Modelling credit card usage for individual card-holders

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Submitted in total fulfillment of the requirements of
the degree of Doctor of Philosophy

August 2016

School of Mathematics & Statistics
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Abstract

The thesis presents a model of the outstanding balance on a credit card account in which purchases and payments made by the card-holder follow marked point processes. We use the model of credit card balance to derive an account-level model of profitability that incorporates ongoing sources of revenue and cost. A key component of the model is a policy that prevents the value of approved purchases from exceeding a level ℓ.

We consider three different policies and specialise to the case where the purchasing process, $A(t)$, $t \geq 0$, is a non-negative marked renewal process with inter-event time distribution G and event size distribution F. We show that, under a policy which permits the card-holder to continue to attempt to make purchases following a declined purchase, the resulting process of approved purchases is bounded above and below by the same marked renewal process under two simpler policies. The upper bound is given by the minimum of the $A(t)$ and ℓ, and the lower bound is given by the truncated supremum

$$\sup\{A(u) : A(u) \leq \ell,\, 0 \leq u \leq t\}.$$

We find expressions for the distribution and expectation of the process of approved purchases under this policy in terms of the Laplace-Stieltjes transforms of G and F.

We adapt the model to two special cases of card-holder payment behaviour: *transacting*, where the card-holder always pays the full amount due, and *partial payment*, where the card-holder only pays a fraction $c \in (0,1)$ of the amount due. In both cases, we derive an optimisation problem to find the credit limit that maximises expected profit, and find bounds on the true optimal limit by solving the optimisation problem using the two simplified balance control policies.

In the case of transacting behaviour, we solve the optimisation problem by numerically inverting the Laplace transform of the expected value of approved purchases. We demonstrate the utility of the model using actual data by fitting a distribution to a single card-holder's transactions and calculating the optimal limit and resulting profitability.

The case of partial payment behaviour is more involved, and we show

that the optimisation problem reduces to the transactor case by regarding the outstanding balance at statement times as a Markov chain and numerically sampling from its invariant measure. We obtain numerical solutions for the expected outstanding balance and optimal limits when the process $A(t)$ is a compound Poisson process.

Declaration

This is to certify that:

i. the thesis comprises only my original work towards the PhD except where indicated in the Preface,

ii. due acknowledgment has been made in the text to all other material used,

iii. the thesis is fewer than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

Signed

Jonathan Keith Budd

Preface

This thesis is concerned with developing mathematical models of how people use credit cards. This is a topic of considerable interest within the financial industry, particularly among credit card-issuing institutions such as retail banks and building societies, but also among financial regulators. Within these organisations, the modelling of customer behaviour has traditionally been the domain of risk management departments, who employ techniques from the fields of statistics, mainly classification via regression models. These techniques have enjoyed immense popularity, and in more recent times, techniques from the fields of machine learning and others have gained interest, particularly given the recent explosion in the availability of data. The field is still relatively young and there are many opportunities still waiting to be seized upon.

My own interest in the area came after several years working in retail credit risk. Through this experience, I became familiar with classification techniques and their short-comings in modelling dynamic phenomena. In particular, I examined how bank customers used their products and the effect this had on measures such as profit. Detailed transaction-level data for credit cards had recently become available at the company I was working for, and I was fortunate enough to work on projects where I had access to the data and could analyse and become familiar with it. I came to realise that the existing credit risk toolkit of techniques was missing a framework for modelling credit card use at the individual customer level, and decided to undertake research to develop such a framework. This thesis is the result.

Modelling at the individual customer level is not the norm in retail banking management, owing to the heterogeneous nature of most lending portfolios. Modelling at the product, segment or portfolio level is far more common. As such, the techniques used in this thesis may be unfamiliar to researchers and practitioners in retail credit risk. I believe this highlights the diversity of research opportunities in the area.

To the best of my knowledge, the models and results in this thesis are new. Of course, the work presented here has benefited from numerous conversations with colleagues in industry and academia throughout my candidature. In

particular, I am thankful to Peter Braunsteins for sharing his intuition that the derivative of the expectation of the process induced by the balance control policy BCP2 could be seen as a path integral. My supervisor, Peter Taylor, taught me the technique of conditioning on the time and value of the first event in a Markovian process. All other work (including the errors) is my own.

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London, United Kingdom
August, 2016

Acknowledgments

First and foremost, I wish to thank my supervisor, Professor Peter Taylor. Your support and guidance throughout this endeavour are what have made it possible. Moreover, you taught me the value of persistence in research. I could not ask for a greater gift from a teacher to his student.

I also thank Professor Daryl Daley and Dr Owen Jones for being on my supervisory committee. Daryl, I am especially grateful that you took the time whenever possible to meet (in both Australia and the U.K.) and provide your encouragement and advice.

The work presented in this thesis has been undertaken at various times in Melbourne, Philadelphia and London. As such, I am grateful to the faculty, staff and students at the institutions that I visited. I thank the faculty at the University of Pennsylvania for making me feel welcome during my brief stay there, including Professor Warren Ewens and Professor Michael Steele, and Professor Robin Pemantle for allowing me to audit his graduate course on Stochastic Processes. I also thank Dr Codina Cotar, who hosted me as a visiting student at University College London during 2015 — 2016.

The staff at the School of Mathematics & Statistics at the University of Melbourne went above and beyond in their assistance with administrative matters, in particular Kirsten Hoak. I am also grateful to the School for providing funding to attend conferences during my candidature through the Research Support Scheme.

I have been fortunate to share several office spaces over the course of my candidature, and even more fortunate to share them with fellow students and researchers who have provided hours of entertaining and stimulating conversation. In Melbourne, I thank Nikki Sonnenberg, Rebecca Chisholm, Yacov Salomon, Peter Braunsteins, Nicholas Read, Ellen Muir, Kate Saunders and Shrupa Shah. Thanks to Sameer Deshpande, Colman Humphrey, Kory Johnston and Matthew Olsen for making me feel welcome during my brief time in Philadelphia, and to Stephen Muirhead, Sam Livingstone and Dr Richard Pymar for inviting me to join their informal reading group at UCL.

I especially thank Shaun McKinlay, whose dining room table in West Hampstead served as a desk for the two of us through the first half of 2015, and Dr Loretta Bartolini, whose advice helped me to keep my perspective.

My friends and family have been a constant source of encouragement and inspiration throughout my studies, and I thank you all. Finally, I am grateful beyond measure to my wife, Sylvie, and my parents, Peter and Cathy, who inevitably joined me on this endeavour. Thank you for your patience, love and support. It means the world to me.

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Notation

$\mathbb{C}$	The set of complex numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{R}_+$	The set of non-negative real numbers
i	$\sqrt{-1}$
$\mathrm{Im}(\omega)$	The imaginary part of the complex number ω
$\mathrm{Re}(\omega)$	The real part of the complex number ω
$\mathbf{X}$	An $m \times n$-dimensional matrix.
$\mathbf{x}$	An n-dimensional vector $(x_1, x_2, \ldots, x_n)$
$f \circ g(x)$	The composition $f(g(x))$
$f * g(x)$	The convolution of f and g
$\tilde{f}$	The Laplace-Stieltjes transform of the function F
$x \vee y$	The maximum of x and y
$x \wedge y$	The minimum of x and y
$\mathbf{1}_{\{x \in \mathcal{X}\}}$	Indicator of the event $\{x \in \mathcal{X}\}$
$\Pr(x \in \mathcal{X})$	Probability of the event $\{x \in \mathcal{X}\}$
F_X	Distribution function of the random variable X
$\mathbf{E}[X]$	Expectation of the random variable X
$\mathrm{Var}(X)$	Variance of the random variable X

$A(t)$	The process of attempted purchases
$P(t)$	The process of payments
$\Psi(t, A, P, \ell)$	The process of approved purchases
$B(t)$	The process of the outstanding balance
$R(t)$	The process of profit
ℓ	Credit limit
s_i	The ith statement date
d_i	The ith due date
c_i	The ith minimum payment due
ρ_i	The ith fraction of the outstanding balance paid
$r(s_i)$	The rate of interest charged at statement date s_i

Chapter 1

Introduction

A brief history of credit cards and an overview of the current industry. A discussion of the role of classification techniques and other modelling methods in credit card management. Finally, an overview of the thesis and a summary of the contributions.

1.1 Credit cards and the retail banking sector

A credit card is a revolving loan facility issued by a financial institution to an individual or business. Revolving lines of credit differ from instalment or fixed term loans such as mortgages in that the funds borrowed by a customer become available to them again upon repayment. For example, if a card-holder has a $5,000 line of credit and they use $1,000, then when the $1,000 is repaid, the full line of credit is again available for use. By contrast, funds repaid on an instalment loan cannot be redrawn. This is a sufficient description for now, and we will revisit the details in Chapter 3.

Credit cards are not the only type of revolving credit product: overdrafts, home equity lines of credit (HELOCS) and mortgage offset accounts are also revolving loans. However, the complexity and flexibility of credit cards as a loan facility furnish them with several features that make them particularly interesting to study. First, credit cards generate variable patterns of borrowing and repayment. They are frequently used as a cash alternative and can be used wherever they are accepted. Credit cards are the default payment option for most online retailers, and also serve as a method to secure reservations for expenses such as hotel bookings or car rentals. The card-issuing bank[1] will request repayment of the borrowed funds at regular times, but the card-holder can make payments as and when they choose, with the caveat that paying anything other than the full amount due on time will incur penalties in the form of interest and fees.

A second feature is that the amount of credit extended to the card-holder is also variable. Banks will base a card-holder's credit limit predominately on continuing assessments of the their credit-worthiness and reserve the legal right to increase or decrease their customers' credit limits at their discretion. If a customer exhibits signs of financial difficulty and the card-issuing bank has reason to believe that the borrowed funds will not be repaid, it will take action to remind the card-holder of their obligation to make the requested payment. In more extreme cases, the card-holder can expect to see their credit line reduced and the card suspended from making purchases. If, on the other hand, the bank believes the customer is worthy of an increased credit line, it is common practice to offer the card-holder the choice of whether or not to accept the additional funds.

Card-issuing banks use credit limit increases to encourage increased expenditure and borrowing. The relationship between credit line increases and increased utilisation was examined in [52], [60] and [116]. This highlights the third feature of credit cards which distinguishes them from other revolving loans: credit cards are designed and marketed to banking customers to incentivise them to use the card. Loyalty schemes are often attached to credit cards that enable card-holders to earn points which can be redeemed at various retailers such as airlines, hotels or supermarkets. Other marketing devices include reduced interest or interest-free periods on balances transferred from other credit cards, cash-back offers and purchase insurance. These programs and offers are apparently very effective. Figure 1.1 shows the monthly outstanding balances and the value of purchases made on personal credit cards in Australia between May 1994 and April 2016.

The growth in purchases and balances is perhaps not so surprising con-

[1]Credit card issuers are not necessarily banks. By definition, a bank is an institution that accepts deposits. American Express and Discover are two examples of card-issuers that are not banks. Furthermore, not all banks are necessarily credit card issuers. The terms "card issuer" and "bank" are used interchangeably throughout this thesis.

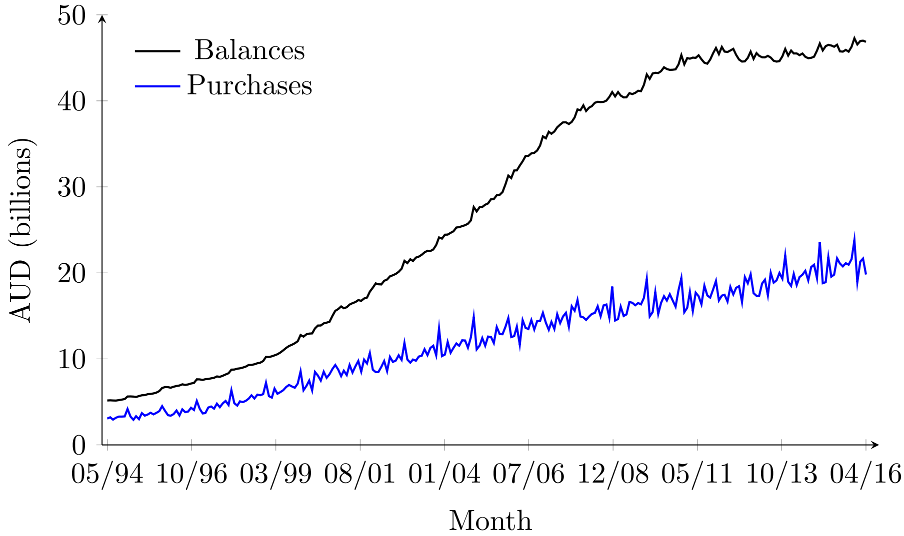

Figure 1.1: Outstanding balances and the value of purchases made on Australia credit cards between May 1994 and April 2016. Source: Reserve Bank of Australia [101].

sidering the provenance of the credit card. A precursor to today's credit cards is the store card. As early as 1914, many U.S. hotels, petrol companies and department stores offered credit to their customers in the form of charge accounts and instalment plans. Store cards were introduced as a way to identify customers with store accounts and record customer purchases. The first general-purpose credit card was not introduced in the United States until 1949 with the creation of the Diner's Club card which, as the name suggests, was marketed to restaurant diners. A similar program was launched in the 1950s in Great Britain with the British Hotel and Restaurant Association. In 1958, American Express, then a traveller's cheque business, partnered with Hilton's private credit card operation, Carte Blanche, to launch a universal credit card that targeted travel and entertainment expenditure. Chase Manhattan and Bank of America also launched credit cards that year.

Credit cards gained widespread use in the 1960s after Bank of America licensed its card, BankAmericard, across the U.S. and globally in partnership with regional banks. Following this, Bank of America began a campaign that mailed millions of unsolicited credit cards to American households. This gave the newly created network a large customer base which encouraged more banks and merchants to join the scheme. By 1978, over 11 000 banks were members of the BankAmericard program, which was later renamed to Visa. Other networks, such as Master Charge (later MasterCard), enjoyed similar growth, but by the late 1970s Visa and American Express had established their dominance in the retail and travel sectors respectively. Today, Visa is by far the largest network. According to [92], Visa processed approximately 56% of all credit card purchase transactions in 2015. A further and more detailed history of the credit card industry up to the early 1990s can be found in [83].

While credit cards have their origin in store cards as a mechanism for encouraging sales and engendering customer loyalty, credit cards are today sold to customers by banks as a convenient payment method and it is common to see them offered together with other loans such as mortgages. To the card-issuing bank, however, they are a financial product like any other and are intended to generate profit. The bank generates profit by charging customers interest when full payment is not made, along with other penalty fees. We examine the effect of interest on credit card profitability in more detail in Chapter 6.

The amount of profit that a card-holder will generate for the card-issuing bank is of course dependent on how the card is used. As we will see later on, a card-holder who only occasionally makes a purchase and routinely pays their statement on time is not very profitable to the bank, whereas a card-holder who borrows heavily will pay interest and generate substantially more revenue, but also carries a significant risk of non-repayment. Card-issuers are well aware of the trade-offs between encouraging credit card use and the associated credit risk, and employ various modelling techniques and analyses to aid in the management of their credit card portfolios. For the applied mathematician or statistician, credit cards offer a rich set of modelling challenges, augmented by the availability of diverse and large data sets.

1.2 Classification techniques in credit card management

Managers of credit card portfolios regularly employ modelling techniques to aid and automate decision-making throughout the customer life-cycle. At the point where the customer is acquired, the decision is whether or not to grant credit, and, if credit is to be granted, what amount. Later in the customer life-cycle, incentives such as an increased limit or a different interest rate may be offered, so the problem is to determine those customers most likely to accept the offer, or those who will generate the most profit if they accept. The set of models and techniques used in making these decisions is referred to as *credit scoring*.

The origins of credit scoring can be traced back to Fisher and his development of linear discriminant analysis in [56]. This statistical technique can be used to detect distinct sub-groups within a sample, and it was Durand in [49] who first used it to identify characteristics distinguishing good and bad instalment loan customers. Credit analysis was a largely manual endeavour guided by heuristics and personal judgement, and the use of statistical techniques and automation did not become common until the introduction of general-purpose credit cards in the 1960s. Driven by the demand for credit, automated credit scoring became the only way to cope with the volume of credit card applications and its widespread adoption was facilitated by the new availability of computers. Consequently, lenders enjoyed dramatically improved repayment rates. See [36] and [89] for accounts of these results. In the U.S., the passage of the Equal Credit Opportunity Act in 1975 and subsequent amendments essentially made credit scoring a legal requirement. The Act, coupled with the success in the credit card business, saw credit scoring applied to other retail products such as personal loans and mortgages starting in the 1980s. See [122], [123] and [124] for discussions on the foundations of credit scoring and its applications in industry.

In general terms, credit scoring is a classification problem. A classification problem is the task of deciding the membership of each of N objects to one of K groups. One typically "solves" a classification problem by constructing a classifier which decides the membership of object i, $1 \leq i \leq N$ based on its associated d-dimensional feature vector $\mathbf{x}_i = (x_{i1}, \ldots, x_{id})$, where the $\{x_{ij}\}$, $1 \leq j \leq d$ are features or characteristics of object i and $d \geq 1$. A *classifier* is the composition of a score function, $s : \mathbb{R}^d \mapsto \mathbb{R}$, which maps a feature vector $\mathbf{x}_i \in \mathbb{R}^d$ to a real number, and a decision rule that determines which group the object belongs to according to the output of the score function and a threshold $t \in \mathbb{R}$. For $K = 2$, we have a binary classification problem with groups $\{0, 1\}$ and the classifier takes the form

$$D \circ s(\mathbf{x}_i) = \begin{cases} 0, & s(\mathbf{x}_i) < t, \\ 1, & s(\mathbf{x}_i) \geq t. \end{cases} \tag{1.1}$$

The choice of form of the score function and its construction is dependent on the specific nature of the classification problem and the available data. In credit scoring, we typically have available a labelled data set

$$[\,\mathbf{Y} \mid \mathbf{X}\,] = \left[\begin{array}{c|cccc} y_1 & x_{11} & x_{12} & \ldots & x_{1d} \\ y_2 & x_{21} & x_{22} & \ldots & x_{2d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_N & x_{N1} & x_{N2} & \ldots & x_{Nd} \end{array}\right] \tag{1.2}$$

which we refer to as the training data set. Here, each row $[\,y_i \mid \mathbf{x}_i\,]$ represents information associated with a single lending account. The labels y_i will take values in $\{0, 1\}$ or perhaps $\{B, G\}$ classifying the accounts as either "good" or "bad" based on some criteria such as whether or not the account was charged-off. Clearly, the number of good accounts, N_G, and bad accounts, N_B, sum to the total number of rows N in the training data set. The features $\mathbf{x}_i$ usually include a mix of information about the account-holder, the lending product and the performance of the loan.

We illustrate how a score function may be constructed with a simple example shown in Figure 1.2. In this example, there are $N = 15$ accounts labelled either B or G, and each account has only two characteristics (x_1, x_2) and so $d = 2$. A linear score function has been constructed which divides the plane into two regions. The decision rule is

$$D \circ s(x_1, x_2) = \begin{cases} B, & x_1 - \beta x_2 < \alpha, \\ G, & x_1 - \beta x_2 \geq \alpha, \end{cases} \tag{1.3}$$

where $\alpha, \beta \in \mathbb{R}$. According to this rule, accounts falling below the line $x_2 = \alpha + \beta x_1$ will be classified as good, and bad otherwise. Note that the rule incorrectly classifies five out of the fifteen accounts. Indeed, in this problem no linear classifier would be able to correctly classify all the accounts. We will have more to say on this shortly.

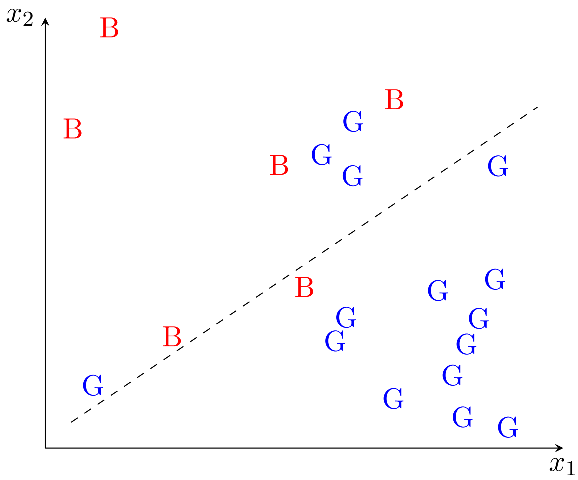

Figure 1.2: Credit scoring is a classification problem. In this two-dimensional example, a linear score function has been applied. Accounts situated below the dashed line are classified G and accounts above as B. Five out of the fifteen accounts are incorrectly classified.

With the data structure (1.2), it is common to assume a *generalised linear model* (GLM) for the score function. A GLM is a statistical model of the form

$$g(\mathbf{Y}) = \mathbf{X}\boldsymbol{\beta} \tag{1.4}$$

where $\boldsymbol{\beta}$ is a d-dimensional vector of parameters associated with each feature $x_{\cdot j}$ and the link function g is chosen so that

$$\mathbf{E}[\mathbf{Y}] = g^{-1}(\mathbf{X}\boldsymbol{\beta}). \tag{1.5}$$

The logistic function is a standard choice for g in the credit scoring industry and under this choice, Equation (1.4) takes the form

$$\log\left(\frac{p}{1-p}\right) = \mathbf{X}\boldsymbol{\beta} \tag{1.6}$$

where $p = N_G/N$ is the proportion of good accounts in the sample data. The parameter vector $\boldsymbol{\beta}$ can be estimated using maximum likelihood estimation and, once this is done, the score function can now be applied to a feature vector as

$$s(\mathbf{x}_i) = \frac{1}{1 + \exp\{-\mathbf{x}_i \cdot \boldsymbol{\beta}\}}, \tag{1.7}$$

to give the estimated probability of account i belonging to the class G. The GLM permits several other link functions, but the logistic function is preferred in credit scoring owing to its relative ease of implementation and the interpretation of the score function as a probability.

Having constructed the score function, the task of determining the threshold t remains. The determination of t is also problem-dependent and there are a number of methods for deriving a threshold that is optimal in some sense relative to the problem. Continuing with the example of binary classification, what is often observed is that the conditional score distributions

$$F_G(z) = \Pr\left(s(\mathbf{x}) \leq z \mid G\right) \quad \text{and} \quad F_B(z) = \Pr\left(s(\mathbf{x}) \leq z \mid B\right) \tag{1.8}$$

have intersecting support. The region of overlap is given by

$$\mathcal{Z} := \{z : f_G(z) > 0 \cap f_B(z) > 0\}. \tag{1.9}$$

where $f_G(z)$ and $f_B(z)$ are the marginal density functions satisfying

$$F_G(z) = \int_0^z f_G(u)\,\mathrm{d}u \quad \text{and} \quad F_B(z) = \int_0^z f_B(u)\,\mathrm{d}u, \tag{1.10}$$

which we assume to exist. For a binary classification problem, the number of correctly classified and incorrectly classified accounts can be summarised in a 2×2 confusion matrix as shown in Figure 1.3.

Misclassification is unavoidable, since choosing $t \leq \inf \mathcal{Z}$ (if, indeed, this is positive) will result in a classifier that correctly identifies all the good accounts, but also mistakenly classifies a number of bad accounts as good. Such a classifier will have a high false positive rate. Conversely, setting $t \geq \sup \mathcal{Z}$ will yield the opposite result and the classifier will have a high false negative rate. As such, t is often set within $\mathcal{Z}$ subject to the relative costs of accepting false positives and false negatives. Figure 1.4 illustrates the overlap of two example marginal densities and the classification errors that setting the threshold t induces.

The receiver operating characteristic (ROC) curve plots the false positive rate, $1 - F_B(t)$, against the true positive rate, $1 - F_G(t)$, as functions of

Classifier label

Actual label		G	B	**Total**
	G	True positive	False negative	N_G
	B	False positive	True negative	N_B
Total		$\hat{N}_B$	$\hat{N}_B$	

Figure 1.3: Confusion matrix for a binary classification problem. The rows sum to the actual numbers of good and bad accounts while the columns sum to the number of good and bad accounts estimated by the classifier.

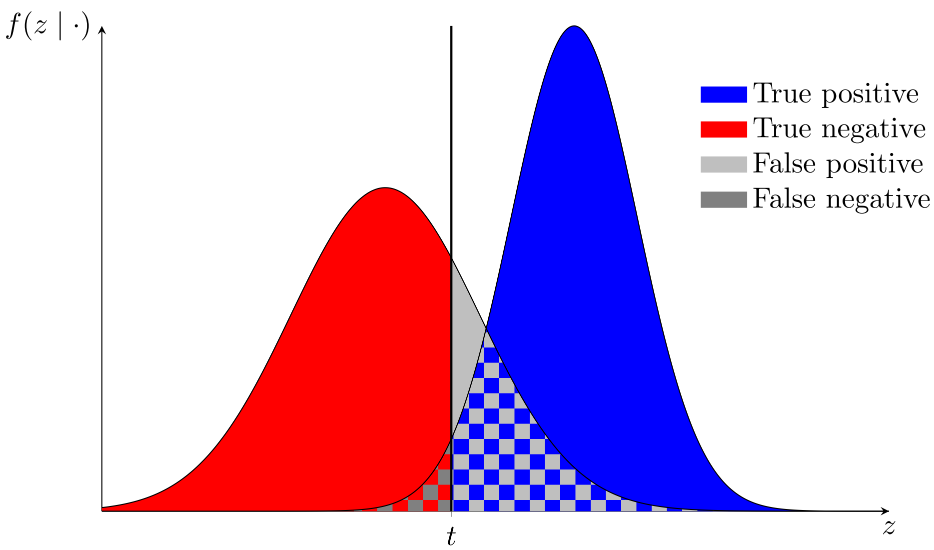

Figure 1.4: Overlap of the marginal score densities. The red density is $f_B(z)$, the score density conditional on the account being bad, and the blue density is $f_G(z)$, the score density conditional on the account being good. The region of intersecting support make classification errors unavoidable.

the threshold t. The ROC is a common tool for determining the threshold t and assessing the ability of the classifier to discriminate. The area under the receiver operating characteristic curve (AUROC) also provides a useful summary statistic of the performance of the classifier since a larger value for the AUROC corresponds to greater separation between the marginal distributions. Figure 1.5 shows the ROC curve and the corresponding AUROC.

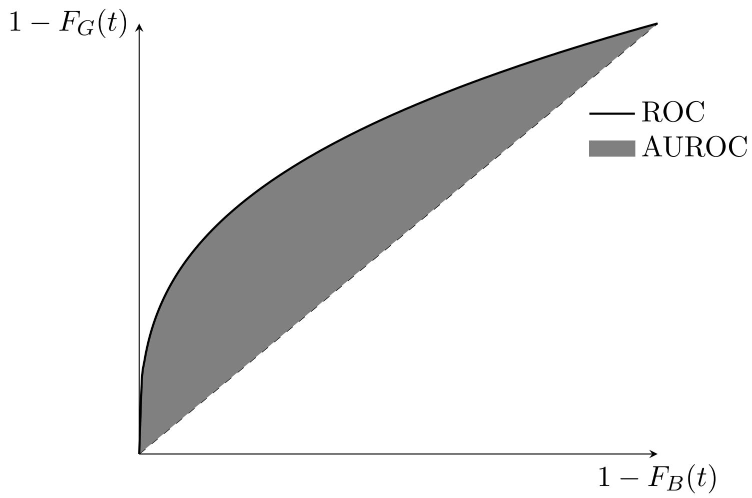

Figure 1.5: The receiver operating characteristic curve (ROC) and the area under the receiver operating characteristic curve (AUROC) are examples of methods used to measure classifier performance.

The AUROC is equivalent to several other measures of classifier performance. See [122, Chapter 2] for a detailed exposition of these and their relationships to each other. An excellent introduction and reference for the construction of classifiers and their assessment is [62]. A discussion of classifier assessment with a focus on classifiers in consumer credit risk is given in [64]. The number of classifier performance measures is indeed very large, and a useful discussion on choosing a performance measure appropriate to the classification problem under consideration is [61]. More recently, the H-measure was proposed in [63] (with further discussion in [65]) as a performance measure that assigns a prior distribution to misclassification costs.

As mentioned earlier, the use of classifiers is widespread in retail banking and is not limited to credit cards. Surveys of the methods used in credit scoring are available in [41], [104] and [67]. Practical information on the development, assessment, and maintenance of credit scoring models can be found in [123], [122] and [110]. Note that [122] and [123] also include chapters on the application of credit scoring techniques to business problems other than risk, such as predicting response to marketing campaigns.

Earlier, it was mentioned that the available data in a credit scoring problem is usually labelled. Within the machine learning field, this is considered to be an instance of a *supervised learning problem*, since the labels guide the learning algorithm. In the absence of labels, we have an *unsupervised learning problem*. An excellent reference for both types of problems is [69]. Recently, techniques in both supervised and unsupervised learning have been brought to bear on problems in retail credit management. For example, in [77], machine learning algorithms were use to incorporate recent credit card transaction data to produce more responsive credit scores. Despite the interest in applying these techniques, logistic regression has remained dominant in credit scoring, largely due to the ease of interpretation.

1.3 Other modelling techniques

The success of credit scoring has made classification techniques the dominant modelling paradigm in retail credit management. It is now common to find both supervised and unsupervised learning techniques employed in the aid of automated decision making throughout the various parts of the lending business, from acquisition to debt management and collection. Indeed, classification techniques excel when it is possible to clearly define categories that are meaningful for the problem under consideration. Examples include default risk models (where the outcome label is default or non-default) or marketing campaign response models (where the outcome label is response or non-response). Combinations of classifiers and other models are also used. For example, an application scorecard used to automate the process of selecting new credit card customers may combine the output of a default risk classifier with a regression model for predicting the customer's credit card balance. A bank using this model may only accept customers who have a low probability of default and a high predicted balance. A practical example combining a classifier and other models for credit card acquisitions is in [118].

The prevalence of regression and classification techniques has led to many problems in retail lending and, in particular, credit cards being adapted to fit into these frameworks. A comparison of regression models of several financial measures (such as revenue) with continuous response against binary classification is is given in [55]. Indeed, there are many problems in credit card lending where other modelling techniques have been successfully applied. For example, Markov chains have been used as models of account repayment behaviour (we will examine these in more detail in Chapters 3 and 5) and survival analysis models have been used to predict time to certain events, such as default. In [27], a portfolio-level framework was proposed using a generalised additive model (GAM). The measure of interest, such as default rate, was specified as a function of portfolio vintage, account age and an exogenous economic effect. The fitted GAM was then used to create forecasts. More recently, credit card balances were modelled directly using several forms of regression in [71].

Credit risk models may be used in conjunction with the output of other models, such as those which forecast profitability or payment behaviour. Indeed, behaviour, or usage, is linked to profitability; it was mentioned earlier that a card-holder who makes regular payments is less profitable than a card-holder who borrows funds on their credit card and pays interest. Of course, such behaviours are recognised by managers of credit card portfolios. Card-holders who regularly pay the full amount due on their credit card and consequently do not pay interest are called *transactors*, while card-holders who don't pay the full amount due and incur charges for interest (among others) are referred to as *revolvers*. Revolving behaviour can manifest in many different ways and in [3], clustering techniques were used to identify broad categories of repayment behaviour among credit card-holders using data on repayment status. Pattern search was also used to identify distinctive behaviour among smaller groups. We will revisit repayment behaviour in Chapters 3 and 6.

There have been several recent studies linking account behaviour with credit risk and profitability. In [6], the authors combined the output of two models predicting default and purchase propensity, to create an estimate of profitability for revolving accounts. More recently, the authors in [113] developed an application model for the profitability of new customers which factored in the effect of whether or not a customer exhibited transacting or revolving behaviour. For delinquent accounts in collections, the authors in [33] used a model of continuous-time repayment behaviour where the payment times follow a point process and the repayment amounts are correlated to create account-specific forecasts of payback probability and valuation.

Modelling profitability in retail lending is an area of interest by itself. Profit scoring was explored in detail in [122, Chap. 4] and [123, Chap. 14]. From an optimisation perspective, the problem of choosing a profit-maximising interest rate for a segment of consumer loans was studied in [96].

One particular area of credit card management that has applied modelling techniques other than classification methods is the area of credit limit assignment. A dynamic programming model was formulated in [19] in which the decision variables were whether or not to grant credit and what amount. In this formulation, the amount of credit offered was linked to the probability of non-payment via an exponentially declining relationship. This model was later extended in [47] to remove the assumption that the expected future payoff from period i onward is zero if no payment is made. More recently, a Markov decision process (MDP) was developed in [128] in which the objective was to optimise customer lifetime value by either changing the customer's credit limit or interest rate. An MDP was also used in [114] to generate a dynamic credit limit policy where the state space was the account behaviour score. In [115], a database of card-holders was classified into 16 groups according to their credit limits using information about their usage and credit-worthiness. Regression models of default and credit card balance were input to an expression for profit, and a genetic algorithm was used to find the profit-maximising allocation of limits across these groups.

Perhaps owing to its dynamic nature, *account behaviour* does not lend itself to the classification paradigm. By account behaviour, we mean how a card-holder uses their credit card. This includes how frequently the card-holder uses the credit card to make purchases, how much each purchase is for, how often they make payments, and, whether or not the card-holder pays interest. Note that behavioural scorecards, which predict credit risk based on account behaviour, are widespread in the credit card industry, but building models of behaviour itself is not as common.

Interest in modelling account behaviour has been spurred by the increased availability of transaction-level data, which has soared in the last decade with the growing prevalence of data warehouses in financial institutions. Despite this, there have been relatively few attempts made to utilise this data to develop new models for account management. The use of data mining techniques to reveal spending patterns in databases of UK credit card transactions was explored in [23], [66], [68], [125] and [126]. The key output of this research was to analyse transactions made at particular retailers, such as petrol stations and supermarkets, and to fit distributions to the purchase values and inter-transaction times. Transaction data was also used in a case study in [93] to improve the targeting of a direct marketing campaign for financial products.

In summary, the use of classification techniques is well-established in retail credit management. Credit cards permit a wide range of spending and repayment behaviours and as such, managers of credit card portfolios employ classification and other techniques to model more dynamic phenomena, particularly those associated with account behaviour. This thesis builds on the existing research which identifies account behaviour and patterns in transaction data, and develops a model that explicitly links an individual's purchasing and payment patterns to their credit card balance and profitability.

1.4 Overview of the thesis

The central hypothesis in this thesis is that an individual credit card-holder's purchases and payments can be modelled as continuous-time marked point processes. The thesis presents a model of the outstanding balance on a credit card account in which inter-event times follow renewal processes and event values are independent and identically distributed draws from non-negative random variables. We then consider three balance control policies that prevent the balance from exceeding a credit limit — these are simplified models of real-world policies. The policies control the balance by declining part or all of the attempted purchases that would result in a balance greater than the credit limit. We show that under the policy which permits a card-holder to continue to attempt to make purchases following a declined purchase (BCP1), the balance is bounded below and above by the balance under two other policies, one which will reject all further purchases following a declined purchase (BCP2), and another that permits the difference between the balance at the time of the attempted purchase and the credit limit (BCP3).

The thesis uses the model of credit card balance to derive an account-level model of profitability that incorporates ongoing sources of revenue and cost. We then find profit-maximising credit limits for individual card-holders based on their account behaviour. It was noted in [71] and the references therein that there have been few attempts in the literature to model credit card balances and, as shown in Section 1.3, most studies have used either Markov chains or a statistical approach such as regression. To the best of the author's knowledge, the models developed in this thesis are the first attempts to model the outstanding balance and profitability of a credit card as continuous-time stochastic processes.

The problem of determining the optimal credit limit bears similarities to problems in determining optimal stock or inventory levels — in particular, the model analysed in [8], which is more commonly referred to as the *newsvendor model* and originally attributed to [50]. Indeed, the model of credit card balance can be thought of as a type of *stochastic clearing system*. A stochastic clearing system is a type of input-output system, in which inputs to the system with capacity Q are random in both arrival time and size. These inputs are removed from the system at random times determined by some clearing rule. Stochastic clearing systems were first described and analysed in general in [120], where the clearing rule was to empty the system at the first time that the cumulative input since the last clearing reaches or exceeds the capacity Q.

Stochastic clearing systems have been used extensively in inventory theory and a survey of their applications was given in [119]. A particular application of relevance is shipment or freight consolidation and a typical problem is as follows: items arrive at a storage facility at random times with random weight or volume. These items are held at the facility until they are consolidated into a single shipment and dispatched at a time determined by a dispatch policy. Shipping the items typically incurs a cost and as such, it is desirable to find a policy which minimises the number of shipments in a given time frame. Three common policies were analysed and compared in [25]:

1. Consolidate the items and dispatch shipment at times iT, $i \geq 1$.

2. Hold shipment until the consolidated weight or volume reaches or exceeds some capacity Q, or,

3. Dispatch the shipment at the earlier of 1. or 2.

The policy BCP1 can be interpreted as the following dispatch policy: the consolidated weight cannot exceed Q, and if an arriving item will result in a consolidated weight greater than Q, reject it. Otherwise, accept the item and dispatch the consolidated shipment at times iT, $i \geq 1$.

We show that finding the optimal credit limit under BCP3 is equivalent to finding the optimal stock level in the newsvendor model. This limit provides an upper bound on the optimal limit under BCP1. We find a lower bound by determining the optimal limit under BCP2, which requires an expression for the expectation of the balance under this policy. This expression is found via a method similar to that used in [34], and is given in terms of the Laplace transforms of the inter-purchase time and purchase value distributions. We invert the expression numerically using the `EULER` algorithm described in [2] in order to solve the optimisation problem.

Transacting behaviour, where the full balance is paid regularly, is a special case in our model, and it is possible to model several varieties of revolving account behaviour by making appropriate choices for the distribution of payment sizes. Both profitability and optimal limits can be assessed under different behaviours by choosing distributions for the inter-purchase times, purchase values and payment sizes. We illustrate the use of the model and its properties using tractable distributions for the process of attempted purchases.

The thesis is structured as follows —

Chapter 2 gives a brief review of Laplace-Stieltjes transforms, which we use to derive several key results in the thesis.

Chapter 3 introduces notation for the essential components of the models of outstanding balance and profit, and discusses some examples of actual credit card usage. The chapter details the sources of revenue and cost for a credit card business, and specifies general models of both balance and profit. These models account for a wide range of account behaviours, but prove difficult to work with. To overcome this, we make several restricting assumptions on interest, fees and payments and the form of the functions used in calculating profit.

Chapter 4 examines balance control policies. A balance control policy restricts the value of purchases that a card-holder can make, and the key result established in this chapter is that a cumulative process under a policy that permits retrials following a rejection is bounded above and below by the same process under two simpler policies. We explore the policies BCP1 — BCP3 in detail and derive the distribution and expectation of a marked renewal process under BCP2 in terms of the Laplace-Stieltjes transforms of the underlying inter-event time and event size distributions. We also derive these expressions via sample paths, and give a probabilistic explanation for the expression of the derivative of the expectation. We illustrate the behaviour of a marked renewal process under BCP2 using a compound Poisson process with jump sizes following an exponential distribution.

Chapter 5 adapts the balance and profit models to the case of transacting behaviour and derives an optimisation problem for determining the credit limit which maximises expected profit. We calculate bounds on the optimal limit and decline probabilities for varying parameters of a compound Poisson process using numerical inversion of the expressions from Chapter 4. Using actual credit card transaction data, we fit a distribution to a single card-holder's supermarket purchase values and again calculate the bounds on the optimal limit and decline probabilities. The chapter concludes with a discussion of the practicalities of setting the limit at its optimal level, considering both the effect on profitability and the probability that the customer will experience a rejected purchase.

Chapter 6 adapts the model to a type of revolving behaviour termed *partial payment behaviour*, where the card-holder pays a fixed fraction c of the outstanding balance. We model the evolution of the outstanding balance as a discrete-time continuous state space Markov chain. By analysing the behaviour of the chain in its stationary regime, we show that the problem of determining optimal limits reduces to a modified version of the problem under transacting behaviour. We calculate the expectation of the Markov chain numerically by regarding the chain as an iterated random function system and sampling from its invariant measure. The numerical results indicate that the bounds established in Chapters 4 and 5 are still valid, and we again calculate bounds on the optimal limit and decline probabilities for a compound Poisson process.

Chapter 7 summarises the results and findings of the previous chapters and gives a critical review of the models and their assumptions with particular emphasis on practical use. We also propose avenues for extending the research.

Appendices A, B and C contain supporting material. This includes extended derivations, computer code used for numerical calculations and the data set used in the example.

1.5 Summary of contributions

The main contributions of this thesis are mathematical models of an individual card-holder's balance and profitability based on account behaviour. We demonstrate the utility of the models by applying them to derive profit-maximising credit limits under several types of account behaviour. In developing these models, we also make several other contributions. The contributions of Chapters 3 — 6 are summarised below.

Chapter 3 provides the mathematical framework that links models of individual patterns of purchasing and payment behaviour to credit card balance and profitability. We develop simplified models from which useful results can be derived in later chapters.

Chapter 4 describes three balance control policies as functionals of a cumulative process and establishes analytic formulæ for the distribution and expectation of a marked renewal process under a balance control policy in terms of the Laplace-Stieltjes transforms of the inter-event time and jump size distributions. We show that the policy which permits further jumps following a rejected jump to be bounded by the other two, and that the derivative of the expectation of a cumulative process under BCP3 has an interpretation as a path integral.

Chapter 5 derives an optimisation problem for calculating optimal limits for transactors and solves this using numerical inversion. By means of an example, we also establish new two-dimensional Laplace transform identities.

Chapter 6 shows that optimal limits for card-holders exhibiting partial payment behaviour can be calculated in the same manner.

