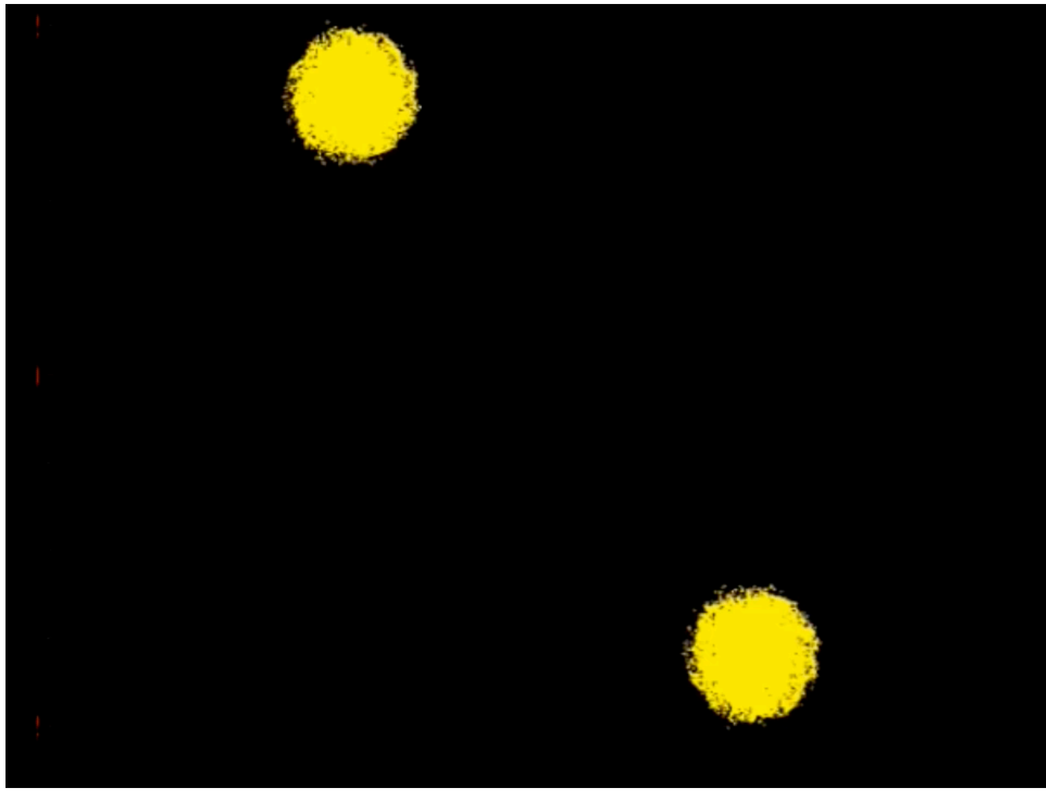
NagBody lectures: Interpolation



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Seminario de investigación,
Departamento de Física,
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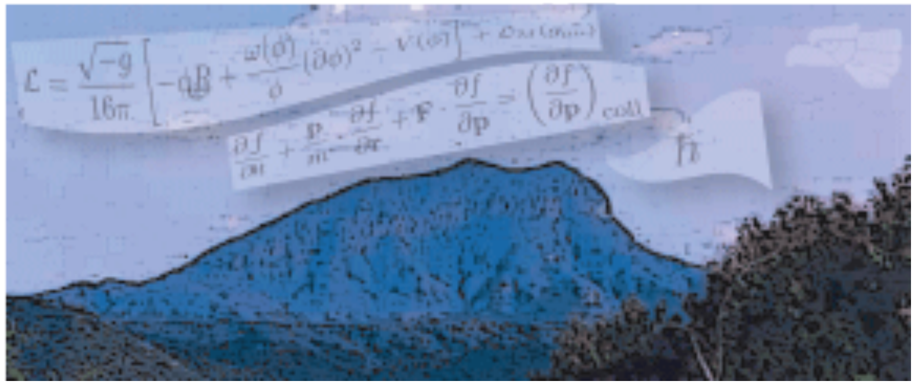
3 de febrero al XX de junio de 2022
Sesiones virtuales (Zoom, Meet, etcétera)

quintessence Group

INSTITUTO AVANZADO DE COSMOLOGIA



References and material

- Cosmología numérica y estadística: NagBody kit (http://bitbucket.org/rodriguezmeza). Mario A. Rodríguez-Meza.

