Tópicos Especiais em Matemática Aplicada - 2025-1
UERJ

05 - Metodo dos Elementos Finitos - Caso unidimensional

Rodrigo Madureira
rodrigo.madureira@ime.uerj.br
Github: https://github.com/rodrigolrmadureira/ElementosFinitos

Sumário

Método dos Elementos Finitos (MEF) - Caso 1D

No caso unidimensional, cada subintervalo $[x_e, x_{e+1}]$, $e = 1, 2, \ldots, m-1$, do domínio $\Omega = [x_1, x_m]$ é um elemento finito de tamanho $h = x_{e+1} - x_e$.

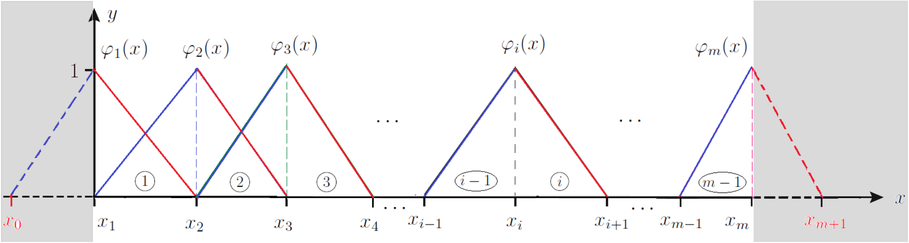


Figura: Funções de base de Lagrange linear $\varphi_i(x)$

Método dos Elementos Finitos (MEF) - Caso 1D

Ao invés de "olhar" para todo o domínio, vamos "olhar" para um único elemento e.

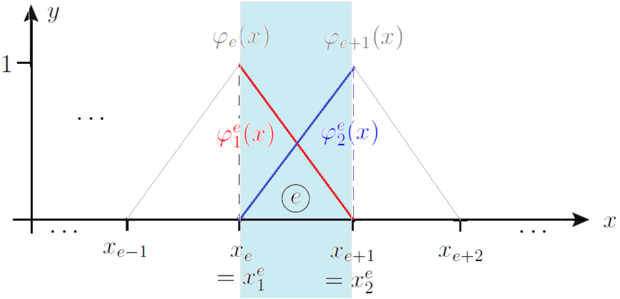


Figura: Funções do elemento e: $\varphi_1^e(x)$ e $\varphi_2^e(x)$

Podemos referenciar um nó da discretização do domínio de duas formas:

• x_i, onde i é o índice da discretização do intervalo $[x_1, x_m]$, $i = 1, 2, \ldots, m$. Neste caso, i é a **numeração global**. Assim, x_i é o **nó global**.

• x_a^e, onde a é o índice do nó no elemento finito e, $a = 1, 2$. Neste caso, a é a **numeração local**. Assim, x_a^e é o **nó local**.

Método dos Elementos Finitos (MEF) - Caso 1D

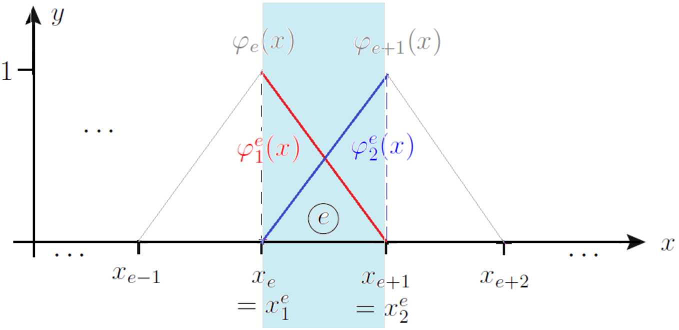


Figura: Funções do elemento e: $\varphi_1^e(x)$ e $\varphi_2^e(x)$

$$\begin{cases} \varphi_1^e(x) = \dfrac{x_{e+1} - x}{x_{e+1} - x_e} = \dfrac{x_{e+1} - x}{x_2^e - x_1^e} = \dfrac{x_{e+1} - x}{h}, \\ \varphi_2^e(x) = \dfrac{x - x_e}{x_{e+1} - x_e} = \dfrac{x - x_e}{x_2^e - x_1^e} = \dfrac{x - x_e}{h}. \end{cases} \tag{1}$$

