EE512 – Applied Biomedical Signal Processing

Instantaneous Frequency Estimation

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OUTLINE/OBJECTIVES

Instantaneous frequency
- concept
- definition
- applications

Methods for estimation of instantaneous frequency
- applicability
- advantages
- drawbacks

Lab session on real-world biomedical signals.
- interpret data
- evaluate the results

Specific learning outcomes:

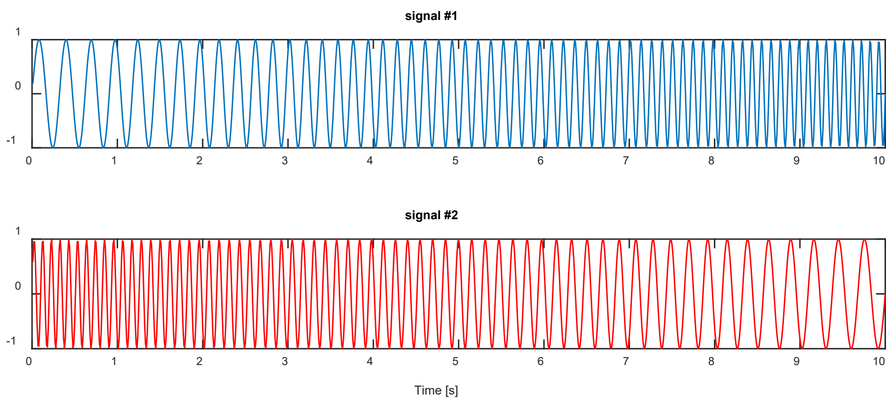
- Understand the theory, mathematical definition, and role of IF in signal processing.
- Distinguish and select appropriate method for IF estimation
- Assess and interpret IF estimates in real-world signal contexts.

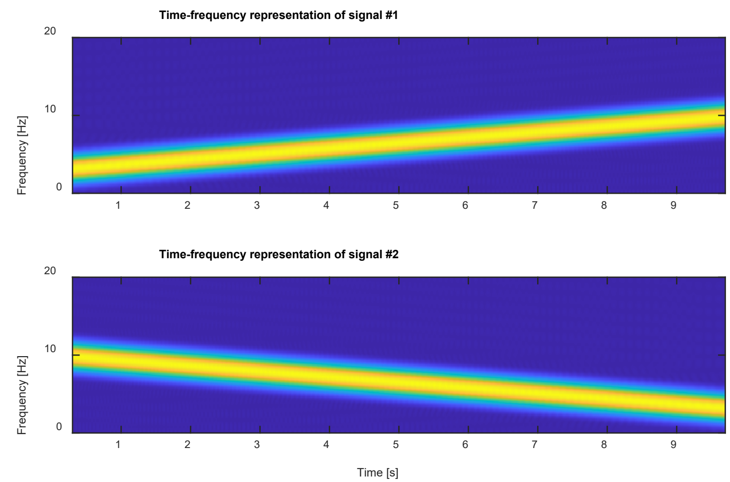
Motivation for IF estimation

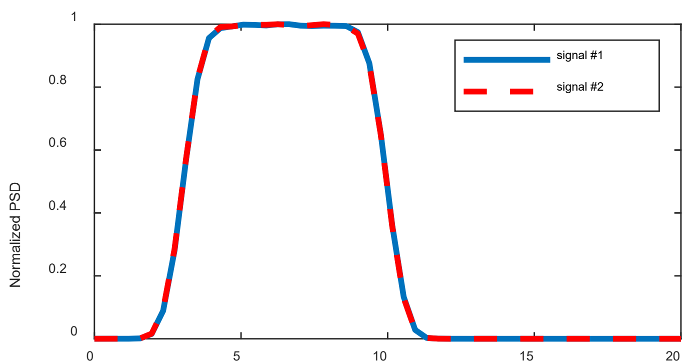
- Many natural and man-made signals are non-stationary and exhibit time-varying frequency, amplitude or phase.
- Power spectral density (PSD) estimation is the main technique to analyze the frequency content.
- A very important requirement of all approaches for PSD estimation is that the signal has to be stationary.

Motivation for IF estimation

Example: two sinusoids whose frequencies vary linearly over time, respectively from 3 to 10 Hz (signal #1) and from 10 to 3Hz (signal #2).



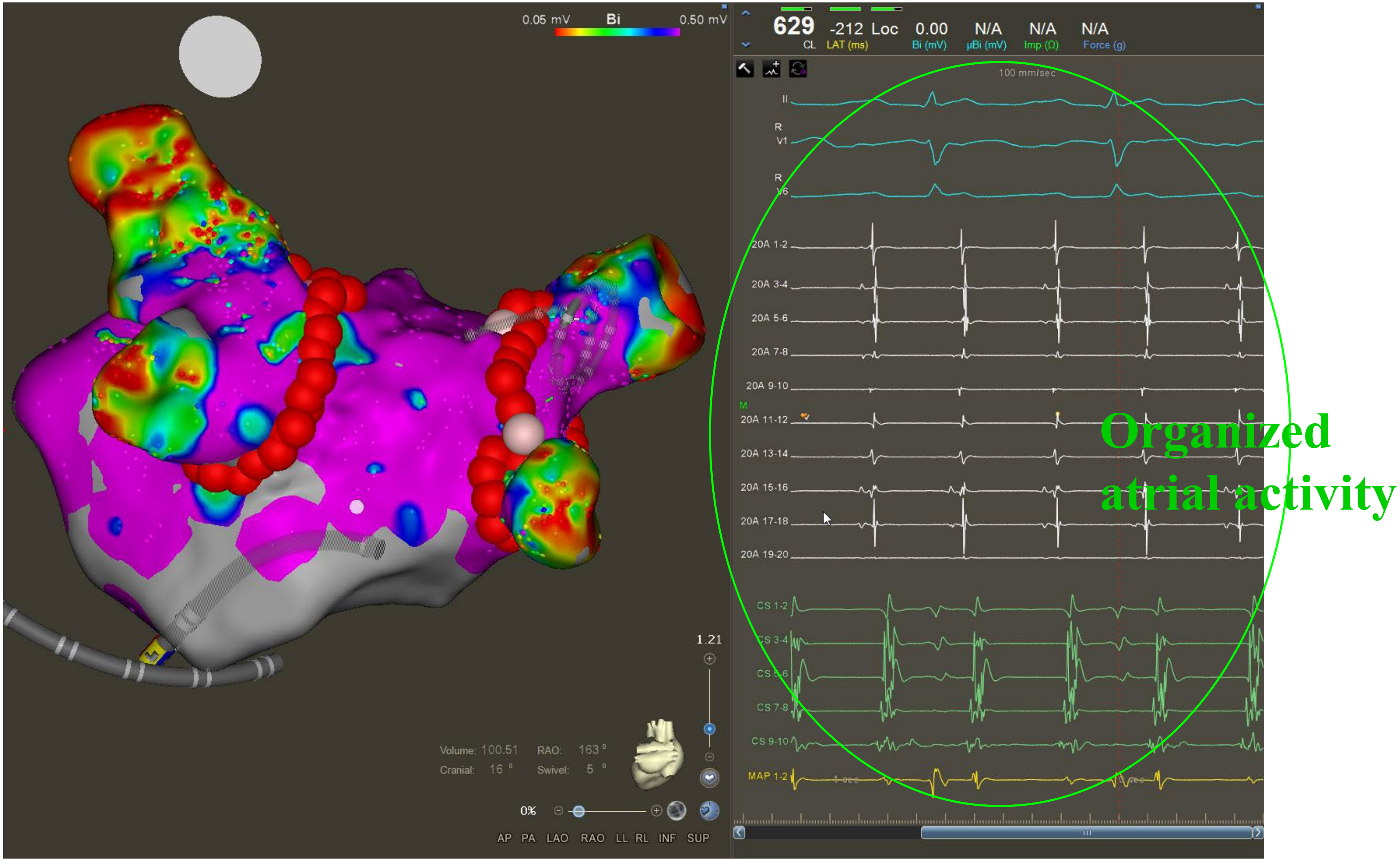




- **Temporal information is lost in a PSD estimate**
- **PSD gives the false idea that there is power in range 3-10Hz for the whole duration of the signal**

