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EQUIVARIANT SK INVARIANTS ON $\mathbb{Z}_{2^r}$ MANIFOLDS WITH BOUNDARY

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0. Introduction

Let G be a finite abelian group. Throughout this paper, by a G manifold we mean an unoriented compact smooth manifold (which may have boundary) with smooth G action. In [**2**], we have studied an equivariant cutting and pasting theory (SK theory) $SK_*^G(pt, pt)$ based on G manifolds by using the notion of G slice types. We now consider a map T for G manifolds which takes values in the ring $\mathbb{Z}$ of rational integers and is additive with respect to the disjoint union of G manifolds. We call T a G-SK invariant if it is invariant under equivariant cuttings and pastings [**3, 4**]. Then it induces an additive homomorphism $T : SK_*^G(pt, pt) \to \mathbb{Z}$.

The main object of this paper is to study such invariants when G is the cyclic group G_r of order 2^r ($r \geq 0$). Here G_0 is the trivial group $\{1\}$.

In Section 1, we first define an SK group $SK_*^G[\mathcal{F}, \mathcal{F}]$ obtained by G manifolds of type $\mathcal{F}$, where $\mathcal{F}$ is a family of G slice types. We always write our theory by using a pair $(\mathcal{F}, \mathcal{F})$ to distinguish it from the theory $SK_*^G[\mathcal{F}]$ for closed G manifolds in [**7**]. In particular, we have $SK_*^G(pt, pt) = SK_*^G[\mathcal{F}, \mathcal{F}]$ if $\mathcal{F}$ is the family $St(G)$ of all G slice types. There is a total ordering on $St(G_r)$ which gives a basis of $SK_*^{G_r}(pt, pt)$ as a free SK_* module, where SK_* is the SK ring of closed manifolds in [**4**] (Proposition 1.12).

Section 2 is concerned with G_r-SK invariants. Let $\mathcal{I}_*^{G_r}$ be the set consisting of all these invariants. We first obtain a basis for $\mathcal{I}_*^{G_r}$ as a free $\mathbb{Z}$ module by using the characteristic $\overline{\chi}(M) = \chi(M) - \chi(\partial M)$ for the pair $(M, \partial M)$ of a manifold and its boundary, where χ is the Euler characteristic (Theorem 2.3). As a result, we see that an element $[M]$ in $SK_*^{G_r}(pt, pt)$ is determined by the class $\{\overline{\chi}(M_\sigma) \mid \sigma \in St(G_r)\}$, where each M_σ is the invariant submanifold of M with slice type containing σ (Proposition 2.5). Using this, we present a multiplicative structure on $SK_*^{G_r}(pt, pt)$ given by the cartesian product $M \times N$ of G_r manifolds M and N (Proposition 2.8). Let $SK_*^{G_r}$ be the SK theory for closed G_r manifolds, then we see that the natural inclusion $i_* : SK_*^{G_r} \to SK_*^{G_r}(pt, pt)$ is injective. As an application of this result, we give an SK relation between the real and complex projective spaces with G_r action by performing an SK process in the theory $SK_*^{G_r}(pt, pt)$ (Proposition 2.9 and Example 2.12).

In Section 3, we consider an invariant T which is multiplicative in the sense that $T(M \times N) = T(M)T(N)$ for any G_r manifolds M and N. We say that such a T is of type (s) if $T(G_r/G_t) = 0$ for each zero-dimensional G_r manifold G_r/G_t with $0 \leq t \leq s-1$ and $T(G_r/G_s) \neq 0$. For example, $\overline{\chi}^{G_s}$ and χ^{G_s} are of type (s), where $\overline{\chi}^{G_s}(M) = \overline{\chi}(M^{G_s})$, $\chi^{G_s}(M) = \chi(M^{G_s})$ and $M^{G_s} = \{x \in M \mid gx = x \text{ for any } g \in G_s\}$. In this case, we see that T is determined by the values on the unit disk D^1 (with trivial action) and G_r manifolds $G_r \times_{G_s} D(V_i)$ with $0 \leq i < 2^{s-1}$, where $\{V_i\}$ is a complete set of non-trivial irreducible G_s modules and $\{D(V_i)\}$ are the associated disks (Theorem 3.12).

1. SK groups for G_r manifolds

Let M^n be an n-dimensional smooth G manifold with boundary ∂M and let $(L, \partial L) \subset (M, \partial M)$ be a submanifold which satisfies the following properties:

(1) $(L, \partial L)$ is a G invariant codimension one submanifold of $(M, \partial M)$. Here we admit the case $\partial L = \emptyset$ and $\partial M \neq \emptyset$; and

(2) the normal bundle of $(L, \partial L)$ in $(M, \partial M)$ is G equivalent to the trivial bundle $(L, \partial L) \times \mathbb{R}$ with trivial action of G on the set $\mathbb{R}$ of real numbers.

We assume that L separates M; that is, $M = N_1 \cup N_2$ (pasting along the common parts L) for some G invariant submanifolds N_i of codimension zero. There is no gain in generality to drop this condition, since the union of L with a second copy of L, suitably embedded near L, will separate M. We denote this decomposition simply by $M = N_1 \cup N_2$.

Definition 1.1. We say that n-dimensional G manifolds M_1 and M_2 are obtained from each other by a G equivariant cutting and pasting if M_1 has been obtained from M_2 by the step as mentioned above; that is, $M_1 = N_1 \cup_\varphi N_2$ and $M_2 = N_1 \cup_\psi N_2$ pasting along the common parts $L \subset M_i$ by some G diffeomorphisms φ and $\psi : L \to L$ $(i = 1, 2)$.

If H is a subgroup of G, then an H module U is a finite-dimensional real vector space together with a linear action of H on it. If M is a G manifold and $x \in M$, then there is a G_x module $\overline{U}_x$ which is equivariantly diffeomorphic to a G_x neighbourhood of x, where $G_x = \{g \in G \mid gx = x\}$ is the isotropy subgroup at x. This module $\overline{U}_x$ decomposes as $\overline{U}_x = \mathbb{R}^p \oplus U_x$, where G_x acts trivially on $\mathbb{R}^p$ and $U_x^{G_x} = \{0\}$. We refer to the pair $[G_x; U_x]$ as the slice type of $x \in M$. By a G slice type, we mean a pair $[H; U]$ of a subgroup H and an H module U such that $U^H = \{0\}$ in general. A family $\mathcal{F}$ of G slice types is a collection of G slice types satisfying the condition that if $[H; U] \in \mathcal{F}$ and $x \in G \times_H U$ then the slice type $[G_x; U_x]$ of x belongs to $\mathcal{F}$.

Definition 1.2. Let $\mathcal{F}$ be a family of G slice types. Let M_1 and M_2 be n-dimensional G manifolds of type $\mathcal{F}$; that is, $[G_x; U_x] \in \mathcal{F}$ for each $x \in M_i$. We say that M_1 and M_2 are G-SK equivalent if M_1 can be obtained from M_2 by a finite sequence of equivariant cuttings and pastings (G-SK processes). This is an equivalence relation on the set of n-dimensional G manifolds of type $\mathcal{F}$. The set of equivalence classes forms an abelian semigroup if we use disjoint union $+$ of G manifolds as addition, and has a zero represented by $\emptyset$. The class containing a G manifold M is denoted by $[M]$. We define by $SK_n^G[\mathcal{F}, \mathcal{F}]$ the Grothendieck group of this semigroup. By defining $SK_*^G[\mathcal{F}, \mathcal{F}] = \bigoplus_{n \geq 0} SK_n^G[\mathcal{F}, \mathcal{F}]$ we have a graded SK_* module with multiplication given by cartesian product of manifolds. When $\mathcal{F} = St(G)$, we write $SK_*^G[\mathcal{F}, \mathcal{F}] = SK_*^G(pt, pt)$. In the case when G is the trivial group $\{1\}$, $SK_*^{\{1\}}(pt, pt)$ is the theory $SK_*(pt, pt)$ studied in [**6**].

In general, let $SK_*(X, X)$ be the theory for singular manifolds in arcwise connected space X, then we have the following lemma.

LEMMA 1.3. (cf. [**6**, Theorem 1.2] and [**2**, Lemma 1.10]) *For any $n \geq 0$, $\chi = (-1)^n\overline{\chi} : SK_n(X, X) \cong SK_n(pt, pt) \cong \mathbb{Z}$, and $SK_n(X, X)$ is generated by $[D^n; *]$, where D^n is the n-disk and $* : D^n \to X$ is the constant map. Further, $SK_*(X, X) = \bigoplus_{n \geq 0} SK_n(X, X)$ is a free SK_* module with basis $[pt; *]$ and $[D^1; *]$.*

In the above, if $\dim(M) = n$ is odd, then $\chi(\partial M) = 2\chi(M)$ by applying χ to the double $DM = M \cup M$. Hence we have $\chi = (-1)^n\overline{\chi}$ for any $n \geq 0$.

The ring SK_* is a polynomial ring over $\mathbb{Z}$ on the class $\alpha = [\mathbb{R}P^2]$, where $\mathbb{R}P^2$ is the real projective plane (cf. [**7**, 2.5.1]). Now let $i_* : SK_* \to SK_*(pt, pt)$ be the natural inclusion. Then it follows from the above lemma that $i_*(\alpha^k) = [D^{2k}]$ in $SK_*(pt, pt)$ and therefore i_* is injective.

Example 1.4. Let $\mathbb{C}P^k$ be the complex projective space, then we see that $[\mathbb{C}P^k] = (k+1)[D^{2k}]$ by applying $\overline{\chi}$ to both sides. An SK process between these elements is as follows. Consider $N_i = A_i + B_i$ $(i = 1, 2)$, where $A_1 = D^{2k}$, $A_2 = D^2 \times_{S^1} S^{2k-1}$, $B_1 = [a, c] \times S^{2k-1}$ and $B_2 = [c, b] \times S^{2k-1}$ $(a < c < b)$. Further, let $L = L' + L''$ where $L' = S^{2k-1} = \partial A_i$ and $L'' = \{c\} \times S^{2k-1}$. By pasting N_1 to N_2 along the common parts L in two ways naturally, we have

$$[\mathbb{C}P^k] + [[a, b] \times S^{2k-1}] = [D^{2k}] + [D^2 \times_{S^1} S^{2k-1}].$$

Then $[\mathbb{C}P^k] = [D^{2k}] + [D^2 \times \mathbb{C}P^{k-1}]$ because $[S^{2k-1}] = [S^1 \times \mathbb{C}P^{k-1}] = 0$ and $[D^2 \times_{S^1} S^{2k-1}] = [D^2 \times \mathbb{C}P^{k-1}]$ (cf. [**4,** Lemma 1.5(i) and (iii)]). Therefore we have $[\mathbb{C}P^k] = (k+1)[D^{2k}]$ by induction on k. Similarly, the equality $[\mathbb{R}P^{2k}] = [D^{2k}]$ is obtained. Hence $[\mathbb{C}P^k] = (k+1)\alpha^k$ holds in SK_{2k} via the injection i_*.

