

The Borel Complexity of Isomorphism for Complete Theories of Linear Orders With Unary Predicates

Richard Rast

February 11, 2015

Abstract

We prove the theorem that if A is a linear order, then $\text{Th}(A)$ is either \aleph_0 -categorical or is Borel-complete in the sense of [1]. A large portion of the proof is based on an argument by Rubin [7] as outlined in Rosenstein [6], where it was shown that $\text{Th}(A)$ is either \aleph_0 -categorical or has continuum-many countable models.

We generalize this result to colored linear orders. If we allow finitely many colors, we get the same result. If we allow countably many, it is also possible for $\text{Th}(A)$ to have finitely many countable models (any number except two is possible), to be exactly equivalent to \cong_1 (equality for real numbers), or to be exactly equal to \cong_2 (equality of image for sequences of real numbers). All these cases are possible, and we compute precise model theoretic conditions characterizing each possibility. This complements work in [5] where approximately the same result was shown for o-minimal theories.

1 Introduction

We are concerned with computing the Borel complexity of isomorphism for (complete theories of) colored linear orders¹. Approximately this is investigating the following problem: given a theory T , we would like to compute invariants from the countable models of T which completely determine the isomorphism class of the model you started with. Since the isomorphism class of the model is itself an invariant for the model, this can always be done. Of course this is not very helpful, so the question we ask is: how complicated must such an invariant be?

We will make this precise later, using the language of Borel reductions of equivalence relations (explained in Subsection 1.2). Our main theorem is the following: if T is a complete theory of a colored linear order, then either the isomorphism problem for T is “maximally difficult” (that is, Borel complete) or “fairly easy.” In particular, if there are only finitely many colors in the language, then “fairly easy” means T has only one countable model.

The proof goes as follows. We first work with “self-additive” structures. We define a condensation (that is, a convex equivalence class) on these structures, and show that either they are \aleph_0 -categorical or admit “ \mathbb{Z} -like behavior” – allowing a certain type-definable quotient order to have any order type we like – and thus are Borel complete. This is defined and proven in Sections 4 and 5 in two parts, depending on whether or not $S_1(T)$ is infinite.

We then show that general structures can be definably decomposed into indecomposable convex types, the realizations of which are nearly self-additive. If one of these types can be Borel complete (as a substructure) then T is also; otherwise, all of them are \aleph_0 -categorical. We then use fine information about \aleph_0 -categorical theories to show that countable models of T are determined up to isomorphism by a finite choices, so the isomorphism relation can be reduced to a large number of finite invariants, so that T is not too difficult. In

¹Those readers who are not interested in colors need not be concerned about needless generalization – the proof for linear orders without colors is essentially identical to the “finite language” case, and even that is only relevant to the argument at a few stages.

the case where the language is finite, we can go further and show that there are no meaningful choices to make, so that T must be \aleph_0 -categorical. This is all accomplished in Section 6.

Before any of this is done, we establish a few basic facts about colored orders and sums of those orders. We also recall a theorem of Rubin about relativizing formulas, and do some detailed work on \aleph_0 -categorical colored orders. Our main proof follows a similar outline as [7], but the exposition and many lemmas are new.

1.1 Conventions and Definitions

A *typical language* is one which contains a binary relation $<$ and countably many unary predicates $\{P_i : i \leq \kappa\}$ where $\kappa \leq \aleph_0$. A *typical theory* in a typical language is one which is complete and consistent, and which makes $<$ a linear order (irreflexive, antisymmetric, and transitive). Capital letters like A, B, C , etc. will refer to models of a typical theory. All languages and theories will be “typical” unless otherwise noted, but this will sometimes be restated anyway for emphasis or clarity.

Throughout, if $A \subset B$ and B has an order $<$, then A is called *convex* when for every $a_1 < a_2$ in A , if $b \in B$ and $a_1 < b < a_2$, then $b \in A$. A set $B \subset A$ is *initial* if it is convex and unbounded below, and likewise $B \subset A$ is *terminal* if it is convex and unbounded above, and likewise with formulas. A formula $\phi(x)$ (with understood typical L -structure A) is convex (initial, terminal) if $\phi(A)$ is.

An equivalence relation whose classes are convex is called a *condensation*; these are precisely the equivalence relations whose classes are canonically ordered by $<$. Following [6], we usually define condensations by a function \mathbf{c} taking elements of A to subsets of A . Such a function is a *condensation* if the following are satisfied:

- For all $x, x \in \mathbf{c}(x)$.
- For all $x, \mathbf{c}(x)$ is convex.
- For all x , if $y \in \mathbf{c}(x)$, then $\mathbf{c}(x) = \mathbf{c}(y)$.

The equivalence of the two definitions is an easy and illustrative exercise:

Proposition 1.1. *The function \mathbf{c} is a condensation if and only if the relation “ $x \in \mathbf{c}(y)$ ” is a condensation.*

1.2 Borel Complexity

We need some formalism in order to discuss “how complex isomorphism is” for countable models of a complete theory. The reader should be aware of these things, but does not need a deep understanding to follow the rest of the paper or to appreciate the result.

Fix a countable language L . An L -structure with universe ω can be considered as a set of functions $f_R : \omega^n \rightarrow 2$, where R is an n -ary relation in L (with constants and functions handled similarly). Alternately, we can consider an L -structure to be an element of the set $\Pi_R 2^{\omega^n}$. We call this space $\text{Mod}(L)$, and give it the “formula topology,” meaning for any L -formula $\phi(\bar{x})$ and any tuple \bar{a} from ω , the set $\{A \in \text{Mod}(L) : A \models \phi(\bar{a})\}$ is open. This is a Polish space (that is, a separable complete metric space) and countable models of T can be considered to be a closed subspace of $\text{Mod}(L)$, which we will call $\text{Mod}(T)$. Therefore, $\text{Mod}(T)$ is also a Polish space, so a standard Borel space (a set without a topology, but which has a set of distinguished “Borel sets” which could conceivably have come from such a topology).

If X and Y are standard Borel spaces, a function $f : X \rightarrow Y$ is *Borel* if the preimage of any Borel subset of Y is a Borel subset of X . In practice, a map is Borel if it can be “completely described” by a reasonable procedure; all maps we construct in this paper are Borel. If E and F are equivalence relations on X and Y (so in particular are subsets of X^2 and Y^2 , respectively), we say $E \leq_B F$ if there is a Borel function $f : X^2 \rightarrow Y^2$ such that for all $x_1, x_2 \in X$, $(x_1, x_2) \in E$ if and only if $(f(x_1), f(x_2)) \in F$. Such a map is called a *Borel reduction*. If $E \leq_B F$ and $F \leq_B E$, we say they are *Borel equivalent*, denoted $E \sim_B F$.

The relation \leq_B is a partial order on the set of invariant equivalence relations (that is, those relations which are the isomorphism relation for some $L_{\omega_1, \omega}$ -sentence for some L). There is a least element, the

relation with exactly one class, corresponding to \aleph_0 -categoricity. Indeed, for any $\kappa \leq \aleph_0$, there is exactly one equivalence relation with exactly κ classes, up to \sim_B , and they form an initial segment of the Borel reduction partial order.

We call the equality relation on a countably infinite standard Borel space \cong_0 . We call the equality relation on an uncountable Borel space \cong_1 ; all uncountable standard Borel spaces are isomorphic, so only one relation arises in this way. It is a theorem that \cong_1 is an immediate successor of \cong_0 , so in particular, is a \leq_B -minimal relation among those relations which have continuum-many classes.

Above \cong_1 (but not immediately) is a relation \cong_2 , defined on the Polish space \mathbb{R}^ω as “equality of image” of the sequences (that is, f and g are \cong_2 -equivalent if and only if $\forall x \exists y \exists z (f(x) = g(y) \wedge g(x) = f(z))$). It is a theorem of Hjorth, Kechris, and Louveau that a relation (X, E) can be Π_3^0 as a subset of X^2 (with respect to *some* Borel equivalent topology on X) if and only if $E \leq_B \cong_2$; see [2] for a proof of this and an explanation of the indexing. It is a theorem of Marker in [3] that if T is not small, then isomorphism for T is *at least* \cong_2 .

These are the “minimal” values that \cong_T can take, subject to various constraints. On the other end, it is perhaps surprising that there is a \leq_B -*maximal* invariant relation: it is a theorem of Friedman and Stanley in [1] that isomorphism on linear orders is one such class, though there are many (isomorphism on graphs, fields, groups, and so on). Any relation which is invariant and which these relations reduces to is called *Borel complete*. Our main theorem is that \cong_T is either Borel complete for a very specific reason, or \cong_T takes one on of the above “minimal values,” subject to its number of countable models and to its number of types.

1.3 Sums, Shuffles, and Fraïssé Games

Here and throughout, L is a typical language, and all L -structures make $<$ a linear order. Unlike the rest of the paper, *we do not assume L is countable*.

Definition 1.2. Let $(X, <)$ be an L -structure, and for each $x \in X$, let $(A_x, <)$ be an L -structure. We define the sum $S = \sum\{A_x : x \in X\}$ to be the set of all pairs (x, a) with $x \in X$ and $a \in A_x$, and say $(x_1, a_1) < (x_2, a_2)$ if and only if $x_1 < x_2$ or $x_1 = x_2$ (so that a_1 and a_2 are comparable) and $a_1 < a_2$. For every i , $S \models P_i((x, a))$ if and only if $A_x \models P_i(a)$. The notation $A + B$ is a shorthand for the obvious finite sum.

The symbols P_i and $<$ are interpreted on the sum precisely so that each of the summands is a convex L -substructure of the sum. This is a useful construction with helpful model-theoretic properties:

Proposition 1.3. *Suppose $(I, <)$ is a linear order and A_i and B_i are L -structures for each $i \in I$. Let $A = \sum\{A_i : i \in I\}$ and $B = \sum\{B_i : i \in I\}$. Then:*

1. *If $A_i \equiv B_i$ for all $i \in I$, then $A \equiv B$.*
2. *If $A_i \prec B_i$ for all $i \in I$, then $A \prec B$.*

Proof. Observe that the second criterion is a special case of the first, by the following argument. For each element of A , add a unary predicate P_a which is interpreted as true on that one element (of that one A_i) and false everywhere else. Since $A_i \prec B_i$ in the original language, $A_i \equiv B_i$ in this expanded language, so by the first criterion, $A \equiv B$ in this expanded language, and therefore $A \prec B$ in the original language. Thus it only remains to show the equivalence condition.

To show this, suppose not; then there is a finite subset of the language on which $A \not\equiv B$, but still $A_i \equiv B_i$ for all $i \in I$. Let $n \in \omega$ be arbitrary. It is sufficient to show player two has a winning strategy on the Ehrenfeucht-Fraïssé game of length n between A and B with regard to this finite sub-language. Since $A_i \equiv B_i$ and the language is finite, there is a winning strategy on the game of length n between A_i and B_i . These can easily be assembled into a winning strategy for A and B ; if player one picks an element from A (or B), it is from some A_i (or B_i , respectively). Follow the strategy for that game and pick the corresponding element from B_i (or A_i , respectively).

This new element satisfies a unary predicate P_i if and only if the original did, is in correct order with the other elements of B_i (or A_i), so preserves the language within the component. Since the components (A_i and B_i) are convex in their orderings, and since we always picked elements from the same index, the order between elements from different components will be automatically preserved. Therefore this is a winning strategy and $A \equiv B$, as desired. \square

If a summand is *type-definable*, we can use sums to produce nonisomorphic models:

Proposition 1.4. *Let I be a linear order and $p(x)$ be a partial 1-type where $p(I)$ is a convex subset of I (so $I = L + A + R$ where $A = p(I)$). If $B \equiv A$, then $L + A + R \equiv L + B + R$ and B is the set of realizations of $p(x)$ in $L + B + R$.*

Proof. Let $P(x)$ be a new unary relation and interpret P on $L + A + R$ as the set of realizations of $p(x)$ (that is, A) and interpret P on $L + B + R$ as B . We will first show that these structures are elementarily equivalent. So let $n \in \omega$ be arbitrary and fix a strategy for player 2 in the Ehrenfeucht-Fraïssé game between A and B , which exists since $A \equiv B$. In the game of the same length for $L + A + R$ and $L + B + R$, if player 1 picks an element of L or R , player 2 picks the same element from the other structure. Otherwise, player 2 can simply follow the strategy between A and B . This strategy clearly preserves order and P , so the enhanced structures are elementarily equivalent.

This shows B is the set of realizations of p in $L + B + R$, as follows. First, in the expanded structure of $L + A + R$, if $\phi(x) \in p(x)$, then $L + A + R \models \forall x (P(x) \rightarrow \phi(x))$, so this holds in $L + B + R$ as well. Consequently, each element of b realizes $p(x)$. On the other hand, for any $c \in L \cup R$, there is a $\phi(x) \in p(x)$ where $L + A + R \models \neg\phi(a)$. The strategy from the preceding paragraph shows that $(L + A + R, a) \equiv (L + B + R, a)$, so $L + B + R \models \neg\phi(a)$ as well, so a does not realize $p(x)$ in $L + B + R$, completing the proof. \square

Corollary 1.5. *Suppose I is a linear order, $p(x)$ is a partial 1-type, and $p(I)$ is a convex subset of I (so $I = L + A + R$ where $A = p(I)$). If $\text{Th}(A)$ is Borel complete, so is $\text{Th}(I)$.*

Proof. The map $A' \mapsto L + A' + R$ is a Borel map from $\text{Mod}(\text{Th}(A))$ to $\text{Mod}(\text{Th}(I))$ by Proposition 1.4. It trivially preserves isomorphism, and since isomorphisms preserve types, also preserves non-isomorphism. So this is a Borel reduction, and since $\text{Th}(A)$ is Borel complete, so is $\text{Th}(I)$. \square

A special case of the general sum is the *shuffle* operation. For a moment, fix some $n \geq 1$, and let L' refer to the language $\{<, D_0, \dots, D_n\}$. The axioms stating $<$ is a dense linear order without endpoints, and that the D_i are disjoint, mutually exhaustive, and dense subsets of the ordering describe a consistent and \aleph_0 -categorical theory for any particular n . We call the countable model of such a theory \mathcal{D}_n .

Definition 1.6. Given an $n \in \omega$ and some countable linear orders $\{A_0, \dots, A_n\}$, define the *shuffle* of the A_i – denoted by $\sigma(A_0, \dots, A_n)$ – as $\sum\{D_q : q \in \mathcal{D}_n\}$, where $D_q \cong A_k$ if and only if $q \in P_k$.

The following properties of the shuffle can be verified directly from the definition and Proposition 1.3:

- If we replace \mathcal{D}_n , A_0 , \dots , and A_n by isomorphic copies (in their respective languages), the resulting shuffle will be isomorphic to the original shuffle.
- If we permute the indexing of the A_i , the resulting shuffle will be isomorphic to the original shuffle.
- $\sigma(A, A, A_1, \dots, A_n) \cong \sigma(A, A_1, \dots, A_n)$. That is, if we add or remove redundant copies of the orders, the isomorphism type of the shuffle does not change.
- $\sigma(A_0, \dots, A_n) \cong \sigma(\sigma(A_0, \dots, A_n))$. That is, if we shuffle the shuffle, nothing happens.
- Suppose $A_i \equiv B_i$ for $i = 1, \dots, n$. Then $\sigma(A_1, \dots, A_n) \equiv \sigma(B_1, \dots, B_n)$.