Conceptually, this thesis demonstrates the utility that techniques in applied probability and stochastic modelling can bring to bear on problems in retail credit management. Previous research has explored facets of the work in this thesis, such as optimising credit limits or modelling purchase behaviour. To the best of the author's knowledge, this work is a novel approach to modelling credit card usage and deriving associated performance measures. The models and results merit study in their own right, but also have applications and it is hoped that they will be of interest to academics and practitioners alike.

Chapter 2

Mathematical preliminaries

A review of the Laplace-Stieltjes transform and its relative, the Laplace transform. A definition of their multi-dimensional analogues and how to numerically invert a transform.

2.1 Laplace-Stieltjes transforms

A number of results in this thesis are derived using Laplace-Stieltjes transforms of non-negative random variables in several dimensions. Here we provide a brief review of Laplace-Stieltjes transforms and their properties. We assume the reader is familiar with the basic concepts of probability theory and stochastic processes. Such background material can be found in [37], [53], [54], [75], [105] and [109].

2.1.1 Definitions

Our discussion of the Laplace-Stieltjes transform will make use of the properties of bounded variation and exponential order, which we now define.

Definition 1 (Bounded variation, [32, p.206])**.** *A real function F is of bounded variation on $[a, b]$ if $V_f[a, b] < \infty$, where for any $x \in [a, b]$,*

$$V_f[a, x] = \sup \left\{ \sum_{i=1}^{n} \left| F(x_i) - F(x_{i-1}) \right| \right\}, \tag{2.1}$$

with the supremum taken over all finite partitions $[a, x]$ with

$$a = x_0 < x_1 < \cdots < x_n = x.$$

A function with bounded variation has finite oscillation or variation on the interval $[a, b]$. Such a function is bounded on $[a, b]$, but not necessarily continuous, nor is a continuous function necessarily of bounded variation.

Definition 2 (Exponential order)**.** *A real function F is said to be of exponential order if there exist finite constants $c, M \in \mathbb{R}$ such that*

$$|F(x)| \leq M\,\mathrm{e}^{cx}, \quad x \geq 0. \tag{2.2}$$

When F is of exponential order, we use the shorthand notation $F(x) \sim O(\mathrm{e}^{cx})$, for the minimal such c.

A function of exponential order cannot grow too fast as $x \to \infty$. We also define the right and left-hand limits

$$a^{+} := \lim_{\epsilon \to 0} a + \epsilon \quad \text{and} \quad a^{-} := \lim_{\epsilon \to 0} a - \epsilon. \tag{2.3}$$

We now define the Laplace-Stieltjes transform.

Definition 3 (Laplace-Stieltjes transform)**.** *Let $F : [0, \infty) \mapsto \mathbb{R}$ be of bounded variation in $[0, R)$, $R \in \mathbb{R}^{+}$. The Laplace-Stieltjes transform $\tilde{f}$ of F is the function $\tilde{f} : \mathbb{R} \mapsto \mathbb{C}$,*

$$\tilde{f}(\omega) := \int_{0^-}^{\infty} \mathrm{e}^{-\omega x}\, F(\mathrm{d}x), \tag{2.4}$$

where $\omega = \sigma + \mathrm{i}\tau$ and $\mathrm{i} = \sqrt{-1}$. The function $\tilde{f}$ is defined for all ω such that the integral in Equation (2.4) *converges.*

Occasionally, we will use the notation $\mathcal{L}\{F(x)\}$ in place of $\tilde{f}(\omega)$. It was shown in [133, pp.36–37] that if the integral in Equation (2.4) converges, it does so in the right half of the complex plane for all ω such that $\sigma > \sigma_c$. The real number σ_c is called the *abscissa of convergence*, and its value is dependent on the properties of F. Theorem 2.1 in [133, pp.38–39] establishes that if the function $F(x) \sim O(\mathrm{e}^{cx})$, then $\sigma_c = c$ and the transform $\tilde{f}(\omega)$ exists. For the remainder of the thesis, whenever we encounter integrals of the form (2.4), we will assume that $F(x) \sim O(\mathrm{e}^{cx})$, for some positive c.

We now specialise to the case where $F(x) = \Pr(X \leq x)$ is the probability distribution function of a non-negative random variable X. Recall that a probability distribution function has the following properties:

i. F is non-decreasing, that is, $a \leq b$ implies $F(a) \leq F(b)$,

ii. F is right-continuous, that is, $F(a) = F(a^+)$, and,

iii. $F(-\infty) = 0$ and $F(\infty) = 1$.

The Laplace-Stieltjes transform (and indeed, Stieltjes integrals) find particular application in probability since a distribution function can be a continuous function, a step function, a combination of both or neither. However, by definition, a distribution function is monotone non-decreasing and since a monotone function is a function of bounded variation (see, for example, [30, p.101] or [32, p.209]), the Laplace-Stieltjes transform of a distribution function always exists.

When the density function f satisfying

$$F(x) = \int_0^x f(u)\,\mathrm{d}u, \quad x \geq 0 \tag{2.5}$$

exists, we may replace $F(\mathrm{d}x)$ with $f(x)\,\mathrm{d}x$ and rewrite Equation (2.4) as

$$\tilde{f}(\omega) = \int_0^\infty \mathrm{e}^{-\omega x} f(x)\,\mathrm{d}x. \tag{2.6}$$

In this case, it is common to refer to $\tilde{f}$ simply as the *Laplace transform* of the function f, although in probability it is sometimes called the Laplace transform of the random variable X. Note that since the expectation of a function h of a random variable X is

$$\mathbf{E}\left[h(X)\right] = \int_0^\infty h(x)F(\mathrm{d}x), \tag{2.7}$$

both the Laplace-Stieljtes and Laplace transforms may be viewed as an expectation with $h(x) = \mathrm{e}^{-\omega x}$.

2.1.2 Properties

The Laplace transform is often convenient to work with since several types of operation on f can be achieved by a simpler operation on $\tilde{f}$ instead. For the moment, we may regard f simply as a non-negative function. The operations of differentiation and integration of f are performed by multiplication and division of $\tilde{f}$ by ω respectively, that is,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \Longleftrightarrow \omega \tilde{f}(\omega) \quad \text{and} \quad \int_0^x f(u)\,\mathrm{d}u \Longleftrightarrow \frac{1}{\omega}\tilde{f}(\omega). \tag{2.8}$$

The notation $\Longleftrightarrow$ is used to denote correspondence between operations on f and $\tilde{f}$ and should be read as "is the Laplace transform of". These correspondences (2.8) also hold in reverse, and we have

$$xf(x) \Longleftrightarrow -\frac{\mathrm{d}}{\mathrm{d}\omega}\tilde{f}(\omega) \quad \text{and} \quad \frac{1}{x}f(x) \Longleftrightarrow \int_{\omega}^{\infty} \tilde{f}(\theta)\,\mathrm{d}\theta. \tag{2.9}$$

Let g also be a non-negative function with Laplace transform $\tilde{g}$. The *convolution* of f and g is given by

$$(f * g)(x) = \int_0^x f(u)g(x-u)\,\mathrm{d}u \Longleftrightarrow \tilde{f}(\omega)\tilde{g}(\omega). \tag{2.10}$$

If f and g are the density functions of random variables X and Y, then their convolution is the density function of the random variable $Z = X + Y$. The correspondence (2.10) is especially useful in probability since the Laplace transform of a sum of independent random variables is simply the product of the Laplace transforms of the random variables. In particular, let f^{*n} denote the n-fold convolution of f with itself. By the correspondence (2.10), we have

$$f^{*n}(x) \Longleftrightarrow \tilde{f}(\omega)^n. \tag{2.11}$$

Proofs of the correspondences (2.8) — (2.10) are elementary and can be found in a reference such as [48].

In addition to simplifying many operations, the Laplace transform also has the following useful properties. By Definition 3, evaluating $\tilde{f}(\omega)$ at $\omega = 0$ is equivalent to integrating the function f over its domain $[0, \infty)$, so that

$$\tilde{f}(0) = \int_0^{\infty} f(x)\,\mathrm{d}x. \tag{2.12}$$

Note that if f is the density function of a proper distribution function, then $\tilde{f}(0) = 1$.

The behaviour of $f(x)$ as x approaches zero or infinity can also be discerned from its Laplace transform via several theorems. Theorems of this type are called Abelian or Tauberian. The results of two Abelian theorems of particular use are

$$\lim_{\omega\to\infty} \omega\tilde{f}(\omega) = \lim_{x\to 0} f(x) \quad \text{and} \quad \lim_{\omega\to 0} \omega\tilde{f}(\omega) = \lim_{x\to\infty} f(x), \tag{2.13}$$

known as the Initial Value theorem and Final Value theorem respectively. Proofs of (2.13) and other theorems of a similar nature can be found in [133, Chap. 5].

Often, we are interested in the tail function $S(x) = 1 - F(x)$ and in many applications, it is easier to obtain an expression for $\tilde{S}(\omega)$, the Laplace transform of $S(x)$, rather than the transforms of the distribution or density functions. By the correspondences (2.8), the transforms of F and f may be expressed in terms of $\tilde{S}(\omega)$ as

$$\tilde{F}(\omega) = \frac{1}{\omega} - \tilde{S}(\omega) \quad \text{and} \quad \tilde{f}(\omega) = 1 - \omega\tilde{S}(\omega). \tag{2.14}$$

As mentioned earlier, the distribution function F may possess a mass or atom at $x = 0$, and this is often the case with problems in applied probability. By working with $\tilde{S}(\omega)$, we can obviate the need to account for the atom.

Using the correspondence (2.9) and the integral property (2.12), the expectation of the random variable X is given by

$$\mathbf{E}\left[X\right] = -\frac{\mathrm{d}}{\mathrm{d}\omega}\tilde{f}(\omega)\Big|_{\omega=0}. \tag{2.15}$$

Higher-order moments can be obtained by evaluating further derivatives of $\tilde{f}$.

$$\mathbf{E}\left[X^n\right] = (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}\omega^n}\tilde{f}(\omega)\Big|_{\omega=0}. \tag{2.16}$$

If X is a non-negative random variable, its expectation can also be computed as the integral of the tail function

$$\mathbf{E}\left[X\right] = \int_0^\infty S(x)\,\mathrm{d}x = \lim_{\omega\to 0}\tilde{S}(\omega), \tag{2.17}$$

where the right-most equality follows again from the integral property (2.12).

2.1.3 Laplace transforms in multiple dimensions

The Laplace-Stieltjes transformation can also be defined for n-dimensional distributions.

Definition 4 (n-dimensional Laplace-Stieltjes transform)**.** *The n-dimensional Laplace-Stieltjes transform of the distribution function $F(x_1, \ldots, x_n)$ is*

$$\tilde{f}(\omega_1, \ldots, \omega_n) := \int_0^\infty \cdots \int_0^\infty \mathrm{e}^{-\sum_{j=1}^n \omega_j x_j}\, F(\mathrm{d}x_1, \ldots, \mathrm{d}x_n), \tag{2.18}$$

where $\omega_j = \sigma_j + \mathrm{i}\tau_j$. The function $\tilde{f}$ is defined for all $\omega = \{\omega_j\}$, $1 \le j \le n$ such that the integral in Equation (2.18) *converges.*

As in the one-dimensional case, occasionally we will use the notation $\mathcal{L}_{\omega_1,\ldots,\omega_n}\big\{F(x_1,\ldots,x_n)\big\}$ in place of $\tilde{f}(\omega_1,\ldots,\omega_n)$.

The definition of bounded variation does not extend easily to multiple dimensions and as such, care must be taken to ensure that F satisfies the definition if it is invoked. Furthermore, the convergence of the integral in Equation (2.18) now holds in the region $\{\omega_j : \sigma_j > \sigma_{c_j}, 1 \le j \le n\}$, where σ_{c_j} are the abscissæ of convergence.

The notion of exponential order can be extended to multi-variable functions. If the function F can be shown to be the sum or product of functions $F_1(x) \sim O(\mathrm{e}^{c_1 x})$ and $F_2(x) \sim O(\mathrm{e}^{c_2 x})$, then F is also of exponential order.

Again, when the density function f satisfying

$$F(x_1,\ldots,x_n) = \int_0^{x_1}\!\!\cdots\int_0^{x_n} f(u_1,\ldots,u_n)\,\mathrm{d}u_1\ldots\mathrm{d}u_n. \tag{2.19}$$

exists, we may write

$$\tilde{f}(\omega_1,\ldots,\omega_n) := \int_0^{\infty}\!\!\cdots\int_0^{\infty} \mathrm{e}^{-\sum_{j=1}^{n}\omega_j x_j}\, f(x_1,\ldots,x_n)\,\mathrm{d}x_1\ldots\,\mathrm{d}x_n, \tag{2.20}$$

and refer to $\tilde{f}$ as the n-dimensional Laplace transform. Equation (2.20) shows that the n-dimensional Laplace transform is nothing more than the Laplace transform applied n times to the function f. As such, several properties of the one-dimensional Laplace transform carry over to the n-dimensional case. The correspondences (2.8) – (2.9) may be replaced by

$$\frac{\partial}{\partial x_j} f(x_1,\ldots,x_n) \Longleftrightarrow \omega_j \tilde{f}(\omega_1,\ldots,\omega_n) \tag{2.21}$$

$$\int_0^{x_j} f(u_1,\ldots,u_n)\,\mathrm{d}u_j \Longleftrightarrow \frac{1}{\omega_j}\tilde{f}(\omega_1,\ldots,\omega_n) \tag{2.22}$$

and their counterparts

$$x_j f(x_1,\ldots,x_n) \Longleftrightarrow -\frac{\partial}{\partial \omega_j}\tilde{f}(\omega_1,\ldots,\omega_n) \tag{2.23}$$

$$\frac{1}{x_j} f(x_1,\ldots,x_n) \Longleftrightarrow \int_{\omega_j}^{\infty} \tilde{f}(\theta_1,\ldots,\theta_n)\,\mathrm{d}\theta_j \tag{2.24}$$

for $1 \le j \le n$. The convolution correspondence (2.10), integral property (2.12) and limit theorems (2.13) hold similarly.

The particular case of the two-dimensional Laplace transform is treated in detail in [132].

2.2 Numerical inversion of Laplace transforms

Given a Laplace transform $\tilde{f}$, the function f can be recovered by means of the Bromwich contour integral

$$f(x) = \frac{1}{2\pi\mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \mathrm{e}^{\omega t}\, \tilde{f}(\omega)\, \mathrm{d}\omega, \tag{2.25}$$

where a is chosen so that any singularities of $\tilde{f}$ lie in the half-plane to the left of a. Inversion using Equation (2.25) can be difficult, and where possible the original function f is recovered by recognising it and its corresponding transform in a reference table of transform pairs. Extensive tables of Laplace transform pairs in one and two dimensions can be found in [48] and [132]. In the case where the Laplace transform does not have a known corresponding function and inversion using Equation (2.25), inversion may still be performed numerically.

2.2.1 The EULER algorithm

Many algorithms for the numerical inversion of transforms exist. Here, we consider the `EULER` algorithm presented in [2]. The basis of the algorithm is the fact that Equation (2.25) can be expressed as

$$f(x) = \frac{2\,\mathrm{e}^{ax}}{\pi} \int_0^\infty \mathrm{Re}\left[\tilde{f}(a+\mathrm{i}u)\right] \cos(ux)\, \mathrm{d}u. \tag{2.26}$$

Approximating the integral in Equation (2.26) using the trapezoidal rule with step size h, we arrive at the series

$$f_h(x) = \frac{h\,\mathrm{e}^{ax}}{\pi} \mathrm{Re}\left[\tilde{f}(a)\right] + \frac{2h\,\mathrm{e}^{ax}}{\pi} \sum_{k=1}^{\infty} \mathrm{Re}\left[\tilde{f}(a+\mathrm{i}kh)\right] \cos(khx). \tag{2.27}$$

Choosing a step size $h = \pi/2x$ and $a = A/2x$ yields the alternating series

$$f_h(x) = \frac{\mathrm{e}^{A/2}}{2x} \mathrm{Re}\left[\tilde{f}\left(\frac{A}{2x}\right)\right] + \frac{\mathrm{e}^{A/2}}{x} \sum_{k=1}^{\infty} (-1)^k \,\mathrm{Re}\left[\tilde{f}\left(\frac{A+2k\pi\mathrm{i}}{2x}\right)\right]. \tag{2.28}$$

The infinite series can be truncated to n terms,

$$s_n(x) = \frac{\mathrm{e}^{A/2}}{2x} \mathrm{Re}\left[\tilde{f}\left(\frac{A}{2x}\right)\right] + \frac{\mathrm{e}^{A/2}}{x} \sum_{k=1}^{n} (-1)^k \,\mathrm{Re}\left[\tilde{f}\left(\frac{A+2k\pi\mathrm{i}}{2x}\right)\right], \tag{2.29}$$

and Euler summation is applied to a further m terms after the initial n so that

$$E(m,n,x) = \sum_{k=0}^{m} \binom{m}{k} 2^{-m} s_{n+k}(x). \tag{2.30}$$

The use of Euler summation accelerates the convergence of the series, and simply a weighting of the last m partial sums by a binomial distribution with parameters m and $p = 1/2$. Pseudo-code implementing the calculations specified in Equations (2.29) – (2.30) is given in Algorithm 1. A function-specific

Algorithm 1 Pseudo-code for the `EULER` algorithm.

> **function** EULER($\tilde{f}, m, n, \gamma, x$)
> $A \leftarrow \gamma \log 10$
> $u \leftarrow \exp(A/2)/x$
> $z \leftarrow A/2x$
> $h \leftarrow \pi/x$
>
> $s \leftarrow \mathrm{Re}\left[\tilde{f}(z,0)\right]/2$
>   **for** $i \leftarrow 1, n$ **do**
> $y \leftarrow ih$
> $s \leftarrow s + (-1)^i \,\mathrm{Re}\left[\tilde{f}(z,y)\right]$
>   **end for**
>
> $e_1 \leftarrow s$
>   **for** $j \leftarrow 1, m$ **do**
> $k \leftarrow n + j$
> $y \leftarrow kh$
> $e_{j+1} \leftarrow e_j + (-1)^k \,\mathrm{Re}\left[\tilde{f}(z,y)\right]$
>   **end for**
>
> $E \leftarrow 0$
>   **for** $l \leftarrow 1, m$ **do**
> $E \leftarrow E + \binom{m}{l} e_l$
>   **end for**
> **return** $2^{-m}E$
> **end function**

implementation in the UBASIC language was given in [2], which used fixed values for the parameters m, n and $A = \gamma \log 10$. Setting the parameter γ results in a discretisation error of at most $10^{-\gamma}$. The pseudo-code in Algorithm 1 takes the parameters n, m, γ as function arguments, as well as the desired point of inversion x and the transform function $\tilde{f}$. The implementation used for calculating the numerical results in this thesis is in Appendix B.1.

We conclude this chapter by noting that numerical inversion can also be applied to transforms of more than one variable. The `EULER` algorithm was extended to the multi-variable case in [35]. An alternative algorithm which uses a representation of the Laplace transform $\tilde{f}$ as a Laguerre generating function is given in [1].

Chapter 3

Modelling credit card profitability

Why credit cards are different from other types of loans. Key concepts and notation for our models. The sources of revenue and expense for a credit card business. Finally, models of the outstanding balance and profit that link behaviour to profitability.

3.1 Behavioural patterns of credit card use

Our study of credit cards and their use is motivated by the wide range of spending and payment behaviours that are possible, and the different ways in which banks make money from these behaviours. Credit cards are a type of revolving loan facility and differ from other loans in several important ways. A typical instalment loan, such as a mortgage, grants the loan holder a fixed amount of credit which must be paid back over a specified period of time, usually according to a predetermined schedule of payments. The value of these payments is determined by the amount of credit granted, the term of the loan and the interest rate, which may be fixed or variable. Figure 3.1 illustrates the evolution of a typical instalment loan balance over time.

Credit cards permit the loan-holder to borrow funds up to the value of their approved credit limit at any time. The loan-issuing bank will bill the loan-holder at regular periods for the borrowed amount and this must be either partially or fully repaid by a specified due date or interest will be charged on the outstanding balance and billed to the customer at the end of the next

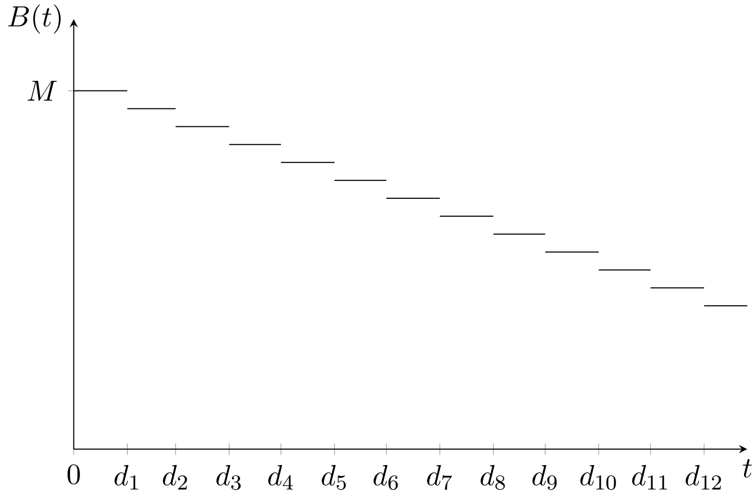

Figure 3.1: Example trajectory of an instalment loan. A amount M is drawn and repaid in instalments at the scheduled times $d_1, d_2, \ldots$. In this example, no payments are missed.

billing cycle. If the customer misses a payment, then penalty fees may also be charged and the line of credit may be reduced or cancelled. These borrowing and repayment terms are sufficient to generate a wide variety of behaviours.

Figure 3.2 provides some examples of the outstanding balance over time from actual credit card accounts. These figures were produced using transaction data provided by an Australian financial institution and emphasise the varying usage patterns that credit cards permit. These transactions occur over a two year observation period beginning on February 1st, 2011 and ending on February 28th, 2013. In order to preserve anonymity, the value of the outstanding balance has been removed, and the vertical axis instead shows the outstanding balance relative to the credit limit, ℓ. In Figures 3.2c and 3.2d, the credit limit is increased during the observation period, and this has also been shown. Note that the accounts in Figures 3.2e — 3.2h and Figures 3.2k — 3.2l were newly opened on February 1st, 2011, and hence the outstanding balance at time $t = 0$ is zero. The other accounts were opened prior to this date, and hence have a non-zero balance at time $t = 0$. Accounts with incomplete data closed during the observation period, resulting in an incomplete time series such as Figure 3.2f.

Several of the accounts show patterns of frequent purchasing and payment, while others show relative inactivity. A particularly stark comparison is the account in Figure 3.2i against the account in Figure 3.2l. Different payment behaviours are also evident. A particular example of transacting behaviour can be seen in Figure 3.2e, where the card-holder makes payments to the approximate value of the outstanding balance roughly every 30 days, corresponding to the approximate length of each billing cycle. The account in Figure 3.2b also demonstrates a form of transacting behaviour; it appears a transaction is regularly made each month (perhaps corresponding to a direct debit), and this is shortly paid in full.

Revolving behaviour can be seen in several figures, notably Figures 3.2a, 3.2c, 3.2f and 3.2h, where the card-holders have periods where either no payment or only small payments are made. The account in Figure 3.2c has a period of about 100 days where only one payment is received. We can further differentiate between those accounts that utilise a large portion of their credit limit against those which use hardly any. Again, the card-holder whose account is shown in Figure 3.2b hardly uses the funds available to them, while the card-holders shown in Figures 3.2c and 3.2h exceed their credit limit. Note that this could be the effect of interest or fees, although some card-issuers do permit card-holders to make purchases that will result in the outstanding balance exceeding their credit limit in some cases. Finally, note that in Figures 3.2j, 3.2l, the account is in credit for short periods. This could be the result of over-payment, or intentional as the card-holder may be planning to make a large purchase that would exceed their credit limit.

From the perspective of the card-issuing bank, a customer essentially makes purchases at random; the bank has no foreknowledge of when the card-holder will use the credit card and for how much, with the exception perhaps of regular direct debits. Similarly, it is unknown to the bank as to when a card-holder will make the requested payment, and whether it will be in full or only for a portion of the outstanding balance. Our aim in this chapter is to model how an individual customer's spending and payment behaviours generate profit for the bank and determine performance measures which the bank may be able to control in order to meet its profitability objectives.

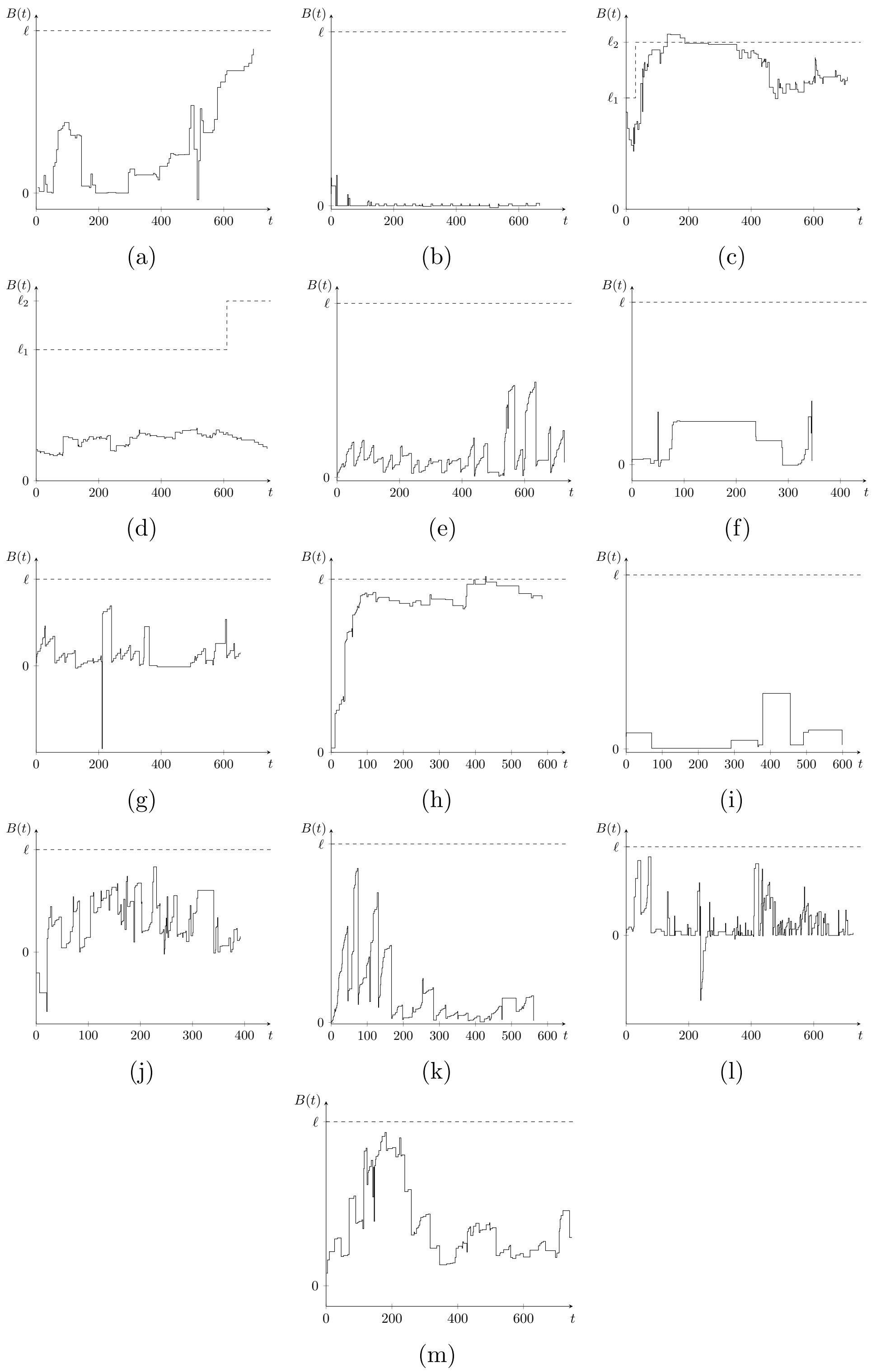

Figure 3.2: Examples of actual credit card balances evolving over time. The figures show the outstanding balance relative to the credit limit for open credit card accounts between February 1st, 2011 and February 28th, 2013. The data was provided by an Australian financial institution.

3.2 Terms of credit card use

In this section, we formalise some of the notions discussed in the previous section and introduce notation which will be used through the remainder of the thesis.

3.2.1 Statement dates and interest-free periods

Consider a credit card with limit $\ell > 0$ which is enforced by an automated card management system (CMS) so that the total value of purchases made may not exceed ℓ at any time. Let $B(t)$ denote the outstanding balance on the credit card at time t, which may include interest charges and fees and hence may exceed ℓ. It is also possible for $B(t)$ to take values less than zero if the card-holder makes payments that are greater than the outstanding balance.

Now, let $0 = s_0 < s_1 < \cdots < s_n < \infty$ be a sequence of statement times where the outstanding balance is automatically billed to the customer as a statement by the CMS and $d_1, \cdots, d_n$ be a sequence of times by which full or partial payment of the outstanding balance is due, with each $d_i \in [s_i, s_{i+1}]$, $1 \leq i \leq n$. The times d_i are commonly referred to as due dates, the interval $[s_i, s_{i+1}]$ as the ith statement period, and the interval $[s_i, d_i]$ as the ith interest-free period. The length of time between the end of the ith statement and interest-free periods is usually a fixed amount of time, although in practice this may vary due to business day conventions.

3.2.2 Minimum payment

Once the ith statement is issued, the card-issuer requires a minimum payment, c_i, to be made before the ith due date in order to avoid the account being considered delinquent and interest being charged. This amount varies between card-issuers and countries, but is usually calculated as the greater of some fixed fraction, $c_f \in (0, 1)$, of the outstanding balance and another fixed dollar amount, $c_d > 0$ or the entire outstanding balance. We write

$$c_i = c_f B(s_i) \vee \big(c_d \wedge B(s_i)\big), \tag{3.1}$$

where $\vee$ and $\wedge$ denote the maximum and minimum respectively. We can alternatively write Equation (3.1) as

$$c\big(B(s_i)\big) = \begin{cases} B(s_i), & B(s_i) < c_d \\ c_d, & c_d \le B(s_i) < c_d/c_f \\ c_f B(s_i), & B(s_i) \ge c_d/c_f. \end{cases} \tag{3.2}$$

Figure 3.3 provides an illustration of the minimum payment as a function of the outstanding balance. To maintain consistency with later notation, we will refer to the sequence $\{c_i\}$, $i \ge 1$ of minimum payments.

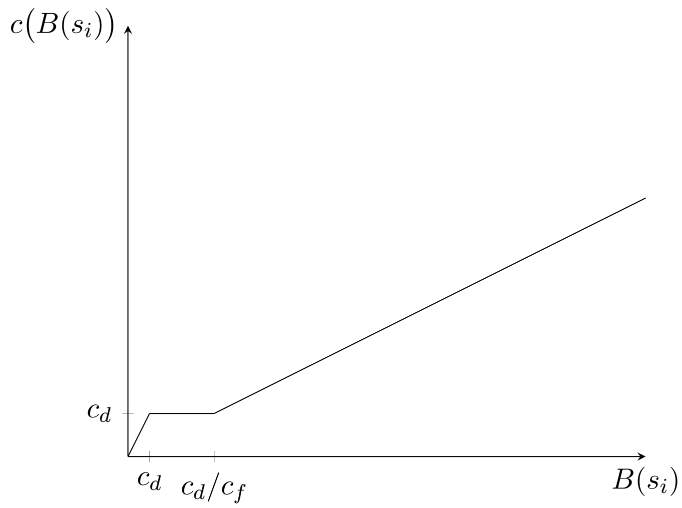

Figure 3.3: Minimum payment as a function of the outstanding balance. The three line segments represent the cases where the minimum payment is either the full outstanding balance, a minimum dollar amount c_d, or a fixed fraction c_f of the outstanding balance.

3.2.3 Payments and interest

Let $\{\rho_i\}$, $i \ge 1$ denote a sequence of random variables where each ρ_i represents the total value of payments made between the end of the $(i-1)$th interest-free period and the end of the ith interest-free period. The amount of interest to be charged is calculated depending on the value of total payments received by the end of the interest-free period. Amongst Australian card-issuing banks, the following three scenarios are possible:

1. $\rho_i = B(s_i)$: the card-holder pays the outstanding balance $B(s_i)$ in full. No interest is charged and the account is considered current. This is commonly referred to as *transacting behaviour*.

2. $c_i \leq \rho_i < B(s_i)$: the card-holder makes payments to the value of at least the minimum payment but less than the outstanding balance. The received payment is deducted from the outstanding balance and applied to purchases in the order in which they were made. Interest is then calculated on the value of the remaining purchases made in $(s_{i-1}, s_i]$ and compounded daily from the date of purchase and will be added to the outstanding balance at the next statement date s_{i+1}. The account is still considered current, but interest will now also be charged on purchases made in $(s_i, s_{i+1}]$ from the date of purchase and compounded daily. In effect, the card-holder loses the interest-free period.

3. $0 \leq \rho_i < c_i$: the card-holder pays less than the minimum payment amount. Interest is calculated as in the second scenario, and the account is now considered delinquent. As such, late charges or other penalty fees may now also be applied to the account.

In either of the first or second scenarios, the card-holder may continue to use the remaining unutilised portion of their credit limit. In the third scenario, the card-holder may also continue to use the credit card, but may find that after two or more consecutive months of non-payment the card-issuing bank will either reduce their available credit limit or prevent the card-holder from making further purchases. Figure 3.4 provides an example illustrating the evolution of the outstanding balance when both full and partial payments are made.

Interest may also be charged differently on different types of purchases. For example, in Australia, cash advances attract interest charges from the date of withdrawal. Conversely, if a card-holder transfers the balance from another credit card, it is common for that amount to be exempt from interest charges for a period of time.

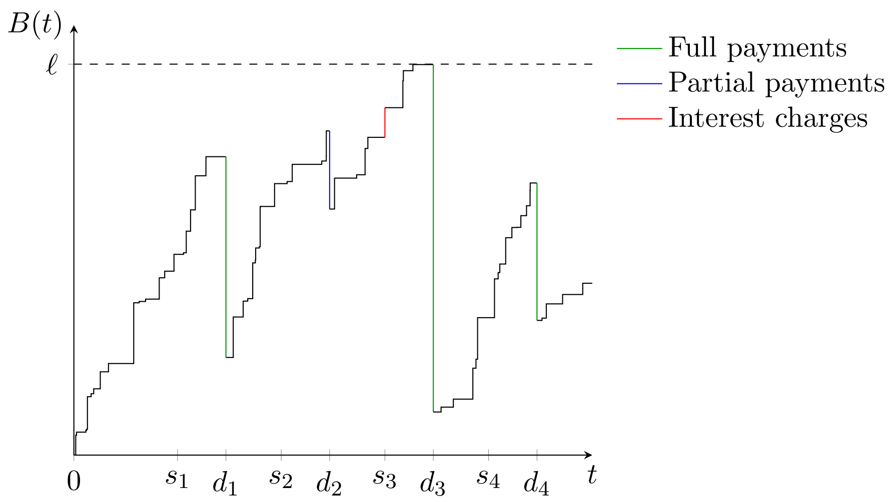

Figure 3.4: Example trajectory of the outstanding balance on a credit card. The card-holder makes full payments at the due dates d_1, d_3 and d_4. Only partial payment is received at d_2 and so interest is charged at s_3.

3.3 Sources of revenue and expense

The previous section examined the features of a credit card and the various usage patterns that these features may lead to. In this section we examine how a card-issuing bank generates profit from a card-holder's use of their credit card.

3.3.1 Revenue

The sources of revenue for a card-issuing bank are interest, interchange, fees (such as annual or penalty fees) and enhancements such as credit insurance. For commercial reasons, card-issuing banks do not make actual breakdowns of their revenue and costs available to the public. However, some publications have made industry-level figures available. Table 3.1 is taken from [111] and details revenue expenses for the U.S. credit card industry in 2004.

3.3.1.1 Interest

As with most loan products, interest charges are the major source of revenue for a credit card business. Interest rates for credit cards are typically very high relative to the rates of other loan products and tend to remain so even

Revenues	**Value (USD in billions)**	**% of total revenue**
Interest	72.13	69.1%
Interchange	16.63	16.0%
Penalty fees	8.10	7.8%
Annual fees	2.41	2.3%
Cash advance fees	4.29	4.1%
Enhancements	0.78	0.7%
Total	104.34	

Table 3.1: Revenue figures for the U.S. credit card industry in 2004.

when interest rates decrease for other loan products. In Australia, interest rates vary between 10 — 21%, barring the inclusion of cards with an interest-free teaser period. The fact that credit card interest rates persist at such high rates despite increasing competition between card issuers is also a topic of ongoing research, studied in [10], [31] [86] and [117].

As mentioned earlier, interest is charged on purchases from the date of purchase when a card-holder fails to make a full payment by the due date. As an example, consider a card-holder who makes the series of purchases in Table 3.2 in their first statement period. The total outstanding balance due

Time	**Value (AUD)**
t_1	100.00
t_2	50.00
t_3	70.00
t_4	180.00
t_5	80.00
Total	500.00

Table 3.2: Interest calculation example. A card-holder makes five purchases in their first statement period.

is \$500.00. Say the card-holder makes payment of \$300 before the due date d_1, and continues to make purchases until the second statement date, s_2 as shown in Table 3.3. The interest owed is calculated at time s_2 as

$$\begin{aligned} I(s_2) = (1+r)^{s_2}\big(&100(1+r)^{-t_4} + 80(1+r)^{-t_5} + 30(1+r)^{-t_6} \\ &+ 120(1+r)^{-t_7} + 50(1+r)^{-t_8}\big), \end{aligned} \tag{3.3}$$

since the payment at time s_2 will be applied to all purchases made prior to s_2

in the order they were made. The first three purchases total \$220, and are paid off at time s_2 when the card-holder makes the payment of \$300. The remaining \$80 is deducted from the purchase at time t_4, leaving an equivalent purchase at time t_4 of \$100. Since the card-holder did not pay the full outstanding amount, interest is calculated on all remaining purchases up to time s_2. This amount will be due by the time d_2.

Time	**Value (AUD)**
t_1	100.00
t_2	50.00
t_3	70.00
t_4	180.00
t_5	80.00
s_1	-300.00
t_6	30.00
t_7	120.00
t_8	50.00
Total	380.00

Table 3.3: Interest calculation example continued. The card-holder makes a partial payment at the end of the first statement period and continues to make purchases. Interest will subsequently be charged.

3.3.1.2 Interchange

Interchange is a fee paid between banks for facilitating credit card transaction, sourced by fees charged to merchants. When a card-holder makes a purchase using their credit card, a discount fee is applied to the purchase and paid by the merchant to the merchant's bank. This fee covers the cost of interchange and other operational costs. The interchange fee is paid to the card-issuer's bank via the credit card association, such as Visa or Mastercard.

An overview of the steps involved in processing a typical credit card transaction was given as an example in [106, p. 295]. The example shows the steps involved in processing a credit card transaction for \$100 when a merchant discount fee of 1.9% and an interchange fee of 1.3% are applied. We reproduce the example here.

An example credit card transaction

1. The card-holder uses a credit card to make a \$100 purchase at a merchant establishment.

2. The merchant submits the charge to the merchant bank at the end of the business day.

3. The merchant bank reimburses the merchant for the purchase minus a fixed discount fee, e.g. 1.9% of the total purchase price. The merchant receives \$98.10.

4. The merchant bank submits the charge to the credit card association.

5. The credit card association forwards the charge to the customer's card-issuing bank.

6. The card-issuing bank submits payment to the credit card association minus a fixed interchange fee, e.g. 1.3% of the total \$100 purchase price. The total payment made is \$98.70.

7. The credit card association forwards payment of \$98.70 to the merchant bank and collects fixed processing fees from both the merchant and the card-issuing bank.

8. The card-issuing bank bills the card-holder for the \$100 purchase.

9. The card-holder pays the card-issuing bank the \$100 or at least some minimum amount with the remaining balance to be paid over time.

The steps in this example are illustrated in Figure 3.5.

This example is greatly simplified, since there are a number of additional processes that are not detailed in Figure 3.5. However, this example serves to illustrate the main parties involved in a credit card transaction and that there are a number of processes and costs that are not apparent to the card-holder in either their use of the credit card or when they receive their statement from their bank.

It is clear from Figure 3.5 that merchants subsidise the operation of the credit card network. While it costs merchants to accept payment via credit card, they also profit from an increased volume of purchases. Thus, it is

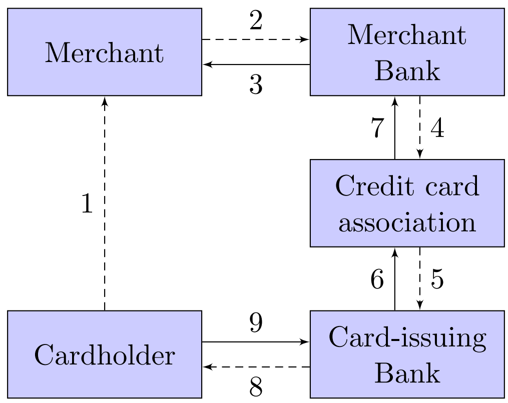

Figure 3.5: Processing steps in a typical credit card transaction. Payments are illustrated by solid lines and charges by dashed lines. The numbers above the arrows correspond to the steps outlined in Section 3.3.1.2.

important for the interchange rate to be set so that merchants will continue to participate in the credit card association and accept payment via credit card. Economic analyses of the interchange rate in credit card networks have been studied in [102].