Let $(\mathcal{F}, \mathcal{F}_0)$ be a pair of families of G slice types such that $\mathcal{F} - \mathcal{F}_0 = \{\rho\}$, $\rho = [H; U]$. If M is a G manifold of type $\mathcal{F}$, then the set M_ρ of all points in $x \in M$ having slice type ρ is a G submanifold (cf. [**4,** p. 37]). We see that a normal bundle $\nu(M_\rho)$ of M_ρ in M is of type ρ. Denote by $SK_*^G[\rho, \rho]$ the SK group resulting from equivariant cuttings and pastings of such a kind of G vector bundles. It is shown that $SK_*^G[\rho, \rho] \cong SK_*(B\Gamma(\rho), B\Gamma(\rho))$ for some space $B\Gamma(\rho)$ (cf. [**7,** 2.2]). Thus $SK_*[\rho, \rho]$ is a free SK_* module with basis $[G \times_H U]$ and $[G \times_H U \times D^1]$ from Lemma 1.3.

PROPOSITION 1.5. (cf. [**2,** Theorem 1.12]) *Let $(\mathcal{F}, \mathcal{F}_0)$ be a pair of families of G slice types such that $\mathcal{F} - \mathcal{F}_0 = \{\rho\}$, then the sequence*

$$0 \longrightarrow SK_*^G[\mathcal{F}_0, \mathcal{F}_0] \xrightarrow{i_*} SK_*^G[\mathcal{F}, \mathcal{F}] \xrightarrow{\nu} SK_*^G[\rho, \rho] \longrightarrow 0 \tag{1.5.1}$$

is a split exact sequence, where i_* is induced by the inclusion $\mathcal{F}_0 \subset \mathcal{F}$ and $\nu([M]) = [\nu(M_\rho)]$. A splitting map s to ν is defined by $s([G \times_H U]) = [G \times_H D(U)]$ and $s([G \times_H U \times D^1]) = [G \times_H D(U \times \mathbb{R})]$.

Let $G = G_r$ be the cyclic group of order 2^r for the remainder of this paper. When $s \geq 1$, the non-trivial irreducible G_s modules are $V_0, V_1, \ldots, V_{2^{s-1}-1}$, where $V_0 = \mathbb{R}$ with a generator of G_s acting by multiplication by -1, while V_j $(j \geq 1)$ is the set $\mathbb{C}$ of complex numbers with a generator of G_s acting by multiplication by $\exp(2\pi i j/2^s)$, $i = \sqrt{-1}$. Then the G_r slice types are of the form

$$\begin{aligned}\sigma^s(A) &= \sigma(a(0), a(1), \ldots, a(2^{s-1}-1)) \\ &= \left[G_s; \prod_j V_j^{a(j)}\right],\end{aligned}$$

where $0 \leq s \leq r$ and $A = (a(0), a(1), \ldots, a(2^{s-1}-1))$ is a 2^{s-1} tuple of non-negative integers. Here we denote $\sigma^0(\emptyset) = [G_0; \{0\}]$ by σ_{-1}.

Let $|\sigma^s(A)| = a(0) + 2\sum_{i\geq 1} a(i)$ be the dimension of $\sigma^s(A)$. We now give a total ordering on the family $St(G_r)$ as follows:

(1) $\sigma_{-1} < \sigma^s(A)$ for all $\sigma^s(A)$ with $s \geq 1$;

(2) $\sigma^s(A) < \sigma^t(B)$ if $|\sigma^s(A)| < |\sigma^t(B)|$;

(3) suppose that $|\sigma^s(A)| = |\sigma^t(B)|$, then $\sigma^s(A) < \sigma^t(B)$ if $s < t$;

(4) suppose that $|\sigma^s(A)| = |\sigma^t(B)|$ and $s = t$, then $\sigma^s(A) < \sigma^s(B)$ if $V^A = \prod_j V_j^{a(j)} < V^B = \prod_j V_j^{b(j)}$ in the ordering of G_s modules induced lexicographically from an ordering in the irreducible G_s modules: $V_0 < V_1 < \cdots < V_{2^{s-1}-1}$ (cf. [**7**, p. 29]).

Definition 1.6. For any $\sigma^t(B) = \sigma(b(0), b(1), \ldots, b(2^{t-1} - 1))$, we define a class $\{\overline{\sigma}^{t-k}\}$ of slice types as follows: $\overline{\sigma}^0 = \sigma_{-1}$, $\overline{\sigma}^t = \sigma^t(B)$ and if $\overline{\sigma}^{t-k} = \sigma(c(0), c(1), \ldots, c(2^{t-k-1} - 1))$ with $1 \le k \le t-1$, then

$$\begin{aligned} c(0) &= 2 \sum_{m=0}^{2^{k-1}-1} b(m2^{t-k} + 2^{t-k-1}) \\ c(i) &= \sum_{m=0}^{2^{k-1}-1} (b(m2^{t-k} + i) + b((m+1)2^{t-k} - i)) \end{aligned} \tag{1.6.1}$$

for $1 \le i < 2^{t-k-1}$. Here we define subsets of $P = \{0, 1, \ldots, 2^{t-1} - 1\}$ as follows:

$$\begin{aligned} P(k; 0) &= \{m2^{t-k} + 2^{t-k-1} \mid 0 \le m < 2^{k-1}\} \\ P(k; i) &= \{m2^{t-k} + i, (m+1)2^{t-k} - i \mid 0 \le m < 2^{k-1}\} \end{aligned} \tag{1.6.2}$$

for $1 \le i < 2^{t-k-1}$. Further, set $P(k; -1) = P \setminus \cup_i P(k; i)$. Then P is a disjoint union of these $P(k; i)$ $(i \ge -1)$.

$$|\overline{\sigma}^{t-k}| = 2 \sum_{i \ne -1} \sum_{j \in P(k;i)} b(j) \quad \text{and} \quad |\sigma^t| - |\overline{\sigma}^{t-k}| = b(0) + 2 \sum_{j \in P(k;-1) \setminus \{0\}} b(j). \tag{1.6.3}$$

LEMMA 1.7. *Let $\{\overline{\sigma}^{t-k}\}$ be the class of slice types for $\sigma = \sigma^t(B)$ stated above. Then $D(\sigma)^{G_{t-k}}$ is a G_t invariant disk $D^{|\sigma| - |\overline{\sigma}^{t-k}|}$ of $D(\sigma)$, and each point of $D(\sigma)^{G_{t-k}} \setminus D(\sigma)^{G_{t-k+1}}$ has a slice type $\overline{\sigma}^{t-k}$.*

Proof. We see that $D(\sigma)^{G_t} = \{\mathbf{0}\}$ and $D(\sigma)$ has a slice type $\sigma = \overline{\sigma}^t$ at the origin $\mathbf{0}$. Next we note that $D(\sigma)^{G_{t-1}} = D(\sigma(b(0), 0, \ldots, 0))$. If $b(0) \ne 0$, then $D(\sigma)$ has a slice type

$$\begin{aligned} \overline{\sigma}^{t-1} = \sigma(&2b(2^{t-2}), b(1) + b(2^{t-1} - 1), b(2) \\ &+ b(2^{t-1} - 2), \ldots, b(2^{t-2} - 1) + b(2^{t-2} + 1)) \end{aligned}$$

at every point of $D^{b(0)} = D(\sigma(b(0), 0, \ldots, 0))$ excluding $\{\mathbf{0}\}$. Here we use the fact that $V_0 \otimes V_c = V_d$, where $d = 2^{t-1} - c$ (cf. [**7**, 4.4]). By using induction on k, in general we have that $D(\sigma)^{G_{t-k}} = D(\sigma(b(0)', b(1)', \ldots, b(2^{t-1} - 1)'))$ in $D(\sigma)$

where $b(j)' = b(j)$ if $j \in P(k; -1)$ and $b(j)' = 0$ if $j \notin P(k; -1)$, and every point of $D(\sigma)^{G_{t-k}}$ excluding $D(\sigma)^{G_{t-k+1}}$ has a slice type $\overline{\sigma}^{t-k}$ in Definition 1.6. Note that $D(\sigma)^{G_{t-k}}$ is an invariant disk $D^{|\sigma|-|\overline{\sigma}^{t-k}|}$ of $D(\sigma)$ by (1.6.3). □

By the lemma, we see that

$$\sigma_{-1} = \overline{\sigma}^0 \prec \overline{\sigma}^1 \prec \cdots \prec \overline{\sigma}^{t-k} \prec \cdots \prec \overline{\sigma}^t = \sigma^t(B) \tag{1.8}$$

since $|\overline{\sigma}^i| \leq |\overline{\sigma}^j|$ if $i < j$.

For the class $\{\overline{\sigma}^{t-k}\}$, we denote the ordering $\overline{\sigma}^{t-k} < \sigma^t$ by $\overline{\sigma}^{t-k} \prec \sigma^t$ from now on.

