Advanced topics in audio processing using deep learning - HW1

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1 Analytical part

1.1 1

1.1.1 Prove $\mathcal{F}(x_1(t) * x_2(t)) = X_1(\omega)X_2(\omega)$

By the definitions of the transform and the convolution operation:

$$\mathcal{F}(x_1(t) * x_2(t)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau \cdot e^{-j\omega t}\,dt$$

By changing order of integration we have:

$$\int_{-\infty}^{\infty} x_1(\tau)\int_{-\infty}^{\infty} x_2(t-\tau)e^{-j\omega t}\,dt\,d\tau = \int_{-\infty}^{\infty} x_1(\tau)\mathcal{F}(x_2(\tau-t))\,d\tau$$

From the time shifting property (that we'll prove on section d) we get:

$$\int_{-\infty}^{\infty} x_1(\tau)\mathcal{F}(x_2(\tau-t))\,d\tau = \int_{-\infty}^{\infty} x_1(\tau)X_2(\omega)e^{-j\omega\tau}\,d\tau = X_2(\omega)\int_{-\infty}^{\infty} x_1(\tau)e^{-j\omega\tau}\,d\tau = X_2(\omega)X_1(\omega)$$

1.1.2 Prove $\mathcal{F}(ax_1(t) + bx_2(t)) = aX_1(\omega) + bX_2(\omega)$

By definition:

$$\mathcal{F}(ax_1(t)+bx_2(t)) = \int_{-\infty}^{\infty}(ax_1(t)+bx_2(t))e^{-j\omega t}\,dt = a\int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}\,dt + b\int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}\,dt$$

$$= aX_1(\omega) + bX_2(\omega)$$

The linearity of integration follows from the convergence of the integral.

1.1.3 Prove: for $a > 0$, $\mathcal{F}(x(at)) = \frac{1}{a}X(\frac{\omega}{a})$

$$\mathcal{F}(x(at) = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} x(at)e^{-j\omega \frac{t}{a}\cdot a}\frac{a}{a}\,dt = \frac{1}{a}\int_{-\infty}^{\infty} x(at)e^{(-j\frac{\omega}{a})t\cdot a}a\,dt$$

Switching variables to $u = at$ we get:

$$\frac{1}{a}\int_{-\infty}^{\infty} x(u)e^{(-j\frac{\omega}{a})u}\,du = \frac{1}{a}X(\frac{\omega}{a})$$

1.1.4 Prove $\mathcal{F}(x(t-t_0)) = X(\omega)e^{-j\omega t_0}$

$$\mathcal{F}(x(t-t_0)) = \int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t+(-j\omega t_0)-(-j\omega t_0)}\,dt = e^{-j\omega t_0}\int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega(t-t_0)}\,dt$$

Switching variables to $u = t - t_0$:

$$e^{-j\omega t_0}\int_{-\infty}^{\infty} x(u)e^{-j\omega u}\,du = e^{-j\omega t_0}X(\omega)$$

As desired.

The effect of time shifting on the amplitude spectrum:

$|\mathcal{F}(x(t-t_0))| = |X(\omega)e^{-j\omega t_0}| = |X(\omega)| \cdot |cos(\omega t_0) + jsin(\omega t_0)| = |X(\omega)|$

As expected, no difference in the amplitude spectrum.

Now for the effect in the phase spectrum we'll use Euler's formula again:

$$< \mathcal{F}(x(t-t_0)) =< (X(\omega)cos(\omega t_0) - X(\omega)jsin(\omega t_0))$$

Meaning there is a shift of ωt_0 in phase.

1.1.5 Prove $rect(\frac{t}{\tau}) = \tau sinc(\frac{\omega\tau}{2})$

$$\mathcal{F}(rect(\frac{t}{\tau})) = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-j\omega t}\,dt = \frac{e^{-j\omega t}}{-j\omega}\Big|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = \frac{1}{-j\omega}(e^{-j\omega\frac{\tau}{2}} - e^{j\omega\frac{\tau}{2}})$$

Now by Euler's formula:

$$\frac{1}{-j\omega}(e^{-j\omega\frac{\tau}{2}} - e^{j\omega\frac{\tau}{2}}) = \frac{1}{-j\omega}(cos(-\omega\frac{\tau}{2}) + jsin(-\omega\frac{\tau}{2}) - cos(\omega\frac{\tau}{2}) - jsin(\omega\frac{\tau}{2}))$$

$$= \frac{-2jsin(\omega\frac{\tau}{2})}{-j\omega} = 2\frac{\tau}{2}\frac{sin(\omega\frac{\tau}{2})}{\frac{\omega\tau}{2}} = \tau sinc(\frac{\omega\tau}{2})$$