Método dos Elementos Finitos (MEF) - Caso 1D

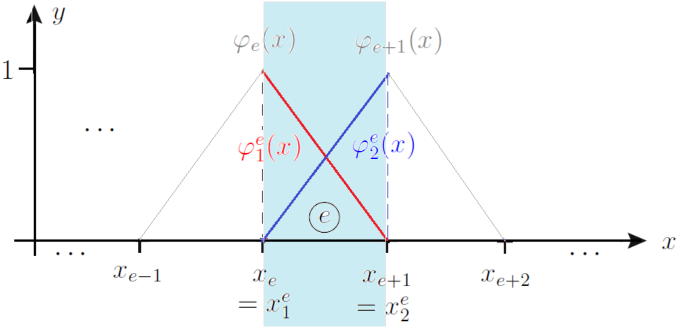


Figura: Funções do elemento e: $\varphi_1^e(x)$ e $\varphi_2^e(x)$

Note que:

$$\begin{cases} \varphi_1^e(x) = \varphi_e(x), \text{ para } x \in [x_e, x_{e+1}], \\ \\ \varphi_2^e(x) = \varphi_{e+1}(x), \text{ para } x \in [x_e, x_{e+1}]. \end{cases} \tag{2}$$

Matriz local K^e

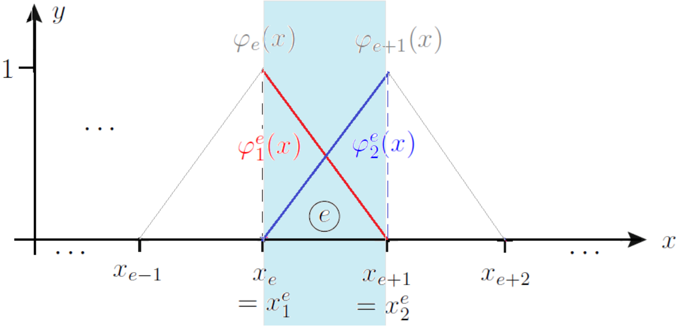


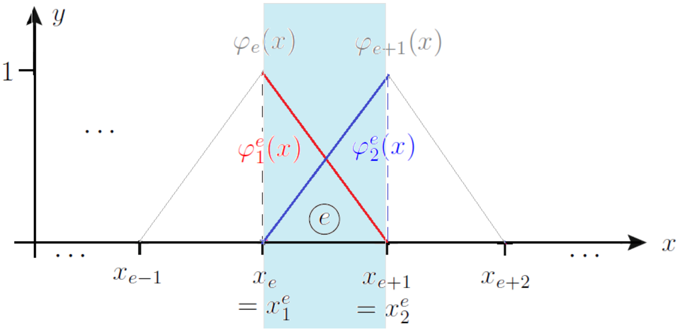
Figura: Funções do elemento e: $\varphi_1^e(x)$ e $\varphi_2^e(x)$

Note que restritos ao elemento finito e, obtemos o elemento da **matriz local** K^e:

$$K_{ab}^e = \alpha(\varphi_{ax}^e, \varphi_{bx}^e) + \beta(\varphi_a^e, \varphi_b^e) = \alpha \int_{x_1^e}^{x_2^e} \varphi_{ax}^e(x)\varphi_{bx}^e(x)dx + \beta \int_{x_1^e}^{x_2^e} \varphi_a^e(x)\varphi_b^e(x)dx,$$

para $a, b = 1, 2.$

Vetor força local F^e



Analogamente, obtemos o elemento do **vetor força local** F^e:

$$F_a^e = (f, \varphi_a^e) = \alpha \int_{x_1^e}^{x_2^e} f(x)\varphi_a^e(x)dx, \text{ para } a = 1, 2.$$

