Tópicos Especiais em Matemática Aplicada - 2025-1
UERJ

04 - Método dos Elementos Finitos

Rodrigo Madureira
rodrigo.madureira@ime.uerj.br
Github: https://github.com/rodrigolrmadureira/ElementosFinitos

Sumário

Problema aproximado

Com a formulação de Galerkin, obtemos o sistema linear:

$$Kc = F, \tag{1}$$

com

$$K = [K_{ij}]_{m\times m} \ \textbf{(matriz de rigidez)},$$
$$F = [F_i]_{m\times 1} \ \textbf{(vetor força)},$$
$$c = [c_i]_{m\times 1} \ \textbf{(vetor solução para } u_h(x)\textbf{)}$$

onde

$$K_{ij} = a(\varphi_i, \varphi_j) = \alpha M_{ij} + \beta N_{ij},$$
$$M_{ij} = (\varphi_{ix}, \varphi_{jx});\ N_{ij} = (\varphi_i, \varphi_j);\ F_j = (f, \varphi_j).$$

Veremos agora duas abordagens para a montagem da matriz K e do vetor F: uma tradicional e outra com **Método dos Elementos Finitos (MEF)**.

Interpolação de Lagrange linear por partes

Obs.: No problema dado, o domínio é $\Omega = [0, 1]$.

Seja V_h um subespaço de $V = H_0^1(\Omega)$ definido como:

$$V_h = [\varphi_1, \ldots, \varphi_m] \tag{2}$$

As funções φ_i escolhidas são **funções de interpolação de Lagrange linear por partes** satisfazendo:

$$\varphi_i(x_j) = \begin{cases} 1, & \text{se } i = j, \\ 0, & \text{se } i \neq j, \end{cases} \tag{3}$$

onde $x_j \in \Omega$ é denominado **nó**.

Os nós são pontos discretos do intervalo Ω distribuídos de forma equidistante.

Tomamos $(m-1)$ divisões do domínio Ω e definimos o passo

$$h = x_{i+1} - x_i, \quad i = 1, \ldots, m. \tag{4}$$

Interpolação de Lagrange linear por partes

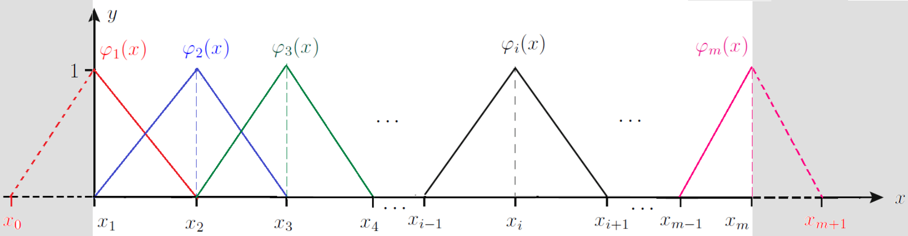


Figura: Funções da base de Lagrange linear

Em cada nó x_i, definimos a função de Lagrange linear por partes $\varphi_i(x)$, satisfazendo a condição (3). Assim, $\varphi_i(x)$ para $i = 1, \ldots, m$ é definida por:

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} = \dfrac{x - x_{i-1}}{h}, & \text{se } x \in [x_{i-1}, x_i], \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} = \dfrac{x_{i+1} - x}{h}, & \text{se } x \in [x_i, x_{i+1}], \\ 0, & \text{se } x \notin [x_{i-1}, x_{i+1}]. \end{cases} \tag{5}$$

Obs.: Os nós x_0 e x_{m+1} estão fora do domínio Ω.