(1.9) For any G_r manifold M and a slice type $\sigma = \sigma^s$, define M_σ to be the set consisting of $x \in M$ whose slice type $\sigma_x = \sigma$ or $\sigma_x \succ \sigma$ in the sense of (1.8). Let $\sigma_x = [G_t; \sigma_x]$ be the slice type of x. Then a suitable neighbourhood $N = G_r \times_{G_t} \sigma_x'$ of the orbit $G_r(x)$ in M gives a chart $N^{G_s} = G_r \times_{G_t} (\sigma_x')^{G_s}$ around $G_r(x)$ in M^{G_s}, where $\sigma_x' = \mathbb{R}^p \times \sigma_x$ for some $p \geq 0$ (cf. [**5**, Theorem 4.14]). Since $\overline{\sigma}_x^s = \sigma$, we see that $\sigma_x^{G_s} = \mathbb{R}^q \times \{0\}$ in σ_x, where $q = |\sigma_x| - |\sigma|$ and every point $y \in \sigma_x^{G_s}$ has a slice type σ_y such that $\sigma_y \succeq \sigma$ by Lemma 1.7. This implies that $N^{G_s} \subset M_\sigma$ and N^{G_s} gives a chart of M_σ. Note $\dim(M_\sigma) = p + q = \dim(M) - |\sigma|$. It is easy to see that $\partial(M_\sigma) = (\partial M)_\sigma$ by using a G_r collar.

Remark 1.10.

(i) If $\sigma^s \neq \sigma'^s$, then $M_{\sigma^s} \cap M_{\sigma'^s} = \emptyset$ by definition, and $M^{G_s} = \cup_{\sigma^s} M_{\sigma^s}$.

(ii) If $\sigma = \sigma_{-1}$, then $M_{\sigma_{-1}} = M$ since $\sigma_{-1} \preceq \sigma^s$ for all σ^s.

(iii) If $\sigma^s(\mathbf{0}) = \sigma^s(0, \ldots, 0)$, then $M_{\sigma^s(\mathbf{0})}$ is the components of M^{G_s} with $\dim(M_{\sigma^s(\mathbf{0})}) = \dim(M) - |\sigma^s(\mathbf{0})| = \dim(M)$. Hence, if M is a manifold with trivial action, then $M_{\sigma^s(\mathbf{0})} = M$ for any s.

Example 1.11. Let $M = G_r \times_{G_t} D(\sigma^t)$ for $t \geq 0$. Then $M_\tau = G_r \times_{G_t} D(\sigma^t)^{G_{t-k}}$ if $\tau = \overline{\sigma}^{t-k} \preceq \sigma^t$ and $0 \leq k \leq t$, or $M_\tau = \emptyset$ otherwise. Therefore we have

$$[M_{\overline{\sigma}}] = 2^{r-t}[D^{|\sigma|-|\overline{\sigma}|}] = \begin{cases} 2^{r-t}\alpha^{(|\sigma|-|\overline{\sigma}|)/2}[pt] & \text{if } |\sigma| \text{ is even,} \\ 2^{r-t}\alpha^{(|\sigma|-|\overline{\sigma}|-1)/2}[D^1] & \text{if } |\sigma| \text{ is odd} \end{cases}$$

in $SK_*^{G_r}(pt, pt)$ from Lemma 1.7 and the identity $[D^2] = \alpha$ in Example 1.4.

We now rename the slice types: $\sigma_{-1} = \rho_0 < \rho_1 < \cdots < \rho_m < \rho_{m+1} < \cdots$ by using the ordering on the family $St(G_r)$ and let $\mathcal{F}_m$ be defined by $\mathcal{F}_m = \{\rho_i \mid 0 \leq i \leq m\}$ for each $m \geq 0$. We see that $\mathcal{F}_m$ is a family of slice types by Lemma 1.7, (1.8) and Example 1.11. If m is sufficiently large compared with n, $SK_n^{G_r}[\mathcal{F}_m, \mathcal{F}_m] =$

$SK_n^{G_r}(pt, pt)$. Hence we have $SK_*^{G_r}(pt, pt) = \oplus_{m\geq 0} SK_*^{G_r}[\rho_m, \rho_m]$ by the exact sequences (1.5.1) when $(\mathcal{F}, \mathcal{F}_0) = (\mathcal{F}_m, \mathcal{F}_{m-1})$, and obtain the following proposition.

PROPOSITION 1.12. *$SK_*^{G_r}(pt, pt)$ is a free SK_* module with basis*

$$\mathcal{B} = \{[G_r \times_{G_s} D(\sigma^s)], [G_r \times_{G_s} D(\sigma^s \times \mathbb{R})] \mid \sigma^s \in St(G_r)\}.$$

2. G_r-SK invariants

Definition 2.1. Let T be a map for n-dimensional G_r manifolds, which is assumed to take values in $\mathbb{Z}$ and to be additive with respect to disjoint union $+$; that is, if $M = M_1 + M_2$ then $T(M) = T(M_1) + T(M_2)$. We call T a G_r-SK invariant if $T(N_1 \cup_\varphi N_2) = T(N_1 \cup_\psi N_2)$ for any G_r diffeomorphisms φ and $\psi : L \to L$ in Definition 1.1. An invariant T induces a homomorphism $T : SK_n^{G_r}(pt, pt) \to \mathbb{Z}$. The set $\mathcal{I}_n^{G_r}$ of all these T is a $\mathbb{Z}$ module under natural addition.

Example 2.2. For each $\sigma \in St(G_r)$, a map $\overline{\chi}_\sigma$ defined by $\overline{\chi}_\sigma(M) = \overline{\chi}(M_\sigma)$ is an invariant. On the other hand, $\overline{\chi}^{G_s}$ mentioned in the introduction is also an invariant. We note that $\overline{\chi}_{\sigma_{-1}} = \overline{\chi}^{G_0} = \overline{\chi}$ (cf. Remark 1.10(ii)). By considering χ instead of $\overline{\chi}$, we also have invariants χ_σ and χ^{G_s}.

Now we divide $St(G_r)$ as $St(G_r) = \{\sigma_{-1}\} \cup \mathcal{S} \cup \{\sigma_* \mid \sigma \in \mathcal{S}\}$, where $\mathcal{S} = \cup_{1\leq s\leq r} \mathcal{S}^s$, $\mathcal{S}^s = \{\sigma^s \mid \sigma^s = \sigma(a(0), a(1), \ldots), a(0); \text{even}\}$ and $\sigma_*^s = \sigma(a(0)+1, a(1), \ldots)$ for $\sigma^s = \sigma(a(0), a(1), \ldots) \in \mathcal{S}^s$.

THEOREM 2.3. *For each $n \geq 0$, the class*

$$\begin{cases} \theta_{\sigma_{-1}} = \dfrac{1}{2^r}\left\{\overline{\chi} + \displaystyle\sum_{1\leq j\leq r}\sum_{\tau\in\mathcal{S}^j} 2^{j-1}(\overline{\chi}_\tau - \overline{\chi}_{\tau_*})\right\}, \\ \theta_\sigma = \dfrac{1}{2^{r-s}}\left\{\overline{\chi}_\sigma + \displaystyle\sum_{s<j\leq r} 2^{j-(s+1)}\left(\sum_{\tau\in\mathcal{S}^j, \sigma\prec\tau} \overline{\chi}_\tau - \overline{\chi}_{\tau_*}\right)\right\}, \\ \theta_{\sigma_*} = \dfrac{1}{2^{r-s}}\overline{\chi}_{\sigma_*} \end{cases}$$

is a basis for $\mathcal{I}_n^{G_r}$, where $\sigma \in \mathcal{S}^s$ with $|\sigma| \leq n$, σ_* with $|\sigma_*| \leq n$ and $1 \leq s \leq r$.

We have obtained the result when $r = 1$ or 2 (cf. [**2,** Propositions 2.3 and 2.4]).

Proof. Let us define an SK_*-homomorphism $g_\sigma : SK_*^{G_r}(pt, pt) \to SK_{*-|\sigma|}(pt, pt)$ by $g_\sigma([M]) = [M_\sigma]$ for each $\sigma \in St(G_r)$. Further, we consider an SK_*-homomorphism $f_* : SK_*^{G_r}(pt, pt) \to L = \sum_{m\geq 0} SK_{*-|\rho_m|}(pt, pt)$ defined by

$f_* = \oplus_{m\geq 0} f_{\rho_m}$, where $St(G_r) = \{\rho_m \mid m \geq 0\}$ which is totally ordered in the previous section and

$$\begin{cases} f_{\sigma_{-1}} = \dfrac{1}{2^r}\left\{g_{\sigma_{-1}} + \displaystyle\sum_{1\leq j\leq r}\sum_{\tau\in\mathcal{S}^j} 2^{j-1}\alpha^{|\tau|/2}(g_\tau + [D^1]g_{\tau_*})\right\}, \\ f_\sigma = \dfrac{1}{2^{r-s}}\left\{g_\sigma + \displaystyle\sum_{s<j\leq r} 2^{j-(s+1)}\left(\sum_{\tau\in\mathcal{S}^j,\sigma\prec\tau}\alpha^{(|\tau|-|\sigma|)/2}(g_\tau + [D^1]g_{\tau_*})\right)\right\}, \\ f_{\sigma_*} = \dfrac{1}{2^{r-s}} g_{\sigma_*} \end{cases} \tag{2.3.1}$$

for $\sigma \in \mathcal{S}^s$ and $1 \leq s \leq r$. The degree of $f_{\sigma_{-1}}$, f_σ or f_{σ_*} is zero, $-|\sigma|$ or $-|\sigma_*| = -(|\sigma|+1)$ respectively. We denote the basis elements of $\mathcal{B}$ for $SK_*^{G_r}(pt, pt)$ as $x_{\sigma_{-1}} = [G_r]$, $x_{\sigma^t} = [G_r \times_{G_t} D(\sigma^t)]$, $\widehat{x}_{\sigma_{-1}} = [G_r \times D^1]$ and $\widehat{x}_{\sigma^t} = [G_r \times_{G_t} D(\sigma^t \times \mathbb{R})]$ (cf. Proposition 1.12), and we give the total orderings on the bases $\mathcal{B} = \{x_{\rho_m}\}$ and $\mathcal{B}' = \cup_{m\geq 0}\mathcal{C}_m$ of L naturally, where $\mathcal{C}_m = \{[pt], [D^1]\}$ is the ordered basis of the mth copy of $SK_*(pt, pt)$ in L (cf. Lemma 1.3). The values of f_{ρ_m} on the elements $x_{\sigma_{-1}}$, x_σ and x_{σ_*} which do not vanish are as follows:

$$\begin{aligned} & f_{\sigma-1} = [pt] \quad \text{on } x_{\sigma_{-1}}, \\ & f_{\sigma_{-1}} = \alpha^{|\sigma^t|/2}[pt], \quad f_{\overline{\sigma}^i} = \alpha^{(|\sigma^t|-|\overline{\sigma}^i|)/2}[pt] \quad (\overline{\sigma}^i \prec \sigma^t, 1 \leq i < t), \\ & f_{\sigma^t} = [pt] \quad \text{on } x_{\sigma^t}, \\ & f_{\sigma_{-1}} = \alpha^{|\sigma^t|/2}[D^1], \quad f_{\overline{\sigma}^i} = \alpha^{(|\sigma^t|-|\overline{\sigma}^i|)/2}[D^1] \quad (\overline{\sigma}^i \prec \sigma^t_*, 1 \leq i < t), \\ & f_{\sigma^t_*} = [pt] \quad \text{on } x_{\sigma^t_*} \end{aligned} \tag{2.3.2}$$

for each $\sigma^t \in \mathcal{S}^t$. For example, if $\overline{\sigma}^i \prec \sigma^t$ with $1 \leq i < t$, then

$$\begin{aligned} f_{\overline{\sigma}^i}(x_{\sigma^t}) &= \frac{1}{2^{r-i}}\left(g_{\overline{\sigma}^i}(x_{\sigma^t}) + \sum_{i<j\leq t} 2^{j-(i+1)}\alpha^{(|\overline{\sigma}^j|-|\overline{\sigma}^i|)/2} g_{\overline{\sigma}^j}(x_{\sigma^t})\right) \\ &= \frac{2^{r-t}}{2^{r-i}}\left(1 + \sum_{i<j\leq t} 2^{j-(i+1)}\right)\alpha^{(|\sigma^t|-|\overline{\sigma}^i|)/2}[pt] \\ &= \alpha^{(|\sigma^t|-|\overline{\sigma}^i|)/2}[pt] \end{aligned}$$

by Example 1.11. For each $\sigma^t \in \mathcal{S}^t$, we note that σ^t and σ^t_* give the same class $\{\overline{\sigma}^i \mid 1 \leq i < t\}$ and each $\overline{\sigma}^i$ belongs to $\mathcal{S}^i$ by (1.6.3). The values on the elements $\widehat{x}_{\sigma_{-1}}$, $\widehat{x}_\sigma$ and $\widehat{x}_{\sigma_*}$ are given by $f_{\rho_m}(\widehat{X}) = [D^1]f_{\rho_m}(X)$. From (2.3.2), we see that f_* is an isomorphism since the matrix relative to the ordered bases $\mathcal{B}$ and $\mathcal{B}'$ is triangular with components 1 on the diagonal. Therefore any invariant T is factorized as

$$T : SK_n^{G_r}(pt, pt) \overset{f_*}{\cong} L_n \overset{\oplus\overline{\chi}}{\cong} \oplus_{|\rho_m| \le n}\mathbb{Z} \xrightarrow{T'} \mathbb{Z} \tag{2.3.3}$$

for some T', where $L_n = \oplus_{|\rho_m| \le n} SK_{n-|\rho_m|}(pt, pt)$. Taking $\overline{\chi} \circ f_{\rho_m} = \theta_{\rho_m}$, we have the result. □

COROLLARY 2.4. *Let M be an n-dimensional G_r manifold such that all $\overline{\chi}_\sigma(M) = 0$. Then $T(M) = 0$ for any G_r-SK invariant.*

By using the isomorphism $\oplus_m \theta_{\rho_m}$ in (2.3.3), we have the following propositions.

PROPOSITION 2.5. *Let $[M_1]$ and $[M_2]$ be elements in $SK_n^{G_r}(pt, pt)$. Then $[M_1] = [M_2]$ if and only if $\overline{\chi}_\sigma(M_1) = \overline{\chi}_\sigma(M_2)$ for any $\sigma \in St(G_r)$ with $|\sigma| \le n$.*

PROPOSITION 2.6. *For each q (≥ 2), let $\mathcal{K}_n(q)$ denote the submodule of $\mathcal{I}_n^{G_r}$ consisting of those invariants T which satisfy that $T(M) \equiv 0 \pmod q$ for any G_r manifolds M (with $\dim(M) = n$). Then $\mathcal{K}_n(q)$ is generated by the class $\{q\theta_\sigma \mid \sigma \in St(G_r), |\sigma| \le n\}$.*

Example 2.7. An invariant T is determined by the values on the basis elements in $\mathcal{B}$ as follows. Suppose that T is written as

$$T = a_{\sigma_{-1}}\theta_{\sigma_{-1}} + \sum_{\sigma \in \mathcal{S}} (a_\sigma \theta_\sigma + a_{\sigma_*}\theta_{\sigma_*}). \tag{2.7.1}$$

Then we have $a_{\sigma_{-1}} = T(x_{\sigma_{-1}})$, $a_{\sigma^s} = T(x_{\sigma^s}) - T(x_{\overline{\sigma}^{s-1}})$ and $a_{\sigma_*^s} = T(x_{\sigma_*^s}) + T(x_{\overline{\sigma}^{s-1}})$ ($\sigma^s \in \mathcal{S}$) from (2.3.2), and $T(\widehat{x}_\tau) = -T(x_\tau)$ for any τ. Now divide M^{G_s} as $M^{G_s} = M_+^{G_s} \cup M_-^{G_s}$, where each component of $M_\varepsilon^{G_s}$ has even codimension in M if $\varepsilon = +$ or has odd codimension if $\varepsilon = -$. We then define by $\overline{\chi}_\varepsilon^{G_s}$ an invariant $\overline{\chi}_\varepsilon^{G_s}(M) = \overline{\chi}(M_\varepsilon^{G_s})$. As an example, we consider the case in which T is a sum of these $\overline{\chi}_\varepsilon^{G_s}$ ($0 \le s \le r$). In this case, for each $\sigma \in \mathcal{S}^s$ the value $T(x_\sigma)$ does not depend on a specific slice type σ but depends on the integer s. Similarly the values $T(x_{\sigma_*})$ are the same for all σ_* with $\sigma \in \mathcal{S}^s$ (cf. Example 1.11). To determine the form of T such that $T \in \mathcal{K}_n(q)$, set $\lambda_0 = q^{-1}T(x_{\sigma_{-1}})$, $\lambda_s = q^{-1}T(x_{\sigma^s})$ and $\mu_s = q^{-1}T(x_{\sigma_*^s})$ for $\sigma^s \in \mathcal{S}^s$ ($1 \le s \le r$). Then $a_{\sigma_{-1}} = q\lambda_0$, $a_{\sigma^s} = q(\lambda_s - \lambda_{s-1})$ and $a_{\sigma_*^s} = q(\mu_s + \lambda_{s-1})$ as mentioned above. Taking these in (2.7.1), we have that

$$T = \frac{q}{2^r}\left\{\lambda_0\overline{\chi} + \sum_{1 \le s \le r} 2^{s-1}(2\lambda_s - \lambda_{s-1})\overline{\chi}_+^{G_s} + \sum_{1 \le s \le r} 2^{s-1}(2\mu_s + \lambda_{s-1})\overline{\chi}_-^{G_s}\right\} \tag{2.7.2}$$

because $\overline{\chi} = \overline{\chi}_{\sigma_{-1}}$, $\overline{\chi}_+^{G_s} = \sum_{\sigma^s \in S^s} \overline{\chi}_{\sigma^s}$ and $\overline{\chi}_-^{G_s} = \sum_{\sigma^s \in S^s} \overline{\chi}_{\sigma_*^s}$. We further assume that T is a sum of $\overline{\chi}^{G_s}$ $(0 \leq s \leq r)$. Apply $T_t = \overline{\chi}^{G_t}$ to the elements x_{σ^s} and $x_{\sigma_*^s}$, then it is easy to check that $T_t(x_{\sigma^s}) = T_t(x_{\sigma_*^s}) + T_t(x_{\overline{\sigma}^{s-1}})$ $(0 \leq t \leq r)$. Hence T also has the same equalities; that is, $\lambda_s = \mu_s + \lambda_{s-1}$ $(1 \leq s \leq r)$. Taking these in (2.7.2), we obtain

$$\begin{aligned} T &= \frac{q}{2^r}\left\{\lambda_0\overline{\chi} + \sum_{1\leq s\leq r} 2^{s-1}(2\lambda_s - \lambda_{s-1})\overline{\chi}^{G_s}\right\} \\ &= \frac{q}{2^r}\left\{\sum_{1\leq s\leq r} 2^{s-1}\lambda_{s-1}(\overline{\chi}^{G_{s-1}} - \overline{\chi}^{G_s}) + 2^r\lambda_r\overline{\chi}^{G_r}\right\} \end{aligned} \tag{2.7.3}$$

because $\overline{\chi}^{G_s} = \overline{\chi}_+^{G_s} + \overline{\chi}_-^{G_s}$. In fact, for a G_r manifold M let us consider the induced $G_{r-s+1} \simeq G_r/G_{s-1}$ action on $M^{G_{s-1}}$. Then we have $\chi(M^{G_{s-1}}) \equiv \chi(M^{G_s}) \pmod{2^{r-s+1}}$ because G_{r-s+1} acts freely on $M^{G_{s-1}} - M^{G_s}$ (cf. [**5**, Theorem 5.24 (2)]). Similarly, we have $\chi(\partial M^{G_{s-1}}) \equiv \chi(\partial M^{G_s}) \pmod{2^{r-s+1}}$. These imply that $\overline{\chi}(M^{G_{s-1}}) \equiv \overline{\chi}(M^{G_s}) \pmod{2^{r-s+1}}$ by the definition of $\overline{\chi}$, and therefore T takes values in $q\mathbb{Z}$. Given t with $0 \leq t \leq r$, suppose that $\lambda_s = 0$ $(s < t)$, $\lambda_s = 1$ $(t \leq s)$ and $q = 2^{r-t}$ for example. Then we have

$$\overline{\chi}^{G_t} + \sum_{t<s\leq r} 2^{s-t-1}\overline{\chi}^{G_s} \equiv 0 \pmod{2^{r-t}}$$

(cf. [**5**, Corollary 5.20]).