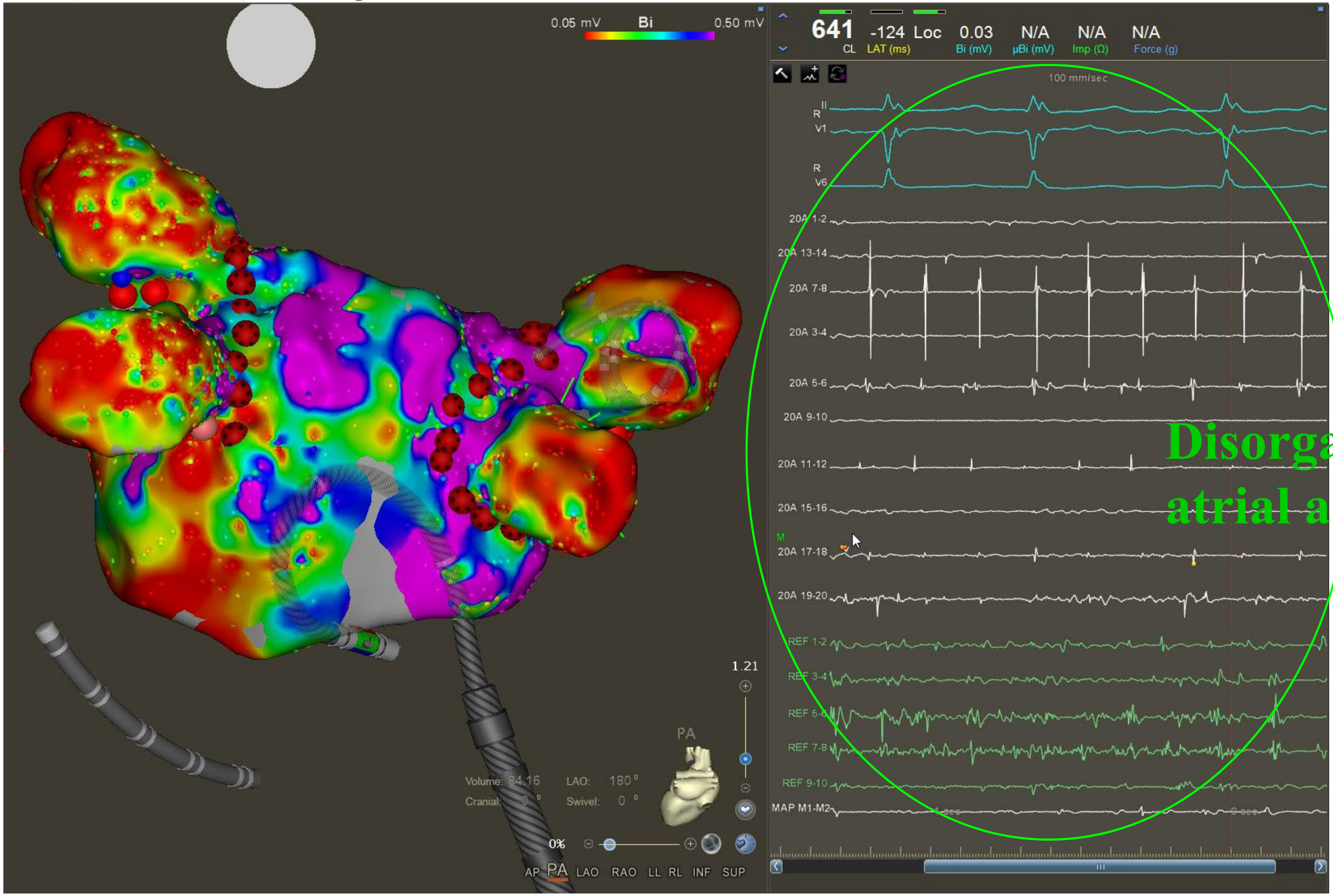
Motivation for IF estimation

Applied example: frequency estimation in atrial fibrillation



Carto® Mapping System, Biosense Webster

Applied example: frequency estimation in atrial fibrillation



Carto® Mapping System, Biosense Webster

Atrial fibrillation – organized activity

disorganized activity

lead V_1

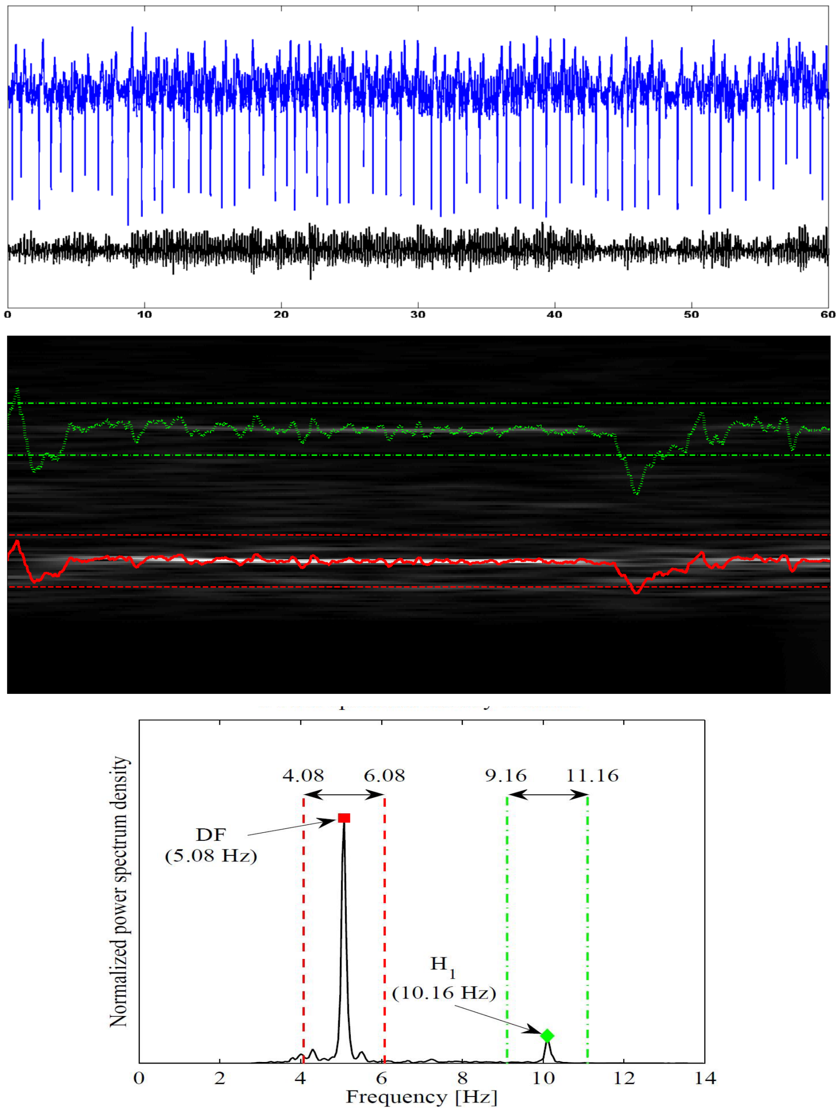
atrial V_1

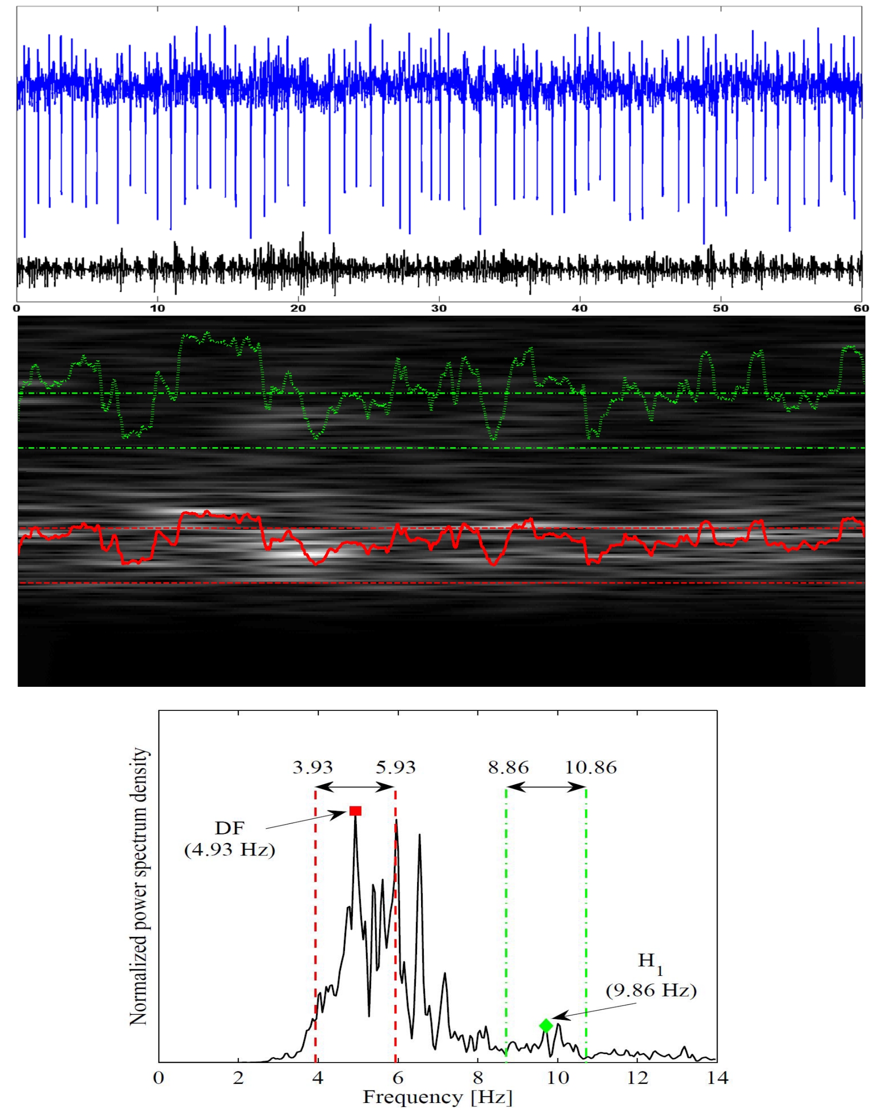
Frequency [Hz]

10

5

0





- The concept of the *period* is connected to the repetitiveness of phenomena or events.
- To determine the *frequency* of a periodic signal one needs to observe the signal over the whole period, that is the *frequency* is not a local feature of the signal.
- For non-stationary signals, one needs to introduce the concept of **instantaneous frequency (IF)** which accounts for the time-varying nature of the process.
- IF can be interpreted as the time evolution of the location of the spectral peak of the signal, i.e. the frequency of the sinusoid that locally fits the signal.