Interpolação de Lagrange linear por partes

De (5), podemos calcular a derivada de $\varphi_i(x)$, obtendo-se:

$$\varphi_{ix} = \frac{d\varphi_i}{dx} = \begin{cases} \frac{1}{h}, & \text{se } x \in [x_{i-1}, x_i], \\ -\frac{1}{h}, & \text{se } x \in [x_i, x_{i+1}], \\ 0, & \text{se } x \notin [x_{i-1}, x_{i+1}]. \end{cases} \tag{6}$$

Observe que para φ_i e φ_j não consecutivos:

$$\varphi_i(x)\varphi_j(x) = \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx} = 0, \qquad \text{se } |i-j| \geq 2. \tag{7}$$

Matriz de rigidez K

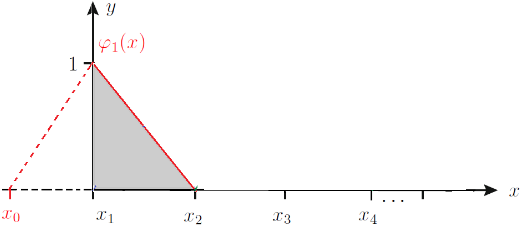
Por definição do problema,

$$K_{ij} = \alpha M_{ij} + \beta N_{ij} = \alpha(\varphi_{ix}, \varphi_{jx}) + \beta(\varphi_i, \varphi_j) = \alpha \int_0^1 \varphi_{ix}\varphi_{jx} dx + \beta \int_0^1 \varphi_i \varphi_j dx$$

Vamos ver o que acontece com a primeira linha de K (K_{1j}, para $j = 1, 2, \ldots, m$):

$$K_{11} = \alpha(\varphi_{1x}, \varphi_{1x}) + \beta(\varphi_1, \varphi_1) = \alpha \int_0^1 (\varphi_{1x})^2 dx + \beta \int_0^1 (\varphi_1(x))^2 dx$$

$$= \alpha \int_{x_1}^{x_2} (\varphi_{1x})^2 dx + \beta \int_{x_1}^{x_2} (\varphi_1(x))^2 dx \neq 0$$

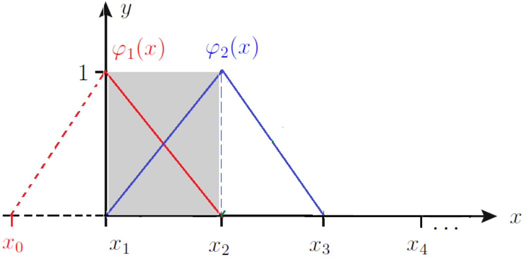


Obs.: x_0 é um nó que não faz parte do domínio, pois $\Omega = [x_1, x_m] = [0, 1]$.

Matriz de rigidez K

$$K_{12} = \alpha(\varphi_{1x}, \varphi_{2x}) + \beta(\varphi_1, \varphi_2) = \alpha \int_0^1 \varphi_{1x}\varphi_{2x} dx + \beta \int_0^1 \varphi_1(x)\varphi_2(x) dx$$

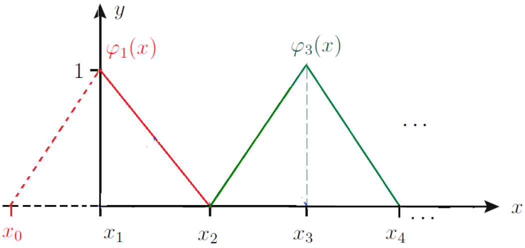
$$= \alpha \int_{x_1}^{x_2} \varphi_{1x}\varphi_{2x} dx + \beta \int_{x_1}^{x_2} \varphi_1(x)\varphi_2(x) dx \neq 0$$



Matriz de rigidez K

Agora, para $j \geq 3$:

$$K_{1j} = \alpha(\varphi_{1x}, \varphi_{jx}) + \beta(\varphi_1, \varphi_j) = \alpha \int_0^1 \varphi_{1x}\varphi_{jx} dx + \beta \int_0^1 \varphi_1(x)\varphi_j(x) dx = 0$$



Então, a primeira linha de K segue o padrão:

$$K_{1j} = \begin{bmatrix} * & * & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

onde os asteriscos são os elementos não nulos K_{11} e K_{12}.

