Tópicos Especiais em Matemática Aplicada - 2025-1
UERJ

05 - Metodo dos Elementos Finitos - Caso 1D estacionário

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Sumário

Método dos Elementos Finitos (MEF) - Caso 1D

No caso unidimensional, cada subintervalo $[x_e, x_{e+1}]$, $e = 1, 2, \ldots, m-1$, do domínio $\Omega = [x_1, x_m]$ é um elemento finito de tamanho $h = x_{e+1} - x_e$.

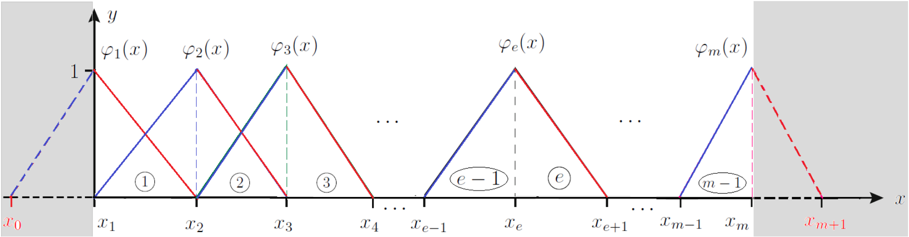


Figura: Funções de base de Lagrange linear $\varphi_i(x)$

Método dos Elementos Finitos (MEF) - Caso 1D

Ao invés de "olhar" para todo o domínio, vamos "olhar" para um único elemento e.

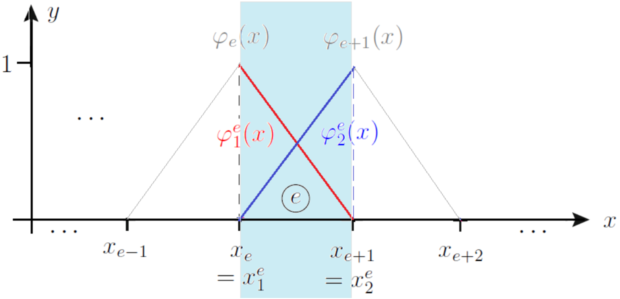


Figura: Funções do elemento e: $\varphi_1^e(x)$ e $\varphi_2^e(x)$

Podemos referenciar um nó da discretização do domínio de duas formas:

- x_i, onde i é o índice da discretização do intervalo $[x_1, x_m]$, $i = 1, 2, \ldots, m$. Neste caso, i é a **numeração global**. Assim, x_i é o **nó global**.
- x_a^e, onde a é o índice do nó no elemento finito e, $a = 1, 2$. Neste caso, a é a **numeração local**. Assim, x_a^e é o **nó local**.

Método dos Elementos Finitos (MEF) - Caso 1D

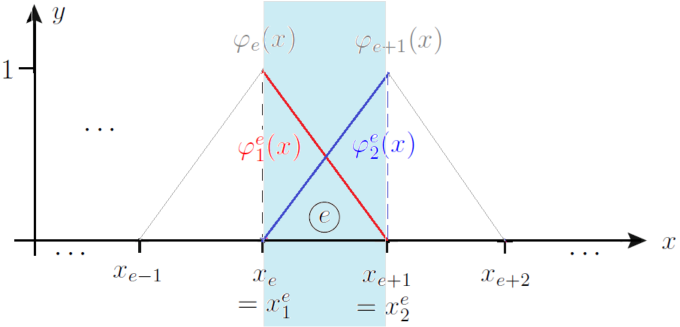


Figura: Funções do elemento e: $\varphi_1^e(x)$ e $\varphi_2^e(x)$

$$
\begin{cases}
\varphi_1^e(x) = \dfrac{x_{e+1} - x}{x_{e+1} - x_e} = \dfrac{x_{e+1} - x}{x_2^e - x_1^e} = \dfrac{x_{e+1} - x}{h}, \\[2ex]
\varphi_2^e(x) = \dfrac{x - x_e}{x_{e+1} - x_e} = \dfrac{x - x_e}{x_2^e - x_1^e} = \dfrac{x - x_e}{h}.
\end{cases}
\tag{1}
$$

