

Tópicos Especiais em Matemática Aplicada - 2025-1 UERJ

05 - Metodo dos Elementos Finitos - Caso 1D estacionário

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Github: <https://github.com/rodrigolrmadureira/ElementosFinitos>

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Método dos Elementos Finitos (MEF) - Caso 1D

No caso unidimensional, cada subintervalo $[x_e, x_{e+1}]$, $e = 1, 2, \dots, m-1$, do domínio $\Omega = [x_1, x_m]$ é um elemento finito de tamanho $h = x_{e+1} - x_e$.

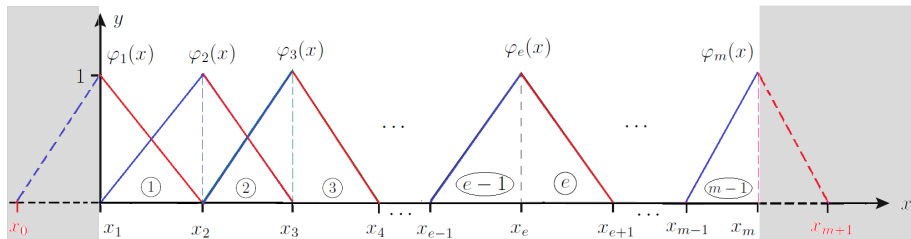


Figura: Funções de base de Lagrange linear $\varphi_i(x)$

Método dos Elementos Finitos (MEF) - Caso 1D

Ao invés de "olhar" para todo o domínio, vamos "olhar" para um único elemento e .

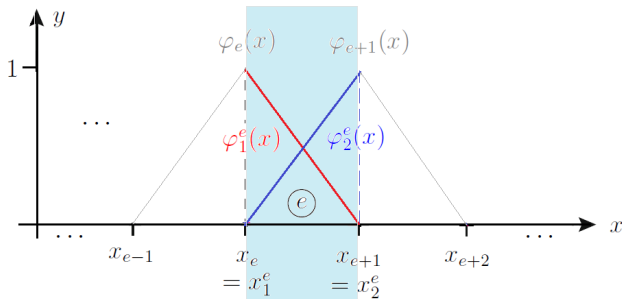


Figura: Funções do elemento e : $\varphi_1^e(x)$ e $\varphi_2^e(x)$

Podemos referenciar um nó da discretização do domínio de duas formas:

- x_i , onde i é o índice da discretização do intervalo $[x_1, x_m]$, $i = 1, 2, \dots, m$. Neste caso, i é a **numeração global**. Assim, x_i é o **nó global**.
- x_α^e , onde α é o índice do nó no elemento finito e , $\alpha = 1, 2$. Neste caso, α é a **numeração local**. Assim, x_α^e é o **nó local**.

Método dos Elementos Finitos (MEF) - Caso 1D

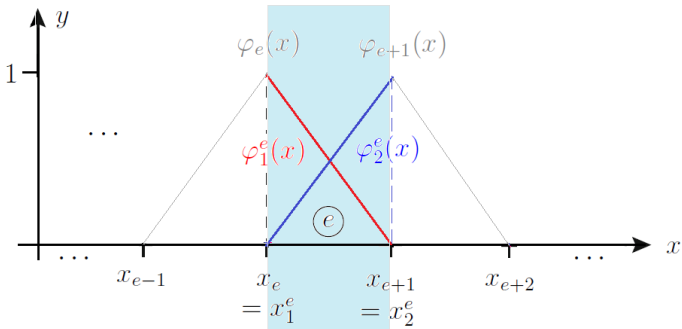


Figura: Funções do elemento e: $\varphi_1^e(x)$ e $\varphi_2^e(x)$

$$\begin{cases} \varphi_1^e(x) = \frac{x_{e+1} - x}{x_{e+1} - x_e} = \frac{x_{e+1} - x}{x_2^e - x_1^e} = \frac{x_{e+1} - x}{h}, \\ \varphi_2^e(x) = \frac{x - x_e}{x_{e+1} - x_e} = \frac{x - x_e}{x_2^e - x_1^e} = \frac{x - x_e}{h}. \end{cases} \quad (1)$$

Método dos Elementos Finitos (MEF) - Caso 1D

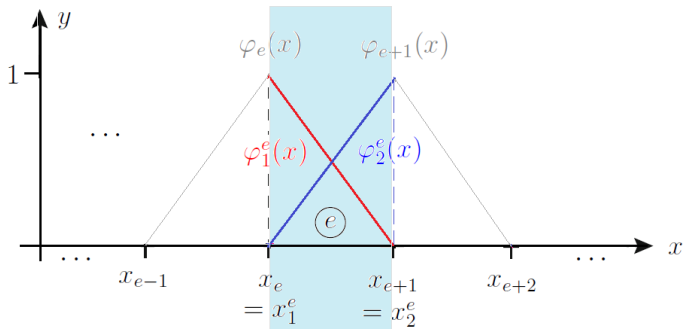


Figura: Funções do elemento e: $\varphi_1^e(x)$ e $\varphi_2^e(x)$

Note que:

$$\begin{cases} \varphi_1^e(x) = \varphi_e(x), \text{ para } x \in [x_e, x_{e+1}], \\ \varphi_2^e(x) = \varphi_{e+1}(x), \text{ para } x \in [x_e, x_{e+1}]. \end{cases} \quad (2)$$