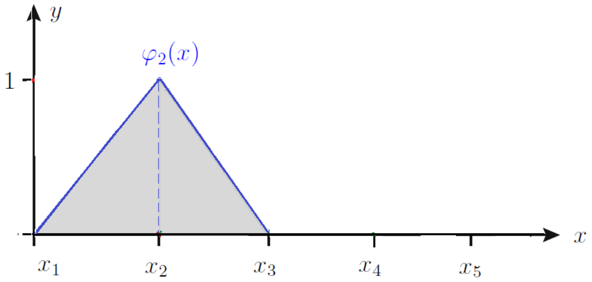
Matriz de rigidez K

Vamos ver o que acontece com a segunda linha de K (K_{2j}, para $j = 1, 2, \ldots, m$):

Como a matriz K é simétrica, $K_{21} = K_{12}$, que já calculamos. Logo, $K_{21} \neq 0$.

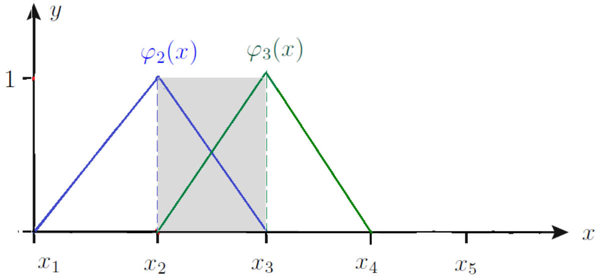
$$K_{22} = \alpha(\varphi_{2x}, \varphi_{2x}) + \beta(\varphi_2, \varphi_2) = \alpha \int_0^1 (\varphi_{2x})^2 dx + \beta \int_0^1 (\varphi_2(x))^2 dx$$

$$= \alpha \int_{x_1}^{x_3} (\varphi_{2x})^2 dx + \beta \int_{x_1}^{x_3} (\varphi_2(x))^2 dx \neq 0$$



Matriz de rigidez K

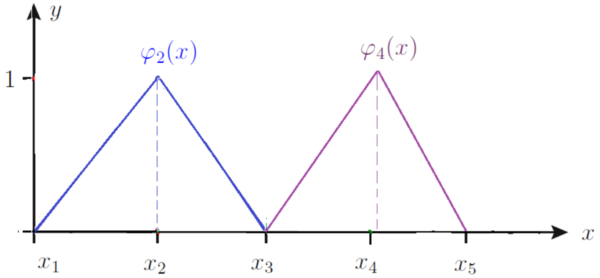
$$K_{23} = \alpha(\varphi_{2x}, \varphi_{3x}) + \beta(\varphi_2, \varphi_3) = \alpha \int_0^1 \varphi_{2x}\varphi_{3x} dx + \beta \int_0^1 \varphi_2(x)\varphi_3(x) dx$$
$$= \alpha \int_{x_2}^{x_3} \varphi_{2x}\varphi_{3x} dx + \beta \int_{x_2}^{x_3} \varphi_2(x)\varphi_3(x) dx \neq 0$$



Matriz de rigidez K

Agora, para $j \geq 4$:

$$K_{2j} = \alpha(\varphi_{2x}, \varphi_{jx}) + \beta(\varphi_2, \varphi_j) = \alpha \int_0^1 \varphi_{2x}\varphi_{jx}dx + \beta \int_0^1 \varphi_2(x)\varphi_j(x)dx = 0$$



Então, a segunda linha de K segue o padrão:

$$K_{2j} = \begin{bmatrix} * & * & * & 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

onde os asteriscos são os elementos não nulos K_{21}, K_{22} e K_{23}.

Matriz de rigidez K

Analogamente, repetindo o processo para as demais linhas, obtemos a matriz quadrada, tridiagonal e esparsa $m \times m$:

$$K = \begin{bmatrix} * & * & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & * & * & * & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & * & * \end{bmatrix}$$

Matriz de rigidez K

$$K = \begin{bmatrix} * & * & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & K_{ii} & K_{i,i+1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & K_{i+1,i} & * & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & * & * \end{bmatrix},$$

Agora, devemos calcular os elementos não nulos, que são:

- **Os elementos da diagonal de** K:

$$K_{ii}, \text{ para } i = 1, 2, \ldots, m.$$

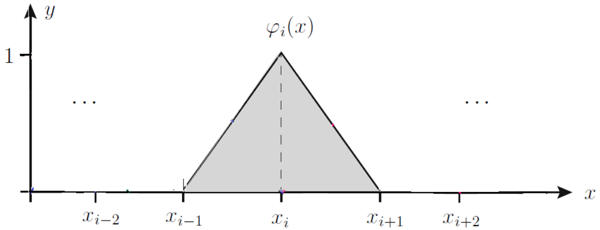
- **Os elementos à direita ou abaixo de** K_{ii}:

$$K_{i,i+1} = K_{i+1,i}, \text{ para } i = 1, 2, \ldots, m.$$

Matriz de rigidez K

Por definição,

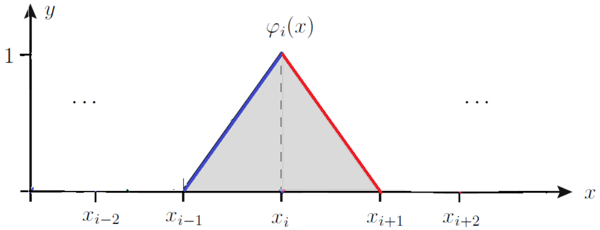
$$
\begin{aligned}
K_{ii} &= \alpha(\varphi_{ix}, \varphi_{ix}) + \beta(\varphi_i, \varphi_i) = \alpha \int_0^1 (\varphi_{ix})^2 dx + \beta \int_0^1 (\varphi_i(x))^2 dx \\
&= \alpha \int_{x_{i-1}}^{x_{i+1}} (\varphi_{ix})^2 dx + \beta \int_{x_{i-1}}^{x_{i+1}} (\varphi_i(x))^2 dx
\end{aligned}
$$



Matriz de rigidez K

Por definição,

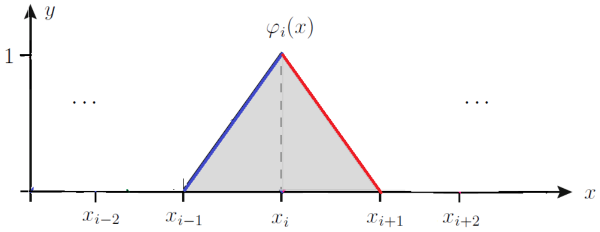
$$K_{ii} = \alpha\left(\int_{x_{i-1}}^{x_i} (\varphi_{ix})^2 dx + \int_{x_i}^{x_{i+1}} (\varphi_{ix})^2 dx\right)$$
$$+ \beta\left(\int_{x_{i-1}}^{x_i} (\varphi_i(x))^2 dx + \int_{x_i}^{x_{i+1}} (\varphi_i(x))^2 dx\right)$$



Matriz de rigidez K

Por definição,

$$
\begin{aligned}
K_{ii} &= \alpha\left(\int_{x_{i-1}}^{x_i}\left(\frac{1}{h}\right)^2 dx + \int_{x_i}^{x_{i+1}}\left(-\frac{1}{h}\right)^2 dx\right) \\
&\quad + \beta\left(\int_{x_{i-1}}^{x_i}\left(\frac{x - x_{i-1}}{h}\right)^2 dx + \int_{x_i}^{x_{i+1}}\left(\frac{x_{i+1} - x}{h}\right)^2 dx\right) \\
&= \frac{\alpha}{h^2}\left(\int_{x_{i-1}}^{x_i} dx + \int_{x_i}^{x_{i+1}} dx\right) \\
&\quad + \beta\left(\int_{x_{i-1}}^{x_i}\left(\frac{x - x_{i-1}}{h}\right)^2 dx + \int_{x_i}^{x_{i+1}}\left(\frac{x_{i+1} - x}{h}\right)^2 dx\right)
\end{aligned}
$$



