# Towards improving the Database of Local Fields

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# Two-paragraph summary

The $p$-adic field section of the LMFDB tabulates degree $n$ extensions of $\mathbb{Q}_p$, including for all $n \leq 15$ and $p \leq 199$. For example, always up to isomorphism, there are 795 nonic extensions $K/\mathbb{Q}_3$ and 1823 octic extensions $K/\mathbb{Q}_2$. Interesting invariants include *visible slopes*, *hidden slopes*, and *Galois groups*.

The main framework for improvement is to **focus first on visible slopes**. Here there is a strong general theory valid for general $K/F$, not just the case $F = \mathbb{Q}_p$. It centers on **Krasner-Monge near-canonical polynomials** for totally ramified extensions $K/F$. These polynomials let one collect all extensions of a given $F$ with given visible slopes into a single parameterized family, and the dependence on $F$ is mild. The family structure then facilitates the investigation of hidden slopes and Galois groups.

# Overview

1. **Introduction**, including a tour of the database.

2. **The ramification invariant** $I$ of an extension $K/F$, as captured in the totally wild degree $p^w$ case by

$$\begin{array}{ll} \textit{heights} & \langle h_1, \ldots, h_w \rangle, \\ \textit{slopes} & [s_1, \ldots, s_w], \\ \text{or } \textit{rams} & (r_1, \ldots, r_w). \end{array}$$

3. **The set $\mathcal{I}$ of possible ramification invariants.**

4. **From ramification invariants to pictures.**

5. **From pictures to near-canonical polynomials.**

6. **Hidden slopes and Galois groups** in two sample families.

## 1.1. Notation for classifyng extensions

Let $n \in \mathbb{Z}_{\geq 1}$ and let $F$ be a field. An important problem is to describe the set $F(n)$ of isomorphism classes of separable field extensions $K/F$ of degree $n$.

Let $G$ run over conjugacy classes of transitive subgroups of $S_n$. Then Galois theory gives a natural decomposition

$$F(n) = \coprod_G F(G).$$

One would like to describe each $F(G)$ individually.

Now let $F$ be a $p$-adic field, i.e. a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, with uniformizer $\pi \in \Pi \subset \mathcal{O} \subset F$ as usual. Then every $K/F$ has a discriminant ideal $\Pi^c$, giving

$$F(n) = \coprod_G \coprod_c F(G, c).$$

The sets $F(G, c)$ are finite and one would like to describe them individually.

# 1.2. Overview of the 795 nonic 3-adic fields

There are 81 nonzero $|\mathbb{Q}_3(G,c)|$ with 22 Galois groups $G$ and 16 discriminant exponents $c$ involved. On the table, groups $G$ are sorted first by the number of cubic subfields: $\geq 2$, 1, and then 0. In the third column, $A = \mathrm{Aut}(K/\mathbb{Q}_3)$ is the centralizer of the Galois group $G$.

| $\lvert G\rvert$ | $G$ | $\lvert A\rvert$ | 0 | 9 | 10 | 12 | 13 | 15 | 16 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 9 | $9T2$ | 9 | | | | *1* | | | | | | | | | | | | |
| 18 | $9T4$ | 3 | | *2* | | *1* | | 6, *3* | 3 | | 9 | | | | | | | |
| 18 | $9T5$ | | | | | | | | | 1 | | | | | | | | |
| 36 | $9T8$ | | | | | 1 | | 2 | | 3 | 3 | | | | | | | |
| 9 | $9T1$ | 9 | **1** | | | *2* | | | | | | | | 9 | | | | |
| 18 | $9T3$ | | | | | | | | | 1 | | | | 1 | | | | 3 |
| 27 | $9T6$ | 3 | | | | *2* | | | | | | | | 6 | | | | |
| 27 | $9T7$ | 3 | | | | *1* | | | 3 | | | | | | | | | |
| 54 | $9T10$ | | | | | | | | 6 | 11 | | | | 8 | | | | 24 |
| 54 | $9T11$ | | | | | 2 | | | 1 | 8 | | | | 9 | | | | |
| 54 | $9T12$ | 3 | | | | | | | | | 9 | | 27 | | | | | |
| 54 | $9T13$ | | | *2* | | *1* | | *2* | | 3 | 3 | | 9 | | | | | |
| 81 | $9T17$ | 3 | | | | *9* | | | | 9 | | | | 18 | | | | |
| 108 | $9T18$ | | | | | | 2 | 4 | | 3 | 12 | | 18 | 9 | | | | |
| 162 | $9T20$ | 3 | | | | | 6 | 12 | | 9 | 45 | | | 27 | | | 81 | |
| 162 | $9T21$ | | | | | | | | | | | 27 | | | | 27 | | 54 |
| 162 | $9T22$ | | | *6* | | *3* | | *6* | | | | | 9 | 9 | 27 | | | |
| 324 | $9T24$ | | | | | | | | 6 | 12 | 9 | 27 | 9 | | 27 | 27 | 27 | |
| 36 | $9T9$ | | | | | 1 | | | 1 | | | | | | | | | |
| 72 | $9T14$ | | | | 1 | | | | | 3 | | | | | | | | |
| 72 | $9T16$ | | | | 1 | | | | | 3 | | | | | | | | |
| 144 | $9T19$ | | | 2 | | 2 | 2 | 6 | 2 | | 6 | | | | | | | |

