Cálculo Numérico - IME/UERJ

Gabarito - Lista de Exercícios 3

Sistemas Lineares - Métodos diretos e iterativos

1. $X = (0.7681, -0.189, -0.6549)^t$

2. A última linha do sistema triangular é nula, portanto é satisfeita para qualquer valor de z, o que significa que o conjunto de soluções do sistema linear é infinito e está definido por:

   $S = \left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; x = \frac{1}{16}(z + 13); y = \frac{1}{8}(11z - 17)\right\}$

3. Há necessidade de troca da primeira e terceira linhas.

   $(x, y, z) = (10/7, -5/3, 9/5) \approx (1.423, -1.667, 1.800)$, usando 4 dígitos e arredondamento.

4.

$$L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}; \; U = \begin{bmatrix} 1 & -4 \\ 0 & 5 \end{bmatrix}.$$

   (a) (i) $X = (6/5, -1/5)^t$

   (ii) $X = (-3/5, -2/5)^t$

   (b) **Primeira maneira:**

$$LU \cdot \underbrace{\begin{bmatrix} | & | \\ d_1 & d_2 \\ | & | \end{bmatrix}}_{D=A^{-1}} = \underbrace{\begin{bmatrix} | & | \\ I_1 & I_2 \\ | & | \end{bmatrix}}_{I}$$

   onde

$$d_1 = \begin{bmatrix} d_{11} \\ d_{21} \end{bmatrix}; \; d_2 = \begin{bmatrix} d_{12} \\ d_{22} \end{bmatrix};$$

$$I_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \; ; \; I_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Assim, resolvemos 4 sistemas:

$$LUd_1 = I_1 \underbrace{\Rightarrow}_{Ud_1 = y_1} \left\{ \begin{array}{ccc} Ly_1 & = & I_1 \quad (1) \Rightarrow \text{Acho } y_1 \\ Ud_1 & = & y_1 \quad (2) \Rightarrow \text{Acho } d_1 \end{array} \right. .$$

$$d_1 = \begin{bmatrix} 1/5 \\ -1/5 \end{bmatrix}.$$

$$LUd_2 = I_2 \underbrace{\Rightarrow}_{Ud_2 = y_2} \left\{ \begin{array}{ccc} Ly_2 & = & I_2 \quad (3) \Rightarrow \text{Acho } y_2 \\ Ud_2 & = & y_2 \quad (4) \Rightarrow \text{Acho } d_2 \end{array} \right. .$$

$$d_2 = \begin{bmatrix} 4/5 \\ 1/5 \end{bmatrix}.$$

Logo,

$$A^{-1} = \begin{bmatrix} 1/5 & 4/5 \\ -1/5 & 1/5 \end{bmatrix}.$$

Segunda maneira: Usando forma escalonada por linhas para achar as inversas de L e U.

Assim,

$A = LU \Rightarrow A^{-1} = U^{-1}L^{-1}$

Cálculo de U^{-1}:

$$U \mid I = \begin{array}{c} \begin{bmatrix} 1 & -4 & | & 1 & 0 \\ 0 & 5 & | & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} L_1 \\ L_2 \end{array}$$

pivô $= a_{22} = 5$;

$$m_{12} = \frac{a_{12}}{a_{22}} = -\frac{4}{5} \Rightarrow L_1 \leftarrow L_1 + \frac{4}{5}L_2$$

$$U^{(1)} \mid D^{(1)} = \begin{bmatrix} 1 & 0 & | & 1 & 4/5 \\ 0 & 5 & | & 0 & 1 \end{bmatrix} \begin{matrix} L_1 \\ L_2 \end{matrix}$$

Dividindo L_2 por 5, obtemos:

$$I \mid U^{-1} = \begin{bmatrix} 1 & 0 & | & 1 & 4/5 \\ 0 & 1 & | & 0 & 1/5 \end{bmatrix} \begin{matrix} L_1 \\ L_2 \end{matrix}$$

Cálculo de L^{-1}:

$$L \mid I = \begin{bmatrix} 1 & 0 & | & 1 & 0 \\ 1 & 1 & | & 0 & 1 \end{bmatrix} \begin{matrix} L_1 \\ L_2 \end{matrix}$$

pivô $= a_{11} = 1$;
$m_{21} = \dfrac{a_{21}}{a_{11}} = 1 \Rightarrow L_2 \leftarrow L_2 - L_1$

$$I \mid L^{-1} = \begin{bmatrix} 1 & 0 & | & 1 & 0 \\ 0 & 1 & | & -1 & 1 \end{bmatrix} \begin{matrix} L_1 \\ L_2 \end{matrix}$$

