

# Tópicos Especiais em Matemática Aplicada - 2025-1

## UERJ

### 05 - Método dos Elementos Finitos - Caso 1D estacionário

Rodrigo Madureira

[rodrigo.madureira@ime.uerj.br](mailto:rodrigo.madureira@ime.uerj.br)

**Github:** <https://github.com/rodrigolrmadureira/ElementosFinitos>

# Sumário

- 1 Método dos Elementos Finitos (MEF) - Caso 1D
- 2 Matriz local  $K^e$
- 3 Vetor local  $F^e$
- 4 Montagem da matriz global  $K$
- 5 Montagem do vetor global  $F$
- 6 Quadratura gaussiana para  $K^e$  e  $F^e$
- 7 Bibliografia

# Método dos Elementos Finitos (MEF) - Caso 1D

No caso unidimensional, cada subintervalo  $[x_e, x_{e+1}]$ ,  $e = 1, 2, \dots, m-1$ , do domínio  $\Omega = [x_1, x_m]$  é um elemento finito de tamanho  $h = x_{e+1} - x_e$ .

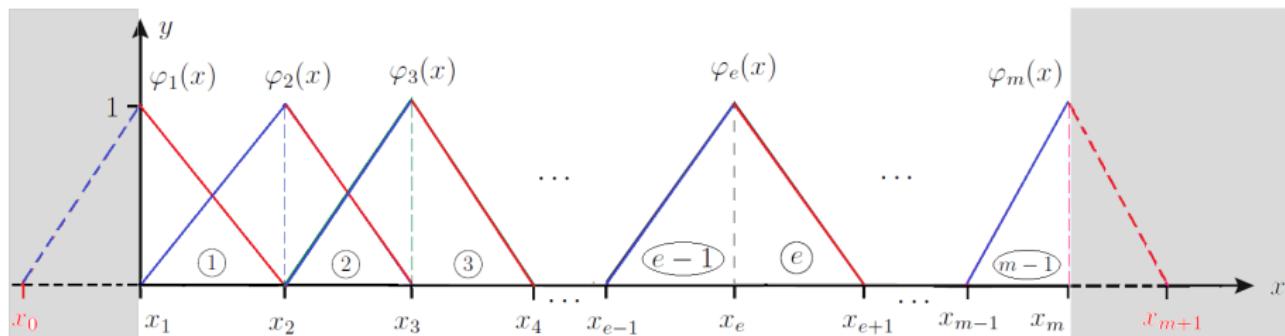


Figura: Funções de base de Lagrange linear  $\varphi_i(x)$

# Método dos Elementos Finitos (MEF) - Caso 1D

Ao invés de "olhar" para todo o domínio, vamos "olhar" para um único elemento  $e$ .

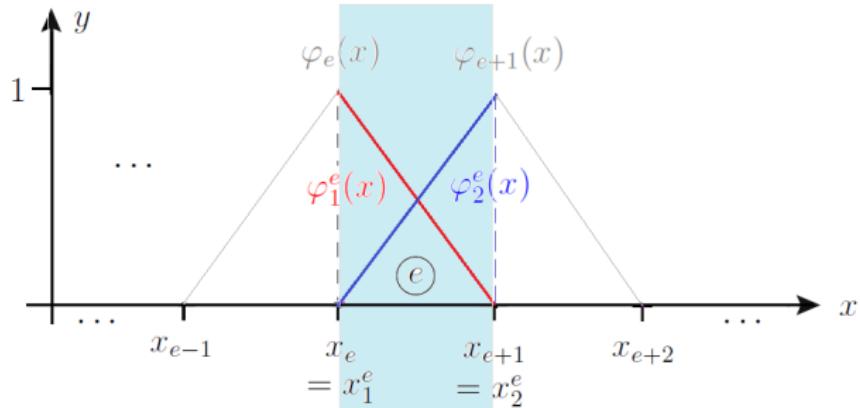


Figura: Funções do elemento  $e$ :  $\varphi_1^e(x)$  e  $\varphi_2^e(x)$

Podemos referenciar um nó da discretização do domínio de duas formas:

- $x_i$ , onde  $i$  é o índice da discretização do intervalo  $[x_1, x_m]$ ,  $i = 1, 2, \dots, m$ . Neste caso,  $i$  é a **numeração global**. Assim,  $x_i$  é o **nó global**.
- $x_a^e$ , onde  $a$  é o índice do nó no elemento finito  $e$ ,  $a = 1, 2$ . Neste caso,  $a$  é a **numeração local**. Assim,  $x_a^e$  é o **nó local**.

# Método dos Elementos Finitos (MEF) - Caso 1D

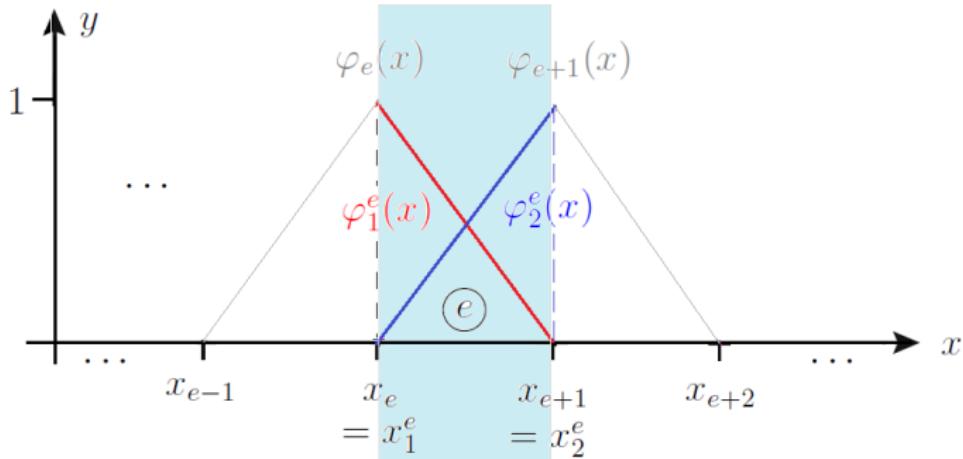


Figura: Funções do elemento  $e$ :  $\varphi_1^e(x)$  e  $\varphi_2^e(x)$

$$\left\{ \begin{array}{l} \varphi_1^e(x) = \frac{x_{e+1} - x}{x_{e+1} - x_e} = \frac{x_{e+1} - x}{x_2^e - x_1^e} = \frac{x_{e+1} - x}{h}, \\ \varphi_2^e(x) = \frac{x - x_e}{x_{e+1} - x_e} = \frac{x - x_e}{x_2^e - x_1^e} = \frac{x - x_e}{h}. \end{array} \right. \quad (1)$$

