Chapter 13 **(Oechslin: 8.3)**

Hahn's *Weltmaschine* in Gotha (1780)

This chapter is a work in progress and is not yet finalized. See the details in the introduction. It can be read independently from the other chapters, but for the notations, the general introduction should be read first. Newer versions will be put online from time to time.

13.1 Introduction

This *Weltmaschine* (figure 13.1) was constructed in 1780 by Philipp Matthäus Hahn (1739-1790) and his brother David Georg (1747-1814)[1] and is located in the Friedenstein Palace in Gotha (Germany).[2]

The *Württembergische Landesbibliothek* in Stuttgart keeps a drawing (figure 13.1) which is related to the Gotha *Weltmaschine*.[3] This drawing may have been made by Johann Christoph Schuster (1759-1823), Hahn's collaborator and future son-in law. It is possible that this drawing was the basis of several machines, the present one, but also another one of which some elements were auctioned in 2023 (figures 13.1 and 13.1).[4]

The Gotha machine is made of a large base in three parts, with a dial in the front, a tellurium on the left side, and an orrery on the right side. Above the central part, there is a celestial globe with an internal work.

This is a rather complete and complex mechanism, and it is to be noted that the orrery was copied in the "astro-skeleton" clock designed by Mark Frank and completed in 2021.

[1]For biographical details on Hahn, I refer the reader to the chapter on the Ludwigsburg *Weltmaschine* (Oechslin 8.1).

[2]For the history of this machine, see especially the 1989 exhibition catalogue [4, p. 393-395, pl .13]. The Gotha machine is mentioned by Zinner [5, p. 354]. Oechslin also described Hahn's *Weltmaschinen*, see especially [2, p. 55-57].

[3]The drawing is reproduced in colour in the 1989 exhibition catalogue [4, pl .5].

[4][3]

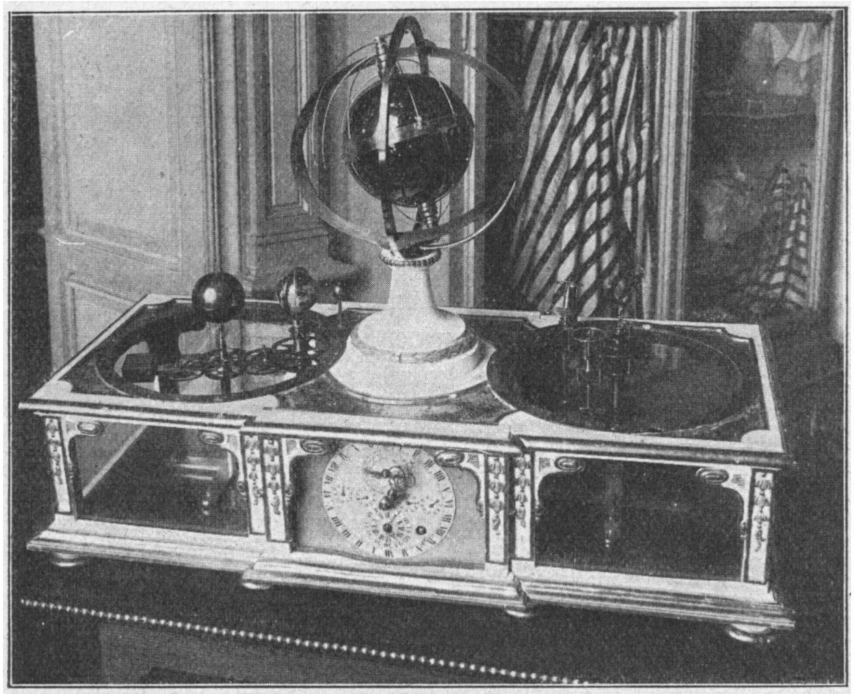

Figure 13.1: Hahn's clock in Gotha. The tellurium is on the left and the orrery on the right. (source: [1])

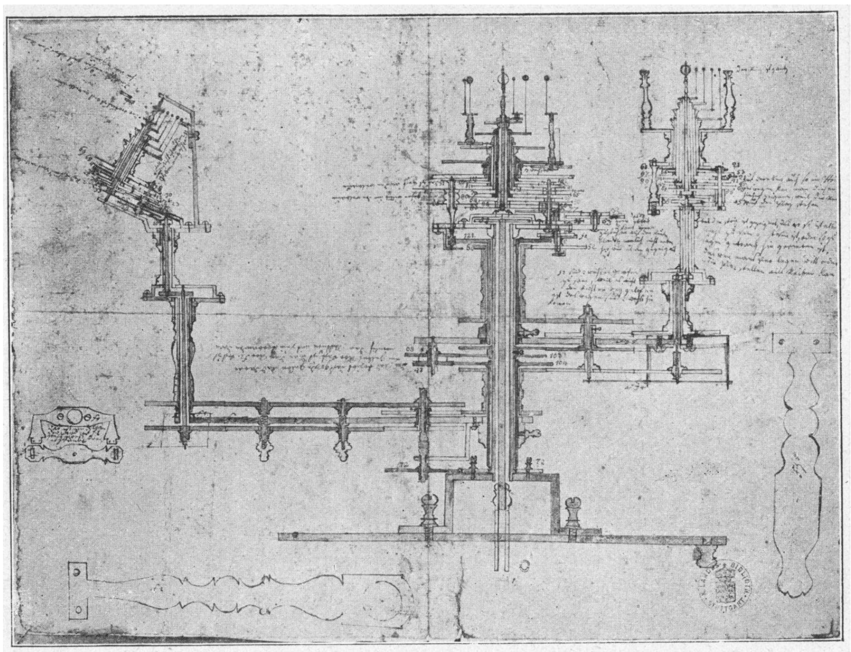

Figure 13.2: Cross-section of a Copernican orrery similar to the Gotha machine, possibly by Johann Christoph Schuster (1759-1823). (source: [1])

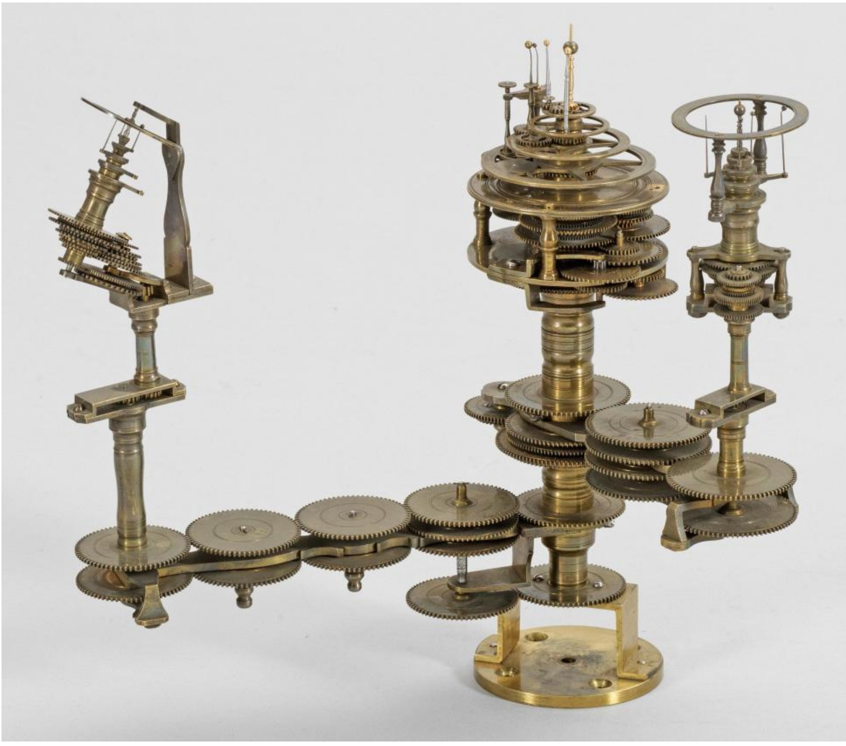

Figure 13.3: The orrery gears auctioned in 2023. (source: Kunstauktionshaus Schloss Ahlden, 6 May 2023, lot 684)

