

Tópicos Especiais em Matemática Aplicada - 2025-1

UERJ

10 - Caso 2D - Elementos Isoparamétricos

Rodrigo Madureira

rodrigo.madureira@ime.uerj.br

Github: <https://github.com/rodrigolrmadureira/ElementosFinitos>

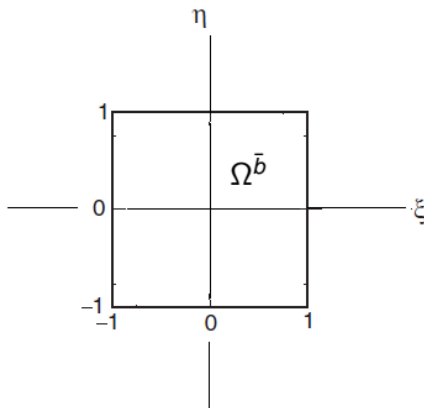
Sumário

- 1 Elementos isoparamétricos
- 2 Cálculo dos elementos da matriz local K^e
- 3 Cálculo dos elementos do vetor local F^e
- 4 Bibliografia

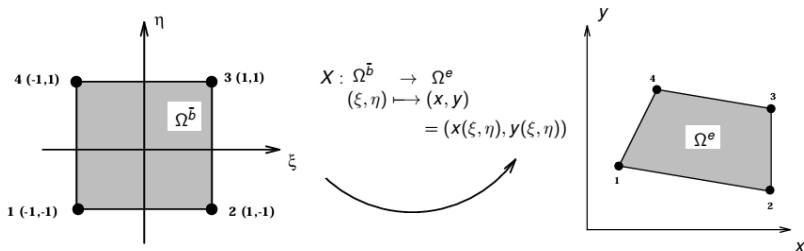
Elementos isoparamétricos

Agora, vamos ver como são calculados K^e e F^e para cada elemento Ω^e .

Seja $\Omega^{\bar{b}} = [-1, 1] \times [-1, 1]$ o elemento finito biunitário como representado na figura abaixo.



Elementos isoparamétricos



Seja

$$X: \Omega^{\bar{b}} \longrightarrow \Omega^e$$

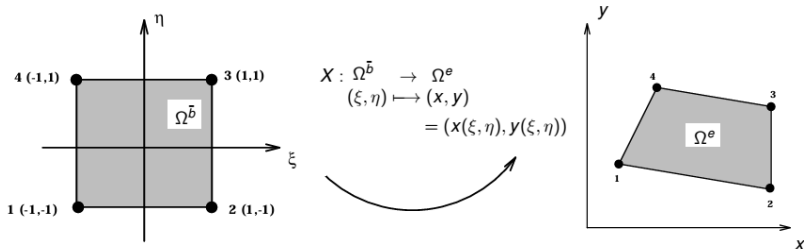
$$(\xi, \eta) \longmapsto (x, y) = (x(\xi, \eta), y(\xi, \eta)),$$

onde

$$x(\xi, \eta) = \sum_{a=1}^4 \varphi_a^{\bar{b}}(\xi, \eta) \cdot x_a^e; \quad y(\xi, \eta) = \sum_{a=1}^4 \varphi_a^{\bar{b}}(\xi, \eta) \cdot y_a^e;$$

$$\varphi_a^{\bar{b}}(\xi_c, \eta_c) = \begin{cases} 1, & \text{se } a = c, \\ 0, & \text{se } a \neq c. \end{cases}$$

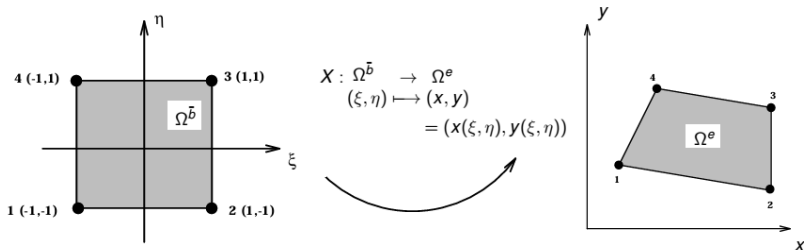
Elementos isoparamétricos



Por exemplo,

$$\begin{aligned}
 x(-1, -1) &= x(\xi_1, \eta_1) = \sum_{a=1}^4 \varphi_a^{\bar{b}}(\xi_1, \eta_1) \cdot x_a^e \\
 &= \underbrace{\varphi_1^{\bar{b}}(\xi_1, \eta_1)}_1 \cdot x_1^e + \cancel{\varphi_2^{\bar{b}}(\xi_1, \eta_1)}^0 \cdot x_2^e + \cancel{\varphi_3^{\bar{b}}(\xi_1, \eta_1)}^0 \cdot x_3^e + \cancel{\varphi_4^{\bar{b}}(\xi_1, \eta_1)}^0 \cdot x_4^e \\
 &= x_1^e
 \end{aligned}$$

Elementos isoparamétricos



Analogamente,

$$x(1, -1) = x(\xi_2, \eta_2) = x_2^e;$$

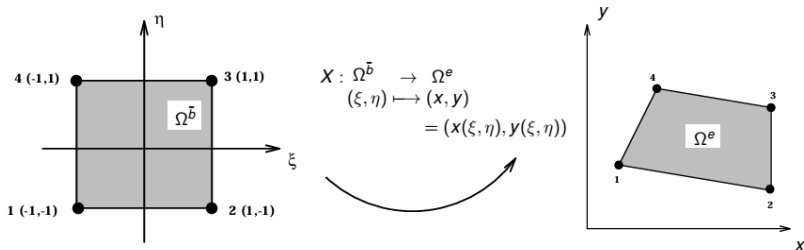
$$x(1, 1) = x(\xi_3, \eta_3) = x_3^e;$$

$$x(-1, 1) = x(\xi_4, \eta_4) = x_4^e$$

Ou seja, para todo $a = 1, 2, 3, 4$,

$$x(\xi_a, \eta_a) = x_a^e; \quad y(\xi_a, \eta_a) = y_a^e$$

Elementos isoparamétricos

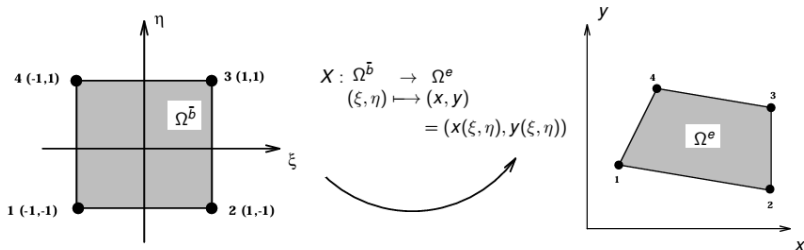


Vamos assumir que $x(\xi, \eta)$ e $y(\xi, \eta)$ são lineares em ξ e η , ou seja:

$$x(\xi, \eta) = a_1 + a_2\xi + a_3\eta + a_4\xi\eta$$

$$y(\xi, \eta) = b_1 + b_2\xi + b_3\eta + b_4\xi\eta$$

Elementos isoparamétricos



Então,

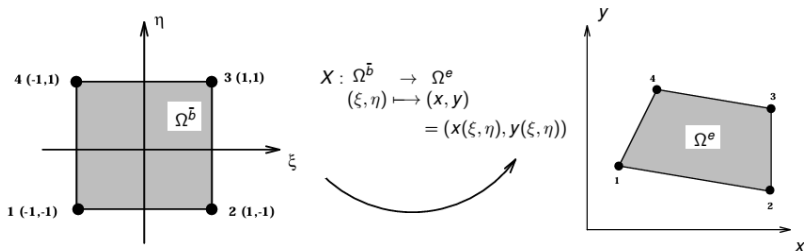
$$x(-1, -1) = x_1^e \Rightarrow a_1 - a_2 - a_3 + a_4 = x_1^e$$

