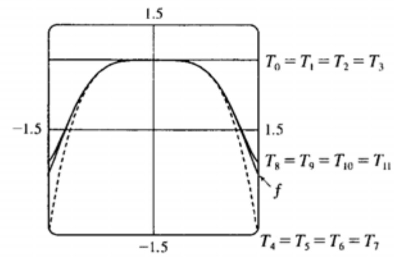
$$(1+2x)^{-1/2} = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n\,1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!}x^n$$

$\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{2n+1}{n+1}|x| = 2|x| < 1$ for convergence, so $R = \frac{1}{2}$.

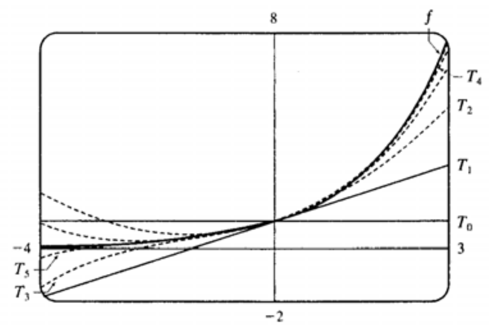
Another method: Use Exercise 31 and differentiate.



33. $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow f(x) = \cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$, $R = \infty$



34. $2^x = \left(e^{\ln 2}\right)^x = e^{x \ln 2} = \sum_{n=0}^{\infty} \frac{(x \ln 2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(\ln 2)^n x^n}{n!}$, $R = \infty$.



35. $\ln(1+x) = \int \frac{dx}{1+x} = \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$ with $C = 0$ and $R = 1$,

so $\ln(1.1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (0.1)^n}{n}$. This is an alternating series with $b_5 = \frac{(0.1)^5}{5} = 0.000002$, so to five decimal places, $\ln(1.1) \approx \sum_{n=1}^{4} \frac{(-1)^{n-1} (0.1)^n}{n} \approx 0.09531$.

36. $3° = \frac{\pi}{60}$ radians and $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, so

$\sin \frac{\pi}{60} = \frac{\pi}{60} - \frac{\left(\frac{\pi}{60}\right)^3}{3!} + \frac{\left(\frac{\pi}{60}\right)^5}{5!} - \cdots = \frac{\pi}{60} - \frac{\pi^3}{1{,}296{,}000} + \frac{\pi^5}{93{,}312{,}000{,}000} - \cdots$. But $\frac{\pi^5}{93{,}312{,}000{,}000} < 10^{-8}$, so by the Alternating Series Estimation Theorem, $\sin \frac{\pi}{60} \approx \frac{\pi}{60} - \frac{\pi^3}{1{,}296{,}000} \approx 0.05234$.

37. $\int \sin\left(x^2\right) dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{\left(x^2\right)^{2n+1}}{(2n+1)!} \, dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!} \, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}$

38. $\frac{\sin x}{x} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$, so

$\int \frac{\sin x}{x} \, dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} \, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$, with $R = \infty$.

39. Using the series from Exercise 31 and substituting x^3 for x, we get

$$\int \sqrt{x^3+1}\,dx = \int \left[1 + \frac{x^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\, 1\cdot 3\cdot 5\cdot \cdots \cdot (2n-3)}{2^n n!} x^{3n}\right] dx$$

$$= C + x + \frac{x^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\, 1\cdot 3\cdot 5\cdot \cdots \cdot (2n-3)}{2^n n!\,(3n+1)} x^{3n+1}$$

40. $\displaystyle\int e^{x^3}\,dx = \int \sum_{n=0}^{\infty} \frac{\left(x^3\right)^n}{n!}\,dx = C + \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)\,n!}$, with $R = \infty$.

41. Using our series from Exercise 37, we get $\displaystyle\int_0^1 \sin\left(x^2\right) dx = \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}\right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)(2n+1)!}$

and $|c_3| = \dfrac{1}{75{,}600} < 0.000014$, so by the Alternating Series Estimation Theorem, we have

$\displaystyle\sum_{n=0}^{2} \frac{(-1)^n}{(4n+3)(2n+1)!} = \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} \approx 0.310$ (correct to three decimal places).

42. $\displaystyle\cos\left(x^2\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^2\right)^{2n}}{(2n)!}$, so

$\displaystyle\int_0^{0.5} \cos\left(x^2\right) dx = \int_0^{0.5} \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}\,dx = \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!}\right]_0^{0.5} = 0.5 - \frac{(0.5)^5}{5\cdot 2!} + \frac{(0.5)^9}{9\cdot 4!} - \cdots$, but

$\dfrac{(0.5)^9}{9\cdot 4!} \approx 0.000009$, so by the Alternating Series Estimation Theorem, $\int_0^{0.5} \cos\left(x^2\right) dx \approx 0.5 - \dfrac{(0.5)^5}{5\cdot 2!} \approx 0.497$ (correct to three decimal places).

43. We first find a series representation for $f(x) = (1+x)^{-1/2}$, and then substitute.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{-1/2}$	1
1	$-\frac{1}{2}(1+x)^{-3/2}$	$-\frac{1}{2}$
2	$\frac{3}{4}(1+x)^{-5/2}$	$\frac{3}{4}$
3	$-\frac{15}{8}(1+x)^{-7/2}$	$-\frac{15}{8}$
$\cdots$	$\cdots$	$\cdots$

$\displaystyle\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3}{4}\left(\frac{x^2}{2!}\right) - \frac{15}{8}\left(\frac{x^3}{3!}\right) + \cdots \;\Rightarrow\; \frac{1}{\sqrt{1+x^3}} = 1 - \tfrac{1}{2}x^3 + \tfrac{3}{8}x^6 - \tfrac{5}{16}x^9 + \cdots \;\Rightarrow$

$\displaystyle\int_0^{0.1} \frac{dx}{\sqrt{1+x^3}} = \left[x - \frac{1}{8}x^4 + \frac{3}{56}x^7 - \frac{1}{32}x^{10} + \cdots\right]_0^{0.1} \approx (0.1) - \frac{1}{8}(0.1)^4$, by the Alternating Series Estimation Theorem, since $\frac{3}{56}(0.1)^7 \approx 0.0000000054 < 10^{-8}$, which is the maximum desired error. Therefore,

$\displaystyle\int_0^{0.1} \frac{dx}{\sqrt{1+x^3}} \approx 0.09998750.$

44. $\displaystyle\int_0^{0.5} x^2 e^{-x^2}\,dx = \int_0^{0.5} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!}\,dx = \sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{2n+3}}{n!\,(2n+3)}\right]_0^{1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+3)\,2^{2n+3}}$ and since $\displaystyle c_2 = \frac{1}{1792} < 0.001$ we use $\displaystyle\sum_{n=0}^{1} \frac{(-1)^n}{n!\,(2n+3)\,2^{2n+3}} = \frac{1}{24} - \frac{1}{160} \approx 0.0354.$

45. $$\lim_{x\to 0} \frac{x - \tan^{-1} x}{x^3} = \lim_{x\to 0} \frac{x - \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots\right)}{x^3} = \lim_{x\to 0} \frac{\frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{1}{7}x^7 - \cdots}{x^3}$$
$$= \lim_{x\to 0}\left(\tfrac{1}{3} - \tfrac{1}{5}x^2 + \tfrac{1}{7}x^4 - \cdots\right) = \tfrac{1}{3}$$
since power series are continuous functions.

46. $$\lim_{x\to 0} \frac{1 - \cos x}{1 + x - e^x} = \lim_{x\to 0} \frac{1 - \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right)}{1 + x - \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots\right)}$$
$$= \lim_{x\to 0} \frac{\frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \frac{1}{6!}x^6 - \cdots}{-\frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{1}{4!}x^4 - \frac{1}{5!}x^5 - \frac{1}{6!}x^6 - \cdots}$$
$$= \lim_{x\to 0} \frac{\frac{1}{2!} - \frac{1}{4!}x^2 + \frac{1}{6!}x^4 - \cdots}{-\frac{1}{2!} - \frac{1}{3!}x - \frac{1}{4!}x^2 - \frac{1}{5!}x^3 - \frac{1}{6!}x^4 - \cdots} = \frac{\frac{1}{2} - 0}{-\frac{1}{2} - 0} = -1$$
since power series are continuous functions.

47. $$\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} = \lim_{x\to 0} \frac{\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right) - x + \frac{1}{6}x^3}{x^5}$$
$$= \lim_{x\to 0} \frac{\frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots}{x^5} = \lim_{x\to 0}\left(\frac{1}{5!} - \frac{x^2}{7!} + \frac{x^4}{9!} - \cdots\right) = \frac{1}{5!} = \frac{1}{120}$$
since power series are continuous functions.

48. $$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots\right) - x}{x^3} = \lim_{x\to 0} \frac{\frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots}{x^3} = \lim_{x\to 0}\left(\tfrac{1}{3} + \tfrac{2}{15}x^2 + \cdots\right) = \tfrac{1}{3}$$
since power series are continuous functions.

49. As in Example 8(a), we have $e^{-x^2} = 1 - \dfrac{x^2}{1!} + \dfrac{x^4}{2!} - \dfrac{x^6}{3!} + \cdots$ and we know that $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots$ from Equation 16. Therefore, $e^{-x^2}\cos x = \left(1 - x^2 + \frac{1}{2}x^4 - \cdots\right)\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots\right)$. Writing only the terms with degree ≤ 4, we get $e^{-x^2}\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - x^2 + \frac{1}{2}x^4 + \frac{1}{2}x^4 + \cdots = 1 - \frac{3}{2}x^2 + \frac{25}{24}x^4 + \cdots$.

50.

$$
\begin{array}{r|l}
 & 1+\frac{1}{2}x^2+\frac{5}{24}x^4+\cdots \\ \hline
1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots & 1 \\
 & \underline{1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots} \\
 & \frac{1}{2}x^2-\frac{1}{24}x^4+\cdots \\
 & \underline{\frac{1}{2}x^2-\frac{1}{4}x^4+\cdots} \\
 & \frac{5}{24}x^4+\cdots \\
 & \underline{\frac{5}{24}x^4+\cdots} \\
 & \cdots
\end{array}
$$

$\sec x = \dfrac{1}{\cos x} = \dfrac{1}{1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots}$.

From the long division above,

$\sec x = 1+\frac{1}{2}x^2+\frac{5}{24}x^4+\cdots$.

51.

$$
\begin{array}{r|l}
 & -x+\frac{1}{2}x^2-\frac{1}{3}x^3+\cdots \\ \hline
1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\cdots & -x-\frac{1}{2}x^2-\frac{1}{3}x^3-\cdots \\
 & \underline{-x-x^2-\frac{1}{2}x^3-\cdots} \\
 & \frac{1}{2}x^2+\frac{1}{6}x^3-\cdots \\
 & \underline{\frac{1}{2}x^2+\frac{1}{2}x^3+\cdots} \\
 & -\frac{1}{3}x^3+\cdots \\
 & \underline{-\frac{1}{3}x^3+\cdots} \\
 & \cdots
\end{array}
$$

From Example 6 in Section 12.9, we have

$\ln(1-x) = -x-\frac{1}{2}x^2-\frac{1}{3}x^3-\cdots$, $|x|<1$.

Therefore,

$y = \dfrac{\ln(1-x)}{e^x} = \dfrac{-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\cdots}{1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\cdots}$. So by the long division above,

$\dfrac{\ln(1-x)}{e^x} = -x+\dfrac{x^2}{2}-\dfrac{x^3}{3}+\cdots$, $|x|<1$.

52. From Example 6 in Section 12.9, we have $\ln(1-x) = -x-\frac{1}{2}x^2-\frac{1}{3}x^3-\cdots$, $|x|<1$. Therefore,

$$
\begin{aligned}
e^x\ln(1-x) &= \left(1+x+\tfrac{1}{2}x^2+\cdots\right)\left(-x-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3+\cdots\right) \\
&= -x-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3-x^2-\tfrac{1}{2}x^3-\tfrac{1}{2}x^3-\cdots \\
&= -x-\tfrac{3}{2}x^2-\tfrac{4}{3}x^3-\cdots,\ |x|<1
\end{aligned}
$$

53. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{n!} = \sum_{n=0}^{\infty}\frac{\left(-x^4\right)^n}{n!} = e^{-x^4}$, by (11).

54. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{\pi}{6}\right)^{2n}}{2n!} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, by (16).

55. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{4^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, by (15).

56. $\displaystyle\sum_{n=0}^{\infty}\frac{3^n}{5^n\,n!} = \sum_{n=0}^{\infty}\frac{(3/5)^n}{n!} = e^{3/5}$, by (11).

57. $\displaystyle 3+\frac{9}{2!}+\frac{27}{3!}+\frac{81}{4!}+\cdots = \frac{3^1}{1!}+\frac{3^2}{2!}+\frac{3^3}{3!}+\frac{3^4}{4!}+\cdots = \sum_{n=1}^{\infty}\frac{3^n}{n!} = \sum_{n=0}^{\infty}\frac{3^n}{n!}-1 = e^3-1$, by (11).

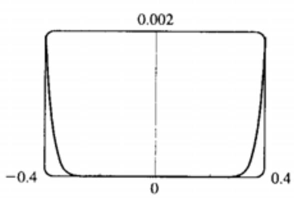
58. $\displaystyle 1-\ln 2+\frac{(\ln 2)^2}{2!}-\frac{(\ln 3)^3}{3!}+\cdots = \sum_{n=0}^{\infty}\frac{(-\ln 2)^n}{n!} = e^{-\ln 2} = \left(e^{\ln 2}\right)^{-1} = 2^{-1} = \frac{1}{2}$, by (11).

59. Assume that $|f'''(x)| \le M$, so $f'''(x) \le M$ for $a \le x \le d$. Now $\int_a^x f'''(t)\,dt \le \int_a^x M\,dt$ $\Rightarrow$ $f''(x) - f''(a) \le M(x-a)$ $\Rightarrow$ $f''(x) \le f''(a) + M(x-a)$. Thus, $\int_a^x f''(t)\,dt \le \int_a^x [f''(a) + M(t-a)]\,dt$ $\Rightarrow$ $f'(x) - f'(a) \le f''(a)(x-a) + \frac{1}{2}M(x-a)^2$ $\Rightarrow$ $f'(x) \le f'(a) + f''(a)(x-a) + \frac{1}{2}M(x-a)^2$ $\Rightarrow$ $\int_a^x f'(t)\,dt \le \int_a^x \left[f'(a) + f''(a)(t-a) + \frac{1}{2}M(t-a)^2\right]dt$ $\Rightarrow$ $f(x) - f(a) \le f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}M(x-a)^3$. So $f(x) - f(a) - f'(a)(x-a) - \frac{1}{2}f''(a)(x-a)^2 \le \frac{1}{6}M(x-a)^3$. But $R_2(x) = f(x) - T_2(x) = f(x) - f(a) - f'(a)(x-a) - \frac{1}{2}f''(a)(x-a)^2$, so $R_2(x) \le \frac{1}{6}M(x-a)^3$. A similar argument using $f'''(x) \ge -M$ shows that $R_2(x) \ge -\frac{1}{6}M(x-a)^3$. So $|R_2(x_2)| \le \frac{1}{6}M|x-a|^3$. Although we have assumed that $x > a$, a similar calculation shows that this inequality is also true if $x < a$.

60. (a) $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$ so

$f'(0) = \lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x\to 0} \dfrac{e^{-1/x^2}}{x} = \lim\limits_{x\to 0} \dfrac{1/x}{e^{1/x^2}} = \lim\limits_{x\to 0} \dfrac{x}{2e^{1/x^2}} = 0$ (using l'Hospital's Rule and simplifying in the penultimate step). Similarly, we can use the definition of the derivative and l'Hospital's Rule to show that $f''(0) = 0$, $f^{(3)}(0) = 0, \ldots, f^{(n)}(0) = 0$, so that the Maclaurin series for f consists entirely of zero terms. But since $f(x) \ne 0$ except for $x = 0$, we see that f cannot equal its Maclaurin series except at $x = 0$.

(b)



From the graph, it seems that the function is extremely flat at the origin. In fact, it could be said to be "infinitely flat" at $x = 0$, since all of its derivatives are 0 there.

12.11 The Binomial Series

1. The general binomial series in (2) is

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots.$$

$$(1+x)^{1/2} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n = 1 + \left(\tfrac{1}{2}\right)x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \cdots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{2^2 \cdot 2!} + \frac{1 \cdot 3 \cdot x^3}{2^3 \cdot 3!} - \frac{1 \cdot 3 \cdot 5 \cdot x^4}{2^4 \cdot 4!} + \cdots$$

$$= 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\, 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)\, x^n}{2^n \cdot n!},\ R = 1$$

2. $\dfrac{1}{(1+x)^4} = (1+x)^{-4} = \sum\limits_{n=0}^{\infty} \dbinom{-4}{n} x^n$. The binomial coefficient is

$$\binom{-4}{n} = \frac{(-4)(-5)(-6)\cdots(-4-n+1)}{n!} = \frac{(-1)^n \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot \cdots \cdot (n+1)(n+2)(n+3)}{2 \cdot 3 \cdot n!}$$
$$= \frac{(-1)^n (n+1)(n+2)(n+3)}{6}$$

so $\dfrac{1}{(1+x)^4} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n (n+1)(n+2)(n+3)}{6} x^n$ for $|x| < 1$, so $R = 1$.

3. $\dfrac{1}{(2+x)^3} = \dfrac{1}{[2(1+x/2)]^3} = \dfrac{1}{8}\left(1+\dfrac{x}{2}\right)^{-3} = \dfrac{1}{8}\sum\limits_{n=0}^{\infty} \dbinom{-3}{n}\left(\dfrac{x}{2}\right)^n$. The binomial coefficient is

$$\binom{-3}{n} = \frac{(-3)(-4)(-5)\cdot \cdots \cdot (-3-n+1)}{n!} = \frac{(-1)^n \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \cdots \cdot (n+1)(n+2)}{2 \cdot n!}$$
$$= \frac{(-1)^n (n+1)(n+2)}{2}$$

so $\dfrac{1}{(2+x)^3} = \dfrac{1}{8}\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n (n+1)(n+2)}{2} \dfrac{x^n}{2^n} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n (n+1)(n+2)\, x^n}{2^{n+4}}$ for $\left|\dfrac{x}{2}\right| < 1 \;\Leftrightarrow\; |x| < 2$, so $R = 2$.

4.
$$\left(1+x^2\right)^{1/3} = \sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n} x^{2n} = 1 + \frac{x^2}{3} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}x^4 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}x^6 + \cdots$$
$$= 1 + \frac{x^2}{3} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} \cdot 2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-4)\, x^{2n}}{3^n\, n!}, \text{ with } R = 1.$$

5.
$$\sqrt[4]{1-8x} = (1-8x)^{1/4} = \sum_{n=0}^{\infty} \binom{\frac{1}{4}}{n} (-8x)^n$$
$$= 1 + \tfrac{1}{4}(-8x) + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{2!}(-8x)^2 + \frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{3!}(-8x)^3 + \cdots$$
$$= 1 - 2x + \sum_{n=2}^{\infty} \frac{(-1)^n (-1)^{n-1} \cdot 3 \cdot 7 \cdot \cdots \cdot (4n-5)\, 8^n}{4^n \cdot n!} x^n$$
$$= 1 - 2x - \sum_{n=2}^{\infty} \frac{3 \cdot 7 \cdot \cdots \cdot (4n-5)\, 2^n}{n!} x^n$$

and $|-8x| < 1 \;\Leftrightarrow\; |x| < \frac{1}{8}$, so $R = \frac{1}{8}$.