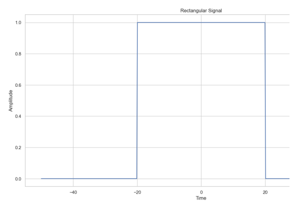

Figure 1: the rect function

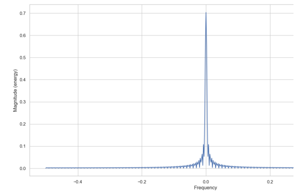

Figure 2: the rect function magnitude

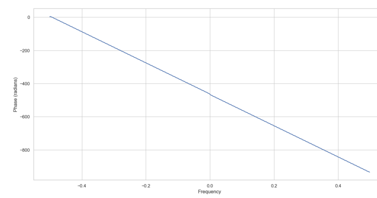

Figure 3: the rect function phase

2

We'll find the values of D_n by the formula: $D_n = \frac{1}{T_0}\int_{T_0} \delta(t)_{T_0} e^{-j\omega_0 t}\, dt$ where $\omega_0 = \frac{2\pi}{T_0}$ and T_0 is the cycle of the function δ_{T_0}

$$D_n = \frac{1}{T_0}\int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} \delta(t)_{T_0} e^{-j\omega_0 t}\, dt = \frac{1}{T_0}\int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} \delta(t) e^{-j\omega_0 t}\, dt = \frac{1}{T_0}$$

The interval $[D_n, D_{n+1}]$ is actually the point $\frac{1}{T_0}$.

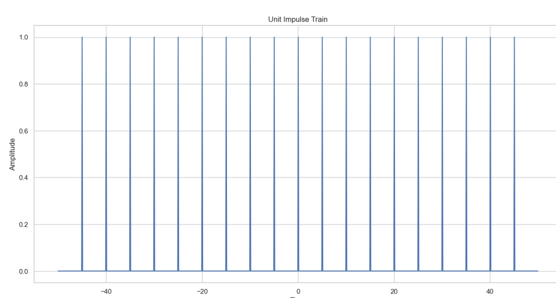

Figure 4: $\delta_{T_0}(t)$

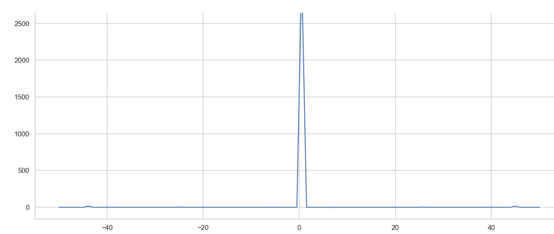

Figure 5: $X(\omega)$

3

We can write the function as the following:

$$x(t) = \sum_{n=-\infty}^{\infty} rect(\frac{t - 2\pi n}{\pi}))$$

From the linearity property:

$$\mathcal{F}(x(t)) = \mathcal{F}(\sum_{n=-\infty}^{\infty} rect(\frac{t-2\pi n}{\pi})) = \sum_{n=-\infty}^{\infty} \mathcal{F}(rect(\frac{t-2\pi n}{\pi}))$$

From the time shifting property:

$$= \sum_{n=-\infty}^{\infty} F(rect(\frac{t}{\pi})) \cdot e^{-j\omega 2n}$$

From 1.e.ii:

$$= \sum_{n=-\infty}^{\infty} \pi sinc(\frac{\omega\pi}{2}) \cdot e^{-j\omega 2n}$$

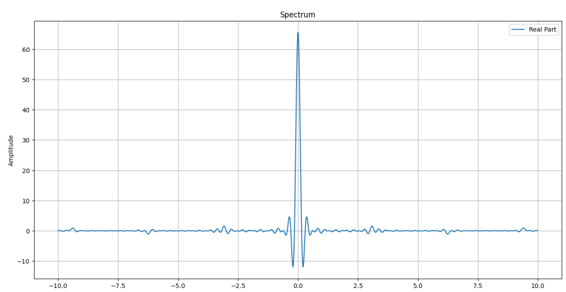

Figure 6: spectrum

4

4.1 Find $\mathcal{F}(x(t))$ when $x(t) = e^{-at}u(t)$

$$\mathcal{F}(x(t)) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\dot{d}t = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\dot{d}t$$

$$= \int_{0}^{\infty} x(t)e^{-(a+j\omega)t}\dot{d}t$$

Let's calculate the integral:

$$\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\Big|_0^\infty = -\frac{e^0}{-(a+j\omega)} = \frac{1}{(a+j\omega)}$$

