# Determinism and termination in the semantics of the Céu programming language

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## 1 The original formulation

The semantics discussed in this section follows as much as possible the original semantics of Céu presented in [**?**]. Any deviations from original definitions are duly noted in the text.

### 1.1 Abstract syntax

The *abstract syntax* of Céu programs is given by the following grammar:

| | |
|---|---|
| $p \in P ::= \texttt{skip}$ | do nothing |
| $\mid v := a$ | assignment |
| $\mid \texttt{await}(e)$ | await event |
| $\mid \texttt{emit}(e)$ | emit event |
| $\mid \texttt{break}$ | break innermost loop |
| $\mid \texttt{if}\, b\, \texttt{then}\, p_1\, \texttt{else}\, p_2$ | conditional |
| $\mid p_1 ; p_2$ | sequence |
| $\mid \texttt{loop}\, p_1$ | repetition |
| $\mid p_1\, \texttt{and}\, p_2$ | par/and |
| $\mid p_1\, \texttt{or}\, p_2$ | par/or |
| $\mid \texttt{fin}\, p_1$ | finalization |
| $\mid \texttt{@awaiting}(e, n)$ | awaiting $e$ since reaction $n$ |
| $\mid \texttt{@emitting}(e, n)$ | emitting $e$ on stack level $n$ |
| $\mid p_1\, \texttt{@loop}\, p_2$ | unwinded loop |

where $n \in N$ is an integer, $v \in V$ is a memory location (variable) identifier, $e \in E$ is an event identifier, $a \in A$ is an arithmetic expression, $b \in B$ is a boolean expression, and $p$, $p_1$, $p_2 \in P$ are programs. We assume the usual structure for arithmetic and boolean expressions, and omit their definition.

### 1.2 The reaction inner-step relation

The *state* of a Céu program within a reaction is represented by a stack of events $\alpha = e_1 e_2 \ldots e_n \in E^*$ together with a memory map $m\colon v \to N \in \mathcal{M}$. A *configuration* is a 4-tuple $\langle p, \alpha, m, n\rangle \in \Delta$ that represents the situation of program $p$ waiting to be evaluated in state $\langle \alpha, m\rangle$ and reaction $n$. Given an initial configuration, each small-step within a program reaction is determined by the reaction-inner-step relation $\rightarrow \in \Delta \times \Delta$ such that $\langle p, \alpha, m, n\rangle \rightarrow \langle p', \alpha', m', n\rangle$ iff a reaction inner-step of program $p$ in state $\langle \alpha, m\rangle$ and reaction number $n$ evaluates to a modified program $p'$ and a modified state $\langle \alpha', m'\rangle$ in the same reaction $(n)$. Since relation $\rightarrow$ can only relate configurations with the same $n$, we shall write $\langle p, \alpha, m\rangle \xrightarrow{n} \langle p', \alpha', m'\rangle$ for $\langle p, \alpha, m, n\rangle \rightarrow \langle p', \alpha', m', n\rangle$.

Relation $\rightarrow$ is defined inductively with the help of the auxiliary functions *eval*, *blocked*, and *clear*. The *eval* function evaluates arithmetic or boolean expressions on a given memory; we omit its definition and assume that such evaluation is deterministic and always terminates. The *blocked* function is a predicate that determines if all trails of a program $p$ are blocked on a given event stack and reaction number. And the *clear* function extracts the body of `fin` blocks from a given program.

**Definition 1.1.** Function $\mathit{blocked}\colon P \times E^* \times N \rightarrow \{0, 1\}$ is defined inductively as follows.

$$
\begin{aligned}
\mathit{blocked}(\texttt{skip}, e\alpha, n) &= 0\\
\mathit{blocked}(v := a, e\alpha, n) &= 0\\
\mathit{blocked}(\texttt{await}(e'), e\alpha, n) &= 0\\
\mathit{blocked}(\texttt{emit}(e'), e\alpha, n) &= 0\\
\mathit{blocked}(\texttt{break}, e\alpha, n) &= 0\\
\mathit{blocked}(\texttt{if}\, b\, \texttt{then}\, p_1\, \texttt{else}\, p_2, e\alpha, n) &= 0\\
\mathit{blocked}(p_1; p_2, e\alpha, n) &= \mathit{blocked}(p_1, e\alpha, n)\\
\mathit{blocked}(\texttt{loop}\, p, e\alpha, n) &= 0\\
\mathit{blocked}(p_1\, \texttt{and}\, p_2, e\alpha, n) &= \mathit{blocked}(p_1, e\alpha, n) \cdot \mathit{blocked}(p_2, e\alpha, n)\\
\mathit{blocked}(p_1\, \texttt{or}\, p_2, e\alpha, n) &= \mathit{blocked}(p_1, e\alpha, n) \cdot \mathit{blocked}(p_2, e\alpha, n)\\
\mathit{blocked}(\texttt{fin}\, p_1, e\alpha, n) &= 0\\
\mathit{blocked}(\texttt{@awaiting}(e', n'), e\alpha, n) &= \begin{cases} 1 & \text{if } e \neq e' \text{ or } n = n'\\ 0 & \text{otherwise}\end{cases}\\
\mathit{blocked}(\texttt{@emitting}(n'), e\alpha, n) &= \begin{cases} 1 & \text{if } |e\alpha| \neq n'\\ 0 & \text{otherwise}\end{cases}\\
\mathit{blocked}(p_1\, \texttt{@loop}\, p_2, e\alpha, n) &= \mathit{blocked}(p_1, e\alpha, n)
\end{aligned}
$$

**Definition 1.2.** Function $\mathit{clear}\colon P \rightarrow P'$, where $P' = \{v := a\}$, is defined inductively as follows.

$$
\begin{aligned}
clear(v := a) &= v := a \\
clear(\texttt{await}(e')) &= \texttt{skip} \\
clear(\texttt{emit}(e')) &= \texttt{skip} \\
clear(\texttt{break}) &= \texttt{skip} \\
clear(\texttt{if}\, v\, \texttt{then}\, p_1\, \texttt{else}\, p_2) &= \texttt{skip} \\
clear(p_1; p_2) &= clear(p_1); clear(p_2) \\
clear(\texttt{loop}\, p) &= clear(p) \\
clear(p_1\, \texttt{and}\, p_2) &= clear(p_1); clear(p_2) \\
clear(p_1\, \texttt{or}\, p_2) &= clear(p_1); clear(p_2) \\
clear(\texttt{fin}\, p) &= p \\
clear(\texttt{@awaiting}(e', n')) &= \texttt{skip} \\
clear(\texttt{@emitting}(n')) &= \texttt{skip} \\
clear(p_1\, \texttt{@loop}\, p_2) &= clear(p_1)
\end{aligned}
$$