3.3.1.3 Fees and enhancements

Card-issuing banks also charge a number of fees to card-holders. Fees can be divided into two types: activity-driven fees and regularly-recurring fees. Activity-driven fees are charged when a card-holder makes or fails to make a certain type of transaction: they include over-limit fees, late fees and cash advance fees. Late payments or cash advances typically reflect an increased likelihood of non-repayment and are charged to cover the costs involved in recovering delinquent balances. Over-limit fees were banned in Australia in July 2012 and are not charged on accounts opened after this date unless the card-holder expressly consents to having their outstanding balance exceed their credit limit rather than have a transaction declined. This reform is detailed in [91, s113BI].

Regularly-recurring fees, such as annual fees, are charged according to a pre-determined schedule. Annual fees are usually paid on premium cards that offer the card-holder the ability to earn points on loyalty schemes and enhancements such as purchase insurance or access to proprietary concierge services.

3.3.2 Expenses

The sources of expense for a bank in running a credit card business are the costs due to charge-offs, capital provisioning, operations and marketing, and the cost of funds and fraud. Table 3.4 shows industry-level expense figures for the U.S. credit card industry in 2004, again taken from [111]. This table does not include the costs due to capital provisioning since capital adequacy standards were not widely adopted by then.

Expenses	**Value (USD in billions)**	**% of total expenses**
Charge-offs	\$31.86	44%
Operations & marketing	\$27.73	39%
Cost of funds	\$11.58	16%
Fraud	\$0.70	1%
Total	\$71.87	

Table 3.4: Expense figures for the U.S. credit card industry in 2004.

3.3.2.1 Charge-offs and loan loss provisioning

Charge-off costs are incurred when the card-issuing bank deems a card-holder unlikely to repay the outstanding balance and chooses to consider the total outstanding amount as a loss. The bank may attempt to recover the charged-off amount and associated costs through legal action or by selling the debt to a debt-recovery agency at a discounted cost.

Card-issuing banks employ several management strategies to minimise the losses due to charge-offs. As mentioned in Section 3.2.3, if a card-holder does not make a minimum payment by the required due date, the account is considered delinquent. An account that has failed to make one minimum payment is usually referred to as one cycle delinquent or thirty days past due. Similarly, an account that has missed n consecutive minimum payments is referred to as n cycles delinquent or $30n$ days past due.

As an account progresses through the stages of delinquency, the bank will attempt to contact the card-holder and encourage repayment of the outstanding balance (including any incurred penalty fees). In the early stages or delinquency, reminder letters or contact via telephone or SMS are widely employed and effective at prompting card-holders to meet the minimum payment owed. The frequency and urgency of contact will increase if the card-holder becomes further delinquent. In cases where the card-holder is experiencing financial impairment, the use of the card may be temporarily suspended and the bank may negotiate for the outstanding balance to be repaid in instalments over a period of time so as to prevent further financial impairment and distress. These methods are referred to as collections strategies and are managed by a bank's collections or recovery department.

If the bank determines that the card-holder will not be able to pay the amount owed (for example, in the instance of personal bankruptcy), or if the cost of further action is greater than the amount owed, then the bank will charge-off the outstanding balance and consider it a loss.

In Australia, an account that is three or more cycles delinquent is said to be in default. The definition of default is can vary between countries; in the U.S. it is usually six or more cycles. The event of default has particular significance for financial regulation and the calculation of regulatory capital, which will be discussed in the next section.

An account that is not delinquent is said to be current or up-to-date. Some banks also make a distinction between current accounts that have made purchases in the prior statement period and paid the outstanding amount in full and accounts which are current because they have had no activity in the prior statement period.

A card-issuing bank will typically provision a fraction of an account's outstanding balance in anticipation of losses (however unlikely) due to default and eventual charge-off. This is usually called a *loan-loss* or *bad and doubtful debt provision*. Note that a credit card account is not necessarily charged-off at the point of default, since a bank may still attempt to recover the outstanding balance via collection strategies as described earlier.

The method for calculating a loan-loss provision will vary between banks, but is usually dependent on product-specific parameters such as the interest rate, length of the interest-free period, and the delinquency state of the account, since industry experience shows that past delinquency is a strong indicator of future delinquency. In recognition of this, a common method for calculating a loan-loss provision is to use historical portfolio data to estimate the fraction of the outstanding balance which will move between delinquency states via a discrete-time and discrete-state Markov chain. There are several studies on the use of Markov chains for this purpose.

The first published Markov chain model for forecasting delinquent balances is [42], in which the authors modelled dollars receivable from retail credit accounts (not necessarily credit card accounts). In [40], the model was extended to account for non-stationarity and seasonality in the transition probabilities using exponential smoothing and a seasonal factor. A further extension was made in [43] to allow for different transition matrices based on the risk profile of the account holder, which may be determined using a behavioural score, for example. In [130], a modification was proposed that allowed for a distinction between the delinquency state of the total amount owed and the delinquency state of different portions of the total amount owed. Recognising the heterogeneity of credit card portfolios, later studies have attempted to account for different payment behaviours. In [58], the authors adapted the mover-stayer model first proposed in [22] to distinguish between account-holders who never leave their initial state (stayers) and those who will leave the initial state and move through delinquency states according to a transition matrix. In [74], the authors proposed alternative state space definitions, one which the delinquency state was sub-divided by the amount owed and another based on transitions between payment behaviours similar to those described in Section 3.2.3.

To illustrate the use of Markov chain methods in calculating a loss provision, we reproduce the model proposed in [42]. In this model, a lending business has lent a total of b dollars to its customers at time i, and classifies this amount into $n+1$ age categories, representing the number of periods past due. For $0 \leq j \leq n-1$, let D_j be the amounts receivable in dollars that are j periods past due, and let D_n be the amounts receivable that are n or more periods past due. The state n represents a charge-off state and dollars moving into this state are deemed irrecoverable. In period $i+1$, some portion of the dollars receivable in each of the age categories will be repaid or move into further delinquency. The amount that is repaid to the lending business is represented by the variable $D_{\bar{0}}$. Using this classification, the movement of dollars receivable between the age categories in a single period is represented by $(n+2) \times (n+2)$ matrix

$$\mathbf{D} = \begin{pmatrix} D_{\bar{0}\bar{0}} & \cdots & D_{\bar{0}k} & \cdots & D_{\bar{0}n} \\ \vdots & & \vdots & & \vdots \\ D_{j\bar{0}} & \cdots & D_{jk} & \cdots & D_{jn} \\ \vdots & & \vdots & & \vdots \\ D_{n\bar{0}} & \cdots & D_{nk} & \cdots & D_{nn} \end{pmatrix}. \tag{3.4}$$

From this matrix, we may obtain a matrix of transition probabilities

$$\mathbf{P} = \begin{pmatrix} p_{\bar{0}\bar{0}} & \cdots & p_{\bar{0}k} & \cdots & p_{\bar{0}n} \\ \vdots & & \vdots & & \vdots \\ p_{j\bar{0}} & \cdots & p_{jk} & \cdots & p_{jn} \\ \vdots & & \vdots & & \vdots \\ p_{n\bar{0}} & \cdots & p_{nk} & \cdots & p_{nn} \end{pmatrix} \tag{3.5}$$

where

$$p_{jk} = \frac{D_{jk}}{\sum\limits_{k \in \mathcal{S}} D_{jk}}, \quad \mathcal{S} = \{\bar{0}, 0, 1, \ldots, n\}. \tag{3.6}$$

The states $\bar{0}$ and n are absorbing, since dollars received by the business cannot move into delinquency and it is assumed that dollars moving into the charge-off state cannot be recovered. A diagram illustrating the Markov chain is given in Figure 3.6. The p_{jk} were estimated using actual data with $n = 6$,

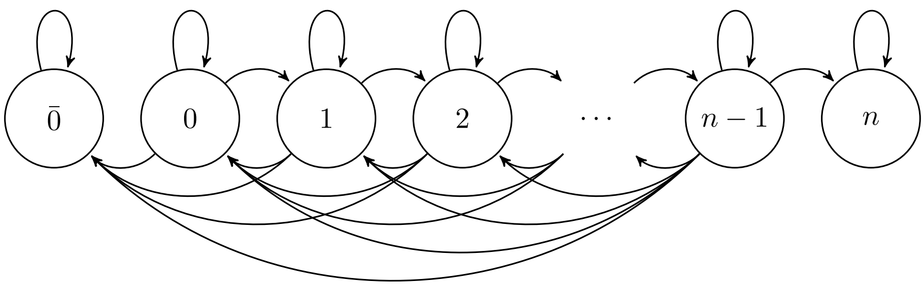

Figure 3.6: A Markov chain model of delinquency.

and the authors arrived at the following one-step transition matrix,

$$\mathbf{P} = \begin{pmatrix} 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.21 & 0.67 & 0.12 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.13 & 0.19 & 0.44 & 0.24 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.13 & 0.08 & 0.20 & 0.36 & 0.23 & 0.00 & 0.00 & 0.00 \\ 0.10 & 0.01 & 0.04 & 0.17 & 0.29 & 0.39 & 0.00 & 0.00 \\ 0.14 & 0.02 & 0.00 & 0.09 & 0.20 & 0.41 & 0.14 & 0.00 \\ 0.09 & 0.01 & 0.02 & 0.01 & 0.10 & 0.12 & 0.47 & 0.18 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 1.00 \end{pmatrix}. \tag{3.7}$$

To estimate the proportion of dollars that will end up in the charge-off, note that the rows of matrix $\mathbf{P}$ can be reordered so that the first and second states are the absorbing states $\bar{0}$ and n respectively, and the remaining transient states are $0, 1, \ldots, n-1$. This reordered matrix, $\mathbf{P}^*$, can be partitioned as

$$\mathbf{P}^* = \left(\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{R} & \mathbf{Q} \end{array}\right) \tag{3.8}$$

where $\mathbf{I}$ is the 2×2 identity matrix, $\mathbf{0}$ is a $2 \times n$ matrix consisting of zeroes, $\mathbf{R}$ is a $n \times 2$ matrix and $\mathbf{Q}$ is an $n \times n$ matrix. The matrix

$$\mathbf{N} = (\mathbf{I} - \mathbf{Q})^{-1} \tag{3.9}$$

exists and is called the fundamental matrix, and the entries of the $n \times 2$ matrix $\mathbf{NR}$ yield the probabilities of absorption in states $\bar{0}$ and n. Let $\mathbf{b}$ be a column vector describing the total outstanding balances in each of the delinquency states. We obtain a loss provision by multiplying $\mathbf{b}$ and the second column of $\mathbf{NR}$. Using the transition matrix $\mathbf{P}$ specified in Equation (3.7), we find

$$\mathbf{NR} = \begin{pmatrix} 0.99 & 0.01 \\ 0.98 & 0.02 \\ 0.97 & 0.03 \\ 0.92 & 0.08 \\ 0.88 & 0.12 \\ 0.62 & 0.38 \end{pmatrix}. \tag{3.10}$$

Setting $\mathbf{b} = (100\,000, 20\,000, 8\,000, 5\,000, 2\,000, 1\,000)$, we arrive at a loss provision of \$2 178.97.

It is common for the loan loss provision to be treated as an on-balance sheet item that is calculated monthly and deducted against total revenue. Applied in this manner, it smooths the realisation of losses due to charge-off.

3.3.2.2 Regulatory capital and liquidity provisioning

In addition to the loan loss provision described in the previous section, banks must also hold a regulatory capital provision. In contrast to the loan loss provision, the regulatory capital provision is not held as a liability, but is rather held as assets in reserve in order to cover the bank against losses. Regulatory capital is provisioned according to the standards set by the Basel Committee on Banking Supervision (BCBS), an international advisory committee set up by Bank of International Settlements (BIS) in 1974. The Committee has published three accords on regulatory capital provisioning and these accords are enforced by local prudential regulators, such as the Australian Prudential Regulatory Authority (APRA).

Under the first Basel accord (Basel I) published in 1998, regulatory capital was not required to be held for credit card loans, since card-issuing banks reserved the right to unconditionally cancel the loan agreement. The second Basel accord (Basel II) [12] was published in 2005 and allowed banks to pursue either a standardised or internal ratings based (IRB) approach to calculating regulatory capital. Again, regulatory capital was not required for credit card loans under the standardised approach. However, the IRB approach requires banks to estimate three risk measures in order to calculate the regulatory capital: probability of default (p_D), exposure at default (e_D) and loss given default (l_D). As mentioned in the previous section, the definition of default can vary between different national regulators, but in most countries is taken as 90 days past due.

The quantities p_D, e_D and l_D are estimated over a twelve month horizon and regulatory capital is calculated as

$$C_\varrho = e_D l_D \big(K(p_D, \varrho) - p_D\big), \tag{3.11}$$

where

$$K(p_D, \varrho) = \Phi\left(\frac{\Phi^{-1}(p_D) - \Phi^{-1}(0.999)\varrho}{1-\varrho}\right) \tag{3.12}$$

and Φ is the cumulative distribution function for the standard normal distribution and Φ^{-1} is its inverse. The quantity ϱ reflects the default correlation between individual loans in a portfolio, and for credit cards is specified as $\varrho = 0.04$. The function K arises from the underlying model of default behaviour that the Basel accord assumes, which was originally due to Merton [85]. A plot of the capital requirement as a function of the probability of default and loss given default is given in Figure 3.7. A derivation of this model, as

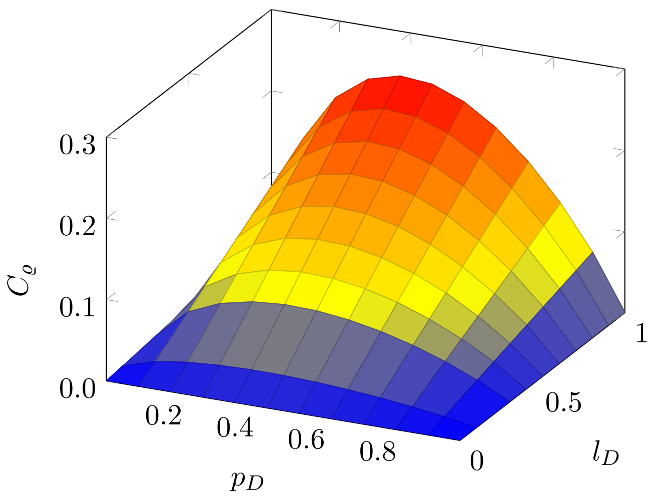

Figure 3.7: Capital requirement for a single dollar of exposure as a function of probability of default and loss given default. In this plot, $\varrho = 0.04$, the prescribed value under Basel II for revolving retail exposures.

well as a concise history of the Basel I and II accords and a discussion of the Basel II risk measures can be found in [122, Chap. 5]. A comparison of capital allocation methods in use prior to the introduction of the Basel II accord against the method in the IRB approach is given in [94].

A standard reference for the estimation and validation of the individual risk measures is [51]. Probabilities of default are commonly estimated using logistic regression, similar to the manner in which behavioural scores are calculated. Alternative approaches to the estimation of probabilities of default for retail loans have been studied in [81], [103] and [135].

The estimation of exposure at default for credit cards and other revolving retail exposures is complicated by the fact that a card-holder may draw further funds (up to the assigned credit limit) in between the time that the probability of default was estimated and the actual point of default. As noted in [90], a reasonable and IRB-compliant approach is to simply take the credit limit ℓ as the exposure at default, although one may argue this is overly conservative. Another approach is to estimate a loan equivalent exposure (LEQ), q, such that

$$e_D = B(t) + q\big(\ell - B(t)\big) = q\ell + (1-q)B(t), \tag{3.13}$$

where t is the time of observation. As noted in [5] and [100], credit card-holders usually default for an amount close to the limit and so typically $q \approx 1$. An early survey of the estimation techniques for exposure at default, including the use of a loan equivalent exposure, can be found in [129]. A more recent survey is in [80], in which a model of exposure at default as a function of credit limit and spending and payment amounts was proposed. As mentioned in Section 1.3, the authors in [71] modelled the unconditional balance of credit cards using several regression methods. A comparison between estimation approaches of exposure at default with and without the use of a loan equivalent exposure is in [127].

Loss given default can also be estimated in a number of ways. Most methods focus on modelling the collections strategies that a bank uses, given the clear relationship between these and recovery rates, particularly for unsecured products such as credit cards. See [51, Chaps. 6–12] for examples. A comparison of several regression algorithms for estimating loss given default is given in [82].

In calculating the regulatory capital requirement, it is important to note that the three quantities p_D, e_D and l_D enter Equation (3.11) as a product. Underlying this is an assumption of independence between the three quantities. The product $p_D e_D l_D$ is referred to as the expected loss and recent work has focused on understanding dependencies between these quantities and the consequences for the calculation of expected loss. The correlation between exposure at default and loss given default in general was investigated in [15], and the effect of correlation between default rate and loss given default on the distribution of expected loss for a credit card portfolio was studied in [14].

The regulatory capital calculated using Equation (3.11) is regarded as a minimum capital requirement: a bank may choose to hold further capital if it deems this necessary. The amount of capital actually held by the bank is referred to as economic capital.

3.3.2.3 Cost of funds

A retail bank usually has several lines of lending business and sources funds for these business lines via an internal treasury that buys funds from the deposit-raising business units and through interbank lending. These funds are then redistributed to the lending arms of the business such as the credit cards lending unit, which will pay a funds transfer price (FTP) to the treasury representing an opportunity cost to the bank. The FTP will also cover a portion of the treasury's cost in managing the interest rate and liquidity risk involved in borrowing funds from either the short or long-term money market.

The cost of funds is paid on the outstanding balances which the credit card business needs in order to pay the credit card association when a transaction is undertaken by a customer (refer to step 6 in Section 3.3.1.2). However, the credit card business will not borrow funds from the treasury every time a purchase is made. Rather, funds will be bought ahead of time and according to a budgeted amount. As noted in [131, p. 9], the FTP paid by the business can also vary depending on the portion of the total balances that are revolving versus transacting.

A stand-alone credit card business which does not have access to deposits will usually fund its activities by securitising future cash flows (receivables) from its loans. An overview of the sources of funding for credit card businesses is given in [131]. A discussion of the calculation of funds transfer prices is in [112].

3.3.2.4 Operations and marketing

Operational costs are direct costs to the bank involved in running and managing a credit card portfolio. These are largely fixed costs that include overheads such as staff wages, property leases and IT infrastructure. Marketing costs include campaigns for acquisitions, credit limit increases and balance transfers. There are also small variable costs associated with producing and mailing statements and processing transactions.

3.3.2.5 Fraud

Losses due to fraud are a comparatively small but nonetheless significant cost to a credit card business. As shown in Table 3.4, the cost to the U.S. credit card industry in fraud losses was USD $700 million in 2004. Recent figures for fraud losses in Australia were published in [9], which showed fraudulent transactions in 2013 totalled AUD $304 million against a total value of AUD $624 billion in credit card transactions. Thus, while the rate of fraud is low (approximately 0.049%), the total value of purchases made means the cost of fraud to a credit card business is still in the hundreds of millions of dollars.

Credit card fraud is perpetrated in a number of ways. Credit card details may be stolen from merchant systems or databases where card and card-holder details are stored and then used to make purchases. This is referred to as *card-not-present* fraud. Credit cards may also be counterfeited using skimming devices which can be surreptitiously fitted to ATMs or POS devices. Lost or stolen cards can be used at merchants for low-value purchases which don't require the card-holder to enter a PIN or signature. Despite the widespread use of PIN and chip technology, signatures may still be accepted by some merchants. Finally, credit cards may be acquired using stolen identity information.

To mitigate and prevent losses from fraud, a card-issuing bank will also operate a fraud prevention unit. This unit will monitor applications for credit cards and the transactions of customers in an effort to identify anomalous and potentially fraudulent activity. An overview of the techniques used in industry in fraud identification is given in [24].

The costs of creating and running a fraud prevention unit include staff and technology costs, amongst others, and will be included in the operations costs mentioned above.

3.3.2.6 Loyalty schemes

As discussed in Section 1.1, credit cards were originally conceived as a device to engender customer loyalty and encourage increased spending at a particular retailer. Card-issuing banks encourage card-holders to use their credit cards as much as possible in order to earn revenue through interchange, and spending at particular merchants is encouraged through merchant-specific loyalty schemes. These schemes offer the card-holder the opportunity to earn points, which may be redeemed for future purchases with the merchant, or receive a cash rebate or discount on certain types of purchases. In general, loyalty schemes can be considered as a form of rebate to the card-holder, and as such they are an expense to the card-issuing bank. Points earned in point-based schemes, such as those associated with airlines, hotel chains or supermarkets, are typically purchased by the card-issuing bank in advance from the merchant. The costs of loyalty schemes to card-issuing banks and their effect on card-holder behaviour is discussed briefly in [76]. A more in-depth discussion on the effect of loyalty schemes on profitability is in [88].

3.4 A model of credit card profitability

As we have seen in the previous section, the profit a bank derives from a credit card is linked to the card-holder's use of the facility. In this section, we formulate a mathematical model of how profit is earned from an individual's use of a credit card. We first describe a general model of an individual card-holder's balance, allowing for a wide variety of behaviours and interest and fee structures. In turn, this model is used as an input to a general model of profit. Due to the complicated nature of how revenue and expenses are calculated, we make some assumptions about how interest, fees and other quantities are calculated. We also specify how a card-holder makes payments. This leads to a model that captures the essential features, yet is easier to manipulate.

3.4.1 A general model of the outstanding balance

As in Section 3.2.1, we consider a credit card account with credit limit $\ell > 0$ and let $B(t)$ be the card-holder's outstanding balance on their credit card at time $t \geq 0$. The outstanding balance can be decomposed into four different quantities, namely, the value of purchases, payments, interest charges and fees. A bank is likely to only take action on a card-holder's behaviour at the pre-defined statement dates or due dates introduced in Section 3.2.1 since this is typically when a CMS will aggregate information about the card-holder's actions. Restricting our observations of the balance to the statement dates s_i, the card-holder's balance at the ith statement date can be written as

$$B(s_i) = B(s_{i-1}) - \big(P(s_i) - P(s_{i-1})\big) + \Psi\big[s_{i-1}, s_i, A, P, \ell - B(s_{i-1})\big]$$
$$+ I(s_i, r(s_i), \mathcal{A}_{s_i}, \mathcal{P}_{s_{i-1}}) + F(s_i, \mathcal{A}_{s_i}, \mathcal{P}_{s_i}), \quad i \geq 1. \tag{3.14}$$

In Equation (3.14),

- $A(s_i)$ and $P(s_i)$ are cumulative stochastic processes describing the total value of purchases attempted and payments made on the credit card up to time s_i.

- $\mathcal{A}_{s_i}$ and $\mathcal{P}_{s_i}$ are the σ-algebras generated by the processes $A(\cdot)$ and $P(\cdot)$ up to time s_i, and may be interpreted as the histories of attempted purchases and payments.

- $\Psi\big[s_{i-1}, s_i, A, P, \ell - B(s_{i-1})\big]$ is the value of approved purchases in the statement period $[s_{i-1}, s_i)$. The card-holder begins the statement period $[s_{i-1}, s_i]$ with available credit $\ell - B(s_{i-1})$ and attempts to make purchases during the period, which the CMS will approve or decline according to the amount of available credit. Any payments made during the period $[s_{i-1}, s_i)$ will result in further credit being made available for purchases. As such, Ψ takes as its arguments the beginning and end of the statement period, s_{i-1} and s_i, the processes $A(u)$ and $P(u)$, $s_{i-1} \leq u \leq s_i$ and the available credit at the beginning of the statement period, $\ell - B(s_{i-1})$. The particular form that Ψ takes is dependent on the *balance control policy* that the bank uses to decide which purchases to decline. We postpone a detailed discussion of balance control policies and expressions for Ψ until Chapter 4.

- $I(s_i, r(s_i), \mathcal{A}_{s_i}, \mathcal{P}_{s_i})$ is the total value of interest charged to the credit card up to time s_i, as a function of the interest rate $r(s_i)$ and the histories of attempted purchases and payments. Note that the interest rate may vary over time, hence we specify $r(s_i)$ as function of the statement date, s_i. As discussed in Section 3.3.1.1, interest calculations can be quite complicated and require careful accounting of the purchases and payments made. Hence, we require the complete histories of purchases and payments up to time s_i.

- $F(s_i, \mathcal{A}_{s_i}, \mathcal{P}_{s_i})$ is the total value of fees (both activity-based and regularly-recurring) charged to the credit card up to time s_i, as a function of the histories of attempted purchases and payments.

Note that the cumulative effect of interest charges and fees can result in a balance that is greater than ℓ. Furthermore, payments can also render $B(t) < 0$.

As noted in Section 3.1, the card-holder uses their card in continuous time and, from the bank's perspective, attempts to make purchases at random. For now, we assume the same of how the card-holder makes payments. Without any loss of generality, we may also assume that $A(t)$ and $P(t)$ are marked point processes of the form

$$A(t) = \sum_{j=1}^{N(t)} \xi_j \quad \text{and} \quad P(t) = \sum_{k=1}^{M(t)} \chi_k \tag{3.15}$$

where $\{\xi_j\}$, $j \geq 1$, and $\{\chi_k\}$, $k \geq 1$, are sequences of random variables that represent the size of attempted purchases and payments made by the card-holder. The processes $N(t)$ and $M(t)$ are counting processes describing the number of attempted purchases and payments in $(0, t]$ respectively. We write $t_j = \inf\{t : N(t) = j\}$ and $\tau_j = t_j - t_{j-1}$, with $t_0 = 0$. Similarly, we write $u_k = \inf\{t : M(t) = k\}$ and $\upsilon_k = u_k - u_{k-1}$, with $u_0 = 0$. The σ-algebras generated by $A(t)$ and $P(t)$ take the form

$$\mathcal{A}_t = \big\{(t_j, \xi_j),\ \forall j : t_j < t\big\} \quad \text{and} \quad \mathcal{P}_t = \big\{(u_k, \chi_k),\ \forall k : u_k < t\big\}. \tag{3.16}$$

At this stage, we do not state any further assumptions on $A(t)$ and $P(t)$. Note that, given knowledge of the statement times s_i, the functions I, F and $r(s_i)$ and the functional Ψ, the balance at time t is completely determined by the realisations of $A(u)$ and $P(u)$, $0 \leq u \leq t$, reflecting the card-holder's purchasing and payment behaviour.

The process of attempted purchases, $A(t)$, can be separated into a superposition of processes for different types of purchases such as retail purchases and cash advances, since interest is typically calculated differently for these types of purchases. Of course, this also introduces another function for the calculation of interest on cash advances and Equation (3.14) can be modified to accommodate this by adding another interest rate function specific to those types of purchases.

Regularly-occurring purchases such as annual fees or direct debits, can be modelled by adding a deterministic component to Equation (3.15). Let $\{v_l\}$, $l \geq 1$ be a sequence of pre-determined times at which the pre-determined amounts $\{\varsigma_l\}$, $l \geq 1$, are charged to the credit card. The process of attempted purchases in Equation (3.15) then becomes

$$A(t) = \sum_{i=1}^{N(t)} \xi_i + \sum_{l:v_l \leq t} \varsigma_l. \tag{3.17}$$

Other credit card features that may affect the outstanding balance can be modelled by suitably modifying Equations (3.14) and (3.15). For example, balance transfers can be treated as a separate type of purchase, and require a separate interest rate function to be added to Equation (3.14). Note that the card-issuing bank may change at its discretion the method by which interest is calculated, the interest rate itself, and the fees which may be charged on the credit card account. As such, different forms for I and F may be required for different time periods, and this is a straight-forward modification to Equation (3.14). Interest rates, interest calculation methods and fee structures typically do not change often, so when considering a relatively short period of time, we will rarely have to deal with more than one function for calculating the balance.

3.4.2 A general model of profit

Let $R(t)$ denote the total profit earned by the bank from an individual credit card account up to time t. Combining the sources of revenue and expense detailed in Section 3.2, we can express the profit earned in the statement period $[s_{i-1}, s_i]$ as

$$\begin{aligned} R(s_i) = {} & R(s_{i-1}) + I(s_i, r(s_i), \mathcal{A}_{s_i}, \mathcal{P}_{s_i}) + F(s_i, \mathcal{A}_{s_i}, \mathcal{P}_{s_i}) \\ & + \gamma\big[\Psi[s_{i-1}, s_i, A, P, \ell - B(s_{i-1})]\big] - \epsilon\big[\Psi[s_{i-1}, s_i, A, P, \ell - B(s_{i-1})]\big] \\ & - \eta\big(B(s_i), \mathbf{w}\big) - \kappa\big(B(s_i), \mathbf{x}\big) - C\big(B(s_i), \ell, \mathbf{y}\big). \end{aligned} \tag{3.18}$$

The new quantities introduced in Equation (3.18) are:

- $\gamma\big[\Psi[s_{i-1}, s_i, A, P, \ell - B(s_{i-1})]\big]$, which describes the interchange charged on the value of approved purchases.

- $\epsilon\big[\Psi[s_{i-1}, s_i, A, P, \ell - B(s_{i-1})]\big]$, the cost of loyalty points or rebates which is also calculated on the value of approved purchases. Typically, these points can only be earned on the first L dollars spent within each statement period and the precise form which the function $\epsilon[\cdot]$ takes will need to account for this.

- $\eta\big(B(s_i), \mathbf{w}\big)$ is a loan-loss provision calculated as a function of the outstanding balance and some vector of characteristics associated with the card-holder's risk profile, $\mathbf{w}$. Such characteristics may include behaviour score or delinquency state;

- $\kappa\big(B(s_i), \mathbf{x}\big)$ is a function describing the cost of funds. As mentioned in Section 3.3.2.3, the cost of funds may depend on whether the balance is revolving or not, hence we specify its dependence on $\mathbf{x}$, representing a vector of characteristics used to determine whether or not to treat the balance as revolving.

- $C\big(B(s_i), \ell, \mathbf{y}\big)$ is the regulatory capital held in reserve. This will be calculated according to Equation (3.11). As noted in Section 3.3.2.2, for revolving loans including credit cards the exposure at default is typically estimated using a loan equivalent exposure which estimates the fraction of the undrawn limit that the card-holder will use in the event of default. Hence we specify dependence on both the balance $B(s_i)$ and, importantly for our later analysis, the credit limit ℓ. We let $\mathbf{y}$ denote the vector of account characteristics that are inputs to the estimation of probability of default, exposure at default and loss given default.

We assume the fixed costs remain constant across statement periods so that in taking the difference between statement periods, we remove their contribution to Equation (3.18). We also assume that variable operational costs are incurred periodically and that they are netted from the periodic fees charged to the account. In addition, we annualise regularly-recurring fees and consider them a fixed cost as well. Hence, $F(s_i, \mathcal{A}_{s_i}, \mathcal{P}_{s_i})$ represents the income from activity-based fees. Furthermore, note that the functions introduced in Equation (3.18) may vary across different credit card products, particularly the functions for the calculation of interchange and loyalty point costs.

It is important to note that although interest and fees are charged to the account at the statement time s_i, these amounts may not be paid by the customer. However, the loan-loss provision function $\eta\big(B(s_i), \mathbf{w}\big)$ estimates the portion of the balance which will eventually be charged-off, and since the balance includes the interest and fees charged, the net effect is an estimate of actual profit from these revenue items. Thus, from the card-issuing bank's accounting, $R(t)$ genuinely reflects the profit earned.

The form of Equation (3.18) generally accords with our expectations of how different payment behaviours drive profit for the bank. In the case of a "pure transacting" card-holder who always pays the full outstanding balance, there will be no interest contribution to profit. Furthermore, transacting card-holders are generally considered to be low risk, and hence the functions $\eta\big(B(s_i), \mathbf{w}\big)$, $\kappa\big(B(s_i), \mathbf{x}\big)$ and $C\big(B(s_i), \ell, \mathbf{y}\big)$ are likely to be close to zero, and income from penalty fees is also likely to be small. As such, revenue will largely be driven by interchange and the costs will be the cost of funds, the loyalty program and the cost of regulatory capital. By contrast, a revolving card-holder will generate profit through interest, and possibly fees as well. However, due to the increased risk associated with revolving behaviour, the functions η, κ and C are likely to be larger and hence profitability is affected by increased loan-loss provisions, cost of funds and capital.

3.4.3 A simplified model

While Equations (3.14) and (3.18) allow for a wide range of spending and payment behaviours, they are, in general, difficult to work with. We now make some assumptions which will lead to a simplified model of profitability.

Our first assumption concerns the method of interest rate calculation.

Assumption 1 (M1)**.** *Interest is only charged on the total outstanding balance* $B(s_i)$ *at the statement dates* $\{s_i\}$, $i \geq 1$.

Assumption M1 greatly simplifies all our calculations involving interest, since in assuming a fixed interest calculation method, we only have to concern ourselves with the interest rate, $r(s_i)$. Typically, the rate is given as an annual percentage rate, and we will have to adjust it in order to apply the rate across a statement period.

The model of the balance specified by Equation (3.14) allows for both approved purchases and payments to occur at random. As such, the card-holder can make multiple payments within a single statement period. The second assumption we make is

Assumption 2 (M2)**.** *The card-holder makes only one payment during the ith statement period, $i \geq 1$. This payment is made at the end of the ith interest-free period.*

This assumption means that we only have to consider a single random variable describing the payments in a statement period. This is not an overly restrictive assumption from a modelling perspective, since rather than modelling the individual payments within a statement period, we can choose to model their aggregate value. With this in mind, let $\{\rho_i\}_{i\geq 1}$ be a sequence of random variables that describe the fraction of the balance $B(s_i)$ paid by the card-holder in the ith interest-free period. Then the value of total payments received by the end of the ith interest-free period is

$$P(d_i) = \sum_{k=1}^{i} \rho_k B(s_k). \tag{3.19}$$

Several payment behaviours can be modelled by how we specify the payment sequence $\{\rho_i\}$, $i \geq 1$. As mentioned in Section 3.2.3, transactors are customers who routinely pay down the full outstanding balance due at the end of each statement period. We model this by setting $\rho_i = 1$, $i \geq 1$. This has two consequences: interest is never charged to the account and the full credit limit is available to the customer to make purchases in each statement period.

Customers that only pay part of their outstanding balance are referred to as *revolvers*. Here, we list some possible models that Assumption M2 allows for.

R1 If we ignore the requirement of a minimum payment and assume that the customer pays a constant fraction $c \in (0, 1)$ of the due balance at each statement date, then we have the case where $\rho_i = c$, $i \geq 1$;

R2 Assume the situation as in R1 above, but with customer either paying the proportion c of the due balance with probability p or the full amount with probability $q = 1 - p$. In this situation, the customer switches between revolving and transacting behaviour;

R3 The customer always pays at least the minimum amount c_i by each due date and that $c_d = 0$, so

$$c_i = c_f B(s_i) \vee 0 = c_f B(s_i), \quad i \geq 1. \tag{3.20}$$

We set $\rho_i = c_f \vee b_i$, where $\{b_i\}$, $i \geq 1$ is a sequence of independent and identically distributed random variables taking values in $[0, 1]$.

R4 The customer pays any amount in $[c, 1]$. In this case, $\rho_i = c \vee b_i$, where each b_i is an independent and identically distributed random variable on the unit interval $[0, 1]$.

R5 The ρ_i are a sequence of independent and identically distributed random variables on the unit interval. This is the case where the card-holder may not always pay the minimum amount.

In the cases R3 — R5, the β-distribution seems a natural choice for the random variables $\{\rho_i\}$. However, there is no particular reason to restrict ourselves to this choice of distribution at this stage.

In the previous section, we stated that regularly-recurring fees are annualised and so the only fees which we cater for are activity-based fees. As noted in Section 3.3.1.3, over-limit fees can't be charged without consent in Australia, which leaves late payment fees. We therefore assume

Assumption 3 (M3)**.** *Only fixed late payment fees are charged and the fee amount, $\phi \geq 0$, is fixed. Fees are charged at statement dates.*

In conjunction with Assumption M2, this means that if $\rho_i < c_i$, an amount ϕ will be charged.

Since interest and fees are charged to the account at the statement dates, s_i, $i \geq 1$, we now introduce another assumption to align the payments with the application of interest and fees.

Assumption 4 (M4). *There is no interest-free period. Consequently,* $s_i = d_i$*,* $i \geq 0$*.*

The combined effect of Assumptions M2 and M4 is that payments only occur at the end of a statement period and cannot increase the available credit during a statement period. As a consequence, the value of payments received at the end of the ith statement period affects the value of approved purchases during the $(i+1)$th statement period only through the opening balance, $B(s_i)$. This allows us to restrict our attention in Chapter 4 to balance control policies that act on increasing processes only.

Our next assumption concerns the functions introduced in Equation (3.18).

Assumption 5 (M5). *The calculation of interchange and cost of loyalty points take the form*

$$\gamma\big[\Psi[s_{i-1}, s_i, A, P, \ell - B(s_{i-1})]\big] = \bar{\gamma}\Psi[s_{i-1}, s_i, A, P, \ell - B(s_i)], \tag{3.21}$$

$$\epsilon\big[\Psi[s_{i-1}, s_i, A, P, \ell - B(s_{i-1})]\big] = \bar{\epsilon}\Psi[s_{i-1}, s_i, A, P, \ell - B(s_i)], \tag{3.22}$$

where $\bar{\gamma}, \bar{\epsilon} > 0$*. Furthermore, the functions describing the calculation of the loan loss provision and cost of funds take the form*

$$\eta\big(B(s_i), \mathbf{w}\big) = \bar{\eta}(\mathbf{w})B(s_i), \tag{3.23}$$

$$\kappa\big(B(s_i), \mathbf{x}\big) = \bar{\kappa}(\mathbf{x})B(s_i), \tag{3.24}$$

where $\bar{\eta}(\mathbf{w}), \bar{\kappa}(\mathbf{x}) > 0$*.*

The function forms in M5 are reasonable assumptions to make, since interchange and the cost of loyalty points are typically calculated as fixed fractions of the value of approved purchases, and these fractions are unlikely to change within a statement period. The loss provision and cost of funds will change based on the card-holder's risk profile, but can be expressed as fractions of the outstanding balance.

Our last assumption at this stage concerns the calculation of regulatory capital.

Assumption 6 (M6). *Exposure at default is calculated using the loan equivalent exposure method.*

This means that the calculation of regulatory capital in Equation (3.11) becomes

$$\begin{aligned} C_\varrho &= \big(B(s_i) + q(\mathbf{y}_1)(\ell - B(s_i))\big)\big(K(p_D(\mathbf{y}_2), \varrho)l_D(\mathbf{y}_3) - p_D(\mathbf{y}_2)l_D(\mathbf{y}_3)\big) \\ &= \nu_\varrho(\mathbf{y}_2, \mathbf{y}_3)\big(B(s_i) + q(\mathbf{y}_1)(\ell - B(s_i)\big), \end{aligned} \tag{3.25}$$

where $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$ contain the sub-elements of $\mathbf{y}$ used in the calculation of q, p_D and l_D respectively. Put another way, $\mathbf{y}$ is the concatenation of $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$. The function $\nu_\varrho(\mathbf{y}_2, \mathbf{y}_3)$ represents the fraction of exposure at default that must be reserved.

By Assumptions M1 — M4, the balance at the ith statement date is

$$B(s_i) = Z(s_{i-1}) + \Psi[s_{i-1}, s_i, A, \ell - Z(s_{i-1})], \quad i \geq 1. \tag{3.26}$$

where

$$Z(s_i) := (1 + \mathbf{1}_{\{\rho_i < 1\}}\, r)(1 - \rho_i)B(s_i^-) + \phi\, \mathbf{1}_{\{\rho_i B(s_i^-) < c_i\}}, \quad i \geq 1, \tag{3.27}$$

represents the simultaneous application of payment, interest charges and fees to the outstanding balance at the ith statement date and is the opening balance of the account. Equation (3.26) implies that the CMS approves purchases to the value of

$$\Psi[s_{i-1}, s_i, A, \ell - Z(s_{i-1})] = B(s_i^-) - Z(s_{i-1}), \quad i \geq 1, \tag{3.28}$$

which is simply the difference of the opening and closing balances of the account.

With assumptions M1 – M6, the profit earned by the bank at the end of the ith statement period is

$$\begin{aligned} R(s_i) = R(s_{i-1}) &+ r\, \mathbf{1}_{\{\rho_i < 1\}}\, B(s_i^-) + \phi\, \mathbf{1}_{\{\rho_i B(s_i^-) < c_i\}} \\ &+ (\bar{\gamma} - \bar{\epsilon})\Psi[s_{i-1}, s_i, A, \ell - Z(s_{i-1})] \\ &- \big(\bar{\eta}(\mathbf{w}) + \bar{\kappa}(\mathbf{x})\big)B(s_i^-) \\ &- \nu_\varrho(\mathbf{y}_2, \mathbf{y}_3)\big(B(s_i^-) + q(\mathbf{y}_1)(\ell - B(s_i^-)\big). \end{aligned} \tag{3.29}$$

Rearranging and gathering terms, we have

$$\begin{aligned} R(s_i) = R(s_{i-1}) &+ (\bar{\gamma} - \bar{\epsilon})\Psi[s_{i-1}, s_i, A, \ell - Z(s_{i-1})] \\ &+ r\, \mathbf{1}_{\{\rho_i < 1\}}\, B(s_i^-) + \phi\, \mathbf{1}_{\{\rho_i B(s_i^-) < c_i\}} - \nu_\varrho(\mathbf{y}_2, \mathbf{y}_3)q(\mathbf{y}_1)\ell \\ &- \big(\bar{\eta}(\mathbf{w}) + \bar{\kappa}(\mathbf{x}) + \nu_\varrho(\mathbf{y}_2, \mathbf{y}_3)(1 - q(\mathbf{y}_1))\big)B(s_i^-). \end{aligned} \tag{3.30}$$

From Equation (3.30), we see that the random quantities affecting profit are the outstanding balance, the value of approved purchases made in the statement period $(s_{i-1}, s_i]$ and the payment made. Thus, the profit that the bank can earn is dependent upon the card-holder's behaviour and by the credit limit it assigns to the card-holder, since the value of approved purchases can not exceed ℓ.

