Tópicos Especiais em Matemática Aplicada - 2025-1
UERJ

10 - Caso 2D - Elementos Isoparamétricos

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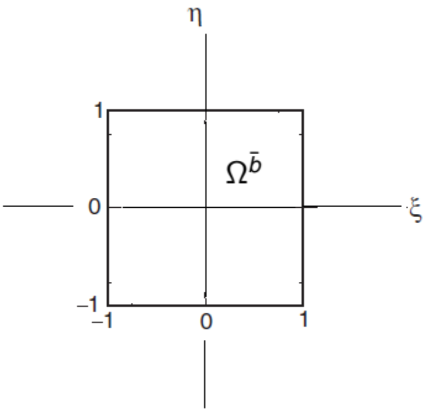
Github: https://github.com/rodrigolrmadureira/ElementosFinitos

Sumário

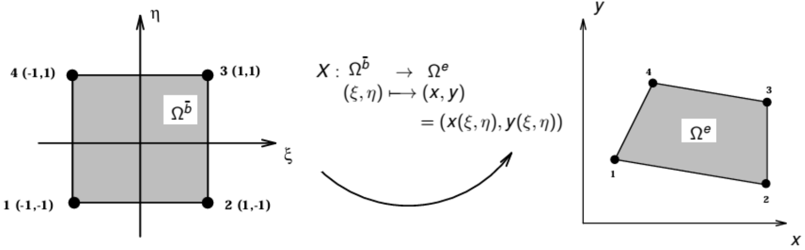
Elementos isoparamétricos

Agora, vamos ver como são calculados K^e e F^e para cada elemento Ω^e.

Seja $\Omega^{\bar{b}} = [-1, 1] \times [-1, 1]$ o elemento finito biunitário como representado na figura abaixo.



Elementos isoparamétricos



Seja

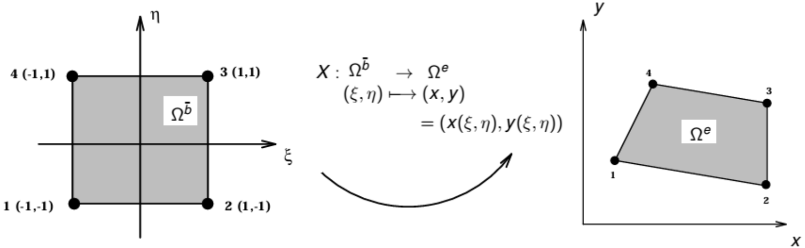
$$X\colon \Omega^{\bar{b}} \longrightarrow \Omega^e$$
$$(\xi,\eta) \longmapsto (x,y) = (x(\xi,\eta), y(\xi,\eta)),$$

onde

$$x(\xi,\eta) = \sum_{a=1}^{4} \varphi_a^{\bar{b}}(\xi,\eta)\cdot x_a^e; \qquad y(\xi,\eta) = \sum_{a=1}^{4} \varphi_a^{\bar{b}}(\xi,\eta)\cdot y_a^e;$$

$$\varphi_a^{\bar{b}}(\xi_c,\eta_c) = \begin{cases} 1, & \text{se } a = c, \\ 0, & \text{se } a \neq c. \end{cases}$$

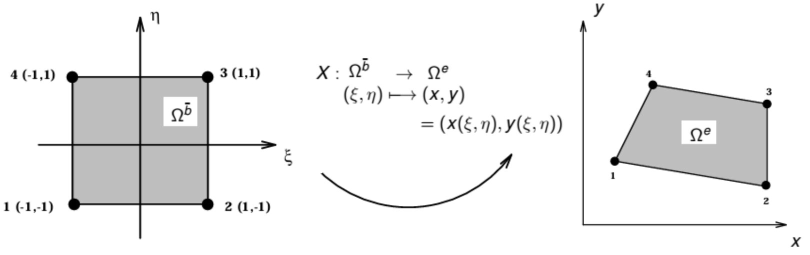
Elementos isoparamétricos



Por exemplo,

$$
\begin{aligned}
x(-1,-1) = x(\xi_1,\eta_1) &= \sum_{a=1}^{4} \varphi_a^{\bar{b}}(\xi_1,\eta_1)\cdot x_a^e \\
&= \underbrace{\varphi_1^{\bar{b}}(\xi_1,\eta_1)}_{1}\cdot x_1^e + \cancelto{0}{\varphi_2^{\bar{b}}(\xi_1,\eta_1)}\cdot x_2^e + \cancelto{0}{\varphi_3^{\bar{b}}(\xi_1,\eta_1)}\cdot x_3^e + \cancelto{0}{\varphi_4^{\bar{b}}(\xi_1,\eta_1)}\cdot x_4^e \\
&= x_1^e
\end{aligned}
$$