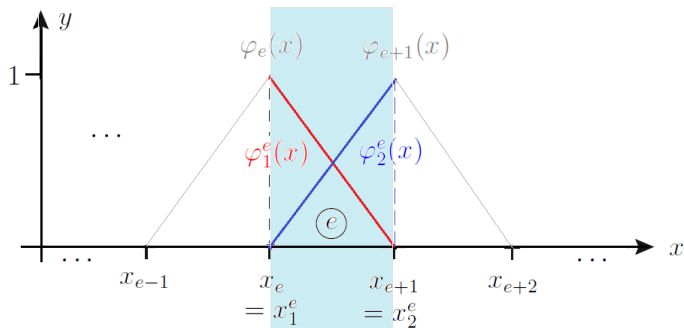
Matriz local K^e 

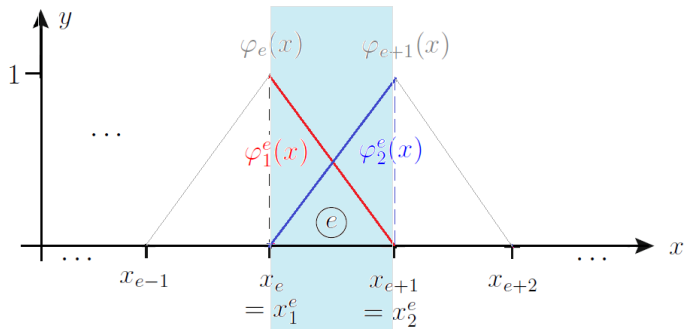
Figura: Funções do elemento e: $\varphi_1^e(x)$ e $\varphi_2^e(x)$

Note que restritos ao elemento finito e, obtemos o elemento da **matriz local** K^e :

$$K_{ab}^e = \alpha(\varphi_{ax}^e, \varphi_{bx}^e) + \beta(\varphi_a^e, \varphi_b^e) = \alpha \int_{x_1^e}^{x_2^e} \varphi_{ax}^e(x) \varphi_{bx}^e(x) dx + \beta \int_{x_1^e}^{x_2^e} \varphi_a^e(x) \varphi_b^e(x) dx,$$

para $a, b = 1, 2$.

Vetor força local F^e



Analogamente, obtemos o elemento do **vetor força local** F^e :

$$F_a^e = (f, \varphi_a^e) = \alpha \int_{x_1^e}^{x_2^e} f(x) \varphi_a^e(x) dx, \text{ para } a = 1, 2.$$

Montagem da matriz global K

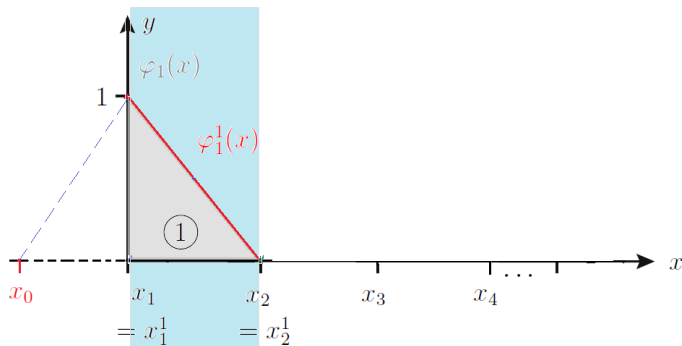
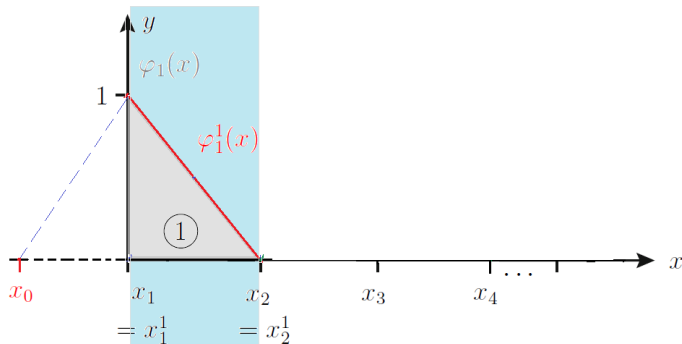


Figura: Funções do elemento e : $\varphi_1^e(x)$ e $\varphi_2^e(x)$

Podemos reescrever a **matriz global** K em termos de elementos da **matriz local** K^e , $e = 1, 2, \dots, m$. Vimos na abordagem anterior que:

$$K_{11} = \alpha(\varphi_{1x}, \varphi_{1x}) + \beta(\varphi_1, \varphi_1) = \alpha \int_{x_1}^{x_2} (\varphi_{1x}(x))^2 dx + \beta \int_{x_1}^{x_2} (\varphi_1(x))^2 dx$$

