Tópicos Especiais em Matemática Aplicada - 2025-1
UERJ

05 - Metodo dos Elementos Finitos - Caso 1D estacionário

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Github: https://github.com/rodrigolrmadureira/ElementosFinitos

Sumário

Método dos Elementos Finitos (MEF) - Caso 1D

No caso unidimensional, cada subintervalo $[x_e, x_{e+1}]$, $e = 1, 2, \ldots, m-1$, do domínio $\Omega = [x_1, x_m]$ é um elemento finito de tamanho $h = x_{e+1} - x_e$.

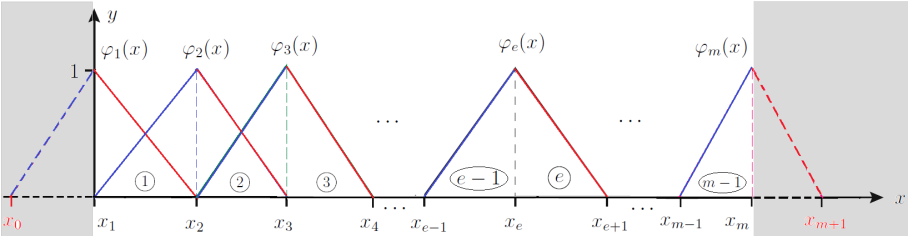


Figura: Funções de base de Lagrange linear $\varphi_i(x)$

Método dos Elementos Finitos (MEF) - Caso 1D

Ao invés de ”olhar” para todo o domínio, vamos ”olhar” para um único elemento *e*.

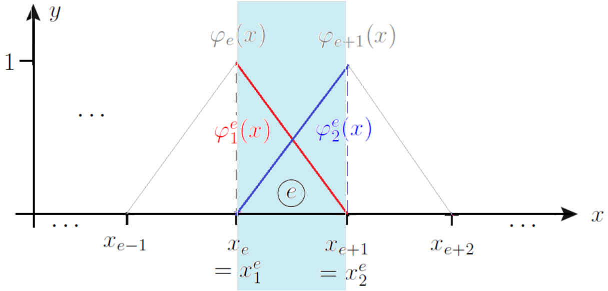


Figura: Funções do elemento e: $\varphi_1^e(x)$ e $\varphi_2^e(x)$

Podemos referenciar um nó da discretização do domínio de duas formas:

- x_i, onde i é o índice da discretização do intervalo $[x_1, x_m]$, $i = 1, 2, \ldots, m$. Neste caso, i é a **numeração global**. Assim, x_i é o **nó global**.
- x_a^e, onde a é o índice do nó no elemento finito e, $a = 1, 2$. Neste caso, a é a **numeração local**. Assim, x_a^e é o **nó local**.

Método dos Elementos Finitos (MEF) - Caso 1D

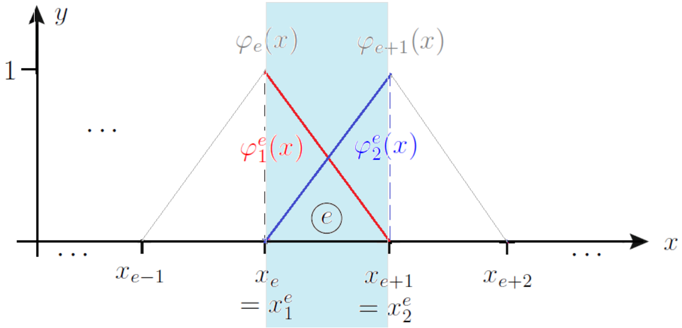


Figura: Funções do elemento e: $\varphi_1^e(x)$ e $\varphi_2^e(x)$

$$\begin{cases} \varphi_1^e(x) = \dfrac{x_{e+1} - x}{x_{e+1} - x_e} = \dfrac{x_{e+1} - x}{x_2^e - x_1^e} = \dfrac{x_{e+1} - x}{h}, \\ \varphi_2^e(x) = \dfrac{x - x_e}{x_{e+1} - x_e} = \dfrac{x - x_e}{x_2^e - x_1^e} = \dfrac{x - x_e}{h}. \end{cases} \tag{1}$$

Método dos Elementos Finitos (MEF) - Caso 1D

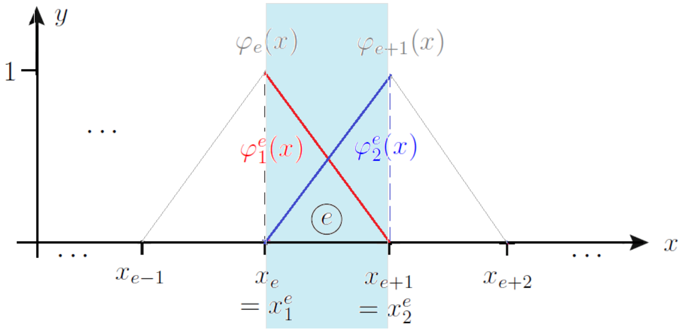


Figura: Funções do elemento e: $\varphi_1^e(x)$ e $\varphi_2^e(x)$

Note que:

$$\begin{cases} \varphi_1^e(x) = \varphi_e(x), \text{ para } x \in [x_e, x_{e+1}], \\ \varphi_2^e(x) = \varphi_{e+1}(x), \text{ para } x \in [x_e, x_{e+1}]. \end{cases} \tag{2}$$

