0.1 Tensor Product of Vector Spaces

A question that popped up once in a while during my bachelor years was "what is a tensor?". Some colleagues would jokingly answer that "a tensor is something that transforms as a tensor". Ok, this kind of answer did not clarify a single thing to me back then. It was not until I enrolled in a course on group theory for physicists that a proper answer was given to me (though I did not understand it the first few times). If you too do not know what a tensor is, here a brief and rough explanation:

Given a bilinear map $\varphi : V_1 \times V_2 \to W$ between real vector spaces, the tensor product $V_1 \otimes V_2$ is the unique real vector space such that there exists a linear map $\tilde{\varphi} : V_1 \otimes V_2 \to W$ that is compatible with φ. A tensor is nothing more than an element of this new vector space.

Ok. There are a few things lacking in the explanation above, sure, but it already clarifies something: tensors are elements of a vector space that satisfies a very important property, *i.e.*, it is the vector space which "linearises" bilinear maps from $V_1 \times V_2$ to any other vector space W. The remainder of this appendix aims to make these ideas more precise. I will try to proceed as carefully as possible during the construction of the tensor product, but there can be some faults on my reasoning, since I am not a mathematician.

Definition 0.1. *Let X be a set. The* ***freely generated vector space by*** X *is defined as the set*

$$\mathcal{F}(X) \doteq \{f : X \to \mathbb{R} |\, \operatorname{supp} f \textit{ is finite}\},$$

where $\operatorname{supp} f$ *denotes the set of values* $x \in X$ *such that* $f(x) \neq 0$.

One could (rightfully) argue that $\mathcal{F}(X)$ is not yet a real vector space. Though that is true, one does not need to be that imaginative to see that $\mathcal{F}(X)$ becomes a real vector space when we define addition and scalar multiplication just as we do with functions.

The set X is a basis for its freely generated vector space because any element of $\mathcal{F}(X)$ has the following decomposition:

$$f = \sum_{p=1}^{r} f(a_p) a_p,$$

for some $r \in \mathbb{N}$, where the first $a_p \in \operatorname{supp} f$ and the second a_p, seen as an element of $\mathcal{F}(X)$, denotes the function that has as support the set $\{a_p\}$.

Proposition 0.1. *Let X be a set, $\mathcal{F}(X)$ be its freely generated vector space, and W be a real vector space. If $T : X \to W$ is a function, then there exists a unique linear map $\tilde{T} : \mathcal{F}(X) \to W$ for which the diagram below is commutative,*

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, hook, "\iota"'] & W \\ \mathcal{F}(X) \arrow[ur, dashed, "\tilde{T}"'] & \end{tikzcd}$$

where $\iota : X \to \mathcal{F}(X)$ *denotes the inclusion map defined as* $\iota(x) = x$ *for all* $x \in X$.

Proof. One just needs to define the map $\tilde{T} : \mathcal{F}(X) \to W$ by

$$\tilde{T}(f) \doteq \sum_{p=1}^{r} f(a_p)T(a_p), \; \forall f \in \mathcal{F}(X),$$

and be a bit careful when proving that $\tilde{T}(f+g) = \tilde{T}(f)+\tilde{T}(g)$. To prove uniqueness, let $M : \mathcal{F}(X) \to W$ be another linear map that satisfies the diagram above. Then, for an arbitrary $f \in \mathcal{F}(X)$:

$$\begin{aligned} M(f) = \sum_{a \in \operatorname{supp} f} f(a)M(a) \text{ and } T(a) = (M \circ \iota)(a) = M(a) \\ \implies M(f) = \sum_{a \in \operatorname{supp} f} f(a)T(a) \\ \implies M(f) = \tilde{T}(f). \end{aligned}$$

Since f was arbitrary, one deduces that $M = \tilde{T}$. ■

The previous proposition was just a restatement of a result usually seen in linear algebra courses: a linear transformation between vector spaces is fully determined by the way it acts on a basis.

Now, let us recall our previous case: we had a set $X = V_1 \times V_2$ and a function $\varphi : V_1 \times V_2 \to W$ which was assumed bilinear if we had treated the set as a vector space. Then, we could be mislead by proposition 0.1 and treat $\mathcal{F}(V_1 \times V_2)$ as the tensor product $V_1 \otimes V_2$, after all, for any function (in particular bilinear ones), the proposition gives us a unique linear map from a new vector space to our target space W. However, if $\tilde{\varphi}$ is linear map, it still is not compatible with the bilinearity of φ, for example:

$$\varphi(v_1 + u_1, v2) = \varphi(v_1, v_2) + \varphi(u_1, v_2)$$

for any $v_1, u_1 \in V_1$ and $v_2 \in V_2$, but in general

$$\tilde{\varphi}((v_1 + u_1, v_2)) \neq \tilde{\varphi}((v_1, v_2)) + \tilde{\varphi}((u_1, v_2)).$$

