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Begründet 1868 durch Alfred Clebsch · Carl Neumann

Fortgeführt durch Felix Klein · David Hilbert
Otto Blumenthal · Erich Hecke

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Tensor Product Mappings

J. R. HOLUB

§ 1. Introduction

Let X_1, X_2, Y_1, and Y_2 be Banach spaces, α a norm defined on the algebraic tensor product $X \otimes Y$, and $X \otimes_\alpha Y$ the completion of $X \otimes Y$ in the α-norm. Then if $S: X_1 \to X_2$ and $T: Y_1 \to Y_2$ are continuous linear maps, the mapping $S \otimes_\alpha T: X_1 \otimes_\alpha Y_1 \to X_2 \otimes_\alpha Y_2$ defined by $S \otimes_\alpha T\left(\sum_i x_i \otimes y_i\right) = \sum_i Sx_i \otimes Ty_i$ is called the *tensor product of* S *and* T or the *tensor product mapping* [3, p. 37].

If ε denotes the least crossnorm on $X \otimes Y$ and π denotes the greatest crossnorm, then $S \otimes_\varepsilon T: X_1 \otimes_\varepsilon Y_1 \to X_2 \otimes_\varepsilon Y_2$ and $S \otimes_\pi T: X_1 \otimes_\pi Y_1 \to X_2 \otimes_\pi Y_2$ are both continuous linear operators [3, p. 93 and 37]. Grothendieck has shown that if S and T are isomorphisms (i.e. linear homeomorphisms) then $S \otimes_\varepsilon T$ is also an isomorphism, but $S \otimes_\pi T$ need not be an isomorphism [3, p. 93]. On the other hand, if S and T are topological homomorphisms onto dense subspaces of X_2 and Y_2 respectively, then $S \otimes_\pi T$ is a topological homorphism which is onto but $S \otimes_\varepsilon T$ is not necessarily so [3, p. 39].

In § 3 we continue the study of tensor product mappings begun in [3] by considering tensor products of p-absolutely summing [6], nuclear [8] and quasi-nuclear [9] maps. In a recent paper [6] Lindenstrauss and Pelczynski have demonstrated the power of the notion of p-absolutely summing mappings in investigating the subspace structure of Banach spaces, while Grothendieck [3] and Pietsch [8], [9], have extensively studied nuclear and quasi- nuclear maps.

The main results of § 3 are (i) $S \otimes_\varepsilon T$ is *p-absolutely summing if and only if S and T are p-absolutely summing* ($1 \leqq p < +\infty$). However an example is given of absolutely summing maps S and T for which $S \otimes_\pi T$ is not p-absolutely summing for any $1 \leqq p < +\infty$. (ii) *If* α *is any crossnorm* ($\varepsilon \leqq \alpha \leqq \pi$) *then* $S \otimes_\alpha T$ *is nuclear if and only if S and T are nuclear.* (iii) $S \otimes_\varepsilon T$ *is quasi-nuclear if and only if S and T are quasi-nuclear.*

Again, an example is given to show that S and T may both be quasi-nuclear and yet $S \otimes_\pi T$ need not be quasi-nuclear.

Examples are given of classes of operators A for which S and T are in A but $S \otimes_\varepsilon T$ is not in A, providing a contrast to the results (i), (ii), and (iii) quoted above.

In § 4 we use the notion of tensor product mapping to show that certain results of Schatten [12] concerning subspaces of $X \otimes_\pi Y$ also hold for $X \otimes_\alpha Y$ where α is any uniform crossnorm.

The author wishes to thank Professor A. Pietsch for his helpful comments concerning portions of this paper.

§ 2. Preliminary Results and Notation

The only spaces considered in this paper are Banach spaces. By a subspace of a space X we will always mean a *closed* subspace.

Let X and Y be Banach spaces and $T: X \to Y$ a continuous linear mapping. Then T is said to be

(i) p-absolutely summing [6] $(1 \leqq p < +\infty)$ if there exists $C > 0$ such that

$$\left(\sum_{i=1}^{n} \|Tx_i\|^p\right)^{\frac{1}{p}} \leqq C \sup_{\substack{\|x^*\| \leqq 1 \\ x^* \in X^*}} \left(\sum_{i=1}^{n} |x^*(x_i)|^p\right)^{\frac{1}{p}} \tag{2.1}$$

for all sequences $(x_i)_{i=1}^n$, $n = 1, 2, \ldots$ in X.

If $p = 1$ we say T is *absolutely summing* rather than "1-absolutely summing". The source of this terminology is the fact that T is absolutely summing if and only if T sends unconditionally convergent sequences in X to absolutely convergent sequences in Y [6, 8, 1].

(ii) *nuclear* [3, 8] if there exist sequences $(x_i^*) \subset X^*$ and $(y_i) \subset Y$ such that

$$T(x) = \sum_{i=1}^{\infty} x_i^*(x) y_i \quad \text{for all} \quad x \in X \tag{2.2}$$

and $\sum_{i=1}^{\infty} \|x_i^*\| \, \|y_i\| < +\infty$.

(iii) *quasi-nuclear* [8, 9] if there exists a sequence $(x_i^*) \subset X^*$ such that $\sum_{i=1}^{\infty} \|x_i^*\| < +\infty$ and

$$\|T(x)\| \leqq \sum_{i=1}^{\infty} |x_i^*(x)| \quad \text{for all} \quad x \in X. \tag{2.3}$$

Let X be a Banach space. Recall that a set $M \subset X^*$ is called *essential* if $\|x^*\| \leqq 1$ for all $x^* \in M$, M is w^*-compact, and $\|x\| = \sup_{x^* \in M} |x^*(x)|$ for all $x \in X$.

Pietsch has given the following basic characterization of p-absolutely summing maps [8] (see also [6]).

Theorem. *Let $1 \leqq p < +\infty$, $T: X \to Y$ a continuous linear mapping, and $M \subset X^*$ an essential set. Then T is p-absolutely summing if and only if there exists a $C < +\infty$ and a positive regular Borel measure μ on M (i.e. an element of $[C(M)]^*$ such that for all $x \in X$*

$$\|T(x)\| \leqq \left(\int_M |x^*(x)|^p \, d\mu(x^*)\right)^{\frac{1}{p}}. \tag{2.4}$$

A sequence (x_i) in X is called a (Schauder) *basis* for X if for each $x \in X$ there exists a unique sequence of scalars (a_i) such that $x = \sum_{i=1}^{\infty} a_i x_i$, convergence in the norm topology of X. If this convergence is unconditional for each $x \in X$, then the basis is called an *unconditional basis*. Otherwise the basis is called a *conditional basis*.

Associated with the basis (x_i) for X is a sequence of continuous linear functionals $(f_i) \subset X^*$ defined by $f_n(x) = f_n\left(\sum_i a_i x_i\right) = a_n$ and called the *sequence of coefficient functionals associated with* (x_i). The basis (x_i) is sometimes denoted (x_i, f_i) to identify the coefficient functionals.

A basis (x_i) for X is said to be *similar* to a basis (y_i) for Y when $\sum_{i=1}^{\infty} a_i x_i$ converges if and only if $\sum_{i=1}^{\infty} a_i y_i$ converges. In this case it follows that there is an isomorphism $T: X \xrightarrow{\text{onto}} Y$ such that $T(x_i) = y_i$.

If (x_i, f_i) is a basis for X then it is well known that (f_i) is a basic sequence in X^* (i.e. (f_i) is a basis for its closed linear span $[f_i]$) and the sequence of coefficient functionals associated with (f_i) is similar to the basis (x_i) for X. We designate this by writing (f_i, x_i).

If X and Y are Banach spaces, $X \otimes Y$ is their algebraic tensor product, and α is a crossnorm on $X \otimes Y$ (i.e. a norm α for which $\alpha(x \otimes y) = \|x\| \, \|y\|$ [12]), then we denote by $X \otimes_\alpha Y$ the completion of $X \otimes Y$ in the α-norm.

In particular we define $X \otimes_\varepsilon Y$ to be the completion of $X \otimes Y$ in the norm

$$\left\| \sum_{i=1}^{n} x_i \otimes y_i \right\| = \sup_{\substack{\|f\| \leq 1,\, f \in X^* \\ \|g\| \leq 1,\, g \in Y^*}} \left| \sum_{i=1}^{n} f(x_i)\, g(y_i) \right| \tag{2.5}$$

and $X \otimes_\pi Y$ to be the completion in the norm

$$\left\| \sum_{i=1}^{n} x_i \otimes y_i \right\| = \inf \left\{ \sum_{j=1}^{k} \|x_j'\| \, \|y_j'\| : \sum_{j=1}^{k} x_j' \otimes y_j' = \sum_{i=1}^{n} x_i \otimes y_i \right\}. \tag{2.6}$$

If $S: X_1 \to X_2$ and $T: Y_1 \to Y_2$ are continuous linear maps then the linear mapping $S \otimes T: X_1 \otimes Y_1 \to X_2 \otimes Y_2$ defined by

$$S \otimes T\left(\sum_{i=1}^{n} x_i \otimes y_i \right) = \sum_{i=1}^{n} S x_i \otimes T y_i \tag{2.7}$$

is called the *tensor product* of S *and* T or the *tensor product map* [3]. If α is a crossnorm on $X \otimes Y$ and $S \otimes T$ is continuous on $X_1 \otimes Y_1$ to $X_2 \otimes Y_2$ (each space under the α-norm) then $S \otimes T$ may be extended to the continuous mapping $S \otimes_\alpha T: X_1 \otimes_\alpha Y_1 \to X_2 \otimes_\alpha Y_2$ which we also call the *tensor product mapping*.

A *uniform crossnorm* α is one for which if $S: X \to X$ and $T: Y \to Y$ are continuous linear maps, then

$$\|S \otimes_\alpha T(z)\|_\alpha \leq \|S\| \, \|T\| \, \|z\|_\alpha$$

for all $z \in X \otimes_\alpha Y$ [12]. It is well known that both ε and π are uniform crossnorms [12].

Now let (x_i, f_i) and (y_i, g_i) be bases for X and Y respectively and let α be a uniform crossnorm. Then the sequence of tensors $(x_i \otimes y_j)$ ordered as

$$\begin{array}{ccc|c}
x_1 \otimes y_1 & x_1 \otimes y_2 & x_1 \otimes y_3 & \cdots \\
x_2 \otimes y_1 & x_2 \otimes y_2 & x_2 \otimes y_3 & \cdots \\
x_3 \otimes y_1 & x_3 \otimes y_2 & x_3 \otimes y_3 & \cdots \\
\vdots & \vdots & \vdots &
\end{array} \tag{2.8}$$

is a basis for $X \otimes_\alpha Y$ called the *tensor product basis* whose sequence of coefficient functionals is the sequence $(f_i \otimes g_j)$ (ordered in the same way as the basis $(x_i \otimes y_j)$) [2]. The subsequence $(x_i \otimes y_i)$ of the basis $(x_i \otimes y_j)$ is called the *tensor diagonal* of the bases (x_i) and (y_i).

By an *ideal* A of operators we will mean a class of operators such that if $T: X \to Y \in A$ and $S: Z \to X$ and $Q: Y \to W$ are bounded linear operators then $T \circ S$ and $Q \circ T$ are both in A. We denote those operators from X to Y which are in A by $A(X, Y)$.

§ 3. Tensor Product Mappings

We first prove the following proposition which will be found useful in showing that the converse to a number of our later results are also valid.

Proposition 3.1. *Let $A(X, Y)$ denote an ideal of operators. Then for any crossnorm $\alpha \geqq \varepsilon$ and operators $S: X_1 \to X_2$, $T: Y_1 \to Y_2$, if $S \otimes_\alpha T \in A(X_1 \otimes_\alpha Y_1, X_2 \otimes_\alpha Y_2)$ and $T \neq 0$ then $S \in A(X_1, X_2)$.*

Proof. Choose $y_0 \in Y_1$ such that $\|T y_0\| = 1$ and $f_0 \in Y_2^*$, $\|f_0\| = 1$, such that $\langle T y_0, f_0 \rangle = 1$. Define the operators $P_0: X_1 \to X_1 \otimes_\alpha Y_1$ and $Q_0: X_2 \otimes_\alpha Y_2 \to X_2$ by

$$P_0(x) = x \otimes y_0,$$
$$Q_0\left(\sum_i x_i \otimes y_i\right) = \sum_i \langle f_0, y_i \rangle x_i.$$

Clearly each of P_0 and Q_0 is a continuous linear operator and $S = Q_0(S \otimes_\alpha T) P_0$. Since $S \otimes_\alpha T \in A(X_1 \otimes_\alpha Y_1, X_2 \otimes_\alpha Y_2)$ and A is an ideal, $S \in A(X_1, X_2)$.

As we have mentioned in the introduction, there are certain properties of mappings $S: X_1 \to X_2$ and $T: Y_1 \to Y_2$ which are carried over to the mapping $S \otimes_\varepsilon T$ but not to $S \otimes_\pi T$, and vice versa. The next theorem shows that p-absolute summability of S and T is such a property and provides a strong contrast between the ε and π topologies.

Theorem 3.2. *Let $S: X_1 \to X_2$ and $T: Y_1 \to Y_2$ be continuous linear mappings. Then $S \otimes_\varepsilon T: X_1 \otimes_\varepsilon Y_1 \to X_2 \otimes_\varepsilon Y_2$ is p-absolutely summing if and only if S and T are p-absolutely summing* $(1 \leqq p < +\infty)$.

Proof. Suppose S and T are p-absolutely summing ($1 \leqq p < +\infty$) and let B_1^* and B_2^* denote the closed unit balls in X_1^* and Y_1^* respectively.

Since S is p-absolutely summing there exists a $\mu \in [C(B_1^*)]^*$ such that

$$\|S(x)\|^p \leqq \int_{B_1^*} |\langle x, x^*\rangle^p \, d\mu(x^*) \quad \text{for all} \quad x \in X_1 \tag{3.2}$$

(see (2.4)).

Similarly, since T is p-absolutely summing there exists a $\nu \in [C(B_2^*)]^*$ such that

$$\|T(y)\|^p \leqq \int_{B_2^*} |\langle y, y^*\rangle|^p \, d\nu(y^*) \quad \text{for all} \quad y \in Y_1 . \tag{3.3}$$

Let $\sum_i x_i \otimes y_i \in X_1 \otimes_\varepsilon Y_1$. Then for any $y^* \in Y_1^*$, $\sum_i y^*(y_i)\, x_i \in X_1$ so by (3.2)

$$\Big\|\sum_i y^*(y_i)\, S(x_i)\Big\|^p \leqq \int_{B_1^*} \Big|\sum_i y^*(y_i)\, x^*(x_i)\Big|^p d\mu(x^*) .$$

Therefore

$$\int_{B_2^*} \Big\|\sum_i y^*(y_i)\, S(x_i)\Big\|^p d\nu(y^*) \leqq \int_{B_2^*} \Big[\int_{B_1^*} \Big|\sum_i y^*(y_i)\, x^*(x_i)\Big|^p d\mu\Big] d\nu . \tag{3.4}$$

Now fix $x^* \in X_2^*$, $\|x^*\| \leqq 1$. Then $\sum_i x^*(S(x_i))\, y_i \in Y_1$ and by (3.3)

$$\Big\|\sum_i x^*(S(x_i))\, T(y_i)\Big\|^p \leqq \int_{B_2^*} \Big|\sum_i x^*(S(x_i))\, y^*(y_i)\Big|^p d\nu(y^*) .$$

But $\Big|\sum_i x^*(S(x_i))\, y^*(y_i)\Big|^p \leqq \Big\|\sum_i S(x_i)\, y^*(y_i)\Big\|^p$ for every $\|x^*\| \leqq 1$ so we have by the above,

$$\Big\|\sum_i x^*(S(x_i))\, T(y_i)\Big\|^p \leqq \int_{B_2^*} \Big\|\sum_i S(x_i)\, y^*(y_i)\Big\|^p d\nu(y^*)$$

for every $\|x^*\| \leqq 1$.

It now follows from (2.5) and (3.4) that

$$\Big\|\sum_i S(x_i) \otimes T(y_i)\Big\|_\varepsilon^p \leqq \int_{B_2^*} \Big[\int_{B_1^*} \Big|\sum_i y^*(y_i)\, x^*(x_i)\Big|^p d\mu(x^*\Big] d\nu(y^*) .$$

Applying Fubini's theorem we have

$$\Big\|\sum_i S(x_i) \otimes T(y_i)\Big\|^p \leqq \int_{B_1^* \times B_2^*} \Big|\sum_i y^*(y_i)\, x^*(x_i)\Big|^p d(\mu \times \nu) . \tag{3.5}$$

Now if Q is the closed unit ball in $(X \otimes_\varepsilon Y)^*$, then $X \otimes_\varepsilon Y \subset C(Q)$ (where Q has the w^*-topology from $X \otimes_\varepsilon Y$) so certainly if K is the w^*-closure of $B_1^* \otimes B_2^*$ (i.e. $K = w^* - \mathrm{cl}\,\{x^* \otimes y^* \mid x^* \in B_1^*, y^* \in B_2^*\} \subset Q$) then $X \otimes_\varepsilon Y \subset C(K)$. Note that K is essential by definition of the ε-topology.

The mapping $\chi: X^* \times Y^* \to X^* \otimes_\pi Y^*$ defined by $\chi(x^*, y^*) = x^* \otimes y^*$ is continuous [11, p. 93]. Therefore the mapping $\chi_1 : B_1^* \times B_2^* \to B_1^* \otimes B_2^*$ (where $B_1^* \otimes B_2^*$ has the induced w^*-topology from $(X \otimes_\varepsilon Y)^*$) defined by $\chi_1(x^*, y^*) = x^* \otimes y^*$ is continuous since the induced topology on $B_1^* \otimes B_2^*$ is weaker than the π-topology. Indeed, the w^*-topology is weaker than the ε'-topology [12, p. 26] induced on $B_1^* \otimes B_2^*$ by $X_1^* \otimes_{\varepsilon'} Y_1^*$, a closed subspace of $(X_1 \otimes_\varepsilon Y_1)^*$ [12, p. 43].

It follows that the mapping $T: C(K) \to C(B_1^* \times B_2^*)$ defined by $T(f) = f \circ \chi_1$ is an isometry into since

$$\|f\| = \sup_{z \in K} |f(z)| = \sup_{x^* \otimes y^* \in B_1^* \otimes B_2^*} |f(x^* \otimes y^*)| = \sup_{(x^*, y^*) \in B_1^* \times B_2^*} |f \circ \chi_1(x^*, y^*)| = \|T(f)\| .$$

Therefore under the adjoint map T^* there corresponds to the product measure $\mu \times \nu$ on $B_1^* \times B_2^*$ a measure γ on K defined by $\gamma = T^*(\mu \times \nu)$ such that for $f \in C(K)$, $\gamma(f) = T^*(\mu \times \nu)(f) = (\mu \times \nu)(Tf) = \int\limits_{B_1^* \times B_2^*} f \circ \chi_1(x^*, y^*)\, d(\mu \times \nu)$;

i.e.

$$\int\limits_K f(z)\, d\gamma = \int\limits_{B_1^* \times B_2^*} f \circ \chi_1(x^*, y^*)\, d(\mu \times \nu)$$

for all $f \in C(K)$.

In particular for $\sum\limits_i x_i \otimes y_i \in C(K)$ the function $f(z) = \left|\left\langle \sum\limits_i x_i \otimes y_i, z \right\rangle\right|^p$ is in $C(K)$ so

$$\int\limits_K \left|\left\langle \sum_i x_i \otimes y_i, z \right\rangle\right|^p d\gamma = \int\limits_{B_1^* \times B_2^*} \left|\sum_i x^*(x_i)\, y^*(y_i)\right|^p d(\mu \times \nu) .$$

Hence by (3.5),

$$\left\| S \otimes_\varepsilon T \left(\sum_i x_i \otimes y_i\right) \right\|_\varepsilon^p \leqq \int\limits_K \left|\left\langle \sum_i x_i \otimes y_i, z \right\rangle\right|^p d\gamma$$

and by (2.4) $S \otimes_\varepsilon T$ is p-absolutely summing.

Conservely, if $S \otimes_\varepsilon T$ is p-absolutely summing, then by Proposition 3.1 and [10] each of S and T is p-absolutely summing.

Gelbaum and de Lamadrid [2] have shown that if (e_i) is the unit vector basis for l^2 then the basis $(e_i \otimes e_j)$ for $l^2 \otimes_\varepsilon l^2$ is conditional (see § 2 for notation). As an application of Theorem 3.2 we prove that the same is true for the basis (e_i) of l^1.

Proposition 3.3 *Let (e_i) denote the unit vector basis of l^1. Then the basis $(e_i \otimes e_j)$ for $l^1 \otimes_\varepsilon l^1$ is conditional.*

Proof. Let $i: l^1 \to l^2$ be the injection mapping which is known to be absolutely summing [7]. By Theorem 3.2 $i \otimes_\varepsilon i: l^1 \otimes_\varepsilon l^1 \to l^2 \otimes_\varepsilon l^2$ is absolutely summing and hence maps unconditionally convergent series in $l^1 \otimes_\varepsilon l^1$ into absolutely convergent series in $l^2 \otimes_\varepsilon l^2$ [6]. If the basis $(e_i \otimes e_j)$ for $l^1 \otimes_\varepsilon l^1$ were unconditional then whenever $\sum\limits_{i,j} a_{ij}\, e_i \otimes e_j \in l^1 \otimes_\varepsilon l^1$, $i \otimes_\varepsilon i \left(\sum\limits_{i,j} a_{ij} e_i \otimes e_j\right) = \sum\limits_{i,j} a_{ij} e_i \otimes e_j$ would be absolutely convergent in $l^2 \otimes_\varepsilon l^2$ and so $\sum\limits_{i,j} |a_{ij}| < +\infty$. But then the basis

$(e_i \otimes e_j)$ in $l^1 \otimes_\varepsilon l^1$ would be similar to (e_i) in l^1 and hence also to $(e_i \otimes e_j)$ in $l^1 \otimes_\pi l^1$ under the mapping $T: l^1 \otimes_\varepsilon l^1 \to l^1 \otimes_\pi l^1$, where $T(e_i \otimes e_j) = e_i \otimes e_j$. By a theorem of Pietsch it would then follow that l^1 is a nuclear space [10], a contradiction since no normed space is nuclear [3]. Hence the basis $(e_i \otimes e_j)$ for $l^1 \otimes_\varepsilon l^1$ must be conditional (*Remark.* Both the result of Gelbaum and de Lamadrid quoted above and Proposition 3.3 are special cases of a recent result of Pelczynski and Kwapien [5]).

In contrast to Theorem 3.2 we show that both S and T may be p-absolutely summing and yet $S \otimes_\pi T$ may not be p-absolutely summing. We will need the following lemmas which are proved in [4] (for the notation, see § 2).

Lemma 3.4. *Let (e_i) denote the unit vector basis for l^1. Then the tensor diagonal $(e_i \otimes e_i)$ in $l^1 \otimes_\pi l^1$ is similar to (e_i) in l^1.*

Lemma 3.5. *Let (e_i) denote the unit vector basis for l^2. Then the tensor diagonal $(e_i \otimes e_i)$ in $l^2 \otimes_\pi l^2$ is similar to (e_i) in l^1.*

Proposition 3.6. *Let $i: l^1 \to l^2$ be the (absolutely summing) injection map. Then $i \otimes_\pi i: l^1 \otimes_\pi l^1 \to l^2 \otimes_\pi l^2$ is not p-absolutely summing for any $1 \leqq p < +\infty$.*

Proof. Suppose $i \otimes_\pi i$ is p-absolutely summing. Then since $i \otimes_\pi i\,(e_i \otimes e_i) = e_i \otimes e_i$ and each of $(e_i \otimes e_i)$ in $l^1 \otimes_\pi l^1$ and $(e_i \otimes e_i)$ in $l^2 \otimes_\pi l^2$ is similar to (e_i) in l^1, we see that the restriction of $i \otimes_\pi i$ to $[e_i \otimes e_i]$ in $l^1 \otimes_\pi l^1$ (a p-absolutely summing map) is an isomorphism which induces in the obvious way an isomorphism $T: l^1 \xrightarrow{\text{onto}} l^1$ which is p-absolutely summing.

However every p-absolutely summing map is weakly compact [6] and hence no p-absolutely summing map on a non-reflexive space could be an isomorphism. This contradiction shows that $i \otimes_\pi i$ is not p-absolutely summing for any $1 \leqq p < +\infty$.

Theorem 3.2 and Proposition 3.6 demonstrate that p-absolute summability of operators differentiates sharply between the ε and π topologies. Our next result shows that nuclearity of operators is completely preserved for every crossnorm α on the tensor product.

Theorem 3.7. *Let $S: X_1 \to X_2$ and $T: Y_1 \to Y_2$ be continuous linear maps. Then for any crossnorm α ($\varepsilon \leqq \alpha \leqq \pi$) the mapping $S \otimes_\alpha T: X_1 \otimes_\alpha Y_1 \to X_2 \otimes_\alpha Y_2$ is nuclear if and only if S and T are nuclear.*

Proof. Suppose S and T are nuclear. Then by definition (2.2) there exist sequences $(x_n^*) \subset X_1^*$, $(y_n^*) \subset Y_1^*$, $(z_n) \subset X_2$, and $(w_n) \subset Y_2$ such that

$$S(x) = \sum_{n=1}^{\infty} x_n^*(x) z_n \quad \text{for all} \quad x \in X_1\,,$$

$$T(y) = \sum_{n=1}^{\infty} y_n^*(y) w_n \quad \text{for all} \quad y \in Y_1\,,$$

$$\sum_{n=1}^{\infty} \|x_n^*\|\,\|z\| < +\infty \quad \text{and} \quad \sum_{n=1}^{\infty} \|y_n^*\|\,\|w_n\| < +\infty\,. \tag{3.7}$$

Let $\sum_{i=1}^{m} x_i \otimes y_i \in X_1 \otimes_\alpha Y_1$. Then

$$\begin{aligned}
S \otimes_\alpha T\left(\sum_{i=1}^{m} x_i \otimes y_i\right) &= \sum_{i=1}^{m} S(x_i) \otimes T(y_i) \\
&= \sum_{i=1}^{m} \left[\left(\sum_{n=1}^{\infty} x_n^*(x_i) z_n\right) \otimes \left(\sum_{n=1}^{\infty} y_n^*(y_i) w_n\right)\right] \\
&= \sum_{i=1}^{m} \left[\sum_{n,k} x_n^*(x_i)\, y_k^*(y_i)\, z_n \otimes w_k\right] \\
&= \sum_{n,k} x_n^* \otimes y_k^* \left(\sum_{i=1}^{m} x_i \otimes y_i\right) z_n \otimes w_k
\end{aligned}$$

(where the summation is taken according to the ordering given in (2.8)). Thus we have

$$S \otimes_\alpha T\left(\sum_{i=1}^{m} x_i \otimes y_i\right) = \sum_{n,k} x_n^* \otimes y_k^* \left(\sum_{i=1}^{m} x_i \otimes y_i\right) z_n \otimes w_k\,. \tag{3.8}$$

Also $\sum_{n,k} \|x_n^* \otimes y_n^*\|\; \|z_n \otimes w_k\| = \sum_{n,k} \|x_n^*\|\; \|y_k^*\|\; \|z_n\|\; \|w_k\|$ (since α is a cross-norm for which $\varepsilon \leqq \alpha \leqq \pi$). By (3.7) this last sum is finite. Hence setting $f_{nk} = x_n^* \otimes y_k^* \in (X_1 \otimes_\alpha Y_1)^*$ and $V_{nk} = z_n \otimes w_k \in X_2 \otimes_\alpha Y_2$ we have

$$S \otimes_\alpha T(z) = \sum_{n,k} f_{nk}(z)\, V_{nk} \quad \text{for all} \quad z = \sum x_i \otimes y_i \in X_1 \otimes_\alpha Y_1$$

and $\sum_{n,k} \|f_{nk}\|\; \|V_{n,k}\| < +\infty$. By definition (2.2) $S \otimes_\alpha T$ is nuclear.

Conservely, if $S \otimes_\alpha T$ is nuclear then by Proposition 3.1 and [8] each of S and T is nuclear.

Theorem 3.8. *Let $S: X_1 \to X_2$ and $T: Y_1 \to Y_2$ be quasi-nuclear maps. Then $S \otimes_\varepsilon T: X_1 \otimes_\varepsilon Y_1 \to X_2 \otimes_\varepsilon Y_2$ is quasi-nuclear if and only if S and T are quasi-nuclear.*

Proof. Suppose S and T are quasi-nuclear. Then by definition there exist sequences $(x_n^*) \subset X_1^*$ and $(y_n^*) \subset Y_1^*$ such that $\sum_{n=1}^{\infty} \|x_n^*\| < +\infty$, $\sum_{n=1}^{\infty} \|y_n^*\| < +\infty$ and

$$\begin{aligned}
\|S(x)\| &\leqq \sum_{n=1}^{\infty} |x_n^*(x)| \quad \text{for} \quad x \in X_1 \\
\|T(y)\| &\leqq \sum_{n=1}^{\infty} |y_n^*(y)| \quad \text{for} \quad y \in Y_1\,.
\end{aligned} \tag{3.9}$$

Let $\sum_{i=1}^{\infty} x_i \otimes y_i \in X_1 \otimes_\varepsilon Y_1$. Then since $I \otimes_\varepsilon T$ is continuous (where I is the identity map on X_1), $\sum_{i=1}^{\infty} x_i \otimes T y_i \in X_1 \otimes_\varepsilon Y_2$. By definition of the ε-norm, $\sum_{i=1}^{\infty} q(T(y_i))\, x_i \in X_1$ for any $q \in Y_2^*$. Hence by (3.9)

$$\left\| \sum_{i=1}^{\infty} q(T(y_i))\, S(x_i) \right\| \leqq \sum_{n=1}^{\infty} \left| \sum_{i=1}^{\infty} q(T(y_i))\, x_n^*(x_i) \right| .$$

Since this holds for all $q \in Y_2^*$ we have

$$\left\| \sum_{i=1}^{\infty} S(x_i) \otimes T(y_i) \right\|_\varepsilon \leqq \sum_{n=1}^{\infty} \left\| \sum_{i=1}^{\infty} x_n^*(x_i)\, T(y_i) \right\| . \tag{3.10}$$

Now for each $n = 1, 2, \ldots$, $\sum_{i=1}^{\infty} x_n^*(x_i)\, y_i$ converges in Y_1. Hence by (3.9)

$$\left\| \sum_{i=1}^{\infty} x_n^*(x_i)\, T(y_i) \right\| \leqq \sum_{m=1}^{\infty} \left| \sum_{i=1}^{\infty} x_n^*(x_i)\, y_m^*(y_i) \right| .$$

Substituting this into (3.10) we obtain

$$\left\| \sum_{i=1}^{\infty} S(x_i) \otimes T(y_i) \right\|_\varepsilon \leqq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left| \sum_{i=1}^{\infty} x_n^*(x_i)\, y_m^*(y_i) \right| . \tag{3.11}$$

Setting

$$\begin{aligned} f_{nm} = x_n^* \otimes y_m^* \in (X_1 \otimes_\varepsilon Y_1)^*, \ \sum_{n,m} \|f_{nm}\| \\ = \sum_n \sum_m \|x_n^* \otimes y_m^*\| = \sum_n \sum_m \|x_n^*\| \; \|y_m^*\| < +\infty \end{aligned}$$

and

$$\|S \otimes_\varepsilon T(z)\|_\varepsilon \leqq \sum_{m,n} |f_{nm}(z)| \quad \text{for all} \quad z \in X_1 \otimes_\varepsilon Y_1$$

(by (3.11)). Therefore $S \otimes_\varepsilon T$ is quasi-nuclear.

Again, if $S \otimes_\varepsilon T$ is quasi-nuclear then by Proposition 3.1 and [9] each of S and T is quasi-nuclear.

In contrast to Theorem 3.8 we prove the following result which is rather surprising in view of the strong similarity (in many respects) between nuclear and quasi-nuclear mappings.

Proposition 3.9. *Let $T : l^1 \to l^2$ be defined by $T\left(\sum_i b_i e_i\right) = \sum_i a_i b_i e_i$ where $(a_i) \in c_0$ but $(a_i) \notin l^4$. Then T is quasi- nuclear but $T \otimes_\pi T$ is not quasi-nuclear.*

Proof. To show that T is quasi-nuclear we need only show it may be approximated arbitrarily closely in absolutely summing norm by finite dimensional maps [9]. Following Pietsch [8] we denote the absolutely summing norm of an operator Q by $\pi(Q)$.

Let $S: l^1 \to l^1$ be defined by $S\left(\sum_{i=1}^{\infty} b_i e_i\right) = \sum_{i=1}^{\infty} a_i b_i e_i$ where (a_i) is as specified above. Clearly S is well defined since $\sum_{i=1}^{\infty} b_i e_i$ converges unconditionally in l^1 and $(a_i) \in c_0$. Let $S_n: l^1 \to l^1$ be defined by $S_n\left(\sum_{i=1}^{\infty} b_i e_i\right) = \sum_{i=1}^{n} a_i b_i e_i$. Then $\|S - S_n\| \xrightarrow{n} 0$ (where $\|\ \|$ denotes the operator norm) and $T = i \circ S$, where $i: l^1 \to l^2$ is the (absolutely summing) injection map. Hence $\pi(T - i \circ S_n) = \pi(i \circ (S - S_n)) \leqq \pi(i) \|S - S_n\|$ [8]. Since $i \circ S_n$ is a finite dimensional map and $\|S - S_n\| \to 0$, according to our earlier comment T is quasi-nuclear. Suppose $T \otimes_\pi T$ is quasi-nuclear.

Then $T \otimes_\pi T: l^1 \otimes_\pi l^1 \to l^2 \otimes_\pi l^2$ has the property that $T \otimes_\pi T(e_i \otimes e_i) = a_i^2 e_i \otimes e_i$ and when restricted to $[e_i \otimes e_i]$ in $l^1 \otimes_\pi l^1$ induces a quasi-nuclear map R from $[e_i \otimes e_i]$ in $l^1 \otimes_\pi l^1$ into $[e_i \otimes e_i]$ in $l^2 \otimes_\pi l^2$. As we have mentioned in Lemmas 3.4 and 3.5, $(e_i \otimes e_i)$ in $l^1 \otimes_\pi l^1$ and $(e_i \otimes e_i)$ in $l^2 \otimes_\pi l^2$ are both similar to (e_i) in l^1. Therefore the mappings $Q_1: l^1 \to [e_i \otimes e_i] \subset l^1 \otimes_\pi l^1$ and $Q_2: [e_i \otimes e_i] \subset l^2 \otimes_\pi l^2 \to l^1$ defined by $Q_1(e_i) = e_i \otimes e_i$ and $Q_2(e_i \otimes e_i) = e_i$ are isomorphisms. It follows that the mapping $\hat{T}: l^1 \to l^1$ defined by $\hat{T} = Q_2 \circ R \circ Q_1$ is quasi-nuclear [9] and therefore $\hat{T} \circ \hat{T}: l^1 \to l^1$ is nuclear [8]. Note that $\hat{T} \circ \hat{T}(e_i) = a_i^4 e_i$ for $i = 1, 2, \ldots$.

Our proof will be complete when we show that a mapping $S: l^1 \to l^1$ with $S e_i = b_i e_i$ is nuclear if and only if $(b_i) \in l^1$. If (b_i) is in l^1, S is clearly nuclear. On the other hand, if S is nuclear then S^* is nuclear [8] and hence absolutely summing [8]. Since $S^*: l^\infty \to l^\infty$ and $S^* e_i = b_i e_i$, the fact that S^* is absolutely summing implies $\sum_{i=1}^{\infty} |b_i| < +\infty$, so $(b_i) \in l^1$.

Since we assumed $(a_i) \notin l^4$ we see that $T \otimes_\pi T$ cannot be quasi-nuclear.

Remark. In view of Proposition 3.1 and Theorems 3.2, 3.7 and 3.8 a natural question is whether there is an ideal A of operators such that S and T are in A but $S \otimes_\varepsilon T$ is not in A. We show here the existence of such an ideal.

Let A denote the class of weakly compact operators. Clearly the identity map $i: l^2 \to l^2$ is in $A(l^2, l^2)$ since l^2 is reflexive. However $i \otimes_\varepsilon i: l^2 \otimes_\varepsilon l^2 \to l^2 \otimes_\varepsilon l^2$ is the identity map on the non-reflexive space $l^2 \otimes_\varepsilon l^2$ [12] and hence cannot be weakly compact.

One can also show that if A denotes the class of Hilbertian mappings (see [6] for the definition) then there exist S and T in A such that $S \otimes_\varepsilon T$ is not in A.

§ 4. Subspaces of Tensor Product Spaces

It is well known that M and N may be closed subspaces of X and Y, respectively, and yet $M \otimes_\pi N$ may not be a closed subspace of $X \otimes_\pi Y$. Schatten [12] has shown that if M and N are each the range of a projection of norm

1 on X and Y, respectively, then $M \otimes_\pi N$ is a closed subspace of $X \otimes_\pi Y$ and moreover the π-norm on $X \otimes Y$ is an extension of the π-norm on $M \otimes N$. More generally, Schatten's proof shows that if M and N are complemented in X and Y, respectively, then $M \otimes_\pi N$ is a closed subspace of $X \otimes_\pi Y$ (though, of course, the norm of an element in $M \otimes N$ computed in $X \otimes_\pi Y$ may differ from its norm in $M \otimes_\pi N$). The proof given by Schatten makes essential use of the particular properties of the π-norm. We show in this section that these results also hold true in the space $X \otimes_\alpha Y$ where α is any uniform crossnorm. Our approach is through the notion of tensor product mappings.

Theorem 4.1. *Let α be a uniform crossnorm and M and N complemented subspaces of X and Y respectively. Then $M \otimes_\alpha N$ is a closed subspace of $X \otimes_\alpha Y$ (being isomorphic to $\overline{M \otimes N}$, closure in $X \otimes_\alpha Y$) and there is a projection $P: X \otimes_\alpha Y \to M \otimes_\alpha N$.*

Proof. Since M is complemented in X and N is complemented in Y there exist projections $P_1 : X \xrightarrow{\text{onto}} M$ and $P_2 : Y \xrightarrow{\text{onto}} N$. Therefore since α is a uniform crossnorm the tensor product mapping $P_1 \otimes_\alpha P_2 : X \otimes_\alpha Y \to M \otimes_\alpha N$ is continuous and is the identity map on $M \otimes N$.

Let I denote the restriction of $P_1 \otimes_\alpha P_2$ to $M \otimes N$. Then I^{-1} is the injection of $M \otimes N$ (with the inherited α-topology from $M \otimes_\alpha N$) into $X \otimes Y$ (with the inherited α-topology from $X \otimes_\alpha Y$) and is clearly continuous being the restriction of the tensor product map $i \otimes_\alpha j : M \otimes_\alpha N \to X \otimes_\alpha Y$, where $i: M \to X$ and $j: N \to Y$ are injection maps.

Therefore I^{-1} is an isomorphism from $M \otimes N$ (with the inherited α-topology from $M \otimes_\alpha N$) onto $M \otimes N$ (with α-topology from $X \otimes_\alpha Y$) and hence may be extended to an isomorphism of $M \otimes_\alpha N$ and $\overline{M \otimes N}$ (where the closure is taken in $X \otimes_\alpha Y$). It follows that $M \otimes_\alpha N$ is a closed subspace of $X \otimes_\alpha Y$ and, as the above proof shows, $P = P_1 \otimes_\alpha P_2$ is a projection of $X \otimes_\alpha Y$ onto $M \otimes_\alpha N$ with $\|P\| = \|P_1\| \cdot \|P_2\|$.

Finally, we show that the converse of Theorem 4.1 is also true.

Theorem 4.2. *Let M be a closed subspace of X, N a closed subspace of Y, and α a uniform crossnorm. Then there exist projections $P_1 : X \to M$ and $P_2 : Y \to N$ if and only if*

(i) *$M \otimes_\alpha N$ is a closed subspace of $X \otimes_\alpha Y$ and*

(ii) *There is a projection $P: X \otimes_\alpha Y \to M \otimes_\alpha N$.*

Proof. If the projections P_1 and P_2 exist, then by Theorem 4.1 both (i) and (ii) hold.

Conservely, suppose (i) and (ii) hold. For any $m_0 \neq 0, m_0 \in M$, there is a projection $Q: M \to [m_0]$. Hence by Theorem 4.1 $[m_0] \otimes_\alpha N$ is a closed subspace of $M \otimes_\alpha N$ and the tensor product mapping $Q \otimes_\alpha I : M \otimes_\alpha N \to [m_0] \otimes_\alpha N$ (where I is the identity map on N) is a projection onto $[m_0] \otimes_\alpha N$.

By the same theorem $[m_0] \otimes_\alpha Y$ is a closed subspace of $X \otimes_\alpha Y$. Therefore the composition map $(Q \otimes I) \circ P : X \otimes_\alpha Y \to [m_0] \otimes_\alpha N$ is a projection which when restricted to $[m_0] \otimes_\alpha Y$ is then a projection of $[m_0] \otimes_\alpha Y$ onto $[m_0] \otimes_\alpha N$. But since α is a crossnorm it is now immediate that there is a projection $P_2 : Y \to N$.

Similarly there is a projection $P_1 : X \to M$ and the theorem is proved.

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P- and z-Embedded Subspaces

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The notion of extending zero sets of a subspace to the entire topological space has proven useful in the theory of rings of continuous functions. If S is a non-empty subspace of a topological space $(X, \mathscr{T})$ then S is said to be z-embedded in the topological space X if every zero set of S is the intersection of S with some zero set of X. In particular, this concept has been shown (see [3]) to be appropriate in realcompactness. It has been used to give sufficient conditions for a space to be realcompact and to assist in the preservation of realcompactness under continuous mappings. In addition, Lindelöf subspaces (that are Tichonov), C^*-embedded subspaces, and cozero-sets are z-embedded. Also it characterizes normal spaces in that a space is normal if and only if every closed subset is z-embedded (see [3]).

It is clear that if S is z-embedded in X, then every cozero-set of S extends to a cozero-set of X. Also if S is C^*-embedded in X then S is z-embedded in X since every zero set of S is the zero set of a bounded continuous function. In perfectly normal spaces, the closed subsets are zero sets. Consequently every subset of a perfectly normal space is a z-embedded subset. Now we have numerous examples of z-embedded, non- C^*-embedded subsets, for example, any non- C^*-embedded subset of the real line.

In this paper the relationship between the extension of continuous pseudometrics from the subspace to the entire space (P-embedding) and z-embedding is discussed. Every P-embedded subset is C-embedded (see [7]) and therefore C^*-embedded and z-embedded.

For P-embedding various characterizations have been given in terms of the extension to the entire space of various covers on the subspace. In a similar way, it is shown that a subspace S is z-embedded in the topological space $(X, \mathscr{T})$ if and only if for every countable normal open cover $\mathscr{U}$ of S there exists a cozero-set G containing S and a refinement of $\mathscr{U}$ that extends to a countable cozero-set cover of G. Using this, a necessary condition for a subset to be z-embedded can be given. For z-embedded subsets, every separable continuous pseudometric on the subset must extend to a continuous pseudometric defined on some countable intersection of cozero-sets containing the subset.

Definitions. A non-empty subset S of a topological space $(X, \mathscr{T})$ is *z-embedded in X* if for every zero set Z of S there is a zero set Z' of X such that $Z' \cap S = Z$. The subset S is *P-embedded in X* if every continuous pseudometric on S extends to a continuous pseudometric on X. A pseudometric d on X is said to be *γ-separable* if there is a subset G of X such that the cardinal number of G is at most γ and G is dense in the topology $\mathcal{T}_d$ generated by d. A pseudometric d is *totally bounded* or *complete* if the topological space $(X, \mathcal{T}_d)$ is totally bounded or complete. If A and B are subsets of the space X, then A and B are *S-separated in* X if there are zero-sets Z and Z' of X such that $A \subset Z, B \subset Z'$ and $Z \cap Z' \cap S = \emptyset$.

We will use I to be a non-empty indexing set and $[I]$ to represent the collection of all finite subsets of I. If S is a subset of X and if $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ is a family of subsets of X, then by $\mathcal{U} | S$ we mean the family $(U_\alpha \cap S)_{\alpha \in I}$.

The definitions of normal cover, partition of unity, etc., can be found in [9]. Other terms, such as the defintions of a uniformity in terms of a collection of pseudometrics can be found in [4].

The following are needed to develop the main results of this paper.

Theorem 1 *(see [6]). Let $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ be an open cover of a topological space. Then the following statements are equivalent:*

(1) *The cover $\mathcal{U}$ is normal.*

(2) *There exists a locally finite cozero-set cover $(V_\alpha)_{\alpha \in I}$ of X such that* $\mathrm{cl}\, V_\alpha$ *is completely separated from U_α for each α in I.*

(3) *The cover $\mathcal{U}$ has a locally finite cozero-set refinement of the same cardinality as $\mathcal{U}$.*

As a corollary it is easy to show the following.

Corollary 1.1. *If $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ is a normal open cover of a topological space X, then there is a locally finite partition of unity $(f_\alpha)_{\alpha \in I}$ (f_α are bounded) on X such that $C\,Z(f_\alpha) \subset U_\alpha$ for all α in I.*

Proof. By Theorem 1, $\mathcal{U}$ has a locally finite cozero set cover $(V_\alpha)_{\alpha \in I}$ of X such that $\mathrm{cl}\, V_\alpha$ is completely separated from $C U_\alpha$ for all α in I. This implies that $V_\alpha \subset U_\alpha$ for all α in I. For each α in I let $V_\alpha = CZ(g_\alpha)$, where we may assume that $0 \leqq g_\alpha(X) \leqq 1$ for all x in X. Let $g = \sum_{\alpha \in I} g_\alpha$. Since $(V_\alpha)_{\alpha \in I}$ is a locally finite cover, g is well-defined and $g(x) > 0$ for all x in X. Also g is continuous on X. For each α in I, let $f_\alpha = (g_\alpha / g) \wedge 1$. Then $(f_\alpha)_{\alpha \in I}$ is a locally finite partition of unity on X such that $CZ(f_\alpha) \subset U_\alpha$ for all α in I and the f_α are bounded.

Corollary 1.2. *Every countable cozero-set cover of a topological space is normal.*

Proof. In Theorem 4.1. of [7], it is shown that countable cozero-set covers have countable locally finite cozero-set refinements. Hence the result follows from the theorem.

The following two lemmas are basic working tools when dealing with continuous pseudometrics, continuous functions, and partitions of unity.

Lemma 2. *Let X be a topological space, let γ be an infinite cardinal number and let f be a continuous function from X into the space Y.*

(1) *If m is a pseudometric on Y and if $f(X)$ is a γ-separable subset of (Y, m), then $d = m \circ (f \times f)$ is a γ-separable continuous pseudometric on X.*

(2) *If* $(Y, \| \|)$ *is a normed space and if* $f(X)$ *is a* γ*-separable subset in its metric topology, then* $d(x, y) = \|f(x) - f(y)\|$ $(x, y$ *in* $X)$ *defines a* γ*-separable continuous pseudometric on* X.

(3) *If* f *is a continuous real-valued function on* X, *then the function* Ψ_f *defined by*

$$\Psi_f(x, y) = |f(x) - f(y)| \qquad (x, y \text{ in } X)$$

is an $\aleph_0$*-separable continuous pseudometric on* X.

Proof. The second and third statements follow from the first. Let $X, (Y, m)$, and f be as in statement 1 of the lemma. It is easy to verify that d is a pseudometric on X. Since $f(X)$ is a γ-separable subset of (Y, m), there is a family $(f(x_\alpha))_{\alpha \in I}$ of power at most γ which is dense in the topology $\mathscr{T}_m$ generated by m restricted to $f(X)$. It is easy to verify that $(x_\alpha)_{\alpha \in I}$ is dense in the topology generated by d. Therefore d is γ-separable. The pseudometric d is the composition of two continuous functions and is therefore continuous.

Lemma 3. *Let* γ *be an infinite cardinal number and let* $(f_\alpha)_{\alpha \in I}$ *be a locally finite partition of unity of cardinality at most* γ *on the topological space* X. *The function* d *defined on* $X \times X$ *by*

$$d(x, y) = \sum_{\alpha \in I} |f_\alpha(x) - f_\alpha(y)| \qquad (x, y \in X)$$

is a bounded γ*-separable continuous pseudometric on* X.

Proof. It is easy to verify that d is a pseudometric on X. Since $\sum_{\alpha \in I} f_\alpha(x) = 1$ for all x in X, it follows that $d(x, y) \leqq 2$ for all x, y in X. Therefore d is bounded. To show that it is continuous, let x_0 and y_0 be in X. There are open sets U and V about x_0 and y_0 respectively and J and K in $[I]$ such that $f_\alpha(x) = 0$ for all x in U and $\alpha \notin J$ and $f_\alpha(y) = 0$ for all y in V and $\alpha \notin K$. Therefore, if (x, y) is in $U \times V$, then $d(x, y) = \sum_{\alpha \in J \cup K} |f_\alpha(x) - f_\alpha(y)|$. Therefore d is a finite sum of continuous functions on a neighborhood of (x_0, y_0) and so is continuous.

It remains to show that d is γ-separable. First observe that if $\{f_1, \ldots, f_n\}$ is a finite set of continuous functions defined on a topological space Z, then $e = \sum_{i=1}^{n} \Psi_{f_i}$ is a $\aleph_0$-separable continuous pseudometric on Z. To see that e is $\aleph_0$-separable, define φ from Z into R^n by $(\varphi(z))_i = f_i(z)$ for $1 \leqq i \leqq n$ and z in Z. The mapping φ is continuous and for x, y in Z, $e(x, y)$ is just $\|\varphi(x) - \varphi(y)\|$, where the norm on R^n is $\|(x_i)\| = \sum_{i=1}^{n} |x_i|$, which generates the topology of R^n. Since R^n is a separable metric space, it is hereditarily separable. By (1) of Lemma 2 it follows that e is $\aleph_0$-separable.

To show that d is γ-separable let Y be the following subset of $\prod_{\alpha \in I} Q_\alpha$, where Q_α is the unit interval for all α in I: $Y = \{(x_\alpha)_{\alpha \in I} : x_\alpha \in Q_\alpha$ for all α in I, $x_\alpha = 0$

for all α not in some $J \in [I]\}$. Define a function m on $Y \times Y$ by $m(x,y) = \sum_{\alpha \in I} |x_\alpha - y_\alpha|$ where $x = (x_\alpha)_{\alpha \in I}$, $y = (y_\alpha)_{\alpha \in I}$ are any elements of Y. It is easy to check that m is a pseudometric on Y. Since m is just $\sum_{\alpha \in I} \Psi_{\pi_\alpha}$, it follows that the projections π_α are continuous with respect to $\mathscr{T}_m$ for all α in I.

Define a function f from X to Y by $(f(x))_\alpha = f_\alpha(x)$ for all x in X and α in I. Observe that $d^0 = m \circ (f \times f)$. Since we already know that d is continuous, it is easy to see that f is continuous with respect to $\mathscr{T}_m$. Since any subspace of a γ-separable pseudometric space is γ-separable (use (1) of Lemma 2 to complete the proof) it suffices to show that (Y, m) is γ-separable.

