# 18MAT31 DISCRETE MATHEMATICS

## UNIT - I

## PROPOSITIONAL CALCULUS

* Propositions
  - Simple (Atomic)
  - Compound (Molecular)

* Logical Connectives
  - Conjuction ($\wedge$)
  - disjunction ($\vee$)
  - Conditional ($\rightarrow$)
  - Biconditional ($\leftrightarrow$)
  - Negation ($\neg$ or $\sim$)

* Truth tables
* Tautologies and Contradictions
* Converse, Inverse and Contrapositive
* Logical equivalences and implications
  - Truth table
  - Laws
* Normal forms
  - PCNF
  - PDNF
* Rules of Inference
* Argument
* Validity of argument.

## Propositions:

A Proposition is a statement that is either true or false but not both. These values - true or false, are called truth values.

Ex:-

(i) Chennai is the capital of Tamil Nadu (True)

(ii) $3+2=4$ (False)

Note:

* True is denoted by T (or) 1
* false is denoted by F (or) 0
* Propositions are denoted using alphabets (Capital or Small)

* Exclamatory, Interrogative, Commands are not propositions.

## Primitive Propositions:-

A Proposition is said to be primitive if it cannot be broken down into simpler propositions.

## Compound propositions:

A Proposition which is composed of atomic propositions connected by logical connectives are called compound propositions.

## LOGICAL CONNECTIVES:-

**Conjuction:** Let p and q be two propositions. The proposition 'p and q' is called the conjuction of p and q and is denoted by '$p \wedge q$'.

Truth table:

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

Disjunction:

Let $p$ and $q$ be two propositions. The proposition '$p$ or $q$' is called the disjunction of $p$ and $q$ and is denoted by '$p \vee q$'.

Truth table:

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

Conditional: If $p$ and $q$ are two propositions then the compound proposition 'if $p$ then $q$' denoted by '$p \rightarrow q$' is called conditional proposition.

Truth table:

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

Biconditional: If $p$ and $q$ are two propositions then the proposition '$p$ if and only if $q$' denoted by '$p \leftrightarrow q$' is called biconditional proposition.

Truth table:

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

Negation: The negation of a statement is generally formed by introducing the word 'not' at a proper place in the statement.

If $P$ denotes a statement then negation of $P$ is denoted by '$\neg P$' or '$\sim p$'

Truth table:

| $p$ | $q$ |
|---|---|
| T | F |
| F | T |

In general,

| $p$ | $q$ | $\neg p$ | $\neg q$ | $p \wedge q$ | $p \vee q$ | $p \rightarrow q$ | $p \leftrightarrow q$ |
|---|---|---|---|---|---|---|---|
| T | T | F | F | T | T | T | T |
| T | F | F | T | F | T | F | F |
| F | T | T | F | F | T | T | F |
| F | F | T | T | F | F | T | T |

Precendence of operators:-

The connectives $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$ have the following order of precedence.

| | |
|---|---|
| I | $\neg$ |
| II | $\wedge$ |
| III | $\vee$ |
| IV | $\rightarrow, \leftrightarrow$ |

For example,

$\neg P \vee Q$ means $(\neg P) \vee Q$

Exercises:-

Find the truth table of the following:

① $(p \wedge q) \rightarrow (p \vee q)$

② $\neg(p \wedge q) \vee \neg(q \leftrightarrow p)$

③ Determine the truth value of "If $30 \div 10 = 3$ then $7 \times 5 = 35$"

④ Consider the following $p$: $x$ is even $q$: $x$ is divisible by 2. State the following in words: $\neg p$, $\neg p \wedge \neg q$, $p \rightarrow q$, $p \leftrightarrow q$.

Tautology:- A proposition which is always true is called a tautology. ie) it contains only 'TRUE' in the last cloumn of its truth table irrespective of the truth values of its atomic propositions.

Eg:- $p \vee \neg p$ is a tautology.

| $p$ | $\neg p$ | $p \vee \neg p$ |
|---|---|---|
| T | F | T |
| F | T | T |

Contradiction:- A proposition which is always false is called a contradiction. (ie) it contains only 'FALSE' in the last column of its truth table

Eg:- $q \wedge \neg q$ is a contradiction.

| $q$ | $\neg q$ | $q \wedge \neg q$ |
|---|---|---|
| T | F | F |
| F | T | F |

Contigency:- neither tautology nor contradiction.

① Check whether the Compound proposition $\neg q \wedge (p \to q) \to \neg p$ is tautology or Contradiction.

| $p$ | $q$ | $\neg q$ | $p \to q$ | $\neg q \wedge (p \to q)$ | $\neg p$ | $\neg q \wedge (p \to q) \to \neg p$ |
|---|---|---|---|---|---|---|
| T | T | F | T | F | F | T |
| T | F | T | F | F | F | T |
| F | T | F | T | F | T | T |
| F | F | T | T | T | T | T |

$\therefore \neg q \wedge (p \to q) \to \neg p$ is tautology.

② Determine the proposition $\neg(q \to r) \wedge r \wedge (p \to q)$ is tautology or Contradiction.

| $p$ | $q$ | $r$ | $q \to r$ | $\neg(q \to r)$ | $\neg(q \to r) \wedge r$ | $p \to q$ | $\neg(q \to r) \wedge r \wedge (p \to q)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | F | F | T | F |
| T | T | F | F | T | F | T | F |
| T | F | T | T | F | F | F | F |
| F | T | T | T | F | F | T | F |
| T | F | F | T | F | F | F | F |
| F | T | F | F | T | F | T | F |
| F | F | T | T | F | F | T | F |
| F | F | F | T | F | F | T | F |

$\therefore \neg(q \to r) \wedge r \wedge (p \to q)$ is Contradiction.