Matriz local K^e

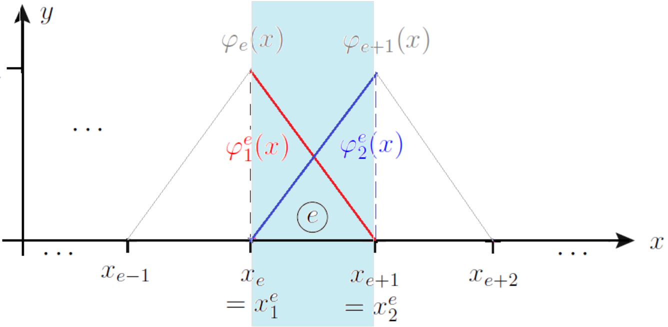


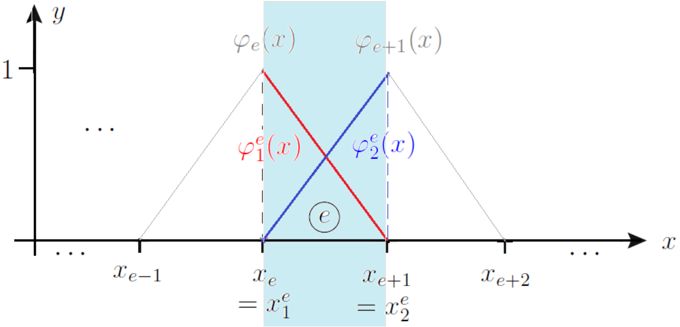
Figura: Funções do elemento e: $\varphi_1^e(x)$ e $\varphi_2^e(x)$

Note que restritos ao elemento finito e, obtemos o elemento da **matriz local** K^e:

$$K_{ab}^e = \alpha(\varphi_{ax}^e, \varphi_{bx}^e) + \beta(\varphi_a^e, \varphi_b^e) = \alpha \int_{x_1^e}^{x_2^e} \varphi_{ax}^e(x)\varphi_{bx}^e(x)dx + \beta \int_{x_1^e}^{x_2^e} \varphi_a^e(x)\varphi_b^e(x)dx,$$

para $a, b = 1, 2$.

Vetor força local F^e



Analogamente, obtemos o elemento do **vetor força local** F^e:

$$F_a^e = (f, \varphi_a^e) = \alpha \int_{x_1^e}^{x_2^e} f(x)\varphi_a^e(x)dx, \text{ para } a = 1, 2.$$

Montagem da matriz global K

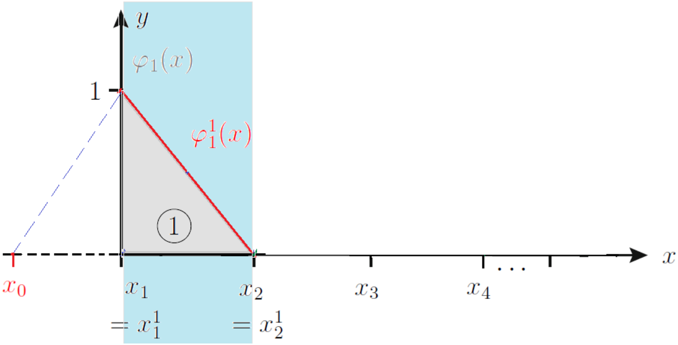
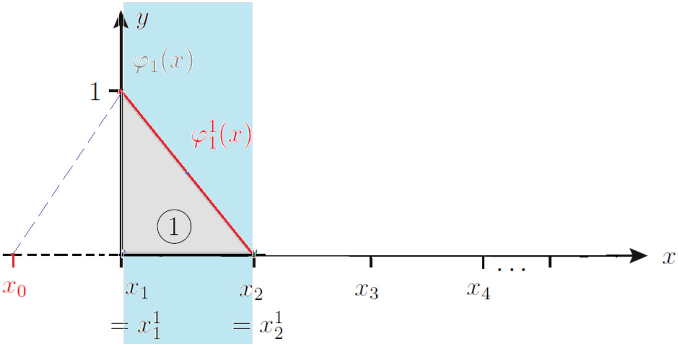


Figura: Funções do elemento e: $\varphi_1^e(x)$ e $\varphi_2^e(x)$

Podemos reescrever a **matriz global** K em termos de elementos da **matriz local** K^e, $e = 1, 2, \ldots, m$. Vimos na abordagem anterior que:

$$K_{11} = \alpha(\varphi_{1x}, \varphi_{1x}) + \beta(\varphi_1, \varphi_1) = \alpha \int_{x_1}^{x_2} (\varphi_{1x}(x))^2 dx + \beta \int_{x_1}^{x_2} (\varphi_1(x))^2 dx$$

Montagem da matriz global K

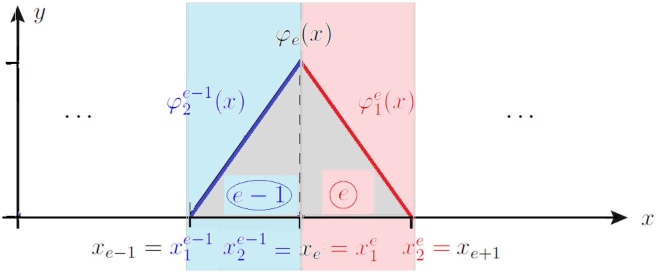


Podemos reescrever K_{11} como:

$$K_{11} = \alpha(\varphi_{1x}, \varphi_{1x}) + \beta(\varphi_1, \varphi_1) = \alpha \int_{x_1}^{x_2} (\varphi_{1x}(x))^2 dx + \beta \int_{x_1}^{x_2} (\varphi_1(x))^2 dx$$

$$= \alpha \int_{x_1^1}^{x_2^1} (\varphi_{1x}^1(x))^2 dx + \beta \int_{x_1^1}^{x_2^1} (\varphi_1^1(x))^2 dx = \alpha(\varphi_{1x}^1, \varphi_{1x}^1) + \beta(\varphi_1^1, \varphi_1^1) = K_{11}^1$$

