Notes on Category Theory

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About these notes

This is a personal notebook that I use to keep my notes on category theory. I mostly use Coecke (2008); Coecke and Paquette (2009); Awodey (2010) as main references for the basic contents. These notes' purpose is to understand how quantum mechanics is formulated in a category-theoretical language (Heunen and Vicary, 2019), and, in particular, how the Choi-Jamiołkowski isomorphism looks like in this framework.

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1 Motivation

From my perspective, category theory is a fascinating topic in and of itself: it allows us to make statements that hold true in different mathematical fields without the need to commit to the specific structures of one of them. More remarkable than that is the fact that we can formulate quantum theory in a category-theoretical language (Heunen and Vicary, 2019). Since quantum theory is a particular kind of a generalized probabilistic theory (GPT), that means we can also describe GPTs within this framework and doing so would help me get a better understanding of how one could formulate a Choi-Jamiołkowski-esque isomorphism in GPTs without resorting to purification (Chiribella et al., 2010). With that in mind and without further ado, let us embark on our category studies.

2 Categories

Definition 1.1: Category

A category $\mathbf{C}$ consists of

1. A family of objects: $A, B, C, \ldots$
2. For any two objects A and B, a set of arrows (also called morphims) $\mathbf{C}(A, B)$. For each arrow $f \in \mathbf{C}(A, B)$, we write it as $A \xrightarrow{f} B$.
3. For any objects A,B and C, a composition rule
$$\circ : \mathbf{C}(A, B) \times \mathbf{C}(B, C) \to \mathbf{C}(A, C),\ (f, g) \mapsto g \circ f,\ such\ that$$
4. For any $f \in \mathbf{C}(A, B)$,$g \in \mathbf{C}(B, C)$ and $h \in \mathbf{C}(C, D)$ the composition is associative:
$$h \circ (g \circ f) = (h \circ g) \circ f;$$
5. For any object A corresponds an arrow $1_A \in \mathbf{C}(A, A)$, called the identity arrow and it satisfies
$$f = f \circ 1_A = 1_B \circ f$$
for any $f \in \mathbf{C}(A, B)$.

By a long shot, that is the most important definition in this entire text. In order to get a better picture of what a category is, a few examples might help.

Example 1.1: Sets

Take the class of all sets as our family of objects, consider the functions between sets as our arrows and let $\circ$ be the ordinary composition rule for functions. For each set A, define $A \xrightarrow{1_A} A$ to be the identity map on A, *i.e.* $1_A \doteq \mathrm{id}_A$. These data constitute the category of sets, which is written as **Sets**.

Example 1.2: Concrete categories

- The category $\mathbf{FdVect}_{\mathbb{K}}$ consists of
  1. Finite-dimensional vector spaces over $\mathbb{K}$ as objects;
  2. Linear maps between vector spaces as arrows;
  3. Ordinary composition of linear maps as $\circ$;
  4. For each object V, the identity map id_V as 1_V.
- The category **Pos** consists of
  1. Partially ordered sets as objects;
  2. Monotone maps as arrows, *i.e.* $a \leqslant a' \implies f(a) \leqslant f(a')$;
  3. Ordinary composition of functions and identity maps as $\circ$ and 1_A, respectively.

A more physically inclined example of category is

Example 1.3: Physical Processes

The category **PhysProc** consists of

1. All physical systems $A, B, C, \ldots$ as objects;
2. All physical processes which take a physical system A into another physical system B as morphisms $A \longrightarrow B$, and
3. Sequential composition of physical processes as $\circ$, and the process that leaves system A invariant as 1_A.

If we want to construct a category whose objects are categories themselves, then we need to tell what are the arrows $\mathbf{C} \xrightarrow{F} \mathbf{D}$ that take a category $\mathbf{C}$ to another category $\mathbf{D}$. For our purposes, the notion of *functor* originates from this necessity.