③ ~~Prove~~ Verify that $(p \to q) \to r$ is tautology or not.

④. Show that $q \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$ is tautology.

⑤ Show that $(q \to p)$ and $(\sim p \wedge q)$ is a Contradiction (and denotes '$\wedge$')

Converse, Inverse, Contrapositive :-

The Converse, inverse and Contrapositive can be associated with the conditional propositions.

Consider the Conditional proposition $p \to q$

(i) Its Converse is $q \to p$

(ii) Its inverse is $\neg p \to \neg q$

(iii) Its Contrapositive is $\neg q \to \neg p$

Eg:-

If a triangle ABC is a right angled triangle then $|AB|^2 + |BC|^2 = |AC|^2$

Converse:

If $|AB|^2 + |BC|^2 = |AC|^2$ then the triangle is a right angled triangle

Inverse:

If ABC is not a right angled triangle then $|AB|^2 + |BC|^2 \neq |AC|^2$

Contrapositive:-

If $|AB|^2 + |BC|^2 \neq |AC|^2$ then the triangle ABC is not a right angled triangle.

① Write the Converse of the Conditional statement "The home team wins whenever it is raining"

② Write down the Converse and Contrapositive of the following statement "If the flood destroys my house or the fire destroys my house, then my insurance Company will Pay me"

Hint:- Given $(p \vee q) \to r$ find $r \to (p \vee q)$ and $\neg r \to \neg(p \vee q)$

③ Write the Converse and Contrapositive of the following statement. "Team India wins whenever Dhoni is a Captain"

④ Write the Contrapositive "If it is raining, then I get wet"

## Logical equivalences and implications :-

**Logically Equivalent :-**

Two propositions are said to be logically equivalent if they have identical truth values.

(or)

Two propositions A and B are said to be logically equivalent iff $A \leftrightarrow B$ is a tautology and it is denoted by

$$A \equiv B \text{ (or) } A \Leftrightarrow B$$

**Logical Implications :-**

Two propositions A and B are said to be logical Implications (logically implies) iff $A \rightarrow B$ is a tautology and it is denoted by

$$A \Rightarrow B$$

① Prove that $p \rightarrow q \equiv \neg p \vee q$

Sol: It's enough to prove $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$ is a tautology.

| $p$ | $q$ | $p \rightarrow q$ | $\neg p$ | $\neg p \vee q$ | $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$ |
|---|---|---|---|---|---|
| T | T | T | F | T | T |
| T | F | F | F | F | T |
| F | T | T | T | T | T |
| F | F | T | T | T | T |

$\therefore (p \rightarrow q) \leftrightarrow (\neg p \vee q)$ is a tautology

$\rightarrow \quad p \rightarrow q \Leftrightarrow \neg p \vee q$

② . prove that $p \Rightarrow p \vee q$

It's enough to prove $p \to (p \vee q)$ is a tautology.

| $p$ | $q$ | $p \vee q$ | $p \to p \vee q$ |
|---|---|---|---|
| T | T | T | T |
| T | F | T | T |
| F | T | T | T |
| F | F | F | T |

$\because$ $p \to p \vee q$ is a tautology.

$\therefore$ $p \Rightarrow p \vee q$.

③ . show that $\sim(p \leftrightarrow q)$ and $(p \wedge \sim q) \vee (\sim p \wedge q)$ are logically equivalent.

Sol:- To prove $\underbrace{\sim(p \leftrightarrow q)}_{A} \leftrightarrow \underbrace{(p \wedge \sim q) \vee (\sim p \wedge q)}_{B}$ is a tautology.

| $p$ | $q$ | $p \leftrightarrow q$ | $\sim(p \leftrightarrow q)$ | $\sim q$ | $p \wedge \sim q$ | $\sim p$ | $\sim p \wedge q$ | $B$ | $A \leftrightarrow B$ |
|---|---|---|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F | F | F | T |
| T | F | F | T | T | T | F | F | T | T |
| F | T | F | T | F | F | T | T | T | T |
| F | F | T | F | T | F | T | F | F | T |

$\therefore$ $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (\sim p \wedge q)$

④ show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

⑤ . S.T $p \to (q \to r) \Leftrightarrow q \to (p \vee r)$

## Law of Algebras :-

1. Commutative law :- $p \vee q \equiv q \vee p$

$p \wedge q \equiv q \wedge p$

2. Associative law :-

$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

3. Distributive law :-

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

4. Demorgan's law :-

$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

5. Compliment law :-

$p \vee \neg p \Leftrightarrow T \; ; \; p \wedge \neg p \Leftrightarrow F$

6. Double Negation law :-

$\neg(\neg p) \Leftrightarrow p$

7. Identity law :-

$p \vee F \Leftrightarrow p \; ; \; p \wedge T \Leftrightarrow p$

8. Dominant law :-

$p \vee T \Leftrightarrow T \; ; \; p \wedge F \Leftrightarrow F$

9. Idempotent law :-

$p \vee p \Leftrightarrow p \; ; \; p \wedge p \Leftrightarrow p$

10. Absorption law :-

$p \vee (p \wedge q) \Leftrightarrow p \; ; \; p \wedge (p \vee q) \Leftrightarrow p$

11. Implication law :-

$p \rightarrow q \Leftrightarrow \neg p \vee q$

12. Equivalence law:-

$$p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$$

$$p \leftrightarrow q \Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$$

$$p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$$

12. Equivalence law:-

$$p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$$
$$p \leftrightarrow q \Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$$
$$p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$$

① . Show that $(p \rightarrow q) \wedge (\neg p \rightarrow q) \Leftrightarrow q$ without using the truth table.