$$x(1, -1) = x_2^e \Rightarrow a_1 + a_2 - a_3 - a_4 = x_2^e$$

$$x(1, 1) = x_3^e \Rightarrow a_1 + a_2 + a_3 + a_4 = x_3^e$$

$$x(-1, 1) = x_4^e \Rightarrow a_1 - a_2 + a_3 - a_4 = x_4^e$$

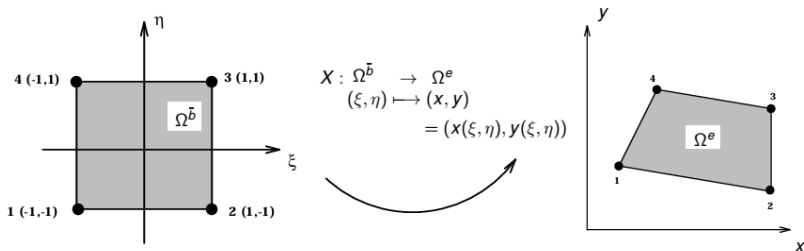
Elementos isoparamétricos



Resolvendo o sistema:

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{bmatrix}$$

Elementos isoparamétricos



encontramos:

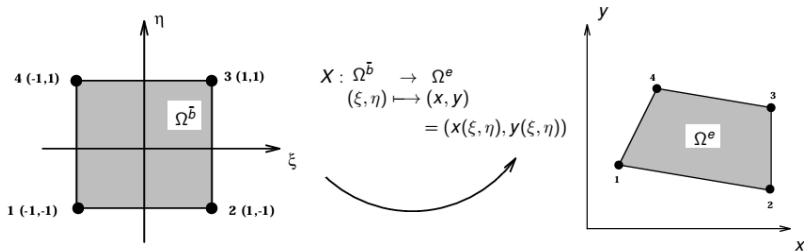
$$a_1 = (x_1^e + x_2^e + x_3^e + x_4^e)/4;$$

$$a_2 = (-x_1^e + x_2^e + x_3^e - x_4^e)/4;$$

$$a_3 = (-x_1^e - x_2^e + x_3^e + x_4^e)/4;$$

$$a_4 = (x_1^e - x_2^e + x_3^e - x_4^e)/4$$

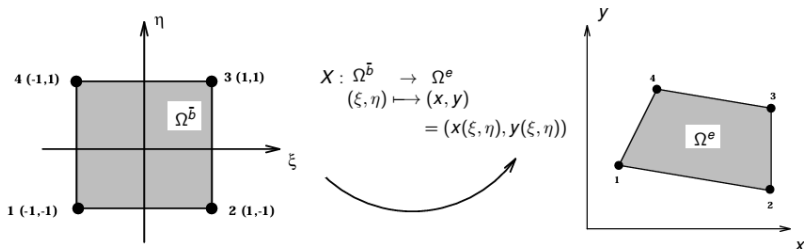
Elementos isoparamétricos



Substituindo a_1 , a_2 , a_3 , a_4 em $x(\xi, \eta)$ e arrumando os termos, obtemos:

$$\begin{aligned}
 x(\xi, \eta) = & \underbrace{\frac{1}{4}(1 - \xi)(1 - \eta) x_1^e}_{\varphi_1^b(\xi, \eta)} + \underbrace{\frac{1}{4}(1 + \xi)(1 - \eta) x_2^e}_{\varphi_2^b(\xi, \eta)} \\
 & + \underbrace{\frac{1}{4}(1 + \xi)(1 + \eta) x_3^e}_{\varphi_3^b(\xi, \eta)} + \underbrace{\frac{1}{4}(1 - \xi)(1 + \eta) x_4^e}_{\varphi_4^b(\xi, \eta)}
 \end{aligned}$$

Elementos isoparamétricos



Assim, temos as quatro funções de base para Q_4 :

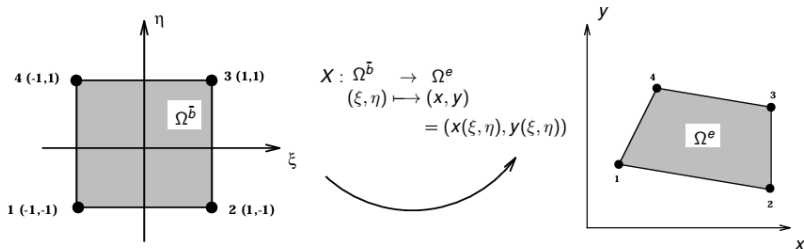
$$\varphi_1^{\bar{b}}(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 - \eta);$$

$$\varphi_2^{\bar{b}}(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 - \eta);$$

$$\varphi_3^{\bar{b}}(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 + \eta);$$

$$\varphi_4^{\bar{b}}(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 + \eta)$$

Elementos isoparamétricos

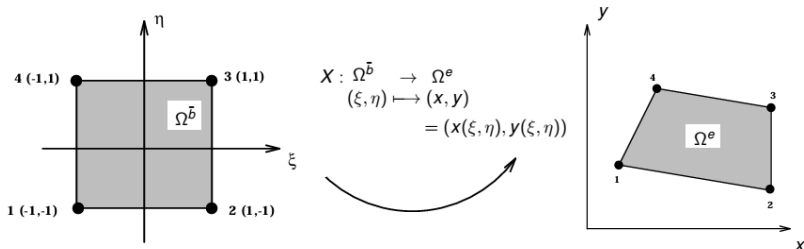


Logo, podemos reescrever:

$$x(\xi, \eta) = \sum_{a=1}^4 \varphi_a^{\bar{b}}(\xi, \eta) \cdot x_a^e$$

$$y(\xi, \eta) = \sum_{a=1}^4 \varphi_a^{\bar{b}}(\xi, \eta) \cdot y_a^e$$

Elementos isoparamétricos



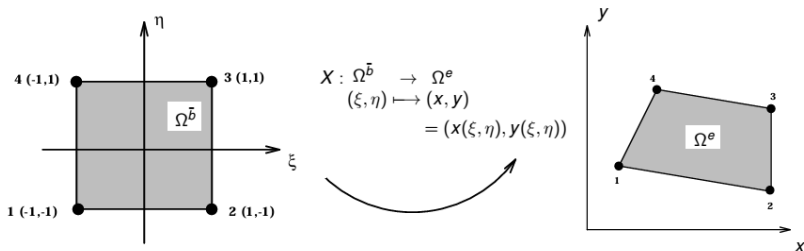
Assim, escrevemos a solução aproximada u_h^e em cada elemento Ω^e como:

$$u_h^e(x, y) = \sum_{a=1}^4 c_a^e \cdot \varphi_a^e(x, y)$$

Com a mudança de variáveis $(x, y) \mapsto (\xi, \eta)$, obtemos:

$$u_h^e(x, y) = u_h^{\bar{b}}(\xi, \eta) = \sum_{a=1}^4 c_a^{\bar{b}} \cdot \varphi_a^{\bar{b}}(\xi, \eta)$$