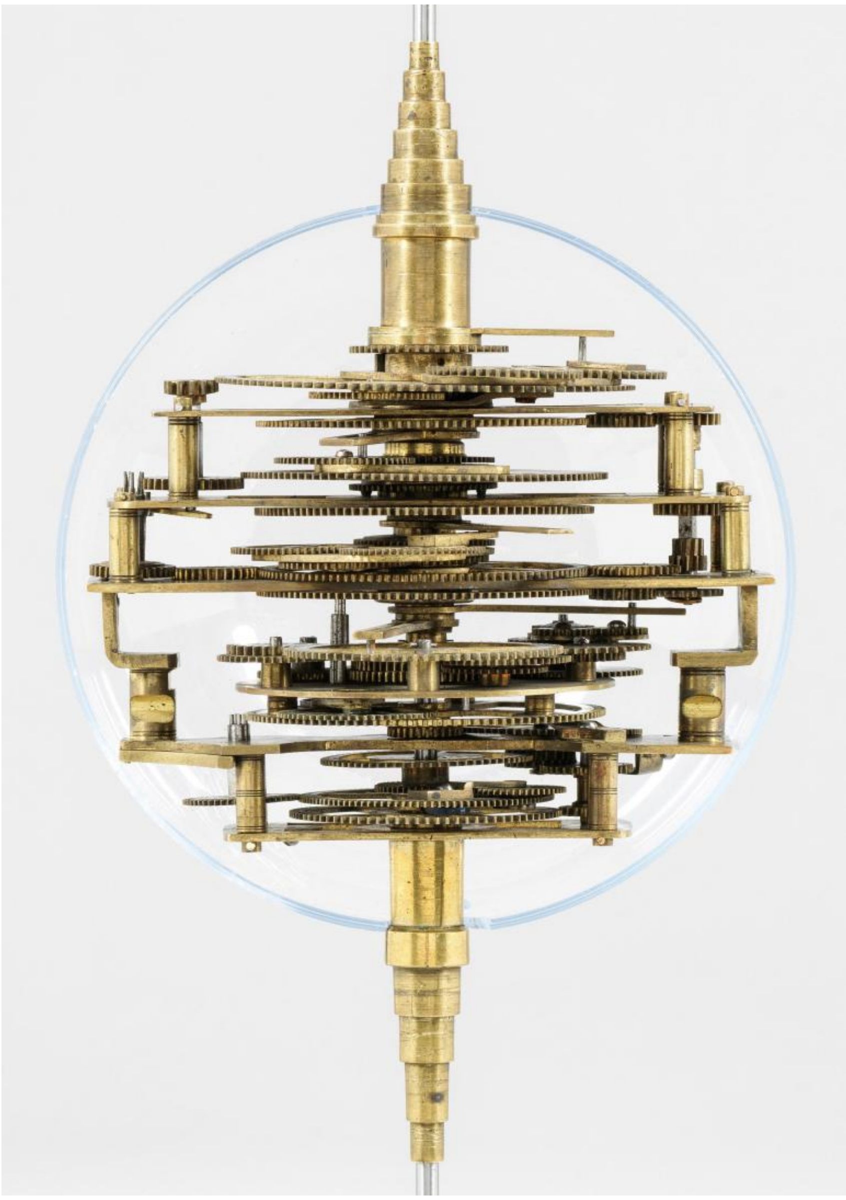

Figure 13.4: The globe gears auctioned in 2023. (source: Kunstauktionshaus Schloss Ahlden, 6 May 2023, lot 684)

There may be two errors in this machine, namely in the gears for the satellites of Saturn, and in the celestial globe for the lunar apsides and nodes. These errors are detailed below.

Oechslin was the first to describe this machine in 1996, but he actually didn't have the opportunity to disassemble it. He based his analysis on archives provided to him by Helmut Sändig and R. Lang from Dresden [2, p. 5].

13.2 The going work

Oechslin does not show the details of the going work, but his drawing shows that one hand on arbor 1 makes a turn in 24 hours.

$$\mathrm{T}_1^0 = 1 \tag{13.1}$$

The rotation is clockwise seen from the front, but from the left on Oechslin's drawing. This motion is then transfered to the central vertical arbor 5:

$$\begin{align}
\mathrm{V}_5^0 &= \mathrm{V}_1^0 \times \left(-\frac{76}{79}\right) \times \left(-\frac{57}{28}\right) \times \left(-\frac{28}{58}\right) \times \left(-\frac{35}{33}\right) \tag{13.2}\\
&= (-1) \times \frac{25270}{25201} = -\frac{25270}{25201} \tag{13.3}\\
\mathrm{P}_5^0 &= -\frac{25201}{25270} = -0.9972\ldots \text{ days} = -86164.0838\ldots \text{ seconds} \tag{13.4}\\
&= -23 \text{ h } 56 \text{ m } 4.083 \text{ s} \tag{13.5}
\end{align}$$

This is the velocity of arbor 5 seen from above. This arbor moves clockwise at the rate of the sidereal day. This ratio is the same as the one used on the globe clock located in Darmstadt. The motion of this arbor is transfered to the celestial globe above.

The motion of arbor 5 is also transfered to two other vertical arbors/tubes, namely tube 67 (tellurium) and arbor 7 (orrery). We have

$$\begin{align}
\mathrm{V}_{67}^0 &= -\mathrm{V}_5^0 \tag{13.6}\\
\mathrm{P}_{67}^0 &= \frac{25201}{25270} \tag{13.7}\\
\mathrm{V}_7^0 &= \mathrm{V}_{67}^0 \tag{13.8}\\
\mathrm{P}_7^0 &= \mathrm{P}_{67}^0 \tag{13.9}
\end{align}$$

These two arbors/tubes rotate counterclockwise in one sidereal day.

We now basically have only vertical arbors to examine, except for the tilted axes of the Earth and of Saturn.

13.3 The tellurium

In the tellurium, the central arbor 70 is fixed and supports the Sun as well as two fixed wheels with 78 and 104 teeth. The rotating tube 67 is used to move the rotating frame 71 supporting the Earth-Moon system. This is done indirectly. In the rotating reference frame, we have

$$\mathrm{V}_{70}^{71} = \mathrm{V}_{67}^{71} \times \left(-\frac{119}{118}\right) \times \left(-\frac{4}{85}\right) \times \left(-\frac{6}{104}\right) = -\mathrm{V}_{67}^{71} \times \frac{21}{7670} = \mathrm{V}_{71}^{67} \times \frac{21}{7670} \tag{13.10}$$

This relation is in the form

$$\mathrm{V}_a^c = \mathrm{V}_c^b \times x \tag{13.11}$$

where the motions of a and b are known, the value of x is known, and the motion of c is sought.

From this equation, we easily derive

$$\mathrm{V}_b^c = \mathrm{V}_b^a \times \frac{1}{1+x} \tag{13.12}$$

and

$$\mathrm{V}_a^c = \mathrm{V}_a^b \times \frac{x}{1+x} \tag{13.13}$$

or

$$\mathrm{V}_c^a = \mathrm{V}_b^a \times \frac{x}{1+x} \tag{13.14}$$

In our case, taking $a = 70$, $b = 67$ and $c = 71$, $x = \frac{21}{7670}$, we have

$$\mathrm{V}_{71}^{70} = \mathrm{V}_{71}^{0} = \mathrm{V}_{67}^{70} \times \frac{21}{7691} = \frac{25270}{25201} \times \frac{21}{7691} = \frac{530670}{193820891} \tag{13.15}$$

$$\mathrm{P}_{71}^{70} = \frac{193820891}{530670} = 365.2380\ldots \text{ days} \tag{13.16}$$

This is an approximation of the tropical year. The same value is given by Oechslin.

Now, Hahn obtains the motion of the frame 73 supporting Earth's axis. We expect this axis to be non rotating (only translating) in the absolute frame, so that this axis does always keep the same orientation.

$$\mathrm{V}_{73}^{71} = \mathrm{V}_{70}^{71} \times \left(-\frac{78}{78}\right) \times \left(-\frac{78}{78}\right) = \mathrm{V}_{70}^{71} \tag{13.17}$$

$$\mathrm{V}_{73}^{0} = \mathrm{V}_{73}^{71} + \mathrm{V}_{71}^{0} = \mathrm{V}_{70}^{71} + \mathrm{V}_{71}^{0} = \mathrm{V}_{70}^{0} = 0 \tag{13.18}$$

The motion of the Earth on its axis 77 in the absolute frame is obtained as follows:

$$\mathrm{V}^0_{77} = \mathrm{V}^0_{75} \times \left(-\frac{12}{8}\right) \times \left(-\frac{8}{12}\right) = \mathrm{V}^0_{75} \tag{13.19}$$

$$\mathrm{V}^0_{75} = \mathrm{V}^{71}_{75} + \mathrm{V}^0_{71} = \mathrm{V}^{71}_{67} \times \left(-\frac{78}{78}\right) \times \left(-\frac{78}{78}\right) + \mathrm{V}^0_{71} \tag{13.20}$$

$$= \mathrm{V}^{71}_{67} + \mathrm{V}^0_{71} = \mathrm{V}^0_{67} = \frac{25270}{25201} \tag{13.21}$$

$$\mathrm{P}^0_{75} = \frac{25201}{25270} \tag{13.22}$$

Hence, the Earth rotates around its axis in one sidereal day.