At this stage, we are now in a position to use Equation (3.30) to maximise profit. We take the expectation to arrive at

$$\begin{aligned}\mathbf{E}\left[R(s_i)\right] &= \mathbf{E}\left[R(s_{i-1})\right] + (\bar{\gamma} - \bar{\epsilon})\mathbf{E}\big[\Psi[s_{i-1}, s_i, A, \ell - Z(s_{i-1})]\big] \\ &\quad + r\Pr\left(\rho_i < 1\right)\mathbf{E}\left[B(s_i^-)\right] + \phi\Pr\left(\rho_i B(s_i^-) < c_i\right) - \nu_\varrho(\mathbf{y}_2, \mathbf{y}_3)q(\mathbf{y}_1)\ell \\ &\quad - \left(\bar{\eta}(\mathbf{w}) + \bar{\kappa}(\mathbf{x}) + \nu_\varrho(\mathbf{y}_2, \mathbf{y}_3)(1 - q(\mathbf{y}_1))\right)\mathbf{E}\left[B(s_i^-)\right]. \end{aligned} \tag{3.31}$$

Since it is the credit limit ℓ that constrains the value of $\mathbf{E}\big[\Psi[s_{i-1}, s_i, A, \ell - Z(s_{i-1})]\big]$, a reasonable approach to take would be to find the credit limit that maximises profit. To proceed further, we require an expression for $\mathbf{E}\big[\Psi[s_{i-1}, s_i, A, \ell - Z(s_{i-1})]\big]$ which will, in turn, give us an expression for $\mathbf{E}\left[B(s_i)\right]$ via Equation (3.28). In the next chapter, we develop expressions for the distribution and expectation of the value of approved purchases.

3.5 Summary

The principal contribution of this chapter are the models of outstanding balance and profit described by Equations (3.14) and (3.18). The proposed models are continuous-time stochastic processes and, to the best of the author's knowledge, are the first such models of credit card balance and profitability. The model of profitability considers revenue earned from interest and fees and other common sources of revenue and cost which are detailed in Section 3.3. In Equation (3.18), the sources of revenue and cost can depend on an account state variables which could represent the account's delinquency status, for example.

Equations (3.14) and (3.18) are difficult to work with in their original form, and so several simplifying assumptions have been made which render further manipulations possible. These assumptions result in Equations (3.26) and (3.29) which form the basis of study for the remainder of the thesis. We postpone a critical discussion of the models and assumptions until Chapter 7.

Chapter 4

Balance control policies and the process of approved purchases

Expressions for the distribution and expectation of approved purchases, Ψ, under three balance control policies: retrial, rejection and partial acceptance. Bounds on the paths and expectation of the process induced by the retrial policy. An example illustrating the properties of the rejection policy.

4.1 Approved purchases

In the previous chapter, we derived equations describing the outstanding balance and profitability of an individual credit card account. After making the simplifying assumptions M1 – M6, the resulting Equations (3.26) and (3.30) show that we require an expression for $\Psi[s_{i-1}, s_i, A, \ell - B(s_i)]$, the value of approved purchases in a statement period. In our original equations for the outstanding balance and profit, we specified that the value of approved purchases in the ith statement period was dependent on the balance at time s_{i-1} (the opening balance), the credit limit ℓ, and the processes of attempted purchases and payments made. However, with Assumptions M2 and M4, we removed the explicit dependency on payments made by the card-holder in the ith statement period, instead incorporating payments into the opening balance for the next statement period.

As noted at the end of the previous chapter, the process describing the value of approved purchases is itself a marked point process. In this chapter, we derive formulæ for the distribution and expectation of Ψ based on the balance control policy enforced by the card-issuing bank. As we shall see, the expression for the distribution function of Ψ under a realistic balance control policy can be complicated, and the contribution of this chapter is to demonstrate that the value of approved purchases under a realistic balance control policy can be bounded above and below using simple policies.

4.2 Balance control policies

At the point at which the card-holder was granted a credit card, a credit limit ℓ was assigned to the account based on an assessment of credit-worthiness. The card-issuing bank will enforce this limit through a balance control policy, which is a set of rules describing which purchases to permit and which to decline. The essential function of the policy is to prevent the outstanding balance from exceeding the credit limit ℓ, and there are a number of practical considerations which make modelling a balance control policy challenging. For example, the card-holder may be barred from making purchases due to suspected fraudulent activity, or they may have requested that purchases from certain types of merchants not be permitted (this can occur with corporate credit accounts). Furthermore, the card-issuing bank may have authorised a shadow limit $\ell_s > \ell$ on the card-holder's account. The shadow limit is usually calculated through an *authorisation strategy* and updated monthly, based on characteristics such as the card-holder's utilisation and behaviour score. The balance control policies that we consider do not allow for special rules based on transaction or merchant type, account status, or for shadow limits.

Applying a balance control policy to the process of attempted purchases during the ith statement period will induce a stochastic process that maps the paths of $A(u)$, $s_{i-1} \leq u \leq s_i$, from the positive half-line $\mathbb{R}_+$ to a sub-set of $\mathbb{R}_+$, resulting in the process of approved purchases. By Assumptions M2 and M4, the available credit is only affected by purchases approved during this period and since payments are received at the statement dates s_i, $i \geq 1$, it suffices to consider the first statement period $[0, s_1)$. Formally, we have that

$$\Psi : \mathcal{D}\big([0, s_1), \mathbb{R}_+\big) \mapsto \mathcal{D}\big([0, s_1), [0, \ell]\big). \tag{4.1}$$

where $\mathcal{D}$ is the space of right-continuous functions with left limits, or càdlàg functions. It is understood that the functional $\Psi[0, s_1, A, \ell]$ acts only on the process $A(u)$, $0 \leq u \leq s_1$ and so, where convenient, we will suppress some notation and write $\Psi(s_1, \ell) \equiv \Psi[0, s_1, A, \ell]$. In the remainder of this chapter, we describe three balance control policies in detail and derive expressions for the distribution function of the value of approved purchases that results from their application.

4.3 BCP 1: A retrial policy

Consider a credit card account with credit limit ℓ where the balance evolves according to Equation (3.26) and $B(0) = 0$ so that the full credit limit is available to the card-holder. As mentioned earlier, we confine our analysis to the first statement period $[0, s_1)$, and let the card-holder attempt to make purchases at the random times $0 < t_1 < \cdots < t_j < \ldots t_n < s_1$, $n \geq 1$. Our first balance control policy is stated as follows:

Balance Control Policy 1 (BCP1)**.** *Accept an attempted purchase if and only if the sum of the current value of approved purchases and the value of the attempted purchase does not exceed the credit limit ℓ.*

Let $\Psi_1(t, \ell)$, $0 \leq t \leq s_1$ denote the value of approved purchases on a credit card under BCP1 with $\Psi_1(t_0, \ell) = \Psi_1(0, \ell) = 0$. The process $A(t)$ can be expressed as a recurrence at the purchase attempt times t_j, $1 \leq j \leq n$ as

$$A(t_j) = A(t_j^-) + \xi_j, \tag{4.2}$$

where ξ_j is the value of the jth attempted purchase. The process $A(t)$ remains constant between the purchase attempt times. Similarly, we can express the value of approved purchases under BCP1 at the purchase attempt times via the recurrence equation

$$\Psi_1(t_j, \ell) = \Psi_1(t_j^-, \ell) + \mathbf{1}_{\{\Psi_1(t_j^-, \ell) + \xi_j \leq \ell\}} \cdot \xi_j, \tag{4.3}$$

where

$$\mathbf{1}_{\{x\in\mathcal{A}\}} = \begin{cases} 0, & x \notin \mathcal{A}, \\ 1, & x \in \mathcal{A}. \end{cases} \tag{4.4}$$

Again, the process $\Psi_1(t,\ell)$ remains constant between the purchase attempt times. Equation (4.3) was described in [95] in the context of queues with a rejection admission policy. Rejection admission policies in queueing systems have been studied in several works, notably [38], [39] and [59]. In [26], an admission policy that gave partial service depending on the workload requested was explored.

Figure 4.1 provides an illustration of the process of attempted purchases and the resulting outstanding balance under BCP1.

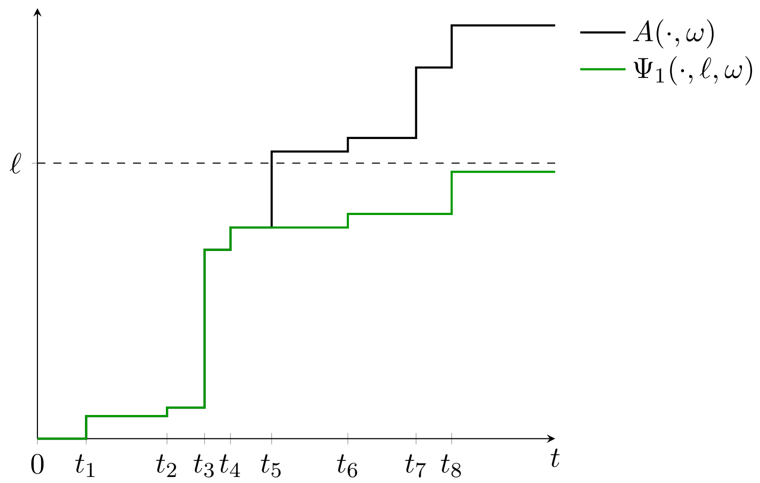

Figure 4.1: A realisation of the process of attempted purchases $A(\cdot,\omega)$ and the process of approved purchases $\Psi_1(\cdot,\ell,\omega)$ resulting from the application of the balance control policy BCP1. Under this policy, the jumps at times t_5 and t_7 are rejected.

4.3.1 An integral equation for the tail function of $\Psi_1(t,\ell)$

We use an approach similar to the method in [34] to derive an expression for the Laplace-Stieltjes transform of the tail function $\Pr(\Psi_1(t,\ell) \in (y,\ell])$. Since $\Psi_1(t,\ell)$ is a non-negative random variable, its expectation is easily computed from the tail function. The method exploits the Markov property by conditioning on the time and value of the first jump in the process. So, to use the method, we require additional assumptions about the process $A(t)$. The first is

Assumption 7 (A1). *The probability* $\Pr(\xi_j = 0) = 0$ *for* $j \geq 1$.

Assumption A1 is justified by the fact that any purchase made by a cardholder will be for an amount greater than zero. We also make the further assumption

Assumption 8 (A2). *The process of attempted purchases $A(t)$ is a marked renewal process and both the renewal distribution, G, and mark size distribution, F, are of exponential order. The sequence of inter-event times $\{\tau_j\}$ and mark sizes $\{\xi_j\}$, $j \geq 1$ are independent.*

With the additional assumptions A1 and A2, the process $A(t)$ is now a Markov process. The Laplace-Stieljtes transforms of F and G are defined as

$$\tilde{f}(\theta) := \int_0^\infty \mathrm{e}^{-\theta z}\, F(\mathrm{d}z) \quad \text{and} \quad \tilde{g}(\omega) := \int_0^\infty \mathrm{e}^{-\omega u}\, G(\mathrm{d}u) \tag{4.5}$$

for $\mathrm{Re}(\theta) > \sigma_F$ and $\mathrm{Re}(\omega) > \sigma_G$, where the respective abscissæ of convergence σ_F and σ_G of $\tilde{f}$ and $\tilde{g}$ are strictly less than zero. As stated in Section 2.1.1, the assumption that F and G are of exponential order is sufficient to guarantee the existence of their Laplace-Stieltjes transforms.

For $0 < y \leq \ell$ and $t > 0$, we attempt to find an expression for the tail function

$$S_1(y, t, \ell) := \Pr\big(\Psi_1(t, \ell) \in (y, \ell]\big) \tag{4.6}$$

by conditioning on the time and value of the first jump of the process. We have four possibilities to consider

i. No jumps occur in the interval $[0, t)$;

ii. The process jumps to some $z \in (0, y]$ at time $u \in (0, t)$ and then regenerates itself at this point. That is, a new process starts at z that behaves like the original one, but shifted by $\ell - z$ in space and $t - u$ in time.

iii. The process jumps to some $z \in (y, \ell]$ at time $u \in (0, t)$. Note that any subsequent jumps that happen in the remaining time interval (u, t) cannot take the process out of the interval $(y, \ell]$.

iv. The process attempts to jump to some $z \in (\ell, \infty)$ at time $u \in (0, t)$. If this occurs, the process regenerates from 0 with time $t - u$ remaining for further jumps to occur.

With $\tau = \tau_1$ and $\xi = \xi_1$ we combine the cases above to derive

$$\mathbf{E}\left[\mathbf{1}_{\{\Psi_1(t,\ell)\in(y,\ell]\}} \mid \tau,\, \xi\right] = \begin{cases} 0, & \tau > t \\ \mathbf{E}\left[\mathbf{1}_{\{\Psi_1(t,\ell)-\Psi_1(\tau,\ell)\in(y-\xi,\ell-\xi]\}}\right], & \tau \le t,\, \xi \le y \\ 1, & \tau \le t,\, y < \xi \le \ell \\ \mathbf{E}\left[\mathbf{1}_{\{\Psi_1(t,\ell)-\Psi_1(\tau,\ell)\in(y,\ell]\}}\right], & \tau \le t,\, \xi > \ell. \end{cases} \tag{4.7}$$

For $0 < \xi \le y$, the distribution of $\Psi_1(t,\ell) - \Psi_1(\tau,\ell)$ is the same as that of $\Psi_1(t-\tau, \ell-\xi)$ by the regenerative property mentioned above. For $\xi > \ell$, the distribution of $\Psi_1(t,\ell) - \Psi_1(\tau,\ell)$ is equal to the distribution of $\Psi_1(t-\tau,\ell)$. So we have

$$\mathbf{E}\left[\mathbf{1}_{\{\Psi_1(t,\ell)\in(y,\ell]\}} \mid \tau,\, \xi\right] = \begin{cases} 0, & \tau > t \\ \mathbf{E}\left[\mathbf{1}_{\{\Psi_1(t-\tau,\ell-\xi)\in(y-\xi,\ell-\xi]\}}\right], & \tau \le t,\, \xi \le y \\ 1, & \tau \le t,\, y < \xi \le \ell \\ \mathbf{E}\left[\mathbf{1}_{\{\Psi_1(t-\tau,\ell)\in(y,\ell]\}}\right], & \tau \le t,\, \xi > \ell. \end{cases} \tag{4.8}$$

We obtain the unconditional tail function by integrating each of the terms in Equation (4.8) with respect to the random variables τ and ξ, which yields

$$\begin{aligned} S_1(y,t,\ell) = &\int_0^t \int_0^y S_1(y-z, t-u, \ell-z) F(\mathrm{d}z)\, G(\mathrm{d}u) \\ &+ G(t)\big(F(\ell) - F(y)\big) + \big(1 - F(\ell)\big) \int_0^t S_1(y, t-u, \ell)\, G(\mathrm{d}u). \end{aligned} \tag{4.9}$$

It follows that

$$\begin{aligned} |S_1(y,t,\ell)| &\le G(t)F(y) + G(t)F(\ell) - G(t)F(y) \\ &\quad + G(t) - G(t)F(\ell) \\ &= G(t), \end{aligned} \tag{4.10}$$

which shows that $S_1(y,t,\ell)$ is of exponential order with respect to t when G is of exponential order. The one-dimensional Laplace transform

$$\tilde{S}_1(y,\omega,\ell) \equiv \int_0^\infty \mathrm{e}^{-\omega t}\, S_1(y,t,\ell)\, \mathrm{d}t \tag{4.11}$$

exists for all ω with $\mathrm{Re}(\omega) > \sigma_G$. Furthermore,

$$\tilde{S}_1(\psi,\omega,\theta) \equiv \int_0^\infty \int_0^\infty \int_0^\ell \mathrm{e}^{-(\omega t+\theta\ell+\psi y)}\, S_1(y,t,\ell)\,\mathrm{d}y\,\mathrm{d}\ell\,\mathrm{d}t. \tag{4.12}$$

exists for all ω with $\mathrm{Re}(\omega) > \sigma_G$, and θ and ψ with $\mathrm{Re}(\theta) > 0$ and $\mathrm{Re}(\psi) > 0$. So, by multiplying the right-hand side of Equation (4.9) by $\mathrm{e}^{-(\omega t+\theta\ell+\psi y)}$ and integrating over $(0,\ell]$ for y and $(0,\infty)$ for ℓ and t, we have

$$\begin{aligned}\tilde{S}_1(\psi,\omega,\theta) = \int_0^\infty \int_0^\infty \int_0^\ell & \mathrm{e}^{-(\omega t+\theta\ell+\psi y)} \\ & \Bigg[\int_0^t \int_0^y S_1(y-z,t-u,\ell-z)F(\mathrm{d}z)\,G(\mathrm{d}u) \\ & \quad + G(t)\big(F(\ell)-F(y)\big) \\ & \quad + \big(1-F(\ell)\big)\int_0^t S_1(y,t-u,\ell)\,G(\mathrm{d}u)\Bigg]\,\mathrm{d}y\,\mathrm{d}\ell\,\mathrm{d}t. \end{aligned} \tag{4.13}$$

When ψ, ω and θ all have a strictly positive real part, we may integrate Equation (4.13) term by term and change the order of integration to arrive at

$$\begin{aligned}\tilde{S}_1(\psi,\omega,\theta) &= \tilde{g}(\omega)\tilde{f}(\theta+\psi)\tilde{S}_1(\psi,\omega,\theta) + \frac{1}{\theta\omega\psi}\tilde{g}(\omega)\big(\tilde{f}(\theta)-\tilde{f}(\theta+\psi)\big) \\ &\quad + \tilde{g}(\omega)\bigg(\int_0^\infty \int_0^\infty \int_0^\ell \mathrm{e}^{-(\omega s+\theta\ell+\psi y)}\, S_1(y,s,\ell)\big(1-F(\ell)\big)\,\mathrm{d}y\,\mathrm{d}\ell\,\mathrm{d}s\bigg). \end{aligned} \tag{4.14}$$

The detailed manipulations can be found in Appendix A.1. Note that the integral in the third term is "almost" the Laplace transform $\tilde{S}_1(\psi,\omega,\theta)$, but the term $1-F(\ell)$ complicates the expression.

We can proceed further by examining a special case. For example, if we assume that ξ is exponentially distributed with parameter μ, then $1-F(\ell) = \mathrm{e}^{-\mu\ell}$, so Equation (4.14) becomes

$$\begin{aligned}\tilde{S}_1(\psi,\omega,\theta) &= \tilde{g}(\omega)\tilde{f}(\theta+\psi)\tilde{S}_1(\psi,\omega,\theta) + \frac{1}{\theta\omega\psi}\tilde{g}(\omega)\big(\tilde{f}(\theta)-\tilde{f}(\theta+\psi)\big) \\ &\quad + \tilde{g}(\omega)\tilde{S}_1(\psi,\omega,\theta+\mu). \end{aligned} \tag{4.15}$$

Rearranging leads to the functional equation

$$\tilde{S}_1(\psi,\omega,\theta) - \frac{\tilde{g}(\omega)\tilde{S}_1(\psi,\omega,\theta+\mu)}{1-\tilde{g}(\omega)\tilde{f}(\theta+\psi)} = \frac{1}{\theta\omega\psi}\frac{\tilde{g}(\omega)\big(\tilde{f}(\theta)-\tilde{f}(\theta+\psi)\big)}{1-\tilde{g}(\omega)\tilde{f}(\theta+\psi)}, \tag{4.16}$$

where $\tilde{f}(\theta) = \mu/(\mu+\theta)$. However, it is not clear to the author how to proceed further from here to obtain an expression for $\tilde{S}_1(\psi, \omega, \theta)$ in terms of $\tilde{g}$ and $\tilde{f}$.

Another route of analysis is available by the Final Value theorem. We have that

$$\lim_{\theta \to 0} \theta \tilde{S}_1(\psi, \omega, \theta) \Longleftrightarrow \lim_{\ell \to \infty} S_1(y, t, \ell). \tag{4.17}$$

The right-hand side of Equation (4.17) is $S_0(y,t) = \Pr\big(A(t) > y\big)$, the tail function of the process of attempted purchases. Therefore, we have

$$\lim_{\theta \to 0} \theta \tilde{S}_1(\psi, \omega, \theta) = \frac{1}{\psi\omega} \frac{\tilde{g}(\omega)\big(1 - \tilde{f}(\psi)\big)}{1 - \tilde{g}(\omega)\tilde{f}(\psi)}, \tag{4.18}$$

where the right-hand side is the two-dimensional Laplace transform

$$\tilde{S}_0(\psi, \omega) \equiv \int_0^\infty \int_0^\infty \mathrm{e}^{-(\omega t + \psi y)} \, S_0(y,t) \, \mathrm{d}y \, \mathrm{d}t, \tag{4.19}$$

which exists with ω and ψ with $\mathrm{Re}(\omega) > 0$ and $\mathrm{Re}(\psi) > 0$ respectively. See Appendix A.2 for the derivation of Equation (4.19). However, multiplying Equation (4.16) by θ on both sides and taking the limit of each term as $\theta \to 0$ gives

$$\frac{1}{\psi\omega} \frac{\tilde{g}(\omega)\big(1 - \tilde{f}(\psi)\big)}{1 - \tilde{g}(\omega)\tilde{f}(\psi)} - \lim_{\theta \to 0} \frac{\theta \tilde{g}(\omega) \tilde{S}_1(\psi, \omega, \theta + \mu)}{1 - \tilde{g}(\omega)\tilde{f}(\theta + \psi)} = \frac{1}{\psi\omega} \frac{\tilde{g}(\omega)\big(1 - \tilde{f}(\psi)\big)}{1 - \tilde{g}(\omega)\tilde{f}(\psi)}, \tag{4.20}$$

where the second term is equal to 0 and so the left and right-hand sides are identical and no further information is forthcoming.

We conclude that the balance control policy BCP1 induces a stochastic process that is difficult to work with. It should be noted that in [34], the authors found the Laplace transform for the expectation of a Markovian process, whereas here we attempted to find the Laplace transform of the tail function. However, applying the same method to the expectation of $\Psi_1(t, \ell)$ runs into similar difficulties and, in particular, results in an incomplete Laplace transform.

Figure 4.1 shows that one could also think of the policy as generating another waiting time in the process whenever an incoming purchase would take the outstanding balance over the credit limit ℓ. This is made clear in the third term of Equation (4.15). This also offers a reason for why it is so difficult to find a tractable expression for the tail function — in the event that a purchase is rejected the process remains indistinguishable from the case that a purchase was never attempted.

In order to compute the distribution or expectation of $\Psi_1(t,\ell)$, we could use simulation, but this has the usual drawbacks of being costly both in terms of time and computational expense. In the next two sections, we examine two more balance control policies that yield information about the process $\Psi_1(t,\ell)$.

4.4 BCP 2: A rejection policy

The balance control policy described in the previous section is a reasonable model for how a CMS would approve purchases without any special conditions for transaction type or account status. However, as we saw, it is difficult to obtain an expression for the tail function of $\Psi_1(t,\ell)$ that we can use directly. In this section, we consider the following policy,

Balance Control Policy 2 (BCP2). *If an attempted purchase ξ_j arriving at time t_j will take the value of approved purchases over the limit ℓ, reject the purchase and accept no further purchases until a payment is received.*

This policy can be expressed as the following functional of the attempted purchase process $A(t)$,

$$\Psi_2(t,\ell) = \sup\big\{A(u) : A(u) \le \ell, u \in (0,t]\big\}. \tag{4.21}$$

The effect of BCP2 is to freeze the process $A(t)$, if it crosses the limit ℓ, at its last value before it jumped above ℓ and prevent the card-holder from making further purchases until a payment is made. Figure 4.2 gives an illustration of policies BCP1 and BCP2. As indicated in the figure, the process $\Psi_1(t,\ell)$ is bounded below by $\Psi_2(t,\ell)$, a useful property which we will exploit later on.

A related functional is the supremum functional

$$\sup_{0\le u\le t} X(u), \quad t > 0, \tag{4.22}$$

which was first studied in [72] in relation to the distribution of the maximum of partial sums of independent random variables. The case where $X(t)$ is a separable stochastic process with stationary and independent increments was

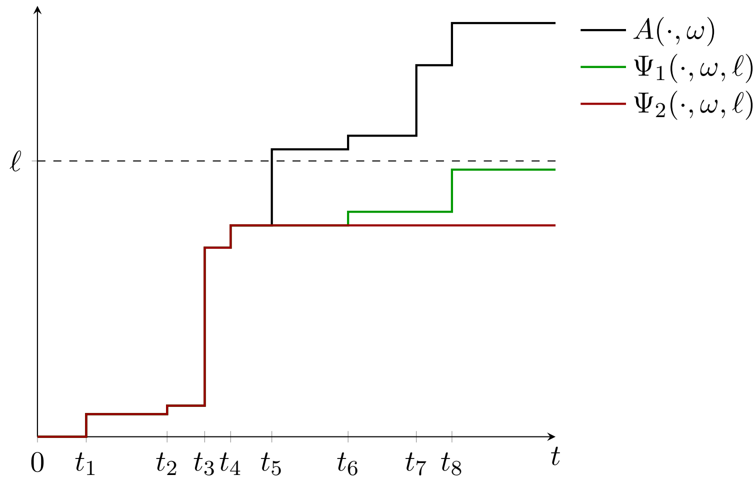

Figure 4.2: Realisations of the attempted purchase process $A(t)$ and the processes induced by BCP1 and BCP2.

studied in [13], where the authors determined the double Laplace-Stieltjes transform of the distribution function

$$\sigma(x, t) = \Pr\left(\sup_{0 \le u \le t} X(u) \le x\right). \tag{4.23}$$

This result was investigated further and extended in several papers, most notably [121], [20], [99] and [70].

4.4.1 An integral equation for the tail function of $\Psi_2(t, \ell)$

Proceeding as we did in Section 4.3.1, we attempt to find an expression for the tail function

$$S_2(y, t, \ell) := \Pr\big(\Psi_2(t, \ell) \in (y, \ell]\big) \tag{4.24}$$

by conditioning on the time and value of the first jump of the process. Under BCP2, we have again have four possibilities to consider for $0 < y < \ell$:

i. No jumps occur in the interval $[0, t)$;

ii. The process jumps to some $z \in (0, y]$ at some time $u \in (0, t)$ and then regenerates itself at this point. That is, a new process starts at z that behaves like the original one, but shifted by $\ell - z$ in space and $t - u$ in time.

iii. The process jumps to some $z \in (y, \ell]$ at some time $u \in (0, t)$ and any subsequent jumps that happen in the remaining time interval (u, t) cannot take the process out of the interval $(y, \ell]$.

iv. The process jumps to some $z \in (\ell, \infty)$ at some time $u \in (0, t)$. If this occurs, the process is stopped at 0 and no further jumps are allowed.

With $\tau = \tau_1$ and $\xi = \xi_1$, we combine the cases above and apply the same reasoning we did previously to derive

$$\mathbf{E}\left[\mathbf{1}_{\{\Psi_2(t,\ell)\in(y,\ell]\}} \mid \tau,\, \xi\right] = \begin{cases} 0, & \tau > t \\ \mathbf{E}\left[\mathbf{1}_{\{\Psi_2(t-\tau,\ell-\xi)\in(y-\xi,\ell-\xi]\}}\right], & \tau \le t,\, \xi \le y \\ 1, & \tau \le t,\, y < \xi \le \ell \\ 0, & \tau \le t,\, \xi > \ell. \end{cases} \tag{4.25}$$

Integrating with respect to the random variables τ and ξ to obtain the unconditional tail function, we find

$$\begin{aligned} S_2(y,t,\ell) = &\int_0^t \int_0^y S_2(y-z, t-u, \ell-z)\, F(\mathrm{d}z)\, G(\mathrm{d}u) \\ &+ G(t)\big(F(\ell) - F(y)\big). \end{aligned} \tag{4.26}$$

Note the similarity to Equation (4.9). In this case however, there are only two terms, since the possibility of retrial following a declined purchase has been removed. It follows that

$$|S_2(y,t,\ell)| \le G(t)F(y) + G(t)F(\ell) - G(t)F(y) = G(t)F(\ell), \tag{4.27}$$

which shows that $S_2(y,t,\ell)$ is of exponential order with respect to t and ℓ when both F and G are, since products of functions of exponential order are also of exponential order.

As in Section 4.3, the one-dimensional Laplace transform

$$\tilde{S}_2(y,\omega,\ell) \equiv \int_0^\infty \mathrm{e}^{-\omega t}\, S_2(y,t,\ell)\, \mathrm{d}t \tag{4.28}$$

exists for ω with $\mathrm{Re}(\omega) > \sigma_G$, and the three-dimensional Laplace transform

$$\tilde{S}_2(\psi,\omega,\theta) \equiv \int_0^\infty \int_0^\infty \int_0^\ell \mathrm{e}^{-(\omega t+\theta\ell+\psi y)}\, S_2(y,t,\ell)\, \mathrm{d}y\, \mathrm{d}\ell\, \mathrm{d}t \tag{4.29}$$

exists for ω with $\mathrm{Re}(\omega) > \sigma_G$, θ with $\mathrm{Re}(\theta) > \sigma_F$, and ψ with $\mathrm{Re}(\psi) > 0$. Proceeding as in Section 4.3, we multiply Equation (4.26) by $\mathrm{e}^{-(\omega t+\theta\ell+\psi y)}$ and integrate over $(0,\ell]$ for y and $(0,\infty)$ for ℓ and t to find

$$\tilde{S}_2(\psi,\omega,\theta) = \frac{1}{\theta\omega\psi}\frac{\tilde{g}(\omega)\big(\tilde{f}(\theta)-\tilde{f}(\theta+\psi)\big)}{1-\tilde{g}(\omega)\tilde{f}(\theta+\psi)}, \tag{4.30}$$

for ω, θ and ψ with strictly positive real part.

To calculate the (two-dimensional) Laplace transform of the expectation of $\Psi_2(t,\ell)$, we note that

$$\begin{aligned}\mathcal{L}_{\omega,\theta}\big\{\,\mathbf{E}\left[\Psi_2(t,\ell)\right]\big\} &\equiv \int_0^\infty\int_0^\infty \mathrm{e}^{-(\omega t+\theta\ell)}\,\mathbf{E}\left[\Psi_2(t,\ell)\right]\mathrm{d}\ell\,\mathrm{d}t\\ &= \int_0^\infty\int_0^\infty \mathrm{e}^{-(\omega t+\theta\ell)}\int_0^\ell \Pr\left(\Psi_2(t,\ell)\in(y,\ell]\right)\mathrm{d}y\,\mathrm{d}\ell\,\mathrm{d}t,\end{aligned} \tag{4.31}$$

which corresponds to evaluating $\tilde{S}_2(0,\omega,\theta)$. We apply l'Hôpital's rule to Equation (4.30) to obtain

$$\begin{aligned}\lim_{\psi\to 0}\tilde{S}_2(\psi,\omega,\theta) &= \lim_{\psi\to 0}\frac{1}{\theta\omega\psi}\frac{\tilde{g}(\omega)\big(\tilde{f}(\theta)-\tilde{f}(\theta+\psi)\big)}{1-\tilde{g}(\omega)\tilde{f}(\theta+\psi)}\\ &= -\frac{\tilde{g}(\omega)}{\theta\omega\big(1-\tilde{g}(\omega)\tilde{f}(\theta)\big)}\frac{\mathrm{d}}{\mathrm{d}\theta}\tilde{f}(\theta).\end{aligned} \tag{4.32}$$

It should be noted that the derivative of $\mathbf{E}\left[\Psi_2(t,\ell)\right]$ may not exist. Indeed, if F is lattice then $\mathbf{E}\left[\Psi_2(t,\ell)\right]$ will be a step function. In the case where the derivative does exist, we obtain its Laplace transform by multiplying (4.32) by θ to yield

$$\mathcal{L}_{\omega,\theta}\left\{\frac{\partial}{\partial\ell}\,\mathbf{E}\left[\Psi_2(t,\ell)\right]\right\} = -\frac{\tilde{g}(\omega)}{\omega\big(1-\tilde{g}(\omega)\tilde{f}(\theta)\big)}\frac{\mathrm{d}}{\mathrm{d}\theta}\tilde{f}(\theta). \tag{4.33}$$

4.4.2 The distribution function of $\Psi_2(t,\ell)$ via paths of $A(t)$

In this section, we show that an expression for the distribution function $F_{\Psi_2}(y) \equiv F_{\Psi_2}(y,t,\ell) = \Pr\left(\Psi_2(t,\ell)\le y\right)$ can also be derived directly by considering the mappings between the sample paths of $A(t)$ and the paths of $\Psi_2(t,\ell)$. From this expression, we can also derive expressions for $\mathbf{E}\left[\Psi_2(t,\ell)\right]$ and its derivative with respect to ℓ. The resulting expression for the derivative reveals a interesting connection to the paths of $A(t)$.

For $n \geq 0$, let $p_n(t) = \Pr(N(t) = n)$ denote the probability mass function for the random variable $N(t)$ and $Q_n(t) = \Pr(N(t) > n)$ denote its tail function. We continue to let F denote the common distribution function of ξ_j, $j \geq 1$. The process $\Psi_2(t, \ell)$ will only remain at 0 if the process $A(t)$ has no jumps in $(0, t]$ or if its first jump is greater than ℓ. So for $y = 0$, we have

$$F_{\Psi_2}(0) = p_0(t) + \bigl(1 - F(\ell)\bigr)Q_0(t). \tag{4.34}$$

Now, let $\Xi_n = \xi_1 + \cdots + \xi_n$ denote the partial sums of the ξ_n, $n \geq 1$, with $F_{\Xi_n}(y) = \Pr(\Xi_n \leq y)$, $y > 0$. We obtain the conditional distribution function for $0 < y \leq \ell$ by conditioning on the event $\{N(t) = n\}$ for $n \geq 1$. If $n = 1$, then $\Psi_2(t, \ell) \in (0, y]$ if $\Xi_1 \in (0, y]$. If $n = 2$, then $\Psi_2(t, \ell) \in (0, y]$ if $\Xi_2 \in (0, y]$, or if $\Xi_1 \in (0, y]$ and $\Xi_2 > \ell$. Continuing in this manner, we see that

$$\begin{aligned} F_{\Psi_2}(y \mid N(t) = n) &= F_{\Xi_1}(y)\bigl(1 - F_{\Xi_2}(\ell \mid \Xi_1 \leq y)\bigr) \\ &\qquad + F_{\Xi_2}(y)\bigl(1 - F_{\Xi_3}(\ell \mid \Xi_2 \leq y)\bigr) + \cdots + F_{\Xi_n}(y) \\ &= F_{\Xi_n}(y) + \sum_{k=1}^{n-1} F_{\Xi_k}(y)\bigl(1 - F_{\Xi_{k+1}}(\ell \mid \Xi_k \leq y)\bigr). \end{aligned} \tag{4.35}$$

Multiplying Equation (4.35) above by $p_n(t)$ and summing over $n \geq 1$, we arrive at the following expression for the distribution function,

$$F_{\Psi_2}(y) = \sum_{n=1}^{\infty}\left(p_n(t)F_{\Xi_n}(y) + p_{n+1}(t)\sum_{k=1}^{n}\bigl(1 - F_{\Xi_{k+1}}(\ell \mid \Xi_k \leq y)\bigr)F_{\Xi_k}(y)\right). \tag{4.36}$$

The structure of $F_{\Psi_2}(y)$ for $0 < y \leq \ell$ is now clear: it is a mixture distribution composed of the paths of $A(t)$ which haven't crossed the level ℓ by time t and those which have. Conditional on the first jump being less than the level ℓ, the first term on the right-hand side of Equation (4.36) corresponds to the distribution function of a compound process, while the second term describes the paths of $A(t)$ which crossed the level ℓ at the $(k+1)$th jump, for $k \geq 1$.

The expectation of $\Psi_2(t, \ell)$ is given by the Lebesgue-Stieljtes integral

$$\mathbf{E}\left[\Psi_2(t, \ell)\right] = \int_0^{\ell} y\, F_{\Psi_2}(\mathrm{d}y). \tag{4.37}$$

If the density function f_{Ψ_2} satisfying

$$F_{\Psi_2}(y) = \int_0^{y} f_{\Psi_2}(x)\,\mathrm{d}x \tag{4.38}$$

exists, we may rewrite Equation (4.37) as

$$\mathbf{E}\left[\Psi_2(t,\ell)\right] = \int_0^\ell y\, f_{\Psi_2}(y)\,\mathrm{d}y. \tag{4.39}$$

Now, if the distribution function $F_{\Xi_n} = F^{*n}$ is absolutely continuous with respect Lebesgue measure, it has a density function $f_{\Xi_n} = f^{*n}$. We may then differentiate Equation (4.36) with respect to y and use the independence of the ξ_j to arrive at

$$f_{\Psi_2}(y) = \sum_{n=1}^{\infty} p_n(t) f^{*n}(y) + \big(1 - F(\ell - y)\big) p_{n+1}(t) \sum_{k=1}^{n} f^{*k}(y), \quad 0 < y \leq \ell. \tag{4.40}$$

We find a more compact form of Equation (4.40) by changing the order of summation to yield

$$f_{\Psi_2}(y) = \sum_{k=1}^{\infty} f^{*k}(y)\big(Q_{k-1}(t) - F(\ell - y)Q_k(t)\big), \quad 0 < y \leq \ell. \tag{4.41}$$

Both forms of f_{Ψ_2} are useful — Equation (4.40) conveys the mixture nature of the density while Equation (4.41) is easier to implement in numerical routines.

Substituting Equation (4.41) into Equation (4.39), we obtain for the expectation of $\Psi_2(t,\ell)$,

$$\mathbf{E}\left[\Psi_2(t,\ell)\right] = \sum_{k=1}^{\infty} Q_{k-1}(t)\int_0^\ell y f^{*k}(y)\,\mathrm{d}y - Q_k(t)\int_0^\ell y f^{*k}(y) F(\ell - y)\,\mathrm{d}y. \tag{4.42}$$

The second integral on the right-hand side of Equation (4.42) cannot be evaluated in general. Nevertheless, if $\mathbf{E}\left[\Psi_2(t,\ell)\right]$ is a differentiable function of ℓ at the point t, we may take the partial derivative with respect to ℓ of Equation

(4.42), to obtain

$$
\begin{aligned}
\frac{\partial}{\partial \ell} \mathbf{E}\left[\Psi_2(t,\ell)\right] &= \sum_{k=1}^{\infty} Q_{k-1}(t) \frac{\partial}{\partial \ell} \int_0^{\ell} y f^{*k}(y)\,\mathrm{d}y \\
&\quad - Q_k(t) \frac{\partial}{\partial \ell} \int_0^{\ell} y f^{*k}(y) F(\ell - y)\,\mathrm{d}y \\
&= \sum_{k=1}^{\infty} Q_{k-1}(t) \ell f^{*k}(\ell) \\
&\quad - Q_k(t) \left(\ell f^{*k}(\ell) F(0) + \int_0^{\ell} y f^{*k}(y) \frac{\partial}{\partial \ell} F(\ell - y)\,\mathrm{d}y \right) \\
&= \sum_{k=1}^{\infty} Q_{k-1}(t) \ell f^{*k}(\ell) - Q_k(t) \int_0^{\ell} y f^{*k}(y) \frac{\partial}{\partial \ell} \int_0^{\ell - y} f(s)\,\mathrm{d}s\,\mathrm{d}y \\
&= \sum_{k=1}^{\infty} Q_{k-1}(t) \ell f^{*k}(\ell) - Q_k(t) \int_0^{\ell} y f^{*k}(y) f(\ell - y)\,\mathrm{d}y. \qquad (4.43)
\end{aligned}
$$

To compute the integral in Equation (4.43), we can use integration by parts. However, there is a connection between the integral and the partial sums of the ξ_j, and instead we will use the following two elementary lemmas.

