Approximation using Fourier Frames

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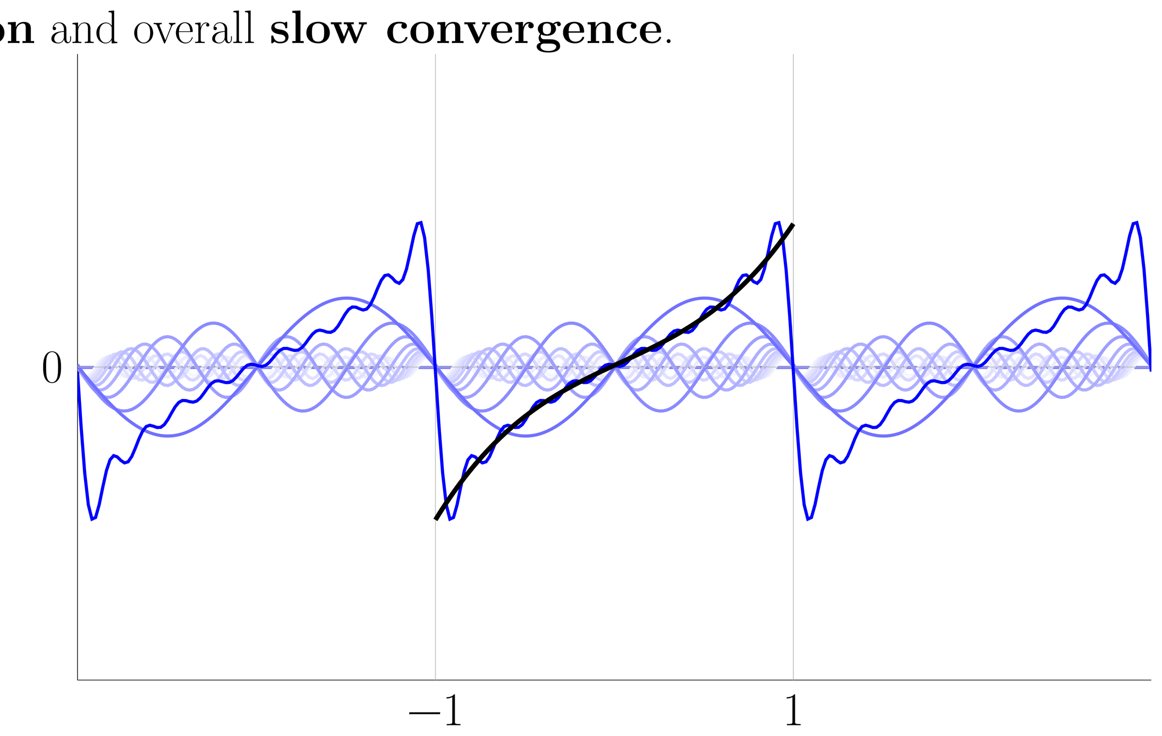
Frame approximation