There is also a frame showing the limit of the lit and unlit parts of the Earth and this frame is fixed on the rotating frame 71. It is always in the same orientation towards the Sun.

We finally come to the motion of the Moon. The direction of the Moon is determined by the frame tied to tube 80. This frame has the following velocity relative to the rotating Earth frame:

$$\mathrm{V}^{71}_{80} = \mathrm{V}^{71}_{67} \times \left(-\frac{78}{78}\right) \times \left(-\frac{12}{87}\right) \times \left(-\frac{41}{41}\right) \times \left(-\frac{41}{167}\right) \tag{13.23}$$

$$= \mathrm{V}^{71}_{67} \times \frac{164}{4843} \tag{13.24}$$

$$= \left(\mathrm{V}^0_{67} + \mathrm{V}^{71}_0\right) \times \frac{164}{4843} \tag{13.25}$$

$$= \left(\frac{25270}{25201} - \frac{530670}{193820891}\right) \times \frac{164}{4843} = \frac{31786627600}{938674575113} \tag{13.26}$$

$$\mathrm{P}^{71}_{80} = \frac{938674575113}{31786627600} = 29.5304\ldots \text{ days} \tag{13.27}$$

This is an approximation of the synodic month. The same value is given by Oechslin.

The tropical month can be computed as follows:

$$\mathrm{V}^0_{80} = \mathrm{V}^{71}_{80} + \mathrm{V}^0_{71} = \frac{31786627600}{938674575113} + \frac{530670}{193820891} = \frac{34356662410}{938674575113} \tag{13.28}$$

$$\mathrm{P}^0_{80} = \frac{938674575113}{34356662410} = 27.3214\ldots \text{ days} \tag{13.29}$$

The same value is given by Oechslin.

Finally, the Moon is a globe half lit, half unlit, and it should always have the same orientation with respect to frame 71:

$$\mathrm{V}^{71}_{83} = \mathrm{V}^{80}_{83} + \mathrm{V}^{71}_{80} = \mathrm{V}^{80}_{81} \times \left(-\frac{53}{53}\right) \times \left(-\frac{53}{53}\right) + \mathrm{V}^{71}_{80} = \mathrm{V}^{80}_{81} + \mathrm{V}^{71}_{80} = \mathrm{V}^{71}_{81} = 0 \tag{13.30}$$

13.4 The orrery

The orrery gets its input from arbor 7 mentioned above. This arbor makes one turn counterclockwise as seen from above in one sidereal day:

$$V_7^0 = \frac{25270}{25201} \tag{13.31}$$

Part of the structure of the orrery is fixed to the reference frame. This includes a tube 20 which carries three wheels and the entire frame housing the gears for Mercury, Venus, the Earth and the Moon, and Mars. The Jupiter and Saturn systems rotate around this central system.

13.4.1 The mean motions

We will first concentrate on the internal system. Like the general orrery, its input is only the motion of arbor 7. The mean motion of Mercury is that of tube 11 and is obtained as follows:

$$V_{11}^0 = V_7^0 \times \left(-\frac{52}{52}\right) \times \left(-\frac{6}{53}\right) \times \left(-\frac{32}{77}\right) \times \left(-\frac{20}{83}\right) = V_7^0 \times \frac{3840}{338723} \tag{13.32}$$

$$= \frac{25270}{25201} \times \frac{3840}{338723} = \frac{13862400}{1219451189} \tag{13.33}$$

$$P_{11}^0 = \frac{1219451189}{13862400} = 87.9682\ldots \text{ days} \tag{13.34}$$

This is the tropical revolution of Mercury. The same value is given by Oechslin.

The motion of Mercury is used to obtain the motion of Venus:

$$V_{13}^0 = V_{11}^0 \times \left(-\frac{54}{50}\right) \times \left(-\frac{29}{80}\right) \tag{13.35}$$

$$= \frac{13862400}{1219451189} \times \frac{783}{2000} = \frac{935712}{210250205} \tag{13.36}$$

$$P_{13}^0 = \frac{210250205}{935712} = 224.6954\ldots \text{ days} \tag{13.37}$$

The same value is given by Oechslin.

The motion of Mercury is also used to obtain the motion of the Moon:

$$V_{19}^0 = V_{11}^0 \times \left(-\frac{84}{85}\right) \times \left(-\frac{101}{31}\right) \tag{13.38}$$

$$= \frac{13862400}{1219451189} \times \frac{8484}{2635} = \frac{23521720320}{642650776603} \tag{13.39}$$

$$P_{19}^0 = \frac{642650776603}{23521720320} = 27.3215\ldots \text{ days} \tag{13.40}$$

This is the tropical month (also given by Oechslin) and the value is slightly different from that used in the tellurium.

The motion of Venus is then used to obtain the motion of the Earth:

$$\mathrm{V}_{15}^{0} = \mathrm{V}_{13}^{0} \times \left(-\frac{79}{65}\right) \times \left(-\frac{41}{81}\right) = \mathrm{V}_{13}^{0} \times \frac{3239}{5265} \tag{13.41}$$

$$= \frac{935712}{210250205} \times \frac{3239}{5265} = \frac{473632}{172990675} \tag{13.42}$$

$$\mathrm{P}_{15}^{0} = \frac{172990675}{473632} = 365.2427\ldots \text{ days} \tag{13.43}$$

The same value is given by Oechslin.

And the motion of the Earth is used to obtain the motion of Mars:

$$\mathrm{V}_{17}^{0} = \mathrm{V}_{15}^{0} \times \left(-\frac{116}{33}\right) \times \left(-\frac{18}{119}\right) = \mathrm{V}_{15}^{0} \times \frac{696}{1309} \tag{13.44}$$

$$= \frac{473632}{172990675} \times \frac{696}{1309} = \frac{329647872}{226444793575} \tag{13.45}$$

$$\mathrm{P}_{17}^{0} = \frac{226444793575}{329647872} = 686.9293\ldots \text{ days} \tag{13.46}$$

The same value is given by Oechslin.

The motion of Mars is used to obtain the motion of Jupiter:

$$\mathrm{V}_{28}^{0} = \mathrm{V}_{17}^{0} \times \left(-\frac{119}{31}\right) \times \left(-\frac{5}{121}\right) = \mathrm{V}_{17}^{0} \times \frac{595}{3751} \tag{13.47}$$

$$= \frac{329647872}{226444793575} \times \frac{595}{3751} = \frac{329647872}{1427553648235} \tag{13.48}$$

$$\mathrm{P}_{28}^{0} = \frac{1427553648235}{329647872} = 4330.5410\ldots \text{ days} \tag{13.49}$$

The same value is given by Oechslin.

Finally, the motion of Saturn is obtained from the motion of Jupiter. First, in the rotating frame of Jupiter:

$$\mathrm{V}_{45}^{28} = \mathrm{V}_{20}^{28} \times \left(-\frac{103}{68}\right) \times \left(-\frac{41}{104}\right) = \mathrm{V}_{20}^{28} \times \frac{4223}{7072} \tag{13.50}$$

$$= \mathrm{V}_{0}^{28} \times \frac{4223}{7072} = -\mathrm{V}_{28}^{0} \times \frac{4223}{7072} \tag{13.51}$$

$$= \left(-\frac{329647872}{1427553648235}\right) \times \frac{4223}{7072} = -\frac{43503217608}{315489356259935} \tag{13.52}$$

and then in the absolute frame:

$$\mathrm{V}_{45}^{0} = \mathrm{V}_{45}^{28} + \mathrm{V}_{28}^{0} = \left(-\frac{43503217608}{315489356259935}\right) + \frac{329647872}{1427553648235} \tag{13.53}$$

$$= \frac{2668087464}{28680850569085} \tag{13.54}$$

$$\mathrm{P}_{45}^{0} = \frac{28680850569085}{2668087464} = 10749.5915\ldots \text{ days} \tag{13.55}$$

The same value is given by Oechslin.

