Tópicos Especiais em Matemática Aplicada - 2025-1
UERJ

10 - Caso 2D - Elementos Isoparamétricos

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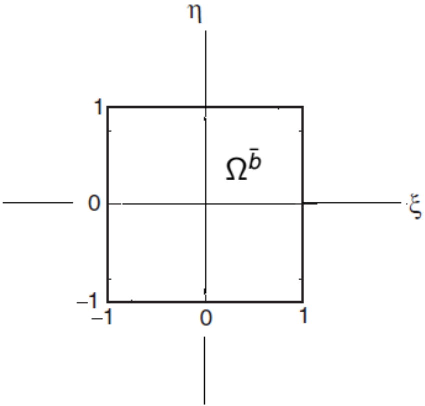
Github: https://github.com/rodrigolrmadureira/ElementosFinitos

Sumário

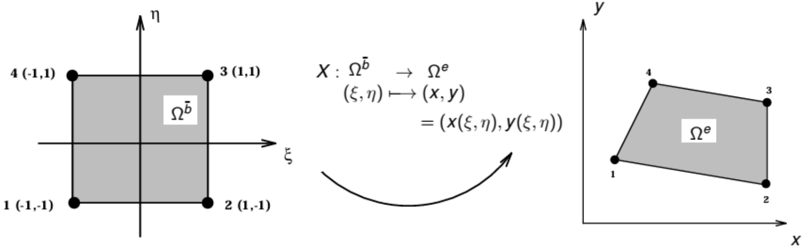
Elementos isoparamétricos

Agora, vamos ver como são calculados K^e e F^e para cada elemento Ω^e.

Seja $\Omega^{\bar{b}} = [-1, 1] \times [-1, 1]$ o elemento finito biunitário como representado na figura abaixo.



Elementos isoparamétricos



Seja

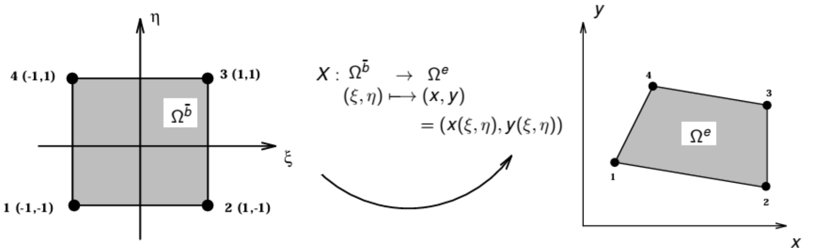
$$X \colon \Omega^{\bar{b}} \longrightarrow \Omega^e$$
$$(\xi,\eta) \longmapsto (x,y) = (x(\xi,\eta), y(\xi,\eta)),$$

onde

$$x(\xi,\eta) = \sum_{a=1}^{4} \varphi_a^{\bar{b}}(\xi,\eta) \cdot x_a^e; \qquad y(\xi,\eta) = \sum_{a=1}^{4} \varphi_a^{\bar{b}}(\xi,\eta) \cdot y_a^e;$$

$$\varphi_a^{\bar{b}}(\xi_c,\eta_c) = \begin{cases} 1, \text{ se } a = c, \\ 0, \text{ se } a \neq c. \end{cases}$$

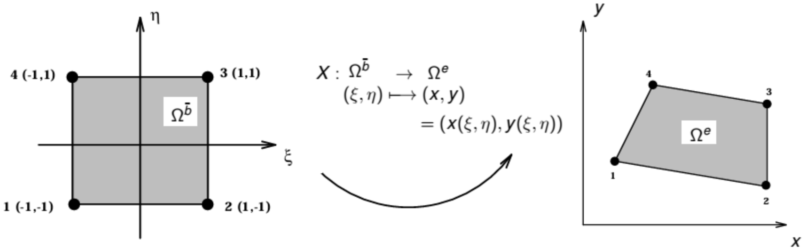
Elementos isoparamétricos



Por exemplo,

$$
\begin{aligned}
x(-1,-1) = x(\xi_1,\eta_1) &= \sum_{a=1}^{4} \varphi_a^{\bar{b}}(\xi_1,\eta_1)\cdot x_a^e \\
&= \underbrace{\varphi_1^{\bar{b}}(\xi_1,\eta_1)}_{1}\cdot x_1^e + \cancelto{0}{\varphi_2^{\bar{b}}(\xi_1,\eta_1)}\cdot x_2^e + \cancelto{0}{\varphi_3^{\bar{b}}(\xi_1,\eta_1)}\cdot x_3^e + \cancelto{0}{\varphi_4^{\bar{b}}(\xi_1,\eta_1)}\cdot x_4^e \\
&= x_1^e
\end{aligned}
$$

Elementos isoparamétricos



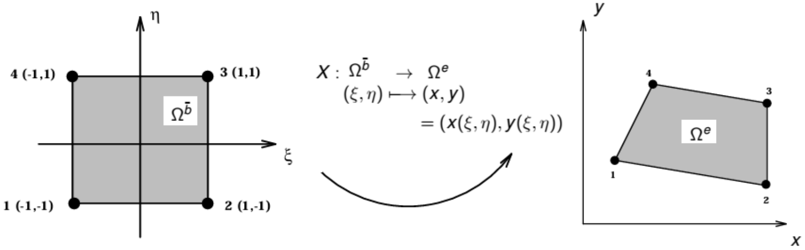
Analogamente,

$$\begin{aligned} x(1,-1) &= x(\xi_2,\eta_2) = x_2^e; \\ x(1,1) &= x(\xi_3,\eta_3) = x_3^e; \\ x(-1,1) &= x(\xi_4,\eta_4) = x_4^e \end{aligned}$$

Ou seja, para todo $a = 1, 2, 3, 4$,

$$x(\xi_a,\eta_a) = x_a^e; \quad y(\xi_a,\eta_a) = y_a^e$$

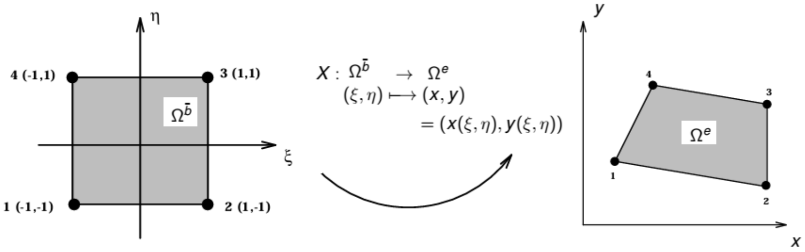
Elementos isoparamétricos



Vamos assumir que $x(\xi,\eta)$ e $y(\xi,\eta)$ são lineares em ξ e η, ou seja:

$$x(\xi,\eta) = a_1 + a_2\xi + a_3\eta + a_4\xi\eta$$
$$y(\xi,\eta) = b_1 + b_2\xi + b_3\eta + b_4\xi\eta$$

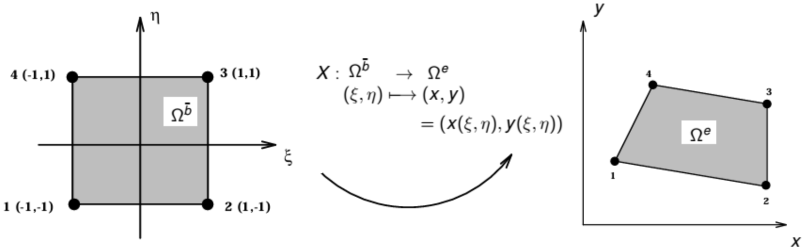
Elementos isoparamétricos



Então,

$$
\begin{aligned}
x(-1,-1) = x_1^e &\Rightarrow a_1 - a_2 - a_3 + a_4 = x_1^e \\
x(1,-1) = x_2^e &\Rightarrow a_1 + a_2 - a_3 - a_4 = x_2^e \\
x(1,1) = x_3^e &\Rightarrow a_1 + a_2 + a_3 + a_4 = x_3^e \\
x(-1,1) = x_4^e &\Rightarrow a_1 - a_2 + a_3 - a_4 = x_4^e
\end{aligned}
$$

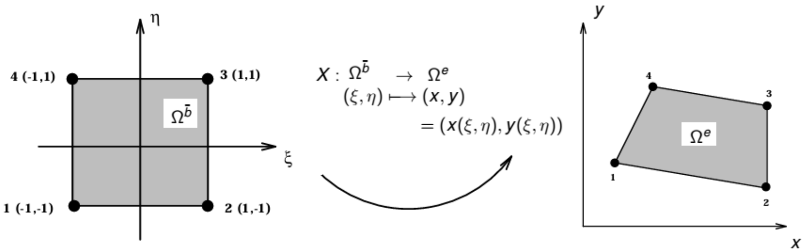
Elementos isoparamétricos



Resolvendo o sistema:

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{bmatrix}$$

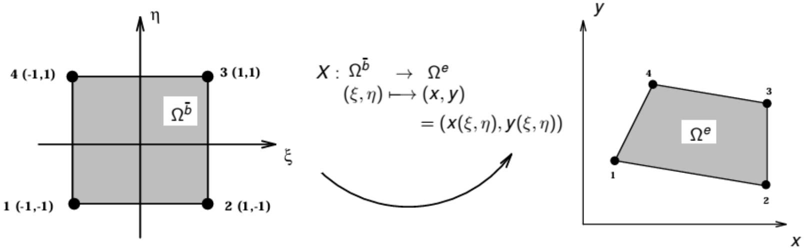
Elementos isoparamétricos



encontramos:

$$
\begin{aligned}
a_1 &= (x_1^e + x_2^e + x_3^e + x_4^e)/4;\\
a_2 &= (-x_1^e + x_2^e + x_3^e - x_4^e)/4;\\
a_3 &= (-x_1^e - x_2^e + x_3^e + x_4^e)/4;\\
a_4 &= (x_1^e - x_2^e + x_3^e - x_4^e)/4
\end{aligned}
$$