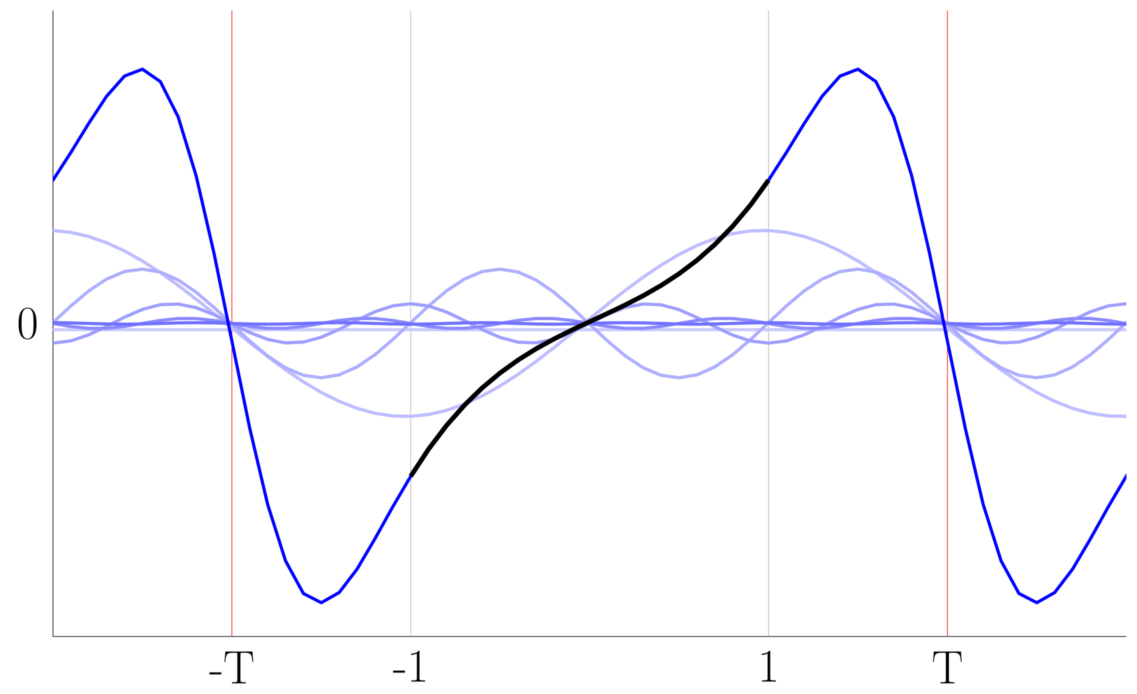
Our aim is to approximate a function $f(x)$ defined on $[-1,1]$, by complex exponentials

$$f(x) \approx \sum_k a_k \phi_k(x), \quad \phi_k(x) = e^{ikcx}$$

Approximation in the traditional Fourier basis on $[-1,1]$ suffers from the **Gibbs-phenomenon** and overall **slow convergence**.



By approximating on an extended interval $[-T,T]$, the periodicity constraint of the approximation is lifted, and it will **converge exponentially** under certain conditions[1].



This approximation is obtained by solving an $M \times N$ least squares problem

$$\begin{bmatrix} f(x_1) \\ \vdots \\ f(x_M) \end{bmatrix} = \begin{bmatrix} \phi_1(x_1) & \cdots & \phi_N(x_1) \\ \vdots & \ddots & \\ \phi_1(x_M) & & \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix}.$$

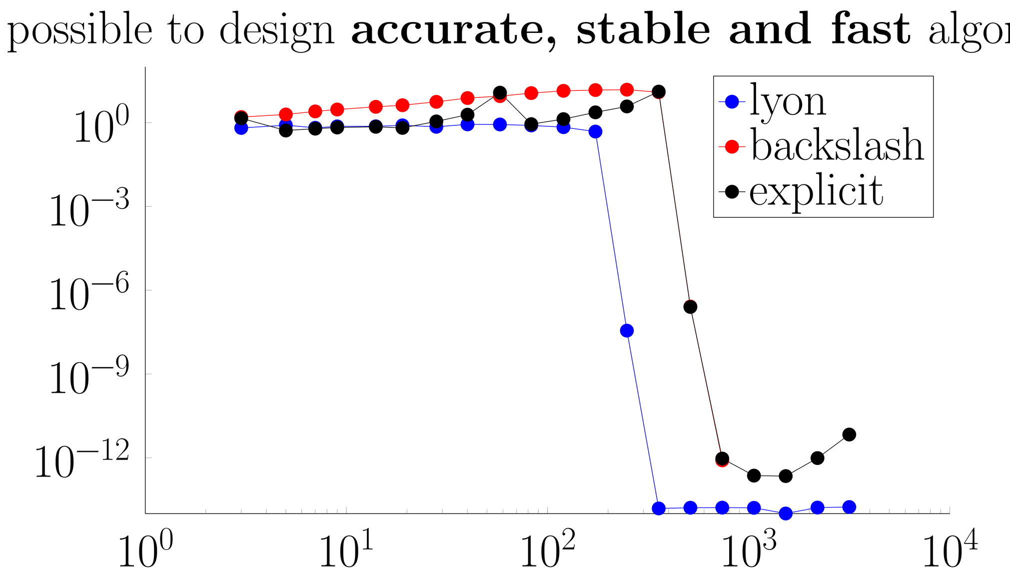
Problem The restriction of the Fourier basis on $[-T,T]$ to $[-1,1]$ constitutes a frame, and the inherent redundancy causes the least squares system to be **severely ill-conditioned**.