Portanto,

$$A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 1 & 4/5 \\ 0 & 1/5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/5 & 4/5 \\ -1/5 & 1/5 \end{bmatrix}$$

5. (a)

$$LU \cdot \underbrace{\begin{bmatrix} | & | & | \\ d_1 & d_2 & d_3 \\ | & | & | \end{bmatrix}}_{D=A^{-1}} = \underbrace{\begin{bmatrix} | & | & | \\ I_1 & I_2 & I_3 \\ | & | & | \end{bmatrix}}_{I}$$

onde

$$d_1 = \begin{bmatrix} d_{11} \\ d_{21} \\ d_{31} \end{bmatrix} ; d_2 = \begin{bmatrix} d_{12} \\ d_{22} \\ d_{32} \end{bmatrix} ; d_3 = \begin{bmatrix} d_{13} \\ d_{23} \\ d_{33} \end{bmatrix} ;$$

$$I_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ; I_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} ; I_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} .$$

Assim, resolvemos 6 sistemas:

$$LUd_1 = I_1 \underbrace{\Rightarrow}_{Ud_1 = y_1} \begin{cases} Ly_1 & = & I_1 & (1) \Rightarrow \text{Acho } y_1 \\ Ud_1 & = & y_1 & (2) \Rightarrow \text{Acho } d_1 \end{cases} .$$

$$LUd_2 = I_2 \underbrace{\Rightarrow}_{Ud_2 = y_2} \begin{cases} Ly_2 & = & I_2 & (3) \Rightarrow \text{Acho } y_2 \\ Ud_2 & = & y_2 & (4) \Rightarrow \text{Acho } d_2 \end{cases} .$$

$$LUd_3 = I_3 \underbrace{\Rightarrow}_{Ud_3 = y_3} \begin{cases} Ly_3 & = & I_3 & (5) \Rightarrow \text{Acho } y_3 \\ Ud_3 & = & y_3 & (6) \Rightarrow \text{Acho } d_3 \end{cases} .$$

Assim,

$$A^{-1} = \begin{bmatrix} | & | & | \\ d_1 & d_2 & d_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ -3 & 2 & -2 \\ 1 & -1 & 2 \end{bmatrix}$$

Segunda maneira: Usando forma escalonada por linhas para achar as inversas de L e U.

Assim,

$A = LU \Rightarrow A^{-1} = U^{-1}L^{-1}$

Cálculo de U^{-1}:

$$U \mid I = \left[\begin{array}{ccccccc} 2 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 0 & 1/2 & | & 0 & 0 & 1 \end{array}\right] \begin{array}{c} L_1 \\ L_2 \\ L_3 \end{array}$$

pivô $= a_{33} = 1/2$;

$m_{23} = \dfrac{a_{23}}{a_{33}} = 2 \Rightarrow L_2 \leftarrow L_2 - 2L_3$

$$U^{(1)} \mid D^{(1)} = \begin{array}{c} \left[\begin{array}{ccccccc} 2 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 1 & -2 \\ 0 & 0 & 1/2 & | & 0 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} L_1 \\ L_2 \\ L_3 \end{array}$$

pivô $= a_{22} = 1$;

$m_{12} = \dfrac{a_{12}}{a_{22}} = 1 \Rightarrow L_1 \leftarrow L_1 - L_2$

$$U^{(2)} \mid D^{(2)} = \left[\begin{array}{ccccccc} 2 & 0 & 0 & | & 1 & -1 & 2 \\ 0 & 1 & 0 & | & 0 & 1 & -2 \\ 0 & 0 & 1/2 & | & 0 & 0 & 1 \end{array}\right] \begin{array}{c} L_1 \\ L_2 \\ L_3 \end{array}$$

Fazendo as operações:

$L_1 \leftarrow \dfrac{1}{2}L_1$;

$L_3 \leftarrow 2L_3$,

obtemos:

$$I \mid U^{-1} = \left[\begin{array}{ccccccc} 1 & 0 & 0 & | & 1/2 & -1/2 & 1 \\ 0 & 1 & 0 & | & 0 & 1 & -2 \\ 0 & 0 & 1 & | & 0 & 0 & 2 \end{array}\right] \begin{array}{c} L_1 \\ L_2 \\ L_3 \end{array}$$