**Definition 1.3** (Reaction inner-step)**.** Relation $\rightarrow \subseteq \Delta \times \Delta$ is defined inductively as follows.
*Await and emit*

$$\langle \texttt{await}(e), \alpha, m\rangle \xrightarrow{n} \langle \texttt{@awaiting}(e, n'), \alpha, m\rangle \quad \text{with } n' = n + 1 \tag{1.1}$$

$$\langle \texttt{@awaiting}(e, n'), e\alpha, m\rangle \xrightarrow{n} \langle \texttt{skip}, e\alpha, m\rangle \quad \text{if } n' \leq n \tag{1.2}$$

$$\langle \texttt{emit}(e), \alpha, m\rangle \xrightarrow{n} \langle \texttt{@emitting}(n'), e\alpha, m\rangle \quad \text{with } n' = |\alpha| \tag{1.3}$$

$$\langle \texttt{@emitting}(n'), \alpha, m\rangle \xrightarrow{n} \langle \texttt{skip}, \alpha, m\rangle \quad \text{if } n' = |\alpha| \tag{1.4}$$

*Conditionals*

$$\langle \texttt{if}\, b\, \texttt{then}\, p_1\, \texttt{else}\, p_2, \alpha, m\rangle \xrightarrow{n} \langle p_1, \alpha, m\rangle \quad \text{if } eval(b, m) = 1 \tag{1.5}$$

$$\langle \texttt{if}\, b\, \texttt{then}\, p_1\, \texttt{else}\, p_2, \alpha, m\rangle \xrightarrow{n} \langle p_2, \alpha, m\rangle \quad \text{if } eval(b, m) = 0 \tag{1.6}$$

*Sequences*

$$\langle \texttt{skip}; p, \alpha, m\rangle \xrightarrow{n} \langle p, \alpha, m\rangle \tag{1.7}$$

$$\langle v := a; p, \alpha, m\rangle \xrightarrow{n} \langle p, \alpha, m'\rangle \quad \text{with } m' = m[v/eval(a)] \tag{1.8}$$

$$\langle \texttt{break}; p, \alpha, m\rangle \xrightarrow{n} \langle \texttt{break}, \alpha, m\rangle \tag{1.9}$$

$$\frac{\langle p_1, \alpha, m\rangle \xrightarrow{n} \langle p_1', \alpha', m'\rangle}{\langle p_1; p_2, \alpha, m\rangle \xrightarrow{n} \langle p_1'; p_2, \alpha', m'\rangle} \quad \text{if } p_1 \neq \texttt{skip}, v := a, \texttt{break} \tag{1.10}$$

*Loops*

(1.11) $\langle \texttt{loop}\ p, \alpha, m\rangle \xrightarrow{n} \langle p\ \texttt{@loop}\ p, \alpha, m\rangle$

(1.12) $\langle \texttt{skip}\ \texttt{@loop}\ p, \alpha, m\rangle \xrightarrow{n} \langle \texttt{loop}\ p, \alpha, m\rangle$

(1.13) $\langle v := a\ \texttt{@loop}\ p, \alpha, m\rangle \xrightarrow{n} \langle \texttt{loop}\ p, \alpha, m'\rangle$ with $m' = m[v/eval(a)]$

(1.14) $\langle \texttt{break}\ \texttt{@loop}\ p, \alpha, m\rangle \xrightarrow{n} \langle \texttt{skip}, \alpha, m\rangle$

(1.15) $$\frac{\langle p_1, \alpha, m\rangle \xrightarrow{n} \langle p_1', \alpha', m'\rangle}{\langle p_1\ \texttt{@loop}\ p_2, \alpha, m\rangle \xrightarrow{n} \langle p_1'\ \texttt{@loop}\ p_2, \alpha', m'\rangle}$$ if $p_1 \neq \texttt{skip}, v := a, \texttt{break}$

*Par/and*

(1.16) $\langle \texttt{skip}\ \textsf{and}\ p, \alpha, m\rangle \xrightarrow{n} \langle p, \alpha, m\rangle$

(1.17) $\langle p\ \textsf{and}\ \texttt{skip}, \alpha, m\rangle \xrightarrow{n} \langle p, \alpha, m\rangle$

(1.18) $\langle v := a\ \textsf{and}\ p, \alpha, m\rangle \xrightarrow{n} \langle p, \alpha, m'\rangle$ with $m' = m[v/eval(a)]$

(1.19) $\langle p\ \textsf{and}\ v := a, \alpha, m\rangle \xrightarrow{n} \langle p, \alpha, m'\rangle$ if $blocked(p, \alpha, n) = 1$, with $m' = m[v/eval(a)]$

(1.20) $\langle \texttt{break}\ \textsf{and}\ p, \alpha, m\rangle \xrightarrow{n} \langle p'; \texttt{break}, \alpha, m\rangle$ if $p \neq \texttt{skip}, v := a$, with $p' = clear(p)$

(1.21) $\langle p\ \textsf{and}\ \texttt{break}, \alpha, m\rangle \xrightarrow{n} \langle p'; \texttt{break}, \alpha, m\rangle$ if $blocked(p, \alpha, n) = 1$, with $p' = clear(p)$

(1.22) $$\frac{\langle p_1, \alpha, m\rangle \xrightarrow{n} \langle p_1', \alpha', m'\rangle}{\langle p_1\ \textsf{and}\ p_2, \alpha, m\rangle \xrightarrow{n} \langle p_1'\ \textsf{and}\ p_2, \alpha', m'\rangle}$$ if $blocked(p_1, \alpha, n) = 0$ and $p_1 \neq \texttt{skip}, v := a, \texttt{break}$

