Determinism and termination in the semantics of the Céu programming language

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1 Abstract syntax

The *abstract syntax* of Céu programs is given by the following grammar:

$p \in P ::= \texttt{skip}$	do nothing
$\mid v := a$	assignment
$\mid \texttt{await}(e)$	await event
$\mid \texttt{emit}(e)$	emit event
$\mid \texttt{break}$	break innermost loop
$\mid \texttt{if}\, b\, \texttt{then}\, p_1\, \texttt{else}\, p_2$	conditional
$\mid p_1 ; p_2$	sequence
$\mid \texttt{loop}\, p_1$	repetition
$\mid p_1\, \texttt{and}\, p_2$	par/and
$\mid p_1\, \texttt{or}\, p_2$	par/or
$\mid \texttt{fin}\, p_1$	finalization
$\mid \texttt{@awaiting}(e, n)$	awaiting e since reaction n
$\mid \texttt{@emitting}(e, n)$	emitting e on stack level n
$\mid p_1\, \texttt{@loop}\, p_2$	unwinded loop

where $n \in N$ is an integer, $v \in V$ is a memory location (variable) identifier, $e \in E$ is an event identifier, $a \in A$ is an arithmetic expression, $b \in B$ is a boolean expression, and p, p_1, $p_2 \in P$ are programs. We assume the usual structure for arithmetic and boolean expressions, and omit their definition.

2 The reaction small-step relation

The *state* of a Céu program within a reaction is represented by a stack of events $\alpha = e_1 e_2 \ldots e_n \in E^*$ together with a memory map $m\colon v \to N \in \mathcal{M}$. A *configuration* is a 4-tuple $\langle p, \alpha, m, n \rangle \in \Delta$ that represents the situation of program p waiting to be evaluated in state $\langle \alpha, m \rangle$ and reaction n. Given an initial configuration, each small-step within a program reaction is determined by the reaction-small-step relation $\to\ \in\ \Delta \times \Delta$ such that $\langle p, \alpha, m, n \rangle \to \langle p', \alpha', m', n \rangle$ iff a small reaction step of program p in state $\langle \alpha, m \rangle$ and reaction number n evaluates to a modified program p' and a modified state $\langle \alpha', m' \rangle$ in the same reaction (n). Since relation $\rightarrow$ can only relate configurations with the same n, we shall write $\langle p, \alpha, m \rangle \xrightarrow{n} \langle p', \alpha', m' \rangle$ for $\langle p, \alpha, m, n \rangle \rightarrow \langle p', \alpha', m', n \rangle$.

Relation $\rightarrow$ is defined inductively on the structure of Céu programs with the help of the auxiliary functions *eval*, *blocked*, and *clear*. The *eval* function evaluates arithmetic and boolean expressions on a given memory; we omit its definition and assume that such evaluation is deterministic and always terminates. The *blocked* function is a predicate that determines if all trails of a program p are blocked on a given event stack and reaction number. And the *clear* function extracts the body of `fin` from a given program.

Definition 1. Function *blocked*: $P \times E^* \times N \rightarrow \{0, 1\}$ is defined inductively as follows.

$$
\begin{aligned}
\mathit{blocked}(v \coloneqq a, e\alpha, n) &= 0 \\
\mathit{blocked}(\texttt{await}(e'), e\alpha, n) &= 0 \\
\mathit{blocked}(\texttt{emit}(e'), e\alpha, n) &= 0 \\
\mathit{blocked}(\texttt{break}, e\alpha, n) &= 0 \\
\mathit{blocked}(\texttt{if}\, v\, \texttt{then}\, p_1\, \texttt{else}\, p_2, e\alpha, n) &= 0 \\
\mathit{blocked}(p_1; p_2, e\alpha, n) &= \mathit{blocked}(p_1, e\alpha, n) \\
\mathit{blocked}(\texttt{loop}\, p, e\alpha, n) &= 0 \\
\mathit{blocked}(p_1\, \texttt{and}\, p_2, e\alpha, n) &= \mathit{blocked}(p_1, e\alpha, n) \cdot \mathit{blocked}(p_2, e\alpha, n) \\
\mathit{blocked}(p_1\, \texttt{or}\, p_2, e\alpha, n) &= \mathit{blocked}(p_1, e\alpha, n) \cdot \mathit{blocked}(p_2, e\alpha, n) \\
\mathit{blocked}(\texttt{fin}\, p_1, e\alpha, n) &= 0 \\
\mathit{blocked}(\texttt{@awaiting}(e', n'), e\alpha, n) &= \begin{cases} 1 & \text{if } e \neq e' \text{ or } n = n' \\ 0 & \text{otherwise} \end{cases} \\
\mathit{blocked}(\texttt{@emitting}(n'), e\alpha, n) &= \begin{cases} 1 & \text{if } |e\alpha| \neq n' \\ 0 & \text{otherwise} \end{cases} \\
\mathit{blocked}(p_1\, \texttt{@loop}\, p_2, e\alpha, n) &= \mathit{blocked}(p_1, e\alpha, n)
\end{aligned}
$$