Elementos isoparamétricos



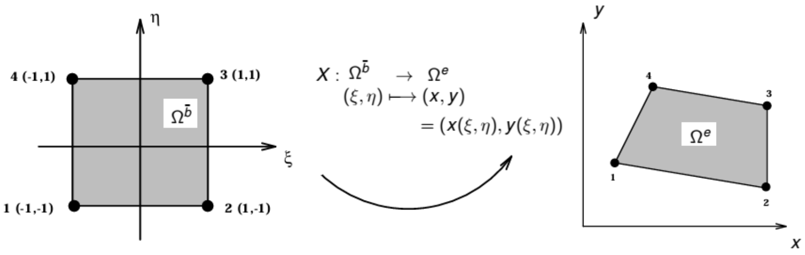
Analogamente,

$$\begin{aligned} x(1,-1) &= x(\xi_2,\eta_2) = x_2^e; \\ x(1,1) &= x(\xi_3,\eta_3) = x_3^e; \\ x(-1,1) &= x(\xi_4,\eta_4) = x_4^e \end{aligned}$$

Ou seja, para todo $a = 1,2,3,4$,

$$x(\xi_a,\eta_a) = x_a^e; \quad y(\xi_a,\eta_a) = y_a^e$$

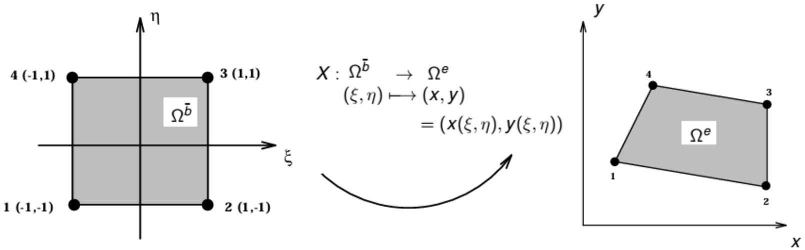
Elementos isoparamétricos



Vamos assumir que $x(\xi,\eta)$ e $y(\xi,\eta)$ são lineares em ξ e η, ou seja:

$$x(\xi,\eta) = a_1 + a_2\xi + a_3\eta + a_4\xi\eta$$
$$y(\xi,\eta) = b_1 + b_2\xi + b_3\eta + b_4\xi\eta$$

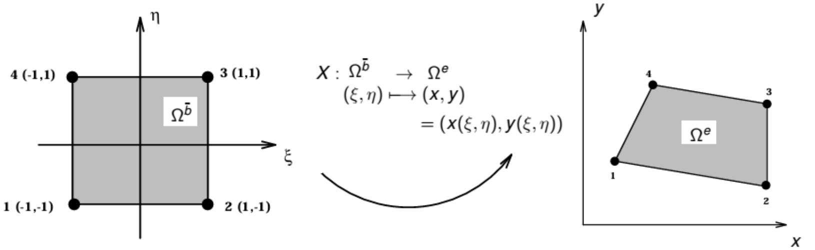
Elementos isoparamétricos



Então,

$$
\begin{aligned}
x(-1,-1) = x_1^e &\Rightarrow a_1 - a_2 - a_3 + a_4 = x_1^e \\
x(1,-1) = x_2^e &\Rightarrow a_1 + a_2 - a_3 - a_4 = x_2^e \\
x(1,1) = x_3^e &\Rightarrow a_1 + a_2 + a_3 + a_4 = x_3^e \\
x(-1,1) = x_4^e &\Rightarrow a_1 - a_2 + a_3 - a_4 = x_4^e
\end{aligned}
$$

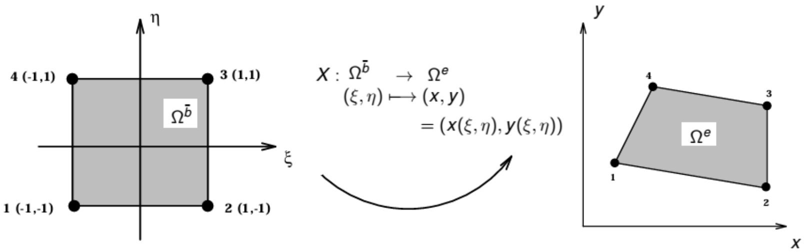
Elementos isoparamétricos



Resolvendo o sistema:

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{bmatrix}$$

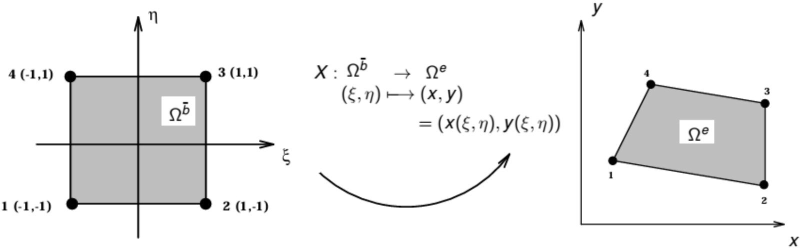
Elementos isoparamétricos



encontramos:

$$
\begin{aligned}
a_1 &= (x_1^e + x_2^e + x_3^e + x_4^e)/4; \\
a_2 &= (-x_1^e + x_2^e + x_3^e - x_4^e)/4; \\
a_3 &= (-x_1^e - x_2^e + x_3^e + x_4^e)/4; \\
a_4 &= (x_1^e - x_2^e + x_3^e - x_4^e)/4
\end{aligned}
$$