Lemma 4.4.1. *Let $f(y)$ be a density function on $(0,\infty)$ and $f^{*k}(y)$ its k-fold convolution. Then*

$$
\int_0^s y f^{*k}(s-y) f(y)\,\mathrm{d}y = \frac{s}{k+1} f^{*(k+1)}(s). \qquad (4.44)
$$

Proof. Let ξ_i, $i = 1,\ldots,k+1$ be a sequence of non-negative independent and identically distributed random variables with density function $f(y)$ and let $\Xi_k = \xi_1 + \cdots + \xi_k$. By the definition of conditional expectation, we have

$$
\begin{aligned}
\mathbf{E}\left[\xi_{k+1} \mid \Xi_{k+1} = s\right] &= \int_0^s y f_{\xi_{k+1}}(y \mid \Xi_{k+1} = s)\,\mathrm{d}y \\
&= \int_0^s y \frac{f_{\xi_{k+1},\Xi_{k+1}}(y,s)}{f_{\Xi_k}(s)}\,\mathrm{d}y \\
&= \frac{1}{f^{*(k+1)}(s)} \int_0^s y f(y) f^{*k}(s-y)\,\mathrm{d}y. \qquad (4.45)
\end{aligned}
$$

Since the ξ_i are independent and identically distributed, we also have that

$$
\mathbf{E}\left[\xi_1 \mid \Xi_{k+1} = s\right] = \mathbf{E}\left[\xi_2 \mid \Xi_{k+1} = s\right] = \cdots = \mathbf{E}\left[\xi_{k+1} \mid \Xi_{k+1} = s\right]. \qquad (4.46)
$$

By linearity of expectation, we have

$$
\sum_{i=1}^{k+1} \mathbf{E}\left[\xi_i \mid \Xi_{k+1} = s\right] = \mathbf{E}\left[\Xi_{k+1} \mid \Xi_{k+1} = s\right] = s. \qquad (4.47)
$$

Combining the previous two statements, we have

$$\mathbf{E}\left[\xi_{k+1} \mid \Xi_{k+1} = s\right] = \frac{s}{k+1}.$$

Substituting this into (4.45) and rearranging, we obtain the identity

$$\int_0^s y f^{*k}(s-y) f(y)\,\mathrm{d}y = \frac{s}{k+1} f^{*(k+1)}(s).$$

□

Lemma 4.4.2. *Let $f(y)$ be a density function on $(0,\infty)$ and $f^{*k}(y)$ its k-fold convolution. Then*

$$\int_0^s y f^{*k}(y) f(s-y)\,\mathrm{d}y = \frac{k}{k+1} s f^{*(k+1)}(s). \tag{4.48}$$

Proof. Let $g(y) = y f^{*k}(y)$. Then by the commutative property of convolution integrals

$$\begin{aligned}\int_0^s y f^{*k}(y) f(s-y)\,\mathrm{d}y &= \int_0^s (s-y) f^{*k}(s-y) f(y)\,\mathrm{d}y \\ &= s\int_0^s f^{*k}(s-y) f(y)\,\mathrm{d}y - \int_0^s y f^{*k}(s-y) f(y)\,\mathrm{d}y,\end{aligned}$$

by the definition of convolution and Lemma 4.4.1

$$\begin{aligned}&= s f^{*(k+1)}(s) - \frac{s}{k+1} f^{*(k+1)}(s) \\ &= \frac{k}{k+1} s f^{*(k+1)}(s).\end{aligned}$$

□

Returning to the integral in Equation (4.43), we apply Lemma 4.4.2 and find

$$\begin{aligned}\frac{\partial}{\partial \ell}\mathbf{E}\left[\Psi_2(t,\ell)\right] &= \sum_{k=1}^{\infty} Q_{k-1}(t)\ell f^{*k}(\ell) - Q_k(t)\frac{k}{k+1}\ell f^{*(k+1)}(\ell) \\ &= \sum_{k=1}^{\infty} Q_{k-1}(t)\frac{\ell}{k} f^{*k}(\ell). \end{aligned} \tag{4.49}$$

Interchanging the order of summation in Equation (4.49) reveals a more probabilistic intuition for the form of the derivative. We have

$$\begin{aligned}\frac{\partial}{\partial \ell}\,\mathbf{E}\left[\Psi_2(t,\ell)\right] &= \sum_{n=1}^{\infty} p_n(t) \sum_{k=1}^{n} f^{*k}(\ell)\frac{\ell}{k} \\ &= p_1(t)\ell f(\ell) + p_2(t)\left(\ell f(\ell) + \frac{\ell}{2} f^{*2}(\ell)\right) \\ &\quad + p_3(t)\left(\ell f(\ell) + \frac{\ell}{2} f^{*2}(\ell) + \frac{\ell}{3} f^{*3}(\ell)\right) + \dots . \end{aligned} \tag{4.50}$$

We now see that the paths of $A(t)$ that will contribute to the derivative of $\mathbf{E}\left[\Psi_2(t,\ell)\right]$ with respect to ℓ are those which lie in $[\ell, \ell + \mathrm{d}\ell)$ at some time $s \in (0, t]$. The contribution of these paths is the jump ξ_k which takes the path into this interval. This has expectation $\mathbf{E}\left[\xi_k \mid \xi_1 + \cdots + \xi_k = \ell\right] = \ell/k$. Figure 4.3 provides an illustration of this remark.

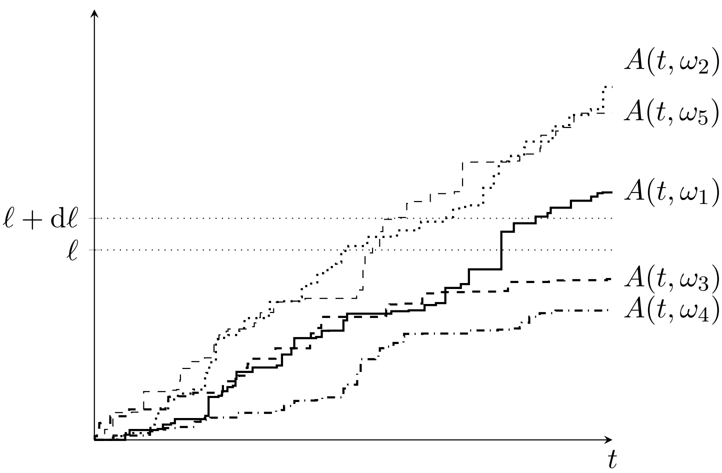

Figure 4.3: The sample paths of $A(t)$ that contribute to the derivative of $\mathbf{E}\left[\Psi_2(t,\ell)\right]$ with respect to ℓ are those which are in $[\ell, \ell + \mathrm{d}\ell)$ at some time $s \in (0, t]$. In this figure, the paths $A(t,\omega_1)$, $A(t,\omega_2)$ and $A(t,\omega_5)$ contribute.

While the expressions we have obtained in this section for the distribution of $\Psi_2(t,\ell)$ and its expectation give us some insight, they involve the use of convolutions which can, in general, be difficult to compute. By contrast, the expressions in Section 4.4 require only the Laplace-Stieltjes transforms of the distributions F and G, and can be inverted numerically if necessary.

4.5 BCP 3: A partial acceptance policy

The third policy we consider is

Balance Control Policy 3 (BCP3). *If an attempted purchase ξ_j arriving at time t_j will take the value of approved purchases over the limit ℓ, then accept an amount $x = \ell - A(t_j^-) \leq \xi_j$.*

This particular balance control policy can be expressed simply as

$$\Psi_3(t, \ell) = A(t) \wedge \ell, \tag{4.51}$$

since accepting a purchase of size $\ell - A(t_j^-)$ when the current value of the process is $A(t_j^-)$ will result in

$$A(t_j) = A(t_j^-) + \ell - A(t_j^-) = \ell. \tag{4.52}$$

This policy will effectively stop the process at ℓ at the time which $A(t)$ first crosses the level ℓ, since once the process reaches ℓ, all future purchases will be rejected. Figure 4.4 compares realisations of the policies BCP1, BCP2 and BCP3, and shows that the path of $\Psi_1(t, \ell)$ is bounded above and below by the paths of $\Psi_3(t, \ell)$ and $\Psi_2(t, \ell)$ respectively.

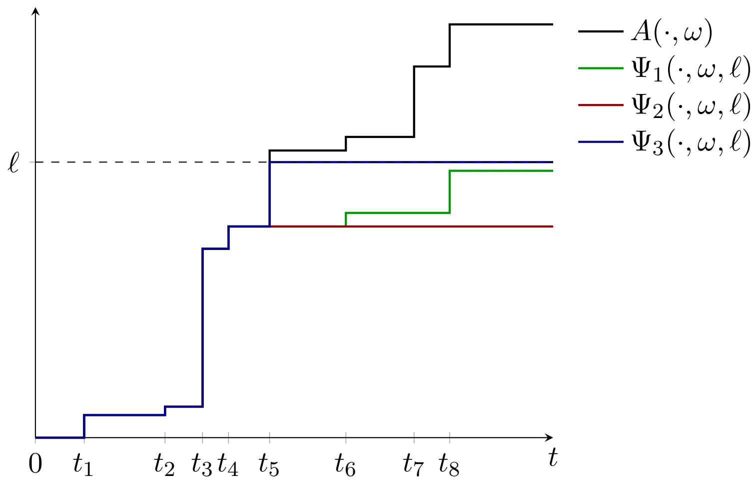

Figure 4.4: Realisations of the attempted purchase process $A(t)$ and the processes induced by the policies BCP1 — BCP3. The process $\Psi_1(t, \ell)$ is bounded below and above by $\Psi_2(t, \ell)$ and $\Psi_3(t, \ell)$ respectively.

The minimum of a non-negative random variable and a constant is well-studied, and it is known that for a non-negative random variable X with distribution function F_X,

$$\mathbf{E}\left[X \wedge c\right] = \int_0^c \big(1 - F_X(x)\big)\,\mathrm{d}x, \quad c > 0. \tag{4.53}$$

See, for example, [75, p. 38]. Since

$$\Pr\left(A(t) > y\right) = \sum_{n=1}^{\infty} p_n(t)\big(1 - F^{*n}(y)\big), \quad y \geq 0, \tag{4.54}$$

we have that

$$\mathbf{E}\left[\Psi_3(t,\ell)\right] = \sum_{n=1}^{\infty} p_n(t) \int_0^{\ell} \big(1 - F^{*n}(y)\big)\,\mathrm{d}y. \tag{4.55}$$

Furthermore, the derivative of $\mathbf{E}\left[\Psi_3(t,\ell)\right]$ with respect to ℓ exists and is given by

$$\frac{\partial}{\partial \ell}\,\mathbf{E}\left[\Psi_3(t,\ell)\right] = \sum_{n=1}^{\infty} p_n(t)\big(1 - F^{*n}(\ell)\big) = \Pr\left(A(t) > \ell\right). \tag{4.56}$$

Equation (4.56) shows that knowledge of the tail function of $A(t)$ is sufficient in order to evaluate the derivative of $\mathbf{E}\left[\Psi_3(t,\ell)\right]$ with respect to ℓ. Evaluating Equation (4.54) requires the evaluation of convolutions and it may be easier to work with its Laplace transform instead, which we derived earlier. Hence, we have the two-dimensional Laplace transform

$$\mathcal{L}_{\omega,\theta}\left\{ \frac{\partial}{\partial \ell}\,\mathbf{E}\left[\Psi_3(t,\ell)\right] \right\} = \frac{1}{\theta\omega}\,\frac{\tilde{g}(\omega)\big(1 - \tilde{f}(\theta)\big)}{1 - \tilde{f}(\theta)\tilde{g}(\omega)}. \tag{4.57}$$

We obtain the two-dimensional Laplace transform of the expectation by dividing Equation (4.57) by θ to arrive at

$$\mathcal{L}_{\omega,\theta}\left\{ \mathbf{E}\left[\Psi_3(t,\ell)\right] \right\} = \frac{1}{\theta^2\omega}\,\frac{\tilde{g}(\omega)\big(1 - \tilde{f}(\theta)\big)}{1 - \tilde{f}(\theta)\tilde{g}(\omega)}. \tag{4.58}$$

4.6 Discussion

In this chapter, we have described three balance control policies, which act on the process of attempted purchases $A(t)$, and the processes they induce. The first process we introduced was the most realistic model of how a CMS might approve purchases, since it allowed for a card-holder to continue to attempt to make purchases following a declined purchase. However, as we saw, it is difficult to discern an expression for the distribution or expectation of $\Psi_1(t,\ell)$. We then introduced two further policies which, as indicated by Figure 4.4, bound the paths of the process $\Psi_1(t,\ell)$. Indeed, for $0<\ell<\infty$ and $0 \leq u \leq t$, if the process $A(u) < \ell$, then

$$\Psi_2(u,\ell) = \Psi_1(u,\ell) = \Psi_3(u,\ell) = A(u). \tag{4.59}$$

On the other hand, if $A(u) \geq \ell$, we will have

$$\Psi_2(u,\ell) \leq \Psi_1(u,\ell) \leq \Psi_3(u,\ell) \leq A(u), \tag{4.60}$$

which implies that

$$\mathbf{E}\left[\Psi_2(t,\ell)\right] \leq \mathbf{E}\left[\Psi_1(t,\ell)\right] \leq \mathbf{E}\left[\Psi_3(t,\ell)\right] \leq \mathbf{E}\left[A(t)\right]. \tag{4.61}$$

The policies BCP2 and BCP3 are unsatisfactory approximations to the true behaviour of the CMS by themselves, since BCP2 involves declining a purchase that would result in exceeding the credit limit ℓ and all further purchases, and BCP3 implies that a merchant accept partial payment for whatever goods were being purchased. In contrast to BCP1, we were able to derive expressions for the distribution, expectation and its derivative under BCP2, and the expectation and its derivative under BCP3. Note that the distribution of $\Psi_3(t,\ell)$ can also be obtained by following the approach in Section 4.4.2 and considering the paths of $A(t)$ that will map to $\Psi_3(t,\ell)$.

The processes $\Psi_2(t,\ell)$ and $\Psi_3(t,\ell)$ are expressed in terms of the event $\{A(t) \geq \ell\}$. If we know whether the process $A(t)$ has crossed ℓ, then we know exactly the value of $\Psi_3(t,\ell)$, and the value of $\Psi_2(t,\ell)$ will be the difference of ℓ and the undershoot of $A(t)$ at the first passage time

$$\tau_\ell = \inf\{t > 0 : A(t) \geq \ell\}. \tag{4.62}$$

The undershoot and overshoot of a process with independent and identically distributed jumps is related to the backward and forward recurrence times in a renewal process, and the joint distribution of these times as $t \to \infty$ was studied in [134]. Expressions for the limiting joint distributions as $\ell \to \infty$ of the undershoot and overshoot of a subordinator (a non-decreasing Lévy process) at first passage were derived via Lévy-process methods in [18]. Classical results for the joint distribution of the undershoot and overshoot using renewal theory are given in [54, Chap. 11]. For a concise and modern treatment of the first passage of subordinators, see [79, Chap. 5]. Although marked renewal processes and their undershoots are well-studied, the author is not aware of a place where the Equations (4.30), (4.32) and (4.33) have appeared. Furthermore, although elementary, Lemmas 4.4.1 and 4.4.2 appear to be new as well.

We conclude this chapter with an example to demonstrate the properties of $\Psi_2(t,\ell)$ using a compound Poisson process.

4.7 A compound Poisson process under BCP2

Suppose that $N(t)$ is a Poisson process with rate parameter λ and the ξ_i are exponentially distributed with parameter μ. Then we have

$$p_n(t) = \mathrm{e}^{-\lambda t}\,\frac{(\lambda t)^n}{n!}, \qquad Q_n(t) = \mathrm{e}^{-\lambda t}\sum_{i=n+1}^{\infty}\frac{(\lambda t)^i}{i!},$$

$$F(y) = 1-\mathrm{e}^{-\mu y}, \qquad F^{*k}(y) = \mathrm{e}^{-\mu y}\sum_{i=k}^{\infty}\frac{(\mu y)^i}{i!}.$$

In this case, the input process $A(t)$ is a compound Poisson process with distribution function

$$\Pr\left(A(t)\le y\right) = \mathrm{e}^{-\lambda t} + \mathrm{e}^{-\lambda t-\mu y}\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}\sum_{i=n}^{\infty}\frac{(\mu y)^i}{i!}. \tag{4.63}$$

This is the so-called *loi des fuites* distribution. The provenance of the name was discussed in [29], and the naming of the distribution is attributed to the authors in [16], who had named it the *law of leaks* since it had been used to describe the distribution of escape flows of gas conduits. The distribution had been studied previously, notably in [57], and has been applied extensively in meteorology and hydrology.

For the distribution function of $\Psi_2(t,\ell)$, we obtain

$$F_{\Psi_2}(0) = \mathrm{e}^{-\lambda t} + \mathrm{e}^{-\mu\ell} - \mathrm{e}^{-(\lambda t+\mu\ell)}, \quad y = 0, \tag{4.64}$$

and, for $y < 0 \leq \ell$,

$$F_{\Psi_2}(y) = \mathrm{e}^{-(\lambda t+\mu y)}\left(\sum_{k=1}^{\infty}\frac{(\lambda t)^k}{k!}\sum_{i=k}^{\infty}\frac{(\mu y)^i}{i!} + \mathrm{e}^{-\mu(\ell-y)}\sum_{k=1}^{\infty}\frac{(\mu y)^k}{k!}\sum_{n=k+1}^{\infty}\frac{(\lambda t)^n}{n!}\right). \tag{4.65}$$

We verify that $F_{\Psi_2}(\ell) = 1$ by interchanging the order of summation in the first sum and evaluating at $y = \ell$,

$$\begin{aligned} F_{\Psi_2}(\ell) &= \mathrm{e}^{-\lambda t} + \mathrm{e}^{-\mu\ell} - \mathrm{e}^{-(\lambda t+\mu\ell)} \\ &\quad + \mathrm{e}^{-(\lambda t+\mu\ell)}\left(\sum_{i=1}^{\infty}\frac{(\mu\ell)^i}{i!}\sum_{k=1}^{i}\frac{(\lambda t)^k}{k!} + \sum_{k=1}^{\infty}\frac{(\mu\ell)^k}{k!}\sum_{n=k+1}^{\infty}\frac{(\lambda t)^n}{n!}\right) \\ &= \mathrm{e}^{-\lambda t} + \mathrm{e}^{-\mu\ell} - \mathrm{e}^{-(\lambda t+\mu\ell)} + (1-\mathrm{e}^{-\mu\ell})(1-\mathrm{e}^{-\lambda t}) \\ &= 1. \end{aligned} \tag{4.66}$$

In this particular example, the distribution function F and its k-fold convolution, F^{*k} are absolutely continuous and differentiable for $k \geq 1$. Thus, the density function f and its k-fold convolution f^{*k} also exist. Substituting

$$f^{*k}(y) = \frac{\mu^k y^{k-1}}{(k-1)!}\,\mathrm{e}^{-\mu y}, \tag{4.67}$$

into Equation (4.40) and making some minor manipulations, we have for $0 < y \leq \ell$,

$$f_{\Psi_2}(y) = \sqrt{\frac{\lambda t\mu}{y}}\,\mathrm{e}^{-(\lambda t+\mu y)}\,\mathcal{I}_1(2\sqrt{\lambda t\mu y}) + \mathrm{e}^{-(\lambda t+\mu\ell)}\sum_{k=1}^{\infty}\frac{\mu^k y^{k-1}}{(k-1)!}\sum_{n=k+1}^{\infty}\frac{(\lambda t)^n}{n!}, \tag{4.68}$$

where

$$\mathcal{I}_1(x) = \sum_{m=0}^{\infty}\frac{(x/2)^{2m+1}}{(m+1)!\,m!} \tag{4.69}$$

is a modified Bessel function of the first kind.

Since the density function f^{*k} exists, the expected balance at time t can be expressed as

$$\mathbf{E}\left[\Psi_2(t,\ell)\right] = \frac{\mathrm{e}^{-(\lambda t+\mu\ell)}}{\mu}\sum_{k=2}^{\infty}\frac{(\mu\ell)^k}{k!}\left(\sum_{n=1}^{k-1}\frac{(\lambda t)^n}{n!}n + k\sum_{n=k}^{\infty}\frac{(\lambda t)^n}{n!}\right). \tag{4.70}$$

Finally, we have for the derivative of the expectation

$$\begin{aligned}\frac{\partial}{\partial \ell}\mathbf{E}\left[\Psi_2(t,\ell)\right] &= \sum_{k=1}^{\infty} \mathrm{e}^{-\mu\ell}\,\frac{\mu^k \ell^{k-1}}{(k-1)!}\,\frac{\ell}{k}\sum_{n=k}^{\infty}\mathrm{e}^{-\lambda t}\,\frac{(\lambda t)^n}{n!} \\ &= \mathrm{e}^{-(\lambda t+\mu\ell)}\sum_{k=1}^{\infty}\frac{(\mu\ell)^k}{k!}\sum_{n=k}^{\infty}\frac{(\lambda t)^n}{n!}. \end{aligned} \tag{4.71}$$

The form of the derivative is particularly straightforward in this example and can be implemented using a routine for the calculation of the regularized lower incomplete Γ-function,

$$\mathcal{P}(a,x) = \frac{1}{\Gamma(a)}\int_0^x \mathrm{e}^{-s}\, s^{a-1}\,\mathrm{d}s. \tag{4.72}$$

Note that this corresponds to calculating the tail function of the Poisson distribution. Most statistical software packages provide this function and the code used to generate these figures is in Appendix B.2. Figures 4.5 and 4.6 provide illustrations of the expectation and its derivative respectively. Note that as $\ell \to \infty$, we see that $\mathbf{E}\left[\Psi_2(t,\ell)\right] \to \mathbf{E}\left[A(t)\right]$ in accordance with the results we obtained in Section 4.4. Figure 4.6 displays a sharp increase for small values of ℓ, which is not evident in Figure 4.5 due to the scale on the vertical axis. The steep increase in the derivative of $\mathbf{E}\left[\Psi_2(t,\ell)\right]$ for small values of ℓ is due to the effect of the balance control policy and the choice of the mark distribution in this example. Indeed, for small ℓ, the process $\Psi_2(t,\ell)$ will become trapped at 0 with high probability since even a single jump is likely to exceed the limit. As such, the increase in $\mathbf{E}\left[\Psi_2(t,\ell)\right]$ is gradual for small ℓ, but quickly approaches linearity.

4.8 Summary

In this chapter, it was shown that a cumulative process under a policy that permits retrials following a rejection is bounded above and below by the same process under two simpler policies. Laplace transforms for the distribution and expectation of a marked renewal process under BCP2 were derived in terms of the Laplace-Stieltjes transforms of the underlying inter-event time and event size distributions. These expressions were also derived via sample paths, and a probabilistic explanation for the expression of the derivative

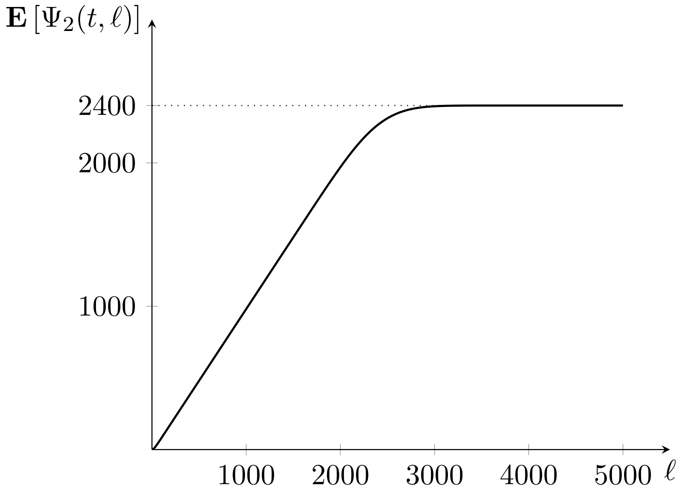

Figure 4.5: $\mathbf{E}[\Psi(t,\ell)]$ plotted for ℓ between 0 and 5000 with $\lambda = 4$, $\mu = 0.05$, and $t = 30$. Note that the expectation approaches the expected value for a compound Poisson process $\lambda t/\mu = 2400$.

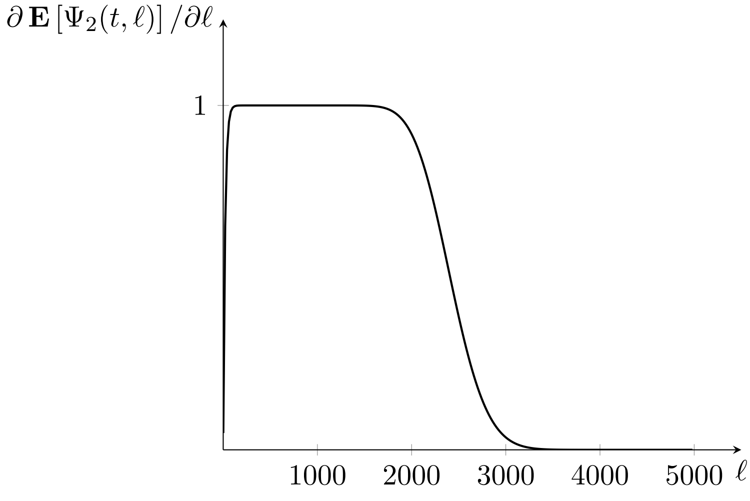

Figure 4.6: $\partial\,\mathbf{E}[\Psi_2(t,\ell)]/\partial\ell$ plotted for ℓ between 0 and 5000 with $\lambda = 4$, $\mu = 0.05$, and $t = 30$.

of the expectation was given. The behaviour of a marked renewal process under BCP2 was illustrated using a compound Poisson process with jump sizes following an exponential distribution.

Chapter 5

Optimal limit calculations for transacting behaviour

Bounds on the optimal limits for transactors using the models of outstanding balance and profit. A scaling property of the optimal limit. New two-dimensional Laplace transform identities. A theoretical example and an example using actual credit card data.

In the previous chapter, we established bounds on the process of approved purchases, $\Psi_1(t, \ell)$. We now return to the model of profitability described in Chapter 3 and derive bounds on the profit-maximising credit limit in the special case of transacting behaviour.

5.1 Transacting behaviour

As described in Chapter 1, transactors are customers who pay the full outstanding balance each month. For most card-issuing banks, transactors make up the majority of their credit card portfolio. This can be attributed to the high interest rates offered on most credit cards and the fact that banks encourage card-holders to pay the outstanding balance in full each month in order to avoid interest charges and late fees.

Industry experience shows that transacting customers typically exhibit the behaviour consistently and for a long time, and this has been observed in several studies. In [126], the authors used data from a major U.K. card-issuing bank to estimate the transition probabilities of credit card accounts between delinquency states using a time-homogeneous Markov chain. The data described the delinquency state of 87,577 credit card accounts over twelve months, and each of the accounts had been open for one year prior. The delinquency state represents the number of consecutive monthly payments that have been missed, with 0 corresponding to no missed payments and 8 being the maximum number of missed payments allowed before the bank initiates a recovery program or charges off the account. It is possible for an account to miss more than 8 payments, and these transitions were aggregated into state 8. The transition probabilities p_{ij}, $0 \leq i, j \leq 8$ were estimated using the maximum likelihood estimator

$$\hat{p}_{ij} = \frac{n_{ij}}{\sum_{j=0}^{8} n_{ij}}, \quad 0 \leq i, j \leq 8. \tag{5.1}$$

Here, n_{ij} is the total number of transitions between each delinquency state over the twelve month period. The total number of transitions originating in each state is given by the vector

$$\mathbf{n} = \begin{pmatrix} 847{,}724 & 69{,}224 & 20{,}834 & 9{,}337 & 5{,}790 & 3{,}876 & 2{,}688 & 1{,}678 & 2{,}196 \end{pmatrix}. \tag{5.2}$$

From the data, the authors found the following empirical transition matrix

$$\hat{\mathbf{P}} = \begin{pmatrix} 0.94 & 0.06 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.56 & 0.18 & 0.26 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.33 & 0.18 & 0.11 & 0.38 & 0 & 0 & 0 & 0 & 0 \\ 0.19 & 0.08 & 0.07 & 0.12 & 0.53 & 0 & 0 & 0 & 0 \\ 0.14 & 0.04 & 0.03 & 0.06 & 0.13 & 0.60 & 0 & 0 & 0 \\ 0.12 & 0.02 & 0.02 & 0.02 & 0.04 & 0.11 & 0.66 & 0 & 0 \\ 0.16 & 0.01 & 0.01 & 0.00 & 0.02 & 0.04 & 0.13 & 0.62 & 0 \\ 0.19 & 0.01 & 0.00 & 0.00 & 0.00 & 0.00 & 0.03 & 0.11 & 0.64 \\ 0.27 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.01 & 0.71 \end{pmatrix}. \tag{5.3}$$

Note that the zeroes in the upper-right of the matrix are structural zeroes, that is, transitions between those states are not permitted. The authors made the following observations about the transition matrix $\hat{\mathbf{P}}$,

1. The majority of transitions are from state 0 to itself, representing payment of at least the minimum amount. Indeed, approximately 83% of all transitions were from state 0 to itself.

2. The two highest values in each row are reversions to state 0 (typically arising from payment of the full balance) and increases of one state (payment of less than the minimum amount, typically no payment).

3. For low state levels, reversion to state 0 dominates, but for higher levels, an increasing state dominates.

Thus, card-holders in state 0 are likely to stay in state 0. If an account leaves state 0, it will transition into state 1 and is very likely to return to state 0. If, however, a further payment is missed, the account will transition into state 2 and the probability of further delinquency increases. Beyond state 2, the probability of further delinquency is larger than the probability of reversion to state 0, likely representing a deteriorating financial situation.

In addition to the observations above, the authors used the matrix $\hat{\mathbf{P}}$ to compute the the expected number of months spent in state i, given that an account enters state i. This is otherwise known as the *sojourn time*. For a discrete time Markov chain, the sojourn time for state i, m_i, is a geometrically distributed random variable with parameter $1-p_{ii}$. The expected sojourn time in state i is therefore given by

$$\mathbf{E}\left[m_i\right] = \sum_{k=1}^{\infty} k\, p_{ii}^{k-1}(1-p_{ii}) = \frac{1}{1-p_{ii}}. \tag{5.4}$$

See, for example, [107, p. 85]. From the diagonal entries of $\hat{\mathbf{P}}$, the authors found the vector of sojourn times

$$\hat{\mathbf{m}} = \begin{pmatrix} 17 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 \end{pmatrix}. \tag{5.5}$$

The entries of $\hat{\mathbf{m}}$ have been rounded to the nearest integer. From $\hat{\mathbf{m}}$, we see that a customer entering state 0 will spend, on average, a further 16 months in that state before exiting. The only state it can transition to is state 1, where it will spend on average 1 month, before likely transitioning back to state 0, where it will again spend, on average, a further 16 months.

Lastly, the authors in [126] computed the stationary distribution of $\hat{\mathbf{P}}$. The stationary distribution of a discrete time Markov chain with transition matrix $\mathbf{P}$ is the vector $\boldsymbol{\pi}$ satisfying

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}. \tag{5.6}$$

Note that the matrix $\hat{\mathbf{P}}$ is finite and irreducible, and hence the stationary distribution exists and is unique (see, for example, [37, p. 153]). The authors found

$$\hat{\boldsymbol{\pi}} = \begin{pmatrix} 0.871 & 0.073 & 0.022 & 0.010 & 0.007 & 0.005 & 0.004 & 0.003 & 0.005 \end{pmatrix}, \tag{5.7}$$

which shows that an account spends, on average, about 87% of its time in state 0. This is further evidence that transacting behaviour is a long-term and consistent behaviour.

5.2 Modelling transacting behaviour

As we noted in Section 3.4.3, we can model transacting behaviour by setting the sequence $\rho_i = 1$, $i \geq 1$. This has the following immediate consequences for Equations (3.26) and (3.29):

1. No interest or late fees are charged, so $\mathbf{1}_{\{\rho_i<1\}} = 0$ and $\mathbf{1}_{\{\rho B(s_i)<c_i\}} = 0$.
2. The full credit limit ℓ is available for new purchases. This results from the fact that $Z(s_i) = 0$, $i \geq 1$ and so $\Psi\big(s_{i-1}, s_i, A, \ell - Z(s_i)\big) = \Psi\big(s_{i-1}, s_i, A, \ell\big)$.
3. The expected balance at the end of each statement period will simply be the expected value of purchases that were approved within the statement period. So $\mathbf{E}\left[B(s_i)\right] = \mathbf{E}\left[\Psi\big(s_{i-1}, s_i, A, \ell\big)\right]$.

An illustration of how the balance evolves under this model of transacting behaviour is given in Figure 5.1. Since a customer that exhibits transacting behaviour is likely to continue to exhibit the same behaviour, the functions $\bar{\eta}(\mathbf{w})$, $\bar{\kappa}(\mathbf{x})$, $\nu(\mathbf{y})$, and $q(\mathbf{y})$ are likely to remain the same between periods. Hence, we assume them to be constant. Furthermore, since statement periods are of approximately equal length, we will analyse a single period, with assumed length $T > 0$. The expected profit is therefore now

$$\begin{aligned}\mathbf{E}\left[R(T)\right] &= \left[\bar{\gamma} - \bar{\epsilon} - \bar{\eta}(\mathbf{w}) - \bar{\kappa}(\mathbf{x}) - \nu(\mathbf{y})\big(1 - q(\mathbf{y})\big)\right] \mathbf{E}\left[\Psi\big(T, A, \ell\big)\right] \\ &\qquad - \nu(\mathbf{y})q(\mathbf{y})\ell \\ &= \gamma^* \, \mathbf{E}\left[\Psi\big(T, A, \ell\big)\right] - \nu^*\ell. \end{aligned} \tag{5.8}$$

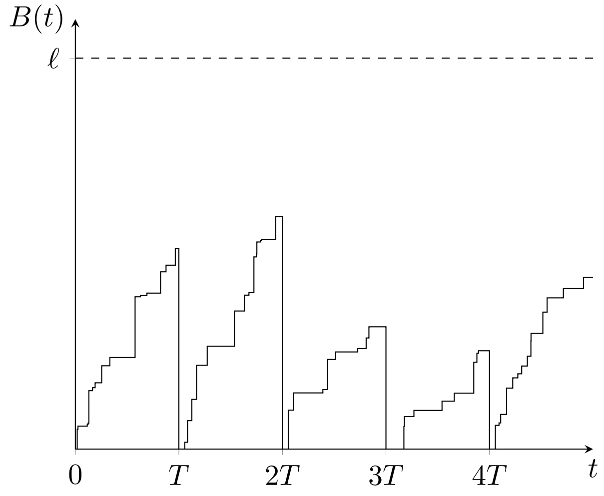

Figure 5.1: A sample trajectory of transacting payment behaviour. At the times $T, 2T, \ldots$, the card-holder pays the outstanding balance in full.

The quantities γ^* and ν^* can be regarded as a net revenue margin and capital charge respectively, and as such we should have $\gamma^* > \nu^*$. Since we are treating γ^*, ν^* and T as fixed, we can maximise the expected profit by finding

$$\hat{\ell} = \arg\max_{\ell \in \Lambda} \Big\{ \gamma^* \, \mathbf{E} \left[\Psi\big(T, A, \ell\big) \right] - \nu^* \ell \Big\}, \tag{5.9}$$

where Λ is a set of permissible limits.

Equation (5.9) has a unique maximum. As discussed in Section 4.6, for values of ℓ for which the process $A(u)$, $0 \leq u \leq T$ is likely to cross ℓ, we have that

$$(\gamma^* - \nu^*)\ell - \gamma^* \, \mathbf{E}\,[U] \leq \mathbf{E}\,[R(T)] \leq (\gamma^* - \nu^*)\ell, \tag{5.10}$$

where U is the undershoot at the τ_ℓ defined in Equation (4.62). So $\mathbf{E}\,[R(T)]$ is bounded below and above by two strictly increasing functions and hence is also a strictly increasing function in ℓ. However, as ℓ becomes large,

$$\mathbf{E}\,[R(T)] = \gamma^* \, \mathbf{E}\,[A(T)] - \nu^* \ell, \tag{5.11}$$

which is strictly decreasing in ℓ. So there exists a number $\ell_c > 0$, an interval $[0, \ell_c)$ on which $\mathbf{E}\,[R(T)]$ is strictly increasing and an interval (ℓ_c, ∞) on which $\mathbf{E}\,[R(T)]$ is strictly decreasing. This implies that $\mathbf{E}\,[R(T)]$ is a concave down function and hence a unique maximum exists.

If Λ is a finite set, then we may determine $\hat{\ell}$ by simply evaluating the right-hand side of Equation (5.9) at each point in Λ. The situation where Λ is not countable requires some knowledge of the properties of $\mathbf{E}\left[\Psi\big(T, A, \ell\big)\right]$. If $\mathbf{E}\left[\Psi\big(T, A, \ell\big)\right]$ happens to be a differentiable function of ℓ at the point T, and if the maximum in Equation (5.9) occurs in the interior of Λ, we can determine $\hat{\ell}$ by differentiating Equation (5.8) and setting the right-hand side equal to 0. This yields

$$\frac{\nu^*}{\gamma^*} = \frac{\partial}{\partial \ell}\,\mathbf{E}\left[\Psi\big(T, A, \ell\big)\right]. \tag{5.12}$$

The task is now to find the limit $\hat{\ell}$ that will render the right-hand side of Equation (5.12) equal to the left-hand side. Whether we are solving Equation (5.9) in the general case, or determining $\hat{\ell}$ via Equation (5.12), we require an expression for $\mathbf{E}\left[\Psi\big(T, A, \ell\big)\right]$ or its derivative if it exists. In the following sections, we use the expressions developed in Chapter 4 to calculate bounds on the optimal limit that would result from BCP1.

5.3 The newsvendor model

If we set $\Psi\big(T, \ell\big) = \Psi_3\big(T, \ell\big) = A(T) \wedge \ell$, then Equation (5.9) becomes

$$\hat{\ell} = \arg\max_{\ell \in \Lambda} \Big\{ \gamma^*\,\mathbf{E}\left[A(T) \wedge \ell\right] - \nu^* \ell \Big\}. \tag{5.13}$$

This is the equation that has to be solved when analysing the well-known *newsvendor problem*. In the setting of the newsvendor problem, the quantity of interest is the number of newspapers which a newsvendor should order so as to maximise the expected profit over a single period of fixed length $T > 0$ when the demand for the newspapers is a random variable with distribution function F_A. Each newspaper costs the newsvendor ν^* to buy and will earn the newsvendor γ^* if it is sold. Given the newsvendor has ordered ℓ newspapers, they will either sell $A(T)$ newspapers if the demand $A(T) < \ell$, or ℓ newspapers if the demand $A(T) \geq \ell$.

The modern formulation of the newsvendor problem was developed in [8] with an additional fixed charge for exceeding the chosen inventory level, but the original formulation of the problem is attributed to [50]. In [8], the authors showed that the optimal stock level can be calculated as a quantile of the demand distribution. A concise derivation of the solution is given in [97], which we reproduce here. Set

$$Q(\ell) = \gamma^* \, \mathbf{E}\left[A(T) \wedge \ell\right] - \nu^* \ell = \gamma^* \left(\int_0^\ell y \, F_A(\mathrm{d}y) + \ell \int_\ell^\infty F_A(\mathrm{d}y) \right) - \nu^* \ell. \quad (5.14)$$

By differentiating $Q(\ell)$ with respect to ℓ, we obtain

$$\begin{aligned} \frac{\partial}{\partial \ell} Q(\ell) &= \gamma^* \left(\ell \, F_A(\ell) + \int_\ell^\infty F_A(\mathrm{d}y) - \ell \, F_A(\ell) \right) - \nu^* \\ &= \gamma^* \big(1 - F_A(\ell)\big) - \nu^*. \end{aligned} \quad (5.15)$$

Setting the left-hand side of Equation (5.15) to 0 and rearranging, we obtain

$$F_A(\ell) = \frac{\gamma^* - \nu^*}{\gamma^*}. \quad (5.16)$$

Hence, given γ^*, ν^* and the distribution of the demand level, F_A, we can calculate the optimal stock level as the quantile

$$\hat{\ell} = \inf_{\ell \in \Lambda} \left\{ F_A(\ell) \geq \frac{\gamma^* - \nu^*}{\gamma^*} \right\}. \quad (5.17)$$

If the stock level is set to the optimal level according to Equation (5.17), then the probability that the newsvendor sells out of newspapers is $1 - F_A(\hat{\ell}) = \nu^*/\gamma^* > 0$. This is a somewhat surprising result since it indicates that in order to maximise profit, the stock level must be set so that there is a non-zero probability that the newsvendor will run out of newspapers and forego some revenue from sales. Of course, the formulation of the problem (5.13) implicitly assumes that running out of newspapers will not have an effect on the demand distribution, whereas in reality it is plausible that a high probability of stock-out will result in reduced demand from customers as they will buy their newspapers elsewhere.

5.4 The optimal limit under BCP2

If we take $\Psi\big(T, A, \ell\big) = \Psi_2(T, \ell) = \sup\{A(u) : A(u) \leq \ell, 0 \leq u \leq t\}$, then Equation (5.9) becomes

$$\hat{\ell} = \arg\max_{\ell \in \Lambda} \left\{ \gamma^* \, \mathbf{E}\left[\Psi_2(T, \ell)\right] - \nu^* \ell \right\}. \quad (5.18)$$

If the expectation is differentiable with respect to ℓ at the point T, then we may proceed as in the case of the newsvendor model and find the solution to (5.9) as

$$\frac{\partial}{\partial \ell}\mathbf{E}\left[\Psi_2(T,\ell)\right] = \frac{\nu^*}{\gamma^*}. \tag{5.19}$$

In Section 4.6, we showed that the expectation of the process of approved purchases under BCP1 was bounded below and above by the expectation of the processes induced by BCP2 and BCP3. It follows directly from Equation (4.61) that

$$\hat{\ell}_{\Psi_3} \leq \hat{\ell}_{\Psi_1} \leq \hat{\ell}_{\Psi_3}, \tag{5.20}$$

since a policy with a higher expected value for all values of ℓ will find a solution to the optimisation problem (5.9) at a lower ℓ.

In Sections 4.4.1 and 4.4.2, we developed expressions for the expectation and its derivative and the corresponding two-dimensional Laplace transforms. We now use these expressions to calculate the optimal limit under the assumption of transacting behaviour.