Definition 2. Function $clear\colon P \to P$ is defined inductively as follows.

$$
\begin{aligned}
clear(v := a) &= \texttt{skip} \\
clear(\texttt{await}(e')) &= \texttt{skip} \\
clear(\texttt{emit}(e')) &= \texttt{skip} \\
clear(\texttt{break}) &= \texttt{skip} \\
clear(\texttt{if}\, v\, \texttt{then}\, p_1\, \texttt{else}\, p_2) &= \texttt{skip} \\
clear(p_1; p_2) &= clear(p_1); clear(p_2) \\
clear(\texttt{loop}\, p) &= clear(p) \\
clear(p_1\ \texttt{and}\ p_2) &= clear(p_1); clear(p_2) \\
clear(p_1\ \texttt{or}\ p_2) &= clear(p_1); clear(p_2) \\
clear(\texttt{fin}\, p) &= p \\
clear(\texttt{@awaiting}(e', n')) &= \texttt{skip} \\
clear(\texttt{@emitting}(n')) &= \texttt{skip} \\
clear(p_1\, \texttt{@loop}\, p_2) &= clear(p_1)
\end{aligned}
$$

Definition 3 (Reaction small-step)**.** Relation $\rightarrow \subseteq \Delta \times \Delta$ is defined inductively as follows.

Await and emit

(R_1) $\quad \langle \texttt{await}(e), \alpha, m\rangle \xrightarrow{n} \langle \texttt{@awaiting}(e, n'), \alpha, m\rangle \quad$ with $n' = n + 1$

(R_2) $\quad \langle \texttt{@awaiting}(e, n'), e\alpha, m\rangle \xrightarrow{n} \langle \texttt{skip}, e\alpha, m\rangle \quad$ if $n' \leq n$

(R_3) $\quad \langle \texttt{emit}(e), \alpha, m\rangle \xrightarrow{n} \langle \texttt{@emitting}(n'), e\alpha, m\rangle \quad$ with $n' = |\alpha|$

(R_4) $\quad \langle \texttt{@emitting}(n'), \alpha, m\rangle \xrightarrow{n} \langle \texttt{skip}, \alpha, m\rangle \quad$ if $n' = |\alpha|$

Conditionals

(R_5) $\quad \langle \texttt{if}\, b\, \texttt{then}\, p_1\, \texttt{else}\, p_2, \alpha, m\rangle \xrightarrow{n} \langle p_1, \alpha, m\rangle \quad$ if $eval(b, m) = 1$

(R_6) $\quad \langle \texttt{if}\, b\, \texttt{then}\, p_1\, \texttt{else}\, p_2, \alpha, m\rangle \xrightarrow{n} \langle p_2, \alpha, m\rangle \quad$ if $eval(b, m) = 0$

Sequences

(R_7) $\quad \langle v := a; p, \alpha, m\rangle \xrightarrow{n} \langle p, \alpha, m'\rangle \quad$ with $m' = m[v/eval(a)]$

(R_8) $\quad \langle \texttt{break}; p, \alpha, m\rangle \xrightarrow{n} \langle \texttt{break}, \alpha, m\rangle$