Montagem da matriz global K



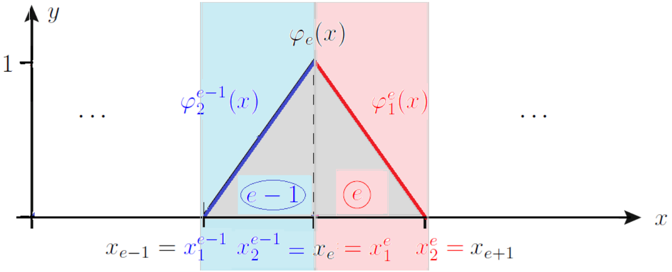
Podemos reescrever K_{ee} como:

$$K_{ee} = \alpha(\varphi_{ex}, \varphi_{ex}) + \beta(\varphi_e, \varphi_e) = \alpha \int_{x_{e-1}}^{x_{e+1}} (\varphi_{ex}(x))^2 dx + \beta \int_{x_{e-1}}^{x_{e+1}} (\varphi_e(x))^2 dx$$

$$= \alpha\Big(\int_{x_{e-1}}^{x_e} (\varphi_{ex}(x))^2 dx + \int_{x_e}^{x_{e+1}} (\varphi_{ex}(x))^2 dx\Big)$$

$$+ \beta\Big(\int_{x_{e-1}}^{x_e} (\varphi_e(x))^2 dx + \int_{x_e}^{x_{e+1}} (\varphi_e(x))^2 dx\Big)$$

Montagem da matriz global K

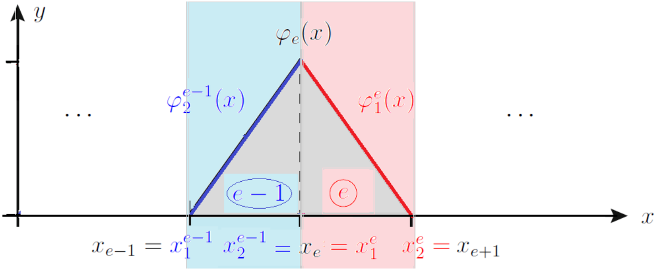


Podemos reescrever K_{ee} como:

$$K_{ee} = \alpha\Big(\int_{x_1^{e-1}}^{x_2^{e-1}} (\varphi_{2x}^{e-1}(x))^2 dx + \int_{x_1^e}^{x_2^e} (\varphi_{1x}^e(x))^2 dx\Big)$$

$$+\beta\Big(\int_{x_1^{e-1}}^{x_2^{e-1}} (\varphi_2^{e-1}(x))^2 dx + \int_{x_1^e}^{x_2^e} (\varphi_1^e(x))^2 dx\Big)$$

Montagem da matriz global K



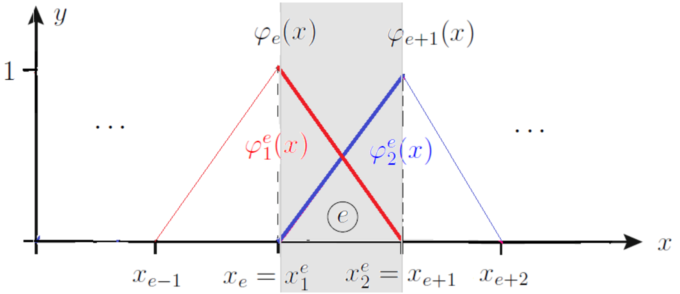
Podemos reescrever K_{ee} como:

$$K_{ee} = \alpha \int_{x_1^{e-1}}^{x_2^{e-1}} (\varphi_{2x}^{e-1}(x))^2 dx + \beta \int_{x_1^{e-1}}^{x_2^{e-1}} (\varphi_{2}^{e-1}(x))^2 dx$$

$$+ \alpha \int_{x_1^{e}}^{x_2^{e}} (\varphi_{1x}^{e}(x))^2 dx + \beta \int_{x_1^{e}}^{x_2^{e}} (\varphi_{1}^{e}(x))^2 dx$$

$$= \alpha(\varphi_{2x}^{e-1}, \varphi_{2x}^{e-1}) + \beta(\varphi_{2}^{e-1}, \varphi_{2}^{e-1}) + \alpha(\varphi_{1x}^{e}, \varphi_{1x}^{e}) + \beta(\varphi_{1}^{e}, \varphi_{1}^{e}) = K_{22}^{e-1} + K_{11}^{e}$$

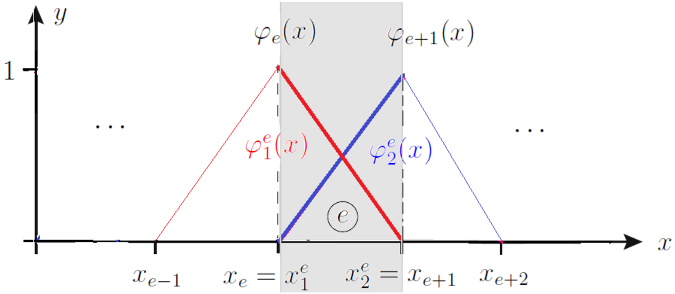
Montagem da matriz global K



Podemos reescrever $K_{e,e+1}$ como:

$$
\begin{aligned}
K_{e,e+1} &= \alpha(\varphi_{ex}, \varphi_{(e+1)x}) + \beta(\varphi_e, \varphi_{e+1}) \\
&= \alpha \int_{x_e}^{x_{e+1}} \varphi_{ex}(x)\varphi_{(e+1)x}(x)dx + \beta \int_{x_e}^{x_{e+1}} \varphi_e(x)\varphi_{e+1}(x)dx \\
&= \alpha \int_{x_1^e}^{x_2^e} \varphi_{1x}^e(x)\varphi_{2x}^e(x)dx + \beta \int_{x_1^e}^{x_2^e} \varphi_1^e(x)\varphi_2^e(x)dx \\
&= \alpha(\varphi_{1x}^e, \varphi_{2x}^e) + \beta(\varphi_1^e, \varphi_2^e) = K_{12}^e
\end{aligned}
$$