- As such, the underlying assumption for IF is that the signal is locally mono-component.
- This is rarely the case in practice (for instance if harmonics are present) and thus some frequency components separation (band-pass filtering) should be performed before retrieving their IFs.
- Note that this may cause problems: if indeed the frequency changes with time, how can one be sure that the filter passband always contains the IF?

- The concept of IF was firstly introduced with complex amplitude and frequency-modulated signals

- For amplitude and frequency-modulated sinusoids:

$$x(t) = a(t)\cos(\Phi(t))$$

where $a(t)$ and $\Phi(\mathrm{t})$ are time-varying amplitude and phase respectively.

- The IF was defined as the derivative of the phase with respect to time:

$$\omega(t) = \frac{d\Phi(t)}{dt}$$

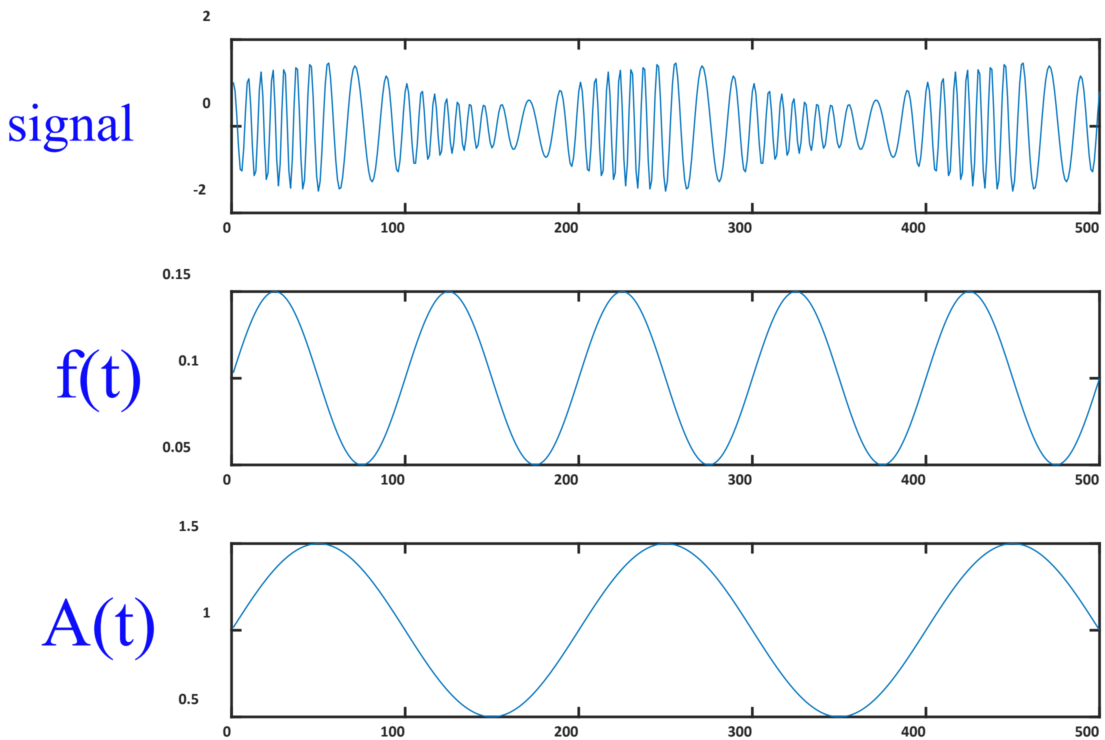
- For a complex exponential $Ae^{j2\pi ft}$, the IF is just the frequency *f*. If one defines $\Phi(t) = 2\pi ft$, then

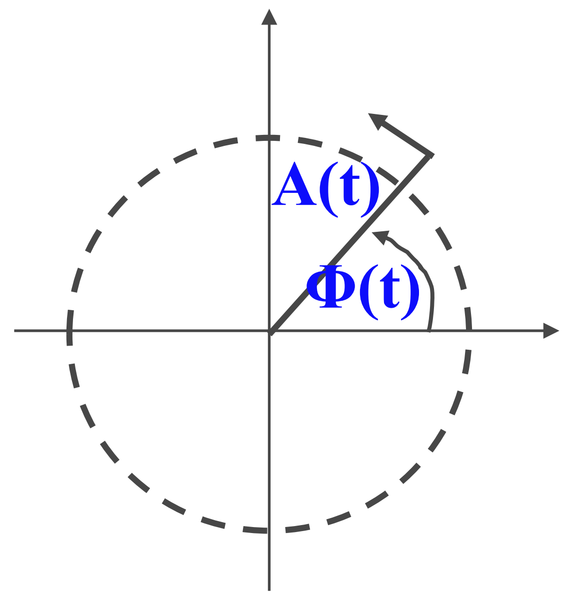
$$f = \Phi'(t)/2\pi$$

- Thus the instantaneous frequency of $Ae^{j\Phi(t)}$, with ϕ(t) not necessarily a linear function of *t*, is defined in the same way: $f(t) = \Phi'(t)/2\pi$

IF is a generalization of the definition of constant frequency, i.e. it is the rate of the change of phase angle at time *t*

- **Example on synthetic signal**: sinusoid with amplitude and frequency modulation





$$x(t) = A(t)e^{j\Phi(t)}$$

Approaches for instantaneous frequency estimation:

- Hilbert transform
- Teager-Kaiser operator
- Adaptive frequency tracking

- There is a need for a method for extracting the phase from any type of signals and not only sinusoids
- The analytic signal is defined as:

$$x_a(t) = x(t) + \mathrm{j}\mathrm{H}\{x(\mathrm{t})\}$$

- HT is defined as:

$$x_h(t) = H\{x(t)\} = \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau$$

Convolution of $x(t)$ with $\frac{1}{\pi t}$

- In the Fourier domain: $X_h(f) = \mathrm{F}\left\{\frac{1}{\pi t}\right\} X(f)$

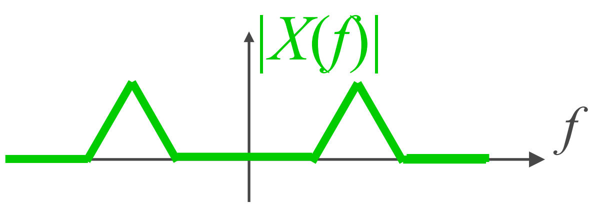
$$\frac{1}{t} \overset{F}{\leftrightarrow} -j\pi \mathrm{sgn}(f)$$

- $X_h(f)$ can be expressed as:

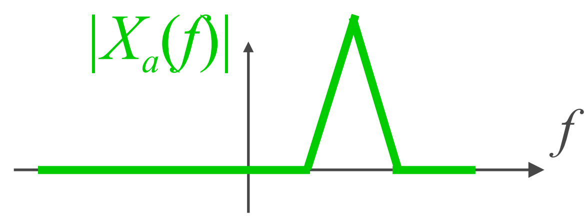
$$X_h(f) = \begin{cases} jX(f) & for\ f < 0 \\ 0 & for\ f = 0 \\ -jX(f) & for\ f > 0 \end{cases}$$

- The Fourier transform of the analytic signal:

$$X_a(f) = X(f) + \mathrm{j}H(f)X(f) = \begin{cases} X(f) - \mathrm{j}^2 X(f) = 2X(f) & \text{for } f > 0 \\ X(f) + \mathrm{j}^2 X(f) = 0 & \text{for } f < 0 \end{cases}$$



Real signal with symmetric amplitude spectrum



Analytic (complex) signal with non-symmetric amplitude spectrum

- Let us apply the Hilbert filter to the $x(t) = A\cos(2\pi f_0 t)$ using its complex representation $x(t) = A\,\frac{e^{j2\pi f_0 t}+e^{-j2\pi f_0 t}}{2}$

$$F\{HT[x(t)]\}=A\frac{-j}{2}\delta(f-f_0)+A\frac{+j}{2}\delta(f+f_0)$$

$$\Big\downarrow F^{-1}$$

$$HT[x(t)] = A\,\frac{-je^{2\pi f_0 t}+je^{-j2\pi f_0 t}}{2} = A\,\frac{e^{j2\pi f_0 t}-e^{-j2\pi f_0 t}}{2j}$$

$$= A\sin(2\pi f_0 t)$$

Hilbert filtering is basically a phasing operation, transforming a cosine into a sine.

