Análisis de circuito RC con Laplace

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Analysis

Sea el circuito RC que se muestra en la siguiente figura, el cual es excitado con una fuente senoidal $V_m \sin(\omega t)$V. Asuma que el capacitor C tiene un voltaje inicial de V_0 V

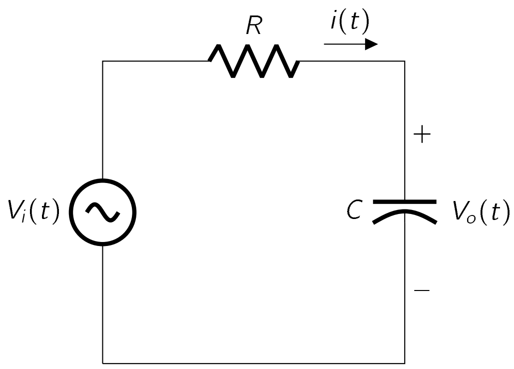


Figure 1: Circuito RC

Solución

Transformamos el circuito de la figura 1 en el dominio de s.

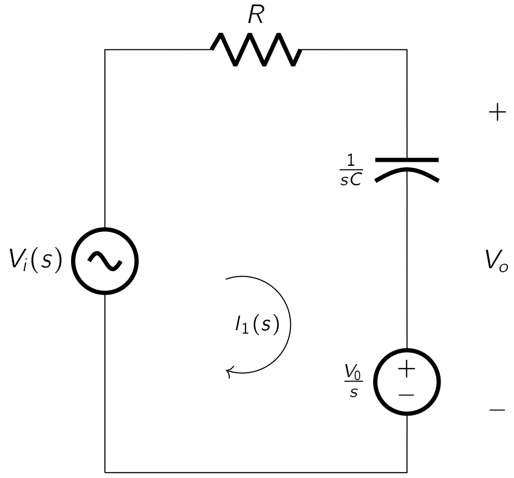


Figure 2: Circuito RC en el dominio de s

Empleamos la LTK sobre la malla del circuito de la figura 2

$$
\begin{aligned}
V_i(s) - RI(s) - \frac{1}{sC}I(s) - \frac{V_0}{s} &= 0 \\
-\left(R + \frac{1}{sC}\right) I(s) &= -V_i(s) + \frac{V_0}{s} \\
\left(\frac{RCs+1}{sC}\right) I(s) &= \frac{V_m\omega}{s^2+\omega^2} - \frac{V_0}{s} \\
\left(\frac{RCs+1}{sC}\right) I(s) &= \frac{\omega V_m s - V_0(s^2+\omega^2)}{s(s^2+\omega^2)} \\
I(s) = \left(\frac{sC}{RCs+1}\right) &\left[\frac{\omega V_m s - V_0(s^2+\omega^2)}{s(s^2+\omega^2)}\right] \\
I(s) = \left[\frac{sC}{RC(s+\frac{1}{RC})}\right] &\left[\frac{\omega V_m s - V_0 s^2 - \omega^2 V_0}{s(s^2+\omega^2)}\right] \\
I(s) = \left[\frac{\frac{1}{R}}{s+\frac{1}{RC}}\right] &\left[\frac{\omega V_m s - V_0 s^2 - \omega^2 V_0}{s^2+\omega^2}\right] \qquad (1)
\end{aligned}
$$

Usamos la expansión en fracciones parciales para separar $I(s)$ de la ecuación (1) en términos más simples. Sea

$$
I(s) = \frac{-\frac{V_0}{R}s^2 + \frac{\omega V_m}{R}s - \frac{\omega^2 V_0}{R}}{\left(s+\frac{1}{RC}\right)(s^2+\omega^2)} = \frac{\alpha_1}{s+\frac{1}{RC}} + \frac{\alpha_2 s + \omega\alpha_3}{s^2+\omega^2} \qquad (2)
$$

Donde α_1, α_2 y α_3 son las constantes por determinar. Utilizamos el método algebraico y multiplicamos ambos lados de la ecuación (2) por $(s+1/RC)(s^2+\omega^2)$ el cual produce:

$$
\begin{aligned}
-\frac{V_0}{R}s^2 + \frac{\omega V_m}{R}s - \frac{\omega^2 V_0}{R} &= \alpha_1(s^2+\omega^2) + (\alpha_2 s + \omega\alpha_3)\left(s + \frac{1}{RC}\right) \\
&= \alpha_1 s^2 + \omega^2\alpha_1 + \alpha_2 s^2 + \frac{1}{RC}\alpha_2 s + \omega\alpha_3 s + \frac{\omega}{RC}\alpha_3 \\
&= (\alpha_1+\alpha_2)s^2 + \left(\frac{1}{RC}\alpha_2 + \omega\alpha_3\right)s + \left(\omega^2\alpha_1 + \frac{\omega}{RC}\alpha_3\right) \qquad (3)
\end{aligned}
$$