If this looks a bit confusing, remember that in the LHS above $(v_1 + u_1, v_2)$ is the function that is zero everywhere but in $(v_1 + u_1, v_2)$, while in the RHS (v_1, v_2) and (u_1, v_2) are functions defined analogously as the function $(v_1 + u_1, v_2)$. Unless $v_2 = 0$, these functions have little to do with each other, and so $\tilde{\varphi}$ has little to no right to mimic the bilinearity of φ. In order to fix this problem (and similar ones), first we have to consider the vector subspace $\mathcal{R} \subseteq \mathcal{F}(V_1 \times V_2)$ that consists of linear combinations of the following elements:

1. $(v_1 + u_1, v_2) - (v_1, v_2) - (u_1, v_2)$,
2. $(v_1, v_2 + u_2) - (v_1, v_2) - (v_1, u_2)$,
3. $\alpha(v_1, v_2) - (\alpha v_1, v_2)$,
4. $\alpha(v_1, v_2) - (v_1, \alpha v_2)$,

for all $v_1, u_1 \in V_1$, $v_2, u_2 \in V_2$ and $\alpha \in \mathbb{R}$.

Second, recall that given a vector space V and a subspace $S \subseteq V$, one can always construct a new vector space V/S, called the quotient space, by introducing the equivalence relation $v \sim u \iff v - u \in S$ for $u, v \in V$. In our specific scenario, we introduce the following notation:

$$\mathcal{F}(V_1 \times V_2)/\mathcal{R} \doteq V_1 \otimes V_2.$$

Now, the elements of this vector space are equivalence classes of elements of $\mathcal{F}(V_1 \times V_2)$ under $\sim$. By construction of the quotient space as a vector space, it follows, for example, that:

$$[(v_1 + u_1, v_2)] = [(v_1, v_2)] + [(u_1, v_2)],$$

because $(v_1 + u_1, v_2) - (v_1, v_2) - (u_1, v_2)$ clearly belongs to $\mathcal{R}$. If we introduce the notation $[(v, v')] \doteq v \otimes v'$, then the equality above translates to

$$(v_1 + u_1) \otimes v_2 = v_1 \otimes v_2 + u_1 \otimes v_2,$$

which is a bit more familiar to someone who has been previously exposed to tensors. Similarly, one can also show that:

1. $v_1 \otimes (v_2 + u_2) = v_1 \otimes v_2 + v_1 \otimes u_2$,
2. $\alpha(v_1 \otimes v_2) = (\alpha v_1) \otimes v_2$,
3. $\alpha(v_1 \otimes v_2) = v_1 \otimes (\alpha v_2)$.

Third, define $i : V_1 \times V_2 \to V_1 \otimes V_2$ as $i(v_1, v_2) \doteq v_1 \otimes v_2$. It is not that hard to show that it is a bilinear map. After all these steps, the story so far can be described by the following diagram:

$$\begin{tikzcd}
 & \mathcal{F}(V_1 \times V_2) \arrow[d, dashed, "\tilde{\varphi}"] \\
V_1 \times V_2 \arrow[ur, hook, "\iota"] \arrow[r, "\varphi"] \arrow[dr, "i"'] & W \\
 & V_1 \otimes V_2.
\end{tikzcd}$$

We are missing one arrow! In order to obtain this arrow, we make use of the following lemma, whose proof we momentarily skip for sake of text continuity:

Lemma 0.1. *Let $T : U \to W$ be a linear map between vector spaces and $\mathcal{R} \subseteq U$ a subspace satisfying $\mathcal{R} \subseteq \ker T$. Denote the canonical projection map $u \mapsto [u]$ by π. There exists a unique linear transformation $\tilde{T} : U/R \to W$ such that the following diagram*

$$\begin{tikzcd}
U \arrow[r, "T"] \arrow[d, hook, "\pi"] & W \\
U/\mathcal{R} \arrow[ur, dashed, "\tilde{T}"'] &
\end{tikzcd}$$

commutes.

By virtue of the this lemma, if we set $U = \mathcal{F}(V_1 \times V_2)$ and $T = \tilde{\varphi}$, then there exists a unique linear map $\tilde{T} \equiv \varphi_\otimes : V_1 \otimes V_2 \to W$ such that

$$\begin{tikzcd}
 & \mathcal{F}(V_1 \times V_2) \arrow[d, dashed, "\exists! \tilde{\varphi}"'] \arrow[dd, bend left=60, "\pi"] \\
V_1 \times V_2 \arrow[ur, hook, "\iota"] \arrow[r, "\varphi"] \arrow[dr, "i"'] & W \\
 & V_1 \otimes V_2. \arrow[u, dashed, "\exists! \varphi_\otimes"]
\end{tikzcd}$$

In other words, we have essentially proven the existence part of the following theorem:

Theorem 0.1. *Let V_1, V_2 be real vector spaces. There exists a unique vector space $V_1 \otimes V_2$ and a unique bilinear map $i : V_1 \times V_2 \to V_1 \otimes V_2$ with the following universal property:*

$$\begin{tikzcd}
V_1 \times V_2 \arrow[r, "\forall \text{ bilinear } \varphi"] \arrow[d, hook, "i"] & \forall W \\
V_1 \otimes V_2 \arrow[ur, dashed, "\exists! \text{ linear } \varphi_\otimes"'] &
\end{tikzcd}$$

Proof of theorem 0.1. All that is left is to prove uniqueness. Assume there's another vector space M and other bilinear map $m : V_1 \times V_2 \to M$ that also satisfies the diagram above. This means that for $\varphi = i$ and $W = V_1 \otimes V_2$, there exists a unique linear map $i' : M \to V_1 \otimes V_2$ such that $i' \circ m = i$. Similarly, since $(i, V_1 \otimes V_2)$ has the same universal property, there exists a unique linear map $m' : V_1 \otimes V_2 \to M$ such that $m' \circ i = m$. Pictorially, we have the following commutative diagram:

$$\begin{tikzcd}
 & V_1 \otimes V_2 \arrow[d, dashed, "m'"] \\
V_1 \times V_2 \arrow[ur, hook, "i"] \arrow[r, "m"] \arrow[dr, hook, "i"'] & M \arrow[d, dashed, "i'"] \\
 & V_1 \otimes V_2.
\end{tikzcd}$$

This means that $i' \circ m' : V_1 \otimes V_2 \to V_1 \otimes V_2$ is a linear map such that $(i' \circ m') \circ i = i$. By making use of the universal property of the pair $(i, V_1 \otimes V_2)$ to itself, it follows that $i' \circ m' = 1_{V_1 \otimes V_2}$, so m' has a left inverse, which means that it is an injective map. By similar reasoning, $m' \circ i' = 1_M$, so m' has a right inverse, which means it is a surjective map. Since the left and right inverses are the same and m' is linear, we have that m' is a vector space isomorphism. ■

Now, all that is left is to prove the lemma aforementioned.

Proof of lemma 0.1. Define the map $\tilde{T} : U/\mathcal{R} \to W$ as

$$\tilde{T}([u]) \doteq T(u).$$

If $\tilde{T}$ is well-defined, then, first of all, $\tilde{T}$ is a linear map that obeys the commutative diagram. To show that it is indeed well-defined, let $u' \in [u]$:

$$\begin{aligned}
u' \in [u] &\iff u - u' \in \mathcal{R} \subseteq \ker T \\
&\implies T(u - u') = 0_W \\
&\implies T(u) = T(u') \\
&\implies \tilde{T}([u]) = \tilde{T}([u']).
\end{aligned}$$

Hence, $\tilde{T}$ does not depend on the representative of the equivalence class. To prove uniqueness, let $M : U/\mathcal{R} \to W$ be another linear map that obeys the diagram. Then, for any $[u]$, $M([u]) = T(u) \doteq \tilde{T}([u])$. Thus, $M = \tilde{T}$. ■

All work done so far can easily be generalised to the case where we have $k \in \mathbb{N}$ real vector spaces $V_1, \ldots, V_k$ and a multilinear map $\varphi : V_1 \times \ldots \times V_k \to W$. I will just state the theorem, but the idea of the proof is entirely analogous to what we did to the case $k = 2$.

Theorem 0.2. *Let $V_1, \ldots V_k$, and W be real vector spaces, $k \in \mathbb{N}$. There exists a unique vector space $V_1 \otimes \ldots \otimes V_k$ and a unique multilinear map $\varphi_\otimes : V_1 \otimes \ldots \otimes \ldots V_k \to W$ with the following universal property:*

$$\begin{tikzcd}
V_1 \times \ldots \times V_k \arrow[r, "\forall\ \mathtt{multilinear}\ \varphi"] \arrow[d, hook, "i"] & \forall W \\
V_1 \otimes \ldots \otimes V_k. \arrow[ur, dashed, "\exists!\ \mathtt{linear}\ \varphi_\otimes"'] &
\end{tikzcd}$$

So far so good. Now, it is time to verify how these objects look like when we introduce ordered basis in each of the vector spaces $V_1, \ldots, V_k$. For simplicity, from now on all vector spaces under consideration will be **finite dimensional**.

Proposition 0.2 (A basis for $V_1 \otimes \ldots \otimes V_k$)**.** *Let $V_1, \ldots, V_k$ be real vector spaces and for each $1 \leqslant j \leqslant k$, let $(e^{(j)}_1, \ldots, e^{(j)}_{n_j})$ be an ordered basis for the n_j-dimensional vector space V_j. The set*

$$\mathcal{B} = \{e^{(1)}_{i_1} \otimes \ldots \otimes e^{(k)}_{i_k} : i_1 \in [n_1], \ldots, i_k \in [n_k]\}$$

defines a basis in $V_1 \otimes \ldots \otimes V_k$.