Elementos isoparamétricos

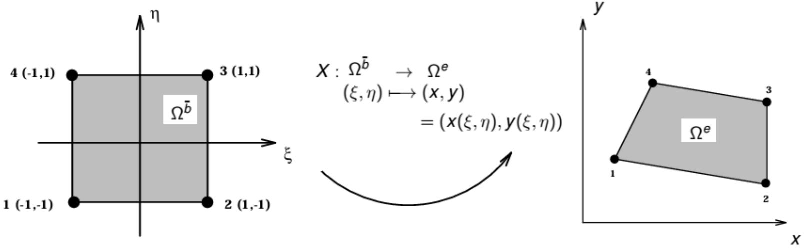


Substituindo a_1, a_2, a_3, a_4 em $x(\xi,\eta)$ e arrumando os termos, obtemos:

$$x(\xi,\eta) = \underbrace{\frac{1}{4}(1-\xi)(1-\eta)}_{\varphi_1^{\bar{b}}(\xi,\eta)}\, x_1^e + \underbrace{\frac{1}{4}(1+\xi)(1-\eta)}_{\varphi_2^{\bar{b}}(\xi,\eta)}\, x_2^e$$

$$+ \underbrace{\frac{1}{4}(1+\xi)(1+\eta)}_{\varphi_3^{\bar{b}}(\xi,\eta)}\, x_3^e + \underbrace{\frac{1}{4}(1-\xi)(1+\eta)}_{\varphi_4^{\bar{b}}(\xi,\eta)}\, x_4^e$$

Elementos isoparamétricos



Assim, temos as quatro funções de base para Q_4:

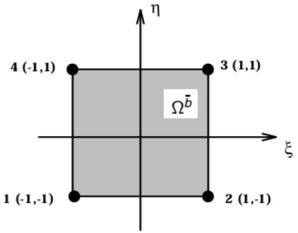
$$\varphi_1^{\bar{b}}(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta);$$

$$\varphi_2^{\bar{b}}(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta);$$

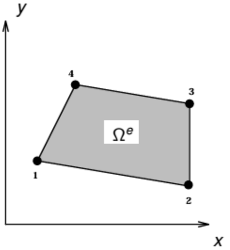
$$\varphi_3^{\bar{b}}(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta);$$

$$\varphi_4^{\bar{b}}(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta)$$

Elementos isoparamétricos



$$\begin{aligned} X : \Omega^{\bar{b}} &\rightarrow \Omega^{e} \\ (\xi,\eta) &\longmapsto (x,y) \\ &= (x(\xi,\eta), y(\xi,\eta)) \end{aligned}$$

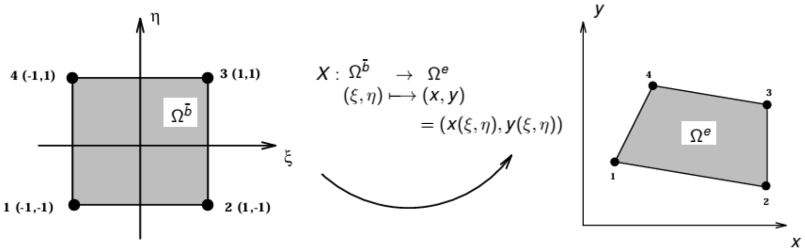


Logo, podemos reescrever:

$$x(\xi,\eta) = \sum_{a=1}^{4} \varphi_a^{\bar{b}}(\xi,\eta) \cdot x_a^e$$

$$y(\xi,\eta) = \sum_{a=1}^{4} \varphi_a^{\bar{b}}(\xi,\eta) \cdot y_a^e$$

Elementos isoparamétricos



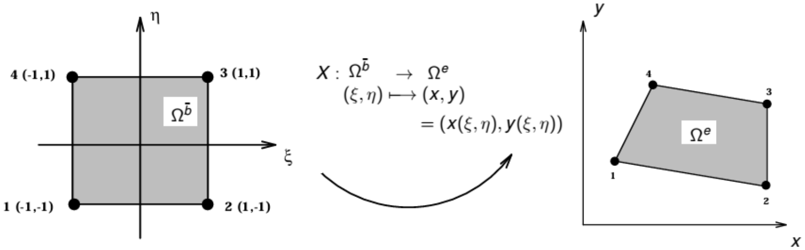
Assim, escrevemos a solução aproximada u_h^e em cada elemento Ω^e como:

$$u_h^e(x,y) = \sum_{a=1}^{4} c_a^e \cdot \varphi_a^e(x,y)$$

Com a mudança de variáveis $(x,y) \longmapsto (\xi,\eta)$, obtemos:

$$u_h^e(x,y) = u_h^{\bar{b}}(\xi,\eta) = \sum_{a=1}^{4} c_a^{\bar{b}} \cdot \varphi_a^{\bar{b}}(\xi,\eta)$$

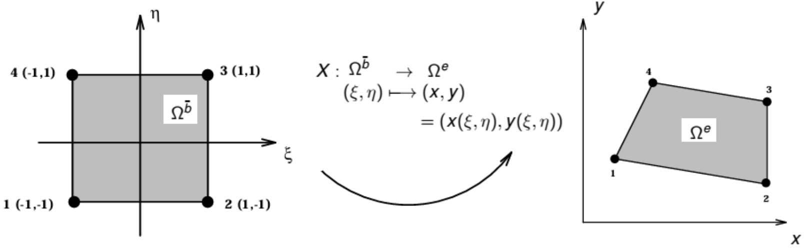
Elemento da matriz local K^e



Também temos para cada elemento Ω^e, ou seja, para todo $a, b = 1, 2, 3, 4$:

$$K^e_{ab} = a(\varphi^e_a, \varphi^e_b) = (\nabla\varphi^e_a, k \cdot \nabla\varphi^e_b) = \int_{\Omega^e} \nabla\varphi^e_a \cdot k \cdot \nabla\varphi^e_b \, dxdy$$

Elemento do vetor local F^e



Também temos para cada elemento Ω^e, ou seja, para todo $a = 1, 2, 3, 4$:

$$
\begin{aligned}
F_a^e &= (f, \varphi_a^e) - (\bar{q}, \varphi_a^e)_{\Gamma_q} - \sum_{b=1}^{4} a(\varphi_a^e, \varphi_b^e) p_b^e \\
&= \int_{\Omega^e} f \varphi_a^e \, dxdy - \int_{\Gamma_q} \bar{q} \varphi_a^e \, d\Gamma_q - \sum_{b=1}^{4} a(\varphi_a^e, \varphi_b^e) p_b^e
\end{aligned}
$$

Cálculo de K^e

Pela definição do vetor gradiente no $\mathbb{R}^2$, temos que:

$$\nabla\varphi_a^e(x,y) = \begin{bmatrix} \dfrac{\partial\varphi_a^e}{\partial x} \\ \dfrac{\partial\varphi_a^e}{\partial y} \end{bmatrix}$$

Usando notação matricial do produto interno (ou escalar) de dois vetores x, y no $\mathbb{R}^2$, temos que:

$$x \cdot y = x^T y$$

Logo, podemos reescrever K_{ab}^e para todo a, $b = 1, 2, 3, 4$ como:

$$K_{ab}^e = a(\varphi_a^e, \varphi_b^e) = (\nabla\varphi_a^e, k\cdot\nabla\varphi_b^e) = \int_{\Omega^e} \nabla\varphi_a^e \cdot k \cdot \nabla\varphi_b^e \, dxdy \tag{1}$$

$$= \int_{\Omega^e} (\nabla\varphi_a^e)^T \, k \, \nabla\varphi_b^e \, dxdy = \int_{\Omega^e} \begin{bmatrix} \dfrac{\partial\varphi_a^e}{\partial x} & \dfrac{\partial\varphi_a^e}{\partial y} \end{bmatrix} k \begin{bmatrix} \dfrac{\partial\varphi_a^e}{\partial x} \\ \dfrac{\partial\varphi_a^e}{\partial y} \end{bmatrix} dxdy$$

Cálculo de K^e

De forma análoga ao que vimos no caso 1D, devemos usar mudança de variáveis através da transformação isoparamétrica

$$
\begin{aligned}
T_{\xi\eta} : \Omega^e &\longrightarrow \Omega^{\bar{b}} \\
(x, y) &\longmapsto (\xi, \eta) = (\xi(x, y), \eta(x, y)),
\end{aligned}
$$

para aproximar a integral dupla com quadratura de Gauss.