6.
$$\frac{1}{\sqrt[5]{32-x}} = \frac{1}{2\sqrt[5]{1-x/32}} = \frac{1}{2}\left(1-\frac{x}{32}\right)^{-1/5} = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{5}}{n}\left(-\frac{x}{32}\right)^n = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{5}}{n}\frac{(-1)^n x^n}{2^{5n}}$$
$$= \frac{1}{2}\left[1 + \left(-\frac{1}{5}\right)\left(-\frac{x}{2^5}\right) + \frac{\left(-\frac{1}{5}\right)\left(-\frac{6}{5}\right)}{2!}\frac{x^2}{2^{10}} + \frac{\left(-\frac{1}{5}\right)\left(-\frac{6}{5}\right)\left(-\frac{11}{5}\right)}{3!}\left(-\frac{x^3}{2^{15}}\right) + \cdots\right]$$
$$= \frac{1}{2} + \frac{1}{5 \cdot 2^6}x + \frac{1 \cdot 6}{5^2 \cdot 2! \cdot 2^{11}}x^2 + \frac{1 \cdot 6 \cdot 11}{5^3 \cdot 3! \cdot 2^{16}}x^3 + \cdots = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 \cdot 6 \cdot \cdots \cdot (5n-4)}{5^n 2^{5n+1}\, n!}x^n$$

The radius of convergence is 32.

7. $\dfrac{x}{\sqrt{4+x^2}} = \dfrac{x}{2\sqrt{1+x^2/4}} = \dfrac{x}{2}\left(1+\dfrac{x^2}{4}\right)^{-1/2} = \dfrac{x}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\dfrac{x^2}{4}\right)^n$

$$= \frac{x}{2}\left[1+\left(-\tfrac{1}{2}\right)\frac{x^2}{4}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x^2}{4}\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x^2}{4}\right)^3+\cdots\right]$$

$$= \frac{x}{2}+\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}{n!\,2^{3n+1}}x^{2n+1} \text{ and } \frac{x^2}{4}<1 \iff \frac{|x|}{2}<1 \iff |x|<2, \text{ so } R=2.$$

8. $\dfrac{x^2}{\sqrt{2+x}} = \dfrac{x^2}{\sqrt{2\,(1+x/2)}} = \dfrac{x^2}{\sqrt{2}}\left(1+\dfrac{x}{2}\right)^{-1/2} = \dfrac{x^2}{\sqrt{2}}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\dfrac{x}{2}\right)^n$

$$= \frac{x^2}{\sqrt{2}}\left[1+\left(-\tfrac{1}{2}\right)\left(\frac{x}{2}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x}{2}\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x}{2}\right)^3+\cdots\right]$$

$$= \frac{x^2}{\sqrt{2}}+\frac{x^2}{\sqrt{2}}\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}{n!\,2^{2n}}x^n$$

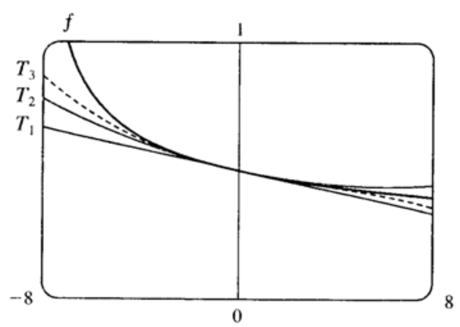
$$= \frac{x^2}{\sqrt{2}}+\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}{n!\,2^{2n+1/2}}x^{n+2} \text{ and } \left|\frac{x}{2}\right|<1 \iff |x|<2, \text{ so } R=2.$$

9. $\dfrac{1}{\sqrt[3]{8+x}} = (8+x)^{-1/3} = 8^{-1/3}\left(1+\dfrac{x}{8}\right)^{-1/3} = \dfrac{1}{2}\left(1+\dfrac{x}{8}\right)^{-1/3}$

$$= \frac{1}{2}\left[1+\left(-\tfrac{1}{3}\right)\left(\frac{x}{8}\right)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2!}\left(\frac{x}{8}\right)^2+\cdots\right]$$

$$= \frac{1}{2}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n\,1\cdot4\cdot7\cdot\cdots\cdot(3n-2)}{3^n\cdot n!\,8^n}x^n\right] \text{ and } \left|\frac{x}{8}\right|<1 \iff |x|<8, \text{ so } R=8.$$

The three Taylor polynomials are $T_1(x) = \frac{1}{2}-\frac{1}{48}x$, $T_2(x) = \frac{1}{2}-\frac{1}{48}x+\frac{1}{576}x^2$, and
$T_3(x) = \frac{1}{2}-\frac{1}{48}x+\frac{1}{576}x^2-\frac{4\cdot7}{2\cdot27\cdot6\cdot512}x^3 = \frac{1}{2}-\frac{1}{48}x+\frac{1}{576}x^2-\frac{7}{41{,}472}x^3$.

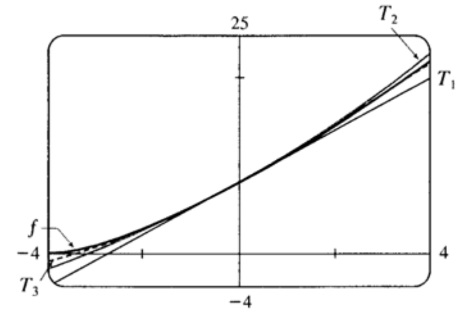


10. $(4+x)^{3/2} = 8\left(1+\frac{x}{4}\right)^{3/2} = 8\sum_{n=0}^{\infty}\binom{\frac{3}{2}}{n}\left(\frac{x}{4}\right)^n$

$$= 8\left[1+\tfrac{3}{2}\left(\frac{x}{4}\right)+\frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{2!}\left(\frac{x}{4}\right)^2+\frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{3!}\left(\frac{x}{4}\right)^3+\cdots\right]$$

$$= 8+3x+\sum_{n=2}^{\infty}\frac{(3)(1)(-1)\cdot\cdots\cdot(5-2n)\,x^n}{8^{n-1}\cdot n!} \text{ and } \left|\frac{x}{4}\right|<1 \Leftrightarrow |x|<4, \text{ so } R=4.$$

The three Taylor polynomials are $T_1(x) = 8+3x$, $T_2(x) = 8+3x+\frac{3}{16}x^2$, and $T_3(x) = 8+3x+\frac{3}{16}x^2-\frac{1}{128}x^3$.



11. (a) $\left[1+(-x^2)\right]^{-1/2} = 1+\left(-\frac{1}{2}\right)(-x^2)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x^2)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x^2)^3+\cdots$

$$= 1+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n\cdot n!}x^{2n}$$

(b) $\sin^{-1}x = \int\frac{1}{\sqrt{1-x^2}}\,dx = C+x+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(2n+1)\,2^n\cdot n!}x^{2n+1}$

$$= x+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(2n+1)\,2^n\cdot n!}x^{2n+1} \text{ since } 0=\sin^{-1}0=C.$$

12. (a) $\left(1+x^2\right)^{-1/2} = \sum_{n=0}^{\infty}\binom{-1/2}{n}x^{2n} = 1+\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)\,x^{2n}}{2^n\cdot n!}$

(b) $\sinh^{-1}x = \int\frac{dx}{\sqrt{1+x^2}} = C+x+\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)\,x^{2n+1}}{2^n\cdot n!\,(2n+1)}$, but $C=0$ since $\sinh^{-1}0=0$,

so $\sinh^{-1}x = x+\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)\,x^{2n+1}}{2^n\cdot n!\,(2n+1)}$, $R=1$.

13. (a) $(1+x)^{-1/2} = 1+\left(-\frac{1}{2}\right)x+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^3+\cdots$

$$= 1+\sum_{n=1}^{\infty}\frac{(-1)^n\,1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n\cdot n!}x^n$$

(b) Take $x=0.1$ in the above series. $\frac{1\cdot 3\cdot 5\cdot 7}{2^4 4!}(0.1)^4 < 0.00003$, so

$$\frac{1}{\sqrt{1.1}} \approx 1-\frac{0.1}{2}+\frac{1\cdot 3}{2^2\cdot 2!}(0.1)^2-\frac{1\cdot 3\cdot 5}{2^3\cdot 3!}(0.1)^3 \approx 0.953.$$

14. (a) $(8+x)^{1/3} = 2\left(1+\frac{x}{8}\right)^{1/3} = 2\sum_{n=0}^{\infty}\binom{\frac{1}{3}}{n}\left(\frac{x}{8}\right)^n$

$$= 2\left[1+\tfrac{1}{3}\left(\frac{x}{8}\right)+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\left(\frac{x}{8}\right)^2+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(\frac{x}{8}\right)^3+\cdots\right]$$

$$= 2\left[1+\frac{x}{24}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\cdot 2\cdot 5\cdot\cdots\cdot(3n-4)\,x^n}{24^n\cdot n!}\right]$$

(b) $(8+0.2)^{1/3} = 2\left[1+\frac{0.2}{24}-\frac{(0.2)^2}{24^2}+\frac{2\cdot 5\,(0.2)^3}{24^3\cdot 3!}-\cdots\right] \approx 2\left[1+\frac{0.2}{24}-\frac{(0.2)^2}{24^2}\right]$ since

$2\cdot\frac{2\cdot 5\,(0.2)^3}{24^3\cdot 3!}\approx 0.000002$, so $\sqrt[3]{8.2}\approx 2.0165$.

15. (a) $[1+(-x)]^{-2} = 1+(-2)(-x)+\frac{(-2)(-3)}{2!}(-x)^2+\frac{(-2)(-3)(-4)}{3!}(-x)^3+\cdots$

$$= 1+2x+3x^2+4x^3+\cdots = \sum_{n=0}^{\infty}(n+1)\,x^n,$$

so $\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)\,x^{n+1} = \sum_{n=1}^{\infty}nx^n$.

(b) With $x=\frac{1}{2}$ in part (a), we have $\sum_{n=1}^{\infty}\frac{n}{2^n} = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2$.

16. (a) $[1+(-x)]^{-3} = \sum_{n=0}^{\infty}\binom{-3}{n}(-x)^n$

$$= 1+(-3)(-x)+\frac{(-3)(-4)}{2!}(-x)^2+\frac{(-3)(-4)(-5)}{3!}(-x)^3+\cdots$$

$$= 1+\sum_{n=1}^{\infty}\frac{3\cdot 4\cdot 5\cdot\cdots\cdot(n+2)}{n!}x^n = \sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^n \quad\Rightarrow$$

$$\left(x+x^2\right)[1+(-x)]^{-3} = \sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^{n+1}+\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^{n+2}$$

$$= x+\sum_{n=2}^{\infty}\left[\frac{n(n+1)}{2}+\frac{(n-1)\,n}{2}\right]x^n = x+\sum_{n=2}^{\infty}n^2x^n = \sum_{n=1}^{\infty}n^2x^n,\ -1<x<1$$

(b) Setting $x=\frac{1}{2}$ in the last series above gives the required series, so $\sum_{n=1}^{\infty}\frac{n^2}{2^n} = \frac{\frac{1}{2}+\left(\frac{1}{2}\right)^2}{\left(1-\frac{1}{2}\right)^3} = 6$.

17. (a) $\left(1+x^2\right)^{1/2} = 1+\left(\frac{1}{2}\right)x^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(x^2\right)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(x^2\right)^3+\cdots$

$$= 1+\frac{x^2}{2}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\,1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n\cdot n!}x^{2n}$$

(b) The coefficient of x^{10} (corresponding to $n=5$) in the above Maclaurin series is $\frac{f^{(10)}(0)}{10!}$, so

$\frac{f^{(10)}(0)}{10!} = \frac{(-1)^4\cdot 1\cdot 3\cdot 5\cdot 7}{2^5\cdot 5!} \quad\Rightarrow\quad f^{(10)}(0) = 10!\left(\frac{1\cdot 3\cdot 5\cdot 7}{2^5\cdot 5!}\right) = 99{,}225.$

18. (a) $\left(1+x^3\right)^{-1/2} = \sum\limits_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(x^3\right)^n$

$$= 1 + \left(-\tfrac{1}{2}\right)\left(x^3\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(x^3\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(x^3\right)^3 + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n\, 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)\, x^{3n}}{2^n \cdot n!}$$

(b) The coefficient of x^9 in the preceding series is $\dfrac{f^{(9)}(0)}{9!}$, so $\dfrac{f^{(9)}(0)}{9!} = \dfrac{(-1)^3\, 1 \cdot 3 \cdot 5}{2^3 \cdot 3!} \quad \Rightarrow$

$f^{(9)}(0) = -\dfrac{9! \cdot 5}{8 \cdot 2} = -113{,}400.$

19. (a) $g(x) = \sum\limits_{n=0}^{\infty} \binom{k}{n} x^n \quad \Rightarrow \quad g'(x) = \sum\limits_{n=1}^{\infty} \binom{k}{n} n x^{n-1}$, so

$$(1+x)\, g'(x) = (1+x) \sum_{n=1}^{\infty} \binom{k}{n} n x^{n-1} = \sum_{n=1}^{\infty} \binom{k}{n} n x^{n-1} + \sum_{n=1}^{\infty} \binom{k}{n} n x^n$$

$$= \sum_{n=0}^{\infty} \binom{k}{n+1} (n+1)\, x^n + \sum_{n=0}^{\infty} \binom{k}{n} n x^n \qquad \begin{bmatrix}\text{Replace } n \text{ with } n+1 \\ \text{in the first series}\end{bmatrix}$$

$$= \sum_{n=0}^{\infty} (n+1)\, \frac{k(k-1)(k-2)\cdots(k-n+1)(k-n)}{(n+1)!}\, x^n$$

$$+ \sum_{n=0}^{\infty} \left[(n)\, \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}\right] x^n$$

$$= \sum_{n=0}^{\infty} \frac{(n+1)\, k(k-1)(k-2)\cdots(k-n+1)}{(n+1)!}\, [(k-n)+n]\, x^n$$

$$= k \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}\, x^n = k \sum_{n=0}^{\infty} \binom{k}{n} x^n = k g(x)$$

Thus, $g'(x) = \dfrac{k g(x)}{1+x}$.

(b) $h(x) = (1+x)^{-k} g(x) \quad \Rightarrow$

$$\begin{aligned} h'(x) &= -k(1+x)^{-k-1} g(x) + (1+x)^{-k} g'(x) \quad \text{[Product Rule]} \\ &= -k(1+x)^{-k-1} g(x) + (1+x)^{-k} \frac{k g(x)}{1+x} \quad \text{[from part (a)]} \\ &= -k(1+x)^{-k-1} g(x) + k(1+x)^{-k-1} g(x) = 0 \end{aligned}$$

(c) From part (b) we see that $h(x)$ must be constant for $x \in (-1, 1)$, so $h(x) = h(0) = 1$ for $x \in (-1, 1)$. Thus, $h(x) = 1 = (1+x)^{-k} g(x) \quad \Leftrightarrow \quad g(x) = (1+x)^k$ for $x \in (-1, 1)$.

20. (a) $4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-k^2\sin^2 x}}=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1+\left(-k^2\sin^2 x\right)\right]^{-1/2}dx$

$$=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1-\frac{1}{2}\left(-k^2\sin^2 x\right)+\frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}\left(-k^2\sin^2 x\right)^2-\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}\left(-k^2\sin^2 x\right)^3+\cdots\right]dx$$

$$=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1+\left(\frac{1}{2}\right)k^2\sin^2 x+\left(\frac{1\cdot 3}{2\cdot 4}\right)k^4\sin^4 x+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)k^6\sin^6 x+\cdots\right]dx$$

[split up the integral and use the result from Exercise 8.1.40]

$$=4\sqrt{\frac{L}{g}}\left[\frac{\pi}{2}+\left(\frac{1}{2}\right)\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)\left(\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2}\right)k^4+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{\pi}{2}\right)k^6+\cdots\right]$$

$$=2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1^2}{2^2}k^2+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}k^4+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}k^6+\cdots\right]$$

(b) The first of the two inequalities is true because all of the terms in the series are positive. For the second,

$$T=2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1^2}{2^2}k^2+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}k^4+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}k^6+\frac{1^2\cdot 3^2\cdot 5^2\cdot 7^2}{2^2\cdot 4^2\cdot 6^2\cdot 8^2}k^8+\cdots\right]$$

$$\le 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1}{4}k^2+\frac{1}{4}k^4+\frac{1}{4}k^6+\frac{1}{4}k^8+\cdots\right]$$

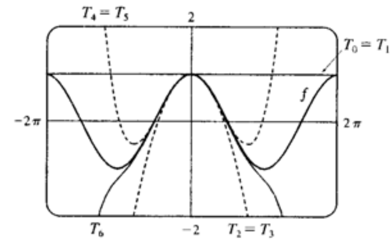
The terms in brackets (after the first) form a geometric series with $a=\frac{1}{4}k^2$ and $r=k^2=\sin^2\left(\frac{1}{2}\theta_0\right)<1$. So $T\le 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{k^2/4}{1-k^2}\right]=2\pi\sqrt{\frac{L}{g}}\frac{4-3k^2}{4-4k^2}$.

(c) We substitute $L=1$, $g=9.8$, and $k=\sin(10°/2)\approx 0.08716$, and the inequality from part (b) becomes $2.01090\le T\le 2.01093$, so $T\approx 2.0109$. The estimate $T\approx 2\pi\sqrt{L/g}\approx 2.0071$ differs by about 0.2%. If $\theta_0=42°$, then $k\approx 0.35837$ and the inequality becomes $2.07153\le T\le 2.08103$, so $T\approx 2.0763$. The one-term estimate is the same, and the discrepancy between the two estimates increases to about 3.7%.

12.12 Applications of Taylor Polynomials

1. (a)

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$T_n(x)$
0	$\cos x$	1	1
1	$-\sin x$	0	1
2	$-\cos x$	-1	$1-\frac{1}{2}x^2$
3	$\sin x$	0	$1-\frac{1}{2}x^2$
4	$\cos x$	1	$1-\frac{1}{2}x^2+\frac{1}{24}x^4$
5	$-\sin x$	0	$1-\frac{1}{2}x^2+\frac{1}{24}x^4$
6	$-\cos x$	-1	$1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6$



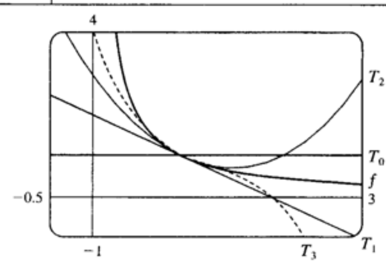
(b)

x	f	$T_0 = T_1$	$T_2 = T_3$	$T_4 = T_5$	T_6
$\frac{\pi}{4}$	0.7071	1	0.6916	0.7074	0.7071
$\frac{\pi}{2}$	0	1	-0.2337	0.0200	-0.0009
π	-1	1	-3.9348	0.1239	-1.2114

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

2. (a)

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$T_n(x)$
0	x^{-1}	1	1
1	$-x^{-2}$	-1	$1-(x-1)=2-x$
2	$2x^{-3}$	2	$1-(x-1)+(x-1)^2=x^2-3x+3$
3	$-6x^{-4}$	-6	$1-(x-1)+(x-1)^2-(x-1)^3=-x^3+4x^2-6x+4$



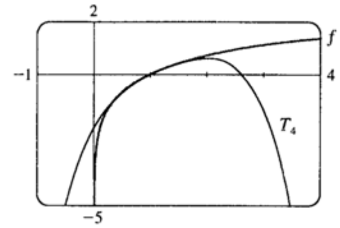
(b)

x	f	T_0	T_1	T_2	T_3
0.9	$1.\overline{1}$	1	1.1	1.11	1.111
1.3	0.7692	1	0.7	0.79	0.763

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

3.

