Stability, resonance, and frequency response identification of second species fractional transfer functions

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Abstract: This paper is a literature review of the stability, resonance, and frequency response identification of fractional order transfer functions of second species, without zeros.

Keywords: Frequency domain identification; Fractional systems; Stability; Resonance.

1. INTRODUCTION

This paper is concerned with the following explicit transfer function:

$$F_{\nu+\mu}(s) = \frac{1}{1 + 2\zeta\left(\frac{s}{\omega_0}\right)^{\nu} + \left(\frac{s}{\omega_0}\right)^{\nu+\mu}}, \tag{1}$$

Orders ν and μ are always considered positive, and we assume $\omega_0 > 0$. After this general case (1), three important particular cases are then considered separately:

$$F_{2\nu}(s) = \frac{1}{1 + 2\zeta\left(\frac{s}{\omega_0}\right)^{\nu} + \left(\frac{s}{\omega_0}\right)^{2\nu}}, \tag{2}$$

$$F_{2}(s) = \frac{1}{1 + 2\zeta\left(\frac{s}{\omega_0}\right)^{\nu} + \left(\frac{s}{\omega_0}\right)^{2}}, \tag{3}$$

$$F_{\nu+1}(s) = \frac{1}{1 + 2\zeta\left(\frac{s}{\omega_0}\right)^{\nu} + \left(\frac{s}{\omega_0}\right)^{\nu+1}}. \tag{4}$$

In (2), $\mu = \nu$ and the result is a commensurate transfer function; in (3), $\mu = 2 - \nu$, and, in (4), $\mu = 1$. Both these cases (3)–(4), as well as (1), are, in general, non-commensurate.

Remark. The arc of tangent, which appears when calculating the phase of a complex number, must be handled considering that the signs of both the imaginary and real parts are known, to obtain the correct angle not restricted to $[-\frac{\pi}{2}, +\frac{\pi}{2}]$. This applies to all arcs of tangent throughout the paper.

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2. GENERAL EXPLICIT SECOND SPECIES TRANSFER FUNCTION (1)

This case was studied in (Zhang et al., 2020), and can also be handled with the methodologies of (Turkulov et al., 2023; Malti et al., 2024).

2.1 Frequency response

The frequency response of (1) is

$$\begin{aligned} &F_{\nu+\mu}(j\omega) = \\ &= \frac{1}{1 + 2\zeta\left(\frac{j\omega}{\omega_0}\right)^{\nu} + \left(\frac{j\omega}{\omega_0}\right)^{\nu+\mu}} \\ &= \frac{1}{\begin{array}{c}1 + 2\zeta\cos\frac{\nu\pi}{2}\left(\frac{\omega}{\omega_0}\right)^{\nu} + \cos\frac{(\nu+\mu)\pi}{2}\left(\frac{\omega}{\omega_0}\right)^{\nu+\mu} \\ +j2\zeta\sin\frac{\nu\pi}{2}\left(\frac{\omega}{\omega_0}\right)^{\nu} + j\sin\frac{(\nu+\mu)\pi}{2}\left(\frac{\omega}{\omega_0}\right)^{\nu+\mu}\end{array}}. \end{aligned} \tag{5}$$

It is now expedient to make $\Omega = \frac{\omega}{\omega_0}$, since neither stability nor the existence of resonance peaks depend on ω_0, as we will see. This non-dimensional Ω may be called relative frequency. The corresponding gains (in decibel) and phases are

$$\begin{aligned} &20\log_{10}|F_{\nu+\mu}(\Omega)| = \\ &= -20\log_{10}\left|1 + 2\zeta\cos\frac{\nu\pi}{2}\Omega^{\nu} + \cos\frac{(\nu+\mu)\pi}{2}\Omega^{\nu+\mu}\right. \\ &\qquad \left. +j2\zeta\sin\frac{\nu\pi}{2}\Omega^{\nu} + j\sin\frac{(\nu+\mu)\pi}{2}\Omega^{\nu+\mu}\right| \end{aligned}$$

$$\begin{aligned}
&= -10\log_{10}\Bigg(1+4\zeta^2\cos^2\frac{\nu\pi}{2}\Omega^{2\nu} \\
&\qquad + \cos^2\frac{(\nu+\mu)\pi}{2}\Omega^{2\nu+2\mu} + 4\zeta\cos\frac{\nu\pi}{2}\Omega^{\nu} \\
&\qquad + 2\cos\frac{(\nu+\mu)\pi}{2}\Omega^{\nu+\mu} \\
&+ 4\zeta\cos\frac{\nu\pi}{2}\cos\frac{(\nu+\mu)\pi}{2}\Omega^{2\nu+\mu} + 4\zeta^2\sin^2\frac{\nu\pi}{2}\Omega^{2\nu} \\
&+4\zeta\sin\frac{\nu\pi}{2}\sin\frac{(\nu+\mu)\pi}{2}\Omega^{2\nu+\mu} + \sin^2\frac{(\nu+\mu)\pi}{2}\Omega^{2\nu+2\mu}\Bigg) \\
&= -10\log_{10}\Bigg(1+4\zeta^2\Omega^{2\nu}+\Omega^{2\nu+2\mu}+4\zeta\cos\frac{\nu\pi}{2}\Omega^{\nu} \\
&\qquad +2\cos\frac{(\nu+\mu)\pi}{2}\Omega^{\nu+\mu}+4\zeta\cos\frac{\mu\pi}{2}\Omega^{2\nu+\mu}\Bigg) \qquad (6)
\end{aligned}$$

$$\begin{aligned}
&\angle F_{\nu+\mu}(\Omega) = \\
&= \angle 1 - \angle\Bigg(1+2\zeta\cos\frac{\nu\pi}{2}\Omega^{\nu}+\cos\frac{(\nu+\mu)\pi}{2}\Omega^{\nu+\mu} \\
&\qquad +j2\zeta\sin\frac{\nu\pi}{2}\Omega^{\nu}+j\sin\frac{(\nu+\mu)\pi}{2}\Omega^{\nu+\mu}\Bigg) \\
&= -\arctan\frac{2\zeta\sin\frac{\nu\pi}{2}\Omega^{\nu}+\sin\frac{(\nu+\mu)\pi}{2}\Omega^{\nu+\mu}}{1+2\zeta\cos\frac{\nu\pi}{2}\Omega^{\nu}+\cos\frac{(\nu+\mu)\pi}{2}\Omega^{\nu+\mu}}. \qquad (7)
\end{aligned}$$

Fig. 1. Second species transfer function $F_{\nu+\mu}(s)$, given by (1), as a closed-loop, with $H(s)$ defined by (8) in the open-loop.