Sol:

$(p \rightarrow q) \wedge (\neg p \rightarrow q) \Leftrightarrow (\neg p \vee q) \wedge (\neg(\neg p) \vee q)$ (by Implication Law)

$\Leftrightarrow (\neg p \vee q) \wedge (p \vee q)$ (by double Negation law)

$\Leftrightarrow (q \vee \neg p) \wedge (q \vee p)$ (by Commutative law)

$\Leftrightarrow q \vee (\neg p \wedge p)$ (by distributive law)

$\Leftrightarrow q \vee F$ (by Compliment law)

$\Leftrightarrow q$ (by Identity law)

② . Show that $(p \vee q) \wedge \neg p \Leftrightarrow \neg p \wedge q$

Sol:

$(p \vee q) \wedge \neg p \Leftrightarrow (p \wedge \neg p) \vee (q \wedge \neg p)$ (by distributive law)

$\Leftrightarrow F \vee (q \wedge \neg p)$ (by Compliment law)

$\Leftrightarrow (q \wedge \neg p) \vee F$ (by Commutative law)

$\Leftrightarrow q \wedge \neg p$ (by Identity law)

$\Leftrightarrow \neg p \wedge q$ (by Commutative law)

③. S.T $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p) \Leftrightarrow (p \vee q) \to (p \wedge q)$

Sol:- To prove (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii)

$p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$ (by equivalence law)

$\Leftrightarrow \underbrace{(\neg p \vee q)}_{A} \wedge \underbrace{(\neg q}_{B} \vee \underbrace{p)}_{C}$ (by implication law)

$\Leftrightarrow ((\neg p \vee q) \wedge \neg q) \vee (\neg p \vee q \wedge p)$ (by distributive law)

$\Leftrightarrow (\neg p \wedge \neg q) \vee (q \wedge \neg q) \vee (\neg p \wedge p) \vee (q \wedge p)$ (by distributive law)

$\Leftrightarrow (\neg p \wedge \neg q) \vee F \vee F \vee (q \wedge p)$ (by compliment law)

$\Leftrightarrow (\neg p \wedge \neg q) \vee (q \wedge p)$ (by Identity law)

$\Leftrightarrow \neg(p \vee q) \vee (p \wedge q)$ (by Demorgan's law and commutative law)

$\Leftrightarrow (p \vee q) \to (p \wedge q)$ (by Implication law)

④. P.T $\neg(p \leftrightarrow q) \Leftrightarrow (p \vee q) \wedge \neg(p \wedge q) \Leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q)$

Sol:-

$\neg(p \leftrightarrow q) \Leftrightarrow \neg((p \to q) \wedge (q \to p))$ (by equivalence law)

$\Leftrightarrow \neg(p \to q) \vee \neg(q \to p)$ (by demorgan's law)

$\Leftrightarrow \neg(\neg p \vee q) \vee \neg(\neg q \vee p)$ (by Implication law)

$\Leftrightarrow (p \wedge \neg q) \vee (q \wedge \neg p)$ (by Demorgan's law)

$\Leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q)$ (by Commutative law)

(i) $\Leftrightarrow$ (iii)

$(p \vee q) \wedge \neg(p \wedge q) \Leftrightarrow (p \vee q) \wedge (\neg p \vee \neg q)$ (by Demorgan's law)

$\Leftrightarrow ((p \vee q) \wedge \neg p) \vee ((p \vee q) \wedge \neg q)$ (by Distributive law)

$\Leftrightarrow (p \wedge \neg p) \vee (q \wedge \neg p) \vee (p \wedge \neg q) \vee (q \wedge \neg q)$ //

$\Leftrightarrow F \vee (q \wedge \neg p) \vee \cancel{F \vee (q \wedge \neg p)} (p \wedge \neg q) \vee F$

(by Compliment law)

$\Leftrightarrow (q \wedge \neg p) \vee (p \wedge \neg q)$ (by Identity law)

$\Leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q)$ (by Commutative law)

(ii) $\Leftrightarrow$ (iii)

Since (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iii) $\therefore$ (i) $\Leftrightarrow$ (ii)

(5). without using truth table prove that $p \wedge q \Rightarrow p$

Sol:

To prove $(p \wedge q) \rightarrow p$ is a tautology.

$(p \wedge q) \rightarrow p \Leftrightarrow \neg(p \wedge q) \vee p$ (by Implication law)

$\Leftrightarrow (\neg p \vee \neg q) \vee p$ (by Demorgan's law)

$\Leftrightarrow (\neg p \vee p) \vee \neg q$ (by Associative law)

$\Leftrightarrow T \vee \neg q$ (by Compliment law)

$\Leftrightarrow T$ (by Dominant law)

$\therefore p \wedge q \Rightarrow p$.

⑥. S.T $p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$

Sol:-

$p \to (q \to r) \Leftrightarrow \neg p \vee (q \to r)$ (by Implication law)

$\Leftrightarrow \neg p \vee (\neg q \vee r)$ //

$\Leftrightarrow (\neg p \vee \neg q) \vee r$ (by associative law)

$\Leftrightarrow \neg(p \wedge q) \vee r$ (by Demorgan's law)

$\Leftrightarrow (p \wedge q) \to r$ (by Implication law)

~~⑦. $(p \wedge q)$ $(p \to q) \wedge (\neg p \to q) \Leftrightarrow q$~~

~~Sol:-~~

~~To prove $(p \to q) \wedge (\neg p \to q) \Leftrightarrow$~~

HW

①. S.T $(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \Leftrightarrow p \wedge q$

②. S.T $\neg(p \vee (\neg p \wedge q)) \Leftrightarrow \neg p \wedge \neg q$

③. P.T $p \wedge q \Rightarrow p \vee q$

## Normal Forms:

A well formed formula is denoted by $A(p_1, p_2 \ldots p_n)$ where $p_1, p_2 \ldots p_n$ are atomic propositions. Hence a well formed formula is a compound proposition containing atomic propositions connected by connectives.

