Advanced topics in audio processing using deep learning - HW1

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1 Analytical part

1.1 1

1.1.1 Prove $\mathcal{F}(x_1(t) * x_2(t)) = X_1(\omega)X_2(\omega)$

By the definitions of the transform and the convolution operation:

$$\mathcal{F}(x_1(t) * x_2(t)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau \cdot e^{-j\omega t}\,dt$$

By changing order of integration we have:

$$\int_{-\infty}^{\infty} x_1(\tau)\int_{-\infty}^{\infty} x_2(t-\tau)e^{-j\omega t}\,dt\,d\tau = \int_{-\infty}^{\infty} x_1(\tau)\mathcal{F}(x_2(\tau-t))\,d\tau$$

From the time shifting property (that we'll prove on section d) we get:

$$\int_{-\infty}^{\infty} x_1(\tau)\mathcal{F}(x_2(\tau-t))\,d\tau = \int_{-\infty}^{\infty} x_1(\tau)X_2(\omega)e^{-j\omega\tau}\,d\tau = X_2(\omega)\int_{-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau}\,d\tau = X_2(\omega)X_1(\omega)$$

1.1.2 Prove $\mathcal{F}(ax_1(t) + bx_2(t)) = aX_1(\omega) + bX_2(\omega)$

By definition:

$$\mathcal{F}(ax_1(t)+bx_2(t)) = \int_{-\infty}^{\infty}(ax_1(t)+bx_2(t))e^{-j\omega t}\,dt = a\int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}\,dt + b\int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}\,dt$$

$$= aX_1(\omega) + bX_2(\omega)$$

The linearity of integration follows from the convergence of the integral.

1.1.3 Prove: for $a > 0$, $\mathcal{F}(x(at)) = \frac{1}{a}X(\frac{\omega}{a})$

$$\mathcal{F}(x(at) = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} x(at)e^{-j\omega\frac{t}{a}\cdot a}\frac{a}{a}\,dt = \frac{1}{a}\int_{-\infty}^{\infty} x(at)e^{(-j\frac{\omega}{a})t\cdot a}a\,dt$$

Switching variables to $u = at$ we get:

$$\frac{1}{a}\int_{-\infty}^{\infty} x(u)e^{(-j\frac{\omega}{a})u}\,du = \frac{1}{a}X(\frac{\omega}{a})$$

1.1.4 Prove $\mathcal{F}(x(t-t_0)) = X(\omega)e^{-j\omega t_0}$

$$\mathcal{F}(x(t-t_0)) = \int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t+(-j\omega t_0)-(-j\omega t_0)}\,dt = e^{-j\omega t_0}\int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega(t-t_0)}\,dt$$

Switching variables to $u = t - t_0$:

$$e^{-j\omega t_0}\int_{-\infty}^{\infty} x(u)e^{-j\omega u}\,du = e^{-j\omega t_0}X(\omega)$$

As desired.

The effect of time shifting on the amplitude spectrum:

$|\mathcal{F}(x(t-t_0))| = |X(\omega)e^{-j\omega t_0}| = |X(\omega)| \cdot |cos(\omega t_0) + jsin(\omega t_0)| = |X(\omega)|$

As expected, no difference in the amplitude spectrum.

Now for the effect in the phase spectrum we'll use Euler's formula again:

$$< \mathcal{F}(x(t-t_0)) =< (X(\omega)cos(\omega t_0) - X(\omega)jsin(\omega t_0))$$

Meaning there is a shift of ωt_0 in phase.

1.1.5 Prove $rect(\frac{t}{\tau}) = \tau sinc(\frac{\omega\tau}{2})$

$$\mathcal{F}(rect(\frac{t}{\tau})) = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-j\omega t}\,dt = \frac{e^{-j\omega t}}{-j\omega}\Big|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = \frac{1}{-j\omega}(e^{-j\omega\frac{\tau}{2}} - e^{j\omega\frac{\tau}{2}})$$

Now by Euler's formula:

$$\frac{1}{-j\omega}(e^{-j\omega\frac{\tau}{2}} - e^{j\omega\frac{\tau}{2}}) = \frac{1}{-j\omega}(cos(-\omega\frac{\tau}{2}) + jsin(-\omega\frac{\tau}{2}) - cos(\omega\frac{\tau}{2}) - jsin(\omega\frac{\tau}{2}))$$

$$= \frac{-2jsin(\omega\frac{\tau}{2})}{-j\omega} = 2\frac{\tau}{2}\frac{sin(\omega\frac{\tau}{2})}{\frac{\omega\tau}{2}} = \tau sinc(\frac{\omega\tau}{2})$$