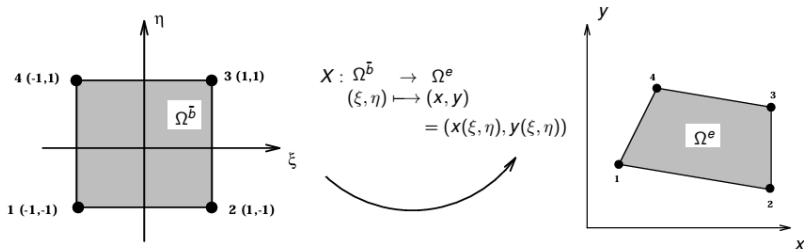
Elemento da matriz local K^e



Também temos para cada elemento Ω^e , ou seja, para todo $a, b = 1, 2, 3, 4$:

$$K_{ab}^e = a(\varphi_a^e, \varphi_b^e) = (\nabla \varphi_a^e, k \cdot \nabla \varphi_b^e) = \int_{\Omega^e} \nabla \varphi_a^e \cdot k \cdot \nabla \varphi_b^e \, dx dy$$

Elemento do vetor local F^e



Também temos para cada elemento Ω^e , ou seja, para todo $a = 1, 2, 3, 4$:

$$\begin{aligned}
 F_a^e &= (f, \varphi_a^e) - (\bar{q}, \varphi_a^e)_{\Gamma_q} - \sum_{b=1}^4 a(\varphi_a^e, \varphi_b^e) p_b^e \\
 &= \int_{\Omega^e} f \varphi_a^e \, dx dy - \int_{\Gamma_q} \bar{q} \varphi_a^e \, d\Gamma_q - \sum_{b=1}^4 a(\varphi_a^e, \varphi_b^e) p_b^e
 \end{aligned}$$

Cálculo de K^e

Pela definição do vetor gradiente no \mathbb{R}^2 , temos que:

$$\nabla \varphi_a^e(x, y) = \begin{bmatrix} \frac{\partial \varphi_a^e}{\partial x} \\ \frac{\partial \varphi_a^e}{\partial y} \end{bmatrix}$$

Usando notação matricial do produto interno (ou escalar) de dois vetores x, y no \mathbb{R}^2 , temos que:

$$x \cdot y = x^T y$$

Logo, podemos reescrever K_{ab}^e para todo $a, b = 1, 2, 3, 4$ como:

$$\begin{aligned} K_{ab}^e &= a(\varphi_a^e, \varphi_b^e) = (\nabla \varphi_a^e, k \cdot \nabla \varphi_b^e) = \int_{\Omega^e} \nabla \varphi_a^e \cdot k \cdot \nabla \varphi_b^e \, dx dy \\ &= \int_{\Omega^e} (\nabla \varphi_a^e)^T k \nabla \varphi_b^e \, dx dy = \int_{\Omega^e} \begin{bmatrix} \frac{\partial \varphi_a^e}{\partial x} & \frac{\partial \varphi_a^e}{\partial y} \end{bmatrix} k \begin{bmatrix} \frac{\partial \varphi_b^e}{\partial x} \\ \frac{\partial \varphi_b^e}{\partial y} \end{bmatrix} \, dx dy \end{aligned} \quad (1)$$

Cálculo de K^e

De forma análoga ao que vimos no caso 1D, devemos usar mudança de variáveis através da transformação isoparamétrica

$$T_{\xi\eta}: \Omega^e \longrightarrow \Omega^{\bar{b}}$$

$$(x, y) \longmapsto (\xi, \eta) = (\xi(x, y), \eta(x, y)),$$

para aproximar a integral dupla com quadratura de Gauss.

Então,

$$\varphi_a^e(x, y) = \varphi_a^{\bar{b}}(\xi, \eta) = \varphi_a^{\bar{b}}(\xi(x, y), \eta(x, y))$$

e usando a Regra da Cadeia, obtemos as derivadas parciais:

$$\frac{\partial \varphi_a^e}{\partial x} = \frac{\partial \varphi_a^{\bar{b}}}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial \varphi_a^{\bar{b}}}{\partial \eta} \frac{\partial \eta}{\partial x}$$

$$\frac{\partial \varphi_a^e}{\partial y} = \frac{\partial \varphi_a^{\bar{b}}}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial \varphi_a^{\bar{b}}}{\partial \eta} \frac{\partial \eta}{\partial y}$$

Cálculo de K^e

Passando as duas equações para a forma matricial, obtemos:

$$\underbrace{\begin{bmatrix} \frac{\partial \varphi_a^e}{\partial x} \\ \frac{\partial \varphi_a^e}{\partial y} \end{bmatrix}}_{=\nabla \varphi_a^e(x,y)} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} \underbrace{\begin{bmatrix} \frac{\partial \varphi_a^{\bar{b}}}{\partial \xi} \\ \frac{\partial \varphi_a^{\bar{b}}}{\partial \eta} \end{bmatrix}}_{=\nabla \varphi_a^{\bar{b}}(\xi,\eta)}$$

Usando a notação $\xi_x = \frac{\partial \xi}{\partial x}$ para as derivadas parciais da matriz, temos:

$$\nabla \varphi_a^e(x, y) = \begin{bmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{bmatrix} \nabla \varphi_a^{\bar{b}}(\xi, \eta) \quad (2)$$

Cálculo de K^e

Do estudo de mudança de variáveis em integrais duplas, sabemos que:

$$\int_{\Omega^e} g(x, y) \, dx dy = \int_{\Omega^{\bar{b}}} g(\xi, \eta) |J(\xi, \eta)| \, d\xi d\eta, \quad (3)$$

onde

$$J(\xi, \eta) = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix}$$

é a **matriz jacobiana** da transformação isoparamétrica

$$\begin{aligned} T_{xy}: \Omega^{\bar{b}} &\longrightarrow \Omega^e \\ (\xi, \eta) &\longmapsto (x, y) = (x(\xi, \eta), y(\xi, \eta)) \end{aligned}$$

e

$$|J(\xi, \eta)| = \det(J(\xi, \eta)) = x_\xi y_\eta - x_\eta y_\xi$$

é o **jacobiano** dessa transformação.

Cálculo de K^e

Usando os resultados de (2) e (3) na Eq. (1), obtemos:

$$\begin{aligned}
 K_{ab}^e &= \int_{\Omega^e} (\nabla \varphi_a^e)^T k \nabla \varphi_b^e \, dx dy \\
 &= \int_{\Omega^{\bar{b}}} (\nabla \varphi_a^{\bar{b}}(\xi, \eta))^T \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} k \begin{bmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{bmatrix} \nabla \varphi_b^{\bar{b}}(\xi, \eta) |J(\xi, \eta)| \, d\xi d\eta
 \end{aligned}
 \tag{4}$$

Cálculo de K^e

Note que se:

$$T_{xy}: \Omega^{\bar{b}} \longrightarrow \Omega^e$$

$$(\xi, \eta) \longmapsto (x, y) = (x(\xi, \eta), y(\xi, \eta))$$

é a transformação onde a matriz jacobiana é

$$J(\xi, \eta) = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix},$$

a transformação inversa é dada por

$$T_{xy}^{-1} = T_{\xi\eta}: \Omega^e \longrightarrow \Omega^{\bar{b}}$$

$$(x, y) \longmapsto (\xi, \eta) = (\xi(x, y), \eta(x, y)),$$

onde a matriz jacobiana é

$$J(x, y) = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = J^{-1}(\xi, \eta) \quad (5)$$

Cálculo de K^e

Portanto, usando a definição de $J^{-1}(\xi, \eta)$ de (5) na Eq. (4), obtemos:

Elemento K_{ab}^e da matriz local K^e (Materiais isotrópicos: $Q = kl$)

$$\begin{aligned} K_{ab}^e &= \int_{\Omega^e} (\nabla \varphi_a^e)^T k \nabla \varphi_b^e dx dy \\ &= \int_{\Omega^{\bar{e}}} (\nabla \varphi_a^{\bar{e}}(\xi, \eta))^T \cdot J^{-1} \cdot k \cdot (J^{-1})^T \cdot \nabla \varphi_b^{\bar{e}}(\xi, \eta) \cdot |J| d\xi d\eta, \end{aligned} \quad (6)$$

para $a, b = 1, 2, 3, 4$, onde

$$J = J(\xi, \eta) = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix}.$$

Cálculo de K^e

Expandindo em colunas o vetor gradiente

$$\nabla \varphi_a^{\bar{b}}(\xi, \eta) = \begin{bmatrix} \frac{\partial \varphi_a^{\bar{b}}}{\partial \xi} \\ \frac{\partial \varphi_a^{\bar{b}}}{\partial \eta} \end{bmatrix}$$

para todo $a = 1, 2, 3, 4$, obtemos a matriz

$$N = \begin{bmatrix} \nabla \varphi_1^{\bar{b}}(\xi, \eta) & \nabla \varphi_2^{\bar{b}}(\xi, \eta) & \nabla \varphi_3^{\bar{b}}(\xi, \eta) & \nabla \varphi_4^{\bar{b}}(\xi, \eta) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial \varphi_1^{\bar{b}}}{\partial \xi} & \frac{\partial \varphi_2^{\bar{b}}}{\partial \xi} & \frac{\partial \varphi_3^{\bar{b}}}{\partial \xi} & \frac{\partial \varphi_4^{\bar{b}}}{\partial \xi} \\ \frac{\partial \varphi_1^{\bar{b}}}{\partial \eta} & \frac{\partial \varphi_2^{\bar{b}}}{\partial \eta} & \frac{\partial \varphi_3^{\bar{b}}}{\partial \eta} & \frac{\partial \varphi_4^{\bar{b}}}{\partial \eta} \end{bmatrix}$$

Cálculo de K^e

Portanto,

Matriz local K^e (Materiais isotrópicos: $Q = kI$)

$$K^e = \int_{\Omega^{\bar{b}}} N^T \cdot J^{-1} \cdot k \cdot (J^{-1})^T \cdot N \cdot |J| d\xi d\eta, \quad (7)$$

onde

$$J = J(\xi, \eta) = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix}.$$

Cálculo de K^e

Sabendo que

$$x(\xi, \eta) = \sum_{a=1}^4 \varphi_a^{\bar{b}}(\xi, \eta) \cdot x_a^e = \begin{bmatrix} \varphi_1^{\bar{b}}(\xi, \eta) & \varphi_2^{\bar{b}}(\xi, \eta) & \varphi_3^{\bar{b}}(\xi, \eta) & \varphi_4^{\bar{b}}(\xi, \eta) \end{bmatrix} \cdot \begin{bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{bmatrix}$$

$$y(\xi, \eta) = \sum_{a=1}^4 \varphi_a^{\bar{b}}(\xi, \eta) \cdot y_a^e = \begin{bmatrix} \varphi_1^{\bar{b}}(\xi, \eta) & \varphi_2^{\bar{b}}(\xi, \eta) & \varphi_3^{\bar{b}}(\xi, \eta) & \varphi_4^{\bar{b}}(\xi, \eta) \end{bmatrix} \cdot \begin{bmatrix} y_1^e \\ y_2^e \\ y_3^e \\ y_4^e \end{bmatrix}$$

e usando as funções de base do Q_4 ,

$$\varphi_1^{\bar{b}}(\xi, \eta) = (1 - \xi)(1 - \eta)/4;$$

$$\varphi_2^{\bar{b}}(\xi, \eta) = (1 + \xi)(1 - \eta)/4;$$

$$\varphi_3^{\bar{b}}(\xi, \eta) = (1 + \xi)(1 + \eta)/4;$$

$$\varphi_4^{\bar{b}}(\xi, \eta) = (1 - \xi)(1 + \eta)/4,$$

Cálculo de K^e

obtemos

$$x(\xi, \eta) = \frac{1}{4} \begin{bmatrix} (1-\xi)(1-\eta) & (1+\xi)(1-\eta) & (1+\xi)(1+\eta) & (1-\xi)(1+\eta) \end{bmatrix} \begin{bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{bmatrix}$$

$$y(\xi, \eta) = \frac{1}{4} \begin{bmatrix} (1-\xi)(1-\eta) & (1+\xi)(1-\eta) & (1+\xi)(1+\eta) & (1-\xi)(1+\eta) \end{bmatrix} \begin{bmatrix} y_1^e \\ y_2^e \\ y_3^e \\ y_4^e \end{bmatrix}$$

e os elementos da matriz jacobiana J são:

$$x_\xi = (1/4) \cdot \begin{bmatrix} -(1-\eta) & 1-\eta & 1+\eta & -(1+\eta) \end{bmatrix} \begin{bmatrix} x_1^e & x_2^e & x_3^e & x_4^e \end{bmatrix}^T$$

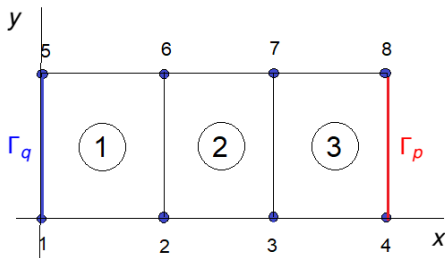
$$y_\xi = (1/4) \cdot \begin{bmatrix} -(1-\eta) & 1-\eta & 1+\eta & -(1+\eta) \end{bmatrix} \begin{bmatrix} y_1^e & y_2^e & y_3^e & y_4^e \end{bmatrix}^T$$

$$x_\eta = (1/4) \cdot \begin{bmatrix} -(1-\xi) & -(1+\xi) & 1+\xi & 1-\xi \end{bmatrix} \begin{bmatrix} x_1^e & x_2^e & x_3^e & x_4^e \end{bmatrix}^T$$

$$y_\eta = (1/4) \cdot \begin{bmatrix} -(1-\xi) & -(1+\xi) & 1+\xi & 1-\xi \end{bmatrix} \begin{bmatrix} y_1^e & y_2^e & y_3^e & y_4^e \end{bmatrix}^T$$

Cálculo de K^e

- **Exemplo:** Considere o domínio Ω



onde $0 \leq x \leq 1$, $0 \leq y \leq 1/3$.

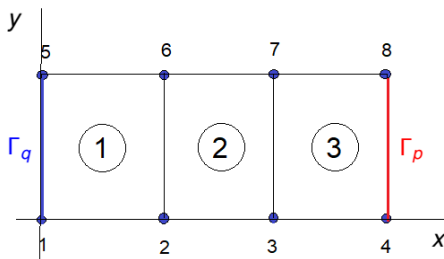
Note que:

Número de subintervalos de x : $nelx = 3$

Número de subintervalos de y : $nely = 1$

Total de elementos: $nelx \cdot nely = 3$.

Cálculo de K^e



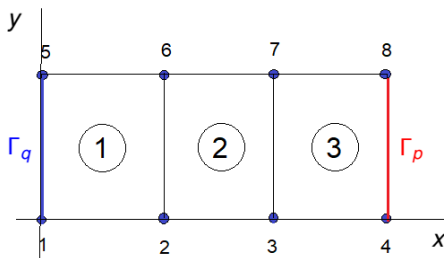
Número de nós de x : $Nnosx = nelx + 1 = 4$.

Tamanho dos subintervalos de x : $h_x = (x_f - x_1)/nelx$, onde x_f é o nó final e x_1 é o nó inicial em x . Neste exemplo, $h_x = (1 - 0)/3 = 1/3$.

Número de nós de y : $Nnosy = nely + 1 = 2$.