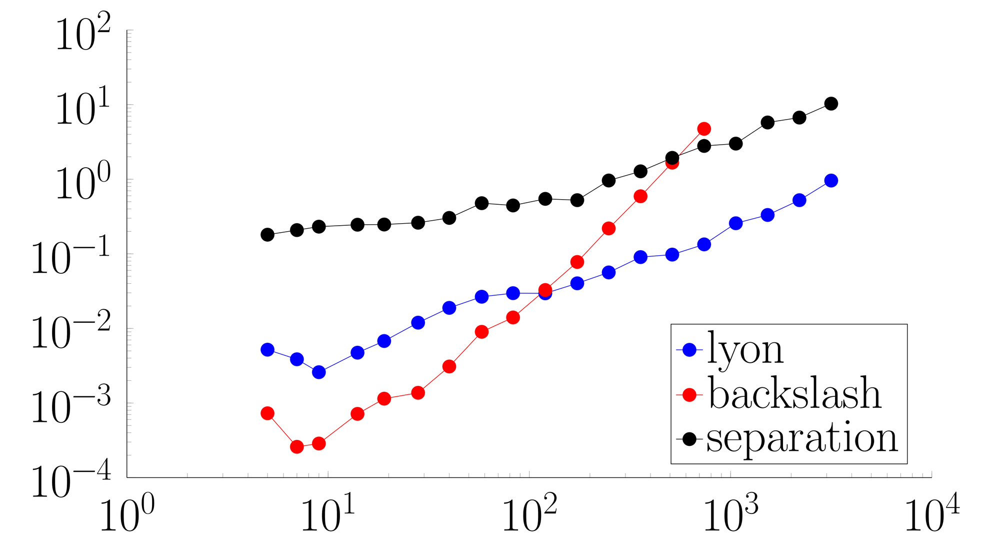
Numerical results

The benefits of the frame approximation are apparent:

- **Fast convergence**, when compared to traditional Fourier methods.
- **Good resolution power**, when compared to Chebychev polynomials.
- **Equispaced data points** avoid severe time-step restrictions when discretising in Chebychev points.

Furthermore, it is possible to design **accurate, stable and fast** algorithms.