Então,

$$\varphi_a^e(x, y) = \varphi_a^{\bar{b}}(\xi, \eta) = \varphi_a^{\bar{b}}(\xi(x, y), \eta(x, y))$$

e usando a Regra da Cadeia, obtemos as derivadas parciais:

$$
\begin{aligned}
\frac{\partial \varphi_a^e}{\partial x} &= \frac{\partial \varphi_a^{\bar{b}}}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \varphi_a^{\bar{b}}}{\partial \eta}\frac{\partial \eta}{\partial x} \\
\frac{\partial \varphi_a^e}{\partial y} &= \frac{\partial \varphi_a^{\bar{b}}}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \varphi_a^{\bar{b}}}{\partial \eta}\frac{\partial \eta}{\partial y}
\end{aligned}
$$

Cálculo de K^e

Passando as duas equações para a forma matricial, obtemos:

$$\underbrace{\begin{bmatrix} \dfrac{\partial \varphi_a^e}{\partial x} \\ \dfrac{\partial \varphi_a^e}{\partial y} \end{bmatrix}}_{=\nabla \varphi_a^e(x,y)} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \underbrace{\begin{bmatrix} \dfrac{\partial \varphi_a^{\bar{b}}}{\partial \xi} \\ \dfrac{\partial \varphi_a^{\bar{b}}}{\partial \eta} \end{bmatrix}}_{=\nabla \varphi_a^{\bar{b}}(\xi,\eta)}$$

Usando a notação $\xi_x = \dfrac{\partial \xi}{\partial x}$ para as derivadas parciais da matriz, temos:

$$\nabla \varphi_a^e(x,y) = \begin{bmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{bmatrix} \nabla \varphi_a^{\bar{b}}(\xi,\eta) \tag{2}$$

Cálculo de K^e

Do estudo de mudança de variáveis em integrais duplas, sabemos que:

$$\int_{\Omega^e} g(x,y)\, dxdy = \int_{\Omega^{\bar{b}}} g(\xi,\eta)\, |J(\xi,\eta)|\, d\xi d\eta, \tag{3}$$

onde

$$J(\xi,\eta) = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix}$$

é a **matriz jacobiana** da transformação isoparamétrica

$$\begin{aligned} T_{xy} : \Omega^{\bar{b}} &\longrightarrow \Omega^e \\ (\xi,\eta) &\longmapsto (x,y) = (x(\xi,\eta), y(\xi,\eta)) \end{aligned}$$

e

$$|J(\xi,\eta)| = det(J(\xi,\eta)) = x_\xi y_\eta - x_\eta y_\xi$$

é o **jacobiano** dessa transformação.

Cálculo de K^e

Usando os resultados de (2) e (3) na Eq. (1), obtemos:

$$
\begin{aligned}
K_{ab}^e &= \int_{\Omega^e} (\nabla \varphi_a^e)^T \; k \; \nabla \varphi_b^e \; dxdy \\
&= \int_{\Omega^{\bar{b}}} (\nabla \varphi_a^{\bar{b}}(\xi,\eta))^T \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} k \begin{bmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{bmatrix} \nabla \varphi_b^{\bar{b}}(\xi,\eta) \, |J(\xi,\eta)| \, d\xi d\eta
\end{aligned}
\tag{4}
$$

Cálculo de K^e

Note que se:

$$
\begin{aligned}
T_{xy} : \Omega^{\bar{b}} &\longrightarrow \Omega^e \\
(\xi, \eta) &\longmapsto (x, y) = (x(\xi, \eta), y(\xi, \eta))
\end{aligned}
$$

é a transformação onde a matriz jacobiana é

$$
J(\xi, \eta) = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix},
$$

a transformação inversa é dada por

$$
\begin{aligned}
T_{xy}^{-1} = T_{\xi\eta} : \Omega^e &\longrightarrow \Omega^{\bar{b}} \\
(x, y) &\longmapsto (\xi, \eta) = (\xi(x, y), \eta(x, y)),
\end{aligned}
$$

onde a matriz jacobiana é

$$
J(x, y) = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = J^{-1}(\xi, \eta) \tag{5}
$$

Cálculo de K^e

Portanto, usando a definição de $J^{-1}(\xi, \eta)$ de (5) na Eq. (4), obtemos:

Elemento K^e_{ab} da matriz local K^e (Materiais isotrópicos: $Q = kI$)

$$
\begin{aligned}
K^e_{ab} &= \int_{\Omega^e} (\nabla\varphi^e_a)^T \; k \; \nabla\varphi^e_b \; dxdy \qquad (6)\\
&= \int_{\Omega^{\bar{b}}} (\nabla\varphi^{\bar{b}}_a(\xi,\eta))^T \cdot J^{-1} \cdot k \cdot (J^{-1})^T \cdot \nabla\varphi^{\bar{b}}_b(\xi,\eta) \cdot |J| \; d\xi d\eta,
\end{aligned}
$$

para $a, b = 1, 2, 3, 4$, onde

$$
J = J(\xi,\eta) = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix}.
$$

Cálculo de K^e

Expandindo em colunas o vetor gradiente

$$\nabla\varphi_a^{\bar{b}}(\xi,\eta) = \begin{bmatrix} \dfrac{\partial\varphi_a^{\bar{b}}}{\partial\xi} \\ \dfrac{\partial\varphi_a^{\bar{b}}}{\partial\eta} \end{bmatrix}$$

para todo $a = 1, 2, 3, 4$, obtemos a matriz

$$N = \begin{bmatrix} | & | & | & | \\ | & | & | & | \\ \nabla\varphi_1^{\bar{b}}(\xi,\eta) & \nabla\varphi_2^{\bar{b}}(\xi,\eta) & \nabla\varphi_3^{\bar{b}}(\xi,\eta) & \nabla\varphi_4^{\bar{b}}(\xi,\eta) \\ | & | & | & | \\ | & | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\partial\varphi_1^{\bar{b}}}{\partial\xi} & \dfrac{\partial\varphi_2^{\bar{b}}}{\partial\xi} & \dfrac{\partial\varphi_3^{\bar{b}}}{\partial\xi} & \dfrac{\partial\varphi_4^{\bar{b}}}{\partial\xi} \\ \dfrac{\partial\varphi_1^{\bar{b}}}{\partial\eta} & \dfrac{\partial\varphi_2^{\bar{b}}}{\partial\eta} & \dfrac{\partial\varphi_3^{\bar{b}}}{\partial\eta} & \dfrac{\partial\varphi_4^{\bar{b}}}{\partial\eta} \end{bmatrix}$$

Cálculo de K^e

Portanto,

Matriz local K^e (Materiais isotrópicos: $Q = kI$)

$$K^e = \int_{\Omega^{\bar{b}}} N^T \cdot J^{-1} \cdot k \cdot (J^{-1})^T \cdot N \cdot |J| \, d\xi d\eta, \tag{7}$$

onde

$$J = J(\xi, \eta) = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix}.$$

Cálculo de K^e

Sabendo que

$$x(\xi,\eta) = \sum_{a=1}^{4} \varphi_a^{\bar{b}}(\xi,\eta) \cdot x_a^e = \begin{bmatrix} \varphi_1^{\bar{b}}(\xi,\eta) & \varphi_2^{\bar{b}}(\xi,\eta) & \varphi_3^{\bar{b}}(\xi,\eta) & \varphi_4^{\bar{b}}(\xi,\eta) \end{bmatrix} \cdot \begin{bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{bmatrix}$$

$$y(\xi,\eta) = \sum_{a=1}^{4} \varphi_a^{\bar{b}}(\xi,\eta) \cdot y_a^e = \begin{bmatrix} \varphi_1^{\bar{b}}(\xi,\eta) & \varphi_2^{\bar{b}}(\xi,\eta) & \varphi_3^{\bar{b}}(\xi,\eta) & \varphi_4^{\bar{b}}(\xi,\eta) \end{bmatrix} \cdot \begin{bmatrix} y_1^e \\ y_2^e \\ y_3^e \\ y_4^e \end{bmatrix}$$

e usando as funções de base do Q_4,

$$\begin{aligned} \varphi_1^{\bar{b}}(\xi,\eta) &= (1-\xi)(1-\eta)/4; \\ \varphi_2^{\bar{b}}(\xi,\eta) &= (1+\xi)(1-\eta)/4; \\ \varphi_3^{\bar{b}}(\xi,\eta) &= (1+\xi)(1+\eta)/4; \\ \varphi_4^{\bar{b}}(\xi,\eta) &= (1-\xi)(1+\eta)/4, \end{aligned}$$

Cálculo de K^e

obtemos

$$x(\xi,\eta) = \frac{1}{4}\begin{bmatrix}(1-\xi)(1-\eta) & (1+\xi)(1-\eta) & (1+\xi)(1+\eta) & (1-\xi)(1+\eta)\end{bmatrix}\begin{bmatrix}x_1^e\\ x_2^e\\ x_3^e\\ x_4^e\end{bmatrix}$$

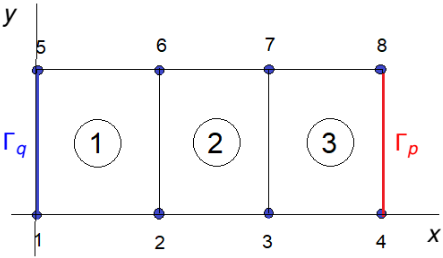
$$y(\xi,\eta) = \frac{1}{4}\begin{bmatrix}(1-\xi)(1-\eta) & (1+\xi)(1-\eta) & (1+\xi)(1+\eta) & (1-\xi)(1+\eta)\end{bmatrix}\begin{bmatrix}y_1^e\\ y_2^e\\ y_3^e\\ y_4^e\end{bmatrix}$$

e os elementos da matriz jacobiana J são:

$$\begin{aligned}
x_\xi &= (1/4)\cdot\begin{bmatrix}-(1-\eta) & 1-\eta & 1+\eta & -(1+\eta)\end{bmatrix}\begin{bmatrix}x_1^e & x_2^e & x_3^e & x_4^e\end{bmatrix}^T\\
y_\xi &= (1/4)\cdot\begin{bmatrix}-(1-\eta) & 1-\eta & 1+\eta & -(1+\eta)\end{bmatrix}\begin{bmatrix}y_1^e & y_2^e & y_3^e & y_4^e\end{bmatrix}^T\\
x_\eta &= (1/4)\cdot\begin{bmatrix}-(1-\xi) & -(1+\xi) & 1+\xi & 1-\xi\end{bmatrix}\begin{bmatrix}x_1^e & x_2^e & x_3^e & x_4^e\end{bmatrix}^T\\
y_\eta &= (1/4)\cdot\begin{bmatrix}-(1-\xi) & -(1+\xi) & 1+\xi & 1-\xi\end{bmatrix}\begin{bmatrix}y_1^e & y_2^e & y_3^e & y_4^e\end{bmatrix}^T
\end{aligned}$$

Cálculo de K^e

- **Exemplo**: Considere o domínio Ω

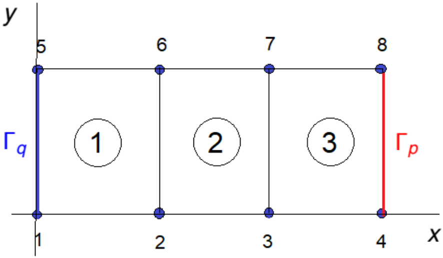


onde $0 \leq x \leq 1$, $0 \leq y \leq 1/3$.