(1.23) $$\frac{\langle p_2, \alpha, m\rangle \xrightarrow{n} \langle p_2', \alpha', m'\rangle}{\langle p_1\ \textsf{and}\ p_2, \alpha, m\rangle \xrightarrow{n} \langle p_1\ \textsf{and}\ p_2', \alpha', m'\rangle}$$ if $blocked(p_1, \alpha, n) = 1$ and $p_2 \neq \texttt{skip}, v := a, \texttt{break}$

*Par/or*

(1.24) $\langle \texttt{skip or } p, \alpha, m\rangle \xrightarrow{n} \langle p', \alpha, m\rangle$ $\quad$ $p' = clear(p)$

(1.25) $\langle p \texttt{ or skip}, \alpha, m\rangle \xrightarrow{n} \langle p', \alpha, m\rangle$ $\quad$ if $blocked(p, \alpha, n) = 1$, and $p' = clear(p)$

(1.26) $\langle v := a \texttt{ or } p, \alpha, m\rangle \xrightarrow{n} \langle p', \alpha, m'\rangle$ $\quad$ with $m' = m[v/eval(a)]$ and $p' = clear(p)$

(1.27) $\langle p \texttt{ or } v := a, \alpha, m\rangle \xrightarrow{n} \langle p', \alpha, m'\rangle$ $\quad$ if $blocked(p, \alpha, n) = 1$, with $m' = m[v/eval(a)]$ and $p' = clear(p)$

(1.28) $\langle \texttt{break or } p, \alpha, m\rangle \xrightarrow{n} \langle p'; \texttt{break}, \alpha, m\rangle$ $\quad$ with $p' = clear(p)$

(1.29) $\langle p \texttt{ or break}, \alpha, m\rangle \xrightarrow{n} \langle p'; \texttt{break}, \alpha, m\rangle$ $\quad$ if $blocked(p, \alpha, n) = 1$, with $p' = clear(p)$

(1.30) $$\frac{\langle p_1, \alpha, m\rangle \xrightarrow{n} \langle p_1', \alpha', m'\rangle}{\langle p_1 \texttt{ or } p_2, \alpha, m\rangle \xrightarrow{n} \langle p_1' \texttt{ or } p_2, \alpha', m'\rangle}$$ if $blocked(p_1, \alpha, n) = 0$ and $p_1 \neq \texttt{skip}, v := a, \texttt{break}$

(1.31) $$\frac{\langle p_2, \alpha, m\rangle \xrightarrow{n} \langle p_2', \alpha', m'\rangle}{\langle p_1 \texttt{ or } p_2, \alpha, m\rangle \xrightarrow{n} \langle p_1 \texttt{ or } p_2', \alpha', m'\rangle}$$ if $blocked(p_1, \alpha, n) = 1$ and $p_2 \neq \texttt{skip}, v := a, \texttt{break}$

The next theorem establishes that the reaction inner-step relation is deterministic, i.e., that it is in fact a *partial* function.

**Theorem 1.4** (Determinism of the inner-step relation)**.** *For all $p$, $p_1$, $p_2 \in P$, $\alpha$, $\alpha_1$, $\alpha_2 \in E^*$, $m$, $m_1$, $m_2 \in \mathcal{M}$, and $n \in N$,*

$$\begin{aligned} \textit{if} \quad \langle p, \alpha, m\rangle \xrightarrow{n} \langle p_1, \alpha_1, m_1\rangle \quad \textit{and} \quad \langle p, \alpha, m\rangle \xrightarrow{n} \langle p_2, \alpha_2, m_2\rangle, \\ \textit{then} \quad \langle p_1, \alpha_1, m_1\rangle = \langle p_2, \alpha_2, m_2\rangle. \end{aligned}$$

*Proof.* By induction on the structure of derivations. Suppose

$$d_1 \Vdash \langle p, \alpha, m\rangle \xrightarrow{n} \langle p_1, \alpha_1, m_1\rangle \quad \text{and} \quad d_2 \Vdash \langle p, \alpha, m\rangle \xrightarrow{n} \langle p_2, \alpha_2, m_2\rangle,$$

for some derivations $d_1$ and $d_2$. Then there are ten possibilities depending on the structure of $p$. (Note that $p$ cannot be equal to `skip`, $v := a$, or `break`, as there are no rules to evaluate such programs.)

[Case 1] $p = \texttt{await}(e)$, for some $e \in E$. Then $d_1$ and $d_2$ are instances of axiom (1.1), and as such, $p_1 = p_2 = \texttt{@awaiting}(e, n')$ with $n' = n + 1$, and $\alpha_1 = \alpha_2 = \alpha$ and $m_1 = m_2 = m$.

[Case 2] $p = \texttt{@awaiting}(e, n')$, for some $e \in E$ and $n' \in N$. Then $d_1$ and $d_2$ are instances of axiom (1.2), with $n' \leq n$. Thus $p_1 = p_2 = \texttt{skip}$, $\alpha_1 = \alpha_2 = \alpha$, and $m_1 = m_2 = m$.

[Case 3] $p = \mathtt{emit}(e)$, for some $e \in E$. Then $d_1$ and $d_2$ are instances of axiom (1.3), and as such, $p_1 = p2 = \mathtt{@emitting}(n')$ with $n' = |\alpha|$, and $\alpha_1 = \alpha_2 = e\alpha$ and $m_1 = m_2 = m$.

[Case 4] $p = \mathtt{@emitting}(e, n')$, for some $e \in E$ and $n' \in N$. Then $d_1$ and $d_2$ are instances of axiom (1.4) with $n' = |\alpha|$. Thus $p_1 = p_2 = \mathtt{skip}$, $\alpha_1 = \alpha_2 = \alpha$ and $m_1 = m_2 = m$.

[Case 5] $p = \mathtt{if}\, b\, \mathtt{then}\, p'\, \mathtt{else}\, p''$, for some $b \in B$ and $p', p'' \in P$.

[Case 5.1] $\mathit{eval}(b, m) = 1$. Then $d_1$ and $d_2$ are instances of axiom (1.5), and as such, $p_1 = p_2 = p'$, $\alpha_1 = \alpha_2 = \alpha$, and $m_1 = m_2 = m$.