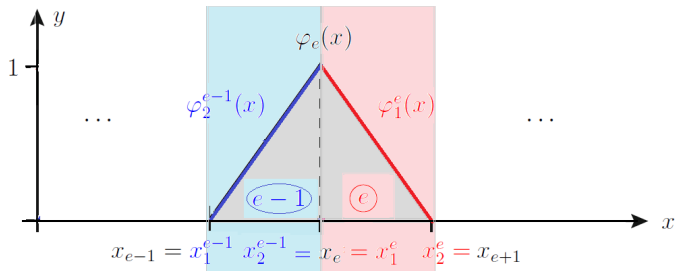
Montagem da matriz global K



Podemos reescrever K_{11} como:

$$\begin{aligned}
 K_{11} &= \alpha(\varphi_{1x}, \varphi_{1x}) + \beta(\varphi_1, \varphi_1) = \alpha \int_{x_1}^{x_2} (\varphi_{1x}(x))^2 dx + \beta \int_{x_1}^{x_2} (\varphi_1(x))^2 dx \\
 &= \alpha \int_{x_1^1}^{x_2^1} (\varphi_{1x}^1(x))^2 dx + \beta \int_{x_1^1}^{x_2^1} (\varphi_1^1(x))^2 dx = \alpha(\varphi_{1x}^1, \varphi_{1x}^1) + \beta(\varphi_1^1, \varphi_1^1) = K_{11}^1
 \end{aligned}$$

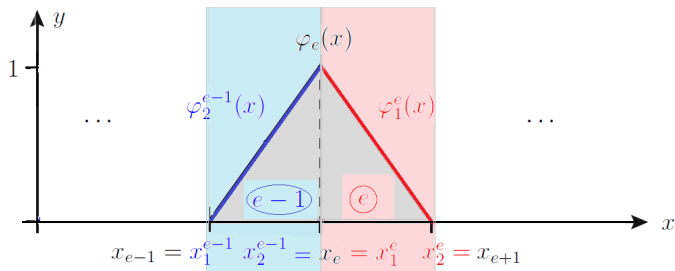
Montagem da matriz global K



Podemos reescrever K_{ee} como:

$$\begin{aligned}
 K_{ee} &= \alpha(\varphi_{ex}, \varphi_{ex}) + \beta(\varphi_e, \varphi_e) = \alpha \int_{x_{e-1}}^{x_{e+1}} (\varphi_{ex}(x))^2 dx + \beta \int_{x_{e-1}}^{x_{e+1}} (\varphi_e(x))^2 dx \\
 &= \alpha \left(\int_{x_{e-1}}^{x_e} (\varphi_{ex}(x))^2 dx + \int_{x_e}^{x_{e+1}} (\varphi_{ex}(x))^2 dx \right) \\
 &\quad + \beta \left(\int_{x_{e-1}}^{x_e} (\varphi_e(x))^2 dx + \int_{x_e}^{x_{e+1}} (\varphi_e(x))^2 dx \right)
 \end{aligned}$$

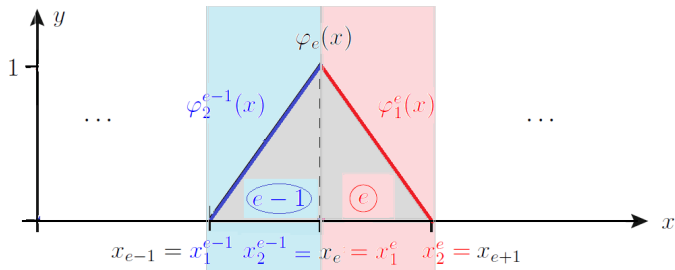
Montagem da matriz global K



Podemos reescrever K_{ee} como:

$$K_{ee} = \alpha \left(\int_{x_1^{e-1}}^{x_2^{e-1}} (\varphi_{2x}^{e-1}(x))^2 dx + \int_{x_1^e}^{x_2^e} (\varphi_{1x}^e(x))^2 dx \right) + \beta \left(\int_{x_1^{e-1}}^{x_2^{e-1}} (\varphi_2^{e-1}(x))^2 dx + \int_{x_1^e}^{x_2^e} (\varphi_1^e(x))^2 dx \right)$$

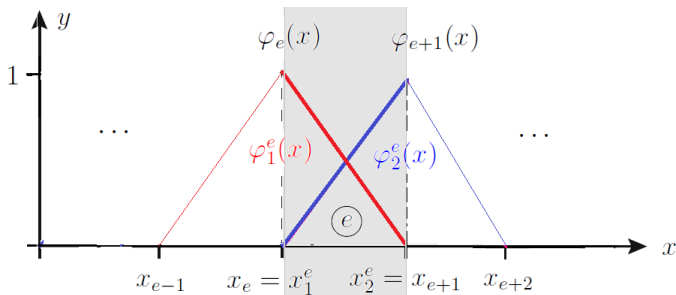
Montagem da matriz global K



Podemos reescrever K_{ee} como:

$$\begin{aligned}
 K_{ee} &= \alpha \int_{x_1^{e-1}}^{x_2^{e-1}} (\varphi_{2x}^{e-1}(x))^2 dx + \beta \int_{x_1^{e-1}}^{x_2^{e-1}} (\varphi_2^{e-1}(x))^2 dx \\
 &\quad + \alpha \int_{x_1^e}^{x_2^e} (\varphi_{1x}^e(x))^2 dx + \beta \int_{x_1^e}^{x_2^e} (\varphi_1^e(x))^2 dx \\
 &= \alpha(\varphi_{2x}^{e-1}, \varphi_{2x}^{e-1}) + \beta(\varphi_2^{e-1}, \varphi_2^{e-1}) + \alpha(\varphi_{1x}^e, \varphi_{1x}^e) + \beta(\varphi_1^e, \varphi_1^e) = K_{22}^{e-1} + K_{11}^e
 \end{aligned}$$