Any well formed formula can be reduced to any one of the following normal forms.

(i) Disjunctive Normal Form (DNF)
(ii) Conjunctive Normal Form (CNF)
(iii) Principle Disjunctive Normal Form (PDNF)
(iv) Principle Conjunctive Normal Form (PCNF)

## Disjunctive Normal Form (DNF)

**Elementary product:** A product of variables and their negations is called elementary product.

If $p$ and $q$ are two variables then $p \wedge q$, $p \wedge \neg q$, $q \wedge \neg p$ are some elementary product. $p \wedge \neg p$, $q \wedge \neg q$, $(p \wedge \neg q \wedge q)$

**DNF:** A formula which is equivalent to a given formula and which consists of a sum of elementary products is called a DNF.

DNF = Sum of elementary products

$$= \begin{pmatrix}\text{elementary}\\ \text{product}\end{pmatrix} \vee \begin{pmatrix}\text{elementary}\\ \text{product}\end{pmatrix} \vee \ldots \vee \begin{pmatrix}\text{elementary}\\ \text{product}\end{pmatrix}$$

## Conjuctive Normal Form: (CNF)

**Elementary Sum:** Sum of variables and their negations is called elementary sum

Eg:- $p \vee q$, $p \vee \neg q$, $q \vee \neg p$, $\neg p \vee \neg q$ are some examples of elementary sum in two variables. $(p \vee \neg q \vee q)$, $q \vee \neg q$, $p \vee \neg p$

**CNF:**

A formula which is equivalent to a given formula and which consists of a product of elementary sum is called as CNF

CNF = product of elementary sum

$$= \begin{pmatrix}\text{elementary}\\ \text{sum}\end{pmatrix} \wedge \begin{pmatrix}\text{elementary}\\ \text{sum}\end{pmatrix} \wedge \begin{pmatrix}\text{elementary}\\ \text{sum}\end{pmatrix} \cdots$$

① Find the disjunctive normal form of $p \wedge (p \rightarrow q)$

**Sol:**

$p \wedge (p \rightarrow q) \Leftrightarrow p \wedge (\neg p \vee q)$ (Implication law)

$\Leftrightarrow (p \wedge \neg p) \vee (p \wedge q)$ (Distributive law)

which is DNF

② Find the CNF and DNF of $q \rightarrow (q \rightarrow p)$

**Sol:**

$q \rightarrow (q \rightarrow p) \Leftrightarrow q \rightarrow (\neg q \vee p)$ Implication law

$\Leftrightarrow \neg q \vee (\neg q \vee p)$ "

$\Leftrightarrow (\neg q \vee \neg q) \vee p$ (Associative law)

$\Leftrightarrow \neg q \vee p$ (Idempotent law)

~~which is DNF~~

$$q \to (q \to p) \Leftrightarrow \neg q \vee p$$
$$\Leftrightarrow \neg q \vee (p \wedge p)$$
$$\Leftrightarrow (\neg q \vee p) \wedge (\neg q \vee p)$$

which is CNF

(3). Find the DNF of $(p \to \neg q) \to (\neg p \wedge r)$

Sol:: $(p \to \neg q) \to (\neg p \wedge r) \Leftrightarrow (\neg p \vee \neg q) \to (\neg p \wedge r)$ (Implication law)

$\Leftrightarrow \neg(\neg p \vee \neg q) \vee (\neg p \wedge r)$ "

$\Leftrightarrow (p \wedge q) \vee (\neg p \wedge r)$ (Demorgan's law)

DNF

Principle Disjunctive Normal Form:- (PDNF)

Minterms:- Let $p$ and $q$ be two variables. The product of the variables and their negations such that none of the terms contain a variable and its negation are called minterms

For $p$ and $q$ the minterms are $p \wedge q$, $p \wedge \neg q$, $\neg p \wedge q$, $\neg p \wedge \neg q$

For three variables there will be 8 minterms.

PDNF:: A formula which is equivalent to a given formula consisting of sum of minterms only is called PDNF.

## Principle Conjuctive Normal Form: (PCNF)

**Maxterms:** Let $p$ and $q$ be two variables. The sum of the variables and their negations such that none of the terms contain a variable and its negation.

For $p$ and $q$ the max terms are $p \vee q$, $p \vee \neg q$, $\neg p \vee q$, $\neg p \vee \neg q$

For $p, q$ and $r$ the max terms are

$p \vee q \vee r$, $p \vee q \vee \neg r$, $p \vee \neg q \vee r$, $\neg p \vee q \vee r$, $\neg p \vee \neg q \vee r$, $\neg p \vee q \vee \neg r$, $p \vee \neg q \vee \neg r$, $\neg p \vee \neg q \vee \neg r$.

**PCNF:** A formula which is equivalent to the given formula and consisting of product of maxterms only is called PCNF.

① Obtain PDNF and PCNF of $p \to q$

| $p$ | $q$ | $p \to q$ | minterms | Maxterms |
|---|---|---|---|---|
| T | T | T | $p \wedge q$ | – |
| T | F | F | – | $\neg p \vee q$ |
| F | T | T | $\neg p \wedge q$ | – |
| F | F | T | $\neg p \wedge \neg q$ | – |

PDNF is $(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$

PCNF is $\neg p \vee q$.

(OR)

Let $S \Leftrightarrow p \to q$

$\Leftrightarrow \neg p \vee q$ (PCNF)

$\neg S \Leftrightarrow (p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q)$

$S \Leftrightarrow \neg(\neg S)$

$\Leftrightarrow \neg((p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q))$

$\Leftrightarrow (\neg p \wedge \neg q) \vee (\neg p \wedge q) \vee (p \wedge q)$ (PDNF).