Since we can remove redundant elements from the shuffle without altering the isomorphism type, if we only care about the theory, we may assume the elements of the shuffle are all pairwise elementarily inequivalent.

2 \aleph_0 -Categorical Theories

In [6], the class of \aleph_0 -categorical linear orders was completely characterized. In [4], the same characterization was shown to apply to linear orders with finitely many colors². With this in mind, fix a typical language L for the remainder of the section, and let $\{P_n : n \leq \kappa\}$ be the unary predicates of L .

We define an ascending family \mathcal{M}_n (for $n \in \omega$) of sets of L -structures as follows. Each \mathcal{M}_n is the smallest family such that each of the following is satisfied:

1. \mathcal{M}_0 contains a one-point order with the P_i interpreted however is desired.
2. If $A, B \in \mathcal{M}_n$, then $A + B \in \mathcal{M}_{n+1}$.
3. If $A_1, \dots, A_k \in \mathcal{M}_n$, then $\sigma(A_1, \dots, A_k) \in \mathcal{M}_{n+1}$.

Let $\mathcal{M} = \bigcup_n \mathcal{M}_n$.

Theorem 2.1 (Mwesigye, Truss). *Let A be a countable L -structure. Then $\text{Th}(A)$ is \aleph_0 -categorical if and only if $A \cong B$ for some $B \in \mathcal{M}$.*

We will need more specific information about \aleph_0 -categorical theories, though. In particular, to take advantage of finiteness of the language, we need the following theorem from [7]:

Theorem 2.2 (Rubin). *Let \mathcal{M}^2 be the closure of \mathcal{M} under the map $M \mapsto \mathbb{Z} \times M$. Then an L -structure M (with $T = \text{Th}(M)$) has $|S_1(T)| < \aleph_0$ if and only if $M \equiv N$ for some $N \in \mathcal{M}$. Further, if L is fixed and finite, then every such theory is finitely axiomatizable.*

Definition 2.3. Let A be a countable, \aleph_0 -categorical L -structure. Let $r(A)$ be the least n such that $A \in \mathcal{M}_n$.

Proposition 2.4. *Suppose L is finite. Then, for each n , \mathcal{M}_n is finite.*

Proof. By induction on n . First: \mathcal{M}_0 contains exactly $2^{|L|-1}$ elements, depending on which colors are “true” on the single point. Then \mathcal{M}_{n+1} is the union of three sets: \mathcal{M}_n , the set of sums of pairs from \mathcal{M}_n , and the set of shuffles of subsets of \mathcal{M}_n . Let $k = |\mathcal{M}_n|$. Then there are at most k^2 (isomorphism types of) sums of pairs from \mathcal{M}_n . And finally, since repeats in a shuffle do not affect the isomorphism type, there are at most $2^k - 1$ (the number of nonempty subsets of \mathcal{M}_n) possible shuffles from \mathcal{M}_n , so each element of the union is finite, so \mathcal{M}_{n+1} is finite. \square

More interesting is the following:

Proposition 2.5. *Let L be finite or infinite. Let A be an \aleph_0 -categorical structure, and say $r(A) = n$. If $B \subset A$ is nonempty and convex, then $r(B) \leq 2n + 1$. In particular, $\text{Th}(B)$ is \aleph_0 -categorical.*

Proof. The base case $r(A) = 0$ is trivial; if A is a single point and $B \subset A$ is convex and nonempty, then B is a single point and $r(B) = 0 \leq 2 \cdot 0 + 1$.

Now for the step; suppose $r(A) = n + 1$; we need to show $r(B) \leq 2n + 3$. A is either a sum or shuffle. If $A = A_1 + A_2$ for some $A_1, A_2 \in \mathcal{M}_n$ and $B \subset A$ is convex, then $B \subset A_1$, $B \subset A_2$, or $B = (B \cap A_1) + (B \cap A_2)$ where both parts are nonempty. In the first two cases $r(B) \leq 2r(A_i) + 1 \leq 2n + 1 < 2n + 3$; in the latter, $r(B \cap A_1) \leq 2n + 1$ and $r(B \cap A_2) \leq 2n + 1$, so $r(B) \leq (2n + 1) + 1 \leq 2(n + 1) + 1$. Thus the proposition holds for sums.

Finally, suppose $A = \sigma(A_1, \dots, A_k)$ for some $A_1, \dots, A_k \in \mathcal{M}_n$. If B is a convex subset of A , then B contains three parts: a (right unbounded) convex subset of A_i , followed by $\sigma(A_1, \dots, A_k)$, followed by a (left unbounded) convex subset of A_j . Any of these may be empty, but the rank is highest when they are not. The left part has rank at most $2n + 1$, the middle has rank $n + 1$, and the right has rank at most $2n + 1$. Thus the left plus the middle has rank at most $2n + 2$, and that plus the right has rank at most $2n + 3$, as desired. This completes the proof. \square

²If there are infinitely many inequivalent unary predicates, $S_1(T)$ is infinite, so T is not \aleph_0 -categorical.

For our purposes, the point is a sufficient condition for *not* being \aleph_0 -categorical, which we will use frequently later:

Corollary 2.6. *Let L be finite or infinite. Suppose A is a countable, \aleph_0 -categorical L -structure. Then A has only finitely many pairwise nonisomorphic convex subsets.*

Proof. First we address the case where L is finite. Suppose $r(A) = n$. Then for all i , B_i is a countable \aleph_0 -categorical order and $r(B_i) \leq 2n + 1$. There are only finitely many \aleph_0 -categorical typical L -theories of a certain fixed rank, so the B_i cannot all be pairwise nonisomorphic.

Now we address the case where L is infinite. But again, since A is already known to be \aleph_0 -categorical, there is an $N \in \omega$ and a function $i : \omega \rightarrow N$ where for all $n \in \omega$, $A \models \forall x (P_n(x) \leftrightarrow P_{i(n)}(x))$. Since this is a universal sentence, it also applies to all $B \subset A$. So let $L_0 = \{<\} \cup \{P_i : i \leq N\}$. If $B, C \subset A$ are nonisomorphic as L -structures, they are also nonisomorphic as L_0 -structures; so if the conclusion fails for A , it also fails for the reduct of A to L_0 , contradicting the positive result we already have for finite languages. \square

We will also use the notion of rank, together with Proposition 2.4, to show that certain convex types are not \aleph_0 -categorical.

Lemma 2.7. *Let L be finite or infinite. Suppose A is \aleph_0 -categorical. Then there is a pair $a \leq b \in A$ where $[a, b]$ contains a convex subset B which is isomorphic to A .*

Proof. If $r(A) = 0$ then A consists of exactly one point a ; the closed interval $[a, a]$ has the desired property.

Next suppose $A = \sigma(D_1, \dots, D_t)$. Let a be some element of some representative of D_1 , and $b > a$ be an element of some representative of some later iteration of D_1 . Then $[a, b]$ contains an isomorphic copy of A , as desired.

Finally, suppose $A = B + C$. If B can be represented as a shuffle, do so; otherwise either B is a single point or can be broken $B = B_1 + B_2$ where both B_1 and B_2 have rank strictly below $r(B)$. This process can be repeated until $A = B_1 + \dots + B_k$ where $k \geq 2$ and each B_i is either a single point or a shuffle. If B_1 is a single point, let a be that point; otherwise let $a \in B_1$ be arbitrary. Similarly, if B_k is a single point, let b be that point; otherwise let $b \in B_k$ be arbitrary. Clearly $B_2 + \dots + B_{k-1} \subset [a, b]$. If B_1 is a single point then $B_1 \subset [a, b]$; otherwise B_1 is a shuffle, and $(a, \infty) \cap B_1$ contains an isomorphic copy of B_1 . The same holds for B_k , completing the proof. \square

Proposition 2.8. *Let L be finite or infinite. Suppose A is \aleph_0 -categorical and $r(A) \geq 2n + 1$. Then there is a closed interval $[a, b] \subset A$ where $r([a, b]) \geq n$.*

Proof. Let $[a, b]$ be as in Lemma 2.7. Then $[a, b]$ contains an isomorphic copy of A as a convex subset – call it B . Then $r(B) \leq 2r([a, b]) + 1$, but $B \cong A$, so $r(B) = r(A) \geq 2n + 1$, so $2n + 1 \leq 2r([a, b]) + 1$, so $r([a, b]) \geq n$, as desired. \square

Proposition 2.9. *Let A be an \aleph_0 -categorical L -structure. If $r(A) \leq n$, then A admits at most 2^n pairwise disjoint nonempty convex formulas.*

Proof. If $r(A) = 0$, then A is a single point, so has exactly $1 = 2^0$ convex formula, $x = x$, as desired..

If $r(A) = n + 1$, then either A is either a shuffle or a sum. If $A = \sigma(B_1, \dots, B_k)$, then suppose $\phi(x)$ is a proper consistent initial formula. Then there is a B_1 component contained in $\phi(A)$ and a B_1 component contained in $\neg\phi(A)$. But there is an isomorphism of A taking the former to the latter, because of the homogeneity of k -colored dense linear orders. This isomorphism must preserve ϕ , but does not; therefore, no such ϕ exists, and A has no proper initial formulas. Evidently this means A has no proper convex formulas either, so the proposition holds here.

Finally, suppose $A = B + C$ where $r(B), r(C) \leq n$. Expand A to include a new unary predicate P which defines B . Any convex formula definable in A is also definable in this expansion, and any convex formula in this expansion divides into a B part and a C part, so any collection of pairwise disjoint convex formulas induces one which is at least as large and whose elements are completely contained in either B or C . There

are at most 2^n disjoint convex formulas in B and likewise in C , so this expansion of A admits at most 2^{n+1} disjoint convex formulas, meaning A does as well. \square

3 Relativizing Formulas

The following are theorems of Rubin which we use to talk about definability inside convex subsets of the structure. They are Corollaries 2.3 and 2.4, respectively, in [7]:

Theorem 3.1. *Let A be a typical L -structure, B a convex subset of A , and $a_1, \dots, a_m \in A \setminus B$. Then for each formula $\phi(\bar{v}; w_1, \dots, w_m)$, there is a formula $\phi^*(\bar{v})$ such that for all \bar{b} from B ,*

$$A \models \phi(\bar{b}; a_1, \dots, a_m) \text{ if and only if } B \models \phi^*(\bar{b})$$

Corollary 3.2. *Let A be a typical L -structure, B a convex subset of A that is definable over c_1, \dots, c_t , and let $a_1, \dots, a_m \in A \setminus B$. Then for every $\phi(\bar{v}, \bar{w})$, there is a formula $\phi^\#(\bar{v}, \bar{x})$ such that for every $\bar{b} \in A$,*

$$A \models \phi^\#(\bar{b}, \bar{c}) \text{ if and only if both } \bar{b} \in B \text{ and } A \models \phi(\bar{b}, \bar{c})$$

Proof. By Theorem 3.1, there is a formula $\phi^*(\bar{v})$ where for all \bar{b} in B , $B \models \phi^*(\bar{b})$ if and only if $A \models \phi(\bar{b}, \bar{a})$. Since B is \bar{c} -definable, the formula

$$\bar{b} \in B \wedge B \models \phi^*(\bar{b})$$

is a formula over \bar{c} , completing the proof. \square

We will always use ϕ^* for the output of Theorem 3.1 and $\phi^\#$ for the output of Corollary 3.2. The following elementary embedding lemmas are due to Rubin which we use occasionally:

Lemma 3.3. *Let A and B be typical L -structures. Suppose $A + B \prec C$ and $A \prec C$ and $B \prec C$. Then $A \prec A + B$ and $B \prec A + B$.*

Proof. Define $C_1 = \{c \in C : (\exists a \in A)(c \leq a)\}$, $C_3 = \{c \in C : (\exists b \in B)(c \geq b)\}$, and $C_2 = \{c \in C : (\forall a \in A \forall b \in B)(a < c < b)\}$. Then $C = C_1 + C_2 + C_3$. We want to show $A \prec A + B$ and $B \prec A + B$; we first establish a sufficient condition through the use of relativized formulas:

Claim: If $A \prec A + C_2 + B$ and $B \prec A + C_2 + B$, then $A + B \prec A + C_2 + B$

Proof: Suppose $d_1 < \dots < d_k \in A \cup B$ and $c \in C_2$, such that $A + C_2 + B \models \phi(\bar{d}, c)$; we need to find a $c' \in A + B$ where $A + C_2 + B \models \phi(\bar{d}, c')$. Let a be the largest of the d_i in A and b the smallest of the d_i which is in B ; if there is no such a , use “ $-\infty$ ” in the obvious ways where necessary, and likewise with b . By Corollary 3.2, there is a selecting formula $\phi^\#$ for ϕ in $[a, b]$. That is, a formula $\phi^\#(x_1, x_2, v)$ such that for all d , $A + C_2 + B \models \phi^\#(a, b, d)$ if and only if $a \leq d \leq b$ and $A + C_2 + B \models \phi(\bar{d}, d)$. It therefore suffices to find a $c' \in A$ where $A + C_2 + B \models \phi^\#(a, b, c')$. The role of this formula is to simply our reliance on the parameters \bar{d} (and the resulting notation), and we will henceforth assume that ϕ is $\phi^\#$.

Since A is a convex subset of $A + C_2 + B$ and $b \notin A$, we can use Theorem 3.1 to get a testing formula $\phi^*(x_1, x_2)$ for $\phi(x_1, b, x_2)$. That is, for all $a', c' \in A$, $A \models \phi^*(a', c')$ if and only if $A + C_2 + B \models \phi(a', b, c')$. So we only need to find a $c' \in A$ where $A \models \phi^*(a, c')$. So, for sake of contradiction, **suppose there is no such c'** . That is, $A \models \neg(\exists z) \phi^*(a, z)$. Since $A \prec A + C_2 + B$,

$$A + C_2 + B \models \phi(a, b, c) \wedge \neg(\exists z) \phi^*(a, z)$$

Therefore,

$$A + C_2 + B \models (\exists x_1, x_2) [x_1 < x_2 \leq c \wedge \phi(x_1, b, x_2) \wedge \neg(\exists z) \phi^*(x_1, z)]$$

where in this case x_1 and x_2 are witnessed by a and c , respectively. Call this sentence $\psi(b, c)$. Observe that c is only acting as an upper bound on x_2 , so for every element $d \geq c$, $A + C_2 + B \models \psi(b, d)$. In particular, for all $b' \in B$, $A + C_2 + B \models \psi(b, b')$. Since $B \prec A + C_2 + B$ and $\psi(b, b')$ is a sentence in B , $B \models \psi(b, b')$ as well. Therefore, $B \models \forall z \psi(b, z)$, and since $B \prec A + C_2 + B$, $A + C_2 + B \models \forall z \psi(b, z)$.