2.2 Stability

Second species transfer function $F_{\nu+\mu}(s)$, given by (1), can be seen as the transfer function of a closed-loop as shown in Fig. 1, consisting of transfer function

$$H(s) = \frac{1}{\left(\frac{s}{\omega_0}\right)^{\nu}\left(\left(\frac{s}{\omega_0}\right)^{\mu}+2\zeta\right)} \qquad (8)$$

in the direct branch of the open-loop, and of negative unit feedback, since

$$\frac{H(s)}{H(s)+1} = \frac{\dfrac{1}{\left(\frac{s}{\omega_0}\right)^{\nu+\mu}+2\zeta\left(\frac{s}{\omega_0}\right)^{\nu}}}{\dfrac{1}{\left(\frac{s}{\omega_0}\right)^{\nu+\mu}+2\zeta\left(\frac{s}{\omega_0}\right)^{\nu}}+1} = F_{\nu+\mu}(s). \qquad (9)$$

In this way, the stability of $F_{\nu+\mu}$ can be studied using the Nyquist criterion applied to the frequency response of open-loop H, which, together with the corresponding gain and phase, is

$$H(j\omega) = \frac{1}{\left(\frac{j\omega}{\omega_0}\right)^{\nu}\left(\left(\frac{j\omega}{\omega_0}\right)^{\mu}+2\zeta\right)} \Rightarrow$$

$$H(\Omega) = \frac{1}{j^{\nu}\Omega^{\nu}\left(j^{\mu}\Omega^{\mu}+2\zeta\right)} \qquad (10)$$

$$\begin{aligned}
|H(\Omega)| &= \frac{1}{\Omega^{\nu}\sqrt{\left(2\zeta+\Omega^{\mu}\cos\frac{\mu\pi}{2}\right)^2+\left(\Omega^{\mu}\sin\frac{\mu\pi}{2}\right)^2}} \\
&= \frac{1}{\Omega^{\nu}\sqrt{\Omega^{2\mu}+4\zeta^2+4\zeta\Omega^{\mu}\cos\frac{\mu\pi}{2}}} \qquad (11)
\end{aligned}$$

$$\angle H(\Omega) = -\frac{\nu\pi}{2}-\arctan\frac{\Omega^{\mu}\sin\frac{\mu\pi}{2}}{2\zeta+\Omega^{\mu}\cos\frac{\mu\pi}{2}}. \qquad (12)$$

We also need the phase crossover frequency Ω_{pc} of $H(\Omega)$, which could be obtained from phase (12). However, calculations turn out to be simpler by splitting expression (10) into real and imaginary parts, namely

$$\begin{aligned}
H(j\omega) &= \frac{1}{j^{\nu+\mu}\Omega^{\nu+\mu}+2\zeta j^{\nu}\Omega^{\nu}} \\
&= \frac{j^{-\nu-\mu}\Omega^{\nu+\mu}+2\zeta j^{-\nu}\Omega^{\nu}}{\left(j^{\nu+\mu}\Omega^{\nu+\mu}+2\zeta j^{\nu}\Omega^{\nu}\right)\left(j^{-\nu-\mu}\Omega^{\nu+\mu}+2\zeta j^{-\nu}\Omega^{\nu}\right)} \\
&= \frac{\left[\Omega^{\nu+\mu}\left(j\sin\frac{(-\nu-\mu)\pi}{2}+\cos\frac{(-\nu-\mu)\pi}{2}\right) +2\zeta\Omega^{\nu}\left(j\sin\frac{-\nu\pi}{2}+\cos\frac{-\nu\pi}{2}\right)\right]}{\Omega^{2\nu}\left(\Omega^{2\mu}+4\zeta^2+4\zeta\Omega^{\mu}\cos\frac{\mu\pi}{2}\right)} \\
&= \frac{\left[\left(\Omega^{\mu}\cos\frac{(\nu+\mu)\pi}{2}+2\zeta\cos\frac{\nu\pi}{2}\right) -j\left(\Omega^{\mu}\sin\frac{(\nu+\mu)\pi}{2}+2\zeta\sin\frac{\nu\pi}{2}\right)\right]}{\Omega^{\nu}\left(\Omega^{2\mu}+4\zeta^2+4\zeta\Omega^{\mu}\cos\frac{\mu\pi}{2}\right)},
\end{aligned} \qquad (13)$$

and then equating the imaginary part $\Omega^{\mu}\sin\frac{(\nu+\mu)\pi}{2}+2\zeta\sin\frac{\nu\pi}{2}$ to zero:

$$\Omega_{pc} = \left(\frac{-2\zeta\sin\frac{\nu\pi}{2}}{\sin\frac{(\nu+\mu)\pi}{2}}\right)^{\frac{1}{\mu}}. \qquad (14)$$

In the Nyquist diagram:

- at low frequencies, i.e. $\Omega\to 0^+$, the phase tends to $-\frac{\nu\pi}{2}$ (with a shift of π if $\zeta<0$ or $\mu>2$),
- at high frequencies, i.e. $\Omega\to+\infty$, the phase tends to $-\frac{(\nu+\mu)\pi}{2}$ (with a shift of π if $\mu>1$).
- the curve at infinity has an amplitude of $\nu\pi$.

Applying the Nyquist stability criterion, stability is obtained in the cases shown in Fig. 2, and can be summed up as follows — second species transfer function $F_{\nu+\mu}(s)$ is stable only in the following cases:

(1) $\zeta>0$ and $\nu+\mu\le 2$;
(2) $\zeta>0$, $\nu<2$, $\nu+\mu>2$, and

$$|H(\Omega_{pc})|<1 \Leftrightarrow \Omega_{pc}^{2\nu}\left(\Omega_{pc}^{2\mu}+4\zeta^2+4\zeta\Omega_{pc}^{\mu}\cos\frac{\mu\pi}{2}\right)>1, \qquad (15)$$

where Ω_{pc} is given by (14);
(3) $\zeta<0$, $\nu+\mu<2$, and

$$|H(\Omega_{pc})| > 1 \Leftrightarrow \Omega_{pc}^{2\nu}\left(\Omega_{pc}^{2\mu} + 4\zeta^2 + 4\zeta\Omega_{pc}^{\mu}\cos\frac{\mu\pi}{2}\right) < 1, \tag{16}$$

where Ω_{pc} is given by (14).

2.3 Resonance peaks

The frequency response of $F_{\nu+\mu}(s)$, given by (1), may have two resonance frequencies, or only one, or none. This can be found from (6), solving $\frac{\mathrm{d}}{\mathrm{d}\Omega}|F_{\nu+\mu}(\Omega)| = 0$. There is no known analytical solution, and thus numerical results such as those in Fig. 3 are computed.

2.4 System identification

The value of $\nu + \mu$ can be found from the behaviour at high frequencies, where:

- the phase is constant and equal to $-(\nu+\mu)\frac{\pi}{2}$;
- the gain varies linearly with the logarithm of the frequency, with a slope of $-20(\nu+\mu)$ dB/decade.

Even knowing $\nu + \mu$, the value of each order must be obtained numerically, e.g. with a metaheuristic. When setting up the metaheuristic, for each possible pair of values of ν and μ, the other parameters of the model can be found with the Levy method (Valério and Tejado, 2015).