For each $K \in [I]$, let π_K denote the projection of (Y, m) onto $\prod_{\alpha \in K} Q_\alpha$. By the remarks above about continuity of the projections, it follows that π_K is continuous. Also by the remarks above, the pseudometric $\sum_{\alpha \in K} \Psi_{\pi_\alpha}$ on Y is $\aleph_0$-separable. We wish to pick the countable dense subset in a special way. Let A_K be a countable dense subset of $\pi_K(Y)$ in $\prod_{\alpha \in K} Q_\alpha$. For each $a \in A_K$, define $y(a)$ in Y by $(y(a))_\alpha = a_\alpha$ if α is in K and 0 otherwise.

Consider the set $\{y(a) : a \in A_K$ and $K \in [I]\}$. This set has cardinality at most γ and is dense in (Y, m). Let x_0 be in Y and let $\varepsilon > 0$. There is a $K \in [I]$ such that $(x_0)_\alpha = 0$ for all $\alpha \notin K$. There is an a in $A_K \cap S_{\|\,\|}(\pi_K(x_0),)$, where $\|(x_\alpha)\| = \sum_{\alpha \in K} |x_\alpha|$. It follows that $y(a)$ is in $S_m(x_0, \varepsilon)$.

In the results to follow we will also need a characterization of extending γ-separable continuous pseudometrics in terms of extending open covers. The following result is implicit in the proof of Theorem 2.1 of [7]. Here we give a direct and shorter proof of this result.

Theorem 4. *Suppose that S is a subspace of a topological space X, that γ is an infinite cardinal number and that d is a γ-separable continuous pseudometric on S with a dense subset $(x_\alpha)_{x \in I}$ of cardinality at most γ. Then the following statements are equivalent:*

(1) *The pseudometric d can be extended to a γ-separable continuous pseudometric on X.*

(2) *For each $n \in N$, the cover $(S_d(x, 1/n))_{x \in S}$ of S has a refinement that can be extended to a locally finite cozero-set cover of X of cardinality at most γ.*

(3) *Every open cover of S of the form $(S_d(x_\alpha, 1/n))_{\alpha \in I}$ for a fixed n in N has a refinement that extends to a locally finite cozero-set cover of X.*

Proof. We first show that (1) *implies* (2). Suppose that d can be extended to a continuous (γ-separable) pseudometric d^* on X, let n be a natural number and observe that $\mathscr{U} = (S_d(x_\alpha, 1/n))_{\alpha \in I}$ covers S and refines $(S_d(x, 1/n))_{x \in S}$. If we set $\mathscr{U}^* = (S_{d^*}(x, 1/2n))_{x \in X}$ then $\mathscr{U}^* | S$ is a refinement of $\mathscr{U}$. Moreover $\mathscr{U}^*$ is an open cover of the paracompact space $(X, \mathscr{T}_{d^*})$ and hence has a locally finite cozero-set refinement $\mathscr{V}$. For each α in I let

$$W_\alpha = \cup \{V \in \mathscr{V} : V \cap S \subset S_d(x_\alpha, 1/n)\}.$$

One easily verifies that $\mathscr{W}=(W_\alpha)_{\alpha\in I}$ is a locally finite cozero-set over of X of cardinality at most γ such that $\mathscr{W}|S$ refines $\mathscr{U}$. Therefore (2) holds.

The implication (2) *implies* (3) is immediate.

Finally we show that (3) *implies* (1). For each fixed n in N the family $(S_d(x_\alpha, 1/n))_{\alpha\in I}$ is a (normal) open cover of S of cardinality at most γ and therefore, by (3), there is a locally finite cozero-set cover $\mathscr{C}_n$ of X such that $\mathscr{C}_n|S$ refines $(S_d(x_\alpha, 1/n))_{\alpha\in I}$. By (3) of Theorem 1, $\mathscr{C}_n$ is a normal cover and therefore by [7, Proposition 2.5] there is a normal open cover $\mathscr{W}^n=(W^n_\alpha)_{\alpha\in I}$ of X such that $W^n_\alpha\cap S\subset S_d(x_\alpha, 1/n)$ for each α in I. By Corollary 1.1 there is a locally finite partition of unity $(f^n_\alpha)_{\alpha\in I}$ on X such that $CZ(f^n_\alpha)\subset W^n_\alpha$ for all α in I.

Define the function d_n on $X\times X$ by

$$d_n(x, y)=\sum_{\alpha\in I}|f^n_\alpha(x)-f^n_\alpha(y)| \qquad (x, y\in X).$$

Then Lemma 3 shows that for each n in N, d_n is a γ-separable continuous pseudometric on X. Let $\mathscr{D}$ be the uniformity on X generated by $\{d_n : n\in N\}$. This uniformity includes the pseudometric d. For let $\varepsilon>0$ and choose i in N such that $1/i\leqq\varepsilon$. Select d_{2i} and $\delta=1/2$. If $s_{2i}(x, y)\leqq 1/2$ for x, y in S then x and y are in $CZ(f_\alpha^{2i})$ for some α in I. Since $CZ(f_\alpha^{2i})\cap S\subset S_d(x_\alpha, 1/2i)$ it follows that $d(x, y)\leqq\varepsilon$ and that $d\in\mathscr{D}|S\times S$ (see [4], page 217).

In Theorem 1 of [8] it is shown that every uniformly continuous pseudometric on a uniform subspace of a uniform space has a continuous extension. Let d^* be such an extension. Since each d_n is γ-separable, the topology generated by $\mathscr{D}$ is also γ-separable. Hence d^* must be γ-separable and continuous with respect to this topology, a subcollection of the original topology. Hence d^* is the required extension. The proof is now complete.

We observe that a result similar to Theorem 4 for totally bounded continuous pseudometrics is implicit in the proof of Theorem 2.7 of [2] or could be given using appropriate modifications in Theorem 4 and preceeding results. For completeness we state it as Theorem 5.

Theorem 5. *Suppose that S is a subspace of a topological space X and that d is a totally bounded continuous pseudometric on S. Then d can be extended to a totally bounded continuous pseudometric on X if and only if for each n in N the cover $(S_d(x, 1/n))_{x\in S}$ of S has a refinement that can be extended to a finite cozero-set cover of X.*

Subspaces that are z-embedded need not be Lindelöf subspaces nor cozero-sets. In the first place any non-Lindelöf (Tichonov) space is z-embedded in its Stone-Čech compactification. Furthermore, the closed unit interval is a cozero-set. The following Theorem however shows that Lindelöf subspaces (of Tichonov spaces) and cozero-sets are always z-embedded.

Theorem 6. *If S is a non-empty subspace of a topological space X, then the following statements hold.*

(1) *The subspace S is z-embedded in X if and only if any two completely separated subsets of S are S-separated in X.*

(2) *If S is Lindelöf and if X is a completely regular T_1-space, then S is z-embedded in X.*

(3) *If S is a cozero subset of X then it is z-embedded in X.*

Proof. To show (1) first recall that two subsets are completely separated if and only if they are contained in disjoint zero sets [4, page 17]. If S is z-embedded in X and if A and B are completely separated subsets of S, then A and B are contained in disjoint zero-sets Z_1 and Z_2 of S. Furthermore, there exist zero sets Z_3 of X such that $Z_3 \cap S = Z_1$ and $Z_4 \cap S = Z_2$. Therefore A is contained in Z_3, B is contained in Z_4 and $Z_3 \cap Z_4 \cap S = \emptyset$. To show the converse, let Z be a zero set of S, say $Z = Z(f)$. For each n in N, let $A_n = \{x \in S : |f(x)| \geqq 1/n\}$. Since they are disjoint zero sets, A_n and Z are completely separated in S for each n in N. Hence there are zero sets Z_n and Z'_n in X such that $Z \subset Z_n$, $A_n \subset Z'_n$, and $Z_n \cap Z'_n \cap S = \emptyset$ for each n in N. Then $Z^* = \bigcap_{n \in N} Z_n$ is a zero set of X and $Z^* \cap S = Z$.

To prove (2) let S be a Lindelöf subspace of a completely regular T_1 space X and let A be a zero set of S. Since $S - A$ is a cozero set in S, it is an F_σ in S, that is a countable union of closed subsets of S. Let $\mathscr{F} = \{(S - A) \cap Z : Z$ is a zero set of X such that $A \subset Z\}$. The family $\mathscr{F}$ is a collection of closed sets in $S - A$. We will show that $\cap \mathscr{F} = \emptyset$. Suppose that x is in $S - A$. Since any open set in X whose intersection with S is $S - A$ will be disjoint from A, it follows that x is not in $\mathrm{cl}_X A$. Hence by complete regularity there is a real-valued continuous function f on X such that $f(y) = 0$ for all y in $\mathrm{cl}_X A$ and $f(x) = 1$. The point x is not in $Z(f) \cap (S - A)$, an element of $\mathscr{F}$. Since F_σ subsets of Lindelöf spaces are Lindelöf, $S - A$ is Lindelöf. Thus there is a countable family $(Z_n)_{n \in N}$ of zero sets on X such that $Z_n \cap (S - A)$ is in $\mathscr{F}$ for all n, but $\bigcap_{n \in N} (Z_n \cap (S - A)) = \left(\bigcap_{n \in N} Z_n\right) \cap (S - A) = \emptyset$. Let $Z = \bigcap_{n \in N} Z_n$; then Z is a zero set on X. The set $A \subset Z_n$ for all n in N, therefore $A \subset Z$, and $Z \cap (S - A) = \emptyset$. Hence $Z \cap S = A$.

To show (3) let S be $X - Z(f)$, where we may assume that $f \leqq 0$, $f \in C(X)$, and let $Z(g)$, g in $C(S)$, $g \geqq 0$ be a zero set of S. Define a function h on X by $h(x) = 0$ if x is in $Z(f)$ and $h(x) = (f \wedge g)(x)$ if x is in S. Since $Z(h) \cap S = Z(g)$, the proof will be completed once we show that h is continuous on X. The continuity of h at points of S is clear, since h is the infimum of two continuous functions on S. Let x be in $Z(f)$ and $\varepsilon > 0$. The set $\{x \in X : h(x) < \varepsilon\} = \{(x \in X : f(x) < \varepsilon\} \cup \{x \in S : g(x) < \varepsilon\}$. The first set in this union is open in X and the second is open in S, hence is open in X.

It is well known (see for example [4], Theorem 1.18) that a C^*embedded subset of a topological space is C-embedded if and only if it is completely separated from every zero set disjoint from it. In the next proposition we show that Lindelöf subspaces satisfying the above condition are P-embedded.

Proposition 7. *Let X be a completely regular T_1 space. Every Lindelöf subspace of X that is completely separated from every zero set disjoint from it is P-embedded in X.*

Proof. Let S be a Lindelöf subspace of X that is completely separated from every zero set disjoint from it and let $\mathscr{U}$ be a locally finite cozero-set cover of S. By Theorem 2.1 of [7], it is sufficient to show that $\mathscr{U}$ has a refinement that can be extended to a normal open cover of X. Since S is Lindelöf, $\mathscr{U}$ has a countable subcover $(U_i)_{i\in N}$. By (2) of Theorem 6, S is z-embedded in X. Hence for each i in N, there exists a cozero-set W_i on X such that $W_i \cap S = U_i$. The subset $G = \bigcap_{i\in N} CW_i$ is a zero set that is disjoint from S. By hypothesis there is a continuous function g on X such that $g(x)=1$ for all x in G and $g(y)=0$ for all y in S. The family $\mathscr{W} = (W_i)_{i\in N} \cup (CZ(g))$ is a countable cozero-set cover of X, which by Corollary 1.2 is normal and clearly $\mathscr{W}|S$ refines $\mathscr{U}$. Therefore S is P-embedded in X.

Corollary 7.1. *Every Lindelöf, C-embedded subset of a completely regular T_1 space is P-embedded.*

Proof. This result follows immediately from ([4], Theorem 1.18) and the proposition.

A zero set of a topological space is completely separated from any zero set disjoint from it. Using the above proposition, the following corollary is now immediate.

Corollary 7.2. *Every Lindelöf zero set of a completely regular T_1 space is P-embedded.*

We now come to one of the main results of this paper. Normal covers have been utilized in many ways. Here we can use them to give a characterization for z-embedded subsets of a topological space.

Theorem 8. *Let S be a subspace of a topological space X. The subspace S is z-embedded in X if and only if whenever $\mathscr{U}$ is a normal open cover of S, card $\mathscr{U} \leqq \aleph_0$, there exists a cozero set G containing S and a refinement of $\mathscr{U}$ that can be extended to a cozero-set cover $\mathscr{V}^*$ of G such that card $\mathscr{U} =$ card $\mathscr{V}^*$.*

Proof. Suppose that S is z-embedded in X and let $\mathscr{U}$ be a normal open cover of S, card $\mathscr{U} \leqq \aleph_0$. By Theorem 1, $\mathscr{U}$ has a locally finite cozero-set refinement $\mathscr{V} = (V_i)_{i\in A}$, card $A =$ card $\mathscr{U}$. Since S is z-embedded in X, for each i in A there is a cozero-set V_i^* on X such that $V_i^* \cap S = V_i$. The subset $G = \bigcup_{i\in A} V_i^*$ is a cozero-set of X that contains S. Since V_i^* is a cozero-set of X and $V_i^* \subset G$, the set V_i^* is a cozero-set of G for all i in A. The family $(V_i^*)_{i\in A}$ is the required cozero-set cover of G whose restriction to S refines $\mathscr{U}$.

To show the converse, let A and B be two completely separated sets in S. By (1) of Theorem 6, it is sufficient to show that A and B are S-separated in X. By ([4], Theorem 1.15) there are disjoint zero sets Z_1 and Z_2 of S such that

$A \subset Z_1$ and $B \subset Z_2$. Let $\mathscr{U} = (CZ_1, CZ_2)$ and note that $\mathscr{U}$ is a finite normal open cover of S (Corollary 1.2). Therefore by hypothesis there exists a cozero-set G containing S and a finite cozero-set cover $\mathscr{V} = (V_i)_{i \in N}$ of G such that $\mathscr{V} | S$ refines $\mathscr{U}$. Let $U_j = \cup \{V_i : V_i \cap S \subset CZ_j\}$ for $j = 1, 2$. The sets U_1 and U_2 are cozero-sets of G and $CU_j \cap S \supset Z_j$ for $j = 1, 2$. By (3) of Theorem 6 the cozero-set G is z-embedded in X. Therefore, there exist cozero-sets U_i^* of X such that $U_j^* \cap G = U_j$ for $j = 1, 2$. It is easy to verify that $A \subset CU_1^*$, $B \subset CU_2^*$ and $CU_1^* \cap CU_2^* \cap S = \emptyset$. Therefore A and B are S-separated in X.

Suppose that S is a subspace of a topological space X. We know that S being P-embedded implies C-embedded implies C^*-embedded implies z-embedded and that in general these implications do not reverse. We now study some specific extension theorems. In Theorem 9 we show that every complete totally bounded continuous pseudometric on S can be extended to a continuous pseudometric on X. In Theorem 10 we prove that if S is z-embedded in X then every separable continuous pseudometric on S can be extended to a set G containing S that is a countable intersection of cozero-sets. This last result has been independently observed by R. L. Blair and A. Hager. We then give an example to show that the converse of Theorem 10 does not hold and observe that the converse does hold if X is a P-space.

Theorem 9. *Suppose that S is a subspace of a topological space $(X, \mathscr{T})$. Then every complete totally bounded continuous pseudometric on S can be extended to a continuous pseudometric on X.*

Proof. Let d be a complete totally bounded continuous pseudometric on S, let $\mathscr{T}_d$ be the pseudometric topology on S generated by d and let $\mathscr{G} = \{G \in \mathscr{T} : G \cap S \in \mathscr{T}_d\}$. Observe that $\mathscr{G}$ is a topology on X that is coarser than $\mathscr{T}$. Now $(S. \mathscr{T}_d)$ is a compact subspace of $(X, \mathscr{G})$ and since $(S, \mathscr{T}_d)$ is absolutely P-embedded (see [7], Theorem 3.10), $(S, \mathscr{T}_d)$ is P-embedded in $(X, \mathscr{G})$. Therefore d can be extended to a continuous pseudometric d^* on $(X, \mathscr{G})$. But $\mathscr{G} \subset \mathscr{T}$ implies that d^* is continuous with respect to $(X, \mathscr{T})$.

Theorem 10. *Suppose that S is a subspace of a topological space X, then* (1) *implies* (2) *and* (2) *implies* (3) *below.*

(1) *For each totally bounded continuous pseudometric d on S there exists a cozero-set G_d containing S such that d can be extended to a continuous pseudometric on G_d.*

(2) *The subspace S is z-embedded in X.*

(3) *For each separable continuous pseudometric d on S there exists a set G_d containing S such that G_d is a countable intersection of cozero-sets and d can be extended to a continuous pseudometric on G_d.*

Proof. (1) *implies* (2). Suppose that Z is a non-empty zero-set in S. Let f be a real-valued continuous bounded function on S such that $Z = Z_S(f)$ and $f \geqq 0$. Let Ψ_f be the pseudometric associated with f and note that Ψ_f is totally bounded (see Lemma 3.2 in [2]). Therefore by (1) there exists a cozero-set G

containing S and a continuous pseudometric d on G such that $d|S\times S=\Psi_f$. Let $g:G\to R$ be defined by $g(x)=\inf\{d(x,z):z\in Z_S(f)\}$ and let $Z'=Z_G(g)$. Then Z' is a zero-set on G and $Z'\cap S=Z$. Moreover, by (3) of Theorem 6, G is z-embedded in X and hence there exists a zero-set Z^* on X such that $Z^*\cap G=Z'$. Clearly $Z^*\cap S=Z$ and therefore S is z-embedded in X.

(2) *implies* (3). Let d be a separable continuous pseudometric on S with countable dense subset $(x_n)_{n\in N}$. For each n in N, the family $(S_d(x_i,1/n))_{i\in N}$ is a normal open cover of S. By Theorem 8 for each n in N there exists a cozero-set G_n containing S and a countable cozero-set cover $\mathscr{V}_n$ of G_n such that $\mathscr{V}_n|S$ refines the cover $(S_d(x_i,1/n))_{i\in N}$.

The set S is contained in $G=\bigcap_{n\in N} G_n$. By Theorem 4 we need only show that for n in N every cover of S of the form $(S_d(x_i,1/n))_{i\in N}$ has a refinement that extends to a locally finite cozero-set cover of G. But $\mathscr{V}_n|G$ covers G and by Corollary 1.2 every countable cozero-set cover is normal. Hence by Theorem 1, $\mathscr{V}_n|G$ has a locally finite cozero-set refinement whose restriction to S refines $(S_d(x_i,1/n))_{i\in N}$.

Note that in general the converse of (2) *implies* (3) of Theorem 10 does not hold. The space for this example is Γ, known as Niemytski space and discussed in problem $3K$ of [4]. The space Γ is the subset $\{(x,y):y\geqq 0\}$ of $R\times R$ with the usual neighborhoods of points (x,y) with $y>0$, and with a base of open neighborhoods for a point $(x,0)$ being the open tangent spheres at that point together with the point. Let $D=\{(x,0):x\in R\}$; D is a discrete subset of Γ of cardinality c. If we define f to be the usual R^2 metric distance from a point in Γ to the subset D, then it is easy to see that f is continuous and $Z(f)=D$. Hence D is a countable intersection of cozero-sets in Γ, namely $D=\bigcap_{n\in N}\{x\in\Gamma:f(x)<1/n\}$. Therefore the condition that every separable continuous pseudometric on D extend to a countable intersection of cozero sets containing D is automatically satisfied. The example will be completed once we show that D is not z-embedded in Γ. But for zero sets z-embedding is equivalent to C-embedding. Thus it is sufficient to show that D is not C-embedded in Γ. Now the space Γ is separable but D is a discrete subset of cardinality c, hence not every continuous function on D can extend to Γ.

A completely regular T_1 space X is a *P-space* in case every prime ideal in $C(X)$ is maximal. In a P-space every cozero-set is closed and every G_δ is open (see [4, 4J]). Thus one can easily verify that (1), (2), and (3) of Theorem 10 are equivalent if X is a P-space.

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Isomorphisms of $C_0(Y)$ with Y Discrete

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For any locally compact Hausdorff space Y, we denote by $C_0(Y)$ the Banach space of continuous scalar-valued functions vanishing at infinity on Y, provided with the usual supremum norm. (Here the scalar field is understood to be either the set of reals or the set of complex numbers. If we consider a mapping between two such function spaces, it is further understood that the same field of scalars is associated with both spaces.) If Y is actually compact, so that $C_0(Y)$ consists of all continuous scalar functions on Y, we will, whenever it is convenient to do so, represent this function space by $C(Y)$.

Now in [3] it was shown that if c denotes the space of convergent sequences, and c_0 the space of sequences convergent to zero, then any continuous isomorphism φ of c onto c_0 must satisfy $\|\varphi\| \, \|\varphi^{-1}\| \geqq 3$. Moreover, there exists a linear isomorphism φ of c onto c_0 with $\|\varphi\| \, \|\varphi^{-1}\| = 3$. This result gave a complete solution to a problem initially suggested by Banach in [1, p. 242]. Partial solutions to this problem had previously been given by the author in [2], and, quite independently, by Gurariĭ in [4]. The object of this paper is to note that the result of [3] is, in fact, a particular case of a theorem concerning spaces of continuous functions.

If N denotes the set of positive integers, and N^* denotes the one-point compactification of this set, then $c = C(N^*)$ and $c_0 = C_0(N)$. Thus the result cited above concerning isomorphisms of sequence spaces is a consequence of the following

Theorem. *Let Y be an infinite set with the discrete topology.*

(A) *If X is any locally compact Hausdorff space which contains a point of accumulation, and if φ is any continuous isomorphism of $C_0(X)$ onto $C_0(Y)$, then $\|\varphi\| \, \|\varphi^{-1}\| \geqq 3$.*

(B) *If Y^* denotes the one-point compactification of Y, there exists an isomorphism φ of $C(Y^*)$ onto $C_0(Y)$ satisfying $\|\varphi\| \, \|\varphi^{-1}\| = 3$.*

Proof. (A) It suffices to show that if φ is any norm-increasing isomorphism of $C_0(X)$ onto $C_0(Y)$ (i.e. $\|f\| \leqq \|\varphi(f)\|$, $f \in C_0(X)$), then $\|\varphi\| \geqq 3$. To this end, we assume the existence of a norm-increasing isomorphism φ of $C_0(X)$ onto $C_0(Y)$ satisfying $\|\varphi\| < 3$, and arrive at a contradiction.

Thus suppose such a φ exists and choose a positive real number ε satisfying

$$\|\varphi\| < (3 - \varepsilon)/(1 + \varepsilon) . \tag{*}$$

Let x_0 be an accumulation point of X and let U be a compact neighborhood of x_0. Let f be an element of $C_0(X)$ such that $1 = \|f\| = f(x)$, for all $x \in U$.

Since $\varphi(f)$ is zero at infinity on Y, there exists a finite subset $S \subsetneqq Y$ such that $|(\varphi(f))(y)| < \varepsilon$ for all $y \in Y - S$.

Now define $g \in C_0(Y)$ by $g = \varphi(f)\chi_S$ and choose a positive real number δ with $\delta < [3 - (\varepsilon + \|\varphi\|)]/2$. (Note that $(*)$, together with the fact that $\|\varphi\| \geqq 1$, insures that $\varepsilon < 1$, and hence the right-hand side of the last inequality is indeed strictly positive. Also $\|\varphi\| \geqq 1$ implies $\delta < 1$.) Next, employing the Riesz representation theorem, we consider the elements of the dual spaces $C_0(X)^*$ and $C_0(Y)^*$ as measures, and use the customary notation for unit point masses. We then note that for each $y \in Y$, the set $\{x \in X : |\varphi^* \mu_y(\{x\})| \geqq \delta\}$ is finite. Thus since S is a finite subset of Y, the set $T = \bigcup_{y \in S} \{x \in X : |\varphi^* \mu_y(\{x\})| \geqq \delta\}$ is a finite subset of X.

Since $\operatorname{Int}(U)$ is infinite, we may choose a point $x_1 \in \operatorname{Int}(U) - T$. Then $|\varphi^* \mu_y(\{x_1\})| < \delta$ for all $y \in S$, and since the measures $\varphi^* \mu_y$, $y \in S$, are regular and finite in number, there exists an open set V such that $x_1 \in V \subsetneqq U$, and $|\varphi^* \mu_y|(V) < \delta$ for all $y \in S$.

We next choose a real-valued element $f_1 \in C_0(X)$ such that $0 \leqq f_1(x) \leqq 1 = f_1(x_1)$ for all $x \in V$, and $f_1(x) = 0$ for all $x \in X - V$. Let M denote the maximum set of $|\varphi(f_1)|$: $M = \{y \in Y : |(\varphi(f_1))(y)| = \|\varphi(f_1)\|\}$. Since φ is norm-increasing, $|(\varphi(f_1))(y)| \geqq 1$ for $y \in M$, while for $y \in S$ we have

$$\begin{aligned} |(\varphi(f_1))(y)| &= |\textstyle\int \varphi(f_1)\, d\mu_y| = |\textstyle\int f_1\, d(\varphi^* \mu_y)| \\ &= \left|\int_V f_1\, d(\varphi^* \mu_y)\right| \leqq \|f_1\|\, |\varphi^* \mu_y|(V) < \delta < 1, \end{aligned}$$

which shows that $M \cap S = \emptyset$.

Now consider the element $g + 2\varphi(f_1) \in C_0(Y)$. Since $g(y) = 0$ if $y \in Y - S$, for such y we have $|g(y) + 2(\varphi(f_1))(y)| \leqq 2\|\varphi(f_1)\|$, with equality holding for $y \in M$. Thus $\|g + 2\varphi(f_1)\| = \max\{2\|\varphi(f_1)\|, |g(y) + 2(\varphi(f_1))(y)| : y \in S\}$. Next note that $\|g - \varphi(f)\| < \varepsilon$, so that, since φ^{-1} is norm-decreasing, $\|\varphi^{-1}(g) - f\| < \varepsilon$. Thus $\|\varphi^{-1}(g + 2\varphi(f_1))\| \geqq |(\varphi^{-1}(g))(x_1) + 2f_1(x_1)| > |f(x_1) + 2f_1(x_1)| - \varepsilon = 3 - \varepsilon$. Then again using the fact φ^{-1} is norm-decreasing, we conclude that $\|g + 2\varphi(f_1)\| > 3 - \varepsilon$. Thus at least one of the following conditions must be satisfied:

(i) $2\|\varphi(f_1)\| > 3 - \varepsilon$, or

(ii) $|g(y) + 2(\varphi(f_1))(y)| > 3 - \varepsilon$, for some $y \in S$.

We first suppose that (i) is true and consider the element $\varphi^{-1}(g) - 2f_1$ of $C_0(X)$. If $x \in X - V$, $f_1(x) = 0$, so that for such x $|(\varphi^{-1}(g))(x) - 2f_1(x)| = |(\varphi^{-1}(g))(x)| < |f(x)| + \varepsilon \leqq 1 + \varepsilon$. Moreover for $x \in V$, $|1 - 2f_1(x)| \leqq 1$. And since $V \subsetneqq U$, for all $x \in V$ we have $|(\varphi^{-1}(g))(x) - 2f_1(x)| < |f(x) - 2f_1(x)| + \varepsilon \leqq 1 + \varepsilon$. Thus $\|\varphi^{-1}(g) - 2f_1\| < 1 + \varepsilon$. Now recalling that $g(y) = 0$ for $y \in Y - S$, and that $M \cap S = \emptyset$, for $y \in M$ we have $|(\varphi(\varphi^{-1}(g) - 2f_1))(y)| = 2\|\varphi(f_1)\| > 3 - \varepsilon$. It thus follows that the function h defined by $h = (\varphi^{-1}(g) - 2f_1)/(1 + \varepsilon)$ is an element of $C_0(X)$ with $\|h\| < 1$, while $\|\varphi(h)\| > (3 - \varepsilon)/(1 + \varepsilon)$ which contradicts our choice of ε.

Next suppose that (ii) is true. Then for some $y \in S$ we would have $|g(y)| + 2|(\varphi(f_1))(y)| \geqq |g(y) + 2(\varphi(f_1))(y)| > 3 - \varepsilon$, and consequently

$|g(y)| > 3-\varepsilon-2|(\varphi(f_1))(y)| > 3-\varepsilon-2\delta > \|\varphi\|$. But since φ is norm-increasing and $\varepsilon<1$, the maximum set of $|\varphi(f)|$ is contained in S. And as $\varphi(f)=g$ on S, we have $\|f\|=1$ while $\|\varphi(f)\| = \|g\| \geqq |g(y)| > \|\varphi\|$, which again is a contradiction. Hence there can be no norm-increasing isomorphism φ of $C_0(X)$ onto $C_0(Y)$ with $\|\varphi\|<3$.

(B) Denote by y_∞ the point at infinity in Y^*, and let $\{y_k : k=1,2,\ldots\}$ be a sequence of distinct points of Y. Then necessarily $y_k \to y_\infty$ in Y^*. Define $\varphi: C(Y^*) \to C_0(Y)$ by

$$\begin{aligned}
(\varphi(f))(y_1) &= 3f(y_\infty),\\
(\varphi(f))(y_k) &= \tfrac{3}{2}(f(y_{k-1})-f(y_\infty)), \quad k\geqq 2,\\
(\varphi(f))(y) &= \tfrac{3}{2}(f(y)-f(y_\infty)), \quad y\in Y-\{y_k\},
\end{aligned}$$

for $f\in C(Y^*)$. Then φ is an isomorphism of $C(Y^*)$ onto $C_0(Y)$ with $\|\varphi\|=3$ and $\|\varphi^{-1}\|=1$.

Added in Proof. After submitting this article, the author received a preprint of a paper by Y. Gordon: On the distance coefficient between isomorphic function spaces. Israel J. Math. (to appear). Part (A) of the theorem in this article follows from the main theorem of Gordon's paper.

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Über die i-ten Koeffizienten der Kreisteilungspolynome

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1. Einleitung

Das n-te Kreisteilungspolynom $\Phi_n(x)$ hat genau die $\varphi(n)$ primitiven n-ten Einheitswurzeln als einfache Nullstellen[1]. Bezeichnet $a_i(n)$ für $i=0, \ldots, \varphi(n)$ den Koeffizienten von $x^{\varphi(n)-i}$ in $\Phi_n(x)$, so ist $(-1)^i a_i(n)$ die i-te elementarsymmetrische Funktion der primitiven n-ten Einheitswurzeln und damit eine Exponentialsumme mit $\binom{\varphi(n)}{i}$ Summanden. Für $i>\varphi(n)$ wird $a_i(n):=0$ gesetzt.

Es ist seit langem bekannt, daß $a_i(n)$ stets ganzzahlig ist und daß $a_0(n)=1$, $a_1(n)=-\mu(n)$ für alle n sowie $a_i(n)=a_{\varphi(n)-i}(n)$ für $n>1$ gilt, wobei $\mu(n)$ die Möbiusfunktion bezeichnet. I. Schur zeigte 1931 (siehe [5]), daß $a_i(n)$ für wachsendes i unbeschränkt ist. D. H. Lehmer [4] hat 1966 eine allgemeine Formel für $a_i(n)$ angegeben und $a_i(n)$ für $i \leqq 10$ berechnet. Weitere Ergebnisse für die Koeffizienten mit festem Index i sind bisher nicht bekannt geworden.

In der vorliegenden Arbeit wird zunächst eine neue explizite Formel für die Koeffizienten $a_i(n)$ der Kreisteilungspolynome $\Phi_n(x)$ hergeleitet; $a_i(n)$ wird dabei durch eine Summe dargestellt, die über alle Partitionen von i erstreckt ist und deren Summanden nur die Werte 0 und ± 1 annehmen. Aus dieser Formel ergibt sich die Abschätzung $|a_i(n)| \leqq p(i)-p(i-2)$ für alle n, wobei $p(i)$ die Anzahl der Partionen von i ist. Außerdem wird mit Hilfe dieser Formel bewiesen, daß $i^{-m} a_i(n)$ für alle $m \in \mathbb{N}$ bei wachsendem i unbeschränkt ist. Diese überraschende Tatsache zeigt, daß kein Polynom in i Schrankenfunktion für $a_i(n)$ sein kann. Schließlich werden die erzeugenden Funktionen der Dirichletschen Reihen $\sum_{n=1}^{\infty} a_i(n) n^{-s}$ berechnet und daraus Eigenschaften der (asymptotischen) Mittelwerte $\lim_{x\to\infty} x^{-1} \sum_{n \leqq x} a_i(n)$ abgeleitet.

Die folgenden Identitäten für die Kreisteilungspolynome werden als bekannt vorausgesetzt (siehe z. B. [4]):

$$\Phi_n(x)=\prod_{d \mid n}(x^d-1)^{\mu(n/d)}\,; \tag{1}$$

$$\Phi_{2n}(x)=\Phi_n(-x), \quad \text{wenn } n>1 \text{ ungerade ist}\,; \tag{2}$$

$$\Phi_n(x)=\Phi_{\tilde{n}}(x^{n/\tilde{n}}) \quad \text{mit} \quad \tilde{n}:=\prod_{p \mid n} p\,. \tag{3}$$

[1] Zur Vereinheitlichung der Schreibweise wird $\Phi_n(x)$ vorgeschlagen, da Φ an „Kreisteilung“ erinnert.

2. Darstellungen der Koeffizienten

Da die Koeffizienten $a_i(n)$ bis auf das Vorzeichen elementarsymmetrische Funktionen der primitiven n-ten Einheitswurzeln sind, können sie mit Hilfe der sog. Newtonschen Formeln als Polynome der Ramanujanschen Summen $c_n(m) = \sum_{\substack{k=1\\(k,n)=1}}^{n} e^{2\pi ikm/n}$ geschrieben werden, für die nach Ramanujan

$$c_n(m) = \sum_{d\,|\,D} d\mu(n/d), \qquad D = (n, m), \tag{4}$$

bzw. nach O. Hölder

$$c_n(m) = \mu(n/D)\,\varphi(n)/\varphi(n/D) \tag{5}$$

gilt. Zur Vereinfachung der folgenden Formeln werden einige Abkürzungen benötigt:

Definition. Sei $M \subset \mathbb{N}$ und $i \in \mathbb{N}_0$. $\mathfrak{p} = (n_k)_{k\in M}$ heißt eine *Partition von i in Summanden aus M*, wenn $n_k \in \mathbb{N}_0$ und $\sum_{k\in M} k n_k = i$ ist. $\mathfrak{P}_i(M)$ bzw. $p(i, M)$ bezeichnen die Menge bzw. die Anzahl der Partitionen von i in Summanden aus M. Statt $\mathfrak{P}_i(\mathbb{N})$ bzw. $p(i, \mathbb{N})$ wird stets $\mathfrak{P}_i$ bzw. $p(i)$ geschrieben. Für jedes $\mathfrak{p} \in \mathfrak{P}_i(M)$ sei

$$I := I(\mathfrak{p}) = \{k \in M;\, n_k = (\mathfrak{p})_k > 0\}\,.$$

Damit gilt (siehe [4]):

$$a_i(n) = \sum_{\mathfrak{p}\in\mathfrak{P}_i} \left\{\prod_{k\in I} [-c_n(k)]^{n_k}/(n_k!\,k^{n_k})\right\} \tag{6}$$

für alle $i, n \in \mathbb{N}$.

Für die Untersuchung der Koeffizienten ist (6) wenig geeignet, da die Summanden im allgemeinen nicht ganz sind und ihre Anzahl schnell wächst. Eine ähnliche Formel für $a_i(n)$, bei der die Summanden aber nur die Werte 0 und ± 1 annehmen, läßt sich ebenfalls mit Hilfe der Newtonschen Formeln und (4) durch vollständige Induktion herleiten; doch ermöglicht der folgende Satz 1 einen wesentlich einfacheren, direkten Beweis dieser grundlegenden Formel in Satz 2.

Satz 1. *Sei E der Operator, der das Argument einer Funktion um* 1 *erhöht:* $Ef(i) = f(i+1)$ *und entsprechend für alle* $s \in \mathbb{Z}$ $E^s f(i) = f(i+s)$. *Ferner seien* M_n^+ *bzw.* M_n^- *die Mengen der Teiler d von n, für die* $\mu\left(\frac{n}{d}\right) = +1$ *bzw.* -1 *ist. Dann gilt für alle i und alle* $n > 1$*:*

$$a_i(n) = \prod_{d\in M_n^+} (E^0 - E^{-d})\, p(i, M_n^-)\,. \tag{7}$$

Beweis. Für $n > 1$ ist $\sum_{d\,|\,n} \mu\left(\frac{n}{d}\right) = 0$; damit folgt aus (1):

$$\Phi_n(x) = \prod_{d\,|\,n} (1 - x^d)^{\mu(n/d)} = \prod_{d\in M_n^+} (1 - x^d) \prod_{d\in M_n^-} (1 - x^d)^{-1}\,.$$

Nun ist $\prod_{d\in M_n^-}(1-x^d)^{-1}$ die erzeugende Funktion von $p(i, M_n^-)$. Also gilt:

$$\begin{aligned}\Phi_n(x) &= \prod_{d\in M_n^+}(1-x^d)\sum_{i=0}^{\infty} p(i, M_n^-)\,x^i \\ &= \sum_{i=0}^{\infty}\left[\prod_{d\in M_n^+}(E^0-E^{-d})\,p(i, M_n^-)\right]x^i,\end{aligned}$$

da $(1-x^d)\sum_{i=0}^{\infty} p(i, M_n^-)\,x^i = \sum_{i=0}^{\infty}[(E^0-E^{-d})\,p(i, M_n^-)]\,x^i$ und der Operator distributiv ist. Koeffizientenvergleich ergibt dann die Behauptung.

Satz 2. *Sei $\mu(r)$ für $r\in\mathbb{N}$ die Möbiusfunktion und $\mu(r):=0$, wenn $r\notin\mathbb{N}$. Dann gilt für alle $i, n\in\mathbb{N}$:*

$$a_i(n) = \sum_{\mathfrak{p}\in\mathfrak{P}_i}\left\{\prod_{k\in I}(-1)^{n_k}\binom{\mu\left(\frac{n}{k}\right)}{n_k}\right\} \tag{8}$$

mit $\binom{z}{n} = \prod_{m=0}^{n-1}(z-m)/n!$.

Beweis. Für $n=1$ ergibt (8) $a_1(1)=-1$ und $a_i(1)=0$ für $i>1$. Für $n>1$ folgt aus Satz 1, wenn man einen bekannten Satz über Binome auf den Operator $\prod_{d\in M_n^+}(E^0-E^{-d})$ anwendet:

$$\begin{aligned}a_i(n) &= \sum_{K\subset M_n^+}(-1)^{\operatorname{card}K}\,p\Big(i-\sum_{k\in K}k, M_n^-\Big) \\ &= \sum_{K\subset M_n^+}\ \sum_{\substack{(n_k)_{k\in M_n^-}\\ n_k\in\mathbb{N}_0\\ \sum_{k\in M_n^-}kn_k = i-\sum_{k\in K}k}}(-1)^{\operatorname{card}K}.\end{aligned}$$

Hier kann man die beiden Summationen zusammenfassen, indem man $(-1)^{\operatorname{card}K}$ durch

$$\prod_{k\in M_n}(-1)^{n_k}\binom{\mu\left(\frac{n}{k}\right)}{n_k}\quad\text{mit}\quad M_n := M_n^-\cup M_n^+$$

ersetzt und über alle Partitionen $\mathfrak{p}\in\mathfrak{P}_i(M_n)$ summiert; denn einerseits ist

$$(-1)^{n_k}\binom{\mu\left(\frac{n}{k}\right)}{n_k} = 1\quad\text{für alle}\quad k\in M_n^-,$$

und andererseits ist für $k\in M_n^+$

$$\binom{\mu\left(\frac{n}{k}\right)}{n_k} = \begin{cases}1, & \text{wenn}\quad n_k = 0, 1,\\ 0, & \text{wenn}\quad n_k > 1,\end{cases}$$

so daß

$$\prod_{k \in M_n^+} (-1)^{n_k} \binom{\mu\left(\frac{n}{k}\right)}{n_k} = \begin{cases} 0, & \text{wenn mindestens ein } n_k > 1 \text{ ist}, \\ (-1)^{\Sigma n_k}, & \text{wenn alle } n_k \leqq 1 \text{ sind}. \end{cases}$$

Der Summation über $K \subset M_n^+$ entspricht dann die Summation über alle Partitionen $\mathfrak{p} \in \mathfrak{P}_i(M_n)$, für die $\{k \in M_n^+ ; n_k = 1\} = K$ ist. In den von Null verschiedenen Summanden wird damit

$$\sum_{k \in M_n^+} n_k = \operatorname{card} K,$$

und es gilt

$$a_i(n) = \sum_{\mathfrak{p} \in \mathfrak{P}_i(M_n)} \left\{ \prod_{k \in I} (-1)^{n_k} \binom{\mu\left(\frac{n}{k}\right)}{n_k} \right\}.$$

Summiert man über alle Partitionen $\mathfrak{p} \in \mathfrak{P}_i$, so ändert sich der Wert der Summe nicht, da $\mu\left(\frac{n}{k}\right) = 0$ ist für $k \notin M_n$, und man erhält die Koeffizientenformel (8), in der die Summation von n unabhängig ist.

3. Abschätzungen der *i*-ten Koeffizienten

Aus (8) ergibt sich leicht folgende obere Abschätzung:

$$|a_i(n)| \leqq p(i) - p(i-2) \quad \text{für alle} \quad i, n \in \mathbb{N}. \tag{9}$$

Bei dem Beweis unterscheidet man die Fälle $\mu(n) = 0$, $+1$ und $\mu(n) = -1$. Im ersten Fall verschwinden in (8) mindestens alle Summanden, die zu Partitionen mit $n_1 \geqq 2$ gehören; also gilt hierfür

$$|a_i(n)| \leqq \sum_{\mathfrak{p} \in \mathfrak{P}_i} 1 - \sum_{\substack{\mathfrak{p} \in \mathfrak{P}_i \\ n_1 \geqq 2}} 1 = p(i) - p(i-2),$$

da $\operatorname{card}\{\mathfrak{p} \in \mathfrak{P}_i ; n_1 \geqq 2\} = p(i-2)$ ist.

Aus (2) folgt für ungerades $n > 1$, daß $a_i(2n) = (-1)^i a_i(n)$ ist; damit gilt (9) auch für $\mu(n) = -1$, da $\mu(2n) = 1$ oder $\mu\left(\frac{n}{2}\right) = 1$ ist.

Die folgende Tabelle für $M(i) := \max_{n \in \mathbb{N}} |a_i(n)|$, $i \leqq 30$, zeigt allerdings, daß die Abschätzung (9) sehr grob ist (z. B. wird $p(30) - p(28) = 1886$):

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$M(i)$	1	1	1	1	1	1	2	1	1	1	2	1	2	2	2

i	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$M(i)$	2	3	3	3	3	3	3	4	3	3	3	3	4	4	5

Diese Tabelle wurde mit Hilfe von (8) berechnet, indem zunächst der genaue Wertebereich von $a_i(n)$ für alle n mit $\mu(n) = -1$ durch Fallunterscheidung bei $\mu\left(\frac{n}{p}\right)$, $p = 2, 3, 5$, bestimmt und dann mit (2) und (3) für die übrigen Werte von n ergänzt wurde. Weitere Maxima werden demnächst mit einer Rechenanlage errechnet.

Es gilt nicht immer $\left|\min_{n\in\mathbb{N}} a_i(n)\right| = \left|\max_{n\in\mathbb{N}} a_i(n)\right|$; Ausnahmen für $i \leqq 30$ sind $\min a_{18}(n) = -2$, $\max a_{20}(n) = 2$, $\min a_{28}(n) = \min a_{30}(n) = -3$.

Die durch die obige Tabelle nahegelegte Vermutung, daß $|a_i(n)| \leqq i$ für alle $i, n \in \mathbb{N}$ gilt, wird durch folgendes Beispiel widerlegt:

$$a_{1820}\Bigl(\prod_{599 \leqq p \leqq 1217} p\Bigr) = 2121\,. \tag{10}$$

Obwohl hier $n \approx 7{,}55 \cdot 10^{267}$ ist, läßt sich der Wert dieses Koeffizienten doch mit Hilfe von (8) durch einfaches Abzählen von Primzahlen berechnen.

In dem folgenden Satz soll nun darüber hinaus gezeigt werden, daß $M(i)$ für hinreichend großes i größer als jede feste Potenz von i wird.

Satz 3. *Zu jedem $m \in \mathbb{N}$ gibt es ein $i_0 = i_0(m) \in \mathbb{N}$, so daß für alle $i \geqq i_0$ gilt:*

$$M(i) = \max_{n\in\mathbb{N}} |a_i(n)| > i^m\,. \tag{11}$$

Für den Beweis dieses Satzes werden folgende Bezeichnungen und ein Hilfssatz benötigt:

p, q, p_k, q_k bezeichnen stets Primzahlen; $p, p', \ldots, p^{(k)}$ sind aufeinanderfolgende Primzahlen und $p + \cdots + p^{(k)}$ ihre Summe. Ferner sei

$$S_k(q, Q; i) := \sum_{\substack{q < q_1 < \cdots < q_k \leqq Q \\ q_1 + \cdots + q_k \leqq i}} 1$$

und O das Landau-Symbol, das im folgenden nur von N abhängt.

Hilfssatz 4. *Sei $N \in \mathbb{N}$ fest gewählt, $i \in \mathbb{N}$, $Q \in \mathbb{R}$ mit $\frac{i}{N} \leqq Q \leqq i$, $q = q(N, i)$ $:= \max\{p; p + \cdots + p^{(N)} \leqq i\}$ sowie $n = \prod_{q \leqq p \leqq Q} p$. Dann gilt für alle i mit $qq' > i$ und alle $r \in \mathbb{N}$ mit $\mu(nr) = -1$ und $\Bigl(r, \prod_{p \leqq i} p\Bigr) = 1$:*

$$a_i(nr) =' \sum_{k=1}^{N} (-1)^k \{S_k(q, Q; i) - S_k(q, Q; i - q)\}\,. \tag{12}$$

Beweis. Wegen $qq' > i$ und $\mu(nr/p) = 0$ oder 1 tragen in (8) nur Partitionen von i zu $a_i(nr)$ bei, deren Summanden außer 1 verschiedene Primteiler von n

sind ($n_p \leqq 1$):

$$
\begin{aligned}
a_i(nr) &= 1 + \sum_{k=1}^{N+1} \sum_{\substack{q \leqq q_1 < \cdots < q_k \leqq Q \\ q_1 + \cdots + q_k \leqq i}} (-1)^k \\
&= 1 - 1 + \sum_{k=2}^{N+1} (-1)^k \sum_{\substack{q = q_1 < \cdots < q_k \leqq Q \\ q + q_2 + \cdots + q_k \leqq i}} 1 + \sum_{k=1}^{N+1} (-1)^k S_k(q, Q; i) \\
&= \sum_{k=1}^{N} (-1)^{k+1} S_k(q, Q; i-q) + \sum_{k=1}^{N} (-1)^k S_k(q, Q; i),
\end{aligned}
$$

da $q' + \cdots + q^{(N+1)} > i$ ist. q.e.d.

Aus der letzten Ungleichung folgt wegen $q' < 2q$, daß $q > (2^{N+2} - 2)^{-1} i$ gilt. Damit ist die Bedingung $qq' > i$ für alle hinreichend großen i erfüllt.

Relativ große Koeffizienten ergeben sich u. a. für $Q = \dfrac{2i}{N+1}$ und $Q = i - q' - \cdots - q^{(N-1)}$. Der folgende Beweis vereinfacht sich jedoch, wenn $Q = \dfrac{i}{N}$ gewählt wird.

Beweis von Satz 3. Sei $m \in \mathbb{N}$ fest vorgegeben und $N = m + 1$ sowie i, q, n und r wie in Hilfssatz 4 mit $Q = \dfrac{i}{N}$. Dann folgt aus (12) für hinreichend großes i:

$$|a_i(nr)| = S_N(q, Q; i) - S_N(q, Q; i-q) + R(N)$$

mit

$$R(N) = \sum_{k=1}^{N-1} (-1)^{k+N} \{S_k(q, Q; i) - S_k(q, Q; i-q)\} = \mathrm{O}\left(\frac{i^{N-1}}{\log^{N-1} i}\right),$$

da $0 \leqq S_k(q, Q; i) - S_k(q, Q; i-q) \leqq S_k(q, Q; i) = \dbinom{\pi(Q) - \pi(q)}{k}$

$$\leqq \binom{\pi(i)}{k} \leqq c \frac{i^k}{\log^k i}$$

und $k \leqq N - 1$ ist, wobei $\pi(x)$ die Primzahlfunktion bedeutet. $S_N(q, Q; i-q)$ läßt sich folgendermaßen abschätzen:

$$
\begin{aligned}
S_N(q, Q; i-q) &= S_N(q', Q; i-q) + S_{N-1}(q', Q; i-q-q') \\
&= S_N(q', Q; i-q) + \mathrm{O}\left(\frac{i^{N-1}}{\log^{N-1} i}\right).
\end{aligned}
$$

Wegen $i - q' < q'' + \cdots + q^{(N+1)} \leqq q_1 + \cdots + q_N$ ist

$$
\begin{aligned}
S_N(q', Q; i-q) &= \sum_{k=i-q'+1}^{i-q} \sum_{\substack{q' < q_1 < \cdots < q_N \leqq Q \\ q_1 + \cdots + q_N = k}} 1 \\
&\leqq (q' - q) \sum_{q_1 + \cdots + q_N = i} 1 .
\end{aligned}
$$

Für die Differenz aufeinanderfolgender Primzahlen gilt nach [2]:

$$q' - q < q^{5/8} = O(i^{5/8}),$$

und für die Anzahl der Lösungen der Gleichung $i = q_1 + \cdots + q_N$ ergibt sich aus [1], Satz 11 (S. 77)

$$\sum_{q_1 + \cdots + q_N = i} 1 = O\left(\frac{i^{N-1}}{\log^N i}\right).$$

Damit wird $S_N(q, Q; i-q) = O\left(i^{5/8} \frac{i^{N-1}}{\log^N i}\right) + O\left(\frac{i^{N-1}}{\log^{N-1} i}\right)$

$$= O\left(\frac{i^{N-3/8}}{\log^N i}\right).$$

Andererseits ist wegen $Q = \frac{i}{N}$ und $q < \frac{i}{N+1}$

$$\pi(Q) - \pi(q) \geqq \pi\left(\frac{i}{N}\right) - \pi\left(\frac{i}{N+1}\right) = \pi\left(\left(1 + \frac{1}{N}\right) \frac{i}{N+1}\right) - \pi\left(\frac{i}{N+1}\right)$$

$$> C_1 \frac{i}{N+1} \log^{-1}\left(\frac{i}{N+1}\right) > C_2 \frac{i}{\log i} \quad \text{mit} \quad C_1 > C_2 > 0$$

für hinreichend großes i. Also gilt wegen $NQ = i$

$$S_N(q, Q; i) = \binom{\pi(Q) - \pi(q)}{N} > C_3 \frac{i^N}{\log^N i} \quad \text{mit} \quad C_3 > 0.$$

Faßt man die obigen Abschätzungen zusammen, so folgt für die in Hilfssatz 4 definierten Koeffizienten:

Es gibt Konstanten $C > 0$ und B sowie $i_1 \in \mathbb{N}$, so daß für alle $i \geqq i_1$ gilt

$$|a_i(nr)| > C \frac{i^N}{\log^N i} + B \frac{i^{N-3/8}}{\log^N i}.$$

Damit wird für $N = m + 1$

$$\max_{n \in \mathbb{N}} |a_i(n)| > i^m \left(C \frac{i}{\log^{m+1} i} + B \frac{i^{5/8}}{\log^{m+1} i}\right) > i^m$$

für alle $i \geqq i_0$, wenn man $i_0 \geqq i_1$ so groß wählt, daß die Klammer für $i = i_0$ größer als 1 wird, was wegen $C > 0$ für alle m und B möglich ist.