- The analytic signal is:

$$x_a(t) = A\cos(2\pi f_0 t) + A\, j\sin(2\pi f_0 t) = Ae^{j2\pi f_o t}$$

$|x_a(t)| = A$ is the signal envelope, and $\angle\, x_a(t) = 2\pi f_0 t$ is the phase.

- For a signal $x(t) = A(t)\cos\big(\phi(t)\big)$ with slowly time-varying amplitude and frequency, the analytic signals is $x_a(t) \approx A(t)\, e^{j\phi(t)}$

- One gets: $|x_a(t)| = A(t)$ - instantaneous envelope (amplitude)
  $\angle\, x_a(t) = \phi(t)$ - instantaneous phase

The instantaneous frequency $f(t) = \frac{1}{2\pi}\frac{d\phi(t)}{dt}$.

Similar development in discrete-time:

For $x[n] = A[n]\cos(\Phi[n]) \rightarrow x_a[n] \approx A[n]e^{j\Phi[n]}$

- $|x_a(n)| \approx A[n]$ – instantaneous envelope
- $\arg[x_a(n)] \approx \phi[n]$ – instantaneous phase

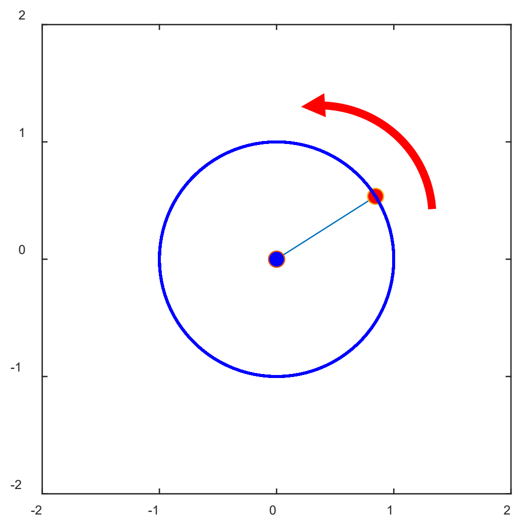
signal → analytic signal → instantaneous phase → phase unwrapping → differentiation → IF

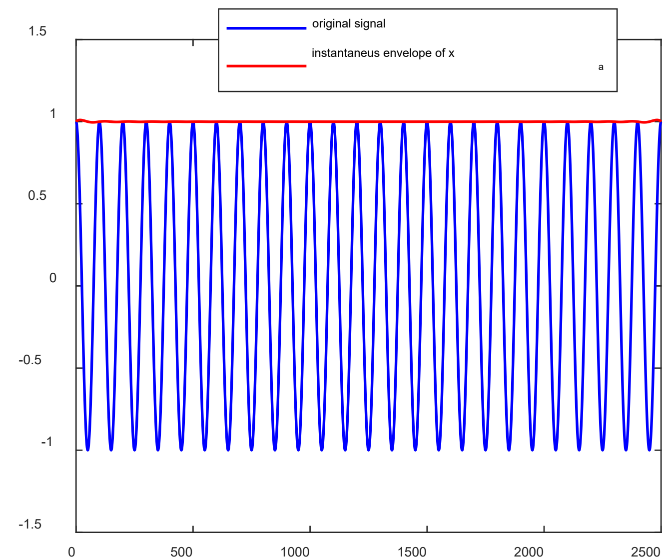
! the phase must be unwrapped

! the estimation of IF (in Hz) must take into account the sampling frequency

- As mentioned, this development is directly applicable to discrete-time signals. In Matlab®, the function hilbert(x) directly computes the analytic signal.
- Exemple 1: pure sinusoid

Analytic signal x_a

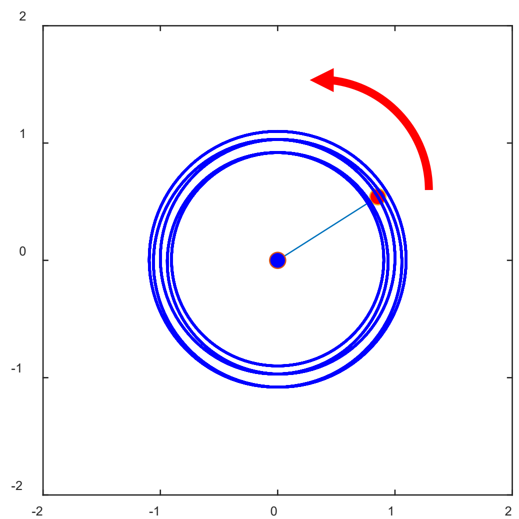


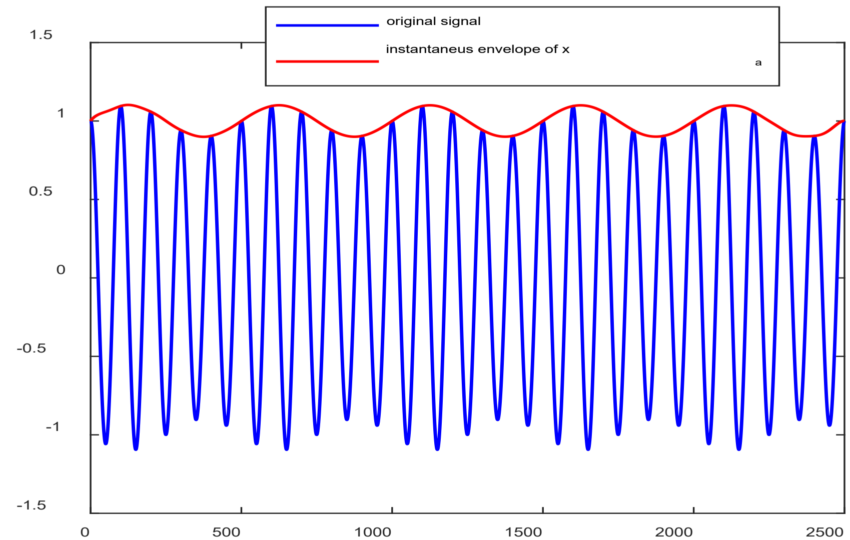


***Animated illustrations using the homemade Matlab routine «sc_hilbert» by courtesy of Jean-Marc Vesin, PhD**

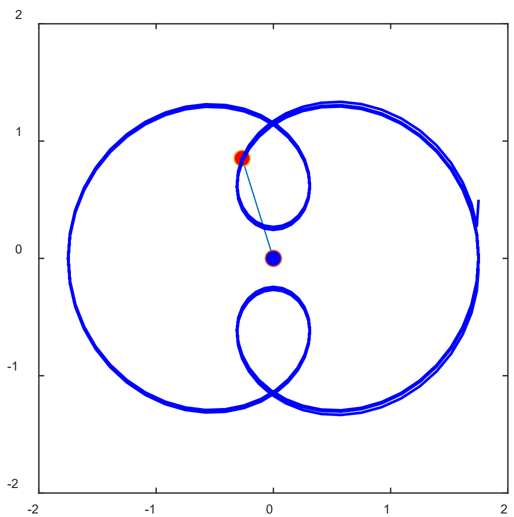
- Exemple 2: sinusoid with amplitude modulation

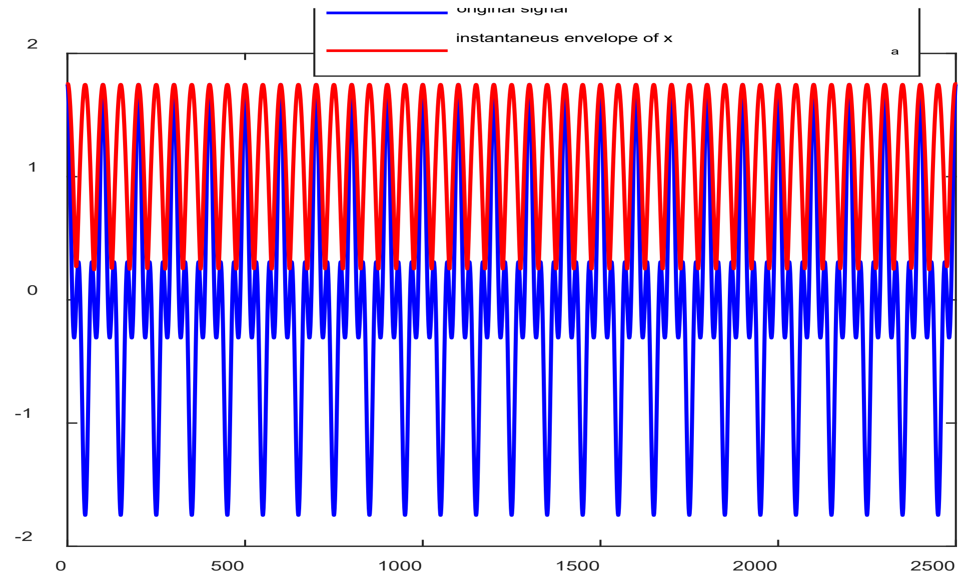
Analytic signal x_a





- Exemple 3: weighted sum of two sinusoids (i.e. the signal is not narrowband)