**Bold=Unramified** *Italic=partially ramified* Regular=totally ramified

# 1.3. Tour of the $p$-adic section of the LMFDB

As said earlier, the LMFDB currently contains the sets $\mathbb{Q}_p(n)$ for all $p \leq 199$ and all $n \leq 15$, with information on each field $K$. E.g., the field $K$ labeled 3.9.21.20 is defined by $x^9 + 12x^6 + 18x^4 + 3$.

The degree $n$ of any field factors as $utp^w$ with $u$ and $t$ its unramified and tamely ramified parts. There are $w$ wild slopes $\hat{s}_1 \leq \cdots \leq \hat{s}_w$, as introduced in the next section. The "slope content" of our example (not directly given in the LMFDB) is $[[\hat{s}_1, \hat{s}_2]]_t^u = [[2, \frac{17}{6}]]_1^1 = [[2, \frac{17}{6}]]$.

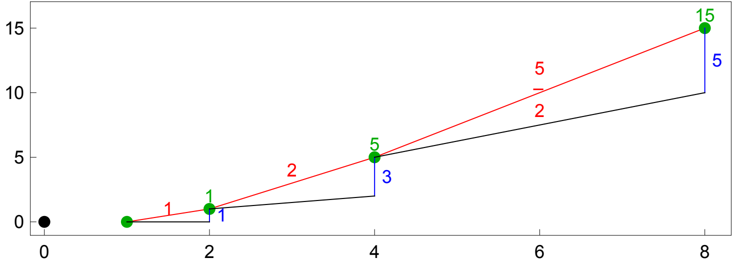
The LMFDB does however always gives the much-harder-to-calculate slope content of the Galois closure $L$. In our example $G = 9T24$ has $324 = 2^2 3^4$ elements and the slope content of $L$ is

$$\left[\left[\frac{3}{2}, \mathbf{2}, \frac{5}{2}, \mathbf{\frac{17}{6}}\right]\right]_2^2.$$

At this level, the wild slopes are breaks in the Artin upper numbering of the ramification filtration on $G$. They consist of the wild slopes already **visible** in $K$, and also some more *hidden* slopes.

# 2.1. The canonical filtration of a $p$-adic extension

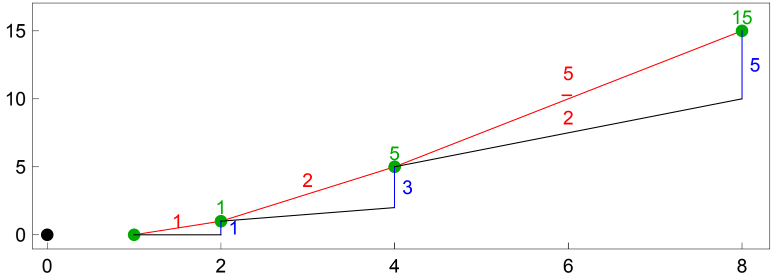
Any $K/F \in F(n)$ has a canonical filtration obtained by climbing from $F$ to $K$ via suitable minimally ramified subextensions. To focus on the main phenomena, we first restrict to $u = t = 1$ so $n = p^w$.




As we'll see, for any 2-adic field $F$, the picture arises from many octic extensions $K/F$, e.g. from 32 in the case $F = \mathbb{Q}_2$. The filtration takes the form

$$F = K_0 \subset K_1 \subset K_2 \subset K_3 = K$$

with each $[K_i : K_{i-1}] = 2$.




