Accuracy of “original” MPM

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TU Delft

Outline

- “Original” MPM
- Numerical accuracy
- Benchmarks
  - Vibrating bar
  - Oedometer
- Sources of spatial errors
  - Analogy with FEM
  - Grid crossing
- Outlook

“Original” MPM

- Modified Lagrangian algorithm
- Euler-Cromer time integration
- Piecewise linear basis functions
- No additional interventions

“Original” MPM

- Modified Lagrangian algorithm
- Euler-Cromer time integration
- Piecewise linear basis functions
- No additional interventions
- Own MATLAB implementation
- 1D (UL)FEM/MPM
- Simplified version of Deltares’ code

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Numerical accuracy

Numerical Approximation

$$u_{ex} = u_{num} + \mathcal{O}(\Delta x^n) + \mathcal{O}(\Delta t^m)$$

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Numerical accuracy

Numerical Approximation

$$u_{ex} = u_{num} + \mathcal{O}(\Delta x^n)$$

Spatial accuracy

n	Grid size	Error
	Δx	E
1	$\Delta x/2$	$E/2$

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Numerical accuracy

Numerical Approximation

$$u_{ex} = u_{num} + \mathcal{O}(\Delta x^n)$$

Spatial accuracy

n	Grid size	Error
	Δx	E
2	$\Delta x/2$	$E/4$

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Numerical accuracy

Numerical Approximation

$$u_{ex} = u_{num} + \mathcal{O}(\Delta x^n)$$

Spatial accuracy

n	Grid size	Error
	Δx	E
k	$\Delta x/2$	$E/2^k$

Numerical accuracy

Numerical Approximation

$$u_{ex} = u_{num} + \mathcal{O}(\Delta x^n)$$

Spatial accuracy

n	Grid size	Error
	Δx	E
k	$\Delta x/2$	$E/2^k$

RMS Error

$$Error_{RMS} = \sqrt{\frac{1}{n_p}\sum_{p=1}^{n_p}\left(u_{num}(x_p,t) - u_{ex}(x_p,t)\right)^2}$$

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Numerical Accuracy

Temporal accuracy

MPM is first order accurate in time, i.e. $m = 1$.

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Temporal accuracy

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Spatial accuracy

Order of spatial accuracy	Source
2	Gong (2015); Steffen (2008)
0.5 - 1	Tran (2010)
lack of spatial convergence	Gong (2015); Steffen (2008)

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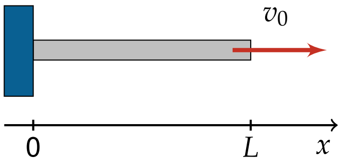
Vibrating bar



$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2}$$

Boundary conditions:

$$u(0,t) = 0$$

$$\frac{\partial u}{\partial x}(L,t) = 0$$

Initial conditions:

$$u(x,0) = 0$$

$$\frac{\partial u}{\partial t}(x,0) = v_0 \sin\left(\frac{\pi x}{2L}\right)$$

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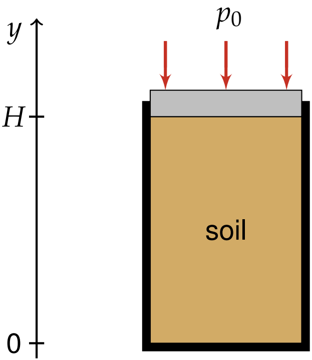
Oedometer



$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial^2 y} - g$$

Boundary conditions:

$$u(0,t) = 0$$

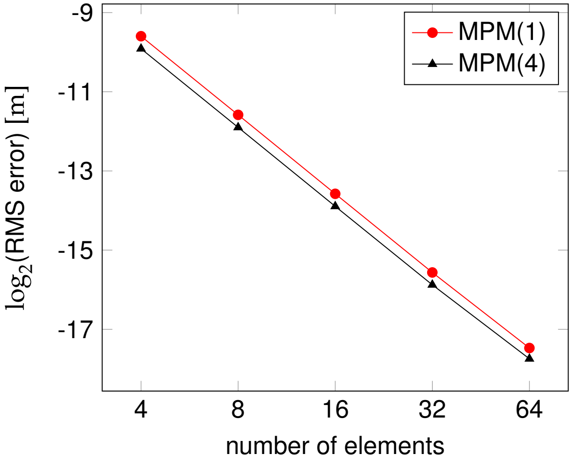
$$\frac{\partial u}{\partial y}(H,t) = \frac{p_0}{E}$$