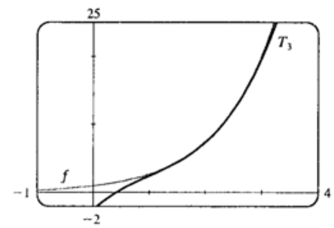
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln x$	0
1	$1/x$	1
2	$-1/x^2$	-1
3	$2/x^3$	2
4	$-6/x^4$	-6



$$T_4(x) = \sum_{n=0}^{4} \frac{f^{(n)}(1)}{n!}(x-1)^n = 0 + (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4$$

4.

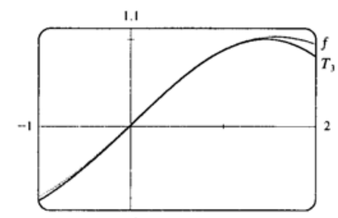
n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	e^x	e^2
1	e^x	e^2
2	e^x	e^2
3	e^x	e^2



$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(2)}{n!}(x-2)^n = e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^2}{6}(x-2)^3$$

5.

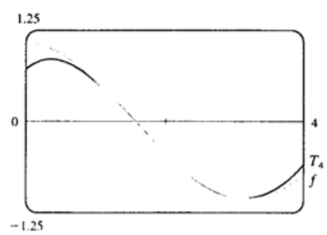
n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{6}\right)$
0	$\sin x$	$\frac{1}{2}$
1	$\cos x$	$\frac{\sqrt{3}}{2}$
2	$-\sin x$	$-\frac{1}{2}$
3	$-\cos x$	$-\frac{\sqrt{3}}{2}$



$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}\left(\frac{\pi}{6}\right)}{n!}\left(x-\tfrac{\pi}{6}\right)^n = \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\left(x-\tfrac{\pi}{6}\right) - \tfrac{1}{4}\left(x-\tfrac{\pi}{6}\right)^2 - \tfrac{\sqrt{3}}{12}\left(x-\tfrac{\pi}{6}\right)^3$$

6.

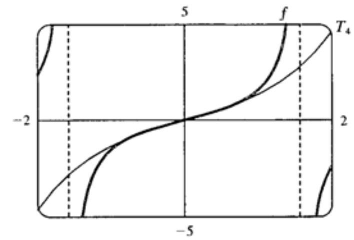
n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{2\pi}{3}\right)$
0	$\cos x$	$-\frac{1}{2}$
1	$-\sin x$	$-\frac{\sqrt{3}}{2}$
2	$-\cos x$	$\frac{1}{2}$
3	$\sin x$	$\frac{\sqrt{3}}{2}$
4	$\cos x$	$-\frac{1}{2}$



$$T_4(x) = \sum_{n=0}^{4} \frac{f^{(n)}\left(\frac{2\pi}{3}\right)}{n!}\left(x-\tfrac{2\pi}{3}\right)^n = -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}\left(x-\tfrac{2\pi}{3}\right) + \tfrac{1}{4}\left(x-\tfrac{2\pi}{3}\right)^2 + \tfrac{\sqrt{3}}{12}\left(x-\tfrac{2\pi}{3}\right)^3 - \tfrac{1}{48}\left(x-\tfrac{2\pi}{3}\right)^4$$

7.

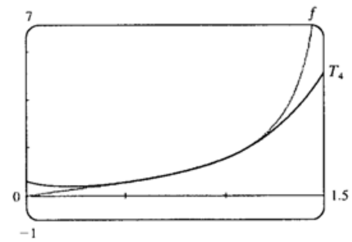
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\tan x$	0
1	$\sec^2 x$	1
2	$2\sec^2 x \tan x$	0
3	$4\sec^2 x \tan^2 x + 2\sec^4 x$	2
4	$8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$	0



$$T_4(x) = \sum_{n=0}^{4} \frac{f^{(n)}(0)}{n!} x^n = x + \frac{2x^3}{3!} = x + \frac{x^3}{3}$$

8.

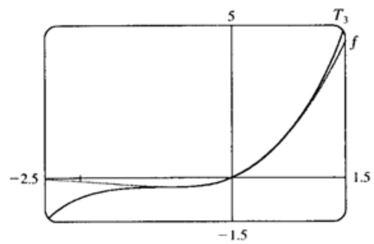
n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{4}\right)$
0	$\tan x$	1
1	$\sec^2 x$	2
2	$2\sec^2 x \tan x$	4
3	$4\sec^2 x \tan^2 x + 2\sec^4 x$	16
4	$8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$	80



$$T_4(x) = \sum_{n=0}^{4} \frac{f^{(n)}\left(\frac{\pi}{4}\right)}{n!}\left(x - \tfrac{\pi}{4}\right)^n = 1 + 2\left(x - \tfrac{\pi}{4}\right) + 2\left(x - \tfrac{\pi}{4}\right)^2 + \tfrac{8}{3}\left(x - \tfrac{\pi}{4}\right)^3 + \tfrac{10}{3}\left(x - \tfrac{\pi}{4}\right)^4$$

9.

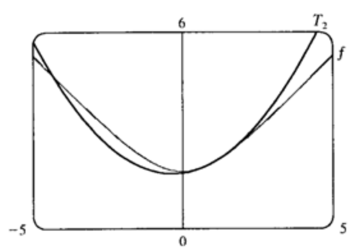
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$e^x \sin x$	0
1	$e^x(\sin x + \cos x)$	1
2	$2e^x \cos x$	2
3	$2e^x(\cos x - \sin x)$	2



$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(0)}{n!} x^n = x + x^2 + \tfrac{1}{3}x^3$$

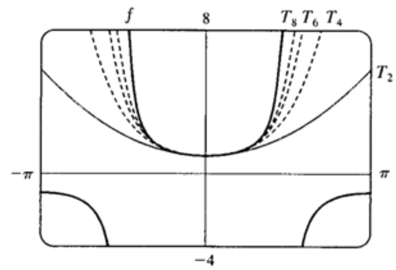
10.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$(3+x^2)^{1/2}$	2
1	$x(3+x^2)^{-1/2}$	$\frac{1}{2}$
2	$3(3+x^2)^{-3/2}$	$\frac{3}{8}$

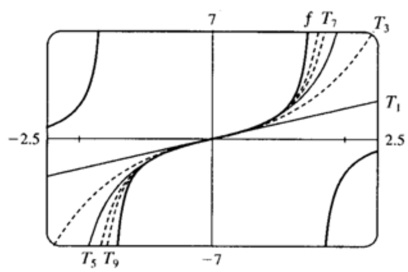


$$T_2(x) = \sum_{n=0}^{2} \frac{f^{(n)}(1)}{n!}(x-1)^n = 2 + \tfrac{1}{2}(x-1) + \tfrac{3/8}{2}(x-1)^2 = 2 + \tfrac{1}{2}(x-1) + \tfrac{3}{16}(x-1)^2$$

11. In Maple, we can find the Taylor polynomials by the following method: first define `f:=sec(x);` and then set `T2:=convert(taylor(f,x=0,3),polynom);`, `T4:=convert(taylor(f,x=0,5),polynom);`, etc. (The third argument in the `taylor` function is one more than the degree of the desired polynomial). We must `convert` to the type `polynom` because the output of the `taylor` function contains an error term which we do not want. In Mathematica, we use `Tn:=Normal[Series[f,{x,0,n}]]`, with `n=2, 4`, etc. Note that in Mathematica, the "degree" argument is the same as the degree of the desired polynomial. In Derive, author $\sec x$, then enter `Calculus,Taylor,8,0`; and then simplify the expression. The eighth Taylor polynomial is $T_8(x) = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8$.



12. See Exercise 11 for the CAS commands used to generate the Taylor polynomials. The ninth Taylor polynomial for $\tan x$ is $T_9(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9$.



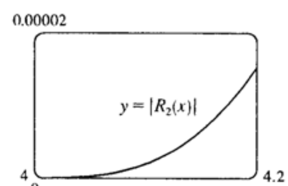
13.

$$\begin{aligned} f(x) &= \sqrt{x} & f(4) &= 2 \\ f'(x) &= \tfrac{1}{2}x^{-1/2} & f'(4) &= \tfrac{1}{4} \\ f''(x) &= -\tfrac{1}{4}x^{-3/2} & f''(4) &= -\tfrac{1}{32} \\ f'''(x) &= \tfrac{3}{8}x^{-5/2} \end{aligned}$$

(a) $\sqrt{x} \approx T_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1/32}{2!}(x-4)^2$
$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$

(b) $|R_2(x)| \le \dfrac{M}{3!}|x-4|^3$, where $|f'''(x)| \le M$. Now $4 \le x \le 4.2 \Rightarrow |x-4| \le 0.2 \Rightarrow |x-4|^3 \le 0.008$. Since $f'''(x)$ is decreasing on $[4, 4.2]$, we can take $M = |f'''(4)| = \frac{3}{8}4^{-5/2} = \frac{3}{256}$, so $|R_2(x)| \le \frac{3/256}{6}(0.008) = \frac{0.008}{512} = 0.000015625$.

(c)



From the graph of $|R_2(x)| = |\sqrt{x} - T_2(x)|$, it seems that the error is less than 1.52×10^{-5} on $[4, 4.2]$.

14. $f(x) = x^{-2}$ $\quad f(1) = 1$

$f'(x) = -2x^{-3}$ $\quad f'(1) = -2$

$f''(x) = 6x^{-4}$ $\quad f''(4) = 6$

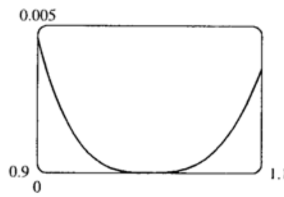
$f'''(x) = -24x^{-5}$

(a) $x^{-2} \approx T_2(x) = 1 - 2(x-1) + \frac{6}{2!}(x-1)^2$
$= 1 - 2(x-1) + 3(x-1)^2$

(b) $|R_2(x)| \leq \dfrac{M}{3!}|x-1|^3$, where $|f'''(x)| \leq M$. Now $0.9 \leq x \leq 1.1 \Rightarrow |x-1| \leq 0.1 \Rightarrow |x-1|^3 \leq 0.001$. Since $f'''(x)$ is decreasing on $[0.9, 1.1]$, we can take $M = |f'''(0.9)| = \frac{24}{(0.9)^5}$, so

$|R_2(x)| \leq \frac{24/(0.9)^5}{6}(0.001) = \frac{0.004}{0.59049}$
≈ 0.00677404

(c)



From the graph of $|R_2(x)| = |x^{-2} - T_2(x)|$, it seems that the error is less than 0.0046 on $[0.9, 1.1]$.

15. $f(x) = \sin x$ $\quad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(x) = \cos x$ $\quad f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f''(x) = -\sin x$ $\quad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f'''(x) = -\cos x$ $\quad f'''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f^{(4)}(x) = \sin x$ $\quad f^{(4)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

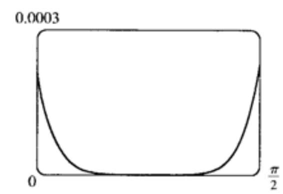
$f^{(5)}(x) = \cos x$ $\quad f^{(5)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f^{(6)}(x) = -\sin x$

(a) $\sin x \approx T_5(x)$
$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3$
$+ \frac{\sqrt{2}}{48}\left(x - \frac{\pi}{4}\right)^4 + \frac{\sqrt{2}}{240}\left(x - \frac{\pi}{4}\right)^5$

(b) $|R_5(x)| \leq \dfrac{M}{6!}\left|x - \frac{\pi}{4}\right|^6$, where $|f^{(6)}(x)| \leq M$. Now $0 \leq x \leq \frac{\pi}{2} \Rightarrow \left(x - \frac{\pi}{4}\right)^6 \leq \left(\frac{\pi}{4}\right)^6$, and letting $x = \frac{\pi}{2}$ gives $M = 1$, so $|R_5(x)| \leq \frac{1}{6!}\left(\frac{\pi}{4}\right)^6 = \frac{1}{720}\left(\frac{\pi}{4}\right)^6 \approx 0.00033$.

(c)



From the graph of $|R_5(x)| = |\sin x - T_5(x)|$, it seems that the error is less than 0.00026 on $\left[0, \frac{\pi}{2}\right]$.

16. $f(x) = \cos x$ $\quad f\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$f'(x) = -\sin x$ $\quad f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$f''(x) = -\cos x$ $\quad f''\left(\frac{\pi}{3}\right) = -\frac{1}{2}$

$f'''(x) = \sin x$ $\quad f'''\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

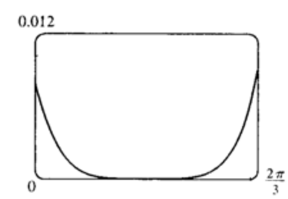
$f^{(4)}(x) = \cos x$ $\quad f^{(4)}\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$f^{(5)}(x) = -\sin x$

(a) $\cos x \approx T_4(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2$
$+ \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3 + \frac{1}{48}\left(x - \frac{\pi}{3}\right)^4$

(b) $|R_4(x)| \leq \dfrac{M}{5!}\left|x - \frac{\pi}{3}\right|^5$, where $|f^{(5)}(x)| \leq M$. Now $0 \leq x \leq \frac{2\pi}{3} \Rightarrow \left(x - \frac{\pi}{3}\right)^5 \leq \left(\frac{\pi}{3}\right)^5$, and letting $x = \frac{\pi}{2}$ gives $M = 1$, so $|R_4(x)| \leq \frac{1}{5!}\left(\frac{\pi}{3}\right)^5 \approx 0.0105$.

(c)



From the graph of $|R_4(x)| = |\cos x - T_4(x)|$, it seems that the error is less than 0.01 on $\left[0, \frac{2\pi}{3}\right]$.

17. $f(x) = \tan x$ $\qquad f(0) = 0 \qquad f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x \qquad f'''(0) = 2$

$f'(x) = \sec^2 x \qquad f'(0) = 1 \qquad f^{(4)}(x) = 8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$

$f''(x) = 2\sec^2 x \tan x \qquad f''(0) = 0$

(a) $\tan x \approx T_3(x) = x + \frac{1}{3}x^3$

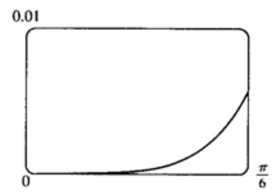
(b) $|R_3(x)| \le \dfrac{M}{4!}|x|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now $0 \le x \le \frac{\pi}{6}$

$\Rightarrow \quad x^4 \le \left(\frac{\pi}{6}\right)^4$, and letting $x = \frac{\pi}{6}$ gives

$$|R_3(x)| \le \frac{8\left(\frac{2}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right)^3 + 16\left(\frac{2}{\sqrt{3}}\right)^4\left(\frac{1}{\sqrt{3}}\right)}{4!}\left(\frac{\pi}{6}\right)^4$$

$$= \tfrac{4\sqrt{3}}{9}\left(\tfrac{\pi}{6}\right)^4 \approx 0.057859$$

(c)



From the graph, it seems that the error is less than 0.006 on $[0, \pi]$.

18. $f(x) = \left(1 + x^2\right)^{1/3} \qquad f(0) = 1 \qquad f'''(x) = \dfrac{8x^3 - 72x}{27\left(1 + x^2\right)^{8/3}}$

$f'(x) = \frac{2}{3}x\left(1 + x^2\right)^{-2/3} \qquad f'(0) = 0$

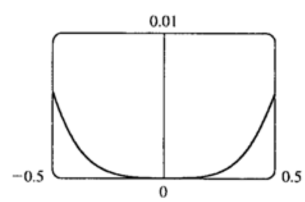
$f''(x) = \frac{2}{3}\left(1 - \frac{1}{3}x^2\right)\left(1 + x^2\right)^{-5/3} \qquad f''(0) = \frac{2}{3}$

(a) $\sqrt[3]{1 + x^2} \approx T_2(x) = 1 + \frac{1}{3}x^2$

(b) $|R_2(x)| \le \dfrac{M}{3!}|x|^3$, where $\left|f'''(x)\right| \le M$. By examining a graph of $\left|f'''(x)\right|$, we see that its maximum is approximately 0.71495314. Thus,

$$|R_2(x)| \le \frac{0.71495314}{3!}(0.5)^3 \approx 0.014895.$$

(c)



It seems that the error is less than 0.0061 on $[-0.5, 0.5]$.

19. $f(x) = e^{x^2}$ $\quad f(0) = 1$ $\quad f'''(x) = e^{x^2}(12x + 8x^3)$ $\quad f'''(0) = 0$

$f'(x) = e^{x^2}(2x)$ $\quad f'(0) = 0$ $\quad f^{(4)}(x) = e^{x^2}(12 + 48x^2 + 16x^4)$

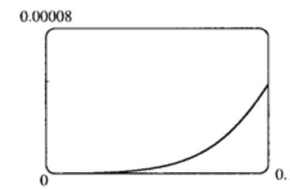
$f''(x) = e^{x^2}(2 + 4x^2)$ $\quad f''(0) = 2$

(a) $e^{x^2} \approx T_3(x) = 1 + \frac{2}{2!}x^2 = 1 + x^2$

(b) $|R_3(x)| \le \dfrac{M}{4!}|x|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now $0 \le x \le 0.1$

$\Rightarrow \quad x^4 \le (0.1)^4$, and letting $x = 0.1$ gives

$|R_3(x)| \le \dfrac{e^{0.01}(12 + 0.48 + 0.0016)}{24}(0.1)^4 \approx 0.00006.$

(c) 

From the graph of $|R_3(x)| = \left|e^{x^2} - (1 + x^2)\right|$, it appears that the error is less than 0.000051 on $[0, 0.1]$.

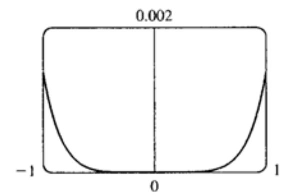
20. (a) Clearly $f^{(2n)}(0) = 1$ and $f^{(2n+1)}(0) = 0$, so

$\cosh x \approx T_5(x) = 1 + \dfrac{x^2}{2} + \dfrac{x^4}{24}.$

(b) $|R_5(x)| \le \dfrac{M}{6!}|x|^6$, where $\left|f^{(6)}(x)\right| \le M$. Since

$f^{(6)}(x) = \cosh x$ and $\cosh x$ attains its maximum on $[-1, 1]$ at both endpoints, we let $x = 1$ and get

$|R_5(x)| \le \dfrac{\cosh 1}{6!}(1)^6 \approx 0.002143.$

(c) 

It appears that the error is less than 0.0015 on $(-1, 1)$.