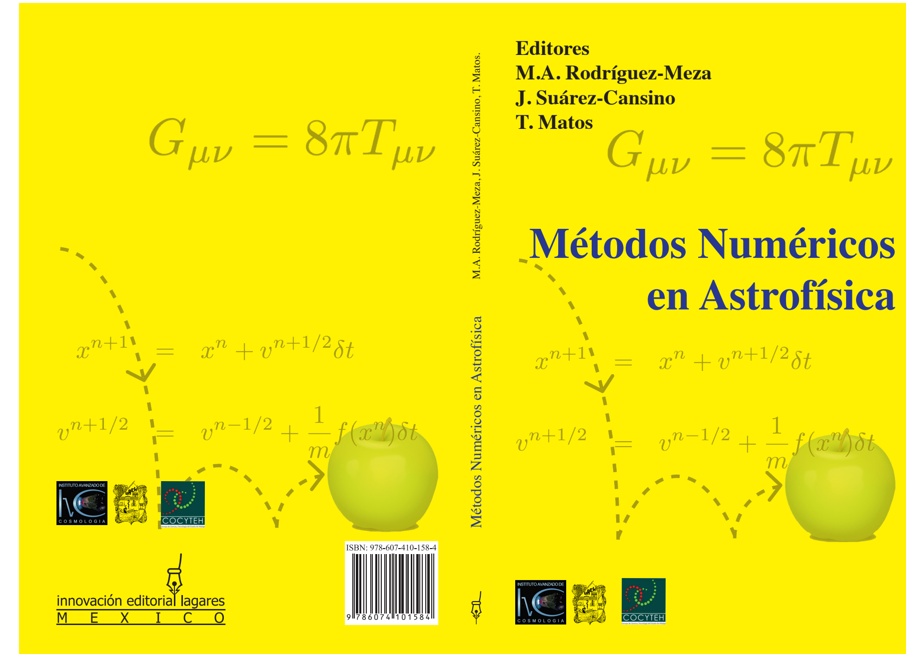
- Métodos numéricos en astrofísica, capítulo 1, Método de N-cuerpos en astrofísica. (https://www.researchgate.net/publication/316582859_Metodo_de_N-Cuerpos_en_Astrofisica)

- La estructura a gran escala del universo. Capítulo 22 en Travesuras cosmológicas de Einstein et al. https://www.researchgate.net/publication/316582400_La_estructura_a_gran_escala_del_universo_simulaciones_numericas

- https://www.researchgate.net/profile/Mario_Rodriguez-Meza

- https://www.researchgate.net/publication/314281416_Los_agujeros_negros_y_las_ondas_del_Dr_Einstein

- M.A. Rodriguez-Meza, Adv. Astron. 2012, 509682 (2012). arXiv: 1112.5201. (https://www.researchgate.net/publication/51967093_A_Scalar_Field_Dark_Matter_Model_and_Its_Role_in_the_Large-Scale_Structure_Formation_in_the_Universe)



Content: Introduction to numerical interpolation

- We know some times a function in tabular form.
- We ask if we can compute its value for any value of x.
- If x is in the inside the interval of the x-table, we call it interpolation, if not extrapolation.
- Polynomials are often used.

Content: Introduction to numerical interpolation

- Interpolation versus function approximation.
- In interpolation the function is given at a set of x values. You do not chose the set.
- In function approximation you are not given a set of values. Instead you can compute a value at a given x using an approximation to the function.

Content: Introduction to numerical interpolation

- Patological cases:

$$f(x) = 3x^2 + \frac{1}{\pi^4}\ln\left[(\pi - x)^2\right] + 1$$

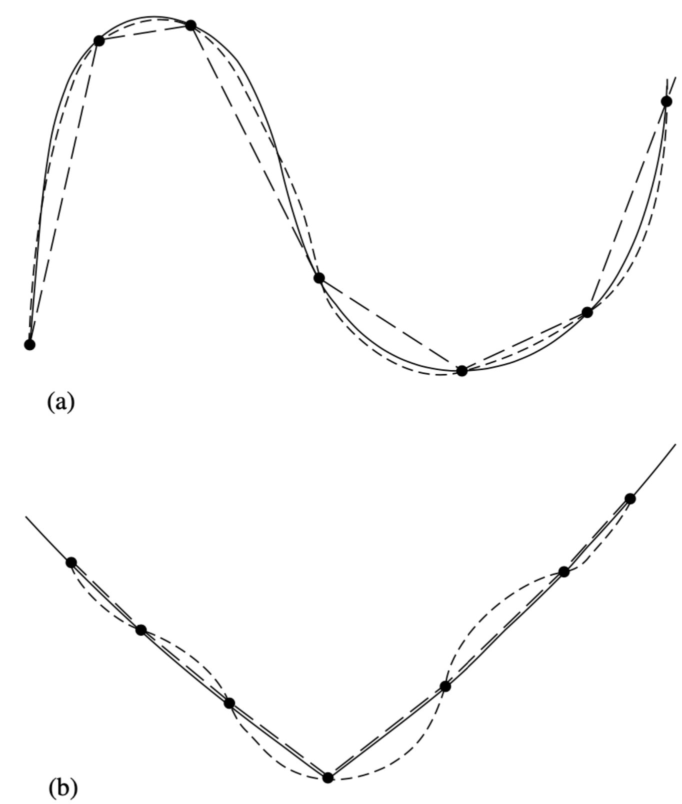
- Interpolation should provide an estimate of its own error.

Content: Introduction to numerical interpolation

- Interpolation steps:
  a. Fit once an interpolating function to the data.
  b. Evaluate that interpolating function at a target value of x.

Content: Introduction to numerical interpolation

- Interpolation steps (case two):
  a. Find the right starting position in the table (x's).
  b. Perform the interpolation using M nearby values around given x.
- This is a local interpolation. And do not in general give interpolated values that are not continuous in its first or higher order derivatives. Because crosses the x tabulated values and the interpolation scheme switches which tabulated points are the local one.



(a) (b)

Content: Introduction to numerical interpolation

- Spline interpolation: continuous first and second derivatives.
- Is a polynomial interpolation between two pair of interpolation points.
- Coefficients are determined non locally.
- Spline tend to be less oscillating than polynomials or other interpolation schemes.

Content: Introduction to numerical interpolation

- In a polynomial interpolation. Where coefficients are determined non locally.
- We chose a number M of local point to be used. This procedure takes $O(M2)$ operations, $M << N$.
- $M-1$ is the order of the interpolation method. Increasing the order not necessarily increase the accuracy.
- The non locality is designed to guarantee global smoothness in the interpolated function up to some derivative.

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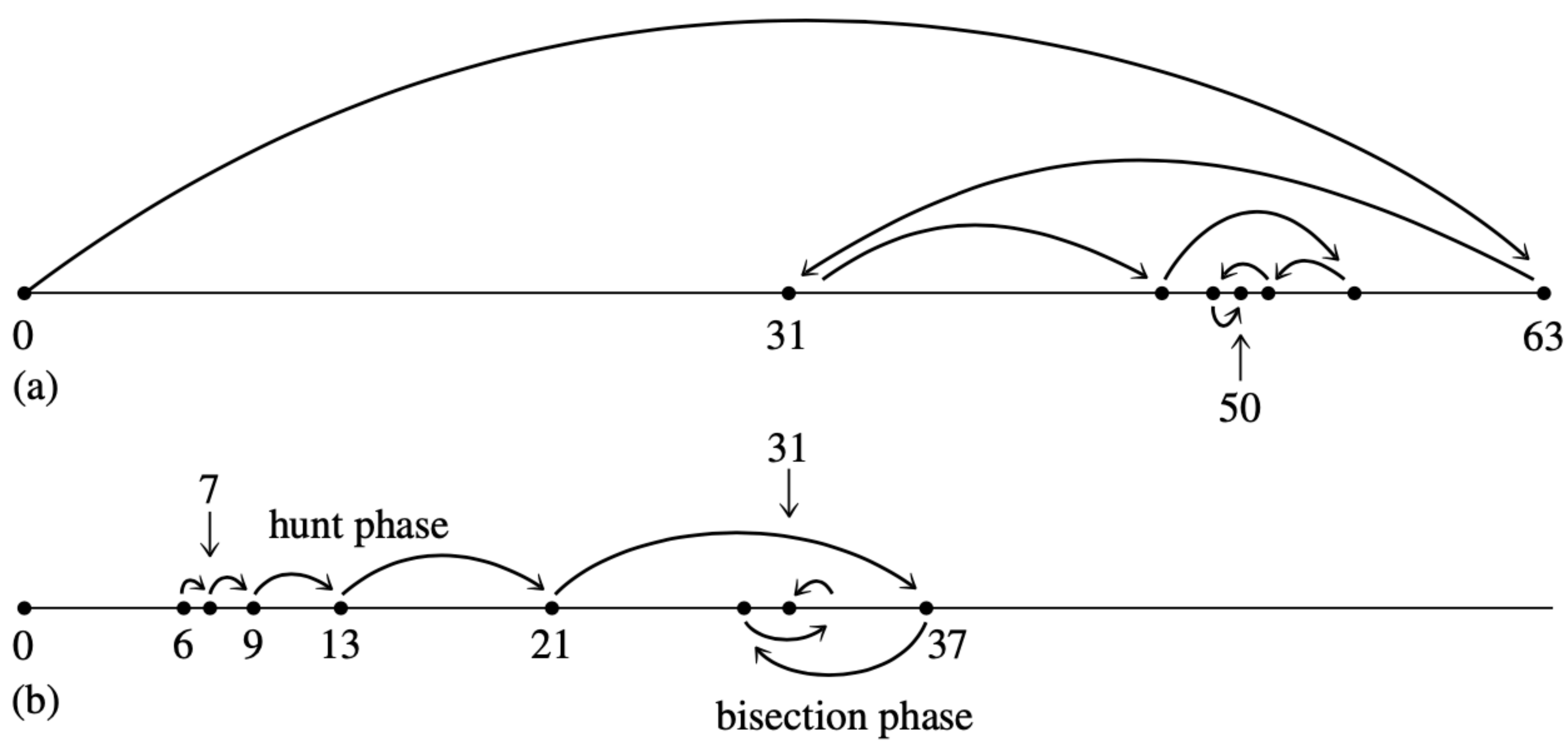
Searching algorithms