Tamanho dos subintervalos de y : $h_y = (y_f - y_1)/nely$, onde y_f é o nó final e y_1 é o nó inicial em y . Neste exemplo,
 $h_y = (y_f - x_0)/nely = ((1/3) - 0)/1 = 1/3$.

Cálculo de K^e



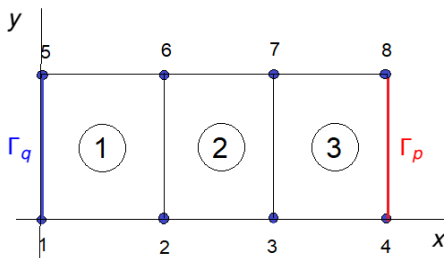
Cada nó de x é dado por: $x_j = x_1 + (j - 1) \cdot h_x$, para todo $j = 1, 2, \dots, Nnosx$.

Neste exemplo, os nós de x são: $x_1 = 0$, $x_2 = 1/3$, $x_3 = 2/3$, $x_4 = 1$.

Cada nó de y é dado por: $y_k = y_1 + (k - 1) \cdot h_y$, para todo $k = 1, 2, \dots, Nnosy$.

Neste exemplo, os nós de y são: $y_1 = 0$, $y_2 = 1/3$.

Cálculo de K^e



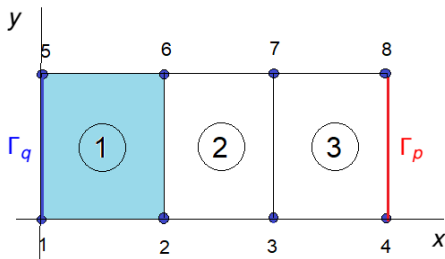
Neste exemplo, o número de nós total da malha é:

$$Nnos = Nosx \cdot Nnosy = (nelx + 1) \cdot (nely + 1) = 4 \cdot 2 = 8.$$

Logo, cada ponto do domínio Ω é (x_j, y_k) , para $j = 1, 2, \dots, Nnosx$, $k = 1, 2, \dots, Nnosy$. Neste exemplo, temos os seguintes pontos:

- 1 : $(x_1, y_1) = (0, 0)$; • 2 : $(x_2, y_1) = (1/3, 0)$; • 3 : $(x_3, y_1) = (2/3, 0)$;
- 4 : $(x_4, y_1) = (1, 0)$; • 5 : $(x_1, y_2) = (0, 1/3)$; • 6 : $(x_2, y_2) = (1/3, 1/3)$;
- 7 : $(x_3, y_2) = (2/3, 1/3)$; • 8 : $(x_4, y_2) = (1, 1/3)$.

Cálculo de K^e

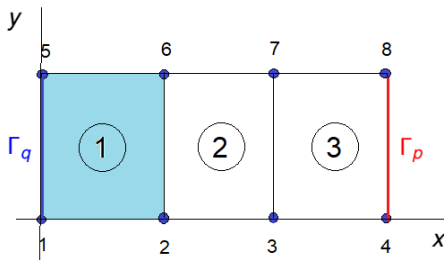


- **Exemplo:** Cálculo dos elementos da matriz local

$$K^1 = \begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 & K_{14}^1 \\ K_{21}^1 & K_{22}^1 & K_{23}^1 & K_{24}^1 \\ K_{31}^1 & K_{32}^1 & K_{33}^1 & K_{34}^1 \\ K_{41}^1 & K_{42}^1 & K_{43}^1 & K_{44}^1 \end{bmatrix}$$

para o elemento $e = 1$ do domínio Ω .

Cálculo de K^e

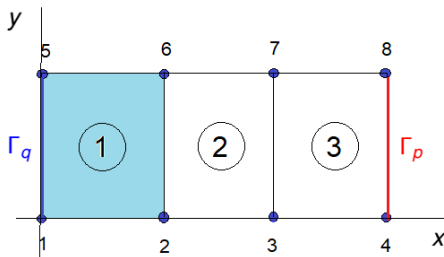


Para $e = 1$, temos:

$$\begin{aligned} K_{ab}^1 &= \int_{\Omega^1} (\nabla \varphi_a^1(x, y))^T k \nabla \varphi_b^1(x, y) \, dx dy \\ &= \int_0^{1/3} \int_0^{1/3} (\nabla \varphi_a^1(x, y))^T k \nabla \varphi_b^1(x, y) \, dx dy, \end{aligned}$$

para $a, b = 1, 2, 3, 4$.

Cálculo de K^e

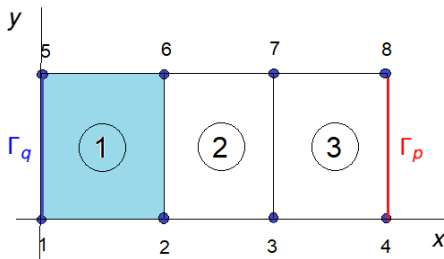


Com a mudança de variáveis na transformação isoparamétrica, temos:

$$\begin{aligned}
 K_{ab}^1 &= \int_{\Omega^1} (\nabla \varphi_a^1(\xi, \eta))^T \cdot J^{-1} \cdot k \cdot (J^{-1})^T \cdot \nabla \varphi_b^1(\xi, \eta) \cdot |J| \, d\xi d\eta \\
 &= \int_{-1}^1 \int_{-1}^1 (\nabla \varphi_a^1(\xi, \eta))^T \cdot J^{-1} \cdot k \cdot (J^{-1})^T \cdot \nabla \varphi_b^1(\xi, \eta) \cdot |J| \, d\xi d\eta,
 \end{aligned}$$

para $a, b = 1, 2, 3, 4$.

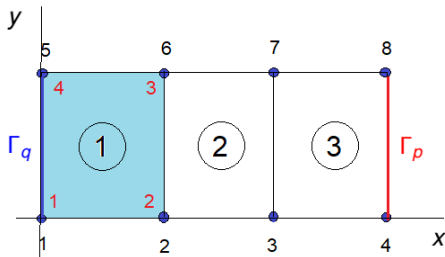
Cálculo de K^e



Para o elemento K_{11}^1 da matriz local K^1 , fazemos $a = b = 1$ na integral. Assim, obtemos:

$$\begin{aligned} K_{11}^1 &= \int_{\Omega^1} (\nabla \varphi_1^1(\xi, \eta))^T \cdot J^{-1} \cdot k \cdot (J^{-1})^T \cdot \nabla \varphi_1^1(\xi, \eta) \cdot |J| \, d\xi d\eta \\ &= \int_{-1}^1 \int_{-1}^1 (\nabla \varphi_1^1(\xi, \eta))^T \cdot J^{-1} \cdot k \cdot (J^{-1})^T \cdot \nabla \varphi_1^1(\xi, \eta) \cdot |J| \, d\xi d\eta \end{aligned}$$

Cálculo de K^e



• Cálculo da matriz jacobiana J :

Neste exemplo, no elemento Ω^1 , temos:

$$(x^1)^T = [x_1^1 \quad x_2^1 \quad x_3^1 \quad x_4^1] = [0 \quad 1/3 \quad 1/3 \quad 0];$$

$$(y^1)^T = [y_1^1 \quad y_2^1 \quad y_3^1 \quad y_4^1] = [0 \quad 0 \quad 1/3 \quad 1/3].$$