Proof. It is evident that $\mathrm{span}(\mathcal{B}) \subseteq V_1 \otimes \ldots \otimes V_k$. To show the opposite inclusion, consider a tensor of the form $v_1 \otimes \ldots \otimes v_k$. By expanding each v_j in terms of the ordered basis chosen in V_j, it is evident that $v_1 \otimes \ldots \otimes v_k \in \mathrm{span}(\mathcal{B})$. Since any tensor in $V_1 \otimes \ldots \otimes V_k$ is a linear combination of elements like $v_1 \otimes \ldots \otimes v_k$, it is clear that the converse inclusion must also hold. Hence, $V_1 \otimes \ldots \otimes V_k = \mathrm{span}(\mathcal{B})$.

Now, we must prove linear independence. To avoid typing sums, we adopt the Einstein summation convention (repeated upper and lower indices are being summed).

Let $\alpha^{i_1 \ldots i_k} \in \mathbb{R}$ be coefficients such that

$$\alpha^{i_1 \ldots i_k} e^{(1)}_{i_1} \otimes \ldots \otimes e^{(k)}_{i_k} = 0.$$

Now, for each $k - \mathrm{uple}$ $(m_1, \ldots, m_k$ we construct a multilinear map $\psi^{m_1, \ldots, m_k} : V_1 \times \ldots \times V_k \to \mathbb{R}$ as

$$\psi^{m_1 \ldots m_k}(v_1, \ldots, v_k) \doteq e^{m_1}_{(1)}(v_1) \ldots e^{m_k}_{(k)}(v_k),$$

where $e^{i_j}_{(j)}$ represents an element of the dual basis to $e^{(j)}_{i_j}$. By virtue of theorem 0.2, there is a linear map $\tilde{\psi}^{m_1 \ldots m_k} : V_1 \otimes \ldots \otimes V_k \to \mathbb{R}$ that $\tilde{\psi}^{m_1, \ldots, m_k} \circ i = \psi^{m_1 \ldots m_k}$. Hence, by virtue of each such map:

$$\begin{aligned}
\tilde{\psi}^{m_1 \ldots m_k}(\alpha^{i_1 \ldots i_k} e^{(1)}_{i_1} \otimes \ldots \otimes e^{(k)}_{i_k}) &= \alpha^{i_1 \ldots i_k} \psi^{m_1 \ldots m_k}(e^{(1)}_{i_1}, \ldots, e^{(k)}_{i_k}) \\
&\doteq \alpha^{i_1 \ldots i_k} e^{m_1}_{(1)}(e^{(1)}_{i_1}) \ldots e^{m_k}_{(k)}(e^{(k)} i_k) \\
&= \alpha^{i_1 \ldots i_k} \delta^{m_1}_{i_1} \ldots \delta^{m_k}_{i_k} \\
&= \alpha^{m_1 \ldots m_k} \\
&= 0.
\end{aligned}$$

Linear independence easily follows from that. ∎

Corollary 0.1. $\dim(V_1 \otimes \ldots \otimes V_k) = \prod_{j=1}^{k} n_j.$

0.2 Multilinear Maps and (p, q)-tensors

Hitherto, tensors are extremely abstract objects obtained by taking quotients of vector spaces. In the case where all these spaces are finite-dimensional, the tensor product is actually isomorphic to "more concrete" space, name the vector space of real-valued multilinear maps $L(V_1, \dots, V_k; \mathbb{R})$.

Proposition 0.3. *Let $V_1, \dots, V_k$ be real vector spaces and for each $1 \leqslant j \leqslant k$, let $(e_1^{(j)}, \dots, e_{n_j}^{(j)})$ be an ordered basis for the n_j-dimensional vector space V_j. Denote the ordered dual basis for V_j^* as $(e^1_{(j)}, \dots, e^{n_j}_{(j)})$. The set*

$$\mathcal{B}^* \doteq \{e^{i_1}_{(1)} \otimes \dots \otimes e^{i_k}_{(k)} : i_1 \in [n_1], \dots, i_k \in [n_k]\}$$

is a basis for $L(V_1, \dots, V_k; \mathbb{R})$.

Corollary 0.2. $\dim L(V_1, \dots, V_k; \mathbb{R}) = \prod_{j=1}^{k} n_j$.

Proposition 0.4 (Tensors as Multilinear Maps)**.** *Let $V_1, \dots, V_k$ be finite-dimensional real vector spaces. It is true that*

$$V_1^* \otimes \dots \otimes V_k^* \simeq L(V_1, \dots, V_K; \mathbb{R}). \tag{0.1}$$

Proof. Let $V_1, \dots, V_k$ be vector spaces as stated above. By corollaries 0.1 and 0.2, $V_1 \otimes \dots \otimes V_k$ and $L(V_1, \dots, V_k; \mathbb{R})$ are both finite-dimensional real vector spaces with the same dimension. Consequently, they are isomorphic. ∎

By virtue of the canonical isomorphism $V_j^{**} \simeq V_j$, proposition 0.4 also implies that

$$V_1 \otimes \dots \otimes V_k \simeq L(V_1^*, \dots, V_k^*; \mathbb{R}). \tag{0.2}$$

We are now ready to talk what covariant, contravariant and mixed tensors are. These definitions are extremely important in general relativity contexts, for they make it clear what kind of object we are working with[1].