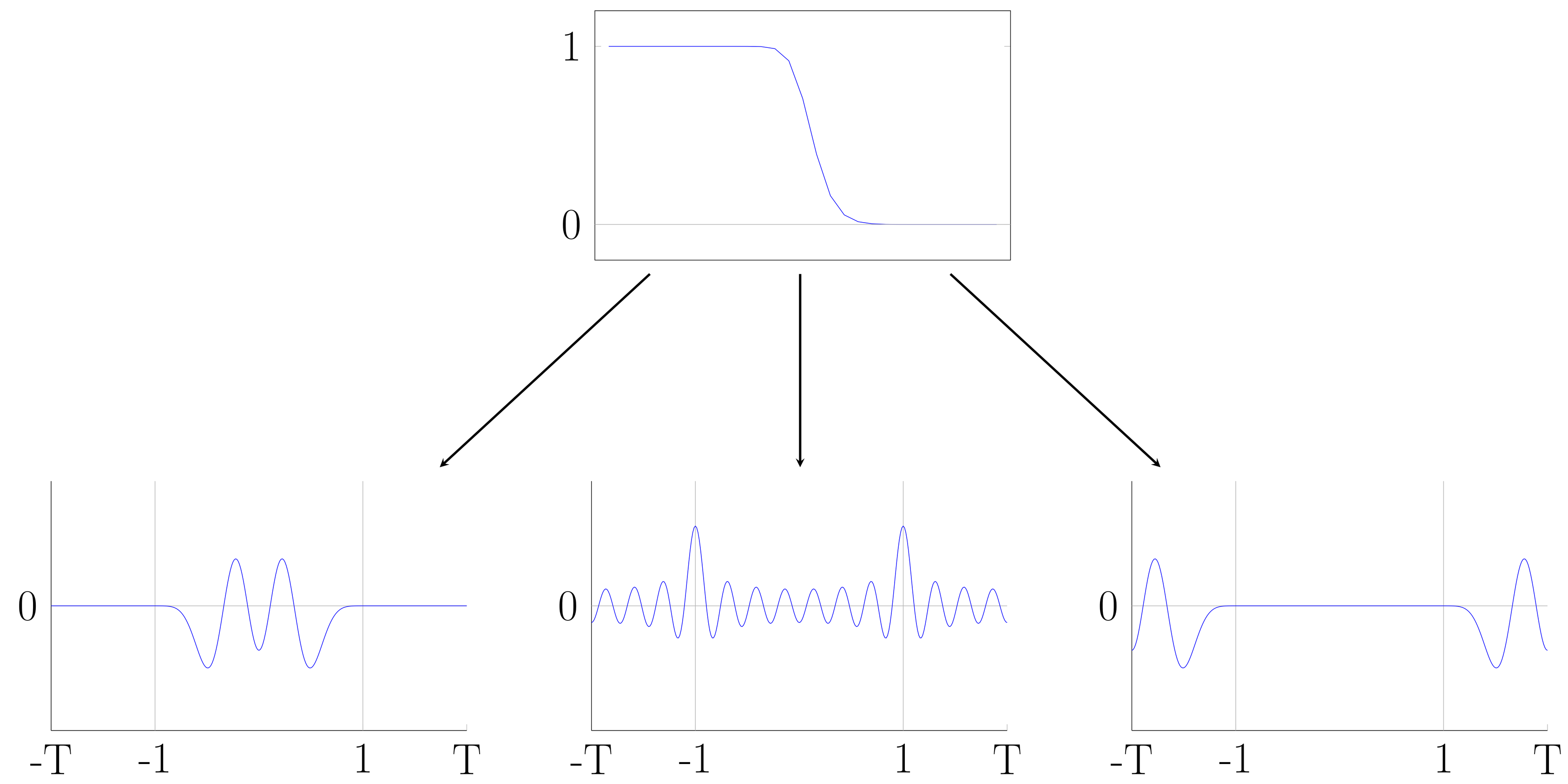
The fastest algorithms are on par with the $O(N \log N)$ complexity of the FFT.

Fast Algorithms

The key to fast algorithms lies in the SVD of the least squares matrix:

$$\begin{bmatrix} \phi_1(x_1) & \cdots & \phi_N(x_1) \\ \vdots & \ddots & \\ \phi_1(x_M) & & \end{bmatrix} = U\Sigma V^*$$

The ill-conditioning is explained by the **singular values** and the associated **singular vectors**:

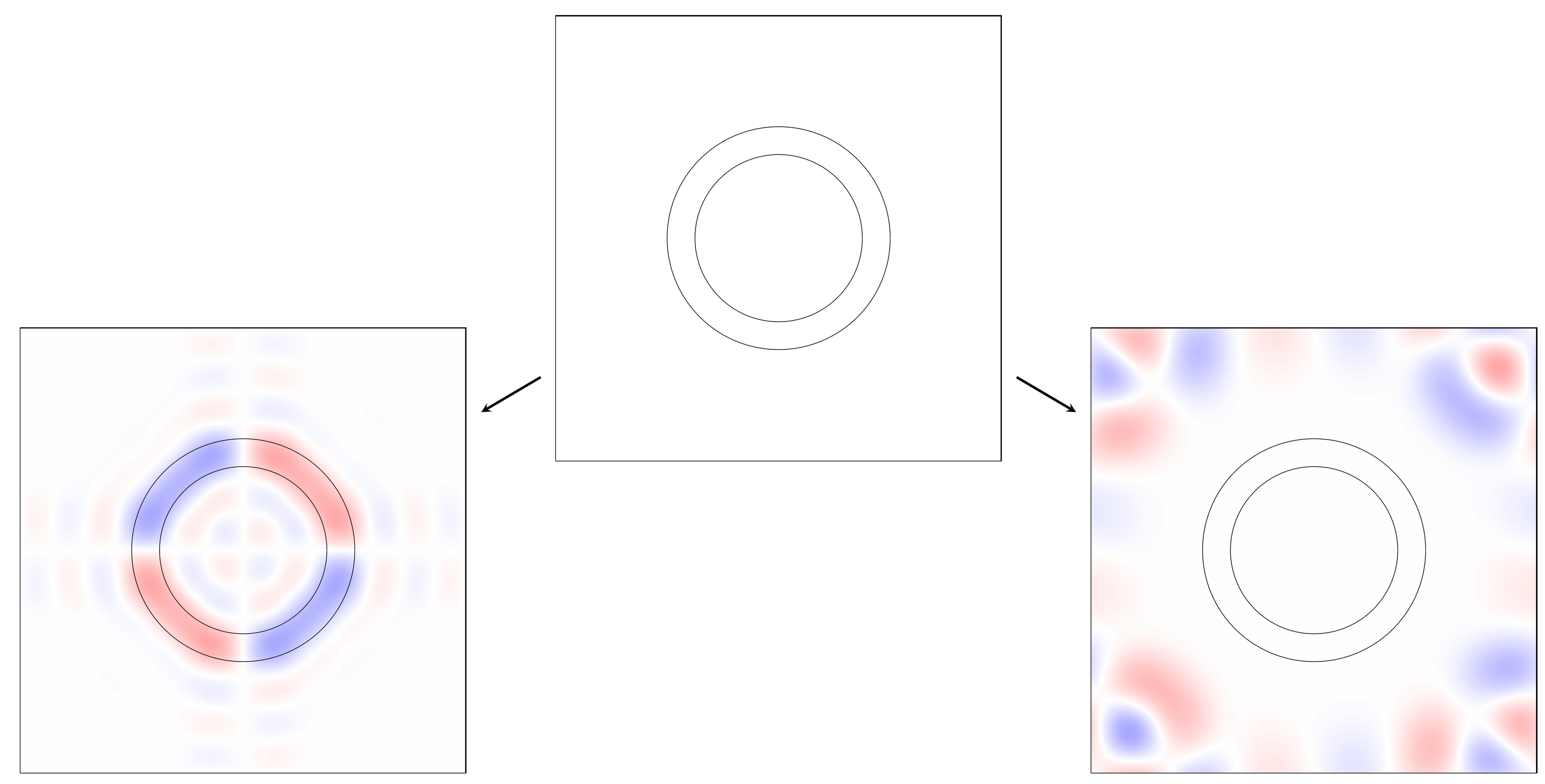


These singular values and vectors are studied in **Prolate Spheroidal Wave** theory[2], and have a number of interesting properties:

- Double orthogonality, over the periodicity interval and it's restriction
- Compact frequency support
- Number of intermediate singular values grows logarithmically

The compact frequency support allows us to **separate** the "good" from the "bad" singular values. The cost of such algorithms when implemented with FFTs grows with the number of intermediate eigenvalues $O(N \log N)$.

Extensions to 2D The 2D problem produces singular vectors with similar properties.



Problem The intermediate singular values grow as $\sqrt{N}$, restricting the minimal complexity to $O(N^{3/2})$.

Outlook

- Applications in (1D) PDE solvers
- Circumventing 2D restrictions

References

[1] Daan Huybrechs. On the Fourier extension of nonperiodic functions. *SIAM Journal on Numerical Analysis*, 47(6):4326–4355, 2010.

[2] D Slepian. Prolate spheroidal wave functions, Fourier analysis, and uncertainty -V: The Discrete Case. *Bell Syst. Tech. J*, 1978.

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