Método dos Elementos Finitos (MEF) - Caso 1D

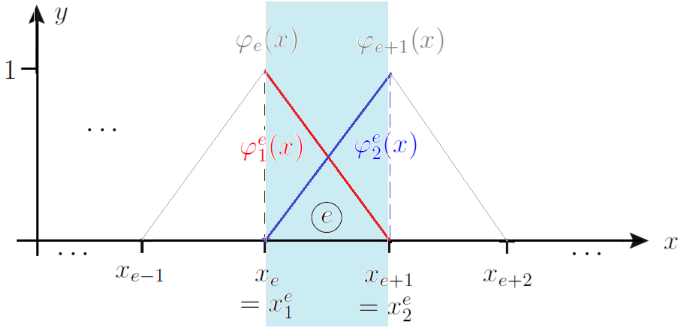


Figura: Funções do elemento e: $\varphi_1^e(x)$ e $\varphi_2^e(x)$

Note que:

$$\begin{cases} \varphi_1^e(x) = \varphi_e(x), \text{ para } x \in [x_e, x_{e+1}], \\ \\ \varphi_2^e(x) = \varphi_{e+1}(x), \text{ para } x \in [x_e, x_{e+1}]. \end{cases} \tag{2}$$

Matriz local K^e

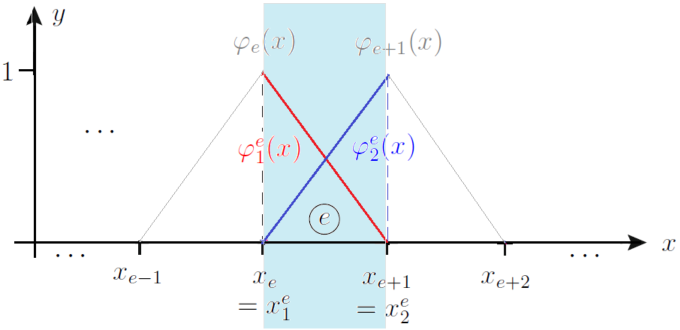


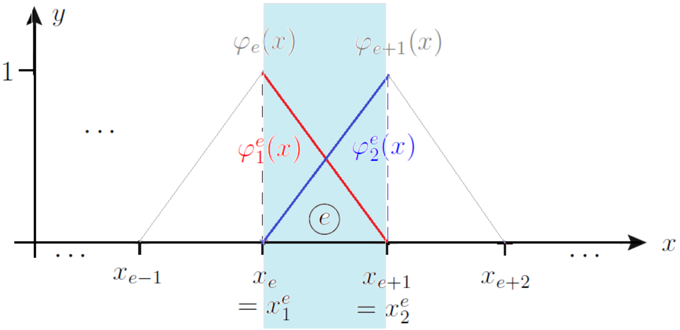
Figura: Funções do elemento e: $\varphi_1^e(x)$ e $\varphi_2^e(x)$

Note que restritos ao elemento finito e, obtemos o elemento da **matriz local** K^e:

$$K_{ab}^e = \alpha(\varphi_{ax}^e, \varphi_{bx}^e) + \beta(\varphi_a^e, \varphi_b^e) = \alpha \int_{x_1^e}^{x_2^e} \varphi_{ax}^e(x)\varphi_{bx}^e(x)dx + \beta \int_{x_1^e}^{x_2^e} \varphi_a^e(x)\varphi_b^e(x)dx,$$

para $a, b = 1, 2.$

Vetor força local F^e



Analogamente, obtemos o elemento do **vetor força local** F^e:

$$F_a^e = (f, \varphi_a^e) = \alpha \int_{x_1^e}^{x_2^e} f(x)\varphi_a^e(x)dx, \text{ para } a = 1, 2.$$