Al igualar los coeficientes de potencias iguales de s se obtiene

$$
\left[\begin{array}{ccc|c}
1 & 1 & 0 & -\frac{V_0}{R} \\
0 & \frac{1}{RC} & \omega & \frac{\omega V_m}{R} \\
\omega^2 & 0 & \frac{\omega}{RC} & -\frac{\omega^2 V_0}{R}
\end{array}\right] R_3 \to R_3 - \omega^2 R_1
$$

$$
\sim \left[\begin{array}{ccc|c}
1 & 1 & 0 & -\frac{V_0}{R} \\
0 & \frac{1}{RC} & \omega & \frac{\omega V_m}{R} \\
0 & -\omega^2 & \frac{\omega}{RC} & 0
\end{array}\right] R_2 \to RCR_2
$$

$$
\sim \left[\begin{array}{ccc|c}
1 & 1 & 0 & -\frac{V_0}{R} \\
0 & 1 & \omega RC & \omega C V_m \\
0 & -\omega^2 & \frac{\omega}{RC} & 0
\end{array}\right] \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + \omega^2 R_2 \end{array}
$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\omega RC & -\frac{V_0}{R}-\omega CV_m \\ 0 & 1 & \omega RC & \omega CV_m \\ 0 & 0 & \frac{\omega+\omega^3R^2C^2}{RC} & \omega^3 CV_m \end{array}\right] R_3 \to \left(\frac{RC}{\omega+\omega^3R^2C^2}\right) R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -\omega RC & -\frac{V_0}{R}-\omega CV_m \\ 0 & 1 & \omega RC & \omega CV_m \\ 0 & 0 & 1 & \frac{\omega^2 RC^2 V_m}{1+\omega^2R^2C^2} \end{array}\right] \begin{array}{l} R_1 \to R_1 + \omega RC R_3 \\ R_2 \to R_2 - \omega RC R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{V_0}{R}-\omega CV_m\left(1-\frac{\omega^2R^2C^2}{1+\omega^2R^2C^2}\right) \\ 0 & 1 & \omega RC & \omega CV_m \\ 0 & 0 & 1 & \frac{\omega^2 RC^2 V_m}{1+\omega^2R^2C^2} \end{array}\right] a$$

Así

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\dfrac{V_0}{R}-\omega CV_m\left(1-\dfrac{\omega^2R^2C^2}{1+\omega^2R^2C^2}\right) \\ 0 & 1 & 0 & \omega CV_m\left(1-\dfrac{\omega^2R^2C^2}{1+\omega^2R^2C^2}\right) \\ 0 & 0 & 1 & \dfrac{\omega^2 RC^2 V_m}{1+\omega^2R^2C^2} \end{array}\right]$$

Se puede ver de inmediato que la solución es

$$\alpha_1 = -\frac{V_0}{R}-\omega CV_m\left(1-\frac{\omega^2R^2C^2}{1+\omega^2R^2C^2}\right) \tag{4}$$

$$\alpha_2 = \omega CV_m\left(1-\frac{\omega^2R^2C^2}{1+\omega^2R^2C^2}\right) \tag{5}$$

$$\alpha_3 = \frac{\omega^2 RC^2 V_m}{1+\omega^2R^2C^2} \tag{6}$$

Al sustituir (4), (5) y (6) en (2) obtenemos

$$\begin{aligned} I(s) = &\left[-\frac{V_0}{R}-\omega CV_m\left(1-\frac{\omega^2R^2C^2}{1+\omega^2R^2C^2}\right)\right]\left(\frac{1}{s+\frac{1}{RC}}\right) \\ &+\left[\omega CV_m\left(1-\frac{\omega^2R^2C^2}{1+\omega^2R^2C^2}\right)\right]\left(\frac{s}{s^2+\omega^2}\right) \\ &+\left(\frac{\omega^2 RC^2 V_m}{1+\omega^2R^2C^2}\right)\left(\frac{\omega}{s^2+\omega^2}\right) \end{aligned} \tag{7}$$

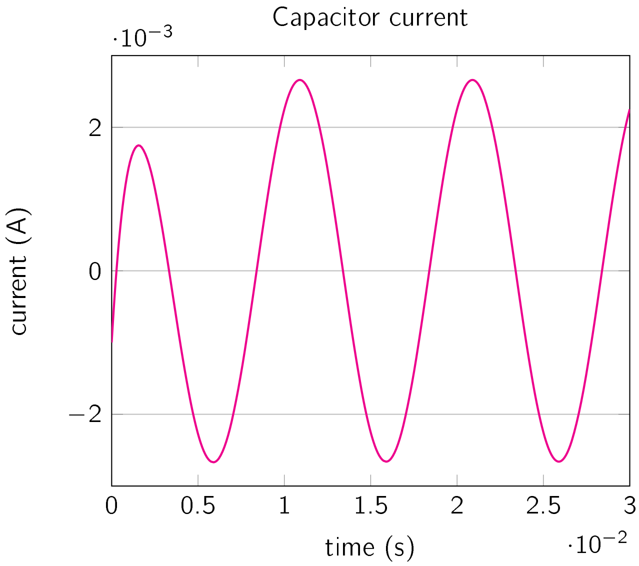
Al tomar la transformada inversa de Laplace en (7) obtenemos