5.5 Optimal limits with a compound Poisson process

We continue the example from Section 4.7 and calculate the optimal limit under both BCP2 and BCP3 when the process $A(t)$ is a compound Poisson process with rate λ and exponentially distributed jump size with parameter μ. To facilitate our calculation of the optimal limit, we make use of Equations (4.32) and (4.33) which can be inverted numerically. These equations require the Laplace-Stieltjes transforms of the distribution functions of G and F which are, in this case,

$$\tilde{g}(\omega) = \frac{\lambda}{\lambda+\omega} \quad \text{and} \quad \tilde{f}(\theta) = \frac{\mu}{\mu+\theta}. \tag{5.21}$$

Substituting the above into Equation (4.32), we find the Laplace transform of the expectation is

$$\mathcal{L}_{\omega,\theta}\big\{\Psi_2(T,\ell)\big\} = \frac{\lambda\mu}{\theta\omega(\theta+\mu)\big(\mu\omega+\theta(\lambda+\omega)\big)}. \tag{5.22}$$

Multiplying Equation (5.22) by θ, we obtain the Laplace transform of the derivative of the expectation,

$$\mathcal{L}_{\omega,\theta}\left\{\frac{\partial}{\partial \ell}\Psi_2(t,\ell)\right\} = \frac{\lambda\mu}{\omega(\theta+\mu)\big(\mu\omega+\theta(\lambda+\omega)\big)}. \tag{5.23}$$

Equations (5.22) and (5.23) can be inverted analytically to yield

$$\mathcal{L}\left\{\mathbf{E}\left[\Psi_2(t,\ell)\right]\right\} = \frac{1}{\theta^2}\left(\frac{\mu}{\mu+\theta}\right)\left(1-\exp\left\{\lambda t\left(\frac{\mu}{\mu+\theta}-1\right)\right\}\right) \tag{5.24}$$

and

$$\mathcal{L}\left\{\frac{\partial}{\partial \ell}\mathbf{E}\left[\Psi_2(t,\ell)\right]\right\} = \frac{1}{\theta}\left(\frac{\mu}{\mu+\theta}\right)\left(1-\exp\left\{\lambda t\left(\frac{\mu}{\mu+\theta}-1\right)\right\}\right). \tag{5.25}$$

Further analytic inversion is not straight forward, so we resort to numerical inversion using the `EULER` algorithm. To calculate the optimal limit under BCP3, we require the tail function of $A(t)$ or, equivalently, its Laplace transform, which we derived in Section 4.3.1. Substituting Equations (5.21) into Equation (4.19) and inverting analytically from ω to T, we have

$$\tilde{A}(\psi) = \frac{1}{\psi}\left(1-\exp\left\{\lambda T\left(\frac{\mu}{\mu+\psi}-1\right)\right\}\right), \quad \mathrm{Re}(\psi) > -\mu. \tag{5.26}$$

We calculate the optimal limits under BCP2 and BCP3 using a net revenue margin $\gamma^* = 0.0054$, a capital charge $\nu^* = 0.0007$ and a statement period length $T = 30$. We take $\Lambda = (0, 50000]$ and use a bisection search to solve Equation (5.19). Table 5.1 shows the optimal limits calculated for a range of values of λ and μ, along with the expected value of purchases attempted by the card-holder. The table shows that the optimal limit is always higher than the expected value of attempted purchases for the values of γ^* and ν^* in this example. Furthermore, the optimal limit can be different for the same expected value of $A(T)$.

Similar to the event of stocking-out in the newsvendor model, an important factor in the assignment of credit limits is the card-holder's experience of having a purchase declined due to insufficient available credit. This is an important performance measure for a card-issuing bank, since a declined purchase attempt is likely to cause considerable frustration and embarrassment

		$1/\mu$				
		20	40	60	80	100
	1	[776.78, 798.22]	[1,553.55, 1,596.44]	[2,330.33, 2,394.65]	[3,107.10, 3,192.87]	[3,883.88, 3,991.09]
		600.00	1,200.00	1,800.00	2,400.00	3,000.00
	2	[1,449.38, 1,470.40]	[2,898.76, 2,940.80]	[4,348.14, 4,411.21]	[5,797.52, 5,881.61]	[7,246.90, 7,352.01]
		1,200.00	2,400.00	3,600.00	4,800.00	6,000.00
λ	3	[2,105.02, 2,125.86]	[4,210.04, 4,251.72]	[6,315.06, 6,377.57]	[8,420.09, 8,503.43]	[10,525.11, 10,629.29]
		1,800.00	3,600.00	5,400.00	7,200.00	9,000.00
	4	[2,751.91, 2,772.63]	[5,503.81, 5,545.26]	[8,255.72, 8,317.90]	[11,007.63, 11,090.53]	[13,759.54, 13,863.16]
		2,400.00	4,800.00	7,200.00	9,600.00	12,000.00
	5	[3,393.20, 3,413.85]	[6,786.41, 6,827.70]	[10,179.61, 10,241.55]	[13,572.81, 13,655.41]	[16,966.02, 17,069.26]
		3,000.00	6,000.00	9,000.00	12,000.00	15,000.00

Table 5.1: Table of values for the optimal limit. The values were calculated using a statement period length of $T = 30$, with $\gamma^* = 0.0054$ and $\nu^* = 0.0007$. The lower and upper bounds result from calculating the optimal limit under BCP3 and BCP2 respectively. The expected value of attempted purchases $\mathbf{E}\left[A(T)\right]$ is given immediately below each set of bounds.

for a card-holder and lead to a complaint, or worse, the loss of the card-holder's account. As mentioned earlier, the probability of a purchase being declined when the credit limit is ℓ is given by evaluating the tail function of $A(T)$ at ℓ. In this example, we may use the Laplace transform (5.26). The results of inverting Equation (5.26) at the upper bounds of the optimal limits calculated in Table 5.1 are presented in Table 5.2. Note that since the lower bounds are the result of calculating the optimal limit under BCP3 (the news-vendor model), the decline probabilities using the lower bounds are the same and equal to $\nu^*/\gamma^* = 0.129\,629\,63$.

The results in Table 5.2 show that, under BCP2, the probability of a declined purchase remains constant when the rate parameter of the purchase size distribution changes. Indeed, by scaling the purchase size distribution by some $\alpha \in \mathbb{R}_+$, we scale the input process $A(t)$ and, as evidenced by Table 5.1, the optimal limit. The following proposition formalises this result.

Proposition 5.5.1 (Scaling property of the optimal limit)**.** *Let*

$$X(t) = \sum_{i=1}^{N(t)} \xi_i \quad \text{and} \quad Y(t) = \sum_{j=1}^{M(t)} \zeta_j$$

be marked renewal processes such that $Y(t) \stackrel{d}{=} \alpha X(t)$ *and let*

$$\Psi\big(t, X, \ell\big) = \sup_{0 \le u \le t} \{X(u) : X(u) \le \ell\}$$

and

$$\Psi\big(t, Y, \ell\big) = \sup_{0 \le u \le t} \{Y(u) : Y(u) \le \ell\}.$$

Then the solution to the optimisation problem

$$\begin{aligned}\hat{\ell}_Y &= \arg\max_{\ell \in \Lambda} \left(\gamma^* \,\mathbf{E}\left[\Psi\big(t, Y, \ell\big)\right] - \nu\ell\right) \\ &= \alpha \arg\max_{\ell \in \Lambda} \left(\gamma \,\mathbf{E}\left[\Psi\big(t, X, \ell\big)\right] - \nu\ell\right) \\ &= \alpha \hat{\ell}_X,\end{aligned}$$

as long as $\alpha\hat{\ell}_X \in \Lambda$*. Furthermore, we have that*

$$\Pr\left(Y(t) > \hat{\ell}_Y\right) = \Pr\left(X(t) > \hat{\ell}_X\right).$$

		$1/\mu$				
		20	40	60	80	100
	1	0.105 916 58	0.105 916 58	0.105 916 58	0.105 916 58	0.105 916 58
	2	0.112 186 71	0.112 186 71	0.112 186 71	0.112 186 71	0.112 186 71
λ	3	0.115 131 27	0.115 131 27	0.115 131 27	0.115 131 27	0.115 131 27
	4	0.116 938 21	0.116 938 21	0.116 938 21	0.116 938 21	0.116 938 21
	5	0.118 194 13	0.118 194 13	0.118 194 13	0.118 194 13	0.118 194 13

Table 5.2: Probability of the credit card customer experiencing a declined purchase when assigned the optimal limit under BCP2. These results were obtained using $T = 30$, $\gamma^* = 0.0054$, and $\nu^* = 0.0007$. Under BCP3, the decline probabilities are constant at $\nu^*/\gamma^* = 0.129\,629\,63$

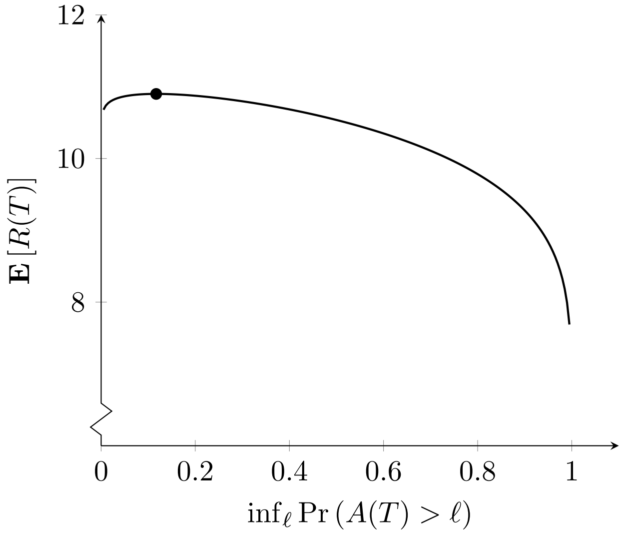

Figure 5.2: Expected profit as a function of decline probability. In this example, the input process $A(T)$ is a compound Poisson process with $\lambda = 4.00$ and $\mu = 0.05$. We use a statement period length $T = 30$, $\gamma^* = 0.0054$ and $\nu^* = 0.0007$. The decline probability at the optimal limit under BCP2 is indicated by the mark.

Proof. See Appendix A.3. □

Similar to the solution of the newsvendor problem, the optimal limit under BCP2 carries with it a significant chance that the card-holder will experience a declined purchase. A card-issuing bank may not wish to reduce this risk, and Figure 5.2 shows the expected profit as a function of the quantile $\inf_\ell \Pr(A(T) > \ell)$. The figure shows that in the region of the optimal limit, the probability of decline can be reduced by increasing the limit with a relatively small reduction to the expected profit.

5.6 New two-dimensional Laplace transform identities

In Section 4.7, we derived the expectation of the process $\sup\{A(u) : A(u) \leq \ell, 0 \leq u \leq t\}$ and its derivative when the process $A(t)$ is a compound Poisson process with exponentially distributed jump sizes. Equations (5.22) and (5.23) are the corresponding two-dimensional Laplace transforms. Since continuous functions have a unique Laplace transform (see [48, p. 21]), we have

$$\frac{\mathrm{e}^{-(\lambda t+\mu\ell)}}{\mu}\sum_{k=2}^{\infty}\frac{(\mu\ell)^k}{k!}\left(\sum_{n=1}^{k-1}\frac{(\lambda t)^n}{(n-1)!}+k\sum_{n=k}^{\infty}\frac{(\lambda t)^n}{n!}\right)\Longleftrightarrow\frac{\lambda\mu}{\theta\omega(\theta+\mu)\big(\mu\omega+\theta(\lambda+\omega)\big)},\tag{5.27}$$

and

$$\mathrm{e}^{-(\lambda t+\mu\ell)}\sum_{k=1}^{\infty}\frac{(\mu\ell)^k}{k!}\sum_{n=k}^{\infty}\frac{(\lambda t)^n}{n!}\Longleftrightarrow\frac{\lambda\mu}{\omega(\theta+\mu)\big(\mu\omega+\theta(\lambda+\omega)\big)},\tag{5.28}$$

where $\Longleftrightarrow$ denotes the correspondence between the function and its (two-dimensional) Laplace transform. Neither of the identities listed above appear in the tables of two-dimensional Laplace transform correspondences in [132]. To the best of the author's knowledge, these correspondences are new.

5.7 An example using credit card transaction data

In this section, we apply the model we have developed to actual data from a credit card customer. Two data sets of anonymised credit card transactions were made available to the author for the purposes of this research. The first data set holds posted transactions, which are the approved purchases and payments, and also includes interest charges, fees, reversals and other automated transactions. The second data set describes authorisations, which includes the purchases and payments attempted by customers.

The posted transactions data set describes the value and processing dates of 771,457 transactions made between 8 February 2011 and 27 February 2013 by 3,734 customers holding 3,971 accounts. Of the 771,457 transactions, 511,969 are retail purchase transactions and 84,503 are payments. In addition to the above, the data set also contains identifying merchant information which allows us to categorise transactions by store type.

The authorisations data set describes the value and transaction times, accurate to the second, of 405,844 transactions made between 7 February 2011 and 27 February 2013. The data set also contains account credit limits and describes whether or not the transaction was approved or declined and, in the case of a decline, a code describing the reason, such as insufficient funds or an incorrectly entered PIN. These transactions were made by the same 3,734 customers across 4,333 credit card accounts. Due to issues encountered during the data extraction process, it was only possible to match the authorisation transaction records with the posted transaction records of 2,246 customers. These customers attempted 288,423 purchases, of which 223,804 were approved.

For the purposes of illustrating the transactor model, a single customer was identified as a transactor through the absence of interest charges to their account over the period. Their transactions were extracted and filtered to include only those made at supermarkets since they account for a large proportion of purchases made on the credit card (306 out of 732) and are easily identified in both the authorisations and posted transactions data set. The time series was modified to exclude transactions that were declined due to a POS device error or an incorrect PIN entry. A preliminary analysis of the supermarket transactions of several card-holders revealed occasional clustering of transactions in time. This could be explained by a number of customer behaviours. For example, a customer may visit a supermarket only to find that some of the items they intended on purchasing are not available, so they buy the items that are in stock and then visit another supermarket nearby to purchase the remaining items. It could also be due to a customer forgetting some items, and quickly returning to the same store to purchase them. With this in mind, transactions made within an hour of each other were combined into a single transaction with the total value of those transactions.

The customer made 306 purchases at various supermarkets over a period of 473 days which totalled \$11,469.44. This equates to approximately \$37.36 per transaction or \$24.25 per day. In a 30-day period, this totals \$727.50 in purchases, which is far less than the account credit limit of \$5000.

We fit a Γ-distribution to the purchase values of the modified time series and estimated the shape and scale parameters using maximum likelihood estimation. Using the two-sided Kolmogorov-Smirnov test statistic

$$D_n = \sup_x |F_n(x) - F(x)| \tag{5.29}$$

where $F_n(x)$ is the empirical distribution function and $F(x)$ is the distribution function of the fitted Γ-distribution, we find the fit to be statistically significant at the 0.05 level as evidenced by the result in Table 5.3. Finding an appropriate distribution for the inter-transaction times was not so straight-forward, so for the purposes of this example, we assume the inter-transaction times follow

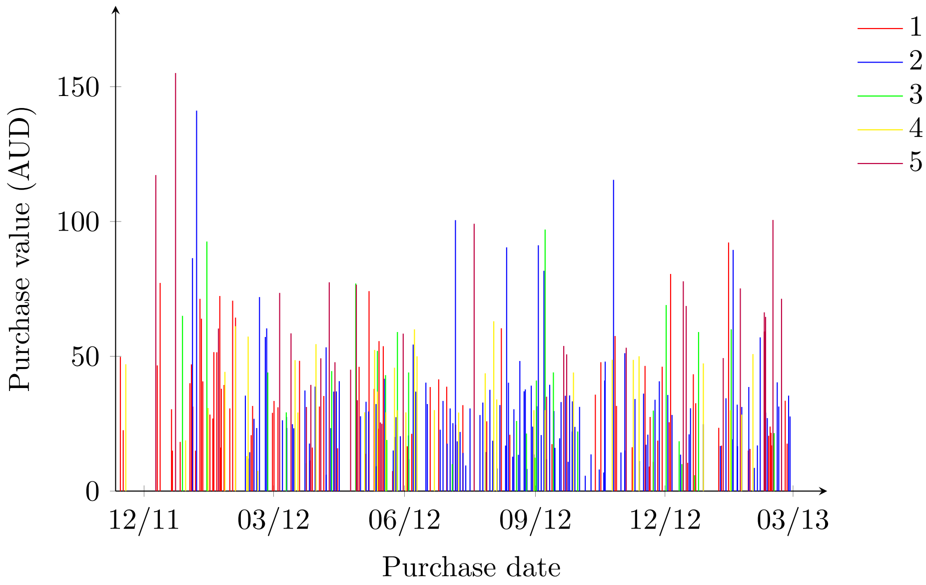

Figure 5.3: Time series of supermarket purchases from a transacting customer. Colours correspond to purchases from different supermarkets, and purple represents trips to two supermarkets within an hour.

an exponential distribution with parameter λ, which was estimated from the reciprocal of the mean of the inter-purchase times to be $\hat{\lambda} = 0.6451 \pm 0.0369$. Independence between the purchase values and the inter-purchase times is also assumed, but note that this assumption could be tested by computing the coherence between the inter-purchase times and the purchase values (see Theorem 4.4 in [28]). Some degree of dependence between the inter-purchase times and purchase values is likely, particularly with supermarket transactions, since a large inter-purchase time could indicate that a customer has not visited a supermarket for a while, and hence the next purchase is likely to be a large one.

Statistic	Estimate
D_n	0.0350
p-value	0.8623
$\hat{k}$ (shape)	2.8946 ± 0.2258
$\hat{\mu}$ (scale)	0.0769 ± 0.0065

Table 5.3: Kolmogorov-Smirnov test statistics and Γ-distribution shape and scale parameter estimates for the purchase value distribution.

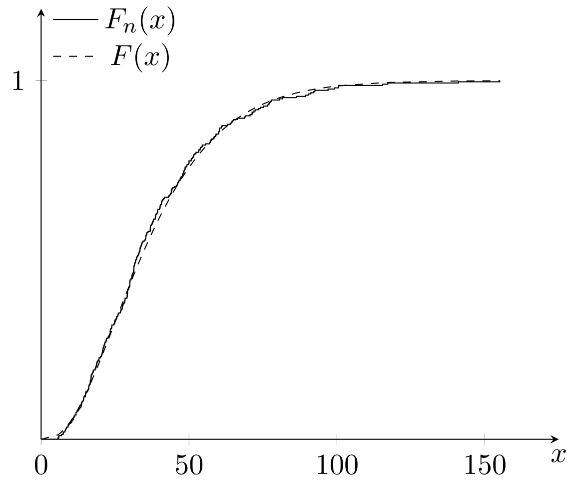

Figure 5.4: The empirical distribution function $F_n(x)$ for the purchase values and the distribution function $F(x)$ of the fitted Γ-distribution with estimated parameters $\hat{\mu} = 2.8946$ and $\hat{k} = 0.0769$

Substituting

$$\tilde{g}(\omega;\lambda) = \left(\frac{\lambda}{\lambda+\omega}\right) \quad \text{and} \quad \tilde{f}(\theta;\mu,k) = \left(\frac{\mu}{\mu+\theta}\right)^k \tag{5.30}$$

into Equation (4.33) and inverting once from ω to t, we have for the Laplace transform of the expectation and its derivative

$$\mathcal{L}\{\, \mathbf{E}\,[\Psi_2(t,\ell)]\,\} = \frac{k}{\theta(\mu+\theta)}\left(\frac{\mu}{\mu+\theta}\right)^k \frac{1-\mathrm{e}^{\lambda t\left(\left(\frac{\mu}{\mu+\theta}\right)^k - 1\right)}}{1-\left(\frac{\mu}{\mu+\theta}\right)^k} \tag{5.31}$$

and

$$\mathcal{L}\left\{\frac{\partial}{\partial \ell}\,\mathbf{E}\,[\Psi_2(t,\ell)]\right\} = \frac{k}{\mu+\theta}\left(\frac{\mu}{\mu+\theta}\right)^k \frac{1-\mathrm{e}^{\lambda t\left(\left(\frac{\mu}{\mu+\theta}\right)^k - 1\right)}}{1-\left(\frac{\mu}{\mu+\theta}\right)^k}. \tag{5.32}$$

For calculations using the newsvendor model, we use the Laplace transform of the tail function of the compound Poisson process with Γ-distributed jumps,

$$\tilde{A}_\Gamma(\psi) = \frac{1}{\psi}\left(1-\exp\left\{\lambda T\left(\left(\frac{\mu}{\mu+\psi}\right)^k - 1\right)\right\}\right), \quad \mathrm{Re}(\psi) > -\mu \tag{5.33}$$

and

$$\mathcal{L}\{\, \mathbf{E}\,[A(T)\wedge \ell]\,\} = \int_0^\infty \mathrm{e}^{-\theta\ell}\int_0^\ell \Pr\left(A(T) > y\right) \mathrm{d}y\,\mathrm{d}\ell = \frac{1}{\theta}\tilde{A}_\Gamma(\theta). \tag{5.34}$$

Again we use a net revenue margin $\gamma^* = 0.0054$, a capital charge $\nu^* = 0.0007$ and statement period length $T = 30$. Substituting the estimated parameters $\hat{\lambda}$, $\hat{\mu}$ and $\hat{k}$ into Equations (5.32) and (5.33), we obtain the results in Table 5.4 by again using a bisection search and the `EULER` algorithm to calculate the optimal limits. The table shows the expected balance, expected profit and probability of a declined purchase at the original limit, the upper and lower bounds of the optimal limit and a revised limit. The bounds on the optimal limit accord with the average monthly supermarket spend of the customer. Recall that we stated that the customer spent on average \$727.50 in supermarkets in a 30-day period; the upper and lower limits yield an expected balance of just over \$714.

	Original	Optimal	Revised
Limit	\$5,000.00	[\$947.83, \$973.81]	\$1,000.00
Expected balance	\$728.64	[\$714.06, \$714.09]	[\$717.13, \$719.64]
Expected profit	\$0.44	[\$3.17, \$3.19]	[\$3.17, \$3.19]
Probability of decline	0.0000	[0.1060, 0.1296]	0.0857

Table 5.4: Expected balance, expected profit and probability of a declined purchase at the original limit, upper and lower bounds of the optimal limit and a proposed revised limit.

The results in Table 5.4 show a marked increase in profitability as a result of lowering the credit limit to the optimal level. In this particular case, the increase in profit is driven by a reduction in cost as it is expensive for the bank to maintain a high limit if the customer does not make full use of it. However, reducing the credit limit to the optimal level suggested by the analysis would substantially increase the probability that the customer will experience a declined purchase if their purchasing behaviour remains unchanged. This is undoubtedly a poor outcome for the customer and likely to be unacceptable in practice. The right-most column of Table 5.4 shows that by changing the limit to \$1000, the probability of decline is reduced while preserving the increased profitability. The revised limit is proposed since most card-issuers offer limits in multiples of \$500. As shown in the table, the deviation from profit at optimality is negligible, but there is slightly smaller chance of the customer experiencing a declined purchase. This suggests one possible implementation strategy: calculate the optimal limit and then increase it until a satisfactory probability of decline is achieved.

Another possible implementation strategy might be to contact card-holders and recommend that they allow their credit limit to be reduced to the optimal level with the option to reinstate the original limit within 12 to 24 months without requiring additional authorisation or credit assessment. This would allow the card-issuer to immediately realise some cost savings from decreased capital costs while minimising the negative impact to card-holders, since only those card-holders who are happy with a reduced limit will agree to take one.

The increase in profitability is substantial, but note that this is somewhat artificial given the analysis has been restricted to those purchases made at supermarkets. To make a complete assessment of the effect to profitability, we would need to fit a distribution to the complete data set of purchases. In Section 3.4.1, we noted that the process of attempted purchases $A(t)$, could be thought of as a superposition of marked point processes. If we assume that each of the component processes in the superposition is a marked renewal process, independent of the other processes, then the process of attempted purchases can be expressed as

$$A(t) = \sum_{m \in \mathcal{M}} A_m(t), \tag{5.35}$$

where

$$A_m(t) = \sum_{i=1}^{N_m(t)} \xi_i^{(m)} \tag{5.36}$$

represents the total value of attempted purchases of type m made in $(0, t]$. Here, each $N_m(t)$ is an independent renewal process with inter-event time distribution G_m. Similarly, the sequence $\{\xi_i^{(m)}\}$, $i \geq 1$ is a sequence of independent non-negative random variables with common distribution function F_m.

The set $\mathcal{M}$ of purchase types can be determined in a number of ways. The transaction data used in this example identified the retailer at which the purchase was made, and one possible categorisation is to simply group transactions made at similar retailers together (for example, supermarkets, hotels, and petrol stations). However, since the optimisation problem we have developed requires the Laplace transforms of the distributions G and F, another approach may be to group transactions with similar purchase values and inter-event times together in order to find a good distributional fit for G and F regardless of the merchant type. The optimisation problem (5.9) can then be solved for each purchase type and then combined into a pooled credit limit.

5.8 Summary

In this chapter, Equations (3.26) and (3.29) were specialised to the case of transacting behaviour and an optimisation problem for determining the credit limit which maximises expected profit was derived. Under the assumption that the process of approved purchases $A(t)$ is a compound Poisson process with rate λ and exponentially distributed marks with parameter μ, bounds on the optimal limit and decline probabilities for varying values of λ and μ were calculated via numerical inversion of Equations (4.32) and (4.33). These expressions are the Laplace transforms of the expectation and derivative of the process $\sup\{A(u) : A(u) \leq \ell, 0 \leq u \leq t\}$, which were calculated directly in Section 4.7. To the best of the author's knowledge, these two-dimensional Laplace transform correspondences are new.

Chapter 6

Optimal limit calculations for partial payment behaviour

An extension of the transactor model to a type of revolving behaviour. The outstanding balance as a Markov chain and an iterated random function system. Numerical results for the expected balance, and a useful approximation for calculating bounds on the optimal limit.

6.1 Interest charges and credit card profitability

In this chapter, we extend the model of transacting payment behaviour to *partial payment* behaviour where the card-holder pays a fraction $\rho_i \in (0, 1)$, $i \geq 1$ of the balance due. As described in Section 3.3.1.1, when a card-holder only pays part of the outstanding balance due, the card-issuing bank will charge interest to the card-holder's account. While the actual calculation of interest is particular to the bank and product, interest charges are a salient feature of credit card profitability. As shown in Table 3.1, interest makes up a significant amount of revenue for the credit card industry, contributing almost 70% of the total revenue in the U.S. in 2004. More recent figures indicate that interest charges continue to contribute significantly to credit card profitability. Figure 6.1 shows monthly interest-accruing balances for credit and charge cards in Australia between August 2002 and April 2016. Interest rates for credit cards in Australia range between 10% — 20% and, according to a recent survey of credit cards in [45], the average rate is approximately

17%. Based on the most recent figure in [101], this puts interest charges at approximately AUD $456 million per month or AUD $5.55 billion annually.

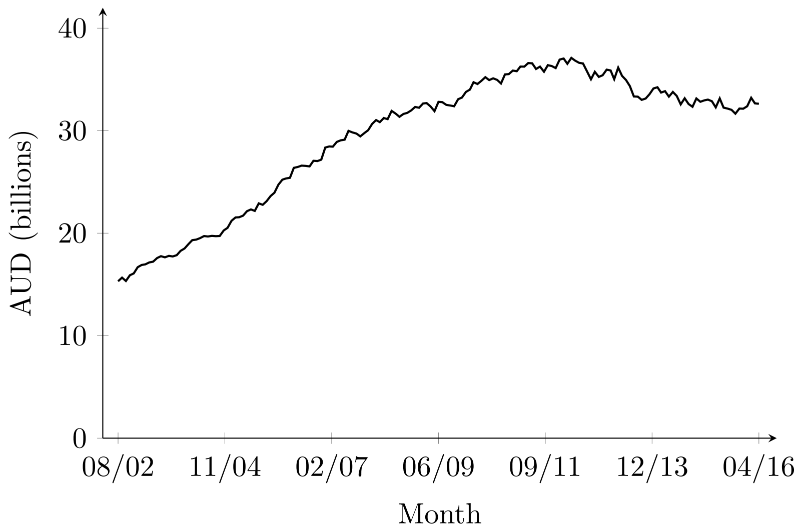

Figure 6.1: Australia credit card balances accruing interest between August 2002 and April 2016. Source: Reserve Bank of Australia, [101].

While interest charges provide the majority of the profit for a card-issuing bank, it is a minority of card-holders in a bank's credit card portfolio that are actually charged interest in a single month. Figures detailing the composition of credit card portfolios by payment behaviour are not readily available, but the data made available to researchers shows that transacting behaviour is the norm for the majority of customers. The Markov chain model reviewed in Section 5.1 indicated that revolving behaviour is either short-term, and the card-holder soon returns to a pattern of transacting, or an early symptom of financial difficulty, in which case the card-holder moves into further stages of delinquency and eventually default. While a card-holder's behaviour may only generate interest charges for a short time, it is worth studying how this behaviour will affect profitability given the substantive contribution of interest towards it.

The questions of why credit card interest rates are so high and why card-holders would choose to borrow at those rates are the subject of on-going research in economics and finance. It was proposed in [10] that card-holders underestimate the probability that they will borrow on their cards and that card-issuing lenders maintain high interest rates in order to discourage consumers who are actively looking to borrow funds. It was also proposed that the costs of searching for and switching to a lending product with a lower interest rate are strong disincentives, in spite of the considerable savings. Empirical evidence for these hypotheses was presented in [31]. Consumer borrowing at high lending rates has also been placed in the economic framework of the hyperbolic consumption model proposed in [7], which posits that present or immediate preferences may be vastly different to future preferences. The bias towards the present in decision-making is of course more commonly referred to as a lack of self-control and has been studied in psychology for some time. A standard reference for this and other biases and heuristics in decision-making is [73]. The relationship between present bias and credit card borrowing was studied empirically in [84].

Another model was proposed in [17], and presented credit card borrowing as a rational game between two competing economic entities, the prudent accountant and the impulsive shopper, who co-exist within a card-holder or the card-holder and their household. The model attempted to explain why house-holds with liquid assets might choose to take on high-interest credit card debt. Indeed, in [108], the authors surveyed card-holders in the U.S. and classified them as either low control minimum payers, high control minimum payers, full balance paying multiple card-holders, or full balance paying single card-holders. The group identified as high control minimum payers use credit card debt deliberately and purposefully to pay other debts, which seemingly contradicts the perception of revolving customers as undisciplined. Finally, a data-driven approach was adopted in [78] in which the authors used regression methods to model the outstanding balance for revolving credit card customers and identify the determining factors.

While the reasons underpinning credit card borrowing may not be fully understood, the framework we have developed enables us to model the effect that such behaviour has on a card-holder's balance and the resulting profit that a card-issuing bank will earn. In this chapter, we show that optimal limits can be calculated for card-holders who exhibit partial payment behaviour. The task of finding an optimal limit essentially reduces to the case of the transactor model, but we require the expectation of a first order autoregressive process with independent, but not identically distributed innovations. Using the characterisation of this process as an iterated random function system, we show that its expectation under the stationary distribution can be evaluated numerically, and we present a routine for doing so which is an implementation of the *coupling from the past* algorithm. For the case where the purchasing process is a compound Poisson process with exponential marks, we show that the bounds that were established in Chapter 3 still hold. We conclude the chapter with an example in which we calculate optimal limits for a card-holder exhibiting partial-payment behaviour.

6.2 Partial payment behaviour

In Section 3.4.3, we introduced some simplifying assumptions to the models of the outstanding balance and profit and showed that these assumptions still provided for several types of revolving behaviour. For now, we focus on the case described in R1. Instead of paying the full balance due at the end of each statement period, consider a card-holder who pays a constant fraction $c \in (0, 1)$ of the due balance and that the minimum payment amount $c_d = 0$. In this case, no late fees will be charged and the account can not become delinquent or enter default. Equation (3.26) becomes

$$\begin{aligned} B(s_i) = {} & (1+r)(1-c)B(s_{i-1}^-) \\ & + \Psi\big(s_{i-1}, s_i, A, \ell - (1+r)(1-c)B(s_{i-1}^-)\big). \end{aligned} \tag{6.1}$$

Similarly, Equation (3.29) becomes

$$\begin{aligned} \mathbf{E}\left[R(s_i)\right] = {} & \mathbf{E}\left[R(s_{i-1})\right] \\ & + (\bar{\gamma} - \bar{\epsilon})\, \mathbf{E}\left[\Psi\big(s_{i-1}, s_i, A, \ell - (1+r)(1-c)B(s_{i-1}^-)\big)\right] \\ & + \left[r - \bar{\eta}(\mathbf{w}) - \bar{\kappa}(\mathbf{x}) - \nu(\mathbf{y})\big(1 - q(\mathbf{y})\big)\right] \mathbf{E}\left[B(s_i^-)\right] \\ & - \nu(\mathbf{y}) q(\mathbf{y}) \ell. \end{aligned} \tag{6.2}$$

Figure 6.2 below illustrates how the outstanding balance of a card-holder following partial payment behaviour might evolve. Using Equation (6.2), we can proceed as we did in Section 5.2 and attempt to find a credit limit that will maximise profit in the statement period $[s_{i-1}, s_i]$ but we quickly run into difficulties. In the transactor model, we were able to exploit the fact that the total

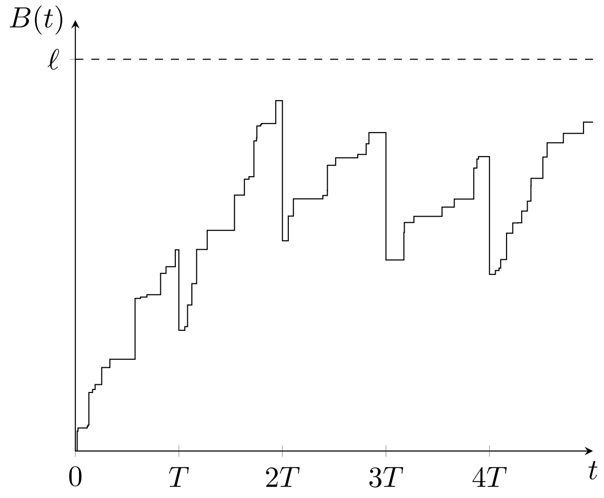

Figure 6.2: A sample trajectory of partial payment behaviour. At the times $T, 2T, \ldots,$ interest is charged on the balance and the card-holder pays an amount $cB(T^{-})$, $0 < c < 1$.

payments made at the statement dates s_i, $i \geq 1$ were equal to the outstanding balance $B(s_i)$. This induced a regenerative structure which allowed us to confine our analysis to a single period, since at the beginning of each period, the full credit limit ℓ was available to the card-holder to make purchases. The approach we used with transacting behaviour is not readily applicable to partial payment behaviour since we cannot assume that the outstanding balance, B or the value of new purchases, Ψ, are identically distributed across the statement periods.

6.3 The outstanding balance $B(s_i)$ as a Markov chain

We restrict our observations of the process to the statement times s_i^-, $i \geq 0$ and let $a = (1 + r)(1 - c)$, where $0 < r < 1$ is the interest rate charged and $0 < c < 1$ is the payment fraction. We assume that, given r, the card-holder always chooses a payment fraction $c > r/1+r$ so that $(1+r)(1-c) < 1$. The role of this restriction will become apparent. For now, Figure 6.3 illustrates this region. The embedded Markov chain $B(s_i)$, $i \geq 1$, is a discrete-time

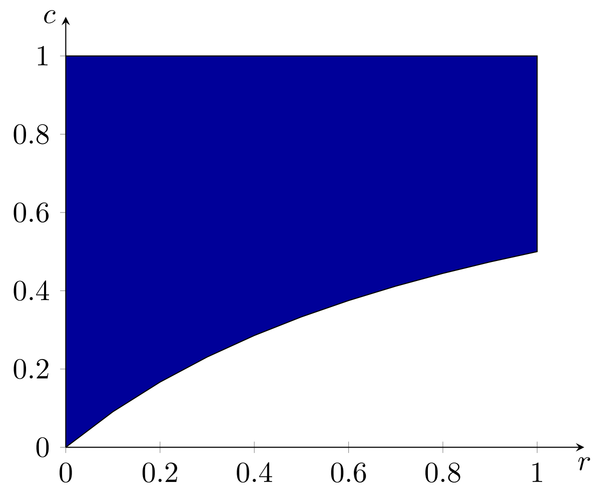

Figure 6.3: Minimum payment fraction c as a function of the interest rate r. The shaded region shows the values of r and c that render $(1+r)(1-c) < 1$.

Markov chain on the space $\mathcal{X} = (0, \ell]$ with one-step transition probability

$$\begin{aligned} P_i(x, \mathcal{A}) &= \Pr\left(B(s_i) \in \mathcal{A} \mid B(s_{i-1}) = x\right) \\ &= \Pr\left(ax + \Psi\big(s_{i-1}, s_i, A, \ell - ax\big) \in \mathcal{A}\right), \quad i \geq 1, \end{aligned} \tag{6.3}$$

where $\mathcal{A} \subseteq \mathcal{X}$. Writing out the successive values of the process

$$\begin{aligned} B(s_0) &= x_0, \\ B(s_1) &= aB(s_0) + \Psi\big(s_0, s_1, A, \ell - aB(s_0)\big), \\ B(s_2) &= aB(s_1) + \Psi\big(s_1, s_2, A, \ell - aB(s_1)\big), \\ &= \Psi\big(s_1, s_2, A, \ell - aB(s_1)\big) \\ &\quad + a\Psi(s_0, s_1, A, \ell - aB(s_0)\big) + a^2 B(s_0), \\ &\vdots \\ B(s_i) &= \sum_{k=1}^{i} a^{i-k}\Psi\big(s_{k-1}, s_k, A, \ell - aB(s_{k-1})\big) + a^i B(s_0), \end{aligned} \tag{6.4}$$

we see that $B(s_i)$ is an autoregressive process of first order, where the innovations introduced at each time step are independent, but not identically distributed, since their value is dependent on the previous state and the length of the previous statement period.

If we assume that statements periods are of equal length T and that the process $A(t)$ is stationary, then we may write $B(s_i) = B(i)$ and Equation (6.4) becomes

$$B(i) = \sum_{k=1}^{i} a^{i-k} \Psi\big(T, A_k, \ell - aB(k-1)\big) + a^i B(0), \tag{6.5}$$

where

$$A_k(T) = \sum_{j=1}^{N_k(T)} \xi_{j,k}, \tag{6.6}$$

is the accumulated value of attempted purchases in the kth period of length T. The sequence of random variables $\{N_k(T)\}$, $k \geq 1$ are independent and identically distributed copies of the counting process $N(T)$, and the $\{\xi_{j,k}\}$, $1 \leq j \leq N_k(T)$, $k \geq 1$ are independent and identically distributed with distribution F. The one-step transition probability now becomes

$$\begin{aligned} P_i(x, \mathcal{A}) &= \Pr\left(B(i) \in \mathcal{A} \mid B(i-1) = x\right) \\ &= \Pr\left(ax + \Psi\big(T, A_i, \ell - ax\big) \in \mathcal{A}\right), \quad i \geq 1. \end{aligned} \tag{6.7}$$

Figure 6.4 illustrates the embedded discrete-time Markov chain described by Equation (6.5). In our analysis of the transactor model, we relied upon the

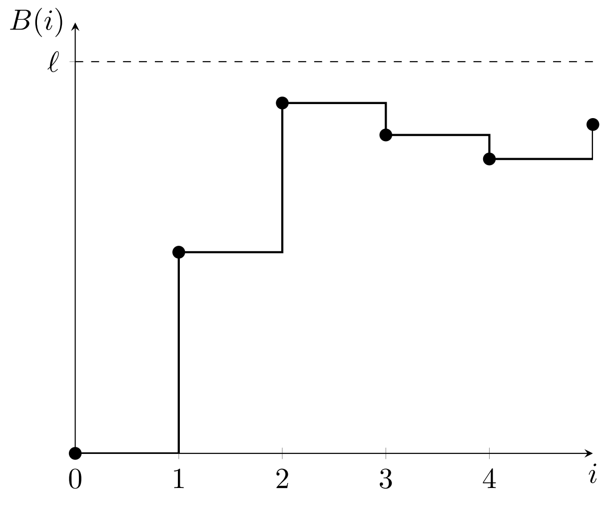

Figure 6.4: The embedded Markov chain $B(i)$ of the continuous time process $B(t)$ at the payment instants $1, 2, \ldots$.

fact that the process regenerates from 0 at the instants iT, $i \geq 1$. If the process $B(i)$ converges in distribution as $i \to \infty$, the limiting distribution will satisfy the equation

$$B \stackrel{d}{=} aB + \Psi\big(T, A, \ell - aB\big), \tag{6.8}$$

where the distribution function of B is given by

$$\pi(\mathcal{A}) = \int_{\mathcal{X}} P(x, \mathcal{A})\, \pi(\mathrm{d}x). \tag{6.9}$$

Here, the transition probability is

$$P(x, A) = \Pr\big(ax + \Psi\big(T, A, \ell - ax\big) \in A\big)\,. \tag{6.10}$$

Equation (6.8) provides us with a useful relationship akin to the regeneration property that we used for transacting behaviour. Taking expectations of (6.8) and rearranging, we find that

$$(1-a)\,\mathbf{E}\,[B] = \mathbf{E}\left[\Psi\big(T, A, \ell - aB\big)\right]. \tag{6.11}$$

Thus, if $B(i)$ is in its stationary regime, then we may set $B(i) = B$ and replace $\mathbf{E}\left[\Psi\big(T, A, \ell - aB\big)\right]$ in Equation (6.2) to obtain

$$\begin{aligned}
\mathbf{E}\,[R(T)] &= (\bar{\gamma} - \bar{\epsilon})(1-a)\,\mathbf{E}\,[B] \\
&\quad + \left[r - \bar{\eta}(\mathbf{w}) - \bar{\kappa}(\mathbf{w}) - \nu(\mathbf{y})\big(1 - q(\mathbf{y})\big)\right] \mathbf{E}\,[B] \\
&\quad - \nu(\mathbf{y})q(\mathbf{y})\ell \\
&= \Big\{(\bar{\gamma} - \bar{\epsilon})(1-a) + r - \bar{\eta}(\mathbf{w}) - \bar{\kappa}(\mathbf{x}) - \nu(\mathbf{y})\big(1 - q(\mathbf{y})\big)\Big\}\,\mathbf{E}\,[B] \\
&\quad - \nu(\mathbf{y})q(\mathbf{y})\,\ell.
\end{aligned} \tag{6.12}$$

Proceeding as we did with the transactor model, we can maximise profitability by finding

$$\hat{\ell} := \underset{\ell \in \Lambda}{\arg\max} \Big\{\big(r + (\bar{\gamma} - \bar{\epsilon})(1-a) - \bar{\chi}(\mathbf{w}, \mathbf{x}, \mathbf{y})\big)\,\mathbf{E}\,[B] - \nu^* \ell\Big\}, \tag{6.13}$$

where Λ is again some set of permissable limits. Equation (6.13) shows that the problem of finding the optimal limit for partial payment behaviour now reduces to the problem of finding the limit for transacting behaviour, but with two important modifications,

1. The multiplier $\gamma^* = r + (\bar{\gamma} - \bar{\epsilon})(1-a) - \bar{\chi}(\mathbf{w}, \mathbf{x}, \mathbf{y})$ now incorporates interest charges and the effect of partial payments on the interchange rate $\bar{\gamma}$ and cost of funds $\bar{\epsilon}$, and;

2. The expectation $\mathbf{E}[B]$ is the expectation of the embedded discrete-time Markov chain $B(i)$ under the invariant measure π.