Matriz de rigidez K

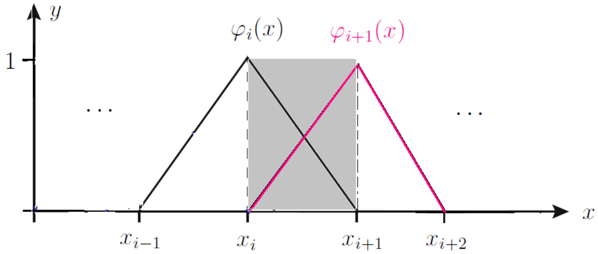
Por definição,

$$
\begin{aligned}
K_{ii} &= \frac{\alpha}{h^2}(h+h) + \beta\left(\int_{x_{i-1}}^{x_i}\left(\frac{x-x_{i-1}}{h}\right)^2 dx + \int_{x_i}^{x_{i+1}}\left(\frac{x_{i+1}-x}{h}\right)^2 dx\right) \\
&= \frac{2\alpha}{h} + \beta\left(\int_{x_{i-1}}^{x_i}\left(\frac{x-x_{i-1}}{h}\right)^2 dx + \int_{x_i}^{x_{i+1}}\left(\frac{x_{i+1}-x}{h}\right)^2 dx\right) \\
&= \frac{2\alpha}{h} + \frac{\beta}{h^2}\left(\int_{x_{i-1}}^{x_i}(x-x_{i-1})^2 dx + \int_{x_i}^{x_{i+1}}(x_{i+1}-x)^2 dx\right) \\
&= \frac{2\alpha}{h} + \frac{\beta}{h^2}\left(\left[\frac{(x-x_{i-1})^3}{3}\right]_{x_{i-1}}^{x_i} + \left[-\frac{(x_{i+1}-x)^3}{3}\right]_{x_i}^{x_{i+1}}\right) \\
&= \frac{2\alpha}{h} + \frac{\beta}{h^2}\left(\left[\frac{(x_i-x_{i-1})^3}{3} - \cancelto{0}{\frac{(x_{i-1}-x_{i-1})^3}{3}}\right]\right. \\
&\qquad\qquad \left. + \left[-\cancelto{0}{\frac{(x_{i+1}-x_{i+1})^3}{3}} + \frac{(x_{i+1}-x_i)^3}{3}\right]\right) \\
&= \frac{2\alpha}{h} + \frac{\beta}{h^2}\left(\frac{h^3}{3} + \frac{h^3}{3}\right) = \frac{2\alpha}{h} + \frac{\beta}{h^2}\left(\frac{2h^3}{3}\right) = \frac{2\alpha}{h} + \frac{2\beta h}{3} = 2\left(\frac{\alpha}{h} + \frac{\beta h}{3}\right).
\end{aligned}
$$

Matriz de rigidez K

Agora, calculamos:

$$
\begin{aligned}
K_{i,i+1} &= \alpha(\varphi_{ix}, \varphi_{(i+1)x}) + \beta(\varphi_i, \varphi_{i+1}) \\
&= \alpha \int_0^1 \varphi_{ix}\varphi_{(i+1)x} dx + \beta \int_0^1 \varphi_i(x)\varphi_{i+1}(x) dx \\
&= \alpha \int_{x_i}^{x_{i+1}} \varphi_{ix}\varphi_{(i+1)x} dx + \beta \int_{x_i}^{x_{i+1}} \varphi_i(x)\varphi_{i+1}(x) dx \\
&= \alpha \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right)\left(\frac{1}{h}\right) dx + \beta \int_{x_i}^{x_{i+1}} \left(\frac{x_{i+1}-x}{h}\right)\left(\frac{x-x_i}{h}\right) dx
\end{aligned}
$$



Matriz de rigidez K

$$
\begin{aligned}
K_{i,i+1} &= -\frac{\alpha}{h^2}\int_{x_i}^{x_{i+1}} dx + \beta\int_{x_i}^{x_{i+1}} \left(\frac{x_{i+1}-x}{h}\right)\left(\frac{x-x_i}{h}\right)dx \\
&= -\frac{\alpha}{h^2}h + \beta\int_{x_i}^{x_{i+1}} \left(\frac{x_{i+1}-x}{h}\right)\left(\frac{x-x_i}{h}\right)dx \\
&= -\frac{\alpha}{h} + \frac{\beta}{h^2}\int_{x_i}^{x_{i+1}} (x_{i+1}-x)(x-x_i)dx
\end{aligned}
$$

Vamos aplicar substituição de variáveis na segunda integral:

$z = x - x_i \Rightarrow dx = dz$

Sabemos que $x \in [x_i, x_{i+1}]$. Assim, $x = x_i \Rightarrow z = 0$ e $x = x_{i+1} \Rightarrow z = h$. Logo, $z \in [0, h]$.