This is to say:

$$A + C_2 + B \models (\forall v \exists x_1, x_2) [x_1 < x_2 \leq v \wedge \phi(x_1, b, x_2) \wedge \neg (\exists z) \phi^*(x_1, z)]$$

Now, pick any $e \in A$. There is an $a_1 < a_2 \leq e \in A + C_2 + B$ (and therefore $a_1, a_2 \in A$) such that

$$A + C_2 + B \models \phi(a_1, b, a_2) \wedge \neg (\exists z) \phi^*(a_1, z)$$

Since $A + C_2 + B \models \neg (\exists z) \phi^*(a_1, z)$, then because $A \prec A + C_2 + B$, $A \models \neg (\exists z) \phi^*(a_1, z)$. But also, by construction of ϕ^* , since $A + C_2 + B \models \phi(a_1, b, a_2)$, then $A \models \phi^*(a_1, a_2)$. These two facts about A are inconsistent, giving the desired contradiction. \square

By this claim, it is enough to show $A \prec A + C_2 + B$ and $B \prec A + C_2 + B$. We will do this by showing the following claim:

Claim: $C_1 \prec C_1 + C_2 + C_3 = C$ and $C_3 \prec C_1 + C_2 + C_3$.

Proof: We show $C_1 \prec C$; the case for C_3 is similar. So suppose \bar{a} is from C_1 , $c \in C_2 + C_3$, and $C \models \phi(\bar{a}, c)$. Let $a \in A$ be strictly greater than all the \bar{a} (possible since A bounds C_1 and has no greatest element). By Corollary 3.2 there is a selection formula $\phi^\#$ for ϕ such that for all $d \in C$, $C \models \phi^\#(a, d)$ if and only if $d \geq a$ and $C \models \phi(\bar{a}, d)$. Then $C \models \phi^\#(a, c)$, so $C \models \exists v \phi^\#(a, v)$. Since $A \prec C$, $A \models \exists v \phi^\#(a, v)$, so for some $a' \in A$, $A \models \phi^\#(a, a')$. Again by elementarity, $C \models \phi^\#(a, a')$, so $C \models \phi(\bar{a}, a')$. Since $a' \in A \subset C_1$, this shows $C_1 \prec C$, as desired. \square

By this last claim, since $A \subset C_1 \subset C$, $A \prec C$, and $C_1 \prec C$, we have $A \prec C_1$. Similarly, $B \prec C_3$. By Proposition 1.3, this means $A + C_2 + B \prec C_1 + C_2 + C_3$. By the first claim, $A \prec A + B$ and $B \prec A + B$, as desired. \square

We will use the following later as well:

Lemma 3.4. *Suppose $A \prec B$. Let $C \subset A$ be convex, and let $D \subset B$ be the “convex hull” of C in B , defined as the set of all $b \in B$ satisfying $\exists c_1, c_2 \in C (c_1 \leq b \leq c_2)$. Then $C \prec D$.*

Proof. Let $\phi(\bar{c}, y)$ be a C -formula and let $d \in D$ be such that $D \models \phi(\bar{c}, d)$. We need a $c' \in C$ where $D \models \phi(\bar{c}, c')$. If $d \in C$ we're done; otherwise, we may pick $c_1 < c_2$ from C where $c_1 < d < c_2$ and where every element of \bar{c} is outside (c_1, c_2) . Let $\phi^*(y)$ be a relativization of $\phi(\bar{c}, y)$ to (c_1, c_2) ³ using Theorem 3.1, so that for any $c' \in (c_1, c_2)$, $D \models \phi(\bar{c}, c')$ if and only if $(c_1, c_2) \models \phi^*(y)$.

Of course $(c_1, c_2) \models \exists y \phi^*(y)$. Let $\psi(u, v, y)$ state “ $u < y < v$ and $(u, v) \models \phi^*(y)$.” Then $B \models \exists y \psi(c_1, c_2, y)$ with d a witness for y . Since $c_1, c_2 \in A$ and $A \prec B$, $A \models \exists y \psi(c_1, c_2, y)$, so there is a $c' \in A$ where $c_1 < c' < c_2$ and $A \models \psi(c_1, c_2, c')$. Then $B \models \psi(c_1, c_2, c')$ as well, so $(c_1, c_2) \models \phi^*(c')$, so $D \models \phi(\bar{c}, c')$, as desired. \square

4 Self-Additive Linear Orders

The most essential concept to this proof is the concept of a self-additive structure:

Definition 4.1. A typical L -structure A is called *self-additive* if both canonical embeddings $A \rightarrow A + A$ are elementary.

Theorem 4.2. *Let I be a typical L -structure with more than one point and let $T = Th(I)$. Then the following are equivalent:*

1. I is self-additive.
2. Every model of T is self-additive.
3. If $A \models T$ and $B \models T$, then the embeddings $A \rightarrow A + B$ and $B \rightarrow A + B$ are elementary.

³To avoid confusion, for this proof, the interval notation will only refer to convex subsets of B .

4. I has no definable convex sets where both the set and its complement are nonempty.

Note that if $|I| = 1$, then (1), (2), and (3) fail, but (4) succeeds for trivial reasons.

Proof. Trivially, (3) implies (2) and (2) implies (1). Also, (1) implies (4), since the existence of a proper convex set (defined by ϕ) implies the existence of a proper terminal set ψ (if ϕ itself is not terminal, then the set of elements above every element of ϕ comprise a proper terminal set). However, the canonical injection $I \rightarrow I + I$ onto the first element means ψ holds on some element of the first component and is terminal, so holds for all elements of the second component. However, some element of the second component makes ψ false, since the second component is isomorphic to the first. This contradiction shows why ϕ cannot exist (in fact, no proper *bounded* set can be definable, by the same argument).

The meat of the proof is to show (4) implies (3). Observe that since T is complete, (4) applies to all $A \models T$. Say $A, B \models T$; we show the canonical embeddings $A \rightarrow A + B$ and $B \rightarrow A + B$ are elementary. To do this, first consider the set Σ consisting of $\Sigma(A)$, $\Sigma(B)$ (the elementary diagrams of A and B , respectively), and the set $\{a < b : a \in A, b \in B\}$.

Claim: The set Σ is consistent.

Proof: Suppose not; then there is a tuple $b_1 < \dots < b_n$ and a formula $\phi(\bar{x})$ such that $\phi(\bar{b}) \in \Sigma(B)$, along with an element $a \in A$ such that $\Sigma(A) \vdash a < b_1 \rightarrow \neg\phi(\bar{b})$. Since the \bar{b} do not appear in $\Sigma(A)$, this means $\Sigma(A) \vdash \forall \bar{x} (a < x_1 < \dots < x_n \rightarrow \neg\phi(\bar{x}))$. Trivially, we can replace a by any $a' > a$ in A and the sentence will still be true. Therefore formula

$$\psi(y) := \forall \bar{x} (y < x_1 < \dots < x_n \rightarrow \neg\phi(\bar{x}))$$

defines a terminal subset of A which is nonempty (since $A \models \psi(a)$). By (4), this means $A \models \forall y \psi(y)$. Since A has no first element (as that would contradict (4)), $A \models \forall \bar{x} (x_1 < \dots < x_n \rightarrow \neg\phi(\bar{x}))$. But of course $B \models \exists \bar{x} (x_1 < \dots < x_n \wedge \phi(\bar{x}))$, since $B \models \phi(\bar{b})$. Therefore $A \not\models B$, contradicting hypothesis. This proves the claim. \square

So now, let $C \models \Sigma$. Observe that $A \prec C$, $B \prec C$, and $A + B \subset C$. By Lemma 3.3, this shows $A \prec A + B$ and $B \prec A + B$, completing the proof. \square

So self-additivity is actually a property of (complete) L -theories. The additivity also applies to arbitrary sums:

Theorem 4.3. *Suppose I is self-additive. Let $Y \subset X$ be nonempty linear orders, and for each $x \in X$, $I_x \equiv I$. Let $I_Y = \sum\{I_y : y \in Y\}$ and let $I_X = \sum\{I_x : x \in X\}$. Then the canonical embedding $I_Y \rightarrow I_X$ is elementary and both structures model $\text{Th}(I)$.*

Proof. Note: the conclusion “both structures model $\text{Th}(I)$ ” follows from a special case of the rest, where Y is a singleton. Then $I \cong I_Y \prec I_X$, so $I \equiv I_X$. Therefore it is enough to show that $I_Y \prec I_X$ always.

First, **assume X is finite**. We prove this by induction on $|X| = k$. The base case $k = 0$ and $k = 1$ are trivial (k can never be zero, and if $k = 1$, then $Y = X$). If $k = 2$, this is exactly self-additivity. So we may assume $k \geq 3$. We assume $X = \{n : n \leq k\}$ (so $|X| = k + 1$) and that $A = \sum\{I_x : x \in X\}$ is $I_0 + \dots + I_k$. Let $B = \sum\{I_x : x \in Y\}$.

Suppose $A \models \phi(b_1, \dots, b_n, a)$ where $b_1, \dots, b_n \in B$ and $a \in A$. In fact we assume $a \notin I_k$ (otherwise we do a symmetric argument, exchanging the roles of I_0 and I_k). If $Y = \{k\}$, then $I_0 + \dots + I_{k-1} \equiv I$ by inductive hypothesis, the embedding $B = I_k \rightarrow A$ is elementary by self-additivity. So we assume $Y \cap \{0, \dots, k-1\}$ is nonempty.

Decompose the b_i into $b_1, \dots, b_t \in I_0 + \dots + I_{k-1}$ and $b_{t+1}, \dots, b_n \in I_k$. Since $I_0 + \dots + I_{k-1}$ is convex, we can apply Theorem 3.1 to get a testing formula $\phi^*(v_1, \dots, v_t)$ for ϕ . That is, for all $c_1, \dots, c_t, c \in I_0 + \dots + I_{k-1}$,

$$I_0 + \dots + I_{k-1} \models \phi^*(c_1, \dots, c_t) \text{ if and only if } A \models \phi(c_1, \dots, c_t, b_{t+1}, \dots, b_n, c)$$

In particular $I_0 + \dots + I_{k-1} \models \phi^*(b_1, \dots, b_t, a)$. By inductive hypothesis, $B \cap I_0 + \dots + I_{k-1} \prec I_0 + \dots + I_{k-1}$, so there is a $c \in B \cap I_0 + \dots + I_{k-1}$ where $I_0 + \dots + I_{k-1} \models \phi^*(b_1, \dots, b_t, c)$. Then $A \models \phi(b_1, \dots, b_t, b_{t+1}, \dots, b_n, c)$, and $c \in B$, as desired. This completes the argument.

Now, **assume X is infinite**. We again proceed by induction on $\kappa = |X|$. Enumerate X as $\{x_\alpha : \alpha < \kappa\}$. For each α , let $A_\alpha = \sum\{I_{x_i} : i < \alpha\}$ and $B_\alpha = \sum\{I_{x_i} : x_i \in Y \text{ and } i < \alpha\}$. By induction hypothesis, for all $\alpha < \beta < \kappa$, $B_\alpha \prec B_\beta$, $B_\alpha \prec A_\alpha$, and $A_\alpha \prec A_\beta$.

Let $A = \bigcup_\alpha A_\alpha$. As the union of an elementary chain, $A_\alpha \prec A$ for all α . Similarly, let $B = \bigcup_\alpha B_\alpha$, so that $B_\alpha \prec B$ for all α . We want to show $B \prec A$. So, let $\bar{b} \in B$, $a \in A$, and suppose $A \models \phi(\bar{b}, a)$. Since κ is a limit ordinal, there is an $\alpha < \kappa$ such that $\bar{b} \in B_\alpha$ and $a \in A_\alpha$. Since $A_\alpha \prec A$, $A_\alpha \models \phi(\bar{b}, a)$. Since $B_\alpha \prec A_\alpha$, there is a $b' \in B_\alpha \subset B$ where $A_\alpha \models \phi(\bar{b}, b')$. Since $A_\alpha \prec A$, $A \models \phi(\bar{b}, b')$, so $B \prec A$, as desired. \square

Finally, for the sake of understanding self-additive orders, we need to define a certain condensation:

Definition 4.4. Let I be an L -structure and $a \in I$. Let $\mathbf{c}_D(a)$ be the union of all convex bounded sets which contain a and are definable over a in I .

While this is not quite a definable set in general, it is the disjunction of definable sets, so is preserved under elementary maps when phrased appropriately:

Remark 4.5. Suppose $f : A \rightarrow B$ is elementary and $a, b \in A$. Then $a \in \mathbf{c}_D(b)$ if and only if $f(a) \in \mathbf{c}_D(f(b))$. In particular, if $A \prec B$ and f is the identity function, then $a \in \mathbf{c}_D(b)$ (interpreted in A) if and only if $a \in \mathbf{c}_D(b)$ (interpreted in B).

Consequently, if $f : A \rightarrow B$ is an isomorphism, then for every $a \in A$, $f(\mathbf{c}_D(a)) = \mathbf{c}_D(f(a))$.

It is especially well-behaved when we consider convex elementary substructures (e.g. when working with a sum of models of a self-additive theory).

Proposition 4.6. Say $A \prec B$ is convex. If $a \in A$ and $b \in B$ and $b \in \mathbf{c}_D(a)$, then $b \in A$.

Consequently, $\mathbf{c}_D(a)$ (calculated in A) is exactly equal to $\mathbf{c}_D(a)$ (calculated in B), and the latter is completely contained in A .

Proof. The “consequently” follows immediately. So suppose $b \in B \setminus A$ and $b \in \mathbf{c}_D(a)$. We may assume $b > a$. Let $\phi(x, y)$ be such that $\phi(a, B)$ is convex, bounded, contains a , and contains b . Then for all $a' > a$ in A , $B \models \phi(a, a')$ by convexity. Since $A \prec B$, $A \models \phi(a, a')$ for all such a' . But then $A \models \forall y > a \phi(a, y)$, so $B \models \forall y > a \phi(a, y)$, contradicting assumption that $\phi(a, B)$ is bounded. This contradiction proves the result. \square

For any L -structure A and $a \in A$, $\mathbf{c}_D(a)$ is convex and contains a , but in general, \mathbf{c}_D is not a condensation. For example in the order $\mathbb{N} + \mathbb{Z}$, if a is from the \mathbb{N} part, then $\mathbf{c}_D(a) = \mathbb{N}$, but if b is from the \mathbb{Z} part, then $\mathbf{c}_D(b) = \mathbb{N} + \mathbb{Z}$. So we have a and b where $a \in \mathbf{c}_D(b)$ but $\mathbf{c}_D(a) \neq \mathbf{c}_D(b)$. If the structure is self-additive, this doesn't happen:

Theorem 4.7. If I is self-additive, \mathbf{c}_D is a condensation.

Proof. We want to show that if $b \in \mathbf{c}_D(a)$, then $\mathbf{c}_D(a) = \mathbf{c}_D(b)$. We prove several partial results toward what we're looking for, then assemble them into the complete result.

Claim: If $a < b < c$ and $c \in \mathbf{c}_D(a)$, then $c \in \mathbf{c}_D(b)$ as well.