[Case 5.2] $\mathit{eval}(b, m) = 0$. Then $d_1$ and $d_2$ are instances of axiom (1.6), and as such, $p_1 = p_2 = p''$, $\alpha_1 = \alpha_2 = \alpha$, and $m_1 = m_2 = m$.

[Case 6] $p = p'; p''$, for some $p', p'' \in P$.

[Case 6.1] $p' = \mathtt{skip}$. Then $d_1$ and $d_2$ are instances of axiom (1.7), and as such, $p_1 = p_2 = p''$, $\alpha_1 = \alpha_2 = \alpha$ and $m_1 = m_2 = m$.

[Case 6.2] $p' = v \coloneqq a$, for some $v \in V$ and $a \in A$. Then $d_1$ and $d_2$ are instances of axiom (1.8), and as such, $p_1 = p_2 = p''$, $\alpha_1 = \alpha_2 = \alpha$ and, as $\mathit{eval}$ is a total function, $m_1 = m_2 = m[v/\mathit{eval}(a)]$.

[Case 6.3] $p' = \mathtt{break}$. Then $d_1$ and $d_2$ are instances of axiom (1.9), and as such, $p_1 = p_2 = p'$, $\alpha_1 = \alpha_2 = \alpha$ and $m_1 = m_2 = m$.

[Case 6.4] $p' \neq \mathtt{skip}, v \coloneqq a, \mathtt{break}$. Then $d_1$ and $d_2$ are instances of rule (1.10). Thus there are derivations $d_1'$ and $d_2'$ such that

$$d_1' \Vdash \langle p', \alpha, m \rangle \xrightarrow{n} \langle p_1', \alpha_1, m_1 \rangle \quad \text{and} \quad d_2' \Vdash \langle p', \alpha, m \rangle \xrightarrow{n} \langle p_2', \alpha_2, m_2 \rangle,$$

for some $p_1', p_2' \in P$. Since $d_1' \prec d_1$ and $d_2' \prec d_2$, by induction hypothesis, $\alpha_1 = \alpha_2$, $m_1 = m_2$, and $p_1' = p_2'$, which implies

$$p_1 = p_1'; p'' = p_2'; p'' = p_2.$$

[Case 7] $p = \mathtt{loop}\, p'$, for some $p' \in P$. Then $d_1$ and $d_2$ are instances of axiom (1.11), and as such, $p_1 = p_2 = p'\, \mathtt{@loop}\, p'$, $\alpha_1 = \alpha_2 = \alpha$, and $m_1 = m_2 = m$.

[Case 8] $p = p'\, \mathtt{@loop}\, p''$, for some $p', p'' \in P$.

[Case 8.1] $p = \mathtt{skip}\, \mathtt{@loop}\, p'$, for some $p' \in P$. Then $d_1$ and $d_2$ are instances of axiom (1.12), and as such, $p_1 = p_2 = \mathtt{loop}\, p'$, $\alpha_1 = \alpha_2 = \alpha$, and $m_1 = m_2 = m$.

[Case 8.2] $p = v \coloneqq a\, \mathtt{@loop}\, p'$, for some $a \in A$, $v \in V$, and $p' \in P$. Then $d_1$ and $d_2$ are instances of axiom (1.13), and as such, $p_1 = p_2 = \mathtt{loop}\, p'$, $\alpha_1 = \alpha_2 = \alpha$, and as $\mathit{eval}$ is a total function, $m_1 = m_2 = m[v/\mathit{eval}(a)]$.

[Case 8.3] $p = \mathtt{break}\, \mathtt{@loop}\, p'$, for some $p' \in P$. Then $d_1$ and $d_2$ are instances of axiom (1.14), and as such, $p_1 = p_2 = \mathtt{skip}$, $\alpha_1 = \alpha_2 = \alpha$, and $m_1 = m_2 = m$.

[Case 8.4] $p = p' \,\texttt{@loop}\, p''$, for some $p', p'' \in P$ such that $p' \neq \texttt{skip}, v := a, \texttt{break}$. Then $d_1$ and $d_2$ are instances of rule (1.15). Thus there are derivations $d_1'$ and $d_2'$ such that

$$d_1' \Vdash \langle p', \alpha, m\rangle \xrightarrow{n} \langle p_1', \alpha_1, m_1\rangle \quad \text{and} \quad d_2' \Vdash \langle p', \alpha, m\rangle \xrightarrow{n} \langle p_2', \alpha_2, m_2\rangle,$$

for some $p_1'$, $p_2' \in P$. Since $d_1' \prec d_1$ and $d_2' \prec d_2$, by induction hypothesis, $\alpha_1 = \alpha$, $m_1 = m_2$, and $p_1' = p_2'$, which implies

$$p_1 = p_1' \,\texttt{@loop}\, p'' = p_2' \,\texttt{@loop}\, p'' = p_2.$$

[Case 9] $p = p' \,\texttt{and}\, p''$, for some $p', p'' \in P$.

[Case 9.1] $p = \texttt{skip and}\, p'$, for some $p' \in P$. Then $d_1$ and $d_2$ are instances of axiom (1.16), and as such, $p_1 = p_2 = p'$, $\alpha_1 = \alpha_2 = \alpha$, and $m_1 = m_2 = m$.

[Case 9.2] $p = p' \,\texttt{and skip}$, for some $p' \in P$. Similar to Case 9.1.

[Case 9.3] $p = v := a \,\texttt{and}\, p'$, for some $v \in V$, $a \in A$ and $p' \in P$. Then $d_1$ and $d_2$ are instances of axiom (1.18), and as such, $p_1 = p_2 = p'$, $\alpha_1 = \alpha_2 = \alpha$, and as such, as *eval* is a total function, $m_1 = m_2 = m[v/eval(a)]$.

[Case 9.4] $p = p' \,\texttt{and}\, v := a$, for some $v \in V$, $a \in A$ and $p' \in P$.
Then either $blocked(p') = 0$ or $blocked(p') = 1$. If $blocked(p') = 0$ then this case becomes Case 9.7. Otherwise, if $blocked(p') = 1$, then $d_1$ and $d_2$ are instances of axiom (1.19), and as such, $p_1 = p_2 = p'$, $\alpha_1 = \alpha_2 = \alpha$, and as *eval* is a total function, $m_1 = m_2 = m[v/eval(a)]$.