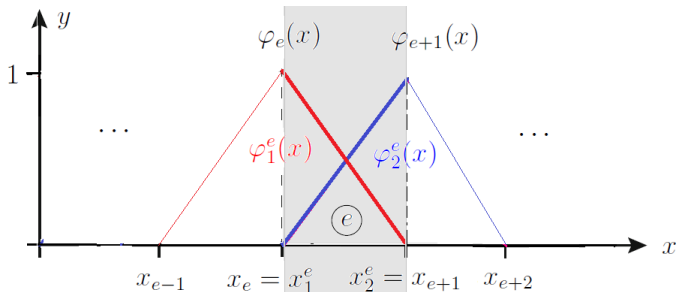
Montagem da matriz global K



Podemos reescrever $K_{e,e+1}$ como:

$$\begin{aligned}
 K_{e,e+1} &= \alpha(\varphi_{ex}, \varphi_{(e+1)x}) + \beta(\varphi_e, \varphi_{e+1}) \\
 &= \alpha \int_{x_e}^{x_{e+1}} \varphi_{ex}(x) \varphi_{(e+1)x}(x) dx + \beta \int_{x_e}^{x_{e+1}} \varphi_e(x) \varphi_{e+1}(x) dx \\
 &= \alpha \int_{x_1^e}^{x_2^e} \varphi_{1x}^e(x) \varphi_{2x}^e(x) dx + \beta \int_{x_1^e}^{x_2^e} \varphi_1^e(x) \varphi_2^e(x) dx \\
 &= \alpha(\varphi_{1x}^e, \varphi_{2x}^e) + \beta(\varphi_1^e, \varphi_2^e) = K_{12}^e
 \end{aligned}$$

Montagem da matriz global K



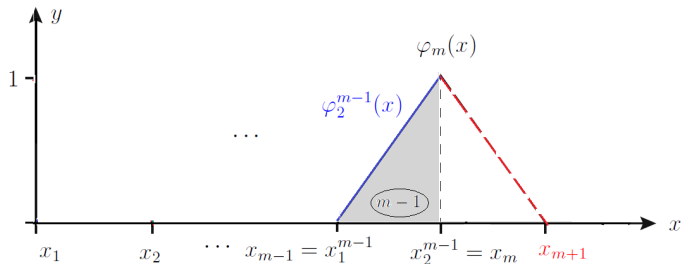
De forma análoga, obtemos:

$$K_{e+1,e} = K_{21}^e$$

Como K é simétrica ($K = K^T$), logo:

$$K_{12}^e = K_{21}^e$$

Montagem da matriz global K



Podemos reescrever K_{mm} como:

$$\begin{aligned}
 K_{mm} &= \alpha(\varphi_{mx}, \varphi_{mx}) + \beta(\varphi_m, \varphi_m) \\
 &= \alpha \int_{x_{m-1}}^{x_m} (\varphi_{mx}(x))^2 dx + \beta \int_{x_{m-1}}^{x_m} (\varphi_m(x))^2 dx \\
 &= \alpha \int_{x_1^{m-1}}^{x_2^{m-1}} (\varphi_{2x}^{m-1}(x))^2 dx + \beta \int_{x_1^{m-1}}^{x_2^{m-1}} (\varphi_2^{m-1}(x))^2 dx \\
 &= \alpha(\varphi_{2x}^{m-1}, \varphi_{2x}^{m-1}) + \beta(\varphi_2^{m-1}, \varphi_2^{m-1}) = K_{22}^{m-1}
 \end{aligned}$$

Montagem do vetor global F

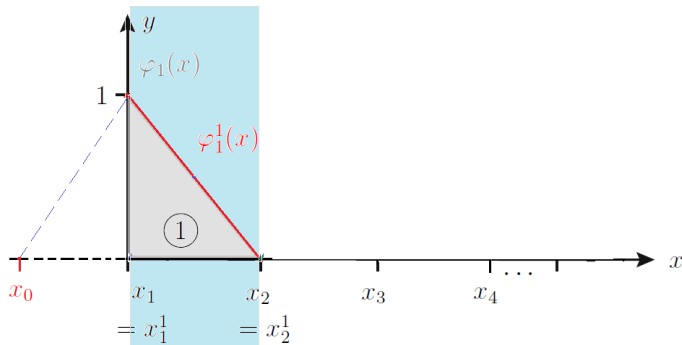
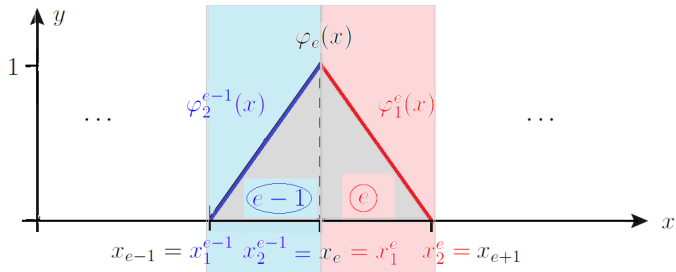


Figura: Funções do elemento e : $\varphi_1^e(x)$ e $\varphi_2^e(x)$

Podemos reescrever a **vetor global** F em termos de elementos do **vetor local** F^e , $e = 1, 2, \dots, m$. Vimos na abordagem anterior que:

$$F_1 = (f, \varphi_1) = \int_{x_1}^{x_2} f(x) \varphi_1(x) dx = \int_{x_1^1}^{x_2^1} f(x) \varphi_1^1(x) dx = (f, \varphi_1^1) = F_1^1$$