② Obtain PCNF of $\neg(p \to q)$

| $p$ | $q$ | $p \to q$ | $\neg(p \to q)$ | Max terms |
|---|---|---|---|---|
| T | T | T | F | $\neg p \vee \neg q$ |
| T | F | F | T | – |
| F | T | T | F | $p \vee \neg q$ |
| F | F | T | F | $p \vee q$ |

PCNF is $(\neg p \vee \neg q) \wedge (p \vee \neg q) \wedge (p \vee q)$

(OR)

$S \Leftrightarrow \neg(p \to q)$

$\Leftrightarrow \neg(\neg p \vee q)$

$\Leftrightarrow p \wedge \neg q$

$\Leftrightarrow (p \vee F) \wedge (\neg q \vee F)$

$$\Leftrightarrow (p \vee (q \wedge \neg q)) \wedge (\neg q \vee (p \wedge \neg p))$$

$$\Leftrightarrow (p \vee q) \wedge (p \wedge \neg q) \wedge (\neg q \vee p) \wedge (\neg q \wedge \neg p)$$

$$\Leftrightarrow (p \vee q) \wedge (p \wedge \neg q) \wedge (\neg q \wedge \neg p).$$

③. Obtain the PCNF and PDNF of $(\neg p \rightarrow r) \wedge (q \leftrightarrow p)$

Sol:

Let $S \Leftrightarrow (\neg p \rightarrow r) \wedge (q \leftrightarrow p)$

| $p$ | $q$ | $r$ | $\neg p$ | $\neg p \rightarrow r$ | $q \leftrightarrow p$ | $S$ | Minterms | Maxterms |
|---|---|---|---|---|---|---|---|---|
| T | T | T | F | T | T | T | $p \wedge q \wedge r$ | – |
| T | T | F | F | T | T | T | $p \wedge q \wedge \neg r$ | – |
| T | F | T | F | T | F | F | – | $\neg p \vee q \vee \neg r$ |
| F | T | T | T | T | F | F | – | $p \vee \neg q \vee \neg r$ |
| T | F | F | F | T | F | F | – | $\neg p \vee q \vee r$ |
| F | T | F | T | F | F | F | – | $p \vee \neg q \vee r$ |
| F | F | T | T | T | T | T | $\neg p \wedge \neg q \wedge r$ | – |
| F | F | F | T | F | T | F | – | $p \vee q \vee r$ |

PDNF is $(p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$

PCNF is $(\neg p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee r)$

(OR)

$$S \Leftrightarrow (\neg p \rightarrow r) \wedge (q \leftrightarrow p)$$

$$\Leftrightarrow (\neg p \rightarrow r) \wedge (q \rightarrow p) \wedge (p \rightarrow q)$$

$$\Leftrightarrow (\neg(\neg p) \vee r) \wedge (\neg q \vee p) \wedge (\neg p \vee q)$$

$\Leftrightarrow (p \vee r) \wedge (\neg q \vee p) \wedge (\neg p \vee q)$

$\Leftrightarrow (p \vee r \vee F) \wedge (\neg q \vee p \vee F) \wedge (\neg p \vee q \vee F)$

$\Leftrightarrow (p \vee r \vee (q \wedge \neg q)) \wedge (\neg q \vee p \vee (r \wedge \neg r)) \wedge (\neg p \vee q \vee (r \wedge \neg r))$

$\Leftrightarrow (p \vee r \vee q) \wedge (p \vee r \vee \neg q) \wedge (\neg q \vee p \vee r) \wedge (\neg q \vee p \vee \neg r)$
$\wedge (\neg p \vee q \vee r) \wedge (\neg p \vee q \vee \neg r)$

$\Leftrightarrow (p \vee q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee q \vee \neg r)$
(PCNF)

$\neg S \Leftrightarrow (p \vee q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$

$S \Leftrightarrow \neg(\neg S)$
$\Leftrightarrow (\neg p \wedge \neg q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge q \wedge r)$
(PDNF)

(4). Find PDNF and PCNF of $(p \to (q \wedge r)) \wedge (\neg p \to (\neg q \wedge \neg r))$

(5) Find PDNF and PCNF of $p \vee (\neg p \to (q \vee (\neg q \to r)))$

(6) Obtain PDNF of $(p \wedge q) \vee (p \wedge r) \vee (q \wedge r)$

## Inference theory:-

Inference theory is concerned with the inferring of a conclusion from certain hypothesis or basic assumptions, called premises, by applying certain principles of reasoning, called rules of inference.

In inference theory we check the validity of the arguments.

## Argument:-

An argument is an assertion that a given set of propositions yields another proposition called the conclusion. Such an argument is denoted by

$$p_1, p_2, \ldots p_n \Rightarrow q \quad (\text{or})$$

$$p_1 \wedge p_2 \wedge \ldots \wedge p_n \Rightarrow q$$

Here $p_1, p_2, \ldots p_n$ are called premises, $q$ is conclusion.

## Valid arguments:-

An argument $p_1, p_2 \ldots p_n$ is said to be valid if $q$ is true whenever all the premises $p_1, p_2, \ldots p_n$ are true.

## Methods to check validity of the arguments:-

The following methods are ~~check~~ used to check the validity of the arguments

(i) Direct Proof

(ii) Indirect Proof.

**Direct proof:-** A direct proof is the one in which the truth of the premises are shown to directly imply the truth of the conclusion.

Indirect proof:

In indirect proof we show that the assumption of the negated conclusion results in a contradiction.

Inconsistent:

A set of premises (formulas) $p_1, p_2, \ldots p_n$ is said to be inconsistent, if their conjuction implies a contradiction.