[Case 9.5] $p = \texttt{break and}\, p'$, for some $p' \in P$. Then $d_1$ and $d_2$ are instances of axiom (1.20), and as such, $\alpha_1 = \alpha_2 = \alpha$, $m_1 = m_2 = m$, and as *clear* is a total function, $p_1 = p_2 = clear(p'); \texttt{break}$.

[Case 9.6] $p = p' \,\texttt{and break}$, for some $p' \in P$. Then either $blocked(p') = 0$ or $blocked(p') = 1$. If $blocked(p') = 0$ then this case becomes Case 9.7. Otherwise, if $blocked(p') = 1$, then $d_1$ and $d_2$ are instances of axiom (1.21), and as such, $\alpha_1 = \alpha_2 = \alpha$, $m_1 = m_2 = m$, and as *clear* is a total function, $p_1 = p_2 = clear(p'); \texttt{break}$.

[Case 9.7] $p = p' \,\texttt{and}\, p''$, for some $p'$ and $p'' \in P$. Then there are two possibilities. If $blocked(p') = 0$ then $d_1$ and $d_2$ are instances of (1.22). Thus there are derivations $d_1'$ and $d_2'$ such that

$$d_1' \Vdash \langle p', \alpha, m\rangle \xrightarrow{n} \langle p_1', \alpha_1, m_1\rangle \quad \text{and} \quad d_2' \Vdash \langle p', \alpha, m\rangle \xrightarrow{n} \langle p_2', \alpha_2, m_2\rangle,$$

for some $p_1'$, $p_2' \in P$. Since $d_1' \prec d_1$ and $d_2' \prec d_2$, by induction hypothesis, $\alpha_1 = \alpha_2$, $m_1 = m_2$, and $p_1' = p_2'$, which implies

$$p_1 = (p_1' \,\texttt{and}\, p'') = (p_2' \,\texttt{and}\, p'') = p_2.$$

If, however, $blocked(p') = 1$ and $p'' \neq \texttt{skip}, v := a, \texttt{break}$, then $d_1$ and $d_2$ are instances of (1.23). Thus there are derivations $d_1''$ and $d_2''$ such that

$$d_1'' \Vdash \langle p'', \alpha, m\rangle \xrightarrow{n} \langle p_1'', \alpha_1, m_1\rangle \quad \text{and} \quad d_2'' \Vdash \langle p'', \alpha, m\rangle \xrightarrow{n} \langle p_2'', \alpha_2, m_2\rangle,$$

for some $p_1'', p_2'' \in P$. Since $d_1'' \prec d_1$ and $d_2'' \prec d_2$, by induction hypothesis, $\alpha_1 = \alpha_2$, $m_1 = m_2$, and $p_1'' = p_2''$, which implies

$$p_1 = (p' \text{ and } p_1'') = (p' \text{ and } p_2'') = p_2.$$

[Case 10] $p = p' \texttt{ or } p''$, for some $p', p'' \in P$.

[Case 10.1] $p = \texttt{skip or } p'$, for some $p' \in P$. Then $d_1$ and $d_2$ are instances of axiom (1.24), and as such, $\alpha_1 = \alpha_2 = \alpha$, $m_1 = m_2 = m$, and as *clear* is a total, $p_1 = p_2 = clear(p')$.

[Case 10.2] $p = p' \texttt{ or skip}$, for some $p' \in P$. Then either $blocked(p') = 0$ or $blocked(p') = 1$. If $blocked(p') = 0$ then this case becomes Case 10.7. Otherwise, if $blocked(p') = 1$, then $d_1$ and $d_2$ are instances of axiom (1.25), and as such, $\alpha_1 = \alpha_2 = \alpha$, $m_1 = m_2 = m$, and as *clear* is a total function, $p_1 = p_2 = clear(p')$.

[Case 10.3] $p = v := a \texttt{ or } p'$, for some $v \in V$, $a \in A$ and $p' \in P$. Then $d_1$ and $d_2$ are instances of axiom (1.26), and as such, $\alpha_1 = \alpha_2 = \alpha$, as *eval* is a total function, $m_1 = m_2 = m[v/eval(a)]$, and as *clear* is a total function, $p_1 = p_2 = clear(p')$.

[Case 10.4] $p = p' \texttt{ or } v := a$, for some $v \in V$, $a \in A$ and $p' \in P$. Then either $blocked(p') = 0$ or $blocked(p') = 1$. If $blocked(p') = 0$ then this case becomes Case 10.7. Otherwise, if $blocked(p') = 1$, then $d_1$ and $d_2$ are instances of axiom (1.27), and as such, $\alpha_1 = \alpha_2 = \alpha$, as *eval* is a total function, $m_1 = m_2 = m[v/eval(a)]$, and as *clear* is a total function, $p_1 = p_2 = clear(p')$.

[Case 10.5] $p = \texttt{break or } p'$, for some $p' \in P$. Then $d_1$ and $d_2$ are instances of axiom (1.28), and as such, $\alpha_1 = \alpha_2 = \alpha$, $m_1 = m_2 = m$, and as *clear* is a total function, $p_1 = p_2 = clear(p'); \texttt{break}$.

[Case 10.6] $p = p' \texttt{ or break}$, for some $p' \in P$. Then either $blocked(p') = 0$ or $blocked(p') = 1$. If $blocked(p') = 0$ then this case becomes Case 10.7. Otherwise, if $blocked(p') = 1$, then $d_1$ and $d_2$ are instances of axiom (1.29), and as such, $\alpha_1 = \alpha_2 = \alpha$, $m_1 = m_2 = m$, and as *clear* is a total function, $p_1 = p_2 = clear(p'); \texttt{break}$.