Cálculo de K^e

Logo, os elementos da matriz jacobiana J são:

$$x_\xi = (1/4) \cdot \begin{bmatrix} -(1-\eta) & 1-\eta & 1+\eta & -(1+\eta) \end{bmatrix} \begin{bmatrix} 0 \\ 1/3 \\ 1/3 \\ 0 \end{bmatrix}$$

$$= \frac{1}{4} \left[\frac{1-\eta}{3} + \frac{1+\eta}{3} \right] = \frac{1}{6},$$

e verifique que:

$$y_\xi = 0; \quad x_\eta = 0; \quad y_\eta = \frac{1}{6}.$$

Portanto, a matriz jacobiana J e seu jacobiano ($|J| = \det(J)$) são dados por:

$$J = J(\xi, \eta) = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} = \begin{bmatrix} 1/6 & 0 \\ 0 & 1/6 \end{bmatrix};$$

$$|J| = 1/36.$$

Cálculo de K^e

A inversa da matriz J é dada por:

$$J^{-1} = \frac{1}{|J|} \begin{bmatrix} 1/6 & 0 \\ 0 & 1/6 \end{bmatrix} = \frac{1}{1/36} \begin{bmatrix} 1/6 & 0 \\ 0 & 1/6 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

Portanto, no integrando de K_{11}^1 , já temos que:

$$\begin{aligned} J^{-1} \cdot k \cdot (J^{-1})^T &= \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \cdot k \cdot \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \\ &= 6 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot k \cdot 6 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= 36 \cdot k \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Cálculo de K^e

Substituindo em K_{11}^1 , obtemos:

$$\begin{aligned} K_{11}^1 &= \int_{-1}^1 \int_{-1}^1 (\nabla \varphi_1^{\bar{1}}(\xi, \eta))^T \cdot 36 \cdot k \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \nabla \varphi_1^{\bar{1}}(\xi, \eta) \cdot \frac{1}{36} d\xi d\eta \\ &= \int_{-1}^1 \int_{-1}^1 (\nabla \varphi_1^{\bar{1}}(\xi, \eta))^T \cdot k \cdot \nabla \varphi_1^{\bar{1}}(\xi, \eta) d\xi d\eta \\ &= k \int_{-1}^1 \int_{-1}^1 (\nabla \varphi_1^{\bar{1}}(\xi, \eta))^T \cdot \nabla \varphi_1^{\bar{1}}(\xi, \eta) d\xi d\eta \end{aligned}$$

• **Cálculo do gradiente:** No integrando, sabendo que

$\varphi_1^{\bar{1}}(\xi, \eta) = (1/4) \cdot (1 - \xi)(1 - \eta)$, temos:

$$\nabla \varphi_1^{\bar{1}}(\xi, \eta) = \begin{bmatrix} \frac{\partial \varphi_1^{\bar{1}}}{\partial \xi} \\ \frac{\partial \varphi_1^{\bar{1}}}{\partial \eta} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -(1 - \eta) \\ -(1 - \xi) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \eta - 1 \\ \xi - 1 \end{bmatrix}$$

Cálculo de K^e

Portanto,

$$\begin{aligned}
 K_{11}^1 &= k \int_{-1}^1 \int_{-1}^1 \frac{1}{4} [\eta - 1 \quad \xi - 1] \cdot \frac{1}{4} \begin{bmatrix} \eta - 1 \\ \xi - 1 \end{bmatrix} d\xi d\eta \\
 &= k \int_{-1}^1 \int_{-1}^1 \frac{1}{16} [(\eta - 1)^2 + (\xi - 1)^2] d\xi d\eta \\
 &= \frac{k}{16} \int_{-1}^1 \int_{-1}^1 [(\eta - 1)^2 + (\xi - 1)^2] d\xi d\eta
 \end{aligned}$$

Devemos aproximar esta integral dupla usando quadratura gaussiana. Aqui, vamos usar apenas $npg = 2$.

Neste caso, os pontos de Gauss serão: $-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}$,

enquanto os pesos de Gauss serão iguais a 1.

Cálculo de K^e

Na integral dupla, denotamos a função que está no integrando por:

$$g(\xi, \eta) = (\eta - 1)^2 + (\xi - 1)^2.$$

Assim, calculamos em

$$\int_{-1}^1 \int_{-1}^1 g(\xi, \eta) d\xi d\eta = \int_{-1}^1 \left[\int_{-1}^1 g(\xi, \eta) d\xi \right] d\eta$$

a integral que está entre colchetes primeiro. Note que nesta integral, somente ξ está variando no intervalo $[-1, 1]$.

Outra alternativa é calcular em

$$\int_{-1}^1 \int_{-1}^1 g(\xi, \eta) d\xi d\eta = \int_{-1}^1 \left[\int_{-1}^1 g(\xi, \eta) d\eta \right] d\xi$$

, onde a integral que está entre colchetes possui somente η variando em $[-1, 1]$.

Cálculo de K^e

Calculando a integral entre colchetes pela primeira alternativa usando quadratura gaussiana, obtemos:

$$\begin{aligned}\int_{-1}^1 g(\xi, \eta) d\xi &= \sum_{j=1}^{npg} g(\xi_j, \eta) \cdot w_j = \sum_{j=1}^2 g(\xi_j, \eta) \cdot w_j \\ &= g(\xi_1, \eta) \cdot w_1 + g(\xi_2, \eta) \cdot w_2\end{aligned}$$

Como $\xi_1 = -\frac{\sqrt{3}}{3}$, $\xi_2 = \frac{\sqrt{3}}{3}$, $w_1 = w_2 = 1$, obtemos:

$$\begin{aligned}\int_{-1}^1 g(\xi, \eta) d\xi &= g\left(-\frac{\sqrt{3}}{3}, \eta\right) \cdot 1 + g\left(\frac{\sqrt{3}}{3}, \eta\right) \cdot 1 \\ &= \left[(\eta - 1)^2 + \left(-\frac{\sqrt{3}}{3} - 1\right)^2\right] \cdot 1 + \left[(\eta - 1)^2 + \left(\frac{\sqrt{3}}{3} - 1\right)^2\right] \cdot 1 \\ &= \frac{8}{3} + 2(\eta - 1)^2\end{aligned}$$

Cálculo de K^e

Agora, calculamos por quadratura gaussiana

$$\int_{-1}^1 \left[\frac{8}{3} + 2(\eta - 1)^2 \right] d\eta,$$

onde denotamos $\bar{g}(\eta) = \frac{8}{3} + 2(\eta - 1)^2$. Assim, obtemos:

$$\int_{-1}^1 \bar{g}(\eta) d\eta = \sum_{k=1}^{npg} \bar{g}(\eta_k) \cdot w_k = \sum_{k=1}^2 \bar{g}(\eta_k) \cdot w_k = \bar{g}(\eta_1) \cdot w_1 + \bar{g}(\eta_2) \cdot w_2$$

Como $\eta_1 = -\frac{\sqrt{3}}{3}$, $\eta_2 = \frac{\sqrt{3}}{3}$, $w_1 = w_2 = 1$, obtemos:

$$\begin{aligned} \int_{-1}^1 \bar{g}(\eta) d\eta &= \bar{g}\left(-\frac{\sqrt{3}}{3}\right) \cdot 1 + \bar{g}\left(\frac{\sqrt{3}}{3}\right) \cdot 1 \\ &= \left[\frac{8}{3} + 2\left(-\frac{\sqrt{3}}{3} - 1\right)^2 \right] + \left[\frac{8}{3} + 2\left(\frac{\sqrt{3}}{3} - 1\right)^2 \right] = \frac{32}{3} \end{aligned}$$

Cálculo de K^e

Logo, obtemos:

$$\int_{-1}^1 \int_{-1}^1 [(\eta - 1)^2 + (\xi - 1)^2] d\xi d\eta = \frac{32}{3},$$

e portanto,

$$K_{11}^1 = \frac{k}{16} \int_{-1}^1 \int_{-1}^1 [(\eta - 1)^2 + (\xi - 1)^2] d\xi d\eta = \frac{k}{16} \cdot \frac{32}{3} = \frac{2}{3}k$$