The numerical invariants are captured in three equivalent ways:

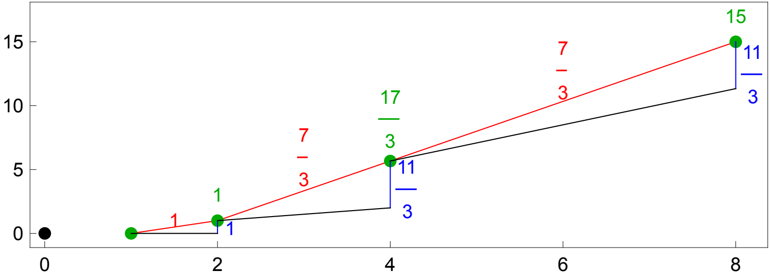
$$\begin{array}{lrcl} \textit{Heights} & \langle\langle H_1, H_2, H_3\rangle\rangle & = & \langle\langle 1, 5, 15\rangle\rangle, \\ \textit{slopes} & [s_1, s_2, s_3] & = & [1, 2, \frac{5}{2}], \\ \text{or } \textit{Rams} & ((R_1, R_2, R_3)) & = & ((1, 3, 5)). \end{array}$$

We switch to focusing on wild ramification only, writing

$$\mathrm{cond}(K_i/K_j) = c(K_i/K_j) - \deg(K_i/K_j) + 1.$$

The definitions are then $H_i = \mathrm{cond}(K_i/K_0)$ and $R_i = \mathrm{cond}(K_i/K_{i-1})$. Accordingly, we also switch from the website's Artin-Fontaine slopes $\hat{s}_i$ to the Serre-Swan slopes $s_i$ via $\hat{s}_i = s_i + 1$.

When the canonical filtration has fewer than $w$ steps we keep a uniform notation that refers to nonexistent subfields:




Here the filtration is $F = K_0 \subset K_1 \subset K_2 \subset K_3 = K$. The invariants are

$$I = \langle\langle 1, \frac{17}{3}, 15 \rangle\rangle = \left[1, \frac{7}{3}, \frac{7}{3}\right] = \left(\left(1, \frac{11}{3}, \frac{11}{3}\right)\right).$$

When $K_i$ is nonexistent, $H_i$ is no longer forced to be integral. Similarly too, a ram $R_j$ repeated $\rho$ times is now only forced to have a denominator dividing $P_\rho(p) := p^{\rho-1} + p^{\rho-2} + \cdots + p + 1$.

# 2.2.Including unramified and tame steps

A general extension $K/F$ contains its canonical tower $F \overset{u}{\subset} F_{\mathrm{un}} \overset{t}{\subset} K_0 \subset \cdots \subset K$ with the $\cdots$ representing wild steps as before. Its ramification invariant is a symbol $I = W_t^u$, with $W$ expressible as a $w$-tuple in three equivalent ways, as before. The more general pictures now include $t$ but not $u$. An example for $p = 2$ is

In this example, $I = \langle\langle H_1, H_2\rangle\rangle_3^u = [s_1, s_2]_3^u = ((R_1, R_2))_3^u$. For every base $F$ and every positive integer $u$ there are fields with invariant $I$. For example, for $F = \mathbb{Q}_2$ and $u = 1$ there are ?? such fields.

# 2.3. Conversions formulas among $H_k$, $R_k$, and $s_k$

The three ways of describing a ramification invariant $I$ interrelate via

$$H_k \overset{2a}{=} t\sum_{j=1}^{k}\phi(p^j)s_j, \qquad H_k \overset{2b}{=} \sum_{j=1}^{k}p^{k-j}R_j,$$

$$s_k \overset{1a}{=} \frac{H_k - H_{k-1}}{t\phi(p^k)}, \qquad ts_k \overset{3}{=} \frac{R_k}{\phi(p^k)} + \sum_{j=1}^{k-1}\frac{R_j}{p^j},$$

$$R_k \overset{1b}{=} H_k - pH_{k-1}, \quad R_k \overset{4}{=} t\phi(p^k)s_k - t\phi(p)\sum_{j=1}^{k-1}\phi(p^j)s_j.$$

1a captures the definition of slope as rise/run. 1b emphasizes that $R_k$ measures how $K_k/K_{k-1}$ is more ramified than the unramified algebra $K_{k-1}^p/K_{k-1}$. 2a and 2b are their inversions, each intuitive in its own right. $3 = 1a \circ 2b$ and $4 = 1b \circ 2a$ are less directly intuitive.

