

BELIEFS: STATE UNCERTAINTY

AA228/CS238 DECISION MAKING UNDER UNCERTAINTY¹

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¹Mykel J. Kochenderfer, Tim A. Wheeler, and Kyle H. Wray. *Algorithms for Decision Making*. MIT Press, 2022.

CONTENT OUTLINE

- Introduction to POMDPs
- Belief representations
- Algorithms to update beliefs (discrete/continuous)
- Interactive Julia notebooks:
 - Crying baby POMDP (from scratch)
 - Crying baby POMDP (using `POMDPs.jl`)
 - Kalman filtering
 - Particle filtering

POMDPs AND BELIEFS

- A POMDP² is an MDP with *state uncertainty*

MDP: $\langle \mathcal{S}, \mathcal{A}, T, R, \gamma \rangle$

POMDP: $\langle \mathcal{S}, \mathcal{A}, \mathcal{O}, T, R, \gamma \rangle$

- The agent receives an *observation* of the current state rather than the true state (potentially imperfect observations)
- Using past observations, the agent builds a *belief* of their underlying state
 - Which can be represented by a probability distribution over true states
- Remember, a POMDP is a *problem formulation* and not an *algorithm*
 - A POMDP formulation enables the use of solution methods, i.e. algorithms.

²Partially observable Markov decision process. “Partially observable” is key in understanding beliefs.

OBSERVATION SPACE

- The agent receives an observation o , which belongs to some *observation space* \mathcal{O}
- The probability of observing o given action a and next state s' is: $O(o \mid a, s')$
 - If \mathcal{O} is continuous, then $O(o \mid a, s')$ is a probability density

DYNAMIC DECISION NETWORK FOR POMDPs

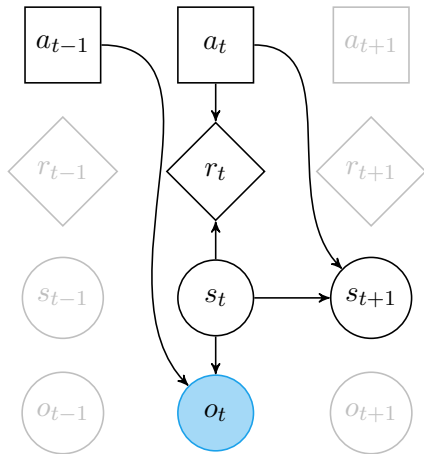


Figure: A dynamic decision network for the POMDP problem formulation.

BELIEF REPRESENTATION

Beliefs can be represented in different ways:

- **Parametric:** The belief distribution is represented by a set of parameters for a fixed distribution family
 - E.g., Categorical distribution³ or multivariate normal (Gaussian) distribution
- **Non-parametric:** The belief distribution is represented by particles (or points sampled from the state space)

Depending on the representation, different algorithms can be used to update beliefs.

³A probability mass is assigned to each discrete category.

ALGORITHMS FOR UPDATING BELIEFS

Various algorithms can update the current belief:

- If the state space is *discrete* (or certain linear Gaussian assumptions are met), then we can perform *exact belief updates*:⁴
 - Recursive Bayesian estimation (i.e. discrete state filter)
 - Kalman filter
- Otherwise, we can use approximations based on linearization or sampling:
 - Extended Kalman filter
 - Unscented Kalman filter
 - Particle filter
 - Particle filter with rejection
 - Injection particle filter
 - Adaptive injection particle filter

⁴Meaning we arrive at an analytical solution without approximations.

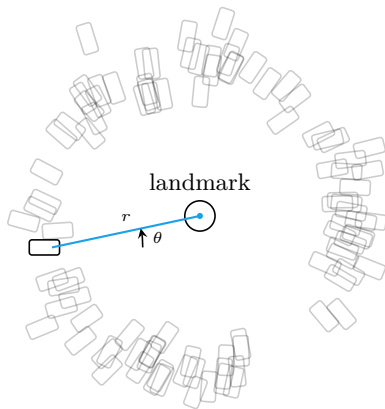
BELIEF INITIALIZATION

Before any actions or observations, we start with an initial belief distribution

- We can encode prior knowledge in the initial distribution
- Generally want to use diffuse (i.e. spread out) initial distributions to avoid over confidence in the absence of information
 - In non-parametric representations, a diffuse initial prior may cause difficulties
 - Thus, we may wait until an informative observation is made to initialize beliefs

EXAMPLE: LANDMARK BELIEF INITIALIZATION

Figure: Localization of an autonomous car using a landmark (Example 19.1).