Montagem da matriz global K



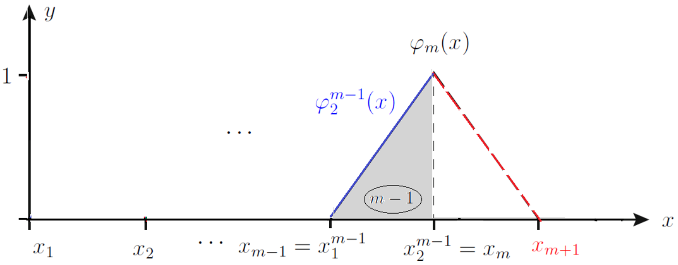
De forma análoga, obtemos:

$$K_{e+1,e} = K_{21}^e$$

Como K é simétrica ($K = K^T$), logo:

$$K_{12}^e = K_{21}^e$$

Montagem da matriz global K



Podemos reescrever K_{mm} como:

$$
\begin{aligned}
K_{mm} &= \alpha(\varphi_{mx}, \varphi_{mx}) + \beta(\varphi_m, \varphi_m) \\
&= \alpha \int_{x_{m-1}}^{x_m} (\varphi_{mx}(x))^2 dx + \beta \int_{x_{m-1}}^{x_m} (\varphi_m(x))^2 dx \\
&= \alpha \int_{x_1^{m-1}}^{x_2^{m-1}} (\varphi_{2x}^{m-1}(x))^2 dx + \beta \int_{x_1^{m-1}}^{x_2^{m-1}} (\varphi_2^{m-1}(x))^2 dx \\
&= \alpha(\varphi_{2x}^{m-1}, \varphi_{2x}^{m-1}) + \beta(\varphi_2^{m-1}, \varphi_2^{m-1}) = K_{22}^{m-1}
\end{aligned}
$$

Montagem do vetor global F

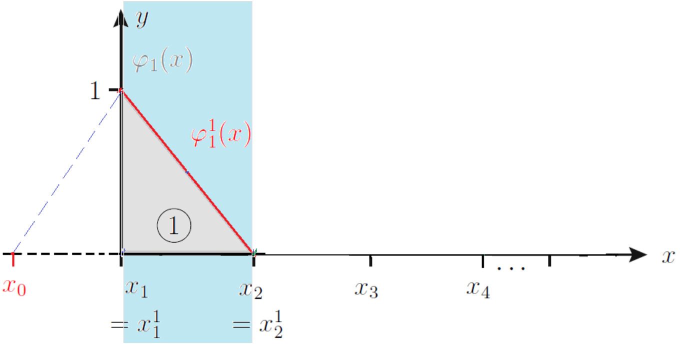
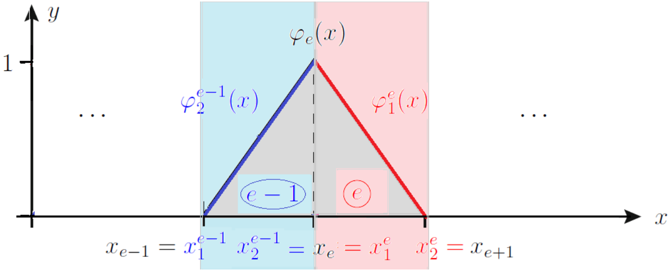


Figura: Funções do elemento e: $\varphi_1^e(x)$ e $\varphi_2^e(x)$

Podemos reescrever a **vetor global** F em termos de elementos do **vetor local** F^e, $e = 1, 2, \ldots, m$. Vimos na abordagem anterior que:

$$F_1 = (f, \varphi_1) = \int_{x_1}^{x_2} f(x)\varphi_1(x)dx = \int_{x_1^1}^{x_2^1} f(x)\varphi_1^1(x)dx = (f, \varphi_1^1) = F_1^1$$

Montagem do vetor global F

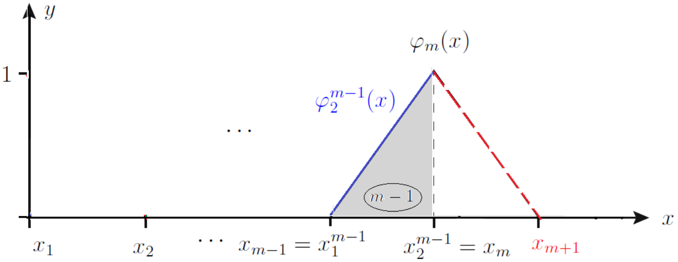


Podemos reescrever F_e como:

$$F_e = (f, \varphi_e) = \int_{x_{e-1}}^{x_{e+1}} f(x)\varphi_e(x)dx = \int_{x_{e-1}}^{x_e} f(x)\varphi_e(x)dx + \int_{x_e}^{x_{e+1}} f(x)\varphi_e(x)dx$$

$$= \int_{x_1^{e-1}}^{x_2^{e-1}} f(x)\varphi_2^{e-1}(x)dx + \int_{x_1^e}^{x_2^e} f(x)\varphi_1^e(x)dx = (f, \varphi_2^{e-1}) + (f, \varphi_1^e) = F_2^{e-1} + F_1^e$$