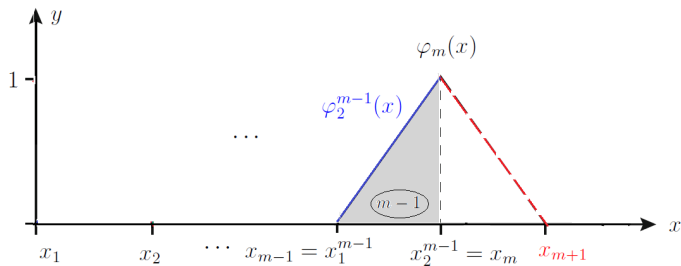
Montagem do vetor global F



Podemos reescrever F_e como:

$$\begin{aligned}
 F_e = (f, \varphi_e) &= \int_{x_{e-1}}^{x_{e+1}} f(x) \varphi_e(x) dx = \int_{x_{e-1}}^{x_e} f(x) \varphi_e(x) dx + \int_{x_e}^{x_{e+1}} f(x) \varphi_e(x) dx \\
 &= \int_{x_1^{e-1}}^{x_2^{e-1}} f(x) \varphi_2^{e-1}(x) dx + \int_{x_1^e}^{x_2^e} f(x) \varphi_1^e(x) dx = (f, \varphi_2^{e-1}) + (f, \varphi_1^e) = F_2^{e-1} + F_1^e
 \end{aligned}$$

Montagem do vetor global F



Podemos reescrever F_m como:

$$\begin{aligned}
 F_m = (f, \varphi_m) &= \int_{x_{m-1}}^{x_m} f(x) \varphi_m(x) dx = \int_{x_1^{m-1}}^{x_2^{m-1}} f(x) \varphi_2^{m-1}(x) dx \\
 &= (f, \varphi_2^{m-1}) = F_2^{m-1}
 \end{aligned}$$

Resultados

Matriz de rigidez K:

$$K_{11} = K_{11}^1;$$

$$K_{ee} = K_{22}^{e-1} + K_{11}^e, \text{ para } e = 2, 3, \dots, m-1; \quad (3)$$

$$K_{e,e+1} = K_{12}^i \Rightarrow K_{e+1,e} = K_{21}^i, \text{ para } e = 1, 2, 3, \dots, m-1; \text{ (**Simetria: } K = K^T \text{)}**$$

$$K_{mm} = K_{22}^{m-1}.$$

Vetor força F:

$$F_1 = F_1^1;$$

$$F_e = F_2^{e-1} + F_1^e, \text{ para } i = 2, 3, \dots, m-1;$$

$$F_m = F_2^{m-1}. \quad (4)$$

Resultados

A matriz de rigidez global K,

$$\begin{bmatrix}
 K_{11}^1 & K_{12}^1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
 K_{21}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
 0 & K_{21}^2 & K_{22}^2 + K_{11}^3 & K_{12}^3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
 0 & 0 & K_{21}^3 & K_{22}^3 + K_{11}^4 & K_{12}^4 & 0 & \cdots & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & K_{21}^4 & K_{22}^4 + K_{11}^5 & K_{12}^5 & \cdots & 0 & 0 & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & * & * & * \\
 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & * & K_{22}^{m-1}
 \end{bmatrix}$$

é montada a partir das **matrizes locais**

$$K^e = \begin{bmatrix} K_{11}^e & K_{12}^e \\ K_{21}^e & K_{22}^e \end{bmatrix},$$

para $e = 1, 2, \dots, m - 1$.

Resultados

O **vetor força global**,

$$F = \begin{bmatrix} F_1^1 \\ F_2^1 + F_1^2 \\ F_2^2 + F_1^3 \\ F_2^3 + F_1^4 \\ \vdots \\ F_2^{m-2} + F_1^{m-1} \\ F_2^{m-1} \end{bmatrix}$$

é montado a partir dos **vetores locais**

$$F^e = \begin{bmatrix} F_1^e \\ F_2^e \end{bmatrix},$$

para $e = 1, 2, \dots, m - 1$.

Quadratura gaussiana para K^e e F^e

A regra da **Quadratura Gaussiana** para integração numérica consiste em aproximar

$$\int_{-1}^1 g(\xi) d\xi \approx \sum_{i=1}^{npg} w_i \cdot g(\xi_i),$$

onde npg é o **número de pontos de Gauss**,

ξ_i , $i = 1, 2, \dots, npg$, são os **pontos de Gauss**,

w_i , $i = 1, 2, \dots, npg$, são os **pesos de Gauss**.

Comparação com outros métodos de integração numérica:

Regra de integração numérica	Número de pontos (nós)	Grau do polinômio com integração exata
Trapézios	2	1
Simpson	3	2
Quadratura Gaussiana	2	3
	n	$2n - 1$

Quadratura gaussiana para K^e e F^e

Com $n = 2$, é possível mostrar que:

Pontos de Gauss: $\xi_1 = -\sqrt{3}/3$, $\xi_2 = \sqrt{3}/3$.

Pesos de Gauss: $w_1 = 1$, $w_2 = 1$.