A set of premises is said to be consistent, if it is not inconsistent.

Basic rules of inference:-

Rule P:- A premise may be introduced at any step in the derivation.

Rule T:- A formula S may be introduced in the derivation, if S is tautologically implied by one or more preceding formulas in the derivation.

Rule CP (or) Rule of Conditional Proof:-

If we can derive S from R and a set of premises then we can derive $R \to S$ from the set of premises alone.

Some valid inferences: Rules in tautological form:-

1. $p \wedge q \Rightarrow p$, $p \wedge q \Rightarrow q$ (simplification)
2. $p \Rightarrow p \vee q$, $q \Rightarrow p \vee q$ (addition)
3. $p \wedge (p \to q) \Rightarrow q$ (modus ponens)

4. $\neg q \wedge (p \to q) \Rightarrow \neg p$ (modus tollens)

5. $(p \to q) \wedge (q \to r) \Rightarrow p \to r$ (~~disjunctive~~ hypothetical syllogism or Chain rule)

6. $\neg p \wedge (p \vee q) \Rightarrow q$ (disjunctive syllogism)

7. $(p \vee q) \wedge (\neg p \vee r) \Rightarrow q \vee r$ (Resolution)

8. $(p \vee q) \wedge (p \to r) \wedge (q \to r) \Rightarrow r$ (Dilemma)

9. $(p) \wedge (q) \Rightarrow p \wedge q$ (Conjuction)

① Prove that $R \wedge (P \vee Q)$ is a valid Conclusion from the premises $P \vee Q$, $Q \to R$, $P \to M$ and $\neg M$

Sol:

To prove $P \vee Q, Q \to R, P \to M, \neg M \Rightarrow R \wedge (P \vee Q)$

| Step | premises | Reason |
|---|---|---|
| 1. | $\neg M$ | Rule P |
| 2. | $P \to M$ | Rule P |
| 3. | $\neg P$ | Rule T, step 1, 2 modus tollens |
| 4. | $P \vee Q$ | Rule P |
| 5. | $Q$ | Rule T, Step 3, 4 disjunctive syllogism |
| 6. | $Q \to R$ | Rule P |
| 7. | $R$ | Rule T, step 5, 6 modus ponens |
| 8. | $R \wedge (P \vee Q)$ | Rule T, Step 4, 7 Conjuction |

## Equivalences Involving Conditionals:-

1. $p \to q \Leftrightarrow \neg p \vee q$ (Implication law)
2. $p \to q \Leftrightarrow \neg q \to \neg p$ (Contrapositive)
3. $(p \to q) \wedge (p \to r) \Leftrightarrow p \to (q \wedge r)$
4. $(p \to r) \wedge (q \to r) \Leftrightarrow (p \vee q) \to r$
5. $(p \to q) \vee (p \to r) \Leftrightarrow p \to (q \vee r)$
6. $(p \to r) \vee (q \to r) \Leftrightarrow (p \wedge q) \to r$

## Equivalences Involving biconditionals:-

1. $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$
2. $p \leftrightarrow q \Leftrightarrow \neg p \leftrightarrow \neg q$
3. $p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$

(2). Derive $R \vee S$ follows logically from $C \vee D$, $(C \vee D) \to \neg H$, $\neg H \to (A \wedge \neg B)$ and $(A \wedge \neg B) \to (R \vee S)$

Sol:- To prove $C \vee D, (C \vee D) \to \neg H, \neg H \to (A \wedge \neg B), (A \wedge \neg B) \to (R \vee S) \Rightarrow R \vee S$

| Step | Statement | Reason. |
|---|---|---|
| 1. | $C \vee D$ | Rule P |
| 2. | $(C \vee D) \to \neg H$ | Rule P |
| 3. | $\neg H$ | Rule T, 1,2 Modus ponens |
| 4. | $\neg H \to (A \wedge \neg B)$ | Rule P |
| 5. | $A \wedge \neg B$ | Rule T, 3,4 Modus ponens |
| 6. | $(A \wedge \neg B) \to (R \vee S)$ | Rule P |
| 7. | $R \vee S$ | Rule T, 5,6 Modus ponens. |

③. Show that $t \wedge s$ can be derived from the premises $p \vee (t \wedge s)$, $p \to q$, $q \to \neg r$ and $r$.

Sol:

To prove $p \vee (t \wedge s), p \to q, q \to \neg r, r \Rightarrow t \wedge s$

| Step | Statement | Reason |
|---|---|---|
| 1. | $p \to q$ | Rule P |
| 2. | $q \to \neg r$ | Rule P |
| 3. | $p \to \neg r$ | Rule T 1, 2 Chain rule |
| 4. | $r \to \neg p$ | Rule T, 3 Equivalence law |
| 5. | $r$ | Rule P |
| 6. | $\neg p$ | Rule T, 4, 5 Modus ponens |
| 7. | $p \vee (t \wedge s)$ | Rule P |
| 8. | $t \wedge s$ | Rule T, 6, 7 disjunctive syllogism. |

④. Prove by direct method $p \vee q, p \to r, q \to s \Rightarrow s \vee r$

| Step | Statement | Reason |
|---|---|---|
| 1. | $p \vee q$ | Rule P |
| 2. | $p \to r$ | Rule P |
| 3. | $q \to s$ | Rule P |
| 4. | $\neg p \to q$ | Rule T, 1, implication |
| 5. | $\neg r \to \neg p$ | Rule T, 2 Contrapositive |
| 6. | $\neg s \to \neg q$ | Rule T, 3 Contrapositive |
| 7. | $\neg p \to s$ | Rule T, 3, 4 Chain rule |
| 8. | $\neg s \to p$ | Rule T, 7 Contrapositive |
| 9. | $\neg s \to r$ | Rule T, 8, 2 Chain rule |

| | | |
|---|---|---|
| 10. | $\neg r \to s$ | Rule T, 9 Contrapositive |
| 11. | $\neg(\neg r) \vee s$ | Rule T, 10 Implication |
| 12. | $r \vee s$ | Rule T, 11, double Negation law. |

(5). Prove that $p \to q$, $q \to r$, $s \to \neg r$ and $p \wedge s$ are inconsistent.