Initial conditions:

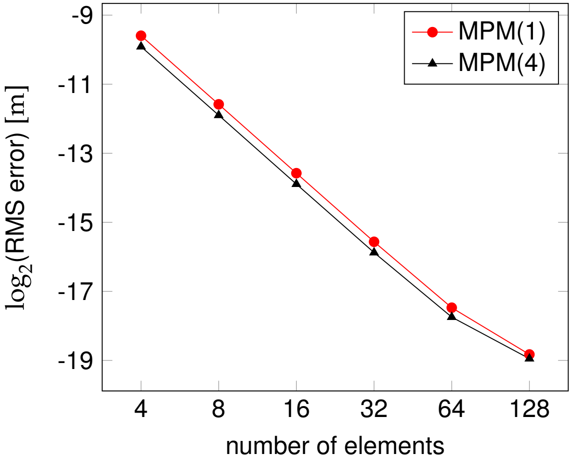
$$u(y,0) = 0$$

$$\frac{\partial u}{\partial t}(y,0) = 0$$

Accuracy: vibrating bar



Accuracy: vibrating bar

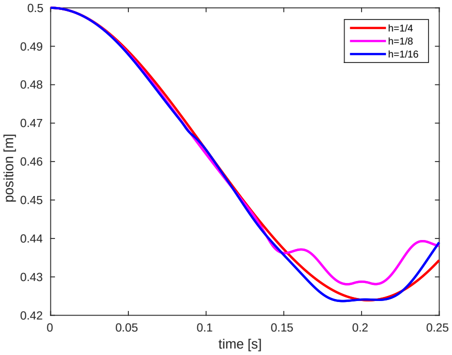


Accuracy: oedometer

Richardson's extrapolation

The order of accuracy n is obtained from

$$\frac{u_{num}(2h) - u_{num}(4h)}{u_{num}(h) - u_{num}(2h)} = 2^n.$$



Accuracy: oedometer

Richardson's extrapolation

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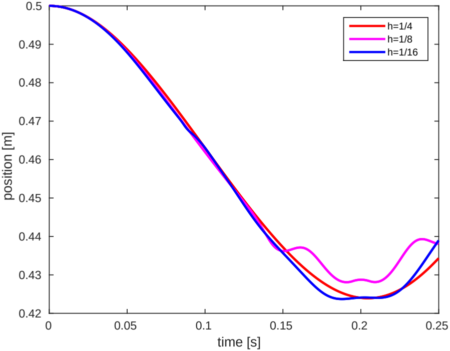
$$\frac{u_{num}(2h) - u_{num}(4h)}{u_{num}(h) - u_{num}(2h)} = 2^n.$$

Conclusion

Lack of spatial convergence.



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FEM: oedometer

Theoretical order of accuracy

2nd order accurate for PW basis functions when integrated using Gauss rule with 1 Gauss point per element[1].

[1] Van Kan (2008)

FEM: oedometer

Theoretical order of accuracy

2nd order accurate for PW basis functions when integrated using Gauss rule with 1 Gauss point per element[1].