4. Erzeugende Funktionen

Mit der Grundformel (8) lassen sich erzeugende Funktionen für $a_i(n)$ und sogar für beliebige Polynome $P(a_i(n))$ bei festem i (also auch für spezielle Untermengen der a_i) berechnen.

Satz 5. *Ist* $i \in \mathbb{N}$ *fest und* $F(i,s) := \sum_{n=1}^{\infty} a_i(n) n^{-s}$ $(\operatorname{Re} s > 1)$ *die erzeugende Funktion der Koeffizienten* $a_i(n)$, *so gilt:*

$$F(i,s) = E_0(i,s) \frac{\zeta(s)}{\zeta(2s)} - E_1(i,s) \frac{1}{\zeta(s)} \tag{13}$$

mit

$$E_r(i,s) = \sum_{\mathfrak{p} \in \mathfrak{P}_i} S_r(\mathfrak{p}) t^{-s} \prod_{p \mid \frac{v}{t}} [p^s + (-1)^r]^{-1}, \qquad r = 0, 1,$$

$t := \operatorname{ggT}(I)$ *(größter gemeinsamer Teiler)*, $v := \operatorname{kgV}(I)$ *(kleinstes Vielfaches) und*

$$S_r(\mathfrak{p}) = \prod_{k \in I} m_k^{-1} \sum_{\substack{(\alpha_k)_{k \in I} \\ \alpha_k \in \mathbb{N},\ 1 \leqq \alpha_k \leqq m_k \\ \sum_{k \in I} \alpha_k \equiv r \pmod 2}} \left[\prod_{k \in I} \mu^{\alpha_k}\left(\frac{v}{k}\right) \right],$$

$m_k := \min\{2, n_k\}$. $\zeta(s)$ *ist die Riemannsche Zetafunktion.*

Beweis. Zunächst werden einige Gleichungen zusammengestellt:

$$(-1)^m \binom{\mu(n)}{m} = \begin{cases} 1 & \text{für } m = 0, \\ -\mu(n) & \text{für } m = 1, \\ \frac{1}{2}\mu^2(n) - \frac{1}{2}\mu(n) & \text{für } m \geqq 2. \end{cases} \tag{14}$$

Bei Fallunterscheidung $\mu(n) = 0$, $+1$ bzw. $\mu(n) = -1$ sind diese Beziehungen offensichtlich. Die folgenden beiden Formeln sind Spezialfälle der Produktdarstellung von Dirichletschen Reihen mit multiplikativen Koeffizienten:

$$\sum_{\substack{n=1 \\ (n,k)=1}}^{\infty} \mu(n) n^{-s} = \prod_{\substack{p \\ (p,k)=1}} (1 - p^{-s}) = \prod_{p \mid k} (1 - p^{-s})^{-1} \frac{1}{\zeta(s)}, \tag{15}$$

$$\sum_{\substack{n=1 \\ (n,k)=1}}^{\infty} \mu^2(n) n^{-s} = \prod_{\substack{p \\ (p,k)=1}} (1 + p^{-s}) = \prod_{p \mid k} (1 + p^{-s})^{-1} \frac{\zeta(s)}{\zeta(2s)}. \tag{16}$$

Es folgen noch zwei Relationen, die zur Vereinfachung von (13) dienen; dabei ist $I \subset \mathbb{N}$ eine endliche Zahlenmenge und v, t wie oben definiert:

$$\operatorname{kgV}\left(\frac{v}{k}\right)_{k \in I} = \operatorname{kgV}\left(\frac{k}{t}\right)_{k \in I} = \frac{v}{t}, \tag{17}$$

$$\prod_{k \in I} \mu^2\left(\frac{v}{k}\right) = \prod_{k \in I} \mu^2\left(\frac{k}{t}\right) = \mu^2\left(\frac{v}{t}\right). \tag{18}$$

Sei $k=\prod_p p^{\varepsilon_p(k)}$ mit $\varepsilon_p(k)\geqq 0$ die Primzahldarstellung für $k\in\mathbb{N}$ und $\delta_p:=\min_{k\in I}\varepsilon_p(k)$, $\gamma_p:=\max_{k\in I}\varepsilon_p(k)$. Dann gilt $\varepsilon_p(t)=\delta_p$ und $\varepsilon_p(v)=\gamma_p$ für alle p sowie $\varepsilon_p\left(\frac{v}{k}\right)=\gamma_p-\varepsilon_p(k)$ und $\varepsilon_p\left(\frac{k}{t}\right)=\varepsilon_p(k)-\delta_p$; also ist

$$\varepsilon_p\left[\operatorname{kgV}\left(\frac{v}{k}\right)_{k\in I}\right]=\max_{k\in I}\{\gamma_p-\varepsilon_p(k)\}=\gamma_p-\min_{k\in I}\varepsilon_p(k)=\gamma_p-\delta_p$$

sowie

$$\varepsilon_p\left[\operatorname{kgV}\left(\frac{k}{t}\right)_{k\in I}\right]=\max_{k\in I}\{\varepsilon_p(k)-\delta_p\}=\gamma_p-\delta_p=\varepsilon_p\left(\frac{v}{t}\right).$$

(18) folgt unmittelbar aus (17), da $\frac{v}{t}$ genau dann quadratfrei ist, wenn dieses für alle $\frac{v}{k}$ bzw. $\frac{k}{t}$, $k\in I$, gilt.

Nach (14) ist für jede Partition $\mathfrak{p}\in\mathfrak{P}_i$

$$\prod_{k\in I}(-1)^{n_k}\binom{\mu\left(\frac{n}{k}\right)}{n_k}=\sum_{\substack{(\alpha_k)_{k\in I}\\ \alpha_k\in\mathbb{N},\ 1\leqq\alpha_k\leqq m_k}}\left[\prod_{k\in I}(-1)^{\alpha_k}m_k^{-1}\mu^{\alpha_k}\left(\frac{n}{k}\right)\right]. \tag{19}$$

Außerdem gilt für jede Teilsumme mit $\alpha_k\in\mathbb{N}$

$$\begin{aligned} F(s):=\sum_{n=1}^{\infty}\left[\prod_{k\in I}\mu^{\alpha_k}\left(\frac{n}{k}\right)\right]n^{-s}&=\sum_{m=1}^{\infty}\left[\prod_{k\in I}\mu^{\alpha_k}\left(\frac{v}{k}m\right)\right](vm)^{-s} &(20)\\ &=\prod_{k\in I}\mu^{\alpha_k}\left(\frac{v}{k}\right)v^{-s}\sum_{\substack{m=1\\ (m,\frac{v}{t})=1}}^{\infty}\mu^{\Sigma\alpha_k}(m)m^{-s}, \end{aligned}$$

wobei $\frac{v}{t}$ nach (17) für $\operatorname{kgV}\left(\frac{v}{k}\right)_{k\in I}$ steht. Also wird mit $\sum_{k\in I}\alpha_k\equiv r(\operatorname{mod}2)$, $r=0,1$, und

$$Z_r(s):=\begin{cases}\zeta(s)/\zeta(2s) & \text{für}\quad r=0,\\ 1/\zeta(s) & \text{für}\quad r=1\end{cases}$$

nach (15), (16) und (18)

$$\begin{aligned} F(s)&=\prod_{k\in I}\mu^{\alpha_k}\left(\frac{v}{k}\right)v^{-s}\prod_{p\mid\frac{v}{t}}[1+(-1)^r p^{-s}]^{-1}Z_r(s) &(20)\\ &=\prod_{k\in I}\mu^{\alpha_k}\left(\frac{v}{k}\right)t^{-s}\prod_{p\mid\frac{v}{t}}[p^s+(-1)^r]^{-1}Z_r(s)\,. \end{aligned}$$

Faßt man nun für jede Partition $\mathfrak{p} \in \mathfrak{P}_i$ in (19) die Summanden mit $\sum_{k \in I} \alpha_k \equiv r (\mathrm{mod}\, 2)$, $r = 0, 1$, zusammen und beachtet, daß wegen der absoluten Konvergenz der gegebenen Dirichletschen Reihen für $\mathrm{Re}\, s > 1$ Summationen vertauscht werden können, so folgt (13), wobei die Vorzeichen von $E_r(i, s)$ durch $(-1)^{\sum \alpha_k} = (-1)^r$ bestimmt sind. q.e.d.

Auf die gleiche Weise können auch erzeugende Funktionen für $G(i, s) := \sum_{n=1}^{\infty} P(a_i(n)) n^{-s}$ berechnet werden, wenn $P(x)$ ein Polynom ist; denn $G(i, s)$ ist dann Summe von endlich vielen Dirichletschen Reihen der Form (20). So kann man z. B. für gegebenes w aus dem endlichen Wertebereich W_i der i-ten Koeffizienten die Anzahlfunktion

$$Z(i, w; x) := \mathrm{card}\, \{n \in \mathbb{N};\, n \leqq x \quad \text{und} \quad a_i(n) = w\}$$

als summatorische Funktion der folgenden Reihe erzeugen:

$$\sum_{\substack{n=1 \\ a_i(n) = w}}^{\infty} n^{-s} = \sum_{n=1}^{\infty} \frac{P_w(a_i(n))}{P_w(w)} n^{-s} \tag{21}$$

mit $P_w(x) := \prod_{a \in W_i - \{w\}} (x - a)$.

5. Mittelwerte

Mit Hilfe von (13) lassen sich die (asymptotischen) Mittelwerte

$$m(i) := \lim_{x \to \infty} \frac{1}{x} \sum_{n \leqq x} a_i(n) \tag{22}$$

bestimmen, falls gezeigt werden kann, daß der Limes existiert; denn es gilt (siehe [3], Bd. I, § 31): Wenn (22) existiert, so ist

$$m(i) = \lim_{s \to 1+0} (s-1)\, F(i, s) . \tag{23}$$

Wegen (8), (9), (14), (19) und (20) genügt es zu zeigen, daß $\lim_{x \to \infty} \frac{1}{x} \sum_{\substack{n \leqq x \\ (n,k) = 1}} \mu^r(n)$ für $r = 1, 2$ und jedes feste $k \in \mathbb{N}$ existiert. Da $\frac{1}{x} \sum_{n \leqq x} a_i(n)$ durch endlich viele solche Summanden dargestellt werden kann, existiert dann auch der Limes (22) für alle $i \in \mathbb{N}$. Nun ist

$$\frac{1}{x} \sum_{\substack{n \leqq x \\ (n,k) = 1}} \mu^r(n) = \frac{1}{x} \sum_{\substack{l=1 \\ (l,k) = 1}}^{k-1} \sum_{\substack{n \leqq x \\ n \equiv l (\mathrm{mod}\, k)}} \mu^r(n) ,$$

und nach [3] (Bd. II, § 173 und 174) existieren die Grenzwerte $\lim_{x\to\infty} \frac{1}{x} \sum_{\substack{n\leq x \\ n\equiv l(k)}} \mu^r(n)$ für $r=1,2$ und alle l, k mit $(l,k)=1$. Wegen (23) und $\lim_{s\to 1+0} (s-1)\zeta(s)=1$ ist damit der folgende Satz über die (asymptotischen) Mittelwerte $m(i)$ bewiesen:

Satz 6. *Für alle $i\in\mathbb{N}$ gilt*

$$m(i)=e_i\frac{1}{\zeta(2)}=6e_i\pi^{-2}, \tag{24}$$

wobei

$$e_i := E_0(i,1)=\sum_{\mathfrak{p}\in\mathfrak{P}_i} \frac{S_0(\mathfrak{p})}{t\prod\limits_{p\mid\frac{v}{t}}(p+1)} \tag{25}$$

wie in (13) *mit $s=1$ definiert ist.*

Hier läßt sich $S_0(\mathfrak{p})$ noch wesentlich vereinfachen:

Satz 7. *Sei für jedes $\mathfrak{p}\in\mathfrak{P}_i$ und für $\sigma,\tau\in\{1,2\}$*

$$I_{\sigma\tau} := \left\{k\in I;\, m_k=\sigma \quad \text{und} \quad \mu\left(\frac{k}{t}\right)=(-1)^\tau\right\}, \qquad c_{\sigma\tau} := \operatorname{card} I_{\sigma\tau};$$

$$I_0 := \left\{k\in I;\, \mu\left(\frac{k}{t}\right)=0\right\}, \qquad c_0 := \operatorname{card} I_0 .$$

Dann gilt für alle $i\in\mathbb{N}$ und alle $\mathfrak{p}\in\mathfrak{P}_i$

$$S_0(\mathfrak{p})=\begin{cases} (-1)^{c_{11}}, & \text{wenn } c_0+c_{21}+c_{22}=0 \text{ und } c_{11}+c_{12} \text{ gerade};\\ \frac{1}{2}(-1)^{c_{1\tau}}, & \text{wenn } c_0+c_{2\tau}=0 \text{ und } c_{21}+c_{22}>0,\ \tau=1,2;\\ 0 & \text{sonst}. \end{cases} \tag{26}$$

Beweis. Nach Satz 5 ist

$$S_0(\mathfrak{p})=\prod_{k\in I} m_k^{-1} \sum_{\substack{(\alpha_k)_{k\in I} \\ \alpha_k\in\mathbb{N},\ 1\leq\alpha_k\leq m_k \\ \sum\alpha_k\equiv 0\,(\mathrm{mod}\,2)}} \left[\prod_{k\in I}\mu^{\alpha_k}\left(\frac{v}{k}\right)\right]. \tag{27}$$

Zunächst werden die Summanden umgeformt:

$$\prod_{k\in I}\mu^{\alpha_k}\left(\frac{v}{k}\right)=\mu^r\left(\frac{v}{t}\right)\prod_{k\in I}\mu^{\alpha_k}\left(\frac{k}{t}\right), \qquad r=\sum_{k\in I}\alpha_k . \tag{28}$$

Für $\mu\left(\frac{v}{t}\right)=0$ ist (28) wegen (18) richtig. Ist $\frac{v}{t}$ quadratfrei, so gilt $\frac{v}{t}=\frac{v}{k}\,\frac{k}{t}$ mit $\left(\frac{v}{k},\frac{k}{t}\right)=1$ für alle $k\in I$, also $\omega\left(\frac{v}{t}\right)=\omega\left(\frac{v}{k}\right)+\omega\left(\frac{k}{t}\right)$, wobei $\omega(n)$ die

Anzahl der verschiedenen Primfaktoren von n ist. Damit wird

$$\prod_{k\in I}\mu^{\alpha_k}\left(\frac{v}{k}\right)=(-1)^{\sum\limits_{k\in I}\alpha_k\omega\left(\frac{v}{k}\right)}=(-1)^{\sum\limits_{k\in I}\alpha_k\left\{\omega\left(\frac{v}{t}\right)-\omega\left(\frac{k}{t}\right)\right\}}$$

$$=\mu^r\left(\frac{v}{t}\right)\prod_{k\in I}\mu^{\alpha_k}\left(\frac{k}{t}\right).$$

Wegen $r\equiv 0(\mathrm{mod}\,2)$ können in (27) die Summanden also durch $\prod\limits_{k\in I}\mu^{\alpha_k}\left(\frac{k}{t}\right)$ ersetzt werden.

Nun läßt sich (26) durch Fallunterscheidung beweisen, wobei wegen $\mathrm{ggT}\left(\frac{k}{t}\right)_{k\in I}=1$ o. B. d. A. $t=1$ angenommen werden kann.

1. Fall. $c_0+c_{21}+c_{22}=0$ und $c_{11}+c_{12}$ gerade, d. h. $\mathfrak{p}$ ist eine Partition von i in eine gerade Anzahl verschiedener quadratfreier Summanden; dann ist $S_0(\mathfrak{p})=\prod\limits_{k\in I_{11}\cup I_{12}}\mu(k)=(-1)^{c_{11}}$.

2. Fall. $c_0+c_{21}=0$ und $c_{22}>0$, d. h. $\mathfrak{p}$ ist eine Partition von i in quadratfreie Summanden, wobei mindestens ein Summand k mit $\mu(k)=1$ aber kein Summand mit $\mu(k)=-1$ mehrfach vorkommt; dann haben alle Summanden von $S_0(\mathfrak{p})$ das Vorzeichen $(-1)^{c_{11}}$, und ihre Anzahl ist wegen $\sum\limits_{k\in I}\alpha_k\equiv 0\ (\mathrm{mod}\,2)$ gleich $\frac{1}{2}\prod\limits_{k\in I}m_k$.

3. Fall. $c_0+c_{22}=0$ und $c_{21}>0$, d. h. $\mathfrak{p}$ ist eine Partition von i in quadratfreie Summanden, wobei mindestens ein Summand k mit $\mu(k)=-1$ aber kein Summand mit $\mu(k)=1$ mehrfach vorkommt; dann haben alle Summanden von $S_0(\mathfrak{p})$ das Vorzeichen $(-1)^{c_{11}}(-1)^{c_{11}+c_{12}}=(-1)^{c_{12}}$, und ihre Anzahl ist wieder $\frac{1}{2}\prod\limits_{k\in I}m_k$.

4. Fall. $c_0+c_{21}+c_{22}=0$ und $c_{11}+c_{12}$ ungerade; d. h. $\mathfrak{p}$ ist eine Partition von i in eine ungerade Anzahl verschiedener quadratfreier Summanden; dann kann die Bedingung $\sum\limits_{k\in I}\alpha_k\equiv 0\,(\mathrm{mod}\,2)$ nicht erfüllt werden; also ist $S_0(\mathfrak{p})=0$.

5. Fall. $c_0=0$, $c_{21}>0$, $c_{22}>0$; d. h. $\mathfrak{p}$ ist eine Partition von i in quadratfreie Summanden, wobei sowohl Summanden k mit $\mu(k)=1$ als auch mit $\mu(k)=-1$ mehrfach vorkommen; dann sind die Anzahlen der positiven und der negativen Summanden von $S_0(\mathfrak{p})$ gleich, nämlich $2^{c_{21}-1}2^{c_{22}-1}$; also ist $S_0(\mathfrak{p})=0$.

6. Fall. $c_0>0$, d. h. die Partition $\mathfrak{p}$ enthält mindestens einen Summanden k mit $\mu(k)=0$; dann sind alle Summanden von $S_0(\mathfrak{p})$ gleich Null. q.e.d.

Zum Abschluß wird noch eine Tabelle der e_i für $1\leqq i\leqq 20$ angegeben, die demnächst mit Hilfe der IBM 7090 in Bonn wesentlich erweitert werden soll, um vor allem die folgenden beiden Vermutungen zu überprüfen:

(a) $0\leqq e_i\leqq\frac{1}{2}$ für alle $i\in\mathbb{N}$,

(b) $(-1)^i(e_i-e_{i+1})>0$ für alle $i\in\mathbb{N}$.

i	1	2	3	4	5	6	7	8	9	10
e_i	0	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{8}$	$\frac{7}{24}$	$\frac{1}{18}$	$\frac{7}{24}$	$\frac{19}{144}$	$\frac{319}{1440}$
$h \cdot e_i$	0	20160	6720	13440	5040	11760	2240	11760	5320	8932

i	11	12	13	14	15	16	17	18	19	20
e_i	$\frac{1}{16}$	$\frac{55}{192}$	$\frac{13}{288}$	$\frac{61}{288}$	$\frac{2287}{20160}$	$\frac{733}{2016}$	$\frac{667}{8064}$	$\frac{79}{336}$	$\frac{55}{1344}$	$\frac{221}{960}$
he_i	2520	11550	1820	8540	4574	14660	3335	9480	1650	9282

mit $h = 40320$.

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Construction of Quasi-Uniformities

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Let (X, σ) be a proximity space. It is well known [1] that there is a unique totally bounded uniform structure $\mathfrak{U}_\omega$ compatible with σ, and $\mathfrak{U}_\omega$ is the coarsest uniform structure compatible with σ. Theorem 1 extends this result to quasi-proximity spaces [5, 6].

Let $(X, \mathscr{T})$ be a topological space. W. Pervin [4] has given a quasi-uniformity $\mathfrak{U}_P$ compatible with $\mathscr{T}$ with a subbase consisting of all sets of the form

$$S_G = G \times G \cup (X - G) \times X ,$$

where $G \in \mathscr{T}$. R. Nielsen and C. Sloyer [3] have given a quasi-uniformity $\mathfrak{U}_N$ compatible with $\mathscr{T}$ with a subbase consisting of all sets of the form

$$U(\varepsilon, f) = \{(x, y) : f(x) - \varepsilon < f(y)\} ,$$

where ε is positive and f is a bounded real valued lower semi-continuous function on X. It is observed in [3] that $\mathfrak{U}_P \subset \mathfrak{U}_N$; in Theorem 2 we show that $\mathfrak{U}_P = \mathfrak{U}_N$.

Given a quasi-uniformity $\mathfrak{U}$ on X the quasi-proximity δ induced by $\mathfrak{U}$ is defined as follows: $A \,\delta\, B$ if and only if $A \times B \cap U \neq \emptyset$ for every U in $\mathfrak{U}$. In the sequel

$$T(A, B) = X \times X - A \times B = (X - A) \times X \cup X \times (X - B) .$$

For any quasi-proximity ϱ we write $A \,\varrho^-\, B$ whenever $A \,\varrho\, B$ is false.

Lemma 1. *Let δ be a quasi-proximity on X. If $A \,\delta\, B$ and $E \,\delta^-\, F$, then either $A \cap (X - E) \,\delta\, B$ or $A \,\delta\, B \cap (X - F)$.*

Proof. Suppose $A \cap (X - E) \,\delta^-\, B$ and $A \,\delta^-\, B \cap (X - F)$. Since $(A \cap (X - E)) \cup (A \cap E) = A$, $(B \cap (X - F)) \cup (B \cap F) = B$, and $A \,\delta\, B$ we have $(A \cap E) \,\delta\, B$ and $A \,\delta\, (B \cap F)$. If $A \cap (X - E) \,\delta\, (B \cap F)$, then $A \cap (X - E) \,\delta\, B$. Hence we must have $(A \cap E) \,\delta\, (B \cap F)$ which implies that $E \,\delta\, F$.

Theorem 1. *Let (X, δ) be a quasi-proximity space. The collection $\mathfrak{S}$ of all sets of the form $T(A, B)$, where $A \,\delta^-\, B$, is a subbase for a totally bounded quasi-uniformity $\mathfrak{U}_\beta$ which is compatible with δ. Moreover, $\mathfrak{U}_\beta$ is the coarsest quasi-uniformity compatible with δ and is the only totally bounded quasi-uniformity compatible with δ.*

Proof. *$\mathfrak{S}$ is a subbase for a quasi-uniformity.* Let $T(A, B) \in \mathfrak{S}$. Since $A\ \delta^-\ B$ there exist sets E and F such that $A\ \delta^-\ (X-E)$, $(X-F)\ \delta^-\ B$, and $E \cap F = \emptyset$. Finally we have $[T(A, X-E) \cap T(X-F, B)]^2 \subset T(A, B)$.

$\mathfrak{U}_\beta$ is totally bounded. Let $U = \cap\{T(A_i, B_i): 1 \leqq i \leqq n\}$ be in $\mathfrak{U}_\beta$. For each $i, 1 \leqq i \leqq n, (X-A_i) \cup (X-B_i) = X$, and $(X-A_i) \times (X-A_i) \cup (X-B_i) \times (X-B_i) \subset T(A_i, B_i)$. From this it follows that the collection of all sets of the form $\cap\{C_j: 1 \leqq j \leqq n\}$, where $C_j \in \{X-A_j, X-B_j\}$, $1 \leqq j \leqq n$, is a cover of X, and for each set E in the cover we have $E \times E \subset U$.

$\mathfrak{U}_\beta$ is compatible with δ. Let α be the quasi-proximity induced on X by $\mathfrak{U}_\beta$. If $A\ \delta^-\ B$, then $A \times B \cap T(A, B) = \emptyset$ and hence $A\ \alpha^-\ B$. Assume $A\ \delta\ B$ and $A\ \alpha^-\ B$. Then there exist $T(E_i, F_i)$, $1 \leqq i \leqq n$, in $\mathfrak{S}$ such that

$$A \times B \cap (\cap\{T(E_i, F_i): 1 \leqq i \leqq n\}) = \emptyset .$$

We use induction to show that no such n exists. Suppose $n=1$. Assume $A\ \delta\ B$ and there exists $T(E, F) \in \mathfrak{S}$ such that $A \times B \cap T(E, F) = \emptyset$. Then $A \subset E$ and $B \subset F$, and hence $E\ \delta$ F. This contradicts $E\ \delta^-\ F$. Assume that for any pair of subsets A, B with $A\ \delta\ B$, no intersection of $n-1$ subbasic entourages of the form $T(E_i, F_i)$ has an empty intersection with $A \times B$. Let $M = A \times B \cap G_n$, where

$$G_k = \cap\{T(E_i, F_i): 1 \leqq i \leqq k\}\,, \quad 1 \leqq k \leqq n\,.$$

Then

$$\begin{aligned} M &= [(A \times B \cap (X-E_n) \times X) \cap G_{n-1}] \cup [(A \times B \cap X \times (X-F_n)) \cap G_{n-1}] \\ &= [([A \cap (X-E_n)] \times B) \cap G_{n-1}] \cup [(A \times [B \cap (X-F_n)]) \cap G_{n-1}]\,. \end{aligned}$$

M cannot be empty since by Lemma 1 and the inductive hypothesis, one of the terms in the above union must be nonempty.

$\mathfrak{U}_\beta$ is the coarsest quasi-uniformity compatible with δ. Let $\mathfrak{U}$ be any quasi-uniformity compatible with δ. If $A\ \delta^-\ B$ then there exists $U \in \mathfrak{U}$ such that $A \times B \cap U = \emptyset$. Since $U \subset T(A, B)$, we have $\mathfrak{U}_\beta \subset \mathfrak{U}$.

$\mathfrak{U}_\beta$ is the unique totally bounded quasi-uniformity compatible with δ. Let $\mathfrak{U}$ be any totally bounded quasi-uniformity compatible with δ. Let $U \in \mathfrak{U}$ and choose $W \in \mathfrak{U}$ such that $W^2 \subset U$. Since $\mathfrak{U}$ is totally bounded there exists a cover $\{A_i : 1 \leqq i \leqq n\}$ of X such that $A_i \times A_i \subset W$, $1 \leqq i \leqq n$. Since $A_i \times (X - W(A_i)) \cap W = \emptyset$, we have $A_i\ \delta^-\ (X - W(A_i))$. Let $V_i = T(A_i, X - W(A_i))$ and let $V = \cap\{V_i: 1 \leqq i \leqq n\}$. $V \in \mathfrak{U}_\beta$ and

$$V \subset \cup\{A_i \times W(A_i): 1 \leqq i \leqq n\} \subset W^2 \subset U\,.$$

Therefore $U \in \mathfrak{U}_\beta$ and $\mathfrak{U} \subset \mathfrak{U}_\beta$. This completes the proof.

Corollary. *Let (X, δ) be a quasi-proximity space and let $\mathfrak{U}_\beta$ be the totally bounded quasi-uniformity compatible with δ. A base for $\mathfrak{U}_\beta$ is the collection $\mathfrak{B}$ of all sets of the form $\cup\{B_i \times A_i: 1 \leqq i \leqq n\}$, where $\{B_i\}$ is a finite cover of X and $B_i\ \delta^-\ X - A_i$ for every i, $1 \leqq i \leqq n$.*

Proof. Let $U = \cup\{B_i \times A_i : 1 \leqq i \leqq n\}$ be an element of $\mathfrak{B}$. Then for each i, $V_i = T(B_i, X - A_i)$ is in $\mathfrak{U}_\beta$. Let $V = \cap\{V_i : 1 \leqq i \leqq n\}$, $V \in \mathfrak{U}_\beta$, $V(B_i) \subset A_i$. Therefore $V \subset U$ and $\mathfrak{B} \subset \mathfrak{U}_\beta$. Let $U \in \mathfrak{U}_\beta$ and choose $W \in \mathfrak{U}_\beta$ such that $W^2 \subset U$. Let $\{A_i : 1 \leqq i \leqq n\}$ be a cover of X such that $A_i \times A_i \subset W$, $1 \leqq i \leqq n$. The proof that $U \in \mathfrak{B}$ now follows as in the last paragraph of the proof of Theorem 1.

It is clear that if δ is a proximity then $\mathfrak{U}_\beta = \mathfrak{U}_\omega$ [1].

Theorem 2. $\mathfrak{U}_N = \mathfrak{U}_P$.

Proof. By Theorem 1, it suffices to show that $\mathfrak{U}_N$ and $\mathfrak{U}_P$ are totally bounded, and induce the same quasi-proximity on X. Since $S_G^2 \subset S_G$ $(U(\varepsilon/2, f)^2 \subset U(\varepsilon, f))$, and each $S_G(U(\varepsilon, f))$ is totally bounded it follows that $\mathfrak{U}_P(\mathfrak{U}_N)$ is totally bounded. Let the quasi-proximities induced by $\mathfrak{U}_N$ and $\mathfrak{U}_P$ be denoted by δ and α respectively. Let ϱ denote the quasi-proximity defined by $A \,\varrho\, B$ if and only if $A \cap \bar{B} \neq \emptyset$ [5]. Since $\mathfrak{U}_P \subset \mathfrak{U}_N$ it follows that $A \,\alpha\, B$ whenever $A \,\delta\, B$. It is well known [2] that $A \,\alpha\, B$ implies $A \,\varrho\, B$. Finally, assume $A \,\varrho\, B$ and $A \,\delta^-\, B$. Let $x \in A \cap \bar{B}$; since $A \,\delta^-\, B$ there exist entourages $U(\varepsilon_i, f_i)$, $1 \leqq i \leqq n$, in $\mathfrak{U}_N$ such that $(\cap\{U(\varepsilon_i, f_i) : 1 \leqq i \leqq n\}) \cap \{x\} \times B = \emptyset$. If $y \in B$, then for some i, $1 \leqq i \leqq m$, $f_i(y) \leqq f_i(x) - \varepsilon_i$; hence by the lower semi-continuity of each f_i, $\bar{B}$ is a subset of $\cup\{z : f_i(z) \leqq f_i(x) - \varepsilon_i,\ 1 \leqq i \leqq n\}$. In this case for some i, $f_i(x) \leqq f_i(x) - \varepsilon_i$, which is impossible. Therefore $\delta = \alpha = \varrho$. This completes the proof.

Corollary. *Let $(X, \mathcal{T})$ be a topological space, and let f be a bounded real valued function on X. Then f is quasi-uniformly lower semi-continuous with respect to $\mathfrak{U}_P$ if and only if f is lower semi-continuous.*

Let $(X, \mathcal{T})$ be a topological space. The proof of Proposition 1.2 [3] shows that the collection of all $U(\varepsilon, f)$, where ε is positive and f is a real valued lower semi-continuous function, is a subbase for a quasi-uniformity $\mathfrak{U}$ compatible with $\mathcal{T}$. The quasi-uniformity $\mathfrak{U}$ is in general different from $\mathfrak{U}_P$ since $\mathfrak{U}$ is not necessarily totally bounded. The following theorem yields a totally bounded quasi-uniformity which is in general coarser than $\mathfrak{U}_P$.

Theorem 3. *If $(X, \mathcal{T})$ is a topological space and $\mathcal{F}$ is a filter without adherent point, then there exists a totally bounded quasi-uniformity $\mathfrak{U}$ compatible with $\mathcal{T}$ such that $\mathcal{F}$ is $\mathfrak{U}$-Cauchy.*

Proof. A subbase for $\mathfrak{U}$ is the collection of all sets of the form S_G where $G \in \mathcal{T}$ and $G \cap F = \emptyset$ for some $F \in \mathcal{F}$.

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Idempotents and Blocks in Artinian d.g. Near-rings with Identity Element

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1. Preliminaries

A *near-ring* R with identity is a system with two binary operations $+$ and $\cdot$ called addition and multiplication such that R is a (not necessarily abelian) group under addition and a monoid under multiplication, satisfying one distributive law

$$(a+b)c = ac + bc \quad \text{for all} \quad a, b, c \in R. \tag{1.1}$$

0 will denote the zero-element of the additive group, 1 the identity element of the multiplicative monoid of R. We say that R is a *distributively generated (d.g.) near-ring* if, in addition, there exists a set S of elements in R such that S generates R additively, and

$$s(a+b) = sa + sb \quad \text{for all} \quad s \in S,\ a, b \in R. \tag{1.2}$$

An *R-group* G is a group G such that S acts on G as a set of endomorphisms, i.e.

$$s(g_1 + g_2) = sg_1 + sg_2 \quad \text{for all} \quad s \in S,\ g_1, g_2 \in G, \tag{1.3}$$

and if $r = \varepsilon_1 s_1 + \varepsilon_2 s_2 + \cdots + \varepsilon_n s_n$, $\varepsilon_i = \pm 1$, $s_i \in S$, $i = 1, 2, \ldots, n$, then $rg \in G$ is a well-defined element of G, given by $rg = \varepsilon_1 s_1 g + \varepsilon_2 s_2 g + \cdots + \varepsilon_n s_n g$. This implies

$$(r_1 + r_2)g = r_1 g + r_2 g \quad \text{for all} \quad r_1, r_2 \in R,\ g \in G. \tag{1.4}$$

A subgroup H of G is called an *R-subgroup* of G if $rh \in H$ for all $r \in R$, $h \in H$. Throughout this paper, we will assume that R-groups G are *unitary*, i.e.

$$lg = g \quad \text{for all} \quad g \in G. \tag{1.5}$$

The additive group of R can be considered as an R-group by means of the multiplication in R. We call R an *artinian* near-ring if R satisfies the minimum condition for R-subgroups of R. If $a \in R$ then $\ell(a) = \{r \in R \mid ra = 0\}$ is called the *left annihilator* of a. The aim of this note is to obtain a theory of primitive idempotents and block decomposition for artinian d.g. near-rings with identity element as it is well-known for artinian rings ([1]).

2. Primitive Idempotents

Throughout this section R will always denote an artinian d.g. near-ring with identity element. An R-subgroup U of R is called *minimal nonnilpotent* if U is nonnilpotent, i.e. for any positive integer n there exists a sequence of elements $u_1, u_2, \ldots, u_n$ in U such that $u_1 u_2 \ldots u_n \neq 0$, and every proper R-subgroup of U is nilpotent. Basic for minimal nonnilpotent R-subgroups U of R is the property that they contain an idempotent u such that $Ru = U$ ([4]). An idempotent $e \in R$ is called a *primitive idempotent* of R if there does not exist an idempotent $f \in R$ such that $fe = f$, $ef \neq e$.

2.1. Proposition. *An R-subgroup U of R is minimal nonnilpotent if and only if $U = Re$ where e is a primitive idempotent.*

Proof. Let U be a minimal nonnilpotent R-subgroup of R. Then $U = Re$ for some idempotent $e \in R$. If $f \in R$ is an idempotent with $fe = f$ then $Rf = Rfe \subseteq Re$ implying $Rf = Re$ since Re is minimal nonnilpotent. Moreover $(ef - e)ef = efef - ef = ef - ef = 0$ whence $(ef - e)e \in \ell(f) \cap Re = \ell(f) \cap Rf = 0$. Thus $ef - e \in \ell(e) \cap Re = 0$, and $ef = e$. Conversely, let $U = Re$, e being a primitive idempotent. If U is not minimal nonnilpotent then there exists an idempotent $f \in R$ such that $U \supset Rf$. But then $f = re$ for some $r \in R$ and $fe = ree = re = f$. If $ef = e$ then $U = Re = Ref \subseteq Rf$, contradiction.

2.2. Definition. A *DPI-set* in R (decomposition set of primitive idempotents) is an ordered set $\{e_1, e_2, \ldots, e_i\}$ of primitive idempotents in R with $e_l e_k = 0$ for $l > k$. A *maximal* DPI-set K in R is a DPI-set in R such that $\{K, e\}$ is not a DPI-set in R for any primitive idempotent e in R.

2.3. Proposition. *Every DPI-set in R is finite.*

Proof. Assume $\{e_1, e_2, e_3, \ldots\}$ is an infinite DPI-set in R. Then by minimum condition, the chain $\ell(e_1) \supseteq \ell(e_1) \cap \ell(e_2) \supseteq \ell(e_1) \cap \ell(e_2) \cap \ell(e_3) \supseteq \cdots$ becomes stationary i.e. there exists some positive integer i such that

$$\ell(e_1) \cap \ell(e_2) \cap \cdots \cap \ell(e_i) \subseteq \ell(e_{i+1})$$

implying $e_{i+1} \in \ell(e_{i+1})$ and $e_{i+1}^2 = 0$, contradiction.

2.4. Proposition. *Let $\{e_1, e_2, \ldots, e_i\}$ be a DPI-set in R. Then every $r \in R$ can be written uniquely as*

$$r = r_1 + r_2 + \cdots + r_i + \bar{r},\ r_k \in Re_k,\ \bar{r} \in \bigcap_{k=1}^{i} \ell(e_k), \qquad k = 1, 2, \ldots, i. \tag{2.1}$$

Proof. By induction on i. For $i = 1$, the proposition follows from $R = Re_1 + \ell(e_1)$ and $Re_1 \cap \ell(e_1) = 0$. Let $i > 1$. By induction, $r = r_1 + r_2 + \cdots + r_{i-1} + s$, $r_k \in Re_k$, $s \in \bigcap_{k=1}^{i-1} \ell(e_k)$, $k = 1, \ldots, i-1$. We can write $s = se_i + (-se_i + s)$, and put $r_i = se_i \in Re_i$, $\bar{r} = -se_i + s$. Clearly $\bar{r} \in \ell(e_i)$, but also $re_j = (-se_i + s)e_j = -se_i e_j + se_j = 0$ for $j < i$. Thus $\bar{r} \in \bigcap_{k=1}^{i} \ell(e_k)$. Let $r = r_1 + r_2 + \cdots + r_i + \bar{r} = r_1' + r_2' + \cdots + r_i' + \bar{r}'$ be two decompositions of r with $r_k, r_k' \in Re_k$, $\bar{r}, \bar{r}' \in \bigcap_{k=1}^{i} \ell(e_k)$. Then $(r_i + \bar{r})e_j = (r_i e_i + \bar{r})e_j = r_i e_i e_j + \bar{r} e_j = 0$ for $j < i$. Similarly $(r_i' + \bar{r}')e_j = 0$ for

$j<i$. Thus $r_i+\bar{r},\ r'_i+\bar{r}' \in \bigcap_{k=1} \ell(e_k)$, and by induction, $r_1=r'_1, \ldots, r_{i-1}=r'_{i-1}$, $r_i+\bar{r}=r'_i+\bar{r}'$. But $-r'_i+r_i=\bar{r}'-\bar{r}\in Re_i\cap\ell(e_i)=0$. Therefore $r_i=r'_i$, $\bar{r}=\bar{r}'$, Q.E.D.

We call an R-subgroup U of R a *left ideal* of R if U is a normal subgroup of the additive group of R. Clearly, if $a\in R$ then $\ell(a)$ is a left ideal of R.

2.5. Proposition. *A DPI-set $\{e_1, e_2, \ldots, e_n\}$ in R is maximal if and only if $\bigcap_{k=1}^{n} \ell(e_k)$ is a nilpotent left ideal of R.*

Proof. Let $L=\bigcap_{k=1}^{n} \ell(e_k)$ be nonnilpotent. Then L contains a primitive idempotent e by 2.1, and $\{e_1, \ldots, e_n, e\}$ is a DPI-set. Conversely, let $\{e_1, e_2, \ldots, e_n\}$ be a DPI-set in R that is not maximal. Then there exists a primitive idempotent $e\in R$ and $e\in\bigcap_{k=1}^{n} \ell(e_k)=L$ so that L is nonnilpotent.

3. Near-rings with Nilpotent Radical

A left ideal U of R is called *strictly maximal* if U is maximal even as an R-subgroup of R. The *radical* $J(R)$ of R is the intersection of all strictly maximal left ideals U of R, and we know from [6] that the set of the strictly maximal left ideals of R is non-empty. Thus $J(R)$ is a proper two-sided ideal of R, i.e. $J(R)\neq R$, $J(R)$ is a left ideal of R, and $J(R)r\subseteq J(R)$ for all $r\in R$ (cf. [5]). From now on, we will make the assumption:

$$J(R) \quad \textit{is a nilpotent ideal} \quad \text{(cf. [2, 4, 6, 7])}. \tag{3.1}$$

Beidleman ([4]) has shown that, under condition (3.1), each minimal nonnilpotent R-subgroup Re of R, e being a primitive idempotent, has a unique maximal R-subgroup that can be written as $J(R)e$. For brevity sake we will write just J instead of $J(R)$. By [6], every nilpotent R-subgroup of R is contained in J.

3.1. Lemma. *Let $\{e_1, \ldots, e_i\}$ be a DPI-set in R. Then*

$$Re_j+\bigcap_{k=1}^{j} \ell(e_k)=\bigcap_{k=1}^{j-1} \ell(e_k) \quad \textit{for} \quad j=2, 3, \ldots, i. \tag{3.2}$$

Proof. $R=Re_j+\ell(e_j)$, $e_j\in\bigcap_{k=1}^{j-1} \ell(e_k)$. Thus $\bigcap_{k=1}^{j-1} \ell(e_k)=\bigcap_{k=1}^{j-1} \ell(e_k)\cap(Re_j+\ell(e_j))$ $=Re_j+\bigcap_{k=1}^{j} \ell(e_k)$, by the modular law.

An R-group G is called *irreducible* if G has no proper normal R-subgroup. A series

$$G=G_0\supset G_1\supset\cdots\supset G_n=0 \tag{3.3}$$

of R-groups is called a *Jordan-Hölder series* of G if G_i is normal in G_{i-1}, and G_{i-1}/G_i is an irreducible R-group, for $i=1,2,\ldots,n$.

3.2. Theorem. *Any two maximal DPI-sets in R have the same number of elements.*

Proof. Let $\{e_1, e_2, \ldots, e_n\}$ be a maximal DPI-set in R. Then

$$\begin{aligned} R/J = Re_1 + \ell(e_1) + J/J \supseteq Re_2 + \bigcap_{i=1}^{2} \ell(e_i) + J/J \supseteq \cdots \\ \supseteq Re_n + \bigcap_{i=1}^{n} \ell(e_i) + J/J \supseteq J/J \end{aligned} \tag{3.4}$$

is a chain of R-subgroups of R/J which are normal in R/J by 3.1. Moreover, for $k=1,2,\ldots,n$

$$Re_k + \bigcap_{i=1}^{k} \ell(e_i) + J/Re_{k+1} + \bigcap_{i=1}^{k+1} \ell(e_k) + J \cong Re_k/Re_k \cap \left(\bigcap_{i=1}^{k} \ell(e_i) + J\right),$$

by 3.1. If $Re_k \subsetneq \bigcap_{i=1}^{k} \ell(e_i) + J$ then $R = Re_k + \ell(e_k) = \ell(e_k) + J = \ell(e_k)$ by Laxton ([6]), contradiction. Also $Re_k \cap \left(\bigcap_{i=1}^{k} \ell(e_i) + J\right) \supseteq Re_k \cap J = Je_k$, hence $Re_k \cap \left(\bigcap_{i=1}^{k} \ell(e_i) + J\right) = Je_k$, and (3.4) is a Jordan-Hölder series of R/J whose length is independent of the chosen DPI-set in R (cf. [3] (1.1)), Q.E.D. Roth ([8], cf. also [3] (1.2)) has proved, that an R-group G has a Jordan-Hölder series if and only if G satisfies the minimum condition for R-subgroups and if H is a term in a normal series of R-subgroups of G then every ascending series of normal R-subgroups $H_0 \subsetneq H_1 \subsetneq \cdots$ of H becomes, for some $n \geqq 0$, stationary after finitely many steps, i.e. $H_i = H_n$ for all $i \geqq n$ (maximum condition).

3.3. Proposition. *If R possesses a Jordan-Hölder series, and e is an idempotent of R then every R-epiendomorphism of Re is an R-automorphism.*

Proof. Let $\phi: Re \to Re$ be an R-epiendomorphism. Then $\phi(e) = ere$ for some $r \in R$, and $\phi(x) = \phi(xe) = x\phi(e) = xere = xre$ for all $x \in Re$, hence $Re = Rere$. Thus there exists $y \in R$ such that $yere = e$. We consider the ascending chain of normal R-subgroups of R, $\ell(re) \subseteq \ell((re)^2) \subseteq \cdots$, and the maximum condition yields a positive integer i such that $\ell((re)^i) = \ell((re)^{i+1})$ and hence $Re \cap \ell((re)^i) = Re \cap \ell((re)^{i+1})$. Let $ue \in Re \cap \ell((re)^i)$, then $0 = ue(re)^i = uyere(re)^i = uye(re)^{i+1}$, hence $uye(re)^i = 0$, and $ue = uyere \in Re \cap \ell((re)^{i-1})$. By repeating this argument, we arrive at $Re \cap \ell(re) = 0$ whence ϕ is an R-automorphism.

3.4. Proposition. *Let R possess a Jordan-Hölder series, and e, f be two primitive idempotents of R. Then Re and Rf are R-isomorphic if and only if $Re/Je \cong Rf/Jf$ as R-groups.*

Proof. The "only if" is obvious. Let $\eta: Re/Je \to Rf/Jf$ be an R-isomorphism. Then $\eta(e + Je) = eu + Jf$, $eu \notin J$ for some $u \in Rf$. Hence $Reu = Rf$.

Similarly, we can find $v \in Re$ such that $Rfv = Re$, and $\phi : x \to xuv$ is an R-epiendomorphism of Re, by 3.3, ϕ is an R-automorphism. $\psi : Re \to Rf$ given by $\psi(x) = xu$ is an R-epimorphism. Let $x \in \ker \psi$, then $\psi(x) = xu = 0$. ϕ has an inverse ϕ^{-1}, let $\phi^{-1}(e) = ew$. Then $\phi^{-1}(y) = yw$ for all $y \in Re$, and so $y = \phi^{-1}(\phi(y)) = \phi^{-1}(yuv) = yuvw$. In particular, $x = xuvw = 0$. Thus ψ is an R-isomorphism.

3.5. Proposition. *Let M be an R-group with a Jordan-Hölder series, and e a primitive idempotent of R. Then M has an R-composition factor V containing an R-subgroup U isomorphic to Re/Je if and only if $eM \neq 0$.*

Proof. Let $M = M_1 \supset M_2 \supset \cdots \supset M_t \supset M_{t+1} = 0$ be an R-composition series of M, and $eM \neq 0$. Then $eV \neq 0$ for some $V = M_i/M_{i+1}$, otherwise $eM = e^t M = 0$. Hence $U = Rev \neq 0$ for some $v \in V$. We can assume that U has no proper non-trivial R-subgroups. Since $x \to xv$, $x \in Re$, is an R-epimorphism from Re to U,, its kernel must be the maximal R-subgroup of Re. Hence $Re/Je \cong U$. Conversely let $\psi : Re/Je \to V = M_i/M_{i+1}$ be an R-monomorphism. Then $\psi(e + Je) \neq 0$, and there exists $v \in M_i \backslash M_{i+1}$ such that $\psi(e + Je) = v + M_{i+1}$. Hence $ev \notin M_{i+1}$ implying $eM \neq 0$.

3.6. Proposition. *Every R-group M with no proper non-trivial R-subgroups is R-isomorphic to some Re/Je where e is a primitive idempotent of R.*

Proof. This proposition is a special case of a result due to Laxton (cf. [6]). We give a straight-forward proof: Assume $eM = 0$ for all primitive idempotents e of R, and let $L = \bigcap_{i=1}^{n} \ell(e_i)$ where $\{e_1, \ldots, e_n\}$ is a maximal DPI-set in R. Then 2.4 and 2.5 imply $M = RM = Re_1M + Re_2M + \cdots + Re_nM + LM = LM \supseteq JM = 0$ since the radical annihilates every R-group with no proper non-trivial R-subgroups (cf. [6]), contradiction. If $K = \{e_1, e_2, \ldots, e_n\}$ is a maximal DPI-set in R we will write $\mathscr{L}_K(e_i)$ for the left ideal $\bigcap_{k \neq i} \ell(e_k)$.

3.7. Lemma. *Let $K = \{e_1, e_2, \ldots, e_n\}$ be a maximal DPI-set in R. Then $\mathscr{L}_K(e_i)e_i$ is a nonnilpotent R-subgroup of R, $i = 1, 2, \ldots, n$, and $\mathscr{L}_K(e_i) + \ell(e_i) = R$.*

Proof. $\mathscr{L}_K(e_i)$ is certainly not nilpotent else $\{e_1, e_2, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n\}$ would be a maximal DPI-set in R, by 2.5. Let $x \in \mathscr{L}_K(e_i)$, then $x = xe_i + (-xe_i + x)$. Suppose $xe_i \in J$. Then we can write $x = j_i + (-j_i + x)$, $j_i \in J$, $-j_i + x \in \bigcap_{k=1}^{i} \ell(e_k)$. Assume we have already shown that there exist $j_r \in J$, $r = i, i+1, \ldots, s$, such that $-j_r + x \in \bigcap_{k=1}^{r} \ell(e_k)$. Then put $j_{s+1} = j_s - j_s e_{s+1} \in J$. We obtain $-j_{s+1} + x = j_s e_{s+1} - j_s + x \in \bigcap_{k=1}^{s+1} \ell(e_k)$, and hence $-j_n + x \in \bigcap_{k=1}^{n} \ell(e_k) \subseteq J$, by 2.5. Thus $x \in J$. This yields the first statement in the lemma. Since $\mathscr{L}_K(e_i)e_i$ is nonnilpotent we therefore conclude $\mathscr{L}_K(e_i)e_i = Re_i$. Clearly $R/\ell(e_i) \cong Re_i$, the isomorphism is explicitely given by $r + \ell(e_i) \leftrightarrow re_i$. Hence $\mathscr{L}_K(e_i) + \ell(e_i) = R$.