Thus, to calculate $\hat{\ell}$, we will need to establish the convergence of $B(i)$ and calculate $\mathbf{E}[B]$. A first approach here might be to establish the distribution of B, but such a direct approach quickly runs into difficulties. To illustrate, consider an autoregressive system started at $X_0 = x_0$ with

$$X_i = \alpha X_{i-1} + \xi_i, \quad i \geq 1, \tag{6.14}$$

where $|\alpha| < 1$ and the ξ_i are independent and identically distributed. By the independent and identically distributed property of the ξ_i, the stationary distribution is given by the distribution function of

$$X_\infty = \sum_{i=0}^{\infty} \alpha^i \xi_i \tag{6.15}$$

and satisfies

$$X_\infty \overset{d}{=} \alpha X_\infty + \xi. \tag{6.16}$$

Despite the simplicity of the distributional equality (6.16), we have no guarantee that an expression for the distribution function of X_∞ can be found. In the case that the ξ_i are exponentially distributed with parameter μ, then $\alpha^i \xi_i$ is also exponentially distributed with parameter μ/α^i. Thus, the Laplace transform of the density function of $\alpha^i \xi_i$ is

$$\mathcal{L}(\alpha^i \xi_i) = \left(\frac{\mu}{\mu + \alpha^i \theta} \right). \tag{6.17}$$

The Laplace transform of the density function of X_∞ is then simply the infinite product

$$\mathcal{L}(X_\infty) = \prod_{i=0}^{\infty} \frac{\mu}{\mu + \alpha^i \theta}, \tag{6.18}$$

which does not have a closed form expression. Indeed, it is not clear whether the right hand side of Equation (6.18) converges since the convergence of

$$\sum_{i=0}^{\infty} \log \left(\frac{\mu}{\mu + \alpha^i \theta} \right) \tag{6.19}$$

is indeterminate. However, we may determine the convergence of the series (6.16) through the following theorem.

Theorem 6.3.1 (Two-series theorem)**.** *A sufficient condition for the convergence of the series*

$$\sum_{k=0}^{\infty} Z_k$$

of independent random variables, with probability 1, is that both series

$$\sum_{k=0}^{\infty} \mathbf{E}\left[Z_k\right] \quad \textit{and} \quad \sum_{k=0}^{\infty} \mathrm{Var}[Z_k]$$

converge. If $\Pr\left(|Z_k| \leq c\right) = 1$*, the condition is also necessary.*

Proof. See [109]. □

By linearity of expectation and the assumption that $|\alpha| < 1$,

$$\sum_{n=0}^{\infty} \mathbf{E}\left[\alpha^n \xi_n\right] = \mathbf{E}\left[\xi\right] \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \mathbf{E}\left[\xi\right]. \tag{6.20}$$

Similarly,

$$\sum_{n=0}^{\infty} \mathrm{Var}[\alpha^n \xi_n] = \mathrm{Var}[\xi] \sum_{n=0}^{\infty} \alpha^{2n} = \frac{1}{1-\alpha^2} \mathrm{Var}[\xi]. \tag{6.21}$$

Thus, the series in Equation (6.15) converges if $\mathbf{E}\left[\xi\right] < \infty$ and $\mathrm{Var}[\xi] < \infty$. Furthermore, Equation (6.20) gives us the expectation of the limiting distribution.

The Markov chain $B(i)$ satisfying the recursion (6.5) is not so straightforward owing to the non-identically distributed innovations. We can appeal to the last condition in Theorem 6.3.1 to establish convergence, but this still leaves us with the task of deriving the distribution of B or even its expectation.

6.4 An iterated random function system representation

At this stage, we make use of the fact that the Markov chain $B(i)$ can be viewed as an iterated random function system. We follow the terminology in [46], and say that a Markov chain admits a representation as an iterated random function system if there exists a family $\{f_\xi : \xi \in \Xi\}$ of functions that map the state space of the chain, $\mathcal{X}$, onto itself, and a measure ϑ on Ξ, such that the Markov chain may be written as the sequential composition of functions $X_0 = x_0$, $X_1 = f_{\xi_1}(x_0)$, $X_2 = (f_{\xi_2} \circ f_{\xi_1})(x_0)$ and so forth, so that we have for the $(i+1)$th iteration of the chain

$$X_{i+1} = f_{\xi_{i+1}}(X_i), \tag{6.22}$$

where $\xi_1, \xi_2, \ldots$ are independent draws from ϑ and represent the randomness introduced into the system at the ith iteration. Iterating the chain in this manner is running the chain "forward", while applying the composition in reverse,

$$Y_{i+1} = (f_{\xi_1} \circ f_{\xi_2} \circ \cdots \circ f_{\xi_{i+1}})(x_0) \tag{6.23}$$

is referred to as the backward iteration. Note that there is no restriction placed on the dimension of Ξ, and we could modify our definition to accommodate a family of functions $\{f_{\boldsymbol{\xi}} : \boldsymbol{\xi} \in \Xi\}$ where $\boldsymbol{\xi}$ is a vector of random variables.

Let X_∞ be the limiting random variable of the Markov chain X_i. If the time

$$\min\{i \geq 1 : X_i \stackrel{d}{=} X_\infty\} \tag{6.24}$$

is finite with probability 1, we can sample from the stationary measure π by running the chain forward until stationarity is reached. This is the basis of Markov chain Monte Carlo algorithms. However, depending on the particular Markov chain, this may take a long time and prove too costly for simulation. In [98], the authors proposed the *coupling from the past* algorithm (CFTP) which has the desirable property of being able to determine when convergence to the invariant measure has occurred and return an exact sample from the invariant measure. The core of the algorithm is the backward iteration.

To understand the usefulness of the backward iteration in this context, consider again the autoregressive process described by Equation (6.14). The forward process, given by

$$X_1 = \alpha x_0 + \xi_1, \quad X_2 = \alpha^2 x_0 + \alpha\xi_1 + \xi_2, \quad X_3 = \alpha^3 x_0 + \alpha^2\xi_1 + \alpha\xi_2 + \xi_3, \tag{6.25}$$

and so on, will continue to move at random due to the new randomness introduced at each iteration. The backward iteration, again started at the initial point x_0, will evolve as

$$Y_1 = \alpha x_0 + \xi_1, \quad Y_2 = \alpha^2 x_0 + \xi_1 + \alpha\xi_2, \quad Y_3 = \alpha^3 x_0 + \xi_1 + \alpha\xi_2 + \alpha^2\xi_3 \tag{6.26}$$

and so forth. The backward iteration converges almost surely to a point, since the randomness introduced at each iteration is damped by the factor α. Moreover, this point will be independent of the initial point x_0 since $\alpha^i x_0 \to 0$ as $i \to \infty$ and, due to the identical distribution of the ξ_i, it will have the same law as the forward iteration. It should now be clear why the backward iteration is of particular use in obtaining a sample from π.

Returning our attention to $B(i)$, its forward iteration is given by Equation (6.5). The first few forward iterations are given by

$$\begin{aligned} B(0) &= x_0, \\ B(1) &= ax_0 + \Psi\big(T, A_1 T, \ell - ax_0\big), \\ B(2) &= aB(1) + \Psi\big(T, A_2, \ell - aB(1)\big) \\ &= a^2 x_0 + a\Psi\big(T, A_1, \ell - ax_0\big), \\ &\quad + \Psi\big(T, A_2, \ell - a^2 x_0 - a\Psi(T, A_1, \ell - ax_0)\big). \end{aligned} \tag{6.27}$$

The backward iterations are given by

$$\begin{aligned} \bar{B}(0) &= x_0, \\ \bar{B}(1) &= ax_0 + \Psi\big(T, A_1, \ell - ax_0\big), \\ \bar{B}(2) &= a^2 x_0 + a\Psi\big(T, A_2, \ell - ax_0\big), \\ &\quad + \Psi\big(T, A_1, \ell - a^2 x_0 - a\Psi(T, A_2, \ell - ax_0)\big), \end{aligned} \tag{6.28}$$

and so on. It is not readily apparent from the iterations of $\bar{B}(i)$ that the forward and backward iterations should have the same distribution. However, the following theorem and proposition give us a set of conditions that can be used to show that the backward iteration will converge and that the point it yields will be from the stationary distribution.

Theorem 6.4.1 ([46]). *Let $(\mathcal{X}, d)$ be a complete separable metric space. Let $\{f_\xi : \xi \in \Xi\}$ be a family of Lipschitz functions on $\mathcal{X}$ and let ϑ be a probability distribution on Ξ. Suppose that $\int K_\xi \, \vartheta(\mathrm{d}\xi) < \infty$, $\int d[f_\xi(x_0), x_0] \, \vartheta(\mathrm{d}\xi) < \infty$ for some $x_0 \in \mathcal{X}$, and $\int \log K_\xi \, \vartheta(\mathrm{d}\xi) < 0$.*

(i) *The induced Markov chain has a unique stationary distribution π.*

(ii) *$d[P_n(x, \cdot), \pi] \leq A_x \varepsilon^n$ for constants A_x and ε with $0 < A_x < \infty$ and $0 < \varepsilon < 1$; the bound holds for all times n and all starting states x.*

(iii) *The constant ε does not depend on n or x; the constant A_x does not depend on n, and $A_x < a + bd(x, x_0)$ where $0 < a, b < \infty$.*

Proof. See Theorem 5.1 in [46]. □

The quantity K_ξ is the *Lipschitz constant* and is defined as any $K_\xi \geq 0$ such that

$$d[f_\xi(x), f_\xi(y)] \leq K_\xi \, d[x, y], \quad \forall x, y \in \mathcal{X}. \tag{6.29}$$

Note that the Lipschitz constant is relative to the metric d. The first condition in Theorem 6.4.1 characterises the concept of contracting on average. The metric in (ii) is the Prokhorov metric on probabilities.

If the Markov chain of interest satisfies the conditions of Theorem 6.4.1, then the following proposition applies.

Proposition 6.4.2 ([46])**.** *Under the regularity conditions of Theorem* (6.4.1)*, the backward iterations converge almost surely to a limit, at an exponential rate. The limit has the unique stationary distribution π.*

Proof. See Proposition 5.1 in [46]. □

It was shown in Section 6.1 of [46] that the autoregressive process given by Equation (6.14) satisfies the conditions of Theorem 6.4.1 since the function $f_\xi(x) = \alpha x + \xi$ has Lipschitz constant $K_\xi = \alpha$, hence $\int K_\xi \, \vartheta(\mathrm{d}\xi) < \infty$ and $\int \log K_\xi \, \vartheta(\mathrm{d}\xi) < 0$. The condition that $\int d[f_\xi(x_0), x_0] \, \vartheta(\mathrm{d}\xi) < \infty$ for some $x_0 \in \mathcal{X}$ is satisfied if the distribution of the ξ_i has an algebraic tail, that is, there are positive, finite constants β, δ with $\Pr(|\xi| > u) < \beta/u^\delta$ for all $u > 0$.

The process $B(i)$ is slightly more complicated since the function being iterated is, for a single random variable, ξ

$$f_\xi(x) = ax + \Psi(\xi, \ell - ax). \tag{6.30}$$

The Lipschitz constant is $K_\xi = a$ and the second condition is satisfied trivially since the functional Ψ limits each innovation in its size to be less than ℓ. Thus, Proposition 6.4.2 applies and we may use the backward iteration given by Equation (6.28) to sample the invariant measure.

6.5 Numerical results for $\mathbf{E}[B]$

Having determined that the backward iteration of the Markov chain $B(i)$ can be used to sample the invariant measure, we can now obtain numerical results for the expectation $\mathbf{E}[B]$ under the invariant measure π using a suitable algorithm to create the backward iteration. Pseudo-code for calculating $\mathbf{E}[B]$ is given in Algorithm 2.

Algorithm 2 Pseudo-code for estimating $\mathbf{E}[B]$ using the backward iteration

```
procedure BACKWARD ITERATION
    for k ← 1, N do
        M ← 2
        stopCondition ← false
        while (stopCondition = false) do
            seed ← σ_k
            GENERATE(ζ_1, ..., ζ_M)
            X_T ← ℓ
            X_B ← 0
            i ← M
            while i > 0 do
                X_T ← aX_T + Ψ(ζ_i, ℓ − aX_T)
                X_B ← aX_B + Ψ(ζ_i, ℓ − aX_B)
                i ← i − 1
            end while
            if |X_T − X_B| < ε then
                stopCondition ← true
            else
                M ← 2M
            end if
        end while
        total ← total + X_B
    end for
return total / N
end procedure
```

The algorithm is a modification of the CFTP algorithm proposed in [98] and exploits the fact that if the backward iteration is known to converge, then for a given sequence of random functions $f_{\xi_1}, \ldots, f_{\xi_M}$, it will converge to the same value regardless of the initial point. Note that convergence may not take place at the same time for all initial points. It was shown in [98] that if the function being iterated is monotonic, that is,

$$x \leq y \Longrightarrow f(x) \leq f(y), \quad \forall x, y \in \mathcal{X}, \tag{6.31}$$

and $\mathcal{X}$ is a bounded space, then if any two chains started at $x, y \in \mathcal{X}$, $x < y$ have converged to the same value, then all in chains started in between x and y will have converged as well.

The chain $B(i)$ does not strictly satisfy the property (6.31) for all $i \geq 1$. We require that for $x \leq y$

$$ax + \Psi(T, A_i, \ell - ax) \leq ay + \Psi(T, A_i, \ell - ay). \tag{6.32}$$

It suffices to consider the value of the first jump $\xi_{1,i} = \xi$ in the sequence of jumps in the ith iteration of the chain. If $\xi > \ell - ax$, then we also have $\xi > \ell - ay$ under BCP1 and BCP2, and so $f_\xi(x) = ax$ and $f_\xi(y) = ay$. Under BCP3, $f_\xi(x) = ax + \ell - ax = \ell$ and $f_\xi(y) = \ell$ as well. So under all policies we have $f_\xi(x) \leq f_\xi(y)$.

Conversely, if $\xi \leq \ell - ay$, then $\xi \leq \ell - ax$ and under all balance control policies $f_\xi(x) = ax + \xi$ and $f_\xi(y) = ay + \xi$. Once again, this results in $f_\xi(x) \leq f_\xi(y)$.

If $\xi \leq \ell - ax$ and $\xi > \ell - ay$, then $f_\xi(x) = ax + \xi$ but $f_\xi(y) = ay$ under both BCP1 and BCP2. In this case, $ax + \xi \in [ay, \ell]$ and $f_\xi(x) \geq f_\xi(y)$. Observe however that if $x = y$, then this case simply reduces to the second and we will have $f_\xi(x) = f_\xi(y)$. Now, since the chain $B(i)$ satisfies the conditions of Theorem 6.4.1, it is contracting. Therefore, as $i \to \infty$, the distance between successive iterations approaches zero and the monotonicity property will eventually hold under all balance control policies. Thus, we can exploit the property for testing convergence of the system by starting two chains at 0 and ℓ and evolving them with the same random sequence until they converge.

As noted in [87, p. 483], the main difficulty in applying the property (6.31) when the state space $\mathcal{X}$ is continuous is that paths started from different values will not converge in finite time, although they may become arbitrarily close. As such, the algorithm does not produce an exact draw from the stationary distribution. However, it can produce an approximate draw to arbitrary precision which is sufficient for the purposes of this thesis.

Table 6.1 below shows the numerical results obtained for the expectation of $B(i)$ under the balance control policies described in Chapter 4. Recall that BCP1 allows for the card-holder to continue to attempt to make purchases following a declined purchase, while BCP2 will freeze the process of at its last value before it attempted to exceed ℓ and BCP3 will take the process to ℓ.

The results were obtained for a compound Poisson process with parameters $\lambda = 4.00$ and exponentially distributed jumps with rate $\mu = 0.05$. The length of the statement period is $T = 30.00$ and the payment fraction $a = 0.4$. The results were generated using 10,000 simulations. A specific implementation for generating the random variables ζ_i and calculating the expectation under BCP1 is in Appendix B.3.

	Estimated $\mathbf{E}[B]$				
ℓ	Ψ_1	Ψ_2	Ψ_3	$\mathbf{E}\left[\Psi_2(T,\ell)\right]$	$\mathbf{E}\left[\Psi_3(T,\ell)\right]$
500	499.80	479.82	500.00	480.00	500.00
1,000	999.76	980.29	1,000.00	980.00	1,000.00
1,500	1,499.71	1,479.89	1,500.00	1,480.00	1,500.00
2,000	1,999.62	1,980.32	2,000.00	1,980.00	2,000.00
2,500	2,499.42	2,479.80	2,499.97	2,480.00	2,500.00
3,000	2,996.82	2,978.08	2,998.11	2,979.65	2,999.58
3,500	3,470.27	3,454.91	3,472.35	3,463.96	3,482.09
4,000	3,833.84	3,825.55	3,836.69	3,830.27	3,840.47
4,500	3,981.81	3,980.37	3,983.79	3,975.18	3,977.52
5,000	4,000.06	4,000.00	4,001.94	3,998.51	3,998.71
5,500	4,000.75	4,000.75	4,002.42	3,999.96	3,999.97
6,000	4,000.76	4,000.76	4,002.42	4,000.00	4,000.00
6,500	4,000.76	4,000.76	4,002.42	4,000.00	4,000.00
7,000	4,000.76	4,000.76	4,002.42	4,000.00	4,000.00

Table 6.1: Numerical results for $\mathbf{E}[B]$ obtained for a compound Poisson process with $\lambda = 4.00$, $\mu = 0.05$ and $T = 30.00$ for all balance control policies using Algorithm 2 with 10 000 simulations. The two right-most columns show the expectation of the value of approved purchases in a single-period when the parameter λ is multipled by $1/(1-a)$.

The numerical results obtained using Algorithm 2 show a close similarity with the analytical results obtained for the transactor model in Section 4.7. Figure 6.5 plots the numerical results for $\mathbf{E}[B]$ under BCP2. These results show that as ℓ becomes large, $\mathbf{E}[B]$ approaches a limiting value. For the transactor model, this limiting value is the expectation of the input process

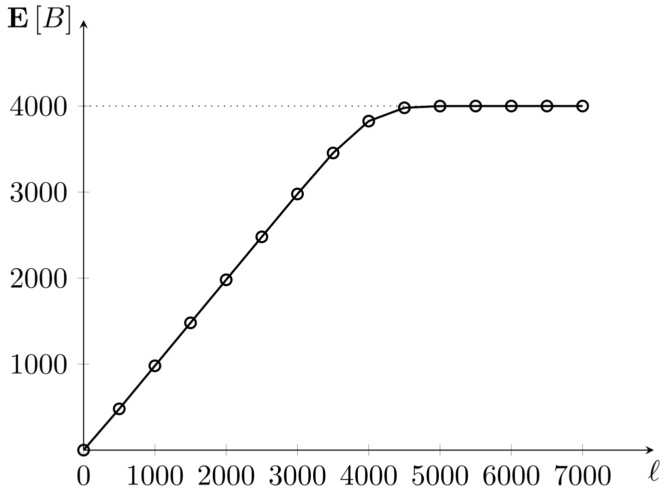

Figure 6.5: Plot of the numerical results obtained for $\mathbf{E}[B]$ under Ψ_2. Note the similarity to the expectation of $\Psi_2\big(T, \ell\big)$ in the transactor model.

$A(T)$. For the Markov chain $B(i)$, as $\ell \to \infty$, the process will become an autoregressive process with independent and identically distributed innovations and evolve according to Equation (6.14) with $\xi_i = A_i(T)$. In the case of a compound Poisson process with rate λ and exponentially distributed mark sizes with parameter μ, the conditions of Theorem 6.3.1 are satisfied since

$$\sum_{k=0}^{\infty} \mathbf{E}\left[\alpha^k A_k(T)\right] = \mathbf{E}[A(T)] \sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha} \frac{\lambda T}{\mu}, \tag{6.33}$$

and

$$\sum_{k=0}^{\infty} \mathrm{Var}[\alpha^k A_k(T)] = \mathrm{Var}[A(T)] \sum_{k=0}^{\infty} \alpha^{2k} = \frac{1}{1-\alpha^2} \frac{2\lambda T}{\mu^2}. \tag{6.34}$$

The Laplace transform of the density of $B = \lim_{n\to\infty} B(n)$ is

$$\begin{aligned} \mathbf{E}\left[\mathrm{e}^{-\psi X_\infty}\right] &= \prod_{k=0}^{\infty} \exp\left\{\lambda T \left(\frac{\mu}{\mu + \alpha^k \psi} - 1\right)\right\} \\ &= \exp\left\{\lambda T \sum_{k=0}^{\infty} \frac{\alpha^k \psi}{\mu + \alpha^k \psi}\right\}. \end{aligned} \tag{6.35}$$

The series in the exponent is absolutely convergent for $|\alpha| < 1$, since the ratio

$$\lim_{n\to\infty} \left| \frac{\frac{\alpha^{n+1}\psi}{\mu+\alpha^{n+1}\psi}}{\frac{\alpha^n \psi}{\mu+\alpha^n\psi}} \right| = \lim_{n\to\infty} \left| \frac{\alpha(\mu + \alpha^n \psi)}{\mu + \alpha^{n+1}\psi} \right| = \alpha. \tag{6.36}$$

However, a closed-form expression for the series is not immediately forthcoming. Indeed, it is not clear from Equation (6.35) that the resulting density is the density of a compound Poisson process, although it has the same expectation as a compound Poisson process with either the parameter λ multiplied by $1/(1-\alpha)$ or the parameter μ multiplied by $(1-\alpha)$. The same does not hold true for the variance of B and $A(T)$, since multiplying the rate parameter λ by $1/(1-\alpha)$ yields

$$\mathrm{Var}[A(T)] = \frac{1}{1-\alpha}\frac{2\lambda T}{\mu^2}. \tag{6.37}$$

and multiplying the jump size parameter μ by $(1-\alpha)$ results in

$$\mathrm{Var}[A(T)] = \frac{1}{(1-\alpha)^2}\frac{2\lambda T}{\mu^2}. \tag{6.38}$$

For small values of ℓ, the expectation of BCP1 and BCP3 is well-approximated by ℓ. Similarly, BCP2 is well-approximated by $\ell - 1/\mu$. Again, this accords with our understanding of the behaviour of the balance policies under the transactor model. If the probability that the input process $A(T)$ will exceed ℓ is high, then the balance control is likely to be activated. For BCP3, this means that $B(i)$ will likely be equal to ℓ. For BCP2, it means that $B(i)$ will likely be equal to $\ell - U$, where U is the random undershoot. Of course, BCP1 will be bounded between the two again.

We can determine the distribution of $\ell - U$ from the expression for the tail function given by Equation (4.30). By the Final Value theorem, we have that

$$\lim_{\omega\to\infty} \omega\, \tilde{S}(\theta,\omega,\psi) = \frac{1}{\theta\psi}\frac{\tilde{f}(\theta) - \tilde{f}(\theta+\psi)}{1-\tilde{f}(\theta+\psi)}. \tag{6.39}$$

Note that this is equivalent to taking the limit as $t \to \infty$. In the case of exponentially distributed jumps with parameter μ, we have

$$\lim_{\omega\to\infty} \omega\, \tilde{S}(\theta,\omega,\psi) = \frac{\mu}{\theta(\theta+\mu)(\theta+\psi)}, \tag{6.40}$$

and taking the limit as $\psi \to 0$ of Equation (6.40) above yields the transform of the expectation of $\ell - U$

$$\frac{\mu}{\theta^2(\theta+\mu)}, \tag{6.41}$$

which can be inverted analytically to give

$$\mathbf{E}[\ell - U] = \ell - \frac{1}{\mu}\big(1 - \mathrm{e}^{-\mu\ell}\big). \tag{6.42}$$

These observations on the behaviour of the numerical results lead us to surmise that we may be able to bound the expectation of the balance under partial payment behaviour by using the transactor model with the input process $A(t)$ modified accordingly so that $\mathbf{E}[B] = \mathbf{E}[A(T)]$ for large ℓ and the size of the undershoot is preserved under BCP3. In the case of the compound Poisson process, scaling the rate parameter λ by $1/(1-\alpha)$ will give us the right behaviour for small ℓ, since this will maintain the undershoot size, and result in the correct expectation for large ℓ.

The expectation of $\Psi\big(T, \ell\big)$ under BCP2 and BCP3 with the modified input process is given in the last two columns of Table 6.1. The results show a close match with the numerical results for small ℓ and large ℓ. When ℓ is close to the expected value of the scaled input process, the approximation is not so good. However, we also notice that $\mathbf{E}[B]$ under BCP1 is bounded above and below by the analytical results for BCP2 and BCP3, except for the estimates calculated at $\ell \geq 4{,}500$. Indeed, the error here is small and likely due to the small number of draws used in generating the numerical results.

These observations suggest that, for partial payment behaviour, the upper and lower bounds on the optimal limit under BCP1 can be approximated using BCP2 and BCP3 by scaling the rate parameter λ of the compound Poisson process $A(t)$ and solving the optimisation problem (6.13) as we did in Section 5.5.

6.6 Optimal limit calculations

We extend the example from Sections 4.7 and 5.5 and calculate bounds on the optimal limit for a card-holder exhibiting partial payment behaviour. We continue to use a compound Poisson process with rate λ and exponential jumps with parameter μ. Under our model of partial payment behaviour, the optimal limit is given by the solution to Equation (6.13), which now requires the additional parameters r and c. We use the values listed in Table 6.2 as inputs to Equation (6.13). For simplicity and ease of comparison to the transactor model, we use the same net revenue and capital charge multipliers and assume that the function $\bar{\chi}(\mathbf{w}, \mathbf{x}, \mathbf{y}) = 0$ independently of the capital charge multiplier $\nu^* = \nu(\mathbf{y})q(\mathbf{y})$. The interest rate r is given as an annual percentage rate of 19.99% and then scaled to yield a 30-day rate.

Parameter	Symbol	Value
Net revenue multiplier	γ^*	0.0054
Capital charge multiplier	ν^*	0.0007
Statement period length	T	30
Interest rate	r	$0.1999 \times 30/365.25$
Payment fraction	c	0.40
Scaling factor	a	0.39

Table 6.2: Parameters for the optimal limit calculations

To solve the optimisation problem (6.13), we used numerical inversion again to invert Equations (5.25) and (5.26) and obtain the upper and lower bounds on the optimal limit. We multiplied the rate parameter λ of the input compound Poisson process $A(t)$ by $1/(1-a)$, where $a = (1+r)(1-c)$, and left the jump size parameter of the process, μ, unchanged. The results are displayed for varying values of λ and μ in Table 6.3.

In addition to the bounds on the optimal limit, we can also calculate bounds on the probability that a card-holder will experience a declined purchase. Under the assumption that the scaled purchasing process $A(t)$ is the correct process, we can proceed as we did in Section 5.5 and numerically invert Equation (5.26) at the limits given in Table 6.3 to obtain the probability of a declined purchase.

In Section 6.5, it was shown that as $\ell \to \infty$, $B(i)$ becomes a first order autoregressive process with independent and identically distributed innovations, which we denoted X_i. In the case of a compound Poisson process with exponentially distributed jumps, the density of $\lim_{i\to\infty} X_i = X_\infty$ is given by Equation (6.35), which does not have a closed-form expression. Since a first order autoregressive process with independent and identically distributed innovations satisfies the conditions of Theorem 6.4.1, the backward iteration will yield a draw from the stationary density and we can use Algorithm 2 to numerically sample the stationary density to arbitrary precision and calculate the tail probability $\Pr(X_\infty > \hat{\ell})$.

Table 6.4 shows the probability of decline estimated from both the stationary density of $B(i)$ as $\ell \to \infty$ and the scaled version of $A(t)$. Note that

		$1/\mu$				
		20	40	60	80	100
	1	[1,999.49, 2,020.88]	[3,998.98, 4,041.76]	[5,998.47, 6,062.63]	[7,997.95, 8,083.51]	[9,997.44, 10,104.39]
		1,537.88	3,075.75	4,613.63	6,151.50	7,689.38
	2	[3,720.12, 3,741.11]	[7,440.24, 7,482.22]	[11,160.36, 11,223.33]	[14,880.48, 14,964.45]	[18,600.60, 18,705.56]
		3,075.75	6,151.50	9,227.25	12,303.00	15,378.75
λ	3	[5,398.17, 5,418.98]	[10,796.33, 10,837.96]	[16,194.50, 16,256.94]	[21,592.66, 21,675.91]	[26,990.83, 27,094.89]
		4,613.63	9,227.25	13,840.88	18,454.50	23,068.13
	4	[7,054.19, 7,074.89]	[14,108.38, 14,149.79]	[21,162.57, 21,224.68]	[28,216.76, 28,299.58]	[35,270.94, 35,374.47]
		6,151.50	12,303.00	18,454.50	24,606.00	30,757.51
	5	[8,696.14, 8,716.78]	[17,392.29, 17,433.55]	[26,088.43, 26,150.33]	[34,784.57, 34,867.10]	[43,480.72, 43,583.88]
		7,689.38	15,378.75	23,068.13	30,757.51	38,446.88

Table 6.3: Optimal limits for partial payment behaviour. The upper and lower bounds are given within the brackets and the expected value of the purchasing process $A(T)$ is given immediately below. The values were calculated for a compound Poisson process with rate λ and exponentially distributed purchase sizes with parameter μ. The parameters used in the optimal limit calculation are in Table 6.2.

the probabilities obtained using the scaled process are higher. This is due to the fact that the scaled process has a larger variance.

6.7 Discussion

The optimal limits and decline probabilities presented in Tables 6.3 and 6.4 demonstrate the effect that partial payment behaviour and the resulting revenue from interest have on the profit generated by the card-holder. In contrast to the optimal limits calculated in the case of transacting behaviour, the optimal limit for a card-holder who only pays a fraction of the due balance can be substantially higher and result in a much lower probability of decline. This reflects that it is more profitable for a card-issuing bank to allow a revolving customer a higher limit in order to maximise the combined revenue from interest and interchange, particularly since credit card interest rates are high relative to the costs associated with the increased risk of non-payment, such as a higher loan-loss provision and cost of funding and capital.

We did not explore the effects of increasing these costs in the example given in Section 6.6, and as such the optimal limits presented in Table 6.3 may overstate the effect that interest revenue has on profitability. Nonetheless, industry-level statistics show that interest is the principal component of revenue and the figures given in Tables 3.1 and 3.4 indicate that the total revenue from interest alone exceeds the total cost incurred from charge-offs and cost of funding. Therefore, the results presented in the previous section are broadly in line with what is observed in industry.

In Section 5.2, we assumed that the functions $\bar{\eta}(\mathbf{w})$, $\bar{\kappa}(\mathbf{x})$, $\nu(\mathbf{y})$ and $q(\mathbf{y})$ were small and constant between statement periods. This may not be a valid assumption with partial-payment behaviour, even if the card-holder exhibits it consistently, and further research to understand the relationship between changes in payment behaviour and these functions would be useful in testing the validity of the model.

Although the results indicate that a card-holder who only pays a fraction of the balance should receive a higher limit than a transacting customer (depending on the specific values of r and c), card-issuing banks do not usually offer higher credit limits to revolving customers. On the contrary, a common

		$1/\mu$				
		20	40	60	80	100
	1	[0.011 60, 0.031 99]	[0.011 60, 0.031 99]	[0.011 60, 0.031 99]	[0.011 60, 0.031 99]	[0.011 60, 0.031 99]
	2	[0.010 70, 0.033 53]	[0.010 70, 0.033 53]	[0.010 70, 0.033 53]	[0.010 70, 0.033 53]	[0.010 70, 0.033 53]
λ	3	[0.011 50, 0.034 25]	[0.011 50, 0.034 25]	[0.011 50, 0.034 25]	[0.011 50, 0.034 25]	[0.011 50, 0.034 25]
	4	[0.011 70, 0.034 69]	[0.011 70, 0.034 69]	[0.011 70, 0.034 69]	[0.011 70, 0.034 69]	[0.011 70, 0.034 69]
	5	[0.011 60, 0.035 00]	[0.011 60, 0.035 00]	[0.011 60, 0.035 00]	[0.011 60, 0.035 00]	[0.011 60, 0.035 00]

Table 6.4: Probabilities of decline for the optimal limits in Table 6.3. The lower values in each entry were calculated using Algorithm 2 with $n = 10\,000$ draws and the upper values were calculated via Laplace inversion.

strategy is to increase the credit lines of card-holders with a strong history of transacting behaviour. As discussed in Chapter 1, this is primarily because most automated credit limit management strategies are driven by risk measures such as the probability of default, rather than an assessment of profitability through the card-holder's behaviour. As such, the methods developed in this chapter offer an alternative framework for determining credit limits for card-holder's exhibiting partial payment behaviour.

We note, however, that the assignment of credit limits is also driven by business strategy. Credit cards are often sold in conjunction with other financial products such as mortgages. A retail bank may thus choose to sacrifice some profitability for the credit card business by giving a customer a high credit limit if it means that the profit can be recouped via another product.

The optimal limits in Table 5.1 were calculated after we were able to demonstrate that the expectation of the stationary process B is bounded by the expectations of $\Psi_2\big(T,\ell\big)$ and $\Psi_3\big(T,\ell\big)$ when the rate parameter λ of the purchasing process $A(T)$ is multiplied by $1/(1-a)$. Ideally, we would have made use of the scaling result in Proposition 5.5.1, but this result does not apply in the example we considered since multiplying the rate parameter λ in a compound Poisson process does not result in a straight-forward scaling of the distribution. Nevertheless, Algorithm 2 provides us with a numerical method for evaluating $\mathbf{E}\left[B\right]$ to arbitrary precision, assuming that routines for generating random draws from the inter-event time and jump size distributions are available.

Depending on the desired precision and the speed at which the particular implementation of Algorithm 2 runs, it may be expedient to use the bounds provided by $\Psi_2\big(T,\ell\big)$ and $\Psi_3\big(T,\ell\big)$ (with $A(T)$ modified accordingly) to narrow the range of points to be evaluated by the algorithm to a subset of Λ, rather than attempting to cover all of Λ. Assuming that $A(t)$ is a marked renewal process as described in Section 3.4.1, then the expectation of $A(T)$ can be calculated as the product of the expectation of the underlying renewal process $N(t)$ and the expectation of the jump sizes ξ.

Algorithm 2 can also be adapted to the case where the payment fractions ρ_i are a sequence of independent and identically distributed random variables. Rather than just generating the innovation ζ_i at the ith iteration, we generate the pair (ρ_i, ζ_i). In this case, some conditions on the ρ_i have to be satisfied in order for $B(i)$ to fulfil the conditions of Theorem 6.4.1. These conditions are given in [46] and the references therein.

In Section 5.7, it was demonstrated how a card-holder's time series of purchases could be used to estimate the distribution of purchase sizes. This could also be done for the card-holder's payments, although one would typically expect that time series to be much shorter since payments typically occur less frequently than purchases. If a paucity of data means that distribution-fitting is not an option, an alternative may be to assume that $\rho_i = c$, $i \geq 1$ and estimate c using some modification of the standard techniques employed in parameter estimation for autoregressive processes such as ordinary least-squares or the method of moments.

Since Algorithm 2 is only accurate up to arbitrary precision, it might be desirable to modify the algorithm to generate an exact sample from the stationary distribution of B. In [87], the authors describe a variety of scenarios for which the CFTP algorithm may be applied to a (non-monotone) continuous state space and detail methods for generating a perfect sample from the stationary distribution which are based on gamma-coupling and rejection sampling. For the purposes of this research, Algorithm 2 performs well and the loss of accuracy is a reasonable trade-off against the additional complexity and computing time introduced in attaining a perfect sample.

A further question is the applicability of the model since we explicitly require the process $B(i)$ to reach stationarity in order for our calculations to hold. Our model of partial-payment behaviour treats the behaviour as long-term and consistent, but a card-holder may exhibit partial-payment behaviour only for one or two statement periods since the interest incurred may impel them to return to transacting behaviour. A customer could also use a different type of payment strategy such as making a payment that is the maximum of the minimum payment and some fixed amount. Therefore, analysis of $B(i)$ in its transient phase would be useful in modelling the profitability of those customers who only occasionally exhibit partial-payment behaviour.

Chapter 7

Conclusion

A summary of the work done in this thesis and a critical review of the models and their underlying assumptions. Thoughts on practical considerations, and future research to extend this work.

7.1 Summary

This thesis has developed a model of the outstanding balance and profit for an individual credit card-holder. The model assumes that a card-holder's purchases and payments can be modelled as stochastic processes, in particular, marked point processes. It is not entirely new, but relatively uncommon, to apply stochastic processes directly to individual card-holders. This thesis is an effort to provide a mathematical framework which not only models credit card usage, but reflects the dynamic nature of individual behaviour and allows important performance measures to be derived and estimated using actual data. As we saw in Chapter 3, the data at the level of individual transactions, which reflects card-holder behaviour, motivates the choice of stochastic processes as our modelling tools.

Stochastic processes feature prominently in finance as models of various assets. Arithmetic Brownian motion was proposed as a model of stock prices by Bachelier in [11] and since then, many other stochastic processes have been studied as models of asset prices. For example, the Black-Scholes-Merton model for pricing derivative securities described in [21] assumes a geometric Brownian motion for the value of the underlying asset. In addition, actuaries rely heavily on stochastic process theory for the calculation of risk premiums, which depends on estimation of the likelihood of randomly occurring events, such as automobile accidents. In both these examples, the phenomenon being modelled is both random and dynamic in its nature and card-holder behaviour is no different.

Chapter 1 discussed the prevalence of classification methods in retail banking management and their shortcomings in modelling individual behaviour. Card-issuing banks are increasingly interested in behavioural analyses, particularly at the level of individual accounts. The recent ability to warehouse large amounts of data, including the transaction histories of individual accounts, has spurred this interest. Financial regulators are also encouraging banks to better understand and model customer behaviour. Yet most financial institutions at present are not modelling at such a granular level, and for good reason: we have seen that modelling the behaviour of an individual credit card-holder is both mathematically challenging and computationally intensive. To justify the additional effort, there must be some tangible benefit.

We demonstrated the value of the modelling at the individual level in two ways. First, we showed that the both the outstanding balance and profitability of a individual card-holder can be calculated when the processes describing their purchasing and payment behaviour can be modelled. As such, the Equations (3.14) and (3.18), and their simplified versions, Equations (3.26) and (3.30), provide a link between a card-holder's behaviour and their profitability. This can be used to value individual card-holders using their transaction history, and simulate or infer future behaviour and profitability.

Second, we demonstrated that optimal limits can be derived at the individual level for both transacting and partial payment behaviour, and examined the effects that setting the credit limit at its optimal level had on both profitability and the probability that the card-holder would experience a declined purchase.

Modelling at the individual level can create further differentiation within certain populations encountered in credit card portfolios. It is quite common to see large numbers of low-risk accounts (often transactors) with similar behaviour scores and since this is often a key input into account management strategies, these accounts will receive identical treatment. Analysis of the individual transaction patterns of such accounts can provide increased differentiation and lead to the development of more profitable strategies. Used this way, the models developed in this thesis can be a useful complement to existing scorecard models.

Although we have emphasised the differences between the approach we have taken in this thesis and current industry practice, the motivation behind this research is not to displace classification and scoring methods. Rather, we seek to extend the retail portfolio management toolkit.

7.2 Critical review

Every model is, in a sense, an approximation. As such, we must be aware and critical of its limitations, particularly when modelling something as complex as human economic behaviour. As the aphorism by George E. P. Box goes — "All models are wrong, but some are useful."

7.2.1 Point process assumptions

In Chapter 3, we made the explicit assumption that an individual card-holder's purchases and payments follow a marked point process. We provided evidence for this through actual transaction data. We also posited in Chapters 3 and 5 that the process of attempted purchases, $A(t)$, could be modelled as the superposition of independent processes describing attempts of different purchase types. This approach works well with the small sample of data used in this thesis, but more research is required to assess the suitability of the approach in general.

We made the further assumption in Chapter 4 that the purchase process could be modelled by a marked renewal process in which the inter-event times and event sizes where independent. This assumption aids greatly in facilitating calculations, and is not without precedent. As mentioned in Chapter 1, renewal models were fitted to credit card transaction times in both [125] and [126]. We also noted in Section 5.7 that the assumption of independence between the time and value of events can be tested. Again, part of the appeal of the approach of using point processes to model card-holder behaviour is the ability to statistically assess the suitability of the model through hypothesis tests, as we demonstrated in Section 5.7.

A marked renewal process is a *simple* point process, meaning that multiple events cannot occur simultaneously. Since card-holders can only make one purchase at a time, this is a reasonable model to use. Indeed, one could argue that in the case of a credit card account with an additional card-holder, this property may fail to hold. However, even in this situation, the probability of observing simultaneous transactions is small.

The renewal processes considered in this thesis have been homogeneous and thus held constant intensity. This assumption is somewhat questionable given the strong evidence that purchasing frequency and value varies within a calendar year, with a clear peak in December (see Figure 1.1). A remedy could be to adopt a time-varying or stochastic intensity function for $A(t)$. Stochastic intensity models have been applied in other areas of finance, particularly in the valuation of credit derivatives.