Definition 0.2. *Let p and q be non-negative integers and V be a real vector space. The (p, q)-tensor product of V is defined as the tensor product*

$$T^{(p,q)}(V) \doteq \underbrace{V \otimes \dots \otimes V}_{p \text{ times}} \otimes \underbrace{V^* \otimes \dots \otimes V^*}_{q \text{ times}}.$$

An element of such tensor product is called a (p,q)-tensor or, alternatively, a rank (p,q) tensor.

Covariant tensors are the $(0, q)$-tensors and **contravariant tensors** are the $(p, 0)$-tensors. We denote these tensor products respectively as

$$T^{(0,q)}(V) = T^q(V^*),$$

$$T^{(p,0)}(V) = T^p(V).$$

Since these tensors products can be reinterpreted as multilinear maps, we also have that

$$T^{(0,q)}(V) \simeq L(\underbrace{V, \dots, V}_{q \text{ times}}; \mathbb{R}),$$

$$T^{(p,0)}(V) \simeq L(\underbrace{V^*, \dots, V^*}_{p \text{ times}}; \mathbb{R}).$$

This means that **covariant tensors eat vectors**, while **contravariant tensors eat covectors**.

[1] Besides, some objects, such the electromagnetic field tensor, may change how they look like if we work in terms of covariant or contravariant components.

Definition 0.3. *Let* V *be finite-dimensional real vector space. A rank k covariant tensor* $A \in T^k(V^*)$ *is* ***symmetric*** *if and only if for any* $v_1, \ldots, v_k \in V$*:*

$$A(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = \alpha(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

whenever $1 \leqslant i < j \leqslant k$.

The set of all symmetric, rank k, covariant tensors is a vector subspace of $T^k(V^*)$.

Let S_k be the permutation group of $\{1, 2, \ldots, k\}$. We can always turn a tensor $A \in T^k(V^*)$ into a symmetric one by virtue of the Symmetrization operator $\mathrm{Sim} : T^k(V^*) \to \Sigma^k(V^*)$, which is defined via:

$$\mathrm{Sym}\, A \doteq \frac{1}{k!} \sum_{\sigma \in S_k} \sigma A.$$

Proposition 0.5. *Let* V *be a real vector space and* $A \in T^k(V^*)$*. The following statements are equivalent:*

a) A *is symmetric.*

b) For any $v_1, \ldots, v_k \in V$, $A(v_1, \ldots, v_k)$ *remains unaltered by any rearrangement of* $v_1, \ldots, v_k$.

c) In any basis for V*, the components* $A_{i_1 \ldots i_k}$ *remain unaltered by any rearrangement of their indices.*

Exercise 0.1. *Let* V *be a real vector space and* $A \in T^k(V^*)$*. Then*

a) $\mathrm{Sym}\, A$ *is symmetric.*

b) $\mathrm{Sym}\, A = A \iff A \in \Sigma^k(V^*)$.

Let $A \in \Sigma^k(V^*)$ and $B \in \Sigma^l(V^*)$, there is no guarantee that $A \otimes B \in T^{(k+l)}(V^*)$ will be a symmetric tensor too. However, by virtue of the operator Sym, we can define a so-called **symmetric product**, which acts on A and B as:

$$AB \doteq \mathrm{Sym}(A \otimes B).$$

This means that

$$\alpha\beta(v_1, \ldots, v_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}).$$

For example, if A and B are rank 1 covariant tensors (which are always symmetric), then

$$AB = \frac{1}{2}(A \otimes B + B \otimes A),$$

is a symmetric (0,2)-tensor.

0.3 Antisymmetric Tensors

Just like we can define symmetric covariant tensors, we can define antisymmetric tensors. Such tensors are somewhat special[2] and deserve their own section in this appendix.

Definition 0.4. *Let* V *be a finite-dimensional real vector space. A rank* k *covariant tensor* $A \in T^k(V^*)$ *is* ***antisymmetric*** *if and only if for any* $v_1, \ldots, v_k \in V$*:*

$$A(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -A(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k),$$

whenever $1 \leqslant i < j \leqslant k$.

These objects go under many other names: k-covectors, covectors or exterior forms, for example. The set of all such tensors is denoted by $\Lambda^k(V^*)$ and it is also a vector subspace of $T^k(V^*)$.

[2] They will help us define differential forms, which are the objects that one integrates on a manifold.