If $c < b < a$ and $c \in \mathbf{c}_D(a)$, then $c \in \mathbf{c}_D(b)$ as well.

Proof: We prove the first statement; the second is exactly symmetric.

Let $\phi(x, y)$ be such that $\phi(a, I)$ is a bounded convex set with minimum a and which includes c . Use Corollary 3.2 to get a formula $\phi^\#(b, y)$ such that $I \models \phi^\#(b, d)$ if and only if $b \leq d$ and $I \models \phi(a, d)$. Of course this means $\phi^\#(b, I)$ is a bounded convex set with minimum b and which includes c , since it is precisely $\phi(a, I) \cap [b, \infty)$. Therefore $c \in \mathbf{c}_D(b)$. \square

Claim: If $a < b < c$ and $c \in \mathbf{c}_D(b)$, then $c \in \mathbf{c}_D(a)$ as well.

If $c < b < a$ and $c \in \mathbf{c}_D(b)$, then $c \in \mathbf{c}_D(a)$ as well.

Proof: We prove the first statement; the second is exactly symmetric.

Let $\psi(x, y)$ be such that $\psi(a, I)$ is a bounded convex set including a and b and has minimum a . Let $\phi(x, y)$ be such that $\phi(b, I)$ is a bounded convex set with minimum b and which includes c . By preparing

ϕ , we may assume that for all c , $\phi(c, I)$ is a bounded convex set with minimum c , admitting the possibility that $\phi(c, I) = \{c\}$.

Define the formula

$$\theta(x, y) := \psi(x, y) \vee \exists z (\psi(x, z) \wedge \phi(z, y))$$

Certainly $\theta(a, I)$ includes all of $\psi(a, I)$, so in particular includes all of $[a, b]$. It also includes all of $\phi(b, I)$, so in particular includes all of $[b, c]$. Moreover, $\theta(a, I)$ has minimum a and is convex – it is the union of the convex set $\psi(a, I)$ together with convex sets $\phi(d, I)$, all of whom touch $\psi(a, I)$. If $\theta(a, I)$ is bounded above, $c \in \mathbf{c}_D(a)$ and the claim is proved. So suppose not, so that $I \models \forall y (y \geq a \rightarrow \theta(a, y))$.

So let $\Sigma(v)$ be the type $\Sigma(I) \cup \{v > a : a \in I\} \cup \{\neg\phi(c, v) : a \in I\}$. For each $c \in I$, $\phi(c, I)$ is bounded, so any finite subset of $\Sigma(v)$ is satisfiable in I (which has no last element by self-additivity). So there is a model $C \succ I$ which realizes Σ . Decompose Σ as $C_1 + C_2$, where $C_1 = \{c \in C : \exists a \in I (c \leq a)\}$. As in the final claim of the proof of Theorem 4.2, $C_1 \prec C$, so we may conclude $I \prec C_1$, and therefore $I + C_2 \prec C_1 + C_2$ by Proposition 1.3. Notably, since $\psi(a, I)$ is bounded by some element of I , it has no new realizations in $I + C_2$.

Thus, there is an element $d \in I + C_2$ such that $I + C_2 \models \neg\phi(c, d)$ for all $c \in I$ with $I \models \psi(a, c)$, and therefore, $I + C_2 \models \forall y (\psi(a, y) \rightarrow \neg\phi(y, d))$, so $I + C_2 \models \neg\theta(a, d)$. However, I (and thus $I + C_2$) says $\theta(a, y)$ is a terminal formula, so $I + C_2 \models \theta(a, d)$, a contradiction. \square

Claim: If $b \in \mathbf{c}_D(a)$, then $a \in \mathbf{c}_D(b)$.

Proof: As usual we will assume $a < b$; the $b < a$ case is symmetric. Assume $\mathbf{c}_D(a)$ is bounded above; if not, pass to $I + I'$. This is an elementary extension by self-additivity, so does not alter the truth of the relation $x \in \mathbf{c}_D(y)$. Further, since $\mathbf{c}_D(a)$ is a union of intervals which are bounded in I , no new elements are in $\mathbf{c}_D(a)$ in the new structure, so every element of I' is an upper bound on $\mathbf{c}_D(a)$.

We may also doctor $\psi(x, y)$ so that for every $a' \in I$, $\psi(a', I)$ is a bounded convex set with minimum a' . Let $\tau(x)$ be the formula $(\forall z \exists y) (z < y \wedge \psi(z, x))$. **Suppose** $I \models \tau(b)$. Since I is self-additive and the set of strict upper bounds is a terminal formula not including b , it must be empty. That is, the set $\tau(I)$ is unbounded above. So let $b' > \mathbf{c}_D(a)$ satisfy τ . Then for some $a' < a$, $I \models \psi(a', b')$. Since $\psi(a', I)$ is a bounded convex set including a' , $b' \in \mathbf{c}_D(a')$. Since $a' < a < b'$, by the first claim, $b' \in \mathbf{c}_D(a)$ as well, a contradiction of construction of b' .

Thus our assumption is wrong; $I \models \neg\tau(b)$, so $I \models (\exists z \forall y) (z < y \rightarrow \neg\psi(z, b))$. Choose some $e \in I$ where for all $c < e$, $I \models \neg\psi(c, e)$. Define

$$\phi(b, v) := (b \geq v \wedge \exists y (y \leq v \wedge \psi(y, b)))$$

Then $\phi(b, I)$ has maximum b , is bounded below by e , contains a , and is convex (since if you have a solution for y , any v with $y \leq v \leq b$ is in the set). Thus $a \in \mathbf{c}_D(b)$, as desired. \square

We now assemble this into the desired result. As before we assume $a < b$; the other case is symmetric. We need to show that $\mathbf{c}_D(a) = \mathbf{c}_D(b)$. If $c > b$, then $c \in \mathbf{c}_D(a)$ if and only if $c \in \mathbf{c}_D(b)$ by the first and second claims' first forms; if $c < a$, then $c \in \mathbf{c}_D(a)$ if and only if $c \in \mathbf{c}_D(b)$ by the first and second claims' second forms. By the third claim, $a \in \mathbf{c}_D(b)$, and by hypothesis $b \in \mathbf{c}_D(a)$. Since both sets are convex, if $a \leq c \leq b$, then c is in $\mathbf{c}_D(a)$ and $\mathbf{c}_D(b)$. These cases are exhaustive, completing the proof. \square

With this technology assembled, we prove the following “nice model” theorem. Essentially, if $p \in S_1(T)$ is nonisolated, there is a model of T in which there is exactly one \mathbf{c}_D class which contains a realization of p . Consequently, T is Borel complete (as we will see in a moment).

Lemma 4.8. *Suppose I is self-additive, $T = \text{Th}(I)$, and $P \in S_1(T)$ is nonisolated. Then there is a countable $B \equiv I$ with exactly one \mathbf{c}_D class in B that contains a realization of P .*

Proof. Since $S_1(T)$ is infinite, there is a non-principal 1-type $P(x)$. Let $I_1 \models T$ be countable and omit P , and let $I_2 \models T$ be countable and realize P . Let $e \in I_2$ realize P , and let $B = I_1 + \mathbf{c}_D(e) + I_3$, where $\mathbf{c}_D(e)$ is computed in I_2 and $I_3 \cong I_1$. We will show that this B is as desired. Most of the content of the proof is in the following claim:

Claim: $B \prec I_1 + I_2 + I_3$.

Proof: Let $A = I_1 + I_2 + I_3$. Suppose $b_1, \dots, b_k \in B$, $a \in A$, and $A \models \psi(b_1, \dots, b_k, a)$. We must produce a $b \in B$ where $A \models \psi(b_1, \dots, b_k, b)$. We may assume $a \in I_2 \setminus \mathbf{c}_D(e)$ (as otherwise $a \in B$) and that $e < a$ (the case $a < e$ is symmetric). Further, assume $b_1 < \dots < b_k$, and b_P is the first b_i which is from I_3 (if there is no such b_P , use ∞ in the appropriate places). Let $b' \in \mathbf{c}_D(e)$ such that $b' \geq b_{P-1}$. Since \mathbf{c}_D classes are convex, this means $b' < a < b_P$.

Use Corollary 3.2 to produce a $\psi^\#$ for ψ in $[b', b_P]$, so that for all c , $A \models \psi^\#(b', b_P, c)$ if and only if $b' < c < b_P$ and $A \models \psi(b_1, \dots, b_k, c)$. It is therefore sufficient to find a c satisfying $\psi^\#(b', b_P, v)$; as a consequence, we will never again make reference to the original ψ or the b_1, \dots, b_k , so we will (for sake of notation) assume ψ is $\psi^\#$.

By Theorem 3.1 there is a testing formula $\psi^*(x, v)$ for ψ in I_3 . That is, for any $c_P, c \in I_3$, $I_3 \models \psi^*(c_P, c)$ if and only if $A \models \psi(b', c_P, c)$. **Suppose** there is no $c \in I_3$ such that $A \models \psi(b', b_P, c)$ (if this is false, the claim is true). Then there is no $c \in I_3$ such that $I_3 \models \psi^*(b_P, c)$, so $I_3 \models \forall v (\neg \psi^*(b_P, v))$. Since $I_3 \prec A$, we have

$$A \models \psi(b', b_P, a) \wedge \forall v (\neg \psi^*(b_P, v))$$

Consider the above as a formula satisfied by b' . We will use it to construct a formula $\theta(u, w)$ to make it hold on an interval. Define $\theta(u, w)$ by:

$$w \geq u \wedge (\exists x, y) [x \geq w \wedge y \geq w \wedge \psi(u, x, y) \wedge \forall v (\neg \psi^*(x, v))]$$

Then $\theta(b', A)$ has minimum b' , is convex (since the formula is really the set of lower bounds on witnesses x and y), and is satisfied by a (let $x = b_P$ and $y = a$). By self-additivity, $I_2 \prec A$, so for any $d \in I_2$, $I_2 \models \theta(b', d)$ if and only if $A \models \theta(b', d)$. Since $a \notin \mathbf{c}_D(b') = \mathbf{c}_D(e)$, the interval $\theta(b', I_2)$ must not be bounded in I_2 . Thus $I_2 \models \forall w (w \geq b' \rightarrow \theta(b', w))$. Since $I_2 \prec A$, $A \models \forall w (w \geq b' \rightarrow \theta(b', w))$.

This in mind, let $d \in I_3$ be arbitrary. Then $d > b'$, so $A \models \theta(b', d)$. Thus, for some $b'_P \geq d$ and some $a' \geq d$, $A \models \psi(b', b'_P, a')$ and $A \models \forall v (\neg \psi^*(b'_P, v))$. Since $b'_P \in I_3$ and $I_3 \prec A$, this means $I_3 \models \forall v (\neg \psi^*(b'_P, v))$. In particular, $I_3 \models \neg \psi^*(b'_P, a')$. However, $I_3 \models \psi(b', b'_P, a')$, contradicting construction of ψ^* and proving the claim. \square

Since $B \prec A = I_1 + I_2 + I_3$, $B \models T$, as desired. We next show the condensation class works as desired:

Claim: $\mathbf{c}_D(e)$ (interpreted in I_2) is equal to $\mathbf{c}_D(e)$ (interpreted in B)

Proof: Suppose $\phi(e, v)$ defines a bounded convex set (containing e) in B (or I_2); this fact is a statement about e , and since $B \prec A$ (resp. $I_2 \prec A$), also holds in A . And since $I_2 \prec A$ (resp. $B \prec A$), this fact also holds in I_2 .

So if $d \in \mathbf{c}_D(e)$ (in I_2), there is such a $\phi(e, v)$ in I_2 . It still works in B , and since $B \models \phi(e, d)$ if and only if $I_2 \models \phi(e, d)$, we have $d \in \mathbf{c}_D(e)$ (in B). Conversely, if $d \in \mathbf{c}_D(e)$ (in B), there is a formula $\phi(e, v)$ which is bounded, convex, and contains both d and e . If $d \in I_2$, great. But suppose $d \in I_1$ (the case $d \in I_3$ is similar). Then the formula $\phi(e, d)$ also holds in $I_1 + I_2 + I_3$ and is still a convex set containing e , so it contains all of I_2 below e . Thus I_2 says $\phi(e, v)$ is unbounded below, so B does too, contradicting construction of ϕ . This shows there is no such d , so the computations of \mathbf{c}_D agree, as desired. \square

Further, since $I_2 \prec I_1 + I_2 + I_3 = A$ and $e \in I_2$ realizes P in I_2 , e also realizes P in A . Since $B \prec A$ and $e \in B$, e realizes P in B . Furthermore, if $a \in A$ is I_1 , then the type of a in I_1 is the same as in A , so is *not* P (and likewise with $a \in I_3$). Thus, the only \mathbf{c}_D class which contains a realization of P is exactly $\mathbf{c}_D(e)$. \square

This gives us our main theorem for the section:

Theorem 4.9. *Suppose I is self-additive and $T = Th(I)$. If $S_1(T)$ is infinite, then T is Borel complete.*

Proof. Let $p \in S_1(T)$ be nonisolated, and let $B \equiv I$ be as produced by Lemma 4.8. The map $L \mapsto L \times B$ is a Borel map from linear orders to $\text{Mod}(T, \omega)$ by self-additivity of T . Clearly, if $L \cong L'$ then $L \times B \cong L' \times B$. More interestingly, for any L , the set of all \mathbf{c}_D classes of $L \times B$ which contains a realization of p is precisely order isomorphic to L . This is an isomorphism invariant of models of T , so if $L \times B \cong L' \times B$, then $L \cong L'$. Therefore $L \mapsto L \times B$ is a Borel reduction from the (Borel complete) class of linear orders to $\text{Mod}(T)$, so T is Borel complete. \square

5 Structures with Finitely Many 1-Types

For the remainder of this section, we will be concerned with typical theories which have only finitely many 1-types. The theorem at the end of this section will show that such a theory must always be either \aleph_0 -categorical or Borel complete. Along the way, we will show that such theories are built up as sums of self-additive orders, so that we can use the technology of the previous section.

The following lemma helps simplify the discussion about defining \mathbf{c}_D classes:

Lemma 5.1. *Suppose I is self-additive and $a, b \in I$. Suppose that $\text{tp}(a) = \text{tp}(b)$, both are isolated, and $\mathbf{c}_D(a)$ is definable by $\phi(a, y)$. Then $\mathbf{c}_D(b)$ is definable by $\phi(b, y)$.*

Also, if $\mathbf{c}_D(a)$ is the first or last \mathbf{c}_D class, it is the only \mathbf{c}_D class. Therefore, either $\phi(a, y)$ is $y = y$ or is a bounded convex set containing a .

Proof. First; if $c \in \mathbf{c}_D(b)$, then there is a $\psi(b, y)$ whose realizations are a bounded convex set containing b (and c). The same holds for $\psi(a, y)$ and a , so if $I \models \psi(a, c')$, then $c' \in \mathbf{c}_D(a)$, so $I \models \phi(a, c')$. That is, $I \models \forall y (\psi(a, y) \rightarrow \phi(a, y))$. The same holds for b (since a and b realize the same type) so $I \models \forall y (\psi(b, y) \rightarrow \phi(b, y))$, so in particular $I \models \phi(b, c)$. Thus, $\mathbf{c}_D(b)$ is contained in the set of realizations of $\phi(b, y)$.