# 2.4. Conversions formulas among $h_k$, $r_k$, and $s_k$

It is often better to scale down quantities associated with $K_k/F$ by the ramification degree $[K_k : F_{\mathrm{un}}] = tp^k$, writing $h_k = H_k/(tp^k)$ and $r_k = R_k/(tp^k)$. With the abbreviation $\epsilon = 1/(p-1)$, one has

$$h_k \stackrel{2a}{=} \frac{1}{1+\epsilon} \sum_{j=1}^{k} \frac{s_j}{p^{k-j}}, \quad h_k \stackrel{2b}{=} \sum_{j=1}^{k} r_j,$$

$$s_k \stackrel{1a}{=} (1+\epsilon)h_k - \epsilon h_{k-1}, \qquad s_k \stackrel{3}{=} (1+\epsilon)r_k + \sum_{j=1}^{k-1} r_j,$$

$$r_k \stackrel{1b}{=} h_k - h_{k-1}, \qquad r_k \stackrel{4}{=} \frac{s_k}{1+\epsilon} - \frac{p-1}{1+\epsilon} \sum_{j=1}^{k-1} \frac{s_j}{p^{k-j}}.$$

The integrality properties of $H_k$ and $R_k$ are obscured, but now the quantities are roughly on similar scales, with $r_k \in [0, te]$ as we'll discuss, and thus $h_k \in [0, kte]$ and $s_k \in [0, (k+\epsilon)te]$. In particular, (2b) and (3) give the nice formula $s_k = h_k + \epsilon r_k$.

# 3.1. Allowed $R_1 = \cdots = R_\rho$ in one-step extensions

The set of $R_1$ arising in one-step degree $p^\rho$ extensions of a field $F$ is very simple and depends only on the ramification index $e = \mathrm{ord}_\Pi(p)$!

In the case of $e = \infty$, i.e. function fields, it is

$$\mathcal{R}_{p,\infty,\rho} = \frac{\mathbb{Z}_{\geq 1} - p\mathbb{Z}_{\geq 1}}{P_\rho(p)} = \frac{\mathbb{Z}_{\geq 1} - p\mathbb{Z}_{\geq 1}}{p^{\rho-1} + p^{\rho-2} + \cdots p + 1}.$$

In the case $e < \infty$, i.e. extensions of $\mathbb{Q}_p$, it is

$$\mathcal{R}_{p,e,\rho} = (\mathcal{R}_{p,\infty,\rho} \cap (0, pe)) \cup \begin{cases} \{pe\} & \text{if } \rho = 1, \\ \{\ \} & \text{if } \rho > 1. \end{cases}$$

The conversion formulas are trivial in the one-step extension context, $s_1 = \epsilon R_1$ and $H_\rho = P_\rho(p) R_1$.

# 3.2. Example: One-step extensions for $p = 2$

From the previous slide for multiplicities $\rho = 1$ and $\rho = 2$,

$$\begin{array}{rcl} \mathcal{R}_{2,\infty,1} & = & \{ \quad 1, \quad \mid \quad 3, \quad \ldots \quad \}, \\ \mathcal{R}_{2,\infty,2} & = & \{ \tfrac{1}{3}, \quad 1, \quad \tfrac{5}{3}, \mid \tfrac{7}{3}, \quad 3, \quad \ldots \quad \}. \end{array}$$

The cutoff for $e = 1$ is indicated and so the corresponding sets are

$$\begin{array}{rcl} \mathcal{R}_{2,1,1} & = & \{ \quad 1, \quad 2 \}, \\ \mathcal{R}_{2,1,2} & = & \{ \tfrac{1}{3}, \quad 1, \quad \tfrac{5}{3} \}. \end{array}$$

Over $\mathbb{Q}_2$, the quadratic fields for the Rams 1 and 2 are respectively $\mathbb{Q}_2(\sqrt{d})$ for $d \in \{-1, -1*\}$ and $d \in \{2, 2*, -2, -2*\}$, with say $* = 5$. The quartic fields appear on the LMFDB as

| $(R_1, R_2) = (1/3, 1/3)$ | $(R_1, R_2) = (1, 1)$ | $(R_1, R_2) = (5/3, 5/3)$ |
|---|---|---|
| $x^4 + 2x + 2$ $[4/3, 4/3]_3^2$ $S_4$ | $x^4 + 2x^3 + 2x^2 + 2$ $[2, 2]^3$ $A_4$ | $x^4 + 4x^2 + 4x + 2$ $[8/3, 8/3]_3^2$ $S_4$ |
| | $x^4 + 2x^3 + 2$ $[2, 2]^2$ $D_4$ | $x^4 + 4x^2 + 2$ $[8/3, 8/3]_3^2$ $S_4$ |
| | $x^4 + 2x^3 + 6$ $[2, 2]^2$ $D_4$ | |

# 3.3. Occurring invariants $I$ in multistep extensions

Fix a ground field $F$ with $\mathrm{ord}_\Pi(p) = e$ and consider its totally ramified extensions of degree $p^w$.