# Método dos Elementos Finitos (MEF) - Caso 1D

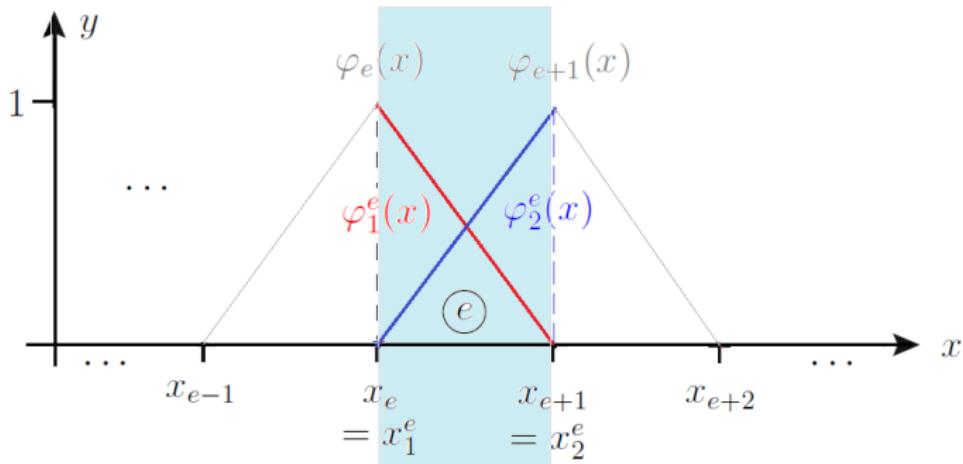


Figura: Funções do elemento  $e$ :  $\varphi_1^e(x)$  e  $\varphi_2^e(x)$

Note que:

$$\begin{cases} \varphi_1^e(x) = \varphi_e(x), \text{ para } x \in [x_e, x_{e+1}], \\ \varphi_2^e(x) = \varphi_{e+1}(x), \text{ para } x \in [x_e, x_{e+1}]. \end{cases} \quad (2)$$

# Matriz local $K^e$

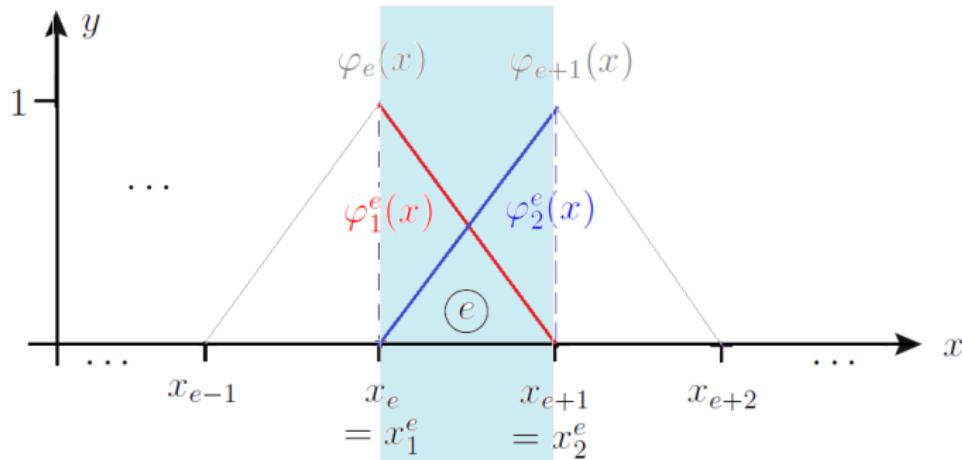


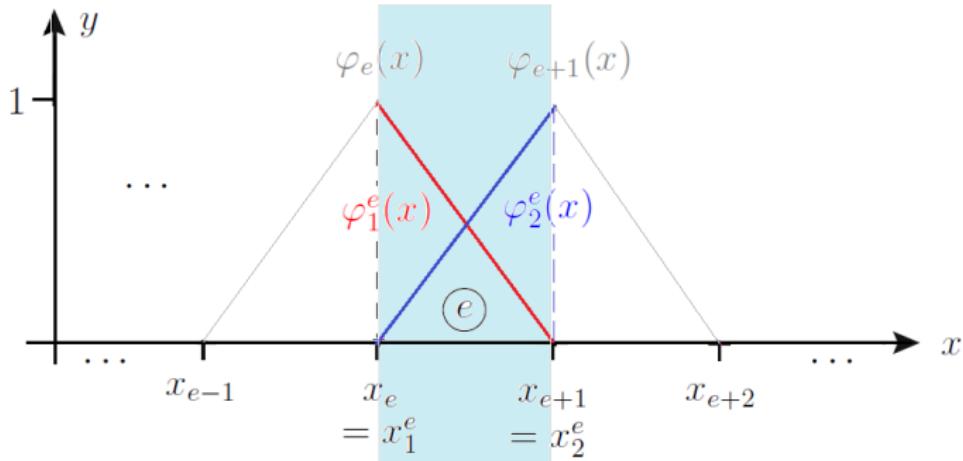
Figura: Funções do elemento  $e$ :  $\varphi_1^e(x)$  e  $\varphi_2^e(x)$

Note que restritos ao elemento finito  $e$ , obtemos o elemento da **matriz local  $K^e$** :

$$K_{ab}^e = \alpha(\varphi_{ax}^e, \varphi_{bx}^e) + \beta(\varphi_a^e, \varphi_b^e) = \alpha \int_{x_1^e}^{x_2^e} \varphi_{ax}^e(x) \varphi_{bx}^e(x) dx + \beta \int_{x_1^e}^{x_2^e} \varphi_a^e(x) \varphi_b^e(x) dx,$$

para  $a, b = 1, 2$ .

# Vetor força local $F^e$



Analogamente, obtemos o elemento do **vetor força local  $F^e$** :

$$F_a^e = (f, \varphi_a^e) = \alpha \int_{x_1^e}^{x_2^e} f(x) \varphi_a^e(x) dx, \text{ para } a = 1, 2.$$

# Montagem da matriz global $K$

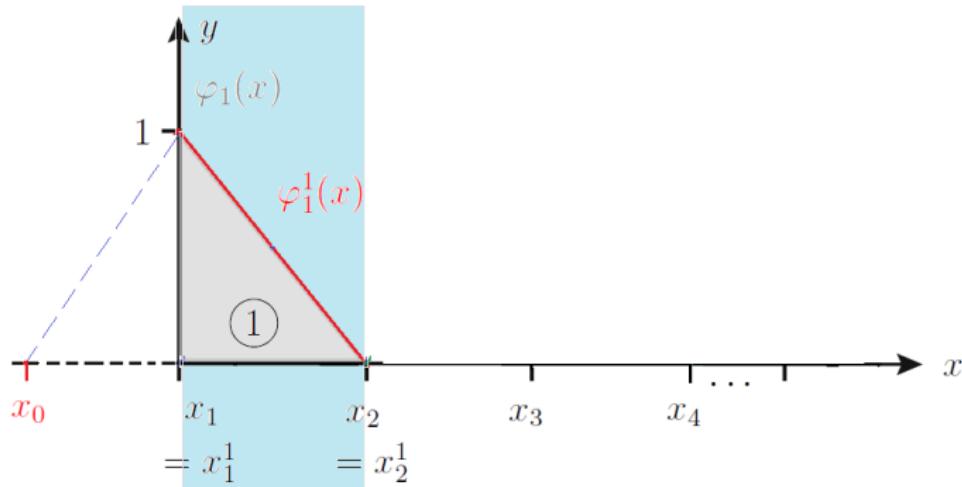
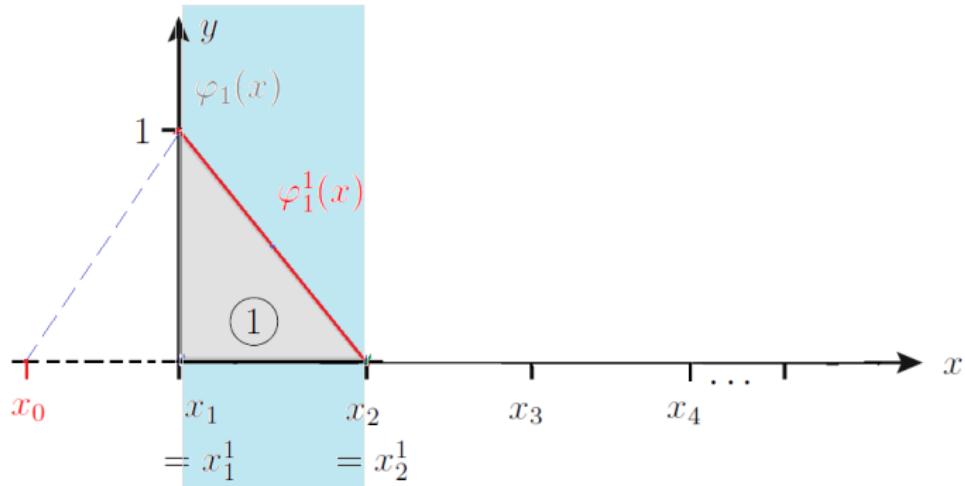