Montagem do vetor global F



Podemos reescrever F_m como:

$$F_m = (f, \varphi_m) = \int_{x_{m-1}}^{x_m} f(x)\varphi_m(x)dx = \int_{x_1^{m-1}}^{x_2^{m-1}} f(x)\varphi_2^{m-1}(x)dx$$

$$= (f, \varphi_2^{m-1}) = F_2^{m-1}$$

Resultados

Matriz de rigidez K:

$$K_{11} = K_{11}^{1};$$

$$K_{ee} = K_{22}^{e-1} + K_{11}^{e}, \text{ para } e = 2, 3, \ldots, m-1; \tag{3}$$

$$K_{e,e+1} = K_{12}^{i} \Rightarrow K_{e+1,e} = K_{21}^{i}, \text{ para } e = 1, 2, 3, \ldots, m-1; \textbf{ (Simetria: } K = K^{T}\textbf{)}$$

$$K_{mm} = K_{22}^{m-1}.$$

Vetor força F:

$$F_{1} = F_{1}^{1};$$

$$F_{e} = F_{2}^{e-1} + F_{1}^{e}, \text{ para } i = 2, 3, \ldots, m-1;$$

$$F_{m} = F_{2}^{m-1}. \tag{4}$$

Resultados

A **matriz de rigidez global** K,

$$\begin{bmatrix} K^1_{11} & K^1_{12} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ K^1_{21} & K^1_{22}+K^2_{11} & K^2_{12} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & K^2_{21} & K^2_{22}+K^3_{11} & K^3_{12} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & K^3_{21} & K^3_{22}+K^4_{11} & K^4_{12} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K^4_{21} & K^4_{22}+K^5_{11} & K^5_{12} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & * & K^{m-1}_{22} \end{bmatrix}$$

é montada a partir das **matrizes locais**

$$K^e = \begin{bmatrix} K^e_{11} & K^e_{12} \\ K^e_{21} & K^e_{22} \end{bmatrix},$$

para $e = 1, 2, \ldots, m-1$.

Resultados

O **vetor força global**,

$$F = \begin{bmatrix} F_1^1 \\ F_2^1 + F_1^2 \\ F_2^2 + F_1^3 \\ F_2^3 + F_1^4 \\ \vdots \\ F_2^{m-2} + F_1^{m-1} \\ F_2^{m-1} \end{bmatrix}$$

é montado a partir dos **vetores locais**

$$F^e = \begin{bmatrix} F_1^e \\ F_2^e \end{bmatrix},$$

para $e = 1, 2, \ldots, m-1$.

Quadratura gaussiana para K^e e F^e

A regra da **Quadratura Gaussiana** para integração numérica consiste em aproximar

$$\int_{-1}^{1} g(\xi) d\xi \approx \sum_{i=1}^{npg} w_i \cdot g(\xi_i),$$

onde *npg* é o **número de pontos de Gauss**,

ξ_i, $i = 1, 2, \ldots, npg$, são os **pontos de Gauss**,

w_i, $i = 1, 2, \ldots, npg$, são os **pesos de Gauss**.

Comparação com outros métodos de integração numérica:

Regra de integração numérica	**Número de pontos (nós)**	**Grau do polinômio com integração exata**
Trapézios	2	1
Simpson	3	2
Quadratura Gaussiana	**2**	**3**
	n	**2n - 1**

Quadratura gaussiana para K^e e F^e

Com $n = 2$, é possível mostrar que:

Pontos de Gauss: $\xi_1 = -\sqrt{3}/3$, $\xi_2 = \sqrt{3}/3$.
Pesos de Gauss: $w_1 = 1$, $w_2 = 1$.

Demonstração: Se com $npg = 2$, o resultado da integração numérica é exato para um polinômio de grau 3,

$$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3,$$

então,

$$\int_{-1}^{1} P_3(x)dx = \sum_{i=1}^{2} w_i \cdot P_3(\xi_i) = w_1 \cdot P_3(\xi_1) + w_2 \cdot P_3(\xi_2)$$

$$\Rightarrow \int_{-1}^{1} (a_0 + a_1 x + a_2 x^2 + a_3 x^3)dx = w_1 \cdot (a_0 + a_1 \xi_1 + a_2 \xi_1^2 + a_3 \xi_1^3)$$

$$+ w_2 \cdot (a_0 + a_1 \xi_2 + a_2 \xi_2^2 + a_3 \xi_2^3)$$

Quadratura gaussiana para K^e e F^e

$$
\begin{aligned}
&\Rightarrow a_0 \int_{-1}^{1} dx + a_1 \int_{-1}^{1} x dx + a_2 \int_{-1}^{1} x^2 dx + a_3 \int_{-1}^{1} x^3 dx \\
&= (a_0 w_1 + a_1 w_1 \xi_1 + a_2 w_1 \xi_1^2 + a_3 w_1 \xi_1^3) \\
&\quad + (a_0 w_2 + a_1 w_2 \xi_2 + a_2 w_2 \xi_2^2 + a_3 w_2 \xi_2^3) \\
&= a_0 (w_1 + w_2) + a_1 (w_1 \xi_1 + w_2 \xi_2) + a_2 (w_1 \xi_1^2 + w_2 \xi_2^2) + a_3 (w_1 \xi_1^3 + w_2 \xi_2^3)
\end{aligned}
$$

Igualando os coeficientes de a_0, a_1, a_2, a_3, obtemos:

$$
\begin{aligned}
\int_{-1}^{1} dx &= w_1 + w_2 \Rightarrow w_1 + w_2 = 2; \\
\int_{-1}^{1} x dx &= w_1 \xi_1 + w_2 \xi_2 \Rightarrow w_1 \xi_1 + w_2 \xi_2 = 0; \\
\int_{-1}^{1} x^2 dx &= w_1 \xi_1^2 + w_2 \xi_2^2 \Rightarrow w_1 \xi_1^2 + w_2 \xi_2^2 = 2/3; \\
\int_{-1}^{1} x^3 dx &= w_1 \xi_1^3 + w_2 \xi_2^3 \Rightarrow w_1 \xi_1^3 + w_2 \xi_2^3 = 0.
\end{aligned}
$$

Resolvendo este sistema, obtemos: $w_1 = 1$, $w_2 = 1$, $\xi_1 = -\sqrt{3}/3$, $\xi_2 = \sqrt{3}/3$.