Elementos isoparamétricos



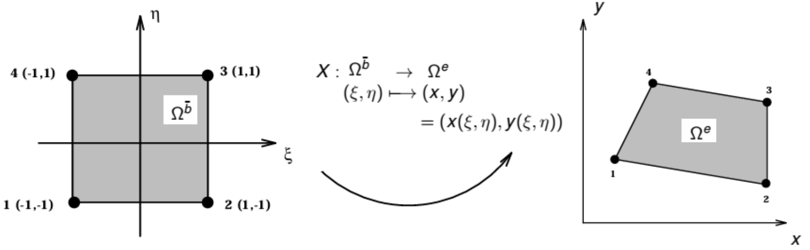
$$\begin{aligned} X:\ \Omega^{\bar{b}} &\to \Omega^e \\ (\xi,\eta) &\longmapsto (x,y) \\ &= (x(\xi,\eta), y(\xi,\eta)) \end{aligned}$$

Substituindo a_1, a_2, a_3, a_4 em $x(\xi,\eta)$ e arrumando os termos, obtemos:

$$x(\xi,\eta) = \underbrace{\frac{1}{4}(1-\xi)(1-\eta)}_{\varphi_1^{\bar{b}}(\xi,\eta)} x_1^e + \underbrace{\frac{1}{4}(1+\xi)(1-\eta)}_{\varphi_2^{\bar{b}}(\xi,\eta)} x_2^e$$

$$+ \underbrace{\frac{1}{4}(1+\xi)(1+\eta)}_{\varphi_3^{\bar{b}}(\xi,\eta)} x_3^e + \underbrace{\frac{1}{4}(1-\xi)(1+\eta)}_{\varphi_4^{\bar{b}}(\xi,\eta)} x_4^e$$

Elementos isoparamétricos



Assim, temos as quatro funções de base para Q_4:

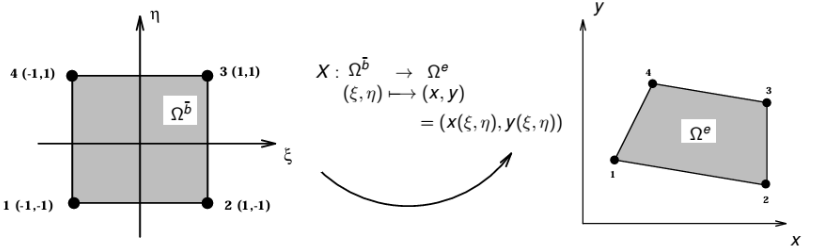
$$\varphi_1^{\bar{b}}(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta);$$

$$\varphi_2^{\bar{b}}(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta);$$

$$\varphi_3^{\bar{b}}(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta);$$

$$\varphi_4^{\bar{b}}(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta)$$

Elementos isoparamétricos



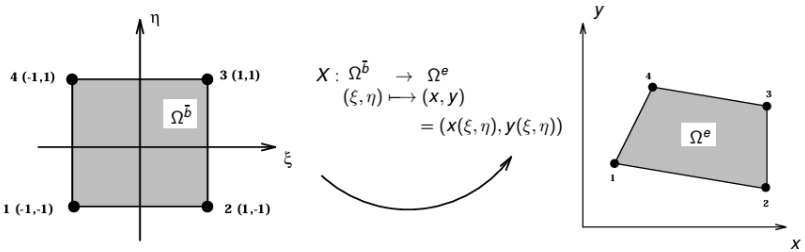
Assim, escrevemos a solução aproximada u_h^e em cada elemento Ω^e como:

$$u_h^e(x,y) = \sum_{a=1}^{4} c_a^e \cdot \varphi_a^e(x,y)$$

Com a mudança de variáveis $(x,y) \longmapsto (\xi,\eta)$, obtemos:

$$u_h^e(x,y) = u_h^{\bar{b}}(\xi,\eta) = \sum_{a=1}^{4} c_a^{\bar{b}} \cdot \varphi_a^{\bar{b}}(\xi,\eta)$$

