ODE 2 Report

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September 2020

Abstract

This lab explores various ways of solving and analysing differential equations that can model various dynamical systems. In particular, we studied Euler's Method, Runge-Kutta, Heuns Method, and some modifications to these methods as well as comparing them to each other.

Introduction

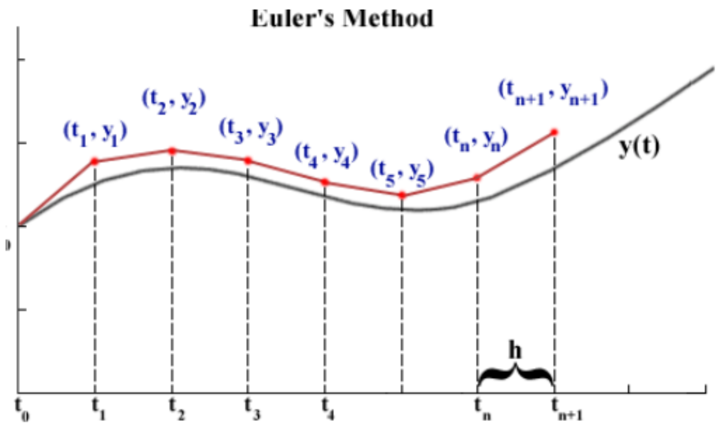
When dealing with an in ital value problem, we know at least one solution to the differential equation. From here, we want to find more. Some Initial Value Problems, or IVPs, are easy to solve analytically but some are impossible to solve with current methods and require a numerical approximation. The methods described in the report are possible ways to gain accurate approximations that could be used in real life like modeling fluid flow or rocket flight.

Euler's Method

The method approximates the solution to the differential equation rather than finding an analytical one, which is much more applicable to more complicated systems. The algorithm is defined by the function

$$x_{k+1} = x_k + hf(t_k, x_k)$$

Since we are working with an first order iterative function, the error is proportional to the square of the iteration step. Therefore for small values of n, it can be pretty accurate for a lot of iterations.



Modified Euler's

One could improve the accuracy of the previous method by reducing the step size, which would be achieved by reducing the parameter h. However, this would increase computational power by a lot. Another way to increase accuracy would be to average and midpoint method in combination with the Euler method, achieved by the formula

$$\begin{aligned} x_0 &= \alpha \\ x_{i+1} &= x_i + \frac{h}{2}\Big[f(t_i, x_i) + f\Big(t_{i+1}, x_i + hf(t_i, x_i)\Big)\Big] \end{aligned}$$

Huen's Method

This method also uses the Euler Method but it has a greater weight on the midpoint evaluation, defined by the formula

$$\begin{aligned} x_0 &= \alpha \\ x_{i+1} &= x_i + \frac{h}{4}\Big[f(t_i, x_i) + 3f\Big(t_i + \frac{2}{3}h, x_i + \frac{2}{3}hf(t_i, x_i)\Big)\Big] \end{aligned}$$

Runge- Kutta

Essentially the above methods remove errors corresponding to Taylor series terms of various order. This method uses this technique to derive methods for higher order terms without needing to compute high order derivatives. The method is defined as

$$
\begin{aligned}
x_0 &= \alpha \\
k_1 &= hf(t_i, x_i) \\
k_2 &= hf\left(t_i + \frac{h}{2}, x_i + \frac{1}{2}k_1\right) \\
k_3 &= hf\left(t_i + \frac{h}{2}, x_i + \frac{1}{2}k_2\right) \\
k_4 &= hf(t_{i+1}, x_i + k_3) \\
x_{i+1} &= x_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)
\end{aligned}
$$

Error Analysis

METHOD	RELATIVE ERROR
Euler	0.722245
Midpoint	0.00268413
Modified	0.00921544
Heun	0.00486124
4th-order R-K	1.40903e-07

By computing the difference between the analytically solved map and each method we arrive at the above result. We can see that in this case the 4th order R-K method is more accurate, but due to its many iterations of the function requirement it is computationally more expensive.

Conclusion

In conclusion different methods were analysed and compared against each other in order to further understand the process that goes into analysing complicated dynamical systems. The trade off between accuracy and computational power was further discussed.