Claim 1. *Let V be a real vector space and $A \in T^k(V^*)$. The following statements are equivalent:*

a) *A is antisymmetric.*

b) *For any $v_1, \dots, v_k \in V$ and any element $\sigma \in S_k$:*

$$A(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = (\operatorname{sgn} \sigma) A(v_1, \dots, v_k),$$

with $\operatorname{sgn} \sigma \doteq (-1)^m$, where m is the number of transpositions obtained by decomposing σ.

c) *In any basis for V, the components $A_{i_1 \dots i_k}$ change sign for any transposition of indices.*

Furthermore, these are also equivalent:

a') *A is antisymmetric.*

b') *$A(v_1, \dots, v_k) = 0$ if the k-uple $(v_1, \dots, v_k)$ is linearly dependent.*

c') *$A(v_1, \dots, v_k) = 0$ whenever two entries of $(v_1, \dots, v_k)$ are equal.*

The version of the Sym operator for antisymmetric tensors is the operator $\mathrm{Antisym} : T^k(V^*) \to \Lambda^k(V^*)$, named **antisymmetrization**, defined by:

$$\mathrm{Antisym}\, A \doteq \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)(\sigma A).$$

Explicitly, this means that:

$$(\mathrm{Antisym}\, A)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) A(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

Clearly, for $A \in T^{(0,2)}(V^*)$:

$$(\mathrm{Antisym}\, A)(v, w) = \frac{1}{2}(A(v, w) - A(w, v)).$$

Exercise 0.2. *Let V be a real vector space and $A \in T^k(V^*)$. Then*

a) *$\mathrm{Antisym}\, A$ is antisymmetric.*

b) *$\mathrm{Antisym}\, A = A \iff A \in \Lambda^k(V^*)$.*

Some Antisymmetric Tensors

A **k multi-index** is defined as a ordered k-uple $(i_1, \dots, i_k)$ of positive integers. If $\sigma \in S_k$, define I_σ as

$$I_\sigma \doteq (i_{\sigma(1)}, \dots, i_{\sigma(k)}).$$

Fix an ordered basis $(e^1, \dots, e^n)$ on V^*. For any k multi-index I, define the (0,k)-tensor ε^I as follows:

$$\varepsilon^I(v_1, \dots, v_k) \doteq \det \begin{pmatrix} e^{i_1}(v_1) & \dots & e^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ e^{i_k}(v_1) & \dots & e^{i_k}(v_k) \end{pmatrix} = \det \begin{pmatrix} v_1^{i_1} & \dots & v_k^{i_1} \\ \vdots & \ddots & \vdots \\ v_1^{i_k} & \dots & v_k^{i_k} \end{pmatrix}.$$

Clearly that is an antisymmetric tensor, let us call it an elementary alternating tensor.

For k multi-indices I e J a **generalized Kronecker delta** by setting

$$\delta^I_J \doteq \det \begin{pmatrix} \delta^{i_1}_{j_1} & \dots & \delta^{i_1}_{j_k} \\ \vdots & \ddots & \vdots \\ \delta^{i_k}_{j_1} & \dots & \delta^{i_k}_{j_k} \end{pmatrix}.$$

One can show that it satisfies

$$\delta^I_J = \begin{cases} \operatorname{sgn}\sigma, & \text{if neither I nor J have repeated indices and } J = I_\sigma,\ \sigma \in S_k. \\ 0, & \text{otherwhise.} \end{cases}$$

A way do prove the property above is by rewriting the RHS of the definition of δ^I_J as:

$$\delta^I_J = \sum_{\eta \in S_k} (\operatorname{sgn}\eta)\delta^{i_1}_{j_{\eta(1)}} \dots \delta^{i_k}_{j_{\eta(k)}}.$$

Claim 2. *Fix a basis $\{e_i\}_{i=1}^n$ on V and let $\{e^i\}_{i=1}^n$ be its dual basis. For any k multi-indices I and J:*

a) If I has repeated indices, then $\varepsilon^I = 0$.

b) If $J = I_\sigma$ for some $\sigma \in S_k$, then $\varepsilon^I = (\operatorname{sgn}\sigma)\varepsilon^J$.

c) For any k-uple of basis elements

$$\varepsilon^I(e_{j_1}, \dots, e_{j_k}) = \delta^I_J.$$

Proposition 0.6. *Let V be a real vector space. If $\{e^i\}_{i=1}^n$ is a basis for V^*, then for every positive integer $k \leqslant \dim V = n$, the set*

$$\mathcal{E} = \{\varepsilon^I :\ i_1 < \dots < i_k\}$$

is a basis for $\Lambda^k(V^*)$ and

$$\dim \Lambda^k(V^*) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Proof.

Let $k > n$, then by statement b′ from claim 1, we have $\mathcal{E} = \{0\}$, for any k-uple of vectors is linearly dependent in this situation.

Now consider the case $k \leqslant n$. If $A \in \Lambda^k(V^*)$ and $I = (i_1, \dots, i_k)$, define

$$a_I \doteq A(e_{i_1}, \dots, e_{i_k}).$$

Thus, for any other k multi-index J,

$$\sum_I{}' a_I \varepsilon^I(e_{j_1}, \dots, e_{j_k}) \doteq \sum_{\{I:\ i_1<\dots<i_k\}} a_I \varepsilon^I(e_{j_1}, \dots, e_{j_k}) = a_J.$$

This means that $\sum_I' \alpha_I \varepsilon^I = A$, which implies $\Lambda^k(V^*) = \operatorname{span}\mathcal{E}$.