21. $f(x) = x^{3/4}$ $\quad f(16) = 8$ $\quad f'''(x) = \frac{15}{64}x^{-9/4}$ $\quad f'''(16) = \frac{15}{32{,}768}$

$f'(x) = \frac{3}{4}x^{-1/4}$ $\quad f'(16) = \frac{3}{8}$ $\quad f^{(4)}(x) = -\frac{135}{256}x^{-13/4}$

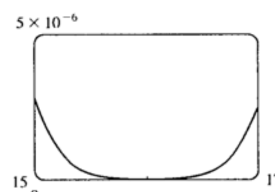
$f''(x) = -\frac{3}{16}x^{-5/4}$ $\quad f''(16) = -\frac{3}{512}$

(a) $x^{3/4} \approx T_3(x) = 8 + \frac{3}{8}(x - 16) - \frac{3}{1024}(x - 16)^2 + \frac{5}{65{,}536}(x - 16)^3$

(b) $|R_3(x)| \le \dfrac{M}{4!}|x - 16|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now

$15 \le x \le 17 \quad \Rightarrow \quad |x - 16|^4 \le 1^4 = 1$, and letting $x = 15$ to minimize the denominator of $f^{(4)}(x)$ gives

$|R_3(x)| \le \dfrac{135/\left[256(15)^{13/4}\right]}{4!}(1) \approx 0.0000033.$

(c) 

It appears that the error is less than 3×10^{-6} on $(15, 17)$.

22. $f(x) = \ln x$ $\quad f(4) = \ln 4$ $\quad f'''(x) = 2x^{-3}$ $\quad f'''(4) = \frac{1}{32}$

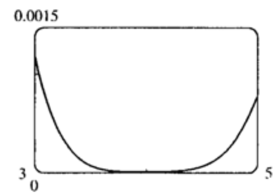
$f'(x) = x^{-1}$ $\quad f'(4) = \frac{1}{4}$ $\quad f^{(4)}(x) = -6x^{-4}$

$f''(x) = -x^{-2}$ $\quad f''(4) = -\frac{1}{16}$

(a) $\ln x \approx T_3(x) = \ln 4 + \frac{1}{4}(x-4) - \frac{1}{32}(x-4)^2 + \frac{1}{192}(x-4)^3$

(b) $|R_3(x)| \le \dfrac{M}{4!}|x-4|^4$, where $|f^{(4)}(x)| \le M$. Now $3 \le x \le 5 \Rightarrow (x-4)^4 \le 1^4 = 1$, and letting $x = 3$ gives $M = 6/3^4$, so $|R_3(x)| \le \frac{6}{4!\,3^4} \cdot 1 = \frac{1}{324} \approx 0.0031$.

(c)



From the graph of $|R_3(x)| = |\ln x - T_3(x)|$, it appears that the error is less than 0.0013 on [3, 5].

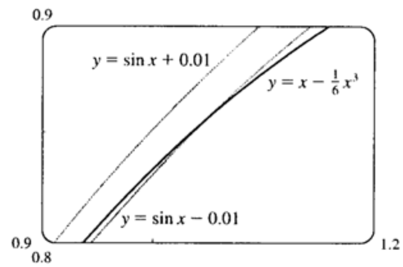
23. From Exercise 5, $\sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3 + R_3(x)$, where $R_3(x) \le \dfrac{M}{4!}\left|x - \frac{\pi}{6}\right|^4$ with $|f^{(4)}(x)| = |\sin x| \le M = 1$. Now $35° = \left(\frac{\pi}{6} + \frac{\pi}{36}\right)$ radians, so the error is $\left|R_3\left(\frac{\pi}{36}\right)\right| \le \dfrac{\left(\frac{\pi}{36}\right)^4}{4!} < 0.000003$. Therefore, to five decimal places, $\sin 35° \approx \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{\pi}{36}\right) - \frac{1}{4}\left(\frac{\pi}{36}\right)^2 - \frac{\sqrt{3}}{12}\left(\frac{\pi}{36}\right)^3 \approx 0.57358$.

24. From Exercise 16, $\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3 + \frac{1}{48}\left(x - \frac{\pi}{3}\right)^4 + R_4(x)$. Now since $x = 69° = \left(\frac{\pi}{3} + \frac{\pi}{20}\right)$ radians, the error is $|R_4(x)| \le \dfrac{\left(\frac{\pi}{20}\right)^5}{5!} < 8 \times 10^{-7}$. Therefore, to five decimal places, $\cos 69° \approx \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\pi}{20}\right) - \frac{1}{4}\left(\frac{\pi}{20}\right)^2 + \frac{\sqrt{3}}{12}\left(\frac{\pi}{20}\right)^3 + \frac{1}{48}\left(\frac{\pi}{20}\right)^4 \approx 0.35837$.

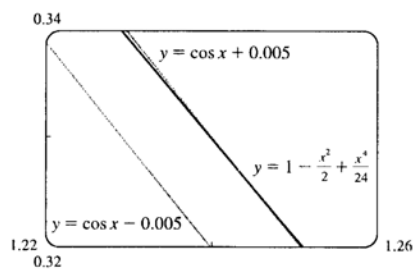
25. All derivatives of e^x are e^x, so $|R_n(x)| \le \dfrac{e^x}{(n+1)!}|x|^{n+1}$, where $0 < x < 0.1$. Letting $x = 0.1$, $R_n(0.1) \le \dfrac{e^{0.1}}{(n+1)!}(0.1)^{n+1} < 0.00001$, and by trial and error we find that $n = 3$ satisfies this inequality since $R_3(0.1) < 0.0000046$. Thus, by adding the three terms of the Maclaurin series for e^x corresponding to $n = 0$, 1, and 2, we can estimate $e^{0.1}$ to within 0.00001.

26. From Exercise 12.10.35, the Maclaurin series for $\ln(1+x)$ is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$. So $\ln 1.4 = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(0.4)^n$. Since this is an alternating series, the error is less than the first neglected term by the Alternating Series Estimation Theorem, and we find that $|a_6| = (0.4)^6/6 \approx 0.0007 < 0.001$. So we need the first five (non-zero) terms of the Maclaurin series for the desired accuracy.

27. $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$. By the Alternating Series Estimation Theorem, the error in the approximation $\sin x = x - \frac{1}{3!}x^3$ is less than $\left|\frac{1}{5!}x^5\right| < 0.01 \quad \Leftrightarrow \quad \left|x^5\right| < 120\,(0.01) \quad \Leftrightarrow$ $|x| < (1.2)^{1/5} \approx 1.037$. The curves intersect at $x \approx 1.043$, so the graph confirms our estimate. Since both the sine function and the given approximation are odd functions, we need to check the estimate only for $x > 0$.



28. $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$. By the Alternating Series Estimation Theorem, the error is less than $\left|-\frac{1}{6!}x^6\right| < 0.005 \quad \Leftrightarrow \quad x^6 < 3.6 \quad \Leftrightarrow$ $|x| < (3.6)^{1/6} \approx 1.238$. The curves intersect at $x \approx 1.244$, so the graph confirms our estimate. Since both the cosine function and the given approximation are even functions, we need to check the estimate only for $x > 0$.



29. Let $s\,(t)$ be the position function of the car, and for convenience set $s\,(0) = 0$. The velocity of the car is $v\,(t) = s'\,(t)$ and the acceleration is $a\,(t) = s''\,(t)$, so the second degree Taylor polynomial is $T_2\,(t) = s\,(0) + v\,(0)\,t + \dfrac{a\,(0)}{2}t^2 = 20t + t^2$. We estimate the distance travelled during the next second to be $s\,(1) \approx T_2\,(1) = 20 + 1 = 21$ m. The function $T_2\,(t)$ would not be accurate over a full minute, since the car could not possibly maintain an acceleration of 2 m/s^2 for that long (if it did, its final speed would be 140 m/s ≈ 315 mi/h!)

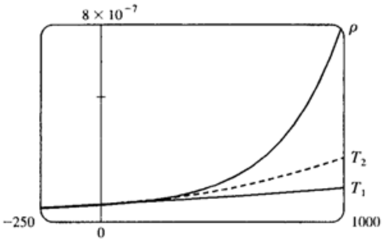
30. (a)

$$\rho(t) = \rho_{20} e^{\alpha(t-20)} \qquad \rho(20) = \rho_{20}$$
$$\rho'(t) = \alpha \rho_{20} e^{\alpha(t-20)} \qquad \rho'(20) = \alpha \rho_{20}$$
$$\rho''(t) = \alpha^2 \rho_{20} e^{\alpha(t-20)} \qquad \rho''(20) = \alpha^2 \rho_{20}$$

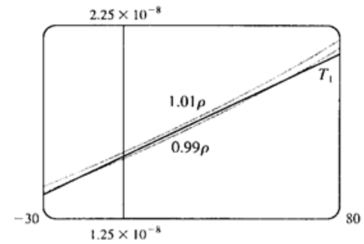
The linear approximation is $T_1(t) = \rho(20) + \rho'(20)(t-20) = \rho_{20}[1 + \alpha(t-20)]$. The quadratic approximation is

$$T_2(t) = \rho(20) + \rho'(20)(t-20) + \frac{\rho''(20)}{2}(t-20)^2 = \rho_{20}\left[1 + \alpha(t-20) + \tfrac{1}{2}\alpha^2(t-20)^2\right]$$

(b)



(c)



From the graph, it seems that $T_1(t)$ is within 1% of $\rho(t)$, that is, $0.99\rho(t) \le T_1(t) \le 1.01\rho(t)$, for $-14\,°\mathrm{C} \le t \le 58\,°\mathrm{C}$.

31. $E = \dfrac{q}{D^2} - \dfrac{q}{(D+d)^2} = \dfrac{q}{D^2} - \dfrac{q}{D^2(1+d/D)^2} = \dfrac{q}{D^2}\left[1 - \left(1 + \dfrac{d}{D}\right)^{-2}\right].$

We use the Binomial Series to expand $(1 + d/D)^{-2}$:

$$E = \frac{q}{D^2}\left[1 - \left(1 - 2\left(\frac{d}{D}\right) + \frac{2\cdot 3}{2!}\left(\frac{d}{D}\right)^2 - \frac{2\cdot 3\cdot 4}{3!}\left(\frac{d}{D}\right)^3 + \cdots\right)\right]$$
$$= \frac{q}{D^2}\left[2\left(\frac{d}{D}\right) - 3\left(\frac{d}{D}\right)^2 + 4\left(\frac{d}{D}\right)^3 - \cdots\right] \approx 2qd \cdot \frac{1}{D^3}$$

when D is much larger than d, that is, when P is far away from the dipole.

32. (a) $\dfrac{n_1}{\ell_o} + \dfrac{n_2}{\ell_i} = \dfrac{1}{R}\left(\dfrac{n_2 s_i}{\ell_i} - \dfrac{n_1 s_o}{\ell_o}\right)$ (Equation 1) where

$$\ell_o = \sqrt{R^2 + (s_o + R)^2 - 2R(s_o + R)\cos\phi} \quad\text{and}\quad \ell_i = \sqrt{R^2 + (s_i - R)^2 + 2R(s_i - R)\cos\phi}$$

Using $\cos\phi \approx 1$ gives

$$\ell_o = \sqrt{R^2 + (s_o + R)^2 - 2R(s_o + R)} = \sqrt{R^2 + s_o^2 + 2Rs_o + R^2 - 2Rs_o - 2R^2} = \sqrt{s_o^2} = s_o$$

and similarly, $\ell_i = s_i$. Thus, Equation 1 becomes $\dfrac{n_1}{s_o} + \dfrac{n_2}{s_i} = \dfrac{1}{R}\left(\dfrac{n_2 s_i}{s_i} - \dfrac{n_1 s_o}{s_o}\right) \Rightarrow \dfrac{n_1}{s_o} + \dfrac{n_2}{s_i} = \dfrac{n_2 - n_1}{R}$.

(b) Using $\cos\phi \approx 1 - \frac{1}{2}\phi^2$ in (2) gives us

$$\ell_o = \sqrt{R^2 + (s_o + R)^2 - 2R\,(s_o + R)\left(1 - \tfrac{1}{2}\phi^2\right)}$$
$$= \sqrt{R^2 + s_o^2 + 2Rs_o + R^2 - 2Rs_o + Rs_o\phi^2 - 2R^2 + R^2\phi^2} = \sqrt{s_o^2 + Rs_o\phi^2 + R^2\phi^2}$$

Anticipating that we will use the binomial series expansion $(1+x)^k \approx 1 + kx$, we can write the last expression as $s_o\sqrt{1 + \phi^2\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)}$ and similarly, $\ell_i = s_i\sqrt{1 - \phi^2\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)}$. Thus,

$$\frac{n_1}{\ell_o} + \frac{n_2}{\ell_i} = n_1\ell_o^{-1} + n_2\ell_i^{-1} = \frac{1}{R}\left(\frac{n_2 s_i}{\ell_i} - \frac{n_1 s_o}{\ell_o}\right) \quad\Leftrightarrow$$

$$\frac{n_1}{s_o}\left[1 + \phi^2\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)\right]^{-1/2} + \frac{n_2}{s_i}\left[1 - \phi^2\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)\right]^{-1/2}$$
$$= \frac{n_2}{R}\left[1 - \phi^2\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)\right]^{-1/2} - \frac{n_1}{R}\left[1 + \phi^2\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)\right]^{-1/2}$$

Approximating the expressions for ℓ_o^{-1} and ℓ_o^{-1} by the first two terms in their binomial series, we get

$$\frac{n_1}{s_o}\left[1 - \tfrac{1}{2}\phi^2\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)\right] + \frac{n_2}{s_i}\left[1 + \tfrac{1}{2}\phi^2\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)\right]$$
$$= \frac{n_2}{R}\left[1 + \tfrac{1}{2}\phi^2\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)\right] - \frac{n_1}{R}\left[1 - \tfrac{1}{2}\phi^2\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)\right] \quad\Leftrightarrow$$

$$\frac{n_1}{s_o} - \frac{n_1\phi^2}{2s_o}\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right) + \frac{n_2}{s_i} + \frac{n_2\phi^2}{2s_i}\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)$$
$$= \frac{n_2}{R} + \frac{n_2\phi^2}{2R}\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right) - \frac{n_1}{R} + \frac{n_1\phi^2}{2R}\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right) \quad\Leftrightarrow$$

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2}{R} - \frac{n_1}{R} + \frac{n_1\phi^2}{2s_o}\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right) + \frac{n_1\phi^2}{2R}\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)$$
$$+ \frac{n_2\phi^2}{2R}\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right) + \frac{n_2\phi^2}{2s_i}\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)$$
$$= \frac{n_2 - n_1}{R} + \frac{n_1\phi^2}{2}\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)\left(\frac{1}{s_o} + \frac{1}{R}\right) + \frac{n_2\phi^2}{2}\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)\left(\frac{1}{R} - \frac{1}{s_i}\right)$$
$$= \frac{n_2 - n_1}{R} + \frac{n_1\phi^2 R^2}{2s_o}\left(\frac{1}{R} + \frac{1}{s_o}\right)\left(\frac{1}{R} + \frac{1}{s_o}\right) + \frac{n_2\phi^2 R^2}{2s_i}\left(\frac{1}{R} - \frac{1}{s_i}\right)\left(\frac{1}{R} - \frac{1}{s_i}\right)$$
$$= \frac{n_2 - n_1}{R} + \phi^2 R^2\left[\frac{n_1}{2s_o}\left(\frac{1}{R} + \frac{1}{s_o}\right)^2 + \frac{n_2}{2s_i}\left(\frac{1}{R} - \frac{1}{s_i}\right)^2\right]$$

From Figure 7, we see that $\sin\phi = h/R$. So if we approximate $\sin\phi$ with ϕ, we get $h = R\phi$ and $h^2 = \phi^2 R^2$ and hence, Equation 4, as desired.

33. (a) If the water is deep, then $2\pi d/L$ is large, and we know that $\tanh x \to 1$ as $x \to \infty$. So we can approximate $\tanh(2\pi d/L) \approx 1$, and so $v^2 \approx gL/(2\pi) \quad \Leftrightarrow \quad v \approx \sqrt{gL/(2\pi)}$.

(b) From the calculations at right, the first term in the Maclaurin series of $\tanh x$ is x, so if the water is shallow, we can approximate $\tanh \dfrac{2\pi d}{L} \approx \dfrac{2\pi d}{L}$, and so $v^2 \approx \dfrac{gL}{2\pi} \cdot \dfrac{2\pi d}{L} \quad \Leftrightarrow \quad v \approx \sqrt{gd}$.

$f(x) = \tanh x$	$f(0) = 0$
$f'(x) = \operatorname{sech}^2 x$	$f'(0) = 1$
$f''(x) = -2\operatorname{sech}^2 x \tanh x$	$f''(0) = 0$
$f'''(x) = 2\operatorname{sech}^2 x\,(3\tanh^2 x - 1)$	$f'''(0) = -2$

(c) Since $\tanh x$ is an odd function, its Maclaurin series is alternating, so the error in the approximation $\tanh \dfrac{2\pi d}{L} \approx \dfrac{2\pi d}{L}$ is less than the first neglected term, which is $\dfrac{|f'''(0)|}{3!}\left(\dfrac{2\pi d}{L}\right)^3 = \dfrac{1}{3}\left(\dfrac{2\pi d}{L}\right)^3$. If $L > 10d$, then $\dfrac{1}{3}\left(\dfrac{2\pi d}{L}\right)^3 < \dfrac{1}{3}\left(2\pi \cdot \dfrac{1}{10}\right)^3 = \dfrac{\pi^3}{375}$, so the error in the approximation $v^2 = gd$ is less than $\dfrac{gL}{2\pi} \cdot \dfrac{\pi^3}{375} \approx 0.0132gL$.

34. $T_n(x) = f(a) + \dfrac{f'(a)}{1!}(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \cdots + \dfrac{f^{(n)}(a)}{n!}(x-a)^n$. Let $0 \le m \le n$. Then

$$T_n^{(m)}(x) = m!\,\frac{f^{(m)}(a)}{m!}(x-a)^0 + (m+1)(m)\cdots(2)\,\frac{f^{(m+1)}(a)}{(m+1)!}(x-a)^1 + \cdots$$
$$+\,n(n-1)\cdots(n-m+1)\,\frac{f^{(n)}(z)}{n!}(x-a)^{n-m}$$

For $x = a$, all terms in this sum except the first one are 0, so $T_n^{(m)}(a) = \dfrac{m!\,f^{(m)}(a)}{m!} = f^{(m)}(a)$.

35. Using $f(x) = T_n(x) + R_n(x)$ with $n = 1$ and $x = r$, we have $f(r) = T_1(r) + R_1(r)$, where T_1 is the first-degree Taylor polynomial of f at a. Because $a = x_n$, $f(r) = f(x_n) + f'(x_n)(r - x_n) + R_1(r)$. But r is a root of f, so $f(r) = 0$ and we have $0 = f(x_n) + f'(x_n)(r - x_n) + R_1(r)$. Taking the first two terms to the left side and dividing by $f'(x_n)$, we have $f'(x_n)(x_n - r) - f(x_n) = R_1(r) \quad \Rightarrow \quad x_n - r - \dfrac{f(x_n)}{f'(x_n)} = \dfrac{R_1(r)}{f'(x_n)}$. By the formula for Newton's method, the left side of the preceding equation is $x_{n+1} - r$, so $|x_{n+1} - r| = \left|\dfrac{R_1(r)}{f'(x_n)}\right|$.