Montagem da matriz global K

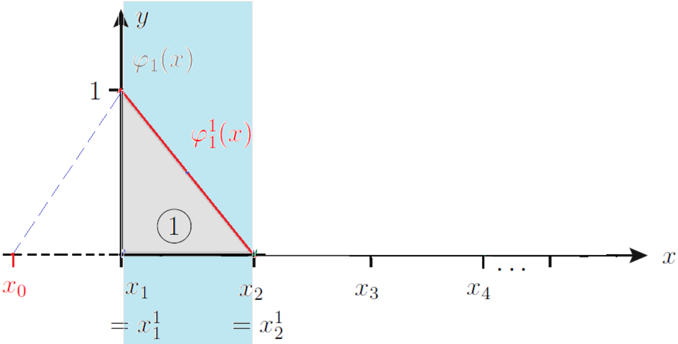
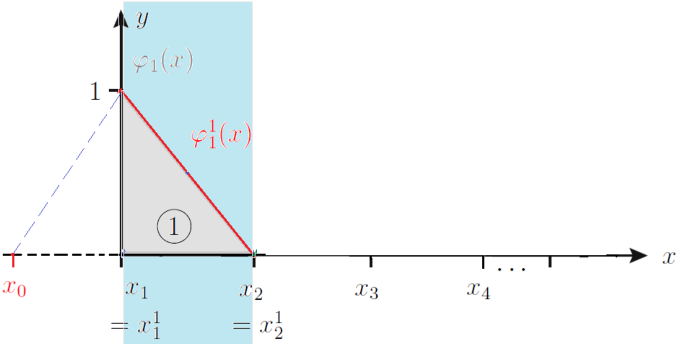


Figura: Funções do elemento e: $\varphi_1^e(x)$ e $\varphi_2^e(x)$

Podemos reescrever a **matriz global** K em termos de elementos da **matriz local** K^e, $e = 1, 2, \dots, m$. Vimos na abordagem anterior que:

$$K_{11} = \alpha(\varphi_{1x}, \varphi_{1x}) + \beta(\varphi_1, \varphi_1) = \alpha \int_{x_1}^{x_2} (\varphi_{1x}(x))^2 dx + \beta \int_{x_1}^{x_2} (\varphi_1(x))^2 dx$$

Montagem da matriz global K

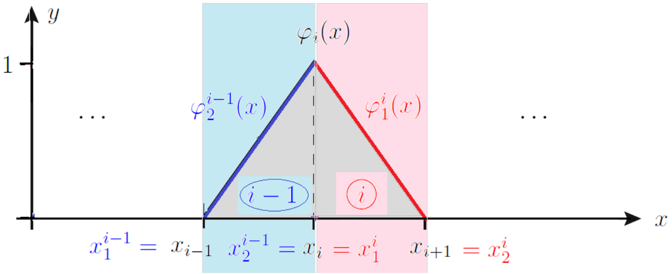


Podemos reescrever K_{11} como:

$$K_{11} = \alpha(\varphi_{1x}, \varphi_{1x}) + \beta(\varphi_1, \varphi_1) = \alpha \int_{x_1}^{x_2} (\varphi_{1x}(x))^2 dx + \beta \int_{x_1}^{x_2} (\varphi_1(x))^2 dx$$

$$= \alpha \int_{x_1^1}^{x_2^1} (\varphi_{1x}^1(x))^2 dx + \beta \int_{x_1^1}^{x_2^1} (\varphi_1^1(x))^2 dx = \alpha(\varphi_{1x}^1, \varphi_{1x}^1) + \beta(\varphi_1^1, \varphi_1^1) = K_{11}^1$$