Montagem da matriz global K

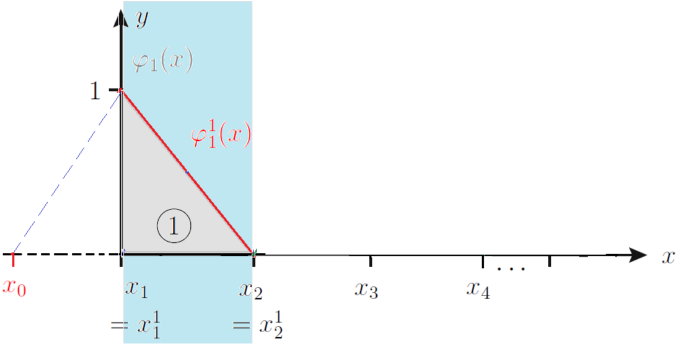
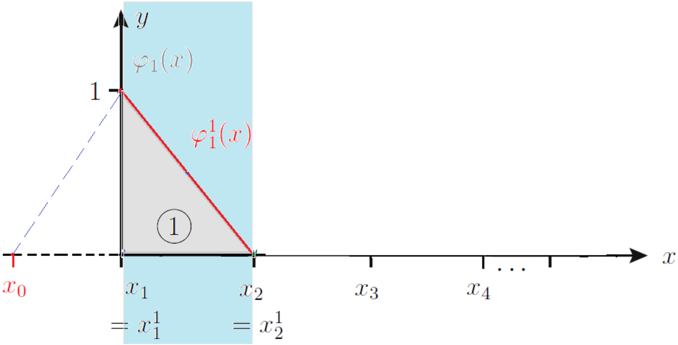


Figura: Funções do elemento e: $\varphi_1^e(x)$ e $\varphi_2^e(x)$

Podemos reescrever a **matriz global** K em termos de elementos da **matriz local** K^e, $e = 1, 2, \ldots, m$. Vimos na abordagem anterior que:

$$K_{11} = \alpha(\varphi_{1x}, \varphi_{1x}) + \beta(\varphi_1, \varphi_1) = \alpha \int_{x_1}^{x_2} (\varphi_{1x}(x))^2 dx + \beta \int_{x_1}^{x_2} (\varphi_1(x))^2 dx$$

Montagem da matriz global K

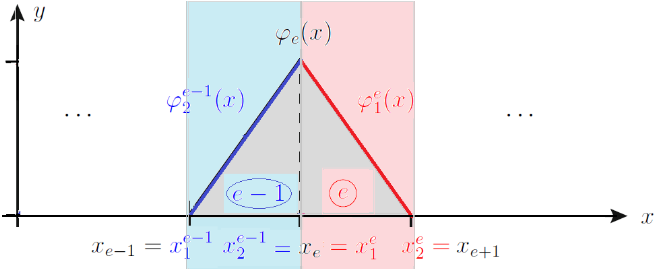


Podemos reescrever K_{11} como:

$$K_{11} = \alpha(\varphi_{1x}, \varphi_{1x}) + \beta(\varphi_1, \varphi_1) = \alpha \int_{x_1}^{x_2} (\varphi_{1x}(x))^2 dx + \beta \int_{x_1}^{x_2} (\varphi_1(x))^2 dx$$

$$= \alpha \int_{x_1^1}^{x_2^1} (\varphi_{1x}^1(x))^2 dx + \beta \int_{x_1^1}^{x_2^1} (\varphi_1^1(x))^2 dx = \alpha(\varphi_{1x}^1, \varphi_{1x}^1) + \beta(\varphi_1^1, \varphi_1^1) = K_{11}^1$$

Montagem da matriz global K



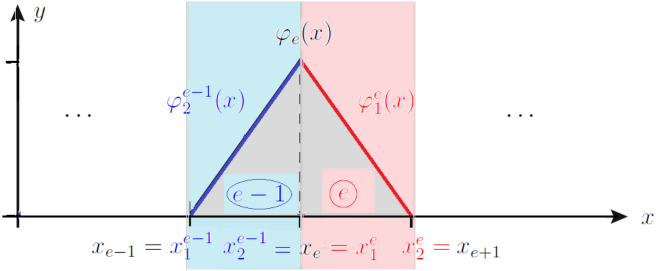
Podemos reescrever K_{ii} como:

$$K_{ee} = \alpha(\varphi_{ex}, \varphi_{ex}) + \beta(\varphi_e, \varphi_e) = \alpha \int_{x_{e-1}}^{x_{e+1}} (\varphi_{ex}(x))^2 dx + \beta \int_{x_{e-1}}^{x_{e+1}} (\varphi_e(x))^2 dx$$

$$= \alpha\left(\int_{x_{e-1}}^{x_e} (\varphi_{ex}(x))^2 dx + \int_{x_e}^{x_{e+1}} (\varphi_{ex}(x))^2 dx\right)$$

$$+ \beta\left(\int_{x_{e-1}}^{x_e} (\varphi_e(x))^2 dx + \int_{x_e}^{x_{e+1}} (\varphi_e(x))^2 dx\right)$$