Taylor's Inequality gives us $|R_1(r)| \le \dfrac{|f''(r)|}{2!}|r - x_n|^2$. Combining this inequality with the facts $|f''(x)| \le M$ and $|f'(x)| \ge K$ gives us $|x_{n+1} - r| \le \dfrac{M}{2K}|x_n - r|^2$.

Applied Project □ Radiation from the Stars

1. If we write $f(\lambda) = \dfrac{8\pi hc\lambda^{-5}}{e^{hc/(\lambda kT)} - 1} = \dfrac{a\lambda^{-5}}{e^{b/(\lambda T)} - 1}$, then as $\lambda \to 0^+$, it is of the form ∞/∞, and as $\lambda \to \infty$ it is of the form $0/0$, so in either case we can use l'Hospital's Rule. First of all,

$$\lim_{\lambda\to\infty} f(\lambda) \overset{\text{H}}{=} \lim_{\lambda\to\infty} \frac{a\left(-5\lambda^{-6}\right)}{-\dfrac{bT}{(\lambda T)^2}e^{b/(\lambda T)}} = 5\frac{aT}{b}\lim_{\lambda\to\infty}\frac{\lambda^2\lambda^{-6}}{e^{b/(\lambda T)}} = 5\frac{aT}{b}\lim_{\lambda\to\infty}\frac{\lambda^{-4}}{e^{b/(\lambda T)}} = 0$$

Also,

$$\lim_{\lambda\to 0^+} f(\lambda) \overset{\text{H}}{=} 5\frac{aT}{b}\lim_{\lambda\to 0^+}\frac{\lambda^{-4}}{e^{b/(\lambda T)}} \overset{\text{H}}{=} 5\frac{aT}{b}\lim_{\lambda\to 0^+}\frac{-4\lambda^{-5}}{-\dfrac{bT}{(\lambda T)^2}e^{b/(\lambda T)}} = 20\frac{aT^2}{b^2}\lim_{\lambda\to 0^+}\frac{\lambda^{-3}}{e^{b/(\lambda T)}}$$

This is still indeterminate, but note that each time we use l'Hospital's Rule, we gain a factor of λ in the numerator, as well as a constant factor, and the denominator is unchanged. So if we use l'Hospital's Rule three more times, the exponent of λ in the numerator will become 0. That is, for some $\{k_i\}$, all constant,

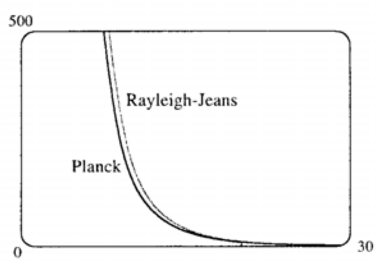
$$\lim_{\lambda\to 0^+} f(\lambda) \overset{\text{H}}{=} k_1\lim_{\lambda\to 0^+}\frac{\lambda^{-3}}{e^{b/(\lambda T)}} \overset{\text{H}}{=} k_2\lim_{\lambda\to 0^+}\frac{\lambda^{-2}}{e^{b/(\lambda T)}} \overset{\text{H}}{=} k_3\lim_{\lambda\to 0^+}\frac{\lambda^{-1}}{e^{b/(\lambda T)}} \overset{\text{H}}{=} k_4\lim_{\lambda\to 0^+}\frac{1}{e^{b/(\lambda T)}} = 0$$

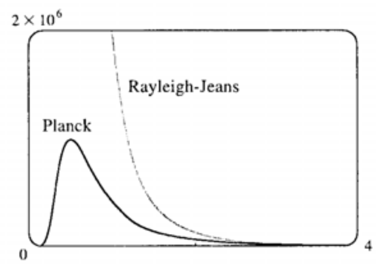
2. We expand the denominator of Planck's Law using the Taylor series $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$ with $x = \dfrac{hc}{\lambda kT}$, and use the fact that if λ is large, then all subsequent terms in the Taylor expansion are very small compared to the first one, so we can approximate using the Taylor polynomial T_1:

$$f(\lambda) = \frac{8\pi hc\lambda^{-5}}{e^{hc/(\lambda kT)} - 1} = \frac{8\pi hc\lambda^{-5}}{\left[1 + \dfrac{hc}{\lambda kT} + \dfrac{1}{2!}\left(\dfrac{hc}{\lambda kT}\right)^2 + \dfrac{1}{3!}\left(\dfrac{hc}{\lambda kT}\right)^3 + \cdots\right] - 1} \approx \frac{8\pi hc\lambda^{-5}}{\left(1 + \dfrac{hc}{\lambda kT}\right) - 1} = \frac{8\pi kT}{\lambda^4}$$

which is the Rayleigh-Jeans Law.

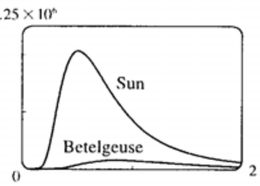
3. To convert to μm, we substitute $\lambda/10^6$ for λ in both laws. The first figure shows that the two laws are similar for large λ. The second figure shows that the two laws are very different for short wavelengths (Planck's Law gives a maximum at $\lambda \approx 0.51$ μm; the Rayleigh-Jeans Law gives no minimum or maximum.).

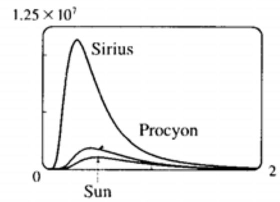




4. From the graph in Problem 3, $f(\lambda)$ has a maximum under Planck's Law at $\lambda \approx 0.51$ μm.

5.





As T gets larger, the total area under the curve increases, as we would expect: the hotter the star, the more energy it emits. Also, as T increases, the λ-value of the maximum decreases, so the higher the temperature, the shorter the peak wavelength (and consequently the average wavelength) of light emitted. This is why Sirius is a blue star and Betelgeuse is a red star: most of Sirius's light is of a fairly short wavelength, that is, a higher frequency, toward the blue end of the spectrum, whereas most of Betelgeuse's light is of a lower frequency, toward the red end of the spectrum.

12 Review

CONCEPT CHECK

1. (a) See Definition 12.1.1.

(b) See Definition 12.2.2.

(c) The terms of the sequence $\{a_n\}$ approach 3 as n becomes large.

(d) By adding sufficiently many terms of the series, we can make the partial sums as close to 3 as we like.

2. (a) See Definition 12.1.9.

(b) A sequence is monotonic if it is either increasing or decreasing.

(c) By Theorem 12.1.10, every bounded, monotonic sequence is convergent.

3. (a) See (4) in Section 12.2.

(b) See (1) in Section 12.3.

4. If $\sum a_n = 3$, then $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} s_n = 3$.

5. (a) See the Test for Divergence (12.2.7).

(b) See the Integral Test on page 749.

(c) See the Comparison Test on page 756.

(d) See the Limit Comparison Test on page 757.

(e) See the Alternating Series Test on page 761.

(f) See the Ratio Test on page 767.

(g) See the Root Test on page 769.

6. (a) See Definition 12.6.1.

(b) By (12.6.3), it is convergent.

(c) See Definition 12.6.2.

7. (a) Use either (2) or (3) in Section 12.3.

(b) See Example 5 in Section 12.4.

(c) By adding terms until you reach the desired accuracy given by the Alternating Series Estimation Theorem on page 763.

8. (a) $\sum_{n=0}^{\infty} c_n (x-a)^n$

(b) Given the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, the radius of convergence is:

(i) 0 if the series converges only when $x = a$

(ii) ∞ if the series converges for all x, or

(iii) a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

(c) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (b), the interval of convergence is: (i) the single point $\{a\}$, (ii) all real numbers, that is, the real number line $(-\infty, \infty)$, or (iii) an interval with endpoints $a - R$ and $a + R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.

9. (a), (b) See Theorem 12.9.2.

10. (a) $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$

(b) $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

(c) $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ [$a = 0$ in part (b)]

(d) See Theorem 12.10.8.

(e) See Taylor's Inequality (12.10.9).

11. (a) – (e) See the table on page 792.

12. See the Binomial Series (12.11.2) for the expansion. The radius of convergence for the binomial series is 1.

TRUE-FALSE QUIZ

1. False. See Note 2 after Theorem 12.2.6.

2. True by Theorem 12.8.3.
Or: Use the Comparison Test to show that $\sum c_n (-2)^n$ converges absolutely.

3. False. For example, take $c_n = (-1)^n / (n6^n)$.

4. True by Theorem 12.8.3.

5. False, since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n^3}{(n+1)^3}\right| = \lim_{n\to\infty}\frac{1}{(1+1/n)^3} = 1.$

6. True, since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty}\frac{1}{n+1} = 0 < 1.$

7. False. See the note after Example 2 in Section 12.4.

8. True, since $\frac{1}{e} = e^{-1}$ and $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$, so $e^{-1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$.

9. True. See (7) in Section 12.1.

10. True, because if $\sum |a_n|$ is convergent, then so is $\sum a_n$ by Theorem 12.6.3.

11. True. By Theorem 12.10.5 the coefficient of x^3 is $\dfrac{f'''(0)}{3!} = \dfrac{1}{3} \Rightarrow f'''(0) = 2$.
Or: Use Theorem 12.9.2 to differentiate f three times.

12. False. Let $a_n = n$ and $b_n = -n$. Then $\{a_n\}$ and $\{b_n\}$ are divergent, but $a_n + b_n = 0$, so $\{a_n + b_n\}$ is convergent.

13. False. For example, let $a_n = b_n = (-1)^n$. Then $\{a_n\}$ and $\{b_n\}$ are divergent, but $a_n b_n = 1$, so $\{a_n b_n\}$ is convergent.

14. True. by Theorem 12.1.10 (the Monotonic Sequence Theorem), since $\{a_n\}$ is decreasing and $0 < a_n \le a_1$ for all $n \Rightarrow \{a_n\}$ is bounded.

15. True by Theorem 12.6.3. $\left[\sum (-1)^n a_n\right.$ is absolutely convergent and hence convergent.$\left.\right]$

16. True. $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} < 1 \Rightarrow \sum a_n$ converges (Ratio Test) $\Rightarrow \lim\limits_{n\to\infty} a_n = 0$ (Theorem 12.2.6).

EXERCISES

1. $\left\{\dfrac{2+n^3}{1+2n^3}\right\}$ converges since $\lim\limits_{n\to\infty} \dfrac{2+n^3}{1+2n^3} = \lim\limits_{n\to\infty} \dfrac{2/n^3+1}{1/n^3+2} = \dfrac{1}{2}$.

2. $a_n = \dfrac{9^{n+1}}{10^n} = 9 \cdot \left(\frac{9}{10}\right)^n$, so $\lim\limits_{n\to\infty} a_n = 9 \lim\limits_{n\to\infty} \left(\frac{9}{10}\right)^n = 9 \cdot 0 = 0$ by (12.1.7).

3. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{n^3}{1+n^2} = \lim\limits_{n\to\infty} \dfrac{n}{1/n^2+1} = \infty$, so the sequence diverges.

4. $\left\{\dfrac{n}{\ln n}\right\}$ diverges, since $\lim\limits_{x\to\infty} \dfrac{x}{\ln x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1}{1/x} = \lim\limits_{x\to\infty} x = \infty$.

5. $\{\sin n\}$ is divergent since $\lim\limits_{n\to\infty} \sin n$ does not exist.

6. $\left\{\dfrac{\sin n}{n}\right\}$ converges, since $-\dfrac{1}{n} \le \dfrac{\sin n}{n} \le \dfrac{1}{n}$ and $\pm\dfrac{1}{n} \to 0$ as $n \to \infty$, so $\lim\limits_{n\to\infty} \dfrac{\sin n}{n} = 0$ by the Squeeze Theorem.

7. $\left\{\left(1+\dfrac{3}{n}\right)^{4n}\right\}$ is convergent. Let $y = \left(1+\dfrac{3}{x}\right)^{4x}$. Then

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} 4x \ln(1+3/x) = \lim_{x\to\infty} \frac{\ln(1+3/x)}{1/(4x)} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\dfrac{1}{1+3/x}\left(-\dfrac{3}{x^2}\right)}{-1/(4x^2)} = \lim_{x\to\infty} \frac{12}{1+3/x} = 12$$

so $\lim\limits_{x\to\infty} y = \lim\limits_{n\to\infty} \left(1+\dfrac{3}{n}\right)^{4n} = e^{12}$.

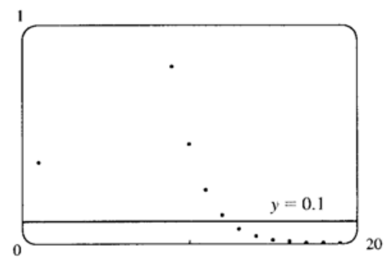
8. $\left\{\dfrac{(-10)^n}{n!}\right\}$ converges, since $\dfrac{10^n}{n!} = \dfrac{10 \cdot 10 \cdot 10 \cdot \cdots \cdot 10}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot 10} \cdot \dfrac{10 \cdot 10 \cdot \cdots \cdot 10}{11 \cdot 12 \cdot \cdots \cdot n} \le 10^{10} \left(\dfrac{10}{11}\right)^{n-10} \to 0$ as $n \to \infty$, so $\lim\limits_{n\to\infty} \dfrac{(-10)^n}{n!} = 0$ (Squeeze Theorem). *Or:* Use (12.10.10).

9. We use induction, hypothesizing that $a_{n-1} < a_n < 2$. Note first that $1 < a_2 = \frac{1}{3}(1+5) = \frac{5}{3} < 2$, so the hypothesis holds for $n = 2$. Now assume that $a_{k-1} < a_k < 2$. Then
$a_k = \frac{1}{3}(a_{k-1}+4) < \frac{1}{3}(a_k+4) < \frac{1}{3}(2+4) = 2$. So $a_k < a_{k+1} < 2$, and the induction is complete. To find the limit of the sequence, we note that $L = \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1} \Rightarrow L = \frac{1}{3}(L+4) \Rightarrow L = 2$.

10. $\lim\limits_{x\to\infty} \dfrac{x^4}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{4x^3}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{12x^2}{e^x}$

$\overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{24x}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{24}{e^x} = 0$

Then we conclude from Theorem 12.1.2 that $\lim\limits_{n\to\infty} n^4e^{-n} = 0$. From the graph, it seems that $12^4e^{-12} > 0.1$, but $n^4e^{-n} < 0.1$ whenever $n > 12$. So the smallest value of N corresponding to $\varepsilon = 0.1$ in the definition of the limit is $N = 12$.



11. $\dfrac{n}{n^3+1} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$, so $\sum\limits_{n=1}^{\infty} \dfrac{n}{n^3+1}$ converges by the Comparison Test with the convergent p-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ $(p = 2 > 1)$.

12. Let $a_n = \dfrac{n^2+1}{n^3+1}$ and $b_n = \dfrac{1}{n}$, so $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^3+n}{n^3+1} = \lim\limits_{n\to\infty} \dfrac{1+1/n^2}{1+1/n^3} = 1 > 0$. Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} a_n$ also diverges by the Limit Comparison Test.

13. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left[\dfrac{(n+1)^3}{5^{n+1}} \cdot \dfrac{5^n}{n^3}\right] = \lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right)^3 \cdot \dfrac{1}{5} = \dfrac{1}{5} < 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n^3}{5^n}$ converges by the Ratio Test.

14. Let $b_n = \dfrac{1}{\sqrt{n+1}}$. Then b_n is positive for $n \ge 1$, the sequence $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt{n+1}}$ converges by the Alternating Series Test.

15. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \dfrac{n}{3n+1} = \dfrac{1}{3} < 1$, so the series converges by the Root Test.

16. $\lim\limits_{n\to\infty} \dfrac{n}{3n+1} = \dfrac{1}{3}$, so $\lim\limits_{n\to\infty} \ln\left(\dfrac{n}{3n+1}\right) = \ln\frac{1}{3} \neq 0$. Thus, $\sum\limits_{n=1}^{\infty} \ln\left(\dfrac{n}{3n+1}\right)$ diverges by the Test for Divergence.

17. $\left|\dfrac{\sin n}{1+n^2}\right| \le \dfrac{1}{1+n^2} < \dfrac{1}{n^2}$ and since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ converges (p-series with $p = 2 > 1$), so does $\sum\limits_{n=1}^{\infty} \left|\dfrac{\sin n}{1+n^2}\right|$ by the Comparison Test, and so does $\sum\limits_{n=1}^{\infty} \dfrac{\sin n}{1+n^2}$ by Theorem 12.6.3.

18. $f(x) = \dfrac{1}{x(\ln x)^2}$ is continuous, positive, and decreasing on $(2, \infty)$, so we can use the Integral Test.

$\displaystyle\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{t\to\infty} \left[\frac{-1}{\ln x}\right]_2^t = \frac{1}{\ln 2}$, so the series converges.

19. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)(2n+1)}{5^{n+1}(n+1)!} \cdot \dfrac{5^n n!}{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)} = \lim\limits_{n\to\infty} \dfrac{2n+1}{5(n+1)} = \dfrac{2}{5} < 1$, so the series converges by the Ratio Test.

20. $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^{2n}}{n^2 9^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{25}{9}\right)^n$. Now $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n^2}{(n+1)^2}\left(\frac{25}{9}\right)^{n+1}\left(\frac{9}{25}\right)^n = \frac{25}{9} > 1$, so the series diverges by the Ratio Test.

21. Let $b_n = \dfrac{\sqrt{n}}{n+1} > 0$. Then $0 \le \displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{\sqrt{n}}{n+1} \le \lim_{n\to\infty}\frac{\sqrt{n}}{n} = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$, so $\displaystyle\lim_{n\to\infty} b_n = 0$. If

$f(x) = \dfrac{\sqrt{x}}{x+1}$ for $x > 0$, then $f'(x) = \dfrac{(x+1)\cdot\frac{1}{2\sqrt{x}} - \sqrt{x}\cdot 1}{(x+1)^2} = \dfrac{(x+1)-2x}{2\sqrt{x}\,(x+1)^2} = \dfrac{1-x}{2\sqrt{x}\,(x+1)^2}$, so $f'(x) < 0$

for $x > 1$. It follows that $f(1) > f(2) > f(3) > \cdots$; that is, $b_n > b_{n+1}$ for all n. Thus, $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{n}}{n+1}$

converges by the Alternating Series Test.

22. Use the Limit Comparison Test with $a_n = \dfrac{\sqrt{n+1}-\sqrt{n-1}}{n} = \dfrac{2}{n\left(\sqrt{n+1}+\sqrt{n-1}\right)}$ (rationalizing the

numerator) and $b_n = \dfrac{1}{n^{3/2}}$. $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2\sqrt{n}}{\sqrt{n+1}+\sqrt{n-1}} = 1$, so since $\sum_{n=1}^{\infty} b_n$ converges ($p = \frac{3}{2} > 1$), $\sum_{n=1}^{\infty} a_n$ converges also.

23. Consider the series of absolute values: $\sum_{n=1}^{\infty} n^{-1/3}$ is a p-series with $p = \frac{1}{3} < 1$ and is therefore divergent. But if we apply the Alternating Series Test we see that $a_{n+1} < a_n$ and $\displaystyle\lim_{n\to\infty} n^{-1/3} = 0$. Therefore $\sum_{n=1}^{\infty}(-1)^{n-1} n^{-1/3}$ is conditionally convergent.