Break up this set of extensions according to their multiplicity vector $m = (m_1, \ldots, m_k)$. Let $M_i = \sum_{j=1}^{i} m_j$. Let $\mathcal{I}_{p,e,m}$ be the set of occurring invariants $I$. Then necessarily $\mathcal{I}_{p,e,m}$ is in

$$\widehat{\mathcal{I}}_{p,e,m} = \{(\overbrace{R(1), \ldots, R(1)}^{m_1}, \ldots, \overbrace{R(k), \ldots, R(k)}^{m_k}) : R(i) \in \mathcal{R}_{p,ep^{M_{i-1}},m_i}\}.$$

The index gymnastics hide a simple Cartesian product! E.g. $\widehat{\mathcal{I}}_{p,e,(3,2)}$ consists of tuples $(R_1, R_1, R_1, R_4, R_4)$ with $(R_1, R_4) \in \mathcal{R}_{p,e,3} \times \mathcal{R}_{p,p^3e,2}$.

The set $\mathcal{I}_{p,e,m}$ is then the subset of $\widehat{\mathcal{I}}_{p,e,m}$ such that the list of rams $R(1), \ldots, R(k)$, or equivalently the list of slopes $s(1), \ldots, s(k)$, is strictly increasing. The elementary nature of $\mathcal{I}_{p,e,w} = \coprod_{m \vdash w} \mathcal{I}_{p,e,m}$ is illustrated by the next slide by $\mathcal{I}_{3,1,2} = \mathcal{I}_{3,1,(1,1)} \coprod \mathcal{I}_{3,1,(2)}$.

# 3.4. Invariants for tot. ram. nonic 3-adic fields

Rams $(R_1, R_2)$ and slopes $[\hat{s}_1, \hat{s}_2]$ with $m = (1,1)$ before $m = (2)$:

| | | | |
|---|---|---|---|
| $(1,1)$ | $(2,1)$ | $(3,1)$ | $(0.25, 0.25)$ |
| $(1,2)$ | $(2,2)$ | $(3,2)$ | $(0.50, 0.50)$ |
| $(1,4)$ | $(2,4)$ | $(3,4)$ | $(1.00, 1.00)$ |
| $(1,5)$ | $(2,5)$ | $(3,5)$ | $(1.25, 1.25)$ |
| $(1,7)$ | $(2,7)$ | $(3,7)$ | $(1.75, 1.75)$ |
| $(1,8)$ | $(2,8)$ | $(3,8)$ | $(2.00, 2.00)$ |
| $(1,9)$ | $(2,9)$ | $(3,9)$ | |
| | | | $(2.50, 2.50)$ |
| | | | $(2.75, 2.75)$ |

| | | | |
|---|---|---|---|
| $[1.5, 1.50]$ | $[2, 1.83]$ | $[2.5, 2.16]$ | $[1.125, 1.125]$ |
| $[1.5, 1.66]$ | $[2, 2.00]$ | $[2.5, 2.33]$ | $[1.250, 1.250]$ |
| $[1.5, 2.00]$ | $[2, 2.33]$ | $[2.5, 2.66]$ | $[1.500, 1.500]$ |
| $[1.5, 1.16]$ | $[2, 2.50]$ | $[2.5, 2.83]$ | $[1.625, 1.625]$ |
| $[1.5, 2.50]$ | $[2, 2.83]$ | $[2.5, 3.16]$ | $[1.875, 1.875]$ |
| $[1.5, 2.66]$ | $[2, 3.00]$ | $[2.5, 3.33]$ | $[2.000, 2.000]$ |
| $[1.5, 2.83]$ | $[2, 3.16]$ | $[2.5, 3.50]$ | |
| | | | $[2.250, 2.250]$ |
| | | | $[2.375, 2.375]$ |

The Cartesian structure of $\widehat{\mathcal{I}}_{3,1,(1,1)}$ is visible in Rams as $\{1,2,3\} \times \{1,2,\ 4,5,\ 7,8,9\}$, but obscured in slopes.

The Cartesian structure on the $(1,1)$ part is still visible in the total masses to the right, where $K/F$ has mass $|1/\mathrm{Aut}(K/F)|$.