Sol:

| Step | Statement | Reason |
|---|---|---|
| 1. | $p \to q$ | Rule P |
| 2. | $q \to r$ | Rule P |
| 3. | $p \to r$ | Rule T, 1,2 Chain rule. |
| 4. | $s \to \neg r$ | Rule P |
| 5. | $r \to \neg s$ | Rule T, 4 Contrapositive |
| 6. | $p \to \neg s$ | Rule T, 3,5 Chain rule |
| 7. | $q \to \neg s$ | Rule T, 2,5 Chain rule |
| 8. | $\neg p \vee \neg s$ | Rule T, 6 Implication law |
| 9. | $\neg(p \wedge s)$ | Rule T, 8 Demorgan's law |
| 10. | $p \wedge s$ | Rule P |
| 11. | $(p \wedge s) \wedge \neg(p \wedge s)$ | Rule T, 9, 10 Conjuction |
| 12. | F | Rule T, 11 Negation law. |

(6). Prove that $p \to q$, $(q \vee r) \to s$, $s \to \neg p$, $p \wedge \neg r$ are inconsistent.

Sol:

| Step | Statement | Reason. |
|---|---|---|
| 1. | $(q \vee r) \to s$ | Rule P |
| 2. | $s \to \neg p$ | Rule P |
| 3. | $(q \vee r) \to \neg p$ | Rule T, 1,2 Chain rule |

| | | |
|---|---|---|
| 4. | $p \to \neg(q \vee r)$ | Rule T, 3 Contrapositive |
| 5. | $p \to q$ | Rule P |
| 6. | $p \wedge \neg r$ | Rule P |
| 7. | $p$ | Rule T, 6, simplification |
| 8. | $\neg r$ | Rule T, 6 " |
| 9. | $q$ | Rule T, 5, 7 Modus ponens |
| 10. | $\neg(q \vee r)$ | Rule T, 4, 7 modus ponens |
| 11. | $\neg q \wedge \neg r$ | Rule T, 10 Demorgan's law |
| 12. | $\neg q$ | Rule T, 11 Simplification |
| 13. | $q \wedge \neg q$ | Rule T, 9, 12 Conjuction |
| 14. | $F$ | Rule T, 13, Compliment law. |

HW

① S.T $(a \to b) \wedge (a \to c), \neg(b \wedge c), d \vee a \Rightarrow d$

② S.T $a \vee b$ follow logically from the premises $p \vee q, (p \vee q) \to \neg r, \neg r \to (s \wedge \neg t)$ and $(s \wedge \neg t) \to (a \vee b)$

③ Prove that $(p \wedge q) \to r, \neg r \vee s, \neg s \Rightarrow \neg p \vee \neg q$

④ Show that the following premises are inconsistent.

If Raja misses many classes then he fails in the final examination. If Raja fails in the final examination, then he is ~~not~~ uneducated. If Raja reads a lot of books, then he is not uneducated. Raja misses many classes and reads a lot of books.

## Indirect Methods:-

① Use the indirect method to show that $r \to \neg q$, $r \vee s$, $s \to \neg q$, $p \to q \Rightarrow \neg p$

Sol:- To use the indirect method, we will include $\neg(\neg p) \Leftrightarrow p$ as an additional premises and Prove a Contradiction.

| Step | Statement | Reason |
|---|---|---|
| 1. | $p$ | Rule P (additional premises) |
| 2. | $p \to q$ | Rule P |
| 3. | $q$ | Rule T, 1, 2 modus ponens |
| 4. | $r \to \neg q$ | Rule P |
| 5. | $s \to \neg q$ | Rule P |
| 6. | $(r \vee s) \to \neg q$ | Rule T 4, 5, Equivalence law |
| 7. | $r \vee s$ | Rule P |
| 8. | $\neg q$ | Rule T, 6, 7 modus ponens |
| 9. | $q \wedge \neg q$ | Rule T, 3, 8 Conjuction |
| 10. | $F$ | Rule T, 9 Compliment law |

② Show that $b$ can be derived from the premises $a \to b$, $c \to b$, $d \to (a \vee c)$, $d$ by the indirect method

Sol:- Let us include additional premises $\neg b$ and prove a Contradiction.

| Step | Statement | Reason |
|---|---|---|
| 1. | $\neg b$ | Rule P (additional) |
| 2. | $a \to b$ | Rule P |
| 3. | $c \to b$ | Rule P |
| 4. | $(a \vee c) \to b$ | Rule T, 2, 3, Equivalence law |

| | | |
|---|---|---|
| 5. | $d \to (a \vee c)$ | Rule P |
| 6. | $d \to b$ | Rule T 4, 5 Chain rule |
| 7. | $d$ | Rule P |
| 8. | $b$ | Rule T, 6, 7 modus ponens |
| 9. | $b \wedge \neg b$ | Rule T, 1, 9 Conjuction |
| 10. | $F$ | Rule T, 9 Compliment law. |

(3). Using indirect method of proof derive $p \to \neg s$ from the premises $p \to (q \vee r)$, $q \to \neg p$, $s \to \neg r$, $p$

Sol:

Let us include additional premises $\neg(p \to \neg s) \Leftrightarrow \neg(\neg p \vee \neg s) \Leftrightarrow p \wedge s$ and prove a Contradiction.