For G_r manifolds M and N, we have the cartesian product $M \times N$ by straightening the angle, then it gives a multiplication on $SK_*^{G_r}(pt, pt)$ naturally.

PROPOSITION 2.8. *The multiplicative relations on the basis elements of $\mathcal{B}$ are given by the following:*

(i) $[G_r]^2 = 2^r[G_r]$;

(ii) $[G_r] \cdot [G_r \times_{G_s} D(\sigma_s)] = 2^{r-s}[D^{|\sigma^s|}][G_r]$;

(iii) $[G_r \times_{G_s} D(\sigma^s)] \cdot [G_r \times_{G_{s+k}} D(\tau^{s+k})] = 2^{r-(s+k)}[D^{|\tau^{s+k}|-|\overline{\tau}^s|}][G_r \times_{G_s} D(\sigma^s \times \overline{\tau}^s)]$;

(iv) $\widehat{y} \cdot z = y \cdot \widehat{z} = \widehat{y \cdot z}$ *and* $\widehat{(\widehat{y})} = \alpha y$ *for any y and z;*

where $\widehat{y} = [D^1]y$ in general, and if $\sigma^s = \sigma(\ldots, a(i), \ldots)$ and $\overline{\tau}^s = \sigma(\ldots, b(i), \ldots)$, then $\sigma^s \times \overline{\tau}^s = \sigma(\ldots, a(i) + b(i), \ldots)$.

Proof. For any G_r manifold, let $M_0 = M$ ignoring the action. Then $[M] \cdot [G_r] = [M_0] \cdot [G_r]$ in $SK_*^{G_r}(pt, pt)$ since each side has $\overline{\chi}_{\sigma_{-1}} = \overline{\chi} = 2^r\overline{\chi}(M_0)$. Thus (i) and (ii) follow by Proposition 2.5. To show (iii), let $\{\overline{\sigma}^i\}$ or $\{\overline{\tau}^j\}$ be the class (1.8) for σ^s

or τ^{s+k} respectively. Then $\sigma^s \times \overline{\tau}^s$ gives the class $\{\overline{\sigma}^i \times \overline{\tau}^i \mid 0 \leq i \leq s\}$ and each side of (iii) has the data

$$\begin{aligned}\overline{\chi}_{\overline{\sigma}^i \times \overline{\tau}^i} &= 2^{r-s}(-1)^{|\sigma^s|-|\overline{\sigma}^i|} \cdot 2^{r-(s+k)}(-1)^{|\tau^{s+k}|-|\overline{\tau}^i|} \\ &= 2^{r-(s+k)}(-1)^{|\tau^{s+k}|-|\overline{\tau}^s|} \cdot 2^{r-s}(-1)^{|\sigma^s \times \overline{\tau}^s|-|\overline{\sigma}^i \times \overline{\tau}^i|} \qquad (2.8.1)\end{aligned}$$

and $\overline{\chi}_\nu = 0$ if $\nu \notin \{\overline{\sigma}^i \times \overline{\tau}^i \mid 0 \leq i \leq s\}$ by Example 1.11. This implies (iii). The identities (iv) are clear. □

Let $SK_*^{G_r}$ be the SK theory for closed G_r manifolds in [**7**]. Then we have the following proposition.

PROPOSITION 2.9. *Let $[M_1]$ and $[M_2]$ be elements in $SK_n^{G_r}$. Then $[M_1] = [M_2]$ if and only if $\chi_\sigma(M_1) = \chi_\sigma(M_2)$ for any $\sigma \in St(G_r)$ with $|\sigma| \leq n$ and $|\sigma| \equiv n$* (mod 2).

Proof. First note that $\chi_\sigma(M_i) = 0$ if $|\sigma| \equiv n + 1$ (mod 2) because $\dim((M_i)_\sigma) = n - |\sigma| \equiv 1$ (mod 2) and $(M_i)_\sigma$ is closed (cf. (1.9)). From Proposition 2.5, it suffices to prove that the inclusion map $i_* : SK_n^{G_r} \to SK_n^{G_r}(pt, pt)$ is injective. To show this, suppose that $i_*(x) = 0$ for $x = [M_1] - [M_2] \in SK_n^{G_r}$, then $[M_1 \times D^1] = [M_2 \times D^1]$ in $SK_{n+1}^{G_r}(pt, pt)$ naturally. Now apply a map ∂_* to this, where $\partial_* : SK_{n+1}^{G_r}(pt, pt) \to SK_n^{G_r}$ is defined by $\partial_*([M]) = [\partial M]$. Then $2x = 0$ and hence $x = 0$ because $SK_n^{G_r}$ has no torsion (cf. [**7**, Theorem 5.5.1]). Therefore i_* is injective. □

Remark 2.10. For a closed G_r manifold M and $\sigma \in St(G_r)$, define $M^\sigma = \{x \in M \mid \sigma_x = \sigma \text{ or } \sigma_x \prec \sigma\}$. Then it is a codimension zero open invariant submanifold of M if $M^\sigma \neq \emptyset$ by the slice theorem and (1.8). Note that $M^\sigma \cap M_\sigma = \{x \in M \mid \sigma_x = \sigma\}$. Now let χ^σ be a G_r-SK invariant given by $\chi^\sigma(M) = \chi(M^\sigma)$, then the values $\{\chi^\sigma(M)\}$ also determine the class $[M]$ in $SK_*^{G_r}$ (cf. [**7**, 5.2 and Corollary 5.5.2]).

Hence, to obtain an SK relation between closed G_r manifolds, it is sufficient to consider it in the theory $SK_*^{G_r}(pt, pt)$. For the rest of this section, we give some examples from this point of view.

Let $H = S^1$ (the circle group) or G_1 and let M_1 and M_2 be $G_r \times H$ manifolds such that H acts freely on them. Although $G_r \times S^1$ is not a finite group, we can similarly define the notion of equivariant cutting and pasting of $G_r \times S^1$ manifolds as in Definition 1.1. If there is a $G_r \times H$-SK equivalence between them in the sense of Definition 1.2, then we write it as $M_1 \overset{H}{\sim} M_2$. Let U_i $(i = 1, 2)$ be of the form $\mathbb{C}^{a(0)} \times \prod_j V_j^{a(j)}$, where V_j is a two-dimensional irreducible G_r module $(j \geq 1)$.

Then an obvious $G_r \times H$-SK process gives that

$$S(U_1 \times U_2) + [a,b] \times S(U_1) \times S(U_2) \overset{H}{\sim} S(U_1) \times D(U_2) + D(U_1) \times S(U_2), \tag{2.11}$$

where H acts on the associated sphere $S(Y)$ ($Y = U_i$ or $U_1 \times U_2$) or $D(U_i)$ naturally and G_r (or H) acts trivially on the interval $[a,b]$. If $M_1 \overset{H}{\sim} M_2$ and N is an H manifold, then $[N \times_H M_1] = [N \times_H M_2]$ in $SK_*^{G_r}(pt, pt)$ by the induced G_r-SK process, where a G_r action on $N \times_H M_i$ is given by that on M_i. In particular, we have $[\overline{M}_1] = [\overline{M}_2]$, where $\overline{M}_i = M_i/H$ is the orbit space of M_i.

Example 2.12. (i) Let $\sigma = [G_r; U]$ ($r \geq 2$), where U is a product of two-dimensional irreducible G_r modules V_j as above. Then $S(U)_{\overline{\sigma}^s} = S^{|\sigma|-|\overline{\sigma}^s|-1}$ from Lemma 1.7, and $|\sigma| \equiv |\overline{\sigma}^s| \equiv 0 \pmod 2$ from (1.6.3). Thus $\chi_\tau(S(U)) = 0$ for any τ, and $[S(U)] = 0$ in $SK_*^{G_r}$ from Proposition 2.9. More precisely, we have $S(U) + K \overset{G_1}{\sim} K$ for some $G_r \times G_1$ manifold K, where $S(U)$ is regarded as a $G_r \times G_1$ manifold with G_1 acting via $G_1 \subset G_r$. In particular, $[N \times_{G_1} S(U)] = 0$ for any G_1 manifold N. To show this, first consider the case $U = V_j$ and divide $S(V_j) = S^1$ into four $G_r \times G_1$ invariant parts $A_u = G_r\{\exp(2\pi i t) \mid (u-1)/2^{r+2} \leq t \leq u/2^{r+2}\}$ ($1 \leq u \leq 4$), where $G_r\{\cdots\} = \cup_{x \in \{\cdots\}} G_r(x)$. Let $N_1 = A_1 + A_3$ and $N_2 = A_2 + A_4$, and let $\partial N_1 = \{p_j\}$ or $\partial N_2 = \{q_j\}$, where $p_j = q_j = \exp(2\pi i j/2^{r+2})$ ($0 \leq j < 2^{r+2}$). We define a $G_r \times G_1$ equivariant identification φ or $\psi : \partial N_1 \to \partial N_2$ by $\varphi(p_j) = q_j$ or $\psi(p_{2j}) = q_{2j}$, $\psi(p_{2j+1}) = q_{2j+3}$ ($q_{2^{r+2}+1} = q_1$) respectively. Then $N_1 \cup_\varphi N_2 = S(V_j)$ and $N_1 \cup_\psi N_2 = 2S(V_j)$, which implies that $S(V_j) + K_j \overset{G_1}{\sim} K_j$ by putting $K_j = S(V_j)$. In general, let U decompose as $U = V_j \times V$. The result is proved by induction on $\dim(V)$. Suppose that $S(V) + K' \overset{G_1}{\sim} K'$ for some K', then we also have $S(U) + K \overset{G_1}{\sim} K$ by using the $G_r \times G_1$-SK equivalence (2.11) when $(U_1, U_2) = (V_j, V)$, where $K = [a,b] \times K_j \times S(V) + K_j \times D(V) + D(V_j) \times K'$ for example.