Montagem da matriz global K

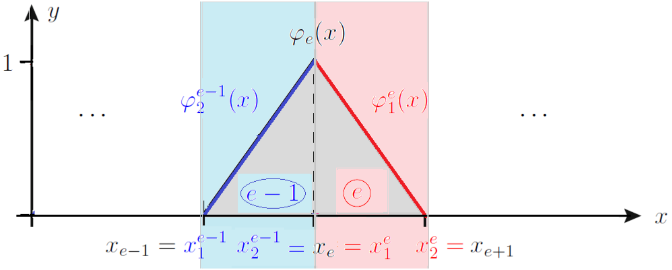


Podemos reescrever K_{ee} como:

$$K_{ee} = \alpha\left(\int_{x_1^{e-1}}^{x_2^{e-1}} (\varphi_{2x}^{e-1}(x))^2 dx + \int_{x_1^e}^{x_2^e} (\varphi_{1x}^{e}(x))^2 dx\right)$$

$$+ \beta\left(\int_{x_1^{e-1}}^{x_2^{e-1}} (\varphi_{2}^{e-1}(x))^2 dx + \int_{x_1^e}^{x_2^e} (\varphi_{1}^{e}(x))^2 dx\right)$$

Montagem da matriz global K



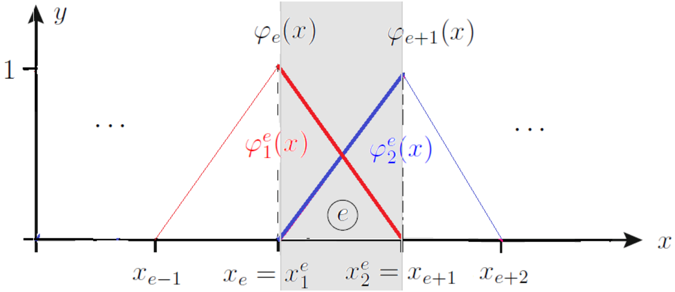
Podemos reescrever K_{ee} como:

$$K_{ee} = \alpha \int_{x_1^{e-1}}^{x_2^{e-1}} (\varphi_{2x}^{e-1}(x))^2 dx + \beta \int_{x_1^{e-1}}^{x_2^{e-1}} (\varphi_2^{e-1}(x))^2 dx$$

$$+ \alpha \int_{x_1^e}^{x_2^e} (\varphi_{1x}^e(x))^2 dx + \beta \int_{x_1^e}^{x_2^e} (\varphi_1^e(x))^2 dx$$

$$= \alpha(\varphi_{2x}^{e-1}, \varphi_{2x}^{e-1}) + \beta(\varphi_2^{e-1}, \varphi_2^{e-1}) + \alpha(\varphi_{1x}^e, \varphi_{1x}^e) + \beta(\varphi_1^e, \varphi_1^e) = K_{22}^{e-1} + K_{11}^e$$

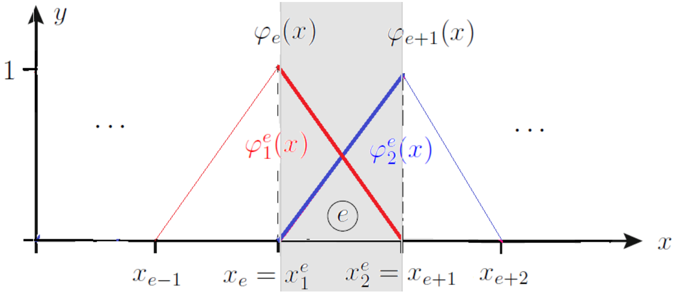
Montagem da matriz global K



Podemos reescrever $K_{e,e+1}$ como:

$$
\begin{aligned}
K_{e,e+1} &= \alpha(\varphi_{ex}, \varphi_{(e+1)x}) + \beta(\varphi_e, \varphi_{e+1}) \\
&= \alpha \int_{x_e}^{x_{e+1}} \varphi_{ex}(x)\varphi_{(e+1)x}(x)dx + \beta \int_{x_e}^{x_{e+1}} \varphi_e(x)\varphi_{e+1}(x)dx \\
&= \alpha \int_{x_1^e}^{x_2^e} \varphi_{1x}^e(x)\varphi_{2x}^e(x)dx + \beta \int_{x_1^e}^{x_2^e} \varphi_1^e(x)\varphi_2^e(x)dx \\
&= \alpha(\varphi_{1x}^e, \varphi_{2x}^e) + \beta(\varphi_1^e, \varphi_2^e) = K_{12}^e
\end{aligned}
$$