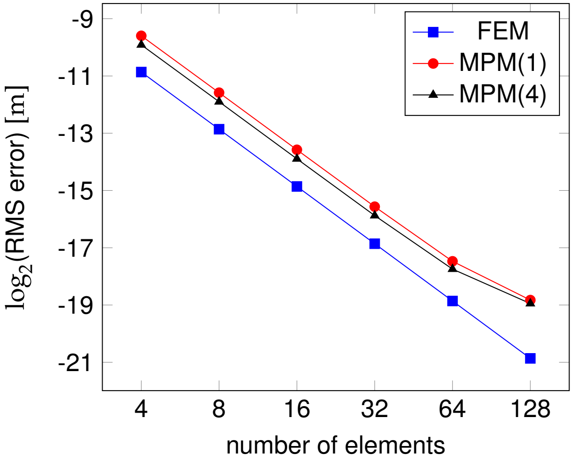
Observations

- Lack of spatial convergence
- Problems arise due to external forces

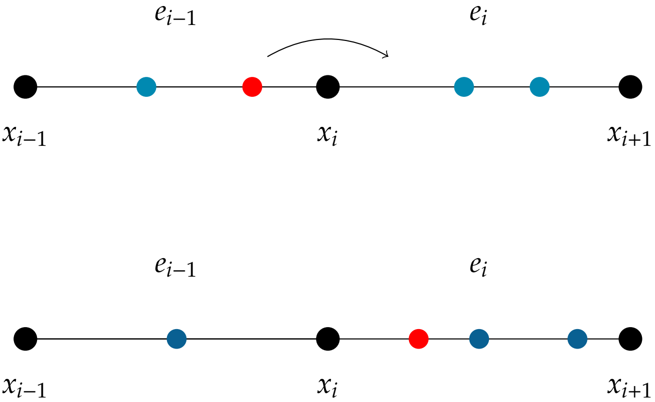
$$\mathbf{M}\frac{d\mathbf{v}}{dt} = \mathbf{F}_{ext} - \mathbf{F}_{int}$$

[1] Van Kan (2008)

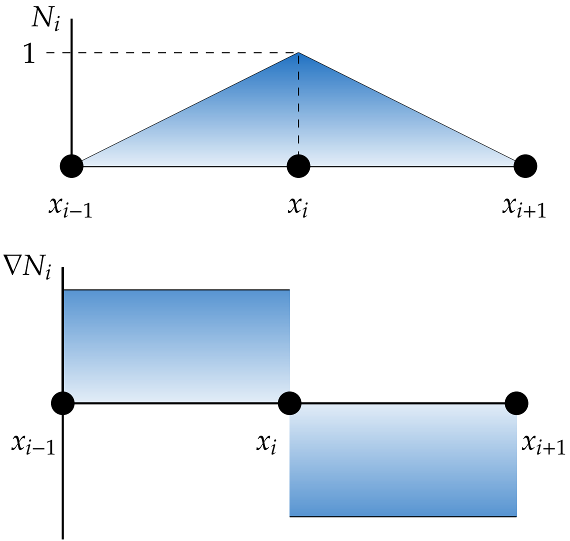
FEM: vibrating bar



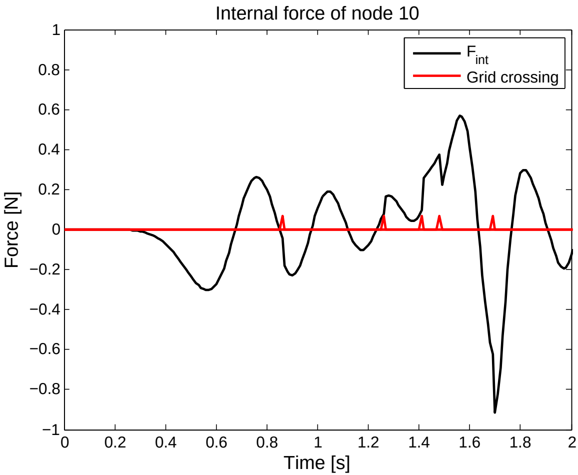
Grid crossing



Grid crossing: properties of shape functions



Grid crossing: internal force



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Grid crossing: vibrating bar

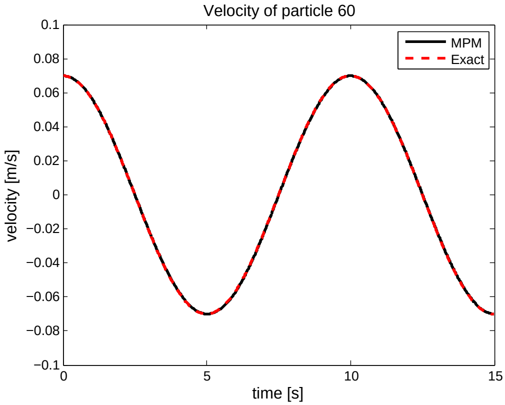


Figure: No grid crossing (30 elements).