Definition 1.2: Functor

Let $\mathbf{C}$ and $\mathbf{D}$ be categories and denote the family of their objects by $\mathcal{O}bj(\mathbf{C})$ and $\mathcal{O}bj(\mathbf{D})$, respectively. A functor $F: \mathbf{C} \to \mathbf{D}$ consists of

1. A mapping
$$F : \mathcal{O}bj(\mathbf{C}) \to \mathcal{O}bj(\mathbf{D}),\ A \mapsto F(A) \text{ and}$$
2. For any $A, B, C \in \mathcal{O}bj(\mathbf{C})$, a mapping
$$F: \mathbf{C}(A, B) \to \mathbf{D}(F(A), F(B)),\ f \mapsto F(f)$$
that preserves identites and compositions.

In terms of diagrams, functors behave as follows:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] & & F(A) \arrow[r, "F(f)"] \arrow[dr, "F(g\circ f)"'] & F(B) \arrow[d, "F(g)"] \\
& C & & & F(C) \\
\mathbf{C} \arrow[rrrr, "F"] & & & & \mathbf{D}
\end{tikzcd}$$

Functor composition works as expected, that is, if $F : \mathbf{C} \to \mathbf{D}$ and $G: \mathbf{D} \to \mathbf{E}$ are functors, then we define the functor $G \circ F: \mathbf{C} \to \mathbf{E}$ via the maps that define F and G, *i.e*

$$A \in \mathcal{O}bj(\mathbf{C}) \mapsto (G \circ F)(A) \doteq G(F(A)) \in \mathcal{O}bj(\mathbf{E}),\ \text{and}$$
$$f \in \mathbf{C}(A, B) \mapsto (G \circ F)(f) \doteq G(F(f)) \in \mathbf{E}(G(F(A)), G(F(B))).$$

It is fairly easy to show that the functor composition as defined above is associative, and preserves morphisms compositions and identities. Furthermore, for each category $\mathbf{C}$, one can construct an identity functor $1_{\mathbf{C}}$. From all that, we have that the collection of all categories and all functors constitutes a category which we denote by **Cat**[1].

Groups

Due to Noether's theorem (be its classical version or its quantum-mechanical one), we know that groups play a fundamental role in physics. Hence, it seems like a very basic request to have a category-theoretical description of groups if we want to talk about symmetries in quantum mechanical systems in a category-theoretical fashion. In order to do this, we first have to introduce the notion of a *monoid*.

Definition 1.3: Monoid

A monoid is a set M equipped with a binary relation $\cdot : M \times M \to M$ that is associative and admits a unit, *i.e.* for all $x, y, z \in M$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

and there is an (unique) element $1 \in M$ such that

$$1 \cdot x = x \cdot 1 = x.$$

[1] I did not mention the composition rule nor the identity, but they should be fairly obvious at this point.

A simple example of a monoid is the set of arrows from A to A, denoted by $\mathrm{Hom}_{\mathbf{C}}(A, A)$, where A is an object of a category $\mathbf{C}$. Besides, given that monoid homomorphisms preserve the monoid structure, we can construct the category **Mon** whose objects are monoids and whose arrows are monoid homomorphisms. What is more interesting is that we can interpret a monoid as a category itself, as it is illustrated in the following example.

Example 1.4: Monoids as categories

If M is a monoid, we can identify it with a category $\mathbf{M}$ that has a single object $*$, whose morphisms consists are arrows $* \xrightarrow{m} *$, where $m \in M$. The composition between arrows is given by the monoid product $\cdot$ and the identity $* \xrightarrow{1_*} *$ is associated to the unit element $1 \in M$.

Now, recall that a group G is just a monoid (in the sense of definition 1.3) such that every $g \in G$ admits an (unique) inverse $g^{-1} \in G$. So, at the category theory level, we would expect groups to the monoids (in the sense of example 1.4) whose arrows, in a sense, also have inverses. This is made precisely clear through the following definition.

Definition 1.4: Isomorphism

Let $\mathbf{C}$ be a category. Two objects $A, B \in \mathcal{O}b j(\mathbf{C})$ are isomorphic if there are morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} B$ such that

$$g \circ f = 1_A \text{ and } f \circ g = 1_B.$$

In this case, f is called an isomorphism and $g \equiv f^{-1}$ is called the inverse of f.

Example 1.5: Groups as categories

Putting together example 1.4 and definition 1.4, we conclude that a group G is a category with one object and whose morphisms are all isomorphisms.

3 Building new categories

4 Hilbert Spaces

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