Demonstração: Se com $npg = 2$, o resultado da integração numérica é exato para um polinômio de grau 3,

$$P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3,$$

então,

$$\int_{-1}^1 P_3(x) dx = \sum_{i=1}^2 w_i \cdot P_3(\xi_i) = w_1 \cdot P_3(\xi_1) + w_2 \cdot P_3(\xi_2)$$

$$\Rightarrow \int_{-1}^1 (a_0 + a_1x + a_2x^2 + a_3x^3) dx = w_1 \cdot (a_0 + a_1\xi_1 + a_2\xi_1^2 + a_3\xi_1^3) + w_2 \cdot (a_0 + a_1\xi_2 + a_2\xi_2^2 + a_3\xi_2^3)$$

Quadratura gaussiana para K^e e F^e

$$\begin{aligned}
 &\Rightarrow a_0 \int_{-1}^1 dx + a_1 \int_{-1}^1 x dx + a_2 \int_{-1}^1 x^2 dx + a_3 \int_{-1}^1 x^3 dx \\
 &= (a_0 w_1 + a_1 w_1 \xi_1 + a_2 w_1 \xi_1^2 + a_3 w_1 \xi_1^3) \\
 &\quad + (a_0 w_2 + a_1 w_2 \xi_2 + a_2 w_2 \xi_2^2 + a_3 w_2 \xi_2^3) \\
 &= a_0 (w_1 + w_2) + a_1 (w_1 \xi_1 + w_2 \xi_2) + a_2 (w_1 \xi_1^2 + w_2 \xi_2^2) + a_3 (w_1 \xi_1^3 + w_2 \xi_2^3)
 \end{aligned}$$

Igualando os coeficientes de a_0 , a_1 , a_2 , a_3 , obtemos:

$$\begin{aligned}
 \int_{-1}^1 dx &= w_1 + w_2 \Rightarrow w_1 + w_2 = 2; \\
 \int_{-1}^1 x dx &= w_1 \xi_1 + w_2 \xi_2 \Rightarrow w_1 \xi_1 + w_2 \xi_2 = 0; \\
 \int_{-1}^1 x^2 dx &= w_1 \xi_1^2 + w_2 \xi_2^2 \Rightarrow w_1 \xi_1^2 + w_2 \xi_2^2 = 2/3; \\
 \int_{-1}^1 x^3 dx &= w_1 \xi_1^3 + w_2 \xi_2^3 \Rightarrow w_1 \xi_1^3 + w_2 \xi_2^3 = 0.
 \end{aligned}$$

Resolvendo este sistema, obtemos: $w_1 = 1$, $w_2 = 1$, $\xi_1 = -\sqrt{3}/3$, $\xi_2 = \sqrt{3}/3$.

Quadratura gaussiana para K^e e F^e

npg	Pontos	Pesos
1	0	2
2	$\pm\sqrt{3}/3 \approx \pm 0.5773502692$	1
3	$0, \pm\sqrt{3/5} \approx \pm 0.7745966692$	0.8888888889, 0.5555555556
4	$\pm 0.8611363116, \pm 0.3399810436$	0.3478548451, 0.6521451549
5	$\pm 0.9061798459, \pm 0.5384693101, 0$	0.2369268851, 0.4786286705, 0.5688888889
6	$\pm 0.9324695142, \pm 0.6612093865, \pm 0.2386191861$	0.1713244924, 0.3607615730, 0.4679139346
7	$\pm 0.9491079123, \pm 0.7415311856, \pm 0.4058451514, 0$	0.1294849662, 0.2797053915, 0.3818300505, 0.4179591837
8	$\pm 0.9602898565, \pm 0.7966664774, \pm 0.5255324099, \pm 0.1834346425$	0.1012285363, 0.2223810345, 0.3137066459, 0.3626837834

Tabela: Tabela para Quadratura Gaussiana

Quadratura gaussiana para K^e e F^e

Os elementos da **matriz local** K^e são dados por:

$$\begin{aligned} K_{ab}^e &= a(\varphi_a^e, \varphi_b^e) = \alpha(\varphi_{ax}^e, \varphi_{bx}^e) + \beta(\varphi_a^e, \varphi_b^e) \\ &= \alpha \int_{x_1^e}^{x_2^e} \varphi_{ax}^e(x) \varphi_{bx}^e(x) dx + \beta \int_{x_1^e}^{x_2^e} \varphi_a^e(x) \varphi_b^e(x) dx, \text{ para } a, b \in \{1, 2\}. \end{aligned}$$

Para usar a **quadratura gaussiana** nessas integrais, devemos usar mudança de variáveis do domínio $[x_1^e, x_2^e]$ para o domínio $[-1, 1]$:



As distâncias são proporcionais nas duas retas. Logo,

$$\frac{x - x_1^e}{x_2^e - x_1^e} = \frac{\xi - (-1)}{1 - (-1)} \Rightarrow \frac{x - x_1^e}{h} = \frac{\xi + 1}{2} \Rightarrow x(\xi) = x_1^e + h \left(\frac{\xi + 1}{2} \right)$$

$$\text{Logo, } \frac{dx}{d\xi} = \frac{h}{2} \Rightarrow dx = \frac{h}{2} d\xi.$$