Figura: Funções do elemento e:  $\varphi_1^e(x)$  e  $\varphi_2^e(x)$

Podemos reescrever a **matriz global  $K$**  em termos de elementos da **matriz local  $K^e$** ,  $e = 1, 2, \dots, m$ . Vimos na abordagem anterior que:

$$K_{11} = \alpha(\varphi_{1x}, \varphi_{1x}) + \beta(\varphi_1, \varphi_1) = \alpha \int_{x_1}^{x_2} (\varphi_{1x}(x))^2 dx + \beta \int_{x_1}^{x_2} (\varphi_1(x))^2 dx$$

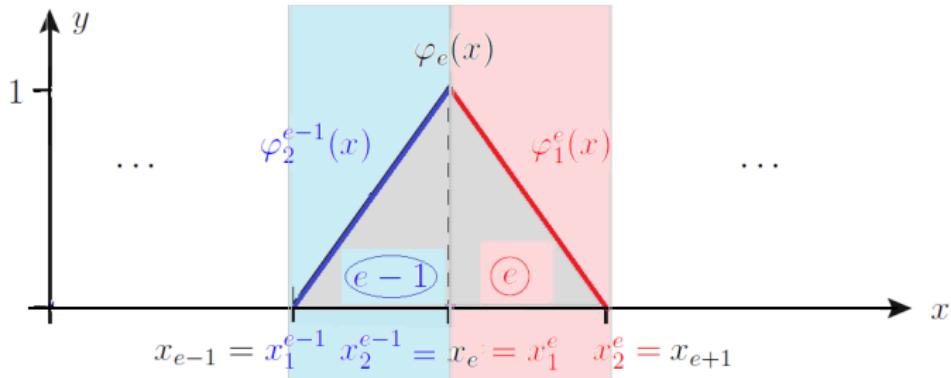
# Montagem da matriz global $K$



Podemos reescrever  $K_{11}$  como:

$$\begin{aligned}
 K_{11} &= \alpha(\varphi_{1x}, \varphi_{1x}) + \beta(\varphi_1, \varphi_1) = \alpha \int_{x_1}^{x_2} (\varphi_{1x}(x))^2 dx + \beta \int_{x_1}^{x_2} (\varphi_1(x))^2 dx \\
 &= \alpha \int_{x_1^1}^{x_2^1} (\varphi_{1x}^1(x))^2 dx + \beta \int_{x_1^1}^{x_2^1} (\varphi_1^1(x))^2 dx = \alpha(\varphi_{1x}^1, \varphi_{1x}^1) + \beta(\varphi_1^1, \varphi_1^1) = K_{11}^1
 \end{aligned}$$

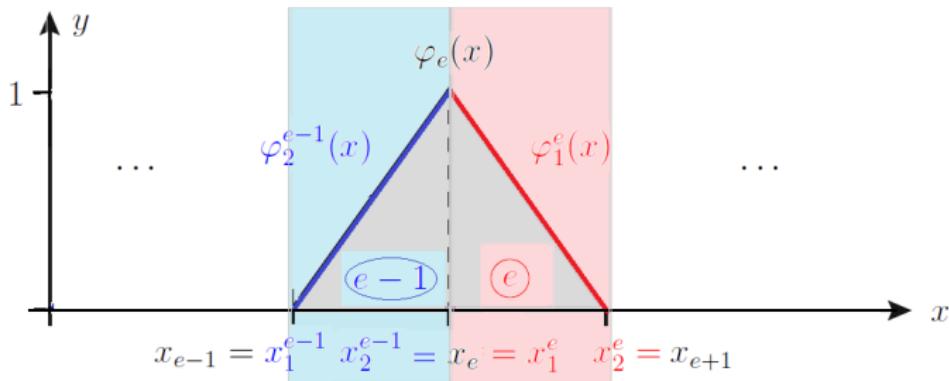
# Montagem da matriz global $K$



Podemos reescrever  $K_{ee}$  como:

$$\begin{aligned}
 K_{ee} &= \alpha(\varphi_{ex}, \varphi_{ex}) + \beta(\varphi_e, \varphi_e) = \alpha \int_{x_{e-1}}^{x_{e+1}} (\varphi_{ex}(x))^2 dx + \beta \int_{x_{e-1}}^{x_{e+1}} (\varphi_e(x))^2 dx \\
 &= \alpha \left( \int_{x_{e-1}}^{x_e} (\varphi_{ex}(x))^2 dx + \int_{x_e}^{x_{e+1}} (\varphi_{ex}(x))^2 dx \right) \\
 &\quad + \beta \left( \int_{x_{e-1}}^{x_e} (\varphi_e(x))^2 dx + \int_{x_e}^{x_{e+1}} (\varphi_e(x))^2 dx \right)
 \end{aligned}$$

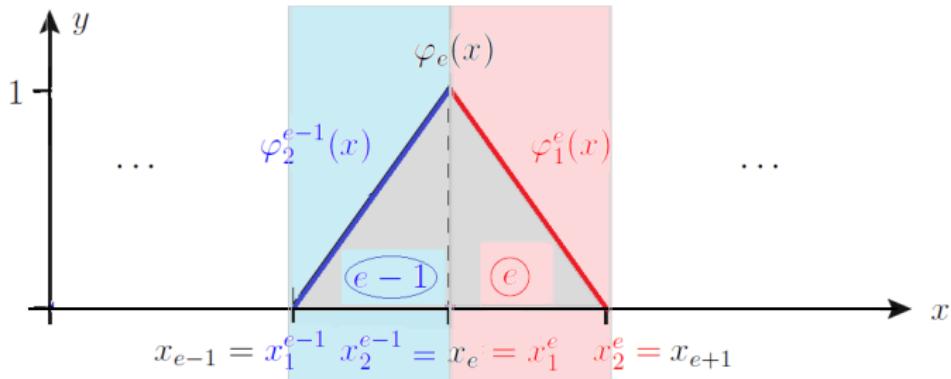
# Montagem da matriz global $K$



Podemos reescrever  $K_{ee}$  como:

$$\begin{aligned}
 K_{ee} = & \alpha \left( \int_{x_1^{e-1}}^{x_2^{e-1}} (\varphi_{2x}^{e-1}(x))^2 dx + \int_{x_1^e}^{x_2^e} (\varphi_{1x}^e(x))^2 dx \right) \\
 & + \beta \left( \int_{x_1^{e-1}}^{x_2^{e-1}} (\varphi_2^{e-1}(x))^2 dx + \int_{x_1^e}^{x_2^e} (\varphi_1^e(x))^2 dx \right)
 \end{aligned}$$

