

# Accuracy of original MPM

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December 13, 2015

# Outline

- “Original” MPM
- Numerical accuracy
- Benchmarks
  - Vibrating bar
  - Oedometer
- Sources of spatial errors
  - Analogy with FEM
  - Grid crossing
- Outlook

# “Original” MPM

- Modified Lagrangian algorithm
- Euler-Cromer time integration
- Piecewise linear basis functions
- No additional interventions

# “Original” MPM

- Modified Lagrangian algorithm
- Euler-Cromer time integration
- Piecewise linear basis functions
- No additional interventions
- Own MATLAB implementation
- 1D (UL)FEM/MPM
- Simplified version of Deltares' code

# Numerical accuracy

## Numerical Approximation

$$u_{ex} = u_{num} + O(\Delta x^n) + O(\Delta t^m)$$

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## Example

$n$	Grid size	Error
	$\Delta x$	$E$
1	$\Delta x/2$	$E/2$

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## Example

$n$	Grid size $\Delta x$	Error $E$
2	$\Delta x/2$	$E/4$

## RMS Error

$$Error_{RMS} = \sqrt{\frac{1}{n_p} \sum_{p=1}^{n_p} \left( u_{num}(x_p, t) - u_{ex}(x_p, t) \right)^2}$$



# Numerical Accuracy

## Temporal accuracy

MPM is first order accurate in time, i.e.  $m = 1$ .

# Numerical Accuracy

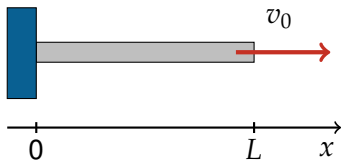
## Temporal accuracy

MPM is first order accurate in time, i.e.  $m = 1$ .

## Spatial accuracy

Order of spatial accuracy	Source
2	Gong (2015); Steffen (2008)
0.5 - 1	Tran (2010)
lack of spatial convergence	Gong (2015); Steffen (2008)

# Vibrating bar



$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u}{\partial x^2}$$

Boundary conditions:

$$u(0, t) = 0$$

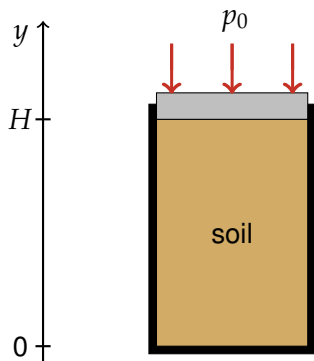
$$\frac{\partial u}{\partial x}(L, t) = 0$$

Initial conditions:

$$u(x, 0) = 0$$

$$\frac{\partial u}{\partial t}(x, 0) = v_0 \sin\left(\frac{\pi x}{2L}\right)$$

# Oedometer



$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u}{\partial y^2} - g$$

Boundary conditions:

$$u(0, t) = 0$$

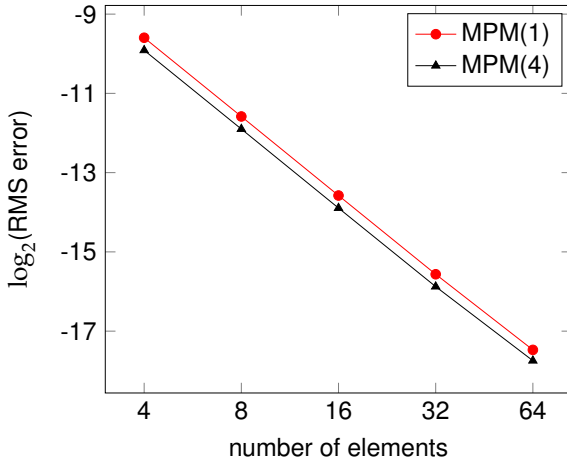
$$\frac{\partial u}{\partial y}(H, t) = \frac{p_0}{E}$$

Initial conditions:

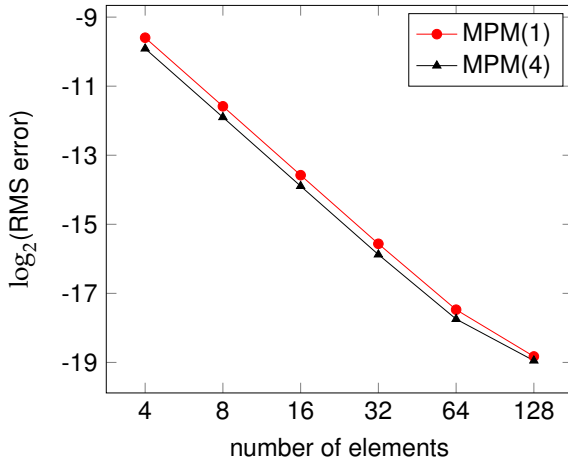
$$u(y, 0) = 0$$

$$\frac{\partial u}{\partial t}(y, 0) = 0$$

# Accuracy: vibrating bar



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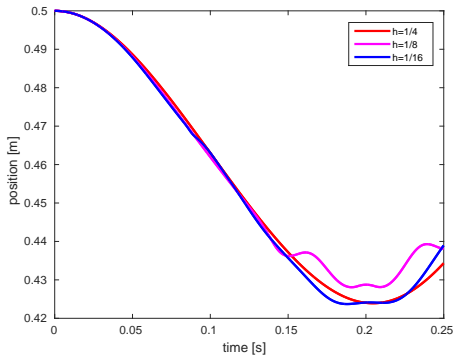


# Accuracy: oedometer

## Richardson's extrapolation

The order of accuracy  $n$  is obtained from

$$\frac{u_{num}(2h) - u_{num}(4h)}{u_{num}(h) - u_{num}(2h)} = 2^n.$$



# Accuracy: oedometer

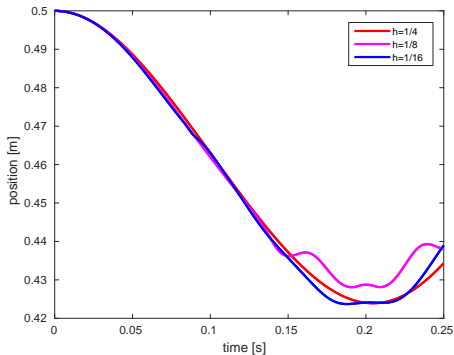
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## Conclusion

Lack of spatial convergence.





# FEM: oedometer

## Theoretical order of accuracy

$k + 1$ , where  $k$  is the order of the interpolating polynomials<sup>1</sup>.

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<sup>1</sup>Van Kan (2008)

# FEM: oedometer

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## Observations

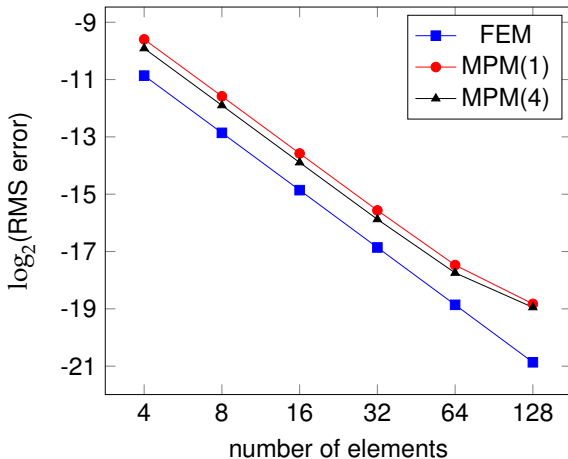
- Lack of spatial convergence
- Problems arise due to external forces

$$\mathbf{M} \frac{d\mathbf{v}}{dt} = \mathbf{F}_{ext} - \mathbf{F}_{int},$$

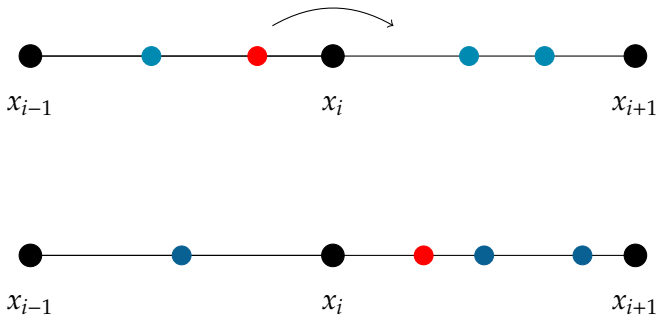
$$\text{where } \mathbf{F}_{ext} = \mathbf{N}(H)^T p_0 - \int_0^H \mathbf{N}^T \rho g dy$$

<sup>1</sup>Van Kan (2008)