Grid crossing: vibrating bar

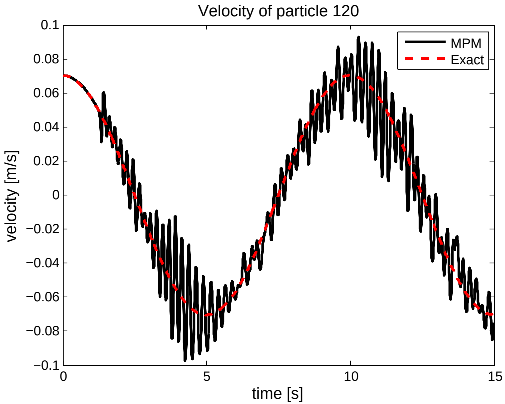
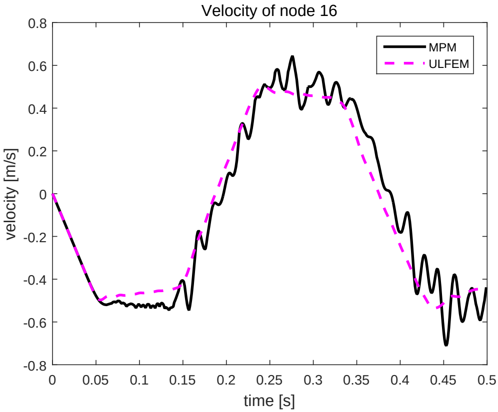
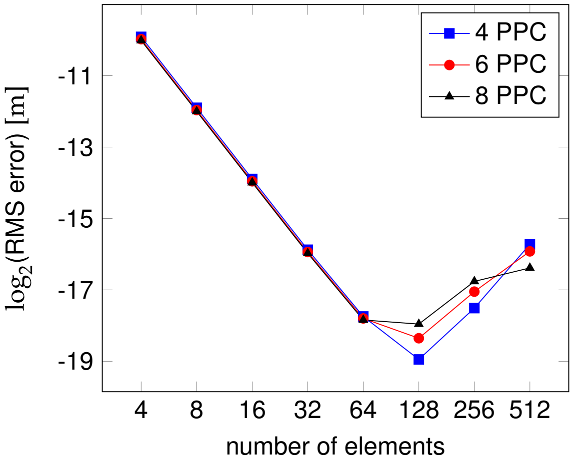


Figure: Grid crossing (60 elements).

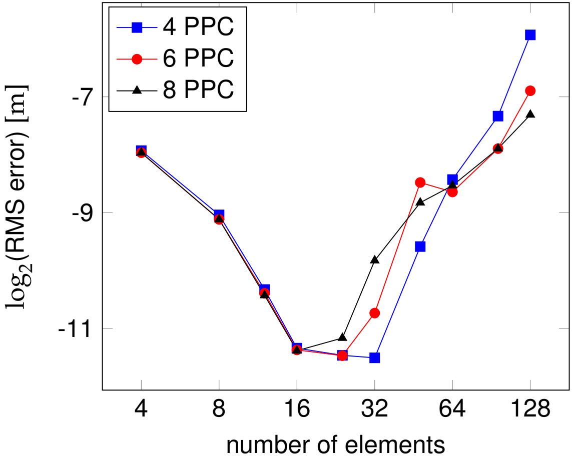
Grid crossing: oedometer



Depenence on PPC: vibrating string



Depenence on PPC: oedometer



Main sources of spatial errors

Presented today

- Errors arising due to external forces
- Grid crossing errors

Other sources[2]

- Mass mapping error
- Momentum mapping error
- Force mapping error

[2] Tran (2010)

Conclusions

Vibrating bar

Second order accuracy: unless particles cross element boundaries.

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Oedometer

Lack of convergence: due to external forces and grid crossing.

Conclusions

Vibrating bar

Second order accuracy: unless particles cross element boundaries.

Oedometer

Lack of convergence: due to external forces and grid crossing.

Both problems

Other error sources can be involved.

Outlook

- External forces: further analysis
- Projection errors: MLS
- Grid crossing: higher order interpolation functions
- 2D MPM code in MATLAB
- Deltares' implementation: analysis and recommendations

References

Gong M. *Improving the Material Point Method.* The University of New Mexico, July, 2015.