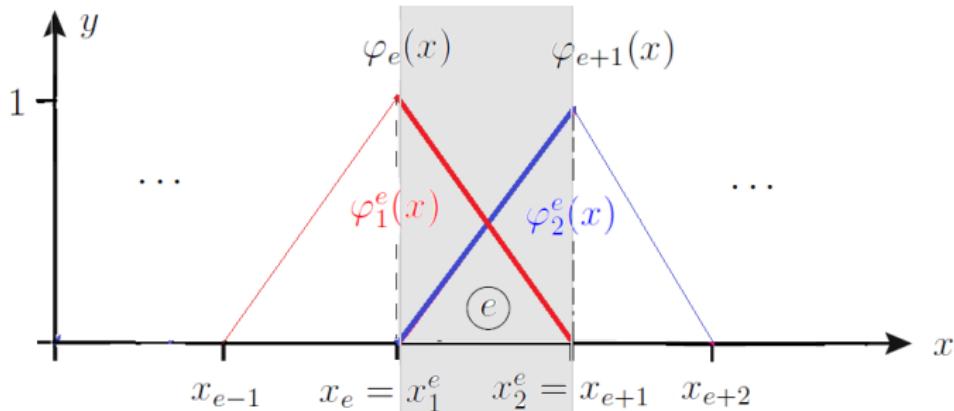
# Montagem da matriz global $K$



Podemos reescrever  $K_{ee}$  como:

$$\begin{aligned}
 K_{ee} &= \alpha \int_{x_1^{e-1}}^{x_2^{e-1}} (\varphi_{2x}^{e-1}(x))^2 dx + \beta \int_{x_1^{e-1}}^{x_2^{e-1}} (\varphi_2^{e-1}(x))^2 dx \\
 &\quad + \alpha \int_{x_1^e}^{x_2^e} (\varphi_{1x}^e(x))^2 dx + \beta \int_{x_1^e}^{x_2^e} (\varphi_1^e(x))^2 dx \\
 &= \alpha(\varphi_{2x}^{e-1}, \varphi_{2x}^{e-1}) + \beta(\varphi_2^{e-1}, \varphi_2^{e-1}) + \alpha(\varphi_{1x}^e, \varphi_{1x}^e) + \beta(\varphi_1^e, \varphi_1^e) = K_{22}^{e-1} + K_{11}^e
 \end{aligned}$$

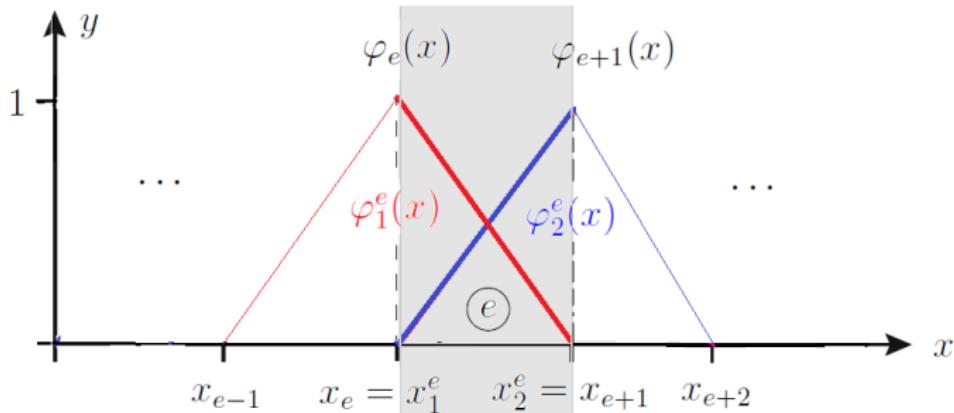
# Montagem da matriz global $K$



Podemos reescrever  $K_{e,e+1}$  como:

$$\begin{aligned}
 K_{e,e+1} &= \alpha(\varphi_{ex}, \varphi_{(e+1)x}) + \beta(\varphi_e, \varphi_{e+1}) \\
 &= \alpha \int_{x_e}^{x_{e+1}} \varphi_{ex}(x) \varphi_{(e+1)x}(x) dx + \beta \int_{x_e}^{x_{e+1}} \varphi_e(x) \varphi_{e+1}(x) dx \\
 &= \alpha \int_{x_1^e}^{x_2^e} \varphi_{1x}^e(x) \varphi_{2x}^e(x) dx + \beta \int_{x_1^e}^{x_2^e} \varphi_1^e(x) \varphi_2^e(x) dx \\
 &= \alpha(\varphi_{1x}^e, \varphi_{2x}^e) + \beta(\varphi_1^e, \varphi_2^e) = K_{12}^e
 \end{aligned}$$

# Montagem da matriz global $K$



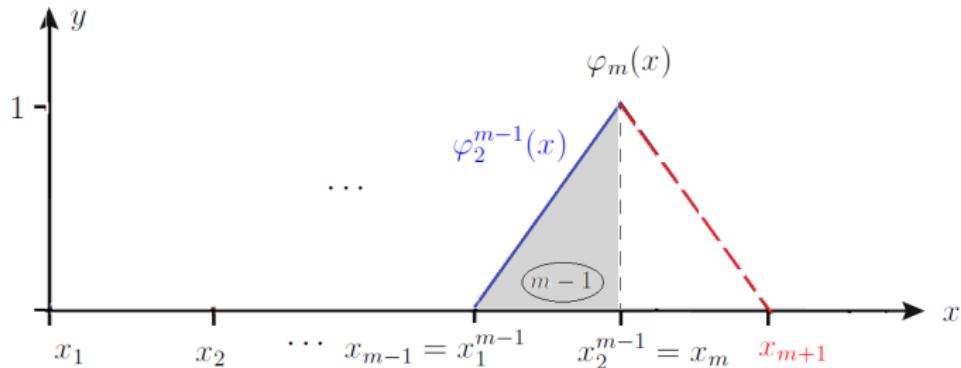
De forma análoga, obtemos:

$$K_{e+1,e} = K_{21}^e$$

Como  $K$  é simétrica ( $K = K^T$ ), logo:

$$K_{12}^e = K_{21}^e$$

# Montagem da matriz global $K$



Podemos reescrever  $K_{mm}$  como:

$$\begin{aligned}
 K_{mm} &= \alpha(\varphi_{mx}, \varphi_{mx}) + \beta(\varphi_m, \varphi_m) \\
 &= \alpha \int_{x_{m-1}}^{x_m} (\varphi_{mx}(x))^2 dx + \beta \int_{x_{m-1}}^{x_m} (\varphi_m(x))^2 dx \\
 &= \alpha \int_{x_1^{m-1}}^{x_2^{m-1}} (\varphi_{2x}^{m-1}(x))^2 dx + \beta \int_{x_1^{m-1}}^{x_2^{m-1}} (\varphi_2^{m-1}(x))^2 dx \\
 &= \alpha(\varphi_{2x}^{m-1}, \varphi_{2x}^{m-1}) + \beta(\varphi_2^{m-1}, \varphi_2^{m-1}) = K_{22}^{m-1}
 \end{aligned}$$