[Case 10.7] $p = p' \texttt{ or } p''$, for some $p'$ and $p'' \in P$. Then there are two possibilities. If $blocked(p') = 0$ then $d_1$ and $d_2$ are instances of (1.30). Thus there are derivations $d_1'$ and $d_2'$ such that

$$d_1' \Vdash \langle p', \alpha, m\rangle \xrightarrow{n} \langle p_1', \alpha_1, m_1\rangle \quad \text{and} \quad d_2' \Vdash \langle p', \alpha, m\rangle \xrightarrow{n} \langle p_2', \alpha_2, m_2\rangle,$$

for some $p_1', p_2' \in P$. Since $d_1' \prec d_1$ and $d_2' \prec d_2$, by induction hypothesis, $\alpha_1 = \alpha_2$, $m_1 = m_2$, and $p_1' = p_2'$, which implies

$$p_1 = (p_1' \texttt{ or } p'') = (p_2' \texttt{ or } p'') = p_2.$$

If, however, $blocked(p') = 1$ and $p'' \neq \texttt{skip}, v := a, \texttt{break}$, then $d_1$ and $d_2$ are instances of (1.31). Thus there are derivations $d_1''$ and $d_2''$ such that

$$d_1'' \Vdash \langle p'', \alpha, m\rangle \xrightarrow{n} \langle p_1'', \alpha_1, m_1\rangle \quad \text{and} \quad d_2'' \Vdash \langle p'', \alpha, m\rangle \xrightarrow{n} \langle p_2'', \alpha_2, m_2\rangle,$$

for some $p_1''$, $p_2'' \in P$. Since $d_1'' \prec d_1$ and $d_2'' \prec d_2$, by induction hypothesis, $\alpha_1 = \alpha_2$, $m_1 = m_2$, and $p_1'' = p_2''$, which implies

$$p_1 = (p' \texttt{ or } p_1'') = (p' \texttt{ or } p_2'') = p_2.$$

□

The next lemma establishes that given a program either it is possible to advance it by an inner-step or all its trails are blocked, but not both.

**Lemma 1.5.** *For all $p \in P$, $e, \alpha \in E^*$, $m \in \mathcal{M}$, and $n \in N$, if $p \neq \texttt{skip}, v := a, \texttt{break}$ then either*

$$\exists \delta \in \Delta(\langle p, e\alpha, m\rangle \xrightarrow{n} \delta) \quad \textit{or} \quad blocked(p, e\alpha, n) = 1,$$

*but not both.*

*Proof.* By induction on the structure of programs. There are ten possibilities depending on the structure of $p$.

[Case 1] $p = \texttt{await}(e)$, for some $e \in E$. Then we can derive $\delta$ using the axiom (1.1), and as such, $\delta = \langle \texttt{@awaiting}(e, n'), e\alpha, m\rangle$, with $n' = n + 1$, and according to definition 1.1, $blocked(await(e'), e\alpha, n) = 0$.

[Case 2] $p = \texttt{@awaiting}(e, n')$, for some $e \in E$ and $n' \in N$.
Aqui tive um problema: $blocked(\texttt{@awaiting}(e', n'), e\alpha, n) = 1$ se $n = n'$. Porém, o axioma (1.2) define uma regra para a configuração $\langle \texttt{@awaiting}(e, n'), e\alpha, m\rangle$ se $n' \leq n$.

[Case 3] $p = \texttt{emit}(e)$, for some $e \in E$. Then we can derive $\delta$ using the axiom (1.3), and as such, $\delta = \langle \texttt{@emitting}(n'), e\alpha, m\rangle$ with $n' = |\alpha|$, and according to definition 1.1, $blocked(\texttt{emit}(e'), e\alpha, n) = 0$.

[Case 4] $p = \texttt{@emitting}(e, n')$, for some $e \in E$ and $n' \in N$.

[Case 4.1] For $n' = |e\alpha|$. Then we can derive $\delta$ using the axiom (1.4), and as such, $\delta = \langle \texttt{skip}, e\alpha, m\rangle$, and according to definition 1.1, $blocked(\texttt{@emitting}(e, n'), e\alpha, n) = 0$.

[Case 4.2] For $n' \neq |e\alpha|$. Then there is no rule we can use to derive $\delta$ and according to definition 1.1, $blocked(\texttt{@emitting}(e, n'), e\alpha, n) = 1$.

[Case 5] $p = \texttt{if}\, b\, \texttt{then}\, p'\, \texttt{else}\, p''$, for some $b \in B$ and $p', p'' \in P$. According to definition 1.1, $blocked(\texttt{if}\, b\, \texttt{then}\, p'\, \texttt{else}\, p'', e\alpha, n) = 0$

[Case 5.1] $eval(b, m) = 1$. Then we can derive $\delta$ using the axiom (1.5), and as such, $\delta = \langle p', e\alpha, m\rangle$.

[Case 5.2] $eval(b, m) = 0$. Then we can derive $\delta$ using the axiom (1.6), and as such, $\delta = \langle p'', e\alpha, m\rangle$.

[Case 6] $p = p'; p''$, for some $p', p'' \in P$. According to definition 1.1, $blocked(p'; p'', e\alpha, n) = blocked(p', e\alpha, n)$.

[Case 6.1] $p' = \texttt{skip}$. Then we can derive $\delta$ using the axiom (1.7), and as such, $\delta = \langle p'', e\alpha, m\rangle$ and according to definition 1.1, $blocked(\texttt{skip}, e\alpha, n) = 0$.

[Case 6.2] $p' = v := a$, for some $v \in V$ and $a \in A$. Then we can derive $\delta$ using the axiom (1.8), and as such, $\delta = \langle p'', e\alpha, m'\rangle$, with $m' = m[v/eval(a)]$, and according to definition 1.1, $blocked(v := a, e\alpha, n) = 0$.

[Case 6.3] $p' = \texttt{break}$. Then we can derive $\delta$ using the axiom (1.9), and as such, $\delta = \langle \texttt{break}, e\alpha, m\rangle$, and according to definition 1.1, $blocked(\texttt{break}, e\alpha, n) = 0$.

[Case 6.4] $p' \neq \texttt{skip}, v := a, \texttt{break}$. [TODO: induction hypothesis]

[Case 7] $p = \texttt{loop}\, p'$, for some $p' \in P$. Then we can derive $\delta$ using the axiom (1.11), and as such, $\delta = \langle p' \,\texttt{@loop}\, p', e\alpha, m\rangle$, and according to definition 1.1, $blocked(\texttt{loop}\, p', e\alpha, n) = 0$.

[Case 8] $p = p' \,\texttt{@loop}\, p''$, for some $p'$, $p'' \in P$. According to definition 1.1, $blocked(p' \,\texttt{@loop}\, p'', e\alpha, n) = blocked(p', e\alpha, n)$.