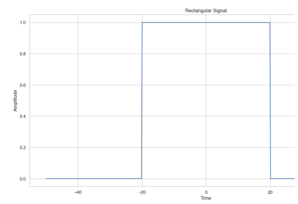

Figure 1: the rect function

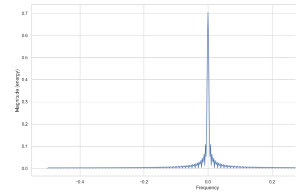

Figure 2: the rect function magnitude

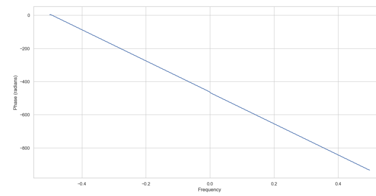

Figure 3: the rect function phase

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We'll find the values of D_n by the formula: $D_n = \frac{1}{T_0}\int_{T_0} \delta(t)_{T_0} e^{-j\omega_0 t}\, dt$ where $\omega_0 = \frac{2\pi}{T_0}$ and T_0 is the cycle of the function δ_{T_0}

$$D_n = \frac{1}{T_0}\int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} \delta(t)_{T_0} e^{-j\omega_0 t}\, dt = \frac{1}{T_0}\int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} \delta(t) e^{-j\omega_0 t}\, dt = \frac{1}{T_0}$$

The interval $[D_n, D_{n+1}]$ is actually the point $\frac{1}{T_0}$.

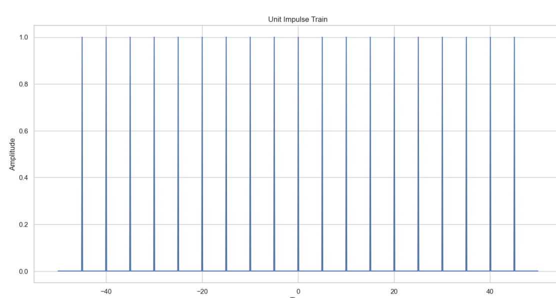

Figure 4: $\delta_{T_0}(t)$

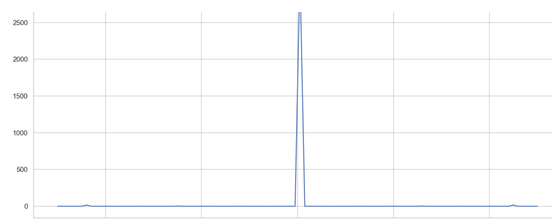

Figure 5: $X(\omega)$

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We can write the function as the following:

$$x(t) = \sum_{n=-\infty}^{\infty} rect(\frac{t - 2\pi n}{\pi}))$$

From the linearity property:

$$\mathcal{F}(x(t)) = \mathcal{F}(\sum_{n=-\infty}^{\infty} rect(\frac{t-2\pi n}{\pi})) = \sum_{n=-\infty}^{\infty} \mathcal{F}(rect(\frac{t-2\pi n}{\pi}))$$

From the time shifting property:

$$= \sum_{n=-\infty}^{\infty} F(rect(\frac{t}{\pi})) \cdot e^{-j\omega 2n}$$

From 1.e.ii:

$$= \sum_{n=-\infty}^{\infty} \pi sinc(\frac{\omega\pi}{2}) \cdot e^{-j\omega 2n}$$

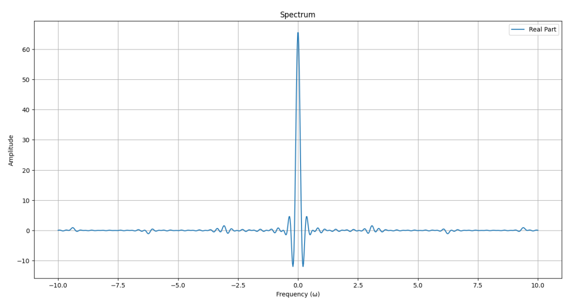

Figure 6: spectrum

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4.1 Find $\mathcal{F}(x(t))$ when $x(t) = e^{-at}u(t)$

$$\mathcal{F}(x(t)) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\dot{d}t = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\dot{d}t$$

$$= \int_{0}^{\infty} x(t)e^{-(a+j\omega)t}\dot{d}t$$

Let's calculate the integral:

$$\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\Big|_0^\infty = -\frac{e^0}{-(a+j\omega)} = \frac{1}{(a+j\omega)}$$