Usando $k = 1$, obtemos:

$$K_{11}^1 = \frac{2}{3}.$$

Exercício: verifique para os demais elementos de K^1 com $k = 1$ que:

$$K_{12}^1 = K_{14}^1 = K_{21}^1 = K_{23}^1 = K_{32}^1 = K_{34}^1 = -\frac{1}{6}; \quad K_{13}^1 = K_{24}^1 = K_{31}^1 = K_{42}^1 = -\frac{1}{3};$$

$$K_{22}^1 = K_{33}^1 = K_{44}^1 = \frac{2}{3}.$$

Cálculo de F^e

Também temos para cada elemento Ω^e , ou seja, para todo $a = 1, 2, 3, 4$:

$$F_a^e = (f, \varphi_a^e) - (\bar{q}, \varphi_a^e)_{\Gamma_q} - \sum_{b=1}^4 a(\varphi_a^e, \varphi_b^e) p_b^e = \underbrace{(f, \varphi_a^e)}_{f_a^e} - \underbrace{(\bar{q}, \varphi_a^e)_{\Gamma_q}}_{q_a^e} - \underbrace{\sum_{b=1}^4 K_{ab}^e p_b^e}_{\bar{p}_a^e}$$

• Cálculo de f_a^e :

$$f_a^e = (f, \varphi_a^e) = \int_{\Omega^e} f(x, y) \cdot \varphi_a^e(x, y) \, dx dy$$

Usando mudança de variáveis $(x, y) \mapsto (\xi, \eta)$, obtemos:

$$f_a^e = (f, \varphi_a^e) = \int_{\Omega^{\bar{b}}} f(\xi, \eta) \cdot \varphi_a^{\bar{b}}(\xi, \eta) |J| \, d\xi d\eta$$

Logo,

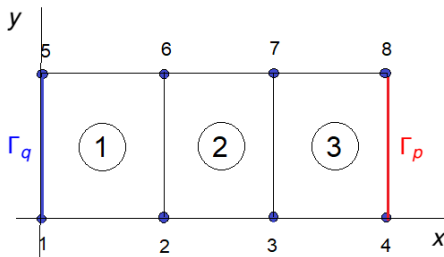
$$f^e = \begin{bmatrix} f_1^e \\ f_2^e \\ f_3^e \\ f_4^e \end{bmatrix}, \text{ para } e = 1, 2, 3.$$

Cálculo de F^e

- Cálculo de q_a^e :

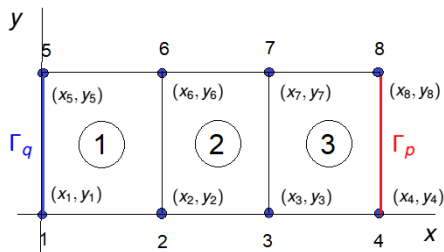
$$q_a^e = (\bar{q}, \varphi_a^e)_{\Gamma_q} = \int_{\Gamma_q} \bar{q}(s) \cdot \varphi_a^e(s) ds, \text{ onde } s \in \Gamma_q$$

- Exemplo de cálculo de q_a^e :



Note que: $\Gamma_q \in \Omega^1$

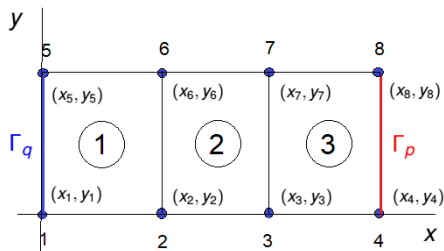
Cálculo de F^e



Aqui, os nós 1 e 5 da malha pertencem a $\Gamma_q = \{(0, y); y_1 \leq y \leq y_5\}$.
Logo,

$$q_a^e = (\bar{q}, \varphi_a^e)_{\Gamma_q} = \int_{\Gamma_q} \bar{q}(0, y) \cdot \varphi_a^e(0, y) dy = \int_{y_1}^{y_5} \bar{q}(0, y) \cdot \varphi_a^e(0, y) dy$$

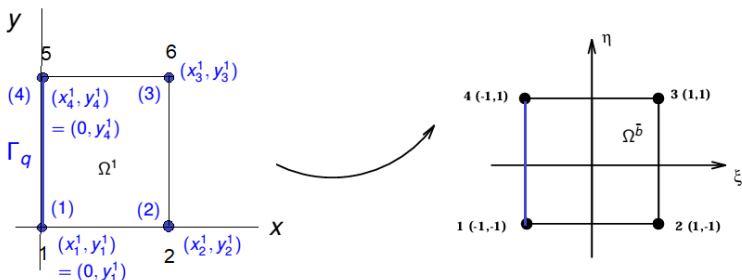
Cálculo de F^e



Aqui, os nós 1 e 5 da malha pertencem a $\Gamma_q = \{(0, y); y_1 \leq y \leq y_5\}$.

Logo,

$$q_a^e = (\bar{q}, \varphi_a^e)_{\Gamma_q} = \int_{\Gamma_q} \bar{q}(0, y) \cdot \varphi_a^e(0, y) dy = \int_{y_1}^{y_5} \bar{q}(0, y) \cdot \varphi_a^e(0, y) dy$$

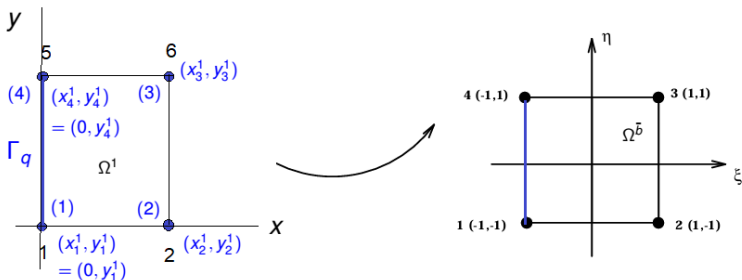
Cálculo de F^e 

Na mudança de variáveis, $(0, y) \mapsto (-1, \eta)$.

Logo, para $a = 1, 4$:

$$q_a^e = (\bar{q}, \varphi_a^e)_{\Gamma_q} = \int_{y_1^1=y_1}^{y_4^1=y_5} \bar{q}(0, y) \cdot \varphi_a^e(0, y) dy = \int_{-1}^1 \bar{q}(-1, \eta) \cdot \varphi_a^{\bar{b}}(-1, \eta) \cdot y_\eta d\eta$$

Para $a = 2, 3$, $q_a^e = 0$.

Cálculo de F^e 

Lembre-se de que:

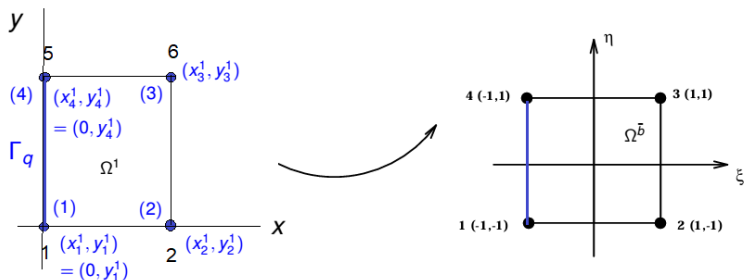
$$\bar{\varphi}_1^{\bar{b}}(-1, \eta) = (1/4)(1 - (-1))(1 - \eta) = (1 - \eta)/2;$$

$$\bar{\varphi}_2^{\bar{b}}(-1, \eta) = (1/4)(1 + (-1))(1 - \eta) = 0;$$

$$\bar{\varphi}_3^{\bar{b}}(-1, \eta) = (1/4)(1 + (-1))(1 + \eta) = 0;$$

$$\bar{\varphi}_4^{\bar{b}}(-1, \eta) = (1/4)(1 - (-1))(1 + \eta) = (1 + \eta)/2$$

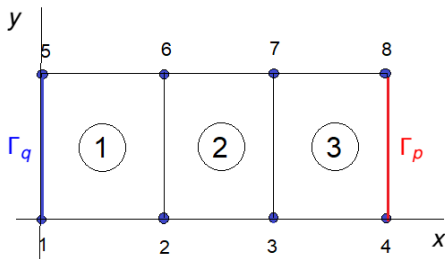
Portanto, no elemento Ω^1 , só serão usadas as funções de base locais $\bar{\varphi}_1^{\bar{b}}$, $\bar{\varphi}_4^{\bar{b}}$.