The supports of Jupiter and Saturn actually carry two gear trains each, in order to take care of the eccentric motions of Jupiter and Saturn, and of the motions of the satellites. In the similar orrery of Furtwangen (Oechslin 8.12) constructed in 1774, these two gear trains were replaced by horizontal arbors.

13.4.2 Taking the excentricities into account

The Earth and Venus are shown revolving on circles centered on the Sun, but for the other planets, Hahn took the excentricities of their orbits into account.

Mercury is placed on an excentric axis on wheel 26, whose velocity is

$$\mathrm{V}_{26}^{11} = \mathrm{V}_{20}^{11} \times \left(-\frac{12}{12}\right) \times \left(-\frac{12}{12}\right) = \mathrm{V}_{20}^{11} \tag{13.56}$$

and

$$\mathrm{V}_{26}^{0} = \mathrm{V}_{26}^{11} + \mathrm{V}_{11}^{0} = \mathrm{V}_{20}^{11} + \mathrm{V}_{11}^{0} = \mathrm{V}_{0}^{11} + \mathrm{V}_{11}^{0} = 0 \tag{13.57}$$

In other words, the orientation of the wheel supporting Mercury's axis is fixed in the absolute reference frame and therefore when Mercury is closest to the Sun at some point, it will be farthest half a period later. Mercury does that way go through an excentric cercle, approximating the elliptical motion of the planet.

The situation is exactly the same for Mars:

$$\mathrm{V}_{22}^{17} = \mathrm{V}_{20}^{17} \times \left(-\frac{40}{40}\right) \times \left(-\frac{40}{40}\right) = \mathrm{V}_{20}^{17} \tag{13.58}$$

and

$$\mathrm{V}_{22}^{0} = \mathrm{V}_{22}^{17} + \mathrm{V}_{17}^{0} = \mathrm{V}_{20}^{17} + \mathrm{V}_{17}^{0} = \mathrm{V}_{20}^{0} = 0 \tag{13.59}$$

It is also similar for Jupiter:

$$\mathrm{V}_{32}^{28} = \mathrm{V}_{20}^{28} \times \left(-\frac{92}{82}\right) \times \left(-\frac{82}{92}\right) = \mathrm{V}_{20}^{28} \tag{13.60}$$

and

$$\mathrm{V}_{32}^{0} = \mathrm{V}_{32}^{28} + \mathrm{V}_{28}^{0} = \mathrm{V}_{20}^{28} + \mathrm{V}_{28}^{0} = \mathrm{V}_{20}^{0} = 0 \tag{13.61}$$

And also for Saturn:

$$\mathrm{V}_{49}^{45} = \mathrm{V}_{20}^{45} \times \left(-\frac{72}{72}\right) \times \left(-\frac{66}{66}\right) \times \left(-\frac{66}{66}\right) \times \left(-\frac{66}{66}\right) = \mathrm{V}_{20}^{45} \tag{13.62}$$

and

$$\mathrm{V}_{49}^{0} = \mathrm{V}_{49}^{45} + \mathrm{V}_{45}^{0} = \mathrm{V}_{20}^{45} + \mathrm{V}_{45}^{0} = \mathrm{V}_{20}^{0} = 0 \tag{13.63}$$

13.4.3 The satellites

13.4.3.1 The Moon

The Moon is located on a 30-teeth wheel on tube 24. This tube rotates around an axis located on the Earth frame 15. Its velocity on the Earth frame 15 is

$$\begin{aligned}
\mathrm{V}_{24}^{15} &= \mathrm{V}_{19}^{15} \times \left(-\frac{30}{30}\right) \times \left(-\frac{30}{30}\right) = \mathrm{V}_{19}^{15} && (13.64)\\
&= \mathrm{V}_{19}^{0} + \mathrm{V}_{0}^{15} = \frac{23521720320}{642650776603} - \frac{473632}{172990675} = \frac{7072716193376}{208861502395975} && (13.65)\\
\mathrm{P}_{24}^{15} &= \frac{208861502395975}{7072716193376} = 29.5305\ldots \text{ days} && (13.66)
\end{aligned}$$

This is an approximation of the synodic month, different from the one used in the tellurium. This value is not given by Oechslin.

The tropical month is then given by:

$$\begin{aligned}
\mathrm{V}_{24}^{0} &= \mathrm{V}_{24}^{15} + \mathrm{V}_{15}^{0} = \frac{7072716193376}{208861502395975} + \frac{473632}{172990675} = \frac{23521720320}{642650776603} && (13.67)\\
\mathrm{P}_{24}^{0} &= \frac{642650776603}{23521720320} = 27.3215\ldots \text{ days} && (13.68)
\end{aligned}$$

which is the same value given by Oechslin. We also have

$$\mathrm{V}_{24}^{0} = \mathrm{V}_{19}^{0} \qquad (13.69)$$

which was seen earlier. The tropical month in the orrery, as already mentioned, differs from the tropical month used in the tellurium.

13.4.3.2 Jupiter's satellites

The motion of the satellites of Jupiter is obtained by a number of gears located on the rotating frame 32, which is itself pivoting on frame 28. However, since we saw that $\mathrm{V}_{32}^{0} = 0$, this frame is actually only in translation with respect to the absolute frame 0.

The input to the system of Jupiter's satellites is arbor 35. The motion of arbor 35 eventually goes back to tube 50, which replicates the motion of tube 30. Tube 30, in turn, replicates the motion of arbor 7, but in the opposite direction:

$$\mathrm{V}_{50}^{0} = \mathrm{V}_{30}^{0} = -\mathrm{V}_{7}^{0} = -\frac{25270}{25201} \qquad (13.70)$$

We can now compute the velocity of arbor 35. We have

$$
\begin{aligned}
V_{35}^{0} &= V_{35}^{32} + V_{32}^{0} = V_{34}^{32} \times \left(-\frac{12}{12}\right) + V_{32}^{0} && (13.71)\\
&= -V_{34}^{32} + V_{32}^{0} = V_{32}^{34} = V_{32}^{28} - V_{34}^{28} && (13.72)\\
&= V_{20}^{28} \times \left(-\frac{92}{82}\right) \times \left(-\frac{82}{92}\right) - V_{50}^{28} \times \left(-\frac{82}{82}\right) \times \left(-\frac{82}{82}\right) && (13.73)\\
&= V_{20}^{28} - V_{50}^{28} = V_{20}^{50} = -V_{50}^{0} = V_{7}^{0} = \frac{25270}{25201} && (13.74)
\end{aligned}
$$

The motions of the four satellites are then the following:

$$
\begin{aligned}
V_{37}^{0} &= V_{35}^{0} \times \left(-\frac{35}{57}\right) \times \left(-\frac{56}{61}\right) = V_{35}^{0} \times \frac{1960}{3477} && (13.75)\\
&= \frac{25270}{25201} \times \frac{1960}{3477} = \frac{2606800}{4611783} && (13.76)\\
P_{37}^{0} &= \frac{4611783}{2606800} = 1.76913\ldots \text{ days} = 1 \text{ d. } 18.45\ldots \text{ h. (Io)} && (13.77)\\
V_{39}^{0} &= V_{37}^{0} \times \left(-\frac{67}{68}\right) \times \left(-\frac{45}{89}\right) = V_{37}^{0} \times \frac{3015}{6052} && (13.78)\\
&= \frac{2606800}{4611783} \times \frac{3015}{6052} = \frac{654958500}{2325875893} && (13.79)\\
P_{39}^{0} &= \frac{2325875893}{654958500} = 3.55118\ldots \text{ days (Europe)} && (13.80)\\
V_{41}^{0} &= V_{39}^{0} \times \left(-\frac{94}{41}\right) \times \left(-\frac{21}{97}\right) = V_{39}^{0} \times \frac{1974}{3977} && (13.81)\\
&= \frac{654958500}{2325875893} \times \frac{1974}{3977} && (13.82)\\
&= \frac{1292888079000}{9250008426461} && (13.83)\\
P_{41}^{0} &= \frac{9250008426461}{1292888079000} = 7.15453\ldots \text{ days (Ganymede)} && (13.84)\\
V_{43}^{0} &= V_{41}^{0} \times \left(-\frac{66}{53}\right) \times \left(-\frac{21}{61}\right) = V_{41}^{0} \times \frac{1386}{3233} && (13.85)\\
&= \frac{1292888079000}{9250008426461} \times \frac{1386}{3233} && (13.86)\\
&= \frac{162903897954000}{2718661567522583} && (13.87)\\
P_{43}^{0} &= \frac{2718661567522583}{162903897954000} = 16.68874\ldots \text{ days (Callisto)} && (13.88)
\end{aligned}
$$

When Oechslin computed the orbital period of Io, he mistakenly used the gear count 36 instead of 35, even though his formula uses 35, and this resulted in the incorrect period 1.719993 days. This error then propagated to the other satellites, as each motion is computed from the previous one. The same mistake was made in the analysis of the Furtwangen planetarium, but not in the Nuremberg machine.