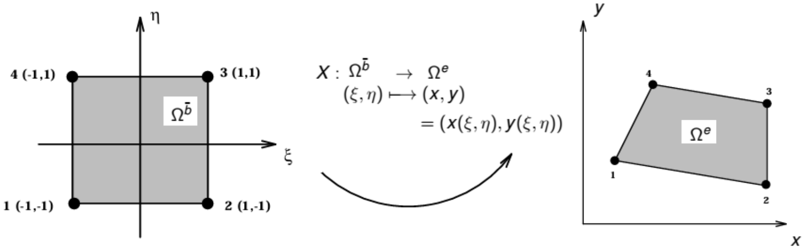
Elemento da matriz local K^e



Também temos para cada elemento Ω^e, ou seja, para todo a, $b = 1, 2, 3, 4$:

$$K^e_{ab} = a(\varphi^e_a, \varphi^e_b) = (\nabla\varphi^e_a, k \cdot \nabla\varphi^e_b) = \int_{\Omega^e} \nabla\varphi^e_a \cdot k \cdot \nabla\varphi^e_b \, dxdy$$

Elemento do vetor local F^e



Também temos para cada elemento Ω^e, ou seja, para todo $a = 1, 2, 3, 4$:

$$
\begin{aligned}
F_a^e &= (f, \varphi_a^e) - (\bar{q}, \varphi_a^e)_{\Gamma_q} - \sum_{b=1}^{4} a(\varphi_a^e, \varphi_b^e) p_b^e \\
&= \int_{\Omega^e} f \varphi_a^e \, dxdy - \int_{\Gamma_q} \bar{q} \varphi_a^e \, d\Gamma_q - \sum_{b=1}^{4} a(\varphi_a^e, \varphi_b^e) p_b^e
\end{aligned}
$$

Cálculo de K^e

Pela definição do vetor gradiente no $\mathbb{R}^2$, temos que:

$$\nabla\varphi_a^e(x,y) = \begin{bmatrix} \dfrac{\partial\varphi_a^e}{\partial x} \\ \dfrac{\partial\varphi_a^e}{\partial y} \end{bmatrix}$$

Usando notação matricial do produto interno (ou escalar) de dois vetores x, y no $\mathbb{R}^2$, temos que:

$$x \cdot y = x^T y$$

Logo, podemos reescrever K_{ab}^e para todo a, $b = 1, 2, 3, 4$ como:

$$K_{ab}^e = a(\varphi_a^e, \varphi_b^e) = (\nabla\varphi_a^e, k \cdot \nabla\varphi_b^e) = \int_{\Omega^e} \nabla\varphi_a^e \cdot k \cdot \nabla\varphi_b^e \, dxdy \tag{1}$$

$$= \int_{\Omega^e} (\nabla\varphi_a^e)^T \, k \, \nabla\varphi_b^e \, dxdy = \int_{\Omega^e} \begin{bmatrix} \dfrac{\partial\varphi_a^e}{\partial x} & \dfrac{\partial\varphi_a^e}{\partial y} \end{bmatrix} k \begin{bmatrix} \dfrac{\partial\varphi_a^e}{\partial x} \\ \dfrac{\partial\varphi_a^e}{\partial y} \end{bmatrix} dxdy$$

Cálculo de K^e

De forma análoga ao que vimos no caso 1D, devemos usar mudança de variáveis através da transformação isoparamétrica

$$\begin{aligned} T_{\xi\eta} : \Omega^e &\longrightarrow \Omega^{\bar{b}} \\ (x, y) &\longmapsto (\xi, \eta) = (\xi(x, y), \eta(x, y)), \end{aligned}$$

para aproximar a integral dupla com quadratura de Gauss.

Então,

$$\varphi_a^e(x, y) = \varphi_a^{\bar{b}}(\xi, \eta) = \varphi_a^{\bar{b}}(\xi(x, y), \eta(x, y))$$

e usando a Regra da Cadeia, obtemos as derivadas parciais:

$$\begin{aligned} \frac{\partial \varphi_a^e}{\partial x} &= \frac{\partial \varphi_a^{\bar{b}}}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \varphi_a^{\bar{b}}}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial \varphi_a^e}{\partial y} &= \frac{\partial \varphi_a^{\bar{b}}}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \varphi_a^{\bar{b}}}{\partial \eta}\frac{\partial \eta}{\partial y} \end{aligned}$$

Cálculo de K^e

Passando as duas equações para a forma matricial, obtemos:

$$\underbrace{\begin{bmatrix} \dfrac{\partial \varphi_a^e}{\partial x} \\ \dfrac{\partial \varphi_a^e}{\partial y} \end{bmatrix}}_{=\nabla \varphi_a^e(x,y)} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \underbrace{\begin{bmatrix} \dfrac{\partial \varphi_a^{\bar{b}}}{\partial \xi} \\ \dfrac{\partial \varphi_a^{\bar{b}}}{\partial \eta} \end{bmatrix}}_{=\nabla \varphi_a^{\bar{b}}(\xi,\eta)}$$

Usando a notação $\xi_x = \dfrac{\partial \xi}{\partial x}$ para as derivadas parciais da matriz, temos:

$$\nabla \varphi_a^e(x,y) = \begin{bmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{bmatrix} \nabla \varphi_a^{\bar{b}}(\xi,\eta) \qquad (2)$$