Van Kan J., Segal A., Vermolen F. *Numerical methods in scientific computing.* Delft University of Technology, 2008.

Steffen M., Kirby R. M., Berzins M. *Analysis and reduction of quadrature errors in the material point method (MPM).* International Journal for Numerical Methods in Engineering 76, pp. 922-946, 2008.

Tran L.T., Kim J., Berzins M. *Solving time-dependent PDEs using the material point method, a case study from gas dynamics.* International Journal for Numerical Methods in Fluids 62, pp. 709-732, 2010.

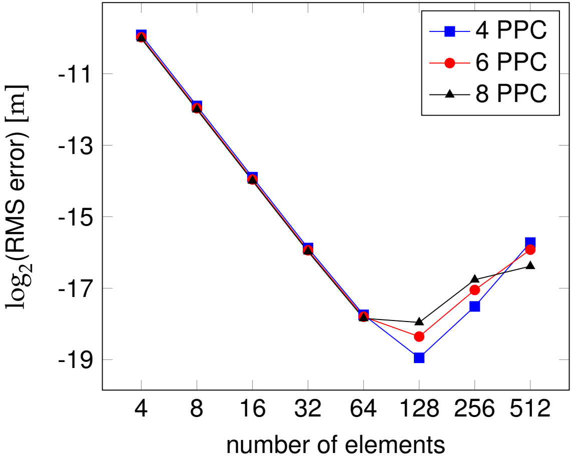
Grid crossing: internal force

$$F_{i+1}^{int} \approx \sum_{p=1}^{n_i} \nabla N_i(\xi_p)\sigma_p\Omega_p + \sum_{p=1}^{n_{i+1}} \nabla N_i(\xi_p)\sigma_p\Omega_p$$

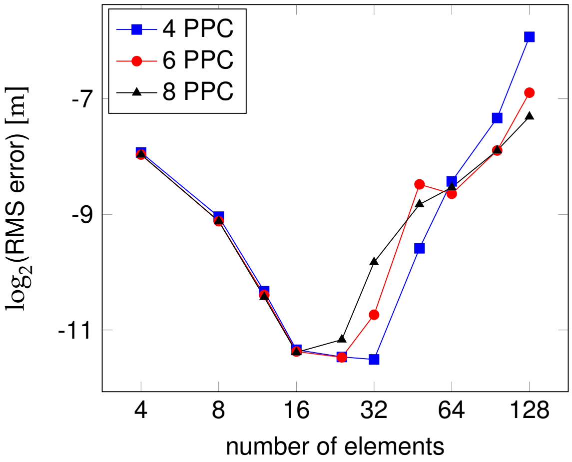
$$F_{i+1}^{int} \approx \sigma\Omega(n_i - n_{i+1})$$

$$\begin{cases} F_{i+1}^{int} = 0, & \text{if } n_i = n_{i+1} \\ F_{i+1}^{int} \neq 0, & \text{otherwise} \end{cases}$$

Depenence on PPC: vibrating string



Depenence on PPC: oedometer



Settings: vibrating bar

	Symbol	Value	Unit
Length	L	25	m
Tension	E	100	Pa
Density	ρ	1	kg/m^3
Maximum velocity	v_0	0.1	m/s
Time step	Δt	$1 \cdot 10^{-3}$	s
Measurement time[1]	t	0.5	s
PPC[2]		4	

Settings: oedometer

	Symbol	Value	Unit
Height	L	1	m
Young's modulus	E	$1 \cdot 10^5$	Pa
Density	ρ	$1 \cdot 10^3$	kg/m^3
Load	p_0	0	Pa
Gravitational acceleration	g	9.81	m/s^2
Time step	Δt	$1 \cdot 10^{-3}$	s
Measurement time[1]	t	0.5	s
Position particle[1]	x_p	≈ 0.5	m
Number of elements[2]		30	
PPC[2]		10	

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Settings: Steffen

	Symbol	Value	Unit
Length	L	1	m
Tension	E	100	Pa
Density	ρ	100	kg/m^3
Load	p_0	0.7	Pa
Gravitational acceleration	g	0	m/s^2
Time step	Δt	$1 \cdot 10^{-2}$	s
Domain		1.15	m
Number of elements		20	
PPC		10	