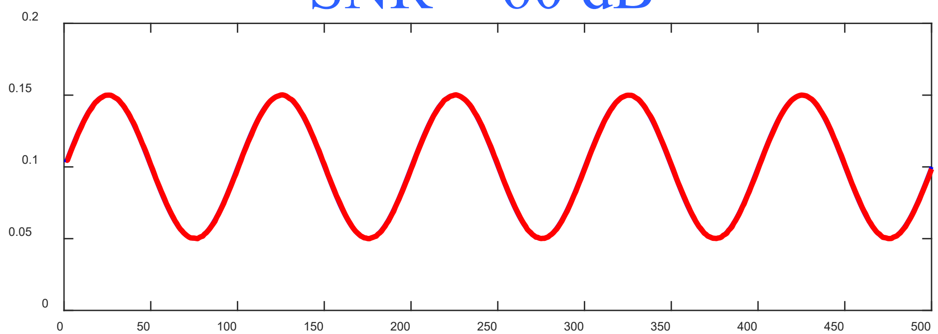




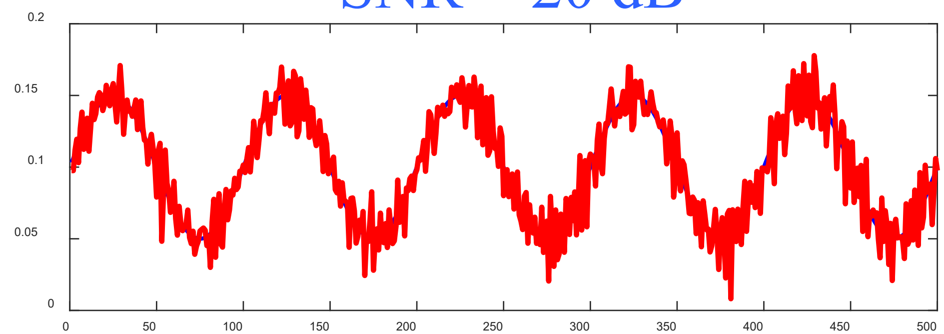
- Problems! The envelope is visibly incorrect, and for x_a one has loops that do not encircle the origin, and would give a negative estimate of the frequency (decrease in the phase).

- IF estimation – highly dependent on SNR ratio

SNR = 60 dB



SNR = 20 dB



- In the case of real-time computation, an FIR filter of length 2L+1 should be used to approximate the Hilbert filter. This introduces a delay of L samples in the output.

- The Teager-Kaiser measures instantaneous energy changes of signals composed of a single time-varying frequency.
- Basically, the operator can be used to estimate the energy required for generating a signal and then separate it into its amplitude and frequency components
- In continuous time, this operator is defined as:

$$\Psi_c[x(t)] = \left[\frac{dx}{dt}\right]^2 - x(t)\frac{d^2x}{dt^2}$$

- For $x(t) = A\cos(2\pi f_0 t+\theta)$, one checks easily that:

$$\Psi_c[x(t)] = (A2\pi f_0)^2 \text{ and } \Psi_c[x'(t)] = A^2(2\pi f_0)^4$$

- For a signal $x(t)$ with slowly time-varying amplitude and frequency, the operator can approximately estimate the squared product of the amplitude and frequency signals:

$$\Psi_c[A(t)\cos(\phi(t)+\theta)] \approx [A(t)\phi'(t)]^2$$

- Applying TK to $x'(t)$: $\Psi_c[x'(t)]] \approx A^2(t)[\phi'(t)]^4$

- By manipulating these two eqs., the instantaneous frequency and the amplitude envelope are estimated as:

$$\phi'(t) \approx \sqrt{\frac{\Psi_c[x'(t)]}{\Psi_c[x(t)]}} \qquad |A(t)| \approx \frac{\Psi_c[x(t)]}{\sqrt{\Psi_c[x'(t)]}}$$

- In the discrete time, the Teager-Kaiser operator becomes:

$$\Psi_d[x(n)] = [x(n)]^2 - x(n+1)\,x(n-1)$$

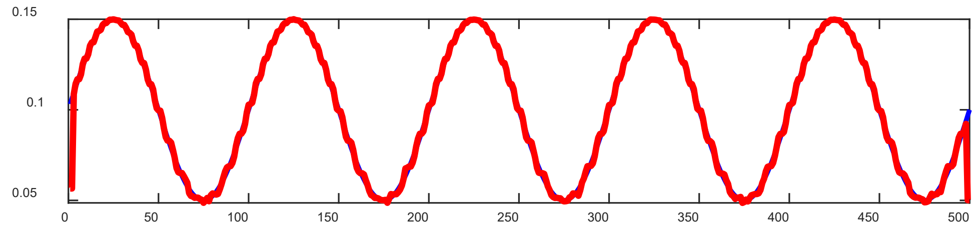
- When Ψ_d is applied to $x[n]$ and its differences, backward $y[n] = x[n] - x[n-1]$ and forward $y[n+1] = x[n+1] - x[n]$:

$$\phi'(n) \approx \arccos\left(1 - \frac{\Psi_d[y(n)] + \Psi_d[y(n+1)]}{4\Psi_d[x(n)]}\right)$$

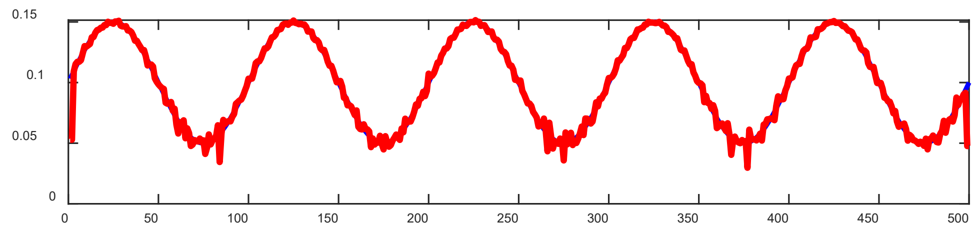
$$|A(n)| \approx \sqrt{\frac{\Psi_d[x(n)]}{1 - \left(1 - \frac{\Psi_d[y(n)] + \Psi_d[y(n+1)]}{4\Psi_d[x(n)]}\right)^2}}$$