Cálculo de F^e 

Logo,

$$q^1 = \begin{bmatrix} q_1^1 \\ q_2^1 \\ q_3^1 \\ q_4^1 \end{bmatrix} = \begin{bmatrix} (\bar{q}, \varphi_1^1)_{\Gamma_q} \\ 0 \\ 0 \\ (\bar{q}, \varphi_4^1)_{\Gamma_q} \end{bmatrix}; \quad q^e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ para } e = 2, 3.$$

Cálculo de F^e



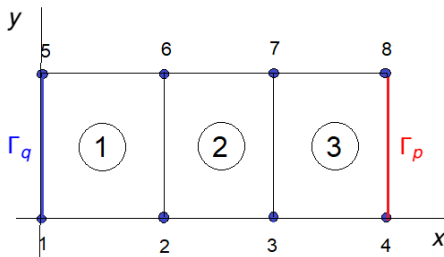
• Cálculo de \bar{p}_a^e :

$$\bar{p}_a^e = \sum_{b=1}^4 K_{ab}^e p_b^e$$

Neste exemplo, os nós prescritos 4 e 8 estão em $\Gamma_p = \{(x_4, y); y_4 \leq y \leq y_8\}$.

Logo, os nós prescritos estão no elemento Ω^3 .

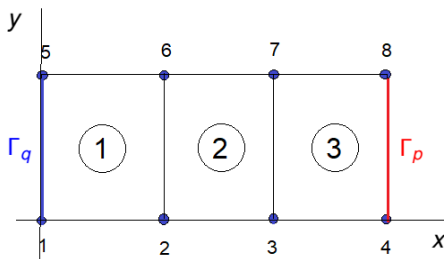
Neste caso, só vamos calcular \bar{p}^3 , enquanto $\bar{p}^e = 0$ para $e = 1, 2$.

Cálculo de F^e 

Logo,

$$\bar{p}_a^3 = \sum_{b=1}^4 K_{ab}^3 p_b^3$$

$$\bar{p}^3 = \begin{bmatrix} \bar{p}_1^3 \\ \bar{p}_2^3 \\ \bar{p}_3^3 \\ \bar{p}_4^3 \end{bmatrix} = \begin{bmatrix} K_{11}^3 & K_{12}^3 & K_{13}^3 & K_{14}^3 \\ K_{21}^3 & K_{22}^3 & K_{23}^3 & K_{24}^3 \\ K_{31}^3 & K_{32}^3 & K_{33}^3 & K_{34}^3 \\ K_{41}^3 & K_{42}^3 & K_{43}^3 & K_{44}^3 \end{bmatrix} \begin{bmatrix} p_1^3 \\ p_2^3 \\ p_3^3 \\ p_4^3 \end{bmatrix}$$

Cálculo de F^e 

Como $p_1^3 = p_4^3 = 0$, obtemos:

$$\bar{p}^3 = \begin{bmatrix} \bar{p}_1^3 \\ \bar{p}_2^3 \\ \bar{p}_3^3 \\ \bar{p}_4^3 \end{bmatrix} = \begin{bmatrix} K_{11}^3 & K_{12}^3 & K_{13}^3 & K_{14}^3 \\ K_{21}^3 & K_{22}^3 & K_{23}^3 & K_{24}^3 \\ K_{31}^3 & K_{32}^3 & K_{33}^3 & K_{34}^3 \\ K_{41}^3 & K_{42}^3 & K_{43}^3 & K_{44}^3 \end{bmatrix} \begin{bmatrix} 0 \\ p_2^3 \\ p_3^3 \\ 0 \end{bmatrix}; \bar{p}^e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ para } e = 1, 2.$$

Cálculo de F^e

Portanto, neste exemplo, para os elementos $e = 1, 2, 3$:

$$F^e = f^e - q^e - \bar{p}^e,$$

onde

$$f^e = \begin{bmatrix} f_1^e \\ f_2^e \\ f_3^e \\ f_4^e \end{bmatrix}, \text{ para } e = 1, 2, 3; \quad q^1 = \begin{bmatrix} q_1^1 \\ 0 \\ 0 \\ q_4^1 \end{bmatrix}; \quad q^e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ para } e = 2, 3;$$

$$\bar{p}^3 = \begin{bmatrix} \bar{p}_1^3 \\ \bar{p}_2^3 \\ \bar{p}_3^3 \\ \bar{p}_4^3 \end{bmatrix} = \begin{bmatrix} K_{11}^3 & K_{12}^3 & K_{13}^3 & K_{14}^3 \\ K_{21}^3 & K_{22}^3 & K_{23}^3 & K_{24}^3 \\ K_{31}^3 & K_{32}^3 & K_{33}^3 & K_{34}^3 \\ K_{41}^3 & K_{42}^3 & K_{43}^3 & K_{44}^3 \end{bmatrix} \begin{bmatrix} 0 \\ p_2^3 \\ p_3^3 \\ 0 \end{bmatrix}; \quad \bar{p}^e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ para } e = 1, 2.$$

Cálculo de F^e

- Para $e = 1$:

$$F^1 = f^1 - q^1 - \bar{p}^1 = \begin{bmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \\ f_4^1 \end{bmatrix} - \begin{bmatrix} q_1^1 \\ 0 \\ 0 \\ q_4^1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1^1 - q_1^1 \\ f_2^1 \\ f_3^1 \\ f_4^1 - q_4^1 \end{bmatrix}$$





- Para $e = 2$:

$$F^2 = f^2 - q^2 - \bar{p}^2 = \begin{bmatrix} f_1^2 \\ f_2^2 \\ f_3^2 \\ f_4^2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1^2 \\ f_2^2 \\ f_3^2 \\ f_4^2 \end{bmatrix}$$

- Para $e = 3$:

$$F^3 = f^3 - q^3 - \bar{p}^3 = \begin{bmatrix} f_1^3 \\ f_2^3 \\ f_3^3 \\ f_4^3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \bar{p}_1^3 \\ \bar{p}_2^3 \\ \bar{p}_3^3 \\ \bar{p}_4^3 \end{bmatrix} = \begin{bmatrix} f_1^3 - \bar{p}_1^3 \\ f_2^3 - \bar{p}_2^3 \\ f_3^3 - \bar{p}_3^3 \\ f_4^3 - \bar{p}_4^3 \end{bmatrix}$$

Referências I

-  Hughes, T.J.R.. **The Finite Element Method - Linear Static and Dynamic Finite Element Analysis**. Prentice-Hall, Inc., 1987.
-  Fish, J.; Belytschko, T.. **A First Course in Finite Elements**. Wiley, 2007.
-  Becker, E. B.; Carey, G. F.; Oden, J. T.. **Finite Elements - An Introduction**. Prentice-Hall, 1981.
-  Liu, I.S.; Rincon, M.A.. **Introdução ao Método de Elementos Finitos, Análise e Aplicação**. IM/UFRJ, 2003.