Cálculo de K^e

Do estudo de mudança de variáveis em integrais duplas, sabemos que:

$$\int_{\Omega^e} g(x,y)\; dxdy = \int_{\Omega^{\bar{b}}} g(\xi,\eta)\; |J(\xi,\eta)|\; d\xi d\eta, \tag{3}$$

onde

$$J(\xi,\eta) = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix}$$

é a **matriz jacobiana** da transformação isoparamétrica

$$\begin{aligned} T_{xy} : \Omega^{\bar{b}} &\longrightarrow \Omega^e \\ (\xi,\eta) &\longmapsto (x,y) = (x(\xi,\eta), y(\xi,\eta)) \end{aligned}$$

e

$$|J(\xi,\eta)| = det(J(\xi,\eta)) = x_\xi y_\eta - x_\eta y_\xi$$

é o **jacobiano** dessa transformação.

Cálculo de K^e

Usando os resultados de (2) e (3) na Eq. (1), obtemos:

$$
\begin{aligned}
K^e_{ab} &= \int_{\Omega^e} (\nabla \varphi^e_a)^T \, k \, \nabla \varphi^e_b \, dxdy \\
&= \int_{\Omega^{\bar{b}}} (\nabla \varphi^{\bar{b}}_a(\xi,\eta))^T \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} k \begin{bmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{bmatrix} \nabla \varphi^{\bar{b}}_b(\xi,\eta) \, |J(\xi,\eta)| \, d\xi d\eta
\end{aligned}
\tag{4}
$$

Cálculo de K^e

Note que se:

$$\begin{aligned} T_{xy} : \Omega^{\bar{b}} &\longrightarrow \Omega^e \\ (\xi, \eta) &\longmapsto (x, y) = (x(\xi, \eta), y(\xi, \eta)) \end{aligned}$$

é a transformação onde a matriz jacobiana é

$$J(\xi, \eta) = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix},$$

a transformação inversa é dada por

$$\begin{aligned} T_{xy}^{-1} = T_{\xi\eta} : \Omega^e &\longrightarrow \Omega^{\bar{b}} \\ (x, y) &\longmapsto (\xi, \eta) = (\xi(x, y), \eta(x, y)), \end{aligned}$$

onde a matriz jacobiana é

$$J(x, y) = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = J^{-1}(\xi, \eta) \tag{5}$$

Cálculo de K^e

Portanto, usando a definição de $J^{-1}(\xi, \eta)$ de (5) na Eq. (4), obtemos:

Elemento K^e_{ab} da matriz local K^e (Materiais isotrópicos: $Q = kI$)

$$
\begin{aligned}
K^e_{ab} &= \int_{\Omega^e} (\nabla \varphi^e_a)^T \; k \; \nabla \varphi^e_b \; dxdy \\
&= \int_{\Omega^{\bar{b}}} (\nabla \varphi^{\bar{b}}_a(\xi, \eta))^T \cdot J^{-1} \cdot k \cdot (J^{-1})^T \cdot \nabla \varphi^{\bar{b}}_b(\xi, \eta) \cdot |J| \; d\xi d\eta,
\end{aligned} \tag{6}
$$

para $a, b = 1, 2, 3, 4$, onde

$$
J = J(\xi, \eta) = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix}.
$$

Cálculo de K^e

Expandindo em colunas o vetor gradiente

$$\nabla \varphi_a^{\bar{b}}(\xi,\eta) = \begin{bmatrix} \dfrac{\partial \varphi_a^{\bar{b}}}{\partial \xi} \\ \dfrac{\partial \varphi_a^{\bar{b}}}{\partial \eta} \end{bmatrix}$$

para todo $a = 1, 2, 3, 4$, obtemos a matriz

$$N = \begin{bmatrix} | & | & | & | \\ \nabla \varphi_1^{\bar{b}}(\xi,\eta) & \nabla \varphi_2^{\bar{b}}(\xi,\eta) & \nabla \varphi_3^{\bar{b}}(\xi,\eta) & \nabla \varphi_4^{\bar{b}}(\xi,\eta) \\ | & | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\partial \varphi_1^{\bar{b}}}{\partial \xi} & \dfrac{\partial \varphi_2^{\bar{b}}}{\partial \xi} & \dfrac{\partial \varphi_3^{\bar{b}}}{\partial \xi} & \dfrac{\partial \varphi_4^{\bar{b}}}{\partial \xi} \\ \dfrac{\partial \varphi_1^{\bar{b}}}{\partial \eta} & \dfrac{\partial \varphi_2^{\bar{b}}}{\partial \eta} & \dfrac{\partial \varphi_3^{\bar{b}}}{\partial \eta} & \dfrac{\partial \varphi_4^{\bar{b}}}{\partial \eta} \end{bmatrix}$$