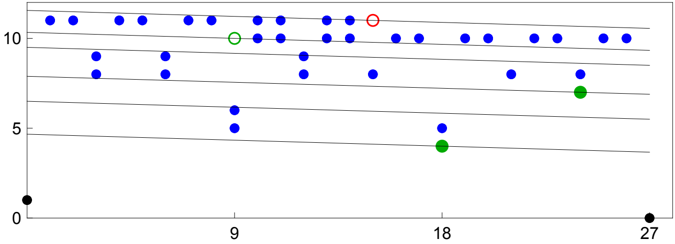
| | | | |
|---|---|---|---|
| | | | 2 |
| 4 | | | 2 |
| 12 | 12 | 18 | 6 |
| 12 | 12 | 18 | 2 |
| 36 | 36 | 54 | 6 |
| 36 | 36 | 54 | 6 |
| 54 | 54 | 81 | |
| | | | 6 |
| | | | 6 |

# 4.1. From an invariant $I$ to its picture

A invariant $I = W_t^u$ with $W = h = s = r$ determines a picture in the window $[0, tp^w] \times [0, \hat{s}_w]$ that does not depend on $u$. For example,

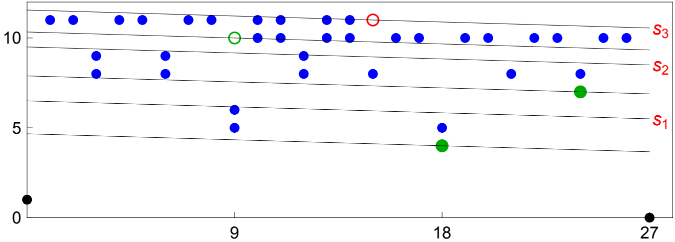
$$W = \langle\langle 11, 62, 252 \rangle\rangle = [5.5, 8.5, 10.\overline{5}] = ((11, 29, 66))$$

with $t = 1$ determines




The points in the $i$-column will give constraints on the coefficient of the $x^i$ term in Eisenstein polynomials, with $x^{tp^w}$ corresponding to the lower right point and the constant term to the lower left point and points above it.

# 4.2. The recipe for drawing the $I$-picture, part 1

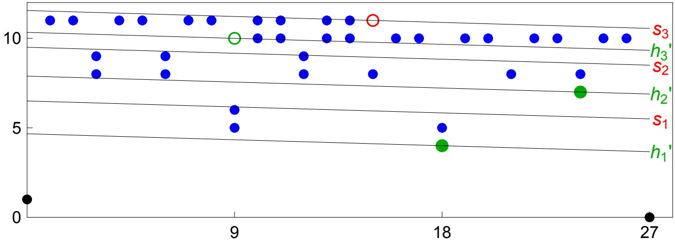



There are $w$ closed bands; in general they may overlap, but they are not overlapping in this example. The top edge of the $k^{\mathrm{th}}$ band $B_k$ decreases from $(0, \hat{s}_k)$ to $(tp^w, s_k)$. All drawn points $(i, j)$ are integral and, besides $(0, 1)$ and $(tp^w, 0)$, occur only in the bands. Then an integral point $(i, j) \in B_k$ is drawn iff it is exactly divisible by $p^{w-k}$ or it's on the boundary. It is drawn solidly iff the first condition holds. If all slopes are different, there is exactly one point on each lower edge, drawn as $\circ$ or $\bullet$. There is at most one point on the upper edge, drawn as $\circ$.

# 4.3. The recipe for drawing the $I$-picture, part 2

Denote the lower case heights $h_k = H_k/(tp^k)$ and the lower case rams $r_k = R_k(tp^k)$ using single delimiters. So the current example becomes

$$I = \langle 3\tfrac{2}{3}, 6\tfrac{8}{9}, 9\tfrac{1}{3} \rangle = [5.5, 8.5, 10.\overline{5}] = (3\tfrac{2}{3}, 9\tfrac{2}{3}, 22).$$



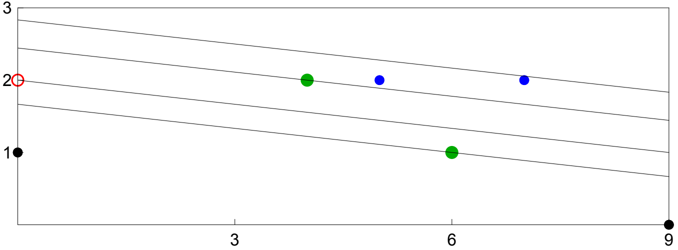

A band has a green point if and only if it is the highest band for its slope. Then it is at is at $(i,j) = (\langle h_k \rangle tp^w, \lceil h_k \rceil)$ so that e.g. the first $\bullet$ comes from $3\frac{2}{3}$ and is at $(i,j) = (\frac{2}{3} \cdot 27, 4) = (18, 4)$. Equivalently, the lower edge of $B_k$ goes through $(tp^w, h_k)$. Also, $B_k$ contains exactly $\lfloor r_k p^{k-1} \rfloor$ $\bullet$'s, and then also a $\bullet$ iff $r_k p^{k-1}$ is nonintegral.