Quadratura gaussiana para K^e e F^e

npg	Pontos	Pesos
1	0	2
2	$\pm\sqrt{3}/3 \approx \pm 0.5773502692$	1
3	$0, \pm\sqrt{3/5} \approx \pm 0.7745966692$	0.8888888889, 0.5555555556
4	$\pm 0.8611363116, \pm 0.3399810436$	0.3478548451, 0.6521451549
5	$\pm 0.9061798459, \pm 0.5384693101$, 0	0.2369268851, 0.4786286705, 0.5688888889
6	$\pm 0.9324695142, \pm 0.6612093865, \pm 0.2386191861$	0.1713244924, 0.3607615730, 0.4679139346
7	$\pm 0.9491079123, \pm 0.7415311856, \pm 0.4058451514$, 0	0.1294849662, 0.2797053915, 0.3818300505, 0.4179591837
8	$\pm 0.9602898565, \pm 0.7966664774, \pm 0.5255324099, \pm 0.1834346425$	0.1012285363, 0.2223810345, 0.3137066459, 0.3626837834

Tabela: Tabela para Quadratura Gaussiana

Quadratura gaussiana para K^e e F^e

Os elementos da **matriz local** K^e são dados por:

$$K^e_{ab} = a(\varphi^e_a, \varphi^e_b) = \alpha(\varphi^e_{ax}, \varphi^e_{bx}) + \beta(\varphi^e_a, \varphi^e_b)$$
$$= \alpha \int_{x_1^e}^{x_2^e} \varphi^e_{ax}(x)\varphi^e_{bx}(x)\, dx + \beta \int_{x_1^e}^{x_2^e} \varphi^e_a(x)\varphi^e_b(x)\, dx, \text{ para } a, b \in \{1, 2\}.$$

Para usar a **quadratura gaussiana** nessas integrais, devemos usar mudança de variáveis do domínio $[x_1^e, x_2^e]$ para o domínio $[-1, 1]$:

As distâncias são proporcionais nas duas retas. Logo,

$$\frac{x - x_1^e}{x_2^e - x_1^e} = \frac{\xi - (-1)}{1 - (-1)} \Rightarrow \frac{x - x_1^e}{h} = \frac{\xi + 1}{2} \Rightarrow x(\xi) = x_1^e + h\left(\frac{\xi + 1}{2}\right)$$

Logo, $\dfrac{dx}{d\xi} = \dfrac{h}{2} \Rightarrow dx = \dfrac{h}{2} d\xi.$

Quadratura gaussiana para K^e e F^e

Fazendo a mudança de variável nas funções locais $\varphi_a^e(x)$ para $a = 1, 2$, obtemos:

$$\varphi_1^e(x) = \frac{x_2^e - x}{x_2^e - x_1^e} = \frac{x_2^e - x}{h}$$

$$\Rightarrow \varphi_1^e(\xi) = \varphi_1^e(x(\xi)) = \frac{x_2^e - x(\xi)}{x_2^e - x_1^e} = \frac{x_2^e - x(\xi)}{h} = \frac{x_2^e - \left(x_1^e + h\left(\frac{\xi+1}{2}\right)\right)}{h}$$

$$\Rightarrow \varphi_1^e(\xi) = \frac{(x_2^e - x_1^e) - h\left(\frac{\xi+1}{2}\right)}{h} = \frac{h - h\left(\frac{\xi+1}{2}\right)}{h} = 1 - \left(\frac{\xi+1}{2}\right) = \frac{1-\xi}{2}$$

De forma análoga, obtemos:

$$\varphi_2^e(x) = \frac{x - x_1^e}{x_2^e - x_1^e} = \frac{x - x_1^e}{h} \Rightarrow \varphi_2^e(\xi) = \frac{1+\xi}{2}$$

Quadratura gaussiana para K^e e F^e

Portanto, as funções da base de Lagrange linear locais no domínio $\xi \in [-1, 1]$ são:

$$\begin{cases} \varphi_1^e(\xi) = \dfrac{1-\xi}{2}, \\ \varphi_2^e(\xi) = \dfrac{1+\xi}{2}, \end{cases} \tag{5}$$

e as suas derivadas são:

$$\begin{cases} \varphi_{1\xi}^e(\xi) = \dfrac{d\varphi_1^e}{d\xi}(\xi) = -\dfrac{1}{2}, \\ \varphi_{2\xi}^e(\xi) = \dfrac{d\varphi_1^e}{d\xi}(\xi) = \dfrac{1}{2}. \end{cases} \tag{6}$$

Quadratura gaussiana para K^e e F^e

Com a mudança de variável, obtemos os elementos de K^e no domínio $[-1,1]$. Lembrando que ao usar a regra da cadeia, para $a, b \in \{1,2\}$,

$$\varphi_{ax}(x) = \varphi_{ax}(\xi(x)) = \frac{d\varphi_a(\xi)}{d\xi}\frac{d\xi}{dx} = \varphi^e_{a\xi}(\xi)\frac{d\xi}{dx} = \varphi^e_{a\xi}(\xi)\frac{2}{h}.$$