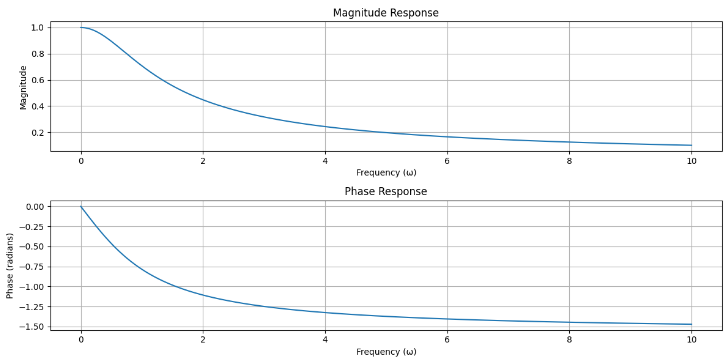

Figure 7: magnitude and phase

4.2 Draw the magnitude and the phase

In Figure 7 we can see the drawing of the magnitude and the phase.

4.3 What kind of filter can it be used for

It behaves as a low-pass filter. The cutoff frequency ω_c of the filter is equal to the parameter α. This type of filter is suitable for applications where high-frequency components need to be attenuated, and lower-frequency components are of interest.

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5.1 Prove $Z[x_1[t] * x_2[t]] = X_1[z] \cdot X_2[z]$

$$Z(x_1[n] * x_2[n]) = \sum_{n=-\infty}^{\infty} x_1[n] * x_2[n] z^{-n} = \sum_{n=-\infty}^{\infty} \sum_{\tau=-\infty}^{\infty} x_1[\tau] x_2[n-\tau] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{\tau=-\infty}^{\infty} x_1[\tau] x_2[n-\tau] z^{-n+\tau-\tau} = \sum_{\tau=-\infty}^{\infty} x_1[\tau] z^{-\tau} \sum_{n=-\infty}^{\infty} x_2[n-\tau] z^{-(-n-\tau)} = X_1[z] \cdot X_2[z]$$

5.2 Prove $Z(a^n x[n]) = X(\frac{z}{n})$

$$Z(a^n x[n]) = \sum_{n=-\infty}^{\infty} x[n] a^n z^{-n} = \sum_{n=-\infty}^{\infty} x[n](a^{-1}z)^{-n} = \sum_{n--\infty}^{\infty} x[n](\frac{z}{a})^{-n} = X[\frac{z}{a}]$$

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6.1 How many samples are there in one period (what is N_0)

We saw in class the calculation for $x[n] = sin(0.1\pi n)$. Same works here. The period of the signal is the smallest positive integer N_0 such that for all n:

$$x[n] = x[n + N_0]$$

Therefore, the period of $x[n]$ is given by:

$$0.1\pi N_0 = 2\pi \rightarrow N_0 = 20$$

6.2 What is the discrete time Fourier transform of $x[n]$

Discrete time Fourier series of $x[n] = cos(0.1\pi n)$:
The spectral components in the fundamental frequency range:

$$x[n] = \sum_{n=-10}^{9} D_r e^{j0.1\pi rn}$$

Now, based on:

$$D_r = \sum_{n=(N_0)} e^{jr\Omega_0 n}$$

We have:

$$D_r = \frac{1}{20} \sum_{n=-10}^{9} cos0.1\pi n \cdot e^{-0.1\pi rn} = \frac{1}{20} \sum_{n=-10}^{9} \frac{1}{2j} (e^{j0.1\pi n} + e^{-j0.1\pi n}) \cdot e^{-0.1\pi rn}$$

$$= \frac{1}{40j} (\sum_{n=-10}^{9} e^{j0.1\pi n(1-r)} + \sum_{n=-10}^{9} e^{j0.1\pi n(1+r)}$$

In these sums, r takes on all values between 10 and 9. From (1), it follows that the first sum on the right-hand side is zero for all values of r except r = 1, when the sum is equal to N_0=20. Similarly, the second sum is zero for all values of r except r = 1, when it is equal to $N0$=20. Therefore,

$$D_1 = \frac{1}{2j}, D_{-1} = \frac{1}{2j}$$

and all other coefficients are 0.

$$x[n] = cos0.1\pi n = \frac{1}{2j}(e^{j0.1\pi n} + e^{-j0.1\pi n})$$

Here the fundamental frequency is 0.1π and there are only two nonzero components:

$$D_1 = \frac{1}{2j} = \frac{e^{\frac{-j\pi}{2}}}{2}, D_{-1} = \frac{1}{2j} = \frac{e^{\frac{j\pi}{2}}}{2}$$

$$\sum_{n=0}^{N_0-1} e^{-jk\Omega_0 n} = \left\{ \begin{array}{lr} N_0 & k = 0, +-N_0, +-2N_0 \\ 0 & else \end{array} \right\} \tag{1}$$