Now we have a few cases. If $\mathbf{c}_D(a)$ is bounded above and below, then $\phi(a, y)$ is a bounded convex set containing a . This also applies to b , so if $I \models \phi(b, c)$, then $c \in \mathbf{c}_D(b)$, proving the lemma. Alternately, if $\mathbf{c}_D(a)$ is unbounded above and below, then $\mathbf{c}_D(a) = I$ and $\phi(a, y)$ is universally satisfied. Also, $b \in \mathbf{c}_D(a)$, so $\mathbf{c}_D(b) = I$ and $\phi(b, y)$ is universally satisfied, so again the lemma is satisfied.

Finally, suppose $C = \mathbf{c}_D(a)$ is unbounded above (the other case is symmetric); we will argue that $\phi(a, y)$ is definable over \emptyset . Let $\alpha(x)$ isolate $\text{tp}(a)$; observe that the realizations of $\alpha(x)$ are unbounded in I by self-additivity. For any a' in $\mathbf{c}_D(a)$ and realizing α , if $I \models \phi(a, c)$, then $c \in \mathbf{c}_D(a)$, so $c \in \mathbf{c}_D(a')$. By the first paragraph, $I \models \phi(a', c)$. Thus:

$$I \models \forall y (\alpha(y) \wedge \phi(a, y) \rightarrow \forall z (\phi(a, z) \rightarrow \phi(y, z)))$$

But this is a sentence true of a , so is also true of all a' satisfying α . Thus if a' realizes α , then $I \models \forall y (\phi(a, y) \leftrightarrow \phi(a', y))$. So the formula

$$\exists x (\alpha(x) \wedge \phi(x, y))$$

defines $\mathbf{c}_D(a)$ over \emptyset . Since this is a nonempty convex set, it must be all of I , proving the lemma. \square

Lemma 5.2. *Suppose I is self-additive and $S_1(T)$ is finite, where $T = \text{Th}(I)$. Suppose further $\mathbf{c}_D(a)$ is definable over a for every $a \in I$. Then the quotient order $\mathbf{c}_D[I]$ is either a singleton or is dense without endpoints. In the latter case, for every 1-type P , the condensation classes which contain a realization of P are dense in $\mathbf{c}_D[I]$.*

Proof. If there is only one \mathbf{c}_D class the lemma is trivially satisfied, so suppose there are multiple classes. We have two things to check: that there is no first or last \mathbf{c}_D class, and if $\mathbf{c}_D(a) < \mathbf{c}_D(b)$, and if P is a 1-type, there is a c realizing P and such that $\mathbf{c}_D(a) < \mathbf{c}_D(c) < \mathbf{c}_D(b)$. The former condition is immediate from Lemma 5.1, since every \mathbf{c}_D class is definable over any of its elements.

For the second condition: fix an a and b where $\mathbf{c}_D(a) < \mathbf{c}_D(b)$ and fix a 1-type P . Suppose the density conclusion fails, so if $\mathbf{c}_D(a) < \mathbf{c}_D(c) < \mathbf{c}_D(b)$, then no element of $\mathbf{c}_D(c)$ realizes P . Let $\psi(a, y)$ define $\{d \in \mathbf{c}_D(a) : d \geq a\}$ and let $\psi'(b, y)$ define $\{d \in \mathbf{c}_D(b) : d \leq b\}$. Let $\alpha(y)$ generate P , and let $\beta(x)$ generate the 1-type of b . Consider the formula $\phi(v, w)$, defined by:

$$w \geq v \wedge (\exists x) [x \geq w \wedge \beta(x) \wedge (\forall y) (v \leq y \leq x \rightarrow \psi(v, y) \vee \psi'(x, y) \vee \neg \alpha(y))]$$

Observe that $\phi(a, w)$ has minimum point a and is convex, since w ranges among the non-strict lower bounds to the x of interest. Also, $I \models \phi(a, b)$, where x is interpreted as b itself. Since $b \notin \mathbf{c}_D(a)$, $\phi(a, y)$ is unbounded above, so by convexity, holds for all $y \geq a$. Since I is self-additive, the realizations of β are unbounded above. So by examination of ϕ , we see that α fails everywhere above $\phi(a, y)$ (which is a bounded

set). But α is a consistent formula, so is realized somewhere; thus the set of points which are below some element of α form a proper initial subset of I , contradicting self-additivity of I . This contradiction proves the existence of the desired c , completing the proof. \square

This lemma can be made into a very precise and useful construction of models of such a theory:

Lemma 5.3. *Suppose I is self-additive and $S_1(T)$ is finite, where $T = Th(I)$. Suppose $\mathbf{c}_D(a)$ is definable over a for every $a \in I$, and furthermore, that there is more than one \mathbf{c}_D class. Then there is a finite set $\{D_1, \dots, D_t\}$ of linear orders satisfying all the following:*

1. $I \equiv \sigma(D_1, \dots, D_t)$.
2. In $\sigma(D_1, \dots, D_t)$, each D_i piece is a \mathbf{c}_D component.
3. For all $1 \leq i < j \leq t$, $D_i \not\equiv D_j$.
4. For each i , let $T_i = Th(D_i)$. Then $\sum_i |S_1(T_i)| \leq |S_1(T)|$.

Proof. This follows from Lemma 5.2 and an explicit construction. First, by passing to an elementary substructure if necessary, assume I is countable. Pick two elements a and b which realize the same 1-type. For any $p \in S_1(T)$, $\mathbf{c}_D(a)$ contains a realization of p if and only if $\mathbf{c}_D(b)$ does, since p is atomic (by finiteness of $S_1(T)$) and so $\mathbf{c}_D(a)$ containing a realization of p is witnessed by the formula $\exists y (p(y) \wedge \phi(a, y))$. Say $|S_1(T)| = n$, and let $\{p_1, \dots, p_n\}$ be the distinct elements of $S_1(T)$. For each i , let $\phi_i(x, y)$ be such that if a realizes p_i , then $\phi_i(a, y)$ defines $\mathbf{c}_D(a_i)$.

Thus, let D_i be $\mathbf{c}_D(a_i)$ where a_i is some element of I which realizes p_i . This is well-defined up to elementary equivalence, as follows. If a and b realize the same type, if ψ is some sentence, and $\phi(x, y)$ is such that $\phi(a, y)$ defines $\mathbf{c}_D(a)$ (and therefore likewise with b), then the formula “ ψ holds on the set of realizations of $\phi(x, -)$ ” is part of the type of a , and thus of b , so $\mathbf{c}_D(a)$ realizes ψ if and only if $\mathbf{c}_D(b)$ does.

Let $\{b_k\}_k$ be a representative subset of I (so they are \mathbf{c}_D -inequivalent, but every $a \in I$ is \mathbf{c}_D -equivalent to some b_k). For each k , let $i(k)$ be the index of $\text{tp}(b_k)$. Then $\mathbf{c}_D(b_k) \equiv D_{i(k)}$. Moreover, $I = \sum_k \mathbf{c}_D(b_k)$. By Proposition 1.3, $I \equiv \sum_k D_{i(k)}$; since we only care about $Th(I)$, we may assume this is an equality. By characterization of the shuffle, to show $I \cong \sigma(D_1, \dots, D_n)$, it's enough to show that $\mathbf{c}_D[I]$ has dense order type without endpoints (guaranteed by Lemma 5.2) and for any $i \leq n$ and any $b_k < b_{k'}$, there is $b_{k''}$ with $b_k < b_{k''} < b_{k'}$ where $\mathbf{c}_D(b_{k''})$ contains a realization of p_i (also guaranteed by Lemma 5.2).

To establish (3), simply eliminate any redundant D_i from the shuffle. This does not change the theory of the result, which is all we need. We now index the D_i as $\{D_1, \dots, D_t\}$. Finally, to establish (4) – for each $i \leq t$, let $A_i = \{a_i^1, \dots, a_i^{k_i}\}$ be a finite subset of (some particular representative of) D_i where each element has a different type in D_i . For each $i \leq t$ and $j \leq k_i$, let $\psi_{i,j}(x)$ be a formula which is true of a_i^j on D_i but false of $a_i^{j'}$ on D_i for $j' \neq j$. Further, let $\phi_i(x)$ say “the \mathbf{c}_D class of x is equivalent to D_i ” (formula-much information since there are only finitely many different \mathbf{c}_D classes). Then let $\tau_{i,j}(x)$ be $\phi_i(x)$, together with the statement “on the \mathbf{c}_D class of x , $\psi_{i,j}$ holds for x .” Evidently the $\tau_{i,j}$ are all pairwise inconsistent, so there are at most $n = |S_1(T)|$ of them. Thus $\sum_i |S_1(T_i)| \leq |S_1(T)|$, completing the proof. \square

That's all fine, but the definability condition is a bit strong. What's interesting is that the alternative is an analogy of Lemma 4.8:

Lemma 5.4. *Suppose I is self-additive and $S_1(T)$ is finite, where $T = Th(I)$. Then either $\mathbf{c}_D(a) \prec I$ for all $a \in I$, or $\mathbf{c}_D(a)$ is definable over a for every $a \in I$.*

Proof. We assume there is a fixed $a \in I$ where $\mathbf{c}_D(a)$ is *not* definable over a (note this implies $\mathbf{c}_D(a)$ is not all of I). We will show that $\mathbf{c}_D(a) \prec I$. First, we strengthen our assumption:

Claim: For all $a' \in \mathbf{c}_D(a)$, $\mathbf{c}_D(a)$ is not definable over a' .

Proof: Suppose not; let $\phi(x, y)$ be such that $\phi(a', y)$ defines $\mathbf{c}_D(a) = \mathbf{c}_D(a')$. By Lemma 5.1, $\mathbf{c}_D(a)$ is neither the first or last \mathbf{c}_D class. Let $\alpha(x)$ generate $\text{tp}(a')$. Let $\psi(a, y)$ be defined as follows:

$$(\exists x) (\alpha(x) \wedge \phi(x, a) \wedge \phi(x, y))$$

We claim $\psi(a, y)$ defines $\mathbf{c}_D(a)$ (over a). For if $c \in \mathbf{c}_D(a)$, then $I \models \psi(a, c)$ with x as a witness. On the other hand, if $I \models \psi(a, c)$, let b be a witness for x . Then b realizes $\text{tp}(a')$, so $\phi(b, y)$ is a bounded convex set containing b which contains both a and c , so $\mathbf{c}_D(c) = \mathbf{c}_D(b) = \mathbf{c}_D(a)$, as desired. This contradicts hypothesis (that $\mathbf{c}_D(a)$ is not definable over a) so proves the claim. \square

So now let \bar{a} be from $\mathbf{c}_D(a)$, let b be from I , and say $I \models \phi(\bar{a}, b)$. Without loss of generality, $b > \mathbf{c}_D(a)$. We need to find a $c \in \mathbf{c}_D(a)$ where $I \models \phi(\bar{a}, c)$. Assume a' is the largest element of \bar{a} and let $\phi^\#$ be a testing formula for ϕ in $[a', \infty)$ as guaranteed by Corollary 3.2; that is, $I \models \phi^\#(a', c)$ if and only if $c \geq a'$ and $I \models \phi(\bar{a}, c)$. This is done to make the parameters into a singleton and to take advantage of the finiteness of $S_1(T)$.

Let $\alpha(x)$ generate $\text{tp}(a')$. Suppose there is no d such that $I \models \phi^*(a', d)$. Let $\psi_1(x, y)$ be

$$x \leq y \wedge \forall v (x \leq v \leq y \rightarrow \neg \phi^*(x, v))$$

Then $I \models \psi_1(a', c)$ for all $c \geq a'$ in $\mathbf{c}_D(a)$, but $I \models \neg \psi_1(a', d)$, so $\psi_1(a', y)$ defines a bounded convex set with minimum a' , so is contained in $\mathbf{c}_D(a)$. By assumption, no element of $\mathbf{c}_D(a)$ realizes $\phi^*(a', y)$, so $\mathbf{c}_D(a) \cap [a', \infty)$ is defined by $\psi_1(a', y)$.

If $\mathbf{c}_D(a)$ is the first \mathbf{c}_D class, then $y \leq a' \vee \psi_1(a', y)$ defines $\mathbf{c}_D(a)$, violating the claim. So we may assume there is a $d < \mathbf{c}_D(a)$, and indeed we may assume that d realizes α (since, by self-additivity, the realizations of α are unbounded below). Since $I \models \phi^*(a', z)$ and a' and d have the same type, $I \models \phi^*(d, z)$. Now suppose, for every $d < \mathbf{c}_D(a)$ realizing α , there is no e between d and a' where $I \models \phi^*(d, e)$. Then the formula

$$(\exists u \leq x \forall v \leq u) [\alpha(v) \rightarrow (\forall z > v) (\phi^*(v, z) \rightarrow z > x)]$$

defines a proper convex subset of I :

This is impossible by self-additivity, so the supposition was wrong; therefore, there is a $d < \mathbf{c}_D(a)$ realizing α and an $e \in (d, a')$ where $I \models \phi^*(d, e)$. Now define the formula $\theta(a', y)$:

$$y \leq a' \wedge (\forall v) [(y \leq v \leq a' \wedge \alpha(v)) \rightarrow (\forall z > v) (\phi^*(v, z) \rightarrow z > a')]$$

By examination this is bounded above by a' , contains a' , is upward closed (by examination of the role in y ; higher y make the condition easier), but does not contain the d produced in the preceding paragraph. So it is bounded, and every realization of $\theta(a', y)$ is in $\mathbf{c}_D(a') = \mathbf{c}_D(a)$. We have already shown that $\mathbf{c}_D(a) \cap [a', \infty)$ is definable over a' , so if this defines $\mathbf{c}_D(a) \cap (-\infty, a']$, we contradict the claim. So there is some $c \in \mathbf{c}_D(a)$ where $I \models \neg \theta(a', c)$.

Therefore, there is a $d' \in [c, a']$ and an $e' \in (d', a']$ where $I \models \phi^*(d', e')$. In particular $d' \in \mathbf{c}_D(a)$, so $a' \in \mathbf{c}_D(d')$, so there is a formula $\psi(x, y)$ where $\psi(d', y)$ defined a convex bounded subset of I containing d' , e' , and a' , and is consequently contained in $\mathbf{c}_D(a)$. Then $I \models \exists z (\phi^*(d', z) \wedge \psi(d', z))$, so this holds for any realization of α , of which a' is one. So $I \models \exists z (\phi^*(a', z) \wedge \psi(a', z))$. Such a z falls into $\psi(a', z)$, a bounded convex a' -definable set which is therefore contained in $\mathbf{c}_D(a') = \text{boc}_D(a)$, so is what we've been looking for.