3. EXPLICIT COMMENSURATE SECOND SPECIES TRANSFER FUNCTION (2)

3.1 Frequency response

The frequency response of (2), described in (Malti et al., 2011), is

$$F_{2\nu}(j\omega) = \frac{1}{1 + 2\zeta\left(\frac{j\omega}{\omega_0}\right)^{\nu} + \left(\frac{j\omega}{\omega_0}\right)^{2\nu}} \tag{17}$$

$$= \frac{1}{\left[\begin{array}{l}1 + 2\zeta\cos\frac{\nu\pi}{2}\left(\frac{\omega}{\omega_0}\right)^{\nu} + \cos(\nu\pi)\left(\frac{\omega}{\omega_0}\right)^{2\nu}\\ +j2\zeta\sin\frac{\nu\pi}{2}\left(\frac{\omega}{\omega_0}\right)^{\nu} + j\sin(\nu\pi)\left(\frac{\omega}{\omega_0}\right)^{2\nu}\end{array}\right]}.$$

It is now expedient to make $\Omega = \frac{\omega}{\omega_0}$, since neither stability nor the existence of resonance peaks depend on ω_0, as we will see. Again, this non-dimensional Ω may be called relative frequency. The corresponding gains (in decibel) and phases are

$$\begin{aligned}
&20\log_{10}|F_{2\nu}(\Omega)| = \\
&= -20\log_{10}\Big|1 + 2\zeta\cos\frac{\nu\pi}{2}\Omega^{\nu} + \cos(\nu\pi)\Omega^{2\nu} \\
&\qquad +j2\zeta\sin\frac{\nu\pi}{2}\Omega^{\nu} + j\sin(\nu\pi)\Omega^{2\nu}\Big| \\
&= -10\log_{10}\Big(1 + 4\zeta^2\cos^2\frac{\nu\pi}{2}\Omega^{2\nu} + \cos^2(\nu\pi)\Omega^{4\nu} \\
&\qquad + 4\zeta\cos\frac{\nu\pi}{2}\Omega^{\nu} + 2\cos(\nu\pi)\Omega^{2\nu} \\
&\qquad + 4\zeta\cos\frac{\nu\pi}{2}\cos(\nu\pi)\Omega^{3\nu} + 4\zeta^2\sin^2\frac{\nu\pi}{2}\Omega^{2\nu} \\
&\qquad +4\zeta\sin\frac{\nu\pi}{2}\sin(\nu\pi)\Omega^{3\nu} + \sin^2(\nu\pi)\Omega^{4\nu}\Big) \\
&= -10\log_{10}\Big(1 + 4\zeta^2\Omega^{2\nu} + \Omega^{4\nu} + 4\zeta\cos\frac{\nu\pi}{2}\Omega^{\nu} \\
&\qquad +2\cos(\nu\pi)\Omega^{2\nu} + 4\zeta\cos\frac{\nu\pi}{2}\Omega^{3\nu}\Big),
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\angle F_{c1}(\Omega) = \\
&= \angle 1 - \angle\Big(1 + 2\zeta\cos\frac{\nu\pi}{2}\Omega^{\nu} + \cos(\nu\pi)\Omega^{2\nu} \\
&\qquad +j2\zeta\sin\frac{\nu\pi}{2}\Omega^{\nu} + j\sin(\nu\pi)\Omega^{2\nu}\Big) \\
&= -\arctan\frac{2\zeta\sin\frac{\nu\pi}{2}\Omega^{\nu} + \sin(\nu\pi)\Omega^{2\nu}}{1 + 2\zeta\cos\frac{\nu\pi}{2}\Omega^{\nu} + \cos(\nu\pi)\Omega^{2\nu}}.
\end{aligned} \tag{19}$$

3.2 Stability

The results of Section 2.2 apply, making $\nu = \mu$, and thus $F_{2\nu}(j\omega)$, given by (2), is stable only in the following cases:

(1) $\zeta > 0$ and $\nu \le 1$;
(2) $\zeta > 0$, $1 < \nu < 2$, and

$$\begin{aligned}
\Omega_{pc} &= \left(\frac{-2\zeta\sin\frac{\nu\pi}{2}}{\sin(\nu\pi)}\right)^{\frac{1}{\nu}} \\
&= \left(\frac{-2\zeta\sin\frac{\nu\pi}{2}}{2\sin\frac{\nu\pi}{2}\cos\frac{\nu\pi}{2}}\right)^{\frac{1}{\nu}} = \left(\frac{-\zeta}{\cos\frac{\nu\pi}{2}}\right)^{\frac{1}{\nu}} \Rightarrow \\
&\left(\frac{-\zeta}{\cos\frac{\nu\pi}{2}}\right)^{2}\left[\left(\frac{-\zeta}{\cos\frac{\nu\pi}{2}}\right)^{2} \right.\\
&\qquad \left. +4\zeta^2 + 4\zeta\left(\frac{-\zeta}{\cos\frac{\nu\pi}{2}}\right)\cos\frac{\mu\pi}{2}\right] > 1 \\
&\Leftrightarrow \left(\frac{-\zeta}{\cos\frac{\nu\pi}{2}}\right)^{4} > 1
\end{aligned} \tag{20}$$

Since $1 < \nu < 2$, the cosine in the denominator is negative, and thus we arrive at $\zeta > -\cos\frac{\nu\pi}{2}$;

(3) $\zeta < 0$, $\nu < 1$, and

$$\begin{aligned}
\Omega_{pc} &= \left(\frac{-\zeta}{\cos\frac{\nu\pi}{2}}\right)^{\frac{1}{\nu}} \Rightarrow \\
&\left(\frac{-\zeta}{\cos\frac{\nu\pi}{2}}\right)^{2}\left[\left(\frac{-\zeta}{\cos\frac{\nu\pi}{2}}\right)^{2} \right.\\
&\qquad \left. +4\zeta^2 + 4\zeta\left(\frac{-\zeta}{\cos\frac{\nu\pi}{2}}\right)\cos\frac{\mu\pi}{2}\right] < 1 \\
&\Leftrightarrow \left(\frac{-\zeta}{\cos\frac{\nu\pi}{2}}\right)^{4} < 1
\end{aligned} \tag{21}$$

Since $\nu < 1$, the cosine in the denominator is positive, and thus we arrive again at $\zeta > -\cos\frac{\nu\pi}{2}$.

The zones of stability are shown in Fig. 4.