(R_9) $\quad \dfrac{\langle p_1, \alpha, m\rangle \xrightarrow{n} \langle p_1', \alpha', m'\rangle}{\langle p_1; p_2, \alpha, m\rangle \xrightarrow{n} \langle p_1'; p_2, \alpha', m'\rangle} \quad$ if $p_1 \neq v := a, \texttt{break}$

Loops

$$(R_{10}) \quad \langle \texttt{loop}\ p, \alpha, m\rangle \xrightarrow{n} \langle p\ \texttt{@loop}\ p, \alpha, m\rangle$$

$$(R_{11}) \quad \langle v := a\ \texttt{@loop}\ p, \alpha, m\rangle \xrightarrow{n} \langle \texttt{loop}\ p, \alpha, m'\rangle \qquad \text{with } m' = m[v/\mathit{eval}(a)]$$

$$(R_{12}) \quad \langle \texttt{break}\ \texttt{@loop}\ p, \alpha, m\rangle \xrightarrow{n} \langle \texttt{skip}, \alpha, m\rangle$$

$$(R_{13}) \quad \frac{\langle p_1, \alpha, m\rangle \xrightarrow{n} \langle p_1', \alpha', m'\rangle}{\langle p_1\ \texttt{@loop}\ p_2, \alpha, m\rangle \xrightarrow{n} \langle p_1'\ \texttt{@loop}\ p_2, \alpha', m'\rangle} \qquad \text{if } p_1 \neq v := a, \texttt{break}$$

Par/and

$$(R_{14}) \quad \langle v := a\ \texttt{and}\ p, \alpha, m\rangle \xrightarrow{n} \langle p, \alpha, m'\rangle \qquad \text{with } m' = m[v/\mathit{eval}(a)]$$

$$(R_{15}) \quad \langle p\ \texttt{and}\ v := a, \alpha, m\rangle \xrightarrow{n} \langle p, \alpha, m'\rangle \qquad \text{with } m' = m[v/\mathit{eval}(a)]$$

$$(R_{16}) \quad \langle \texttt{break}\ \texttt{and}\ p, \alpha, m\rangle \xrightarrow{n} \langle p'; \texttt{break}, \alpha, m\rangle \qquad \text{if } p \neq v := a, \text{ with } p' = \mathit{clear}(p)$$

$$(R_{17}) \quad \langle p\ \texttt{and}\ \texttt{break}, \alpha, m\rangle \xrightarrow{n} \langle p'; \texttt{break}, \alpha, m\rangle \qquad \text{if } \mathit{blocked}(p, \alpha, n) = 1, \text{ with } p' = \mathit{clear}(p)$$

$$(R_{18}) \quad \frac{\langle p_1, \alpha, m\rangle \xrightarrow{n} \langle p_1', \alpha', m'\rangle}{\langle p_1\ \texttt{and}\ p_2, \alpha, m\rangle \xrightarrow{n} \langle p_1'\ \texttt{and}\ p_2, \alpha', m'\rangle} \qquad \text{if } \mathit{blocked}(p_1, \alpha, n) = 0 \text{ and } p_1 \neq v := a, \texttt{break}$$

$$(R_{19}) \quad \frac{\langle p_2, \alpha, m\rangle \xrightarrow{n} \langle p_2', \alpha', m'\rangle}{\langle p_1\ \texttt{and}\ p_2, \alpha, m\rangle \xrightarrow{n} \langle p_1\ \texttt{and}\ p_2', \alpha', m'\rangle} \qquad \text{if } \mathit{blocked}(p_1, \alpha, n) = 1 \text{ and } p_2 \neq v := a, \texttt{break}$$

