Generalized probabilistic theories in a nutshell

Rodrigo Ramos

December 10, 2023

Structure of this presentation

Operational Theories

- The main primitives of any operational theory are:

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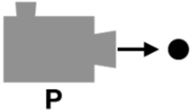

Preparations

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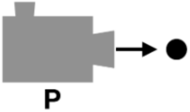

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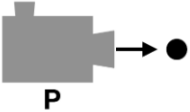

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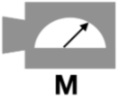

Measurements

Operational Theories

- The main primitives of any operational theory are:

P

Preparations

T

Transformations

M

Measurements

$$p(a \mid P, T, M)$$

Probabilities of outcomes

States and state space

- Let P and P' be distinct preparation procedures. If for any measurement procedure M and outcome a it is true that

$$p(a \mid P, M) = p(a \mid P', M),$$

then we say that P and P' are equivalent.

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Definition

A state is defined as an equivalence class of preparation procedures.

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Let K be a state space. We say that $x \in K$ is an extreme point of K, or equivalently that x is a *pure state*, if for all $y, z \in K$ and $\lambda \in (0, 1)$ such that $x = \lambda y + (1 - \lambda) z$, we have $x = y = z$.

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Naturally, the non-pure states are called *mixed states*.

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Theorem

Let K be a state space and denote by $ext(K)$ the set of pure states of K, then $K = conv(ext(K))$[a]*.*

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In other words, knowing the pure states of a state space is enough to reconstruct it.

Effects and effect algebra

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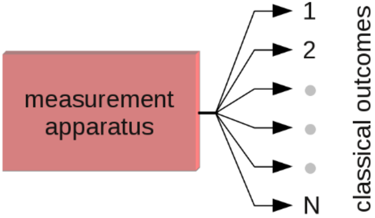

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Definition

Let K be a state space and denote by $A(K)$ the vector space of affine[a] functions on K. The *effect algebra* over K is defined as

$$E(K) \doteq \{f \in A(K) : \; 0 \leq f \leq 1_K\},$$

where 1_K is the constant function $1_K(x) = 1 \; \forall x \in K$.

[a]This means that $f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)$

Effects and effect algebra

- Besides $E(K)$, another important subset of $A(K)$ is the cone of positive functionals:

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and its dual $A(K)^{*+}$.

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and its dual $A(K)^{*+}$.

- One can construct the state space starting from $A(K)^{*+}$:

$$K = \{\varphi \in A(K)^{*+} : \ \langle \varphi | 1_K \rangle = 1\}.$$

Finite-dimensional Classical theory

- In a classical theory with $n \in \mathbb{N}$ independent outcomes, the state space is the simplex

$$S_n \doteq conv(\{s_1, \ldots, s_n\}),$$

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where s_j denotes the deterministic probability distribution associated to the j-th outcome.

- The effect algebra $E(S_n)$ is defined by the effects $b_1, \ldots, b_n$ that satisfy

$$\langle s_i | b_j \rangle = \delta_{ij}.$$

Bipartite systems

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  - BP4) The unit effect $1_{K_{AB}}$ is equivalent to Alice and Bob applying their unit effects 1_{K_A} and 1_{K_B}.
  - BP5) Tomographic locality holds, i.e local effects applied by Alice and Bob are sufficient to distinguish states in K_{AB}.

Bipartite systems

- Another valid bipartite state space is

$$K_A \otimes_{max} K_B \doteq \{\varphi \in K^*(A) \otimes K^*(B) : \langle \varphi | f_A \otimes f_B \rangle \geq 0,$$
$$\langle \varphi | 1_{K_A} \otimes 1_{K_B} \rangle = 1, \ \forall f_A \in E(K_A), f_B \in E(K_B)\}$$

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- For any valid bipartite system:

Theorem

$K_A \otimes_{min} K_B \subseteq K_{AB} \subseteq K_A \otimes_{max} K_B$.

Bipartite systems

- What is really interesting about entanglement in composite scenarios is that

Theorem

Let K be any state space and S_n be a classical state space. Then:

$$K_A \otimes_{min} S_n = K \otimes_{max} S_n{}^{a}.$$

[a] See the proof in M. Plávala. General probabilistic theories: An introduction. Physics Reports, 1033:1–64, sep 2023. doi: 10.1016/j.physrep.2023.09.001.

Bipartite systems

- Before we proceed, we must say that in the GPT framework an *n-outcome measurement* is an affine map $m : K \to S_n$.