We have now shown that if $\mathbf{c}_D(a)$ is not definable over a , then $\mathbf{c}_D(a) \prec I$. So, let $b \in I$ be arbitrary; we want to show $\mathbf{c}_D(b) \prec I$. Let $\alpha(x)$ isolate $\text{tp}(b)$. Since $\mathbf{c}_D(a) \prec I$ and $I \models \exists x \alpha(x)$, $\mathbf{c}_D(a)$ contains a realization of α ; call it b' . If $\mathbf{c}_D(b)$ were definable over b , then by Lemma 5.1, $\mathbf{c}_D(b')$ would be definable over b' , violating the claim; so by contrapositive, $\mathbf{c}_D(b)$ is not definable over b , so $\mathbf{c}_D(b) \prec I$, completing the proof. \square

The really important take-away is this; if $\mathbf{c}_D(a) \prec I$, then we have a model of T with exactly one \mathbf{c}_D class, which means we have completely control over the order type of $\mathbf{c}_D[I]$, implying Borel completeness. This, along with the construction in Lemma 5.3 and induction, is all we need:

Theorem 5.5. *Let I be an L -structure, $T = \text{Th}(I)$, such that $S_1(T)$ is finite. Then either T is \aleph_0 -categorical or Borel complete.*

Proof. We go by induction on $n = |S_1(T)|$. First consider the base case $n = 1$, so every element has the same type. If $|I| = 1$, it's \aleph_0 -categorical, so assume not; then I is infinite without endpoints. Now suppose

there is some pair $a, b \in I$ where b is the immediate successor of a . Then every element has an immediate successor and an immediate predecessor. Then I is a discrete linear order without endpoints, well-known to be an axiomatization of $\text{Th}(\mathbb{Z}, <)$, which is Borel-complete by the reduction $L \mapsto \mathbb{Z} \times L$. On the other hand, if there is no such pair, then I is dense without endpoints, so is equivalent to $\text{Th}(\mathbb{Q}, <)$, which is \aleph_0 -categorical by a back-and-forth argument. So the theorem is true for the base case.

So suppose $|S_1(T)| = n + 1$ and the theorem has been proven for all smaller type-spaces. We have several cases:

Case: I is not self-additive.

Let $\phi(x)$ be a proper initial formula, so I definably decomposes into $A + B$, where $A = \phi(I)$ and $B = \neg\phi(I)$. If both $\text{Th}(A)$ and $\text{Th}(B)$ are \aleph_0 -categorical, so is T by Theorem 2.1. Otherwise, by inductive hypothesis, either $\text{Th}(A)$ or $\text{Th}(B)$ is Borel complete. By Corollary 1.5, this makes T Borel complete.

Case: I is self-additive, and for some $a \in I$, $\mathbf{c}_D(a) \prec I$.

We may assume (since we only care about T) that $I = \mathbf{c}_D(a)$, so that I has only a single \mathbf{c}_D class. The function $L \mapsto L \times I$ is a Borel function from the class of linear orders to the models of T (since I is self-additive). Since elementarity preserves \mathbf{c}_D , the quotient order $\mathbf{c}_D[L \times I]$ is isomorphic to L (indeed, the function $\ell \mapsto \mathbf{c}_D(\ell, a)$ is an isomorphism). Consequently, if L and L' are linear orders, then $L \cong L'$ if and only if $L \times I \cong L' \times I$, so this is a Borel reduction and T is Borel complete.

Case: I is self-additive, and for every $a \in I$, $\mathbf{c}_D(a) \not\prec I$.

Then I has multiple \mathbf{c}_D classes, and all of them are definable over any of their elements. Since $S_1(T)$ is finite, this means the relation “ $x \in \mathbf{c}_D(y)$ ” is definable by a single formula. Also, up to elementarity, I is $\sigma(D_1, \dots, D_t)$ for some finite set of linear orders as produced in Lemma 5.3. If all of them are \aleph_0 -categorical, then so is I by Theorem 2.1. If (say) D_1 is Borel complete, then the map $D \mapsto \sigma(D, D_2, \dots, D_t)$ is a Borel map from $\text{Mod}(\text{Th}(D_1))$ to $\text{Mod}(T)$. Since the “ x is in the D_1 component of y ” is a definable set (since it’s equivalent to $x \in \mathbf{c}_D(y)$), it’s preserved by isomorphisms, so this is a Borel reduction, and T is Borel complete.

In particular, if we can apply the inductive hypothesis to all the D_i , the theorem is proved. This occurs if $|S_1(\text{Th}(D_i))| < n + 1$ for all i . If not, then $t = 1$ so $I = \sigma(D_1)$. Since D_1 is its own \mathbf{c}_D class in I and I is a sum of copies of D_1 , but the \mathbf{c}_D classes are not elementary substructures of I , D_1 is not self-additive. By a previous case, either D_1 is \aleph_0 -categorical (so T is) or D_1 is Borel complete (so T is, by the preceding paragraph).

These cases are exhaustive, so the theorem is proven. \square

We can combine this theorem with the one from the previous section to get a combined result that completely answers the question for self-additive orders, and which we will use heavily in the following section:

Corollary 5.6. *If I is self-additive, then T is either \aleph_0 -categorical or Borel complete.*

Proof. If $S_1(T)$ is infinite, then T is Borel complete by Theorem 4.9; otherwise $S_1(T)$ is finite so Theorem 5.5 applies. Either way, the corollary holds. \square

6 The General Proof

Let I be a typical L -structure and T be $\text{Th}(I)$. We now consider the case where $|S_1(T)|$ is not necessarily finite and I is not necessarily self-additive.

Definition 6.1. Let $IT(T)$ be the space of “interval types;” that is, the set of complete consistent sets of convex formulas.

This can be seen as a “quotient” of $S_1(T)$, since if two elements have the same type, they also have the same interval type, but not conversely for most orders (consider, for instance, the possibility that P_1 is dense and codense). We borrow some terminology: $\Phi \in IT(T)$ is *isolated* (or *principal* or *atomic*) if, for some convex formula $\phi(x) \in \Phi(x)$, $\phi(x) \vdash \Phi(x)$. Observe that the types in $IT(T)$ are naturally ordered by $<$.

Proposition 6.2. *Suppose $\{\Phi_n : n \in \omega\}$ is a strictly ascending (or descending) sequence in $IT(T)$. Then there is a “limit type” $\Phi \in IT(T)$ – sometimes denoted $\lim \Phi_n$ – which is exactly the supremum (or infimum, respectively) of the sequence $\{\Phi_n : n \in \omega\}$. Furthermore, if $\phi \in \Phi$ is convex, then for cofinitely many $n \in \omega$, $\phi \in \Phi_n$.*

Proof. Let Φ be the set of all convex formulas ψ which are in Φ_n for cofinitely many n . Evidently Φ is consistent, since any finite conjunction of these formulas is also in cofinitely many Φ_n . We argue that Φ is complete and that it is the least upper bound for the Φ_n .

Next, let ϕ be an initial formula which is in Φ_n but whose negation is in Φ_{n+1} . Since $\neg\phi$ is terminal it is also in Φ_k for every $k > n$, so it’s in Φ , so $\Phi > \Phi_n$. That is, Φ is an upper bound to the sequence. Suppose Ψ is another upper bound to the sequence and $\Psi < \Phi$. Let ψ be an initial formula where $\psi \in \Psi$ and $\neg\psi \in \Phi$. Since $\Psi > \Phi_n$ and ψ is initial, $\psi \in \Phi_n$ for every n , so $\psi \in \Phi$ as well, contradicting consistency of Ψ as above. So Φ truly is the least upper bound of the sequence.

Finally, suppose ϕ is convex and $\phi \notin \Phi$. Then the set of $\{n : \Phi_n \vdash \neg\phi\}$ is infinite and thus unbounded in ω . Since it’s also convex in ω (since ϕ is a convex formula), it’s cofinite, so $\neg\phi \in \Phi$. Thus Φ is complete, proving the lemma. \square

$IT(T)$ is highly reminiscent of the space $S_1(T)$. Along these lines, the following facts are immediate from the above lemma, compactness, and countability of the language, where appropriate:

Proposition 6.3. *As a linear order with the order topology, $IT(T)$ is second countable, separable, and complete.*

$\Phi \in IT(T)$ is *principal* if and only if Φ is an isolated point in $IT(T)$, if and only if Φ has an immediate successor and predecessor in $IT(T)$.

Limit types in $IT(T)$ are not principal in $IT(T)$, nor are they implied by any isolated type $p \in S_1(T)$. Conversely, every nonprincipal element of $IT(T)$ is a limit of some countable sequence from $IT(T)$.

If $\Phi < \Psi \in IT(T)$, then there is an initial formula ϕ where $\phi \in \Phi$ and $\neg\phi \in \Psi$.

The following is immediate from Corollary 1.5:

Remark 6.4. Let $\Phi \in IT(T)$ and let $I \models T$ be arbitrary. View $\Phi(I)$ as its own L -structure. If $\text{Th}(\Phi(I))$ is Borel complete, so is $\text{Th}(I)$.

Consequently, we can immediately generalize our theorem from the previous section in this way:

Corollary 6.5. *If $IT(T)$ is finite, then T is either \aleph_0 -categorical or Borel complete. Further, T is Borel complete if and only if, for some $\Phi \in IT(T)$ and some $A \models T$, $\text{Th}(\Phi(A))$ is Borel complete.*

Proof. Let ϕ_1, \dots, ϕ_n be convex formulas which partition the formula $x = x$ and isolate the elements of $IT(T)$. For each $I \models T$, $I = A_1 + \dots + A_n$ where A_i is the (convex) set of realizations of ϕ_i . Let $T_i = \text{Th}(A_i)$. Evidently, each T_i is either a single point or a self-additive linear order, so by Corollary 5.6, is either \aleph_0 -categorical or Borel complete. If every A_i is \aleph_0 -categorical, so is T ; if one of them is Borel complete, so is T . \square

Therefore, we concentrate on the case when $IT(T)$ is infinite. Fix an $S \models T$ which is \aleph_0 -saturated and strongly \aleph_0 -homogeneous (that is, if \bar{a} and \bar{b} are finite and have the same type, there is an automorphism of S taking \bar{a} to \bar{b}) – we will refer to this condition as “sufficient saturation.” The following proposition is why “self-additivity” is interesting in general – the interval types decompose S into convex, self-additive pieces:

Proposition 6.6. *Let $\Phi \in IT(T)$ be arbitrary and let S be sufficiently saturated. Then either $\Phi(S)$ is a single point or is self-additive.*

Proof. We build up several facts before we prove the result directly.

Claim: Let $\phi(x)$ be some formula where for $S \models \phi(a)$ for some $a \in \Phi(S)$. Then $\{b \in \Phi(S) : S \models \phi(b)\}$ is unbounded in $\Phi(S)$.

Proof: Let $c \in \Phi(S)$. Suppose for some $\psi(x) \in \Phi(x)$, $S \models \neg\phi(b)$ whenever $S \models \psi(b) \wedge b \geq c$, the formula $(\exists y)(x \leq y \wedge \phi(y) \wedge \psi(y))$ is satisfied by all elements of S below a but no elements of S beyond c . This is a \emptyset -definable formula properly dividing Φ , contradicting completeness of Φ , so our assumption was wrong. So for every $\psi(x) \in \Phi$ there is a $b \geq c$ realizing $\psi(y) \wedge \phi(y)$. By compactness, it is consistent that there is a $b \geq c$ realizing ϕ in Φ , and so by \aleph_0 -saturation, there is a $b \in \Phi(S)$ with $b \geq c$ and $S \models \phi(b)$. \square

Again by compactness and \aleph_0 -saturation of S , for any 1-type $P(x) \in S_1(T)$ and any $\Phi \in IT(T)$, if P is realized in Φ , then the realizations of $P(x)$ are unbounded in $\Phi(S)$. So now suppose $\Phi(S)$ (has more than one point and) is not self-additive. There is a formula $\phi(x)$ which is a proper initial formula in $\Phi(S)$. Let $a \in \Phi(S)$ and $b \in \Phi(S)$ such that $\Phi(S) \models \phi(a) \wedge \neg\phi(b)$. By the strengthening of the claim, there is a $c \geq b$ in $\Phi(S)$ realizing the same type as a . By strong \aleph_0 -homogeneity, there is an automorphism f of S taking a to c . Of course f preserves Φ , so induces an automorphism of $\Phi(S)$. But then f does not preserve ϕ in the structure $\Phi(S)$, which is impossible. This contradiction shows $\Phi(S)$ is self-additive. \square

Therefore, saturated models of T are interesting, but without countability of the type space, such a model is uncountable. We use the following proposition frequently to transfer information about S to information about countable models of T , saying roughly that we can capture the behavior of $\Phi(S)$ in a countable model for countably many $\Phi \in IT(T)$.

Proposition 6.7. *Let $S \models T$ be arbitrary, and let $\{\Phi_n : n \in \omega\} \subset IT(T)$ be countable. Then there is a countable $A \prec S$ such that for all $n \in \omega$, $\Phi_n(A) \prec \Phi_n(S)$. So in particular, $\Phi_n(A) \equiv \Phi_n(S)$.*

Proof. For each n , add a new unary predicate P_n to the language and interpret P_n on S as the realizations of Φ_n . The resulting language is still countable, so let $A \prec S$ be a countable elementary substructure of this expanded structure. Since $P_n(A)$ is a definable subset of A and $A \prec S$, this means $P_n(A) \prec P_n(S)$ as well, so $\Phi_n(A) \prec \Phi_n(S)$ for every n , as desired. \square

Since we've got self-additive pieces, we can conclude the following:

Corollary 6.8. *If, for some $\Phi \in IT(T)$, the theory of $\Phi(S)$ is not \aleph_0 -categorical, then both $Th(\Phi(S))$ and T are Borel complete.*

Proof. If $\Phi(S)$ is not \aleph_0 -categorical, then it's not a single point, so is self-additive. By self-additivity, it's Borel complete. Pick some countable $A \prec S$ with $\Phi(A) \prec \Phi(S)$. If $A \cong B$ then $\Phi(A) \cong \Phi(B)$, and if $D \equiv \Phi(A)$, then we can replace $\Phi(A)$ with D to form $A' \equiv A$ where $\Phi(A') \cong D$. Therefore, we have a Borel reduction from $\text{Mod}(Th(\Phi(A)))$ to $\text{Mod}(T)$, so T is Borel complete, as desired. \square

This is enough to completely settle the issue for finite languages:

Theorem 6.9. *Let L be a finite typical language and T be a typical L -theory. Then T is either \aleph_0 -categorical or Borel complete. Further, T is Borel complete if and only if, for some $\Phi \in IT(T)$ and some $A \models T$, $Th(\Phi(A))$ is Borel complete.*

Proof. If $IT(T)$ is finite, then the conclusion follows from Corollary 6.5. We now show that if $\Phi \in IT(T)$ is not atomic, then $\Phi(S)$ is Borel complete, implying T is also Borel complete. If $IT(T)$ is infinite, there is always such a Φ by completeness of $IT(T)$, and so this will prove the theorem.