Além disso,
$x_{i+1} - x = x_i + h - x = (x_i - x) + h = -(x - x_i) + h = -z + h = h - z$.

Matriz de rigidez K

Após a mudança de variáveis, obtemos:

$$\begin{aligned}
K_{i,i+1} &= -\frac{\alpha}{h} + \frac{\beta}{h^2}\int_0^h (h-z)z\,dz = -\frac{\alpha}{h} + \frac{\beta}{h^2}\int_0^h (hz - z^2)\,dz \\
&= -\frac{\alpha}{h} + \frac{\beta}{h^2}\left[h\frac{z^2}{2} - \frac{z^3}{3}\right]_0^h = -\frac{\alpha}{h} + \frac{\beta}{h^2}\left[h\frac{h^2}{2} - \frac{h^3}{3}\right] \\
&= -\frac{\alpha}{h} + \frac{\beta}{h^2}\left[\frac{h^3}{2} - \frac{h^3}{3}\right] = -\frac{\alpha}{h} + \frac{\beta}{h^2}\left[\frac{h^3}{6}\right] = -\frac{\alpha}{h} + \frac{\beta h}{6}
\end{aligned}$$

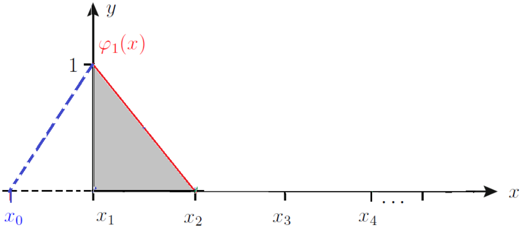
Como K é simétrica,

$$K_{i+1,i} = K_{i,i+1} = -\frac{\alpha}{h} + \frac{\beta h}{6}$$

Matriz de rigidez K

Obs.: x_0 é um nó que não faz parte do domínio $\Omega = [x_1, x_m] = [0, 1]$. Assim,

$$
\begin{aligned}
K_{11} &= \alpha(\varphi_{1x}, \varphi_{1x}) + \beta(\varphi_1, \varphi_1) = \alpha \int_0^1 (\varphi_{1x})^2 dx + \beta \int_0^1 (\varphi_1(x))^2 dx \\
&= \alpha \int_{x_1}^{x_2} \left(-\frac{1}{h}\right)^2 dx + \beta \int_{x_1}^{x_2} \left(\frac{x_2 - x}{h}\right)^2 dx \\
&= \frac{\alpha}{h^2} \int_{x_1}^{x_2} dx + \beta \int_{x_1}^{x_2} \left(\frac{x_2 - x}{h}\right)^2 dx = \frac{\alpha}{h^2} h + \frac{\beta}{h^2} \int_{x_1}^{x_2} (x_2 - x)^2 dx \\
&= \frac{\alpha}{h} + \frac{\beta}{h^2} \left[-\frac{(x_2 - x)^3}{3}\right]_{x_1}^{x_2} = \frac{\alpha}{h} + \frac{\beta h}{3}.
\end{aligned}
$$



Matriz de rigidez K

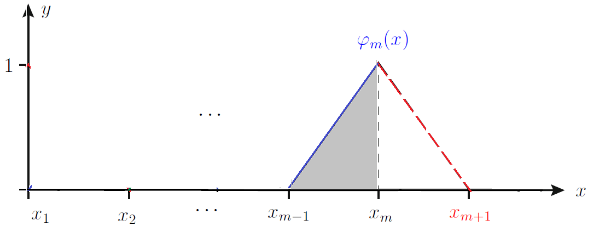
Obs.: x_{m+1} é um nó que não faz parte do domínio $\Omega = [x_1, x_m] = [0, 1]$. Assim,

$$K_{mm} = \alpha(\varphi_{mx}, \varphi_{mx}) + \beta(\varphi_m, \varphi_m) = \alpha \int_0^1 (\varphi_{mx})^2 dx + \beta \int_0^1 (\varphi_m(x))^2 dx$$