Note que:

Número de subintervalos de x: $nelx = 3$
Número de subintervalos de y: $nely = 1$
Total de elementos: $nelx \cdot nely = 3$.

Cálculo de K^e



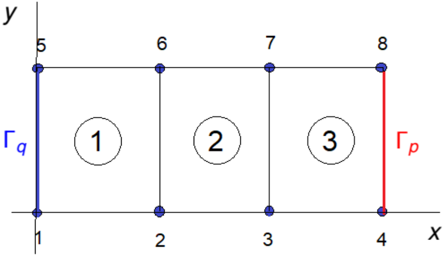
Número de nós de x: $Nnosx = nelx + 1 = 4$.

Tamanho dos subintervalos de x: $h_x = (x_f - x_1)/nelx$, onde x_f é o nó final e x_1 é o nó inicial em x. Neste exemplo, $h_x = (1 - 0)/3 = 1/3$.

Número de nós de y: $Nnosy = nely + 1 = 2$.

Tamanho dos subintervalos de y: $h_y = (y_f - y_1)/nelx$, onde y_f é o nó final e y_1 é o nó inicial em y. Neste exemplo,
$h_y = (y_f - x_0)/nely = ((1/3) - 0)/1 = 1/3$.

Cálculo de K^e



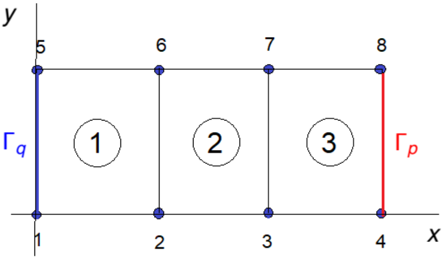
Cada nó de x é dado por: $x_j = x_1 + (j-1) \cdot h_x$, para todo $j = 1, 2, \ldots, Nnosx$.

Neste exemplo, os nós de x são: $x_1 = 0$, $x_2 = 1/3$, $x_3 = 2/3$, $x_4 = 1$.

Cada nó de y é dado por: $y_k = y_1 + (k-1) \cdot h_y$, para todo $k = 1, 2, \ldots, Nnosy$.

Neste exemplo, os nós de y são: $y_1 = 0$, $y_2 = 1/3$.

Cálculo de K^e

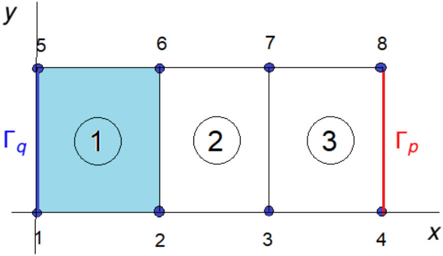


Neste exemplo, o número de nós total da malha é:
$Nnos = Nosx \cdot Nnosy = (nelx + 1) \cdot (nely + 1) = 4 \cdot 2 = 8.$

Logo, cada ponto do domínio Ω é (x_j, y_k), para $j = 1, 2, \ldots, Nnosx$, $k = 1, 2, \ldots, Nnosy$. Neste exemplo, temos os seguintes pontos:

- $1 : (x_1, y_1) = (0, 0)$; • $2 : (x_2, y_1) = (1/3, 0)$; • $3 : (x_3, y_1) = (2/3, 0)$;
- $4 : (x_4, y_1) = (1, 0)$; • $5 : (x_1, y_2) = (0, 1/3)$; • $6 : (x_2, y_2) = (1/3, 1/3)$;
- $7 : (x_3, y_2) = (2/3, 1/3)$; • $8 : (x_4, y_2) = (1, 1/3)$.

Cálculo de K^e

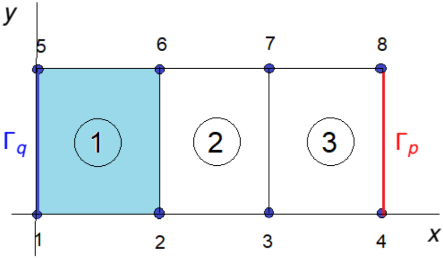


- **Exemplo**: Cálculo dos elementos da matriz local

$$K^1 = \begin{bmatrix} K^1_{11} & K^1_{12} & K^1_{13} & K^1_{14} \\ K^1_{21} & K^1_{22} & K^1_{23} & K^1_{24} \\ K^1_{31} & K^1_{32} & K^1_{33} & K^1_{34} \\ K^1_{41} & K^1_{42} & K^1_{43} & K^1_{44} \end{bmatrix}$$

para o elemento $e = 1$ do domínio Ω.

Cálculo de K^e

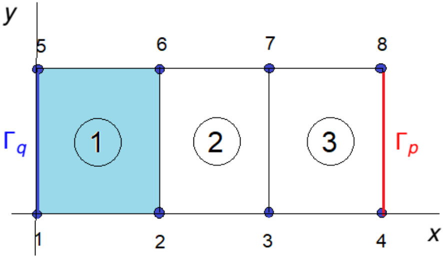


Para $e = 1$, temos:

$$\begin{aligned} K^1_{ab} &= \int_{\Omega^1} (\nabla\varphi^1_a(x,y))^T \, k \, \nabla\varphi^1_b(x,y) \, dxdy \\ &= \int_0^{1/3}\int_0^{1/3} (\nabla\varphi^1_a(x,y))^T \, k \, \nabla\varphi^1_b(x,y) \, dxdy, \end{aligned}$$

para $a, b = 1, 2, 3, 4$.

Cálculo de K^e



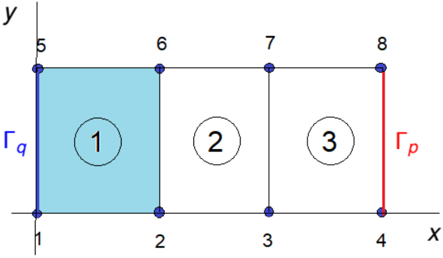
Com a mudança de variáveis na transformação isoparamétrica, temos:

$$K^1_{ab} = \int_{\Omega^{\bar{1}}} (\nabla\varphi^{\bar{1}}_a(\xi,\eta))^T \cdot J^{-1} \cdot k \cdot (J^{-1})^T \cdot \nabla\varphi^{\bar{1}}_b(\xi,\eta) \cdot |J| \; d\xi d\eta$$

$$= \int_{-1}^{1}\int_{-1}^{1} (\nabla\varphi^{\bar{1}}_a(\xi,\eta))^T \cdot J^{-1} \cdot k \cdot (J^{-1})^T \cdot \nabla\varphi^{\bar{1}}_b(\xi,\eta) \cdot |J| \; d\xi d\eta,$$

para $a, b = 1, 2, 3, 4$.

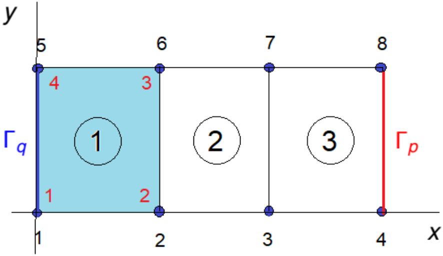
Cálculo de K^e



Para o elemento K_{11}^1 da matriz local K^1, fazemos $a = b = 1$ na integral. Assim, obtemos:

$$K_{11}^1 = \int_{\Omega^{\bar{1}}} (\nabla\varphi_1^{\bar{1}}(\xi,\eta))^T \cdot J^{-1} \cdot k \cdot (J^{-1})^T \cdot \nabla\varphi_1^{\bar{1}}(\xi,\eta) \cdot |J| \; d\xi d\eta$$
$$= \int_{-1}^{1}\int_{-1}^{1} (\nabla\varphi_1^{\bar{1}}(\xi,\eta))^T \cdot J^{-1} \cdot k \cdot (J^{-1})^T \cdot \nabla\varphi_1^{\bar{1}}(\xi,\eta) \cdot |J| \; d\xi d\eta$$