24. $\sum_{n=1}^{\infty}\left|(-1)^{n-1} n^{-3}\right| = \sum_{n=1}^{\infty} n^{-3}$ is a convergent p-series ($p = 3 > 1$.) Therefore, $\sum_{n=1}^{\infty}(-1)^{n-1} n^{-3}$ is absolutely convergent.

25. $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{(-1)^{n+1}(n+2)\,3^{n+1}}{2^{2n+3}}\cdot\dfrac{2^{2n+1}}{(-1)^n(n+1)\,3^n}\right| = \dfrac{n+2}{n+1}\cdot\dfrac{3}{4} = \dfrac{1+(2/n)}{1+(1/n)}\cdot\dfrac{3}{4} \to \dfrac{3}{4} < 1$ as $n\to\infty$, so by the Ratio Test, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)\,3^n}{2^{2n+1}}$ is absolutely convergent.

26. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x}}{\ln x} \overset{H}{=} \lim_{x\to\infty}\frac{1/(2\sqrt{x})}{1/x} = \lim_{x\to\infty}\frac{\sqrt{x}}{2} = \infty$. Therefore, $(-1)^{n+1}\dfrac{\sqrt{n}}{\ln n}$ does not approach 0, so the given series is divergent by the Test for Divergence.

27. Convergent geometric series. $\displaystyle\sum_{n=1}^{\infty}\frac{2^{2n+1}}{5^n} = \sum_{n=1}^{\infty}\frac{\left(2^2\right)^n\cdot 2^1}{5^n} = 2\sum_{n=1}^{\infty}\frac{4^n}{5^n} = 2\left(\frac{\frac{4}{5}}{1-\frac{4}{5}}\right) = 8.$

28. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+3)} = \sum_{n=1}^{\infty}\left[\frac{1}{3n} - \frac{1}{3(n+3)}\right]$ (partial fractions).

$\displaystyle s_n = \sum_{i=1}^{n}\left[\frac{1}{3i} - \frac{1}{3(i+3)}\right] = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} - \frac{1}{3(n+1)} - \frac{1}{3(n+2)} - \frac{1}{3(n+3)}$ (telescoping sum), so

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+3)} = \lim_{n\to\infty} s_n = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{11}{18}.$

29.
$$\begin{aligned}\sum_{n=1}^{\infty}\left[\tan^{-1}(n+1) - \tan^{-1} n\right] &= \lim_{n\to\infty}\left[\left(\tan^{-1}2 - \tan^{-1}1\right) + \left(\tan^{-1}3 - \tan^{-1}2\right) + \cdots\right. \\ &\qquad\qquad \left. + \left(\tan^{-1}(n+1) - \tan^{-1}n\right)\right] \\ &= \lim_{n\to\infty}\left[\tan^{-1}(n+1) - \tan^{-1}1\right] = \tfrac{\pi}{2} - \tfrac{\pi}{4} = \tfrac{\pi}{4}\end{aligned}$$

30. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{2n} n!} = \sum_{n=0}^{\infty} \frac{(-x/4)^n}{n!} = e^{-x/4}$

31. $1.2 + 0.0\overline{345} = \dfrac{12}{10} + \dfrac{345/10{,}000}{1 - 1/1000} = \dfrac{12}{10} + \dfrac{345}{9990} = \dfrac{4111}{3330}$

32. This is a geometric series which converges whenever $|\ln x| < 1 \;\Rightarrow\; -1 < \ln x < 1 \;\Rightarrow\; e^{-1} < x < e$.

33. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} = 1 - \frac{1}{32} + \frac{1}{243} - \frac{1}{1024} + \frac{1}{3125} - \frac{1}{7776} + \frac{1}{16{,}807} - \frac{1}{32{,}768} + \cdots$. Since $\dfrac{1}{32{,}768} < 0.000031$, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} \approx \sum_{n=1}^{7} \frac{(-1)^{n+1}}{n^5} \approx 0.9721$.

34. (a) $s_5 = \displaystyle\sum_{n=1}^{5} \frac{1}{n^6} = 1 + \frac{1}{2^6} + \cdots + \frac{1}{5^6} \approx 1.017305$. The series $\displaystyle\sum_{n=1}^{5} \frac{1}{n^6}$ converges by the Integral Test, so we estimate the remainder R_5 with (12.3.2): $R_5 \le \displaystyle\int_5^{\infty} \frac{dx}{x^6} = \left[-\frac{x^{-5}}{5}\right]_5^{\infty} = \frac{5^{-5}}{5} = 0.000064$. So the error is at most 0.000064.

(b) In general, $R_n \le \displaystyle\int_n^{\infty} \frac{dx}{x^6} = \frac{1}{5n^5}$. If we take $n = 9$, then $s_9 \approx 1.01734$ and $R_9 \le \dfrac{1}{5 \cdot 9^5} \approx 3.4 \times 10^{-6}$. So to five decimal places, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5} \approx \sum_{n=1}^{9} \frac{1}{n^5} \approx 1.01734$.

Another Method: Use (12.3.3) instead of (12.3.2).

35. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2 + 5^n} \approx \sum_{n=1}^{8} \frac{1}{2 + 5^n} \approx 0.18976224$. To estimate the error, note that $\dfrac{1}{2 + 5^n} < \dfrac{1}{5^n}$, so the remainder term is $R_8 = \displaystyle\sum_{n=9}^{\infty} \frac{1}{2 + 5^n} < \sum_{n=9}^{\infty} \frac{1}{5^n} = \frac{1/5^9}{1 - 1/5} = 6.4 \times 10^{-7}$ (geometric series with $a = \frac{1}{5^9}$ and $r = \frac{1}{5}$).

36. (a) $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^{n+1}\,(2n)!}{(2n+2)!\,n^n} = \lim_{n\to\infty} \frac{(n+1)^n\,(n+1)^1}{(2n+2)\,(2n+1)\,n^n}$$
$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \frac{1}{2\,(2n+1)} = e \cdot 0 = 0 < 1$$
so the series converges by the Ratio Test.

(b) The series in part (a) is convergent, so $\lim\limits_{n\to\infty} a_n = 0$ by Theorem 12.2.6.

37. Use the Limit Comparison Test. $\displaystyle\lim_{n\to\infty} \left|\frac{\left(\frac{n+1}{n}\right) a_n}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1 > 0$. Since $\sum |a_n|$ is convergent, so is $\sum \left|\left(\dfrac{n+1}{n}\right) a_n\right|$, by the Limit Comparison Test.

38. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+1)^2\,5^{n+1}}\cdot\dfrac{n^2 5^n}{x^n}\right| = \lim\limits_{n\to\infty}\dfrac{1}{(1+1/n)^2}\dfrac{|x|}{5} = \dfrac{|x|}{5}$, so by the Ratio Test, $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{x^n}{n^2 5^n}$ converges when $|x| < 5$. $R = 5$. When $x = -5$, the series becomes the convergent p-series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ with $p = 2 > 1$. When $x = 5$, the series becomes $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}$, which converges by the Alternating Series Test. Thus, $I = [-5, 5]$.

39. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left[\dfrac{|x+2|^{n+1}}{(n+1)\,4^{n+1}}\cdot\dfrac{n4^n}{|x+2|^n}\right] = \lim\limits_{n\to\infty}\left[\dfrac{n}{n+1}\,\dfrac{|x+2|}{4}\right] = \dfrac{|x+2|}{4} < 1 \;\Leftrightarrow\; |x+2| < 4$, so $R = 4$. $|x+2| < 4 \;\Leftrightarrow\; -4 < x+2 < 4 \;\Leftrightarrow\; -6 < x < 2$. If $x = -6$, then the series becomes $\sum\limits_{n=1}^{\infty}\dfrac{(-4)^n}{n4^n} = \sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n}$, the alternating harmonic series, which converges by the Alternating Series Test. When $x = 2$, the series becomes the harmonic series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$, which diverges. Thus, $I = [-6, 2)$.

40. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{2^{n+1}(x-2)^{n+1}}{(n+3)!}\cdot\dfrac{(n+2)!}{2^n(x-2)^n}\right| = \lim\limits_{n\to\infty}\dfrac{2}{n+3}|x-2| = 0 < 1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{2^n(x-2)^n}{(n+2)!}$ converges for all x. $R = \infty$ and $I = \mathbb{R}$.

41. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+4}}\cdot\dfrac{\sqrt{n+3}}{2^n(x-3)^n}\right| = 2|x-3|\lim\limits_{n\to\infty}\sqrt{\dfrac{n+3}{n+4}} = 2|x-3| < 1 \;\Leftrightarrow\; |x-3| < \frac{1}{2}$, so $R = \frac{1}{2}$. For $x = \frac{7}{2}$, the series becomes $\sum\limits_{n=0}^{\infty}\dfrac{1}{\sqrt{n+3}} = \sum\limits_{n=3}^{\infty}\dfrac{1}{n^{1/2}}$, which diverges ($p = \frac{1}{2} \le 1$), but for $x = \frac{5}{2}$, we get $\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{\sqrt{n+3}}$, which is a convergent alternating series, so $I = \left[\frac{5}{2}, \frac{7}{2}\right)$.

42. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{(2n+2)(2n+1)}{(n+1)(n+1)}|x| = 4|x| < 1$ to converge, so $R = \frac{1}{4}$.

43.

$f(x) = \sin x$	$f\left(\frac{\pi}{6}\right) = \frac{1}{2}$	$f'''(x) = -\cos x$	$f'''\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$
$f'(x) = \cos x$	$f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$	$f^{(4)}(x) = \sin x$	$f^{(4)}\left(\frac{\pi}{6}\right) = \frac{1}{2}$
$f''(x) = -\sin x$	$f''\left(\frac{\pi}{6}\right) = -\frac{1}{2}$	$\cdots$	$\cdots$

$f^{(2n)}\left(\frac{\pi}{6}\right) = (-1)^n\cdot\frac{1}{2}$ and $f^{(2n+1)}\left(\frac{\pi}{6}\right) = (-1)^n\cdot\frac{\sqrt{3}}{2}$.

$$\sin x = \sum_{n=0}^{\infty}\frac{f^{(n)}\left(\frac{\pi}{6}\right)}{n!}\left(x-\frac{\pi}{6}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2\,(2n)!}\left(x-\frac{\pi}{6}\right)^{2n} + \sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{3}}{2\,(2n+1)!}\left(x-\frac{\pi}{6}\right)^{2n+1}$$

44.

$f(x) = \cos x$	$f\left(\frac{\pi}{3}\right) = \frac{1}{2}$	$f'''(x) = \sin x$	$f'''\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$
$f'(x) = -\sin x$	$f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$	$f^{(4)}(x) = \cos x$	$f^{(4)}\left(\frac{\pi}{3}\right) = \frac{1}{2}$
$f''(x) = -\cos x$	$f''\left(\frac{\pi}{3}\right) = -\frac{1}{2}$		

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n\left(x-\frac{\pi}{3}\right)^{2n}}{2\,(2n)!} + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}\sqrt{3}\left(x-\frac{\pi}{3}\right)^{2n+1}}{2\,(2n+1)!}$$

45. $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum\limits_{n=0}^{\infty} (-1)^n x^n$ for $|x| < 1 \Rightarrow \dfrac{x^2}{1+x} = \sum\limits_{n=0}^{\infty} (-1)^n x^{n+2}$ with $R = 1$.

46. $\tan^{-1} x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$ with interval of convergence $[-1, 1]$, so

$\tan^{-1}(x^2) = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{(x^2)^{2n+1}}{2n+1} = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{4n+2}}{2n+1}$, which converges when $x^2 \in [-1, 1] \Leftrightarrow x \in [-1, 1]$.

Therefore, $R = 1$.

47. $\dfrac{1}{1-x} = \sum\limits_{n=0}^{\infty} x^n$ for $|x| < 1 \Rightarrow \ln(1-x) = -\displaystyle\int \frac{dx}{1-x} = -\int \sum_{n=0}^{\infty} x^n\,dx = C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$.

$\ln(1-0) = C - 0 \Rightarrow C = 0 \Rightarrow \ln(1-x) = -\sum\limits_{n=0}^{\infty} \dfrac{x^{n+1}}{n+1} = \sum\limits_{n=1}^{\infty} \dfrac{-x^n}{n}$ with $R = 1$.

48. $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!} \Rightarrow xe^{2x} = \sum\limits_{n=0}^{\infty} \dfrac{x\,(2x)^n}{n!} = \sum\limits_{n=0}^{\infty} \dfrac{2^n x^{n+1}}{n!}$, $R = \infty$

49. $\sin x = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin(x^4) = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n (x^4)^{2n+1}}{(2n+1)!} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{8n+4}}{(2n+1)!}$ for all x, so the radius of convergence is ∞.

50. $10^x = e^{x \ln 10} = \sum\limits_{n=0}^{\infty} \dfrac{(\ln 10)^n x^n}{n!}$, $R = \infty$

51. $f(x) = 1/\sqrt[4]{16-x} = (16-x)^{-1/4} = \frac{1}{2}\left(1 - \frac{1}{16}x\right)^{-1/4}$

$$= \frac{1}{2}\left[1 + \left(-\tfrac{1}{4}\right)\left(-\frac{x}{16}\right) + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{2!}\left(-\frac{x}{16}\right)^2 + \cdots\right]$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{2 \cdot 4^n \cdot n! \cdot 16^n} x^n = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{2^{6n+1}\, n!} x^n$$

for $\left|-\dfrac{x}{16}\right| < 1 \Rightarrow R = 16$.

52. $(1-3x)^{-5} = \sum\limits_{n=0}^{\infty} \dbinom{-5}{n} (-3x)^n = 1 + (-5)(-3x) + \dfrac{(-5)(-6)}{2!}(-3x)^2 + \dfrac{(-5)(-6)(-7)}{3!}(-3x)^3 + \cdots$

$$= 1 + \sum_{n=1}^{\infty} \frac{5 \cdot 6 \cdot \cdots \cdot (n+4) \cdot 3^n x^n}{n!}, \ |-3x| < 1 \text{ so } R = \tfrac{1}{3}.$$

53. $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$ so $\dfrac{e^x}{x} = \dfrac{1}{x} + \sum\limits_{n=1}^{\infty} \dfrac{x^{n-1}}{n!}$ and $\displaystyle\int \frac{e^x}{x}\,dx = C + \ln|x| + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$.

54. $(1+x^4)^{1/2} = \sum\limits_{n=0}^{\infty} \dbinom{\frac{1}{2}}{n} (x^4)^n = 1 + \left(\frac{1}{2}\right)x^4 + \dfrac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(x^4)^2 + \dfrac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(x^4)^3 + \cdots$

$$= 1 + \tfrac{1}{2}x^4 - \tfrac{1}{8}x^8 + \tfrac{1}{16}x^{12}$$

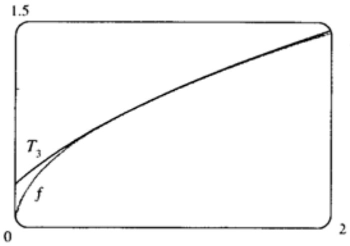
so $\int_0^1 (1+x^4)^{1/2}\,dx = \left[x + \frac{1}{10}x^5 - \frac{1}{72}x^9 + \frac{1}{208}x^{13} - \cdots\right]_0^1 = 1 + \frac{1}{10} - \frac{1}{72} + \frac{1}{208} - \cdots$. This is an alternating series, so by the Alternating Series Test, the error in the approximation $\int_0^1 (1+x^4)^{1/2}\,dx \approx 1 + \frac{1}{10} - \frac{1}{72} \approx 1.086$ is less than $\frac{1}{208}$, sufficient for the desired accuracy. Thus, correct to two decimal places, $\int_0^1 (1+x^4)^{1/2}\,dx \approx 1.09$.

55. (a)

$$f(x) = x^{1/2} \qquad f(1) = 1 \qquad f'''(x) = \tfrac{3}{8}x^{-5/2} \qquad f'''(1) = \tfrac{3}{8}$$
$$f'(x) = \tfrac{1}{2}x^{-1/2} \qquad f'(1) = \tfrac{1}{2} \qquad f^{(4)}(x) = -\tfrac{15}{16}x^{-7/2}$$
$$f''(x) = -\tfrac{1}{4}x^{-3/2} \qquad f''(1) = -\tfrac{1}{4}$$

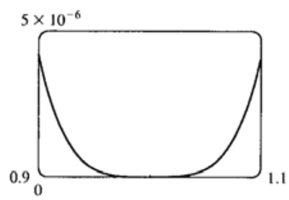
$$\sqrt{x} \approx T_3(x) = 1 + \frac{1/2}{1!}(x-1) - \frac{1/4}{2!}(x-1)^2 + \frac{3/8}{3!}(x-1)^3$$
$$= 1 + \tfrac{1}{2}(x-1) - \tfrac{1}{8}(x-1)^2 + \tfrac{1}{16}(x-1)^3$$

(b)



(c) $|R_3(x)| \le \dfrac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right| \le M$ with $f^{(4)}(x) = -\frac{15}{16}x^{-7/2}$. Now $0.9 \le x \le 1.1 \;\Rightarrow\; (x-1)^4 \le (0.1)^4$, and letting $x = 0.9$ gives $M = \dfrac{15}{16(0.9)^{7/2}}$, so $|R_3(x)| \le \dfrac{15}{16(0.9)^{7/2}\,4!}(0.1)^4 \approx 0.000005648$.

(d)

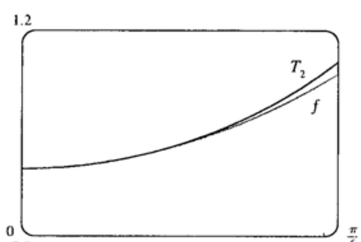


From the graph of $|R_3(x)| = \left|\sqrt{x} - T_3(x)\right|$, it appears that the error is less than 5×10^{-6} on $[0.9, 1.1]$.

56. (a)

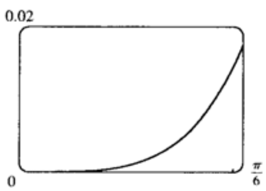
$$f(x) = \sec x \qquad f(0) = 1 \qquad \sec x \approx T_2(x) = 1 + \tfrac{1}{2}x^2$$
$$f'(x) = \sec x \tan x \qquad f'(0) = 0$$
$$f''(x) = \sec x \tan^2 x + \sec^3 x \qquad f''(0) = 1$$
$$f'''(x) = \sec x \tan^3 x + 5\sec^3 x \tan x$$

(b)



(c) $|R_2(x)| \le \dfrac{M}{3!}|x|^3$, where $\left|f^{(3)}(x)\right| \le M$ with $f^{(3)}(x) = \sec x \tan^3 x + 5\sec^3 x \tan x$. Now $0 \le x \le \frac{\pi}{6} \;\Rightarrow\; x^3 \le \left(\frac{\pi}{6}\right)^3$, and letting $x = \frac{\pi}{6}$ gives $M = \frac{14}{3}$, so $|R_2(x)| \le \frac{14}{3 \cdot 6}\left(\frac{\pi}{6}\right)^3 \approx 0.111648$.