Par/or

$$(R_{20}) \quad \langle v := a\ \texttt{or}\ p, \alpha, m\rangle \xrightarrow{n} \langle p', \alpha, m'\rangle \qquad \text{with } m' = m[v/\mathit{eval}(a)] \text{ and } p' = \mathit{clear}(p)$$

$$(R_{21}) \quad \langle p\ \texttt{or}\ v := a, \alpha, m\rangle \xrightarrow{n} \langle p', \alpha, m'\rangle \qquad \text{if } \mathit{blocked}(p, \alpha, n) = 1, \text{ with } m' = m[v/\mathit{eval}(a)] \text{ and } p' = \mathit{clear}(p)$$

$$(R_{22}) \quad \langle \texttt{break}\ \texttt{or}\ p, \alpha, m\rangle \xrightarrow{n} \langle p'; \texttt{break}, \alpha, m\rangle \qquad \text{with } p' = \mathit{clear}(p)$$

$$(R_{23}) \quad \langle p\ \texttt{or}\ \texttt{break}, \alpha, m\rangle \xrightarrow{n} \langle p'; \texttt{break}, \alpha, m\rangle \qquad \text{if } \mathit{blocked}(p, \alpha, n) = 1, \text{ with } p' = \mathit{clear}(p)$$

$$(R_{24}) \quad \frac{\langle p_1, \alpha, m\rangle \xrightarrow{n} \langle p_1', \alpha', m'\rangle}{\langle p_1\ \texttt{or}\ p_2, \alpha, m\rangle \xrightarrow{n} \langle p_1'\ \texttt{or}\ p_2, \alpha', m'\rangle} \qquad \text{if } \mathit{blocked}(p_1, \alpha, n) = 0 \text{ and } p_1 \neq v := a, \texttt{break}$$

$$(R_{25}) \quad \frac{\langle p_2, \alpha, m\rangle \xrightarrow{n} \langle p_2', \alpha', m'\rangle}{\langle p_1\ \texttt{or}\ p_2, \alpha, m\rangle \xrightarrow{n} \langle p_1\ \texttt{or}\ p_2', \alpha', m'\rangle} \qquad \text{if } \mathit{blocked}(p_1, \alpha, n) = 1 \text{ and } p_2 \neq v := a, \texttt{break}$$

The next theorem establishes that the reaction small-step relation is deterministic, i.e., that it is in fact a *partial* function.

Theorem 4 (Determinism of the small-step relation)**.** *For all p, p_1, $p_2 \in P$, α, α_1, $\alpha_2 \in E^*$, m, m_1, $m_2 \in \mathcal{M}$, and $n \in N$,*

$$\begin{gathered}\textit{if} \quad \langle p, \alpha, m\rangle \xrightarrow{n} \langle p_1, \alpha_1, m_1\rangle \quad \textit{and} \quad \langle p, \alpha, m\rangle \xrightarrow{n} \langle p_2, \alpha_2, m_2\rangle, \\ \textit{then} \quad \langle p_1, \alpha_1, m_1\rangle = \langle p_2, \alpha_2, m_2\rangle.\end{gathered}$$

Proof. By induction on the structure of derivations. Suppose

$$d_1 \Vdash \langle p, \alpha, m\rangle \xrightarrow{n} \langle p_1, \alpha_1, m_1\rangle \quad \text{and} \quad d_2 \Vdash \langle p, \alpha, m\rangle \xrightarrow{n} \langle p_2, \alpha_2, m_2\rangle,$$

for some derivations d_1 and d_2. Then there are ten possibilities depending on the structure of p. (Note that p cannot be equal to `skip`, $v \coloneqq a$, or `break`, as there are no rules to evaluate such programs.)

[Case 1] $p = \texttt{await}(e)$, for some $e \in E$. Then d_1 and d_2 are instances of axiom R_1, and as such, $p_1 = p_2 = \texttt{@awaiting}(e, n')$ with $n' = n + 1$, and $\alpha_1 = \alpha_2 = \alpha$ and $m_1 = m_2 = m$.

[Case 2] $p = \texttt{emit}(e)$, for some $e \in E$. Then d_1 and d_2 are instances of axiom R_3, and as such, $p_1 = p2 = \texttt{@emitting}(n')$ with $n' = |\alpha|$, and $\alpha_1 = \alpha_2 = e\alpha$ and $m_1 = m_2 = m$.

[Case 3] $p = \texttt{if}\, b\, \texttt{then}\, p'\, \texttt{else}\, p''$, for some $b \in B$ and p', $p'' \in P$.

[Case 3.1] $\mathit{eval}(b, m) = 1$. Then d_1 and d_2 are instances of axiom R_5, and as such, $p_1 = p_2 = p'$, $\alpha_1 = \alpha_2 = \alpha$, and $m_1 = m_2 = m$.