Cálculo de K^e

Portanto,

Matriz local K^e (Materiais isotrópicos: $Q = kI$)

$$K^e = \int_{\Omega^{\bar{b}}} N^T \cdot J^{-1} \cdot k \cdot (J^{-1})^T \cdot N \cdot |J| \, d\xi d\eta, \tag{7}$$

onde

$$J = J(\xi, \eta) = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix}.$$

Cálculo de F^e

Também temos para cada elemento Ω^e, ou seja, para todo $a = 1, 2, 3, 4$:

$$F_a^e = (f, \varphi_a^e) - (\bar{q}, \varphi_a^e)_{\Gamma_q} - \sum_{b=1}^{4} a(\varphi_a^e, \varphi_b^e)\, p_b^e = \underbrace{(f, \varphi_a^e)}_{f_a^e} - \underbrace{(\bar{q}, \varphi_a^e)_{\Gamma_q}}_{q_a^e} - \underbrace{\sum_{b=1}^{4} K_{ab}^e\, p_b^e}_{\bar{p}_a^e}$$

- **Cálculo de f_a^e:**

$$f_a^e = (f, \varphi_a^e) = \int_{\Omega^e} f(x, y) \cdot \varphi_a^e(x, y)\, dxdy$$

Usando mudança de variáveis $(x, y) \longmapsto (\xi, \eta)$, obtemos:

$$f_a^e = (f, \varphi_a^e) = \int_{\Omega^{\bar{b}}} f(\xi, \eta) \cdot \varphi_a^{\bar{b}}(\xi, \eta)\, |J|\, d\xi d\eta$$

Logo,

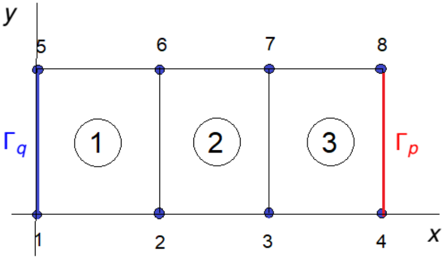
$$f^e = \begin{bmatrix} f_1^e \\ f_2^e \\ f_3^e \\ f_4^e \end{bmatrix}, \text{ para } e = 1, 2, 3.$$

Cálculo de F^e

- **Cálculo de** q_a^e:

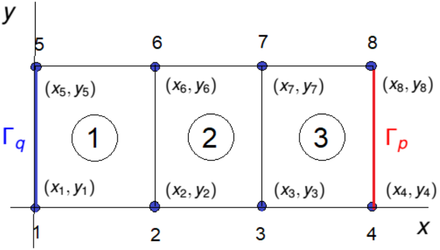
$$q_a^e = (\bar{q}, \varphi_a^e)_{\Gamma_q} = \int_{\Gamma_q} \bar{q}(s) \cdot \varphi_a^e(s)\, ds, \text{ onde } s \in \Gamma_q$$

- **Exemplo de cálculo de** q_a^e:



Note que: $\Gamma_q \in \Omega^1$

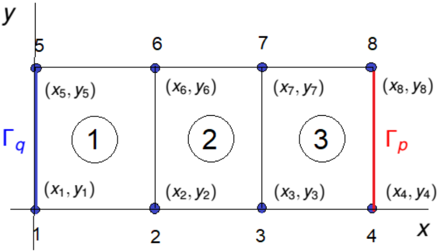
Cálculo de F^e



Aqui, os nós 1 e 5 da malha pertencem a $\Gamma_q = \{(0, y); y_1 \leq y \leq y_5\}$. Logo,

$$q_a^e = (\bar{q}, \varphi_a^e)_{\Gamma_q} = \int_{\Gamma_q} \bar{q}(0, y) \cdot \varphi_a^e(0, y)\, dy = \int_{y_1}^{y_5} \bar{q}(0, y) \cdot \varphi_a^e(0, y)\, dy$$

Cálculo de F^e

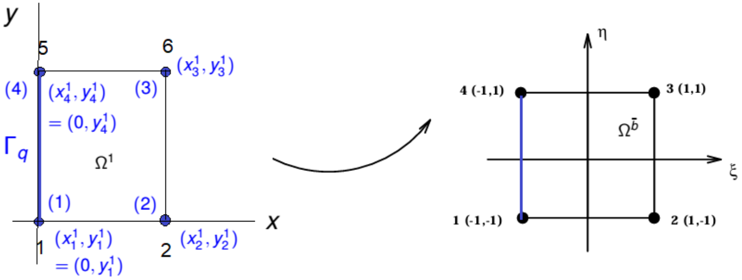


Aqui, os nós 1 e 5 da malha pertencem a $\Gamma_q = \{(0, y); y_1 \leq y \leq y_5\}$.