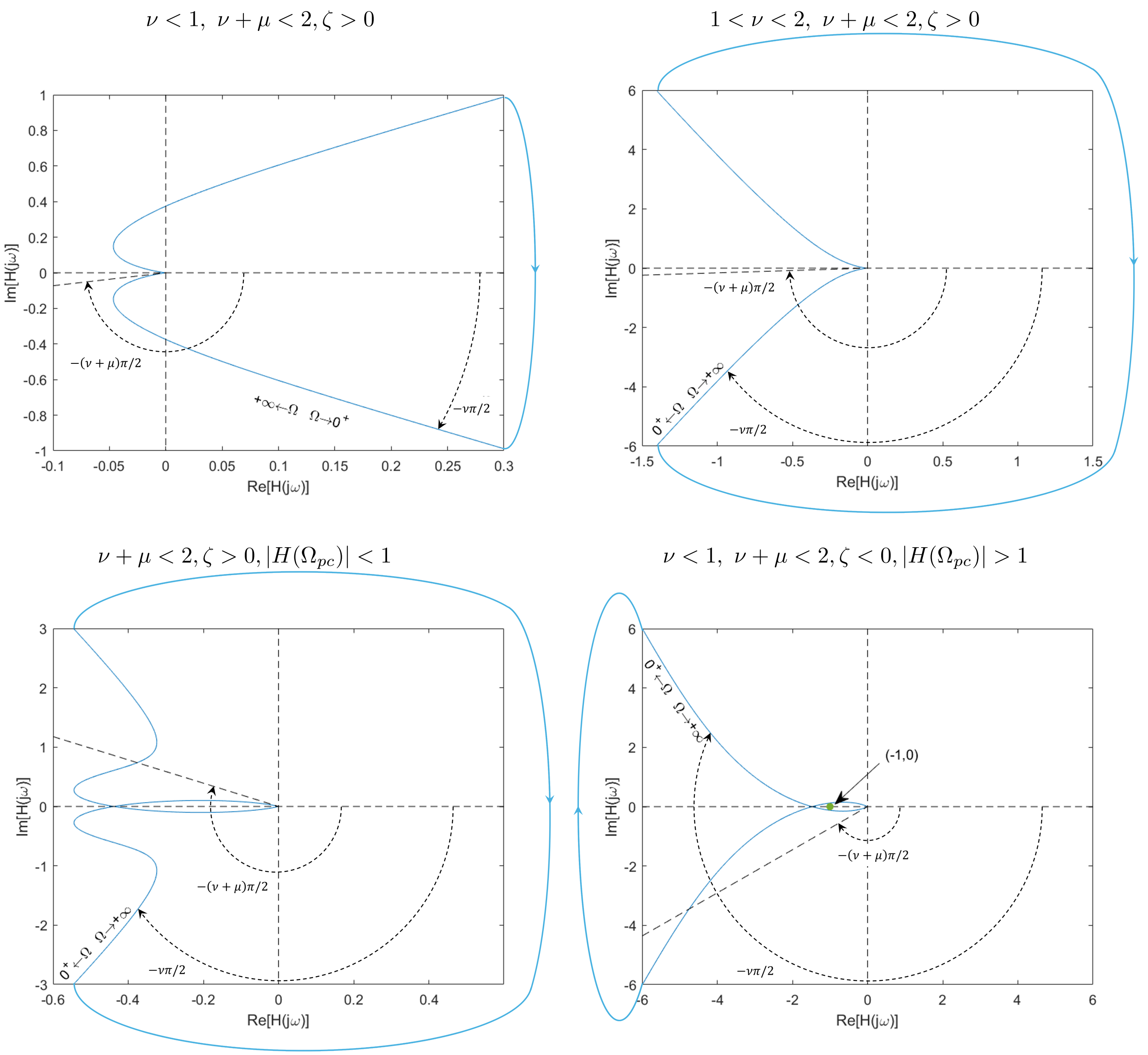

Fig. 2. Cases in which $F_{\nu+\mu}(s)$, given by (1), is stable, as seen from the Nyquist diagram of $H(s)$, given by (8). Top: $H(s)$ is stable, and there are never encirclements of point -1. Thus, $F_{\nu+\mu}(s)$ is stable: this is case 1 in Section 2.2. Bottom left: $H(s)$ is stable, and there is one encirclement, which does not encompass point -1, because the phase crossover frequency Ω_{pc} of $H(\Omega)$ is not large enough. Thus, $F_{\nu+\mu}(s)$ is stable: this is case 2 in Section 2.2. Bottom right: $H(s)$ has 1 unstable pole, and there is one counter-clockwise encirclement of point -1. Thus, $F_{\nu+\mu}(s)$ is stable: this is case 3 in Section 2.2.

Rather than applying the general case to (2), the Matignon theorem could be used (Matignon, 1998); the results are, of course, the same.

3.3 Resonance peaks: frequency

Just like the general case (1), the frequency response of $F_{\nu+\mu}(s)$, given by (1), may have two resonance frequencies, or only one, or none. But there are in fact four possibilities in what resonant frequencies are concerned, as seen in Fig. 4:

(a) there may be no resonant frequency, in which case the slope of the gain is always negative;
(b) there may be one resonant frequency, if the slope of the gain is positive for low frequencies, and negative after the resonance;
(c) there may be one resonant frequency, if the slope of the gain is negative for both low and high frequencies, but positive in-between;
(d) there may be two resonant frequencies, in which case (2) can be written as the product of two first-species transfer functions with the same fractional order ν (otherwise their product would be non-commensurate), and having one resonant peak each. Thus, $\nu > 1$, but this does not suffice: the peaks must occur at frequencies sufficiently far one from

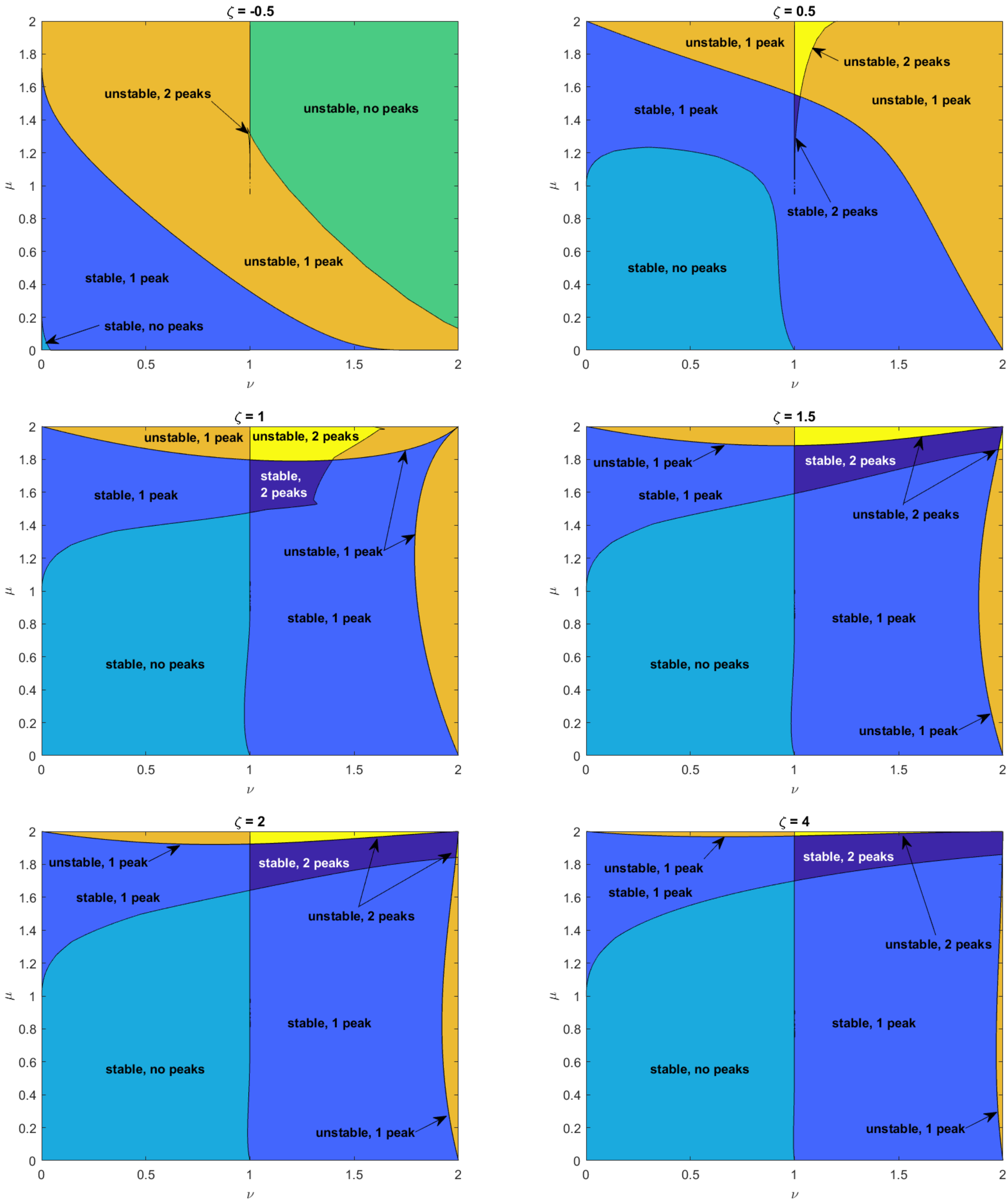

Fig. 3. Number of resonance peaks in $|F_{\nu+\mu}(j\omega)|$, found numerically. Regions where $|F_{\nu+\mu}(j\omega)|$ is unstable were found analytically as shown in section 2.2.