13.4.3.3 Saturn's satellites

Likewise, the motion of the satellites of Saturn is obtained by a number of gears located on the rotating frame 49. However, since we saw that $\mathrm{V}^0_{49} = 0$, this frame is actually only in translation with respect to the absolute frame 0. For this reason, the orientation of Saturn's axis as well as the orbits of the five satellites are fixed in the absolute frame.

The input to the system of Saturn's satellites is arbor 55. We can compute its velocity:

$$\mathrm{V}^0_{55} = \mathrm{V}^{49}_{55} + \mathrm{V}^0_{49} = \mathrm{V}^{49}_{55} = \mathrm{V}^{49}_{54} \times \left(-\frac{16}{16}\right) \tag{13.89}$$

$$= -\mathrm{V}^{49}_{54} = \mathrm{V}^{54}_{49} = \mathrm{V}^{45}_{49} - \mathrm{V}^{45}_{54} \tag{13.90}$$

$$= \mathrm{V}^{45}_{20} \times \left(-\frac{72}{72}\right) \times \left(-\frac{66}{66}\right) \times \left(-\frac{66}{66}\right) \times \left(-\frac{66}{66}\right) \tag{13.91}$$

$$- \mathrm{V}^{45}_{50} \times \left(-\frac{72}{72}\right) \times \left(-\frac{66}{66}\right) \times \left(-\frac{66}{66}\right) \times \left(-\frac{66}{66}\right) \tag{13.92}$$

$$= \mathrm{V}^{45}_{20} - \mathrm{V}^{45}_{50} = \mathrm{V}^{50}_{20} = -\mathrm{V}^0_{50} = \mathrm{V}^0_7 = \frac{25270}{25201} \tag{13.93}$$

Before computing the motions of the five satellites, we must observe that there are two possible problems. First, Oechslin's plan gives a ratio of 86/33 for Tethys, and if we use this value, we do in fact obtain the incorrect periods of 1.4924, 2.1652, 3.5704, 12.5857, 62.2576 days, which are all wrong by the same factor of about 1.27, because the motion of each satellite is obtained from the previous one.

However, in his computations, Oechslin uses the ratio 68/33. Moreover, this satellite system uses the same ratios as the Furtwangen orrery constructed in 1774 (Oechslin 8.12). Both systems seem to use the ratio 86/33 and Gotha's system was obviously copied from the Furtwangen one, without fixing the problem with the periods.

It seems that Oechslin found that the wheel with 86 teeth should have had 68 teeth and this does indeed yield better values of the periods.[5]

There is however another possibility. In the Furtwangen orrery, the ratio used in the motion of Tethys is $\frac{78}{20}$ and this was replaced by $\frac{39}{10}$ in Gotha. It is possible that the cause of the errors is a typo in the Furtwangen orrery calculations that had the teeth count 98 become 78, hence $\frac{98}{20}$ become $\frac{78}{20}$, or, when we simplify, $\frac{49}{10}$ become $\frac{39}{10}$. Another solution to the present problem would then be to replace the 39-teeth wheel by a 49-teeth wheel.

So, the periods given by Oechslin are those obtained when Tethys uses the ratio 68/33. The motions of the five satellites are then the following, where I have added "(a)" to distinguish this choice from the second one "(b)" below.

[5] See [2, p. 204] on this error.

For Tethys:

$$\mathrm{V}^0_{57}(a) = \mathrm{V}^0_{55} \times \left(-\frac{10}{39}\right) \times \left(-\frac{68}{33}\right) = \mathrm{V}^0_{55} \times \frac{680}{1287} \tag{13.94}$$

$$= \frac{25270}{25201} \times \frac{680}{1287} = \frac{17183600}{32433687} \tag{13.95}$$

$$\mathrm{P}^0_{57}(a) = \frac{32433687}{17183600} = 1.8874\ldots \text{ days} \tag{13.96}$$

For Dione:

$$\mathrm{V}^0_{59}(a) = \mathrm{V}^0_{57}(a) \times \left(-\frac{22}{46}\right) \times \left(-\frac{49}{34}\right) = \mathrm{V}^0_{57}(a) \times \frac{539}{782} \tag{13.97}$$

$$= \frac{17183600}{32433687} \times \frac{539}{782} = \frac{24764600}{67815891} \tag{13.98}$$

$$\mathrm{P}^0_{59}(a) = \frac{67815891}{24764600} = 2.7384\ldots \text{ days} \tag{13.99}$$

For Rhea:

$$\mathrm{V}^0_{61}(a) = \mathrm{V}^0_{59}(a) \times \left(-\frac{27}{32}\right) \times \left(-\frac{23}{32}\right) = \mathrm{V}^0_{59}(a) \times \frac{621}{1024} \tag{13.100}$$

$$= \frac{24764600}{67815891} \times \frac{621}{1024} = \frac{9286725}{41934464} \tag{13.101}$$

$$\mathrm{P}^0_{61}(a) = \frac{41934464}{9286725} = 4.5155\ldots \text{ days} \tag{13.102}$$

For Titan:

$$\mathrm{V}^0_{63}(a) = \mathrm{V}^0_{61}(a) \times \left(-\frac{40}{36}\right) \times \left(-\frac{12}{47}\right) = \mathrm{V}^0_{61}(a) \times \frac{40}{141} \tag{13.103}$$

$$= \frac{9286725}{41934464} \times \frac{40}{141} = \frac{15477875}{246364976} \tag{13.104}$$

$$\mathrm{P}^0_{63}(a) = \frac{246364976}{15477875} = 15.9172\ldots \text{ days} \tag{13.105}$$

For Iapetus:

$$\mathrm{V}^0_{65}(a) = \mathrm{V}^0_{63}(a) \times \left(-\frac{68}{39}\right) \times \left(-\frac{8}{69}\right) = \mathrm{V}^0_{63}(a) \times \frac{544}{2691} \tag{13.106}$$

$$= \frac{15477875}{246364976} \times \frac{544}{2691} = \frac{526247750}{41435509401} \tag{13.107}$$

$$\mathrm{P}^0_{65}(a) = \frac{41435509401}{526247750} = 78.7376\ldots \text{ days} \tag{13.108}$$

If instead we replace the ratio 10/39 by 10/49, we obtain the following periods, which are marked "(b)" in order to distinguish them from the previous ones:

For Tethys:

$$V^0_{57}(b) = V^0_{55} \times \left(-\frac{10}{49}\right) \times \left(-\frac{86}{33}\right) = V^0_{55} \times \frac{860}{1617} \tag{13.109}$$

$$= \frac{25270}{25201} \times \frac{860}{1617} = \frac{3104600}{5821431} \tag{13.110}$$

$$P^0_{57}(b) = \frac{5821431}{3104600} = 1.8750\ldots \text{ days} \tag{13.111}$$

For Dione:

$$V^0_{59}(b) = V^0_{57}(b) \times \left(-\frac{22}{46}\right) \times \left(-\frac{49}{34}\right) = V^0_{57}(b) \times \frac{539}{782} \tag{13.112}$$

$$= \frac{3104600}{5821431} \times \frac{539}{782} = \frac{10866100}{29560773} \tag{13.113}$$

$$P^0_{59}(b) = \frac{29560773}{10866100} = 2.7204\ldots \text{ days} \tag{13.114}$$