(ii) The relation between $\mathbb{C}P^k$ and $\mathbb{R}P^{2k}$ in Example 1.4 is generalized to the case in which they have some G_r actions ($r \geq 2$) as follows. Given a slice type $\sigma = \sigma^r(0, a(1), \ldots, a(t))$ ($t = 2^{r-1} - 1$) and $a(0)$ (≥ 0), consider the associated projective space $M = \mathbb{C}P(\mathbb{C}^{a(0)} \times U)$, where $U = \prod_{1 \leq j \leq t} V_j^{a(j)}$. Then we have

$$[M] = \sum_{0 \leq k \leq t} a(k)\alpha^{a(k)-1}[\mathbb{R}P(\mathbb{R} \times U_{(k)})] \tag{2.12.1}$$

in $SK_*^{G_r}$, where $\sigma_{(k)} = [G_r; U_{(k)}]$ is σ if $k = 0$,

$$\sigma^r(0, a(k-1) + a(k+1), a(k-2) + a(k+2), \ldots, a(0) + a(2k),$$
$$a(2k+1), \ldots, a(t), 0, \ldots, 0)$$

if $1 \le k < 2^{r-2}$ or

$$\sigma^r(0, a(k-1) + a(k+1), a(k-2) + a(k+2), \ldots, a(2k-t) + a(t),$$
$$a(2k-t-1), \ldots, a(0), 0, \ldots, 0)$$

if $2^{r-2} \le k \le t$.

We can check the equality (2.12.1) by comparing the data $\{\chi_\tau\}$ of slice types for both sides. On the other hand, it is also obtained by performing an SK process in $SK_*^{G_r}(pt, pt)$ as follows. Consider the $G_r \times S^1$-SK equivalence (2.11) when $(U_1, U_2) = (\mathbb{C}^{a(0)}, U)$. Then we have

$$[M] + [[a, b] \times \overline{S(\mathbb{C}^{a(0)}) \times S(U)}] = [\overline{S(\mathbb{C}^{a(0)}) \times D(U)}] + [\overline{D(\mathbb{C}^{a(0)}) \times S(U)}].$$

Here $[\overline{S(\mathbb{C}^{a(0)}) \times Y(U)}] = [\mathbb{C}P^{a(0)-1} \times Y(U)]$ ($Y = S$ or D), which vanishes if $Y = S$ from (i) (cf. [**7,** Theorem 2.4.1 (iv)]). Continuing this SK process on $S(U)$ inductively, we have

$$\begin{aligned}
[M] &= [\mathbb{C}P^{a(0)-1} \times D(U)] + [\overline{D(\mathbb{C}^{a(0)}) \times S(U)}] \\
&= \sum_{0 \le k < u} [\mathbb{C}P^{a(k)-1} \times D(U_{(k)})] \\
&\quad + [\overline{D(\mathbb{C}^{a(0)} V_1^{a(1)} \ldots V_{u-1}^{a(u-1)}) \times S(V_u^{a(u)} \ldots V_t^{a(t)})}] \\
&= \sum_{0 \le k \le t} a(k) \alpha^{a(k)-1} [D(U_{(k)})] \qquad (2.12.2)
\end{aligned}$$

by the result in (i) and the equality $[\mathbb{C}P^{a(k)-1}] = a(k)\alpha^{a(k)-1}$ in Example 1.4. We may further consider the $G_r \times G_1$-SK equivalence (2.11) when $(U_1, U_2) = (\mathbb{R}, U_{(k)})$ similarly. Then $[\mathbb{R}P(\mathbb{R} \times U_{(k)})] = [D(U_{(k)})] + [D^1 \times_{G_1} S(U_{(k)})] = [D(U_{(k)})]$ from (i). Taking this in (2.12.2), we therefore obtain (2.12.1).

3. Multiplicative invariants

Definition 3.1. Let T be an invariant in Definition 2.1, which is defined for all G_r manifolds. We say that T is multiplicative if $T(M \times N) = T(M)T(N)$ for any G_r manifolds M and N. The map T induces a ring homomorphism $T : SK_*^{G_r}(pt, pt) \to \mathbb{Z}$.

Example 3.2. Invariants $\overline{\chi}^{G_s}$ and χ^{G_s} in Example 2.2 are multiplicative.

Definition 3.3. Let T be an invariant which is not necessarily multiplicative. We define an invariant $T_{(k)}$ by $T_{(k)}(M) = T(M)$ if $k = \dim(M)$ and $T_{(k)}(M) = 0$ if $k \neq \dim(M)$.

From now on, we exclude the trivial invariant $T \equiv 0$. We first consider the case $G_0 = \{1\}$.

PROPOSITION 3.4. *Any multiplicative invariant $T_0 : SK_*(pt, pt) \to \mathbb{Z}$ has a form $T_0 = \sum_{k\geq 0}(-a)^k \overline{\chi}_{(k)}$, where $a = T_0(D^1)$. Here, if $a = 0$, we regard a^0 as* 1.

Proof. It is easy to see that these T_0 are multiplicative. Let T_0 be any multiplicative invariant, then we can write $T_0 = \sum_{k\geq 0} p_k \overline{\chi}_{(k)}$ for some $p_k \in \mathbb{Z}$ by Lemma 1.3. If $k \geq 1$, then $p_k = p_1^k$ since $p_k = (-1)^k T_0(D^k)$. We note that $p_0 = T(pt) = 1$ since if $p_0 = 0$, then T_0 is trivial. Taking $a = -p_1$, we have the desired form. □

Remark 3.5. If $a = 0$, then $T_0 = (-0)^0 \overline{\chi}_{(0)} = \overline{\chi}_{(0)}$. If $a = 1$, then $T_0 = \sum_{k\geq 0}(-1)^k \overline{\chi}_{(k)} = \chi$ by Lemma 1.3. On the other hand, if $a = -1$, then $T_0 = \sum_{k\geq 0} \overline{\chi}_{(k)} = \overline{\chi}$.

Next we consider the invariants on G_r manifolds ($r \geq 1$).

Definition 3.6. Let s be an integer with $0 \leq s \leq r$. We say that a multiplicative invariant T is of type (s) if $T(G_r \times_{G_t} D(\sigma^t(\mathbf{0}))) = 0$ for $0 \leq t \leq s-1$ and $T(G_r \times_{G_s} D(\sigma^s(\mathbf{0}))) = \beta \neq 0$, where $\sigma^t(\mathbf{0}) = \sigma(0, \ldots, 0)$.

Remark 3.7. If $s \geq 1$, applying T to Proposition 2.8(iii) when $\sigma = \tau = \sigma^s(\mathbf{0})$, we have the identity $\beta^2 = 2^{r-s}\beta$. Hence $\beta = 2^{r-s}$. In the same way, if $s = 0$ then $\beta = 2^r$ by using Proposition 2.8(i).

We also have the following lemma.

LEMMA 3.8. *If T is of type (s), then:*
(i) $T(G_r \times_{G_u} D(\sigma^u)) = 0$ *for any σ^u with $u < s$;*
(ii) $T(G_r \times_{G_t} D(\sigma^t(\mathbf{0})) = 2^{r-t}$ *for $s \leq t \leq r$.*

PROPOSITION 3.9. *If T is of type* (0) *and $T(D^1) = a$, then $T(M) = T_0(M_0)$ for any G_r manifold M, where $M_0 = M$ ignoring the action.*

Proof. By Proposition 2.8, we have that $T(M)T(G_r) = T_0(M_0)T(G_r)$. Thus $T(M) = T_0(M_0)$ since $T(G_r) \neq 0$. □

Next we consider the general case $s \geq 1$. For any fixed k with $s+k \leq r$, we need to know the slice type $\overline{\sigma}^s$ in (1.8) for $\sigma^{s+k} = \sigma^{s+k}(\mathbf{e}_j)$, where $\mathbf{e}_j = (b(0), b(1), \dots)$ such that $b(j) = 1$ or zero otherwise.

LEMMA 3.10. *Let $j \in P = \cup_{-1 \leq i < 2^{s-1}} P(k; i)$, where $P(k; i)$ is the subset of $P = \{0, 1, \dots, 2^{s+k-1} - 1\}$ relating to $\overline{\sigma}^s$ (cf. (1.6.2)). Then we have that $\overline{\sigma}^s = \sigma^s(\mathbf{0})$ if $j \in P(k; -1)$, $\sigma^s(2, 0, \dots, 0)$ if $j \in P(k; 0)$ or $\sigma^s(\mathbf{e}_i)$ if $j \in P(k; i)$ with $1 \leq i < 2^{s-1}$. Moreover, if T is of type (s), then*

$$T(G_r \times_{G_{s+k}} D(\sigma^{s+k}(\mathbf{e}_j))) = \begin{cases} 2^{r-s-k}a & \text{if } j = 0 \in P(k; -1), \\ 2^{r-s-k}a^2 & \text{if } j \in P(k; -1)\backslash\{0\}, \\ 2^{-r+s-k}\xi_0^2 & \text{if } j \in P(k; 0), \\ 2^{-k}\xi_i & \text{if } j \in P(k; i), 1 \leq i < 2^{s-1}, \end{cases}$$

where $a = T(D^1)$ and $\xi_i = T(G_r \times_{G_s} D(\sigma^s(\mathbf{e}_i)))$ for $0 \leq i < 2^{s-1}$.

Proof. We write $\overline{\sigma}^s$ as $\overline{\sigma}^s = \sigma(c(0), c(1), \dots)$ for $\sigma^{s+k} = \sigma^{s+k}(\mathbf{e}_j)$. If $j \in P(k; i) = \{m2^s + i, (m+1)2^s - i \mid 0 \leq m < 2^{k-1}\}$ with $1 \leq i < 2^{s-1}$, then $c(i) = 1$ and $c(p) = 0$ $(p \neq i)$ by (1.6.1). Hence we have $\overline{\sigma}^s = \sigma^s(\mathbf{e}_i)$. The converse is obtained similarly. In the same way, we see that $\overline{\sigma}^s = \sigma^s(2, 0, \dots, 0)$ if and only if $j \in P(k; 0)$. From these we have that $\overline{\sigma}^s = \sigma^s(\mathbf{0})$ if and only if $j \in P(k; -1)$.