How to search an ordered list of elements:

- Bisection
- Search with correlated values



Linear interpolation

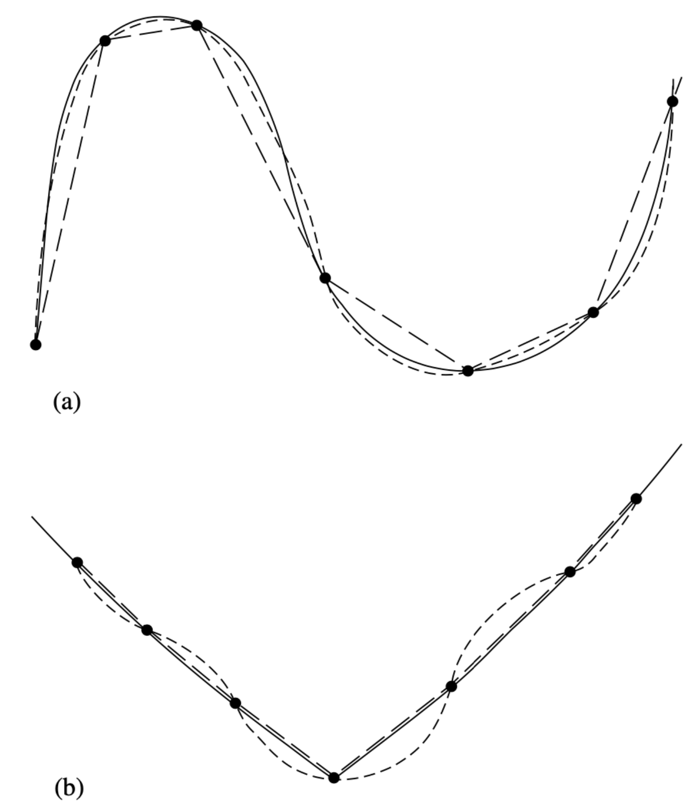
Using the linear form:

$$y = Ay_j + By_{j+1}$$

where

$$A \equiv \frac{x_{j+1} - x}{x_{j+1} - x_j} \qquad B \equiv 1 - A = \frac{x - x_j}{x_{j+1}}$$

- Discontinuity problems



Interpolation with rational functions

- Through any two points there is a unique line.
- Through any three points there is a unique quadratic.
- Etcetera.

$$P(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_{M-1})}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_{M-1})}y_0 + \frac{(x-x_0)(x-x_2)\dots(x-x_{M-1})}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_{M-1})}y_1 + \cdots + \frac{(x-x_0)(x-x_1)\dots(x-x_{M-2})}{(x_{M-1}-x_0)(x_{M-1}-x_1)\dots(x_{M-1}-x_{M-2})}y_{M-1}$$