(d)

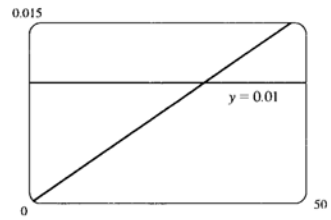


From the graph of $|R_2(x)| = |\sec x - T_2(x)|$, it appears that the error is less than 0.02 on $\left[0, \frac{\pi}{6}\right]$.

57. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, so $\sin x - x = -\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ and $\frac{\sin x - x}{x^3} = -\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \cdots$ and $\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0}\left(-\frac{1}{6} + \frac{x^2}{120} - \frac{x^4}{5040} + \cdots\right) = -\frac{1}{6}$.

58. (a) $F = \frac{mgR^2}{(R+h)^2} = \frac{mg}{(1+h/R)^2} = mg\sum_{n=0}^{\infty}\binom{-2}{n}\left(\frac{h}{R}\right)^n$ (Binomial Series)

(b) We expand $F = mg\left[1 - 2(h/R) + 3(h/R)^2 - \cdots\right]$. This is an alternating series, so by the Alternating Series Estimation Theorem, the error in the approximation $F = mg$ is less than $2mgh/R$, so for accuracy within 1% we want $\left|\frac{2mgh/R}{mgR^2/(R+h)^2}\right| < 0.01 \quad\Leftrightarrow\quad \frac{2h(R+h)^2}{R^3} < 0.01$. This inequality would be difficult to solve for h, so we substitute $R = 6{,}400$ km and plot both sides of the inequality. It appears that the approximation is accurate to within 1% for $h < 31$ km.



59. $f(x) = \sum_{n=0}^{\infty} c_n x^n \quad\Rightarrow\quad f(-x) = \sum_{n=0}^{\infty} c_n(-x)^n = \sum_{n=0}^{\infty}(-1)^n c_n x^n$

(a) If f is an odd function, then $f(-x) = -f(x) \quad\Rightarrow\quad \sum_{n=0}^{\infty}(-1)^n c_n x^n = \sum_{n=0}^{\infty} -c_n x^n$. The coefficients of any power series are uniquely determined (by Theorem 12.10.5), so $(-1)^n c_n = -c_n$. If n is even, then $(-1)^n = 1$, so $c_n = -c_n \quad\Rightarrow\quad 2c_n = 0 \quad\Rightarrow\quad c_n = 0$. Thus, all even coefficients are 0.

(b) If f is even, then $f(-x) = f(x) \quad\Rightarrow\quad \sum_{n=0}^{\infty}(-1)^n c_n x^n = \sum_{n=0}^{\infty} c_n x^n \quad\Rightarrow\quad (-1)^n c_n = c_n$. If n is odd, then $(-1)^n = -1$, so $-c_n = c_n \quad\Rightarrow\quad 2c_n = 0 \quad\Rightarrow\quad c_n = 0$. Thus, all odd coefficients are 0.

60. $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \quad\Rightarrow\quad e^{x^2} = \sum_{n=0}^{\infty}\frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{2n}}{n!} = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k \quad\Rightarrow\quad \frac{f^{(2n)}(0)}{(2n)!} = \frac{1}{n!} \quad\Rightarrow\quad f^{(2n)}(0) = \frac{(2n)!}{n!}$.

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Problems Plus

1. It would be far too much work to compute 15 derivatives of f. The key idea is to remember that $f^{(n)}(0)$ occurs in the coefficient of x^n in the Maclaurin series of f. We start with the Maclaurin series for sin:

$\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots$. Then $\sin(x^3) = x^3 - \dfrac{x^9}{3!} + \dfrac{x^{15}}{5!} - \cdots$ and so the coefficient of x^{15} is $\dfrac{f^{(15)}(0)}{15!} = \dfrac{1}{5!}$. Therefore, $f^{(15)}(0) = \dfrac{15!}{5!} = 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 = 10{,}897{,}286{,}400$.

2. $|AP_2|^2 = 2$, $|AP_3|^2 = 2 + 2^2$, $|AP_4|^2 = 2 + 2^2 + (2^2)^2$, $|AP_5|^2 = 2 + 2^2 + (2^2)^2 + (2^3)^2, \ldots,$

$$|AP_n|^2 = 2 + 2^2 + \left(2^2\right)^2 + \cdots + \left(2^{n-2}\right)^2 \quad (\text{for } n \geq 3) \quad = 2 + \left(4 + 4^2 + 4^3 + \cdots + 4^{n-2}\right)$$

$$= 2 + \frac{4\left(4^{n-2} - 1\right)}{4 - 1} \quad (\text{finite geometric sum with } a = 4,\ r = 4) \quad = \frac{6}{3} + \frac{4^{n-1} - 4}{3} = \frac{2}{3} + \frac{4^{n-1}}{3}$$

So $\tan \angle P_n A P_{n+1} = \dfrac{|P_n P_{n+1}|}{|AP_n|} = \dfrac{2^{n-1}}{\sqrt{\dfrac{2}{3} + \dfrac{4^{n-1}}{3}}} = \dfrac{\sqrt{4^{n-1}}}{\sqrt{\dfrac{2}{3} + \dfrac{4^{n-1}}{3}}} = \dfrac{1}{\sqrt{\dfrac{2}{3 \cdot 4^{n-1}} + \dfrac{1}{3}}} \to \sqrt{3}$ as $n \to \infty$, so $\angle P_n A P_{n+1} \to \frac{\pi}{3}$ as $n \to \infty$.

3. (a) From Formula 14a in Appendix D, with $x = y = \theta$, we get $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$, so $\cot 2\theta = \dfrac{1 - \tan^2\theta}{2\tan\theta} \Rightarrow$

$2\cot 2\theta = \dfrac{1 - \tan^2\theta}{\tan\theta} = \cot\theta - \tan\theta$. Replacing θ by $\frac{1}{2}x$, we get $2\cot x = \cot\frac{1}{2}x - \tan\frac{1}{2}x$, or $\tan\frac{1}{2}x = \cot\frac{1}{2}x - 2\cot x$.

(b) From part (a), $\tan\dfrac{x}{2^n} = \cot\dfrac{x}{2^n} - 2\cot\dfrac{x}{2^{n-1}}$, so the nth partial sum of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}\tan\frac{x}{2^n}$ is

$$s_n = \frac{\tan(x/2)}{2} + \frac{\tan(x/4)}{4} + \frac{\tan(x/8)}{8} + \cdots + \frac{\tan(x/2^n)}{2^n}$$

$$= \left[\frac{\cot(x/2)}{2} - \cot x\right] + \left[\frac{\cot(x/4)}{4} - \frac{\cot(x/2)}{2}\right] + \left[\frac{\cot(x/8)}{8} - \frac{\cot(x/4)}{4}\right] + \cdots$$

$$+ \left[\frac{\cot(x/2^n)}{2^n} - \frac{\cot\left(x/2^{n-1}\right)}{2^{n-1}}\right] = -\cot x + \frac{\cot(x/2^n)}{2^n} \quad (\text{telescoping sum})$$

Now $\dfrac{\cot(x/2^n)}{2^n} = \dfrac{\cos(x/2^n)}{2^n \sin(x/2^n)} = \dfrac{\cos(x/2^n)}{x} \cdot \dfrac{x/2^n}{\sin(x/2^n)} \to \dfrac{1}{x} \cdot 1 = \dfrac{1}{x}$ as $n \to \infty$ since $x/2^n \to 0$ for $x \neq 0$. Therefore, if $x \neq 0$ and $x \neq n\pi$, then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}\tan\frac{x}{2^n} = \lim_{n\to\infty}\left(-\cot x + \frac{1}{2^n}\cot\frac{x}{2^n}\right) = -\cot x + \frac{1}{x}$. If $x = 0$, then all terms in the series are 0, so the sum is 0.

4. We use the problem-solving strategy of taking cases:

Case (i): If $|x| < 1$, then $0 \le x^2 < 1$, so $\lim_{n\to\infty} x^{2n} = 0$ (see Example 8 in Section 12.1)

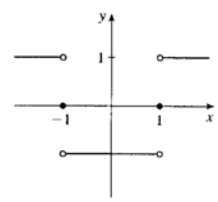
and $f(x) = \lim_{n\to\infty} \dfrac{x^{2n} - 1}{x^{2n} + 1} = \dfrac{0 - 1}{0 + 1} = -1$.

Case (ii): If $|x| = 1$, that is, $x = \pm 1$, then $x^2 = 1$, so $f(x) = \lim_{n\to\infty} \dfrac{1 - 1}{1 + 1} = 0$.

Case (iii): If $|x| > 1$, then $x^2 > 1$, so $\lim_{n\to\infty} x^{2n} = \infty$ and

$$f(x) = \lim_{n\to\infty} \frac{x^{2n} - 1}{x^{2n} + 1} = \lim_{n\to\infty} \frac{1 - (1/x^{2n})}{1 + (1/x^{2n})} = \frac{1 - 0}{1 + 0} = 1.$$

$$\text{Thus, } f(x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } x = -1 \\ -1 & \text{if } -1 < x < 1 \\ 0 & \text{if } x = 1 \\ 1 & \text{if } x > 1 \end{cases}$$



The graph shows that f is continuous everywhere except at $x = \pm 1$.

5. (a) At each stage, each side is replaced by four shorter sides, each of length $\frac{1}{3}$ of the side length at the preceding stage. Writing s_0 and ℓ_0 for the number of sides and the length of the side of the initial triangle, we generate the table at right. In general, we have $s_n = 3 \cdot 4^n$ and $\ell_n = \left(\frac{1}{3}\right)^n$, so the length of the perimeter at the nth stage of construction is $p_n = s_n \ell_n = 3 \cdot 4^n \cdot \left(\frac{1}{3}\right)^n = 3 \cdot \left(\frac{4}{3}\right)^n$.

$s_0 = 3$	$\ell_0 = 1$
$s_1 = 3 \cdot 4$	$\ell_1 = 1/3$
$s_2 = 3 \cdot 4^2$	$\ell_2 = 1/3^2$
$s_3 = 3 \cdot 4^3$	$\ell_3 = 1/3^3$
$\cdots$	$\cdots$

(b) $p_n = \dfrac{4^n}{3^{n-1}} = 4\left(\dfrac{4}{3}\right)^{n-1}$. Since $\frac{4}{3} > 1$, $p_n \to \infty$ as $n \to \infty$.

(c) The area of each of the small triangles added at a given stage is one-ninth of the area of the triangle added at the preceding stage. Let a be the area of the original triangle. Then the area $a_n$of each of the small triangles added at stage n is $a_n = a \cdot \dfrac{1}{9^n} = \dfrac{a}{9^n}$. Since a small triangle is added to each side at every stage, it follows that the total area A_n added to the figure at the nth stage is

$A_n = s_{n-1} \cdot a_n = 3 \cdot 4^{n-1} \cdot \dfrac{a}{9^n} = a \cdot \dfrac{4^{n-1}}{3^{2n-1}}$. Then the total area enclosed by the snowflake curve is

$A = a + A_1 + A_2 + A_3 + \cdots = a + a \cdot \dfrac{1}{3} + a \cdot \dfrac{4}{3^3} + a \cdot \dfrac{4^2}{3^5} + a \cdot \dfrac{4^3}{3^7} + \cdots$. After the first term, this is a geometric series with common ratio $\frac{4}{9}$, so $A = a + \dfrac{a/3}{1 - \frac{4}{9}} = a + \dfrac{a}{3} \cdot \dfrac{9}{5} = \dfrac{8a}{5}$. But the area of the original equilateral triangle with side 1 is $a = \frac{1}{2} \cdot 1 \cdot \sin\frac{\pi}{3} = \frac{\sqrt{3}}{4}$. So the area enclosed by the snowflake curve is $\frac{8}{5} \cdot \frac{\sqrt{3}}{4} = \frac{2\sqrt{3}}{5}$.

6. Let the series be S. Then every term in S is of the form $\dfrac{1}{2^m 3^n}$, $m, n \geq 0$, and furthermore each term occurs only once. So we can write

$$S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2^m 3^n} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2^m} \frac{1}{3^n} = \sum_{m=0}^{\infty} \frac{1}{2^m} \sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1-\frac{1}{2}} \cdot \frac{1}{1-\frac{1}{3}} = 2 \cdot \tfrac{3}{2} = 3$$

7. (a) Let $a = \arctan x$ and $b = \arctan y$. Then, from Formula 14b in Appendix D,

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b} = \frac{\tan(\arctan x) - \tan(\arctan y)}{1 + \tan(\arctan x)\tan(\arctan y)} = \frac{x-y}{1+xy} \quad \Rightarrow$$

$$\arctan x - \arctan y = a - b = \arctan \frac{x-y}{1+xy} \quad \text{since } -\tfrac{\pi}{2} < \arctan x - \arctan y < \tfrac{\pi}{2}$$

(b) From part (a) we have

$$\arctan \tfrac{120}{119} - \arctan \tfrac{1}{239} = \arctan \frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \cdot \frac{1}{239}} = \arctan \frac{\frac{28{,}561}{28{,}441}}{\frac{28{,}561}{28{,}441}} = \arctan 1 = \tfrac{\pi}{4}$$

(c) Replacing y by $-y$ in the formula of part (a), we get $\arctan x + \arctan y = \arctan \dfrac{x+y}{1-xy}$. So

$$4 \arctan \tfrac{1}{5} = 2\left(\arctan \tfrac{1}{5} + \arctan \tfrac{1}{5}\right) = 2 \arctan \frac{\frac{1}{5} + \frac{1}{5}}{1 - \frac{1}{5} \cdot \frac{1}{5}} = 2 \arctan \tfrac{5}{12} = \arctan \tfrac{5}{12} + \arctan \tfrac{5}{12}$$

$$= \arctan \frac{\frac{5}{12} + \frac{5}{12}}{1 - \frac{5}{12} \cdot \frac{5}{12}} = \arctan \tfrac{120}{119}$$

Thus, from part (b), we have $4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \arctan \frac{120}{119} - \arctan \frac{1}{239} = \frac{\pi}{4}$.

(d) From Example 7 in Section 12.9 we have $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dfrac{x^9}{9} - \dfrac{x^{11}}{11} + \cdots$, so

$$\arctan \frac{1}{5} = \frac{1}{5} - \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} - \frac{1}{7 \cdot 5^7} + \frac{1}{9 \cdot 5^9} - \frac{1}{11 \cdot 5^{11}} + \cdots$$

This is an alternating series and the size of the terms decreases to 0, so by the Alternating Series Estimation Theorem, the sum lies between s_5 and s_6, that is, $0.197395560 < \arctan \frac{1}{5} < 0.197395562$.

(e) From the series in part (d) we get $\arctan \dfrac{1}{239} = \dfrac{1}{239} - \dfrac{1}{3 \cdot 239^3} + \dfrac{1}{5 \cdot 239^5} - \cdots$. The third term is less than 2.6×10^{-13}, so by the Alternating Series Estimation Theorem, we have, to nine decimal places, $\arctan \frac{1}{239} \approx s_2 \approx 0.004184076$. Thus, $0.004184075 < \arctan \frac{1}{239} < 0.004184077$.

(f) From part (c) we have $\pi = 16 \arctan \frac{1}{5} - 4 \arctan \frac{1}{239}$, so from parts (d) and (e) we have $16(0.197395560) - 4(0.004184077) < \pi < 16(0.197395562) - 4(0.004184075) \Rightarrow$ $3.141592652 < \pi < 3.141592692$. So, to 7 decimal places, $\pi \approx 3.1415927$.

8. (a) Let $a = \operatorname{arccot} x$ and $b = \operatorname{arccot} y$. Then

$$\cot(a-b) = \frac{1+\cot a \cot b}{\cot b - \cot a} = \frac{1+\cot(\operatorname{arccot} x)\cot(\operatorname{arccot} y)}{\cot(\operatorname{arccot} y) - \cot(\operatorname{arccot} x)} = \frac{1+xy}{y-x} \quad \Rightarrow$$

$$\operatorname{arccot} x - \operatorname{arccot} y = a - b = \operatorname{arccot}\frac{1+xy}{y-x}$$

(b) Applying the identity in part (a) with $x = n$ and $y = n+1$, we have

$$\operatorname{arccot}\left(n^2+n+1\right) = \operatorname{arccot}(1+n(n+1)) = \operatorname{arccot}\frac{1+n(n+1)}{(n+1)-n} = \operatorname{arccot} n - \operatorname{arccot}(n+1)$$

Thus, we have a telescoping series with nth partial sum

$$s_n = [\operatorname{arccot} 0 - \operatorname{arccot} 1] + [\operatorname{arccot} 1 - \operatorname{arccot} 2] + \cdots + [\operatorname{arccot} n - \operatorname{arccot}(n+1)] = \operatorname{arccot} 0 - \operatorname{arccot}(n+1)$$

Thus, $\sum_{n=0}^{\infty} \operatorname{arccot}\left(n^2+n+1\right) = \lim_{n\to\infty} [-\operatorname{arccot}(n+1)] = \frac{\pi}{2}$.

9. We start with the geometric series $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, $|x| < 1$, and differentiate:

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} \text{ for } |x| < 1 \quad \Rightarrow \quad \sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = \frac{x}{(1-x)^2} \text{ for}$$

$|x| < 1$. Differentiate again:

$$\sum_{n=1}^{\infty} n^2x^{n-1} = \frac{d}{dx}\frac{x}{(1-x)^2} = \frac{(1-x)^2 - x\cdot 2(1-x)(-1)}{(1-x)^4} = \frac{x+1}{(1-x)^3} \quad \Rightarrow \quad \sum_{n=1}^{\infty} n^2x^n = \frac{x^2+x}{(1-x)^3} \quad \Rightarrow$$

$$\sum_{n=1}^{\infty} n^3x^{n-1} = \frac{d}{dx}\frac{x^2+x}{(1-x)^3} = \frac{(1-x)^3(2x+1) - \left(x^2+x\right)3(1-x)^2(-1)}{(1-x)^6} = \frac{x^2+4x+1}{(1-x)^4} \quad \Rightarrow$$

$\displaystyle\sum_{n=1}^{\infty} n^3x^n = \frac{x^3+4x^2+x}{(1-x)^4}$, $|x| < 1$. The radius of convergence is 1 because that is the radius of convergence for the geometric series we started with. If $x = \pm 1$, the series is $\sum n^3(\pm 1)^n$, which diverges by the Test For Divergence, so the interval of convergence is $(-1, 1)$.

10. (a)
$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\left(2\sin\frac{\theta}{4}\cos\frac{\theta}{4}\right)\cos\frac{\theta}{2} = 2\left(2\left(2\sin\frac{\theta}{8}\cos\frac{\theta}{8}\right)\cos\frac{\theta}{4}\right)\cos\frac{\theta}{2}$$
$$= \cdots = 2\left(2\left(2\left(\cdots\left(2\left(2\sin\frac{\theta}{2^n}\cos\frac{\theta}{2^n}\right)\cos\frac{\theta}{2^{n-1}}\right)\cdots\right)\cos\frac{\theta}{8}\right)\cos\frac{\theta}{4}\right)\cos\frac{\theta}{2}$$
$$= 2^n\sin\frac{\theta}{2^n}\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cdots\cos\frac{\theta}{2^n}$$

(b) $\displaystyle\sin\theta = 2^n\sin\frac{\theta}{2^n}\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cdots\cos\frac{\theta}{2^n} \quad\Leftrightarrow\quad \frac{\sin\theta}{\theta}\cdot\frac{\theta/2^n}{\sin(\theta/2^n)} = \cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cdots\cos\frac{\theta}{2^n}$.