We now prove the second part. Let us write $\lambda_j = T(G_r \times_{G_{s+k}} D(\sigma^{s+k}(\mathbf{e}_j)))$ for convenience. We first consider the case $j \in P(k; i)$ with $i \neq -1$. Apply T to Proposition 2.8(iii) when $\sigma = \sigma^s(\mathbf{0})$ and $\tau = \sigma^{s+k}(\mathbf{e}_j)$, then we have $\lambda_j = 2^{-k}T(G_r \times_{G_s} D(\overline{\sigma}^s))$ since $|\sigma^{s+k}(\mathbf{e}_j)| = |\overline{\sigma}^s|$ $(= 2)$ by the above result and $\beta = 2^{r-s}$ (cf. Remark 3.7). Therefore, if $1 \leq i < 2^{s-1}$, the result follows since $\overline{\sigma}^s = \sigma^s(\mathbf{e}_i)$. When $i = 0$, we further use the identity $T(G_r \times_{G_s} D(\sigma^s(2, 0, \dots, 0))) = 2^{-r+s}T(G_r \times_{G_s} D(\sigma^s(\mathbf{e}_0)))^2$ and we have the result. On the other hand, if $j \in P(k; -1)$, we have $\lambda_j = 2^{r-s-k}T(D^{|\sigma^{s+k}(\mathbf{e}_j)|})$ in a similar way, and obtain the result. □

LEMMA 3.11. $T(G_r \times_{G_{s+k}} D(\sigma^{s+k})) = 2^{r-(s+k)}a^{|\sigma^{s+k}|-|\overline{\sigma}^s|}\gamma_{\overline{\sigma}^s}$ *for any slice type σ^{s+k}, where $\overline{\sigma}^s = \overline{\sigma}(c(0), c(1), \dots, c(2^{s-1}-1))$ is the slice type in (1.8) for σ^{s+k}. Further, $\gamma_{\overline{\sigma}^s} = \prod_i \gamma_i^{c(i)}$, where γ_i is the integer such that $\gamma_i = (1/2^{r-s})\xi_i$. If $a = 0$ or $\gamma_i = 0$ for some i, then we regard a^0 or γ_i^0 as 1 respectively.*

Proof. Let us write σ^{s+k} as $\sigma^{s+k} = \sigma(b(0), b(1), \dots, b(2^{s+k-1}-1))$ and again the values in Lemma 3.10 as $\lambda_j = T(G_r \times_{G_{s+k}} D(\sigma^{s+k}(\mathbf{e}_j)))$. Note that

$$T(G_r \times_{G_{s+k}} D(\sigma^{s+k}(0,\dots,0,b(j),0,\dots,0))) = \left(\frac{1}{2^{r-s-k}}\right)^{b(j)-1} \lambda_j^{b(j)} \quad (3.11.1)$$

for each j with $b(j) \geq 0$ by using Proposition 2.8(iii) when $\sigma = \sigma^{s+k}(\mathbf{e}_j)$ and $\tau = \sigma^{s+k}(0,\dots,0,b(j),0,\dots,0)$ inductively. Hence we have

$$T(G_r \times_{G_{s+k}} D(\sigma^{s+k})) = \left(\frac{1}{2^{r-s-k}}\right)^{l(\sigma)-1} \prod_{j\in P} \lambda_j^{b(j)} \quad (3.11.2)$$

by induction, where $l(\sigma) = \sum_i b(i)$ and $P = \{0, 1, \dots, 2^{s+k-1} - 1\}$.

Further, we set $L_i = \prod_{j\in P_i} \lambda_j^{b(j)}$, where $P_i = P(k;i)$ are the subsets of P in Lemma 3.10. Then we have

$$\begin{aligned} L_{-1} &= (2^{r-s-k})^{l(bP_{-1})} a^{|\sigma|-|\overline{\sigma}|}, \\ L_0 &= (2^{-r+s-k})^{l(bP_0)} \xi_0^{2l(bP_0)}, \\ L_* &= (2^{-k})^{l(bP_*)} \prod_{i\neq 0} \xi_i^{l(bP_i)} \end{aligned} \quad (3.11.3)$$

by Lemma 3.10 and (1.6.3), where $P_* = \cup_{i\neq -1,0} P_i$ and $l(bP_i) = \sum_{j\in P_i} b(j)$. Hence

$$T(G_r \times_{G_{s+k}} D(\sigma^{s+k})) = 2^{r-(s+k)} \left(\frac{1}{2^{r-s}}\right)^{2l(bP_0)+l(bP_*)} a^{|\sigma|-|\overline{\sigma}|} \xi_0^{2l(bP_0)} \prod_{i\neq 0} \xi_i^{l(bP_i)} \quad (3.11.4)$$

by (3.11.2), (3.11.3) and the fact that $l(\sigma) = l(bP_{-1}) + l(bP_0) + l(bP_*)$. Note that $c(0) = 2l(bP_0)$, $c(i) = l(bP_i)$ for $i \geq 1$ and $l(\overline{\sigma}) = 2l(bP_0) + l(bP_*)$ by (1.6.1) and (1.6.3). Therefore,

$$T(G_r \times_{G_{s+k}} D(\sigma^{s+k})) = 2^{r-(s+k)} \frac{a^{|\sigma|-|\overline{\sigma}|}}{(2^{r-s})^{l(\overline{\sigma})}} \xi_{\overline{\sigma}}. \quad (3.11.5)$$

In (3.11.1), if $b(j) = 0$, then the integer λ_j does not appear in (3.11.2). We thus regard λ_j^0 as 1 if $\lambda_j = 0$; in other words $a^0 = 1$ if $a = 0$ (when $j \in P(k;-1)$) or $\xi_j^0 = 1$ if $\xi_j = 0$ (when $j \in P(k;i)$ with $0 \leq i < 2^{s-1}$) by Lemma 3.10. Consider again (3.11.1) when $k = 0$. Then we note that $(1/2^{r-s})^{b(j)-1}\xi_j^{b(j)} \in \mathbb{Z}$ for any $b(j) \geq 0$. Hence we have $\xi_j = 2^{r-s}\gamma_j$ for some integer γ_j, and the lemma follows since $\xi_{\overline{\sigma}} = (2^{r-s})^{l(\overline{\sigma})}\gamma_{\overline{\sigma}}$. □

THEOREM 3.12. *Let T be any multiplicative invariant of type (s) with $s \geq 1$, and let $\{a, \gamma_i\}$ be the class of integers in Lemmas 3.10 and 3.11. Then T has the form*

$$T = \sum_{n,\sigma} (-a)^n \gamma_\sigma \overline{\chi}_{\sigma,(n+|\sigma|)},$$

where the sum is taken over all slice types $\sigma = \sigma^s$ and $n \geq 0$, and $\overline{\chi}_{\sigma,(j)} = (\overline{\chi}_\sigma)_{(j)}$ is the invariant defined in Definition 3.3. Further, if a or $\gamma_i = 0$ for some i, then we regard a^0 or γ_i^0 as 1 respectively.

Remark. We may consider that $\overline{\chi}_{\sigma,(j)}$ is defined for $j \geq |\sigma|$ since $\overline{\chi}_{\sigma,(j)}(M) = 0$ if $\dim(M) < |\sigma|$. Hence we write $j = n + |\sigma|$ with $n \geq 0$.

Proof of the theorem. We see that such T is multiplicative for the basis elements of $\mathcal{B}$ for $SK_*^{G_r}(pt, pt)$ and so it is for any G_r manifolds. For example, if $M = G_r \times_{G_s} D(\sigma_s)$ and $N = G_r \times_{G_{s+k}} D(\tau^{s+k})$, then we have

$$\overline{\chi}_{\sigma,(|\sigma|)}(M) \cdot \overline{\chi}_{\overline{\tau},(n+|\overline{\tau}|)}(N) = \overline{\chi}_{\sigma\times\overline{\tau},(n+|\sigma\times\overline{\tau}|)}(M \times N) \tag{3.12.1}$$

by (2.8.1) (when $i = s$), where $n = |\tau| - |\overline{\tau}|$. Therefore $T(M)T(N) = T(M \times N)$ by definition and the identity $\gamma_\sigma\gamma_{\overline{\tau}} = \gamma_{\sigma\times\overline{\tau}}$. Note that $T(D^n) = a^n$ by Remark 1.10(iii) and the fact that $\gamma_{\sigma^s(\mathbf{0})} = 1$. Further, T is of type (s) since $T(G_r \times_{G_s} D(\sigma^s(\mathbf{0}))) = (-a)^0\gamma_{\sigma^s(\mathbf{0})} \cdot 2^{r-s} = 2^{r-s}$ and $T(G_r \times_{G_u} D(\sigma^u(\mathbf{0}))) = 0$ if $u < s$ by definition. Now let T be any invariant which is multiplicative and of type (s). By Theorem 2.3, we can write T as

$$T = \sum_n a_{(n)}\theta_{\sigma_{-1},(n)} + \sum_{n,\sigma} b_{\sigma,(n+|\sigma|)}\theta_{\sigma,(n+|\sigma|)} + \sum_{n,\sigma_*} b_{\sigma_*,(n+|\sigma_*|)}\theta_{\sigma_*,(n+|\sigma_*|)} \tag{3.12.2}$$

where $n \geq 0$, $a_{(n)}$, $b_{\sigma,(j)}$, $b_{\sigma_*,(j)} \in \mathbb{Z}$ and $\sigma \in \mathcal{S}^t$ $(1 \leq t \leq r)$. By assumption $T(G_r) = 0$,

$$0 = T(D^n \times G_r) = a_{(n)}\theta_{\sigma_{-1},(n)}(D^n \times G_r) = a_{(n)} \cdot (-1)^n. \tag{3.12.3}$$

Hence $a_{(n)} = 0$ for any $n \geq 0$. Further, if σ^t is any slice type such that $t < s$ and $M = D^n \times G_r \times_{G_t} D(\sigma^t)$, then

$$0 = T(M) = \sum_{\overline{\sigma}^u \preceq \sigma^t} b_{\overline{\sigma},(n+|\sigma|)}\theta_{\overline{\sigma},(n+|\sigma|)}(M) \tag{3.12.4}$$

by Lemma 3.8(i), Example 1.11 and the definition of $\theta_{\sigma,(j)}$. By induction on t, suppose that $b_{\tau,(m+|\tau|)} = 0$ for any τ^u with $1 \leq u < t$ and $m \geq 0$; then