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Theorem

If $m : K \to S_n$ is an n-outcome measurement, then m belongs to

$$A(K)^+ \otimes_{min} A(S_n)^{*+} = conv(cone\{f \otimes s : f \in E(K), s \in E(S_n)\}).$$

Bipartite systems

Corollary

An n-outcome measurement $m : K \to S_n$ is uniquely defined by effects $\{f_1, \ldots, f_n\} \subset E(K)$ such that

$$\sum_{j=1}^{n} f_j = 1_K.$$

Finite-dimensional quantum theory

Let $\mathcal{H}$ be a finite-dimensional Hilbert space and denote the set of its self-adjoint operators by $\mathcal{B}_H(\mathcal{H})$. As we know, the state space in quantum theory is

$$\mathcal{D}(\mathcal{H}) \doteq \{\rho \in \mathcal{B}_H^+(\mathcal{H}) : \ \mathrm{Tr}(\rho) = 1\},$$

where

$$\mathcal{B}_H^+(\mathcal{H}) \equiv \text{set of all positive semi-definite operators on } \mathcal{H}.$$

Finite-dimensional quantum theory

- The space of affine functions is $A(\mathcal{D}(\mathcal{H})) = \mathcal{B}_H(\mathcal{H})$, since the map $X \in \mathcal{B}_H(\mathcal{H}) \mapsto f_X \in A(\mathcal{D}(\mathcal{H}))$,

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  is an isomorphism.
- As consequence:

$$A^+(\mathcal{D}(\mathcal{H})) = \mathcal{B}_H^+(\mathcal{H}).$$

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- Recall that $(\mathcal{B}_H(\mathcal{H}), \langle\cdot|\cdot\rangle_{H-S})$ is a Hilbert space. As Hilbert spaces are self-dual, due to the previous isomorphism it also holds that
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The equality above tells us that quantum theory obeys what is called **strong self-duality**.

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$$\sum_{k=1}^{n} E_j = \mathbb{1}. \quad (POVMS!)$$

Where do GPTs show up?

PRX QUANTUM **2,** 010331 (2021)

Characterization of Noncontextuality in the Framework of Generalized Probabilistic Theories

David Schmid,[1,*] John H. Selby,[1,2,†] Elie Wolfe,[1] Ravi Kunjwal,[1,3] and Robert W. Spekkens[1]

[1] *Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario N2L 2Y5 Canada*

[2] *ICTQT, University of Gdańsk, Wita Stwosza 63, Gdańsk 80-308, Poland*

[3] *Centre for Quantum Information and Communication, Ecole polytechnique de Bruxelles, CP 165, Université libre de Bruxelles, Brussels 1050, Belgium*

(Received 27 January 2020; revised 28 July 2020; accepted 11 January 2021; published 25 February 2021)

Where do GPTs show up?

ARTICLE OPEN

Check for updates

Post-quantum steering is a stronger-than-quantum resource for information processing

Paulo J. Cavalcanti[1], John H. Selby[1], Jamie Sikora[2]✉, Thomas D. Galley[3] and Ana Belén Sainz[1]

We present the first instance where post-quantum steering is a stronger-than-quantum resource for information processing – remote state preparation. In addition, we show that the phenomenon of post-quantum steering is not just a mere mathematical curiosity allowed by the no-signalling principle, but it may arise within compositional theories beyond quantum theory, hence making its study fundamentally relevant. We show these results by formulating a new compositional general probabilistic theory – which we call Witworld – with strong post-quantum features, which proves to be a intuitive and useful tool for exploring steering and its applications beyond the quantum realm.

npj Quantum Information (2022)8:76 ; https://doi.org/10.1038/s41534-022-00574-8

Where do GPTs show up?

Quantum Darwinism and the spreading of classical information in non-classical theories

Roberto D. Baldijão*[,1,2], Marius Krumm*[,2,3], Andrew J. P. Garner[2], and Markus P. Müller[2,4,5]

[1]Instituto de Física Gleb Wataghin, Universidade Estadual de Campinas, Campinas, SP 13083-859, Brazil
[2]Institute for Quantum Optics and Quantum Information, Austrian Academy of Sciences, Boltzmanngasse 3, A-1090 Vienna, Austria
[3]Faculty of Physics, University of Vienna, Boltzmanngasse 5, A-1090 Vienna, Austria
[4]Vienna Center for Quantum Science and Technology (VCQ), Faculty of Physics, University of Vienna, Vienna, Austria
[5]Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, ON N2L 2Y5, Canada

Where do GPTs show up?

A no-go theorem on the nature of the gravitational field beyond quantum theory

Thomas D. Galley[§1], Flaminia Giacomini[§1], and John H. Selby[§2]

[1]Perimeter Institute for Theoretical Physics, 31 Caroline St. N, Waterloo, Ontario, N2L 2Y5, Canada
[2]ICTQT, University of Gdańsk, Wita Stwosza 63, 80-308 Gdańsk, Poland
03-08-2022

Take-home message

- GPTs help us perceive quantum theory from a bird's eye-view (no restriction hypothesis, local tomography, strong self-duality).

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- GPTs help us perceive quantum theory from a bird's eye-view (no restriction hypothesis, local tomography, strong self-duality).
- They serve as a tool to tinker around nonclassical theories beyond quantum.