Montagem da matriz global K



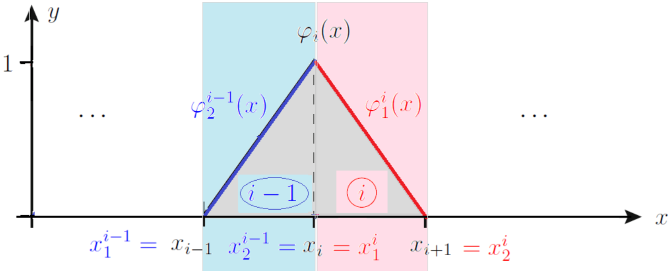
Podemos reescrever K_{ii} como:

$$K_{ii} = \alpha(\varphi_{ix}, \varphi_{ix}) + \beta(\varphi_i, \varphi_i) = \alpha \int_{x_{i-1}}^{x_{i+1}} (\varphi_{ix}(x))^2 dx + \beta \int_{x_{i-1}}^{x_{i+1}} (\varphi_i(x))^2 dx$$

$$= \alpha\left(\int_{x_{i-1}}^{x_i} (\varphi_{ix}(x))^2 dx + \int_{x_i}^{x_{i+1}} (\varphi_{ix}(x))^2 dx\right)$$

$$+ \beta\left(\int_{x_{i-1}}^{x_i} (\varphi_i(x))^2 dx + \int_{x_i}^{x_{i+1}} (\varphi_i(x))^2 dx\right)$$

Montagem da matriz global K

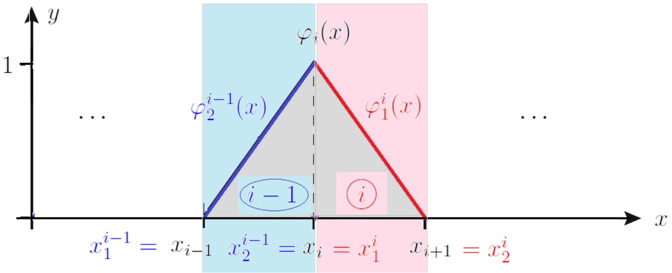


Podemos reescrever K_{ii} como:

$$K_{ii} = \alpha\left(\int_{x_1^{i-1}}^{x_2^{i-1}} (\varphi_{2x}^{i-1}(x))^2 dx + \int_{x_1^i}^{x_2^i} (\varphi_{1x}^{i}(x))^2 dx\right)$$

$$+ \beta\left(\int_{x_1^{i-1}}^{x_2^{i-1}} (\varphi_{2}^{i-1}(x))^2 dx + \int_{x_1^i}^{x_2^i} (\varphi_{1}^{i}(x))^2 dx\right)$$

Montagem da matriz global K



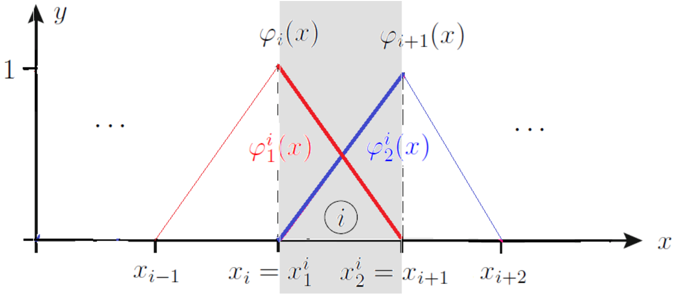
Podemos reescrever K_{ii} como:

$$K_{ii} = \alpha\int_{x_1^{i-1}}^{x_2^{i-1}} (\varphi_{2x}^{i-1}(x))^2 dx + \beta\int_{x_1^{i-1}}^{x_2^{i-1}} (\varphi_2^{i-1}(x))^2 dx$$

$$+ \alpha\int_{x_1^{i}}^{x_2^{i}} (\varphi_{1x}^{i}(x))^2 dx + \beta\int_{x_1^{i}}^{x_2^{i}} (\varphi_1^{i}(x))^2 dx$$

$$= \alpha(\varphi_{2x}^{i-1}, \varphi_{2x}^{i-1}) + \beta(\varphi_2^{i-1}, \varphi_2^{i-1}) + \alpha(\varphi_{1x}^{i}, \varphi_{1x}^{i}) + \beta(\varphi_1^{i}, \varphi_1^{i}) = K_{22}^{i-1} + K_{11}^{i}$$