$0 = T(M) = b_{\sigma^t,(n+|\sigma^t|)}\theta_{\sigma^t,(n+|\sigma^t|)}(M)$ by (3.12.4). Hence we have that each $b_{\sigma^t,(n+|\sigma^t|)} = 0$ since $\theta_{\sigma^t,(n+|\sigma^t|)}(M) = (-1)^n$. Therefore we may write T as

$$T = \sum_{n,\sigma^s} \alpha_{\sigma^s,(n+|\sigma^s|)}\overline{\chi}_{\sigma^s,(n+|\sigma^s|)} + \sum_{s<t}\sum_{m,\tau^t} \beta_{\tau^t,(m+|\tau^t|)}\overline{\chi}_{\tau^t,(m+|\tau^t|)} \qquad (3.12.5)$$

by the definition of θ_σ and θ_{σ_*}. If $M = D^n \times G_r \times_{G_s} D(\sigma^s)$, then

$$T(M) = \alpha_{\sigma^s,(n+|\sigma^s|)}(-1)^n 2^{r-s} = a^n 2^{r-s}\gamma_{\sigma^s}$$

by Lemma 3.11 (when $k = 0$) and the assumption $T(D^n) = a^n$. Therefore we have

$$\alpha_{\sigma^s,(n+|\sigma^s|)} = (-a)^n\gamma_{\sigma^s}. \qquad (3.12.6)$$

To complete the proof, we must show that $\beta_{\tau^t,(m+|\tau^t|)} = 0$. Set $\tau = \tau^{s+k}$ and $M = D^n \times G_r \times_{G_{s+k}} D(\tau)$. By (3.12.5), we have $T(M) = p + q$, where

$$\begin{cases} p = \alpha_{\overline{\tau}^s,(n+|\tau|)}\overline{\chi}_{\overline{\tau}^s,(n+|\tau|)}(M), \\ q = \displaystyle\sum_{s<t\le s+k} \beta_{\overline{\tau}^t,(n+|\tau|)}\overline{\chi}_{\overline{\tau}^t,(n+|\tau|)}(M). \end{cases}$$

Here

$$\begin{aligned} p &= (-a)^{n+|\tau|-|\overline{\tau}^s|}\gamma_{\overline{\tau}^s} \cdot 2^{r-(s+k)}(-1)^{n+|\tau|-|\overline{\tau}^s|} \\ &= a^n \cdot 2^{r-(s+k)}a^{|\tau|-|\overline{\tau}^s|}\gamma_{\overline{\tau}^s} \end{aligned}$$

by (3.12.6) and Example 1.11, which is equal to $T(M)$ by Lemma 3.11 (when $\sigma = \tau^{s+k}$). Therefore,

$$q = \sum_{s<t\le s+k} \beta_{\overline{\tau}^t,(n+|\tau|)}2^{r-(s+k)}(-1)^{n+|\tau|-|\overline{\tau}^t|} = 0.$$

From this, we have $\beta_{\tau^t,(m+|\tau^t|)} = 0$ by induction on k. This completes the proof. □

Remark 3.13. If $r = 0$, then $St(G_0) = \{\sigma_{-1}\}$. Regarding $\gamma_{\sigma_{-1}}$ as 1 in the above theorem, we have $T = \sum_n(-a)^n\overline{\chi}_{\sigma_{-1},(n+|\sigma_{-1}|)} = \sum_n(-a)^n\overline{\chi}_{(n)} = T_0$ in Proposition 3.4.

Example 3.14. Suppose that $a = \gamma_i = 1$ for any i with $0 \le i < 2^{s-1}$. Let M be a G_r manifold with dimension m, and set $M^{G_s} = \cup_{\sigma^s} M_\sigma$ (cf. Remark 1.10(i)). Then $T(M) = \sum_{n+|\sigma^s|=m}(-1)^n\overline{\chi}_{\sigma,(n+|\sigma|)}(M) = \sum_{n+|\sigma^s|=m}(-1)^n\overline{\chi}(M_\sigma)$. Since $\dim(M_\sigma) = m - |\sigma| = n$, we have $T(M) = \sum_{\sigma^s}\chi(M_\sigma) = \chi^{G_s}(M)$ (cf. Lemma 1.3). Similarly, we have $T = \overline{\chi}^{G_s}$ if $a = -1$ and $\gamma_i = 1$ for any i.

Finally, we consider a mod 2 invariant; that is, a G_r-SK invariant T which takes values in the field $\mathbb{Z}_2 = \{0, 1\}$ of integers modulo 2. Let $\mathcal{I}_{2,*}^{G_r} = \mathrm{Hom}(SK_*^{G_r}(pt, pt), \mathbb{Z}_2)$ be the set consisting of all these invariants. Then a natural map $i_* : \mathcal{I}_*^{G_r} \to \mathcal{I}_{2,*}^{G_r}$ induced by $i : \mathbb{Z} \to \mathbb{Z}_2$ is epic because $SK_*^{G_r}(pt, pt)$ is a direct sum of copies of the integers $\mathbb{Z}$ (cf. Lemma 1.3 and (2.3.3)). Such T is also said to be multiplicative if $T(M \times N) \equiv T(M)T(N)$ (mod 2) for any G_r manifolds M and N. We note that $T(pt) \equiv 1$ if T is non-trivial. In this case, $T(G_r \times_{G_t} D(\sigma^t)) \equiv 0$ for any σ^t with $t < r$ because $T(G_r \times_{G_t} D(\sigma^t))^2 \equiv 2^{r-t}T(G_r \times_{G_t} D(\sigma^t \times \sigma^t)) \equiv 0$. Therefore T is always of type (r) in the sense of Definition 3.6. Now let $\Gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{2^{r-1}-1})$ be a 2^{r-1} tuple of elements in $\mathbb{Z}_2$ and set $\Gamma_+ = \{i \mid \gamma_i = 1\}$. For a G_r manifold M, define by $M^{G_r,\Gamma}$ the components of M^{G_r} whose slice types are of the form $[G_r; \prod_{i\in\Gamma_+} V_i^{a(i)}]$. Further, define by $M_0^{G_r,\Gamma}$ the set of isolated points in $M^{G_r,\Gamma}$. If $\Gamma_+ = \emptyset$, then we see that $M^{G_r,\Gamma} = M_{\sigma^r(\mathbf{0})}$ and $M_0^{G_r,\Gamma}$ is empty if $\dim(M) > 0$ (cf. Remark 1.10(iii)). We then have the following proposition.

PROPOSITION 3.15. *For each r (≥ 0), a (non-trivial)* mod 2 *multiplicative invariant T is $\overline{\chi}_0^{G_r,\Gamma}$ if $a \equiv 0$ or $\overline{\chi}^{G_r,\Gamma}$ if $a \equiv 1$ for some Γ, where $a \equiv T(D^1)$ and $\overline{\chi}_0^{G_r,\Gamma}$ (or $\overline{\chi}^{G_r,\Gamma}$) is defined by $\overline{\chi}_0^{G_r,\Gamma}(M) = \overline{\chi}(M_0^{G_r,\Gamma})$ (or $\overline{\chi}^{G_r,\Gamma}(M) = \overline{\chi}(M^{G_r,\Gamma})$) respectively.*

Proof. First we have that

$$T \equiv \sum_{n,\sigma} a^n \gamma_\sigma \overline{\chi}_{\sigma,(n+|\sigma|)} \quad (\mathrm{mod}\ 2), \tag{3.15.1}$$

where the sum is taken over all slice types $\sigma = \sigma^r$ and $n \geq 0$. We also regard a^0 (or γ_i^0) as 1 if $a \equiv 0$ (or $\gamma_i \equiv 0$) respectively. To see this, suppose that $r = 0$ (cf. Remark 3.13). We can write T as $T \equiv \sum_n p_n \overline{\chi}_{(n)}$ for some $p_n \in \mathbb{Z}_2$ by using the surjection $i_* : \mathcal{I}_*^{G_0} \to \mathcal{I}_{2,*}^{G_0}$. Therefore the multiplicative property of T gives us that $T \equiv \overline{\chi}_{(0)}$ if $a \equiv 0$ or $\overline{\chi}$ if $a \equiv 1$ in the same way as the original case in Proposition 3.4 (cf. Remark 3.5). Next we consider the case $r > 0$. By using the map i_*, we may write T as

$$T \equiv \sum_{n,\tau} a_{\tau,(n+|\tau|)} \theta_{\tau,(n+|\tau|)} \quad (\mathrm{mod}\ 2),$$

where τ is σ_{-1}, σ or σ_* in (3.12.2). Set $M = D^n \times G_r \times_{G_t} D(\tau^t)$ with $n \geq 0$ and $t < r$. Since $T(G_r \times_{G_t} D(\tau^t)) \equiv 0$, we have

$$0 \equiv T(M) \equiv \sum_{\overline{\tau}^u \preceq \tau^t} a_{\overline{\tau}^u,(n+|\tau|)} \theta_{\overline{\tau}^u,(n+|\tau|)}(M) \equiv \sum_{\overline{\tau}^u \preceq \tau^t} a_{\overline{\tau}^u,(n+|\tau|)}$$

because $\theta_{\overline{\tau}^u,(n+|\tau|)}(M) \equiv 1$ from (2.3.2). This implies that $a_{\tau^t,(n+|\tau|)} \equiv 0$ by the same induction as in (3.12.4). We therefore obtain the form (3.15.1) because $a_{\sigma,(n+|\sigma|)} \equiv T(D^n \times D(\sigma)) \equiv a^n\gamma_\sigma$ and $\theta_\sigma = \overline{\chi}_\sigma$ for any $\sigma = \sigma^r$ and $n \geq 0$. Now put $\Gamma_+ = \{i \mid \gamma_i = 1\}$; then $\gamma_\sigma = \prod_i \gamma_i^{a(i)} \equiv 0$ if $a(i) > 0$ for some $i \notin \Gamma_+$. Hence we have $T \equiv \sum_{n,\sigma} a^n\overline{\chi}_{\sigma,(n+|\sigma|)}$, where the sum is taken over all σ of the form $[G_r; \prod_{i\in\Gamma_+} V_i^{a(i)}]$. If $a \equiv 1$, then $T \equiv \overline{\chi}^{G_r,\Gamma}$ by definition. On the other hand, if $a \equiv 0$, then $T \equiv \sum_\sigma \overline{\chi}_{\sigma,(|\sigma|)}$ because $0^0 \equiv 1$. Hence $T \equiv \overline{\chi}_0^{G_r,\Gamma}$. □

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