curl2 - An extension of Apophysis' curl-variation

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1 Specification

Our aim is to create a variation working in a higher dimension than *curl*. Important is, that we don't understand "higher dimension" as something like *"curl3D"* (which, by the way, already exists), but instead, a higher order in a polynomial equation within the complex number space.

Before we begin, we have to say a word about how complex numbers are utilized in Apophysis usually. The reader should have basic knowledge about what a complex number is and how it works.

A complex number z can be written as:

$$\mathrm{z = a + bi}$$

where a is the real part, b the imaginary part and i the imaginary unit $i = \sqrt{1}$.

In Apophysis, it is habit to interpret the real part as the x-coordinate and the imaginary part as the y-coordinate. This is why we will write x and y, not a and b in the rest of this document.
Now back to the variation. Right now, *curl* is defined as:

$$f(\mathrm{z}) = \frac{z}{c_2 z^2 + c_1 z + 1} \text{ where } z = x + iy$$

As we can see, the denominator of the fraction represents a polynomial equation of the second order similar to $y = ax^2 + bx + c$. What we would do now is that we raise the denominator to a polynomial equation of the third order $y = ax^3 + bx^2 + cx + d$:

$$f(\mathrm{z}) = \frac{z}{c_3 z^3 + c_2 z^2 + c_1 z + 1} \text{ where } z = x + iy$$

2 Approach

Essentially, we have to take the polynomial equation above and break it down. We first determine z^2, then z^3 and finally, we compose it into our new variation equation.

To square a complex number z, we use the first binomial formula:

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b$$

Therefore:

$$z^2 = (x+iy)^2 = x^2 + 2xyi + (yi)^2$$

We assume $i^2 = -1$ because $i = \sqrt{-1}$ and simplify to:

$$z^2 = x^2 + 2xyi - y^2$$

Cubing a complex number works the same way:

$$(a+b)^3 = (a+b)(a+b)(a+b) = a^3 + 3a^2b + 3ab^2 + b^3$$

Therefore:

$$z^3 = (x+iy)^3 = x^3 + 3x^2yi + 3x(yi)^2 + (yi)^3$$

Assuming $i^3 = -i$ because $i^2 = -1$ and $i^3 = i^2 \cdot i$:

$$z^3 = x^3 + 3x^2yi - 3xy^2 - y^3i$$

Technically, we could start building our formula for the new variation now but before we insert into our fraction from above, I would like to take it apart:

$$\text{f(z)} = \tfrac{z}{c_3z^3+c_2z^2+c_1z+1} = \tfrac{z}{q}$$

and focus on q for simplicity since we won't change the nominator. Let's expand the denominator then:

$$q = c_3z^3 + c_2z^2 + c_1z + 1$$
$$q = c_3(x^3 + 3x^2yi - 3xy^2 - y^3i) + c_2(x^2 + 2xyi - y^2) + c_1(x + yi) + 1$$
$$q = c_3x^3 + c_3x^2yi - 3c_3xy^2 - c_3y^3i + c_2x^2 + 2c_2xyi - c_2y^2 + c_1x + c_1yi + 1$$

Finally, we substitute back into the fraction $f(z) = \frac{z}{q}$:

$$f(z) = \frac{x+yi}{(c_3x^3-3c_3xy^2+c_2x^2-c_2y^2+c_1x+1)+(c_3x^2y-c_3y^3+2c_2xy+c_1y)*i}$$

To separate the real from the imaginary part, we have to factor out i from the whole fraction. If we look closely, we have a fraction in the form of:

$$f(z) = \frac{a+bi}{c+di}$$

We now have a division of two complex numbers $a+bi$ and $c+di$ so we determine the conjugate of $c+di$:

$$\overline{c+di} = c - di$$

We multiply numerator and denominator by the conjugate:

$$\begin{aligned} f(z) &= \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} \\ &= \frac{ac-adi+bci-bdi^2}{c^2-cdi+cdi-d^2i^2} \\ &= \frac{ac-adi+bci+bd}{c^2+d^2} \\ &= \frac{ac+bd}{c^2+d^2} + i \cdot \frac{bc-ad}{c^2+d^2} \end{aligned}$$

Now we will have to substitute the below back into our resulting fraction sum:

$$\begin{aligned} &a = x, \\ &b = y, \\ &c = c_3x^3 - 3c_3xy^2 + c_2x^2 - c_2y^2 + c_1x + 1, \\ &d = c_3x^2y - c_3y^3 + 2c_2xy + c_1y \end{aligned}$$

We can imagine, that actually writing out the entire fraction with the expressions above substituted back into it would take a lot of writing work and paper space. This would be the point where it makes sense to remember that we are writing a computer program.

In an Apophysis extension, we would calculate the result of c and d as well as $c^2 + d^2$ in the first step and then proceed with the rest of the expression. There is no necessity to provide the entire result in a single expression like we would do in the academical-mathematical world.

In order to extract a meaningful output for eventual plotting and/or the input of the next iteration, we would use the following method:

$$x' = x + \omega * Re(z')$$

$$y' = y + \omega * Im(z')$$

We define $\omega :=$ as the variation's density and assume it a constant at this point. Then we use some knowledge from *Inverse Geometry* (Frank Morley and son, 1933) and assume:

$$Re(z') = \frac{z' + \overline{z'}}{2}$$

$$Im(z') = \frac{z' - \overline{z'}}{2i}$$

Remembering our expression from above:

$$z' = f(z) = \frac{ac+bd}{c^2+d^2} + i \cdot \frac{bc-ad}{c^2+d^2}$$

We will receive:

$$Re(z') = \frac{ac+bd}{c^2+d^2}$$

$$Im(z') = \frac{bc+ad}{c^2+d^2}$$

And therefore:

$$x' = x + \omega \cdot \frac{ac+bd}{c^2+d^2}$$

$$y' = y + \omega \cdot \frac{bc+ad}{c^2+d^2}$$

The above set of equations (together with the correct substitutions for a, b, c, d) are fit to be used in the source code of an Apophysis extension.

3 Optimizations

The resulting variation should be easy to implement with simple algebra. Like with *curl*, several optimizations can be made. The most straightforward approach is to separate the following cases outside the iteration loop:

a.) $c_1 = 0$
b.) $c_2 = 0$
c.) $c_1 = c_2 = 0$

Precalculating $3c_3$ and $2c_2$ is reasonable as well since, while being extremely simple operations, these two multiplications can have a reasonable weight considering that they would be potentially executed multiple million times per second.