4.2 Draw the magnitude and the phase

In Figure 7 we can see the drawing of the magnitude and the phase. The calculations: Denote $\mathcal{F}(x) = \frac{a-j\omega}{a^2-j\omega^2}$

$$|\mathcal{F}(x)| = |\frac{a-j\omega}{a^2-j\omega^2}| = \sqrt{(\frac{a}{a^2+\omega^2} + \frac{-\omega}{a^2+\omega^2})} = \sqrt{(\frac{a^2+\omega^2)}{(a^2+\omega^2)^2}} = \sqrt{(\frac{1}{(a^2+\omega^2)}} = \frac{1}{\sqrt{(a^2+\omega^2)}}$$

phase:

$$(\frac{0}{1}) - tan^{-1}(\frac{\omega}{a}) = -tan^{-1}(\frac{\omega}{a}) = arctan(\frac{\omega}{a}))$$

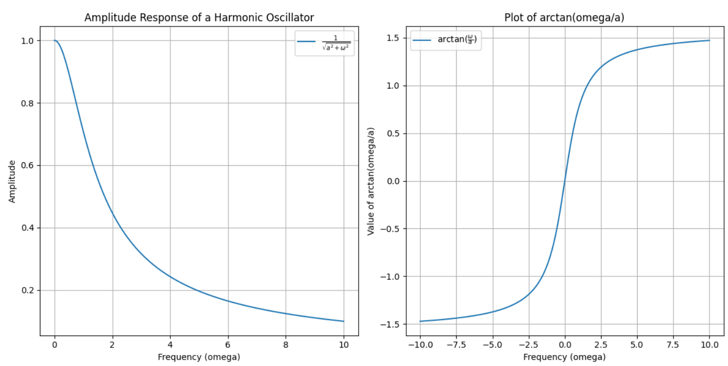

Figure 7: magnitude and phase

4.3 What kind of filter can it be used for

It behaves as a low-pass filter. The cutoff frequency ω_c of the filter is equal to the parameter α. This type of filter is suitable for applications where high-frequency components need to be attenuated, and lower-frequency components are of interest.

5

5.1 Prove $Z[x_1[t] * x_2[t]] = X_1[z] \cdot X_2[z]$

$$Z(x_1[n] * x_2[n]) = \sum_{n=-\infty}^{\infty} x_1[n] * x_2[n] z^{-n} = \sum_{n=-\infty}^{\infty} \sum_{\tau=-\infty}^{\infty} x_1[\tau] x_2[n-\tau] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{\tau=-\infty}^{\infty} x_1[\tau] x_2[n-\tau] z^{-n+\tau-\tau} = \sum_{\tau=-\infty}^{\infty} x_1[\tau] z^{-\tau} \sum_{n=-\infty}^{\infty} x_2[n-\tau] z^{-(-n-\tau)} = X_1[z] \cdot X_2[z]$$

5.2 Prove $Z(a^n x[n]) = X(\frac{z}{n})$

$$Z(a^n x[n]) = \sum_{n=-\infty}^{\infty} x[n] a^n z^{-n} = \sum_{n=-\infty}^{\infty} x[n](a^{-1}z)^{-n} = \sum_{n--\infty}^{\infty} x[n](\frac{z}{a})^{-n} = X[\frac{z}{a}]$$

6

6.1 How many samples are there in one period (what is N_0)

We saw in class the calculation for $x[n] = sin(0.1\pi n)$. Same works here. The period of the signal is the smallest positive integer N_0 such that for all n:

$$x[n] = x[n + N_0]$$

Therefore, the period of $x[n]$ is given by:

$$0.1\pi N_0 = 2\pi \rightarrow N_0 = 20$$

6.2 What is the discrete time Fourier transform of $x[n]$

Discrete time Fourier series of $x[n] = cos(0.1\pi n)$:
The spectral components in the fundamental frequency range:

$$x[n] = \sum_{n=-10}^{9} D_r e^{j0.1\pi rn}$$

Now, based on:

$$D_r = \sum_{n=(N_0)} e^{jr\Omega_0 n}$$

We have:

$$D_r = \frac{1}{20} \sum_{n=-10}^{9} cos 0.1\pi n \cdot e^{-0.1\pi rn} = \frac{1}{20} \sum_{n=-10}^{9} \frac{1}{2j} (e^{j0.1\pi n} + e^{-j0.1\pi n}) \cdot e^{-0.1\pi rn}$$

$$= \frac{1}{40j} (\sum_{n=-10}^{9} e^{j0.1\pi n(1-r)} + \sum_{n=-10}^{9} e^{j0.1\pi n(1+r)}$$