A further extension could be to assume a conditional intensity function for $A(t)$ of the form

$$\lambda(t, \boldsymbol{\Theta} \mid \mathscr{F}_t) = \lim_{\delta \to 0} \frac{1}{\delta} \Pr\left(N(t, t+\delta) > 0 \mid \mathscr{F}_t\right), \tag{7.1}$$

where $\boldsymbol{\Theta} = (\theta_1, \ldots, \theta_m)$ is a collection of parameters and $\mathscr{F}_t$ is the σ-algebra of the process as defined by Equation (3.16). A standard reference for conditional intensity functions and point processes in general is [44]. Of course, the introduction of additional parameters and past or state-dependence greatly complicates matters. In the interest of maintaining simplicity, one could assume a constant intensity within each month, estimate the monthly intensity from data and take the maximum. This way, the optimal limits and decline probabilities anticipate the card-holder's behaviour for the most profitable month of the year.

7.2.2 Modelling assumptions

In Chapter 3.4, we made several assumptions that simplified the models of balance and profit. The assumption M1 that interest is only charged on the outstanding balance considerably simplified calculations concerning revolving behaviour, but is not very important for transacting behaviour since interest is never charged. Nonetheless, it may be a useful exercise to quantify the difference in performance measures under the assumed interest rate calculation and the actual one that is used for Australian credit cards described in Section 3.3.1.1.

The assumptions M2 and M4 have the important consequence that the value of approved purchases in a statement period does not depend on payments made by the card-holder during the statement period, which greatly simplified our calculations for the process of approved purchases. Of course, card-holders can and do make payments throughout the statement period, and as such, the model describing the process of approved purchases underestimates the true value of approved purchases.

The assumptions M5 and M6 are less contentious. Interchange, loyalty scheme costs and the costs of funds do not change frequently. The loan-loss and capital provision are indeed dependent on account characteristics, but note that these quantities are usually calculated monthly as part of regular risk management activities. However, as we originally specified in Equation (3.18), these quantities could be dependent on the outstanding balance in a more complicated manner than the assumptions allow for.

The model proposed in Chapter 3 and developed throughout the thesis does not consider the interaction between the credit limit and the card-holder's purchasing or payment behaviours. Thus, the processes $A(t)$ and $P(t)$ are assumed to be independent of the credit limit ℓ. However, it is an observed phenomenon that both credit card purchasing and payment behaviour are influenced by the credit limit. Most credit card customers hold more than one card, and a plausible behaviour for a customer with two credit cards might be to charge more to the card with the higher limit. In this case, raising the limit on the card with the lower limit might lead to more purchase activity. Certainly many credit card issuers set limits in the belief that this is the case. Furthermore, lowering a credit limit may discourage a card-holder from making purchases. These observations imply that the process of attempted purchases should be a function of both time t and the credit limit ℓ. Ignoring this interaction between $A(t)$ and ℓ is likely one of the reasons that the optimal credit limits calculated in Table 5.4 correspond to such high probabilities of decline.

It is also conceivable that the repayment behaviour of a card-holder would depend upon the credit limit. A customer who repays their credit card in full each month when the limit is \$1,000 may not do so when the limit is \$5,000. This is a real consideration for card issuers when considering whether or not to raise the credit limit on an account. Furthermore, it has also been observed that larger credit limits are associated with higher probabilities of default – not just higher levels of loss given default – due to moral hazard or adverse selection. For an example and a discussion of the phenomenon, see [4]. This implies that the process $P(t)$ should also be a function of ℓ. Incorporating these dependencies is an avenue for further research.

The assumptions underlying the framework presented in thesis can all be tested to varying degrees but, at a more general level, the validity of the framework rests on its ability to assist practitioners in making real decisions. As such, until the framework is applied and tested under "real-world" scenarios, it is difficult to gauge which assumptions are reasonable, which are essential, and which can be done away with or replaced.

7.2.3 Comparison with classification methods

The models of credit card balance and profitability developed in Chapter 3 were used to calculate profit-maximising limits for individual credit card accounts. In Sections 1.2 and 1.3, several modelling techniques were described which have also been applied to model measures of interest and solve various problems, including optimal limit allocation. Indeed, regression could be used to calculate a profit-maximising limit for a credit card based on a vector $\mathbf{x}$ of account and card-holder features. For example, one could take a data set of records that each have a vector of card-holder characteristics $\mathbf{x}_i$ for each card-holder i along with the corresponding credit limit ℓ_i and realised profitability π_i over some period of time. Regression (or some other approach) could then be used to determine how π_i varied as a function of $\mathbf{x}$ and ℓ. This relationship in turn could be used to estimate an optimal credit limit as a function of $\mathbf{x}_i$.

One benefit of the models developed in this thesis over using an approach such as that described above is that they directly link the performance measure of interest to the account usage of the card-holder *at the individual level*. Regression (and other similar methods) provide a way to *estimate* the relationship between a dependent variable and a collection of independent variables based on observed data. The models developed in this thesis explicitly specify that the outstanding balance and profit are driven by the card-holder's purchasing and payment behaviour, which we regard as stochastic processes. Various assumptions can be made about the form of these processes, or the distributional forms can be estimated from the card-holder's transaction history as demonstrated in Section 5.7. As such, the results obtained using the approach developed in this thesis can be said to be tailored to the individual card-holder's behaviour.

A typical constraint of classifiers or scorecards is that they model a single outcome over a fixed period of time, such as the likelihood of default or campaign response. The individual-level models we have developed are dynamic. As such, several measures can be be derived from them. The probability of default, for example, can be derived from the model as the probability of observing three or more consecutive statement periods where no payment is received. Exposure at default is then the balance at the end of that third statement period. By modelling at the individual-level, we shift the paradigm of modelling from one where measures of interest are defined through labels for classification, to one where performance measures are events based on the sample paths of stochastic processes.

As mentioned in Section 1.2, one of the reasons that analysts in credit risk management continue to use logistic regression is the relative ease with which it can be interpreted. In our proposed framework, interpretation is also straight-forward. Changes in the performance measures of interest, such as profit or decline probabilities, can be linked to changes in the underlying stochastic processes or the set of the parameters used.

7.2.4 Practical considerations for implementation

The models and methods used in this thesis will likely be unfamiliar to most practitioners in retail credit risk management. As such, the framework has many aspects which require consideration before application.

Use of the framework presupposes that transaction data is available for the relevant card-holders. Transaction data essentially forms a time series and the volume of data is therefore much greater than the data traditionally used for scorecard modelling, which introduces computational and storage issues. To infer the behaviour of a card-holder, the time series of both attempted and completed transactions is required. As demonstrated in Section 5.7, in order to calculate optimal limits for transacting customers, distributions for the inter-event time and purchase size distributions need to be estimated from the time series. Across a portfolio of even modest size — for example, one million accounts — it quickly becomes infeasible to manually fit distributions to the data and this will need to be automated in some way. Once this is done, however, the models can be updated with new data as and when it arrives, and new optimal limits easily calculated.

Practitioners will also need to consider feedback effects. Credit card application scorecards must be monitored and occasionally rebuilt to ensure that they continue to discriminate within prescribed tolerances as the applying population changes, both organically and in response to the application criteria enforced by the scorecard. Similarly, one should expect that taking action based on the output of the framework will result in changes in card-holder behaviour. Changing card usage in response to credit limit changes is a known phenomenon in industry, and it may be that setting a card-holder's limit to the optimal value as calculated by the models presented in this thesis may result in behaviour that renders the new limit far from optimal. Card-holders may also hold other products at a bank, and the effect on profitability across these products should also be considered.

7.3 Avenues for future work

There are many avenues for extending the research in this thesis, with the key opportunities outlined below.

Point process models It would be a worthwhile exercise to investigate the use of more sophisticated point process models which may better capture the complex nature of credit card purchasing and payment behaviour. Several alternative approaches were outlined in Section 7.2.1.

Further models of revolving behaviour We noted at the end of in Chapter 6 that the model of partial payment be modified in a straight-forward manner to allow for random payment fractions. It was also argued that revolving behaviour is often a short-term behaviour for most card-holders. Analysis of the transient behaviour of the Markov chain model would be useful for assessing the profitability of partial payment behaviour when it is only exhibited for brief periods.

Relax the assumption of a single payment per statement period A further extension would be to relax the assumption of a single payment per statement period to allow for multiple payments and assess the difference this makes to performance measures such as profit and optimal limits. As noted in Section 7.2.2, the assumption M2 results in an underestimation of the expected value of approved purchases. Of course, the introduction of a payment process will complicate the derivation of the expressions derived in Chapter 4.

Derive other performance measures As discussed in Section 7.2.3, several other performance measures can be derived within the framework in addition to optimal limits. A measure of particular interest to derive is the optimal interest rate. In order to derive this measure, the outstanding balance needs to be expressed as a function of the interest rate r. As discussed in Chapter 6, the exact relationship between interest rates and credit card purchasing behaviour is still an active area of research. Mathematically modelling this relationship within the framework developed in this thesis represents an interesting challenge.

Aggregate modelling and effects This thesis has focused on building a model of individual-level behaviour, but several interesting questions arise when considering the application of these models across a portfolio. There is undoubtedly some correlation between the purchasing and payment behaviours of card-holders — consider the increased purchasing value and frequency observed in the month of December. Modelling the relationships between card-holders and the portfolio-level effects represents a further challenge. Indeed, we have seen that revenue in a credit card business is primarily divided between two streams: regular income from transactors who produce interchange income and more volatile but high-value interest income from revolvers. An interesting question to investigate is how the proportions of revenue are split across these two populations when an optimal limit is calculated and assigned to each customer.

Applications to other revolving products Finally, it was argued in the introduction that credit cards are unique among retail lending products for the wide variety of behaviour they permit and generate. Nevertheless, the framework presented could be extended to other revolving lines of credit such as overdrafts, home equity lines of credit (HELOCs), or commercial line of credit agreements.

Appendix A

Derivations

A.1 The three-dimensional Laplace transform of $S_1(y,t,\ell)$

For ψ, ω and θ with strictly positive real part, we may multiply Equation (4.9) by $\mathrm{e}^{-(\omega t+\theta\ell+\psi y)}$ and integrate term by term over $(0,\ell]$ for y and $(0,\infty)$ for ℓ and t. We may change the order of integration in the first term to find

$$\begin{aligned}&\int_0^\infty\int_0^\infty\int_0^\ell \mathrm{e}^{-(\omega t+\theta\ell+\psi y)}\int_0^t\int_0^y S_1(y-z,t-u,\ell-z)\,F(\mathrm{d}z)\,G(\mathrm{d}u)\,\mathrm{d}y\,\mathrm{d}\ell\,\mathrm{d}t\\ &=\int_0^\infty\int_0^\infty\int_u^\infty\int_z^\infty\int_z^\ell \mathrm{e}^{-(\omega t+\theta\ell+\psi y)}\ S_1(y-z,t-u,\ell-z)\,\mathrm{d}y\,\mathrm{d}\ell\,\mathrm{d}t\,F(\mathrm{d}z)\,G(\mathrm{d}u).\end{aligned}$$

By making the substitutions $w=y-z$, $v=t-u$ and $x=\ell-z$, we have

$$\begin{aligned}&\Big(\int_0^\infty \mathrm{e}^{-\omega u}\,G(\mathrm{d}u)\Big)\Big(\int_0^\infty \mathrm{e}^{-(\theta+\psi)z}\ F(\mathrm{d}z)\Big)\\ &\qquad\Big(\int_0^\infty\int_0^\infty\int_0^x \mathrm{e}^{-(\omega v+\theta x+\psi w)}\,S_1(w,v,x)\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}v\Big)\\ &\qquad=\tilde{g}(\omega)\tilde{f}(\theta+\psi)\tilde{S}_1(\psi,\theta,\omega).\end{aligned}\tag{A.1}$$

The second term becomes

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^\ell \mathrm{e}^{-(\omega t+\theta\ell+\psi y)}\; G(t)\big(F(\ell)-F(y)\big)\,\mathrm{d}y\,\mathrm{d}\ell\,\mathrm{d}t\\
&=\bigg(\int_0^\infty \mathrm{e}^{-\omega t}\; G(t)\,\mathrm{d}t\bigg)\bigg(\int_0^\infty \mathrm{e}^{-\theta\ell}\int_0^\ell \mathrm{e}^{-\psi y}\,\big(F(\ell)-F(y)\big)\,\mathrm{d}y\,\mathrm{d}\ell\bigg)\\
&=\frac{1}{\omega}\tilde g(\omega)\bigg(\int_0^\infty \mathrm{e}^{-\theta\ell}\int_0^\ell \mathrm{e}^{-\psi y}\,F(\ell)\,\mathrm{d}y\,\mathrm{d}\ell-\int_0^\infty \mathrm{e}^{-\theta\ell}\int_0^\ell \mathrm{e}^{-\psi y}\,F(y)\,\mathrm{d}y\,\mathrm{d}\ell\bigg)\\
&=\frac{1}{\omega}\tilde g(\omega)\bigg(\int_0^\infty \mathrm{e}^{-\theta\ell}\,F(\ell)\int_0^\ell \mathrm{e}^{-\psi y}\;\mathrm{d}y\,\mathrm{d}\ell-\int_0^\infty \mathrm{e}^{-\theta\ell}\int_0^\ell \mathrm{e}^{-\psi y}\,F(y)\,\mathrm{d}y\,\mathrm{d}\ell\bigg)\\
&=\frac{1}{\omega}\tilde g(\omega)\bigg(\frac{1}{\psi}\int_0^\infty \mathrm{e}^{-\theta\ell}\,F(\ell)(1-\mathrm{e}^{-\psi\ell})\,\mathrm{d}\ell-\int_0^\infty \mathrm{e}^{-\psi y}\,F(y)\int_y^\infty \mathrm{e}^{-\theta\ell}\;\mathrm{d}\ell\,\mathrm{d}y\bigg)\\
&=\frac{1}{\omega}\tilde g(\omega)\bigg(\frac{1}{\psi}\int_0^\infty \big(\mathrm{e}^{-\theta\ell}-\mathrm{e}^{-(\theta+\psi)\ell}\big)F(\ell)\,\mathrm{d}\ell-\frac{1}{\theta}\int_0^\infty \mathrm{e}^{-(\theta+\psi)y}\,F(y)\,\mathrm{d}y\bigg)\\
&=\frac{1}{\omega}\tilde g(\omega)\bigg(\frac{1}{\theta\psi}\tilde f(\theta)-\frac{1}{\psi(\theta+\psi)}\tilde f(\theta+\psi)-\frac{1}{\theta(\theta+\psi)}\tilde f(\theta+\psi)\bigg)\\
&=\frac{1}{\theta\omega\psi}\tilde g(\omega)\big(\tilde f(\theta)-\tilde f(\theta+\psi)\big). \qquad\qquad \text{(A.2)}
\end{aligned}
$$

We immediately change the order of integration in the third term to find

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^\ell \mathrm{e}^{-(\omega t+\theta\ell+\psi y)}\int_0^t\int_\ell^\infty S_1(y,t-u,\ell)\,F(\mathrm{d}z)\,G(\mathrm{d}u)\,\mathrm{d}y\,\mathrm{d}\ell\,\mathrm{d}t\\
&=\int_0^\infty\int_0^\infty\int_0^\ell \mathrm{e}^{-(\omega t+\theta\ell+\psi y)}\,\big(1-F(\ell)\big)\int_0^t S_1(y,t-u,\ell)\,G(\mathrm{d}u)\,\mathrm{d}y\,\mathrm{d}\ell\,\mathrm{d}t.
\end{aligned}
$$

Making the substitution $s=t-u$, this becomes

$$
\begin{aligned}
&=\bigg(\int_0^\infty \mathrm{e}^{-\omega u}\; G(\mathrm{d}u)\bigg)\\
&\qquad\bigg(\int_0^\infty\int_0^\infty\int_0^\ell \mathrm{e}^{-(\omega s+\theta\ell+\psi y)}\;S_1(y,s,\ell)\big(1-F(\ell)\big)\,\mathrm{d}y\,\mathrm{d}\ell\,\mathrm{d}s\bigg)\\
&=\tilde g(\omega)\bigg(\int_0^\infty\int_0^\infty\int_0^\ell \mathrm{e}^{-(\omega s+\theta\ell+\psi y)}\;S_1(y,s,\ell)\big(1-F(\ell)\big)\,\mathrm{d}y\,\mathrm{d}\ell\,\mathrm{d}s\bigg). \qquad \text{(A.3)}
\end{aligned}
$$

Combining (A.1), (A.2) and (A.3), we arrive at

$$
\begin{aligned}
\tilde S_1(\psi,\omega,\theta)&=\tilde g(\omega)\tilde f(\theta+\psi)\tilde S_1(\psi,\omega,\theta)+\frac{1}{\theta\omega\psi}\tilde g(\omega)\big(\tilde f(\theta)-\tilde f(\theta+\psi)\big)\\
&\quad+\tilde g(\omega)\bigg(\int_0^\infty\int_0^\infty\int_0^\ell \mathrm{e}^{-(\omega s+\theta\ell+\psi y)}\;S_1(y,s,\ell)\big(1-F(\ell)\big)\,\mathrm{d}y\,\mathrm{d}\ell\,\mathrm{d}s\bigg).
\end{aligned}
\qquad \text{(A.4)}
$$

A.2 The two-dimensional Laplace transform of $S_0(y,t)$

We proceed as we did in Section 4.3.1 to find the two-dimensional Laplace transform

$$\tilde{S}_0(\psi,\omega) \equiv \int_0^\infty \int_0^\infty \mathrm{e}^{-(\omega t+\psi y)}\, S_0(y,t)\,\mathrm{d}y\,\mathrm{d}t, \tag{A.5}$$

where $S_0(y,t) = \Pr\bigl(A(t) > y\bigr)$ is the tail function of the process $A(t)$ under Assumptions A1 and A2. Let $\tau = \tau_1$ and $\xi = \xi_1$ be the time and value of the first jump in the process $A(t)$, $t > 0$. We have the following three cases

$$\mathbf{E}\left[\mathbf{1}_{\{A(t)>y\}} \mid \tau,\, \xi\right] = \begin{cases} 0, & \tau > t \\ \mathbf{E}\left[\mathbf{1}_{\{A(t-\tau)>y-\xi\}}\right], & \tau \le t,\, \xi \le y \\ 1, & \tau \le t,\, \xi > y. \end{cases} \tag{A.6}$$

Integrating (A.6) with respect to F and G to obtain the unconditional tail function, we find the following integral equation

$$S_0(y,t) = \int_0^t \int_0^y S_0(y-z, t-u)\, F(\mathrm{d}z)\, G(\mathrm{d}u) + G(t)\bigl(1 - F(y)\bigr). \tag{A.7}$$

It follows directly that

$$|S_0(y,t)| \le G(t)F(y) + G(t) - G(t)F(y) = G(t), \tag{A.8}$$

which shows that $S_0(y,t)$ is of exponential order in t when G is of exponential order. The one-dimensional Laplace transform

$$\tilde{S}_0(y,\omega) \equiv \int_0^\infty \mathrm{e}^{-\omega t}\, S_0(y,t)\,\mathrm{d}t \tag{A.9}$$

exists for ω with $\mathrm{Re}(\omega) > \sigma_G$, and the two-dimensional Laplace transform (A.5) exists for ω with $\mathrm{Re}(\omega) > \sigma_G$ and ψ with $\mathrm{Re}(\psi) > 0$. So, we may multiply the right-hand side of Equation (A.7) by $\mathrm{e}^{-(\omega t+\psi y)}$ and integrate y and t over $(0,\infty)$

$$\begin{aligned} \tilde{S}_0(\psi,\omega) = \int_0^\infty \int_0^\infty & \mathrm{e}^{-(\omega t+\psi y)} \\ & \Biggl[\int_0^t \int_0^y S_0(y-z,t-u)\, F(\mathrm{d}z)\, G(\mathrm{d}u) \\ & \quad + G(t)\bigl(1-F(y)\bigr)\Biggr]\mathrm{d}y\,\mathrm{d}t. \end{aligned} \tag{A.10}$$

When both ω and ψ have a strictly positive real part, we may integrate the right-hand side of (A.10) term by term and change the order of integration to find

$$\begin{aligned}
\tilde{S}_0(\psi,\omega) &= \int_0^\infty\int_0^\infty\int_0^t\int_0^y \mathrm{e}^{-(\psi y+\omega t)}\, S_0(y-z,t-u)\, F(\mathrm{d}z)\, G(\mathrm{d}u)\,\mathrm{d}y\,\mathrm{d}t \\
&\quad + \int_0^\infty\int_0^\infty \mathrm{e}^{-(\psi y+\omega t)}\, G(t)\big(1-F(y)\big)\mathrm{d}y\,\mathrm{d}t \\
&= \int_0^\infty\int_0^\infty\int_z^\infty\int_u^\infty \mathrm{e}^{-(\psi y+\omega t)}\, S_0(y-z,t-u)\,\mathrm{d}y\,\mathrm{d}t\, F(\mathrm{d}z)\, G(\mathrm{d}u) \\
&\quad + \frac{1}{\psi\omega}\tilde{g}(\omega)\big(1-\tilde{f}(\psi)\big) \\
&= \tilde{g}(\omega)\tilde{f}(\psi)\tilde{S}_0(\psi,\omega) + \frac{1}{\psi\omega}\tilde{g}(\omega)\big(1-\tilde{f}(\psi)\big).
\end{aligned} \tag{A.11}$$

Rearranging (A.11) above gives

$$\tilde{S}_0(\psi,\omega) = \frac{1}{\psi\omega}\frac{\tilde{g}(\omega)\big(1-\tilde{f}(\psi)\big)}{1-\tilde{g}(\omega)\tilde{f}(\psi)}, \quad \mathrm{Re}(\omega),\, \mathrm{Re}(\psi) > 0. \tag{A.12}$$

A.3 Proof of the scaling property of $\hat{\ell}$

By the assumption that $Y(t) \overset{d}{=} \alpha X(t)$,

$$\begin{aligned}
\hat{\ell}_Y &= \arg\max_{\ell_Y}\Big\{\gamma\,\mathbf{E}\left[\Psi(t,Y,\ell)\right] - \nu\ell_Y\Big\} \\
&= \arg\max_{\ell_Y}\Big\{\gamma\,\mathbf{E}\Big[\sup_{0\le u\le t}\{Y(u) : Y(u)\le \ell_Y\}\Big] - \nu\ell_Y\Big\} \\
&= \alpha\arg\max_{\ell_Y}\left\{\gamma\,\mathbf{E}\left[\sup_{0\le u\le t}\left\{X(u) : X(u)\le\frac{\ell_Y}{\alpha}\right\}\right] - \nu\frac{\ell_Y}{\alpha},\right\}.
\end{aligned}$$

Making the substitution $\ell_X = \ell_Y/\alpha$,

$$\begin{aligned}
\hat{\ell}_Y &= \alpha\arg\max_{\ell_X}\left\{\gamma\,\mathbf{E}\left[\sup_{0\le u\le t}\{X(u) : X(u)\le \ell_X\}\right] - \nu\ell_X\right\} \\
&= \alpha\arg\max_{\ell_X}\Big\{\gamma\,\mathbf{E}\left[\Psi(t,X,\ell_X)\right] - \nu\ell\Big\} = \alpha\hat{\ell}_X,
\end{aligned}$$

which shows that the optimal limit scales with α. Since $\hat{\ell}_Y = \alpha\hat{\ell}_X$, we have

$$\Pr\left(Y(t) > \hat{\ell}_Y\right) = \Pr\left(\alpha X(t) > \alpha\hat{\ell}_X\right) = \Pr\left(A(t) > \hat{\ell}_X\right),$$

which shows that the decline probabilities remain the same.

Appendix B

Code

B.1 The EULER algorithm

The following listings of code are an implementation of the `EULER` algorithm in `C++`. The algorithm is implemented as a template and uses the Boost library for the calculation of the binomial coefficients. The template takes a function as its argument, along with value to invert the function at and the parameters m, n and γ. A test function and routine demonstrating the use of the template is also given. The code was compiled using `g++` (Apple LLVM version 7.0.2 – clang-700.1.81) on a 2009 Macbook Pro.

```
#include <iostream>
#include <cmath>
#include <vector>
#include <boost/math/special_functions/binomial.hpp>

template<class T,
   double (T::*Value)(double, double) const >
double EulerInversion(double theInversionTime,
          const T& theFunctionObject,
          unsigned long m,
          unsigned long n,
          unsigned long discretise)
{
  double t = theInversionTime;
  unsigned long binomialTerms = m;

```

```
std::vector<double> coefficients(binomialTerms + 1);
std::vector<double> SU(binomialTerms + 2);

for (unsigned long i = 0; i < coefficients.size(); ++i){
  coefficients[i] = boost::math::binomial_coefficient<double>(
      binomialTerms, i);
}

const double pi = M_PI;

double A = discretise*std::log(10.00);
unsigned long Ntr = n; // This is the parameter Abate & Whitt
    suggest increasing.
double U = std::exp(0.5*A) / t;
double X = A / (2.0 * t);
double H = pi / t;

double theSum = (theFunctionObject.*Value)(X,0) / 2.0;

for(unsigned long i = 1; i <= Ntr; ++i)
  {
    double Y = i*H;
    int theSign = (i % 2 == 0) ? 1 : -1;
    theSum += theSign*(theFunctionObject.*Value)(X,Y);
  }

SU[0] = theSum;

for(unsigned long i = 0; i < SU.size(); ++i)
  {
    unsigned long N = Ntr + (i + 1);
    double Y = N*H;
    int theSign = (N % 2 == 0) ? 1 : -1;
    SU[i+1] = SU[i] + theSign*(theFunctionObject.*Value)(X,Y);
  }

// double Avgsu = 0.0;
double Avgsu1 = 0.0;

for (unsigned long i=0; i < coefficients.size(); ++i)
  {
```

```
        //  Avgsu += coefficients[i]*SU[i];
        Avgsu1 += coefficients[i]*SU[i+1];
    }

  //  double Fun = U*Avgsu/2048;
  double Fun1 = (U*Avgsu1) / std::pow(2.00,binomialTerms);

  //  double Errt = std::fabs(Fun - Fun1) / 2.0;

  return Fun1;

}
```

code/Euler.hpp

```
#ifndef LAPLACETRANSFORM_H
#define LAPLACETRANSFORM_H

class LaplaceTransform
{
public:
  LaplaceTransform(double rho_, double mu_);

  double RealValue(double x, double y) const;

private:
  double rho;
  double mu;
};

#endif
```

code/LaplaceTransform.hpp

```
#include "LaplaceTransform.hpp"
#include <complex>
#include <cmath>

using namespace std;

LaplaceTransform::LaplaceTransform(double rho_, double mu_)
  : rho(rho_),
```

```
    mu(mu_)
{}

double LaplaceTransform::RealValue(double x, double y) const
{
  complex<double> z(x,y);
  complex<double> transformFunction = 1.0 / sqrt(1.0 + 2.0*z); //
      Insert function here.
  complex<double> Gse = (1.0 - transformFunction) / (mu * z);
  complex<double> Fs = (1.0 - Gse) / (z * (1.0 - rho*Gse));

  double result = Fs.real();

  return result;

}
```

code/LaplaceTransform.cpp

```
#include "Euler.hpp"
#include "LaplaceTransform.hpp"
#include <iostream>
#include <iomanip>
#include <cmath>

using namespace std;

int main(){

  // Parameters for the Laplace transform.

  double rho = 0.75;
  double mu = 1.0;

  // Invert the transform for this value.

  double value = 10.0;

  LaplaceTransform theFunction(rho,mu);

  double invertedValue;

```

```
  invertedValue = EulerInversion<LaplaceTransform, &
      LaplaceTransform::RealValue>(value, theFunction,21,25,10);

  cout << invertedValue << endl;

  return 0;

}
```

code/Laplaceinversion.cpp

B.2 Calculation of $\mathbf{E}[\Psi_2(t,\ell)]$ and its derivative

The following `R` code was used to produce Figures 4.5 and 4.6.

```
### These functions calculate the expectation of the process \Psi_
    2(t,\ell)
### and its derivative.

PsiExpectation <- function(level,t,lambda,mu){

    thisSumOne <- 0
    thisSumTwo <- 0

    for (k in 1:10000){
        thisSumOne <- thisSumOne + k*dpois(k,lambda*t)*ppois(k,mu*
            level,lower.tail=FALSE)
        thisSumTwo <- thisSumTwo + k*dpois(k+1,mu*level)*ppois(k,
            lambda*t,lower.tail=FALSE)
    }

    theResult <- (thisSumOne + thisSumTwo) / mu

    return(theResult)
}

expectationDerivative <- function(level,time,lambda,mu,sigma){

    thisSum <- 0

    for (k in 1:10000)
```

```
        thisSum <- thisSum + (level / k)*dnorm(level,k*mu,sqrt(k)*
            sigma)*ppois(k-1,lambda*time,lower.tail=FALSE)

    theResult <- thisSum

    return (theResult)
}

# Plot the graph of the derivative

x <- seq(0,5000,50)
drv <- rep(NA,length(x))
for (n in 1:length(x))
    drv[n] <- expectationDerivative(x[n],30,4,10,sqrt(10))

plot(x,drv,type="l",pch="")
```

code/expectation.r

B.3 An implementation of the backward iteration for BCP1

In Section 6.5, we described pseudo-code for calculating the expectation of the Markov chain $B(i)$ when the chain is in its stationary regime using the backward iteration. The following listing is an implementation for BCP1, again written in C++. This particular example uses a compound Poisson process with parameters $\lambda = 4$, $\mu = 0.05$ and $T = 30$, and calculates the expectation when $\ell = 7000$ using $N = 10{,}000$ simulations. Again, we make use of the random number and distributional routines from the Boost library.

```
#include <iostream>
#include <vector>
#include <cmath>
#include <array>
#include <algorithm>
#include <boost/random/mersenne_twister.hpp>
#include <boost/random/poisson_distribution.hpp>
#include <boost/random/exponential_distribution.hpp>
#include <boost/random/uniform_01.hpp>
```

```
int main(){

  unsigned long const bigN = 10000;
  unsigned long const littleN = 100;

  double bottom = 0.0;
  double tolerance = 1.0e-12;
  double alpha = 0.4;

  double theLimit = 7000.00;
  double lambda = 120.00;
  double mu = 0.05;

  std::vector<double> draws;
  double thetotal = 0.0;
  double theAverage;

  boost::mt19937 randomGenerator;
  randomGenerator.seed(456789);
  boost::random::poisson_distribution<> poissonDistribution(lambda);
  boost::random::exponential_distribution<> exponentialDistribution(mu);

  for (unsigned long k=0; k < bigN; ++k){

     std::vector<unsigned long> jumps(littleN);
    for (unsigned long j=0; j < littleN; ++j){
      jumps[j] = poissonDistribution(randomGenerator);
    }

    std::vector< std::vector<double> > purchases;

    for (unsigned long i = 0; i < littleN; ++i){

      std::vector<double> thisPurchaseVector;
      double thisRandomDraw(0.0);

      for (unsigned long j = 0; j < jumps[i]; ++j){
  thisRandomDraw = exponentialDistribution(randomGenerator);
```

```
      thisPurchaseVector.push_back(thisRandomDraw);
    }

    purchases.push_back(thisPurchaseVector);

  }

  unsigned long m = 1;
  bool stopCondition(false);

  double bottomEvolve, topEvolve;
  double bottomShrink, topShrink;
  double bottomInnovation, topInnovation;


  while (stopCondition == false){

    topEvolve = theLimit;
    bottomEvolve = bottom;

    for (unsigned long i = m; i > 0; --i){

topShrink = topEvolve*alpha;
bottomShrink = bottomEvolve*alpha;

if (jumps[i] == 0){
  ++m;
  continue;
}

else {

  unsigned long j = 0;
  bottomInnovation = 0.0;
  while (j < jumps[i]){
    bottomInnovation = (bottomInnovation + purchases[i][j] >
        theLimit - bottomShrink) ? bottomInnovation :
        bottomInnovation + purchases[i][j];
    j++;
  }

```

```
    unsigned long l = 0;
    topInnovation = 0.0;
    while (l < jumps[i]){
      topInnovation = (topInnovation + purchases[i][l] > theLimit
          - topShrink) ? topInnovation : topInnovation + purchases
          [i][l];
      l++;
    }

  }

      bottomEvolve = bottomShrink + bottomInnovation;
  topEvolve = topShrink + topInnovation;

      }

      if ( (std::fabs(topEvolve - bottomEvolve) < tolerance) || (m
          >= littleN) ){
      stopCondition = true;
      draws.push_back(bottomEvolve);
      }
      else{
      m++;
      }

    }

    thetotal += draws.back();

   }

  theAverage = thetotal / bigN;

  std::cout.setf(std::ios_base::fixed);
  std::cout.precision(12);
  std::cout << theAverage << std::endl;

  return 0;

}
```

code/backwardChain_bcp1.cpp

Appendix C

Data

In Section 5.7, a distribution was fitted to the values of supermarket purchases for a single card-holder. The data set is given in the listing below.

```
supermarket value date time timeSinceStart
1 49.78 14/11/2011 18:06  3.357048611
1 22.59 16/11/2011 18:02  5.3540625
4 47.05 18/11/2011 13:18  7.157025463
5 117.22  09/12/2011 15:19  28.24130787
1 46.62 10/12/2011 17:30  29.29493056
1 77.21 12/12/2011 17:44  31.30425926
1 30.36 20/12/2011 17:30  39.29458334
1 14.98 21/12/2011 08:28  39.91800926
5 155.07  23/12/2011 13:32  42.12964121
1 18.25 26/12/2011 20:20  45.38244213
3 65.01 28/12/2011 11:05  46.99726852
4 18.82 30/12/2011 15:49  49.1944213
1 40.02 02/01/2012 16:33  52.22469908
1 47  03/01/2012 17:34  53.26766204
2 86.41 04/01/2012 11:36  54.01900463
1 31.33 04/01/2012 17:05  54.24733797
1 15  06/01/2012 15:29  56.18034723
2 141.12  07/01/2012 09:05  56.91377315
1 71.31 09/01/2012 18:06  59.28956019
1 63.94 10/01/2012 18:17  60.29737269
1 40.7  11/01/2012 17:57  61.28306713
3 92.57 14/01/2012 14:19  64.131875
4 30.82 15/01/2012 10:21  64.96689815
1 28.38 16/01/2012 18:00  66.28564815
```

```
1 26.91 18/01/2012 17:39 68.27072917
1 51.54 19/01/2012 12:06 69.03989584
1 51.49 21/01/2012 13:32 71.09928241
5 60.3 22/01/2012 17:11 72.25136574
1 72.36 23/01/2012 17:32 73.23138889
1 16.12 24/01/2012 08:02 73.8350463
1 37.96 24/01/2012 19:47 74.325
1 39.36 26/01/2012 16:57 76.20674769
4 44.22 27/01/2012 07:37 76.81776621
1 30.67 30/01/2012 18:18 80.26304398
1 70.62 01/02/2012 17:59 82.24965278
4 61.16 03/02/2012 15:17 84.13745371
1 64.35 03/02/2012 18:08 84.25587963
2 35.41 10/02/2012 16:00 91.16707176
4 13 11/02/2012 17:06 92.21277778
4 57.31 12/02/2012 15:15 93.13577547
2 12.06 12/02/2012 18:27 93.26934028
2 14.41 13/02/2012 17:31 94.23020834
1 20.74 14/02/2012 18:10 95.25736112
1 31.58 15/02/2012 18:15 96.26119213
2 26.81 16/02/2012 18:10 97.25778936
2 23.39 18/02/2012 15:07 99.13028936
4 7.51 19/02/2012 08:58 99.87400463
2 71.96 20/02/2012 14:35 101.1080208
2 57.16 24/02/2012 16:44 105.1981713
2 60.36 25/02/2012 15:52 106.1618287
3 44 26/02/2012 09:56 106.9142014
1 29.03 29/02/2012 17:09 110.2152778
1 33.43 01/03/2012 19:37 111.3177546
1 31.08 04/03/2012 14:04 114.0865625
5 73.51 05/03/2012 18:32 115.2731019
2 26.27 07/03/2012 17:45 117.2351273
3 29.2 10/03/2012 10:12 119.9204745
2 27.48 10/03/2012 14:15 120.0896065
2 23.35 10/03/2012 18:27 120.2643519
5 58.51 13/03/2012 18:12 123.254375
2 24.82 14/03/2012 18:17 124.2504051
2 23.29 15/03/2012 17:25 125.2145139
4 48.57 16/03/2012 14:51 126.1071412
4 29.15 18/03/2012 14:33 128.0950347
1 48.31 19/03/2012 17:58 129.2372569
```

```
2 37.32 23/03/2012 13:16  133.0415857
3 10.34 24/03/2012 12:18  134.0008912
5 31.2  24/03/2012 13:57  134.0695949
2 17.63 26/03/2012 17:25  136.2028588
1 11.37 27/03/2012 12:28  136.9970486
5 39.39 27/03/2012 17:14  137.1953704
1 16.15 28/03/2012 17:38  138.2103588
2 38.76 30/03/2012 17:51  140.2199653
4 54.51 31/03/2012 07:48  140.8009722
1 31.37 02/04/2012 17:43  143.2142245
5 49.25 03/04/2012 17:27  144.2028125
1 35.27 05/04/2012 20:35  146.2979514
2 53.33 07/04/2012 11:25  147.9160995
2 6.01  07/04/2012 13:40  148.0096065
5 77.45 09/04/2012 16:20  150.1206944
2 23.36 10/04/2012 17:26  151.1486574
3 44.51 11/04/2012 09:41  151.8258102
2 36.91 12/04/2012 19:59  153.255162
5 47.8  13/04/2012 17:02  154.132037
2 36.97 14/04/2012 16:49  155.1069329
1 15.88 15/04/2012 13:06  155.9520023
2 40.78 16/04/2012 17:57  157.1539583
5 45.02 24/04/2012 17:25  165.1317014
3 77  28/04/2012 08:32  168.7363889
5 76.56 28/04/2012 18:22  169.1460995
5 33.7  29/04/2012 12:04  169.8715162
1 46.09 30/04/2012 17:53  171.0890741
2 27.76 01/05/2012 17:46  172.0842708
4 29.24 04/05/2012 16:50  175.0454282
1 13.83 05/05/2012 12:47  175.8765278
2 33.13 05/05/2012 15:08  175.9749306
2 29.54 07/05/2012 10:09  177.7671065
1 74.17 07/05/2012 17:40  178.0802431
1 37.91 11/05/2012 07:25  181.6534375
4 52.35 11/05/2012 16:04  182.0134028
2 32.28 12/05/2012 15:57  183.0087732
2 9.17  12/05/2012 18:29  183.1143171
4 36.9  13/05/2012 12:45  183.8750463
3 52  13/05/2012 14:36  183.9522338
1 23.08 14/05/2012 12:37  184.8700347
1 55.63 14/05/2012 17:56  185.0915046
```

```
5 25.46 15/05/2012 17:05  186.0558449
5 24.97 16/05/2012 18:06  187.0774653
1 53.74 17/05/2012 18:18  188.0727431
2 41.67 18/05/2012 16:24  188.9935764
3 43.01 19/05/2012 08:23  189.6594907
4 29.21 19/05/2012 13:35  189.8762963
3 18.96 20/05/2012 06:19  190.57375
1 7.44  24/05/2012 08:36  194.6690625
2 15.06 24/05/2012 17:49  195.0530787
4 45.77 25/05/2012 15:40  195.9629745
2 20.08 25/05/2012 19:23  196.1177546
2 27.41 26/05/2012 17:45  197.0498727
4 30.01 27/05/2012 10:56  197.7657292
3 59  27/05/2012 17:05  198.0222222
2 20.35 29/05/2012 17:58  200.0593982
5 58.4  31/05/2012 18:24  202.0768519
4 29.21 02/06/2012 16:01  203.9774884
1 16.62 03/06/2012 15:09  204.9414352
2 20.63 04/06/2012 11:47  205.8008565
3 44  04/06/2012 14:06  205.897338
2 11.8  04/06/2012 17:58  206.0585764
1 21.24 06/06/2012 18:47  208.0929977
2 54.35 07/06/2012 18:48  209.0935764
4 60.01 08/06/2012 16:09  209.9827431
2 36.88 09/06/2012 15:09  210.9412616
4 50  10/06/2012 11:33  211.7913542
2 40.22 16/06/2012 16:56  218.0159144
2 32.29 17/06/2012 17:11  219.0263079
1 38.59 19/06/2012 17:50  221.0529167
4 30.05 22/06/2012 16:02  223.9781944
1 41.45 25/06/2012 17:11  227.0259607
2 22.77 26/06/2012 17:45  228.0497569
2 33.44 28/06/2012 17:59  230.0598148
1 38.72 01/07/2012 08:15  232.6538079
2 17.59 01/07/2012 15:23  232.951412
2 30.66 03/07/2012 18:04  235.0626852
3 10  05/07/2012 06:58  236.6005324
2 25.2  05/07/2012 16:40  237.0046759
2 100.5 07/07/2012 11:35  238.7925347
1 24.61 07/07/2012 13:25  238.8692014
2 18.45 08/07/2012 16:45  240.0079282
```

```
4 29.14 09/07/2012 17:34 241.0418982
2 21.93 10/07/2012 17:51 242.0541551
1 31.88 12/07/2012 16:38 244.0034838
2 14.16 12/07/2012 18:24 244.0771759
2 9.5 14/07/2012 17:24 246.0349653
2 30.66 17/07/2012 17:18 249.0308681
5 99.16 20/07/2012 15:03 251.9369097
2 28.21 24/07/2012 17:45 256.024838
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