# Montagem do vetor global $F$

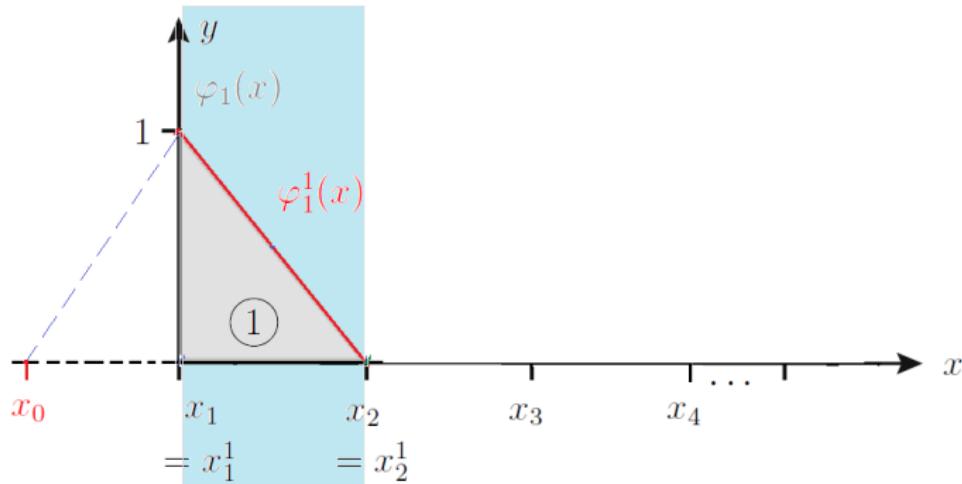
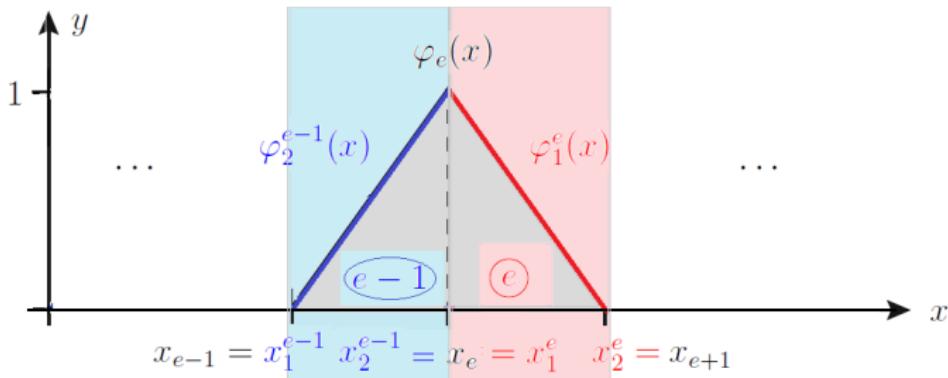


Figura: Funções do elemento  $e$ :  $\varphi_1^e(x)$  e  $\varphi_2^e(x)$

Podemos reescrever a **vetor global**  $F$  em termos de elementos do **vetor local**  $F^e$ ,  $e = 1, 2, \dots, m$ . Vimos na abordagem anterior que:

$$F_1 = (f, \varphi_1) = \int_{x_1}^{x_2} f(x) \varphi_1(x) dx = \int_{x_1^1}^{x_2^1} f(x) \varphi_1^e(x) dx = (f, \varphi_1^e) = F_1^1$$

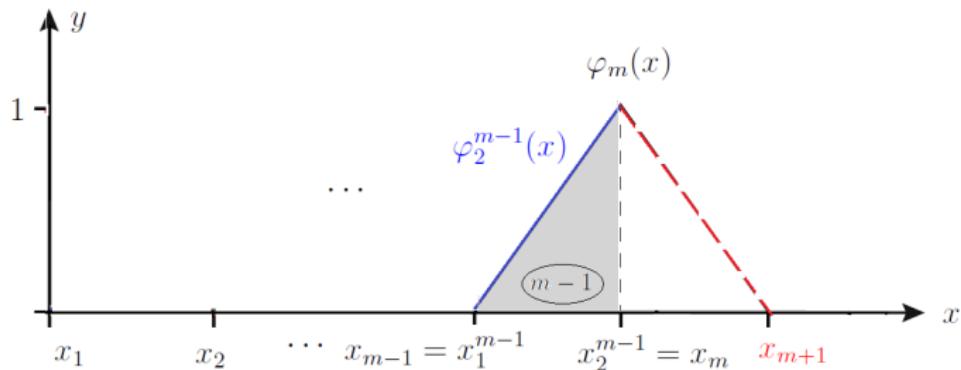
# Montagem do vetor global $F$



Podemos reescrever  $F_e$  como:

$$\begin{aligned}
 F_e &= (f, \varphi_e) = \int_{x_{e-1}}^{x_{e+1}} f(x) \varphi_e(x) dx = \int_{x_{e-1}}^{x_e} f(x) \varphi_e(x) dx + \int_{x_e}^{x_{e+1}} f(x) \varphi_e(x) dx \\
 &= \int_{x_1^{e-1}}^{x_2^{e-1}} f(x) \varphi_2^{e-1}(x) dx + \int_{x_1^e}^{x_2^e} f(x) \varphi_1^e(x) dx = (f, \varphi_2^{e-1}) + (f, \varphi_1^e) = F_2^{e-1} + F_1^e
 \end{aligned}$$

# Montagem do vetor global $F$



Podemos reescrever  $F_m$  como:

$$\begin{aligned}
 F_m = (f, \varphi_m) &= \int_{x_{m-1}}^{x_m} f(x) \varphi_m(x) dx = \int_{x_1^{m-1}}^{x_2^{m-1}} f(x) \varphi_2^{m-1}(x) dx \\
 &= (f, \varphi_2^{m-1}) = F_2^{m-1}
 \end{aligned}$$

# Resultados

**Matriz de rigidez  $K$ :**

$$K_{11} = K_{11}^1;$$

$$K_{ee} = K_{22}^{e-1} + K_{11}^e, \text{ para } e = 2, 3, \dots, m-1; \quad (3)$$

$$K_{e,e+1} = K_{12}^i \Rightarrow K_{e+1,e} = K_{21}^i, \text{ para } e = 1, 2, 3, \dots, m-1; \text{ (**Simetria:** } K = K^T)$$

$$K_{mm} = K_{22}^{m-1}.$$

**Vetor força  $F$ :**

$$F_1 = F_1^1;$$

$$F_e = F_2^{e-1} + F_1^e, \text{ para } i = 2, 3, \dots, m-1;$$

$$F_m = F_2^{m-1}. \quad (4)$$

# Resultados

A matriz de rigidez global  $K$ ,

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ K_{21}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & K_{21}^2 & K_{22}^2 + K_{11}^3 & K_{12}^3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{21}^3 & K_{22}^3 + K_{11}^4 & K_{12}^4 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{21}^4 & K_{22}^4 + K_{11}^5 & K_{12}^5 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & * & * & * & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & * & K_{22}^{m-1} & \end{bmatrix}$$

é montada a partir das **matrizes locais**

$$K^e = \begin{bmatrix} K_{11}^e & K_{12}^e \\ K_{21}^e & K_{22}^e \end{bmatrix},$$

para  $e = 1, 2, \dots, m - 1$ .

# Resultados

O vetor força global,

$$F = \begin{bmatrix} F_1^1 \\ F_2^1 + F_1^2 \\ F_2^2 + F_1^3 \\ F_2^3 + F_1^4 \\ \vdots \\ F_2^{m-2} + F_1^{m-1} \\ F_2^{m-1} \end{bmatrix}$$

é montado a partir dos vetores locais

$$F^e = \begin{bmatrix} F_1^e \\ F_2^e \end{bmatrix},$$

para  $e = 1, 2, \dots, m - 1$ .