So let $\{\Phi_n : n \in \omega\}$ be a sequence of elements of $IT(T)$ which limit on Φ . We may assume this sequence is strictly monotone. Indeed we will assume it is strictly increasing (the other case is similar). Let $\phi \in \Phi$ be arbitrary, so it is a convex formula which includes a tail of the sequence of types. Observe that for any k , there is a pair of elements $a < b$ which both realize ϕ and where $[a, b]$ has at least 2^k pairwise disjoint definable convex pieces in A . Using Theorem 3.1, $[a, b]$ still has at least 2^k pairwise disjoint definable convex pieces when viewed as its own L -structure. Therefore, $[a, b]$ is either not \aleph_0 -categorical or has rank at least k .

Now, since L is finite, for every k , there is a sentence $\psi_k(x, y)$ which says that $x < y$ and $[x, y]$ is not a typical \aleph_0 -categorical L -structure of rank strictly less than k . So consider the type $\Sigma(x, y) = \{x < y\} \cup \Phi(x) \cup \Phi(y) \cup \{\psi_k(x, y) : k \in \omega\}$. This type is finitely satisfied, since for any $\phi \in \Phi$ and $k \in \omega$, the consistency of $(x < y \wedge \phi(x) \wedge \phi(y) \wedge \psi_k(x, y))$ is precisely the content of the previous paragraph. Thus Σ is realized on S . So let $A \models T$ be countable where $A \prec S$ and $\Phi(A) \prec \Phi(S)$. Then $[a, b] \subset \Phi(A)$ is not \aleph_0 -categorical, so $\Phi(A)$ is not \aleph_0 -categorical. Since $\Phi(A) \prec \Phi(S)$ is self-additive, this means it's Borel complete, which completes the proof. \square

The situation for infinite languages is slightly more complicated; we need the following two lemmas. The first says that if there are no Borel complete types, then there are only finitely many choices for each type:

Lemma 6.10. *Suppose $\Phi \in IT(T)$ and $\Phi(S)$ is \aleph_0 -categorical. Then the set $\{Th(\Phi(A)) : A \models T\}$ is finite (note we allow the “theory of no elements” as a possible value). Further, it has exactly one element if and only if Φ is isolated.*

Proof. For the further: if Φ is not isolated, there is $A \models T$ omitting Φ and $B \models T$ realizing it. Clearly $\Phi(A) \neq \Phi(B)$. Conversely, if Φ is isolated by a single element ϕ , then for any $A, B \models T$, $\phi(A) \equiv \phi(B)$ as a definable set, so $\Phi(A) \equiv \Phi(B)$.

More interesting is the main brunt. Suppose $\{A_i : i \in \omega\}$ is a sequence of models of T which all realize Φ and where $\Phi(A_i) \neq \Phi(A_j)$ for $i \neq j$. By saturation of S , we may assume $A_i \prec S$ for all i . Further, let D_i be the convex hull of $\Phi(A_i)$ in $\Phi(S)$. By Lemma 3.4, $\Phi(A_i) \prec D_i$, so $D_i \equiv \Phi(A_i)$, so $D_i \neq D_j$ for all $i \neq j$.

Expand S with symbols Q_i for each D_i , and let B be a countable elementary substructure of this expansion. Let A be the L -reduct of B . Then $\Phi(A) \prec \Phi(S)$, so $\Phi(A) \equiv \Phi(S)$, so is \aleph_0 -categorical. Moreover, $Q_i(A) \prec D_i$, so $Q_i(A) \equiv D_i$, and each $Q_i(A)$ is a convex subset of A . Therefore, $\Phi(A)$ admits infinitely many pairwise inequivalent convex subsets, a contradiction of \aleph_0 -categoricity by Corollary 2.6. \square

The final lemma states that we can make “enough free choices” to generate a large number of models:

Lemma 6.11. *Let $\bar{\Phi}$ be a collection of nonisolated types in $IT(T)$ such that no element of the set is a limit point of the rest of the set. Then, for any $A \models T$ and any $X \subset \omega$, the set $\{a \in A \mid \forall x \in X (A \not\models \Phi_x(a))\}$ is an elementary substructure of A .*

Proof. Let $B = \{a \in A \mid \forall x \in X (A \not\models \Phi_x(a))\}$. Let \bar{b} be from B , a from A , and let $\phi(\bar{x}, y)$ where $A \models \phi(\bar{b}, a)$. We need an $a' \in B$ where $A \models \phi(\bar{b}, a')$. Let $\Phi \in \bar{\Phi}$ be the interval type of a . Then there is a convex formula $\psi(y)$ which is in Φ (and therefore holds of a) but which is inconsistent with each of the other types in $\bar{\Phi}$ and which is also not satisfied by any element of \bar{b} . The former is possible since the points of $\bar{\Phi}$ is mutually isolated; the latter is possible since none of the \bar{b} realize Φ , so have different interval types than a .

By Corollary 3.2, produce a formula $\phi^\#(y)$ where $A \models \phi^\#(a')$ if and only if $A \models \psi(a') \wedge \phi(\bar{b}, a')$. Consider the formula $\tau(y)$, defined by

$$\exists x_1 \exists x_2 (x_1 \leq y \leq x_2 \wedge \phi^\#(x_1) \wedge \phi^\#(x_2))$$

Clearly this is convex, definable over \emptyset , and satisfied by a , but only satisfied by realizations of ψ . Since Φ is a complete interval type, τ is satisfied by all elements of Φ . Since Φ is a limit type, it is also satisfied by some a' which does not realize Φ . This a' must realize ψ , so cannot realize any type in $\bar{\Phi}$, so $a' \in B$ and $A \models \phi(\bar{b}, a')$, as desired. \square

We can now completely classify the possible Borel complexity levels of T :

Theorem 6.12. *Let T be a typical L -theory. If, for some $\Phi \in IT(T)$ and some $A \models T$, $Th(\Phi(A))$ is Borel complete, then T is Borel complete. Otherwise $\Phi(A)$ is \aleph_0 -categorical for every Φ and A . Assume this latter condition for the rest of the cases.*

$T \sim_B \cong_2$ if and only if $IT(T)$ is uncountable (equivalently not scattered).

$T \sim_B \cong_1$ if and only if $IT(T)$ has infinitely many limit types but is countable (equivalently, scattered).

$T \sim_B n$ for some finite $n \geq 3$ if and only if $IT(T)$ is infinite, but has only finitely many limit types.

$T \sim_B 1$ if and only if $IT(T)$ is finite.

Proof. The first clause and the “otherwise” are precisely the content of Corollary 6.8. Therefore we may assume $\Phi(A)$ is \aleph_0 -categorical for all $\Phi \in IT(T)$ and all $A \models T$. Suppose $A, B \models T$. If, for all $\Phi \in IT(T)$, $\Phi(A) \cong \Phi(B)$, then $A \cong B$: since the pieces are convex, we can simply take the union of the local isomorphisms to form a global isomorphism. If A and B are countable then, because of local \aleph_0 -categoricity, we see that $A \cong B$ if and only if $\Phi(A) \equiv \Phi(B)$ for all Φ . Therefore, the countable isomorphism problem for T is only as hard as a choice of theory for each type.

If $IT(T)$ is finite, then every $\Phi \in IT(T)$ is isolated, so there are no choices; therefore, T is \aleph_0 -categorical.

Next, if $IT(T)$ has only finitely many limit points, then the isomorphism type depends on finitely many finite choices, so T has only finitely many countable models.

Next, suppose $IT(T)$ is countable but has infinitely many limit types, enumerated as $\{\Phi_n : n \in \omega\}$. For each n , enumerate the possible theories of Φ_n as an initial subset of ω . For each countable $A \models T$, define the function $t_A : \omega \rightarrow \omega$ where $t_A(n)$ is (the number corresponding to) $\text{Th}(\Phi_n(A))$. As discussed, $A \cong B$ if and only if $t_A = t_B$; therefore, this is a Borel reduction from $\text{Mod}(T)$ to equality on ω^ω , so $\text{Mod}(T) \leq_B \cong_1$.

Conversely, since $IT(T)$ is countable and complete, it's scattered, so every Φ has a CB-rank. Since there are infinitely many limit types (that is, types of CB-rank at least 1), there must be infinitely many types of rank exactly one. The collection $\{\Phi_n : n \in \omega\}$ of all rank-one types clearly satisfies the conditions of Lemma 6.11. So let $A \models T$ realize all of the Φ_n , and for every $X \subset \omega$, let A_X lose precisely the realizations of Φ_n for $n \in X$. Then $A_X \prec A$, so $A_X \models T$, and $A_X \cong A_Y$ if and only if $X = Y$. This gives a Borel reduction from equality on 2^ω to $\text{Mod}(T)$, so combining with the previous paragraph, $\text{Mod}(T) \sim_B \cong_1$, as desired.

Finally, say $IT(T)$ is uncountable. Then so is $S_1(T)$, so by [3], $\cong_2 \leq_B \text{Mod}(T)$. We describe a map in the other direction. To start with, let X be $IT(T) \times \omega$ with the product topology, which is a standard Borel space. For each $\Phi \in IT(T)$, fix a labeling of the possible values of $\text{Th}(\Phi(A))$ as elements of ω . For each model $A \models T$ with universe ω , and for each $n \in \omega$, let (Φ, k) be the interval type of n in A and the theory of $\Phi(A)$. We call this sequence s_A , and have defined a Borel function $\text{Mod}(T) \rightarrow X^\omega$. If $A \cong B$, then $\Phi(A) \equiv \Phi(B)$ for all Φ , so the images of s_A and s_B are equal as sets. Conversely, if these images are equal, then for all A , A realizes Φ if and only if B does, and in that case, $\Phi(A) \equiv \Phi(B)$, so $A \cong B$. This gives a Borel reduction from $\text{Mod}(T)$ to \cong_2 , so $\text{Mod}(T) \sim_B \cong_2$.

The verification that each described reduction is Borel is routine and left to the reader. \square

Proposition 6.13. *All of the above cases can happen:*

$(\mathbb{Q}, <)$ is \aleph_0 -categorical.

For every $n \geq 3$, form the structure \mathcal{M}_n as $(\mathbb{Q}, <)$ with constants for each $i \in \omega$ as well as $n - 2$ dense/codense colors which partition the space. Then $\text{Th}(\mathcal{M}_n)$ has exactly n countable models.

Form the structure \mathcal{B} as $(\mathbb{Q}, <)$ with constants for every rational of the form $m + \frac{1}{n}$, where $m, n \in \mathbb{Z}$ and $n \geq 1$. Then $\text{Th}(\mathcal{B}) \sim_B \cong_1$.

Form the structure \mathcal{A} as $(\mathbb{Q}, <)$ with constants for every rational. Then $\text{Th}(\mathcal{A}) \sim_B \cong_2$.

$(\mathbb{Z}, <)$ is Borel complete.

The verifications are left to the reader.

7 Parameters

An important open problem in this field is the following: suppose we add a constant to the language. What happens to the complexity of the theory? On the one hand, if that constant realizes a nonisolated type, some models of the original theory are no longer models of the new theory, so perhaps we have fewer models and the complexity goes down. On the other hand, a single model may have two nonisomorphic expansions owing to a lack of homogeneity, so we may get more models. To frustrate the intuition, observe that simply making two models distinct does *not* necessarily make the theory more complicated.

Experience with model theory tells us that adding one constant (or finitely many) should not make any essential difference, and this has been borne out in practice, but there is no general theorem available. A special case is very important: suppose T is Borel complete *after* adding finitely many constants. Must T be Borel complete? We don't know in general, although in [5] it was shown that if T is o-minimal, then T

is Borel complete if and only if T_A ⁴ is, for any countable set of constants A . We show the same theorem for typical theories, along with a few more specific pieces of information about what's possible.

Before we get too far, observe that the Borel complexity *can* change by adding a single constant:

Example 7.1. There is a typical theory T in an infinite language and a constant symbol c where $T \not\prec_B T_c$. Specifically, T goes from a finite number of models to a larger (finite) number of models.

Proof. Let A be the structure \mathbb{Q} , together with constants for all rational numbers of the form $\frac{1}{n}$ and $-\frac{1}{n}$ where $n \geq 1$. Let $T = \text{Th}(A)$. Then T has exactly six countable models, depending on the order type of the realizations of the single nonisolated type. If we add a constant for zero, then the new theory T_0 has exactly 9 countable models, depending on the order type of the elements in the two nonisolated types split by zero. \square

In general this is the only change that can happen from a single constant (and, therefore, finitely many), as we will see.

Lemma 7.2. *Suppose $\Phi \in IT(T)$ and $A \subset S$ does not contain any realization of Φ . Then Φ is still a complete interval type over A .*

Proof. Suppose not, so there is some convex formula $\phi(x, \bar{a})$ over A which is neither universally true nor universally false on $\Phi(S)$. By Theorem 3.1, there is a $\phi^*(x)$ where for all $s \in \Phi(S)$, $\Phi(S) \models \phi^*(s)$ if and only if $\phi(s, \bar{a})$. But then ϕ^* is a proper convex formula in $\Phi(S)$, which is either self-additive or a single point. This is impossible, proving the lemma. \square

Lemma 7.3. *Suppose $\Phi \in IT(T)$ and $a \in \Phi(S)$. If $\Phi(S)$ is \aleph_0 -categorical, there are only finitely many complete interval types (over a) extending Φ . Call them Ψ_1, \dots, Ψ_n . Then S is still sufficiently saturated as an $L(a)$ -structure and $\Psi_i(S)$ are all still \aleph_0 -categorical.*

Proof. First – since $\Phi(S)$ is \aleph_0 -categorical, so is its expansion to an $L(a)$ structure. Consequently there are only finitely many 1-formulas, so there is a finite list $\psi_1(x, a), \dots, \psi_n(x, a)$ of convex formulas such that the formulas are pairwise disjoint, partition $\Phi(S)$, and which cannot be divided into definable proper convex pieces.

Let Ψ be a complete interval type over a which extends Φ . Let $\psi(x, a) \in \Psi$ be some convex formula. By Theorem 3.1, there is a $\psi^*(x, a)$ where for all $b \in \Phi(S)$, $\Phi(S) \models \psi^*(b, a)$ if and only if $S \models \psi(b, a)$. On $\Phi(S)$, $\psi^*(x, a)$ is still convex, so is a disjunction of the ψ_i above. By saturation Ψ is completely determined by the elements of $\Phi(S)$ which realize it, and is therefore determined by which of the ψ_i are consistent with it. There are only finitely many choices of this nature to make, so there are only finitely many Ψ .

Finally, observe that $\Psi(S)$ is a convex subset of $\Phi(S)$, so is \aleph_0 -categorical by Proposition 2.5. \square

Lemma 7.4. *Suppose $\Phi \in IT(T)$, $\Phi(S)$ is \aleph_0 -categorical, and a realizes Φ . Then Φ is isolated in $IT(T)$ if and only if all the extensions of Φ to $IT(T_a)$ are isolated in $IT(T_a)$.*

If $\Psi \in IT(T)$ is not equal to Φ , Ψ is a limit type in $IT(T)$ if and only if Ψ is a limit type in $IT(T_a)$.