Montagem da matriz global K



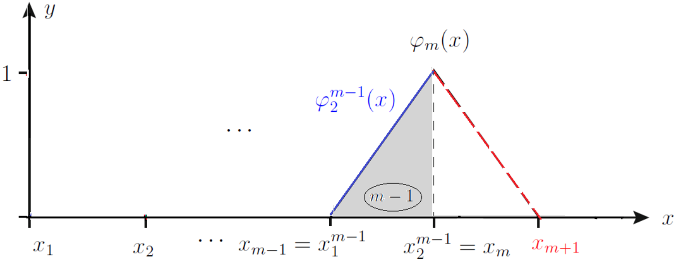
De forma análoga, obtemos:

$$\mathsf{K}_{e+1,e} = \mathsf{K}_{21}^e$$

Como K é simétrica ($\mathsf{K} = \mathsf{K}^T$), logo:

$$\mathsf{K}_{12}^e = \mathsf{K}_{21}^e$$

Montagem da matriz global K



Podemos reescrever K_{mm} como:

$$\begin{aligned} K_{mm} &= \alpha(\varphi_{mx}, \varphi_{mx}) + \beta(\varphi_m, \varphi_m) \\ &= \alpha \int_{x_{m-1}}^{x_m} (\varphi_{mx}(x))^2 dx + \beta \int_{x_{m-1}}^{x_m} (\varphi_m(x))^2 dx \\ &= \alpha \int_{x_1^{m-1}}^{x_2^{m-1}} (\varphi_{2x}^{m-1}(x))^2 dx + \beta \int_{x_1^{m-1}}^{x_2^{m-1}} (\varphi_2^{m-1}(x))^2 dx \\ &= \alpha(\varphi_{2x}^{m-1}, \varphi_{2x}^{m-1}) + \beta(\varphi_2^{m-1}, \varphi_2^{m-1}) = K_{22}^{m-1} \end{aligned}$$

Montagem do vetor global F

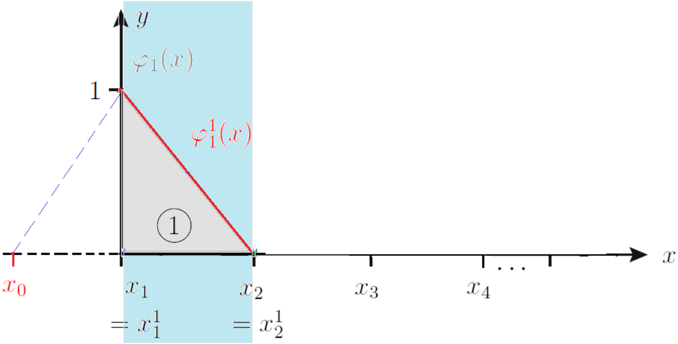
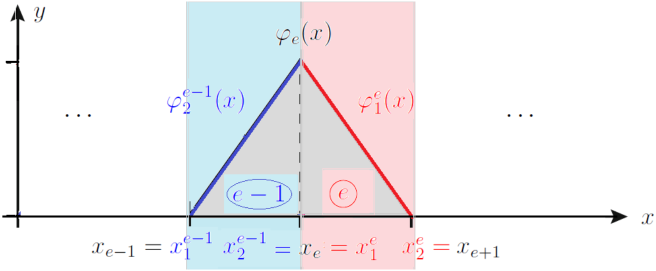


Figura: Funções do elemento e: $\varphi_1^e(x)$ e $\varphi_2^e(x)$

Podemos reescrever a **vetor global** F em termos de elementos do **vetor local** F^e, $e = 1, 2, \ldots, m$. Vimos na abordagem anterior que:

$$F_1 = (f, \varphi_1) = \int_{x_1}^{x_2} f(x)\varphi_1(x)dx = \int_{x_1^1}^{x_2^1} f(x)\varphi_1^1(x)dx = (f, \varphi_1^1) = F_1^1$$