For Rhea:

$$V^0_{61}(b) = V^0_{59}(b) \times \left(-\frac{27}{32}\right) \times \left(-\frac{23}{32}\right) = V^0_{59}(b) \times \frac{621}{1024} \tag{13.115}$$

$$= \frac{10866100}{29560773} \times \frac{621}{1024} = \frac{24448725}{109674752} \tag{13.116}$$

$$P^0_{61}(b) = \frac{109674752}{24448725} = 4.4859\ldots \text{ days} \tag{13.117}$$

For Titan:

$$V^0_{63}(b) = V^0_{61}(b) \times \left(-\frac{40}{36}\right) \times \left(-\frac{12}{47}\right) = V^0_{61}(b) \times \frac{40}{141} \tag{13.118}$$

$$= \frac{24448725}{109674752} \times \frac{40}{141} = \frac{40747875}{644339168} \tag{13.119}$$

$$P^0_{63}(b) = \frac{644339168}{40747875} = 15.8128\ldots \text{ days} \tag{13.120}$$

For Iapetus:

$$V^0_{65}(b) = V^0_{63}(b) \times \left(-\frac{68}{39}\right) \times \left(-\frac{8}{69}\right) = V^0_{63}(b) \times \frac{544}{2691} \tag{13.121}$$

$$= \frac{40747875}{644339168} \times \frac{544}{2691} = \frac{13582625}{1062448959} \tag{13.122}$$

$$P^0_{65}(b) = \frac{1062448959}{13582625} = 78.2211\ldots \text{ days} \tag{13.123}$$

Since this orrery was taken over in Mark Frank's "astro-skeleton," the question arises whether the error on Tethys made its way to Frank's machine or not.

13.5 The celestial globe

The last part of Hahn's *Weltmaschine* is the celestial globe. This globe rotates and shows the geocentric motion of the planets, the Moon and the motion of the lunar nodes. The motions of the planets are shown with their retrogradations and the motions of the Sun and the Moon are shown with their equation of center.

A comparison with the gears of this globe and the globes of the *Weltmaschine* of Stuttgart (Oechslin 8.1) (1769) and the Aschaffenburg globe clock (Oechslin 8.4) (1776/1777), shows that the structures of these three globes are very similar, and there are mainly small differences in teeth counts, but without altering the ratios. There may however be an error in the motion of the lunar apsides and nodes in the present globe, as I will argue below. The globe of the Nuremberg *Weltmaschine* (Oechslin 8.2) follows the same principles, but uses different ratios.

As described above, the celestial globe of the Gotha machine is driven by the vertical arbor 5 which is making one turn clockwise (viewed from above) in one sidereal day:

$$\mathrm{V}_5^0 = -\frac{25270}{25201} \tag{13.124}$$

This motion rotates the entire globe and gives it a motion through the vertical axis with is the axis of the celestial pole (Earth's axis). This apparent motion goes from East to West, hence must be clockwise as seen from above.

However, the axis of all the planets, as well as those of the Sun and the Moon (when neglecting the inclination of its orbit to the ecliptic), is that going through the poles of the ecliptic and it is inclined at an angle of 23.5° to the axis of the celestial poles.

The central axis 94 of the globe is tied to three fixed wheels which will be used to obtain some of the motions displayed.

13.5.1 The motions of the Moon and the Sun

Within the celestial globe, all the motions are produced from the motion of tube 86. The velocity of this tube in the globe reference frame 94 is:

$$\mathrm{V}_{86}^{94} = \mathrm{V}_{84}^{94} \times \left(-\frac{71}{92}\right) \times \left(-\frac{7}{148}\right) = \mathrm{V}_{84}^{94} \times \frac{497}{13616} \tag{13.125}$$

where the arbor/wheel 84 is a fixed wheel on the meridian frame of the globe. Now, since

$$\mathrm{V}_{84}^{94} = \mathrm{V}_0^{94} = -\mathrm{V}_{94}^0 = -\mathrm{V}_5^0 = \frac{25270}{25201} \tag{13.126}$$

it follows that

$$V_{86}^{94} = \frac{25270}{25201} \times \frac{497}{13616} = \frac{6279595}{171568408} \quad (13.127)$$

$$P_{86}^{94} = \frac{171568408}{6279595} = 27.3215\ldots \text{ days} \quad (13.128)$$

This value is given by Oechslin, and also in sidereal days.

The tube 86 actually corresponds to the mean motion of the Moon and the period of its rotation around the globe is one tropical month. This value, incidentally, is different from the two other values already seen for the tropical month, in the tellurium and in the orrery.

This motion is used to obtain the mean motion of the Sun on tube 88:

$$V_{88}^{94} = V_{86}^{94} \times \left(-\frac{112}{53}\right) \times \left(-\frac{4}{113}\right) = V_{86}^{94} \times \frac{448}{5989} \quad (13.129)$$

$$= \frac{6279595}{171568408} \times \frac{448}{5989} = \frac{351657320}{128440399439} \quad (13.130)$$

$$P_{88}^{94} = \frac{128440399439}{351657320} = 365.2430\ldots \text{ days} \quad (13.131)$$

This value is given by Oechslin, and also in sidereal days.

The actual motion of the Sun is on tube 114. This tube has the same motion as another tube S' (erroneously numbered 88 by Oechslin), being connected through axis 107 and the same gear ratios.

The motion of this tube S' is itself the same as that of an unnumbered mobile frame containing a number of gears and which I am calling S. This frame has an oscillating motion around the position of frame 88 which is the mean Sun. This oscillation is produced by an excentric pin on a wheel on arbor 95 located on frame 88. The oscillation period is the time it takes for this wheel to rotate on frame 88:

$$V_{95}^{88} = V_{94}^{88} \times \left(-\frac{48}{48}\right) = -V_{94}^{88} = V_{88}^{94} \quad (13.132)$$

Therefore

$$P_{95}^{88} = P_{88}^{94} \quad (13.133)$$

The oscillation period is exactly that of the tropical year, so that the perihelion (where the Sun is fastest) and aphelion are at fixed locations on the celestial sphere. This oscillation approximates the equation of center. In reality, the apsides are not fixed, but move at a rate of one turn in about 21000 years with respect to the zodiac. That would be the period that Hahn would have had to implement.

Now, the mean motion of the Sun is used to obtain the apsidal precession of the Moon on tube 90:

$$\mathrm{V}_{90}^{94} = \mathrm{V}_{88}^{94} \times \left(-\frac{28}{81}\right) \times \left(-\frac{23}{73}\right) \tag{13.134}$$

$$= \mathrm{V}_{88}^{94} \times \frac{644}{5913} \tag{13.135}$$

$$= \frac{351657320}{128440399439} \times \frac{644}{5913} = \frac{9846404960}{33020351386209} \tag{13.136}$$

$$\mathrm{P}_{90}^{94} = \frac{33020351386209}{9846404960} = 3353.5439\ldots \text{ days} \tag{13.137}$$

The actual value is about 3233 days (8.85 years).

This might seem like a small discrepancy, but it is in fact likely an error. In his computations, Oechslin replaces the value 81 by 78 and obtains the much better approximation:

$$\mathrm{V(h)}_{90}^{94} = \mathrm{V}_{88}^{94} \times \left(-\frac{28}{78}\right) \times \left(-\frac{23}{73}\right) \tag{13.138}$$

$$= \mathrm{V}_{88}^{94} \times \frac{322}{2847} \tag{13.139}$$

$$= \frac{351657320}{128440399439} \times \frac{322}{2847} = \frac{4923202480}{15898687704471} \tag{13.140}$$

$$\mathrm{P(h)}_{90}^{94} = \frac{15898687704471}{4923202480} = 3229.3385\ldots \text{ days} \tag{13.141}$$

where V(h) and P(h) are hypothetical velocities and periods.

It seems therefore that the actual train contains a 81-teeth wheel, but that a better value would have been 78. In fact, in the Stuttgart *Weltmaschine* and the Aschaffenburg globe clock, the first ratio is also $\left(-\frac{28}{78}\right)$ and not $\left(-\frac{28}{81}\right)$.