[Case 8.1] $p = \texttt{skip} \,\texttt{@loop}\, p'$, for some $p' \in P$. Then we can derive $\delta$ using the axiom (1.12), and as such, $\delta = \langle \texttt{loop}\, p', e\alpha, m\rangle$, and according to definition 1.1, $blocked(\texttt{skip}, e\alpha, n) = 0$.

[Case 8.2] $p = v := a \,\texttt{@loop}\, p'$, for some $p' \in P$. Then we can derive $\delta$ using the axiom (1.13), and as such, $\delta = \langle \texttt{loop}\, p', e\alpha, m'\rangle$, with $m' = m[v/eval(a)]$, and according to definition 1.1, $blocked(v := a, e\alpha, n) = 0$.

[Case 8.3] $p = \texttt{break} \,\texttt{@loop}\, p'$, for some $p' \in P$. Then we can derive $\delta$ using the axiom (1.14), and as such, $\delta = \langle \texttt{skip}, e\alpha, m\rangle$, and according to definition 1.1, $blocked(\texttt{break}, e\alpha, n) = 0$.

[Case 8.4] $p = p' \,\texttt{@loop}\, p''$, for some $p', p'' \in P$ such that $p' \neq \texttt{skip}, v := a, \texttt{break}$. [TODO: induction hypothesis]

[Case 9] $p = p' \,\texttt{and}\, p''$, for some $p'$, $p'' \in P$. According to definition 1.1, $blocked(p' \,\texttt{and}\, p'', e\alpha, n) = blocked(p', e\alpha, n) \cdot blocked(p'', e\alpha, n)$.

[Case 9.1] $p = \texttt{skip} \,\texttt{and}\, p'$, for some $p' \in P$. Then we can derive $\delta$ using the axiom (1.16), and as such, $\delta = \langle p', e\alpha, m\rangle$, and according to definition 1.1, $blocked(\texttt{skip}, e\alpha, n) = 0$, therefore $blocked(\texttt{skip} \,\texttt{and}\, p'', e\alpha, n) = 0$.

[Case 9.2] $p = p' \,\texttt{and}\, \texttt{skip}$, for some $p' \in P$. Similar to Case 9.1.

[Case 9.3] $p = v := a$ and $p'$, for some $v \in V$, $a \in A$ and $p' \in P$. Then we can derive $\delta$ using the axiom (1.18), and as such, $\delta =< p', e\alpha, m' >$, with $m' = m[v/eval(a)]$, and according to definition 1.1, $blocked(v := a, e\alpha, n) = 0$, therefore $blocked(v := a \texttt{ and } p'', e\alpha, n) = 0$.

[Case 9.4] $p = p'$ and $v := a$, for some $v \in V$, $a \in A$ and $p' \in P$. Then there are two possibilities.

- if $blocked(p', e\alpha, n) = 1$, then according to definition 1.1, $blocked(v := a, e\alpha, n) = 0$, thus $blocked(p' \texttt{ and } v := a, e\alpha, n) = 0$. In this case, we can derive $\delta$ using the axiom (1.19), and as such, $\delta = \langle p', e\alpha, m' \rangle$, with $m' = m[v/eval(a)]$.
- if $blocked(p', e\alpha, n) = 0$ and $p' \neq \texttt{skip}, v := a, \texttt{break}$, then $blocked(p' \texttt{ and } v := a, e\alpha, n) = 0$. The derivation of $\delta$ becomes Case 9.7

[Case 9.5] $p = \texttt{break and } p'$, for some $p' \in P$. Then we can derive $\delta$ using the axiom (1.20), and as such, $\delta = \langle clear(p'), e\alpha, m \rangle$, and according to definition 1.1, $blocked(\texttt{break}, e\alpha, m) = 0$, therefore $blocked(\texttt{break and } p', e\alpha, n) = 0$

[Case 9.6] $p = p'$ and break, for some $p' \in P$. Then there are two possibilities.

- if $blocked(p', e\alpha, n) = 1$, then according to definition 1.1, $blocked(\texttt{break}, e\alpha, n) = 0$, thus $blocked(p' \texttt{ and break}, e\alpha, n) = 0$. In this case, we can derive $\delta$ using the axiom (1.21), and as such, $\delta = \langle clear(p'); \texttt{break}, e\alpha, m \rangle$.
- if $blocked(p', e\alpha, n) = 0$ and $p' \neq \texttt{skip}, v := a, \texttt{break}$, then $blocked(p' \texttt{ and break}, e\alpha, n) = 0$. The derivation of $\delta$ becomes Case 9.7

[Case 9.7] $p = p'$ and $p''$, for some $p'$ and $p'' \in P$. Then there are two possibilities.

- if $blocked(p') = 0$ and $p' \neq \texttt{skip}, v := a, \texttt{break}$, then $blocked(p' \texttt{ and } p'', e\alpha, m) = 0$. We can derive $\delta$ using the rule (1.22), as such, $\delta = \langle p'_1 \texttt{ and } p'', e\alpha, m' \rangle$.

□

## 1.3 The reaction outer-step relation

From the previous inner-step relation we define an outer-step relation ($\Rightarrow$) that when necessary pops the event stack advances blocked programs. [TODO: Improve this description.]

**Definition 1.6** (Reaction outer-step)**.**

$$\frac{\langle p, \alpha, m \rangle \xrightarrow{n} \langle p', \alpha', m' \rangle}{\langle p, \alpha, m \rangle \overset{n}{\Rightarrow} \langle p', \alpha', m' \rangle} \quad \text{if } blocked(p, \alpha, n) = 0 \tag{1.32}$$

$$\langle p, e\alpha, m \rangle \overset{n}{\Rightarrow} \langle p, \alpha, m \rangle \qquad \text{if } blocked(p, \alpha, n) = 1 \tag{1.33}$$

**Theorem 1.7** (Determinism of the outer-step relation)**.** *For all $p$, $p_1$, $p_2 \in P$, $\alpha$, $\alpha_1$, $\alpha_2 \in E^*$, $m$, $m_1$, $m_2 \in \mathcal{M}$, and $n \in N$,*

$$\begin{aligned} &\textit{if} \quad \langle p, \alpha, m \rangle \overset{n}{\Rightarrow} \langle p_1, \alpha_1, m_1 \rangle \quad \textit{and} \quad \langle p, \alpha, m \rangle \overset{n}{\Rightarrow} \langle p_2, \alpha_2, m_2 \rangle, \\ &\qquad \textit{then} \quad \langle p_1, \alpha_1, m_1 \rangle = \langle p_2, \alpha_2, m_2 \rangle. \end{aligned}$$