To verify linear independence, let β_I be real numbers such that

$$\sum_I{}' \beta_I \varepsilon^I = 0.$$

Then for any increasing k multi-index J:

$$\sum_I{}' \beta_I \varepsilon^I(e_{j_1}, \dots, e_{j_k}) = \beta_J = 0.$$

∎

Proposition 0.7. *Let V be an n-dimensional real vector space and $A \in \Lambda^n(V^*)$. For any linear operator $T :\ V \to V$ and any n-uple $(v_1, \dots, v_n)$,*

$$A(T(v_1), \dots, T(v_n)) = (\det T)A(v_1, \dots, v_k).$$

This result is very similar to the change of coordinates formula for integrals. In fact, I believe it generalises that formula, since the Jacobian matrix is a particular linear operator.

Proof.

Let $\{e_i\}_{i=1}^n$ be a basis for V and $\{e^i\}_{i=1}^n$ be its dual basis. Moreover, let (T^i_j) be the matrix representation of the linear operator T in the basis $\{e_i\}$. Due to multilinearity and antisymmetry of A, we just have to check the equality holds in the case $(v_1, \dots, v_n) = (e_1, \dots, e_n)$.

By proposition 0.6, it follows that $\dim \Lambda^n(V^*) = 1$. Therefore, $A = a\varepsilon^{1\dots n}$ for some $a \in \mathbb{R}$. Hence, we can write the RHS above as:

$$(\det T) a\varepsilon^{1\dots n}(e_1, \dots, e_n) = a(\det T).$$

On the other hand, the LHS can be written as:

$$a\varepsilon^{1\dots n}(T(E_1), \dots, T(E_n)) = a\det\big(\varepsilon^j T(E_i)\big) = a \det T.$$

Since they both agree, the proof is essentially finished. ∎

Exterior Product

O produto exterior é uma operação binária similar ao produto simétrico, o que significa que a ideia básica detrás dela é, a partir de duas formas exteriores de ranks k e l, obter uma forma exterior de rank $k + l$. De forma mais precisa, suponha que $\omega \in \Lambda^k(V^*)$ e $\eta \in \Lambda^l(V^*)$, o produto exterior dessas formas é denotado e definido por

$$\omega \wedge \eta \doteq \frac{(k+l)!}{k!l!} \mathtt{Alt}\,(\omega \otimes \eta). \tag{0.3}$$

A razão para os coeficientes que precedem $\mathtt{Alt}\,(\omega \wedge \eta)$ é justificada pelo

Lemma 0.2. *Seja V um espaço vetorial n-dimensional e seja $\{\varepsilon^i\}_{i=1}^n$ uma base no espaço dual. Para quaisquer multi-índices I e J vale que*

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ}, \tag{0.4}$$

sendo $IJ = (i_1, \dots, i_k, j_1, \dots, j_l)$.

Proof.

Seja $\{E_i\}_{i=1}^n$ uma base em V. Seja $P = (p_1, \dots, p_{k+l}$. Por multilinearidade, é suficiente analisar como $\varepsilon^I \wedge \varepsilon^J$ atua em $(E_{p_1}, \dots, E_{p_{k+l}})$.

Caso I) Se P possui índices repetidos, então pelo claim 1, segue que $\varepsilon^I \wedge \varepsilon^J(E_{p_1}, \dots, E_{p_{k+l}}) = \varepsilon^{IJ}(E_{p_1}, \dots, E_{p_{k+l}}) = 0$.

Caso II) Se P contém um índice que não aparece em I ou J, então

$$\varepsilon^{IJ}(E_{p_1}, \dots, E_{p_{k+l}}) = 0,$$

pois P não é permutação de IJ. O mesmo se aplica para $\varepsilon^I \wedge \varepsilon^J$.

Caso III) $P = IJ$ e não há índices repetidos. Nesta situação, segue que $\varepsilon^{IJ}(E_{p_1}, \dots, E_{p_{k+l}}) = 1$. Quanto ao produto exterior, temos que

$$\begin{aligned}
&\varepsilon^I \wedge \varepsilon^J(E_{p_1}, \dots, E_{p_{k+l}}) = \\
&\frac{(k+l)!}{k!l!} \mathtt{Alt}\,(\varepsilon^I \otimes \varepsilon^J)(E_{p_1}, \dots, E_{p_{k+l}}) = \\
&\frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\mathrm{sgn}\ \sigma) \varepsilon^I(E_{p_{\sigma(1)}}, \dots, E_{p_{\sigma(k)}}) \varepsilon^J(E_{p_{\sigma(k+1)}}, \dots, E_{p_{\sigma(k+l)}}) = 1.
\end{aligned}$$

Caso IV) Se P é permutação de IJ e possui índices repetidos. Nesse caso, basta aplicar a permutação inversa e utilizar o caso III, visto que a aplicação dessa permutação multiplica ambos os lados de equation (0.4) pelo sinal da permutação.