# 5.1. The Krasner-Monge parametrized polynomial

Index a point $(i, j)$ by the integer $m = tp^w(j-1) + i$, so that the $k^{\mathrm{th}}$ $\bullet$ or $\circ$ becomes $m = tp^w h_k$. Introduce variables $a_m$, $b_m$ and $c_m$ for drawn points in bands of the form $\bullet$, $\bullet$, and $\circ$. Form the polynomial

$$\pi + \sum_{(i,j) \text{ as } \bullet} a_m \pi^j x^i + \sum_{(i,j) \text{ as } \bullet} b_m \pi^j x^i + \sum_{(i,j) \text{ as } \circ} c_m \pi^j x^i + x^{tp^w}$$

Our earlier example $I = [1, \frac{11}{6}] = (\frac{2}{3}, 2\frac{1}{3}) = \langle \frac{2}{3}, 1\frac{4}{9} \rangle$ yields




For $\pi = 3$, it's $(3 + 9c_9) + 9a_{13}x^4 + 9b_{14}x^5 + 3a_6x^6 + 9b_{16}x^7 + x^9$.

# Notation for the Krasner-Monge theorem

Let $F$ be a $p$-adic field with residue field $\mathbb{F}_q$ with $q = p^f$.

For $d$ a divisor of $f$, the additive map

$$\mathbb{F}_q \to \mathbb{F}_q : k \mapsto k^{p^d} - k$$

has kernel $\mathbb{F}_{p^d}$ and so image $T_d \subset \mathbb{F}_q$ of index $p^d$.

Choose a uniformizer $\pi$ and a lift $\kappa \subset \mathcal{O}$ of $\mathbb{F}_{p^f}$ . Require $0 \in \kappa$ and write $\kappa^\times = \kappa - \{0\}$. For each divisor $d$ of $f$, choose a lift $\kappa_d \subset \kappa$ of $\mathbb{F}_q/T_d$, so that $|\kappa_d| = p^d$ and $\kappa_f = \kappa$. For $F = \mathbb{Q}_p$, we always just take $\pi = p$ and $\kappa = \{0, 1, \ldots, p-1\}$.

For a ramification invariant $I$, let

- $\alpha$ be its number of $\bullet$'s;
- $\beta$ be its number of $\bullet$'s;.
- $\gamma = \sum_m \gcd(\rho(m), f)$ where $m$ runs over indices of $\circ$'s and $\rho(m)$ it the number of times the corresponding slope is repeated.

# Krasner-Monge theorem

## Theorem

*Let $F$ be a $p$-adic field with $e, f \in \mathbb{Z}_{\geq 1}$ as usual and chosen $\pi$ and $\kappa_d$ as on the previous slide. Let $I \in \mathcal{I}_{p,e}(tp^w)$ be a possible ramification invariant for totally ramified degree $tp^w$ extensions of $F$. Consider the polynomials in the corresponding Krasner-Monge family*

$$\pi + \sum_{(i,j) \text{ as } \bullet} a_m \pi^j x^i + \sum_{(i,j) \text{ as } \bullet} b_m \pi^j x^i + \sum_{(i,j) \text{ as } \circ} c_m \pi^j x^i + x^{tp^w}$$

*with $a_m \in \kappa^\times$, $b_m \in \kappa$, and $c_m \in \kappa_{\gcd(\rho(m),f)}$. Then the corresponding extensions are in $F(I)$, with each $K$ represented $\frac{p^\gamma}{|\mathrm{Aut}(K/F)|}$ times.*

## Corollary

*The total number of extensions in $F(I)$ is $\geq (q-1)^\alpha q^\beta$, with equality if $\gamma = 0$.*

## 6.1 The case $I = [\hat{s}_1, \hat{s}_2] = [2, \frac{17}{6}]$ over $\mathbb{Q}_3$

The database says there are 36 fields falling in four packets of nine. As said before, the family is

$$f(a_6, a_{13}, b_{14}, b_{16}, c_9, x) =$$
$$(3 + 9c_9) + 9a_{13}x^4 + 9b_{14}x^5 + 3a_6x^6 + 9b_{16}x^7 + x^9,$$