- IF estimation

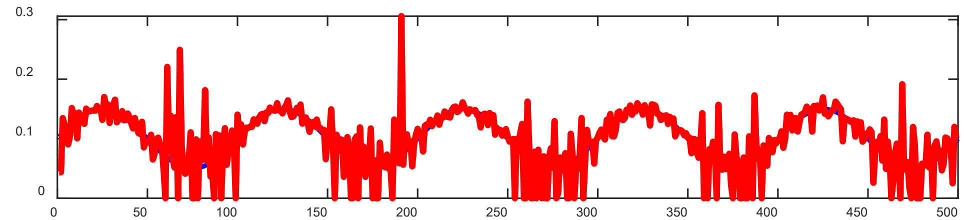
SNR = 60 dB



SNR = 40 dB

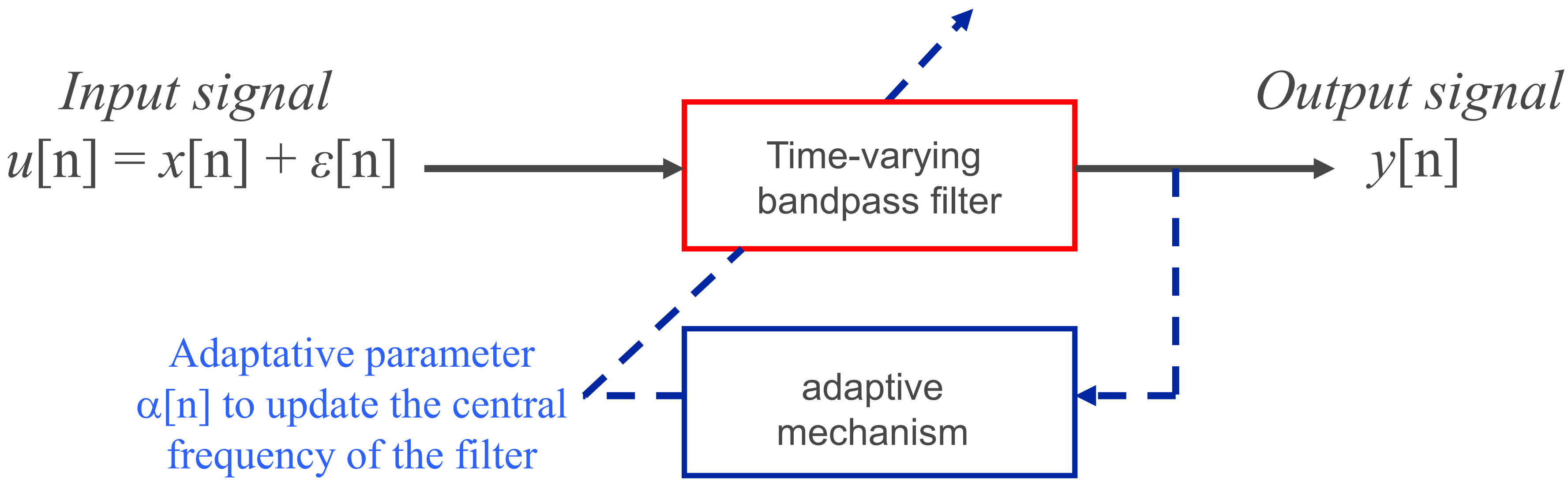


SNR = 20 dB

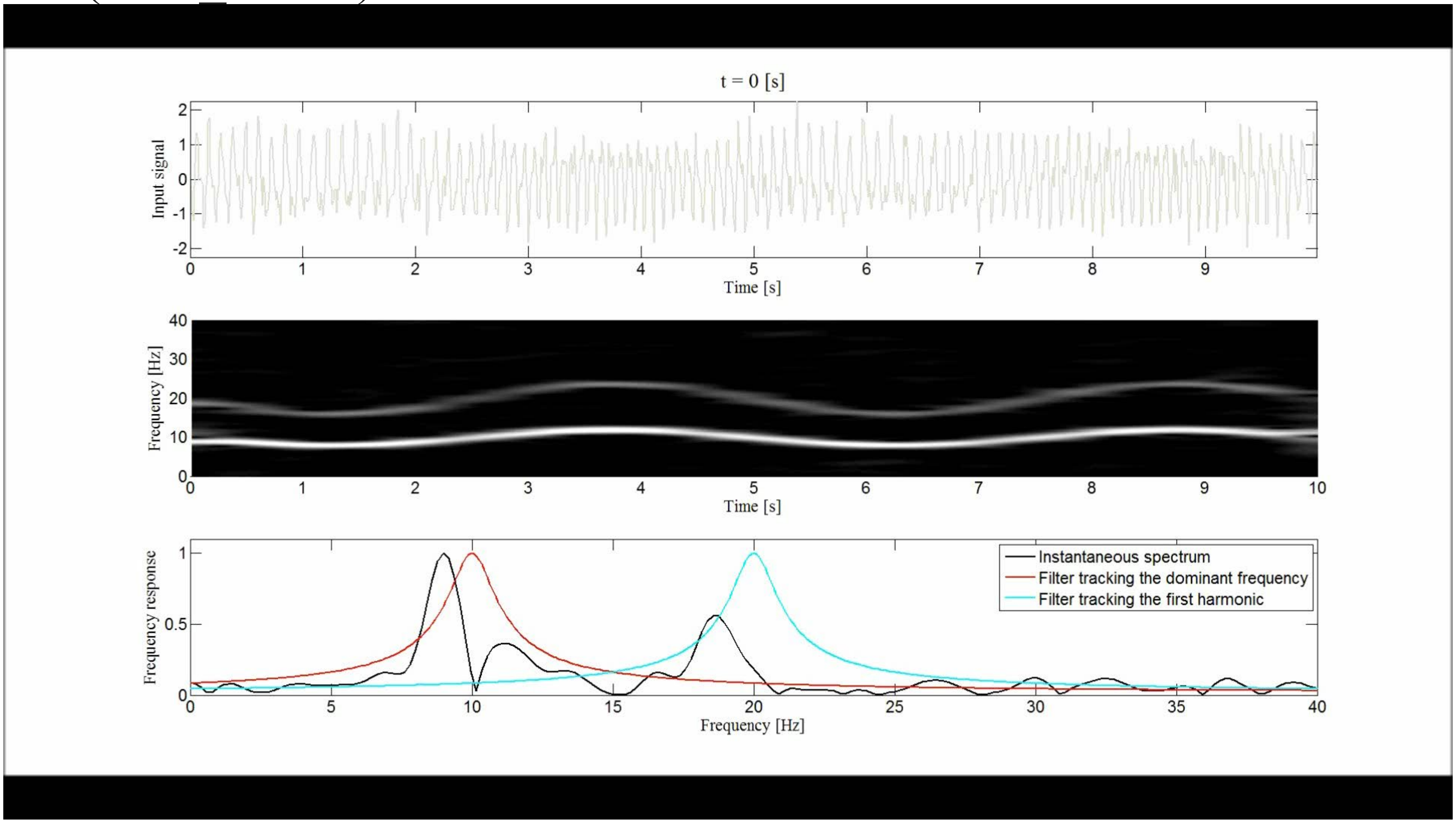


- The Teager-Kaiser operator performs poorly on noisy signals. Why?

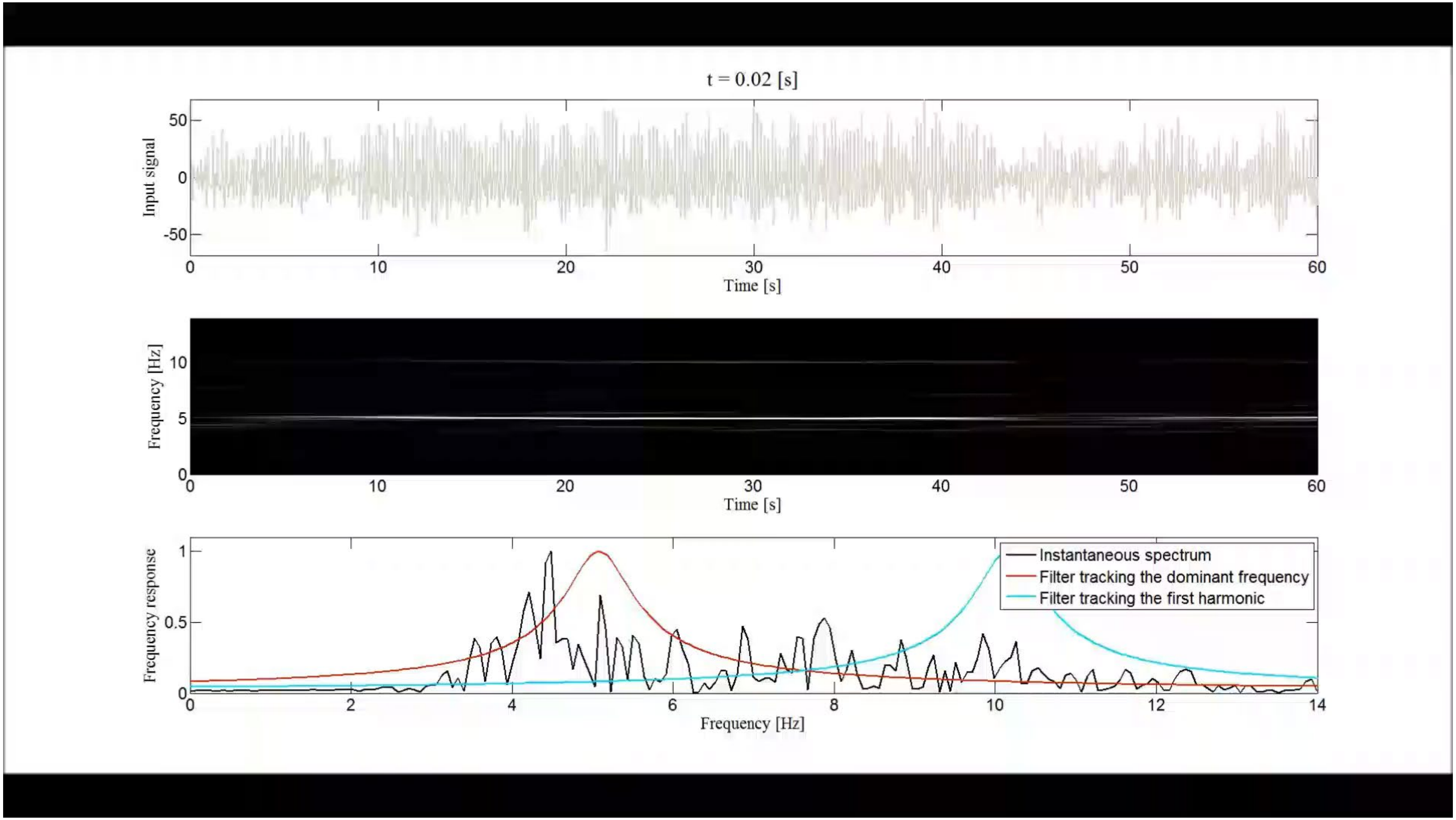
- The idea is to use a time-varying bandpass filter to extract (or enhance) the periodic component in the input signal and an adaptative mechanism for controlling the central frequency of the filter.