This is then used to obtain the apparent motion of the Moon, namely that of the mean Moon plus the equation of center. The equation of center is computed based on the position of the perigee. Now, whereas the equation of center for the Sun rested on a fixed wheel on axis 94, because the perihelion/aphelion were considered fixed on the celestial sphere, the equation of center for the Moon rests on the wheel corresponding to the apsidal precession, because the perigee/apogee are fixed with respect to that wheel. The period of the oscillation caused by the equation of center of the Moon is however not the tropical month:

$$\mathrm{V}_{91}^{86} = \mathrm{V}_{90}^{86} \times \left(-\frac{48}{48}\right) = -\mathrm{V}_{90}^{86} = \mathrm{V}_{86}^{90} \tag{13.142}$$

$$= \mathrm{V}_{86}^{94} + \mathrm{V}_{94}^{90} = \frac{6279595}{171568408} - \frac{9846404960}{33020351386209} \tag{13.143}$$

$$= \frac{220567289199775}{6075744655062456} \tag{13.144}$$

$$\mathrm{P}_{91}^{86} = \frac{6075744655062456}{220567289199775} = 27.54599\ldots \text{ days} \tag{13.145}$$

This is the anomalistic month. This value seems reasonable, but with the corrected value of V_{94}^{90}, we obtain:

$$\mathrm{V(h)}_{91}^{86} = \mathrm{V}_{90}^{86} \times \left(-\frac{48}{48}\right) = -\mathrm{V}_{90}^{86} = \mathrm{V}_{86}^{90} \tag{13.146}$$

$$= \mathrm{V}_{86}^{94} + \mathrm{V(h)}_{94}^{90} = \frac{6279595}{171568408} - \frac{4923202480}{15898687704471} \tag{13.147}$$

$$= \frac{106165514457065}{2925358537622664} \tag{13.148}$$

$$\mathrm{P(h)}_{91}^{86} = \frac{2925358537622664}{106165514457065} = 27.5546\ldots \text{ days} \tag{13.149}$$

As far as the Moon is concerned, the mean motion of the Sun is also used to obtain the precession of the lunar nodes on tube 93:

$$\mathrm{V}_{93}^{94} = \mathrm{V}_{88}^{94} \times \left(-\frac{28}{81}\right) \times \left(-\frac{25}{33}\right) \times \left(-\frac{17}{86}\right) \tag{13.150}$$

$$= \mathrm{V}_{88}^{94} \times \left(-\frac{5950}{114939}\right) = \frac{351657320}{128440399439} \times \left(-\frac{5950}{114939}\right) \tag{13.151}$$

$$= -\frac{2092361054000}{14762811071119221} \tag{13.152}$$

This value is negative, because the nodes have a retrograde motion.

$$\mathrm{P}_{93}^{94} = -\frac{14762811071119221}{2092361054000} = -7055.5753\ldots \text{ days} \tag{13.153}$$

This corresponds to about $\frac{114939}{5950} = 19.3\ldots$ tropical years, whereas the correct value is about 18.6 years.

Again, if we replace 81 by 78, we obtain:

$$\mathrm{V(h)}_{93}^{94} = \mathrm{V}_{88}^{94} \times \left(-\frac{28}{78}\right) \times \left(-\frac{25}{33}\right) \times \left(-\frac{17}{86}\right) \tag{13.154}$$

$$= \mathrm{V}_{88}^{94} \times \left(-\frac{2975}{55341}\right) = \frac{351657320}{128440399439} \times \left(-\frac{2975}{55341}\right) \tag{13.155}$$

$$= -\frac{1046180527000}{7108020145353699} \tag{13.156}$$

$$\mathrm{P(h)}_{93}^{94} = -\frac{7108020145353699}{1046180527000} = -6794.2577\ldots \text{ days} \tag{13.157}$$

The same value is given by Oechslin. We now almost have the correct value of 6798 days for the retrogradation of the lunar nodes.

13.5.2 The motions of the planets

The motion of the true Sun is used to obtain the apparent motions of Mercury and Venus.

13.5.2.1 Venus

I am first considering the motion of Venus, as it is the simplest. Hahn merely has Venus oscillating around the position of the Sun with the period of a wheel on arbor 105. The velocity of this wheel is

$$\mathrm{V}_{105}^{S} = \mathrm{V}_{94}^{S} \times \left(-\frac{62}{40}\right) \times \left(-\frac{52}{44}\right) \times \left(-\frac{28}{82}\right) = \mathrm{V}_{94}^{S} \times \left(-\frac{2821}{4510}\right) \tag{13.158}$$

Now, since

$$\mathrm{V}_{94}^{S} \approx \mathrm{V}_{94}^{88} \tag{13.159}$$

we have

$$\mathrm{V}_{105}^{S} \approx \mathrm{V}_{94}^{88} \times \left(-\frac{2821}{4510}\right) \approx \left(-\frac{351657320}{128440399439}\right) \times \left(-\frac{2821}{4510}\right) \tag{13.160}$$

$$\approx \frac{99202529972}{57926620146989} \tag{13.161}$$

and

$$\mathrm{P}_{105}^{S} \approx \frac{57926620146989}{99202529972} = 583.92281\ldots \text{ days} \tag{13.162}$$

which is the synodic period of Venus. This value is not given by Oechslin. The above value is an average, because it is based on the motion of the true Sun. The oscillation of Venus makes it possible to show the retrogradations of this planet.

From this period, we can obtain the tropical orbit period of Venus which is[6]

$$\text{tropical orbit period} = \frac{1}{\frac{1}{\text{tropical year}} + \frac{1}{\text{synodic period}}} \tag{13.163}$$

$$= \frac{1}{\frac{1}{\mathrm{P}_{88}^{94}} + \frac{1}{\mathrm{P}_{105}^{S}}} \tag{13.164}$$

$$\approx \frac{57926620146989}{257799981292} = 224.6959\ldots \text{ days} \tag{13.165}$$

This value is also given by Oechslin.

13.5.2.2 Mercury

The motion of Mercury is more complex because Hahn both takes into account the retrogradations of Mercury and the excentricity of its orbit (but not the

[6] The derivation of this formula can for instance be found in section 22.4.3.1, in the chapter on Klein's Tychonic clock.

precession of the apsides). If we do not take the excentricity into account, the position of Mercury is that of arbor 102 and it is also merely oscillating around the true Sun.

The velocity of Mercury's axis 102 around its average position (arbor 98) with respect the mean Sun is

$$\mathrm{V}_{98}^{S} = \mathrm{V}_{94}^{S} \times \left(-\frac{58}{39}\right) \times \left(-\frac{58}{34}\right) \times \left(-\frac{41}{33}\right) = \mathrm{V}_{94}^{S} \times \left(-\frac{68962}{21879}\right) \quad (13.166)$$

As before, since

$$\mathrm{V}_{94}^{S} \approx \mathrm{V}_{94}^{88} \quad (13.167)$$

we have

$$\mathrm{V}_{98}^{S} \approx \mathrm{V}_{94}^{88} \times \left(-\frac{68962}{21879}\right) \approx \left(-\frac{351657320}{128440399439}\right) \times \left(-\frac{68962}{21879}\right) \quad (13.168)$$

$$\approx \frac{836241106960}{96901637907789} \quad (13.169)$$

and

$$\mathrm{P}_{98}^{S} \approx \frac{96901637907789}{836241106960} = 115.8776\ldots \text{ days} \quad (13.170)$$

which is the synodic period of Mercury. This value is not given by Oechslin. Like for Venus, the above value is an average, because it is based on the motion of the true Sun. The oscillation of Mercury makes it possible to show the retrogradations of this planet.

Again, we can also deduce the tropical orbit period of Mercury which is

$$\frac{2810147499325881}{31944902606120} = 87.9685\ldots \text{ days} \quad (13.171)$$

This value is also given by Oechslin.

Hahn does however add a second oscillation on top of the first one, in that the pin driving Mercury is excentered from arbor 102. Mercury rotates around the axis 102.

Arbor 102 replicates arbor 100 which replicates arbor 94. Consequently, the crank has a fixed orientation with respect to the globe.

13.5.2.3 Mars

The motions of Mars, Jupiter and Saturn are based on the true motion of the Sun. The motion of the Sun is replicated in three 116-teeth wheels which have the same motion as they are connected with identical gears to arbor 107. The intermediate of these three 116-teeth is the one on tube 114 and is the one carrying the actual Sun, as was already mentioned.