Logo,

$$\begin{aligned}
K^e_{ab} &= a(\varphi^e_a, \varphi^e_b) = \alpha(\varphi^e_{ax}, \varphi^e_{bx}) + \beta(\varphi^e_a, \varphi^e_b) \\
&= \alpha \int_{x_1^e}^{x_2^e} \varphi^e_{ax}(x)\varphi^e_{bx}(x)\, dx + \beta \int_{x_1^e}^{x_2^e} \varphi^e_a(x)\varphi^e_b(x)\, dx \\
&= \alpha \int_{-1}^{1} \varphi^e_{a\xi}(\xi)\frac{2}{h}\varphi^e_{b\xi}(\xi)\frac{2}{h}\,\frac{h}{2}d\xi + \beta \int_{-1}^{1} \varphi^e_a(\xi)\varphi^e_b(\xi)\,\frac{h}{2}d\xi \\
&= \frac{2\alpha}{h}\int_{-1}^{1} \varphi^e_{a\xi}(\xi)\varphi^e_{b\xi}(\xi)\, d\xi + \frac{\beta h}{2}\int_{-1}^{1} \varphi^e_a(\xi)\varphi^e_b(\xi)\, d\xi
\end{aligned}$$

Quadratura gaussiana para K^e e F^e

Então,

$$\begin{aligned} K_{11}^e &= \frac{2\alpha}{h}\int_{-1}^{1}(\varphi_{1\xi}^e(\xi))^2\, d\xi + \frac{\beta h}{2}\int_{-1}^{1}(\varphi_1^e(\xi))^2 d\xi \\ &= \frac{2\alpha}{h}\int_{-1}^{1}\left(-\frac{1}{2}\right)^2 d\xi + \frac{\beta h}{2}\int_{-1}^{1}\left(\frac{1}{2}\right)^2 d\xi \\ &= \frac{\alpha}{2h}\int_{-1}^{1} d\xi + \frac{\beta h}{8}\int_{-1}^{1}(1-\xi)^2\, d\xi \end{aligned}$$

Usando **Quadratura Gaussiana** com $npg = 2$, temos $w_1 = w_2 = 1$, $\xi_1 = -\sqrt{3}/3$, $\xi_2 = \sqrt{3}/3$. Assim, obtemos:

$$\begin{aligned} \int_{-1}^{1} d\xi &= \sum_{i=1}^{2} w_i \cdot g(\xi_i) = w_1 \cdot g(\xi_1) + w_2 \cdot g(\xi_2) \\ &= 1 \cdot g(-\sqrt{3}/3) + 1 \cdot g(\sqrt{3}/3) = 1 \cdot 1 + 1 \cdot 1 = 2 \end{aligned}$$

Quadratura gaussiana para K^e e F^e

$$
\begin{aligned}
\int_{-1}^{1}(1-\xi)^2 d\xi &= \sum_{i=1}^{2} w_i \cdot g(\xi_i) = w_1 \cdot g(\xi_1) + w_2 \cdot g(\xi_2) \\
&= 1 \cdot g(-\sqrt{3}/3) + 1 \cdot g(\sqrt{3}/3) \\
&= 1 \cdot \left(1 - \left(-\frac{\sqrt{3}}{3}\right)\right)^2 + 1 \cdot \left(1 - \left(\frac{\sqrt{3}}{3}\right)\right)^2 \\
&= \left(1 + \frac{\sqrt{3}}{3}\right)^2 + \left(1 - \frac{\sqrt{3}}{3}\right)^2 = \frac{8}{3}
\end{aligned}
$$

Portanto, os elementos de K^e são:

$$K_{11}^e = \frac{\alpha}{2h} \cdot 2 + \frac{\beta h}{8} \cdot \frac{8}{3} = \frac{\alpha}{h} + \frac{\beta h}{3}$$

Analogamente, verifique que:

$$K_{12}^e = K_{21}^e = -\frac{\alpha}{h} + \frac{\beta h}{6}, \text{ pois } K \text{ é simétrica,}$$

$$K_{22}^e = \frac{\alpha}{h} + \frac{\beta h}{3}$$

Quadratura gaussiana para K^e e F^e

Assim, temos a **matriz de rigidez local** K^e **também simétrica**:

$$K^e = \begin{bmatrix} K^e_{11} & K^e_{12} \\ K^e_{21} & K^e_{22} \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{h} + \frac{\beta h}{3} & -\frac{\alpha}{h} + \frac{\beta h}{6} \\ -\frac{\alpha}{h} + \frac{\beta h}{6} & \frac{\alpha}{h} + \frac{\beta h}{3} \end{bmatrix}$$

Para calcular os elementos do vetor local F^e usando **Quadratura Gaussiana** com $npg = 2$, temos:

$$F^e_a = (f, \varphi^e_a) = \int_{x^e_1}^{x^e_2} f(x)\varphi^e_a(x)dx = \int_{-1}^{1} f(\xi)\varphi^e_a(\xi)\frac{h}{2}\, d\xi$$

$$= \frac{h}{2}\int_{-1}^{1} f(\xi)\varphi^e_a(\xi)\, d\xi \approx \frac{h}{2}\sum_{i=1}^{2} w_i \cdot f(\xi_i)\varphi^e_a(\xi_i)$$

$$\approx \frac{h}{2}\left[1 \cdot f\left(-\frac{\sqrt{3}}{3}\right)\varphi^e_a\left(-\frac{\sqrt{3}}{3}\right) + 1 \cdot f\left(\frac{\sqrt{3}}{3}\right)\varphi^e_a\left(\frac{\sqrt{3}}{3}\right)\right]$$

$$\approx \frac{h}{2}\left[f\left(-\frac{\sqrt{3}}{3}\right)\varphi^e_a\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)\varphi^e_a\left(\frac{\sqrt{3}}{3}\right)\right]$$