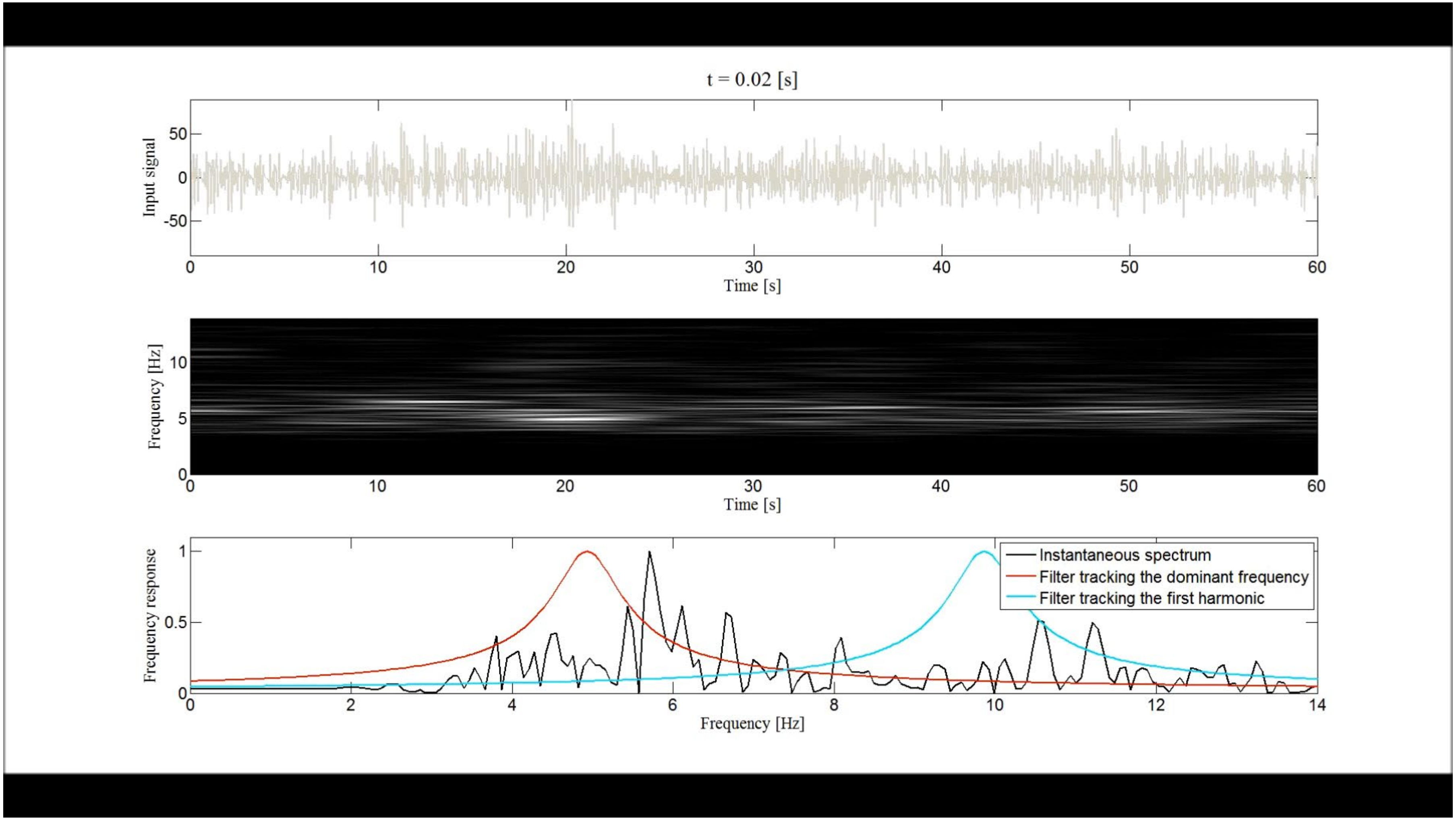
Example 1. Synthetic signal composed of fundamental component and 1st harmonic (HFT_movie)



Example 2. Organized atrial fibrillation (AF_LT_tracking_movie)



Example 3. Highly disorganized atrial fibrillation ablation (AF_NLT_tracking_movie)



- For an input signal, the sinusoidal component x[n] should satisfy the oscillatory equation:

$$x[n] = 2\cos(2\pi f_0)\, x[n-1] - x[n-2] = 2\alpha_0 x[n-1] - x[n-2]$$

Using: $\sin(a+b) + \sin(a-b) = 2\sin(a)\cos(b)$, with $a = 2\pi f(n-1)$ and $b = 2\pi f$

- The bandpass filter is defined by the transfer function:

$$H(z;n) = \frac{1-\beta}{2} \frac{1-z^{-2}}{1-\alpha[n](1+\beta)z^{-1}+\beta z^{-2}}$$

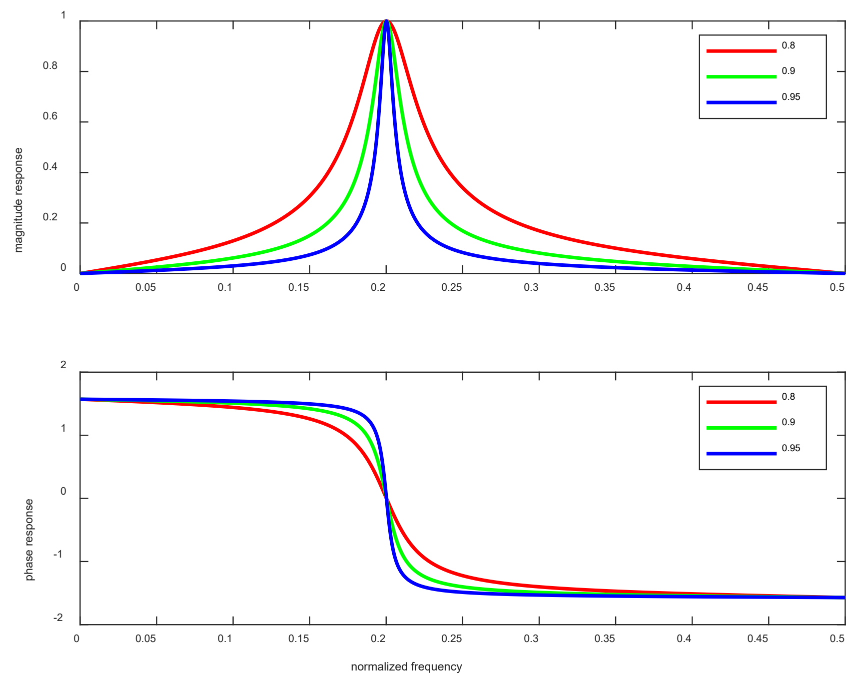
The adaptative coefficient $\alpha[n]$ which tracks $\alpha_0 = \cos(2\pi f_0)$ defines the central frequency $f[n]$ of the filter : $\alpha[n] = \cos(2\pi f[n])$

The parameter $0 < \beta < 1$ controls the filter bandwidth.

- Frequency response (normalized frequency) for a central normalized frequency of 0.2.

  ✓ larger the β (poles closer to the unit circle), the narrow the filter bandwidth.

  ✓ The filter has unitary gain and zero phase shift at the central frequency.



The output signal y(·) is the component of the input signal x(·) at frequency f(·)

- The output output is:

$$y[n] = (1+\beta)\alpha[n]\, y[n-1] - \beta y[n-2] + \frac{1-\beta}{2}(x[n] - x[n-2])$$

- The filter output $y(n)$ is intended to follow the discrete oscillator model, i.e. $y(n)$ is locally as close as possible to a sinusoid:

$$y[n] = 2\cos(2\pi f_0)\, y[n-1] - y[n-2]$$

The goal is to determine $\alpha[n+1] = cos(2\pi f[n+1])$ that satisfies the discrete oscillator model.

- That is to minimize the cost defined as:

$$J = E\{|y[n] - 2\alpha[n+1]y[n-1] + y[n-2]|^2\}$$

- By setting $\partial J / \partial \alpha[n+1] = 0$, the optimal solution is:

$$\alpha[n+1] = \frac{E\{y[n-1](y[n]+y[n-2])\}}{E\{2y^2[n-1]\}} = \frac{Q[n]}{2P[n]}$$

Proof.

$$J = E\{y^2[n] + 4\alpha^2[n+1]\, y^2[n-1] + y^2[n-2]$$
$$-4\alpha[n+1]y[n]y[n-1] - 4\alpha[n+1]y[n-1]y[n-2] - 2y[n]\, y[n-2]\}$$

$$\frac{dJ}{d\alpha[n+1]} = E\{8\alpha[n+1]\, y^2[n-1] - 4y[n]y[n-1] - 4y[n-1]\, y[n-2]\}$$

- The numerator and denominator can be estimated recursively:
  $Q[n] = \delta\, Q[n-1] + (1-\delta)\{y[n-1](y[n]+y[n-2])\}$
  $P[n] = \delta\, P[n-1] + (1-\delta)\, y^2[n-1])$

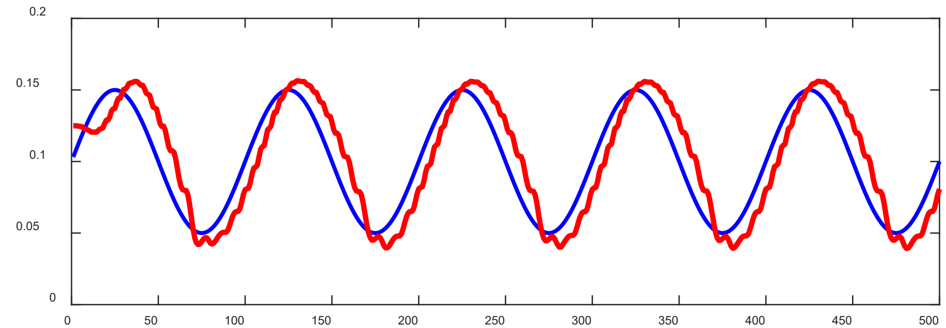
– δ a forgetting parameter ($0 \ll \delta < 1$) to control the estimation update rate.