3.8. Lemma. *Let* $K=\{e_1, e_2, \ldots, e_n\}$ *be a maximal DPI-set in R, and* $L_K=\bigcap_{k=1}^{n} \ell(e_k)$. *Then*

$$R/L_K=\mathscr{L}_K(e_1)/L_K \oplus \mathscr{L}_K(e_2)/L_K \oplus \cdots \oplus \mathscr{L}_K(e_n)/L_K \tag{3.5}$$

as R-groups and

$$\mathscr{L}_K(e_i)/L_K \cong Re_i\,. \tag{3.6}$$

Proof. (3.6) follows from 3.7:

$$Re_i \cong R/\ell(e_i)=\mathscr{L}_K(e_i)+\ell(e_i)/\ell(e_i) \cong \mathscr{L}_K(e_i)/\mathscr{L}_K(e_i)\cap\ell(e_i)=\mathscr{L}_K(e_i)/L_K\,.$$

In order to prove (3.5), we procede as follows:

$$R/L_K=\mathscr{L}_K(e_i)+\ell(e_i)/L_K=\mathscr{L}_K(e_i)/L\oplus\ell(e_i)/L_K, \qquad i=1,2,\ldots,n\,,$$

by 3.7.

$$\begin{aligned}\ell(e_2)/L_K &= \ell(e_2)/L_K\cap(\mathscr{L}_K(e_1)/L_K\oplus\ell(e_1)/L_K)\\ &= \mathscr{L}_K(e_1)/L_K\oplus\ell(e_1)\cap\ell(e_2)/L_K\,.\end{aligned}$$

Thus

$$R/L_K=\mathscr{L}_K(e_2)+\ell(e_2)/L_K=\mathscr{L}_K(e_1)/L_K\oplus\mathscr{L}_K(e_2)/L_K\oplus\ell(e_1)\cap\ell(e_2)/L_K\,.$$

Repeating this argument, we finally arrive at

$$R/L_K=\mathscr{L}_K(e_1)/L_K\oplus\mathscr{L}_K(e_2)/L_K\oplus\cdots\oplus\mathscr{L}_K(e_n)/L_K\,.$$

3.9. Lemma. *Let* $K=\{e_1, e_2, \ldots, e_n\}$ *be a maximal DPI-set in R,* $L_K=\bigcap_{k=1}^{n}\ell(e_k)$, *and suppose R possesses a Jordan-Hölder series. Then* $\mathscr{L}_K(e_i)$ $=R\bar{e}_i+L_K$, $R\bar{e}_i\cap L_K=0$, *for some primitive idempotents* $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n$ *of R. Moreover* $R\bar{e}_i\cong Re_i$ *as R-groups.*

Proof. Since, by (3.6), $\mathscr{L}_K(e_i)/L_K\cong Re_i$, $i=1,2,\ldots,n$, $\mathscr{L}_K(e_i)/L_K$ has a unique maximal R-subgroup. We claim: this is $J\cap\mathscr{L}_K(e_i)/L_K$. Certainly, $J\cap\mathscr{L}_K(e_i)\subset\mathscr{L}_K(e_i)$ since $\mathscr{L}_K(e_i)$ is nonnilpotent (see proof of 3.7). Let $L_K\subseteq T\subset\mathscr{L}_K(e_i)$, T being an R-subgroup of $\mathscr{L}_K(e_i)$, then $Te_i\subset Re_i$. Hence $Te_i\subseteq Je_i\subseteq J$. Let $x\in T$, then, by the same argument as used in the proof of 3.7, $x\in J$ implying $T\subseteq J\cap\mathscr{L}_K(e_i)$. Since $\mathscr{L}_K(e_i)$ is nonnilpotent, there exists a primitive idempotent $\bar{e}_i\in R$ such that $\mathscr{L}_K(e_i)\supseteq R\bar{e}_i$. Since $R\bar{e}_i+L_K$ is nonnilpotent we therefore have $R\bar{e}_i+L_K=\mathscr{L}_K(e_i)$. Moreover $R\bar{e}_i+J/J$ $=\mathscr{L}_K(e_i)+J/J\cong\mathscr{L}_K(e_i)/J\cap\mathscr{L}_K(e_i)\cong Re_i/Je_i$ by what has been said above. By 3.4, $Re_i\cong R\bar{e}_i$. Also, $Re_i\cong\mathscr{L}_K(e_i)/L_K=R\bar{e}_i+L_K/L_K\cong R\bar{e}_i/R\bar{e}_i\cap L_K$. Hence we can find an R-epiendomorphism $R\bar{e}_i\to R\bar{e}_i/R\bar{e}_i\cap L_K\to Re_i\to R\bar{e}_i$, 3.3 implies that this is an R-automorphism. Therefore $R\bar{e}_i\to R\bar{e}_i/R\bar{e}_i\cap L_K$ is an R-isomorphism whence $R\bar{e}_i\cap L_K=0$, Q.E.D.

3.10. Theorem. *Let* $K=\{e_1, e_2, \ldots, e_n\}$ *be a maximal DPI-set in* R, $L_K=\bigcap_{k=1}^{n} \ell(e_k)$, *and suppose* R *possesses a Jordan-Hölder series. Then there exists a set of pairwise orthogonal primitive idempotents* $\{f_1, f_2, \ldots, f_n\}$ *in* R *such that* $L_K=\bigcap_{k=1}^{n} \ell(f_k)$.

Proof. By (3.5) and 3.9, we may write $1+L_K=(r_1\bar{e}_1+L_K)+(r_2\bar{e}_2+L_K)+\cdots$ $\cdots+(r_n\bar{e}_n+L_K)$, where $\bar{e}_i$ has the same meaning as in 3.9, and $r_1, r_2, \ldots, r_n \in R$. If $r\in R$ then

$$r+L_K=r(1+L_K)=(rr_1\bar{e}_1+L_K)+(rr_2\bar{e}_2+L_K)+\cdots+(rr_n\bar{e}_n+L_K),$$

in particular,

$$r_i\bar{e}_i+L_K=(r_i\bar{e}_i\bar{e}_1+L_K)+\cdots+(r_i\bar{e}_ir_i\bar{e}_i+L_K)+\cdots+(r_i\bar{e}_ir_n\bar{e}_n+L_K).$$

Hence by (3.5), $r_i\bar{e}_ir_k\bar{e}_k\in R\bar{e}_k\cap L_K=0$ for $i\neq k$, by (3.5) and 3.9, $r_i\bar{e}_ir_i\bar{e}_i-r_i\bar{e}_i\in R\bar{e}_i\cap L_K=0$, also by (3.5) and 3.9. If we put $f_i=r_i\bar{e}_i$, $i=1,2,\ldots,n$, then $\{f_1, f_2, \ldots, f_n\}$ constitutes a set of pairwise orthogonal idempotents in R. Clearly $Rf_i=R\bar{e}_i$, $i=1,2,\ldots,n$, and the first part of the proof of 2.1 shows that f_i is primitive. Moreover let $s\in\bigcap_{k=1}^{n}\ell(f_k)$. Then $s+L_K=(sf_1+L_K)+\cdots$ $\cdots+(sf_n+L_K)=L_K$, hence $s\in L_K$, and $\bigcap_{k=1}^{n}\ell(f_k)\subseteq L_K$. But, by 3.8 and 3.9

$$R\Big/\bigcap_{k=1}^{n}\ell(f_k)\cong Rf_1\oplus Rf_2\oplus\cdots\oplus Rf_n\cong Re_1\oplus Re_2\oplus\cdots\oplus Re_n\cong R/L_K.$$

This yields an R-epimorphism $rf_i\to rf_ie_i$, $r\in R$, $i=1,2,\ldots,n$, from Rf_i to Re_i by 3.3, this is an R-isomorphism. Thus $\bigcap_{k=1}^{n}\ell(f_k)=L_K$.

4. The Block Decomposition of R

Throughout this section we keep assumption (3.1). Moreover we assume that R has a Jordan-Hölder series.

4.1. Definition. A *maximal orthogonal DPI-set* $\{e_1, e_2, \ldots, e_n\}$ in R is a maximal DPI-set in R with $e_ie_k=0$ for $i\neq k$.

4.2. Definition. Two primitive idempotents $e f\in R$ are *linked* if there exist primitive idempotents $e=e_0, e_1, \ldots, e_s=f$ such that in Re_i and Re_{i+1} there exist R-composition factors V_i and V_{i+1} respectively with R-isomorphic minimal R-subgroups U_i and U_{i+1} respectively (and $U_i\cong Re/Je\cong U_{i+1}$ for some primitive idempotent $e\in R$).

4.3. Proposition. *If $R=B_1\oplus B_2\oplus\cdots\oplus B_t$ is a decomposition of R into indecomposable two-sided ideals of R then each Re, e being a primitive idempotent of R, is contained in exactly one B_i.*

Proof. We have $Re=B_1e\oplus B_2e\oplus\cdots\oplus B_te$, hence $e=b_1e+b_2e+\cdots+b_te$, $b_i\in B_i$, $i=1,2,\ldots,t$. Then $b_ie=b_ieb_1e+b_ieb_2e+\cdots+b_ieb_te$, thus b_ie is an idempotent with $(b_ie)e=b_ie$, or $b_ie=0$. If $eb_ie=e=eb_je$ for $i\neq j$, then $e\in B_i\cap B_j=0$, contradiction. Hence, if $t>1$, $(b_je)e=b_je$, $e(b_je)\neq e$ for all but one j. Since e is primitive, it follows $b_je=0$ for all but one j. Thus $e=b_ie$ for some i implying $Re\subseteq B_i$.

4.4. Proposition. *Let e and f be two linked primitive idempotents in R. Then Re and Rf belong to one and the same two-sided ideal B_i in each decomposition $R=B_1\oplus B_2\oplus\cdots\oplus B_t$ into two-sided ideals.*

Proof. There exist primitive idempotents $e=e_0,e_1,\ldots,e_s=f$ of R such that Re_i has a composition factor V_i containing a minimal R-subgroup U_i, and Re_{i+1} has an R-composition factor V_{i+1} containing a minimal R-subgroup U_{i+1} such that $U_i\cong U_{i+1}$ as R-groups. By 3.5, there exists a primitive idempotent $e'\in R$ such that $e'Re_i\neq 0$ and $e'Re_{i+1}\neq 0$. By 4.3, Re_i is contained in a (uniquely determined) B_j, hence $Re_iR\subset B_j$, and $Re'R\cap Re_iR\neq 0$, in particular, $Re'R\cap B_j\neq 0$. If $Re'\subseteq B_k$, then $Re'R\subseteq B_k$ whence $0\neq Re'R\cap B_j\subseteq B_k\cap B_j$ implying that $Re'\subseteq B_j$. Similarly $Re_{i+1}\subseteq B_j$, Q.E.D.

4.5. Definition. Let $K=\{e_1,e_2,\ldots,e_n\}$ be a maximal orthogonal DPI-set in R, and let us write

$$K=\{e_1,e_2,\ldots,e_n\}=\{e_{11},e_{12},\ldots,e_{1s_1};e_{21},e_{22},\ldots,e_{2s_2};\ldots;e_{r1},e_{r2},\ldots,e_{rs_r}\}$$

where e_{ik} and e_{jl} are linked if and only if $i=j$. We call $T_i=Re_{i1}+Re_{i2}+\cdots+Re_{is_i}+L_K\cap\bigcap_{j\neq i}\ell(Re_{jk})$ a *block* of R where $L_K=\bigcap_{k=1}^{n}\ell(e_k)$ and $\ell(Re_{jk})=\bigcap_{r\in R}\ell(re_{jk})$.

4.6. Proposition. *The blocks of R are two-sided ideals.*

Proof. We first show that $T_{ik}=Re_{ik}+L_K\cap\bigcap_{j\neq i}\ell(Re_{jk})$ is a left ideal of R. Let $xe_{ik}+l\in T_{ik}$, $x\in R$, $l\in L_K\cap\bigcap_{j\neq i}\ell(Re_{jk})$, and $y\in R$. Then $-y+xe_{ik}+l+y=(-y+xe_{ik}+l+y)e_{ik}+(-ye_{ik}-xe_{ik}+ye_{ik}-y+xe_{ik}+l+y)$ and the second bracket expression is an element of $L_K\cap\bigcap_{j\neq i}\ell(Re_{jk})$, since K is an orthogonal DPI-set, and by 3.5. Now let $x\in R$, then $e_{ik}x-(e_{ik}xe_{i1}+e_{ik}xe_{i2}+\cdots+e_{ik}xe_{is_i})\in L_K\cap\bigcap_{j\neq i}\ell(Re_{jk})$, again by 3.5 and orthogonality of K. Hence $e_{ik}x\in T_i$. If $l\in L\cap\bigcap_{j\neq i}\ell(Re_{jk})$, then $lx-(lxe_{i1}+lxe_{i2}+\cdots+lxe_{is_i})\in L_K\cap\bigcap_{j\neq i}\ell(Re_{jk})$, by orthogonality of K. Thus $lx\in T_i$. Therefore T_i is a two-sided ideal of R.

4.7. Theorem. *R is the sum of the blocks obtained from a given maximal orthogonal DPI-set $K=\{e_1,\ldots,e_n\}=\{e_{11},\ldots,e_{rs_r}\}$.*

Proof. As before, let $L_K = \bigcap_{k=1}^{n} \ell(e_k)$. By 2.5, L_K is nilpotent, hence $L_K^m = 0$ for some positive integer m. By orthogonality of K, we can write

$$1 = e_1 + e_2 + \cdots + e_n + \bar{l},\ \bar{l} \in L_K.$$

Let $l \in L_K$, then

$$l = e_1 l + e_2 l + \cdots + e_n l + \bar{l} l$$

$$\bar{l} l = e_1 \bar{l} l + e_2 \bar{l} l + \cdots + e_n \bar{l} l + \bar{l} l^2$$

$$\bar{l} l^{m-2} = e_1 \bar{l} l^{m-2} + e_2 \bar{l} l^{m-2} + \cdots + e_n \bar{l} l^{m-2}.$$

Thus L_K is generated by $\{e_1 L_K, e_2 L_K, \ldots, e_n L_K\}$, and $e_{ik} L_K \subseteq L_K \cap \bigcap_{j \neq i} \ell(Re_{ji})$, Q.E.D.

4.8. Theorem. *Let $\Gamma = \bigcap \ell(Re)$, the intersection taken over all primitive idempotents $e \in R$. (It is sufficient to take the intersection over a fixed given maximal DPI-set in R.) If $T_1, T_2, \ldots, T_r$ are the blocks of R obtained from the DPI-set $K = \{e_{11}, \ldots, e_{rs_r}\}$, there is a decomposition*

$$R/\Gamma = T_1/\Gamma \oplus T_2/\Gamma \oplus \cdots \oplus T_r/\Gamma \tag{4.1}$$

into two-sided ideals of R/Γ.

Proof. Let $t_1 + t_2 + \cdots + t_r \in \Gamma$, $t_i \in T_i$, $i = 1, 2, \ldots, r$. Then $t_j R e_{ik} = 0$ for $i \neq j$, hence $t_i R e_{ik} = 0$ and $t_i \in \Gamma$ for $i = 1, 2, \ldots, r$.

4.9. Theorem. *Let $\Gamma = \bigcap \ell(Re) = 0$, e running through all primitive idempotents of R. Then the blocks obtained from any maximal orthogonal DPI-set $K = \{e_{11}, \ldots, e_{rs_r}\}$ are indecomposable two-sided ideals of R, and if $R = B_1 \oplus \cdots \oplus B_s$ is any decomposition of R into indecomposable two-sided ideals, the direct summands B_i are precisely the blocks of R.*

Proof. Let $e_{i1}, \ldots, e_{is_i}$ be those primitive idempotents of K that belong to B_i. Then, if T_i is the corresponding block as in 4.5, $B_i \cap T_i$ contains the two-sided ideal K_i of R, K_i minimal w.r.t. containing $Re_{i1}, \ldots, Re_{is_i}$. But $R = K_1 + K_2 + \cdots + K_n$ since $L_K \subseteq J$, and J is the intersection of the maximal two-sided ideals of R, by Laxton ([6]). Hence $R = B_1 \cap T_1 \oplus B_2 \cap T_2 \oplus \cdots \oplus B_s \cap T_s$, thus $B_i \cap T_i = B_i = T_i$.

4.10. Theorem. *Let $K = \{e_{11}, \ldots, e_{rs_r}\}$ be a maximal orthogonal DPI-set in R, and $\Gamma = \bigcap_{i,j} \ell(Re_{ij})$. Then*

(a) *$\bar{K} = \{e_{11} + \Gamma, \ldots, e_{rs_r} + \Gamma\}$ is a maximal orthogonal DPI-set in R/Γ, and $\bar{\Gamma} = \bigcap_{i,j} \ell(Re_{ij} + \Gamma) = \Gamma$;*

(b) *(4.1) is the unique decomposition of R/Γ into indecomposable two-sided ideals (hence the blocks obtained from two different maximal orthogonal DPI-sets of R are the same);*

(c) *each block T_i of R contains a unique indecomposable direct summand B_i of R, and*

$$R = B_1 \oplus B_2 \oplus \cdots \oplus B_r$$

is the unique decomposition of R into indecomposable two-sided ideals.

Proof. (a) Clearly, $\bar{K}$ is a set of pairwise orthogonal idempotents of R/Γ. Let $e \in K$ and $\Gamma \subseteq U \subset Re + \Gamma$, U/Γ being a non-nilpotent R/Γ-group. Then also U is a nonnilpotent R-group. But $U = U \cap (Re + \Gamma) = (U \cap Re) + \Gamma$, hence $U \cap Re$ is nonnilpotent, otherwise $U \subset J$. Therefore $Re \subseteq U$, contradiction. Hence U/Γ is nilpotent, and $Re + \Gamma/\Gamma$ is a minimal nonnilpotent R/Γ-subgroup of R/Γ. The argument in the proof of 2.1 shows that $e + \Gamma$ is a primitive idempotent of R/Γ. Let $x \in \bar{\Gamma}$, then $xre_{ij} \in \Gamma$, $r \in R$, for all $e_{ij} \in K$. But $xre_{ij} \in Re_{ij}$ and $Re_{ij} \cap \Gamma \subseteq Re_{ij} \cap \ell(Re_{ij}) \subseteq Re_{ij} \cap \ell(e_{ij}) = 0$, hence $xre_{ij} = 0$ for all $e_{ij} \in K$ and so $x \in \Gamma$. Thus $\bar{K}$ is a maximal DPI-set in R and $\bar{\Gamma} = \Gamma$.

(b) This follows from (a) and 4.9.

(c) Let T_i be the block constructed from $e_{i1}, \ldots, e_{is_i}$, and B_i the indecomposable two-sided ideal of R in a given decomposition of R into a direct sum of two-sided ideals such that $e_{ij} \in B_i$ accordingly to 4.4. Moreover, let K_i be the two-sided ideal minimal w.r.t. containing $e_{i1}, \ldots, e_{is}$. Then $K_i \subseteq B_i \subseteq T_i$. Thus $R = B_1 \oplus \cdots \oplus B_s = K_1 \oplus \cdots \oplus K_s$ and $B_i = K_i$, determining a unique B_i to a given T_i.

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Formulas to Represent Functions by their Derivatives

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1. Introduction

In [8] we announced some formulas for the representation of functions by their partial derivatives and some applications of these formulas to partial differential equations, particularly to coercive inequalities. The purpose of the present article is to give the full proofs, some extensions of the formulas, and some additional applications.

The formulas concern the representation of a function $u \in C_0^\infty(R^n)$ by special integral operators applied to $P_1 u, \ldots, P_N u$, where P_j is a linear homogeneous partial differential operator of order m_j with constant coefficients. In current terminology they assert that if the characteristic polynomials have no common non-trivial complex zero, then the map $u \to (P_1 u, \ldots, P_N u)$ has an inverse which is a convolution pseudo-differential operator with support in any given convex cone Γ. Specifically, the theorem of [8] is as follows.

Theorem I. *If the characteristic polynomials of the P_j have no common non-trivial complex zero, then for any closed convex n-dimensional cone Γ there exist kernels K_j with the following properties:*

(a) *K_j is homogeneous of degree $m_j - n$ and is C^∞ on $R^n - \{0\}$.*

(b) *K_j vanishes identically outside Γ.*

(c) *For every function $u \in C_0^\infty(R^n)$ we have*

$$u = \sum_j P_j u * K_j = \sum_j \int P_j u(x-y)\, K_j(y)\, dy\,. \tag{1}$$

Our interest in such formulas began with the coercive inequalities of N. Aronszajn [3]. If Ω is an open set such that $\Omega - \Gamma \subset \Omega$, then for each $x \in \Omega$ the integral in (1) makes sense when u is defined only on Ω, and the formula remains correct. Differentiation of this formula gives the coercive inequality

$$\|D^\alpha u\| \leqq c \sum \|P_j u\| \quad \textit{for} \quad |\alpha| = m \tag{2}$$

with any of the norms for which singular integrals are known to be bounded operators, e.g. the norms in L^p or L_s^p, $1 < p < \infty$, the Hölder norms, etc. The case studied by Aronszajn is essentially that of the L^2 norm and an Ω with a smooth boundary.

At the first look the condition that $\Omega - \Gamma \subset \Omega$ seems peculiar. However, the applications are local in nature, and the localized condition is simply that each point of $\partial\Omega$ has a neighborhood in which $\partial\Omega$ is the graph of a Lipschitz function

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(in some system of coordinates), and that Ω lies on one side of $\partial\Omega$. An open Ω of this kind is said to satisfy the cone condition, or to be a Lipschitz graph domain. It need not be smooth, but can have corners, edges, and so on. On the other hand, it is easy to show (if somewhat surprising) that very respectable polyhedra are not necessarily Lipschitz graph domains. Consequently, it is interesting to know whether the coercive inequality (2) holds for Ω if it holds for each of $\Omega_1, \ldots, \Omega_k$ with union Ω. This is obvious for the L_s^p norm when s is a positive integer, but it is false in general without supplementary conditions on the intersections $\Omega_i \cap \Omega_j$. This piecing together of Lipschitz graph domains is discussed in detail in [1] in connection with extension theorems, and the conditions that are relevant here appear to be the same. Essentially, the intersections should be non-tangential.

The condition that the characteristic polynomials have no common non-trivial complex zero is necessary for the coercive inequalities, a fortiori for Theorem I. However, if the cone Γ is fixed a priori, then this condition is no longer necessary. For example, we have the following result, which gives the best possible information when the boundary of Ω is smooth.

Theorem II. *Let Γ be a half space. The necessary and sufficient condition for the existence of the kernels of Theorem I is that the characteristic polynomials have no common non-trivial complex zero with imaginary part orthogonal to $\partial\Gamma$.*

Here, however, we get kernels of class $C^r(R^n - \{0\})$ for any $r < \infty$, but not of class C^∞. Probably the latter exist too.

Theorem II is the one that gives the full strength of the Aronszajn inequality when the operators P_j have variable coefficients and the boundary of the domain Ω is smooth. And it gives similar inequalities for L_s^p norms, Hölder norms, etc. The common ground of Theorems I and II is the following statement, which is certainly true, but which we have not yet succeeded in proving except in special cases.

Conjecture. *Let Γ be given. The necessary and sufficient condition for the existence of the kernels of Theorem I is that the characteristic polynomials have no common non trivial complex zero with imaginary part in the dual cone*

$$\Gamma_+ = \{\eta \in R^n : (x, \eta) \geqq 0 \quad \textit{for all} \quad x \in \Gamma\}.$$

Note that the condition remains the same if it is required that there is no common zero with real part in Γ_+, or with real or imaginary part in Γ_-. All four conditions are the same because of the homogeneity of the characteristic polynomials. If ξ is a common zero, then so is $\alpha\xi$ for any complex α.

The necessity of this condition (hence also of the conditions in Theorems I and II) is almost apparent. If K_j has support in Γ, and $\eta \in \Gamma_+$, then the Fourier transform extends to the complex domain so that $\hat{K}_j(\xi - i\tau\eta)$ is an analytic function of τ in the half plane $\operatorname{Re}\tau > 0$. In terms of Fourier transforms the formula (1) becomes

$$1 = (2\pi)^{n/2} \sum p_j(i(\xi - i\tau\eta))\, \hat{K}_j(\xi - i\tau\eta) \tag{3}$$

for $\xi \in R^n$ and $\operatorname{Re}\tau \geqq 0$. It follows that the $p_j(\xi - i\eta)$ cannot all be zero, or in other words that the p_j cannot have a common zero with imaginary part in Γ_-.

It is a simple matter to extend the above theorems to the case where the differential operators P_j act on vector valued functions, i.e.

$$u = (u_1, \ldots, u_M) \quad \textit{and} \quad P_j u = \sum P_{jk} u_k .$$

In this case the condition on the characteristic polynomials is that the matrix $\{p_{jk}(\xi)\}$ has rank M for each non zero $\xi \in C^n$ (whose imaginary part is in Γ_+ in Theorem II and the conjecture). The conclusion is that there are kernels K_{ij} with support in Γ such that

$$u_i = \sum_j P_j u * K_{ij} . \tag{4}$$

The coercive inequalities follow directly as before. Gobert [6] has given a proof of Korn's inequality along lines somewhat like this, but he takes the inequality as the basic point, rather than the formula (4), which appears to be much easier to manage.

The representation formulas have various other applications. The one we will discuss in detail is the application to extension theorems. If each function (or distribution) $P_j u$ has an extension from Ω to R^n, then formula (1) gives an extension of u itself from Ω to R^n. Suppose, for instance, that each P_j has order m, and that $P_j u \in L^p$, $1 < p < \infty$. Then (by the boundedness of singular integrals), the extension given by formula (1), after extending $P_j u$ by 0, is in $L^p_m(R^n)$, which gives simultaneously an extension theorem and a coercive inequality. Indeed, this is the method we will use to prove the coercive inequalites, for it presents them in a strong form where it is not necessary to assume a priori that the function u belongs to $L^p_m(\Omega)$. A. P. Calderón initiated this method of extension in [4]. He considered the case where the P_j consist precisely of all derivatives of a given order m.

A remark about the proofs. There is a sketch of the proof of Theorem I in [8], and this is amplified in the book of S. Agmon [2]. The present proof is a little different, being based on ideal theory in Noetherian rings, rather than the Hilbert Nullstellensatz. It gives slightly more than the original proof, and this could be important in proving the conjecture. The proof of Theorem II is based mainly on Theorem I. For the algebra involved in these proofs almost any good book will do. The one that *I* found most helpful is the work of Hodge and Pedoe, *Methods of Algebraic Geometry*, Cambridge University Press. The homogeneity of various ideals and polynomials is crucial in our proofs, and this can be seen easily in the treatment of Hodge and Pedoe, which is more explicit than that in many modern books. Except in the last two sections, all differential operators are homogeneous with constant coefficients, and all polynomials and polynomial ideals are homogeneous. With the modern techniques for handling elliptic problems, the passage from this case to the general one is fairly routine.

2. Proof of Theorem I

The simplest case of Theorem I, that in which $P_j = D_j = \partial/\partial x_j$, is due to Sobolev [9], who reasoned as follows. If $u \in C_0^\infty(R^n)$ and $|\theta| = 1$, then

$$u(x) = -\int_0^\infty \frac{du(x-t\theta)}{dt}\, dt = \sum \int_0^\infty D_j u(x-t\theta)\theta_j\, dt\,.$$

Let φ be any C^∞ function on the unit sphere with support in Γ and with integral over the sphere equal to 1. Multiply by φ and integrate over the sphere, and then put $y = t\theta$ to get

$$u(x) = \sum \int_S \int_0^\infty D_j u(x-t\theta)\theta_j \varphi(\theta)\, dt\, d\theta = \sum \int_{R^n} D_j u(x-y)\, K_j(y)\, dy$$

with

$$K_j(y) = \frac{1}{\omega_n} \frac{y_j}{y} \varphi\left(\frac{y}{|y|}\right)$$

where ω_n is the surface area of the unit sphere. The general case will be reduced to this one.

Whether the characteristic polynomials p_j have common zeros or not, the existence of the kernels K_j depends only on the ideal $\mathscr{P}$ generated by the p_j. Indeed, let $q_1, \ldots, q_N$ be another system of generators with q_j homogeneous of degree n_j. Then

$$p_j = \sum a_{jk} q_k$$

where a_{jk} is a homogeneous polynomial of degree $m_j - n_k$. For the corresponding differential operators this gives the identity

$$P_j = \sum A_{jk} Q_k\,.$$

Now, if formula (1) in Theorem I holds, then we have

$$u = \sum_j P_j u * K_j = \sum_{j,k} A_{jk} Q_k u * K_j = \sum_k Q_k u * \sum_j A_{jk} K_j\,,$$

which is formula (1) again with the kernels

$$L_k = \sum_j A_{jk} K_j\,.$$

Henceforth we shall say that a homogeneous ideal $\mathscr{P}$ has the property (K_Γ) if for some system of homogeneous generators there exist kernels with the properties (a), (b), and (c) of Theorem I. We have the following theorem, which is somewhat more general than Theorem I.

Theorem 2.1. *Suppose that $\Gamma - \{0\}$ lies in an open half space. If every prime ideal containing $\mathscr{P}$ has the property (K_Γ), then $\mathscr{P}$ itself has the property (K_Γ).*

In the case of Theorem I the only prime ideal containing $\mathscr{P}$ is the ideal of all polynomials which vanish at 0. This is generated by the polynomials

$\xi_1, \ldots, \xi_n$, for which there are the kernels given by Sobolev. Therefore, Theorem 2.1 includes Theorem I. It also shows that the conjecture will be proved if it can be proved for prime ideals. Theorem 2.1 is an easy consequence of a lemma.

Lemma 2.2. *If $\mathscr{P} \supset \mathscr{P}'\mathscr{P}''$ and both $\mathscr{P}'$ and $\mathscr{P}''$ have the property (K_Γ), then so does $\mathscr{P}$.*

Proof of Lemma 2.2. Let $\{p_i\}$, $\{p'_j\}$, $\{p''_k\}$ be sets of generators of $\mathscr{P}$, $\mathscr{P}'$, and $\mathscr{P}''$ respectively. By hypothesis

$$u = \sum_j P'_j u * K'_j = \sum_k P''_k u * K''_k$$

so that

$$u = \sum_{j,k} P'_j P''_k u * K''_k * K'_j .$$

Now, $p'_j p''_k \in \mathscr{P}'\mathscr{P}'' \subset \mathscr{P}$, so that

$$p'_j p''_k = \sum_i a_{ijk} p_i$$

with a_{ijk} homogeneous of degree $m'_j + m''_k - m_i$. Therefore

$$P'_j P''_k = \sum_i A_{ijk} P_i ,$$

which gives

$$u = \sum_{i,j,k} A_{ijk} P_i u * K''_k * K'_j = \sum_i P_i u * K_i$$

with

$$K_i = \sum_{j,k} A_{ijk} K''_k * K'_j .$$

The convolutions which appear in these calculations exist and are associative because both K''_k and K'_j have support in Γ, which is properly contained in an open half space plus the origin. It follows that $K''_k * K'_j$ also has support in Γ and is C^∞ outside the origin. As to the homogeneity, $K''_k * K'_j$ is homogeneous of degree $m''_k + m'_j - n$. Hence K_i is homogeneous of degree

$$(m''_k + m'_j - n) - (m''_k + m'_j - m_i) = m_i - n .$$

Proof of Theorem 2.1. According to the classical ideal theory in Notherian rings we have

$$\mathscr{P} = \mathscr{P}_1 \cap \cdots \cap \mathscr{P}_r \tag{1}$$

where each $\mathscr{P}_j$ is primary. If $\mathscr{R}_j$ is the radical of $\mathscr{P}_j$, then

$$\mathscr{R}_j \supset \mathscr{P}_j \quad \text{and} \quad \mathscr{R}_j^s \subset \mathscr{P}_j . \tag{2}$$

By hypothesis, $\mathscr{R}_j$ has property (K_Γ). Therefore, the lemma and the formula (2) show that $\mathscr{P}_j$ has property (K_Γ), and then the lemma and the formula (1) show that $\mathscr{P}$ has property (K_Γ).

Remark. It is necessary to know that the primaries which occur in the decomposition (1) are homogeneous. This can be avoided by carrying out the proof as follows. Let V be the variety of zeros of $\mathscr{P}$ and write

$$V = V_1 \cup \cdots \cup V_r$$

where each V_j is irreducible. It is clear that V_j is homogeneous, and so is its ideal $\mathscr{R}_j$. If

$$\mathscr{R} = \mathscr{R}_1 \cap \cdots \cap \mathscr{R}_r \tag{3}$$

then by the Hilbert Nullstellensatz we have

$$\mathscr{P} \supset \mathscr{R}^s. \tag{4}$$

Each $\mathscr{R}_j$ is prime, and so by hypothesis has property (K_Γ). According to the lemma and formula (3), $\mathscr{R}$ has property (K_Γ), and then according to the lemma and formula (4), $\mathscr{P}$ has property (K_Γ).

The Hilbert Nullstellensatz is perhaps better known than the primary decomposition that appears in formulas (1) and (2), but the primary decomposition is quite simple, while the Nullstellensatz is quite hard. Of course, the primes $\mathscr{R}_j$ that appear in the two cases are the same.

3. Proof of Theorem II

Now Γ is a half space, and the assumption is that the characteristic polynomials p_j have no common non-trivial complex zero whose imaginary part is orthogonal to $\partial\Gamma$, and we show that there exist kernels with any given regularity $C^r(R^n - \{0\})$, with K_j homogeneous of degree $m_j - n$, and with support in Γ.

Let Γ be the half space $\overline{R^n_+} = \{x : x_n \geqq 0\}$ and write $x = (x', x_n)$ with $x' \in R^{n-1}$. Then the hypothesis is that the p_j have no common non-trivial complex zero ξ with ξ' real. From the classical theory of resultants it follows that there exists a polynomial q' with the following properties:

(a) *q' depends only on ξ'.*

(b) *q' is homogeneous and $q'(\xi') \neq 0$ for $0 \neq \xi' \in R^{n-1}$.*

(c) *There are homogeneous polynomials $q_j(\xi)$ such that*

$$q' = \sum_j q_j p_j.$$

Since at least one of the p_j contains a term $c\xi_n^{m_j}$, $c \neq 0$, there exist homogeneous polynomials $p'_1, \ldots, p'_n$ depending only on ξ' such that $p_1, \ldots, p_N$, $p'_1, \ldots, p'_n$ have no common non-trivial complex zero. For example, $p'_j = \xi_j^{m'_j}$. The degree m'_j of p'_j can be taken as large as desired. For reasons that will appear later we take

$$m'_j \geqq r + m' + 2 \tag{1}$$

where r is the degree of regularity required of the kernels and m' is the degree of q'.

If Γ_0 is any convex n-dimensional cone, then by Theorem I there exist kernels K_j, $K'_j \in C^\infty(R^n - \{0\})$, homogeneous of degrees $m_j - n$ and $m'_j - n$, and with support in Γ_0 such that

$$u = \sum_j P_j u * K_j + \sum_j P'_j u * K'_j . \tag{2}$$

In order to ensure the existence and associativity of the convolutions which appear in the calculations to follow, we choose a $\Gamma_0 \subset R^n_+ \cup \{0\}$ (the open half space plus the origin). The problem now is to get rid of the terms $P'_j u * K'_j$ in formula (2), or, more precisely, to transform them so that they look like the other ones.

Being elliptic, homogeneous, with constant coefficients, the operator Q' has a fundamental solution e' such that

$$v' = Q' v' *' e' \quad \textit{for} \quad v' \in C_0^\infty(R^{n-1}),$$

where $*'$ means that the convolution takes place in R^{n-1}. For any $u \in C_0^\infty(R^n)$ we have

$$\begin{aligned} P'_j u * K'_j &= P'_j(Q' u *' e') * K'_j = ((P'_j Q' u) *' e') * K'_j \\ &= P'_j Q' u * (e' *' K'_j) = \sum_k P'_j Q_k P_k u * (e' *' K'_j) . \end{aligned} \tag{3}$$

All that remains is to show that the operator $P'_j Q_k$ can be transferred to $e' *' K'_j$ and that the resulting kernel has the right homogeneity and regularity.

There is no difficulty in transferring P'_j. It depends only on $D_{x'}$, and for fixed x_n, K'_j is $C_0^\infty(R^{n-1})$. Thus, it is necessary to examine the function

$$L_j = P'_j(e' *' K'_j) .$$

The homogeneity is clear. It is known, for example, from the explicit formulas of Fritz John [7], that e' can be taken almost homogeneous of degree $m' - (n-1)$ in the sense that any derivative of e' of order $k > m' - (n-1)$ is actually homogeneous of degree $m' - (n-1) - k$. Thus, L_j is homogeneous of degree

$$m' - (n-1) - m'_j + (m'_j - n) + n - 1 = m' - n$$

so that $Q_k L_j$, the kernel going along with P_k in formula (3), is homogeneous of degree

$$m' - n - (m' - m_k) = m_k - n$$

as required.

As for the regularity, it is clear that L_j is of class C^∞ on R^n_+ and is identically 0 on R^n_-. Therefore it is sufficient to discuss a neighborhood of a point $(x'_0, 0)$ with $x'_0 \neq 0$.

Choose $\varphi' \in C_0^\infty(R^{n-1})$ to be 1 on a neighborhood of 0 and 0 for $|x'| > |x'_0|/2$, and write

$$L_j = (\varphi' e') *' P'_j K'_j + P'_j\{(1 - \varphi') e'\} *' K'_j .$$

The first term is C^∞ on a neighborhood of $(x'_0, 0)$ because $P'_j K'_j(x'-y', x_n)$ is C^∞ on a neighborhood of this point when y' is in the support of φ'. The second term is of class $C^{m'_j-2}$, for K'_j is homogeneous of degree $m'_j - n$ and can therefore be differentiated $m'_j - 2$ times without destroying its integrability as a function of x', even when $x_n = 0$, while $(1-\varphi')e'$ is of class C^∞.

Thus $Q_k L_j$ is of class C^s, with

$$s = m'_j - 2 - (m' - m_k) \geqq m'_j - m' - 2 \geqq r.$$

4. Systems

It is a routine matter to extend the foregoing formulas to systems. Let P_{ji} be a linear homogeneous differential operator of order $m_j - l_i \geqq 0$, $j = 1, \ldots, N$, $i = 1, \ldots, M$. Put

$$P_j u = \sum_{i=1}^{M} P_{ji} u_i \quad \textit{where} \quad u = (u_1, \ldots, u_M)$$

and let p_{ji} be the characteristic polynomial of P_{ji}.

Theorem 4.1. *If the matrix $\{p_{ji}(\xi)\}$ has rank M at each complex $\xi \neq 0$, then for any closed convex n-dimensional cone Γ there exist kernels K_{ij} with the following properties:*

(a) *K_{ij} is homogeneous of degree $m_j - l_i - n$ and is C^∞ on $R^n - \{0\}$.*
(b) *K_{ij} vanishes identically outside Γ.*
(c) *For every $u \in C_0^\infty(R^n)$ we have*

$$u_i = \sum_j P_j u * K_{ij}$$

Proof. If $J = (j_1, \ldots, j_M)$, $1 \leqq j_k \leqq N$, let $\mathscr{D}_J = \{d_{ki}\}$, where $d_{ki} = p_{j_k i}$, let d_J be the determinant of $\mathscr{D}_J$, and let d_J^{ki} be the determinant of the minor of d_{ki} in $\mathscr{D}_J$, with the proper sign.

The hypothesis of the theorem is that the d_J have no common non-trivial complex zero, so by Theorem I there exist kernels K_J with support in Γ such that

$$v = \sum_J D_J v * K_J$$

for any (scalar) function $v \in C_0^\infty(R^n)$. Hence

$$u_i = \sum_J D_J u_i * K_J = \sum_{m,J} \delta_{im} D_J u_m * K_J. \tag{1}$$

By the usual formulas for determinants,

$$\delta_{im} d_J = \sum_{j_k \in J} p_{j_k m} d_J^{ki}. \tag{2}$$

If we replace the polynomials in (2) by the corresponding differential operators and substitute the result in (1), we get

$$u_i = \sum_{m,J} \sum_{j_k \in J} P_{j_k m} u_m * D_J^{ki} K_J = \sum_J \sum_{j_k \in J} P_{j_k} u * D_J^{ki} K_J.$$

This gives (c) in the theorem if we group the terms in which j_k takes a given value j.

To be sure of the existence of the K_J and to establish (a), the degrees of homogeneity must be checked. Clearly d_J is homogeneous of degree

$$m_J = \sum_{j\in J} m_j - \sum_{i=1}^{M} l_i$$

so that K_J is homogeneous of degree $m_J - n$, while d_J^{ki} is homogeneous of degree $m_J - m_{j_k} + l_i$, so that $D_J^{ki} K_J$ is homogeneous of degree $m_{j_k} - l_i - n$.

Remark 1. In connection with the conjecture in the introduction on the existence of kernels for a given cone Γ, note that the proof above uses only that the d_J have property K_Γ. Hence the corresponding conjecture for systems is true whenever it is true in the scalar case. In particular, this is relevant when Γ is a half space, for which the scalar case was established in the last section. Thus we have

Theorem 4.2. *Let Γ be a half space. If the matrix $\{p_{ji}(\xi)\}$ has rank M at each non-trivial complex ξ whose imaginary part is orthogonal to $\partial\Gamma$, then there exist kernels as in Theorem 4.1 (but of class $C^r(R^n - \{0\})$ for any finite r).*

5. Representation of Distributions

We wish to extend the formula

$$u = \sum_j P_j u * K_j$$

to distributions with bounded support on an open set Ω. First the notion of convolution is extended, which is possible if Ω has the property $\Omega - \Gamma \subset \Omega$.

Suppose that Ω has this property and that K has support in Γ. If v is an integrable function on Ω with bounded support, then the convolution is defined by the integral

$$v * K(x) = \int v(x-y)\, K(y)\, dy \quad \textit{for} \quad x \in \Omega\,.$$

The integral makes sense because $x - y$ belongs to Ω whenever x belongs to Ω and y belongs to Γ, while K vanishes identically outside Γ. If $\varphi \in C_0^\infty(\Omega)$, then

$$\langle v * K, \varphi \rangle = \int\int v(x)\, \varphi(x+y)\, K(y)\, dy\, dx = \langle v, \check{K} * \varphi \rangle = \langle v, \psi(\check{K} * \varphi) \rangle$$

with $\check{K}(y) = K(-y)$ and ψ any function in $C_0^\infty(R^n)$ which is 1 on a neighborhood of the support of v.

Lemma 5.1. *If $\psi \in C_0^\infty(R^n)$, then the map $\varphi \to \psi(\check{K} * \varphi)$ is continuous from $C_0^\infty(\Omega)$ to itself.*

Proof. It is well known (and obvious) that the map is continuous from $C_0^\infty(R^n)$ to itself. If the support of φ is C and the support of ψ is D, then the support of $\psi(\check{K} * \varphi)$ is contained in $D \cap (C - \Gamma)$, which is a compact subset of Ω.

Lemma 5.1 justifies the following definition.

Definition. *If v is a distribution on Ω with bounded support, then $v*K$ is the distribution on Ω (i.e. linear form on $C_0^\infty(\Omega)$)*

$$\varphi \to \langle v, \psi(\check{K}*\varphi)\rangle$$

where $\psi \in C_0^\infty(R^n)$ is 1 on a neighborhood of the support of v.

Lemma 5.2. *If $\tilde{v}$ is an extension of v to R^n with bounded support, then $\tilde{v}*K$ is an extension of $v*K$.*

The proof is obvious from the definition.

Theorem 5.3. *Suppose that $\mathscr{P}=\{p_1,\ldots,p_N\}$ has property (K_Γ) with kernels K_j. If $\Omega-\Gamma\subset\Omega$, then*

$$u=\sum_j P_j u*K_j$$

for every distribution u on Ω with bounded support.

Proof. If $\psi\in C_0^\infty(R^n)$ is 1 on a neighborhood of the support of u and $\varphi\in C_0^\infty(\Omega)$, then we have

$$\left\langle \sum_j P_j u*K_j, \varphi\right\rangle = \left\langle \sum_j P_j u, \psi(\check{K}_j*\varphi)\right\rangle = \sum_j \langle u, \psi(P_j^t \check{K}_j*\varphi)\rangle$$
$$= \langle u, \psi(\check{\delta}*\varphi)\rangle = \langle u, \varphi\rangle .$$

From the theorem and Lemma 5.2 we get the following extension theorem, which will be improved in Section 6. The hypotheses are those of Theorem 5.3.

Corollary 5.4. *Let u be a distribution on Ω with bounded support. If each $P_j u$ has an extension to a distribution f_j on R^n with bounded support, then*

$$\tilde{u}=\sum f_j*K_j$$

is an extension of u to R^n.

6. Lipschitz Graph Domains and Extension of Distributions

Now let us consider the condition

$$\Omega-\Gamma\subset\Omega \tag{1}$$

that appeared in the last section, and the corresponding local condition. We will assume that $\Omega \neq R^n$.

If the coordinates are chosen so that the negative x_n axis is an inner ray of Γ, and if $x=(x', x_n)$, $x'\in R^{n-1}$, then each line $x'=c$ intersects Ω in a half line. Hence

$$\Omega=\{x : x_n > h(x')\}\,, \tag{2}$$

where h is a function on R^{n-1}. Moreover, if θ is the minimum angle between the negative x_n axis and a boundary ray of Γ, then

$$|h(x')-h(y')| \leqq |x'-y'| \cot\theta .$$

Therefore h is Lipschitzian with Lipschitz constant $M=\cot\theta$.

Conversely, if Ω is the open set defined by (2), where h is Lipschitzian with Lipschitz constant $M=\cot\theta$, and if Γ is the circular cone making angle θ around the negative x_n axis, then the condition $\Omega-\Gamma\subset\Omega$ is satisfied.

Thus the open Ω which satisfy the condition $\Omega-\Gamma\subset\Omega$ for some cone Γ are precisely those of the form (2) with h a Lipschitz function. Now localize.

Definition. *An open Ω is a Lipschitz graph domain if for each point $x_0\in\partial\Omega$ there exist a number $r_0>0$, a choice of the coordinates, and a Lipschitz function h such that*

$$B(x_0,r_0)\cap\Omega=B(x_0,r_0)\cap\{x:x_n>h(x')\}\,. \tag{3}$$

By the above remarks, a Lipschitz graph domain is an open set that satisfies a suitable local cone condition, which we shall not write down explicitly.

It is assumed that the Lipschitz function h is defined on the whole space R^{n-1}, but this is immaterial, since a Lipschitz function on any subset of R^{n-1} can be extended to a Lipschitz function on the whole space. Indeed, if h is defined on A, the formula

$$\tilde{h}(x)=\sup_{y\in A}\{h(y)-M|x-y|\}$$

of Whitney [11] gives an extension with the same Lipschitz constant M.

It is clear that when formula (3) holds, then

$$B(x_0,r_0)\cap\partial\Omega=B(x_0,r_0)\cap\{x:x_n=h(x')\}\,. \tag{4}$$

Now, suppose that formula (3) holds, let M be the Lipschitz constant of h, let $\cot\theta>M$, and let Γ be the circular cone making an angle θ about the negative x_n axis.

Lemma 6.1. *If $\Omega_r=(B(x_0,r)\cap\Omega)-\Gamma$, $0<r<r_0$, then*
(a) $\Omega_r-\Gamma\subset\Omega_r$.
(b) $\Omega_r\cap B(x_0,r_0)\subset\Omega$.
(c) $\bar{\Omega}_r\cap B(x_0,r_0)\cap\partial\Omega\subset\bar{B}(x_0,r)$.

Proof. Part (a) is obvious simply from the fact that Γ is a convex cone.

As to (b), if $y\in\Omega_r$, then $y=x-\gamma$, where $x_n>h(x')$ and $-\gamma_n\geqq|\gamma'|\cot\theta$, so that

$$y_n=x_n-\gamma_n>h(x')+|\gamma'|\cot\theta\geqq h(x'-\gamma')=h(y')\,.$$

As to (c), if y is in the set on the left, then from (4) we have $y_n=h(y')$. We also have $y=x-\gamma$, so that

$$h(x'-\gamma')=x_n-\gamma_n\geqq h(x')+|\gamma'|\cot\theta$$

and hence

$$M|\gamma'|\geqq h(x'-\gamma')-h(x')\geqq|\gamma'|\cot\theta\,.$$

Since $\cot\theta > M$, this is possible only if $\gamma' = 0$, in which case

$$x_n \geqq h(x') = h(x' - \gamma') = x_n - \gamma_n \geqq x_n .$$

It follows that $\gamma = 0$ and hence that $y = x \in \bar{B}(x_0, r)$.

Now, let $\psi \in C_0^\infty(B(x_0, r_0))$ be equal to 1 on a neighborhood of $\bar{B}(x_0, r)$. For any distribution u on Ω, let u_ψ be equal to the product ψu on $\Omega_r \cap \Omega$ and to 0 on $\Omega_r - \operatorname{spt} \psi$. By Lemma 6.1 these two sets exhaust Ω_r, and on their intersection the two definitions agree.

Lemma 6.2. *u_ψ is a well defined distribution on Ω_r. It coincides with u on $B(x_0, r) \cap \Omega$. If P is a differential operator such that Pu has an extension to $B(x_0, r_0)$, then Pu_ψ has an extension to R^n.*

Proof. The first two statements are clear. In order to show that Pu_ψ has an extension to R^n it is enough to show that it has an extension to a neighborhood of each boundary point of Ω_r. Let x_1 be such a boundary point, and let v be an extension of Pu to $B(x_0, r_0)$. Define the extension f of Pu to a neighborhood of x_1 as follows:

(a) *If $x_1 \in \Omega$, then $f = P(\psi u)$.*

(b) *If $|x_1| \geqq r_0$, then $f = 0$.*

(c) *If $|x_1| < r_0$ and $x_1 \notin \Omega$, then $f = v$.*

Note that in case (c) formula (4) shows that x_1 must belong to $\partial\Omega$, so that by (c) of Lemma 6.1, $x_1 \in \bar{B}(x_0, r)$. Consequently, $u_\psi = u$ on a neighborhood of x_1.

Lemma 6.2 and Corollary 5.4 give the following theorem.

Theorem 6.3. *Let the characteristic polynomials of $P_1, \ldots, P_N$ have no common non-trivial complex zero, let Ω be a Lipschitz graph domain, and let u be a distribution on Ω. If each $P_j u$ can be extended to a neighborhood of each boundary point of Ω, then u itself can be extended to R^n.*

Proof. It is enough to show that for each point $x_0 \in \partial\Omega$ there is a number $r > 0$ such that the restriction of u to $B(x_0, r) \cap \Omega$ has an extension to some neighborhood of x_0. Form u_ψ as above, and use Lemma 6.2 to extend each $P_j u_\psi$. Then use Corollary 5.4 to extend u_ψ itself. Since u_ψ and u coincide on $B(x_0, r) \cap \Omega$, this gives an extension of u.

7. Extension Theorems for $L_s^p(\Omega)$

The extension described in Theorem 6.3 is bounded relative to all of the norms in which singular integrals are known to be bounded operators (at least all the ones that we know of), so Theorem 6.3 includes both extension theorems and coercive inequalities. Some of the details vary from one space or norm to another, but the basic idea is the same. We will carry out the proof in full for the spaces $L_s^p(\Omega)$ when s is an integer, and then make some remarks about the case when s is not an integer to show how the details vary. It is assumed throughout that $1 < p < \infty$.

When s is a non negative integer, $L_s^p(\Omega)$ consists of all distributions on Ω such that the norm

$$\|u\|_{L_s^p(\Omega)} = \Big(\sum_{|j| \leqq s} \|D_j u\|_{L^p}^p\Big)^{\frac{1}{p}} \tag{1}$$

is finite, and $L_s^p(\Omega; 0)$ is the closure of $C_0^\infty(\Omega)$ in $L_s^p(R^n)$. When s is a negative integer, $L_s^p(\Omega)$ is the dual of $L_{-s}^q(\Omega; 0)$, where q is the conjugate exponent to p.

Note that if $s \geqq 0$, the distributions in $L_s^p(\Omega)$ are actually functions, while if $s < 0$, they are not. Note also that in forming $L_s^p(\Omega; 0)$ we have taken the closure of $C_0^\infty(\Omega)$ in $L_s^p(R^n)$, not in $L_s^p(\Omega)$. In the present case where s is a non negative integer the two closures are effectively the same. When s is not an integer they can be essentially different. The norms induced on $C_0^\infty(\Omega)$ by $L_s^p(R^n)$ and $L_s^p(\Omega)$ are not always equivalent if s is not an integer, even when the boundary of Ω is smooth.