- Neville's algorithm:

$$\begin{array}{lllll}
x_0: & y_0 = P_0 & & & \\
 & & P_{01} & & \\
x_1: & y_1 = P_1 & & P_{012} & \\
 & & P_{12} & & P_{0123} \\
x_2: & y_2 = P_2 & & P_{123} & \\
 & & P_{23} & & \\
x_3: & y_3 = P_3 & & &
\end{array}$$

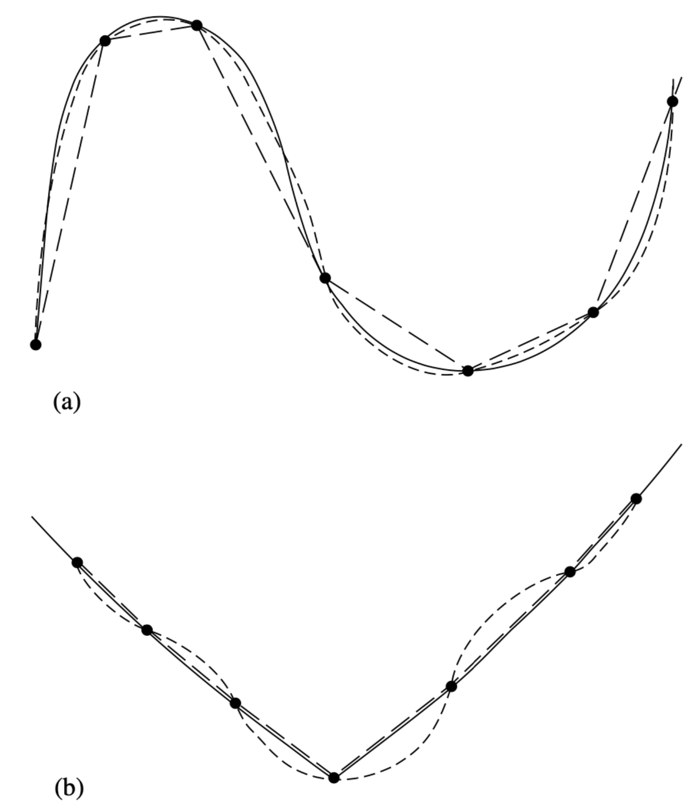
Cubic spline interpolation

AGAIN: Using the linear form:

$$y = Ay_j + By_{j+1}$$

where

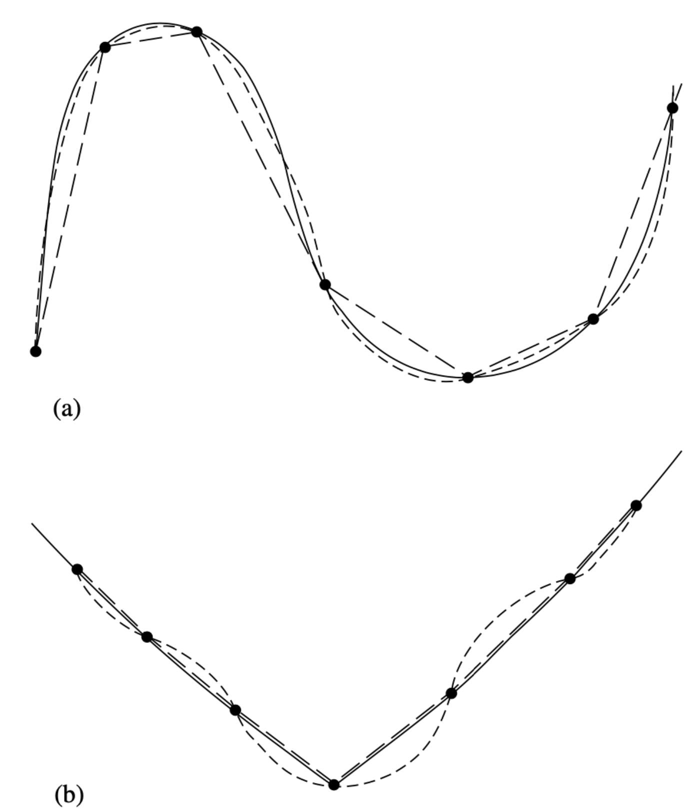
$$A \equiv \frac{x_{j+1} - x}{x_{j+1} - x_j} \qquad B \equiv 1 - A = \frac{x - x_j}{x_{j+1}}$$



- Discontinuity problems: has zero second derivative in the interior of each interval and an undefined, or infinite, second derivative at the abscisas xj.