Quadratura gaussiana para K^e e F^e

Então,

$$
\begin{aligned}
F_1^e = (f, \varphi_1^e) &\approx \frac{h}{2}\Big[f\Big(-\frac{\sqrt{3}}{3}\Big)\Big(\frac{1-(-\sqrt{3}/3)}{2}\Big) + f\Big(\frac{\sqrt{3}}{3}\Big)\Big(\frac{1-(\sqrt{3}/3)}{2}\Big)\Big] \\
&\approx \frac{h}{4}\Big[f\Big(-\frac{\sqrt{3}}{3}\Big)\Big(1+\frac{\sqrt{3}}{3}\Big) + f\Big(\frac{\sqrt{3}}{3}\Big)\Big(1-\frac{\sqrt{3}}{3}\Big)\Big]
\end{aligned}
$$

Analogamente,

$$
\begin{aligned}
F_2^e &\approx \frac{h}{2}\Big[f\Big(-\frac{\sqrt{3}}{3}\Big)\varphi_2^e\Big(-\frac{\sqrt{3}}{3}\Big) + f\Big(\frac{\sqrt{3}}{3}\Big)\varphi_2^e\Big(\frac{\sqrt{3}}{3}\Big)\Big] \\
&\approx \frac{h}{2}\Big[f\Big(-\frac{\sqrt{3}}{3}\Big)\Big(\frac{1+(-\sqrt{3}/3)}{2}\Big) + f\Big(\frac{\sqrt{3}}{3}\Big)\Big(\frac{1+(\sqrt{3}/3)}{2}\Big)\Big] \\
&\approx \frac{h}{4}\Big[f\Big(-\frac{\sqrt{3}}{3}\Big)\Big(1-\frac{\sqrt{3}}{3}\Big) + f\Big(\frac{\sqrt{3}}{3}\Big)\Big(1+\frac{\sqrt{3}}{3}\Big)\Big]
\end{aligned}
$$

Os resultados vão depender do cálculo de $f(x)$.

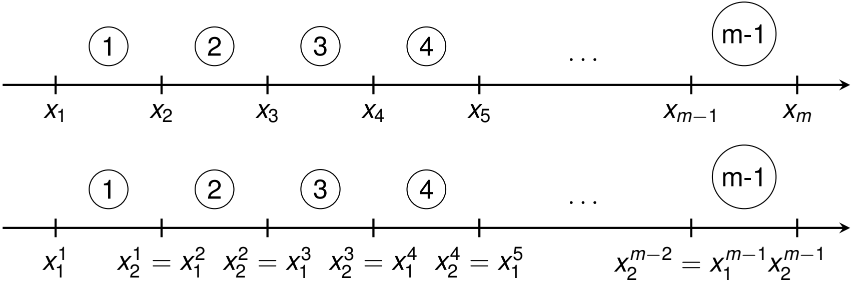
Quadratura gaussiana para K^e e F^e

Assim, temos o **vetor força local** F^e:

$$F^e = \begin{bmatrix} F_1^e \\ F_2^e \end{bmatrix} = \begin{bmatrix} \frac{h}{4}\left[f\left(-\frac{\sqrt{3}}{3}\right)\left(1+\frac{\sqrt{3}}{3}\right)+f\left(\frac{\sqrt{3}}{3}\right)\left(1-\frac{\sqrt{3}}{3}\right)\right] \\ \frac{h}{4}\left[f\left(-\frac{\sqrt{3}}{3}\right)\left(1-\frac{\sqrt{3}}{3}\right)+f\left(\frac{\sqrt{3}}{3}\right)\left(1+\frac{\sqrt{3}}{3}\right)\right] \end{bmatrix}$$

Condições de contorno - Estruturas de dados

Matriz LG: Matriz que associa cada nó local do elemento e ao seu respectivo nó global



a \ e	1	2	3	4	...	m-1
1	1	2	3	4	...	$m-1$
2	2	3	4	5	...	m

Tabela: Matriz LG

Condições de contorno - Estruturas de dados

Vetor EQ: Vetor com as posições das equações de acordo com as condições de contorno.

As posições das colunas são as posições dos nós globais: 1 para x_1, 2 para x_2, . . ., m para x_m .

O vetor será preenchido de maneira a respeitar as condições de contorno.

Condição 1: $u(x_1, t) = u(0, t) = 0 \Rightarrow EQ(1) = 0.$
Condição 2: $u(x_m, t) = u(1, t) = 0 \Rightarrow EQ(m) = 0.$

1	2	3	4	...	$m-1$	m
0	1	2	3	...	$m-2$	0

Tabela: Vetor EQ

Condições de contorno

$$
\begin{bmatrix}
* & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & *
\end{bmatrix}
$$

Por exemplo, nessa matriz global $m \times m$ onde $nel = 16$ e $m = nel + 1 = 17$.

Como

$EQ(LG(1,1)) = EQ(1) = 0;$
$EQ(LG(2, m-1)) = EQ(m) = 0,$

então nem a primeira linha e a primeira coluna, nem a m-ésima linha e a m-ésima coluna entram nos cálculos da matriz global.

Condições de contorno

$$
\begin{bmatrix}
* & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & *
\end{bmatrix}
$$

Com a remoção das linhas e colunas, a matriz global se torna uma matriz $(m-2)\times(m-2)$. Neste exemplo, com $nel = 16$, se transforma numa matriz 15×15.

Referências I

Liu, I.S.; Rincon, M.A.. **Introdução ao Método de Elementos Finitos, Análise e Aplicação**. IM/UFRJ, 2003.