# Quadratura gaussiana para $K^e$ e $F^e$

A regra da **Quadratura Gaussiana** para integração numérica consiste em aproximar

$$\int_{-1}^1 g(\xi) d\xi \approx \sum_{i=1}^{npg} w_i \cdot g(\xi_i),$$

onde  $npg$  é o **número de pontos de Gauss**,

$\xi_i$ ,  $i = 1, 2, \dots, npg$ , são os **pontos de Gauss**,

$w_i$ ,  $i = 1, 2, \dots, npg$ , são os **pesos de Gauss**.

Comparação com outros métodos de integração numérica:

Regra de integração numérica	Número de pontos (nós)	Grau do polinômio com integração exata
Trapézios	2	1
Simpson	3	2
<b>Quadratura Gaussiana</b>	<b>2</b>	<b>3</b>
	<b>n</b>	<b>2n - 1</b>

# Quadratura gaussiana para $K^e$ e $F^e$

Com  $n = 2$ , é possível mostrar que:

**Pontos de Gauss:**  $\xi_1 = -\sqrt{3}/3$ ,  $\xi_2 = \sqrt{3}/3$ .

**Pesos de Gauss:**  $w_1 = 1$ ,  $w_2 = 1$ .

**Demonstração:** Se com  $npg = 2$ , o resultado da integração numérica é exato para um polinômio de grau 3,

$$P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3,$$

então,

$$\int_{-1}^1 P_3(x)dx = \sum_{i=1}^2 w_i \cdot P_3(\xi_i) = w_1 \cdot P_3(\xi_1) + w_2 \cdot P_3(\xi_2)$$

$$\begin{aligned} \Rightarrow \int_{-1}^1 (a_0 + a_1x + a_2x^2 + a_3x^3)dx &= w_1 \cdot (a_0 + a_1\xi_1 + a_2\xi_1^2 + a_3\xi_1^3) \\ &\quad + w_2 \cdot (a_0 + a_1\xi_2 + a_2\xi_2^2 + a_3\xi_2^3) \end{aligned}$$

# Quadratura gaussiana para $K^e$ e $F^e$

$$\begin{aligned}
 & \Rightarrow a_0 \int_{-1}^1 dx + a_1 \int_{-1}^1 xdx + a_2 \int_{-1}^1 x^2 dx + a_3 \int_{-1}^1 x^3 dx \\
 & = (a_0 w_1 + a_1 w_1 \xi_1 + a_2 w_1 \xi_1^2 + a_3 w_1 \xi_1^3) \\
 & \quad + (a_0 w_2 + a_1 w_2 \xi_2 + a_2 w_2 \xi_2^2 + a_3 w_2 \xi_2^3) \\
 & = a_0(w_1 + w_2) + a_1(w_1 \xi_1 + w_2 \xi_2) + a_2(w_1 \xi_1^2 + w_2 \xi_2^2) + a_3(w_1 \xi_1^3 + w_2 \xi_2^3)
 \end{aligned}$$

Igualando os coeficientes de  $a_0$ ,  $a_1$ ,  $a_2$ ,  $a_3$ , obtemos:

$$\int_{-1}^1 dx = w_1 + w_2 \Rightarrow w_1 + w_2 = 2;$$

$$\int_{-1}^1 xdx = w_1 \xi_1 + w_2 \xi_2 \Rightarrow w_1 \xi_1 + w_2 \xi_2 = 0;$$

$$\int_{-1}^1 x^2 dx = w_1 \xi_1^2 + w_2 \xi_2^2 \Rightarrow w_1 \xi_1^2 + w_2 \xi_2^2 = 2/3;$$

$$\int_{-1}^1 x^3 dx = w_1 \xi_1^3 + w_2 \xi_2^3 \Rightarrow w_1 \xi_1^3 + w_2 \xi_2^3 = 0.$$

Resolvendo este sistema, obtemos:  $w_1 = 1$ ,  $w_2 = 1$ ,  $\xi_1 = -\sqrt{3}/3$ ,  $\xi_2 = \sqrt{3}/3$ .

# Quadratura gaussiana para $K^e$ e $F^e$

npg	Pontos	Pesos
1	0	2
2	$\pm\sqrt{3}/3 \approx \pm 0.5773502692$	1
3	$0, \pm\sqrt{3}/5 \approx \pm 0.7745966692$	0.8888888889, 0.5555555556
4	$\pm 0.8611363116, \pm 0.3399810436$	0.3478548451, 0.6521451549
5	$\pm 0.9061798459, \pm 0.5384693101, 0$	0.2369268851, 0.4786286705, 0.5688888889
6	$\pm 0.9324695142, \pm 0.6612093865, \pm 0.2386191861$	0.1713244924, 0.3607615730, 0.4679139346
7	$\pm 0.9491079123, \pm 0.7415311856, \pm 0.4058451514, 0$	0.1294849662, 0.2797053915, 0.3818300505, 0.4179591837
8	$\pm 0.9602898565, \pm 0.7966664774, \pm 0.5255324099, \pm 0.1834346425$	0.1012285363, 0.2223810345, 0.3137066459, 0.3626837834

Tabela: Tabela para Quadratura Gaussiana

# Quadratura gaussiana para $K^e$ e $F^e$

Os elementos da **matriz local**  $K^e$  são dados por:

$$K_{ab}^e = a(\varphi_a^e, \varphi_b^e) = \alpha(\varphi_{ax}^e, \varphi_{bx}^e) + \beta(\varphi_a^e, \varphi_b^e)$$

$$= \alpha \int_{x_1^e}^{x_2^e} \varphi_{ax}^e(x) \varphi_{bx}^e(x) dx + \beta \int_{x_1^e}^{x_2^e} \varphi_a^e(x) \varphi_b^e(x) dx, \text{ para } a, b \in \{1, 2\}.$$

Para usar a **quadratura gaussiana** nessas integrais, devemos usar mudança de variáveis do domínio  $[x_1^e, x_2^e]$  para o domínio  $[-1, 1]$ :



As distâncias são proporcionais nas duas retas. Logo,

$$\frac{x - x_1^e}{x_2^e - x_1^e} = \frac{\xi - (-1)}{1 - (-1)} \Rightarrow \frac{x - x_1^e}{h} = \frac{\xi + 1}{2} \Rightarrow x(\xi) = x_1^e + h \left( \frac{\xi + 1}{2} \right)$$

$$\text{Logo, } \frac{dx}{d\xi} = \frac{h}{2} \Rightarrow dx = \frac{h}{2} d\xi.$$

# Quadratura gaussiana para $K^e$ e $F^e$

Fazendo a mudança de variável nas funções locais  $\varphi_a^e(x)$  para  $a = 1, 2$ , obtemos:

$$\varphi_1^e(x) = \frac{x_2^e - x}{x_2^e - x_1^e} = \frac{x_2^e - x}{h}$$

$$\Rightarrow \varphi_1^e(\xi) = \varphi_1^e(x(\xi)) = \frac{x_2^e - x(\xi)}{x_2^e - x_1^e} = \frac{x_2^e - x(\xi)}{h} = \frac{x_2^e - \left(x_1^e + h\left(\frac{\xi+1}{2}\right)\right)}{h}$$

$$\Rightarrow \varphi_1^e(\xi) = \frac{(x_2^e - x_1^e) - h\left(\frac{\xi+1}{2}\right)}{h} = \frac{h - h\left(\frac{\xi+1}{2}\right)}{h} = 1 - \left(\frac{\xi+1}{2}\right) = \frac{1-\xi}{2}$$