Cálculo de K^e



- **Cálculo da matriz jacobiana J:**

Neste exemplo, no elemento Ω^1, temos:

$$(x^1)^T = \begin{bmatrix} x_1^1 & x_2^1 & x_3^1 & x_4^1 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/3 & 0 \end{bmatrix};$$
$$(y^1)^T = \begin{bmatrix} y_1^1 & y_2^1 & y_3^1 & y_4^1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1/3 & 1/3 \end{bmatrix}.$$

Cálculo de K^e

Logo, os elementos da matriz jacobiana J são:

$$x_\xi = (1/4) \cdot \begin{bmatrix} -(1-\eta) & 1-\eta & 1+\eta & -(1+\eta) \end{bmatrix} \begin{bmatrix} 0 \\ 1/3 \\ 1/3 \\ 0 \end{bmatrix}$$

$$= \frac{1}{4}\left[\frac{1-\eta}{3} + \frac{1+\eta}{3}\right] = \frac{1}{6},$$

e verifique que:

$$y_\xi = 0; \quad x_\eta = 0; \quad y_\eta = \frac{1}{6}.$$

Portanto, a matriz jacobiana J e seu jacobiano ($|J| = det(J)$) são dados por:

$$J = J(\xi,\eta) = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} = \begin{bmatrix} 1/6 & 0 \\ 0 & 1/6 \end{bmatrix};$$

$$|J| = 1/36.$$

Cálculo de K^e

A inversa da matriz J é dada por:

$$J^{-1} = \frac{1}{|J|}\begin{bmatrix} 1/6 & 0 \\ 0 & 1/6 \end{bmatrix} = \frac{1}{1/36}\begin{bmatrix} 1/6 & 0 \\ 0 & 1/6 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

Portanto, no integrando de K^1_{11}, já temos que:

$$\begin{aligned} J^{-1} \cdot k \cdot (J^{-1})^T &= \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \cdot k \cdot \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \\ &= 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot k \cdot 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= 36 \cdot k \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Cálculo de K^e

Substituindo em K^1_{11}, obtemos:

$$K^1_{11} = \int_{-1}^{1}\int_{-1}^{1} (\nabla\varphi^{\bar{1}}_1(\xi,\eta))^T \cdot 36 \cdot k \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \nabla\varphi^{\bar{1}}_1(\xi,\eta) \cdot \frac{1}{36}\, d\xi d\eta$$
$$= \int_{-1}^{1}\int_{-1}^{1} (\nabla\varphi^{\bar{1}}_1(\xi,\eta))^T \cdot k \cdot \nabla\varphi^{\bar{1}}_1(\xi,\eta)\, d\xi d\eta$$
$$= k\int_{-1}^{1}\int_{-1}^{1} (\nabla\varphi^{\bar{1}}_1(\xi,\eta))^T \cdot \nabla\varphi^{\bar{1}}_1(\xi,\eta)\, d\xi d\eta$$

- **Cálculo do gradiente:** No integrando, sabendo que

$\varphi^{\bar{1}}_1(\xi,\eta) = (1/4)\cdot(1-\xi)(1-\eta)$, temos:

$$\nabla\varphi^{\bar{1}}_1(\xi,\eta)) = \begin{bmatrix} \dfrac{\partial\varphi^{\bar{1}}_1}{\partial\xi} \\ \dfrac{\partial\varphi^{\bar{1}}_1}{\partial\eta} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -(1-\eta) \\ -(1-\xi) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \eta - 1 \\ \xi - 1 \end{bmatrix}$$

Cálculo de K^e

Portanto,

$$
\begin{aligned}
K_{11}^{1} &= k\int_{-1}^{1}\int_{-1}^{1}\frac{1}{4}\begin{bmatrix}\eta-1 & \xi-1\end{bmatrix}\cdot\frac{1}{4}\begin{bmatrix}\eta-1\\ \xi-1\end{bmatrix}\,d\xi d\eta\\
&= k\int_{-1}^{1}\int_{-1}^{1}\frac{1}{16}\left[(\eta-1)^2+(\xi-1)^2\right]\,d\xi d\eta\\
&= \frac{k}{16}\int_{-1}^{1}\int_{-1}^{1}\left[(\eta-1)^2+(\xi-1)^2\right]\,d\xi d\eta
\end{aligned}
$$

Devemos aproximar esta integral dupla usando quadratura gaussiana. Aqui, vamos usar apenas $npg = 2$.

Neste caso, os pontos de Gauss serão: $-\dfrac{\sqrt{3}}{3}, \dfrac{\sqrt{3}}{3}$,

enquanto os pesos de Gauss serão iguais a 1.

Cálculo de K^e

Na integral dupla, denotamos a função que está no integrando por:

$$g(\xi, \eta) = (\eta - 1)^2 + (\xi - 1)^2.$$

Assim, calculamos em

$$\int_{-1}^{1} \int_{-1}^{1} g(\xi, \eta)\; d\xi d\eta = \int_{-1}^{1} \left[\int_{-1}^{1} g(\xi, \eta)\; d\xi \right] d\eta$$

a integral que está entre colchetes primeiro. Note que nesta integral, somente ξ está variando no intervalo $[-1, 1]$.
Outra alternativa é calcular em

$$\int_{-1}^{1} \int_{-1}^{1} g(\xi, \eta)\; d\xi d\eta = \int_{-1}^{1} \left[\int_{-1}^{1} g(\xi, \eta)\; d\eta \right] d\xi$$

, onde a integral que está entre colchetes possui somente η variando em $[-1, 1]$.

Cálculo de K^e

Calculando a integral entre colchetes pela primeira alternativa usando quadratura gaussiana, obtemos:

$$\begin{aligned}\int_{-1}^{1} g(\xi,\eta)\, d\xi &= \sum_{j=1}^{npg} g(\xi_j,\eta)\cdot w_j = \sum_{j=1}^{2} g(\xi_j,\eta)\cdot w_j \\ &= g(\xi_1,\eta)\cdot w_1 + g(\xi_2,\eta)\cdot w_2\end{aligned}$$

Como $\xi_1 = -\dfrac{\sqrt{3}}{3}$, $\xi_2 = \dfrac{\sqrt{3}}{3}$, $w_1 = w_2 = 1$, obtemos:

$$\begin{aligned}\int_{-1}^{1} g(\xi,\eta)\, d\xi &= g\Big(-\frac{\sqrt{3}}{3},\eta\Big)\cdot 1 + g\Big(\frac{\sqrt{3}}{3},\eta\Big)\cdot 1 \\ &= \Big[(\eta-1)^2 + \Big(-\frac{\sqrt{3}}{3}-1\Big)^2\Big]\cdot 1 + \Big[(\eta-1)^2 + \Big(\frac{\sqrt{3}}{3}-1\Big)^2\Big]\cdot 1 \\ &= \frac{8}{3} + 2(\eta-1)^2\end{aligned}$$

Cálculo de K^e

Agora, calculamos por quadratura gaussiana

$$\int_{-1}^{1}\left[\frac{8}{3}+2(\eta-1)^2\right]\,d\eta,$$

onde denotamos $\bar{g}(\eta)=\dfrac{8}{3}+2(\eta-1)^2$. Assim, obtemos:

$$\int_{-1}^{1}\bar{g}(\eta)\,d\eta=\sum_{k=1}^{npg}\bar{g}(\eta_k)\cdot w_k=\sum_{k=1}^{2}\bar{g}(\eta_k)\cdot w_k=\bar{g}(\eta_1)\cdot w_1+\bar{g}(\eta_2)\cdot w_2$$

Como $\eta_1=-\dfrac{\sqrt{3}}{3}$, $\eta_2=\dfrac{\sqrt{3}}{3}$, $w_1=w_2=1$, obtemos:

$$\begin{aligned}\int_{-1}^{1}\bar{g}(\eta)\,d\eta&=\bar{g}\left(-\frac{\sqrt{3}}{3}\right)\cdot 1+\bar{g}\left(\frac{\sqrt{3}}{3}\right)\cdot 1\\&=\left[\frac{8}{3}+2\left(-\frac{\sqrt{3}}{3}-1\right)^2\right]+\left[\frac{8}{3}+2\left(\frac{\sqrt{3}}{3}-1\right)^2\right]=\frac{32}{3}\end{aligned}$$

Cálculo de K^e

Logo, obtemos:

$$\int_{-1}^{1}\int_{-1}^{1}\left[(\eta-1)^2+(\xi-1)^2\right]\,d\xi d\eta=\frac{32}{3},$$

e portanto,

$$K_{11}^1=\frac{k}{16}\int_{-1}^{1}\int_{-1}^{1}\left[(\eta-1)^2+(\xi-1)^2\right]\,d\xi d\eta=\frac{k}{16}\cdot\frac{32}{3}=\frac{2}{3}k$$

Usando $k=1$, obtemos:

$$K_{11}^1=\frac{2}{3}.$$

Exercício: verifique para os demais elementos de K^1 com $k=1$ que:

$$K_{12}^1=K_{14}^1=K_{21}^1=K_{23}^1=K_{32}^1=K_{34}^1=-\frac{1}{6};\quad K_{13}^1=K_{24}^1=K_{31}^1=K_{42}^1=-\frac{1}{3};$$

$$K_{22}^1=K_{33}^1=K_{44}^1=\frac{2}{3}.$$

Cálculo de F^e

Também temos para cada elemento Ω^e, ou seja, para todo $a = 1, 2, 3, 4$:

$$F_a^e = (f, \varphi_a^e) - (\bar{q}, \varphi_a^e)_{\Gamma_q} - \sum_{b=1}^{4} a(\varphi_a^e, \varphi_b^e)\, p_b^e = \underbrace{(f, \varphi_a^e)}_{f_a^e} - \underbrace{(\bar{q}, \varphi_a^e)_{\Gamma_q}}_{q_a^e} - \underbrace{\sum_{b=1}^{4} K_{ab}^e\, p_b^e}_{\bar{p}_a^e}$$