$$= \alpha \int_{x_{m-1}}^{x_m} \left(\frac{1}{h}\right)^2 dx + \beta \int_{x_{m-1}}^{x_m} \left(\frac{x - x_{m-1}}{h}\right)^2 dx$$

$$= \frac{\alpha}{h^2} \int_{x_{m-1}}^{x_m} dx + \beta \int_{x_{m-1}}^{x_m} \left(\frac{x - x_{m-1}}{h}\right)^2 dx = \frac{\alpha}{h^2} h + \frac{\beta}{h^2} \int_{x_{m-1}}^{x_m} (x - x_{m-1})^2 dx$$

$$= \frac{\alpha}{h} + \frac{\beta}{h^2} \left[\frac{(x - x_{m-1})^3}{3}\right]_{x_{m-1}}^{x_m} = \frac{\alpha}{h} + \frac{\beta h}{3}.$$



Vetor força F

Componentes do vetor F:

Obs.: $x_0 \notin \Omega = [x_1, x_m] = [0, 1]$. Assim,

$$F_1 = (f, \varphi_1) = \int_0^1 f(x)\varphi_1(x)dx = \int_{x_1}^{x_2} f(x)\varphi_1(x)dx = \int_{x_1}^{x_2} f(x)\varphi_1(x)dx$$
$$= \int_{x_1}^{x_2} f(x)\left(\frac{x_2 - x}{h}\right)dx;$$

Para $i = 2, \ldots, m$:

$$F_i = (f, \varphi_i) = \int_0^1 f(x)\varphi_i(x)dx = \int_{x_{i-1}}^{x_{i+1}} f(x)\varphi_i(x)dx$$
$$= \int_{x_{i-1}}^{x_i} f(x)\varphi_i(x)dx + \int_{x_i}^{x_{i+1}} f(x)\varphi_i(x)dx$$
$$= \int_{x_{i-1}}^{x_i} f(x)\left(\frac{x - x_{i-1}}{h}\right)dx + \int_{x_i}^{x_{i+1}} f(x)\left(\frac{x_{i+1} - x}{h}\right)dx$$

Vetor força F

Obs.: $x_{m+1} \notin \Omega = [x_1, x_m] = [0, 1]$. Assim,

$$F_m = (f, \varphi_m) = \int_0^1 f(x)\varphi_m(x)dx = \int_{x_{m-1}}^{x_m} f(x)\varphi_m(x)dx$$
$$= \int_{x_{m-1}}^{x_m} f(x)\varphi_m(x)dx$$
$$= \int_{x_{m-1}}^{x_m} f(x)\left(\frac{x - x_{m-1}}{h}\right)dx$$

O cálculo vai depender da definição da função $f(x)$.

Resultados

Matriz de rigidez K:

$$K_{11} = \left(\frac{\alpha}{h} + \frac{\beta h}{3}\right);$$

$$K_{ii} = 2\left(\frac{\alpha}{h} + \frac{\beta h}{3}\right), \text{ para } i = 2, 3, \ldots, m-1; \tag{8}$$

$$K_{mm} = \left(\frac{\alpha}{h} + \frac{\beta h}{3}\right).$$

Vetor força F:

$$F_1 = \int_{x_1}^{x_2} f(x)\left(\frac{x_2 - x}{h}\right)dx;$$

$$F_i = \int_{x_{j-1}}^{x_j} f(x)\left(\frac{x - x_{j-1}}{h}\right)dx + \int_{x_j}^{x_{j+1}} f(x)\left(\frac{x_{j+1} - x}{h}\right)dx, \text{ para } i = 2, 3, \ldots, m-1;$$

$$F_m = \int_{x_{m-1}}^{x_m} f(x)\left(\frac{x - x_{m-1}}{h}\right)dx; \tag{9}$$

Veremos a seguir como montar K **e** F **usando MEF.**

Referências I

Liu, I.S.; Rincon, M.A.. **Introdução ao Método de Elementos Finitos, Análise e Aplicação**. IM/UFRJ, 2003.