■

Proposition 0.8 (**Propriedades do produto exterior**). *Suponha que ω, ω', η, η' e ξ, sejam formas exteriores. Então vale que*

a) *Para todos* $a, a' \in \mathbb{R}$*:*

$$(a\omega + a'\omega') \wedge \eta = a(\omega \wedge \eta) + a'(\omega' \wedge \eta)$$

$$\eta \wedge (a\omega + a'\omega') = a(\eta \wedge \omega) + a'(\eta \wedge \omega).$$

b) $\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi$*.*

c) *Se* $\omega \in \Lambda^k(V^*)$ *e* $\eta \in \Lambda^l(V^*)$ *então*

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega.$$

d) *Se* $\{\varepsilon^i\}_{i=1}^n$ *é base para* V^* *e* I *é multi-índice de comprimento k, então*

$$\varepsilon^{i_1} \wedge \ldots \wedge \varepsilon^{i_k} = \varepsilon^I.$$

e) *Se* $\omega^1, \ldots, \omega^k \in V^*$ *e* $v_1, \ldots, v_k \in V$*, então*

$$\omega^1 \wedge \ldots \wedge \omega^k(v_1, \ldots, v_k) = \det\big(\omega^i(v_j)\big).$$

Proof.

a) Segue da bilinearidade do produto tensorial e da linearidade de Alt.

b) Basta verificar o caso para os tensores alternantes elementares, pois o caso geral segue da bilinearidade do produto exterior. Desta forma, sejam I, J e K multi-índices. Temos que

$$(\varepsilon^I \wedge \varepsilon^J) \wedge \varepsilon^K = \varepsilon^{IJ} \wedge \varepsilon^K = \varepsilon^{IJK} = \varepsilon^I \wedge \varepsilon^{JK} = \varepsilon^I \wedge (\varepsilon^J \wedge \varepsilon^K).$$

c) Igual ao item anterior, é suficiente verificar o caso para tensores elementares. Sendo assim, seja τ a permutação que leva IJ em JI. Então:

$$\varepsilon^{IJ} = (sgn\,\tau)\varepsilon^{JI} = (sgn\,\tau)\varepsilon^J \wedge \varepsilon^I.$$

Veja que para obter JI a partir de τ, realizamos k transposições para deslocar j_1 até a posição ocupada por i_1. Repetindo o processo para os demais elementos de J, obtemos um total de kl transposições e portanto $sgn\,\tau = (-1)^{kl}$.

d) Corolário do lemma 0.2, sendo demonstrado por indução.

e) Basta escrever cada ω^i como combinação dos tensores ε^j e utilizar o item acima.

■

As primeiras três propriedades da proposition 0.8 informam que o produto exterior é bilinear, associativo e anti-comutativo. Quando uma forma exterior η pode ser expressa como no item *e*, então diz-se que ela é **decomponível**. Da mesma forma que tensores mais gerais, nem toda forma exterior é decomponível, mas todas são combinações lineares de formas decomponíveis. Além do mais, visto que V tem dimensão finita, podemos definir o seguinte espaço vetorial

$$\Lambda(V^*) = \bigoplus_{k=1}^{n} \Lambda^k(V^*).$$

Pela proposition 0.8, o produto exterior torna este espaço vetorial em uma **álgebra associativa** conhecida como **álgebra exterior** (ou **álgebra de Grassmann**). Além disso, ela é uma álgebra **graduada**, no sentido que ela é uma álgebra cuja decomposição é da forma acima e que satisfaz $\Lambda^k(V^*) \cdot \Lambda^l(V^*) \subseteq \Lambda^{k+l}(V^*)$, onde $\cdot$ representa o produto da álgebra.

A exposição do tema até o momento está sendo bastante abstrata. Talvez as coisas fiquem mais claras quando consideramos $V = \mathbb{R}^n$, como no exemplo a seguir.

Example 0.1. *Sejam* $\omega = x_1 dx_1 + x_2 dx_2 + x_3 dx_3 + x_4 dx_4$ *e* $\varphi = x_1 dx_1 \wedge dx_2 + dx_1 \wedge dx_3$*, onde* x_i *é função componente para cada* i *e* dx_i *é um funcional linear. Pela definição do produto exterior, temos que*

$$\omega \wedge \varphi = (x_1 x_3 - x_2) dx_1 \wedge dx_2 \wedge dx_3.$$

No geral, o produto exterior de um covetor com um 2-covetor assume o seguinte aspecto:

$$\omega \wedge \varphi = (a_1 b_{23} - a_2 b_{13} + a_3 b_{12}) dx_1 \wedge dx_2 \wedge dx_3.$$

Exercise 0.3. *Dada uma forma exterior* ω*, mostre que* $\omega \wedge \omega = 0$ *sempre que k for ímpar.*