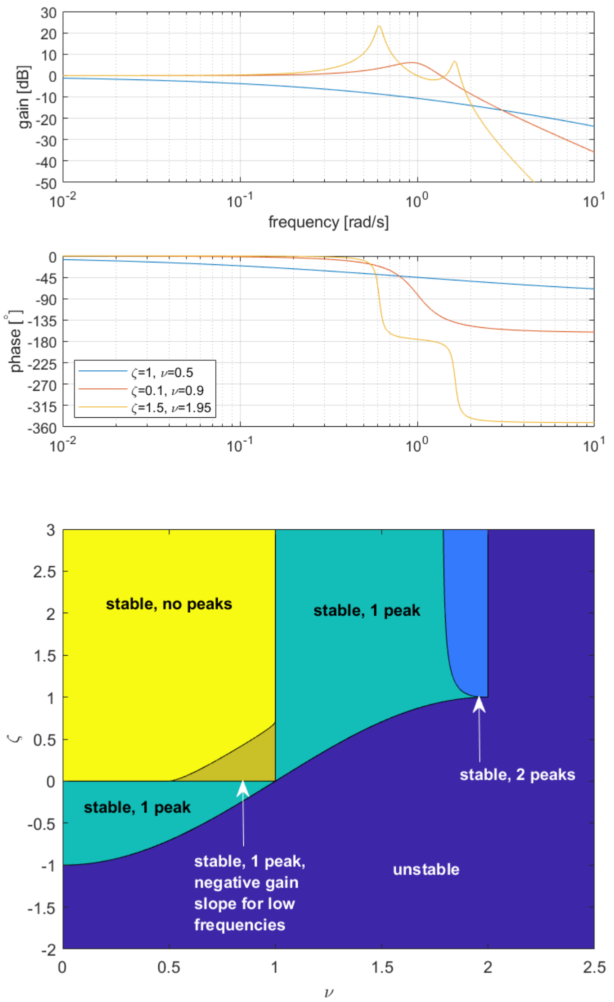

Fig. 4. Top: Bode diagram of (2) for $\omega_0 = 1$ rad/s, and values of ν and ζ as shown; there may be zero, one, or two resonance peaks. Bottom: zones of the (ν, ζ) plane where there are zero or one resonance peaks for (2). See Table 1.

the other (otherwise they merge), and the derivative of the gain close to the peak must be larger than 20ν dB/decade, so as to emerge from the negative slope of -20ν dB/decade caused by the first denominator root. See Fig. 5. This only happens for values of ν close to 2, as shown below.

The derivative of gain (18) is

$$\frac{\mathrm{d}20\log_{10}|F_{i2}(\Omega)|}{\mathrm{d}\Omega} = \frac{-10}{\log_e 10}\frac{\frac{\mathrm{d}}{\mathrm{d}\Omega}P(\Omega)}{P(\Omega)} \tag{22}$$

where

$$P(\Omega) = 1 + 4\zeta^2\Omega^{2\nu} + \Omega^{4\nu} + 4\zeta\cos\frac{\nu\pi}{2}\Omega^{\nu} + 2\cos(\nu\pi)\,\Omega^{2\nu} + 4\zeta\cos\frac{\nu\pi}{2}\Omega^{3\nu}. \tag{23}$$

Notice that, in (22), $\frac{-10}{\log_e 10}$ is a negative constant, and $P(\Omega)$ is positive (since it is the square of an absolute value). Thus, the derivative of gain (18) given by (22) is negative iff $\frac{\mathrm{d}}{\mathrm{d}\Omega}P(\Omega)$ is positive:

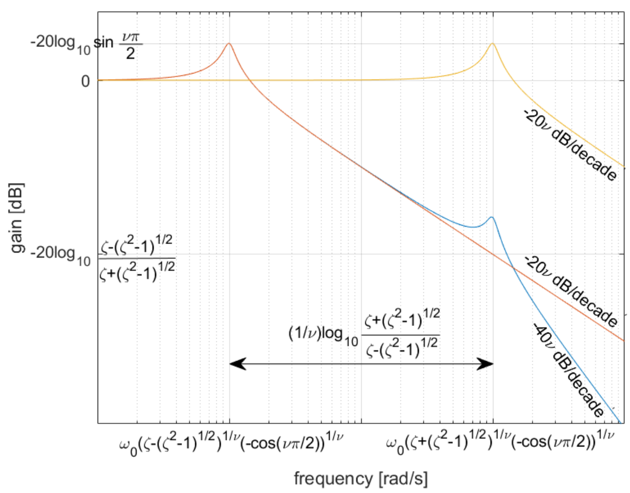

Fig. 5. (2) may have two resonance peaks, if it is the product of two first-species transfer functions with the same fractional order ν having a resonance frequency with a peak which is high enough.

$$\begin{aligned} &8\nu\zeta^2\Omega^{2\nu-1} + 4\nu\Omega^{4\nu-1} + 4\nu\zeta\cos\frac{\nu\pi}{2}\Omega^{\nu-1} \\ &\quad + 4\nu\cos(\nu\pi)\,\Omega^{2\nu-1} + 12\nu\zeta\cos\frac{\nu\pi}{2}\Omega^{3\nu-1} > 0 \Leftrightarrow \\ &\underbrace{\Omega^{3\nu} + 3\zeta\cos\frac{\nu\pi}{2}\Omega^{2\nu} + \left(2\zeta^2 + \cos\nu\pi\right)\Omega^{\nu} + \zeta\cos\frac{\nu\pi}{2}}_{P_3(\Omega^\nu)} > 0. \end{aligned} \tag{24}$$

In cases (a) and (c) above, this holds for low frequencies; making $\Omega \to 0$, we get

$$\zeta\cos\frac{\nu\pi}{2} > 0 \Leftrightarrow \begin{cases}\zeta > 0\\ 0 < \nu < 1\end{cases} \vee \begin{cases}\zeta < 0\\ 1 < \nu < 2\end{cases}. \tag{25}$$

In the last case, however, we know that the transfer function is not stable. Thus, the first must be true.

Notice that in the above the limit case $\nu = 1$ was not considered. In that case, transfer function (2) is integer; (24) reduces to

$$\Omega^3 + \left(2\zeta^2 - 1\right)\Omega > 0 \Leftrightarrow \Omega^2 > 1 - 2\zeta^2. \tag{26}$$

This will be true for low frequencies if

$$1 - 2\zeta^2 < 0 \Leftrightarrow \zeta > \frac{\sqrt{2}}{2}, \tag{27}$$

which is the well known condition for case (a) when the transfer function is integer; otherwise, we will have case (b). The limit case $\zeta = 0$ was also not considered; (24) reduces to

$$\Omega^{3\nu} + \cos(\nu\pi)\Omega^{\nu} > 0 \Leftrightarrow \Omega^{2\nu} + \cos(\nu\pi) > 0. \tag{28}$$

This will be true for low frequencies if

$$\cos(\nu\pi) > 0 \Leftrightarrow 0 < \nu < \frac{1}{2}. \tag{29}$$

If so, we have case (a); otherwise, we will have case (c).