Cubic spline interpolation

- Goal: to get an interpolation formula that is smooth in the first derivative and continuous in the second derivative, both within an interval and at the boundaries.
- Construct the cubic polynomial to have zero values at xj and xj+1
- Discontinuity problems



Cubic spline interpolation

- Replace the linear form by:

$$y = Ay_j + By_{j+1} + Cy''_j + Dy''_{j+1}$$

$$C \equiv \frac{1}{6}(A^3 - A)(x_{j+1} - x_j)^2 \qquad D \equiv \frac{1}{6}(B^3 - B)(x_{j+1} - x_j)^2$$

- Derive twice:

$$\frac{dy}{dx} = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{3A^2 - 1}{6}(x_{j+1} - x_j)y''_j + \frac{3B^2 - 1}{6}(x_{j+1} - x_j)y''_{j+1}$$

$$\frac{d^2y}{dx^2} = Ay''_j + By''_{j+1}$$

- Require that the first derivative be continuos across the boundary between two intervals:

$$\frac{x_j - x_{j-1}}{6}y''_{j-1} + \frac{x_{j+1} - x_{j-1}}{3}y''_j + \frac{x_{j+1} - x_j}{6}y''_{j+1} = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}}$$

Tridiagonal system of equations

- To find the unknowns second derivatives solve the system:

$$\frac{x_j - x_{j-1}}{6} y''_{j-1} + \frac{x_{j+1} - x_{j-1}}{3} y''_j + \frac{x_{j+1} - x_j}{6} y''_{j+1} = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}}$$

- N-2 linear equations in the N unknown, y''i, i=0, 1, … N-1.
- There is a two-parameters family of possible solutions:

a. Set one or two of y''0 and y''N-1 to zero. This is the natural cubic spline: at the boundaries second derivative are zero.

b. Set y''0 and y''N-1 to values calculated using so as to make the first derivative of the interpolation function have a specified value on either or both boundaries

$$\frac{dy}{dx} = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{3A^2 - 1}{6}(x_{j+1} - x_j) y''_j + \frac{3B^2 - 1}{6}(x_{j+1} - x_j) y''_{j+1}$$

Conclusions: Interpolation computation

We have seen:

- The concept of interpolation algorithms. And approximation of functions.
- Searching an ordered list of elements.
- Linear interpolation.
- Rational polynomial interpolation. Neville algorithm.
- Spline interpolation.
- Tridiagonal system of linear equations.

References and material

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- Métodos numéricos en astrofísica, capítulo 1, Método de N-cuerpos en astrofísica. (https://www.researchgate.net/publication/316582859_Metodo_de_N-Cuerpos_en_Astrofisica)
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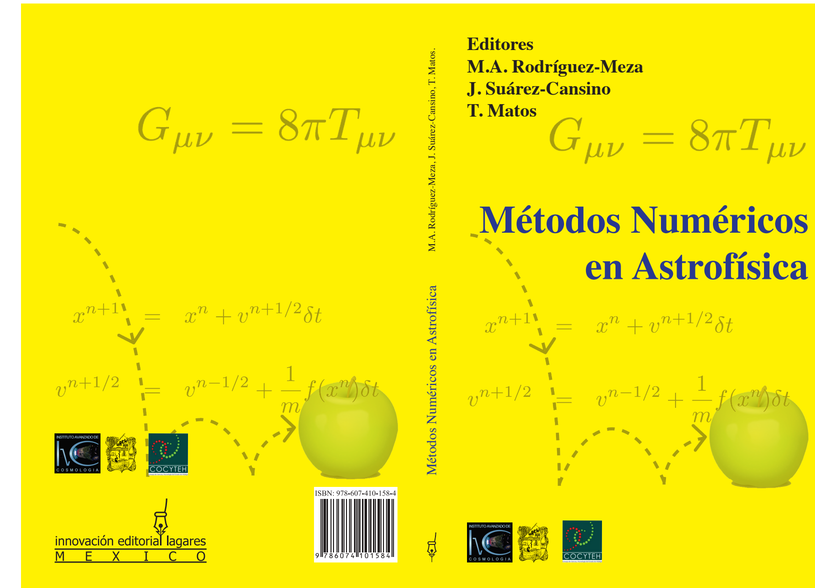

See you!