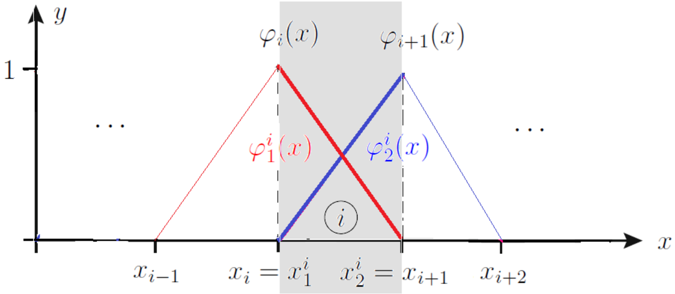
Montagem da matriz global K



Podemos reescrever $K_{i,i+1}$ como:

$$
\begin{aligned}
K_{i,i+1} &= \alpha(\varphi_{ix}, \varphi_{(i+1)x}) + \beta(\varphi_i, \varphi_{i+1}) \\
&= \alpha \int_{x_i}^{x_{i+1}} \varphi_{ix}(x)\varphi_{(i+1)x}(x)dx + \beta \int_{x_i}^{x_{i+1}} \varphi_i(x)\varphi_{i+1}(x)dx \\
&= \alpha \int_{x_1^i}^{x_2^i} \varphi_{1x}^i(x)\varphi_{2x}^i(x)dx + \beta \int_{x_1^i}^{x_2^i} \varphi_1^i(x)\varphi_2^i(x)dx \\
&= \alpha(\varphi_{1x}^i, \varphi_{2x}^i) + \beta(\varphi_1^i, \varphi_2^i) = K_{12}^i
\end{aligned}
$$

Montagem da matriz global K



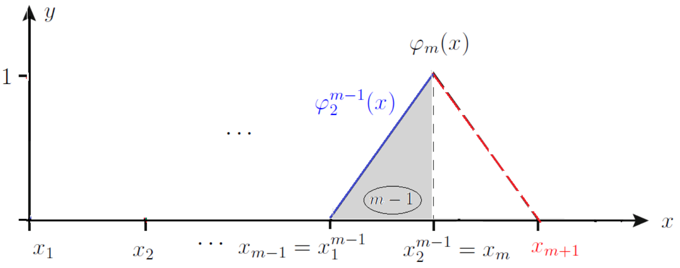
De forma análoga, obtemos:

$$K_{i+1,i} = K_{21}^i$$

Como K é simétrica ($K = K^T$), logo:

$$K_{12}^i = K_{21}^i$$

Montagem da matriz global K



Podemos reescrever K_{mm} como:

$$
\begin{aligned}
K_{mm} &= \alpha(\varphi_{mx}, \varphi_{mx}) + \beta(\varphi_m, \varphi_m) \\
&= \alpha \int_{x_{m-1}}^{x_m} (\varphi_{mx}(x))^2 dx + \beta \int_{x_{m-1}}^{x_m} (\varphi_m(x))^2 dx \\
&= \alpha \int_{x_1^{m-1}}^{x_2^{m-1}} (\varphi_{2x}^{m-1}(x))^2 dx + \beta \int_{x_1^{m-1}}^{x_2^{m-1}} (\varphi_2^{m-1}(x))^2 dx \\
&= \alpha(\varphi_{2x}^{m-1}, \varphi_{2x}^{m-1}) + \beta(\varphi_2^{m-1}, \varphi_2^{m-1}) = K_{22}^{m-1}
\end{aligned}
$$

Montagem do vetor global F

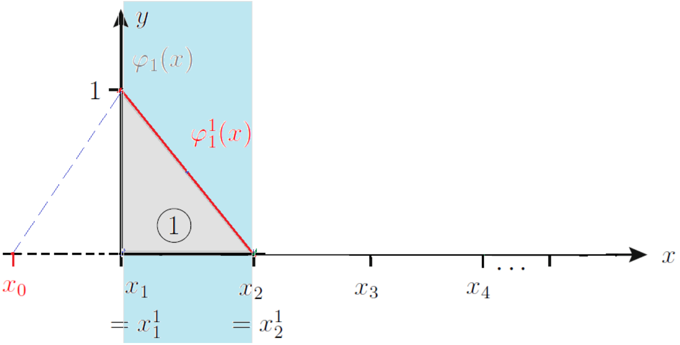
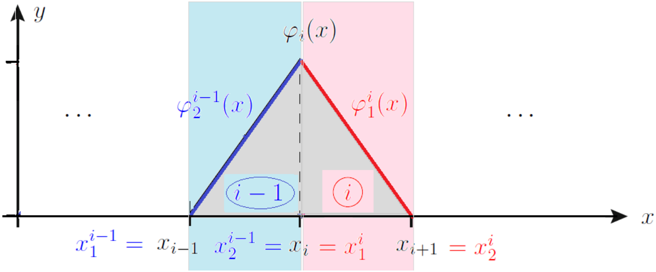