To tell apart case (a) from case (c) when $\zeta > 0 \wedge 0 < \nu < 1$, it is necessary to further examine condition (24), in which $P_3(\Omega^\nu)$ is a third order polynomial in Ω^ν. This equation can be solved analytically, with long calculations. This is done in (Boubidi et al., 2021), resulting in condition

$$\zeta = \cos\frac{\nu\pi}{2}\left(-1+\tan^{\frac{2}{3}}\frac{\nu\pi}{2}\right)\sqrt{\frac{1-\tan^{\frac{4}{3}}\frac{\nu\pi}{2}}{1-2\tan^{\frac{2}{3}}\frac{\nu\pi}{2}}}. \qquad (30)$$

It turns out that (30) separates case (a) from case (c) when $0.5 \leq \nu \leq 1$, and a similar expression

$$\zeta = \cos\frac{\nu\pi}{2}\left(-1+\tan^{\frac{2}{3}}\frac{-\nu\pi}{2}\right)\sqrt{\frac{1-\tan^{\frac{4}{3}}\frac{-\nu\pi}{2}}{1-2\tan^{\frac{2}{3}}\frac{-\nu\pi}{2}}} \qquad (31)$$

separates case case (b) from case (d) when ν is close to 2. The lowest value of ν for which there can be two resonance peaks is the root of the denominator in (31), i.e. the value of ν for which (31) has a vertical asymptote:

$$\begin{aligned} & 1-2\tan^{\frac{2}{3}}\frac{-\nu\pi}{2}=0 \\ \Leftrightarrow\ & \tan\left(\pi-\frac{\nu\pi}{2}\right)=\left(\frac{1}{2}\right)^{\frac{2}{3}} \\ \Leftrightarrow\ & \frac{(2-\nu)\pi}{2}=\arctan\left(\frac{1}{2}\right)^{\frac{2}{3}} \\ \Leftrightarrow\ & \nu = 2-\frac{2}{\pi}\arctan\left(\frac{1}{2}\right)^{\frac{2}{3}}. \end{aligned} \qquad (32)$$

(For details, see (Boubidi et al., 2021), where, however, the expression (31) of case (d) is mistakenly given, and (32) is only found numerically.)

The division of the plane shown in Fig. 4 corresponds to the four cases in Table 1.

3.4 Resonance peaks: magnitude

The maximum value of the gain when there are two resonance peaks is always found at the first, since the two peaks result from the superposition of two systems with one root (as shown in Fig. 5), both with a maximum value of the gain given by (Malti et al., 2011; Valério and Sá da Costa, 2013)

$$\max 20\log_{10}\left|\frac{1}{1+\Omega^\nu}\right| = -20\log_{10}\left|\sin\frac{\nu\pi}{2}\right| \qquad (33)$$

at frequency

$$\omega = \omega_0\left(-\cos\frac{\nu\pi}{2}\right)^{\frac{1}{\nu}}. \qquad (34)$$

Consequently, the peak value can be approximated by (33):

$$\max 20\log_{10}|F_{i2}(\Omega)| > -20\log_{10}\left|\sin\frac{\nu\pi}{2}\right|. \qquad (35)$$

The actual value is always larger, since the contribution of the other pole is being neglected. A better approximation can be found using the exact expression of the gain (18) with an approximation of the frequency peak. Combining the denominator roots of (2)

$$\left(\frac{s}{\omega_0}\right)^\nu = \frac{-2\zeta\pm\sqrt{4\zeta^2-4}}{2} \Leftrightarrow s^\nu = \omega_0^\nu\left(-\zeta\pm\sqrt{\zeta^2-1}\right) \qquad (36)$$

with peak frequency (34) results in

$$\omega_1 \approx \omega_0\left(-\zeta-\sqrt{\zeta^2-1}\right)^{\frac{1}{\nu}}\left(-\cos\frac{\nu\pi}{2}\right)^{\frac{1}{\nu}}, \qquad (37)$$

$$\omega_2 \approx \omega_0\left(-\zeta+\sqrt{\zeta^2-1}\right)^{\frac{1}{\nu}}\left(-\cos\frac{\nu\pi}{2}\right)^{\frac{1}{\nu}}. \qquad (38)$$

The actual values will be below ω_1 and above ω_2, since the contribution of each pole on the other's is being neglected. They are separated by

$$\begin{aligned} \log_{10}\Delta\omega &= \log_{10}\frac{\omega_0\left(-\zeta+\sqrt{\zeta^2-1}\right)^{\frac{1}{\nu}}\left(-\cos\frac{\nu\pi}{2}\right)^{\frac{1}{\nu}}}{\omega_0\left(-\zeta-\sqrt{\zeta^2-1}\right)^{\frac{1}{\nu}}\left(-\cos\frac{\nu\pi}{2}\right)^{\frac{1}{\nu}}} \\ &= \frac{1}{\nu}\log_{10}\frac{-\zeta+\sqrt{\zeta^2-1}}{-\zeta-\sqrt{\zeta^2-1}} \\ &= \frac{2}{\nu}\log_{10}\left(-\zeta+\sqrt{\zeta^2-1}\right) \text{ decades.} \end{aligned} \qquad (39)$$

If $\log_{10}\Delta\omega$ is one decade or more, it is reasonable to approximate the second peak from the gain slope of the first, which is -20ν dB/decade:

$$|G(j\omega_2)| \approx -20\log 10\frac{-\zeta+\sqrt{\zeta^2-1}}{-\zeta-\sqrt{\zeta^2-1}} - 20\log_{10}\left|\sin\frac{\nu\pi}{2}\right|. \qquad (40)$$

Expressions (35)–(37) can be used when there is only one resonance peak, and $\zeta > 1$. (When $\zeta < 1$, results are inaccurate.)

3.5 Identification

The order ν can be found from the from the behaviour at high frequencies, where

- the phase is constant and equal to $-\nu\pi$;
- the gain varies linearly with the logarithm of the frequency, with a -40ν dB/decade slope.

This is the only practical method if there is no resonance peak. If there are two resonance peaks, we can estimate ν from the gain of the first resonance peak, inverting (35):

$$\nu = \frac{2}{\pi}\arcsin\frac{1}{\max|F_{i1}(\Omega)|}. \qquad (41)$$

ν will then be overestimated. If there is only one resonance peak, and if $\zeta > 1$, we can apply the same expression.

The other parameters can then be found with the Levy method (Valério et al., 2008).

4. EXPLICIT NON-COMMENSURATE SECOND SPECIES TRANSFER FUNCTION (3)

This case was studied in (Ivanova et al., 2015).

4.1 Frequency response

The frequency response of (3) is

$$\begin{aligned} F_2(j\omega) &= \frac{1}{1+2\zeta\left(\frac{j\omega}{\omega_0}\right)^\nu+\left(\frac{j\omega}{\omega_0}\right)^2} \\ &= \frac{1}{1+2\zeta\cos\frac{\nu\pi}{2}\left(\frac{\omega}{\omega_0}\right)^\nu-\left(\frac{\omega}{\omega_0}\right)^2+j2\zeta\sin\frac{\nu\pi}{2}\left(\frac{\omega}{\omega_0}\right)^\nu}. \end{aligned} \qquad (42)$$

It is now expedient to make $\Omega = \frac{\omega}{\omega_0}$, since neither stability nor the existence of resonance peaks depend on ω_0, as we will see. Once more, this non-dimensional Ω may be called

Table 1. Analytical conditions for the four cases shown in Fig. 4, adapted and corrected from (Boubidi et al., 2021). If system (2) is not unstable, it can be (a) stable without a resonance peak, (b) stable with one resonance peak and a positive slope of the gain for low frequencies, (c) stable with one resonance peak and a negative slope of the gain for low frequencies, or (d) stable with two resonance peaks.