Cálculo de L^{-1}:

$$L \mid I = \left[\begin{array}{ccccccc} 1 & 0 & 0 & | & 1 & 0 & 0 \\ 2 & 1 & 0 & | & 0 & 1 & 0 \\ 1/2 & 1/2 & 1 & | & 0 & 0 & 1 \end{array}\right] \begin{array}{c} L_1 \\ L_2 \\ L_3 \end{array}$$

pivô $= a_{11} = 1$;

$m_{21} = \dfrac{a_{21}}{a_{11}} = 2 \Rightarrow L_2 \leftarrow L_2 - 2L_1$;

$m_{31} = \dfrac{a_{31}}{a_{11}} = \dfrac{1}{2} \Rightarrow L_3 \leftarrow L_3 - \dfrac{1}{2}L_1$

$$L^{(1)} \mid D^{(1)} = \left[\begin{array}{ccccccc} 1 & 0 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 1/2 & 1 & | & -1/2 & 0 & 1 \end{array}\right] \begin{array}{c} L_1 \\ L_2 \\ L_3 \end{array}$$

pivô $= a_{22} = 1$;

$m_{32} = \dfrac{a_{32}}{a_{22}} = \dfrac{1}{2} \Rightarrow L_3 \leftarrow L_3 - \dfrac{1}{2}L_2$

$$I \mid L^{-1} = \left[\begin{array}{ccccccc} 1 & 0 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & 1/2 & -1/2 & 1 \end{array}\right] \begin{array}{c} L_1 \\ L_2 \\ L_3 \end{array}$$

Portanto,

$$A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1/2 & -1/2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ -3 & 2 & -2 \\ 1 & -1 & 2 \end{bmatrix}$$

(b) **Primeira maneira:**

$$LU \cdot \underbrace{\begin{bmatrix} | & | & | \\ d_1 & d_2 & d_3 \\ | & | & | \end{bmatrix}}_{D=A^{-1}} = \underbrace{\begin{bmatrix} | & | & | \\ P_1 & P_2 & P_3 \\ | & | & | \end{bmatrix}}_{P}$$

onde

$$d_1 = \begin{bmatrix} d_{11} \\ d_{21} \\ d_{31} \end{bmatrix}; d_2 = \begin{bmatrix} d_{12} \\ d_{22} \\ d_{32} \end{bmatrix}; d_3 = \begin{bmatrix} d_{13} \\ d_{23} \\ d_{33} \end{bmatrix};$$

$$P_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; P_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; P_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Assim, resolvemos 6 sistemas:

$$LUd_1 = P_1 \underbrace{\Rightarrow}_{Ud_1=y_1} \left\{ \begin{array}{ccc} Ly_1 & = & P_1 \quad (1) \Rightarrow \text{Acho } y_1 \\ Ud_1 & = & y_1 \quad (2) \Rightarrow \text{Acho } d_1 \end{array} \right. .$$

$$LUd_2 = P_2 \underbrace{\Rightarrow}_{Ud_2=y_2} \left\{ \begin{array}{ccc} Ly_2 & = & P_2 \quad (3) \Rightarrow \text{Acho } y_2 \\ Ud_2 & = & y_2 \quad (4) \Rightarrow \text{Acho } d_2 \end{array} \right. .$$

$$LUd_3 = P_3 \underbrace{\Rightarrow}_{Ud_3=y_3} \left\{ \begin{array}{ccc} Ly_3 & = & P_3 \quad (5) \Rightarrow \text{Acho } y_3 \\ Ud_3 & = & y_3 \quad (6) \Rightarrow \text{Acho } d_3 \end{array} \right. .$$

Assim,

$$A^{-1} = \begin{bmatrix} | & | & | \\ d_1 & d_2 & d_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ -3 & 2 & -2 \\ 1 & -1 & 2 \end{bmatrix}$$

Segunda maneira: Usando forma escalonada por linhas para achar as inversas de L e U.

Assim,

$PA = LU \Rightarrow (PA)^{-1} = (LU)^{-1} \Rightarrow A^{-1}P^{-1} = U^{-1}L^{-1} \Rightarrow A^{-1} = U^{-1}L^{-1}P$

O resultado fica:

$$A^{-1} = U^{-1}L^{-1}P = \begin{bmatrix} 1/2 & -1/2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1/2 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 2 & -1 & 1 \\ -3 & 2 & -2 \\ 1 & -1 & 2 \end{bmatrix}$$

6. $X = (4/7, -1/14, -5/14)^t$

7. (a) $X = (-0.8780, -1.4390, -2.2195, 2.2195)$