Making a range r and bearing θ observation, we initialize our belief around the landmark.

BELIEF INFERENCE

- To infer the unknown belief distribution, we use *recursive Bayesian estimation*
 - Updates belief estimate recursively over time
 - Markov assumption: Only requires the current state, action, and observation
- Let $b(s)$ represent the probability⁵ assigned to state s
 - A particular belief b belongs to a *belief space* \mathcal{B} (containing all possible beliefs)
- For finite state and observation spaces, we can use a *discrete state filter* to perform exact inference

⁵or probability density for continuous state spaces

BELIEF VECTOR

- In the finite state case, we can represent beliefs using a categorical distribution⁶
 - Represented as a *belief vector* \mathbf{b} of length $|\mathcal{S}|$, therefore $\mathcal{B} \subset \mathbb{R}^{|\mathcal{S}|}$
 - Sometimes \mathcal{B} is referred to as a *probability simplex* or *belief simplex*⁷
- The belief vector \mathbf{b} must be strictly non-negative and sum to one:

$$b(s) \geq 0 \text{ for all } s \in \mathcal{S} \quad \sum_s b(s) = 1$$

- In vector notation:

$$\mathbf{b} \geq \mathbf{0} \quad \mathbf{1}^\top \mathbf{b} = 1$$

- In Julia syntax:

$$\text{all}(\mathbf{b} .\geq 0) \ \&\& \ \text{sum}(\mathbf{b}) \approx 1$$

⁶A probability mass is assigned to each discrete state.

⁷Simplex being the generalization of a triangle to arbitrary dimensions.

DISCRETE STATE FILTER: UPDATING BELIEFS

A *filter* is a process that remove noise from data.⁸

Due to the independence assumptions, if an agent with belief b takes an action a and receives an observation o , then the new belief b' becomes:⁹

$$\begin{aligned} b'(s') &= P(s' \mid b, a, o) \\ &\propto P(o \mid b, a, s')P(s' \mid b, a) && \text{(Bayes' rule)} \\ &\propto O(o \mid a, s')P(s' \mid b, a) && \text{(observation definition)} \\ &\propto O(o \mid a, s') \sum_s P(s' \mid b, a, s)P(s \mid b, a) && \text{(law of total probability)} \\ &\propto O(o \mid a, s') \sum_s T(s' \mid s, a)b(s) && \text{(state transition model)} \end{aligned}$$

⁸Often used in signal processing, effectively “filtering” out the noise.

⁹For finite/discrete state and observation spaces.

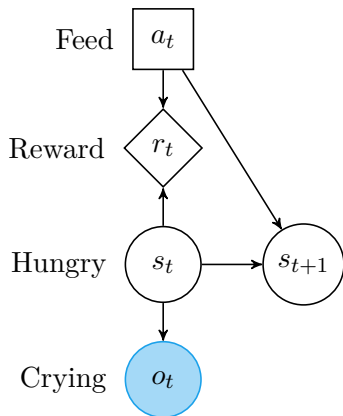
UPDATING BELIEFS: DERIVATION EXPLAINED

$$\begin{aligned} b'(s') &= P(s' \mid b, a, o) && \text{(probability of being in state } s') \\ &\propto P(o \mid b, a, s')P(s' \mid b, a) && \text{(Bayes' rule, dropping normalization)} \\ &\propto O(o \mid a, s')P(s' \mid b, a) && \text{(observation model def., } o \text{ independent of } b) \\ &\propto O(o \mid a, s') \sum_s P(s' \mid b, a, s)P(s \mid b, a) && \text{(law of total probability)} \\ &\propto O(o \mid a, s') \sum_s T(s' \mid s, a)b(s) && \text{(state transition model, belief def.)} \end{aligned}$$

Exact belief updating:¹⁰ $b'(s') \propto O(o \mid a, s') \sum_s T(s' \mid s, a)b(s)$

¹⁰Then normalize so beliefs sum to one.