If $s < 0$, the elements of $L_s^p(\Omega)$ can still be regarded as distributions on Ω. First consider R^n. The identity map is a continuous embedding of C_0^∞ in $L_{-s}^q(R^n)$, and its range is dense. Therefore, its adjoint is a continuous embedding of $L_s^p(R^n)$ in the space of distributions on R^n. For any Ω, general Banach space theory gives

$$L_s^p(\Omega) = L_s^p(R^n)/L_{-s}^q(\Omega; 0)^\perp . \tag{2}$$

It is plain from the definition of $L_{-s}^q(\Omega; 0)$ that two distributions in $L_s^p(R^n)$ are equal modulo $L_{-s}^q(\Omega; 0)^\perp$ if and only if they have equal restrictions to Ω. Thus, each equivalence class on the right side of (2) determines exactly one distribution on Ω.

Moreover, there is no extension problem for $L_s^p(\Omega)$ if $s \leqq 0$. Formula (2) shows that each element of $L_s^p(\Omega)$ (considered as a distribution on Ω) has an extension to R^n which belongs to $L_s^p(R^n)$. This fact will be useful.

In order to deal with unbounded domains it is convenient to introduce one more space. $L_s^p(\tilde{\Omega}; \mathrm{loc})$ consists of the distributions u on Ω such that for each point $x_0 \in \bar{\Omega}$ there are a neighborhood Ω_0 of x_0 and a $v \in L_s^p(\Omega)$ with $v = u$ on $\Omega \cap \Omega_0$. If Ω is bounded, then $L_s^p(\tilde{\Omega}; \mathrm{loc}) = L_s^p(\Omega)$, but if Ω is unbounded, then the use of $L_s^p(\tilde{\Omega}; \mathrm{loc})$ avoids some hypotheses of uniformity in the description of Lipschitz graph domains.

Lemma 7.1. *If $u \in L_s^p(\Omega)$, then any mth derivative of u belongs to $L_{s-m}^p(\Omega)$. If $u \in L_s^p(\tilde{\Omega}; \mathrm{loc})$, then any mth derivative of u belongs to $L_{s-m}^p(\tilde{\Omega}; \mathrm{loc})$.*

In the present case the lemma is perfectly obvious. It is recorded explicitly because it is not obvious when s is not an integer. Indeed, it is not true without additional hypotheses on Ω.

Theorem 7.2. *Let u be a distribution on the Lipschitz graph domain Ω. Let the characteristic polynomials of $P_1, \ldots, P_N$ have no common non trivial complex zero. If $P_j u \in L_{s-m_j}^p(\tilde{\Omega}; \mathrm{loc})$, then u has an extension $\tilde{u} \in L_s^p(R^n; \mathrm{loc})$.*

Proof. It can be assumed that $s - m_j \leqq 0$, for by Lemma 7.1 P_j can be replaced by the set $\{D_k P_j\}$ with all k of some fixed absolute value $\geqq s - m_j$.

In this case each $P_j u$ automatically has an extension to R^n, so by Theorem 6.3 u itself has an extension to R^n. In particular, u is of finite order in a neighborhood of each boundary point of Ω.

Let x_0 be a boundary point, and let $r_0 > 0$ and $t \leqq s-1$ be such that $u \in L_t^p(B(x_0, r_0) \cap \Omega)$ and such that formula (3) in the definition of a Lipschitz graph domain holds. Then, in the notations of Section 6, it follows that $u_\psi \in L_t^p(\Omega_r)$, and therefore that

$$P_j u_\psi \in L^p_{s-m_j}(\Omega_r) + L^p_{t-(m_j-1)}(\Omega_r) \subset L^p_{t-m_j+1}(\Omega_r)$$

because $P_j(\psi u) = \psi P_j u + R_j u$, where R_j has order $< m_j$. Now, let f_j be an extension of $P_j u_\psi$ which lies in $L^p_{t-m_j+1}(R^n)$ and has bounded support. Such an extension exists because $t - m_j + 1 \leqq 0$. By Corollary 5.4,

$$\tilde{u}_\psi = \sum f_j * K_j$$

is an extension of u_ψ, and by the theorems of Calderón and Zygmund [5] on the boundedness of singular integrals, $\tilde{u}_\psi \in L^p_{t+1}(R^n; \text{loc})$. Thus, the number t can be raised step by step up to $s-1$, at which point we have $\tilde{u}_\psi \in L^p_s(R^n; \text{loc})$, and the theorem is proved.

Remark 1. When Ω is bounded the theorem provides an extension in $L^p_s(R^n)$, for the one that has been found can be multiplied by a function in $C_0^\infty(R^n)$ which is 1 on a neighborhood of $\bar{\Omega}$. To get the same result when Ω is not bounded it is necessary to impose a uniformity condition in the Lipschitz graph property.

Corollary 7.3. *Let Ω be a bounded Lipschitz graph domain and let the characteristic polynomials of the P_j have no common non trivial complex zero. If $P_j u \in L^p_{s-m_j}(\Omega)$, then $u \in L^p_s(\Omega)$, and for every t there is a constant $c > 0$ such that*

$$c\,\|u\|_{L^p_s(\Omega)} \leqq \sum_j \|P_j u\|_{L^p_{s-m_j}(\Omega)} + \|u\|_{L^p_t(\Omega)}\,. \tag{3}$$

Proof. If $t \geqq s$, there is nothing to prove, so suppose that $t < s$. Theorem 7.2 and the remark show that the right side of (3) is a complete norm on $L^p_s(\Omega)$, and then the closed graph theorem gives the inequality.

Remark 2. When s is not an integer the statement and proof of Theorem 7.2 remain exactly the same. All that is necessary is to give the definition of $L^p_s(\Omega)$ and the proof of Lemma 7.1. We will do this for the case $p=2$, and refer to Strichartz [10] for the general case. (The definition of $L^p_s(\Omega)$ is a little more complicated in the general case, but the proof of Lemma 7.1 is the same.)

If s is a positive number which is not an integer, write $s = s^* + \alpha$, where s^* is an integer and $0 < \alpha < 1$, and define

$$\|u\|^2_{L^2_s(\Omega)} = \|u\|^2_{L^2_{s^*}(\Omega)} + \frac{1}{C(n,\alpha)} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy\,. \tag{2}$$

$L^2_s(\Omega)$ is the space of all distributions on Ω for which this norm is finite. $C(n,\alpha)$ is a constant that is useful in the general theory of these spaces, but has no relevance here. When s is negative, $L_s^2(\Omega)$ is defined as a dual space just as before.

Lemma 7.4. *If Ω is a Lipschitz graph domain, then Lemma 7.1 holds for non integral as well as integral s.*

Proof. If $s-m \geqq 0$ or if $s \leqq 0$, the result is obvious even without the hypothesis on Ω. Therefore it can be assumed that $0<s<1$ and that $m=1$.

Let x_0 be a boundary point of Ω and form u_ψ as before. Then $u_\psi \in L_s^2(\Omega_r)$, and u_ψ can be extended across the surface $x_n = h(x')$ by the formula

$$\begin{aligned} \tilde{u}_\psi(x) &= u_\psi(x) && \text{if} \quad x_n > h(x'), \\ \tilde{u}_\psi(x) &= u_\psi(x', 2h(x') - x_n) && \text{if} \quad x_n < h(x'). \end{aligned}$$

It is immediately verified that $\tilde{u}_\psi \in L_s^2(R^n)$. Thus we have shown that each $u \in L_s^2(\tilde{\Omega}; \mathrm{loc})$ has an extension in $L_s^2(R^n; \mathrm{loc})$, from which the lemma is clear: (Reflections of this kind, as well as more complicated ones that work simultaneously for all s, are explained in detail in [1]. This particular one is also used in Strichartz [10]. It is also the one to use in treating Hölder norms.)

Remark 3. Theorem 7.2 gives an extension theorem for $L_s^p(\Omega)$. Simply take the P_j to be all derivatives of some fixed order $\geqq s$. This particular extension, which is due to Calderón [4], has some advantages that will be used in the next section. The extension of [1] is quite different, and has quite different advantages that also will be used in the next section.

8. Variable Coefficients—Coercive Inequalities

With the methods that are now available for managing elliptic operators it is a fairly routine matter to extend the foregoing results to variable coefficients. We will continue to state the results for distributions, and so will suppose that the coefficients of the operators are C^∞. When the distributions are limited a priori to some class L_s^p the regularity of the coefficients can be reduced accordingly (and the proofs simplify a little). We begin with some lemmas.

Lemma 8.1. *If $k \in L^1(R^n)$ and $u \in L_s^p(R^n)$, then $k * u \in L_s^p(R^n)$ and $\|k*u\|_{L_s^p} \leqq \|k\|_{L^1} \|u\|_{L_s^p}$.*

Proof. When $s=0$ this is the classical inequality of W. H. Young. When $s>0$ the space L_s^p is the space of all functions of the form $u = G_s * f$, and $\|u\|_{L_s^p} = \|f\|_{L^p}$, where

$$\hat{G}_s(\xi) = (2\pi)^{\frac{-n}{2}} (1+|\xi|^2)^{\frac{-s}{2}}.$$

In this case $k*u = k*G_s*f = G_s*k*f$, so the result follows from the case $s=0$. When $s<0$ it follows by duality.

Lemma 8.2. *If $\varphi \in C^\infty$ with bounded derivatives of all orders and $u \in L_s^p(R^n)$, then $\varphi u \in L_s^p(R^n)$ and*

$$\|u\|_{L_s^p} \leqq c \|\varphi\|_{L_{|s|}^\infty} \|u\|_{L_s^p},$$

the constant c depending only on s, p, and n.

Here $\|\varphi\|_{L^\infty_{|s|}}$ denotes the upper bound of all the derivatives of φ of order $\leqq m$, where m is the first integer $\geqq |s|$. When s is an integer the lemma is obvious. For the general case see [4] or [10].

Lemma 8.3. *Let K be homogeneous of degree $m-n$, $m>0$, and integrable on the unit sphere. If $\varphi, \psi \in C_0^\infty$ and $u \in L_s^p(R^n)$, then $\varphi K * \psi u \in L_s^p(R^n)$. There is a constant c depending only on K, s, p, and n such that*

$$\|\varphi K * \psi u\|_{L_s^p} \leqq c r^m \|\varphi\|_{L^\infty_{|s|}} \|\psi\|_{L^\infty_{|s|}} \|u\|_{L_s^p}$$

if φ and ψ vanish outside the ball $B(0; r)$.

Proof. If $k(x)=K(x)$ for $|x| \leqq 3r$ and $k(x)=0$ for $|x|>3r$, then $\varphi K * \psi u = \varphi k * \psi u$, and we can apply the two lemmas above.

Lemma 8.4. *Lemma 8.3 holds for $m=0$, with the following provisions:* a) *K is bounded on the sphere and has mean value* 0; b) *The convolution is taken in the sense of a Cauchy principal value.*

Proof. This is the theorem of Calderón and Zygmund on the existence of singular integrals. See [4] or [5]. (It is not necessary to take K bounded on the sphere – but not sufficient to take it integrable. In our case it is C^∞.)

Lemma 8.5. *Let K be homogeneous of degree $m-n$, $m>0$, and of class C^m except at 0. If $\varphi, \psi \in C_0^\infty$ and $u \in L_{s-m}^p(R^n)$, then $\varphi K * \psi u \in L_s^p(R^n)$. There is a constant c depending only on K, s, p, and n such that*

$$\|\varphi K * \psi u\|_{L_s^p} \leqq c(1+r^m) \|\varphi\|_{L^\infty_{|s|+2m}} \|\psi\|_{L^\infty_{|s|+m}} \|u\|_{L_{s-m}^p}$$

if φ and ψ vanish outside the ball $B(0; r)$.

Proof. We use the fact that if $f \in L_{s-m}^p$ and $D_k f \in L_{s-m}^p$ for every derivative of order m, then $f \in L_s^p$ and

$$c\|f\|_{L_s^p} \leqq \|f\|_{L_{s-m}^p} + \sum_{|k|=m} \|D_k f\|_{L_{s-m}^p}.$$

The estimate for $f=\varphi K * \psi u$ itself comes from Lemma 8.3, while for the derivatives of f we have

$$D_k f = \sum_{i+j=k} D_i \varphi D_j K * \psi u .$$

If $|j|<m$, we can apply Lemma 8.3 again, and if $|j|=m$, we can apply Lemma 8.4. In the latter case $D_j K$ is a distribution, not a function, but if ∂_j denotes the corresponding pointwise derivative, then $D_j K = \partial_j K + c_j \delta$, where δ is the Dirac measure. Lemma 8.4 applies to the function $\partial_j K$.

According to Lemma 8.2 the norm of a multiplication operator $u \to \varphi u$ depends not only on the size of φ itself, but also on the derivatives. However, the space L_s^p can be re-normed so as to reduce the influence of the derivatives. For $\varrho>0$ put

$$u_\varrho(x) = u(\varrho x) \quad \text{and} \quad \|u\|_{\varrho, L_s^p} = \|u_\varrho\|_{L_s^p}. \tag{1}$$

For each ϱ this norm is equivalent to the original one. If u is a distribution on an open set Ω, put

$$\|u\|_{\varrho, L_s^p(\Omega)} = \inf \|\tilde{u}\|_{\varrho, L_s^p}, \tag{2}$$

taking the inf over all extensions $\tilde{u}$ of u to R^n. If Ω is a Lipschitz graph domain, every $u \in L_s^p(\Omega)$ has such an extension, and for each ϱ the resulting norm is equivalent to the original.

Lemma 8.6. *If $a \in C^\infty(R^n)$ with bounded derivatives and $|k| \leqq m$, then*

$$\|a D_k u\|_{\varrho, L_{s-m}^p(\Omega)} \leqq c \varrho^{-|k|} \|a_\varrho\|_{L_{|s-m|}^\infty} \|u\|_{\varrho, L_s^p(\Omega)}, \qquad \varrho \leqq 1,$$

where c depends only on s, p, m, and n.

Proof. If $\tilde{u}$ is an extension of u, then $aD_k\tilde{u}$ is an extension of $aD_k u$, so it suffices to consider $\Omega = R^n$. In this case $D_k u_\varrho = \varrho^{|k|}(D_k u)_\varrho$, so the result follows from Lemma 8.2.

As a notational convenience we will use the following norms on the differential operator $R = \sum_{|k| \leqq m} a_k D_k$:

$$\|R\|_{L_{|s|}^\infty} = \sum_{|k| \leqq m} \|a_k\|_{L_{|s|}^\infty} \quad \textit{and} \quad |R|_{L_0^\infty} = \sum_{|k|=m} \|a_k\|_{L_0^\infty}. \tag{3}$$

Applying Lemma 8.6 to each term in Ru, and noticing that

$$\|a_\varrho\|_{L_{|s|}^\infty} \leqq \|a\|_{L_{|s|}^\infty} \quad \textit{and} \quad \|a_\varrho\|_{L_{|s|}^\infty} \leqq \|a\|_{L_0^\infty} + \varrho \|a\|_{L_{|s|}^\infty}, \qquad \varrho \leqq 1, \tag{4}$$

we obtain the following.

Lemma 8.7. *If R has order m and $\varrho \leqq 1$, then*

$$\|Ru\|_{\varrho, L_{s-m}^p(\Omega)} \leqq c \varrho^{-m} (|R|_{L_0^\infty} + \varrho \|R\|_{L_{|s-m|}^\infty}) \|u\|_{\varrho, L_s^p(\Omega)},$$

where c depends only on s, p, m, and n.

Lemma 8.8. *Let K be homogeneous of degree $m-n$, $m>0$, and of class C^m except at 0. There is a constant c, depending only on K, s, p, and n such that*

$$\|\varphi K * \psi u\|_{\varrho, L_s^p} \leqq c \varrho^m \|\varphi_\varrho\|_{L_{|s|+2m}^\infty} \|\psi_\varrho\|_{L_{|s|+m}^\infty} \|u\|_{\varrho, L_{s-m}^p}$$

if $\varphi, \psi \in C_0^\infty$ and vanish outside $B(0; \varrho)$, $\varrho \leqq 1$.

Proof. This comes from Lemma 8.5 and the observation that $(K * v)_\varrho = \varrho^m K * v_\varrho$. Note that φ_ϱ and ψ_ϱ vanish outside $B(0; 1)$, so that $r = 1$ in Lemma 8.5.

Now we will consider differential operators $P_1^0, \ldots, P_N^0$ which are homogeneous of order m with constant coefficients and whose characteristic polynomials have no common non trivial complex zero. We will consider a convex n dimensional cone Γ, and kernels K_j homogeneous of degree $m-n$, of class C^m except at 0, with support in Γ, and such that

$$u = \sum K_j * P_j^0 u \quad \textit{for} \quad u \in C_0^\infty.$$

Lemma 8.9. *There is a constant c_0 depending on the P_j^0, the K_j, s, p, m, and n such that if $\Omega - \Gamma \subset \Omega$, then*

$$\|u\|_{\varrho, L_s^p(\Omega)} \leqq c_0 \varrho^m \sum \|P_j^0 u\|_{\varrho, L_{s-m}^p(\Omega)}, \qquad \varrho \leqq \tfrac{1}{2}$$

if u is a distribution on Ω which vanishes outside the ball $B(0;\varrho)$.

Proof. Choose φ and $\psi \in C_0^\infty$ equal to 1 on a neighborhood of $\bar{B}(0;1)$ and to 0 outside a compact set in $B(0;2)$. Given ϱ, choose an extension f_j of $P_j^0 u$ so that

$$\|f_j\|_{\varrho, L_{s-m}^p} = \|P_j^0 u\|_{\varrho, L_{s-m}^p(\Omega)}.$$

Then

$$\tilde{u} = \sum \varphi_{\frac{1}{\varrho}} K_j * \psi_{\frac{1}{\varrho}} f_j$$

is an extension of u, and so by Lemma 8.8

$$\|u\|_{\varrho, L_s^p(\Omega)} \leqq \|\tilde{u}\|_{\varrho, L_s^p} \leqq c\varrho^m \|\varphi\|_{L_{|s|+2m}^\infty} \|\psi\|_{L_{|s|+m}^\infty} \sum \|f_j\|_{\varrho, L_{s-m}^p}$$

which gives the lemma with

$$c_0 = c\|\varphi\|_{L_{|s|+2m}^\infty} \|\psi\|_{L_{|s|+m}^\infty},$$

c being the constant in Lemma 8.8.

Lemma 8.10. *Let c_0 be the constant in Lemma 8.9 and c the constant in Lemma 8.7. If $Q_1, \ldots, Q_N$ are differential operators satisfying*

$$c_0 c \sum |P_j^0 - Q_j|_{L_0^\infty} + \varrho \|P_j^0 - Q_j\|_{L_{|s-m|}^\infty} \leqq \tfrac{1}{2}$$

and $\Omega - \Gamma \subset \Omega$, then

$$\|u\|_{\varrho, L_s^p(\Omega)} \leqq 2c_0 \varrho^m \sum \|Q_j u\|_{\varrho, L_{s-m}^p(\Omega)}, \qquad \varrho \leqq \tfrac{1}{2}$$

if $u \in L_s^p(\Omega;$ loc) vanishes outside $B(0;\varrho)$.

Proof. We can assume that the right side of the inequality is finite. If $u \in L_s^p(\Omega)$, the inequality is immediate from Lemmas 8.9 and 8.7. In the general case let

$$\Omega = \{x : x_n > h(x')\}$$

where h is a Lipschitz function and set

$$\Omega_\varepsilon = \{x : x_n > h(x') + \varepsilon\}.$$

Since $u \in L_s^p(\Omega;$ loc), it follows that the restriction of u to Ω_ε, call it u_ε, belongs to $L_s^p(\Omega_\varepsilon)$. Since $\Omega_\varepsilon - \Gamma \subset \Omega_\varepsilon$, we have the required inequality for each u_ε on Ω_ε.

If $\tilde{v}$ is an extension of v, then clearly it is also an extension of v_ε. Hence

$$\|v_\varepsilon\|_{\varrho, L_s^p(\Omega_\varepsilon)} \leqq \|v\|_{\varrho, L_s^p(\Omega)},$$

so if we write M for the right side of our inequality, we have

$$\|u_\varepsilon\|_{\varrho, L_s^p(\Omega_\varepsilon)} \leqq M$$

for each ε. Let $\tilde{u}_\varepsilon$ be an extension of u_ε with the same norm, so that

$$\|\tilde{u}_\varepsilon\|_{\varrho, L^p_s} \leqq M .$$

If $\tilde{u}$ is a weak limit of the $\tilde{u}_\varepsilon$ in L^p_s, as $\varepsilon \to 0$, then $\tilde{u}$ is an extension of u, so $u \in L^p_s(\Omega)$, and the lemma is proved.

Now it is easy to do the main theorem. We consider a Lipschitz graph domain Ω and differential operators $P_1, \ldots, P_N$ with coefficients of class $C^\infty(\bar{\Omega})$. Because of the extension theorems of either [11] or [1] all reasonable definitions of $C^\infty(\bar{\Omega})$ coincide: a function is in $C^\infty(\bar{\Omega})$ if and only if it has an extension in $C^\infty(R^n)$. We will assume that the coefficients of the P_j are extended to R^n.

Let m_j be the order of P_j. If x_0 is a given point, P_j^0 will denote the operator obtained from P_j by evaluating all coefficients at x_0 and discarding the terms of order $< m_j$. Thus, P_j^0 is homogeneous and has constant coefficients. It has order m_j, unless it is the 0 operator. We consider the following conditions.

A) *For each point $x_0 \in \Omega$ the characteristic polynomials of the P_j^0 have no common non trivial real zero.*

B) *For each point $x_0 \in \partial\Omega$ the characteristic polynomials of the P_j^0 have no common non trivial complex zero.*

The main theorem is as follows.

Theorem 8.11. *Let Ω be a Lipschitz graph domain and let $P_1, \ldots, P_N$ have C^∞ coefficients on $\bar{\Omega}$ and satisfy A) and B). If u is a distribution on Ω such that $P_j u \in L^p_{s-m_j}(\tilde{\Omega}; \mathrm{loc})$, then $u \in L^p_s(\tilde{\Omega}; \mathrm{loc})$. If $P_j u \in C^\infty(\bar{\Omega})$, then $u \in C^\infty(\bar{\Omega})$.*

When Ω is bounded $L^p_s(\tilde{\Omega}; \mathrm{loc}) = L^p_s(\Omega)$, so we have the following corollary, just as in the last section.

Corollary 8.12. *Let Ω be a bounded Lipschitz graph domain and let $P_1, \ldots, P_N$ have C^∞ coefficients on $\bar{\Omega}$ and satisfy* A) *and* B). *If u is a distribution on Ω such that $P_j u \in L^p_{s-m_j}(\Omega)$, then $u \in L^p_s(\Omega)$. For each t there is a constant $c > 0$ such that*

$$c\,\|u\|_{L^p_s(\Omega)} \leqq \sum \|P_j u\|_{L^p_{s-m_j}(\Omega)} + \|u\|_{L^p_t(\Omega)} \,.$$

Proof. To begin with, take an integer $m \geqq$ each m_j and replace P_j by the set $\{D_k P_j\}$ with $|k| = m - m_j$. In this way we can suppose that all P_j have the same order m.

If $P = \sum P_j^* P_j$, then $Pu \in L^p_{s-2m}(\Omega; \mathrm{loc})$, so from the usual theory of interior regularity of solutions of elliptic equations it follows that $u \in L^p_s(\Omega; \mathrm{loc})$. (Note that condition A) is what makes P elliptic.) Consequently it is only necessary to examine u in the neighborhood of a boundary point x_0, and of course it can be assumed that $x_0 = 0$.

Replacing u by the distribution u_ψ of Lemma 6.2, we see that we can assume that u vanishes outside a ball $B(0; r_0)$ with r_0 as small as we please, and that

$$\Omega = \{x : x_n > h(x')\}$$

where h is a Lipschitz function with $h(0) = 0$.

First we choose r_0 so that

$$c_0 c \sum |P_j^0 - P_j|_{L_0^\infty(B(0;2r_0))} < \tfrac{1}{4},$$

where c_0 and c are the constants in Lemma 8.10. The possibility of doing this depends simply on the continuity of the leading coefficients of P_j at the point x_0. Now choose $\varphi \in C^\infty$ so that $\varphi = 1$ on a neighborhood of $\bar{B}(0; r_0)$ and $\varphi = 0$ outside a compact set in $B(0; 2r_0)$ and put

$$Q_j = \varphi P_j + (1-\varphi) P_j^0 .$$

If u vanishes outside $B(0; r_0)$, then $Q_j u = P_j u$. Moreover, $P_j^0 - Q_j = \varphi(P_j^0 - P_j)$, so that

$$c_0 c \sum |P_j^0 - Q_j|_{L_0^\infty} < \tfrac{1}{4}. \tag{5}$$

Now choose $\varrho < r_0$ small enough so that

$$c_0 c \varrho \sum \|P_j^0 - Q_j\|_{L_{|s-m|}^\infty} < \tfrac{1}{4}. \tag{6}$$

According to (5) and (6) the Q_j satisfy the conditions in Lemma 8.10, so that lemma finishes the proof, assuming that u vanishes outside $B(0; \varrho)$. Note that we have already established the required fact that $u \in L_s^p(\Omega; \mathrm{loc})$.

With regard to the statement about $C^\infty(\bar{\Omega})$, we will have to use either the extension theorem of [11] or the extension theorem of [1]. Our present results show that if $P_j u \in C^\infty(\bar{\Omega})$, then for each s, u has an extension which lies in $L_s^p(R^n; \mathrm{loc})$, but do not give one fixed extension which lies in all $L_s^p(R^n; \mathrm{loc})$. The theorems of [11] or [1] do give such a fixed extension.

The corollary follows from the theorem by way of the closed graph theorem, just as in the last section.

Corollary 8.13. *Let Ω be a bounded Lipschitz graph domain and let $P_1, \ldots, P_N$ have C^∞ coefficients on $\bar{\Omega}$ and satisfy* A) *and* B). *The common null space of the P_j is finite dimensional and contained in $C^\infty(\bar{\Omega})$.*

Proof. The fact that the common distribution null space is contained in $C^\infty(\bar{\Omega})$ is a special case of the last part of the theorem. The corollary shows that on this null space the $L_s^p(\Omega)$ and $L_t^p(\Omega)$ norms are equivalent, while it is well known that the $L_t^p(\Omega)$ norm is compact relative to the $L_s^p(\Omega)$ norm if $t < s$ and Ω is a bounded Lipschitz graph domain.

Remark 1. Suppose that the coefficients of $P_1, \ldots, P_N$ are analytic on Ω and that the characteristic polynomials have no common non trivial complex zero at some one point $x_0 \in \Omega$ and no common non trivial real zero at any point of Ω. If Ω is connected, then the common null space of the P_j is finite dimensional and contained in $C^\infty(\Omega)$ (but not $C^\infty(\bar{\Omega})$). Choose a ball $B = B(x_0; r_0)$ on which the characteristic polynomials have no common non trivial complex zero. If N is the common null space, then by the last corollary the set N_B of restrictions of functions in N to B is finite dimensional, so there exist $u_1, \ldots, u_k$

in N whose restrictions to B form a basis of N_B. If $u \in N$, there exist constants $\alpha_1, \ldots, \alpha_k$ such that

$$u - \sum \alpha_j u_j = 0$$

on B. But then the same equality must hold throughout Ω, for all the functions involved are analytic, being solutions of the elliptic equation

$$\sum P_j^* P_j v = 0 .$$

We believe that this remark was due to P. Lax at the time that Aronszajn's inequality first came out. We do not know whether the required unique continuation property holds for solutions of the system $P_j v = 0$ when the coefficients are just C^∞.

Remark 2. In Theorem 8.11 and in the corollaries it is not condition B) per se that plays a role, but rather the existence of the kernels. Thus, for example, if Ω is a half space, then B) can be weakened to

B′) *For each point $x_0 \in \partial\Omega$ the characteristic polynomials of the P_j^0 have no common non trivial complex zero whose imaginary part is orthogonal to $\partial\Omega$ at x_0.*

If Ω has a boundary of class C^∞ and lies on one side of its boundary, then locally Ω can be transformed to a half space, so we get the following.

Theorem 8.14. *If Ω has a boundary of class C^∞ and lies on one side of its boundary, then Theorem 8.11 and the corollaries remain valid under conditions* A) *and* B′).

This points up one advantage of having the conjecture made in the introduction. The assumption that the boundary is of class C^∞ is plainly unnecessary. A boundary of class C^1 should be good enough. It is the transformation to a half space which is unnatural in the problem and which forces the unnatural assumption. With the conjecture one could work directly in Ω and avoid this assumption.

When the distribution u is limited a priori to some $L_t^p(\Omega)$ the regularity can be reduced. Consider, for example, the case where $s = m_j = m$ and u belongs a priori to $L_s^p(\Omega)$, and what is desired is the inequality in Corollary 8.12. This (for $p = 2$) was the initial inequality of N. Aronszajn. In this case it suffices to transform to a half plane by a diffeomorphism of class C^{s+1}. Thus, the a priori coercive inequality holds if $\partial\Omega$ is of class C^{s+1}.

The situation can be improved by replacing the single operator P_j of order s by a system of operators of order 1. (See the next remark.) By this device s is reduced to 1, and the coercive inequality is obtained when $\partial\Omega$ is of class C^2. This is the result of Aronszajn. Note that we do not arrive at C^1, as we should, and, more important, we lose the possibility of considering distributions, or indeed any $L_t^p(\Omega)$ with $|t| > s$, and hence also the possibility of making any assertion about $C^\infty(\bar{\Omega})$. Fortunately the condition B) is usually satisfied in practice if B′) is.

Remark 3. Systems. In Section 4 we have given conditions for the existence of kernels in the case of systems. Once the kernels are available, the same kind of arguments give analogues of Theorem 8.11 and the corollaries.

For $j=1,\dots,N$ and $i=1,\dots,N$, let P_{ji} be a differential operator of order m_j-l_i with coefficients in $C^\infty(\bar{\Omega})$. The analogues of conditions A) and B) are:

A_S) *For each point $x_0\in\Omega$ the matrix $\{p^0_{ji}(\xi)\}$ has rank M if $0\neq\xi\in R^n$.*
B_S) *For each point $x_0\in\partial\Omega$ the matrix $\{p^0_{ji}(\xi)\}$ has rank M if $0\neq\xi\in C^n$.*

If $u=(u_1,\dots,u_M)$, we put

$$P_j u=\sum_{i=1}^{M} P_{ji}u_i,\qquad j=1,\dots,N.$$

The result is as follows.

Theorem 8.15. *Let Ω be a Lipschitz graph domain, and let the P_{ji} have C^∞ coefficients on $\bar{\Omega}$ and satisfy A_S) and B_S). If u is a distribution on Ω such that $P_j u\in L^p_{s-m_j}(\tilde{\Omega};\mathrm{loc})$, then $u_i\in L^p_{s-l_i}(\tilde{\Omega};\mathrm{loc})$. If $P_j u\in C^\infty(\bar{\Omega})$, then $u_i\in C^\infty(\bar{\Omega})$. If Ω is bounded, then for each t there is a constant $c>0$ such that*

$$c\,\|u_i\|_{L^p_{s-l_i}(\Omega)}\leqq\sum_j\|P_j u\|_{L^p_{s-m_j}(\Omega)}+\sum_i\|u_i\|_{L^p_t(\Omega)}\tag{7}$$

and the common null space of the P_j is finite dimensional.

This is the analogue of Theorem 8.11 and Corollaries 8.12 and 8.13. Theorem 8.14 and Remarks 1 and 2 have analogues too, but it does not seem necessary to state them explicitly.

Note that in the coercive inequality (7) the $L^p_t(\Omega)$ norms on the right can be replaced by any semi-norm on

$$L^p_{s-l_1}(\Omega)\times\cdots\times L^p_{s-l_M}(\Omega)$$

which is positive on the finite dimensional common null space of the P_j. This follows from a simple compactness argument. Furthermore, if the inequality is desired only for functions in some closed subspace which does not meet this finite dimensional null space, then the $L^p_t(\Omega)$ norms can be dropped altogether.

Theorem 8.1 can also be stated for differential operators between sections of vector bundles on a differentiable manifold. The conditions A_S) and B_S) are invariant under coordinate transformations, and can be expressed invariantly by taking ξ to be a vector in the covariant tangent space $T^*_{x_0}$ in the case of condition A_S) and in the complexified space $T^*_{x_0}\otimes C$ in the case of condition B_S).

Remark 4. Necessity of the Conditions. It is well known that condition A) on the characteristic polynomials is necessary for results of the kind we have given to hold. If the operators P_j have constant coefficients and the domain Ω is bounded, then the (obviously stronger) condition B) is also necessary, for if ξ is a common non trivial zero of the polynomials p^0_j, then the linearly independent functions $u_t=e^{t(x,\xi)}$ are all in the common null space of the principal parts P_j^0, while if the coercive inequality holds, this null space must be finite dimensional.

When the coefficients are variable, condition B) is no longer necessary, as is shown by Theorem 8.14. In [3] Aronszajn showed that condition B′) is necessary in the case when the boundary of Ω is smooth.

To get a condition which is sufficient and also very close to necessary, the conjecture stated in the introduction will be needed. On the basis of the conjecture, and consideration of the functions $e^{t(x,\xi)}$, it is pretty clear what this condition should be.

9. Quasi-polyhedra

There are simple domains, and even simple polyhedra, in which one can expect results of the kind that we have proved, and yet which are not Lipschitz graph domains. Suppose, for instance, that $\Omega_1, \ldots, \Omega_k$ are Lipschitz graph domains, and put

$$\Omega_0 = \Omega_1 \cup \cdots \cup \Omega_k ,$$

and let

$$\Omega_0 \subset \Omega \subset \bar{\Omega}_0 .$$

It is clear that

$$\|u\|_{L_s^p(\Omega)} \geqq \|u\|_{L_s^p(\Omega_j)}$$

for each j, and that

$$\|u\|_{L_s^p(\Omega)} \leqq c \sum \|u\|_{L_s^p(\Omega_j)} \quad \textit{for} \quad u \in L_s^p(\Omega; \text{loc}) \tag{1}$$

if s is a positive integer.

Theorem 9.1. *Let Ω be as above, and suppose that for each $x_0 \in \bar{\Omega}$ the characteristic polynomials of the P_j^0 have no common non trivial complex zero. Let s be an integer $\geqq 0$. If u is a distribution on Ω such that $P_j u \in L_{s-m_j}^p(\tilde{\Omega}; \text{loc})$, then $u \in L_s^p(\tilde{\Omega}; \text{loc})$. If Ω is bounded, then for each t there is a constant $c > 0$ such that*

$$c\,\|u\|_{L_s^p(\Omega)} \leqq \sum \|P_j u\|_{L_{s-m_j}^p(\Omega)} + \|u\|_{L_t^p(\Omega)} ,$$

and the common null space of the P_j is finite dimensional.

This result is of course obvious from the preceding ones and the inequality (1). Note the differences from the preceding ones however: s is restricted to be an integer $\geqq 0$; and nothing is said about the property $C^\infty(\bar{\Omega})$. Take, for instance, Ω to be the union of two adjacent open intervals. The function which is 0 on one interval and 1 on the other is in the null space of all homogeneous differential operators of positive order and belongs to all $L_s^p(\Omega)$ for s a positive integer, but not, for example, to $L_{\frac{1}{2}}^2(\Omega)$, so the theorem fails for $s = \frac{1}{2}$. Similar examples can be arranged in dimension n by taking Ω to be the union of two open simplexes with a common face of dimension $\leqq n-1$. In these examples the point is that the boundary of Ω separates Ω locally, in which case not much can be expected.

There is an extensive discussion in [1] of what can be done by piecing Lipschitz graph domains together. We will give one example of the effect of combining the results of [1] with the present ones. First a definition.

Definition 9.2. *A quasi-polyhedron is a finite union of compact convex n cells C_j which intersect non tangentially in the sense that*

$$d(x, C_j) + d(x, C_k) \geqq \alpha d(x, C_j \cap C_k), \quad \alpha > 0, \quad \text{if} \quad C_j \cap C_k \neq \emptyset .$$

Of course any ordinary polyhedron is a quasi-polyhedron.

One of the main extension theorems of [1] is the following.

Theorem 9.3. *If Ω is the interior of a quasi-polyhedron and if $\partial\Omega$ does not separate Ω locally, then there is an extension map E from the measurable functions on Ω to the measurable functions on R^n such that*

a) *Eu is an extension of u.*

b) *For every p and every $s \geqq 0$, E is a bounded linear map from $L_s^p(\Omega)$ to $L_s^p(R^n)$.*

c) *For $s \geqq 0$ the inequality* (1) *holds when Ω_j is the interior of the convex cell C_j.*

The theorems of [1] are in fact more general. They allow an infinite number of cells, and more generality in the cells themselves, but in this case the hypotheses become quite complicated, so we will be content with the version above.

Theorem 9.4. *Let Ω be the interior of a quasi-polyhedron such that $\partial\Omega$ does not separate Ω locally, and suppose that for each $x_0 \in \bar{\Omega}$ the characteristic polynomials of the P_j^0 have no common non trivial complex zero. Let $s \geqq 0$. If u is a distribution on Ω such that $P_j u \in L_{s-m_j}^p(\Omega)$, then $u \in L_s^p(\Omega)$. If $P_j u \in C^\infty(\bar{\Omega})$, then $u \in C^\infty(\bar{\Omega})$. For each t there is a constant $c > 0$ such that*

$$c \|u\|_{L_s^p(\Omega)} \leqq \sum \|P_j u\|_{L_{s-m_j}^p(\Omega)} + \|u\|_{L_t^p(\Omega)},$$

and the common null space of the P_j is finite dimensional.

The proof of the theorem is immediate on the basis of the previous results and Theorem 9.3. We have not checked the validity of either theorem in the case $s < 0$.

Theorem 9.4 has an obvious analogue for systems.

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Volumes of Polyhedra Inscribed in the Unit Sphere in E^3

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1. This paper is concerned with the problem of placing n points on the unit sphere in E^3 so as to maximize the volume of their convex hull. A necessary condition is obtained, and this is used to obtain complete solutions when $n=4, \ldots, 8$. The results for $n=7$ and 8 are new. The technique developed also shows that for each of the several polyhedral types considered there is only one relative maximum for the volume function.

Let $p_1, \ldots, p_n$ be points on the unit sphere, S. Denote by $H(p_1, \ldots, p_n)$ the polyhedron P which is the convex hull of the set $\{p_1, \ldots, p_n\}$. Let $C(P)$ be an oriented complex whose vertices are the vertices of P and whose edges are edges of P and diagonals of faces of P, the diagonals chosen such that the faces of $C(P)$ are all triangles. Thus, if every face of P is triangular, there is only one such complex up to orientation, while there is more than one such complex if P has a face which is not triangular.

Let K be a finite oriented complex all of whose faces are triangular and all of whose vertices are on S. If p, q, r are vertices of K which determine a face of K and are in an order determined by the orientation of K then the tetrahedron whose vertices are p, q, r and the origin is a *facial tetrahedron* and its volume is one-sixth of the value of the determinant $|p, q, r|$. Then the volume of K, $vol(K)$, is the sum of the volumes of the facial tetrahedra of K. If $P=H(p_1, \ldots, p_n)$ let $V(p_1, \ldots, p_n)=V(P)$ denote the volume of P. Note that for any oriented complex $C(P)$, $|\mathrm{vol}\,(C(P)|=V(P)$.

If P is a polyhedron, with n vertices, let the *valence* of a vertex of $C(P)$ be the number of edges of $C(P)$ incident with that vertex. By Euler's formula the average of the valences is $6-12/n$. If n is such that $6-12/n$ is an integer then $C(P)$ is *medial* if the valence of each vertex is $6-12/n$. If $6-12/n$ is not an integer then $C(P)$ is *medial* provided the valence of each vertex is either m or $m+1$ where $m<6-12/n<m+1$. P is said to be *medial* provided all faces of P are triangular and $C(P)$ is medial. When considering the isoperimetric problem for polyhedra Goldberg [2] made a conjecture whose dual was formulated by Grace [3]: The polyhedron with n vertices on S whose volume is a maximum is a medial polyhedron provided a medial polyhedron exists for that n.

2. The Necessary Condition. Let $P=H(p_1, \ldots, p_n)$. If for each p_i there is an open set $U_i \subset S$, $p_i \in U_i$, such that if

$$V(p_1, \ldots, p_{i-1}, q, p_{i+1}, \ldots, p_n) \leqq V(p_1, \ldots, p_n)$$

for all $q \in U_i$ then P is said to have *property Z*. By any usual definition of relative maximum, if V has a relative maximum at $(p_1, \ldots, p_n)$ then P has property Z. If p_i and p_j are vertices of P, denote the line segment whose endpoints are p_i and p_j by s_{ij} and its length by $|s_{ij}|$. Also, let $n_{ij} = 1/6\, p_i \times p_j$ where $\times$ denotes the vector product in E^3.

Lemma 1. *Let P with vertices $p_1, \ldots, p_n$ have property Z. Let $C(P)$ be any oriented complex associated with P such that* $\operatorname{vol}(C(P)) \geqq 0$. *Suppose $s_{12}, \ldots, s_{1r}$ are all the edges of $C(P)$ incident with p_1 and that $p_2, p_3, p_1; p_3, p_4, p_1; \ldots; p_r, p_2, p_1$ are orders for faces consistent with the orientation of $C(P)$.*

i) *Then $p_1 = m/|m|$ where $m = n_{23} + n_{34} + \cdots + n_{r2}$.*

ii) *Furthermore, each face of P is triangular.*

Proof. i) From the definition of V,

$$V(p_1, \ldots, p_n) = 1/6\, [|p_2, p_3, p_1| + \cdots + |p_r, p_2, p_1|] + \gamma = p_1 \cdot m + \gamma$$

where γ is the sum of the volumes of the facial tetrahedra of $C(P)$ for which p_1 is not a vertex. Let $q = m/|m|$. Suppose $p_1 \neq q$. Since P has property Z there is an open set U_1 containing p_1 as defined above. Let s be any point in U_1 such that $s \cdot q > p_1 \cdot q$. Let Q be the oriented complex with vertices $s, p_2, \ldots, p_n$ which is isomorphic to $C(P)$ where the isomorphism is such that $p_i \in Q$ corresponds to $p_i \in C(P)$, $i = 2, \ldots, n$, and the isomorphism preserves orientation. Then

$$V(p_1, \ldots, p_n) = p_1 \cdot m + \gamma < s \cdot m + \gamma = \operatorname{vol}(Q) \leqq V(s, p_2, \ldots, p_n)\,.$$

This implies $p_1 = q$.

ii) If P has a quadrilateral face then conclusion i) applied to two different complexes associated with P gives a contradiction and so ii) holds in this special case, and the general case follows from this.

Note 1. If $p_2, \ldots, p_r$ lie in a plane π then p_1 is one of the two antipodal points where the plane tangent to the sphere is parallel to π. Then

$$|s_{12}| = |s_{13}| = \cdots = |s_{1r}|\,.$$

Note 2. If $r = 5$ and β is the sum of the volumes of the facial tetrahedra for which p_1 is a vertex then $\beta = 1/6\, |p_2 - p_4, p_3 - p_5, p_1|$. From this it follows that $p_1 \perp p_2 - p_4$ and $p_1 \perp p_3 - p_5$. Then p_1 lies on the great circles determined by the planes which are the perpendicular bisectors of s_{24} and s_{35}.

By a *double n-pyramid*, $n \geqq 5$, is meant a complex of n vertices with two vertices of valence $n-2$ each of which is connected by an edge to each of the remaining $n-2$ vertices, all of which have valence 4. Note that the $2(n-2)$ faces of a double n-pyramid are all triangular. A polyhedron P is a *double n-pyramid* provided each of its faces is triangular and some $C(P)$ is a double n-pyramid.

Lemma 2. *Let $P = H(p_1, \ldots, p_n)$. If P is a double n-pyramid with property Z then P is unique up to congruence and its volume is* $[(n-2)/3] \sin 2\pi/(n-2)$.

Proof. Let p_{n-1} and p_n be the valence $n-2$ vertices. If $n=6$ let p_{n-1} and p_n be any two vertices not incident with the same edge. Suppose p_i is a vertex of valence 4 connected by edges to p_n, p_{n-1}, p_j, and p_k. From Note 2 it follows that p_i is equidistant from p_n and p_{n-1} and that p_i is equidistant from p_j and p_k. Consequently, the valence 4 vertices form a regular $(n-2)$-gon lying in the plane which is the perpendicular bisector of $s_{n-1\,n}$. It now follows from Note 1 that p_n and p_{n-1} are antipodal. An elementary calculation gives the volume.

3. $n=4, 5, 6$. In this section polyhedra of maximal volume are considered when $n=4$, 5, 6.

For $n=4$ if $P=H(p_1, p_2, p_3, p_4)$ has property Z then, by Note 1, the three edges incident with a vertex all have the same length. Consequently, all six edges have the same length and P is a regular tetrahedron. An elementary calculation gives the volume as $8\sqrt{3}/27$.

For $n=5$, let $P=H(p_1, \ldots, p_5)$ have property Z. Then P is a double 5-pyramid, since the double 5-pyramid is the only complex with five vertices. Then, by Lemma 2, P is unique up to congruence and the volume is $\sqrt{3}/2$.

For $n=6$ it is easily seen that if P is medial then P is a double 6-pyramid. If $P=H(p_1, \ldots, p_6)$ has property Z and is medial, then, by Lemma 2, P is unique up to congruence, the volume of P is 4/3, and P is a regular octahedron. It is well known that if P is not a regular octahedron then its volume is less than 4/3 ([4], p. 264).

Some of these results are summarized in the following theorem.

Theorem 1. *For $n=4$, 5, 6, if $P=H(p_1, \ldots, p_n)$ is a medial polyhedron such that $V(p_1, \ldots, p_n)$ is a relative maximum then P is uniquely determined up to congruence and its volume may be found. In each case such a medial polyhedron exists and its volume is the absolute maximum for V.*

4. $n=7, 8$. For $n=7$ Bowen and Fisk [1] have shown that up to isomorphism there is only one polyhedron with triangular faces having no vertices of valence 3. From Euler's formula it follows that the average valence is 4 2/7. Hence the medial complex is the double 7-pyramid with two valence 5 vertices and five valence 4 vertices. If $P=H(p_1, \ldots, p_7)$ is a double 7-pyramid with property Z then, by Lemma 2, P is unique up to congruence and the volume of P is $5/3 \sin 2\pi/5$, which is approximately 1.58510.

If $P=H(p_1, \ldots, p_7)$ is a double 7-pyramid such that $V(p_1, \ldots, p_7)$ is a relative maximum, then P is unique up to congruence.

Theorem 2. *If $P=H(p_1, \ldots, p_7)$ is such that $V(p_1, \ldots, p_7)$ is an absolute maximum then P is the double 7-pyramid with property Z described in Lemma 2.*

Proof. Let $Q=H(p_1, \ldots, p_7)$. Suppose Q gives the absolute maximum for V. Then Q is convex, has property Z, contains the origin, and has triangular faces. Since the double 7-pyramid is the only complex with 7 vertices none of which has valence 3, assume Q has a vertex of valence 3, say p_1. Then it remains to show that the volume of Q is less than $5/3 \sin 2\pi/5$. Let $s_{1\,2}$, $s_{1\,3}$, $s_{1\,4}$ be the edges incident with p_1, and let T be the triangle with vertices p_2, p_3, p_4. Since Q has

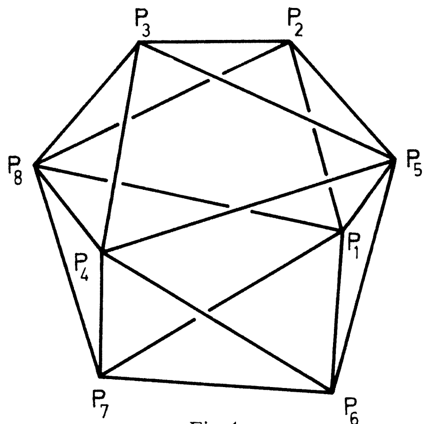

Fig. 1

property Z, the line determined by p_1 and the origin passes through the circumcenter of T. Since Q is convex, this circumcenter lies within T. Then the area of T is less than or equal to $(3\sqrt{3}/4)\,(1-1/3\tan^2(2\pi-\alpha)/6)$ ([4], p. 267), where α is the area of the central projection of T onto the sphere. Let $\theta=4\pi-\alpha$. The central projection of the seven facial tetrahedra not incident with p_1 has total area θ. The total volume of these seven tetrahedra is less than or equal to

$$\begin{aligned}&7/4\tan\frac{(2\pi-\theta/7)}{6}\left[1-1/3\tan^2\frac{(2\pi-\theta/7)}{6}\right]\\&=7/4\tan\frac{(10\pi+\alpha)}{42}\left[1-1/3\tan^2\frac{(10\pi+\alpha)}{42}\right]\end{aligned}$$

([4], p. 275). Then $V(Q)\leqq F(\alpha)$ where

$$\begin{aligned}F(\alpha)=&\sqrt{3}/4\left[1-1/3\tan^2\frac{(2\pi-\alpha)}{6}\right]\\&+7/4\tan\frac{(10\pi+\alpha)}{42}\left[1-1/3\tan^2\frac{(10\pi+\alpha)}{42}\right].\end{aligned}$$

A direct calculation shows that F is concave and that its maximum is less than $5/3\sin 2\pi/5$. Hence, Q does not give the absolute maximum for V.

For $n=8$, by Euler's formula, the average valence is 4 1/2. Bowen and Fisk have shown that there exist only two non-isomorphic complexes which have no vertices of valence 3. One of these is the double 8-pyramid, which by Lemma 2 has maximal volume $\sqrt{3}$. The other has four valence 4 vertices and four valence 5 vertices, as shown in Fig. 1, and therefore it is the medial complex.

Lemma 4. *If $P=H(p_1,\ldots,p_8)$ has property Z and is medial then P is uniquely determined up to congruence and its volume is* $\left[\frac{475+29\sqrt{145}}{250}\right]^{1/2}$.