Since there is just one $c$ and $f = 1$, the ambiguity parameter is $\gamma = 1$ and each field $K$ has $p^\gamma = 3$ near-canonical defining polynomials. The ambiguity is easily resolved by setting a parameter to 0 and the packets are cleanly described:

| | |
|---|---|
| $f(1, 2, 0, b_{16}, c_9, x)$ | gives $9T13$ and hidden slopes $[5/2]_2$ |
| $f(1, 1, b_{14}, b_{16}, 0, x)$ | gives $9T18$ and hidden slopes $[5/2]_2^2$ |
| $f(2, 2, 0, b_{16}, c_9, x)$ | gives $9T22$ and hidden slopes $[3/2, 5/2]_2$ |
| $f(2, 1, b_{14}, b_{16}, 0, x)$ | gives $9T24$ and hidden slopes $[3/2, 5/2]_2^2$ |

# 6.2 The case $I = [\hat{s}_1, \hat{s}_2] = [\frac{5}{2}, \frac{17}{6}]$ over $\mathbb{Q}_3$

The database says that in this case there are 18 fields falling into two packets of nine. The Krasner-Monge family is

$$g(a_{14}, b_{12}, b_{16}, x) = 3 + 9b_{12} + 9a_{14}x^5 + 9b_{16}x^7 + x^9$$

Defining polynomials are in this case unique and

$$\begin{array}{ll} g(2, b_{12}, b_{16}, x) & \text{gives } 9T11 \text{ and hidden slopes } [2]_2 \\ g(1, b_{12}, b_{16}, x) & \text{gives } 9T18 \text{ and hidden slopes } [2]_2^2 \end{array}$$

In general, resolvent constructions should have nice descriptions via the universal families. For example, $9T13$ from the previous slide and $9T11$ are the same abstract group. The bijection between

- the nine $9T13$ fields defined by $f(1, 2, 0, b_{16}, c_9, x)$ and
- the nine $9T11$ fields defined by $g(2, b_{12}, b_{16}, x)$

is given by $c_9 = b_{12}$ and $b_{16} = b_{16} + 1 - {b_{12}}^2$.

## 6.3 The case $I = [\hat{s}_1, \hat{s}_2] = [3/2, \frac{8}{3}]$ over $\mathbb{Q}_3$

The database gives five types of fields. The family is

$$f(a_3, a_{11}, b_{13}, b_{14}, c_{15}) = \\ 3 + 9x^2 a_{11} + 3x^3 a_3 + 9x^4 b_{13} + 9x^5 b_{14} + 9x^6 c_{15} + x^9$$

The five types are

| # | $\mu$ | | |
|---|---|---|---|
| 9 | 3 | $f(1, 2, b_{13}, b_{13}+2, c_{15}, x)$ | gives $9T12$ and h.s. $[5/2]_2$ |
| 18 | 6 | $f(1, 2, b_{13}, b_{13}+ {\scriptstyle 0 \atop 1}, c_{15}, x)$ | gives $9T20$ and h.s $[5/2]_2^3$ |
| 9 | 9 | $f(2, 2, b_{13}, b_{14}, \star, x)$ | gives $9T18$ and h.s. $[3/2]_2^2$ |
| 27 | 9 | $f(2, 1, b_{13}, b_{14}, c_{15}, x)$ | gives $9T20$ and h.s $[3/2, 5/2]_2$ |
| 9 | 9 | $f(1, 1, b_{13}, b_{14}, \star, x)$ | gives $9T24$ and h.d. $[3/2, 2]_2^2$ |

Here $\star$ can be any element of $\{0, 1, 2\}$ without changing the field. Otherwise, different parameters give different fields.

# Commented main references

## Much of this material has origin in:

M. Krasner, *Sur la primitivité des corps p-adiques*, Mathematica (Cluj) 13 (1937) 72–191.

## Krasner's results were modernized in:

P. Deligne, *Les corps locaux de caractéristique p, limites de corps locaux de caractéristique 0*, in Representations of Reductive Groups over a Local Field (1984), pp. 119–157.

## The original database from which the LMFDB database grew:

J. W. Jones and D. P. Roberts, *A database of local fields*, J. Symbolic Comput. 41(1) (2006) 80–97.

## A modernization which, like Krasner, emphasizes polynomials:

M. Monge, *A family of Eisenstein polynomials generating totally ramified extensions, identification of extensions and construction of class fields*. Int. J. Number Theory 10 (2014), no. 7, 1699–1727.