*Proof.* [TODO: By induction on the structure of derivations.] □

**Theorem 1.8** (Termination of the outer-step relation). *For all $p \in P$, $\alpha \in E$, and $m \in \mathcal{M}$, if $p \neq \texttt{skip}, v := a, \texttt{break}$ then*

$$\exists \delta \in \Delta(\langle p, \alpha, m\rangle \overset{n}{\Rightarrow} \delta).$$

*Proof.* [TODO: Directly from Lemma 1.5.] □

### 1.4 The reaction relation

From the reflexive-transitive closure of the outer-step relation ($\overset{*}{\Rightarrow}$) we define the reaction relation $\models \subseteq \Delta \times (P \times \mathcal{M} \times N)$, which computes a full program reaction. Given an initial configuration, the reaction relation evaluates it until the event stack becomes empty.

**Definition 1.9** (Reaction). Let $p, p' \in P$, $\alpha \in E^*$, $m, m' \in \mathcal{M}$. Then

$$\langle p, \alpha, m\rangle \overset{n}{\models} \langle p', m'\rangle \quad \text{iff} \quad \langle p, \alpha, m\rangle \overset{*}{\Rightarrow} \langle p', \varepsilon, m'\rangle.$$

The next two theorems establish, respectively, that reactions are deterministic and always terminate (for the nontrivial programs $p \neq \texttt{skip}, v := a, \texttt{break}$).

**Theorem 1.10** (Determinism of the reaction relation). *For all $p$, $p_1$, $p_2 \in P$, $\alpha \in E^*$, $m$, $m_1$, $m_2 \in \mathcal{M}$, and $n \in N$,*

$$\begin{aligned} \textit{if} \quad \langle p, \alpha, m\rangle \overset{n}{\models} \langle p_1, m_1\rangle \quad \textit{and} \quad \langle p, \alpha, m\rangle \overset{n}{\models} \langle p_2, m_2\rangle, \\ \textit{then} \quad \langle p_1, m_1\rangle = \langle p_2, m_2\rangle. \end{aligned}$$

*Proof.* [TODO: ?] □

**Theorem 1.11** (Termination of the reaction relation). *For all $p \in P$, $\alpha \in E$, and $m \in \mathcal{M}$, if $p \neq \texttt{skip}, v := a, \texttt{break}$ then*

$$\langle p, \alpha, m\rangle \overset{n}{\models} \langle p', m'\rangle,$$

*for some $p' \in P$ and $m' \in \mathcal{M}$.*

*Proof.* [TODO: ?] □

## 2 Big-step version of the original formulation

[TODO: Minha ideia aqui é fazer uma versão big-step da formulação original. E no final comparar as duas versões, i.e., mostrar que são equivalentes.]

**Definition 2.1.** [TODO: Parcial e provavelmente incorreta.]
*Empty program*

$$\langle \varepsilon, \alpha, m, n \rangle \rightsquigarrow \langle \alpha, m, n \rangle \tag{2.1}$$

*Assignment*

$$\langle v := a, \alpha, m, n \rangle \rightsquigarrow \langle \alpha, m[v/eval(a)], n \rangle \tag{2.2}$$

*Conditionals*

$$\frac{\langle p_1, \alpha, m, n \rangle \rightsquigarrow \langle \alpha', m', n' \rangle}{\langle \texttt{if}\, b\, \texttt{then}\, p_1\, \texttt{else}\, p_2, \alpha, m, n \rangle \rightsquigarrow \langle \alpha', m', n' \rangle} \quad \text{if } eval(b, m) = 1 \tag{2.3}$$

$$\frac{\langle p_2, \alpha, m, n \rangle \rightsquigarrow \langle \alpha', m', n' \rangle}{\langle \texttt{if}\, b\, \texttt{then}\, p_1\, \texttt{else}\, p_2, \alpha, m, n \rangle \rightsquigarrow \langle \alpha', m', n' \rangle} \quad \text{if } eval(b, m) = 0 \tag{2.4}$$

*Await*

$$\frac{\langle \texttt{@awaiting}(e, n+1), e\alpha, m, n \rangle \rightsquigarrow \langle \alpha', m', n' \rangle}{\langle \texttt{await}(e), \alpha, m, n \rangle \rightsquigarrow \langle \alpha', m', n' \rangle} \tag{2.5}$$

$$\langle \texttt{@awaiting}(e', n'), e\alpha, m, n \rangle \rightsquigarrow \langle \alpha, m, n \rangle \quad \text{if } e' = e \text{ and } n' < n \tag{2.6}$$

$$\frac{\langle \texttt{@awaiting}(e', n'), \alpha, m, n \rangle \rightsquigarrow \langle \alpha'', m'', n'' \rangle}{\langle \texttt{@awaiting}(e', n'), e\alpha, m, n \rangle \rightsquigarrow \langle \alpha'', m'', n'' \rangle} \quad \text{if } e' \neq e \text{ or } n' \geq n \tag{2.7}$$

*Emit*

$$\frac{\langle \texttt{@emitting}(|\alpha|), e\alpha, m, n \rangle \rightsquigarrow \langle \alpha', m', n' \rangle}{\langle \texttt{emit}(e), \alpha, m, n \rangle \rightsquigarrow \langle \alpha', m', n' \rangle} \tag{2.8}$$

$$\langle \texttt{@emitting}(n'), e\alpha, m, n \rangle \rightsquigarrow \langle \alpha, m, n \rangle \quad \text{if } |e\alpha| = n' \tag{2.9}$$

$$\frac{\langle \texttt{@emitting}(n'), \alpha, m, n \rangle \rightsquigarrow \langle \alpha'', m'', n'' \rangle}{\langle \texttt{@emitting}(n'), e\alpha, m, n \rangle \rightsquigarrow \langle \alpha'', m'', n'' \rangle} \quad \text{if } |e\alpha| \neq n' \tag{2.10}$$