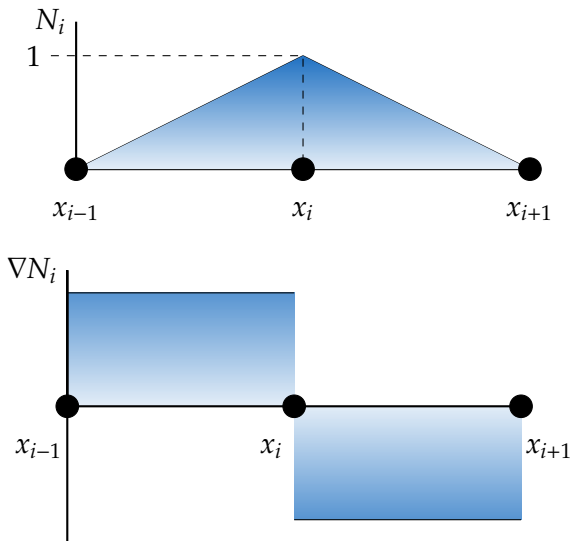
# FEM: vibrating bar



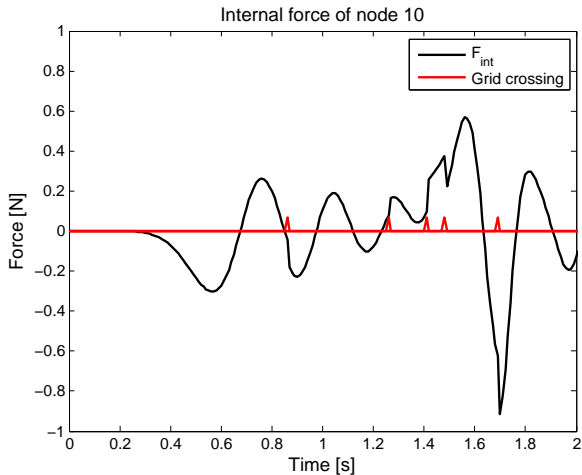
# Grid-crossing



# Grid-crossing: properties of shape functions



# Grid crossing: internal force



# Grid crossing: vibrating bar

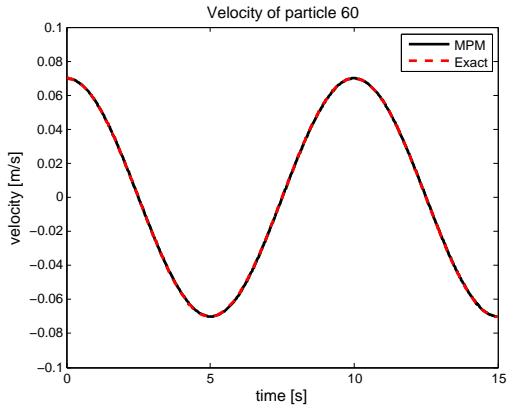


Figure: No grid crossing (30 elements).

# Grid crossing: vibrating bar

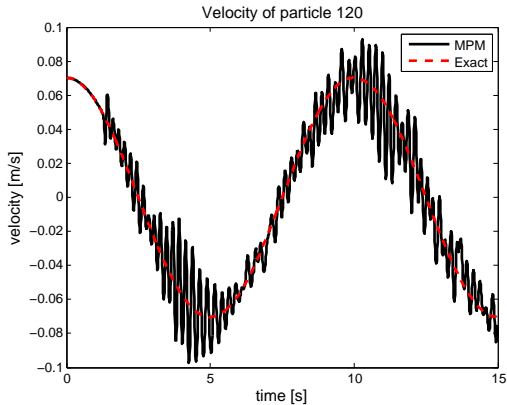
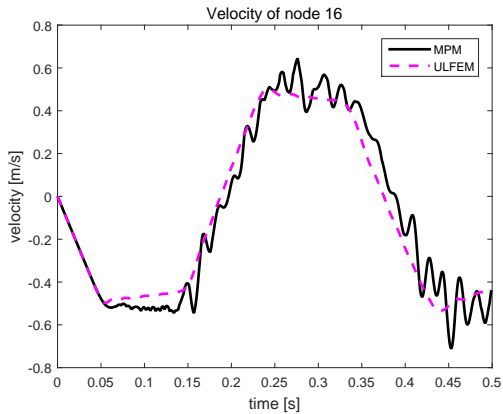


Figure: Grid crossing (60 elements).



# Grid crossing: oedometer



# Main sources of spatial errors

## Presented today

- Errors arising due to external forces
- Grid crossing errors

## Other sources

- Mass mapping error
- Momentum mapping error
- Force mapping error

# Conclusions

## Vibrating bar

Without grid crossing 2nd order accuracy.

## Oedometer

Lack of convergence due to grid crossing and external forces.