| Step | Statement | Reason |
|---|---|---|
| 1. | $p \to (q \vee r)$ | Rule P |
| 2. | $p$ | Rule P |
| 3. | $q \vee r$ | Rule T, 1, 2 Modus ponens |
| 4. | $q \to \neg p$ | Rule P |
| 5. | $p \to \neg q$ | Rule T, 4 Contrapositive |
| 6. | $\neg q$ | Rule T, 2, 5 modus ponens |
| 7. | $s \to \neg r$ | Rule P. |
| 8. | $p \wedge s$ | Rule P (additional) |
| 9. | $p$ | Rule T, 8 simplification |
| 10. | $r$ | Rule T, 3, 6 disjunctive syllogism |
| 11. | $s$ | Rule T, 8 simplification |
| 12. | $\neg r$ | Rule T, 7, 11 modus ponens |
| 13 | $r \wedge \neg r$ | Rule T, 10, 12 Conjuction |
| 14. | $F$ | Rule T, 13, Compliment law. |

(4). Using indirect method of proof, do prove that
$p \to q,\ q \to r,\ \neg(p \wedge r),\ p \vee r \Rightarrow r$

(5). Derive $p \to (q \to s)$ using CP-rule (if necessary) from the premises $p \to (q \to r)$ and $q \to (r \to s)$

Sol:

We shall assume $p$ as an additional premise and then by CP-rule derive $p \to (q \to s)$.

| Step | Statement | Reason |
|---|---|---|
| 1. | $p$ | Rule P (additional) |
| 2. | $p \to (q \to r)$ | Rule P |
| 3. | $q \to r$ | Rule T, 1, 2 modus ponens |
| 4. | $\neg q \vee r$ | Rule T, 3 Implication law |
| 5. | $q \to (r \to s)$ | Rule P |
| 6. | $\neg q \vee (r \to s)$ | Rule T, 5, Implication law |
| 7. | $\neg q \vee (r \wedge (r \to s))$ | Rule T, 4, 6 Distributive law |
| 8. | $\neg q \vee s$ | Rule T, 7 modus ponens |
| 9. | $q \to s$ | Rule T, 8 Implication law |
| 10. | $p \to (q \to s)$ | Rule T, 1, 9 CP-rule. |

(6) Derive $r \to s$ using the CP-rule from the premises
$p \to (q \to s),\ (\neg r \vee p),\ q$.

## Validity of Arguments:

① Construct an argument to show that the following premises imply the conclusion "it rained"

"If it does not rain or if there is no traffic dislocation then the sports day will be held and the cultural programme will go on"; If the sports day is held, the trophy will be awarded" and "the trophy was not awarded.

Sol:

Let $p$ – it rains

$q$ – There is traffic dislocation

$r$ – Sports day will be held

$s$ – Cultural programme will go on

$t$ – The trophy will be awarded.

Then we have to prove that

$$(\neg p \vee \neg q) \to (r \wedge s),\ r \to t,\ \neg t \Rightarrow p$$

| Step | Statement | Reason |
|---|---|---|
| 1. | $(\neg p \vee \neg q) \to (r \wedge s)$ | Rule P. |
| 2. | $(\neg p \to (r \wedge s)) \wedge (\neg q \to (r \wedge s))$ | Rule T, 1 Equivalence law |
| 3. | $\neg p \to (r \wedge s)$ | Rule T, 2, Simplification |
| 4. | $\neg(r \wedge s) \to p$ | Rule T, 3 Contrapositive |
| 5. | $r \to t$ | Rule P |
| 6. | $\neg t \to \neg r$ | Rule T, 5, Contrapositive |
| 7. | $\neg t$ | Rule P |
| 8. | $\neg r$ | Rule T, 6, 7 modus ponens |
| 9. | $\neg r \vee \neg s$ | Rule T, 8 addition |
| 10. | $\neg(r \wedge s)$ | Rule T, 9 Demorgan's law |
| 11 | $p$ | Rule T, 4, 10, modus ponens. |

(2). Construct an argument using rules of inference to show that the hypotheses "Radha works hard", "If Radha works hard, then she is a dull girl" and "If Radha is a dull girl then she will not get the job" imply the Conclusion "Radha will not get the job".

Sol:

Let $p$ - Radha works hard

$q$ - Radha is a dull girl

$r$ - Radha will get the Job

Then we have to prove that

$p, p \to q, q \to \neg r \Rightarrow \neg r$

| Step | Statement | Reason |
|---|---|---|
| 1. | $p \to q$ | Rule P |
| 2. | $q \to \neg r$ | Rule P |
| 3. | $p$ | Rule P |
| 4. | $q$ | Rule T, 1, 3 modus ponens |
| 5. | $\neg r$ | Rule T, 2, 4 modus ponens. |

(3). Show that the following set of permises are inconsistent. If the Contract is valid, then John is liable for penalty. If John is liable for penalty, he will go bankrupt. If the bank will loan him money, he will not go bankrupt. As a matter of fact, the Contract is valid and the bank will loan him money.

Sol: Let $p$ - the Contract is valid.

$q$ - John is liable for penalty

$r$ - he will go bankrupt

$s$ - bank will loan him money

To prove $p \to q$, $q \to r$, $s \to \neg r$, $p \wedge s$ are inconsistent.

④. "It is not sunny this afternoon and it is colder than yesterday", "we will go to the playground only if it is sunny". "If we do not go to the ground, then we will go to a movie" and "If we go to a movie, then we will return home by sunset" lead to the conclusion "we will return home by sunset".

⑤. Show that the following system of premises is inconsistent:
If war is near, then the army would be mobilized. If the army has mobilized then the labour costs are high. However the war is near and yet labour costs are not high.