The mean motion of Mars is obtained first on tube 109:

$$\mathrm{V}_{109}^{94} = \mathrm{V}_{S'}^{94} \times \left(-\frac{116}{22}\right) \times \left(-\frac{22}{22}\right) \times \left(-\frac{12}{32}\right) \times \left(-\frac{32}{119}\right) \tag{13.172}$$

$$\approx \mathrm{V}_{S'}^{94} \times \frac{696}{1309} = \frac{351657320}{128440399439} \times \frac{696}{1309} = \frac{1205682240}{828219127417} \tag{13.173}$$

$$\mathrm{P}_{109}^{94} \approx \frac{828219127417}{1205682240} = 686.9298\ldots \text{ days} \tag{13.174}$$

This is Mars' orbital period and it is also given by Oechslin. Like for Mercury and Venus, the above value is an average, because it is based on the motion of the true Sun.

Hahn then takes the excentricity of the orbit into account, in that he adds the equation of center. This is done exactly in the same way as the equation of center for the Sun. There are two wheels of 52 teeth, one of them located on the frame for the mean motion of Mars, the other fixed inside the celestial globe (frame 94). We have

$$\mathrm{V}_{110}^{109} = \mathrm{V}_{94}^{109} \times \left(-\frac{52}{52}\right) = -\mathrm{V}_{94}^{109} = \mathrm{V}_{109}^{94} \tag{13.175}$$

Therefore

$$\mathrm{P}_{110}^{109} = \mathrm{P}_{109}^{94} \tag{13.176}$$

The oscillation period is exactly that of the tropical orbit period, so that the perihelion (where the Sun is fastest) and aphelion are at fixed locations on the celestial sphere. This oscillation approximates the equation of center. The resulting true motion of Mars is on a frame that Oechslin doesn't seem to number, but that we will name m.

For the geocentric view of the motion of Mars, Hahn proceeds as for Venus, where he corrected the position of Venus using the elongation with the true Sun S. In the case of Mars, the correction is done with respect to S', and we have

$$\mathrm{V}_{112}^{S'} = \mathrm{V}_{112}^{m} + \mathrm{V}_{m}^{S'} = \mathrm{V}_{S'}^{m} \times \left(-\frac{44}{44}\right) \times \left(-\frac{44}{44}\right) + \mathrm{V}_{m}^{S'} \tag{13.177}$$

$$= \mathrm{V}_{S'}^{m} + \mathrm{V}_{m}^{S'} = 0 \tag{13.178}$$

We could also have seen immediately that the motion of arbor 112 replicates that of S', and therefore that $\mathrm{V}_{112}^{S'} = 0$.

The synodic period of Mars corresponds to the velocity

$$\mathrm{V}_{S'}^{m} \approx \mathrm{V}_{88}^{m} \approx \mathrm{V}_{88}^{94} - \mathrm{V}_{m}^{94} \approx \mathrm{V}_{88}^{94} - \mathrm{V}_{109}^{94} \tag{13.179}$$

$$\approx \frac{351657320}{128440399439} - \frac{1205682240}{828219127417} \tag{13.180}$$

$$\approx \frac{30795133880}{24018354695093} \tag{13.181}$$

and therefore

$$\mathrm{P}^m_{S'} \approx \frac{24018354695093}{30795133880} = 779.9399\ldots \text{ days} \tag{13.182}$$

This value is not given by Oechslin.

13.5.2.4 Jupiter

The motion of Jupiter is obtained in a similar way as that of Mars. The mean motion of Jupiter is obtained first on tube 118, from the mean motion of Mars:

$$\mathrm{V}^{94}_{118} = \mathrm{V}^{94}_{109} \times \left(-\frac{119}{31}\right) \times \left(-\frac{31}{31}\right) \times \left(-\frac{5}{34}\right) \times \left(-\frac{34}{121}\right) \tag{13.183}$$

$$= \mathrm{V}^{94}_{109} \times \frac{595}{3751} \approx \frac{1205682240}{828219127417} \times \frac{595}{3751} \tag{13.184}$$

$$\approx \frac{42198878400}{182744114525951} \tag{13.185}$$

$$\mathrm{P}^{94}_{118} \approx \frac{182744114525951}{42198878400} = 4330.5443\ldots \text{ days} \tag{13.186}$$

This is Jupiter's orbital period. This value is also given by Oechslin. Like for Mercury, Venus and Mars, the above value is an average, because it is based on the motion of the true Sun.

Like for Mars, Hahn then takes the excentricity of the orbit into account, in that he adds the equation of center. We have

$$\mathrm{V}^{118}_{119} = \mathrm{V}^{118}_{94} \times \left(-\frac{48}{48}\right) = -\mathrm{V}^{118}_{94} = \mathrm{V}^{94}_{118} \tag{13.187}$$

Therefore

$$\mathrm{P}^{118}_{119} = \mathrm{P}^{94}_{118} \tag{13.188}$$

The oscillation period is exactly that of the tropical year, so that the perihelion (where the Sun is fastest) and aphelion are at fixed locations on the celestial sphere. This oscillation approximates the equation of center. The resulting true motion of Jupiter is on a frame that Oechslin doesn't seem to number, but that we will name j. We have

$$\mathrm{V}^{S}_{121} = \mathrm{V}^{j}_{121} + \mathrm{V}^{S}_{j} = \mathrm{V}^{j}_{S} \times \left(-\frac{48}{48}\right) \times \left(-\frac{48}{48}\right) + \mathrm{V}^{S}_{j} \tag{13.189}$$

$$= \mathrm{V}^{j}_{S} + \mathrm{V}^{S}_{j} = 0 \tag{13.190}$$

In other words

$$\mathrm{V}^{94}_{121} = \mathrm{V}^{S}_{121} + \mathrm{V}^{94}_{S} = \mathrm{V}^{94}_{S} \tag{13.191}$$

We could likewise compute the synodic period of Jupiter, as we did for Mars.

13.5.2.5 Saturn

The motion of Saturn is also obtained similarly. The mean motion of Saturn is obtained first on tube 124, from the mean motion of Jupiter:

$$\mathrm{V}_{124}^{94} = \mathrm{V}_{118}^{94} \times \left(-\frac{121}{34}\right) \times \left(-\frac{12}{106}\right) \tag{13.192}$$

$$= \mathrm{V}_{118}^{94} \times \frac{363}{901} \approx \frac{42198878400}{182744114525951} \times \frac{363}{901} \tag{13.193}$$

$$\approx \frac{126596635200}{1360764026346131} \tag{13.194}$$

$$\mathrm{P}_{118}^{94} \approx \frac{1360764026346131}{126596635200} = 10748.8166\ldots \text{ days} \tag{13.195}$$

This is Saturn's orbital period. This value is also given by Oechslin. Like for Mercury, Venus, Mars and Jupiter, the above value is an average, because it is based on the motion of the true Sun.

Like for Mars and Jupiter, Hahn then takes the excentricity of the orbit into account, in that he adds the equation of center. We have

$$\mathrm{V}_{125}^{124} = \mathrm{V}_{94}^{124} \times \left(-\frac{50}{50}\right) = -\mathrm{V}_{94}^{124} = \mathrm{V}_{124}^{94} \tag{13.196}$$

Therefore

$$\mathrm{P}_{125}^{124} = \mathrm{P}_{124}^{94} \tag{13.197}$$

The oscillation period is exactly that of the tropical year, so that the perihelion (where the Sun is fastest) and aphelion are at fixed locations on the celestial sphere. This oscillation approximates the equation of center. The resulting true motion of Saturn is on a frame that Oechslin doesn't seem to number, but that we will name s. We have

$$\mathrm{V}_{127}^{S} = \mathrm{V}_{127}^{s} + \mathrm{V}_{s}^{S} = \mathrm{V}_{S}^{s} \times \left(-\frac{54}{54}\right) \times \left(-\frac{54}{54}\right) + \mathrm{V}_{s}^{S} \tag{13.198}$$

$$= \mathrm{V}_{S}^{s} + \mathrm{V}_{s}^{S} = 0 \tag{13.199}$$

In other words

$$\mathrm{V}_{127}^{94} = \mathrm{V}_{127}^{S} + \mathrm{V}_{S}^{94} = \mathrm{V}_{S}^{94} \tag{13.200}$$

We could likewise compute the synodic period of Saturn, as we did for Mars.

13.6 References

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