Unstable	$\nu \geq 2$	$\vee$	$\zeta \leq -\cos\frac{\nu\pi}{2}$
(a) No resonant frequencies	$\left(0 < \nu \leq \frac{1}{2}\right.$	$\wedge$	$\left.\zeta \geq 0\right) \vee$
	$\vee\left(\frac{1}{2} < \nu \leq 1\right.$	$\wedge$	$\left.\zeta \geq \cos\frac{\nu\pi}{2}\left(-1+\tan^{\frac{2}{3}}\frac{\nu\pi}{2}\right)\sqrt{\frac{1-\tan^{\frac{4}{3}}\frac{\nu\pi}{2}}{1-2\tan^{\frac{2}{3}}\frac{\nu\pi}{2}}}\right)$
(b) One resonant frequency, positive slope of the gain for low frequencies	When none of the others hold		
(c) One resonant frequency, negative slope of the gain for low frequencies	$\frac{1}{2} < \nu < 1$	$\wedge$	$0 < \zeta < \cos\frac{\nu\pi}{2}\left(-1+\tan^{\frac{2}{3}}\frac{\nu\pi}{2}\right)\sqrt{\frac{1-\tan^{\frac{4}{3}}\frac{\nu\pi}{2}}{1-2\tan^{\frac{2}{3}}\frac{\nu\pi}{2}}}$
(d) Two resonant frequencies	$\frac{2}{\pi}\arctan\left(\left(\frac{1}{2}\right)^{\frac{3}{2}}\right) < \nu < 2$	$\wedge$	$\zeta > \cos\frac{\nu\pi}{2}\left(-1+\tan^{\frac{2}{3}}\frac{-\nu\pi}{2}\right)\sqrt{\frac{1-\tan^{\frac{4}{3}}\frac{-\nu\pi}{2}}{1-2\tan^{\frac{2}{3}}\frac{-\nu\pi}{2}}}$

relative frequency. The corresponding gains (in decibel) and phases are

$$\begin{aligned}
&20\log_{10}|F_2(\Omega)| = \\
&= -20\log_{10}\left|1+2\zeta\cos\frac{\nu\pi}{2}\Omega^\nu - \Omega^2 + j2\zeta\sin\frac{\nu\pi}{2}\Omega^\nu\right| \\
&= -10\log_{10}\left(1+4\zeta^2\cos^2\frac{\nu\pi}{2}\Omega^{2\nu}+\Omega^4\right. \\
&\quad +4\zeta\cos\frac{\nu\pi}{2}\Omega^\nu - 2\Omega^2 \\
&\quad \left.-4\zeta\cos\frac{\nu\pi}{2}\Omega^{\nu+2} + 4\zeta^2\sin^2\frac{\nu\pi}{2}\Omega^{2\nu}\right) \\
&= -10\log_{10}\left(1+4\zeta^2\Omega^{2\nu}+\Omega^4+4\zeta\cos\frac{\nu\pi}{2}\Omega^\nu-2\Omega^2\right. \\
&\quad \left.+4\zeta\cos\frac{\nu\pi}{2}\Omega^{\nu+2}\right),
\end{aligned} \tag{43}$$

$$\begin{aligned}
&\angle F_2(\Omega) = \\
&= \angle 1 - \angle\left(1+2\zeta\cos\frac{\nu\pi}{2}\Omega^\nu-\Omega^2+j2\zeta\sin\frac{\nu\pi}{2}\Omega^\nu\right) \\
&= -\arctan\frac{2\zeta\sin\frac{\nu\pi}{2}\Omega^\nu}{1+2\zeta\cos\frac{\nu\pi}{2}\Omega^\nu-\Omega^2}.
\end{aligned} \tag{44}$$

4.2 Stability

The results of Section 2.2 apply, making $\mu = 2-\nu$, and thus $F_2(j\omega)$, given by (3), is stable only if $\zeta > 0$, which corresponds to case 1. The other two cases do not apply because:

(2) $\zeta > 0$, $\nu < 2$, $\nu + 2 - \nu > 2$ is impossible;
(3) $\zeta < 0$, $\nu + 2 - \nu < 2$ is also impossible.

4.3 Resonance peaks

The frequency response of F_2, given by (3), may have two resonance frequencies, or only one, or none. Just as for (1), in Section 2.3, there is no known analytical solution, and thus numerical results such as those in Fig. 6 have to be found. (The plot in (Ivanova et al., 2015) neglects resonant frequencies with peaks below some unspecified threshold, and thus differs from this one.)

4.4 Identification

The value of ν must be obtained numerically, e.g. with a metaheuristic. For each possible value of ν, the other

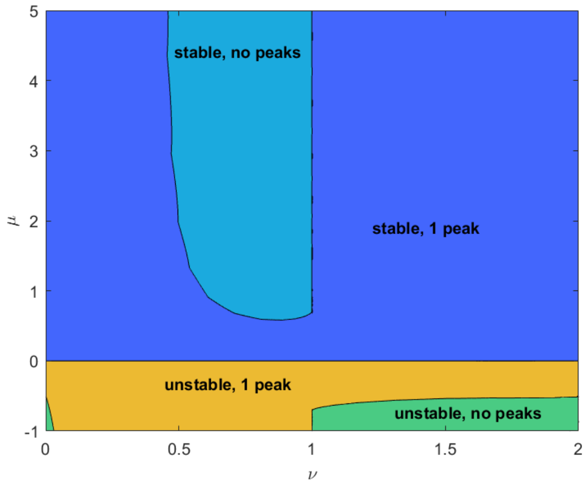

Fig. 6. Number of resonance peaks in $|F_{\nu+\mu}(j\omega)|$, found numerically.

parameters can be found with the Levy method (Valério and Tejado, 2015).

5. EXPLICIT NON-COMMENSURATE SECOND SPECIES TRANSFER FUNCTION (4)

This case was studied in (Hmed et al., 2015).

5.1 Frequency response

The frequency response of (1) is

$$F_{\nu+1}(j\omega) =$$

$$= \frac{1}{1+2\zeta\left(\frac{j\omega}{\omega_0}\right)^\nu + \left(\frac{j\omega}{\omega_0}\right)^{\nu+1}}$$

$$= \frac{1}{1+2\zeta\cos\frac{\nu\pi}{2}\left(\frac{\omega}{\omega_0}\right)^\nu + \cos\left(\frac{\nu\pi}{2}+\frac{\pi}{2}\right)\left(\frac{\omega}{\omega_0}\right)^{\nu+1} + j2\zeta\sin\frac{\nu\pi}{2}\left(\frac{\omega}{\omega_0}\right)^\nu + j\sin\left(\frac{\nu\pi}{2}+\frac{\pi}{2}\right)\left(\frac{\omega}{\omega_0}\right)^{\nu+1}}$$

$$= \frac{1}{1+2\zeta\cos\frac{\nu\pi}{2}\left(\frac{\omega}{\omega_0}\right)^\nu - \sin\frac{\nu\pi}{2}\left(\frac{\omega}{\omega_0}\right)^{\nu+1} + j2\zeta\sin\frac{\nu\pi}{2}\left(\frac{\omega}{\omega_0}\right)^\nu + j\cos\frac{\nu\pi}{2}\left(\frac{\omega}{\omega_0}\right)^{\nu+1}}. \quad (45)$$