Now we let $n\to\infty$, using $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$ with $x = \dfrac{\theta}{2^n}$:

$$\lim_{n\to\infty}\left[\frac{\sin\theta}{\theta}\cdot\frac{\theta/2^n}{\sin(\theta/2^n)}\right] = \lim_{n\to\infty}\left[\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cdots\cos\frac{\theta}{2^n}\right] \quad\Leftrightarrow\quad \frac{\sin\theta}{\theta} = \cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cdots.$$

(c) If we take $\theta = \frac{\pi}{2}$ in the result from part (b) and use the half-angle formula $\cos x = \sqrt{\frac{1}{2}(1 + \cos 2x)}$ (see Formula 17a in Appendix D), we get

$$\frac{\sin \pi/2}{\pi/2} = \cos\frac{\pi}{4}\sqrt{\frac{\cos\frac{\pi}{4}+1}{2}}\sqrt{\frac{\sqrt{\frac{\cos\frac{\pi}{4}+1}{2}}+1}{2}}\sqrt{\frac{\sqrt{\frac{\sqrt{\frac{\cos\frac{\pi}{4}+1}{2}}+1}{2}}+1}{2}}\cdots \quad\Rightarrow$$

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\sqrt{\frac{\frac{\sqrt{2}}{2}+1}{2}}\sqrt{\frac{\sqrt{\frac{\frac{\sqrt{2}}{2}+1}{2}}+1}{2}}\cdots = \frac{\sqrt{2}}{2}\,\frac{\sqrt{2+\sqrt{2}}}{2}\sqrt{\frac{\frac{\sqrt{2+\sqrt{2}}}{2}+1}{2}}\cdots$$

$$= \frac{\sqrt{2}}{2}\,\frac{\sqrt{2+\sqrt{2}}}{2}\,\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots$$

11. $a_{n+1} = \dfrac{a_n + b_n}{2}$, $b_{n+1} = \sqrt{b_n a_{n+1}}$. So $a_1 = \cos\theta$, $b_1 = 1 \;\Rightarrow\; a_2 = \dfrac{1+\cos\theta}{2} = \cos^2\frac{\theta}{2}$,

$b_2 = \sqrt{b_1 a_2} = \sqrt{\cos^2\frac{\theta}{2}} = \cos\frac{\theta}{2}$ since $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then

$a_3 = \frac{1}{2}\left(\cos\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right) = \cos\frac{\theta}{2}\cdot\frac{1}{2}\left(1+\cos\frac{\theta}{2}\right) = \cos\frac{\theta}{2}\cos^2\frac{\theta}{4} \;\Rightarrow$

$b_3 = \sqrt{b_2 a_3} = \sqrt{\cos\frac{\theta}{2}\cos\frac{\theta}{2}\cos^2\frac{\theta}{4}} = \cos\frac{\theta}{2}\cos\frac{\theta}{4} \;\Rightarrow$

$a_4 = \frac{1}{2}\left(\cos\frac{\theta}{2}\cos^2\frac{\theta}{4} + \cos\frac{\theta}{2}\cos\frac{\theta}{4}\right) = \cos\frac{\theta}{2}\cos\frac{\theta}{4}\cdot\frac{1}{2}\left(1+\cos\frac{\theta}{4}\right) = \cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos^2\frac{\theta}{8} \;\Rightarrow$

$b_4 = \sqrt{\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos^2\frac{\theta}{8}} = \cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}$. By now we see the pattern:

$b_n = \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cdot\cdots\cdot\cos\frac{\theta}{2^{n-1}}$ and $a_n = b_n\cos\frac{\theta}{2^{n-1}}$. (This could be proved by mathematical induction.) By Exercise 10(a), $\sin\theta = 2^{n-1}\sin\frac{\theta}{2^{n-1}}\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cdots\cos\frac{\theta}{2^{n-1}}$. So

$b_n = \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cdots\cos\frac{\theta}{2^{n-1}} \to \frac{\sin\theta}{\theta}$ as $n\to\infty$ by Exercise 10(b), and

$a_n = b_n\cos\frac{\theta}{2^{n-1}} \to \frac{\sin\theta}{\theta}\cdot 1 = \frac{\sin\theta}{\theta}$ as $n\to\infty$. So $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \frac{\sin\theta}{\theta}$.

12. Let's first try the case $k = 1$: $a_0 + a_1 = 0 \;\Rightarrow\; a_1 = -a_0 \;\Rightarrow$

$$\lim_{n\to\infty}\left(a_0\sqrt{n} + a_1\sqrt{n+1}\right) = \lim_{n\to\infty}\left(a_0\sqrt{n} - a_0\sqrt{n+1}\right) = a_0\lim_{n\to\infty}\left(\sqrt{n}-\sqrt{n+1}\right)\frac{\sqrt{n}+\sqrt{n+1}}{\sqrt{n}+\sqrt{n+1}}$$

$$= a_0\lim_{n\to\infty}\frac{-1}{\sqrt{n}+\sqrt{n+1}} = 0$$

In general we have $a_0 + a_1 + \cdots + a_k = 0 \;\Rightarrow\; a_k = -a_0 - a_1 - \cdots - a_{k-1} \;\Rightarrow$

$$\lim_{n\to\infty}\left(a_0\sqrt{n} + a_1\sqrt{n+1} + a_2\sqrt{n+2} + \cdots + a_k\sqrt{n+k}\right)$$

$$= \lim_{n\to\infty}\left(a_0\sqrt{n} + a_1\sqrt{n+1} + \cdots + a_{k-1}\sqrt{n+k-1} - a_0\sqrt{n+k} - a_1\sqrt{n+k} - \cdots - a_{k-1}\sqrt{n+k}\right)$$

$$= a_0\lim_{n\to\infty}\left(\sqrt{n}-\sqrt{n+k}\right) + a_1\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n+k}\right) + \cdots + a_{k-1}\lim_{n\to\infty}\left(\sqrt{n+k-1}-\sqrt{n+k}\right)$$

Each of these limits is 0 by the same type of simplification as in the case $k = 1$. So we have

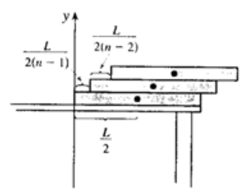
$$\lim_{n\to\infty}\left(a_0\sqrt{n} + a_1\sqrt{n+1} + a_2\sqrt{n+2} + \cdots + a_k\sqrt{n+k}\right) = a_0(0) + a_1(0) + \cdots + a_{k-1}(0) = 0$$

13. Let $f(x) = \sum_{m=0}^{\infty} c_m x^m$ and $g(x) = e^{f(x)} = \sum_{n=0}^{\infty} d_n x^n$. Then $g'(x) = \sum_{n=0}^{\infty} n d_n x^{n-1}$, so nd_n occurs as the coefficient of x^{n-1}. But also

$$\begin{aligned} g'(x) &= e^{f(x)} f'(x) = \left(\sum_{n=0}^{\infty} d_n x^n\right)\left(\sum_{m=1}^{\infty} m c_m x^{m-1}\right) \\ &= \left(d_0 + d_1 x + d_2 x^2 + \cdots + d_{n-1}x^{n-1} + \cdots\right)\left(c_1 + 2c_2 x + 3c_3 x^2 + \cdots + nc_n x^{n-1} + \cdots\right) \end{aligned}$$

so the coefficient of x^{n-1} is $c_1 d_{n-1} + 2c_2 d_{n-2} + 3c_3 d_{n-3} + \cdots + nc_n d_0 = \sum_{i=1}^{n} i c_i d_{n-i}$. Therefore, $nd_n = \sum_{i=1}^{n} i c_i d_{n-i}$.

14. Place the y-axis as shown and let the length of each book be L. We want to show that the center of mass of the system of n books lies above the table, that is, $\overline{x} < L$. The x-coordinates of the centers of mass of the books are $x_1 = \dfrac{L}{2}$, $x_2 = \dfrac{L}{2(n-1)} + \dfrac{L}{2}$,



$x_3 = \dfrac{L}{2(n-1)} + \dfrac{L}{2(n-2)} + \dfrac{L}{2}$, and so on.

Each book has the same mass m, so if there are n books, then

$$\begin{aligned} \overline{x} &= \frac{mx_1 + mx_2 + \cdots + mx_n}{mn} = \frac{x_1 + x_2 + \cdots + x_n}{n} \\ &= \frac{1}{n}\left[\frac{L}{2} + \left(\frac{L}{2(n-1)} + \frac{L}{2}\right) + \left(\frac{L}{2(n-1)} + \frac{L}{2(n-2)} + \frac{L}{2}\right) + \cdots \right. \\ &\qquad \left. + \left(\frac{L}{2(n-1)} + \frac{L}{2(n-2)} + \cdots + \frac{L}{4} + \frac{L}{2} + \frac{L}{2}\right)\right] \\ &= \frac{L}{n}\left[\frac{n-1}{2(n-1)} + \frac{n-2}{2(n-2)} + \cdots + \frac{2}{4} + \frac{1}{2} + \frac{n}{2}\right] = \frac{L}{n}\left[(n-1)\frac{1}{2} + \frac{n}{2}\right] = \frac{2n-1}{2n}L < L \end{aligned}$$

This shows that, no matter how many books are added according to the given scheme, the center of mass lies above the table. It remains to observe that the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots = \frac{1}{2}\sum(1/n)$ is divergent (harmonic series), so we can make the top book extend as far as we like beyond the edge of the table if we add enough books.

15. $u = 1 + \dfrac{x^3}{3!} + \dfrac{x^6}{6!} + \dfrac{x^9}{9!} + \cdots$, $v = x + \dfrac{x^4}{4!} + \dfrac{x^7}{7!} + \dfrac{x^{10}}{10!} + \cdots$, $w = \dfrac{x^2}{2!} + \dfrac{x^5}{5!} + \dfrac{x^8}{8!} + \cdots$. The key idea is to differentiate: $\dfrac{du}{dx} = \dfrac{3x^2}{3!} + \dfrac{6x^5}{6!} + \dfrac{9x^8}{9!} + \cdots = \dfrac{x^2}{2!} + \dfrac{x^5}{5!} + \dfrac{x^8}{8!} + \cdots = w$. Similarly, $\dfrac{dv}{dx} = 1 + \dfrac{x^3}{3!} + \dfrac{x^6}{6!} + \dfrac{x^9}{9!} + \cdots = u$, and $\dfrac{dw}{dx} = x + \dfrac{x^4}{4!} + \dfrac{x^7}{7!} + \dfrac{x^{10}}{10!} + \cdots = v$. So $u' = w$, $v' = u$, and $w' = v$. Now differentiate the left hand side of the desired equation:

$$\begin{aligned} \frac{d}{dx}\left(u^3 + v^3 + w^3 - 3uvw\right) &= 3u^2u' + 3v^2v' + 3w^2w' - 3\left(u'vw + uv'w + uvw'\right) \\ &= 3u^2w + 3v^2u + 3w^2v - 3\left(vw^2 + u^2w + uv^2\right) = 0 \quad \Rightarrow \end{aligned}$$

$u^3 + v^3 + w^3 - 3uvw = C$. To find the value of the constant C, we put $x = 0$ in the last equation and get $1^3 + 0^3 + 0^3 - 3(1 \cdot 0 \cdot 0) = C \quad \Rightarrow \quad C = 1$, so $u^3 + v^3 + w^3 - 3uvw = 1$.

16. First notice that both series are absolutely convergent (p-series with $p > 1$.) Let the given expression be called x. Then

$$x = \frac{1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} = \frac{1 + \left(2 \cdot \frac{1}{2^p} - \frac{1}{2^p}\right) + \frac{1}{3^p} + \left(2 \cdot \frac{1}{4^p} - \frac{1}{4^p}\right) + \cdots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots}$$

$$= \frac{\left(1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots\right) + \left(2 \cdot \frac{1}{2^p} + 2 \cdot \frac{1}{4^p} + 2 \cdot \frac{1}{6^p} + \cdots\right)}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots}$$

$$= 1 + \frac{2\left(\frac{1}{2^p} + \frac{1}{4^p} + \frac{1}{6^p} + \frac{1}{8^p} + \cdots\right)}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} = 1 + \frac{\frac{1}{2^{p-1}}\left(1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots\right)}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} = 1 + 2^{1-p}x$$

Therefore, $x = 1 + 2^{1-p}x \quad \Leftrightarrow \quad x - 2^{1-p}x = 1 \quad \Leftrightarrow \quad x\left(1 - 2^{1-p}\right) = 1 \quad \Leftrightarrow \quad x = \dfrac{1}{1 - 2^{1-p}}$.

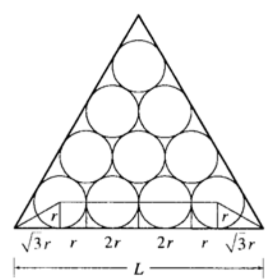
17. If L is the length of a side of the equilateral triangle, then the area is $A = \frac{1}{2}L \cdot \frac{\sqrt{3}}{2}L = \frac{\sqrt{3}}{4}L^2$ and so $L^2 = \frac{4}{\sqrt{3}}A$. Let r be the radius of one of the circles. When there are n rows of circles, the figure shows that $L = \sqrt{3}r + r + (n-2)(2r) + r + \sqrt{3}r = r\left(2n - 2 + 2\sqrt{3}\right)$, so $r = \dfrac{L}{2\left(n + \sqrt{3} - 1\right)}$. The number of circles is $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$ and so the total area of the circles is

$$A_n = \frac{n(n+1)}{2}\pi r^2 = \frac{n(n+1)}{2}\pi \frac{L^2}{4\left(n + \sqrt{3} - 1\right)^2} = \frac{n(n+1)}{2}\pi \frac{4A/\sqrt{3}}{4\left(n + \sqrt{3} - 1\right)^2}$$

$$= \frac{n(n+1)}{\left(n + \sqrt{3} - 1\right)^2} \frac{\pi A}{2\sqrt{3}} \quad \Rightarrow$$

$$\frac{A_n}{A} = \frac{n(n+1)}{\left(n + \sqrt{3} - 1\right)^2} \frac{\pi}{2\sqrt{3}}$$

$$= \frac{1 + 1/n}{\left[1 + \left(\sqrt{3} - 1\right)/n\right]^2} \frac{\pi}{2\sqrt{3}} \to \frac{\pi}{2\sqrt{3}} \text{ as } n \to \infty$$



18. Given $a_0 = a_1 = 1$ and $a_n = \dfrac{(n-1)(n-2)a_{n-1} - (n-3)a_{n-2}}{n(n-1)}$, we calculate the next few terms of the sequence: $a_2 = \dfrac{1 \cdot 0 \cdot a_1 - (-1)a_0}{2 \cdot 1} = \dfrac{1}{2}$, $a_3 = \dfrac{2 \cdot 1 \cdot a_2 - 0a_1}{3 \cdot 2} = \dfrac{1}{6}$, $a_4 = \dfrac{3 \cdot 2 \cdot a_3 - 1a_2}{4 \cdot 3} = \dfrac{1}{24}$. It seems that $a_n = \dfrac{1}{n!}$, so we try to prove this by induction. The first step is done, so assume $a_k = \dfrac{1}{k!}$ and $a_{k-1} = \dfrac{1}{(k-1)!}$. Then

$$a_{k+1} = \frac{k(k-1)a_k - (k-2)a_{k-1}}{(k+1)k} = \frac{\dfrac{k(k-1)}{k!} - \dfrac{k-2}{(k-1)!}}{(k+1)k} = \frac{(k-1)-(k-2)}{[(k+1)(k)](k-1)!} = \frac{1}{(k+1)!}$$

and the induction is complete. Therefore, $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} 1/n! = e$.

19. Call the series S. We group the terms according to the number of digits in their denominators:

$$S = \underbrace{\left(1 + \tfrac{1}{2} + \cdots + \tfrac{1}{8} + \tfrac{1}{9}\right)}_{g_1} + \underbrace{\left(\tfrac{1}{11} + \cdots + \tfrac{1}{99}\right)}_{g_2} + \underbrace{\left(\tfrac{1}{111} + \cdots + \tfrac{1}{999}\right)}_{g_3} + \cdots$$

Now in the group g_n, there are 9^n terms, since we have 9 choices for each of the n digits in the denominator. Furthermore, each term in g_n is less than $\frac{1}{10^{n-1}}$. So $g_n < 9^n \cdot \frac{1}{10^{n-1}} = 9\left(\frac{9}{10}\right)^{n-1}$. Now $\sum_{n=1}^{\infty} 9\left(\frac{9}{10}\right)^{n-1}$ is a geometric series with $a = 9$ and $r = \frac{9}{10} < 1$. Therefore, by the Comparison Test, $S = \sum_{n=1}^{\infty} g_n < \sum_{n=1}^{\infty} 9\left(\frac{9}{10}\right)^{n-1} = \frac{9}{1 - 9/10} = 90$.

20. (a) Since P_n is defined as the midpoint of $P_{n-4}P_{n-3}$, $x_n = \frac{1}{2}(x_{n-4} + x_{n-3})$ for $n \geq 5$. So we prove by induction that $\frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3} = 2$. The case $n = 1$ is immediate, since $\frac{1}{2}0 + 1 + 1 + 0 = 2$. Assume that the result holds for $n = k - 1$, that is, $\frac{1}{2}x_{k-1} + x_k + x_{k+1} + x_{k+2} = 2$. Then for $n = k$,

$$\begin{aligned}\tfrac{1}{2}x_k + x_{k+1} + x_{k+2} + x_{k+3} &= \tfrac{1}{2}x_k + x_{k+1} + x_{k+2} + \tfrac{1}{2}(x_{k+3-4} + x_{k+3-3}) \quad \text{(by above)}\\ &= \tfrac{1}{2}x_{k-1} + x_k + x_{k+1} + x_{k+2} = 2 \quad \text{(by the induction hypothesis)}\end{aligned}$$

Similarly, for $n \geq 5$, $y_n = \frac{1}{2}(y_{n-4} + y_{n-3})$, so the same argument as above holds for y, with 2 replaced by $\frac{1}{2}y_1 + y_2 + y_3 + y_4 = \frac{1}{2}1 + 1 + 0 + 0 = \frac{3}{2}$. So $\frac{1}{2}y_n + y_{n+1} + y_{n+2} + y_{n+3} = \frac{3}{2}$ for all n.

(b) $\lim\limits_{n\to\infty}\left(\frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3}\right) = \frac{1}{2}\lim\limits_{n\to\infty} x_n + \lim\limits_{n\to\infty} x_{n+1} + \lim\limits_{n\to\infty} x_{n+2} + \lim\limits_{n\to\infty} x_{n+3} = 2$. Since all the limits on the left hand side are the same, we get $\frac{7}{2}\lim\limits_{n\to\infty} x_n = 2 \Rightarrow \lim\limits_{n\to\infty} x_n = \frac{4}{7}$. In the same way, $\lim\limits_{n\to\infty} y_n = \frac{3}{7}$, so $P = \left(\frac{4}{7}, \frac{3}{7}\right)$.