Proof. Let the vertices of P be $p_1, \ldots, p_8$ so that the labelling is consistent with that in the figure. It is assumed that the vertices are distinct. From Lemma 1 it follows that

$$\begin{aligned} p_1 &= c_1(n_{25}+n_{56}+n_{67}+n_{78}+n_{82}), \\ p_4 &= c_4(n_{38}+n_{87}+n_{76}+n_{65}+n_{53}), \\ p_5 &= c_5(n_{61}+n_{12}+n_{23}+n_{34}+n_{46}), \end{aligned}$$

where $c_i > 0$, $i=1,4,5$. The vector product of p_1 with the above expression for p_1 gives

$$\begin{aligned} &(p_1\cdot p_5 - p_1\cdot p_8)\,p_2 + (p_1\cdot p_6 - p_1\cdot p_2)\,p_5 + (p_1\cdot p_7 - p_1\cdot p_5)\,p_6 \\ &\quad + (p_1\cdot p_8 - p_1\cdot p_6)\,p_7 + (p_1\cdot p_2 - p_1\cdot p_7)\,p_8 = 0\,. \end{aligned} \tag{1}$$

Similarly, from the expression for p_4, it follows that

$$\begin{aligned} &(p_4\cdot p_8 - p_4\cdot p_5)\,p_3 + (p_3\cdot p_4 - p_4\cdot p_6)\,p_5 + (p_4\cdot p_5 - p_4\cdot p_7)\,p_6 \\ &\quad + (p_4\cdot p_6 - p_4\cdot p_8)\,p_7 + (p_4\cdot p_7 - p_3\cdot p_4)\,p_8 = 0\,. \end{aligned} \tag{2}$$

From Note 2 it follows that $p_1\cdot p_2 = p_2\cdot p_3 = p_3\cdot p_4$ and $p_5\cdot p_6 = p_6\cdot p_7 = p_7\cdot p_8$. The perpendicular bisector of s_{58} contains p_2 and p_3, and the perpendicular bisector of s_{14} contains p_6 and p_7. Consequently, $p_2\cdot p_5 = p_2\cdot p_8$, $p_3\cdot p_5 = p_3\cdot p_8$, $p_1\cdot p_6 = p_4\cdot p_6$, and $p_1\cdot p_7 = p_4\cdot p_7$. Adding (1) and (2) and using these equalities gives

$$\begin{aligned} &(p_1\cdot p_5 - p_1\cdot p_8)\,p_2 + (p_4\cdot p_8 - p_4\cdot p_5)\,p_3 \\ &\quad + (p_4\cdot p_5 - p_1\cdot p_5)\,p_6 + (p_1\cdot p_8 - p_4\cdot p_8)\,p_7 = 0\,. \end{aligned} \tag{3}$$

If none of the coefficients in (3) is zero then p_2, p_3, p_6, and p_7 lie in a plane, π, since the sum of the coefficients is zero. If none of p_1, p_4, p_5, p_8 is on one side of π then either s_{26} or s_{27} is an edge, a contradiction. If one of p_1, p_4, p_5, p_8, without loss of generality say p_8, is on one side of π then s_{68} is an edge, a contradiction. Since s_{58} is not an edge, p_5 and p_8 are on opposite sides of π. Similarly, p_1 and p_4 are on opposite sides of π. Suppose p_1 and p_8 are on the same side of π. Then p_3 and p_6 must be opposite vertices of the quadrilateral $p_2p_3p_7p_6$, since otherwise either s_{13} or s_{68} would be an edge. Suppose p_1 and p_8 lie on the smaller of the two caps of the sphere determined by π. From this and the fact that $p_1\cdot p_2 = p_2\cdot p_3$ it follows that $|s_{18}| \leqq |s_{15}|$, so $p_1\cdot p_8 \geqq p_1\cdot p_5$. If the caps are hemispheres this same result still holds. Then the coefficient of p_2 in (3) is negative. Since s_{27} is a diagonal of the quadrilateral $p_2p_3p_7p_6$, the coefficient of p_7 must also be negative. Thus $p_4\cdot p_8 \geqq p_1\cdot p_8$. Since $p_6\cdot p_7 = p_7\cdot p_8$ it similarly follows that $|s_{18}| \leqq |s_{48}|$. But then $p_4\cdot p_8 \leqq p_1\cdot p_8$, a contradiction. The case where p_1 and p_8 lie on the larger of the two caps gives a similar contradiction. The case where p_1 and p_5 lie on the same side of π also gives a similar contradiction. Thus, at least one of the coefficients in (3) is zero. Suppose only one coefficient is zero, without loss of generality say $p_1\cdot p_8 - p_4\cdot p_8 = 0$. Since the sum of the coefficients in (3) is zero, it follows that p_2, p_3, and p_6 are collinear, which contradicts the fact that they are distinct. Suppose only two coefficients are zero. If the coefficients of p_6 and p_7, are zero then p_2 and p_3 are antipodal. Then $p_1 = p_3$, since $|s_{12}| = |s_{23}|$, a contradiction. If the coefficients of p_2 and p_7 are zero then $p_1 \cdot p_5 = p_1 \cdot p_8 = p_4 \cdot p_8$. It then follows from (3) that $p_3 = p_6$, a contradiction. Other cases are similar. If three coefficients are zero then so is the fourth. Thus,

$$p_1 \cdot p_5 = p_1 \cdot p_8 = p_4 \cdot p_8 = p_4 \cdot p_5, \quad \text{so} \quad |s_{15}| = |s_{18}| = |s_{45}| = |s_{48}|\,.$$

Now let

$$\begin{aligned} p_1 &= (\sin 3\varphi, 0, \cos 3\varphi)\,, & p_5 &= (0, -\sin 3\delta, -\cos 3\delta)\,, \\ p_2 &= (\sin\varphi, 0, \cos\varphi)\,, & p_6 &= (0, -\sin\delta, -\cos\delta)\,, \\ p_3 &= (-\sin\varphi, 0, \cos\varphi)\,, & p_7 &= (0, \sin\delta, -\cos\delta)\,, \\ p_4 &= (-\sin 3\varphi, 0, \cos 3\varphi)\,, & p_8 &= (0, \sin 3\delta, -\cos 3\delta)\,, \end{aligned}$$

where $0 < \varphi, \delta < \pi/3$. Substituting into (1) gives

$$(\cos 2\varphi + \cos 3\varphi \cos\delta) \sin 3\delta - \cos 3\varphi \sin\delta\,(\cos 3\delta - \cos\delta) = 0\,,$$

or

$$3 \cos 3\varphi \sin 2\delta + 2 \cos 2\varphi \sin 3\delta = 0\,.$$

An equation analogous to (1) may be obtained from the expression for p_5 and this in turn yields

$$3 \cos 3\delta \sin 2\varphi + 2 \cos 2\delta \sin 3\varphi = 0\,.$$

Rewriting in terms of functions of single angles gives

$$\begin{aligned} &3 \cos\varphi(4\cos^2\varphi - 3)\cos\delta + (2\cos^2\varphi - 1)(4\cos^2\delta - 1) = 0\,, \\ &3 \cos\delta(4\cos^2\delta - 3)\cos\varphi + (2\cos^2\delta - 1)(4\cos^2\varphi - 1) = 0\,. \end{aligned} \tag{4}$$

Subtracting and factoring yields

$$(2\cos\varphi\cos\delta + 1)(\cos^2\varphi - \cos^2\delta) = 0\,.$$

Then $\delta = \varphi$, so substituting into (4) and solving gives

$$\cos\varphi = [(15 + \sqrt{145})/40]^{1/2}\,.$$

This determines P and the volume now may be found.

Note 3. This volume is approximately 1.815716. Grace [3] describes a polyhedron he obtained with the aid of a computer which is essentially this polyhedron. He pointed out that this gives a relative maximum which may be absolute. It is indeed the case as the following theorem shows.

Theorem 3. *If $P = H(p_1, \ldots, p_8)$ is such that $V(p_1, \ldots, p_8)$ is an absolute maximum then P is the medial polyhedron with property Z described in Lemma 4.*

Proof. Let $Q = H(p_1, \ldots, p_8)$. Suppose Q gives the absolute maximum for V. Then Q is convex, has property Z, contains the origin, and has triangular faces. Since the medial complex and the double 8-pyramid are the only complexes with 8 vertices, none of which has valence 3, assume Q has a vertex of valence 3, say p_1. Using an argument similar to that in the proof of Theorem 2 it can be shown that

$$V(p_1, \ldots, p_8) \leqq F(\alpha) = \frac{\sqrt{3}}{4}\left[1 - \frac{1}{3}\tan^2\frac{2\pi-\alpha}{6}\right] + \frac{9}{4}\tan\frac{14\pi+\alpha}{54}\left[1 - \frac{1}{3}\tan^2\frac{14\pi+\alpha}{54}\right]$$

A direct calculation shows that F is concave and that its maximum is less than $[(475 + 29\sqrt{145})/250]^{1/2}$.

5. Concluding Remarks. In this paper it has been shown that if P is a double n-pyramid, a tetrahedron, or medial with 8 vertices and if P has property Z then P is uniquely determined. This raises the question: For which types of polyhedra does property Z determine a unique polyhedron. More generally, for each isomorphism class of polyhedra is there one and only one polyhedron (up to congruence) which gives a relative maximum for V?

For $n = 4, \ldots, 7$ the duals of the polyhedra of maximum volume are just those polyhedra with n faces circumscribed about the unit sphere of minimum volume, i.e., the solutions to the well known isoperimetric problem. For $n = 8$ the dual of the polyhedron described in Theorem 4 is the best known solution to the isoperimetric problem for polyhedra with 8 faces. The question naturally arises: Is this true in general?

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On the Double Centralizer Condition

J. P. JANS*

In a recent paper [2], Dickson and Fuller noted that all modules over a semisimple ring have the double centralizer property and they wondered which other rings have this property. Concerning these rings we would suggest the following *Conjecture:* If R has minimum condition then every R-module has the *double centralizer condition* if and only if R is a uniserial ring. In this note we will prove the conjecture for finite dimensional algebras over an algebraically closed field.

If M is an R module then M becomes an E_1-module in a natural way where $E_1 = \mathrm{Hom}_R(M, M)$. There is a homomorphism of R into $E_2 = \mathrm{Hom}_{E_1}(M, M)$ since the elements of R cause E_1 homomorphisms of M. M is said to have the double centralizer property if the homomorphism of R to E_2 is onto E_2. See [2, 3, 5, 6] for results on the double centralizer condition.

A left and right Artinian ring R is called uniserial if it is the direct sum of primary rings R_i such that each R_i is generalized uniserial. R_i is generalized uniserial if each indecomposable left (and right) ideal direct summand has exactly one composition series.

In [5] Nakayama showed that R is uniserial if and only if R and every factor of R is a quasi Frobenius ring. In [16], Thrall showed that every faithful module over a quasi Frobenius ring has the double centralizer condition. Since every module over a ring is a faithful module over some factor of the ring, it is clear that every module over a uniserial ring has the double centralizer condition. The difficulty comes in proving the converse. We shall assume in the following that R is left and right Artinian and that all modules over R have the double centralizer condition.

Lemma 1. *If S_1, S_2 are non isomorphic simple R modules then* $\mathrm{Ext}_R{}^1(S_1, S_2) = 0$.

Proof. If $0 \to S_2 \to M \to S_1 \to 0$ is a non split exact sequence and if ϕ is a non zero endomorphism of M then $\mathrm{Ker}\,\phi = 0$. If not, then $\mathrm{Ker}\,\phi = S_2$ and $\mathrm{Im}\,\phi \cong S_1$ is a complement to S_2 in M and the sequence splits.

Thus, the endomorphism ring E_1 of M is a division ring (since the composition length of E is finite $=2$, monomorphisms are isomorphisms). But then there are E_1 homomorphisms of M to M which do not carry S_2 to itself and these cannot be caused by elements of R. Thus, M lacks the double centralizer condition.

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Lemma 2. *If all the composition factors of T are isomorphic to S_1 and $S_1 \not\cong S_2$ then* $\mathrm{Ext}_R{}^1(T, S_2)=0$.

Proof. Induce on the composition length of T and Lemma 1 is the case $n=1$. If the composition length of $T=n+1$ then there is a short exact sequence

$$0 \to \mathrm{X} \to T \to \mathrm{Y} \to 0$$

where the lengths of X and Y are less than $n+1$. The above sequence induces

$$0=\mathrm{Ext}_R{}^1(\mathrm{Y}, S_2) \to \mathrm{Ext}_R{}^1(T, S_2) \to \mathrm{Ext}_R{}^1(\mathrm{X}, S_2)=0$$

so $\mathrm{Ext}_R{}^1(T, S_2)=0$.

Lemma 3. *All the composition factors of the indecomposible projective left ideal Re of R are isomorphic to Re/Ne where N is the radical of R.*

Proof. If not, there exists a short exact sequence

$$0 \to \mathrm{X/Y} \to Re/\mathrm{Y} \to Re/\mathrm{X} \to 0$$

where all the composition factors of Re/X are isomorphic to Re/Ne but X/Y is not. By Lemma 2 the sequence splits. However, Re/Y is indecomposable since it has a unique maximal submodule.

Lemma 4. $R=R_1 \oplus \cdots \oplus R_n$ *ring direct sum where each R_i has only one simple module, equivalently each R_i is primary.*

Proof. Let $1=e_1+\cdots+e_k$ be a decomposition of 1 into orthogonal indecomposable idempotents, then $R=\oplus\sum Re_i$. Re_i is equivalent (actually isomorphic) to Re_j if $Re_i/Ne_i \cong Re_j/Ne_j$. Group the equivalent ones and sum them to obtain $R=R_1 \oplus R_2 \cdots \oplus R_n$ where each R_i is the sum of equivalent Re_k. The claim is that this is a ring decomposition.

It is enough to show that if Re_1 not equivalent to Re_2 then $Re_1\, Re_2=0$. Since no composition factor of Re_1 appears in Re_2 we claim $\mathrm{Hom}_R(Re_1\, Re_2)=0$ and for idempotent generated left ideals this means $e_1\, Re_2=0$. It follows that $Re_1\, Re_2=0$.

Each R_i has only one simple module since each R_i is the direct sum of left ideals, all of which have only one composition factor. Thus each R_i is primary.

We can sum up the results of the four lemmas in the following theorem.

Theorem 1. *If R has minimum condition then every R module has the double centralizer condition if and only if $R=R_1 \oplus \cdots \oplus R_n$ where each R_i is primary and all R_i modules have the double centralizer condition.*

We have reduced the problem to the primary case using only the minimum condition. For the proof of the next theorem we must work with algebras over an algebraically closed field. Our conjecture is that it is true for rings with minimum condition.

Theorem 2. *If R is a primary algebra over an algebraically closed field K with $[R:K]<\infty$ then every R module has the double centralizer condition if and only if R is uniserial.*

Proof. In one direction it follows from the remarks in the introduction.

Let e be an indecomposible idempotent in R and suppose Re does not have a unique composition series. Let N be the radical of R, then Ne is the unique maximal submodule of Re. Thus, every composition series must pass through Ne. There must be a smallest integer r such that $N^r e$ has two (or more) maximal submodules. Otherwise $Re \supset Ne \supset \cdots N^k e = 0$ would be the only composition series.

We may assume that $N^{r+1}=0$ since we are trying to construct an R or R/N^{r+1} module which lacks the double centralizer property. $N^r e$ is now semisimple and is the direct sum of two or more simple submodules $S_1+S_2+\cdots$. Since R is primary, all simple R modules are isomorphic, $S_1 \xrightarrow{\phi} S_2$.

The following construction of a module without the double centralizer condition was used by Dickson and Fuller [2] although our proof uses algebraic closure where they used commutativity of R which we are not assuming.

Let $M=(Re\oplus Re)/S$ where $S=\{(s_1, -\phi(s_1))\, s_1\in S_1, \phi(s_1)\in S_2\}$ and ϕ is a monomorphism of S_1 onto S_2.

Let $\gamma\in E_1=$ the R endomorphism ring of M and let η be the natural map of $Re\oplus Re$ on M. Then by the R-projectivity of $Re\oplus Re$ there is $\bar{\gamma}$ making the following diagram commute

$$\begin{array}{ccc} Re\oplus Re & \xrightarrow{\bar{\gamma}} & Re\oplus Re \\ \eta\downarrow & & \downarrow\eta \\ M & \xrightarrow{\gamma} & M\,. \end{array}$$

The operation of $\bar{\gamma}$ on $Re\oplus Re$ can be represented by a matrix

$$\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$$

where the $r_{ij}\in eRe$. Recall that the endomorphisms of Re are caused by right multiplications by the elements of eRe. The condition that $\bar{\gamma}$ map $\operatorname{Ker}\eta$ into itself is given by the matrix equation

$$(s, -\phi(s))\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = (s', -\phi(s'))$$

or

$$\begin{aligned} sr_{11}-\phi(s)r_{21} &= s'\in S_1\,, \\ sr_{12}-\phi(s)r_{22} &= -\phi(s')\in S_2\,. \end{aligned} \qquad (*)$$

Since $N^r e$ is anihilated on either side by N and eNe, $N^r e$ is an eRe/eNe module. But this latter is the field K since it is the endomorphism ring of the simple R module Re/Ne. It is here that we use the algebraic closure of K.

Therefore, the eRe submodules of $N^r e$ are the K submodules. Since K is in the center of R, the left R submodules of $N^r e$ are also eRe submodules.

Returning now to the Eqs. (*), we see that $sr_{11} \in S_1$ and $\phi(s)r_{22} \in S_2$ so that r_{21} anihilates S_2 and r_{12} anihilates S_1. Since eRe is a local ring that implies that r_{12} and r_{21} are in eNe, the radical of eRe.

Now choose $t_1 \neq 0$, $t_1 \in S_1$ and $t_2 \neq 0$; $t_2 \in S_2$ and consider the additive homomorphism $\bar{\theta}: Re \oplus Re \to Re \oplus Re$ given by $\bar{\theta}(x, y) = (t_1 x, t_2 y)$. We shall show that $\bar{\theta}$ induces θ, an E_1 endomorphism of M and that θ is not caused by an element of R. Thus, M lacks the second centralizer condition.

Note first that $\bar{\theta}(s, -\phi(s)) = 0$ for $s \in S_1 \subsetneqq N^r e$ because $t_1, t_2 \in N^r$. Thus $\bar{\theta}$ does induce an additive homomorphism of M to M. To show that θ is an E_1 endomorphism of M, it is enough to show the following associativity relation

$$\bar{\theta}[(x, y)\bar{\gamma}] = [\bar{\theta}(x, y)]\bar{\gamma} \tag{**}$$

for all $(x, y) \in Re \oplus Re$ and all $\bar{\gamma}$ R-endomorphisms of $Re \oplus Re$ which induce R endomorphisms of M. Using the matrix form of $\bar{\gamma}$ the Eq. (**) becomes

$$t_1(x r_{11} + y r_{21}) = t_1 x r_{11} + t_2 y r_{21}$$

and

$$t_2(x r_{12} + y r_{22}) = t_1 x r_{12} + t_2 y r_{22}\,.$$

Since we have shown r_{21} and r_{12} must be in eNe and we have chosen $t_i \in N^r e$ it follows that

$$0 = t_1 y r_{21} = t_2 y r_{21} = t_2 x r_{12} = t_1 x r_{12}$$

so that the Eq. (**) is satisfied.

Finally, we must show that θ is not induced by an element of R. If θ were induced by $r \in R$, then for each $(x, y) \in Re \oplus Re$

$$(rx - t_1 x, ry - t_2 y) \in S = \{s_1, -\phi(s_1) s_1 \in S_1\}\,.$$

That is, the difference between $\bar{\theta}$ and left multiplication by r sends $Re + Re$ into S. In the above relation let $x = y = xe$. Then

$$rxe - t_1 xe \in S_1 \quad \text{and} \quad rxe - t_2 xe \in S_2\,.$$

Since t_1 and t_2 were chosen in $N^r e$, we have $t_i xe = t_i exe$. Now using the fact that S_1 and S_2 are right eRe modules, we can conclude that $rxe \in S_1 \cap S_2$ or $rxe = 0$. Since this is true for all $x \in R$, r is in the anihilator of Re. But since R is primary, Re is a faithful R module (R itself is the direct sum of isomorphic copies of Re). So now we conclude that $r = 0$. However, the mapping θ is not zero, since $\bar{\theta}$ applied to $(e, 0)$ is $(t_1, 0) \notin S$. This concludes the proof of the theorem.

We note that we used the algebraic closure of the field only to show that left R submodules of $N^r e$ are also eRe submodules. We do not see how to modify the proof so as to avoid this.

Added in proof. Kent Fuller has independently proved Theorem 1 and it will appear in the Proceedings of the American Mathematical Society.

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Hahn-Banach Extensions which are Maximal on a Given Cone

PER ROAR ANDENÆS

1. Let p be a subadditive positively homogeneous functional on a real linear space E and let f be a linear functional defined on a subspace M of E, $f \leqq p|M$. The main result in this paper (Theorem 1) states that for any subset S of E there exists a linear functional F on E extending f, $F \leqq p$, such that F is *(S, p)-maximal*, i.e. if G is another linear functional extending f, $G \leqq p$, such that $G|S \geqq F|S$, then $G|S = F|S$. We also discuss when such maximal extensions are uniquely determined. In Section 3 we show – in some detail – how this strengthened version of the Hahn-Banach theorem can be used systematically to give a unified treatment in proving the existence of representing boundary measures in Choquet theory.

2. We proceed directly to our main result.

Theorem 1. *Let f be a linear functional on a subspace M of a real linear space E and assume that $f \leqq p|M$ where p is a subadditive positively homogenous functional on E. Let S be an arbitrary subset of E. Then there exists a linear functional F on E such that $F|M = f$, $F \leqq p$, and such that F is (S, p)-maximal.*

Remark. Without loss of generality we could – as already indicated in the title of this paper – have assumed S to be a cone. It is perfectly trivial to see that F is (S, p)-maximal if and only if F is $(K(S), p)$-maximal. Here $K(S)$ denotes the cone generated by S.

Proof of Theorem 1. If $S \subset M$, the ordinary Hahn-Banach theorem is applicable, so we assume that $S \setminus M \neq \emptyset$. Let $M(S)$ denote the subspace of E spanned by M and S. We now extend f to $M(S)$. We therefore consider pairs (f', M') where f' is a linear functional on a subspace M' of $M(S)$. These are partially ordered in the usual way: $(f', M') \leqq (f'', M'')$ if $M' \subset M''$ and $f''|M' = f'$. Now, let $\mathscr{A}$ consist of those pairs (f', M') such that

a) $(f, M) \leqq (f', M')$.
b) $f' \leqq p|M'$.
c) f' is maximal on $M' \cap S$ (i.e. f' is $(M' \cap S, p|M')$-maximal).
d) M' is the linear span of M and $M' \cap S$.

Evidently (f, M) satisfies these conditions. Let $\{(f_\alpha, M_\alpha)\}_{\alpha \in A}$ be a chain in $\mathscr{A}$ and define in the usual way f_A on the subspace $M_A = \bigcup_{\alpha \in A} M_\alpha$ by $f_A(x) = f_\alpha(x)$ for some α such that $x \in M_\alpha$. Then evidently (f_A, M_A) will satisfy a), b) and d). Let g_A be a linear functional on M_A such that (g_A, M_A) satisfies a) and b) and such that $g_A|M_A \cap S \geqq f_A|M_A \cap S$. Then $g_A|M_\alpha \cap S \geqq f_\alpha|M_\alpha \cap S$ for each $\alpha \in A$, and from c) it follows that $g_A|M_\alpha \cap S = f_\alpha|M \cap S$. Thus $g_A|M_A \cap S = f_A|M_A \cap S$, and we have proved that $(f_A, M_A) \in \mathscr{A}$. Zorn's lemma then ensures the existence of a $(f_0, M_0) \in \mathscr{A}$ maximal w.r.t. $\lesssim$. Now assume that $M(S) \smallsetminus M_0 \neq \emptyset$, then we also have $S \smallsetminus M_0 \neq \emptyset$. Let $x_1 \in S \smallsetminus M_0$ and denote by M_1 the subspace spanned by M_0 and $\{x_1\}$. Evidently M_1 satisfies condition d) above since M_0 does. We define a linear functional f_1 on M_1 by

$$f_1(x + \lambda x_1) = f_0(x) + \lambda f_1(x_1) \qquad (x \in M_0, \lambda \in \mathbb{R})$$

where $f_1(x_1)$ is defined by

$$f_1(x_1) = \inf\{f_0(x) + p(x_1 - x) \,|\, x \in M_0\} . \tag{1}$$

Obviously we have $(f_0, M_0) \lesssim (f_1, M_1)$, and it is also straightforward to see that $f_1 \leqq p|M_1$. (Actually we have chosen the greatest possible value when defining $f_1(x_1)$.) Let g_1 be a linear functional on M_1 such that $g_1|M = f$, $g_1 \leqq p|M_1$ and such that $g_1|M_1 \cap S \geqq f_1|M_1 \cap S$. Since f_0 is maximal on $M_0 \cap S$, we have $g_1|M_0 \cap S = f_0|M_0 \cap S$. But then g_1 and f_0 coincide on the linear span of M and $M_0 \cap S$, and, because of d), we conclude that $g_1|M_0 = f_0$. We have $g_1(x_1) \geqq f_1(x_1)$. If $g_1(x_1) > f_1(x_1)$, we would have for some $x \in M_0$

$$g_1(x_1) > f_0(x) + p_1(x_1 - x) ,$$

i.e. $g_1(x_1 - x) > p(x_1 - x)$ contradicting the fact that $g_1 \leqq p|M_1$. Thus we also have $g_1(x_1) = f_1(x_1)$, and we have proved that $g_1 = f_1$. In particular it follows that f_1 is maximal on $M_1 \cap S$. Thus we have $(f_1, M_1) \in \mathscr{A}$, $(f_0, M_0) \lneqq (f_1, M_1)$. This contradicts the choice of (f_0, M_0), and we have proved that $M_0 = M(S)$. If $M(S) = E$, there is nothing more to prove. If $M(S) \neq E$, we use the ordinary Hahn-Banach theorem and extend f_0 to a linear functional F on E such that $F \leqq p$. It is trivial to verify that F is maximal on S, and the proof is complete.

Remarks. 1) Let the notation be as in the preceding theorem. Let $\mathscr{F}$ denote the set of all linear functionals F on E extending f with $F \leqq p$. If we define a preorder $\lesssim_S$ on $\mathscr{F}$ by letting $F \lesssim_S G$ if $F(x) \leqq G(x)$ for all $x \in S$, then obviously the result in Theorem 1 can be restated:

$\mathscr{F}$ has at least one element which is maximal w.r.t. $\lesssim_S$.

2) If $M' \supset M$ is a subspace of E, we put $\mathscr{F}_{M'} = \{F|M' \,|\, F \in \mathscr{F}\}$. Evidently $\mathscr{F}_{M'}$ is convex. It is interesting to note that we can strengthen the conclusion of Theorem 1 to the following: *There exists a (S, p)-maximal extension F of f such that $F|M(S)$ is also an extreme point in $\mathscr{F}_{M(S)}$.*

To prove this statement we modify the proof of theorem 1 slightly. We now require that the members (f', M') of $\mathscr{A}$ should also satisfy the following condition e) in addition to the conditions a), b), c) and d)

e) f' is extreme in $\mathscr{F}_{M'}$.

The only "non-trivial" thing to check is that e) is carried through when extending f_0 from M_0 to M_1. We still define $f_1(x_1)$ by (1), and so we know that conditions a)–d) are satisfied. To verify e), assume that $f_1 = \lambda g + (1-\lambda)h$ where $0<\lambda<1$ and g and h are linear functionals on M_1 such that $g|M = h|M = f$, $g \leqq p|M_1$, $h \leqq p|M_1$. By restricting to M_0 we obtain $f_1|M_0 = g|M_0 = h|M_0$ since (f_0, M_0) satisfies e). Combining with (1) we see that $g(x_1) \leqq f_1(x_1)$, $h(x_1) \leqq f_1(x_1)$. Since also $f_1(x_1) = \lambda g(x_1) + (1-\lambda)h(x_1)$, we must conclude that $f_1(x_1) = g(x_1) = h(x_1)$, and we have proved that $f_1 = g = h$, i.e. f_1 is extreme in $\mathscr{F}_{M_1}$.

When applied to Choquet theory this result will provide extreme boundary measures; in this context the above method is due to Vincent-Smith ([5]).

We make some observations which will become useful later. The setting is the same as in Theorem 1. We define $\tilde{p}: E \to \mathbb{R}$ by

$$\tilde{p}(x) = \inf\{f(y) + p(x-y) \mid y \in M\}\,. \tag{2}$$

Observation 1. *If $S = \{x_0\}$, then for any (S, p)-maximal extension $F \in \mathscr{F}$ we have $F(x_0) = \tilde{p}(x_0)$.*

Proof. For any $F \in \mathscr{F}$ we have $F \leqq \tilde{p}$, in particular $F(x_0) \leqq \tilde{p}(x_0)$. Now there exists $F_0 \in \mathscr{F}$ such that $F_0(x_0) = \tilde{p}(x_0)$ (cf. the construction in (1) in the proof of Theorem 1). The conclusion follows from the definition of (S, p)-maximality.

Observation 2. *For each $x_0 \in S$ there exists a (S, p)-maximal extension $F_0 \in \mathscr{F}$ such that $F_0(x_0) = \tilde{p}(x_0)$ and such that $F_0|M(S)$ is extreme in $\mathscr{F}_{M(S)}$.*

Proof. We extend f to a $(\{x_0\}, p)$-maximal linear functional f_0 on the subspace M_0 spanned by M and $\{x_0\}$. Then let F_0 be a (S, p)-maximal extension of f_0 such that $F_0|M(S)$ is extreme in $\mathscr{F}_{0_{M(S)}}$ where $\mathscr{F}_0$ denotes the set of linear functionals F extending f_0 with $F \leqq p$. It is obvious that F_0 is also a (S, p)-maximal extension of f, so it remains to prove that $F_0|M(S)$ is extreme in $\mathscr{F}_{M(S)}$. Suppose that $F_0|M(S) = \lambda G|M(S) + (1-\lambda)H|M(S)$ where $0<\lambda<1$ and $G, H \in \mathscr{F}$. It suffices to prove that $F_0|M_0 = G|M_0 = H|M_0$. From the fact that $F_0|M(S)$ is extreme in $\mathscr{F}_{0_{M(S)}}$ it will then follow that $F_0|M(S) = G|M(S) = H|M(S)$, which is what we want. To see that F_0, G and H coincide on M_0 we need only prove that $F_0(x_0) = G(x_0) = H(x_0)$. We have $F_0(x_0) = \lambda G(x_0) + (1-\lambda)H(x_0)$ where $F_0(x_0) = \tilde{p}(x_0)$, $G(x_0) \leqq \tilde{p}(x_0)$, $H(x_0) \leqq \tilde{p}(x_0)$. This is possible only when $G(x_0) = \tilde{p}(x_0)$ and $H(x_0) = \tilde{p}(x_0)$, and the proof is complete.

We include a result concerning the uniqueness of (S, p)-maximal extensions. The notation is as in Theorem 1.

Theorem 2. *Assume that S is a cone and that $M \subset S$. Then the (S, p)-maximal extensions of f coincide on $M(S)$ if and only if $\tilde{p}$ is additive on S.*

Proof. We first assume that all (S, p)-maximal extensions of f coincide on $M(S)$. According to Observation 2 there exists for each $x \in S$ a (S, p)-maximal

$F_x \in \mathscr{F}$ such that $F_x(x) = \tilde{p}(x)$. This yields

$$\begin{aligned}\tilde{p}(x+y) &= F_{x+y}(x+y)\\ &= F_{x+y}(x) + F_{x+y}(y)\\ &= F_x(x) + F_y(y)\\ &= \tilde{p}(x) + \tilde{p}(y)\,.\end{aligned}$$

Conversely, suppose that $\tilde{p}$ is additive on S. From the definition of $\tilde{p}$ we see that $\tilde{p}|M = f, \tilde{p} \leqq p$, and that $\tilde{p}$ is sub-additive and positively homogenous. Since S is a cone and $M \subset S$ we have $M(S) = S - S$. We define $\tilde{f}$ on $M(S)$ by $\tilde{f}(x-y) = \tilde{p}(x) - \tilde{p}(y)$; $x, y \in S$. It is easily verified that $\tilde{f}$ is well defined, and clearly $\tilde{f}\,|M = f$. We also have

$$\tilde{f}(x-y) = \tilde{p}(x) - \tilde{p}(y) \leqq \tilde{p}(x-y) \leqq p(x-y)$$

for all $x, y \in S$. Thus, $\tilde{f} \leqq p\,|\,M(S)$. An extension $F \in \mathscr{F}$ of $\tilde{f}$ is now clearly (S, p)-maximal since $F\,|\,S = \tilde{p}\,|\,S$. For any $G \in \mathscr{F}$ we have $G\,|\,S \leqq \tilde{p}\,|\,S$, so if G is (S, p)-maximal, we obtain $G\,|\,S = F\,|\,S$, and the theorem is proved.

3. We now show how our results can be used to develop some fundamental facts in Choquet theory.

Let K be a non-empty compact convex subset of a locally convex Hausdorff space over the reals. The set of extreme points in K is denoted $\delta_e K$. We recall some definitions and elementary facts. As usual $C(K)$ denotes the class of continuous functions on K. $C(K)$ is given the usual sup-norm topology and the usual pointwise ordering. We put

$$\begin{aligned}A &= \{f \in C(K)\,|\,f \quad \text{is affine}\}\,,\\ S &= \{f \in C(K)\,|\,f \quad \text{is convex}\}\,.\end{aligned}$$

If $f \in C(K)$, we define the upper envelope $\bar{f}$ of f by $\bar{f} = \inf\{h\,|\,f \leqq h \in A\}$. The lower envelope $\underline{f}$ is defined dually. For any $x \in K$ ε_x is defined by $\varepsilon_x(f) = f(x)$, $f \in C(K)$. We put $\delta_x = \varepsilon_x\,|\,A$. Then δ_x is a positive linear functional on A, and we have $\delta_x(1) = 1$, $\|\delta_x\| = 1$. Denote by $\mathscr{M}_x^+$ the subset of $C(K)^*$ consisting of the extensions μ of δ_x such that $\|\mu\| = 1$. (Then we also have $\mu(f) \geqq 0$ for any $f \in C(K)$, $f \geqq 0$.)

According to the Riesz representation theorem a $\mu \in \mathscr{M}_x^+$ can be considered as a probability measure on K. For an arbitrary $f \in C(K)$ $\mu(\bar{f})$ stands for the integral of $\bar{f}$ w.r.t. this measure. Finally, define p on $C(K)$ by $p(f) = \|f\|$.

Our starting point is the following result due to Bauer ([2]): $x \in \delta_e K$ if and only if $\mathscr{M}_x^+ = \{\varepsilon_x\}$. We first prove a proposition due to Hervé ([3]).

Proposition 1. $\delta_e K = \bigcap\limits_{f \in C(K)} \{x\,|\,\bar{f}(x) = f(x)\}\,.$

Proof. If $x \in \delta_e K$, then $\mathscr{M}_x^+ = \{\varepsilon_x\}$, hence ε_x is the only extension of δ_x to $C(K)$ being dominated by p. Then ε_x is necessarily $(\{f\}, p)$-maximal for each $f \in C(K)$. Since $f \leqq h + \|f - h\| \in A$ for each $h \in A$, we then obtain from

Observation 1:

$$\begin{aligned}\bar{f}(x) &\leqq \inf\{h(x) + \|f-h\| \mid h \in A\}\\ &= \inf\{\delta_x(h) + p(f-h) \mid h \in A\}\\ &= \tilde{p}(f) = \varepsilon_x(f) = f(x).\end{aligned}$$

Thus, $\bar{f}(x) = f(x)$ for every $f \in C(K)$. Conversely, let $\mu \in \mathscr{M}_x^+$ and assume that $\bar{f}(x) = f(x)$ for every $f \in C(K)$. For any $f \in C(K)$ we then have

$$\begin{aligned}f(x) = \underline{f}(x) &= \sup\{\delta_x(h) \mid f \geqq h \in A\}\\ &\leqq \mu(f)\\ &\leqq \inf\{\delta_x(h) \mid f \leqq h \in A\} = \bar{f}(x) = f(x).\end{aligned}$$

It follows that $\mu(f) = f(x)$, i.e. $\mathscr{M}_x^+ = \{\varepsilon_x\}$, and $x \in \delta_e K$.

Remark. In the last proof we observed that $\bar{f} \leqq \inf\{h + \|f-h\| \mid h \in A\}$. In fact we have $\bar{f} = \inf\{h + \|f-h\| \mid h \in A\}$ since $\{h + \|f-h\| \mid h \in A\} = \{h \mid f \leqq h \in A\}$. The elementary proof of this equality is omitted. Though it will not be needed in the sequel we include the following

Proposition 2. *Let $x \in K$, $f \in C(K)$ be arbitrary. Then*

$$\bar{f}(x) = \sup\{\mu(f) \mid \mu \in \mathscr{M}_x^+\}.$$

Proof. For any $\mu \in \mathscr{M}_x^+$ and $h \in A$, $h \geqq f$, we have $\mu(f) \leqq \mu(h) = h(x)$, and it follows that $\sup\{\mu(f) \mid \mu \in \mathscr{M}_x^+\} \leqq \bar{f}(x)$. On the other hand, let $\mu_x \in \mathscr{M}_x^+$ be a $(\{f\}, p)$-maximal extension of δ_x, then

$$\sup\{\mu(f) \mid \mu \in \mathscr{M}_x^+\} \geqq \mu_x(f) = \inf\{\delta_x(h) + \|f-h\| \mid h \in A\} \geqq \bar{f}(x).$$

For each $f \in C(K)$ put $B_f = \{x \mid \bar{f}(x) = f(x)\}$. From the upper semi-continuity of $\bar{f}$ it follows that B_f is a G_δ-set, hence it is Borel-measurable. In Proposition 1 we proved that $\delta_e K = \bigcap \{B_f \mid f \in C(K)\}$. An element $\mu \in C(K)^*$ (with $\mu(1) = 1 = \|\mu\|$) is called a *boundary measure* if $\mu(K \setminus B_f) = 0$ for each $f \in C(K)$; or equivalently, if $\mu(\bar{f}) = \mu(f)$ for each $f \in C(K)$. We are now able to prove

Theorem 3 (Choquet-Bishop-de Leeuw). *For each $x \in K$ there exists a boundary measure $\mu \in \mathscr{M}_x^+$.*

Proof. From Theorem 1 we know that there exists a (S, p)-maximal extension $\mu \in \mathscr{M}_x^+$ of δ_x.

We must prove that $\mu(\bar{f}) = \mu(f)$ for each $f \in C(K)$. Let $f \in C(K)$ be given. We note that p_1 defined by $p_1(g) = \mu(\bar{g})$ is subadditive and positively homogenous on $C(K)$. For $h \in A$ we have $p_1(h) = \mu(\bar{h}) = \mu(h) = \delta_x(h)$, thus $\delta_x \leqq p_1 \mid A$. Let ν be a $(\{f\}, p_1)$-maximal extension of δ_x. For any $g \in C(K)$ we have $\nu(g) \leqq p_1(g) = \mu(\bar{g}) \leqq \mu(\|g\|) = \|g\|$, ($\|g\|$ can be considered as an element of A dominating g). Since $\nu \mid A = \delta_x$, we then have $\|\nu\| = 1$ and $\nu(1) = 1$, thus $\nu \in \mathscr{M}_x^+$.

For any $g \in S$ we know that $\overline{-g} = -g$ (cf. [4], p. 19). We obtain $\nu(-g) \leqq p_1(-g) = \mu(\overline{-g}) = \mu(-g)$, thus $\mu \mid S \leqq \nu \mid S$. Since μ is (S, p)-maximal, we must have $\mu|S=\nu|S$, but then, since $S-S$ is dense in $C(K)$, we have $\mu=\nu$. The $(\{f\}, p_1)$-maximality of ν then gives us

$$\begin{aligned}\mu(f)=\nu(f)&=\inf\{\delta_x(h)+p_1(f-h)\,|\,h\in A\}\\&=\inf\{\mu(\bar{h})+\mu(\overline{f-h})\,|\,h\in A\}\\&\geqq\mu(\bar{f})\,.\end{aligned}$$

On the other hand, $\mu(f)=\nu(f)\leqq p_1(f)=\mu(\bar{f})$, and the proof is complete.

Remark. Using the result of Observation 2 we see that, for a given $x\in K$, the boundary measure can be so chosen that it attains maximum value $(\mu(f)=\bar{f}(x))$ at a given $f\in S$ and is extreme in $\mathcal{M}_x^+$. This includes the result of Vincent-Smith mentioned earlier.

If $\mu_0\in\mathcal{M}_x^+$, we define $\mathcal{M}_{x,\mu_0}^+$ to be the set of those $\mu\in\mathcal{M}_x^+$ for which $\mu\gtrsim_S\mu_0$. Only slight modifications in the proof of Theorem 3 are needed to obtain the following:

Let $\mu_0\in\mathcal{M}_x^+$ be given. Then there exists a boundary measure $\mu\in\mathcal{M}_{x,\mu_0}^+$. Furthermore given $f\in S$, the boundary measure $\mu\in\mathcal{M}_{x,\mu_0}^+$ can be chosen such that $\mu(f)=\mu_0(\bar{f})$ and such that μ is extreme in $\mathcal{M}_{x,\mu_0}^+$.

In fact, we define p_0 on $C(K)$ by $p_0(f)=\mu_0(\bar{f})$. Then let μ be a (S,p_0)-maximal extension of $\delta_x=\mu_0|A$ and proceed as in the proof of Theorem 3. The only new difficulty is to prove that $\mu|S=\nu|S$: A standard integration-theoretic argument gives that $\mu(\bar{f})\leqq\mu_0(\bar{f})$ $(f\in C(K))$ because $\mu|S\geqq\mu_0|S$, thus $p_1\leqq p_0$, and we have $\nu\leqq p_0$. Now we can use the (S,p_0)-maximality of μ to obtain $\mu|S=\nu|S$. Furthermore we observe that

$$\tilde{p}_0(f)=\inf\{\delta_x(h)+\mu_0(\overline{f-h})\,|\,h\in A\}=\mu_0(\bar{f})\,.$$

It is also easily checked that $\nu\gtrsim_S\mu$ $(\nu,\mu\in\mathcal{M}_x^+)$ if and only if $\nu(\bar{f})\leqq\mu(\bar{f})$ for all $f\in C(K)$. Thus $\mathcal{M}_{x,\mu_0}^+$ coincides with the set of linear extensions to $C(K)$ of δ_x dominated by p_0. The last part of the statement above then follows from Observation 2.

For further informations on boundary measures that are extreme in $\mathcal{M}_x^+$ the reader is referred to [1].

Finally, we observe that Theorem 2 on the uniqueness of (S,p)-maximal extensions yields the equivalence of two statements in the Choquet-Meyer theorem charcterizing simplexes (cf. f. ex. [4], p. 66.).

Proposition 3. *The following statements are equivalent:*

(a) *For each $x\in K$ there is only one boundary measure $\mu_x\in\mathcal{M}_x^+$.*

(b) *For any $f,g\in S$ we have $\overline{f+g}=\bar{f}+\bar{g}$.*

We omit the easy proof.

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Meromorphie in der Theorie der Fredholmoperatoren mit Anwendungen auf elliptische Differentialoperatoren

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Herrn Professor Heinz Söhngen zum 60. Geburtstag gewidmet

Wir betrachten analytische Operatorfunktionen $T(z)$, die auf einem Gebiet G von einem oder mehreren komplexen Parametern abhängen und Werte in der Menge der Fredholmoperatoren zwischen Banachräumen haben. Es ergibt sich hier, daß die entsprechenden inversen Operatorfunktionen auf G meromorph sind und im Falle einer komplexen Variablen Hauptteile von endlichem Rang haben (Theorem 11, 12, 13). Für den Fall von Semifredholmoperatoren auf Hilberträumen gelten ähnliche Aussagen (Theorem 11, 12). Die ersten Untersuchungen in dieser Richtung gehen für Integralgleichungen auf Tamarkin [18] zurück; sie wurden fortgesetzt von Atkinson, Gohberg und Sz.-Nagy (vgl. [16], [13], [10]). Daran schließen sich einige Arbeiten von P. H. Müller ([12], [13]) an, in denen ein Linearisierungsverfahren von Wielandt benützt wird, das hier zur Herleitung einer Eigenwertverteilung dient (Th. 17c). Als Fortführung der klassischen Theorie von F. Riesz wurde bei linearer Abhängigkeit ($T(z)=I-zA$) die Meromorphie der „Resolvente“ $(I-zA)^{-1}$ von Ruston, Pietsch und A. Taylor in einer Reihe von Arbeiten untersucht. Weitere Ergebnisse über diesen Problemkreis wurden von Hildebrandt [10], Steinberg [17], Haf [9], Ribaric und Vidav [16] und dem Verfasser [4], [6] gefunden. Durch die Verknüpfung der Resultate von [6] mit einem Verfahren aus [16] (vgl. [11], S. 162) gelingt es, das Meromorphieproblem für Fredholmoperatoren auf Banachräumen zu lösen. Wesentlich im Falle der Semifredholmoperatoren ist das in Theorem 1 eingeführte “Lifting”, wobei von der Nuklearität des F-Raumes der auf einem Gebiet holomorphen Funktionen Gebrauch gemacht wird und ein Ergebnis von Allan [1] über die globale Existenz analytischer einseitiger Inversen eingeht.

Im zweiten Teil betrachten wir elliptische Operatoren, die analytisch von einem Parameter abhängen. Unter Verwendung eines neuen Einbettungssatzes für Sobolevräume ([19], [5]) ergibt sich für eine polynomiale Schar elliptischer Operatoren eine Eigenwertverteilung (Th. 17). Die sich anschließenden Ergebnisse zeigen, daß die Verteilung der Approximationszahlen bzw. der Eigenwerte (elliptischer) abgeschlossener Operatoren nur von dem Einbettungstyp abhängt. Dies ermöglicht die Verschärfung einiger Ergebnisse (vgl. [11], Ch. 7) über analytische Störungen (Th. 22) und liefert ein Analogon zur bekannten Entwicklung Greenscher Operatoren auch bei analytischer

Abhängigkeit. Die hier durchgeführten Untersuchungen sind für einige in der Quantentheorie auftretende Probleme von Interesse (vgl. [17], [16], [9]).

Es ist nun naheliegend das Meromorphieproblem auch für Fredholmoperatoren auf lokalkonvexen Vektorräumen zu betrachten.

Die Meromorphie inverser Operatorfunktionen

Sei $\mathscr{L}$ eine Banachalgebra mit Einselement I, $\mathscr{K} \subset \mathscr{L}$ ein abgeschlossenes zweiseitiges Ideal und $q: \mathscr{L} \to \mathscr{L}/\mathscr{K}$ der kanonische Homomorphismus; ferner seien Γ, Γ^l bzw. Γ^r die Mengen der invertierbaren, links- bzw. rechtsinvertierbaren Elemente von $\mathscr{L}/\mathscr{K}$. Mit Φ, Φ^l bzw. Φ^r bezeichnen wir die gelifteten Mengen $q^{-1}(\Gamma)$, $q^{-1}(\Gamma^l)$ bzw. $q^{-1}(\Gamma^r)$. Ferner sei $\mathscr{H}(G, \mathscr{L})$ der Raum der holomorphen Funktionen auf G mit Werten in $\mathscr{L}$, wobei G ein Gebiet im $\mathbb{C}^N$, ein Holomorphiegebiet oder eine Steinsche Mannigfaltigkeit ist.

$$\begin{array}{ccc} \mathscr{L} & \supset & \Phi^{l,r} \\ q\downarrow & & q\downarrow \\ \mathscr{L}/\mathscr{K} & \supset & \Gamma^{l,r} \end{array} \qquad \text{(vgl. [14], [4])}$$

1. Theorem *(Lifting).* a) *Liegt der Wertebereich der Operatorfunktion $T(z)$ aus $\mathscr{H}(G, \mathscr{L})$ in Φ, dann gibt es ein $H(z) \in \mathscr{H}(G, \mathscr{L})$ mit Werten in Φ, so daß*

$$H(z)\, T(z) = I - K_1(z) \quad \textit{und} \quad T(z)\, H(z) = I - K_2(z),$$

wobei die Funktionen K_j Werte in dem Ideal $\mathscr{K}$ haben.

b)[1] *Sei G ein Holomorphiegebiet oder eine Steinsche Mannigfaltigkeit; ferner habe $T(z) \in \mathscr{H}(G, \mathscr{L})$ Werte in Φ^l bzw. Φ^r, dann gibt es Funktionen $L(z)$ bzw. $R(z) \in \mathscr{H}(G, \mathscr{L})$ mit Werten in Φ^r bzw. Φ^l, so daß*

$$L(z)\, T(z) = I - K(z) \quad \textit{bzw.} \quad T(z)\, R(z) = I - K(z),$$

wobei $K(z)$ Werte in dem Ideal $\mathscr{K}$ hat.

c) *Sei $T(z)$ eine auf G meromorphe Funktion (d. h. lokal existiert eine skalare holomorphe Funktion $f(z)$, so daß $f(z)\, T(z)$ holomorph), so daß $q(T(z))$ Element von $\mathscr{H}(G, \mathscr{L}/\mathscr{K})$ ist und sein Wertebereich in der Gruppe Γ liegt, dann gibt es ein $H(z) \in \mathscr{H}(G, \mathscr{L})$ mit*

$$q(H(z)\, T(z)) = q(T(z)\, H(z)) = q(I), \qquad z \in G.$$

d) *Sei $T(z)$ auf G (Holomorphiegebiet oder Steinsche Mannigfaltigkeit) meromorph, so daß $q(T(z)) \in \mathscr{H}(G, \mathscr{L}/\mathscr{K})$ und $q(T(z)) \in \Gamma^l$ bzw. Γ^r; dann gibt es $L(z)$ bzw. $R(z) \in \mathscr{H}(G, \mathscr{L})$ mit Werten in Φ^r bzw. Φ^l, so daß*

$$q(L(z)\, T(z)) = q(I) \quad \textit{bzw.} \quad q(T(z)\, R(z)) = q(I).$$

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[1] In b) und d) setzen wir G als Holomorphiegebiet bzw. Steinsche Mannigfaltigkeit voraus, da das benützte Resultat von Allan [1], S. 216, für die globale einseitige Invertierbarkeit diese Voraussetzung verlangt.

Beweis. a) Für $a(z)=q(T(z))$ existiert $a^{-1}(z)=b(z)\in\mathscr{H}(G,\mathscr{L}/\mathscr{K})$. Da der F-Raum $\mathscr{H}(G)$ der skalaren holomorphen Funktionen auf G nuklear ist ([8], II, S. 87), gilt wegen $\mathscr{H}(G,\mathscr{L}/\mathscr{K})=\mathscr{H}(G)\hat{\otimes}_\varepsilon\mathscr{L}/\mathscr{K}$ nach [8], II, S. 69, auch $\mathscr{H}(G,\mathscr{L}/\mathscr{K})=\mathscr{H}(G)\hat{\otimes}_\pi\mathscr{L}/\mathscr{K}$. Die Elemente aus $\mathscr{H}(G)\hat{\otimes}_\pi\mathscr{L}/\mathscr{K}$ haben aber eine schnell konvergente Entwicklung ([8], II, S. 87). Daher folgt $b(z)=\sum b_k f_k(z)$, $f_k\in\mathscr{H}(G)$ mit $\sum\|b_k\|\,p_n(f_k)<\infty$ für alle n, wenn $\{p_n\}$ ein abzählbares System von Normen auf $\mathscr{H}(G)$ ist; außerdem kann man $\|b_k\|=1$ setzen. Für jedes b_k wählt man nun ein $B_k\in\mathscr{L}$ mit $q(B_k)=b_k$ und z. B. $\|B_k\|\leqq 2$. Dann konvergiert auch

$$H(z)=\sum_k B_k f_k(z)\in\mathscr{H}(G)\hat{\otimes}_\pi\mathscr{L}=\mathscr{H}(G,\mathscr{L})$$

im obigen Sinne und es gilt $q(H(z))=a^{-1}(z)$, also $q(H(z)\,T(z))=q(I)$, und daher $H(z)\,T(z)=I-K_1(z)$ bzw. $T(z)\,H(z)=I-K_2(z)$.

b) Die Funktion $t(z)=q(T(z))$ hat Werte in Γ^l (bzw. Γ^r) für jedes $z\in G$; für jedes feste z ist $t(z)$ also linksinvertierbar. Nach einem Ergebnis von Allen ([1], S. 215) ist dann $t(z)$ global auf G linksinvertierbar durch ein $l(z)\in\mathscr{H}(G,\mathscr{L}|\mathscr{K})$. Nach dem in a) durchgeführten Lifting gibt es also ein $L(z)\in\mathscr{H}(G,\mathscr{L})$, so daß $q(L(z)\,T(z))=q(I)$ gilt; damit folgt die Behauptung ebenso für $T(z)\in\Phi^r$.

c) Zu dem Element $a(z)=q(T(z))\in\mathscr{H}(G,\mathscr{L}|\mathscr{K})$ gibt es ein $b(z)\in\mathscr{H}(G,\mathscr{L}|\mathscr{K})$ mit $b(z)\,a(z)=q(I)$. Für $B(z)$ (vgl. a)) gilt also $q(B(z)\,T(z))=q(I)$.

d) folgt ebenso wie c).