- Q[n] and P[n] are lowpass-filtered versions of their instantaneous values.

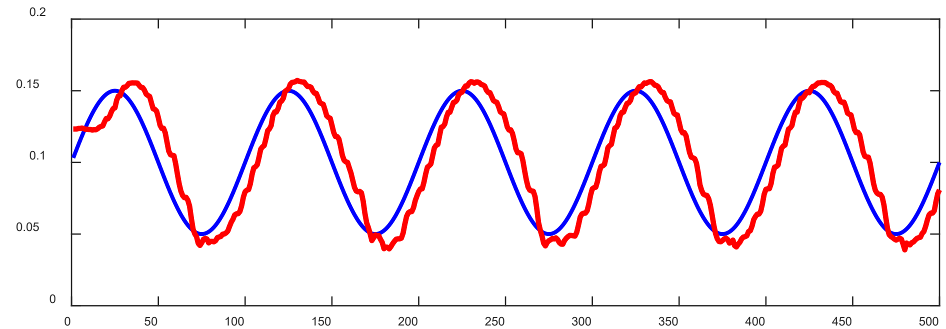
- Finally, the estimate of the instantaneous frequency is computed as $f[n+1] = \arccos(\alpha[n+1]\,)\,/2\pi$

- IF estimation. $f(0) = 0.1$, $\beta = \delta = 0.85$

SNR = 60 dB



SNR = 20 dB



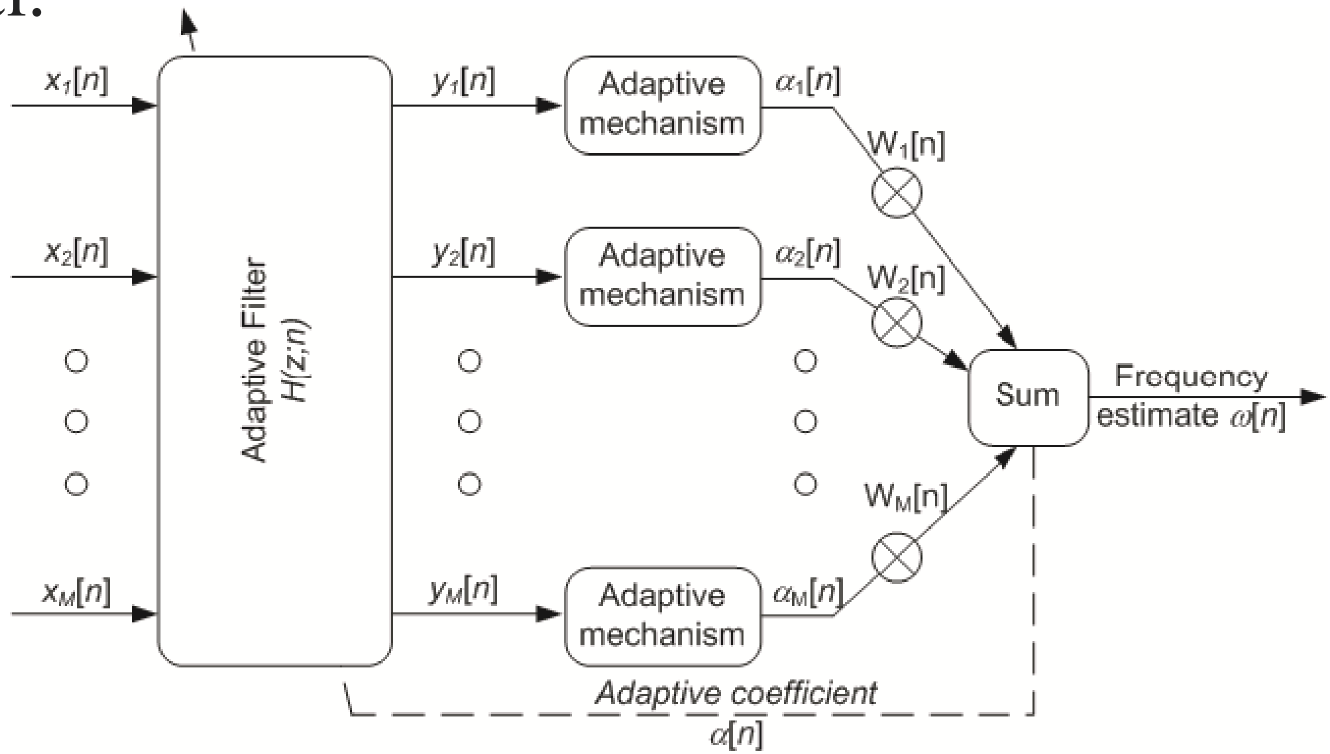
- IF estimation is robust with respect to signal-to-noise ratio.
- Adaptation transient at the beginning of the signal (a delay of about 10 samples in this case)

Extensions of the single adaptative frequency tracking algorithm:

- to estimate the instantaneous frequency of a periodic component present in several signals (e.g. frequency of fibrillatory activity presented in multiple surface ECG leads).

- to track multiple frequency components simultaneously, e.g. the instantaneous fundamental frequency and the harmonic components of a single signal.

- An unique bandpass filter is used and its central frequency is updated based on the individual updates for each signal.



Scheme of the weighted multi-signal approach. Figure from Prudat Y. Adaptive frequency tracking and application to biomedical signals. EPFL. 2009

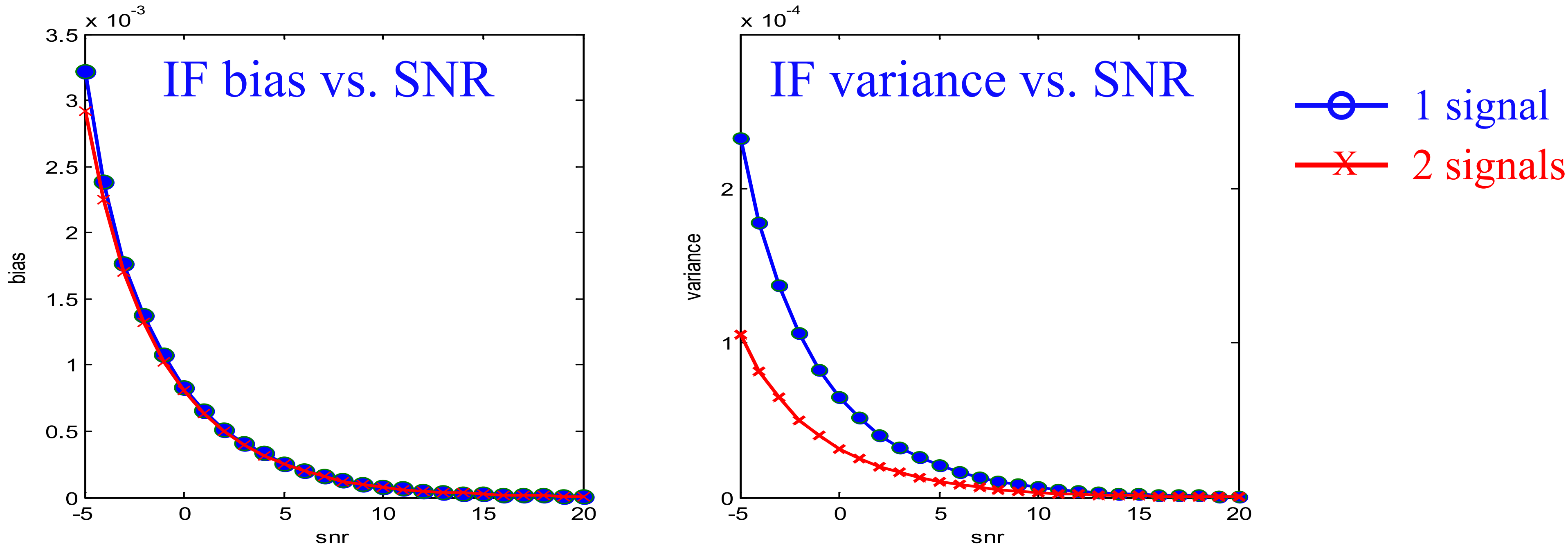
- For M signals the frequency update is given by:

$$\alpha(n+1) = \sum_{i=1}^{M} W_i(n) \frac{Q_i(n)}{2P_i(n)}$$

with:

$$W_i(n) = \frac{\sum_{j \neq i} [y_j(n) - x_j(n)]^2}{(M-1) \sum_{j=1}^{M} [y_j(n) - x_j(n)]^2} \; ; \; \sum_{i=1}^{M} W_i(n) = 1$$

i.e. a larger weight should be given to the signals in which the oscillation (filter output) is closer to the original signal.



- Improved tracking performance in terms of frequency estimation variance (the estimation variance decreases with the number of input signals).

Exam-like questions

1. What are the practical benefits of applying bandpass filtering before estimating the IF?

2. Why does the Teager-Kaiser Operator perform poorly on IF estimation at lower frequencies when the signal-to-noise ratio (SNR) decreases?

3. Why, in discrete-time, one needs to multiply the derivative of the phase by the sampling frequency to obtain the IF in Hz?

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