# Outlook

- External forces: further analysis

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- 2D MPM code in MATLAB

# Outlook

- External forces: further analysis
- Higher order interpolation functions
- 2D MPM code in MATLAB
- Deltares' implementation: analysis and recommendations

# References

- Gong M. *Improving the Material Point Method*. The University of New Mexico, July, 2015.
- Van Kan J., Segal A., Vermolen F. *Numerical methods in scientific computing*. Delft University of Technology, 2008.
- Steffen M., Kirby R. M., Berzins M. *Analysis and reduction of quadrature errors in the material point method (MPM)*. International Journal for Numerical Methods in Engineering 76, pp. 922-946, 2008.
- Tran L.T., Kim J., Berzins M. *Solving time-dependent PDEs using the material point method, a case study from gas dynamics*. International Journal for Numerical Methods in Fluids 62, pp. 709-732, 2010.



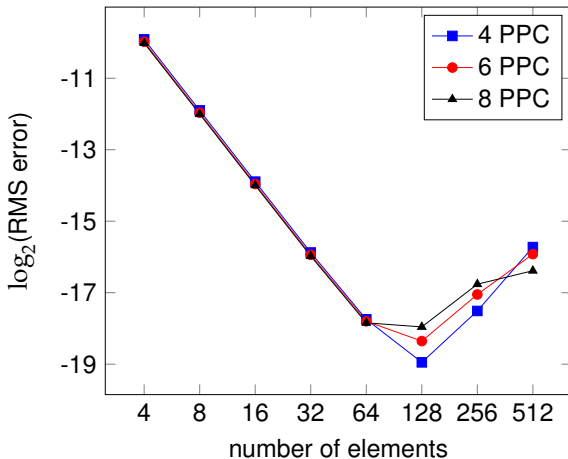
# Grid crossing: internal force

$$F_{i+1}^{int} \approx \sum_{p=1}^{n_i} \nabla N_i(\xi_p) \sigma_p \Omega_p + \sum_{p=1}^{n_{i+1}} \nabla N_i(\xi_p) \sigma_p \Omega_p$$

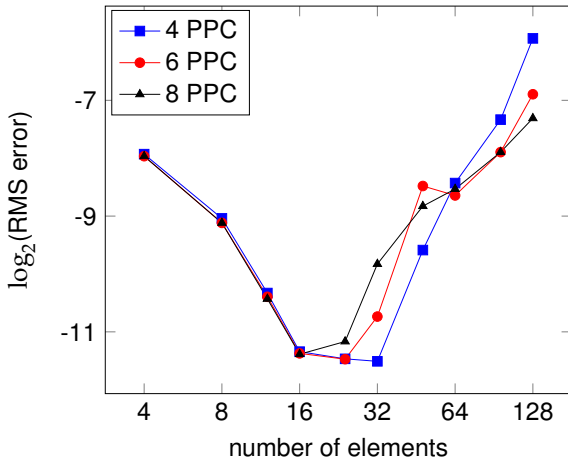
$$F_{i+1}^{int} \approx \sigma \Omega (n_i - n_{i+1})$$

$$\begin{cases} F_{i+1}^{int} = 0, & \text{if } n_i = n_{i+1} \\ F_{i+1}^{int} \neq 0, & \text{otherwise} \end{cases}$$

# Depenence on PPC: vibrating string



# Depenence on PPC: oedometer



## Settings: vibrating bar

	Symbol	Value	Unit
Length	$L$	25	m
Tension	$E$	100	Pa
Density	$\rho$	1	kg/m <sup>3</sup>
Maximum velocity	$v_0$	0.1	m/s
Time step	$\Delta t$	$1 \cdot 10^{-3}$	s
Measurement time <sup>1</sup>	$t$	0.5	s
PPC <sup>2</sup>		4	

## Settings: oedometer

	Symbol	Value	Unit
Height	$L$	1	m
Young's modulus	$E$	$1 \cdot 10^5$	Pa
Density	$\rho$	$1 \cdot 10^3$	kg/m <sup>3</sup>
Load	$p_0$	0	Pa
Gravitational acceleration	$g$	9.81	m/s <sup>2</sup>
Time step	$\Delta t$	$1 \cdot 10^{-3}$	s
Measurement time <sup>1</sup>	$t$	0.5	s
Position particle <sup>1</sup>	$x_p$	$\approx 0.5$	m
Number of elements <sup>2</sup>		30	
PPC <sup>2</sup>		10	

## Settings: Steffen

	Symbol	Value	Unit
Length	$L$	1	m
Tension	$E$	100	Pa
Density	$\rho$	100	kg/m <sup>3</sup>
Load	$p_0$	0.7	Pa
Gravitational acceleration	$g$	0	m/s <sup>2</sup>
Time step	$\Delta t$	$1 \cdot 10^{-2}$	s
Domain		1.15	m
Number of elements		20	
PPC		10	