- **Cálculo de f_a^e:**

$$f_a^e = (f, \varphi_a^e) = \int_{\Omega^e} f(x, y) \cdot \varphi_a^e(x, y)\, dxdy$$

Usando mudança de variáveis $(x, y) \longmapsto (\xi, \eta)$, obtemos:

$$f_a^e = (f, \varphi_a^e) = \int_{\Omega^{\bar{b}}} f(\xi, \eta) \cdot \varphi_a^{\bar{b}}(\xi, \eta)\, |J|\, d\xi d\eta$$

Logo,

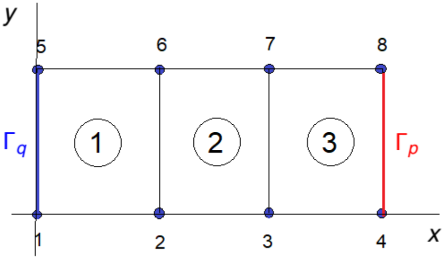
$$f^e = \begin{bmatrix} f_1^e \\ f_2^e \\ f_3^e \\ f_4^e \end{bmatrix}, \text{ para } e = 1, 2, 3.$$

Cálculo de F^e

- **Cálculo de q_a^e:**

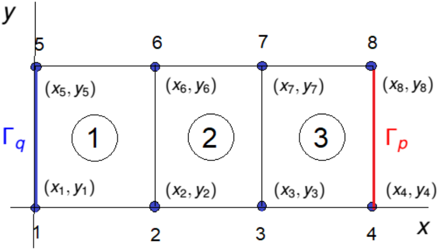
$$q_a^e = (\bar{q}, \varphi_a^e)_{\Gamma_q} = \int_{\Gamma_q} \bar{q}(s) \cdot \varphi_a^e(s)\, ds, \text{ onde } s \in \Gamma_q$$

- **Exemplo de cálculo de q_a^e:**



Note que: $\Gamma_q \in \Omega^1$

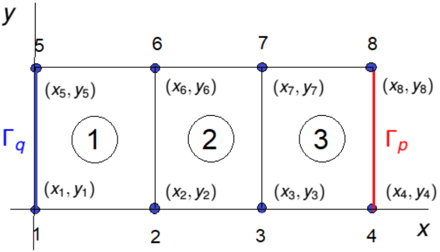
Cálculo de F^e



Aqui, os nós 1 e 5 da malha pertencem a $\Gamma_q = \{(0, y); y_1 \leq y \leq y_5\}$.
Logo,

$$q_a^e = (\bar{q}, \varphi_a^e)_{\Gamma_q} = \int_{\Gamma_q} \bar{q}(0, y) \cdot \varphi_a^e(0, y)\, dy = \int_{y_1}^{y_5} \bar{q}(0, y) \cdot \varphi_a^e(0, y)\, dy$$

Cálculo de F^e

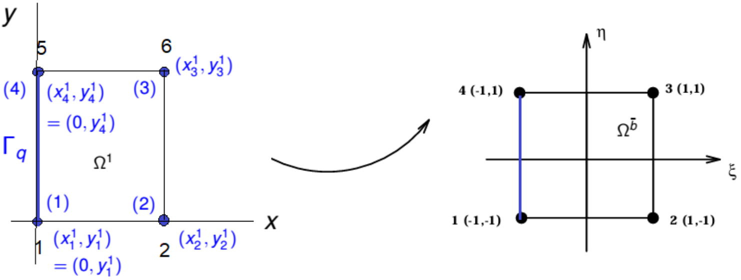


Aqui, os nós 1 e 5 da malha pertencem a $\Gamma_q = \{(0, y); y_1 \leq y \leq y_5\}$.

Logo,

$$q_a^e = (\bar{q}, \varphi_a^e)_{\Gamma_q} = \int_{\Gamma_q} \bar{q}(0, y) \cdot \varphi_a^e(0, y)\, dy = \int_{y_1}^{y_5} \bar{q}(0, y) \cdot \varphi_a^e(0, y)\, dy$$

Cálculo de F^e



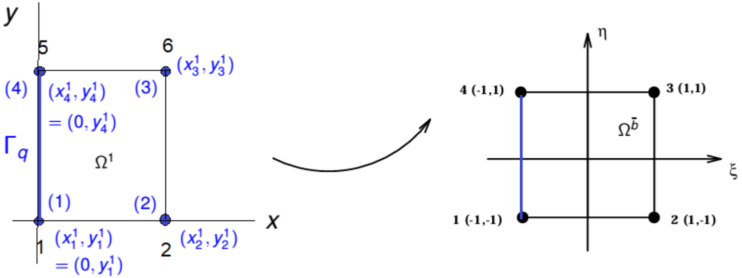
Na mudança de variáveis, $(0, y) \longmapsto (-1, \eta)$.

Logo, para $a = 1, 4$:

$$q_a^e = (\bar{q}, \varphi_a^e)_{\Gamma_q} = \int_{y_1^1 = y_1}^{y_4^1 = y_5} \bar{q}(0, y) \cdot \varphi_a^e(0, y)\, dy = \int_{-1}^{1} \bar{q}(-1, \eta) \cdot \varphi_a^{\bar{b}}(-1, \eta) \cdot y_\eta \, d\eta$$

Para $a = 2, 3$, $q_a^e = 0$.

Cálculo de F^e

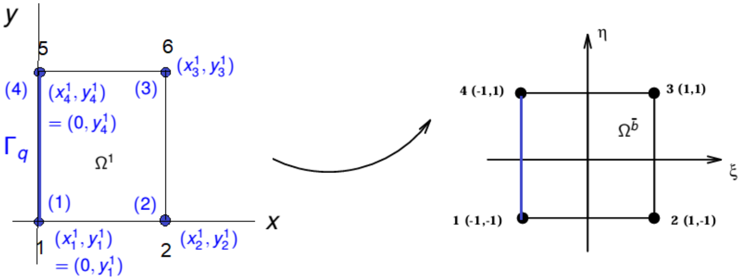


Lembre-se de que:

$$\varphi_1^{\bar{b}}(-1,\eta) = (1/4)(1-(-1))(1-\eta) = (1-\eta)/2;$$
$$\varphi_2^{\bar{b}}(-1,\eta) = (1/4)(1+(-1))(1-\eta) = 0;$$
$$\varphi_3^{\bar{b}}(-1,\eta) = (1/4)(1+(-1))(1+\eta) = 0;$$
$$\varphi_4^{\bar{b}}(-1,\eta) = (1/4)(1-(-1))(1+\eta) = (1+\eta)/2$$

Portanto, no elemento Ω^1, só serão usadas as funções de base locais $\varphi_1^{\bar{b}}$, $\varphi_4^{\bar{b}}$.

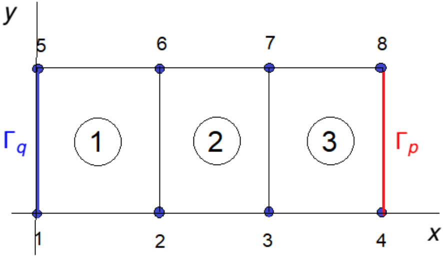
Cálculo de F^e



Logo,

$$q^1 = \begin{bmatrix} q_1^1 \\ q_2^1 \\ q_3^1 \\ q_4^1 \end{bmatrix} = \begin{bmatrix} (\bar{q}, \varphi_1^1)_{\Gamma_q} \\ 0 \\ 0 \\ (\bar{q}, \varphi_4^1)_{\Gamma_q} \end{bmatrix} ; \quad q^e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ para } e = 2, 3.$$

Cálculo de F^e



- **Cálculo de $\bar{p}_a^e$:**

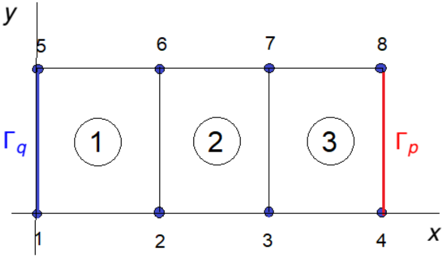
$$\bar{p}_a^e = \sum_{b=1}^{4} K_{ab}^e \, p_b^e$$

Neste exemplo, os nós prescritos 4 e 8 estão em $\Gamma_p = \{(x_4, y); y_4 \leq y \leq y_8\}$.

Logo, os nós prescritos estão no elemento Ω^3.

Neste caso, só vamos calcular $\bar{p}^3$, enquanto $\bar{p}^e = 0$ para $e = 1, 2$.