2. Bemerkung. *Es sei $T(z)$ auf G meromorph, so daß noch $q(T(z))\in\mathscr{H}(G,\mathscr{L}|\mathscr{K})$ gilt und $q(T(z))$ seinen Wertebereich in Γ hat.*

a) *Wenn $T^{-1}(z_0)$ für ein $z_0\in G$ existiert, dann läßt sich $H(z)$* (vgl. Theorem 1) *so wählen, daß $H^{-1}(z_0)$ existiert.*

b) *Ist $T(z_0)$ für ein z_0 links bzw. rechtsinvertierbar, dann läßt sich $H(z)$* (vgl. Theorem 1 a), c)) *so wählen, daß $H(z_0)\,T(z_0)$ bzw. $T(z_0)\,H(z_0)$ invertierbar ist.*

Beweis. a) Sei $H(z)$ eine Funktion wie in Theorem 1 c) mit $H(z)\,T(z)=I-K(z)$. Da $q(T(z_0))^{-1}$ eindeutig ist, gilt $q(H(z_0))=q(T(z_0))^{-1}$. Für $T^{-1}(z_0)$ gilt also $q(T^{-1}(z_0))=q(H(z_0))$; also ist $T^{-1}(z_0)=H(z_0)+K$, $K\in\mathscr{K}$. Die neue Funktion $H(z)+K$ erfüllt die Bedingung unter a).

b) Man benützt die Eindeutigkeit der Inversion in $\mathscr{L}|\mathscr{K}$ und schließt wie in a).

Sei nun $\mathscr{L}=\mathscr{L}(X)$ für einen Banachraum X und $\mathscr{K}$ das abgeschlossene zweiseitige Ideal der kompakten Operatoren in der Banachalgebra $\mathscr{L}(X)$ der stetigen Endomorphismen von X. In diesem Fall ist Φ die Menge der Fredholmoperatoren ($T\in\Phi\Leftrightarrow\dim N(T)<\infty$ und $\operatorname{codim}R(T)<\infty$). Φ^l ist die Menge der Semifredholmoperatoren aus Φ^- ($\dim N(T)<\infty$ und $R(T)=\overline{R(T)}$ $\Leftrightarrow T\in\Phi^-$) mit stetig projiziertem Bild $R(T)$; Φ^r ist die Menge der Semifredholmoperatoren aus Φ^+ ($\operatorname{codim}R(T)<\infty\Leftrightarrow T\in\Phi^+$) mit stetig projiziertem Nullraum $N(T)$. Ist X ein Hilbertraum, so gilt $\Phi^l=\Phi^-$ und $\Phi^r=\Phi^+$ (vgl. [14]).

3. Bemerkung. *Ist* $T(z)$ (wie in Theorem 1b), d)), $T(z) \in \Phi^l(X) \subset \mathscr{L}(X)$ *bzw.* $T(z) \in \Phi^r(X) \subset \mathscr{L}(X)$, *an der Stelle* $z_0 \in G$ *linksinvertierbar bzw. rechtsinvertierbar, dann läßt sich* $L(z)$ *bzw.* $R(z)$ (vgl. Theorem 1b), c)) *so wählen, daß* $L(z_0)\, T(z_0)$ *bzw.* $T(z_0)\, R(z_0)$ *invertierbar ist.*

Beweis. a) Für $L(z)\, T(z) = I - K(z)$ gelte $\dim N(I - K(z_0)) = n > 0$ und $N(T(z_0)) = \{0\}$. Dann ist $\dim N(L(z_0)) \cap R(T(z_0)) = n$ erfüllt; außerdem gilt $\operatorname{codim} R(I - K(z_0)) = n$. Da $R(T(z_0))$ als Bild eines Operators aus Φ^l stetig projiziert ist ([14]), ist auch $N(L(z_0)) \cap R(T(z_0))$ stetig projiziert. Sei nun $X = R(I - K(z_0)) \oplus X_1$, dann gibt es einen stetigen Operator F von endlichem Rang, der $N(L(z_0)) \cap R(T(z_0))$ eineindeutig auf X_1 abbildet und auf X_2 verschwindet, wobei X_2 so gewählt ist, daß $R(T(z_0)) = X_2 \oplus N(L(z_0)) \cap R(T(z_0))$ eine topologisch direkte Zerlegung ist. Die Funktion $L(z) + F$ hat also die Eigenschaft, daß $(L(z_0) + F)\, T(z_0)$ invertierbar ist.

b) Sei $T(z)\, R(z) = I - K(z)$, $T_0 = T(z_0)$, $R_0 = R(z_0)$, $K(z_0) = K_0$. Sei $N(I - K_0) = N(R_0) \oplus \tilde{N}$, $X = N(R_0) \oplus \tilde{N} \oplus X_1$, $X = R(I - K_0) \oplus X_2$ und $X = X_3 \oplus N(T_0)$; in X_3 existiert ein Unterraum X_4 mit $T_0 X_4 = X_2$, da T eine Abbildung auf X ist; es gilt auch die direkte topologische Zerlegung $X = X_5 \oplus X_4 \oplus N(T_0)$, $X_3 = X_5 \oplus X_4$. Es ist $\dim X_4 = \dim N(I - K_0)$ erfüllt. Wir konstruieren nun eine ausgeartete Abbildung F folgendermaßen: Sei $X_4 = \tilde{X}_4 \oplus \tilde{\tilde{X}}_4$, so daß $\dim \tilde{X}_4 = \dim N(R_0)$ und $\dim \tilde{\tilde{X}}_4 = \dim \tilde{N}$. F_1 sei 0 auf $\tilde{N} \oplus X_1$ und bilde $N(R_0)$ auf $\tilde{X}_4$ ab; F_2 sei 0 auf $N(R_0) \oplus X_1$ und es gelte, daß $F_2 + R_0$ eingeschränkt auf $\tilde{N}$ eine Abbildung auf $\tilde{\tilde{X}}_4$ ist. Nun setze man $F = F_1 + F_2$. Dann ist $\operatorname{codim} R(T_0(F + R_0)) = 0$ also auch $\dim N(T_0(F + R_0)) = 0$. Folglich ist $T(z)\,(R(z) + F)$ an der Stelle z_0 invertierbar.

4. Lemma (vgl. [16]). *Seien* $A, B \in \mathscr{L}(X)$ *und* $(I - A)(I - B) = (I - B)(I - A) = I$. *Dann gilt* $AB = BA$, $N(A) = N(B)$, $R(A) = R(B)$; *falls* $R(A) \subset N(A)$ *erfüllt ist, folgt* $B = -A$.

Dies folgt aus $A + B - AB = 0$, $A = B(A - I)$ und der Symmetrie.

Wir wollen im folgenden den Operator B mit Hilfe einer Matrix aus dem Operator A berechnen, wenn A von endlichem Rang ist. Ferner sei $A = A(\lambda)$ von einem (oder mehreren) Parameter(n) abhängig und es gelte, $R(A(\lambda))$ ist in einem festen Unterraum $Y \subset X$ enthalten mit $\dim Y = n$. Seien $e_1, \ldots, e_n$ eine Basis von Y und $f_1, \ldots, f_n$ Elemente des dualen Raumes mit $(f_i, e_j) = \delta_{ij}$. Dann gilt $Ax = \sum (f_k, Ax)\, e_k$ für alle $x \in X$. Es ergibt sich in einfacher Weise eine Matrix $\Lambda = (\delta_{jk} - (f_j, A e_k))_{j,k}$ mit $\det \Lambda \neq 0$, falls $(I - A)^{-1}$ existiert; in diesem Fall gilt (vgl. 4. Lemma) für alle $x \in X$

$$Bx = -\Lambda^{-1} \begin{pmatrix} (f_1, Ax)\, e_1 \\ \vdots \\ (f_n, Ax)\, e_n \end{pmatrix}.$$

5. Lemma. *Sei* Y *ein endlichcodimensionaler Unterraum des Banachraumes* X; $X = Z \oplus Y$, $\dim Z < \infty$. *Für den Durchschnitt* $D = \bigcap_{\lambda} N(A(\lambda))$ *der Kerne der Operatorfamilie* $A(\lambda)$ *gelte* $Y \subset D$. *Dann gilt mit einer Basis* $e_1, \ldots, e_n$ *von* Z

und $f_1, \ldots, f_n \in X'$, $(f_i, e_j) = \delta_{ij}$, $(f_j, x) = 0$ *für* $x \in Y$, *die Gleichung* $A(\lambda)x = \sum_j (f_j, x)\, A(\lambda)\, e_j$.

Ferner gilt, falls $(I - A(\lambda))^{-1}$ *existiert, für* $B(\lambda)$ *die Darstellung* (vgl. 4. Lemma)

$$\begin{pmatrix} B e_1 \\ \vdots \\ B e_n \end{pmatrix} = -\Phi^{-1} \begin{pmatrix} A e_1 \\ \vdots \\ A e_n \end{pmatrix}; \quad Bx = 0,\ x \in Y,$$

wobei für $\Phi = (\delta_{jk} - (f_k, A e_j))_{j,k}$ $\det \Phi \neq 0$ *erfüllt ist* (vgl. [11], S. 161).

Beweis. Es gilt $Bx - BAx = -Ax$, also speziell für jedes j $Be_j - BAe_j = -Ae_j$. Daraus folgt $Be_j - \sum_k (f_k, Ae_j)\, Be_k = -Ae_j$. Für alle stetigen Linearformen gilt daher $(x', Be_j) - \sum_k (f_k, Ae_j)(x', Be_k) = -(x', Ae_j)$. Mit der Matrix Φ^{-1} lassen sich also die Zahlen (x', Be_j) berechnen. Anschließend kann man die Linearformen x' wieder weglassen. $\det \Phi \neq 0$ folgt durch Übergang zur Adjungierten.

6. Bemerkung. Wenn $A(z)$, $z = (z_1, \ldots, z_n)$, meromorph von z abhängt und $(I - A(z))^{-1}$ auf einer überall dichten Teilmenge des Gebietes G, $z \in G$, existiert, dann ist sowohl die Matrix A (nach 4) als auch die Matrix Φ meromorph. In diesen Fällen erhält man auch eine meromorphe Funktion $B(z)$ mit $R(B(z)) \subset Y$ bzw. $Y \subset N(B(z))$.

7. Beispiel (vgl. [16]). Ist $T(z)$ auf einem Gebiet meromorph und an einer Stelle invertierbar, so daß $q(T(z)) \subset \Gamma$, dann braucht $T^{-1}(z)$ auf diesem Gebiet nicht meromorph zu sein: Sei $T(z) = I - z^{-1}K$, wobei K zum Beispiel ein Volterrascher (kompakter) Integraloperator ist mit $K^n \neq 0$ für alle n und $\lim \|K^n\|^{1/n} = 0$. Es existiert $(I - z^{-1}K)^{-1} = \sum (1/z)^n K^n$ für $z \neq 0$ und ist bei $z = 0$ wesentlich singulär.

Um auch den Fall mehrerer komplexer Veränderlichen behandeln zu können, geben wir folgende

8. Definition. *Eine auf dem Gebiet* $G \subset \mathbb{C}^N$ *meromorphe Funktion* $M(z)$ *mit Werten in* $\mathscr{L}(X)$ *heißt links-endlich meromorph bzw. rechts-endlich meromorph, falls für jedes* $z_0 \in G$ *eine Umgebung* $U(z_0)$ *existiert, so daß auf* $U(z_0)$ *die Funktion* $M(z)$ *die Darstellung*

$$M(z) = A(z) + PS(z) \quad \textit{(links-endlich)}$$

bzw.

$$M(z) = A(z) + S(z)P \quad \textit{(rechts-endlich)}$$

hat, wobei P *ein Projektor (oder eine lineare Abbildung) von endlichem Rang ist,* $A(z)$ *auf* $U(z_0)$ *analytisch und der singuläre Teil* $S(z)$ *dort meromorph ist (d. h. es gibt eine skalare analytische Funktion* $f(z)$*, so daß* $f(z)\, S(z)$ *auf* $U(z_0)$ *analytisch).*

9. Lemma. a) *Für eine Variable,* $N = 1$*, ist links-endlich meromorph äquivalent mit rechts-endlich meromorph.* b) *Die im obigen Sinne meromorphen* $(N = 1)$ *Operatorfunktionen sind genau die Funktionen* $M(z)$*, die lokal eine Laurent-*

entwicklung

$$M(z)=\sum_{k=k_0>-\infty}(z-z_0)^k M_k \tag{*}$$

haben, in der die Operatoren M_k für $k<0$ von endlichem Rang sind.

c) *Für $N=1$ bilden die endlich meromorphen Operatorfunktionen eine Algebra.*

Beweis. b) Sei $M(z)$ links-endlich meromorph, dann gilt für $k<0$

$$M_k=\frac{1}{2\pi i}\int_\gamma \frac{P\,S(z)}{(z-z_0)^{k+1}}\,dz=P\,S_k,\quad S_k\in\mathscr{L}(X)$$

und $\dim R(P\,S_k)\leqq \dim R(P)$ und $M_k=0$ für $k<k_0$ bei geeignetem k_0. Im Falle rechts-endlicher Meromorphie gilt $M_k=S_k P$ und man schließt ebenso. Sei umgekehrt $M(z)$ von der Form (*) in einer Umgebung von z_0, dann ist $N=\bigcap_{k=k_0}^{-1} N(M_k)$ von endlicher Kodimension in X, denn der Durchschnitt zweier Unterräume mit jeweils endlicher Kodimension ist von endlicher Kodimension. Ist nun P ein Projektor mit $N(P)=N$, so gilt $M_k P=M_k$ für $k_0\leqq k<0$; also ist $M(z)$ rechts-endlich meromorph. Nun zu links-endlich: Die lineare Hülle L von $\bigcup_{k=k_0}^{-1} R(M_k)$ ist von endlicher Dimension. Für P kann man einen Projektor nehmen, der eingeschränkt auf L die Identität ist.

10. Lemma. *Sei $T(z)$ eine auf dem (zusammenhängenden) Gebiet $G\subset\mathbb{C}^N$ analytische Operatorfunktion mit Werten in der Menge Φ der Fredholmoperatoren eines Banachraumes. Wenn $T(z_0)$ für ein $z_0\in G$ invertierbar ist, dann ist $T(z)$ auf $G\backslash M$ invertierbar, wobei M eine analytische Menge ist (sogar lokal Nullstellenmenge einer skalaren analytischen Funktion).*

Dies beweist man wie den Satz 3.3 in [4] mit einer Methode von Atkinson, die sich auch für mehrere Variablen (vgl. [6], Theorem 2.3) anwenden läßt. Als Spezialfall über Meromorphieaussagen der Inversen von Funktionen $T(z)$ findet man Lemma 10 in [6] (Theorem 2.2–2.5). Das vorangehende Lemma gilt auch für G enthalten in einer Steinschen Mannigfaltigkeit.

11. Theorem. *Sei $T(z)$ links-endlich meromorph auf dem Holomorphiegebiet $G\subset\mathbb{C}^N$ (oder G Steinsche Mannigfaltigkeit) mit Werten in der Menge Φ^r von Semifredholmoperatoren, d. h. insbesondere $q(T(z))\in\Gamma^r$ ist analytisch auf G; ferner sei $T(z)$ für ein $z_0\in G$ eine Abbildung auf ganz X. Dann existiert eine auf G meromorphe Operatorfunktion $R(z)$ mit Werten in Φ^l, so daß $T(z)\,R(z)=I$ auf $G\backslash M$, wobei M eine in G analytische Menge ist. $R(z)$ läßt sich lokal in der Form*

$$R(z)=A(z)+B(z)\,Q\,S(z)$$

darstellen, dabei sind $A(z)$ und $B(z)$ analytisch und Q ein Projektor von endlichem Rang. Für $N=1$ hat $R(z)$ Hauptteile von endlichem Rang.

Beweis. Sei nach 1. Theorem a) $T(z)H(z)=I-K(z)$, dann existiert nach 3. Bemerkung und 9. Lemma die Funktion $(I-K(z))^{-1}$ auf $G\backslash M'$. Es bleibt zu zeigen, daß $H(z)(I-K(z))^{-1}$ das behauptete lokale Verhalten hat. Für beliebiges $z_0\in G$ hat $T(z)$ auf einer geeigneten Umgebung $U(z_0)$ die Darstellung $T(z)=A(z)+PS(z)$, daraus folgt lokal wegen 1., a)

$$T(z)H(z)=A(z)H(z)+PS(z)H(z)=I-K(z).$$

Wegen $\dim R(P)<\infty$ gilt $A(z)H(z)=I-K_1(z)$, wobei $K_1(z)\in\mathscr{H}(U(z_0),\mathscr{K})$. Aus $\dim N(I-K_1(z_0))=\operatorname{codim} R(I-K_1(z_0))$ folgt die Existenz einer Abbildung $F\in\mathscr{L}(X)$ mit $\dim R(F)<\infty$, so daß $I-K_1(z_0)+F$ invertierbar ist. In einer Umgebung $U_1(z_0)$ existiert demnach $(I-K_1(z)+F)^{-1}$. Daraus ergibt sich auf $U_1(z_0)$ mit $C(z)=I-K_1(z)+F$

$$\begin{aligned} T(z)H(z)&=(I-K_1(z)+F)-(F-PS(z)H(z))\\ &=[I-(F-PS(z)H(z))C^{-1}(z)]\,C(z),\\ T(z)H(z)C^{-1}(z)&=I-(F-PS(z)H(z))C^{-1}(z). \end{aligned}$$

Sei nun Q eine Projektion auf die lineare Hülle von $R(F)$ und $R(P)$ (diese ist von endlicher Dimension), dann ergibt sich mit $D(z)=F-PS(z)H(z)$ die Gleichung

$$T(z)H(z)C^{-1}(z)=I-QD(z),$$

wobei $D(z)$ auf $U_1(z_0)$ meromorph ist. In jeder Umgebung von z gibt es für $I-QD(z)$ Punkte mit Invertierbarkeit. Auf $I-QD(z)$ wenden wir 4. Lemma und die darauf folgende Bemerkung an. Wir erhalten also $(I-QD(z))^{-1}=I-QE(z)$ mit einer auf $U_1(z_0)$ meromorphen Funktion $E(z)$. Setzen wir $R(z)=H(z)(I-K(z))^{-1}$, dann hat $R(z)$ die behauptete Eigenschaft, denn es gilt $(I-K(z))^{-1}=C^{-1}(z)(I-QE(z))$ auf $U_1(z_0)$. Die Behauptung für $N=1$ folgt aus 9. Lemma b) und c).

12. Theorem. *Sei $T(z)$ auf dem Holomorphiegebiet $G\subset\mathbb{C}^N$ (oder G Steinsche Mannigfaltigkeit) rechts-endlich meromorph mit Werten in der Menge Φ^l von Semifredholmoperatoren (d. h. insbesondere $q(T(z))\in\Gamma^l$ für alle $z\in G$); ferner sei $T(z)$ für ein $z_0\in G$ linksinvertierbar. Dann existiert eine auf G meromorphe Operatorfunktion $L(z)$ mit Werten in Φ^r, so daß $L(z)T(z)=I$ auf $G\backslash M$, wobei M eine in G analytische Menge ist. $L(z)$ läßt sich lokal in der Form*

$$L(z)=A(z)+S(z)QB(z)$$

darstellen, dabei sind $A(z)$ und $B(z)$ analytisch und Q ein Projektor von endlichem Rang. Für $N=1$ hat $L(z)$ Hauptteile von endlichem Rang.

Beweis (analog zu 11.). Sei $H(z)T(z)=I-K(z)$ auf G und $H(z)T(z)$ an einer Stelle z' invertierbar (3. Bemerkung), dann ist $(I-K(z))^{-1}$ auf $G\backslash M$, M analytische Menge, erklärt. Es gilt lokal auf $U(z_0)$ die Gleichung $T(z)=A(z)+S(z)P$ (8. Def.) und daher

$$H(z)A(z)+H(z)S(z)P=I-K(z)=(I-K_1(z)+F)-(F+H(z)S(z)P).$$

Mit $C(z)=I-K_1(z)+F$ gilt in einer Umgebung $U_1(z_0)$ $C^{-1}(z)H(z)T(z) = I-C^{-1}(z)(F+H(z)S(z)P)=C^{-1}(z)(I-K(z))$. Sei nun Q ein Projektor mit $N(Q)=N(F)\cap N(P)$, also $\dim R(Q)<\infty$, dann ergibt sich mit $D(z)=C^{-1}(z)(F+H(z)S(z)P)$ die Gleichung

$$C^{-1}(z)(I-K(z))=I-D(z)Q.$$

Nach 5. Lemma ist $(I-D(z)Q)^{-1}$ von der Form $I-E(z)Q$, wobei $E(z)$ auf $U_1(z_0)$ meromorph ist. Auf $U_1(z_0)$ gilt also die Darstellung $(I-K(z))^{-1} = (I-E(z)Q)\,C^{-1}(z)$. Daraus folgt auf $U_1(z_0)$ die für $L(z)$ behauptete Darstellung. Für $N=1$ folgt die Behauptung aus 9. Lemma.

Im Falle $N=1$, G (nicht kompakte) Riemannsche Fläche, gilt wegen 9. Lemma ein weiterer Zusammenhang. Mit $\mathfrak{M}(G,\Phi^l)$ bzw. $\mathfrak{M}(G,\Phi^r)$ bezeichnen wir die auf G endlich meromorphen Operatorfunktionen $T(z)$ mit Werten in Φ^l bzw. Φ^r (d. h. insbesondere $q(T(z))\in\Gamma^l$ bzw. Γ^r), die jeweils für wenigstens ein $z\in G$ links- bzw. rechtsinvertierbar sind. Mit $\mathfrak{M}(G,\Phi)_l$ bzw. $\mathfrak{M}(G,\Phi)_r$ bezeichnen wir die Teilmenge von $\mathfrak{M}(G,\Phi^l)$ bzw. $\mathfrak{M}(G,\Phi^r)$, deren Elemente nur Werte in der Menge Φ der Fredholmoperatoren haben. Für $T(z)\in\mathfrak{M}(G,\Phi)_l$ gilt $\operatorname{ind} T(z)\leqq 0$, für $T(z)\in\mathfrak{M}(G,\Phi)_r$ gilt $\operatorname{ind} T(z)\geqq 0$ ($\operatorname{ind} T=\dim N(T)-\operatorname{codim} R(T)$). Die Menge $\mathfrak{M}(G,\Phi)=\mathfrak{M}(G,\Phi^l)\cap\mathfrak{M}(G,\Phi^r)=\mathfrak{M}(G,\Phi)_l\cap\mathfrak{M}(G,\Phi)_r$ enthält nur Operatorfunktionen, die an wenigstens einer Stelle $z\in G$ invertierbar sind. Es zeigt sich nun, daß man durch die Bildung der Inversen nicht aus den angegebenen Mengen heraus kommt.

Mit $[\;]_{l,r}^{-1}$ bezeichnen wir die Menge der links- bzw. rechtsinvertierbaren Operatorfunktionen zu der in der Klammer [ ] stehenden Menge.

13. Theorem. *Es gilt:* a) *Die Menge $\mathfrak{M}(G,\Phi)$ der endlich meromorphen Operatorfunktionen mit Werten in der Menge Φ der Fredholmoperatoren erfüllt*

$$[\mathfrak{M}(G,\Phi)]^{-1}=\mathfrak{M}(G,\Phi).$$

Diese Menge bildet also eine multiplikative Gruppe.

b)[2] $[\mathfrak{M}(G,\Phi^l)]_l^{-1}=\mathfrak{M}(G,\Phi^r)$;
c) $[\mathfrak{M}(G,\Phi^r)_r^{-1}=\mathfrak{M}(G,\Phi^l)$;
d) $[\mathfrak{M}(G,\Phi)_l]_l^{-1}=\mathfrak{M}(G,\Phi)_r$;
e) $[\mathfrak{M}(G,\Phi)_r]_r^{-1}=\mathfrak{M}(G,\Phi)_l$.

Der Beweis folgt unmittelbar aus 9. Lemma und 11., 12. Theorem.

Anwendungen auf elliptische Operatoren

Im folgenden werden obige Ergebnisse so erweitert, daß sie auf analytische Funktionen mit Werten in der Menge der abgeschlossenen Operatoren angewandt werden können (vgl. [11], Ch. VII, 364–426). Insbesondere kommen hier die „holomorphen Familien vom Typ A“ ([11], Ch. VII, § 2) in Frage,

[2] b) bis d) gilt im Sinne der Konstruktion von Theorem 11 und 12.

denn diese Funktionen haben einen festen Definitionsbereich D, so daß diese Operatorfunktionen mittels einer Graphennorm als analytische Funktionen mit Werten in dem Banachraum $\mathscr{L}(D, X)$ der beschränkten linearen Abbildungen des Banachraumes D in den Banachraum X aufgefaßt werden können. Diese Situation liegt ebenfalls bei elliptischen Randwertproblemen (vgl. [10]), Differentialoperatoren bzw. Pseudodifferentialoperatoren (vgl. [3]) vor, wenn sie analytisch von einem Parameter abhängen. In [3] Theorem 27 und 28 werden Pseudodifferentialoperatoren charakterisiert, die Fredholmoperatoren sind.

Mit $\Phi^l(X, Y)$ bzw. $\Phi^r(X, Y)$ bezeichnen wir die Mengen der Semifredholmoperatoren mit stetig projiziertem Kern und Bild von dem Banachraum X in den Banachraum Y. Sind X und Y Hilberträume, so fallen diese Mengen mit den Mengen $\Phi^-(X, Y)$ bzw. $\Phi^+(X, Y)$ aller Semifredholmoperatoren zusammen, denn für Hilberträume ist jeder abgeschlossene Unterraum stetig projiziert. $\Phi(X, Y) = \Phi^l(X, Y) \cap \Phi^r(X, Y) = \Phi^-(X, Y) \cap \Phi^+(X, Y)$ ist die Menge der Fredholmoperatoren von X in Y. Bei Anwendungen in der Theorie der elliptischen Operatoren sind die Räume X und Y Hilberträume bzw. genauer: geeignete Sobolevräume (vgl. [3], [10], [5]).

14. Bemerkung. Für holomorphe (oder meromorphe) Operatorfunktionen $T(z)$ mit Werten in $\Phi^l(X, Y)$ bzw. $\Phi^r(X, Y)$ gelten die Aussagen der Theoreme 11 und 12, wenn Operatoren L bzw. R aus $\mathscr{L}(Y, X)$ existieren, so daß die Funktionen $L\,T(z)$ bzw. $T(z)R$ Werte in $\Phi^l(X)$ bzw. $\Phi^r(Y)$ haben und an einer Stelle z links- bzw. rechtsinvertierbar sind. Dies ist insbesondere der Fall, wenn X und Y topologisch isomorph sind, wie z. B. für separable Hilberträume (Sobolev-Räume), die bei elliptischen Differentialoperatoren verwendet werden.

Für Operatorfunktionen $T(z)$ mit Werten in der Menge $\Phi(X, Y)$ der Fredholmoperatoren entfallen die einschränkenden Forderungen an X und Y. Sei etwa $T(z_0)$ linksinvertierbar, dann existiert $L_0 \in \Phi(Y, X)$ mit $L_0 T(z_0) = I$, also ist die Funktion $L_0 T(z)$ für alle z ein Fredholmoperator, das Entsprechende gilt für $T(z)$, wenn $T(z_0)$ eine Abbildung auf Y ist, dann existiert nämlich ein $R_0 \in \Phi(Y, X)$ mit $T(z_0) R_0 = I$ und $T(z)\, R_0 \in \Phi(Y)$ für alle z. Die Schlußweise kann man für Operatorfunktionen mit Werten in $\Phi^l(X, Y)$ bzw. $\Phi^r(X, Y)$ nicht verwenden.

15. Theorem. a) *Sei $T(z)$ auf dem Gebiet $G \subset \mathbb{C}^N$ rechts-endlich meromorph* (entsprechend 8. Definition) *mit Werten in der Menge $\Phi(X, Y)$ und an einer Stelle linksinvertierbar, also* ind $T(z) \leqq 0$. *Dann existiert eine auf G meromorphe Operatorfunktion $L(z)$ mit Werten in $\Phi(Y, X)$, so daß $L(z)\, T(z) = I_X$ auf $G \backslash M$, wobei M eine in G analytische Menge ist. Es gilt die lokale Darstellung für $L(z)$ wie in 12. Theorem.*

b) *Die Theorem 11. entsprechende Aussage gilt für $T(z) \in \Phi(X, Y)$,* ind $T(z) \geqq 0$, *wenn $T(z)$ für ein $z_0 \in G$ eine Abbildung auf Y ist.*

c) *Für eine Variable z, $N = 1$, sind die Hauptteile der links- bzw. rechtsinversen (*a) *bzw.* b)*) Funktionen von endlichem Rang.*

Bemerkung. Die Aussagen von 13. Theorem a), d) und e) gelten auch für Operatorfunktionen mit Werten in $\Phi(X, Y)$. Die in [4] Satz 3.3 verwendete Methode liefert unter Berücksichtigung von Lemma 4 und 5 eine Beweismöglichkeit, die unabhängig von der hiesigen ist.

Zur Motivierung der folgenden Untersuchungen können elliptische Differentialoperatoren dienen. Es seien

$$T_k = \sum_{r=0}^{m_k} \sum_{|\alpha|=r} a_\alpha^k(x)\, D^\alpha\,{}^3, \qquad k = 0, 1, \ldots, N; \qquad m_k < m_0 \quad \text{für} \quad k \geqq 1,$$

wobei T_0 ein stark elliptischer Differentialoperator auf dem beschränkten Gebiet $\Omega \subset \mathbb{R}^n$ mit Sobolevscher Kegelbedingung ist. Man kann in diesem Zusammenhang auch elliptische Systeme (vgl. [10]) betrachten, die ebenfalls Fredholmoperatoren vermitteln. Die Operatoren T_k sind auf Sobolevräumen $H^m(\Omega), m \geqq m_k$ (vgl. [5]) definiert. Die Randbedingung B legt man durch einen abgeschlossenen Unterraum $H^m(\Omega) \cap B$ fest ($B = H_0^{m/2}$, Dirichlet-Problem).

Nun sei H^m eine Schar von Banachräumen, $m \geqq 0$ reell, mit

$$H^m \subset H^{m'}, \qquad m > m'.$$

Es seien ferner $T_k \in \mathscr{L}(H^{m'+m_k}, H^{m'})$, $m_k \geqq 0$ reell, $k = 0, \ldots, N$ und $m_0 > m_k$ für $k \geqq 1$. m_k heiße dann Ordnung des Operators T_k, wobei m keine ganze Zahl zu sein braucht. Zum Beispiel sei $T_k = (I - \Delta)^s$, s reell, oder $T_k = S_k V_k$, wenn V_k ein Pseudodifferentialoperator und S_k ein singulärer Integraloperator ist. In den Anwendungen gilt meist $H^{m'} = L^2(\Omega, \mu)$ [4].

Wir betrachten die auf H^m, $m = m' + m_0$ definierte Operatorfunktion

$$T(\lambda) = \sum_{k=0} \lambda^k T_k$$

und deren Eigenwerte λ_i (d. h. für ein $f \neq 0$ gilt $T(\lambda_i) f = 0$).

Aufgrund der von Beals, Triebel und dem Verfasser gefundenen Einbettungssätze (vgl. [19], [5]) ist es möglich mit dem in [5] verwendeten Zusammenhang zwischen Einbettungseigenschaften und Eigenwertverteilungen, asymptotische Aussagen für die Eigenwerte von $T(\lambda)$ zu gewinnen. Dazu führen wir in Anlehnung an [15] (S. 107–126) eine Klasse $\mathscr{J}^r(X, Y)$ von kompakten Abbildungen aus $\mathscr{L}(X, Y)$ ein, wenn X und Y Banachräume sind. Sei für $r > 0$

$$\mathscr{J}^r(X, Y) = \left\{ T \in \mathscr{L}(X, Y) : \delta_r(T) = \sup_{j=0,1,2\ldots} (1+j)^r \alpha_j(T) < \infty \right\},$$

dabei sind $\alpha_j(T)$ die in [15], S. 107, definierten Approximationszahlen, die sich für Hilberträume auch mit dem Min-Max-Verfahren von Courant und Weyl

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[3] $D = \dfrac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial x_1^{\alpha_1} + \cdots + \partial x_n^{\alpha_n}}, \qquad |\alpha| = \sum |\alpha_i|.$

[4] $L^2(\Omega, \mu)$ ist der Raum der quadratintegrierbaren Funktionen bezüglich des Lebesgueschen Maßes μ auf $\Omega \subset \mathbb{R}^n$.

bestimmen lassen. Mit den Methoden von [15], S. 111–113, folgt sofort wie für

$$l^p(X, Y) = \{T \in \mathscr{L}(X, Y) : \varrho_p(T) = \sum \alpha_j(T)^p < \infty\}, \qquad 0 < p < \infty,$$

die folgende

16. Bemerkung. *$\mathscr{I}^r(X, Y)$ ist versehen mit der Quasinorm $\delta_r(T)$ ein vollständiger lokalbeschränkter linearer Raum. $\mathscr{I}^r$ besitzt die Idealeigenschaft:*

$$\mathscr{I}^r(X, Y)\,\mathscr{L}(Z, X) \subset \mathscr{I}^r(Z, Y) \quad \textit{und} \quad \mathscr{L}(X, Y)\,\mathscr{I}^r(Z, X) \subset \mathscr{I}^r(Z, X).$$

Außerdem gilt für $p > r^{-1}$ $\mathscr{I}^r(X, Y) \subset l^p(X, Y)$.

Für die Sobolevräume $H^m(\Omega)$ (separable Hilberträume), Ω kompakte differenzierbare Mannigfaltigkeit, wurde in [19] und [5] gezeigt, daß die Einbettungen $I^{m,m'}: H^m(\Omega) \to H^{m'}(\Omega)$ zur Klasse $\mathscr{I}^r$ für $r = (m - m')\,n^{-1}$ gehören, wenn $n = \dim \Omega$. Aus der in [5] benutzten Fourierentwicklung ergibt sich dies mit der Idealeigenschaft von $\mathscr{I}^r$ und einer Zerlegung der Eins nach Palais bzw. mit einem Fortsetzungssatz von Calderon (vgl. [5]) in sehr einfacher Weise (vgl. Beals, Amer. J. Math. **89**, 1967). Das folgende Ergebnis ist eine wesentliche Erweiterung von Satz 2 in [5] und beruht auf einem Kunstgriff von Wielandt (vgl. [13]).

17. Theorem[5]**.** *Sei H^m eine Schar von Banachräumen mit den Einbettungsoperatoren $I^{m,m'} \in \mathscr{I}^r$, $r = (m - m')n^{-1}$. Ferner sei T_0 ein (links-) invertierbarer Fredholmoperator, $T_0 \in \Phi(H^m, H^{m'})$, $m = m' + m_0$.*

a) *Dann besitzt die Operatorfunktion $T(\lambda) = \sum_{k=0}^{N} \lambda^k T_k$ eine in der ganzen komplexen λ-Ebene meromorphe (Links-) Inverse $G(\lambda): H^{m'} \to H^m$ (Greenscher Operator) mit Hauptteilen von endlichem Rang.*

b) *Als Transformation von $H^{m'}$ aufgefaßt liegt bei festem λ die (Links-) Inverse $G(\lambda) = I^{m,m'}\, G(\lambda)$ in $\mathscr{I}^r(H^{m'}, H^{m'})$, $r = (m - m')n^{-1}$; d. h. für die Approximationszahlen gilt*

$$\delta_r(G(\lambda)) = \sup_{j=0,1,\ldots} (1 + j)^r\, \alpha_j(G(\lambda)) < \infty.$$

c) *Sind die Räume H^m Hilberträume, z. B. Sobolevräume $H^m(\Omega)$, so gilt für die Eigenwerte λ_j ($|\lambda_j| \to \infty$), mit Vielfachheit gezählt, $\sum_{j=1}^{\infty} |\lambda_j|^{-p} < \infty$ für jedes $p > nNs^{-1}$, wenn $s = \min_{k>0}(m_0 - m_k) > 0$.*

d) *Für die Anzahlfunktion $N(t)$ der Eigenwerte λ_j (ebenfalls mit Vielfachheit) mit $|\lambda_j| \leqq t$ ist*

$$N(t) = o(t^p) \quad \textit{für jedes} \quad p > nNs^{-1}$$

erfüllt, d. h. $\lim_{t\to\infty} N(t)\, t^{-p} = 0$.

[5] Auch für den Operator $-\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + \lambda\left(a_1(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2} + b(x)\right)$ enthält Theorem 17 eine neue Aussage.

Beweis. $T(\lambda)$ ist für $\lambda=0$ linksinvertierbar und es gilt $T(\lambda)=\sum_{k=0}^{N}\lambda^k I^{m'+m_k,m'}T_k$, so daß die Terme für $k\geqq 1$ kompakt sind. Daraus folgt a) nach obigen (Th. 13) Resultaten. Zu b): Wenn $G(\lambda)$ existiert, gehört $I^{m,m'}G(\lambda)$ zu $\mathscr{J}^r(H^{m'},H^{m'})$ für $r=(m-m')n^{-1}$.

c) Sei $LT_0=I$ auf H^m; dann ist $L(\lambda)=LT(\lambda)=I+\sum_{k=1}^{N}\lambda^k L_k$, $L_k=LT_k$, eine Abbildung von H^m in H^m. Die Operatoren L_k lassen sich folgendermaßen faktorisieren

$$\begin{tikzcd} H^m \arrow[r, "L_k"] \arrow[d, "I^{m,m-s}"'] & H^m \\ H^{m-s} \arrow[r, "T_k"] & H^{m'} \arrow[u, "L"'] \end{tikzcd}\;.$$

Daher gehört L_k zu $\mathscr{J}^r(H^m,H^m)$ für $r=sn^{-1}$. Zunächst substituieren wir $\mu=\lambda^{-1}$ und erhalten

$$L(\lambda)=\mu^{-N}(\mu^N I+\mu^{N-1}L_1+\cdots+L_N)=\mu^{-N}\tilde{L}(\mu)\,.$$

Es genügt nun, die „Eigenwerte" von $\tilde{L}(\mu)$ zu betrachten. Auf $\tilde{L}(\mu)$ wenden wir ein Linearisierungsverfahren von Wielandt (vgl. [13]) an: Sei Π das N-fache Produkt $\Pi=\prod_{r=1}^{N}H^m$ und $W\in\mathscr{L}(\Pi)$ die Matrix

$$W=\begin{bmatrix} 0 & 1 & 0\ldots & & & 0\\ 0 & 0 & 1 & 0\ldots & & 0\\ 0 & 0 & 0 & 1 & 0\;\ldots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0\ldots & & 0 & 1\\ -L_N, & -L_{N-1} & \ldots & & -L_2, & -L_1 \end{bmatrix},$$

dann ist $\mu\neq 0$ genau dann ein Eigenwert mit entsprechender Vielfachheit von $W\in\mathscr{L}(\Pi)$, d. h. $(\mu I_\Pi-W)y=0$ für ein $y\in\Pi$, $y\neq 0$, wenn $\tilde{L}(\mu)x=0$ für ein $x\neq 0$, $x\in H^m$, wie in [13] gezeigt wird. Offensichtlich ist W nicht kompakt aber W^N enthält als Matrixelemente nur noch Linearkombinationen von Produkten der L_k; demnach gehört W^N zu $\mathscr{J}^r(\Pi,\Pi)$ für $r=sn^{-1}$. Für die Approximationszahlen $\alpha_j(W^N)$ gilt also $\delta_r(W^N)=\sup_{j=0,1,\ldots}(j+1)^r\alpha_j(W^N)<\infty$ und daher auch $\sum\alpha_j(W^N)^p<\infty$ für jedes $p>r^{-1}$. Daraus folgt für die Eigenwerte $\varkappa_j(W^N)$ (mit Vielfachheit) nach einer für Hilberträume gültigen Ungleichung von Weyl (vgl. [5], Satz 3; die Ungleichung gilt für $0<p<\infty$)

$$\sum|\varkappa_j(W^N)|^p\leqq\sum\alpha_j(W^N)^p\,,\qquad p>ns^{-1}\,.$$

Die Eigenwerte $\varkappa_j(W^N)$ bringen wir nun mit den Eigenwerten $\mu_j(W)$ in Verbindung (Spektralabbildungssatz). Nach der Rieszschen Theorie sind die Elemente $\neq 0$ des Spektrums von W Eigenwerte endlicher Vielfachheit.

Mit $Q=\sum_{j=0}^{N-1} z^{N-1-j} W^j$ gilt (*) $(z^N I - W^N) = (zI-W)Q = Q(zI-W)$. Wenn $\varkappa_j$ ein Eigenwert von W^N ist, dann sind nur die Werte

$$\mu_{jk} = |\varkappa_j|^{\frac{1}{N}} \exp\left(2\pi i \frac{\varphi(\varkappa_j)+k}{N}\right),$$

$k=0,\ldots,N-1$, als zugeordnete Eigenwerte von W möglich. Wegen der linearen Unabhängigkeit der Eigenvektoren zu verschiedenen Eigenwerten ist die Summe der Vielfachheiten der μ_{jk}, j fest, nach (*) kleiner als die Vielfachheit des Eigenwertes $\varkappa_j(W^N)$. Für die $\mu_j(W)$ gilt also $\sum|\mu_j(W)|^{N\cdot p} \leqq \sum|\varkappa_j(W^N)|^p \leqq \sum \alpha_j(W^N)^p$. Substituieren wir wieder $\mu=\lambda^{-1}$, so folgt $\sum|\lambda_j|^{-p}<\infty$ für $p>nNs^{-1}$.

d) Denken wir uns die $|\lambda_j|$ der Größe nach angeordnet, so folgt ($\sum a_i<\infty$, $a_i \geqq a_{i+1} \geqq 0$, impliziert $i a_i \to 0$) $j|\lambda_j|^{-p}\to 0$ für $j\to\infty$. Setzen wir $j=N(t)$ und $|\lambda_j|=t$, so ergibt sich $\lim_{t\to\infty} N(t)\,t^{-p}=0$ für $p>nNs^{-1}$.

Wir untersuchen nun die Folge der Approximationszahlen ([15], 8.1) Greenscher Operatoren bzw. der Inversen elliptischer Operatoren.

18. Definition. *Die Approximationszahlen $\{\alpha_j(S)\}$ und $\{\alpha_j(T)\}$ von $S, T\in\mathscr{L}(X,Y)$ heißen äquivalent, wenn es positive Konstanten m und M gibt, so daß die Ungleichungen*

$$0<m\leqq \frac{\alpha_j(S)}{\alpha_j(T)} \leqq M$$

für alle $j=0,1,\ldots$ erfüllt sind. Sie heißen asymptotisch äquivalent, wenn die beiden Ungleichungen für $j\geqq j_0(S,T)$ gelten.

19. Satz. *Der Banachraum D sei stetig in den Banachraum X eingebettet. $I^{D,X}$ sei die Einbettung von D in X. Sei $T\in\mathscr{L}(D,X)$ eine linksinvertierbare Abbildung mit $LT=I_D$, $L\in\mathscr{L}(X,D)$. Dann sind die Approximationszahlen der als Transformation von X aufgefaßten Linksinversen $G=I^{D,X}L\in\mathscr{L}(X)$ zu den Approximationszahlen des Einbettungsoperators $I^{D,X}$ äquivalent. Aus der Transitivität der in 18. eingeführten Äquivalenz ergibt sich, daß die Approximationszahlen von $G_i=I^{D,X}\,L_i$, $L_iT_i=I_D$, $T_i\in\mathscr{L}(D,X)$, $L_i\in\mathscr{L}(X,D)$, äquivalent sind. Wenn nur $AB=I_D-F$, $\dim R(F)<\infty$, erfüllt ist, dann sind die Approximationszahlen von $G=I^{D,X}A$ noch asymptotisch äquivalent zu denen von $I^{D,X}$.*

(Bemerkung. Für Hilberträume werden die Approximationszahlen $\alpha_j=\mu_j$ in Dunford und Schwartz, Linear Operators II, S. 1088–1100 behandelt.) Mit $G=I^{D,X}L$ folgt ([15], S. 109) $\alpha_j(G)\leqq\alpha_j(I^{D,X})\,\|L\|_{\mathscr{L}(X,D)}$; andererseits ergibt sich aus $I^{D,X}=I^{D,X}LT$ wegen $G=I^{D,X}L$ die Ungleichung $\alpha_j(I^{D,X})\leqq\alpha_j(G)\,\|T\|_{\mathscr{L}(D,X)}$. Damit folgt die erste Behauptung. Aus $AB=I_D-F$ ergibt sich

$$\alpha_j(I^{D,X}(I-F))\leqq\alpha_j(I^{D,X}A)\,\|B\|_{\mathscr{L}(D,X)}\,.$$

Für $j \geqq \dim R(F)$ folgt (vgl. Definition der Approximationszahlen in [15], 8.1) aus $G = I^{D,X} A$

$$\alpha_j(I^{D,X}) \leqq \alpha_j(G) \, \|B\|_{\mathscr{L}(D,X)}, \qquad j \geqq j_0 .$$

Außerdem hat man $\alpha_j(G) \leqq \alpha_j(I^{D,X}) \, \|A\|_{\mathscr{L}(X,D)}$.

20. Definition. *Eine multiplikative Halbgruppe $\mathscr{H} \subset \mathscr{L}(X)$ heiße ideale (Links- bzw. Rechts-) Halbgruppe, wenn aus $A \in \mathscr{L}(X)$ und $H \in \mathscr{H}$ die Relationen $AH \in \mathscr{H}$ und $HA \in \mathscr{H}$ folgen (bzw. nur $AH \in \mathscr{H}$ bzw. $HA \in \mathscr{H}$).*

Zur Motivierung der folgenden Bemerkung kann Theorem 2.4 aus [11], S. 377, dienen.

21. Bemerkung. *Seien $T_\varkappa$ Isomorphismen aus $\mathscr{L}(D,X)$, dann gehört $I^{D,X} T_{\varkappa_1}^{-1} = G_{\varkappa_1}$ genau dann zur rechtsidealen Halbgruppe $\mathscr{H} \subset \mathscr{L}(X)$, wenn $G_{\varkappa_2} = I^{D,X} T_{\varkappa_2}^{-1}$ zu $\mathscr{H}$ gehört. Das heißt, entweder gehören alle $G_\varkappa$ zu einer vorgegebenen idealen Halbgruppe $\mathscr{H}$ oder kein $G_\varkappa$. Die Approximationszahlen aller $G_\varkappa \in \mathscr{L}(X)$ sind äquivalent.*

Beweis. Es gilt $T_1^{-1} T_1 T_2^{-1} = T_2^{-1}$, denn $T_1^{-1} T_1$ ist die Identität auf D. Da $T_1 T_2^{-1}$ ein Isomorphismus aus $\mathscr{L}(X)$ ist, ergibt sich die Behauptung aus $T_1^{-1}(T_1 T_2^{-1}) = T_2^{-1}$ mit der Symmetrie wie oben.

Jede Belegungsfolge $b = \{b_j\}$, $0 < b_0 \leqq b_j \leqq b_{j+1} \to \infty$, definiert eine ideale Halbgruppe $\mathscr{H}_b = \{T \in \mathscr{L}(X) : q_b(T) = \sup_j b_j \, \alpha_j(T) < \infty\}$, d. h. für $T \in \mathscr{H}_b$ und $A \in \mathscr{L}(X)$ gilt AT und $TA \in \mathscr{H}_b$. Die Halbgruppen brauchen keine linearen Räume zu sein, wie man z. B. für $b = 2^j$ und X einen Hilbertraum leicht einsehen kann.

Sei nun $T(z)$ eine analytische Operatorfunktion auf dem Gebiet $V \subset \mathbb{C}$ mit Werten in $\Phi(D,H)$, D enthalten in dem Hilbertraum H; ferner sei $T(z)$ für ein $z_0 \in V$ invertierbar. Dann ist $G(z) = I^{D,H} T^{-1}(z)$ eine auf G meromorphe Operatorfunktion. In dem Analytizitätsbereich von $G(z)$ gibt es für jedes z zwei Orthonormalsysteme $\{e_j(z)\}$ und $\{f_j(z)\}$, so daß $G(z) \cdot = \sum \alpha_j(z) \, (\cdot, \, e_j(z)) \, f_j(z)$, wobei die $\alpha_j(z)$ eine zu den Approximationszahlen $\alpha_j(I^{D,X})$ äquivalente Folge bilden. Für $D = H^m(\Omega) \cap B$, $H = H^{m'}(\Omega)$ gilt

$$0 < m(z) \leqq \frac{\alpha_j(z)}{(1+j)^{-r}} \leqq M(z), \qquad j = 0, 1, \ldots,$$

für $r = (m - m') n^{-1}$, wenn $\dim \Omega = n$.

Für selbstadjungierte holomorphe Operatorfunktionen im Sinne von Kato [11], S. 385–386, findet man im Gegensatz zum allgemeinen Fall (vgl. die Bemerkung in [11], S. 371) wesentlich weitreichendere Aussagen.

Zur Motivierung des Folgenden kann Theorem 3.9 auf S. 392 von [11] dienen.

22. Theorem. *Sei $T(z)$ eine im Sinne von [11] selbstadjungierte analytische Operatorfunktion mit Werten in $\Phi(D,H)$ auf einem zur reellen Achse $\mathbb{R}$ symmetrischen Gebiet V. Ferner sei der Einbettungsoperator $I^{D,X} : D \to H$, H Hilbertraum, kompakt.*

a) *Dann kann man aus jedem kompakten Intervall J eine endliche Punktmenge S (Singularitäten) herausgreifen, so daß auf jedem kompakten (zusammenhängenden) Intervall* $J_0 \subset (V \cap \mathbb{R}) \setminus S$ *die folgende Darstellung gilt*

$$I^{D,X}\, T^{-1}(z) \cdot = G(z) \cdot = \sum_j \lambda_j(z)\, (\cdot, e_j(z))_H\, e_j(z)\,,$$

wobei auf Umgebungen $U_j(J_0)$ *die* $\lambda_j(z)$ *skalare holomorphe Funktionen, und zwar die Eigenwerte von* $G(z)$ *zu den Eigenvektoren* $e_j(z)$ *sind, die ebenfalls auf* $U_j(J_0)$ *analytisch von z abhängen und ein vollständiges Orthonormalsystem bilden.*

b) *Für eine geeignete von z abhängige Umordnung der Indizes gilt*

$$0 < m(z) \leqq \frac{|\lambda_k(z)|}{\alpha_k(I^{D,H})} \leqq M(z)$$

für $k = 0, 1, 2, \ldots$. *Ist* $I^{D,H} \in \mathcal{J}^r(D, H)$ *(z. B. elliptische Operatoren auf Sobolevräumen), so gilt* $\sum_j |\lambda_j(z)|^p < \infty$ *für* $p > r^{-1}$ *bzw. die obigen Ungleichungen für* $\alpha_k = (1 + k)^{-r}$.