[Case 3.2] $\mathit{eval}(b, m) = 0$. Then d_1 and d_2 are instances of axiom R_6, and as such, $p_1 = p_2 = p''$, $\alpha_1 = \alpha_2 = \alpha$, and $m_1 = m_2 = m$.

[Case 4] $p = p'; p''$, for some p', $p'' \in P$.

[Case 4.1] $p' = v \coloneqq a$, for some $v \in V$ and $a \in A$. Then d_1 and d_2 are instances of axiom R_7, and as such, $p_1 = p_2 = p''$, $\alpha_1 = \alpha_2 = \alpha$ and, as eval is a total function, $m_1 = m_2 = m[v/\mathit{eval}(a)]$.

[Case 4.2] $p' = \texttt{break}$. Then d_1 and d_2 are instances of axiom R_8, and as such, $p_1 = p_2 = p'$, $\alpha_1 = \alpha_2 = \alpha$ and $m_1 = m_2 = m$.

[Case 4.3] $p' \neq v \coloneqq a, \texttt{break}$. Then d_1 and d_2 are instances of rule R_9. Thus there are derivations d'_1 and d'_2 such that

$$d'_1 \Vdash \langle p', \alpha, m\rangle \xrightarrow{n} \langle p'_1, \alpha_1, m_1\rangle \quad \text{and} \quad d'_2 \Vdash \langle p', \alpha, m\rangle \xrightarrow{n} \langle p'_2, \alpha_2, m_2\rangle,$$

for some p'_1, $p'_2 \in P$. Since $d'_1 \prec d_1$ and $d'_2 \prec d_2$, by induction hypothesis, $\alpha_1 = \alpha_2$, $m_1 = m_2$, and $p'_1 = p'_2$, which implies

$$p_1 = p'_1; p'' = p'_2; p'' = p_2.$$

[Case 5] $p = \texttt{loop}\, p'$, for some $p' \in P$. Then d_1 and d_2 are instances of axiom R_{10}, and as such, $p_1 = p_2 = p'\, \texttt{@loop}\, p'$, $\alpha_1 = \alpha_2 = \alpha$, and $m_1 = m_2 = m$.

[Case 6] $p = p'$ and p'', for some $p', p'' \in P$.

[Case 6.1] $p = v \coloneqq a$ and p', for some $v \in V$, $a \in A$ and $p' \in P$. Then d_1 and d_2 are instances of axiom R_{14}, and as such, $p_1 = p_2 = p'$, $\alpha_1 = \alpha_2 = \alpha$, and as such, as *eval* is a total function, $m_1 = m_2 = m[v/eval(a)]$.

[Case 6.2] $p = p'$ and $v \coloneqq a$, for some $v \in V$, $a \in A$ and $p' \in P$. Similar to Case 6.1.

[Case 6.3] $p = \texttt{break}$ andp', for some $p' \in P$. Then d_1 and d_2 are instances of axiom R_{16}, and as such, $\alpha_1 = \alpha_2 = \alpha$, $m_1 = m_2 = m$, and as *clear* is a total function, $p_1 = p_2 = clear(p'); \texttt{break}$.

[Case 6.4] $p = p'$ and $\texttt{break}$, for some $p' \in P$. Then either $blocked(p') = 0$ or $blocked(p') = 1$. If $blocked(p') = 0$ then this case becomes Case 6.5. Otherwise, if $blocked(p') = 1$, then d_1 and d_2 are instances of axiom R_{17}, and as such, $\alpha_1 = \alpha_2 = \alpha$, $m_1 = m_2 = m$, and as *clear* is a total function, $p_1 = p_2 = clear(p'); \texttt{break}$.