EXAMPLE: CRYING BABY PROBLEM



- A simple POMDP with 2 states, 3 actions, and 2 observations:

$$\mathcal{S} = \{\text{hungry}, \text{sated}\}$$

$$\mathcal{A} = \{\text{feed}, \text{sing}, \text{ignore}\}$$

$$\mathcal{O} = \{\text{crying}, \text{quiet}\}$$

- See Pluto notebook:
 - [crying_baby_problem.html](#)

Figure: The crying baby POMDP.

POMDP SOLVERS

A number of ways to solve POMDPs are implemented in the following packages.

Table: POMDPs.jl Solution Methods

Package	Online/Offline	State Spaces	Actions Spaces	Observation Spaces
QMDP.jl	Offline	Discrete	Discrete	Discrete
FIB.jl	Offline	Discrete	Discrete	Discrete
BeliefGridValueIteration.jl	Offline	Discrete	Discrete	Discrete
SARSOP.jl	Offline	Discrete	Discrete	Discrete
BasicPOMCP.jl	Online	Continuous	Discrete	Discrete
ARDESPOT.jl	Online	Continuous	Discrete	Discrete
MCVI.jl	Offline	Continuous	Discrete	Continuous
POMDPSolve.jl	Offline	Discrete	Discrete	Discrete
IncrementalPruning.jl	Offline	Discrete	Discrete	Discrete
POMCPOW.jl	Online	Continuous	Continuous	Continuous
AEMS.jl	Online	Discrete	Discrete	Discrete
PointBasedValueIteration.jl	Offline	Discrete	Discrete	Discrete

When defining your problem, the *type* of state, action, and observation space is very important!

KALMAN FILTER

To update beliefs with *continuous* state spaces, we integrate instead of sum:

$$b'(s') \propto O(o \mid a, s') \int T(s' \mid s, a) b(s) ds$$

A *Kalman filter* assumes that T and O are linear-Gaussian and b is Gaussian:

$$T(\mathbf{s}' \mid \mathbf{s}, \mathbf{a}) = \mathcal{N}(\mathbf{s}' \mid \mathbf{T}_s \mathbf{s} + \mathbf{T}_a \mathbf{a}, \Sigma_s)$$

$$O(\mathbf{o} \mid \mathbf{s}') = \mathcal{N}(\mathbf{o} \mid \mathbf{O}_s \mathbf{s}', \Sigma_o)$$

$$b(\mathbf{s}) = \mathcal{N}(\mathbf{s} \mid \boldsymbol{\mu}_b, \Sigma_b)$$

See Pluto notebook: [StateEstimation.jl/kalman_filter.html](https://pluto.jl/kalman_filter.html)

PARTICLE FILTER

- *Particle filters* represent the belief states as a collection of states.
- Each state in the approximated belief is called a *particle*.
- Useful in problems with large discrete states spaces or continuous problems not well approximated by linear-Gaussian dynamics.

Algorithm 1 Particle filter algorithm.

```
function PARTICLEFILTER( $\mathbf{s}, T, O, a, o$ )  
     $\mathbf{s}' \sim T(\mathbf{s}, a)$  ▷ next states  
     $\mathbf{w} \leftarrow O(o \mid \mathbf{s}', a)$  ▷ weights  
    particles  $\sim \text{SetCategorical} \left( \mathbf{s}', \frac{\mathbf{w}}{\sum_i w_i} \right)$  ▷ sample with normalized weights  
return particles
```

See Pluto notebook: [StateEstimation.jl/particle_filter.html](https://pluto.jl/en/latest/StateEstimation.jl/particle_filter.html)

PARTICLE FILTER VARIANTS

Particle filter with rejection:

- Used in problems with discrete observations.
- Any sampled observation that does not equal the true observation is rejected.
- Problem of *particle deprivation*: lack of particles near the true state.¹¹

Injection particle filter:

- Inject random particles to protect against particle deprivation.

Adaptive injection particle filter:

- Inject particles adaptively based on a ratio of two exponentially moving averages of the mean particle weights (using *fast* and *slow* moving averages).

See Pluto notebook: [StateEstimation.jl/particle_filter.html](https://pluto.jl.org/StateEstimation.jl/particle_filter.html)

¹¹Due to low particle coverage given the stochastic nature of resampling.

QUESTIONS

Any questions?

(feel free to post on Ed or email me: mossr@cs.stanford.edu)

REFERENCES

Kochenderfer, Mykel J., Tim A. Wheeler, and Kyle H. Wray. *Algorithms for Decision Making*. MIT Press, 2022.