Logo,

$$q_a^e = (\bar{q}, \varphi_a^e)_{\Gamma_q} = \int_{\Gamma_q} \bar{q}(0, y) \cdot \varphi_a^e(0, y)\, dy = \int_{y_1}^{y_5} \bar{q}(0, y) \cdot \varphi_a^e(0, y)\, dy$$

Cálculo de F^e



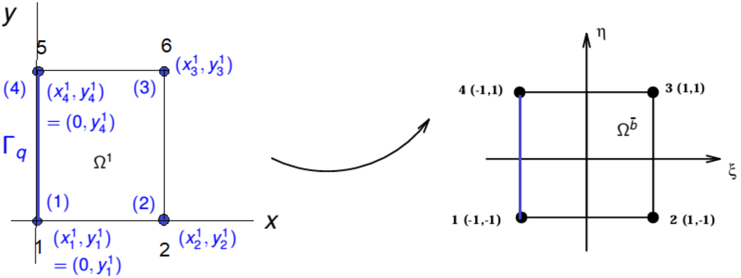
Na mudança de variáveis, $(0, y) \longmapsto (-1, \eta)$.

Logo, para $a = 1, 4$:

$$q_a^e = (\bar{q}, \varphi_a^e)_{\Gamma_q} = \int_{y_1^1 = y_1}^{y_4^1 = y_5} \bar{q}(0, y) \cdot \varphi_a^e(0, y)\, dy = \int_{-1}^{1} \bar{q}(-1, \eta) \cdot \varphi_a^{\bar{b}}(-1, \eta) \cdot y_\eta \, d\eta$$

Para $a = 2, 3$, $q_a^e = 0$.

Cálculo de F^e

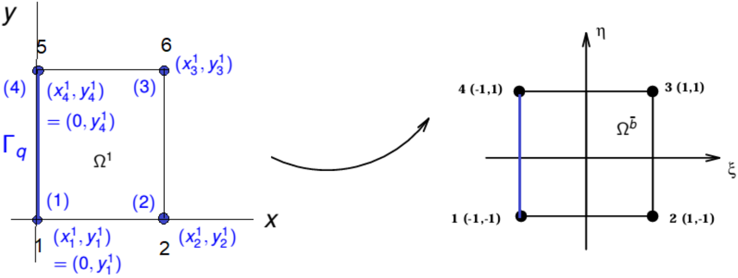


Lembre-se de que:

$$\varphi_1^{\bar{b}}(-1,\eta) = (1/4)(1-(-1))(1-\eta) = (1-\eta)/2;$$
$$\varphi_2^{\bar{b}}(-1,\eta) = (1/4)(1+(-1))(1-\eta) = 0;$$
$$\varphi_3^{\bar{b}}(-1,\eta) = (1/4)(1+(-1))(1+\eta) = 0;$$
$$\varphi_4^{\bar{b}}(-1,\eta) = (1/4)(1-(-1))(1+\eta) = (1+\eta)/2$$

Portanto, no elemento Ω^1, só serão usadas as funções de base locais $\varphi_1^{\bar{b}}$, $\varphi_4^{\bar{b}}$.

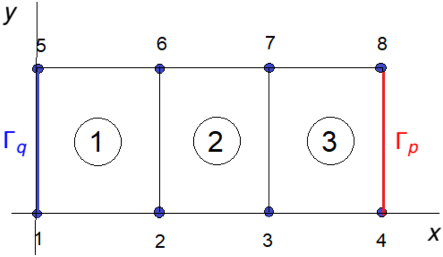
Cálculo de F^e



Logo,

$$q^1 = \begin{bmatrix} q_1^1 \\ q_2^1 \\ q_3^1 \\ q_4^1 \end{bmatrix} = \begin{bmatrix} (\bar{q}, \varphi_1^1)_{\Gamma_q} \\ 0 \\ 0 \\ (\bar{q}, \varphi_4^1)_{\Gamma_q} \end{bmatrix} ; \quad q^e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ para } e = 2, 3.$$

Cálculo de F^e



- **Cálculo de $\bar{p}_a^e$:**

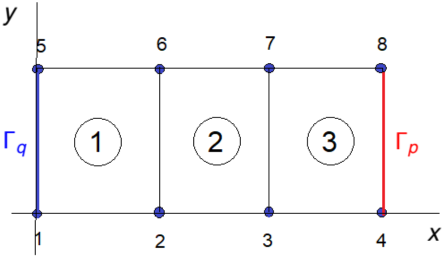
$$\bar{p}_a^e = \sum_{b=1}^{4} K_{ab}^e \, p_b^e$$

Neste exemplo, os nós prescritos 4 e 8 estão em $\Gamma_p = \{(x_4, y); y_4 \leq y \leq y_8\}$.

Logo, os nós prescritos estão no elemento Ω^3.

Neste caso, só vamos calcular $\bar{p}^3$, enquanto $\bar{p}^e = 0$ para $e = 1, 2$.