Cálculo de F^e

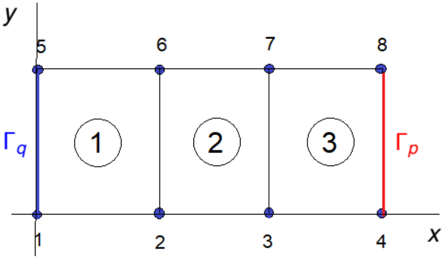


Logo,

$$\bar{p}_a^3 = \sum_{b=1}^{4} K_{ab}^3 \, p_b^3$$

$$\bar{p}^3 = \begin{bmatrix} \bar{p}_1^3 \\ \bar{p}_2^3 \\ \bar{p}_3^3 \\ \bar{p}_4^3 \end{bmatrix} = \begin{bmatrix} K_{11}^3 & K_{12}^3 & K_{13}^3 & K_{14}^3 \\ K_{21}^3 & K_{22}^3 & K_{23}^3 & K_{24}^3 \\ K_{31}^3 & K_{32}^3 & K_{33}^3 & K_{34}^3 \\ K_{41}^3 & K_{42}^3 & K_{43}^3 & K_{44}^3 \end{bmatrix} \begin{bmatrix} p_1^3 \\ p_2^3 \\ p_3^3 \\ p_4^3 \end{bmatrix}$$

Cálculo de F^e



Como $p_1^3 = p_4^3 = 0$, obtemos:

$$\bar{p}^3 = \begin{bmatrix} \bar{p}_1^3 \\ \bar{p}_2^3 \\ \bar{p}_3^3 \\ \bar{p}_4^3 \end{bmatrix} = \begin{bmatrix} K_{11}^3 & K_{12}^3 & K_{13}^3 & K_{14}^3 \\ K_{21}^3 & K_{22}^3 & K_{23}^3 & K_{24}^3 \\ K_{31}^3 & K_{32}^3 & K_{33}^3 & K_{34}^3 \\ K_{41}^3 & K_{42}^3 & K_{43}^3 & K_{44}^3 \end{bmatrix} \begin{bmatrix} 0 \\ p_2^3 \\ p_3^3 \\ 0 \end{bmatrix}; \ \bar{p}^e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ para } e = 1, 2.$$

Cálculo de F^e

Portanto, neste exemplo, para os elementos $e = 1, 2, 3$:

$$F^e = f^e - q^e - \bar{p}^e,$$

onde

$$f^e = \begin{bmatrix} f_1^e \\ f_2^e \\ f_3^e \\ f_4^e \end{bmatrix}, \text{ para } e = 1, 2, 3; \quad q^1 = \begin{bmatrix} q_1^1 \\ 0 \\ 0 \\ q_4^1 \end{bmatrix}; \quad q^e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ para } e = 2, 3;$$

$$\bar{p}^3 = \begin{bmatrix} \bar{p}_1^3 \\ \bar{p}_2^3 \\ \bar{p}_3^3 \\ \bar{p}_4^3 \end{bmatrix} = \begin{bmatrix} K_{11}^3 & K_{12}^3 & K_{13}^3 & K_{14}^3 \\ K_{21}^3 & K_{22}^3 & K_{23}^3 & K_{24}^3 \\ K_{31}^3 & K_{32}^3 & K_{33}^3 & K_{34}^3 \\ K_{41}^3 & K_{42}^3 & K_{43}^3 & K_{44}^3 \end{bmatrix} \begin{bmatrix} 0 \\ p_2^3 \\ p_3^3 \\ 0 \end{bmatrix}; \ \bar{p}^e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ para } e = 1, 2.$$

Cálculo de F^e

- Para $e = 1$:

$$F^1 = f^1 - q^1 - \bar{p}^1 = \begin{bmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \\ f_4^1 \end{bmatrix} - \begin{bmatrix} q_1^1 \\ 0 \\ 0 \\ q_4^1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1^1 - q_1^1 \\ f_2^1 \\ f_3^1 \\ f_4^1 - q_4^1 \end{bmatrix}$$

- Para $e = 2$:

$$F^2 = f^2 - q^2 - \bar{p}^2 = \begin{bmatrix} f_1^2 \\ f_2^2 \\ f_3^2 \\ f_4^2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1^2 \\ f_2^2 \\ f_3^2 \\ f_4^2 \end{bmatrix}$$

- Para $e = 3$:

$$F^3 = f^3 - q^3 - \bar{p}^3 = \begin{bmatrix} f_1^3 \\ f_2^3 \\ f_3^3 \\ f_4^3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \bar{p}_1^3 \\ \bar{p}_2^3 \\ \bar{p}_3^3 \\ \bar{p}_4^3 \end{bmatrix} = \begin{bmatrix} f_1^3 - \bar{p}_1^3 \\ f_2^3 - \bar{p}_2^3 \\ f_3^3 - \bar{p}_3^3 \\ f_4^3 - \bar{p}_4^3 \end{bmatrix}$$

Cálculo de F^e

Exemplo:

$$f(x,y) = f(x) = 25\pi^2 \operatorname{sen}\left(\frac{\pi x}{2}\right)$$

Podemos definir $f(x,y)$ como a interpolação:

$$f(x,y) = \sum_{b=1}^{4} \varphi_b^e(x,y)\cdot f(x_b^e, y_b^e) = 25\pi^2 \cdot \sum_{b=1}^{4} \varphi_b^e(x,y)\cdot \operatorname{sen}\left(\frac{\pi x_b^e}{2}\right)$$

- **Cálculo de f_a^e:**

$$\begin{aligned} f_a^e = (f, \varphi_a^e) &= \int_{\Omega^e} f(x,y)\cdot \varphi_a^e(x,y)\, dxdy \\ &= \int_{\Omega^e} 25\pi^2 \cdot \left[\sum_{b=1}^{4} \varphi_b^e(x,y)\cdot \operatorname{sen}\left(\frac{\pi x_b^e}{2}\right)\right]\cdot \varphi_a^e(x,y)\, dxdy \\ &= \sum_{b=1}^{4} \int_{\Omega^e} \varphi_a^e(x,y)\varphi_b^e(x,y)\cdot \left[25\pi^2 \cdot \operatorname{sen}\left(\frac{\pi x_b^e}{2}\right)\right] dxdy \end{aligned}$$

Cálculo de F^e

No elemento $e = 1$, já vimos que:

$$\begin{bmatrix} x_1^1 & x_2^1 & x_3^1 & x_4^1 \end{bmatrix}^T = \begin{bmatrix} 0 & 1/3 & 1/3 & 0 \end{bmatrix}^T$$

Assim,

$$\begin{aligned} f_a^1 &= \sum_{b=1}^{4} \int_{\Omega^1} \varphi_a^e(x,y)\varphi_b^1(x,y) \cdot \left[25\pi^2 \cdot \operatorname{sen}\left(\frac{\pi x_b^1}{2}\right)\right] \, dxdy \\ &= \int_{\Omega^1} \varphi_a^1(x,y)\varphi_1^1(x,y) \cdot \left[25\pi^2 \cdot \operatorname{sen}\left(\frac{\pi x_1^1}{2}\right)\right] \, dxdy \\ &\quad + \int_{\Omega^1} \varphi_a^1(x,y)\varphi_2^1(x,y) \cdot \left[25\pi^2 \cdot \operatorname{sen}\left(\frac{\pi x_2^1}{2}\right)\right] \, dxdy \\ &\quad + \int_{\Omega^1} \varphi_a^1(x,y)\varphi_3^1(x,y) \cdot \left[25\pi^2 \cdot \operatorname{sen}\left(\frac{\pi x_3^1}{2}\right)\right] \, dxdy \\ &\quad + \int_{\Omega^1} \varphi_a^1(x,y)\varphi_4^1(x,y) \cdot \left[25\pi^2 \cdot \operatorname{sen}\left(\frac{\pi x_4^1}{2}\right)\right] \, dxdy \end{aligned}$$

Cálculo de F^e

Fazendo as substituições dos valores dos nós, obtemos:

$$
\begin{aligned}
f_a^1 &= \int_{\Omega^1} \varphi_a^1(x,y)\varphi_1^1(x,y) \cdot \cancel{\left[25\pi^2 \cdot \text{sen}\left(\frac{\pi \cdot 0}{2}\right)\right]}\, dxdy \\
&+ \int_{\Omega^1} \varphi_a^1(x,y)\varphi_2^1(x,y) \cdot \left[25\pi^2 \cdot \text{sen}\left(\frac{\pi \cdot (1/3)}{2}\right)\right]\, dxdy \\
&+ \int_{\Omega^1} \varphi_a^1(x,y)\varphi_3^1(x,y) \cdot \left[25\pi^2 \cdot \text{sen}\left(\frac{\pi \cdot (1/3)}{2}\right)\right]\, dxdy \\
&+ \int_{\Omega^1} \varphi_a^1(x,y)\varphi_4^1(x,y) \cdot \cancel{\left[25\pi^2 \cdot \text{sen}\left(\frac{\pi \cdot 0}{2}\right)\right]}\, dxdy \\
&= \left[25\pi^2 \cdot \text{sen}\left(\frac{\pi}{6}\right)\right] \cdot \left(\int_{\Omega^1} \varphi_a^1(x,y)\varphi_2^1(x,y)\, dxdy + \int_{\Omega^1} \varphi_a^1(x,y)\varphi_3^1(x,y)\, dxdy\right)
\end{aligned}
$$

Usando mudança de variáveis $(x,y) \longmapsto (\xi,\eta)$, obtemos:

$$
f_a^1 = \left[25\pi^2 \,\text{sen}\left(\frac{\pi}{6}\right)\right] \left(\int_{\Omega^{\bar{1}}} \varphi_a^{\bar{1}}(\xi,\eta)\varphi_2^{\bar{1}}(\xi,\eta)|J|\, d\xi d\eta + \int_{\Omega^{\bar{1}}} \varphi_a^{\bar{1}}(\xi,\eta)\varphi_3^{\bar{1}}(\xi,\eta)|J|\, d\xi d\eta\right)
$$

Cálculo de F^e

Arrumando os termos, obtemos:

$$f_a^1 = \left[25\pi^2 \operatorname{sen}\left(\frac{\pi}{6}\right)\right] \cdot \left(\int_{\Omega^{\bar{1}}} (\varphi_a^{\bar{1}}(\xi,\eta)\varphi_2^{\bar{1}}(\xi,\eta) + \varphi_a^{\bar{1}}(\xi,\eta)\varphi_3^{\bar{1}}(\xi,\eta)) \cdot |J| \; d\xi d\eta\right)$$

Para $a = 1$, temos:

$$f_1^1 = \left[25\pi^2 \operatorname{sen}\left(\frac{\pi}{6}\right)\right] \cdot \left(\int_{\Omega^{\bar{1}}} (\varphi_1^{\bar{1}}(\xi,\eta)\varphi_2^{\bar{1}}(\xi,\eta) + \varphi_1^{\bar{1}}(\xi,\eta)\varphi_3^{\bar{1}}(\xi,\eta)) \cdot |J| \; d\xi d\eta\right)$$

Já vimos que em $e = 1$, $|J| = 1/36$.