In these sums, r takes on all values between 10 and 9. From (1), it follows that the first sum on the right-hand side is zero for all values of r except r = 1, when the sum is equal to N_0=20. Similarly, the second sum is zero for all values of r except r = 1, when it is equal to $N0$=20. Therefore,

$$D_1 = \frac{1}{2j}, D_{-1} = \frac{1}{2j}$$

and all other coefficients are 0.

$$x[n] = cos0.1\pi n = \frac{1}{2j}(e^{j0.1\pi n} + e^{-j0.1\pi n})$$

Here the fundamental frequency is 0.1π and there are only two nonzero components:

$$D_1 = \frac{1}{2j} = \frac{e^{\frac{-j\pi}{2}}}{2}, D_{-1} = \frac{1}{2j} = \frac{e^{\frac{j\pi}{2}}}{2}$$

$$\sum_{n=0}^{N_0-1} e^{-jk\Omega_0 n} = \left\{ \begin{matrix} N_0 & k = 0, +-N_0, +-2N_0 \\ 0 & else \end{matrix} \right\} \tag{1}$$

7 Discrete Signals

7.1 Given $F_s = 8000Hz$ to which frequency $10Khz$ will be aliased to?

According to Nyquist–Shannon Sampling Theorem and "DSP Lecture 2" slides 65 - 68 about Aliasing and Sampling Rate: a. To figure out where 10 kHz goes after aliasing, we can use the Nyquist–Shannon Sampling. It basically says the highest frequency we can accurately catch is half of how quickly we are sampling (sampling frequency). In this case, the Nyquist frequency is 4 kHz (8 kHz / 2). If 10 kHz gets aliased, it jumps back within the Nyquist range. So, the new frequency after aliasing would be 4 kHz - (10 kHz - 4 kHz) = **2 kHz**

7.2 How could you prevent the aliasing if we had the analogue signal? Explain shortly in words

To prevent aliasing in the analog signal, we should ensure that the input signal is bandlimited, meaning that its frequency content is limited to half of the sampling frequency (Nyquist frequency) or lower. This can be achieved by using an anti-aliasing filter before the analog-to-digital conversion process. The anti-aliasing filter removes high-frequency components beyond the Nyquist frequency, preventing them from being aliased into lower frequencies during the sampling process.

7.3 Stereo hearing

We hear a stereo effect in the audio, characterized by periods of silence, followed by audio in the left ear, simultaneous playback in both ears, and finally audio in the right ear. The intentional time shifts / delays make it feel like hearing the sound move around. It's like a journey for the ears, mimicking how we naturally figure out where sounds are coming from. So, when you hear the audio, it's a bit like the ears are playing a little game, following the sound in different directions.

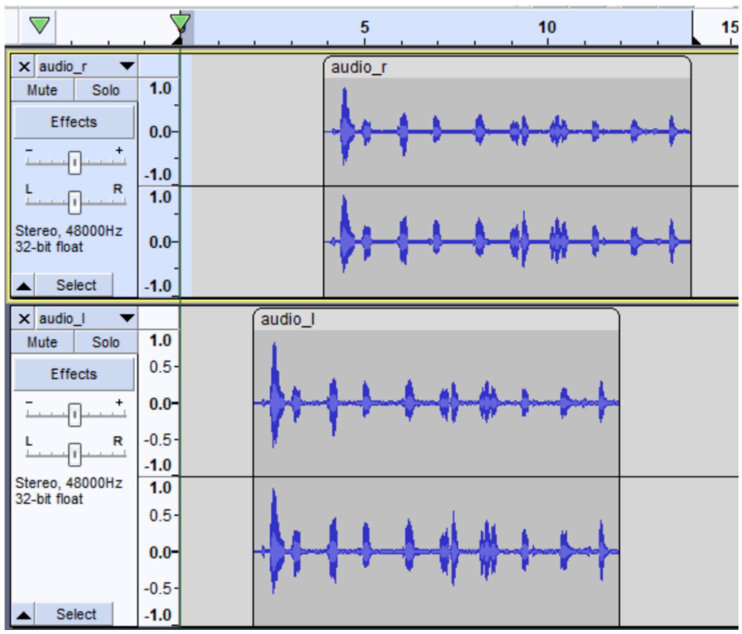

Figure 8: channels plots