Quadratura gaussiana para K^e e F^e

Fazendo a mudança de variável nas funções locais $\varphi_\alpha^e(x)$ para $\alpha = 1, 2$, obtemos:

$$\varphi_1^e(x) = \frac{x_2^e - x}{x_2^e - x_1^e} = \frac{x_2^e - x}{h}$$

$$\Rightarrow \varphi_1^e(\xi) = \varphi_1^e(x(\xi)) = \frac{x_2^e - x(\xi)}{x_2^e - x_1^e} = \frac{x_2^e - x(\xi)}{h} = \frac{x_2^e - \left(x_1^e + h\left(\frac{\xi+1}{2}\right)\right)}{h}$$

$$\Rightarrow \varphi_1^e(\xi) = \frac{(x_2^e - x_1^e) - h\left(\frac{\xi+1}{2}\right)}{h} = \frac{h - h\left(\frac{\xi+1}{2}\right)}{h} = 1 - \left(\frac{\xi+1}{2}\right) = \frac{1-\xi}{2}$$

De forma análoga, obtemos:

$$\varphi_2^e(x) = \frac{x - x_1^e}{x_2^e - x_1^e} = \frac{x - x_1^e}{h} \Rightarrow \varphi_2^e(\xi) = \frac{1+\xi}{2}$$

Quadratura gaussiana para K^e e F^e

Portanto, as funções da base de Lagrange linear locais no domínio $\xi \in [-1, 1]$ são:

$$\begin{cases} \varphi_1^e(\xi) = \frac{1 - \xi}{2}, \\ \varphi_2^e(\xi) = \frac{1 + \xi}{2}, \end{cases} \quad (5)$$

e as suas derivadas são:

$$\begin{cases} \varphi_{1\xi}^e(\xi) = \frac{d\varphi_1^e}{d\xi}(\xi) = -\frac{1}{2}, \\ \varphi_{2\xi}^e(\xi) = \frac{d\varphi_2^e}{d\xi}(\xi) = \frac{1}{2}. \end{cases} \quad (6)$$

Quadratura gaussiana para K^e e F^e

Com a mudança de variável, obtemos os elementos de K^e no domínio $[-1, 1]$. Lembrando que ao usar a regra da cadeia, para $a, b \in \{1, 2\}$,

$$\varphi_{ax}(x) = \varphi_{ax}(\xi(x)) = \frac{d\varphi_a(\xi)}{d\xi} \frac{d\xi}{dx} = \varphi_{a\xi}^e(\xi) \frac{d\xi}{dx} = \varphi_{a\xi}^e(\xi) \frac{2}{h}.$$

Logo,

$$\begin{aligned} K_{ab}^e &= a(\varphi_a^e, \varphi_b^e) = \alpha(\varphi_{ax}^e, \varphi_{bx}^e) + \beta(\varphi_a^e, \varphi_b^e) \\ &= \alpha \int_{x_1^e}^{x_2^e} \varphi_{ax}^e(x) \varphi_{bx}^e(x) \, dx + \beta \int_{x_1^e}^{x_2^e} \varphi_a^e(x) \varphi_b^e(x) \, dx \\ &= \alpha \int_{-1}^1 \varphi_{a\xi}^e(\xi) \frac{2}{h} \varphi_{b\xi}^e(\xi) \frac{2}{h} \frac{h}{2} d\xi + \beta \int_{-1}^1 \varphi_a^e(\xi) \varphi_b^e(\xi) \frac{h}{2} d\xi \\ &= \frac{2\alpha}{h} \int_{-1}^1 \varphi_{a\xi}^e(\xi) \varphi_{b\xi}^e(\xi) \, d\xi + \frac{\beta h}{2} \int_{-1}^1 \varphi_a^e(\xi) \varphi_b^e(\xi) \, d\xi \end{aligned}$$

Quadratura gaussiana para K^e e F^e

Então,

$$\begin{aligned} K_{11}^e &= \frac{2\alpha}{h} \int_{-1}^1 (\varphi_{1\xi}^e(\xi))^2 d\xi + \frac{\beta h}{2} \int_{-1}^1 (\varphi_1^e(\xi))^2 d\xi \\ &= \frac{2\alpha}{h} \int_{-1}^1 \left(-\frac{1}{2}\right)^2 d\xi + \frac{\beta h}{2} \int_{-1}^1 \left(\frac{1}{2}\right)^2 d\xi \\ &= \frac{\alpha}{2h} \int_{-1}^1 d\xi + \frac{\beta h}{8} \int_{-1}^1 (1 - \xi)^2 d\xi \end{aligned}$$

Usando **Quadratura Gaussiana** com $n_{pg} = 2$, temos $w_1 = w_2 = 1$, $\xi_1 = -\sqrt{3}/3$, $\xi_2 = \sqrt{3}/3$. Assim, obtemos:

$$\begin{aligned} \int_{-1}^1 d\xi &= \sum_{i=1}^2 w_i \cdot \xi_i = w_1 \cdot g(\xi_1) + w_2 \cdot g(\xi_2) \\ &= 1 \cdot g(-\sqrt{3}/3) + 1 \cdot g(\sqrt{3}/3) = 1 \cdot 1 + 1 \cdot 1 = 2 \end{aligned}$$