Montagem do vetor global F

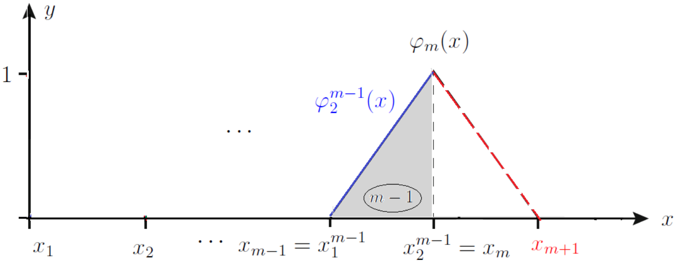


Podemos reescrever F_e como:

$$F_e = (f, \varphi_e) = \int_{x_{e-1}}^{x_{e+1}} f(x)\varphi_e(x)dx = \int_{x_{e-1}}^{x_e} f(x)\varphi_e(x)dx + \int_{x_e}^{x_{e+1}} f(x)\varphi_e(x)dx$$

$$= \int_{x_1^{e-1}}^{x_2^{e-1}} f(x)\varphi_2^{e-1}(x)dx + \int_{x_1^e}^{x_2^e} f(x)\varphi_1^e(x)dx = (f, \varphi_2^{e-1}) + (f, \varphi_1^e) = F_2^{e-1} + F_1^e$$

Montagem do vetor global F



Podemos reescrever F_m como:

$$F_m = (f, \varphi_m) = \int_{x_{m-1}}^{x_m} f(x)\varphi_m(x)dx = \int_{x_1^{m-1}}^{x_2^{m-1}} f(x)\varphi_2^{m-1}(x)dx$$

$$= (f, \varphi_2^{m-1}) = F_2^{m-1}$$

Resultados

Matriz de rigidez K:

$$K_{11} = K_{11}^{1};$$

$$K_{ee} = K_{22}^{e-1} + K_{11}^{e}, \text{ para } e = 2, 3, \ldots, m-1; \tag{3}$$

$$K_{e,e+1} = K_{12}^{i} \Rightarrow K_{e+1,e} = K_{21}^{i}, \text{ para } e = 1, 2, 3, \ldots, m-1; \; \textbf{(Simetria: } K = K^{T}\textbf{)}$$

$$K_{mm} = K_{22}^{m-1}.$$

Vetor força F:

$$F_{1} = F_{1}^{1};$$

$$F_{e} = F_{2}^{e-1} + F_{1}^{e}, \text{ para } i = 2, 3, \ldots, m-1;$$

$$F_{m} = F_{2}^{m-1}. \tag{4}$$

Resultados

A **matriz de rigidez global** K,

$$\begin{bmatrix} K_{11}^{1} & K_{12}^{1} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ K_{21}^{1} & K_{22}^{1}+K_{11}^{2} & K_{12}^{2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & K_{21}^{2} & K_{22}^{2}+K_{11}^{3} & K_{12}^{3} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{21}^{3} & K_{22}^{3}+K_{11}^{4} & K_{12}^{4} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{21}^{4} & K_{22}^{4}+K_{11}^{5} & K_{12}^{5} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & * & K_{22}^{m-1} \end{bmatrix}$$

é montada a partir das **matrizes locais**

$$K^{e} = \begin{bmatrix} K_{11}^{e} & K_{12}^{e} \\ K_{21}^{e} & K_{22}^{e} \end{bmatrix},$$

para $e = 1, 2, \ldots, m-1$.

Resultados

O **vetor força global**,

$$F = \begin{bmatrix} F_1^1 \\ F_2^1 + F_1^2 \\ F_2^2 + F_1^3 \\ F_2^3 + F_1^4 \\ \vdots \\ F_2^{m-2} + F_1^{m-1} \\ F_2^{m-1} \end{bmatrix}$$

é montado a partir dos **vetores locais**

$$F^e = \begin{bmatrix} F_1^e \\ F_2^e \end{bmatrix},$$

para $e = 1, 2, \ldots, m-1$.

Referências I

Liu, I.S.; Rincon, M.A.. **Introdução ao Método de Elementos Finitos, Análise e Aplicação**. IM/UFRJ, 2003.