Proof. First, suppose Φ is nonisolated. By symmetry, we may assume there is an ascending sequence $\{\Phi_n : n \in \omega\}$ in $IT(T)$ which limits on Φ . These are still an ascending sequence in $IT(T_a)$, so clearly have a limit type in $IT(T_a)$. Call it Ψ . If $\phi(x)$ is in cofinitely many of the Φ_n , then $\phi \in \Psi$. If Ψ is not an extension of Φ , there is a convex L -formula $\phi \in \Phi$ whose negation is in Ψ ; such a formula must be in cofinitely many of the Φ_n , so must be in the limit type Ψ , a contradiction. So Ψ extends Φ .

Conversely, suppose Ψ extends Φ and is nonisolated. By symmetry, we may assume there is an ascending sequence $\{\Psi_n : n \in \omega\}$ in $IT(T_a)$ which limits on Ψ . Since the extensions of Φ are $\Phi_1 < \dots < \Phi_t$ for some $t \geq 1$, there cannot be any types in $IT(T_a)$ strictly between any Φ_i and Φ_{i+1} (as this would be another extension of Φ); thus all but Φ_1 are isolated on the left, so Φ_1 is Ψ . Consequently, none of the Ψ_n extend Φ , so are the unique extensions of their reductions to $IT(T)$, and in particular, all their reductions are unique. They clearly limit on Φ , completing the proof. \square

⁴The elementary diagram of A , or equivalently, T expanded to include constants for the elements of A

Lemma 7.5. *Suppose $\Phi \in IT(T)$, $\Phi(S)$ is \aleph_0 -categorical, and a realizes Φ . If $\Psi \in IT(T)$ is not Φ , then Ψ is isolated in $IT(T)$ if and only if Ψ is isolated in $IT(T_a)$.*

Proof. Suppose Ψ is nonisolated in $IT(T_a)$, so we may assume there is an ascending sequence $\{\Psi_n : n \in \omega\}$ in $IT(T_a)$ which limits on Ψ . Since Φ has only finitely many extensions to $IT(T_a)$, there is a tail of the above sequence which is completely to the right of (all extensions of) Φ to $IT(T_a)$; we assume this is the entire sequence. Then each of the Ψ_n , as well as Ψ itself, have distinct reducts to L (since they are all the unique extensions of those reducts) so the Ψ_n limit on Ψ in $IT(T)$ as well. The converse is essentially the same idea. \square

Lemma 7.6. *Let A be a sufficiently saturated typical structure. If A is self-additive and not \aleph_0 -categorical, there is a pair $a < b$ from A where $[a, b]$ is not \aleph_0 -categorical.*

Proof. Suppose A is as described. Let $B \prec A$ be countable, and let D be the convex hull of B in A . Then B is not \aleph_0 -categorical (since A is not) and $B \prec D$ by Lemma 3.4. Then D is not \aleph_0 -categorical since $D \equiv A$. Further, since B is countable and A is sufficiently saturated, there are $a, b \in A$ where $a < B < b$. Then $[a, b]$ contains D as a convex subset, so it is not \aleph_0 -categorical, as desired. \square

Lemma 7.7. *Suppose $\Phi \in IT(T)$ and $a \in \Phi(S)$. If $\Phi(S)$ is not \aleph_0 -categorical, there is a complete interval type $\Psi \supset \Phi$ over a where $\Psi(S)$ is not \aleph_0 -categorical. Specifically, the type “ $\Phi(x)$ and $x < \mathbf{c}_D(a)$ ” satisfies these conditions.*

Proof. Let A be the set elements of $\Phi(S)$ which are strictly below $\mathbf{c}_D(a)$, which is evaluated in $\Phi(S)$. Evidently A is convex. We first show that A is a well-defined, complete interval type over a . Completeness is straightforward – suppose $\phi(x, a)$ is convex and properly divides the set of realizations of $\Phi(x) \wedge (x < \mathbf{c}_D(a))$. We may assume $\phi(x, a)$ is initial in this set, so let $b, b' \in A$ be such that $S \models \phi(b, a) \wedge \neg \phi(b', a)$. Let $\phi^*(x, a)$ be as usual, so it defines an initial subset of $\Phi(S)$ which contains b but not b' . Then the set $x \leq a \wedge \neg \phi^*(x, a)$ is convex, contains a , and is bounded (by b), so $b' \in \mathbf{c}_D(a)$, against construction of A .

To show it’s well-defined, let $\phi(x, a)$ be a formula whose realizations in $\Phi(S)$ are bounded, convex, and include a . We need to show the set $\{b \in \Phi(S) : b \leq a \wedge \Phi(S) \models \phi(b, a)\}$ is definable in S over a . By saturation, this is equivalent to showing it is invariant under automorphisms of S which fix a . So suppose $f : S \rightarrow S$ is such an automorphism and $b \leq a$ and $\Phi(S) \models \phi(b, a)$. Since Φ is type-definable, f fixes Φ , so induces an automorphism of $\Phi(S)$ which fixes a . Consequently f preserves $\phi(x, a)$ on $\Phi(S)$, so $\Phi(S) \models \phi(f(b), a)$ as well.

Thus we have our type. By symmetry, the type “ $\Phi(x)$ and $x > \mathbf{c}_D(a)$ ” (which we will call B) is also well-defined and complete as an interval type over a . Since $\Phi(S) = A + \mathbf{c}_D(a) + B$ is not \aleph_0 -categorical, one of the summands must not be. If it’s A , we’re done. Suppose it is $\mathbf{c}_D(a)$. It is consistent that there is an $a' < \mathbf{c}_D(a)$ where $\text{tp}(a') = \text{tp}(a)$; otherwise some $\phi(x) \in \text{tp}(a)$ is bounded below, dividing $\Phi(S)$. By saturation, there is such an a' . Let $f : S \rightarrow S$ be an automorphism taking a to a' . Since $f : \Phi(S) \rightarrow \Phi(S)$ is also an automorphism, it preserves \mathbf{c}_D , so $\mathbf{c}_D(a') \cong \mathbf{c}_D(a)$, and the former is a convex subset of A . Thus A is not \aleph_0 -categorical, as desired.

Finally, suppose B is not \aleph_0 -categorical. Since it’s the set of realizations of a complete interval type over a , B is self-additive, so there is a pair $b < b'$ in B where $[b, b']$ is not \aleph_0 -categorical. It is consistent that there is a pair c, c' where $\text{tp}(cc') = \text{tp}(bb')$ and $c < c' < \mathbf{c}_D(a)$; otherwise, as before, we violate self-additivity of $\Phi(S)$. By saturation there is such a pair in S , and so $[c, c']$ is a convex subset of A which is not \aleph_0 -categorical, completing the proof. \square

We can now prove the following theorem:

Theorem 7.8. *Suppose T is a typical theory and a is an finite tuple from some model. Let T_a be the theory resulting from adding a (and its complete type) to the language. Then the Borel complexity of T_a follows these rules:*

If T has $n \geq 3$ countable models, then T_a has $m \geq 3$ countable models, though m and n may not be equal. In all other cases, $T \sim_B T_a$.

Proof. Evidently the theorem is invariant under “repetition,” so we may assume a is a singleton. There are five cases for T : \aleph_0 -categoricity, having $n \geq 3$ models, being exactly \cong_1 , being exactly \cong_2 , or being Borel complete.

It is well-known that T is \aleph_0 -categorical if and only if T_a is (this is a general model-theoretic fact).

If T is Borel complete, then there is some Φ where $\Phi(S)$ is not \aleph_0 -categorical. By Lemma 7.7, there is a $\Psi \in IT(T_a)$ where $\Phi(S)$ is not \aleph_0 -categorical, so T_a is Borel complete. Conversely, every $\Psi \in IT(T_a)$ is an extension of some $\Phi \in IT(T)$. If T is not Borel complete, $\Phi(S)$ is \aleph_0 -categorical, and by Lemma 7.3, $\Psi(S)$ is as well. This applies to all Ψ , so T_a is not Borel complete. That is, T is Borel complete if and only if T_a is.

Suppose T has $n \geq 3$ countable models. Then $IT(T)$ has finitely many limit types (not zero). Each of these types which a does not realize is still a limit type in $IT(T_a)$, and the limit type (if any) that a realizes splits into at most two limit types. So $IT(T_a)$ still has only finitely many limit types (not zero), so has $m \geq 3$ countable models. Conversely, if $IT(T_a)$ has only finite many limit types, then $IT(T)$ has at most that many (since the reduct of an isolated type is isolated) but not zero (since the reduct of a nonisolated type is nonisolated). So if T_a has $m \geq 3$ countable models, T has $n \geq 3$ countable models.

Next, T is small if and only if T_a is, since $|S_n(T)| \leq |S_n(T_a)| \leq |S_{n+1}(T)|$ for all n . So if $T \sim_B \cong_2$, T is not small and not Borel complete, so T_a shares these properties, so $T_a \sim_B \cong_2$. This all reverses.

The final case follows by exhaustion; if $T \sim_B \cong_1$, then T does not fall into any of the other cases, so neither does T_a , so $T_a \sim_B \cong_1$, and vice-versa. \square

The situation is very different for infinitely many constants – essentially, one can jump “up” from one of the aforementioned classes to the others, or stay in the same one. But one cannot go down, and one cannot become Borel complete in this way.

Lemma 7.9. *Suppose A is a countable set of constants and T is a typical theory. If $\Phi \in IT(T)$ has $\Phi(S)$ not \aleph_0 -categorical, there is a $\Psi \in IT(T_A)$ where $\Psi(S)$ is not \aleph_0 -categorical and $\Psi \supset \Phi$.*

Proof. Let $\{a_i : i \in \omega\}$ be an enumeration of A . We construct a sequence $\Phi = \Phi_0 \subset \Phi_1 \subset \dots$ where each Φ_n is a Borel complete interval type over $A_n\{a_i : i < n\}$. We do this as follows. If a_i does not realize Φ_i , then Φ_i is a complete interval type over A_{i+1} , so let Φ_{i+1} be Φ_i . Otherwise, let Φ_{i+1} be Φ_i , together with the condition $x < c_D(a_i)$ (evaluated in $\Phi_i(S)$). This is a well-defined element of $IT(T_{A_{i+1}})$ and $\Phi_{i+1}(S)$ is Borel complete by Lemma 7.7.

Finally, let Ψ be the intersection of all the Φ_i . This is a complete consistent interval type over A by compactness. Moreover, let $[b, c]$ be a closed subinterval of $\Phi(S)$ which is not \aleph_0 -categorical. It follows from self-additivity that for each n , it is consistent for there to be a pair $b'c'$, both realizing Φ_n and where $\text{tp}(b'c') = \text{tp}(bc)$. By compactness and saturation of S , there is such a pair $b'c'$ from $\Psi(S)$. Consequently, $\Psi(S)$ is not \aleph_0 -categorical, as desired. \square

Lemma 7.10. *Suppose A is a countable set of constants and T is a typical theory. If $\Phi \in IT(T)$ has $\Phi(S)$ \aleph_0 -categorical, then every $\Psi \in IT(T_A)$ which extends Φ also has $\Psi(S)$ \aleph_0 -categorical.*

Proof. $\Psi(S)$ is a convex subset of $\Phi(S)$. \square

Lemma 7.11. *Suppose A is a countable set of constants and T is a typical theory. If $\Phi \in IT(T)$ is a limit type, there is a $\Psi \in IT(T)$ which extends Φ and is a limit type.*

Proof. Let $\{a_i : i \in \omega\}$ be an enumeration of A . As shown before, there is a sequence $\Phi = \Phi_0 \subset \Phi_1 \subset \dots$ where each Φ_n is a complete interval type of $A_n = \{a_i : i < n\}$ and each Φ_n is a limit type. Let Ψ be the union of the Φ_i . This is a complete, consistent interval type over A . Moreover, if $\phi(x, \bar{a})$ isolates it in $IT(T_A)$, it also isolates it in $IT(T_{A_n})$ where n is sufficient to make $A_n \supset \bar{a}$. This violates the inductive hypothesis and completes the proof. \square

Theorem 7.12. *Suppose A is a countable (possibly infinite) set of constants and T is a typical theory. Adding these constants cannot make us go down the hierarchy in general:*

- If T is \aleph_0 -categorical, then T_A is \aleph_0 -categorical, has $n \geq 3$ countable models, or is Borel equivalent to one of \cong_1 or \cong_2 .
- If T has $n \geq 3$ countable models, then T_A has $m \geq 3$ countable models, or is Borel equivalent to one of \cong_1 or \cong_2 .
- If $T \sim_B \cong_1$, then T_A is Borel equivalent to one of \cong_1 or \cong_2 .
- If $T \sim_B \cong_2$, then $T_A \sim_B \cong_2$.
- If T is Borel complete, T_A is Borel complete.

Proof. The primary thing to note is this: if T is Borel complete, so is T_A , and vice-versa, by Lemma 7.9 and Lemma 7.10, respectively. The rest must go by cases.

If T is \aleph_0 -categorical there is nothing to prove.

If T has $n \geq 3$ countable models, we need only show T_A is not \aleph_0 -categorical. But if it were, then necessarily A is finite (else T_A would have infinitely many 1-types, the elements of A), so by Theorem 7.8, T is \aleph_0 -categorical as well.

If $T \sim_B \cong_1$, then $IT(T)$ has infinitely many limit types. Each of these types has an extension to a limit type in $IT(T_A)$; these are necessarily distinct, so T_A is Borel equivalent to either \cong_1 or \cong_2 .

If $T \sim_B \cong_2$, then T is not small. Evidently $|S_n(T)| \leq |S_n(T_A)|$ for all n , so T_A is also not small. Thus $T_A \sim_B \cong_2$.

If T is \aleph_0 -categorical there is nothing to show except the preceding. If T has $n \geq 3$ countable models, we need only show T □

By an examination of the examples given in the previous section, the reader can easily verify that all of the possibilities in the above theorem can happen. Unfortunately we *cannot* conclude that $T \leq_B T_A$ in all cases (even for finite A) since we cannot answer the following question:

Question 7.13. Suppose T is a typical theory with finitely many countable models. Let A be a countable set of parameters. Must T_A have at least as many countable models?

References

- [1] Harvey Friedman and Lee Stanley. A Borel reducibility theory for classes of countable structures. *The Journal of Symbolic Logic*, 54:894–914, 1989.
- [2] Greg Hjorth, Alexander Kechris, and Alain Louveau. Borel equivalence relations induced by actions of the symmetric group. *Annals of Pure and Applied Logic*, 92:63–112, 1998.
- [3] David Marker. The Borel complexity of isomorphism for theories with many types. *Notre Dame Journal of Formal Logic*, 48(1):93–97, 2007.
- [4] Feresiano Mwesigye and John K. Truss. Countably categorical coloured linear orders. *Mathematical Logic Quarterly*, 56(2):159–163, 2010.
- [5] Richard Rast and Davender Singh Sahota. The Borel complexity of isomorphism for O-Minimal theories. *Journal of Symbolic Logic*, Submitted August 2014.
- [6] Joseph G. Rosenstein. *Linear Orderings*. Academic Press, 1982.
- [7] Matatyahu Rubin. Theories of linear order. *Israel Journal of Mathematics*, 17(4):392–443, 1974.