[Case 6.5] $p = p'$ and p'', for some p' and p''. Then there are two possibilities. If $blocked(p') = 0$ then d_1 and d_2 are instances of R_{18}. Thus there are derivations d_1' and d_2' such that

$$d_1' \Vdash \langle p', \alpha, m\rangle \xrightarrow{n} \langle p_1', \alpha_1, m_1\rangle \quad \text{and} \quad d_2' \Vdash \langle p', \alpha, m\rangle \xrightarrow{n} \langle p_2', \alpha_2, m_2\rangle,$$

for some $p_1', p_2' \in P$. Since $d_1' \prec d_1$ and $d_2' \prec d_2$, by induction hypothesis, $\alpha_1 = \alpha_2$, $m_1 = m_2$, and $p_1' = p_2'$, which implies

$$p_1 = (p_1' \text{ and } p'') = (p_2' \text{ and } p'') = p_2.$$

If, however, $blocked(p') = 1$, then d_1 and d_2 are instances of R_{19}. Thus there are derivations d_1'' and d_2'' such that

$$d_1'' \Vdash \langle p'', \alpha, m\rangle \xrightarrow{n} \langle p_1'', \alpha_1, m_1\rangle \quad \text{and} \quad d_2'' \Vdash \langle p'', \alpha, m\rangle \xrightarrow{n} \langle p_2'', \alpha_2, m_2\rangle,$$

for some $p_1'', p_2'' \in P$. Since $d_1'' \prec d_1$ and $d_2'' \prec d_2$, by induction hypothesis, $\alpha_1 = \alpha_2$, $m_1 = m_2$, and $p_1'' = p_2''$, which implies

$$p_1 = (p' \text{ and } p_1'') = (p' \text{ and } p_2'') = p_2.$$

[Case 7] $p = p'$ $\texttt{or}$ p'', for some $p', p'' \in P$. [TODO: Similar to Case 6 (we hope).]

[Case 8] $p = \texttt{@awaiting}(e, n')$, for some $e \in E$ and $n' \in N$. Then d_1 and d_2 are instances of axiom R_2, with $n' \leq n$. Thus $p_1 = p_2 = \texttt{skip}$, $\alpha_1 = \alpha_2 = \alpha$, and $m_1 = m_2 = m$.

[Case 9] $p = \texttt{@emitting}(e, n')$, for some $e \in E$ and $n' \in N$. Then d_1 and d_2 are instances of axiom R_4 with $n' = |\alpha|$. Thus $p_1 = p_2 = \texttt{skip}$, $\alpha_1 = \alpha_2 = \alpha$ and $m_1 = m_2 = m$.

[Case 10] $p = p'\ \texttt{@loop}\ p''$, for some $p', p'' \in P$.

[Case 10.1] $p = v \coloneqq a \,\texttt{@loop}\, p'$, for some $a \in A$, $v \in V$, and $p' \in P$. Then d_1 and d_2 are instances of R_{11}, and as such, $p_1 = p_2 = \texttt{loop}\, p'$, $\alpha_1 = \alpha_2 = \alpha$, and as *eval* is a total function, $m_1 = m_2 = m[v/\mathit{eval}(a)]$.

[Case 10.2] $p = \texttt{break}\,\texttt{@loop}\, p'$, for some $p' \in P$. Then d_1 and d_2 are instances of axiom R_{12}, and as such, $p_1 = p_2 = \texttt{skip}$, $\alpha_1 = \alpha_2 = \alpha$, and $m_1 = m_2 = m$.

[Case 10.3] $p = p' \,\texttt{@loop}\, p''$, for some $p', p'' \in P$ such that $p' \neq v \coloneqq a, \texttt{break}$. Then d_1 and d_2 are instances of rule R_{13}. Thus there are derivations d_1' and d_2' such that

$$d_1' \Vdash \langle p', \alpha, m\rangle \xrightarrow{n} \langle p_1', \alpha_1, m_1\rangle \quad \text{and} \quad d_2' \Vdash \langle p', \alpha, m\rangle \xrightarrow{n} \langle p_2', \alpha_2, m_2\rangle,$$

for some p_1', $p_2' \in P$. Since $d_1' \prec d_1$ and $d_2' \prec d_2$, by induction hypothesis, $\alpha_1 = \alpha$, $m_1 = m_2$, and $p_1' = p_2'$, which implies

$$p_1 = p_1' \,\texttt{@loop}\, p'' = p_2' \,\texttt{@loop}\, p'' = p_2. \qquad \square$$