De forma análoga, obtemos:

$$\varphi_2^e(x) = \frac{x - x_1^e}{x_2^e - x_1^e} = \frac{x - x_1^e}{h} \Rightarrow \varphi_2^e(\xi) = \frac{1+\xi}{2}$$

# Quadratura gaussiana para $K^e$ e $F^e$

Portanto, as funções da base de Lagrange linear locais no domínio  $\xi \in [-1, 1]$  são:

$$\begin{cases} \varphi_1^e(\xi) = \frac{1 - \xi}{2}, \\ \varphi_2^e(\xi) = \frac{1 + \xi}{2}, \end{cases} \quad (5)$$

e as suas derivadas são:

$$\begin{cases} \varphi_{1\xi}^e(\xi) = \frac{d\varphi_1^e}{d\xi}(\xi) = -\frac{1}{2}, \\ \varphi_{2\xi}^e(\xi) = \frac{d\varphi_2^e}{d\xi}(\xi) = \frac{1}{2}. \end{cases} \quad (6)$$

# Quadratura gaussiana para $K^e$ e $F^e$

Com a mudança de variável, obtemos os elementos de  $K^e$  no domínio  $[-1, 1]$ . Lembrando que ao usar a regra da cadeia, para  $a, b \in \{1, 2\}$ ,

$$\varphi_{ax}(x) = \varphi_{ax}(\xi(x)) = \frac{d\varphi_a(\xi)}{d\xi} \frac{d\xi}{dx} = \varphi_{a\xi}^e(\xi) \frac{d\xi}{dx} = \varphi_{a\xi}^e(\xi) \frac{2}{h}.$$

Logo,

$$\begin{aligned} K_{ab}^e &= a(\varphi_a^e, \varphi_b^e) = \alpha(\varphi_{ax}^e, \varphi_{bx}^e) + \beta(\varphi_a^e, \varphi_b^e) \\ &= \alpha \int_{x_1^e}^{x_2^e} \varphi_{ax}^e(x) \varphi_{bx}^e(x) dx + \beta \int_{x_1^e}^{x_2^e} \varphi_a^e(x) \varphi_b^e(x) dx \\ &= \alpha \int_{-1}^1 \varphi_{a\xi}^e(\xi) \frac{2}{h} \varphi_{b\xi}^e(\xi) \frac{2}{h} \frac{h}{2} d\xi + \beta \int_{-1}^1 \varphi_a^e(\xi) \varphi_b^e(\xi) \frac{h}{2} d\xi \\ &= \frac{2\alpha}{h} \int_{-1}^1 \varphi_{a\xi}^e(\xi) \varphi_{b\xi}^e(\xi) d\xi + \frac{\beta h}{2} \int_{-1}^1 \varphi_a^e(\xi) \varphi_b^e(\xi) d\xi \end{aligned}$$

# Quadratura gaussiana para $K^e$ e $F^e$

Então,

$$\begin{aligned}
 K_{11}^e &= \frac{2\alpha}{h} \int_{-1}^1 (\varphi_{1\xi}^e(\xi))^2 d\xi + \frac{\beta h}{2} \int_{-1}^1 (\varphi_1^e(\xi))^2 d\xi \\
 &= \frac{2\alpha}{h} \int_{-1}^1 \left(-\frac{1}{2}\right)^2 d\xi + \frac{\beta h}{2} \int_{-1}^1 \left(\frac{1}{2}\right)^2 d\xi \\
 &= \frac{\alpha}{2h} \int_{-1}^1 d\xi + \frac{\beta h}{8} \int_{-1}^1 (1-\xi)^2 d\xi
 \end{aligned}$$

Usando **Quadratura Gaussiana** com  $npg = 2$ , temos  $w_1 = w_2 = 1$ ,  $\xi_1 = -\sqrt{3}/3$ ,  $\xi_2 = \sqrt{3}/3$ . Assim, obtemos:

$$\begin{aligned}
 \int_{-1}^1 d\xi &= \sum_{i=1}^2 w_i \cdot g(\xi_i) = w_1 \cdot g(\xi_1) + w_2 \cdot g(\xi_2) \\
 &= 1 \cdot g(-\sqrt{3}/3) + 1 \cdot g(\sqrt{3}/3) = 1 \cdot 1 + 1 \cdot 1 = 2
 \end{aligned}$$

# Quadratura gaussiana para $K^e$ e $F^e$

$$\begin{aligned}
 \int_{-1}^1 (1 - \xi)^2 d\xi &= \sum_{i=1}^2 w_i \cdot g(\xi_i) = w_1 \cdot g(\xi_1) + w_2 \cdot g(\xi_2) \\
 &= 1 \cdot g(-\sqrt{3}/3) + 1 \cdot g(\sqrt{3}/3) \\
 &= 1 \cdot \left(1 - \left(-\frac{\sqrt{3}}{3}\right)\right)^2 + 1 \cdot \left(1 - \left(\frac{\sqrt{3}}{3}\right)\right)^2 \\
 &= \left(1 + \frac{\sqrt{3}}{3}\right)^2 + \left(1 - \frac{\sqrt{3}}{3}\right)^2 = \frac{8}{3}
 \end{aligned}$$

Portanto, os elementos de  $K^e$  são:

$$K_{11}^e = \frac{\alpha}{2h} \cdot 2 + \frac{\beta h}{8} \cdot \frac{8}{3} = \frac{\alpha}{h} + \frac{\beta h}{3}$$

Analogamente, verifique que:

$$K_{12}^e = K_{21}^e = -\frac{\alpha}{h} + \frac{\beta h}{6}, \text{ pois } K \text{ é simétrica,}$$

$$K_{22}^e = \frac{\alpha}{h} + \frac{\beta h}{3}$$

# Quadratura gaussiana para $K^e$ e $F^e$

Assim, temos a **matriz de rigidez local**  $K^e$  também simétrica:

$$K^e = \begin{bmatrix} K_{11}^e & K_{12}^e \\ K_{21}^e & K_{22}^e \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{h} + \frac{\beta h}{3} & -\frac{\alpha}{h} + \frac{\beta h}{6} \\ -\frac{\alpha}{h} + \frac{\beta h}{6} & \frac{\alpha}{h} + \frac{\beta h}{3} \end{bmatrix}$$

- **Cálculo de  $F^e$ :**

Sabendo que na mudança de variável

$$x(\xi) = x_1^e + \frac{h}{2}(\xi + 1) = x_1^e + \frac{x_2^e - x_1^e}{2}(\xi + 1) = x_1^e \left( \frac{1 - \xi}{2} \right) + x_2^e \left( \frac{1 + \xi}{2} \right),$$

temos:

$$\varphi_a^e(x(\xi)) = \varphi_a^e(\xi), \text{ para } a = 1, 2,$$

$$f(x(\xi)) = f \left[ x_1^e \left( \frac{1 - \xi}{2} \right) + x_2^e \left( \frac{1 + \xi}{2} \right) \right],$$

$$\frac{dx}{d\xi} = \frac{h}{2} \Rightarrow dx = \frac{h}{2} d\xi,$$

# Quadratura gaussiana para $K^e$ e $F^e$

para calcular os elementos do vetor local  $F^e$  usando **Quadratura Gaussiana** com  $npg = 2$ , temos:

$$\begin{aligned}
 F_a^e &= (f, \varphi_a^e) = \int_{x_1^e}^{x_2^e} f(x) \varphi_a^e(x) dx = \int_{-1}^1 f(x(\xi)) \varphi_a^e(\xi) \frac{h}{2} d\xi \\
 &= \int_{-1}^1 f \left[ x_1^e \left( \frac{1 - \xi}{2} \right) + x_2^e \left( \frac{1 + \xi}{2} \right) \right] \varphi_a^e(\xi) \frac{h}{2} d\xi \\
 &= \frac{h}{2} \int_{-1}^1 f \left[ x_1^e \left( \frac{1 - \xi}{2} \right) + x_2^e \left( \frac{1 + \xi}{2} \right) \right] \varphi_a^e(\xi) d\xi \\
 &\approx \frac{h}{2} \sum_{i=1}^2 w_i \cdot f \left[ x_1^e \left( \frac{1 - \xi_i}{2} \right) + x_2^e \left( \frac{1 + \xi_i}{2} \right) \right] \varphi_a^e(\xi_i) \\
 &\approx \frac{h}{2} \left\{ f \left[ x_1^e \left( \frac{1 - (-\sqrt{3}/3)}{2} \right) + x_2^e \left( \frac{1 + (-\sqrt{3}/3)}{2} \right) \right] \varphi_a^e \left( -\frac{\sqrt{3}}{3} \right) \right. \\
 &\quad \left. + f \left[ x_1^e \left( \frac{1 - (\sqrt{3}/3)}{2} \right) + x_2^e \left( \frac{1 + (\sqrt{3}/3)}{2} \right) \right] \varphi_a^e \left( \frac{\sqrt{3}}{3} \right) \right\}
 \end{aligned}$$

# Quadratura gaussiana para $K^e$ e $F^e$

Então,

$$\begin{aligned}
 F_1^e = (f, \varphi_1^e) &\approx \frac{h}{2} \left\{ f \left[ x_1^e \left( \frac{1 + (\sqrt{3}/3)}{2} \right) + x_2^e \left( \frac{1 - (\sqrt{3}/3)}{2} \right) \right] \varphi_1^e \left( -\frac{\sqrt{3}}{3} \right) \right. \\
 &\quad \left. + f \left[ x_1^e \left( \frac{1 - (\sqrt{3}/3)}{2} \right) + x_2^e \left( \frac{1 + (\sqrt{3}/3)}{2} \right) \right] \varphi_1^e \left( \frac{\sqrt{3}}{3} \right) \right\} \\
 &\approx \frac{h}{2} \left\{ f \left[ x_1^e \left( \frac{1 + (\sqrt{3}/3)}{2} \right) + x_2^e \left( \frac{1 - (\sqrt{3}/3)}{2} \right) \right] \left( \frac{1 - (\sqrt{3}/3)}{2} \right) \right. \\
 &\quad \left. + f \left[ x_1^e \left( \frac{1 - (\sqrt{3}/3)}{2} \right) + x_2^e \left( \frac{1 + (\sqrt{3}/3)}{2} \right) \right] \left( \frac{1 - (\sqrt{3}/3)}{2} \right) \right\}
 \end{aligned}$$

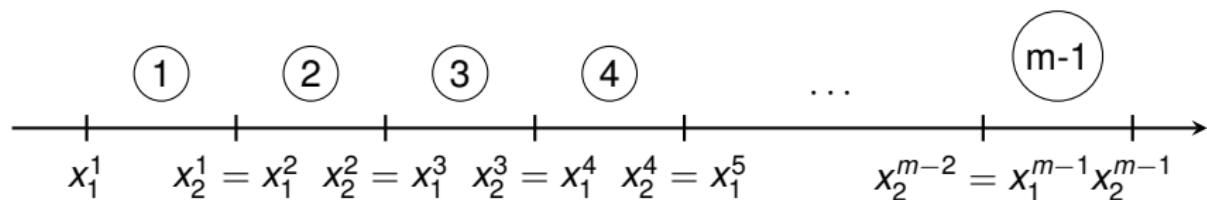
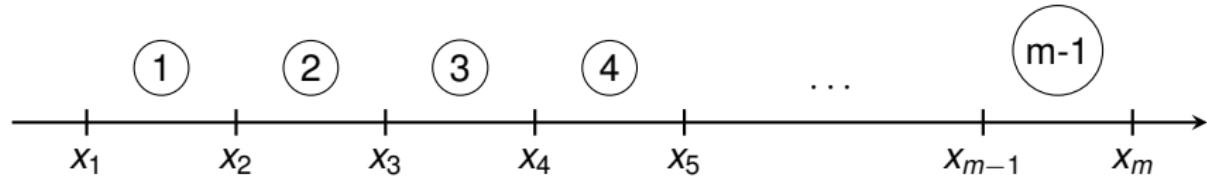
Analogamente,

$$\begin{aligned}
 F_2^e \approx \frac{h}{2} \left\{ f \left[ x_1^e \left( \frac{1 + (\sqrt{3}/3)}{2} \right) + x_2^e \left( \frac{1 - (\sqrt{3}/3)}{2} \right) \right] \left( \frac{1 + (\sqrt{3}/3)}{2} \right) \right. \\
 \left. + f \left[ x_1^e \left( \frac{1 - (\sqrt{3}/3)}{2} \right) + x_2^e \left( \frac{1 + (\sqrt{3}/3)}{2} \right) \right] \left( \frac{1 + (\sqrt{3}/3)}{2} \right) \right\}
 \end{aligned}$$

Assim, temos o **vetor força local**  $F^e = [F_1^e \quad F_2^e]^T$ .

# Condições de contorno - Estruturas de dados

**Matriz LG:** Matriz que associa cada nó local do elemento e ao seu respectivo nó global



a	e	1	2	3	4	...	m-1
1		1	2	3	4	...	$m-1$
2		2	3	4	5	...	$m$

Tabela: Matriz LG

# Condições de contorno - Estruturas de dados

**Vetor EQ:** Vetor com as posições das equações de acordo com as condições de contorno.

As posições das colunas são as posições dos nós globais: 1 para  $x_1$ , 2 para  $x_2$ , ...,  $m$  para  $x_m$ .

O vetor será preenchido de maneira a respeitar as condições de contorno.

**Condição 1:**  $u(x_1, t) = u(0, t) = 0 \Rightarrow EQ(1) = 0$ .

**Condição 2:**  $u(x_m, t) = u(1, t) = 0 \Rightarrow EQ(m) = 0$ .

1	2	3	4	...	$m - 1$	$m$
0	1	2	3	...	$m - 2$	0

Tabela: Vetor EQ

## Condições de contorno

$$\left[ \begin{array}{cccccccccccccc} * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{array} \right]$$

Por exemplo, nessa matriz global  $m \times m$  onde  $nel = 16$  e  $m = nel + 1 = 17$ .

Como

$$EQ(LG(1, 1)) = EQ(1) = 0;$$

$$EQ(LG(2, m - 1)) = EQ(m) = 0,$$

então nem a primeira linha e a primeira coluna, nem a  $m$ -ésima linha e a  $m$ -ésima coluna entram nos cálculos da matriz global.

# Condições de contorno

$$\left[ \begin{array}{cccccccccccccc} * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \end{array} \right]$$

Com a remoção das linhas e colunas, a matriz global se torna uma matriz  $(m - 2) \times (m - 2)$ . Neste exemplo, com  $nel = 16$ , se transforma numa matriz  $15 \times 15$ .

# Referências I

-  Liu, I.S.; Rincon, M.A.. **Introdução ao Método de Elementos Finitos, Análise e Aplicação.** IM/UFRJ, 2003.