It is now expedient to make $\Omega = \frac{\omega}{\omega_0}$, since neither stability nor the existence of resonance peaks depend on ω_0, as we will see. Again, this non-dimensional Ω may be called relative frequency. The corresponding gains (in decibel) and phases are

$$\begin{aligned}
&20\log_{10}|F_{\nu+1}(\Omega)| = \\
&= -20\log_{10}\Big|1+2\zeta\cos\frac{\nu\pi}{2}\Omega^\nu - \sin\frac{\nu\pi}{2}\Omega^{\nu+1} \\
&\quad +j2\zeta\sin\frac{\nu\pi}{2}\Omega^\nu + j\cos\frac{\nu\pi}{2}\Omega^{\nu+1}\Big| \\
&= -10\log_{10}\Big(1+4\zeta^2\cos^2\frac{\nu\pi}{2}\Omega^{2\nu} \\
&\quad +\sin^2\frac{(\nu+\mu)\pi}{2}\Omega^{2\nu+2} + 4\zeta\cos\frac{\nu\pi}{2}\Omega^\nu \\
&\quad -2\sin\frac{\nu\pi}{2}\Omega^{\nu+1} - 4\zeta\cos\frac{\nu\pi}{2}\sin\frac{\nu\pi}{2}\Omega^{2\nu+1} \\
&\quad +4\zeta^2\sin^2\frac{\nu\pi}{2}\Omega^{2\nu} + 4\zeta\sin\frac{\nu\pi}{2}\cos\frac{\nu\pi}{2}\Omega^{2\nu+1} \\
&\quad +\cos^2\frac{\nu\pi}{2}\Omega^{2\nu+2}\Big) \\
&= -10\log_{10}\Big(1+4\zeta^2\Omega^{2\nu}+\Omega^{2\nu+2} \\
&\quad +4\zeta\cos\frac{\nu\pi}{2}\Omega^\nu - 2\sin\frac{\nu\pi}{2}\Omega^{\nu+1}\Big), \qquad (46)
\end{aligned}$$

$$\begin{aligned}
&\angle F_{\nu+1}(\Omega) = \\
&= \angle 1 - \angle\Big(1+2\zeta\cos\frac{\nu\pi}{2}\Omega^\nu - \sin\frac{\nu\pi}{2}\Omega^{\nu+1} \\
&\quad +j2\zeta\sin\frac{\nu\pi}{2}\Omega^\nu + j\cos\frac{\nu\pi}{2}\Omega^{\nu+1}\Big) \\
&= -\arctan\frac{2\zeta\sin\frac{\nu\pi}{2}\Omega^\nu + \cos\frac{\nu\pi}{2}\Omega^{\nu+1}}{1+2\zeta\cos\frac{\nu\pi}{2}\Omega^\nu - \sin\frac{\nu\pi}{2}\Omega^{\nu+1}}. \qquad (47)
\end{aligned}$$

5.2 Stability

Applying the results of Section 2.2 with $\mu = 1$, we can find the stability conditions for $F_{\nu+1}(j\omega)$, given by (4):

(1) $\zeta > 0$ and $\nu+1 \le 2 \Leftrightarrow \nu \le 1$;
(2) $\zeta > 0$, $\nu < 2$, $\nu+1 > 2 \Leftrightarrow \nu > 1$, and

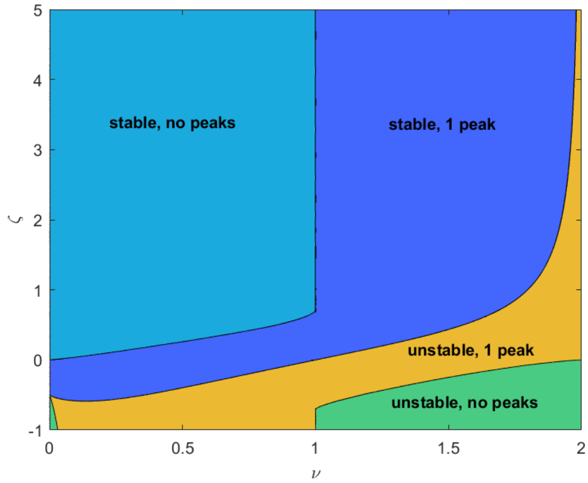

Fig. 7. Number of resonance peaks in $|F_{\nu+1}(j\omega)|$, found numerically. The vertical limit at $\nu = 1$ between the zones with one resonance peak and no resonance peak stops at $\zeta = \frac{\sqrt{2}}{2}$, as seen in (27) for an integer second order system.

$$\Omega_{pc} = \frac{-2\zeta\sin\frac{\nu\pi}{2}}{\sin\left(\frac{\nu+\pi}{2}+\frac{\pi}{2}\right)} = \frac{-2\zeta\sin\frac{\nu\pi}{2}}{\cos\frac{\nu+\pi}{2}} = -2\zeta\tan\frac{\nu\pi}{2}, \qquad (48)$$

$$\begin{aligned}
&|H(\Omega_{pc})| < 1 \Leftrightarrow \\
&\left(-2\zeta\tan\frac{\nu\pi}{2}\right)^{2\nu}\left[\left(-2\zeta\tan\frac{\nu\pi}{2}\right)^2 + 4\zeta^2\right] > 1 \Leftrightarrow \\
&\left(-2\zeta\tan\frac{\nu\pi}{2}\right)^{2\nu+2} + 4\zeta^2\left(-2\zeta\tan\frac{\nu\pi}{2}\right)^{2\nu} > 1; \qquad (49)
\end{aligned}$$

(3) $\zeta < 0$, $\nu+1 < 2$, and $|H(\Omega_{pc})| > 1 \Leftrightarrow \left(-2\zeta\tan\frac{\nu\pi}{2}\right)^{2\nu+2} + 4\zeta^2\left(-2\zeta\tan\frac{\nu\pi}{2}\right)^{2\nu} < 1$.

5.3 Resonance peaks

The frequency response of $F_{\nu+1}(s)$, given by (3), may have one resonance frequency, or none. ((Hmed et al., 2015) prove that there can be no two resonance frequencies.) Just as for (1), in Section 2.3, there is no known analytical solution, and thus numerical results such as those in Fig. 7 have to be found.

5.4 Identification

The order ν can be found from the from the behaviour at high frequencies, where

- the phase is constant and equal to $-(\nu+1)\frac{\pi}{2}$;
- the gain varies linearly with the logarithm of the frequency, with a $-20(\nu+1)$ dB/decade slope.

The other parameters can then be found with the Levy method (Valério and Tejado, 2015).

6. CONCLUSION

This paper presented a literature review of the stability, resonance, and frequency response identification of fractional order transfer functions (1)–(4). Results are summed up in Table 2.

Table 2. Summing up the results of this paper's literature review.

Transfer function	(1)	(2)	(3)	(4)
Bibliographical references	(Zhang et al., 2020) (Turkulov et al., 2023) (Malti et al., 2024)	(Malti et al., 2011)	(Ivanova et al., 2015)	(Hmed et al., 2015)
Frequency response	(5)	(17)	(42)	(45)
Gain in dB	(6)	(18)	(43)	(46)
Phase	(7)	(19)	(44)	(47)
Stability conditions	Three conditions at the end of section 2.2	Three conditions in section 3.2; also see Fig. 4	$\zeta > 0$	Three conditions in section 5.2
Resonance	Fig. 3	Fig. 4, Table 1	Fig. 6	Fig. 7
Order identification	Section 2.4	Section 3.5	Section 4.4	Section 5.4
Levy method	(Valério and Tejado, 2015)	(Valério et al., 2008)	(Valério and Tejado, 2015)	(Valério and Tejado, 2015)

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