Figura: Funções do elemento e: $\varphi_1^e(x)$ e $\varphi_2^e(x)$

Podemos reescrever a **vetor global** F em termos de elementos do **vetor local** F^e, $e = 1, 2, \ldots, m$. Vimos na abordagem anterior que:

$$F_1 = (f, \varphi_1) = \int_{x_1}^{x_2} f(x)\varphi_1(x)dx = \int_{x_1^1}^{x_2^1} f(x)\varphi_1^1(x)dx = (f, \varphi_1^1) = F_1^1$$

Montagem do vetor global F

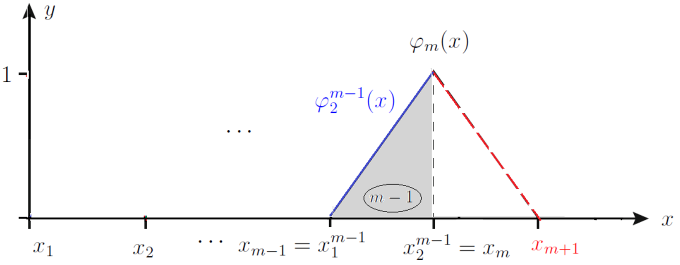


Podemos reescrever F_i como:

$$F_i = (f, \varphi_i) = \int_{x_{i-1}}^{x_{i+1}} f(x)\varphi_i(x)dx = \int_{x_{i-1}}^{x_i} f(x)\varphi_i(x)dx + \int_{x_i}^{x_{i+1}} f(x)\varphi_i(x)dx$$

$$= \int_{x_1^{i-1}}^{x_2^{i-1}} f(x)\varphi_2^{i-1}(x)dx + \int_{x_1^i}^{x_2^i} f(x)\varphi_1^i(x)dx = (f, \varphi_2^{i-1}) + (f, \varphi_1^i) = F_2^{i-1} + F_1^i$$

Montagem do vetor global F



Podemos reescrever F_m como:

$$F_m = (f, \varphi_m) = \int_{x_{m-1}}^{x_m} f(x)\varphi_m(x)dx = \int_{x_1^{m-1}}^{x_2^{m-1}} f(x)\varphi_2^{m-1}(x)dx$$
$$= (f, \varphi_2^{m-1}) = F_2^{m-1}$$

Resultados

Matriz de rigidez K:

$$K_{11} = K^{1}_{11};$$

$$K_{ii} = K^{i-1}_{22} + K^{i}_{11}, \text{ para } i = 2, 3, \ldots, m-1; \tag{3}$$

$$K_{i,i+1} = K^{i}_{12} \Rightarrow K_{i+1,i} = K^{i}_{21}, \text{ para } i = 1, 2, 3, \ldots, m-1; \ \textbf{(Simetria: } K = K^{T}\textbf{)}$$

$$K_{mm} = K^{m-1}_{22}.$$

Vetor força F:

$$F_1 = F^{1}_{1};$$

$$F_i = {\color{blue}F^{i-1}_{2}} + {\color{red}F^{i}_{1}}, \text{ para } i = 2, 3, \ldots, m-1;$$

$$F_m = F^{m-1}_{2}. \tag{4}$$

Referências I

Liu, I.S.; Rincon, M.A.. **Introdução ao Método de Elementos Finitos, Análise e Aplicação**. IM/UFRJ, 2003.