Quadratura gaussiana para K^e e F^e

$$\begin{aligned}
 \int_{-1}^1 (1 - \xi)^2 d\xi &= \sum_{i=1}^2 w_i \cdot \xi_i = w_1 \cdot g(\xi_1) + w_2 \cdot g(\xi_2) \\
 &= 1 \cdot g(-\sqrt{3}/3) + 1 \cdot g(\sqrt{3}/3) \\
 &= 1 \cdot \left(1 - \left(-\frac{\sqrt{3}}{3}\right)\right)^2 + 1 \cdot \left(1 - \left(\frac{\sqrt{3}}{3}\right)\right)^2 \\
 &= \left(1 + \frac{\sqrt{3}}{3}\right)^2 + \left(1 - \frac{\sqrt{3}}{3}\right)^2 = \frac{8}{3}
 \end{aligned}$$

Portanto, os elementos de K^e são:

$$K_{11}^e = \frac{\alpha}{2h} \cdot 2 + \frac{\beta h}{8} \cdot \frac{8}{3} = \frac{\alpha}{h} + \frac{\beta h}{3}$$

Analogamente, verifique que:

$$\begin{aligned}
 K_{12}^e &= K_{21}^e = -\frac{\alpha}{h} + \frac{\beta h}{6}, \text{ pois } K \text{ é simétrica,} \\
 K_{22}^e &= \frac{\alpha}{h} + \frac{\beta h}{3}
 \end{aligned}$$

Quadratura gaussiana para K^e e F^e

Assim, temos a **matriz de rigidez local** K^e também **simétrica**:

$$K^e = \begin{bmatrix} K_{11}^e & K_{12}^e \\ K_{21}^e & K_{22}^e \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{h} + \frac{\beta h}{3} & -\frac{\alpha}{h} + \frac{\beta h}{6} \\ -\frac{\alpha}{h} + \frac{\beta h}{6} & \frac{\alpha}{h} + \frac{\beta h}{3} \end{bmatrix}$$

Para calcular os elementos do vetor local F^e usando **Quadratura Gaussiana** com $npg = 2$, temos:

$$\begin{aligned} F_a^e &= (f, \varphi_a^e) = \int_{x_1^e}^{x_2^e} f(x) \varphi_a^e(x) dx = \int_{-1}^1 f(\xi) \varphi_a^e(\xi) \frac{h}{2} d\xi \\ &= \frac{h}{2} \int_{-1}^1 f(\xi) \varphi_a^e(\xi) d\xi \approx \frac{h}{2} \sum_{i=1}^2 w_i \cdot f(\xi_i) \varphi_a^e(\xi_i) \\ &\approx \frac{h}{2} \left[1 \cdot f\left(-\frac{\sqrt{3}}{3}\right) \varphi_a^e\left(-\frac{\sqrt{3}}{3}\right) + 1 \cdot f\left(\frac{\sqrt{3}}{3}\right) \varphi_a^e\left(\frac{\sqrt{3}}{3}\right) \right] \\ &\approx \frac{h}{2} \left[f\left(-\frac{\sqrt{3}}{3}\right) \varphi_a^e\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) \varphi_a^e\left(\frac{\sqrt{3}}{3}\right) \right] \end{aligned}$$

Quadratura gaussiana para K^e e F^e

Então,

$$\begin{aligned} F_1^e &= (f, \varphi_1^e) \approx \frac{h}{2} \left[f\left(-\frac{\sqrt{3}}{3}\right) \left(\frac{1 - (-\sqrt{3}/3)}{2}\right) + f\left(\frac{\sqrt{3}}{3}\right) \left(\frac{1 - (\sqrt{3}/3)}{2}\right) \right] \\ &\approx \frac{h}{4} \left[f\left(-\frac{\sqrt{3}}{3}\right) \left(1 + \frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) \left(1 - \frac{\sqrt{3}}{3}\right) \right] \end{aligned}$$

Analogamente,

$$\begin{aligned} F_2^e &\approx \frac{h}{2} \left[f\left(-\frac{\sqrt{3}}{3}\right) \varphi_2^e\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) \varphi_2^e\left(\frac{\sqrt{3}}{3}\right) \right] \\ &\approx \frac{h}{2} \left[f\left(-\frac{\sqrt{3}}{3}\right) \left(\frac{1 + (-\sqrt{3}/3)}{2}\right) + f\left(\frac{\sqrt{3}}{3}\right) \left(\frac{1 + (\sqrt{3}/3)}{2}\right) \right] \\ &\approx \frac{h}{4} \left[f\left(-\frac{\sqrt{3}}{3}\right) \left(1 - \frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) \left(1 + \frac{\sqrt{3}}{3}\right) \right] \end{aligned}$$

Os resultados vão depender do cálculo de $f(x)$.

Quadratura gaussiana para K^e e F^e

Assim, temos o **vetor força local** F^e :

$$F^e = \begin{bmatrix} F_1^e \\ F_2^e \end{bmatrix} = \begin{bmatrix} \frac{h}{4} \left[f\left(-\frac{\sqrt{3}}{3}\right) \left(1 + \frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) \left(1 - \frac{\sqrt{3}}{3}\right) \right] \\ \frac{h}{4} \left[f\left(-\frac{\sqrt{3}}{3}\right) \left(1 - \frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) \left(1 + \frac{\sqrt{3}}{3}\right) \right] \end{bmatrix}$$

Referências I



Liu, I.S.; Rincon, M.A.. **Introdução ao Método de Elementos Finitos, Análise e Aplicação**. IM/UFRJ, 2003.