$$\begin{aligned} \mathscr{L}^{-1}\{I(s)\} = &\mathscr{L}^{-1}\left\{\left[-\frac{V_0}{R}-\omega CV_m\left(1-\frac{\omega^2R^2C^2}{1+\omega^2R^2C^2}\right)\right]\left(\frac{1}{s+\frac{1}{RC}}\right)\right\} \\ &+\mathscr{L}^{-1}\left\{\left[\omega CV_m\left(1-\frac{\omega^2R^2C^2}{1+\omega^2R^2C^2}\right)\right]\left(\frac{s}{s^2+\omega^2}\right)\right\} \\ &+\mathscr{L}^{-1}\left\{\left(\frac{\omega^2 RC^2 V_m}{1+\omega^2R^2C^2}\right)\left(\frac{\omega}{s^2+\omega^2}\right)\right\} \end{aligned} \tag{8}$$

$$
\begin{aligned}
i(t) = \left[-\frac{V_0}{R} - \omega C V_m \left(1 - \frac{\omega^2 R^2 C^2}{1+\omega^2 R^2 C^2}\right)\right] e^{-\frac{t}{RC}} + \left[\omega C V_m \left(1 - \frac{\omega^2 R^2 C^2}{1+\omega^2 R^2 C^2}\right)\right] \cos(\omega t) \\
+ \left(\frac{\omega^2 R C^2 V_m}{1+\omega^2 R^2 C^2}\right) \sin(\omega t)
\end{aligned}
\tag{9}
$$

Ejemplo 1

Halle la corriente para el circuito de la figura 1, sea $V_m = 5\,\mathrm{V}$, $R = 1\,\mathrm{k\Omega}$, $C = 1\,\mu\mathrm{F}$, $f = 100\,\mathrm{Hz}$, asuma que el capacitor se encuentra inicialmente descargado, es decir $V_0 = 0\,\mathrm{V}$

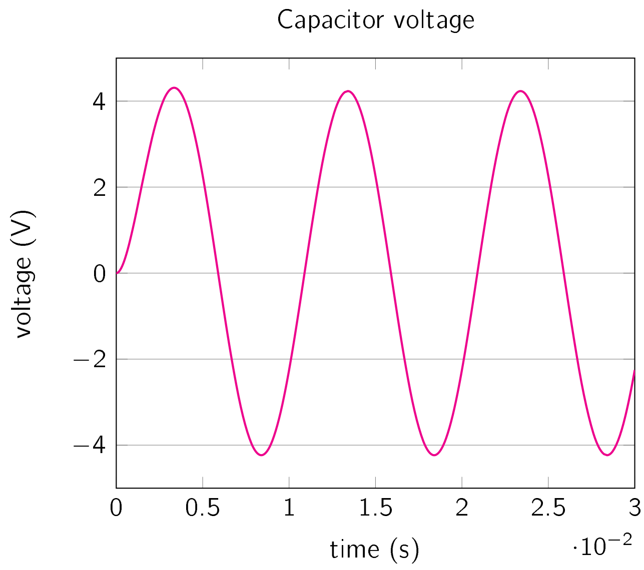


Una vez encontrada la corriente, hallamos el voltaje de salida

$$
\begin{aligned}
V_o(s) &= \frac{V_0}{s} + \frac{1}{sC} I(s) \\
&= \frac{V_0}{s} + \frac{1}{sC} \left\{ \left[-\frac{V_0}{R} - \omega C V_m \left(1 - \frac{\omega^2 R^2 C^2}{1+\omega^2 R^2 C^2}\right)\right] \left(\frac{1}{s + \frac{1}{RC}}\right) \right. \\
&+ \left[\omega C V_m \left(1 - \frac{\omega^2 R^2 C^2}{1+\omega^2 R^2 C^2}\right)\right] \left(\frac{s}{s^2+\omega^2}\right) \\
&+ \left. \left(\frac{\omega^2 R C^2 V_m}{1+\omega^2 R^2 C^2}\right) \left(\frac{\omega}{s^2+\omega^2}\right) \right\}
\end{aligned}
$$

En el dominio del tiempo

$$
\begin{aligned}
V_o(t) = V_0 + R \left[-\frac{V_0}{R} - \omega C V_m \left(1 - \frac{\omega^2 R^2 C^2}{1+\omega^2 R^2 C^2}\right)\right] \left[1 - e^{-\frac{t}{RC}}\right] \\
+ V_m \left(1 - \frac{\omega^2 R^2 C^2}{1+\omega^2 R^2 C^2}\right) \sin(\omega t) \\
+ \left(\frac{\omega^2 R C V_m}{1+\omega^2 R^2 C^2}\right) [1 - \cos(\omega t)]
\end{aligned}
\tag{10}
$$


Capacitor voltage
voltage (V)
4
2
0
−2
−4
0
0.5
1
1.5
2
2.5
3
time (s)
$\cdot 10^{-2}$