Também sabemos que:

$\varphi_1^{\bar{1}}(\xi,\eta) = (1-\xi)(1-\eta)/4;$
$\varphi_2^{\bar{1}}(\xi,\eta) = (1+\xi)(1-\eta)/4;$
$\varphi_3^{\bar{1}}(\xi,\eta) = (1+\xi)(1+\eta)/4.$

Cálculo de F^e

Assim, fazendo as substituições, obtemos:

$$\varphi_1^{\bar{1}}(\xi,\eta)\varphi_2^{\bar{1}}(\xi,\eta) + \varphi_1^{\bar{1}}(\xi,\eta)\varphi_3^{\bar{1}}(\xi,\eta) = \left(\frac{(1-\xi)(1-\eta)}{4}\right)\left(\frac{(1+\xi)(1-\eta)}{4}\right) + \left(\frac{(1-\xi)(1-\eta)}{4}\right)\left(\frac{(1+\xi)(1+\eta)}{4}\right)$$

Desenvolvendo os termos, obtemos:

$$\varphi_1^{\bar{1}}(\xi,\eta)\varphi_2^{\bar{1}}(\xi,\eta) + \varphi_1^{\bar{1}}(\xi,\eta)\varphi_3^{\bar{1}}(\xi,\eta) = \frac{1}{8}(1-\xi^2)(1-\eta)$$

Substituindo na integral, obtemos:

$$\begin{aligned} f_1^1 &= \left[25\pi^2 \operatorname{sen}\left(\frac{\pi}{6}\right)\right] \cdot \frac{1}{8} \cdot \frac{1}{36} \cdot \left(\int_{\Omega^{\bar{1}}} (1-\xi^2)(1-\eta) \cdot \, d\xi d\eta\right) \\ &= \left[25\pi^2 \operatorname{sen}\left(\frac{\pi}{6}\right)\right] \cdot \frac{1}{288} \cdot \left(\int_{\Omega^{\bar{1}}} (1-\xi^2)(1-\eta) \cdot \, d\xi d\eta\right) \end{aligned}$$

Cálculo de F^e

Usando quadratura gaussiana na integral, denotamos:

$$g(\xi, \eta) = (1 - \xi^2)(1 - \eta)$$

e calculamos:

$$\int_{-1}^{1} \left[\int_{-1}^{1} g(\xi, \eta)\, d\xi \right] d\eta$$

Usando $npg = 2$, temos:

$$\begin{aligned}
\int_{-1}^{1} g(\xi, \eta)\, d\xi &= g\left(-\frac{\sqrt{3}}{3}, \eta\right) + g\left(\frac{\sqrt{3}}{3}, \eta\right) \\
&= (1 - (-\sqrt{3}/3)^2)(1 - \eta) + (1 - (\sqrt{3}/3)^2)(1 - \eta) \\
&= 2\left(1 - \frac{1}{3}\right)(1 - \eta) = \frac{4}{3}(1 - \eta)
\end{aligned}$$

Cálculo de F^e

Agora, denotamos

$$\bar{g}(\eta) = \frac{4}{3}(1-\eta)$$

e calculamos por quadratura gaussiana:

$$\int_{-1}^{1} \bar{g}(\eta)\, d\eta = \int_{-1}^{1} \frac{4}{3}(1-\eta)\, d\eta$$

Usando $npg = 2$, temos:

$$\int_{-1}^{1} \bar{g}(\eta)\, d\eta = \int_{-1}^{1} \frac{4}{3}(1-\eta)\, d\eta = \frac{4}{3}(1-(-\sqrt{3}/3)) + \frac{4}{3}(1-(\sqrt{3}/3)) = \frac{8}{3}$$

Portanto, fazendo a substituição, obtemos:

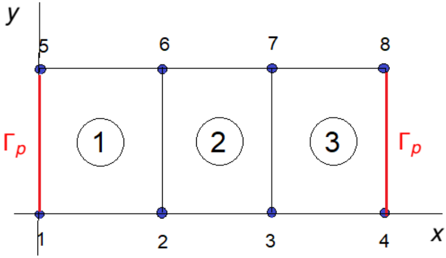
$$f_1^1 = \left[25\pi^2 \operatorname{sen}\left(\frac{\pi}{6}\right)\right] \cdot \frac{1}{288} \cdot \frac{8}{3} \approx 1,14231532$$

Exercício: Mostrar que $f_2^1 = f_3^1 = 2,28463065$; $f_4^1 = 1,14231532$

Cálculo de F^e

Trabalho:

Vamos considerar apenas a região:



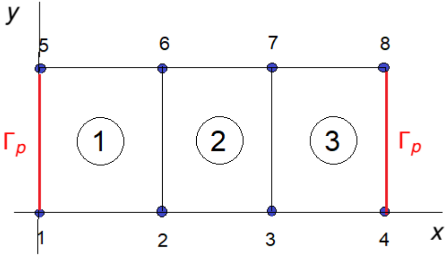
para o problema

$$\begin{cases} -k\Delta u = f, \quad \text{em} \quad \Omega = (0,1)\times(0,1), \\ u = 0^\circ\text{C}, \quad \text{em} \quad \Gamma_1 = \{(0,y);\ 0 \le y \le 1/3\}, \\ u = 100^\circ\text{C}, \quad \text{em} \quad \Gamma_2 = \{(1,y);\ 0 \le y \le 1/3\}, \end{cases}$$

onde $k = 1$ e $u(x,y) = 100\,\text{sen}((\pi/2)x)$.

Cálculo de F^e

Trabalho:
Vamos considerar apenas a região:



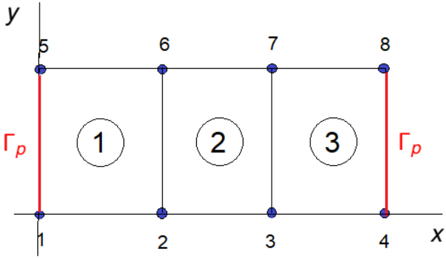
Note que substituindo $u(x, y)$ na equação do problema, obtemos:

$$f(x, y) = 25\pi^2 \operatorname{sen}((\pi/2)x)$$

Cálculo de F^e

Trabalho:

Vamos considerar apenas a região:



(a) Mostre que no elemento $e = 1$:

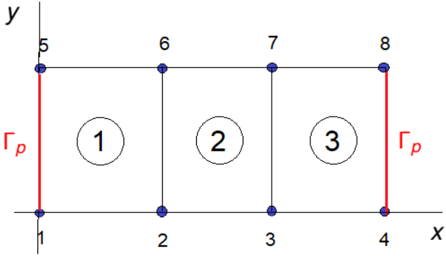
$$K^1 = \begin{bmatrix} 2/3 & -1/6 & -1/3 & -1/6 \\ -1/6 & 2/3 & -1/6 & -1/3 \\ -1/3 & -1/6 & 2/3 & -1/6 \\ -1/6 & -1/3 & -1/6 & 2/3 \end{bmatrix}$$

Dica: a matriz K^1 é simétrica.

Cálculo de F^e

Trabalho:

Vamos considerar apenas a região:



(b) Mostre que:

$$F^1 = \begin{bmatrix} 1,14231532 \\ 2,28463065 \\ 2,28463065 \\ 1,14231532 \end{bmatrix}; \quad F^2 = \begin{bmatrix} 4,26317883 \\ 5,09941168 \\ 5,09941168 \\ 4,26317883 \end{bmatrix}; \quad F^3 = \begin{bmatrix} 56,24172701 \\ -43,45219053 \\ -43,45219053 \\ 56,24172701 \end{bmatrix}$$

Cálculo de F^e

Trabalho:
(c) Sabendo que para os elementos 2 e 3, também temos:

$$K^2 = K^3 = \begin{bmatrix} 2/3 & -1/6 & -1/3 & -1/6 \\ -1/6 & 2/3 & -1/6 & -1/3 \\ -1/3 & -1/6 & 2/3 & -1/6 \\ -1/6 & -1/3 & -1/6 & 2/3 \end{bmatrix}$$

baseie-se nos slides da aula 09 para montar a matriz global K e o vetor global F.

(d) Resolva o sistema linear $Kc = F$ para achar a solução aproximada não prescrita:

$$c = \begin{bmatrix} c_2 \\ c_3 \\ c_6 \\ c_7 \end{bmatrix} = \begin{bmatrix} u_h(x_2, y_2) \\ u_h(x_3, y_3) \\ u_h(x_6, y_6) \\ u_h(x_7, y_7) \end{bmatrix}$$

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