Cálculo de F^e

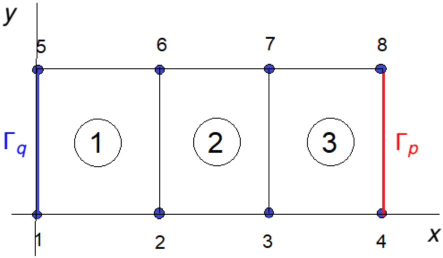


Logo,

$$\bar{p}^3_a = \sum_{b=1}^{4} K^3_{ab}\, p^3_b$$

$$\bar{p}^3 = \begin{bmatrix} \bar{p}^3_1 \\ \bar{p}^3_2 \\ \bar{p}^3_3 \\ \bar{p}^3_4 \end{bmatrix} = \begin{bmatrix} K^3_{11} & K^3_{12} & K^3_{13} & K^3_{14} \\ K^3_{21} & K^3_{22} & K^3_{23} & K^3_{24} \\ K^3_{31} & K^3_{32} & K^3_{33} & K^3_{34} \\ K^3_{41} & K^3_{42} & K^3_{43} & K^3_{44} \end{bmatrix} \begin{bmatrix} p^3_1 \\ p^3_2 \\ p^3_3 \\ p^3_4 \end{bmatrix}$$

Cálculo de F^e



Como $p_1^3 = p_4^3 = 0$, obtemos:

$$\bar{p}^3 = \begin{bmatrix} \bar{p}_1^3 \\ \bar{p}_2^3 \\ \bar{p}_3^3 \\ \bar{p}_4^3 \end{bmatrix} = \begin{bmatrix} K_{11}^3 & K_{12}^3 & K_{13}^3 & K_{14}^3 \\ K_{21}^3 & K_{22}^3 & K_{23}^3 & K_{24}^3 \\ K_{31}^3 & K_{32}^3 & K_{33}^3 & K_{34}^3 \\ K_{41}^3 & K_{42}^3 & K_{43}^3 & K_{44}^3 \end{bmatrix} \begin{bmatrix} 0 \\ p_2^3 \\ p_3^3 \\ 0 \end{bmatrix} ; \ \bar{p}^e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ para } e = 1, 2.$$

Cálculo de F^e

Portanto, neste exemplo, para os elementos $e = 1, 2, 3$:

$$F^e = f^e - q^e - \bar{p}^e,$$

onde

$$f^e = \begin{bmatrix} f_1^e \\ f_2^e \\ f_3^e \\ f_4^e \end{bmatrix}, \text{ para } e = 1, 2, 3; \quad q^1 = \begin{bmatrix} q_1^1 \\ 0 \\ 0 \\ q_4^1 \end{bmatrix}; \quad q^e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ para } e = 2, 3;$$

$$\bar{p}^3 = \begin{bmatrix} \bar{p}_1^3 \\ \bar{p}_2^3 \\ \bar{p}_3^3 \\ \bar{p}_4^3 \end{bmatrix} = \begin{bmatrix} K_{11}^3 & K_{12}^3 & K_{13}^3 & K_{14}^3 \\ K_{21}^3 & K_{22}^3 & K_{23}^3 & K_{24}^3 \\ K_{31}^3 & K_{32}^3 & K_{33}^3 & K_{34}^3 \\ K_{41}^3 & K_{42}^3 & K_{43}^3 & K_{44}^3 \end{bmatrix} \begin{bmatrix} 0 \\ p_2^3 \\ p_3^3 \\ 0 \end{bmatrix}; \; \bar{p}^e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ para } e = 1, 2.$$

Cálculo de F^e

- Para $e = 1$:

$$F^1 = f^1 - q^1 - \bar{p}^1 = \begin{bmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \\ f_4^1 \end{bmatrix} - \begin{bmatrix} q_1^1 \\ 0 \\ 0 \\ q_4^1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1^1 - q_1^1 \\ f_2^1 \\ f_3^1 \\ f_4^1 - q_4^1 \end{bmatrix}$$

- Para $e = 2$:

$$F^2 = f^2 - q^2 - \bar{p}^2 = \begin{bmatrix} f_1^2 \\ f_2^2 \\ f_3^2 \\ f_4^2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1^2 \\ f_2^2 \\ f_3^2 \\ f_4^2 \end{bmatrix}$$

- Para $e = 3$:

$$F^3 = f^3 - q^3 - \bar{p}^3 = \begin{bmatrix} f_1^3 \\ f_2^3 \\ f_3^3 \\ f_4^3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \bar{p}_1^3 \\ \bar{p}_2^3 \\ \bar{p}_3^3 \\ \bar{p}_4^3 \end{bmatrix} = \begin{bmatrix} f_1^3 - \bar{p}_1^3 \\ f_2^3 - \bar{p}_2^3 \\ f_3^3 - \bar{p}_3^3 \\ f_4^3 - \bar{p}_4^3 \end{bmatrix}$$

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