Mit Hilfe der Nuklearität des Raumes $\mathcal{H}(V)$ der auf einem Gebiet V skalaren holomorphen Funktionen wird nun gezeigt, daß man den Greenschen Operator $G(z)$, $z \in V$, einer analytischen Operatorfunktion elliptischer Differentialoperatoren als Element projektiver topologischer (auch nicht lokalkonvexer) Tensorprodukte auffassen kann ([8], II, S. 87; [7], 1.5, 2.5), womit sich eine Analogie zur Entwicklung der Resolvente ergibt.

23. Theorem. *Sei* $T(z)$ *eine analytische Operatorfunktion mit Werten in* $\Phi(H^m(\Omega) \cap B, L^2(\Omega))$ *auf dem Gebiet* $V \subset \mathbb{C}$, *die an einer Stelle* $z_0 \in V$ *invertierbar ist. Die Menge* $P = \{z \in V : T^{-1}(z)$ *existiert nicht*$\}$ *hat keinen Häufungspunkt in V. Es gilt*

a) *Die auf* $W = V \setminus P$ *existierende Operatorfunktion* $I^{D,H}\, T^{-1}(z) = G(z)$, $D = H^m(\Omega) \cap B$, $H = L^2(\Omega)$, *ist Element der projektiven topologischen Tensorprodukte* $\mathcal{J}^r(L^2(\Omega),\ L^2(\Omega))\, \hat{\otimes}_\pi\, \mathcal{H}(W)$ *für* $r = m\, n^{-1}$, $n = \dim \Omega$, *und* $l^p(L^2, L^2)\, \hat{\otimes}_\pi\, \mathcal{H}(W)$ *für* $p > r^{-1}$; *dabei ist* $\mathcal{H}(W)$ *nuklearer F-Raum* ([8], *II*, *S.* 87).

b) $G(z)$ *hat folgende Reihendarstellung*

$$G(z) = \sum_k \lambda_k\, G_k \otimes f_k(z)\,,$$

wobei die Folgen $\{G_k\}$ *bzw.* $\{f_k\}$ *beschränkte Mengen in dem vollständigen lokalbeschränkten Raum* $\mathcal{J}^r(L^2, L^2)$ *(oder* $l^p(L^2, L^2)$*) bzw. in dem F-Raum* $\mathcal{H}(W)$ *sind. Die Folge* $\lambda_k \in \mathbb{C}$ *kann positiv und schnell fallend gewählt werden.*

24. Bemerkung. Da die Operatoren von endlichem Rang in $l^p(L^2, L^2)$ dicht liegen, können für $G(z) \in l^p(L^2, L^2)\, \hat{\otimes}\, \mathcal{H}(W)$ die G_k als Operatoren von endlichem Rang gewählt werden; dann gilt noch $\sum |\lambda_k|^p < \infty$.

Beweis von 23. Aus der Nuklearität von $\mathcal{H}(W)$ erhält man für $T^{-1}(z) \in \mathcal{L}(L^2(\Omega),\ H^m \cap B)\, \hat{\otimes}_\pi\, \mathcal{H}(W)$ nach [8], II, S. 87, die Darstellung

$T^{-1}(z)=\sum \lambda_k T_k \otimes f_k(z)$, wobei $\{\lambda_k\}$ schnell fallend und $\{T_k\}$ bzw. $\{f_k(z)\}$ beschränkte Folgen in $\mathscr{L}(L^2(\Omega), H^m \cap B)$ bzw. in $\mathscr{H}(W)$ sind. Der Einbettungsoperator $I': H^m \to L^2(\Omega)$ führt die Folge T_k in eine beschränkte Folge $G_k = I' T_k$ über $(\alpha_j(I' T_k) \leqq \alpha_j(I') \| T_k \|)$, denn $\mathscr{J}^r(L^2, L^2)$ bzw. $l^p(L^2, L^2)$ sind lokalbeschränkte Räume.

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On the Extreme Points of the Sum of Two Compact Convex Sets

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1. Introduction

If X and Y are non-empty compact convex subsets of a separated (or Hausdorff) locally convex space E, then $X \times Y$ is a set of the same type in $E \times E$, and $\mathrm{Ext}(X \times Y)$, the set of extremal points of $X \times Y$, is easily verified to be $\mathrm{Ext}(X) \times \mathrm{Ext}(Y)$. Since the addition mapping: $(x, y) \to x+y$ of $E \times E$ onto E is continuous and linear, it follows that $X+Y=\{x+y: x \in X, y \in Y\}$ is a non-empty compact convex subset of E, and $\mathrm{Ext}(X+Y) \subsetneqq \mathrm{Ext}(X)+\mathrm{Ext}(Y)$ [6, § 25.1 (9)]. In general, the inclusion is strict, but the following problem still arises:

Given $x \in \mathrm{Ext}(X)$, does there exist $y \in \mathrm{Ext}(Y)$ such that $x+y \in \mathrm{Ext}(X+Y)$?

The purpose of the present note is to investigate this question, and apply it to a problem on vector-valued measures.

Given non-empty compact convex subsets X and Y of a separated locally convex space E, the set $\mathrm{Ext}_Y(X)$ is defined by

$$\mathrm{Ext}_Y(X)=\{x: x \in \mathrm{Ext}(X), \exists y \in \mathrm{Ext}(Y) \quad \text{such that} \quad x+y \in \mathrm{Ext}(X+Y)\}.$$

It is shown in § 2 that $\mathrm{Ext}_Y(X)$ and $\mathrm{Ext}_X(Y)$ are always dense subsets of $\mathrm{Ext}(X)$ and $\mathrm{Ext}(Y)$ respectively (Theorem 1), while if at least one of the sets is finite-dimensional, $\mathrm{Ext}_Y(X)=\mathrm{Ext}(X)$ and $\mathrm{Ext}_X(Y)=\mathrm{Ext}(Y)$ (Theorem 2).

However examples are given in § 3 to show that in general, one or both of $\mathrm{Ext}_Y(X)$ and $\mathrm{Ext}_X(Y)$ may be strictly contained in the corresponding set of extremal points. Finally in § 4 an application is made to the theory of finite-dimensional vector-valued measures.

In general the topological vector space notation of [8] is followed, and the letter P is used to denote the set of positive integers. It is also convenient to observe at this stage that a complex locally convex space always has an associated real locally convex space, obtained by restricting scalar multilication to real scalars. The topological structures are the same, and since convexity depends only on real scalars, it is often sufficient, as in Theorem 2, to give proofs only for the real case.

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2. The Main Results

The following cancellation law is basic to the proof of Theorem 1. It was originally given in [9, Lemma 2] for the case of a real normed space, but the proof is easily seen to be valid under the present conditions.

Lemma 1. *Let X and Y be closed convex subsets of a separated locally convex space E, and suppose that there is a bounded subset Z of E such that $X+Z=Y+Z$. Then $X=Y$.*

Theorem 1. *Let X and Y be non-empty compact convex subsets of a separated locally convex space E. Then $\mathrm{Ext}_Y(X)$ and $\mathrm{Ext}_X(Y)$ are dense subsets of $\mathrm{Ext}(X)$ and $\mathrm{Ext}(Y)$ respectively.*

Proof. Let Z be the closed convex envelope of $\mathrm{Ext}_Y(X)$. Then $\mathrm{Ext}(X+Y) \subseteqq Z+Y \subseteqq X+Y$ so that by the Krein-Milman Theorem [8, p. 138, Theorem 1], $X+Y \subseteqq Z+Y \subseteqq X+Y$, i.e., $Z+Y=X+Y$. Hence, by Lemma 1, $X=Z$. It now follows that $\mathrm{Ext}(X)=\mathrm{Ext}(Z) \subseteqq \mathrm{cl}\, \mathrm{Ext}_Y(X)$, [6, § 25.1 (6)] i.e., $\mathrm{Ext}_Y(X)$ is a dense subset of $\mathrm{Ext}(X)$. Similarly, $\mathrm{Ext}_X(Y)$ is a dense subset of $\mathrm{Ext}(Y)$.

The following is now immediate:

Corollary. *If x is an isolated point of $\mathrm{Ext}(X)$, then $x \in \mathrm{Ext}_Y(X)$ for every non-empty compact convex subset Y of E. Further, if $\mathrm{Ext}(X)$ is topologically discrete, $\mathrm{Ext}_Y(X)=\mathrm{Ext}(X)$ for any such Y.*

Proposition 1 below gives a simple but far-reaching characterization of the extremal points of the sum of two non-empty compact convex sets. The first part of it was given originally by Minkowski [7, p. 181] for certain convex subsets of 3-dimensional space, and later by Fujiwara [2] for convex bodies in n-dimensional space. Minkowski's proof depends on support functions, while that of Fujiwara is of a geometric character, and, apart from the formalism of modern language, it is essentially the same as that given here.

Proposition 1. *Let X and Y be non-empty compact convex subsets of a a separated locally convex space. If $z \in \mathrm{Ext}(X+Y)$, there exist a unique $x \in X$ and a unique $y \in Y$ such that $z=x+y$. Further $x \in \mathrm{Ext}(X)$ and $y \in \mathrm{Ext}(Y)$. Conversely, if an element z of $X+Y$ has a unique representation in the form $z=x+y, x \in X$ and $y \in Y$, and $x \in \mathrm{Ext}(X)$ and $y \in \mathrm{Ext}(Y)$, then $z \in \mathrm{Ext}(X+Y)$.*

Proof. Suppose $z=x+y=x_1+y_1$, where $x, x_1 \in X$, $y, y_1 \in Y$, $x \neq x_1$ and $y \neq y_1$. Then $z=\frac{1}{2}(x+y_1)+\frac{1}{2}(x_1+y)$ and $x+y_1 \neq x_1+y$, for otherwise $x-y=x_1-y_1$, which, combined with $x+y=x_1+y_1$, implies that $x=x_1$ and $y=y_1$. Thus z is not extremal in $X+Y$. Hence if $z \in \mathrm{Ext}(X+Y)$, there exist a unique $x \in X$ and a unique $y \in Y$ such that $z=x+y$. As observed in § 1, $x \in \mathrm{Ext}(X)$ and $y \in \mathrm{Ext}(Y)$.

Now suppose that $z=x+y$, where $x \in X$ and $y \in Y$ are uniquely determined, and $x \in \mathrm{Ext}(X)$ and $y \in \mathrm{Ext}(Y)$. If $z \notin \mathrm{Ext}(X+Y)$, there exist $x_1, x_2 \in X, y_1, y_2 \in Y$ such that $x_1+y_1 \neq x_2+y_2$ and $z=\frac{1}{2}(x_1+y_1)+\frac{1}{2}(x_2+y_2)$. Then $z=\frac{1}{2}(x_1+x_2)+\frac{1}{2}(y_1+y_2)$, which implies that $x=\frac{1}{2}(x_1+x_2)$ and $y=\frac{1}{2}(y_1+y_2)$, since the representation is unique. Since $x \in \mathrm{Ext}(X)$ and $y \in \mathrm{Ext}(Y)$, it then follows that

$x_1 = x_2 = x$ and $y_1 = y_2 = y$. Thus $x_1 + y_1 = x_2 + y_2$, which gives a contradiction. Hence $z \in \mathrm{Ext}(X + Y)$.

Corollary. *If* $\mathrm{Ext}(X + Y)$ *is closed, so also are* $\mathrm{Ext}(X)$ *and* $\mathrm{Ext}(Y)$, *and* $\mathrm{Ext}_Y(X) = \mathrm{Ext}(X)$ *and* $\mathrm{Ext}_X(Y) = \mathrm{Ext}(Y)$.

Proof. Consider the mapping $t: X \times Y \to X + Y$ defined by $t((x, y)) = x + y$. Since t is continuous, $t^{-1}(\mathrm{Ext}(X + Y))$ is closed in $X \times Y$ and therefore compact. Let p_X denote the projection of $X \times Y$ onto X. By Proposition 1, $\mathrm{Ext}_Y(X) = p_X(t^{-1}(\mathrm{Ext}(X + Y)))$ and hence is compact and therefore closed. But by Theorem 1, $\mathrm{Ext}_Y(X)$ is dense in $\mathrm{Ext}(X)$. Thus $\mathrm{Ext}_Y(X) = \mathrm{Ext}(X)$. Similarly, $\mathrm{Ext}_X(Y) = \mathrm{Ext}(Y)$.

The remaining subsidiary results of this section are directed towards the proof of Theorem 2.

Lemma 2. *Let* X *be a non-empty compact convex subset of a separated locally convex space* E, *and let* $x' \in E'$ *(the dual of* E*). If* α *is an extremal point of* $x'(X)$, *then*

$$\mathrm{Ext}(X \cap x'^{-1}(\{\alpha\})) \subseteq \mathrm{Ext}(X) .$$

Proof. Let $x \in \mathrm{Ext}(X \cap x'^{-1}(\{\alpha\}))$, and suppose that $x = \frac{1}{2}x_1 + \frac{1}{2}x_2$, $x_1, x_2 \in X$. Then

$$\alpha = \langle x, x' \rangle = \tfrac{1}{2}\langle x_1, x' \rangle + \tfrac{1}{2}\langle x_2, x' \rangle ,$$

which implies that $\langle x_1, x' \rangle = \langle x_2, x' \rangle = \alpha$, since $\alpha \in \mathrm{Ext}(x'(X))$ i.e., $x_1, x_2 \in X \cap x'^{-1}(\{\alpha\})$. Finally $x_1 = x_2$, since x is extremal in this set.

Lemma 3. *Let* X *and* Y *be non-empty compact convex subsets of a separated locally convex space* E. *Let* $x' \in E'$, *and let* α *and* β *be extremal points of* $x'(X)$ *and* $x'(Y)$ *respectively such that* $\alpha + \beta$ *is an extremal point of* $x'(X + Y)$. *Then, if*

$$X' = \{z : z \in X, \langle z, x' \rangle = \alpha\} ,$$

$$Y' = \{z : z \in Y, \langle z, x' \rangle = \beta\} ,$$

and

$$Z' = \{z \in X + Y, \langle z, x' \rangle = \alpha + \beta\} ,$$

$$Z' = X' + Y' .$$

Proof. By Proposition 1, $\alpha + \beta$ is representable in one and only one way as the sum of an element of $x'(X)$ and an element of $x'(Y)$. Thus if $x \in X'$ and $y \in Y'$, $x + y \in Z'$ if and only if $\langle x, x' \rangle = \alpha$ and $\langle y, x' \rangle = \beta$. This establishes the result.

Corollary. *Let* X *and* Y *be non-empty compact convex subsets of a real separated locally convex space* E. *Let* $x' \in E'$ *and let* X', Y' *and* Z' *be respectively the subsets of* X, Y *and* $X + Y$ *at which* x' *attains its suprema on these sets. Then* $Z' = X' + Y'$.

Proof. It is easily verified that

$$\sup_{z\in X+Y}\langle z, x'\rangle = \sup_{z\in X}\langle z, x'\rangle + \sup_{z\in Y}\langle z, x'\rangle .$$

The result then follows immediately from Lemma 3.

A subset of a vector space is said to have finite dimension if it generates a finite-dimensional vector subspace. However, to associate a translation invariant dimension with an arbitrary subset, one has to consider not the vector subspace, but the linear variety generated by the set, and the dimension of the set is then defined to be that of the generated linear variety [1, §9, N° 3].

Proposition 2. *Let X be a non-empty compact convex subset of a real separated locally convex space E, and let $x\in \operatorname{Ext}(X)$. If X has non-zero finite dimension, there exists $x'\in E'$, $x'\neq 0$, such that*

$$\langle x, x'\rangle = \sup_{z\in X}\langle z, x'\rangle ,$$

and the dimension of

$$X\cap\{z\colon \langle z, x'\rangle = \langle x, x'\rangle\}$$

is strictly less than that of X.

Proof. Suppose that X has dimension $n\geqq 1$. Then the vector subspace F generated by $-x+X$ has dimension n, and $0\in\operatorname{Ext}(-x+X)$. Now there is a linear homeomorphism t of F onto R_n and $0\in\operatorname{Ext}(t(-x+X))$. Hence there exists $y'\in R_n'$, $y'\neq 0$, such that

$$\langle y, y'\rangle \leqq 0 = \langle 0, y'\rangle \quad \text{for all} \quad y\in t(-x+X)$$

[3, p. 12, §4], i.e.,

$$\langle t(-x+z), y'\rangle \leqq 0 \quad \text{for all} \quad z\in X ,$$

or

$$\langle -x+z, t'(y')\rangle \leqq 0 \quad \text{for all} \quad z\in X, \text{ where } t'\colon R_n'\to F'$$

denotes the transpose of t. By the Hahn-Banach Theorem, $t'(y')$ can be extended to an element $x'\in E'$ [8, p. 29, Corollary 1]. Then

$$\langle -x+z, x'\rangle = \langle -x+z, t'(y')\rangle \leqq 0 \quad \text{for all} \quad z\in X ,$$

i.e.,

$$\langle z, x'\rangle \leqq \langle x, x'\rangle \quad \text{for all} \quad z\in X \quad \text{or} \quad \sup_{z\in X}\langle z, x'\rangle = \langle x, x'\rangle .$$

For the final part,

$$\begin{aligned} X\cap\{z\colon \langle z, x'\rangle = \langle x, x'\rangle\} &= x + [(-x+X)\cap\{z\colon \langle z, x'\rangle = 0\}] \\ &= x + [(-x+X)\cap\{z\colon z\in F, \langle z, t'(y')\rangle = 0\}] , \end{aligned}$$

and $\{z\colon z\in F, \langle z, t'(y')\rangle = 0\}$, being a hyperplane in the n-dimensional space F, has dimension $n-1$. It now follows that $X\cap\{z\colon \langle z, x'\rangle = \langle x, x'\rangle\}$ has dimension at most $n-1$.

Theorem 2. *Let X and Y be non-empty compact convex subsets of a separated locally convex space E, at least one of which has finite dimension. Then* $\mathrm{Ext}_Y(X) = \mathrm{Ext}(X)$ *and* $\mathrm{Ext}_X(Y) = \mathrm{Ext}(Y)$.

Proof. A finite-dimensional subset of a complex vector space clearly remains finite-dimensional in the associated real space. Hence, as observed in § 1, it will be enough to establish the result in the case in which E is a real vector space.

Suppose then that E is a real vector space and that X has finite dimension n. It is shown first by induction on the dimension of X that $\mathrm{Ext}_Y(X) = \mathrm{Ext}(X)$. If $n=0$, X consists of a single point x say, and it is then immediate that for every non-empty compact convex subset Y' of E,

$$\mathrm{Ext}(X+Y') = \mathrm{Ext}(x+Y') = x + \mathrm{Ext}(Y').$$

Thus $\mathrm{Ext}_{Y'}(X) = \mathrm{Ext}(X) = \{x\}$ (alternatively apply the corollary of Theorem 1).

Suppose now that it has been established that, for each non-empty compact convex subset X' of dimension at most s, $\mathrm{Ext}_{Y'}(X') = \mathrm{Ext}(X')$ for every non-empty compact convex subset Y' of E. Let X have dimension $s+1$, and let $x \in \mathrm{Ext}(X)$. Then by Proposition 2, there exists $x' \in E'$ such that $\langle x, x' \rangle = \sup_{z \in X} \langle z, x' \rangle$ and $X' = \{z : z \in X, \langle z, x' \rangle = \langle x, x' \rangle\}$ has dimension at most s. Let $Y' = \{z : z \in Y, \langle z, x' \rangle = \sup_{y \in Y} \langle y, x' \rangle\}$. Now $x \in \mathrm{Ext}(X')$ and hence, by the induction hypothesis, there exists $y \in \mathrm{Ext}(Y') \subseteqq \mathrm{Ext}(Y)$ (Lemma 2), such that $x + y \in \mathrm{Ext}(X' + Y')$. By the corollary to Lemma 3, it then follows that $x + y \in \mathrm{Ext}(X+Y)$, therefore $\mathrm{Ext}_Y(X) = \mathrm{Ext}(X)$. The first part now follows by induction.

Now consider Y, and suppose that there exists $y \in \mathrm{Ext}(Y)$ such that for all $x \in \mathrm{Ext}(X)$, $x + y \notin \mathrm{Ext}(X+Y)$. By Proposition 1, for each $x \in \mathrm{Ext}(X)$ there exist $w_x \in X$ and $z_x \in Y$ such that $w_x \neq x$, $z_x \neq y$ and $x + y = w_x + z_x$. Observe that X generates a finite-dimensional vector subspace of E. Also $\{w_x : x \in \mathrm{Ext}(X)\} \subseteqq X$ and $\{w_x + z_x : x \in \mathrm{Ext}(X)\} = \{x + y : x \in \mathrm{Ext}(X)\} = y + \mathrm{Ext}(X) \subseteqq y + X$. Hence $\{w_x : x \in \mathrm{Ext}(X)\}$ and $\{w_x + z_x : x \in \mathrm{Ext}(X)\}$ generate finite-dimensional vector subspace F_1 and F_2 say, respectively. Then $\{z_x : x \in \mathrm{Ext}(X)\} \subseteqq F_1 + F_2$, so that $\{z_x : x \in \mathrm{Ext}(X)\}$ also generates a finite-dimensional vector subspace.

Now let F be the (finite-dimensional) vector subspace generated by $\{z_x : x \in \mathrm{Ext}(X)\} \cup \{y\} \cup X$. Then X and $Y \cap F$ are non-empty compact convex subsets of F, $y \in \mathrm{Ext}(Y \cap F)$ and, by construction, for every $x \in \mathrm{Ext}(X)$, $x + y \notin \mathrm{Ext}(X + (Y \cap F))$. But $Y \cap F$ has finite dimension, and so by the first part, there exists $x \in \mathrm{Ext}(X)$ such that $x + y \in \mathrm{Ext}(X + (Y \cap F))$, which gives a contradiction. Thus $\mathrm{Ext}_X(Y) = \mathrm{Ext}(Y)$.

In particular the conditions of Theorem 2 are clearly satisfied if E itself is finite-dimensional. Thus:

Corollary 1. *Let X and Y be non-empty compact convex subsets of a finite-dimensional space E. Then* $\mathrm{Ext}_Y(X) = \mathrm{Ext}(X)$ *and* $\mathrm{Ext}_X(Y) = \mathrm{Ext}(Y)$.

Corollary 2. *Let X be a non-empty compact convex subset of a separated locally convex space E. Let $x \in \operatorname{Ext}(X)$, and suppose that there exists a closed hyperplane H which contains x and which is a support of X* [5, Sec. 15], *such that $X' = X \cap H$ is finite-dimensional. Then $x \in \operatorname{Ext}_Y(X)$ for every non-empty compact convex subset Y of E.*

Proof. Let K be the scalar field of E. There exist $x' \in E'$ and $\alpha \in K$ such that $H = \{z : \langle z, x' \rangle = \alpha\}$. It follows that α is an extremal point of $x'(X)$, for if $z_1, z_2 \in X$ and $\alpha = \langle x, x' \rangle = \frac{1}{2}\langle z_1, x' \rangle + \frac{1}{2}\langle z_2, x' \rangle$, $\alpha = \langle \frac{1}{2}(z_1 + z_2), x' \rangle$, which implies that $\frac{1}{2}(z_1 + z_2) \in X \cap H$. Since H is a support of X, $z_1, z_2 \in X \cap H$ so that $\langle z_1, x' \rangle = \langle z_2, x' \rangle = \alpha$.

Now let Y be any non-empty compact convex subset of E. Since K is finite-dimensional, by Corollary 1, there exists an extremal point β of $x'(Y)$ such that $\alpha + \beta$ is an extremal point of $x'(X + Y)$. Let $Y' = \{z : z \in Y, \langle z, x' \rangle = \beta\}$. Now by Theorem 2, since $x \in \operatorname{Ext}(X')$, there exists $y \in \operatorname{Ext}(Y')$ such that $x + y \in \operatorname{Ext}(X' + Y')$. But, by Lemma 3, $X' + Y' = \{z : z \in X + Y, \langle z, x' \rangle = \alpha + \beta\}$, and therefore, by Lemma 2, $x + y \in \operatorname{Ext}(X + Y)$ and $y \in \operatorname{Ext}(Y)$. Hence $x \in \operatorname{Ext}_Y(X)$.

In the complex case of Corollary 2, the hyperplane H is assumed to be complex, and therefore H is not a hyperplane of support in the usual sense [6, § 17.5]. However, it is easy to see that the result of Corollary 2 remains valid in the case of a (real) hyperplane of support (consider the associated real vector space).

If a subset of a separated locally convex space has a closed hyperplane of support which intersects it in exactly one point, that point is called an exposed point of the set. It is clear that for a compact convex subset X, the set $\operatorname{Exp}(X)$ of exposed points is contained in $\operatorname{Ext}(X)$. It follows immediately from Corollary 2 and the above discussion that:

Corollary 3. *For every pair X and Y of non-empty compact convex subsets of a separated locally convex space,* $\operatorname{Exp}(X) \subseteq \operatorname{Ext}_Y(X)$ *and* $\operatorname{Exp}(Y) \subseteq \operatorname{Ext}_X(Y)$.

3. Examples

In this section, examples are given to show that in general, denseness cannot be replaced by equality in Theorem 1. Precisely, it is shown that both possibilities

(i) $\operatorname{Ext}_Y(X) \neq \operatorname{Ext}(X)$, $\operatorname{Ext}_X(Y) = \operatorname{Ext}(Y)$,

(ii) $\operatorname{Ext}_Y(X) \neq \operatorname{Ext}(X)$, $\operatorname{Ext}_X(Y) \neq \operatorname{Ext}(Y)$

can occur. The construction of these examples depends on Proposition 1 and Proposition 3 below. The proof of Proposition 3 is essentially the same as that of [6, § 20.9 (6)]. The relevant details of the proof are indicated.

Proposition 3. *Let E be a separated locally convex space which is $\tau(E, E')$-sequentially complete. Let $x_n \to 0$ under $\sigma(E, E')$. Then the closed convex envelope C of $\{x_n : n \in P\}$ is $\sigma(E, E')$-compact, and* $C = \left\{\sum_{n=1}^{\infty} \xi_n x_n : \sum_{n=1}^{\infty} \xi_n \leqq 1, \xi_n \geqq 0, n \in P\right\}$.

Proof. Let K be the scalar field of E. It is shown in [6, § 20.9 (6)] that

(i) if $\xi_n \in K, n \in P$, and $\sum_{n=1}^{\infty} |\xi_n| < \infty$, then the series $\sum \xi_n x_n$ is $\tau(E, E')$-convergent, and

(ii) if l_1 (respectively c_0) denotes the space of sequences (ξ_n) such that $\xi_n \in K, n \in P$, and $\sum_{n=1}^{\infty} |\xi_n| < \infty$ (respectively $\xi_n \to 0$ as $n \to \infty$), the linear mapping $t: l_1 \to E$ defined by $t((\xi_n)) = \sum_{n=1}^{\infty} \xi_n x_n$ is continuous under $\sigma(l_1, c_0)$ and $\sigma(E, E')$.

Now if

$$A = \left\{(\xi_n) \colon \sum_{n=1}^{\infty} \xi_n \leqq 1, \xi_n \geqq 0, n \in P\right\}$$

and

$$B = \left\{(\xi_n) \colon \sum_{n=1}^{\infty} \xi_n \leqq 1, \xi_n \geqq 0, n \in P, \text{ only finitely many } \xi_n \neq 0\right\},$$

A is $\sigma(l_1, c_0)$-compact and B is $\sigma(l_1, c_0)$-dense in A. Let D be the convex envelope of $\{x_n : n \in P\} \cup \{0\}$. Then if $x \in D, x = \lambda 0 + \sum_{i=1}^{m} \lambda_i x_{n(i)}$ for some $\lambda, \lambda_i, x_{n(i)}$ and positive integer m such that $\lambda + \sum_{i=1}^{m} \lambda_i = 1$ and $\lambda, \lambda_i \geqq 0, 1 \leqq i \leqq m$.

$$\text{Put } \xi_n = \begin{cases} \lambda_i & n = n(i)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then $(\xi_n) \in B$ and $t((\xi_n)) = x$. Conversely, if $(\xi_n) \in B, t((\xi_n)) = \sum_{n=1}^{\infty} \xi_n x_n = \left(1 - \sum_{n=1}^{\infty} \xi_n\right) 0 + \sum_{n=1}^{\infty} \xi_n x_n \in D$. Thus $D = t(B)$.

Finally, $\{x_n : n \in P\}$ and $\{x_n : n \in P\} \cup \{0\}$ have the same closed convex envelope C and,

$$t(B) \subseteqq t(A) = t(\operatorname{cl} B) \subseteqq \operatorname{cl} t(B)\ (= C) \subseteqq \operatorname{cl} t(A) = t(A)\,,$$

since $t(A)$ is $\sigma(E, E')$-compact and therefore closed. Thus $t(A) = C$, which establishes the result.

Now let E be a real infinite-dimensional Hilbert space, and let $\{e_n : n \in P\}$ be a countable orthonormal family in E. The sequence

$$-\sum_{n=1}^{\infty} \frac{1}{n} e_n, e_1, \frac{1}{2} e_2, \ldots, \frac{1}{n} e_n, \ldots$$

converges to zero in the norm topology of E. Let X be the closed convex envelope of the set of terms of this sequence. Since E is complete and the sequence converges in the norm topology, X is compact for this topology.

It is now shown that $0 \in \mathrm{Ext}(X)$. Suppose $0 = \frac{1}{2}x_1 + \frac{1}{2}x_2$, $x_1, x_2 \in X$. Then by Proposition 3,

$$0 = \frac{1}{2}\left\{\xi_1\left(-\sum_{n=1}^{\infty} \frac{1}{n} e_n\right) + \sum_{n=1}^{\infty} \xi_{1n} \frac{1}{n} e_n\right\} + \frac{1}{2}\left\{\xi_2\left(-\sum_{n=1}^{\infty} \frac{1}{n} e_n\right) + \sum_{n=1}^{\infty} \xi_{2n} \frac{1}{n} e_n\right\}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \{(\xi_{1n} + \xi_{2n}) - (\xi_1 + \xi_2)\} e_n,$$

where

$$\xi_i + \sum_{n=1}^{\infty} \xi_{in} \leqq 1, \xi_i, \xi_{in} \geqq 0, i = 1,2, n \in P.$$

Thus $\xi_{1n} + \xi_{2n} = \xi_1 + \xi_2$ for all $n \in P$. But $\sum(\xi_{1n} + \xi_{2n})$ converges, so that $\xi_{1n} + \xi_{2n} \to 0$ as $n \to \infty$. Thus $\xi_{1n} + \xi_{2n} = \xi_1 + \xi_2 = 0$ for all $n \in P$, and since all terms are non-negative, this implies that $\xi_{1n} = \xi_{2n} = \xi_1 = \xi_2 = 0$ for all $n \in P$ i.e., $x_1 = x_2 = 0$.

The sequence

$$\sum_{n=1}^{\infty} \frac{1}{n} e_n, \sum_{n=2}^{\infty} \frac{1}{n} e_n, \ldots, \sum_{n=m}^{\infty} \frac{1}{n} e_n, \ldots$$

also converges to zero in the norm topology and, as above, if Y is the closed convex envelope of the set of terms of the sequence, Y is compact in the norm topology. Now

$$\mathrm{Ext}(Y) \subsetneqq A = \left\{\sum_{n=m}^{\infty} \frac{1}{n} e_n : m \in P\right\} \cup \{0\} \qquad [6, \S\, 25.1\ (6)]$$

(actually, as can be verified by using Proposition 3, $\mathrm{Ext}(Y) = A$). It is now shown that no element of $A = 0 + A$ is an element of $\mathrm{Ext}(X + Y)$. For this it is sufficient, by Proposition 1, to show that each element of A can be expressed in two distinct ways as the sum of an element of X and an element of Y. In the following, the first element belongs to X and the second belongs to Y in each case.

$$\sum_{n=m}^{\infty} \frac{1}{n} e_n = 0 + \sum_{n=m}^{\infty} \frac{1}{n} e_n = \frac{1}{m} e_m + \sum_{n=m+1}^{\infty} \frac{1}{n} e_n, m \in P,$$

$$0 = 0 + 0 = \left(-\sum_{n=1}^{\infty} \frac{1}{n} e_n\right) + \sum_{n=1}^{\infty} \frac{1}{n} e_n.$$

Thus $\mathrm{Ext}_Y(X) \neq \mathrm{Ext}(X)$. However, by the corollary to Theorem 1 and Corollary 3 of Theorem 2, $\mathrm{Ext}_X(Y) = \mathrm{Ext}(Y)$, for $\sum_{n=m}^{\infty} \frac{1}{n} e_n$ is clearly an isolated point of $\mathrm{Ext}(Y)$ for each $m \in P$, and 0 is an exposed point of Y. To see that $0 \in \mathrm{Exp}(Y)$, choose $x \in E$ such that the scalar product $(x \mid e_n) > 0$ for all n. Then $\left(\sum_{n=m}^{\infty} \frac{1}{n} e_n \mid x\right) > 0$ for all $m \in P$ and $(0 \mid x) = 0$. Now every extremal point of $\{z : (z \mid x) = 0\} \cap Y$ is an extremal point of Y (Lemma 2). Thus by the Krein-Milman Theorem, $\{z : (z \mid x) = 0\} \cap Y = \{0\}$, and $(z \mid x) \geqq 0$ for all $z \in Y$.

Now let $W = X \times Y$ and $Z = Y \times X$. Then W and Z are non-empty compact convex subsets of $E \times E$ and

$\operatorname{Ext}(W) = \operatorname{Ext}(X) \times \operatorname{Ext}(Y)$ and $\operatorname{Ext}(Z) = \operatorname{Ext}(Y) \times \operatorname{Ext}(X)$. Also $W + Z = (X+Y) \times (X+Y)$, so that $\operatorname{Ext}(W+Z) = \operatorname{Ext}(X+Y) \times \operatorname{Ext}(X+Y)$. Now $(0,0) \in \operatorname{Ext}(W)$. Thus if there existed $(y,x) \in \operatorname{Ext}(Z)$ such that $(0,0)+(y,x) = (y,x) \in \operatorname{Ext}(W+Z)$, $y = 0 + y$ would belong to $\operatorname{Ext}(X+Y)$, which is impossible by the first part. $(0,0)$ is also an element of $\operatorname{Ext}(Z)$, and a similar argument shows that there is no $(x,y) \in \operatorname{Ext}(W)$ such that $(0,0)+(x,y) = (x,y) \in \operatorname{Ext}(W+Z)$. In this case $\operatorname{Ext}_Z(W) \neq \operatorname{Ext} W$ and $\operatorname{Ext}_W(Z) \neq \operatorname{Ext}(Z)$.

Finally it should be observed that even for infinite-dimensional compact convex sets it can still happen that $\operatorname{Ext}_Y(X) = \operatorname{Ext}(X)$ and $\operatorname{Ext}_X(Y) = \operatorname{Ext}(Y)$, e.g., if $X = Y$, $X + Y = 2X$ and $\operatorname{Ext}(X+Y) = 2\operatorname{Ext}(X) = \{x + x : x \in \operatorname{Ext}(X)\}$.

4. An Application

Let S be a set, $\mathscr{M}$ a σ-algebra of subsets of S and let μ_i, $1 \leqq i \leqq n$, be finite non-atomic signed measures on $(S, \mathscr{M})$. Denote by $\boldsymbol{m}$ the vector-valued measure on $(S, \mathscr{M})$ with values in the n-dimensional Euclidean space R_n defined by

$$m(X) = (\mu_1(X), \mu_2(X), \ldots, \mu_n(X)) \quad \text{for all} \quad X \in \mathscr{M},$$

and for each $X \in \mathscr{M}$, let

$$R(X) = \{\boldsymbol{m}(Y) \colon Y \in \mathscr{M}, Y \subseteqq X\}.$$

By Lyapunov's convexity theorem (see e.g. [4]), $R(X)$ is a compact convex subset of R_n for each $X \in \mathscr{M}$.

We give a characterization of extreme points of the range of a measure as follows:

Proposition 4. *In the above notation, for each* $X \in \mathscr{M}$,

$$\operatorname{Ext}(R(X)) = \{\boldsymbol{m}(Y \cap X) \colon Y \in \mathscr{M}, \boldsymbol{m}(Y) \in \operatorname{Ext}(R(S))\}.$$

Proof. Clearly $R(S) = R(X) + R(\complement X)$. Hence if $\boldsymbol{m}(Z) \in \operatorname{Ext}(R(X))$ where $Z \in \mathscr{M}$ and $Z \subseteqq X$, there exists, by Corollary 1 of Theorem 2, $W \in \mathscr{M}$, $W \subseteqq CX$ such that $\boldsymbol{m}(W) \in \operatorname{Ext}(R(CX))$ and $\boldsymbol{m}(Z) + \boldsymbol{m}(W) = \boldsymbol{m}(Z \cup W) \in \operatorname{Ext}(R(S))$. Then $\boldsymbol{m}(Z) = \boldsymbol{m}((Z \cup W) \cap X)$.

Conversely, if $\boldsymbol{m}(Y) \in \operatorname{Ext}(R(S))$, suppose that there exist $W, Z \subseteqq X$, $W, Z \in \mathscr{M}$ such that

$$\boldsymbol{m}(Y \cap X) = \tfrac{1}{2}\boldsymbol{m}(W) + \tfrac{1}{2}\boldsymbol{m}(Z).$$

$$\begin{aligned} \text{Then } \boldsymbol{m}(y) &= \tfrac{1}{2}\boldsymbol{m}(W) + \tfrac{1}{2}\boldsymbol{m}(Z) + \boldsymbol{m}(Y \setminus X) \\ &= \tfrac{1}{2}\{\boldsymbol{m}(W) + \boldsymbol{m}(Y \setminus X)\} + \tfrac{1}{2}\{\boldsymbol{m}(Z) + \boldsymbol{m}(Y \setminus X)\} \\ &= \tfrac{1}{2}\boldsymbol{m}(W \cup (Y \setminus X)) + \tfrac{1}{2}\boldsymbol{m}(Z \cup (Y \setminus X)). \end{aligned}$$

Since $\boldsymbol{m}(Y)$ is an extreme point of $R(S)$, $\boldsymbol{m}(W \cup (Y \setminus X)) = \boldsymbol{m}(Z \cup (Y \setminus X))$, which implies that $\boldsymbol{m}(W) = \boldsymbol{m}(Z)$ so that $\boldsymbol{m}(Y \cap X) \in \operatorname{Ext}(R(X))$.

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A Note on the Mean Value of the Dedekind Zeta-Function of the Quadratic Field

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1. Introduction

Let K be the real quadratic field with the discriminante D, and let $\zeta_k(s)$ be the Dedekind zeta-function of K. Then it is well known that

$$\zeta_k(s) = \zeta(s)\, L(s, \chi) \qquad (s = \sigma + it),$$

where $\zeta(s)$ is the Riemann zeta-function and $L(s, \chi)$ is the Dirichlet L-series for the Legendre symbol χ, and further the functional equation

$$\zeta_k(1-s) = \left(\frac{2}{(2\pi)^s} \cos\frac{\pi s}{2}\, \Gamma(s)\right)^2 \zeta_k(s) \tag{$*$}$$

holds.

In the work [1] Chandrasekharan and Narasimhan have proved an upper estimate for the mean value of the square of $\zeta_k(s)$ on the critical line $\sigma = 1/2$. And also at the end of their work they conjectured an asymptotic equality.

The purpose of this short note is to prove this little conjecture. Namely we will prove the following

Theorem.

$$\int_0^T \left|\zeta_k\left(\frac{1}{2} + it\right)\right|^2 dt = (1 + 0(1)) \frac{6}{\pi^2} \prod_{p \mid D} \left(1 + \frac{1}{p}\right)^{-1} L^2(1, \chi)\, T \log^2 T.$$

Here p runs over all prime divisors of D.

2. Lemmas

Lemma 1. *Let*

$$\zeta_k(s) = \sum_{n=1}^{\infty} a(n) n^{-s} \qquad (\sigma > 1),$$

then we have

$$\sum_{n=1}^{\infty} a^2(n) n^{-s} = \prod_{p \mid D} \left(1 + \frac{1}{p^s}\right)^{-1} \frac{\zeta_k^2(s)}{\zeta(2s)}.$$

This is the analogue of the identity of Ramanujan for $r(n)$, the number of the representation of n as the sum of two squares, and the proof is similar.

Lemma 2. *For sufficiently small $\beta > 0$ we have*

$$\sum_{n=1}^{\infty} \frac{a^2(n)}{n} e^{-n\beta} = \frac{3}{\pi^2} \prod_{p|D} \left(1+\frac{1}{p}\right)^{-1} L^2(1,\chi) \log^2 \frac{1}{\beta} + O\left(\log \frac{1}{\beta}\right).$$

Proof. From Lemma 1 we can find easily that

$$\sum_{n \leqq \xi} \frac{a^2(n)}{n} = \frac{3}{\pi^2} \prod_{p|D} \left(1+\frac{1}{p}\right)^{-1} L^2(1,\chi) \log^2 \xi + O(\log \xi).$$

Hence by the partial summation we get

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{a^2(n)}{n} e^{-n\beta} &= \beta \int_1^{\infty} \sum_{n \leqq \xi} \frac{a^2(n)}{n} e^{-\beta\xi} d\xi \\ &= \frac{3}{\pi^2} \prod_{p|D} \left(1+\frac{1}{p}\right)^{-1} L^2(1,\chi) \int_1^{\infty} \beta \log^2 \xi e^{-\beta\xi} d\xi + O\left(\log \frac{1}{\beta}\right) \\ &= \frac{3}{\pi^2} \prod_{p|D} \left(1+\frac{1}{p}\right)^{-1} L^2(1,\chi) \log^2 \frac{1}{\beta} + O\left(\log \frac{1}{\beta}\right). \end{aligned}$$

Lemma 3. *We have*

$$\sum_{n=1}^{\infty} a^2(n) e^{-n\beta} = O\left(\frac{1}{\beta} \log \frac{1}{\beta}\right)$$

and

$$\sum_{n=1}^{\infty} n^2 a^2(n) e^{-n\beta} = O\left(\frac{1}{\beta^3} \log \frac{1}{\beta}\right).$$

The proof is similar as that of Lemma 2.

3. Proof of the Theorem

Now let consider the integral

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(s) \zeta_k(s) z^{-s} ds \qquad (\mathrm{Re}(z) > 0).$$

Moving the line of integration to $\sigma = \alpha$ $(0 < \alpha < 1)$ we get

$$\begin{aligned} \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \Gamma(s) \zeta_k(s) z^{-s} ds &= \sum_{n=1}^{\infty} a(n) e^{-nz} - z^{-1} L(1,\chi) \\ &= \phi(z), \text{ say}. \end{aligned}$$

Hence as in [2, p. 137] we obtain

$$\int_0^{\infty} \left|\zeta_k\left(\frac{1}{2}+it\right)\right|^2 e^{-2\delta t} dt = \int_0^{\infty} |\phi(ixe^{-i\delta})|^2 dx + O(1)$$

for sufficiently small $\delta > 0$.

Here we remark that

$$\phi\left(\frac{1}{ixe^{-i\delta}}\right)=\frac{2\pi}{\sqrt{D}}\,xe^{-i\delta}\,\phi\left(\frac{4\pi^2}{D}\,ixe^{-i\delta}\right)+O(x^{\alpha}), \tag{$**$}$$

where α may be as small as we please. This transformation-formula can be proved analogously as in [2, p. 142] by the functional equation $(*)$.

Therefore it is sufficient to consider the integral

$$\int_{2\pi/\sqrt{D}}^{\infty}|\phi(ixe^{-i\delta})|^2\,dx, \tag{$***$}$$

since from $(**)$ we get

$$\begin{aligned}\int_0^{2\pi/\sqrt{D}}|\phi(ixe^{-i\delta})|^2\,dx&=\int_{\sqrt{D}/2\pi}^{\infty}\left|\phi\left(\frac{1}{ixe^{-i\delta}}\right)\right|^2\frac{dx}{x^2}\\&=\int_{\sqrt{D}/2\pi}^{\infty}\left|\frac{2\pi}{\sqrt{D}}\,xe^{-i\delta}\,\phi\left(\frac{4\pi^2}{D}\,ixe^{-i\delta}\right)\right|^2\frac{dx}{x^2}+O\left(\frac{1}{\sqrt{\delta}}\log\frac{1}{\delta}\right)\\&=\int_{2\pi/\sqrt{D}}^{\infty}|\phi(ixe^{-i\delta})|^2\,dx+O\left(\frac{1}{\sqrt{\delta}}\log\frac{1}{\delta}\right).\end{aligned}$$

Now the integral $(***)$ is equal to

$$\begin{aligned}&\sum_{n,m=1}^{\infty}a(n)\,a(m)\int_{2\pi/\sqrt{D}}^{\infty}\exp(-inxe^{-i\delta}+imxe^{i\delta})\,dx\\&\quad+2i\,L(1,\chi)\,\mathrm{Im}\left\{ie^{i\delta}\int_{2\pi/\sqrt{D}}^{\infty}\sum_{n=1}^{\infty}a(n)\exp(inxe^{i\delta})\frac{dx}{x}\right\}+\frac{\sqrt{D}}{2\pi}L^2(1,\chi)\\&\quad=A_1(\delta)+2i\,L(1,\chi)\,\mathrm{Im}(A_2(\delta))+O(1),\quad\text{say}.\end{aligned}$$

We have

$$\begin{aligned}|A_2(\delta)|&\leqq\sum_{n=1}^{\infty}\int_{2\pi/\sqrt{D}}^{\infty}a(n)\exp(-nx\sin\delta)\frac{dx}{x}\\&=\left\{\sum_{n<\frac{\sqrt{D}}{2\pi\sin\delta}}+\sum_{n\geqq\frac{\sqrt{D}}{2\pi\sin\delta}}\right\}a(n)\int_{\frac{2\pi\sin\delta}{\sqrt{D}}n}^{\infty}\exp(-x)\frac{dx}{x}\\&=O\left\{\sum_{n<\frac{\sqrt{D}}{2\pi\sin\delta}}a(n)\int_{\frac{2\pi\sin\delta}{\sqrt{D}}n}^{1}\frac{dx}{x}\right\}+O\left\{\frac{1}{\sin\delta}\int_1^{\infty}\sum_{n\leqq\xi}\frac{a(n)}{n}\exp\left(-\frac{2\pi\sin\delta}{\sqrt{D}}\,\xi\right)d\xi\right\}\\&=O\left(\frac{1}{\delta}\log\frac{1}{\delta}\right).\end{aligned}$$

Also we have

$$A_1(\delta)=\frac{1}{2\sin\delta}\sum_{n=1}^{\infty}\frac{a^2(n)}{n}\exp\left(-\frac{4\pi}{\sqrt{D}}n\sin\delta\right)$$

$$+2\sum_{m=2}^{\infty}a(m)\sum_{n=1}^{m-1}a(n)\frac{(m+n)\sin\delta\cos\left\{\frac{2\pi}{\sqrt{D}}(m-n)\cos\delta\right\}}{(m+n)^2\sin^2\delta+(m-n)^2\cos^2\delta}\exp\left(-\frac{2\pi}{\sqrt{D}}(m+n)\sin\delta\right)$$

$$-2\sum_{m=2}^{\infty}a(m)\sum_{n=1}^{m-1}a(n)\frac{(m-n)\cos\delta\sin\left\{\frac{2\pi}{\sqrt{D}}(m-n)\cos\delta\right\}}{(m+n)^2\sin^2\delta+(m-n)^2\cos^2\delta}\exp\left(-\frac{2\pi}{\sqrt{D}}(m+n)\sin\delta\right)$$

$$=\Sigma_1+\Sigma_2-\Sigma_3, \text{ say}.$$

From Lemma 1 we find

$$\Sigma_1=\frac{3}{2\pi^2}\prod_{p|D}\left(1+\frac{1}{p}\right)^{-1}L^2(1,\chi)\frac{1}{\delta}\log^2\frac{1}{\delta}+O\left(\frac{1}{\delta}\log\frac{1}{\delta}\right).$$

As in [2, p. 145] the sums Σ_2 and Σ_3 are less than the value

$$\frac{2\sin\delta}{3\cos^2\delta}\pi^2\left\{\sum_{n=1}^{\infty}n^2a^2(n)\exp\left(-\frac{4\pi}{\sqrt{D}}n\sin\delta\right)\right\}^{\frac{1}{2}}\left\{\sum_{n=1}^{\infty}a^2(n)\exp\left(-\frac{4\pi}{\sqrt{D}}n\sin\delta\right)\right\}^{\frac{1}{2}}$$

$$=O\left(\delta\left(\frac{1}{\delta^3}\log\frac{1}{\delta}\right)^{\frac{1}{2}}\left(\frac{1}{\delta}\log\frac{1}{\delta}\right)^{\frac{1}{2}}\right)$$

$$=O\left(\frac{1}{\delta}\log\frac{1}{\delta}\right)\quad\text{from Lemma 3}.$$

Collecting these estimates we obtain

$$\int_{2\pi/\sqrt{D}}^{\infty}|\phi(ixe^{-i\delta})|^2=\frac{3}{2\pi^2}\prod_{p|D}\left(1+\frac{1}{p}\right)^{-1}L^2(1,\chi)\frac{1}{\delta}\log^2\frac{1}{\delta}+O\left(\frac{1}{\delta}\log\frac{1}{\delta}\right).$$

And this gives

$$\int_0^{\infty}\left|\zeta_k\left(\frac{1}{2}+it\right)\right|^2e^{-\delta t}\,dt=\frac{6}{\pi^2}\prod_{p|D}\left(1+\frac{1}{p}\right)^{-1}L^2(1,\chi)\frac{1}{\delta}\log^2\frac{1}{\delta}+O\left(\frac{1}{\delta}\log\frac{1}{\delta}\right).$$

Now by the assertion of [2, p. 136] we complete the proof of the theorem.

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Extension of Coherent Analytic Subsheaves★

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Recently the theory of extending coherent analytic sheaves and subsheaves has been investigated by Frisch and Guenot [1], Serre [10], Thimm [19, 20], and the authors [12–15, 17, 21, 22, 23]. It is proved in [1] and [15] that a coherent analytic sheaf which is equal to its $(n+1)^{\text{th}}$ absolute gap-sheaf can always be extended through a subvariety of dimension $\leqq n$. The best result for coherent analytic sheaf extension is conjectured to be the following:

(1) If $0 \leqq \varepsilon_i < 1$ $(1 \leqq i \leqq N)$, then a coherent analytic sheaf defined on the Hartogs domain

$$Q' = \{(z_1, \ldots, z_N) \in \mathbb{C}^N \mid |z_i| < 1, 1 \leqq i \leqq N; |z_j| > \varepsilon_j \quad \text{for some} \quad n+1 \leqq j \leqq N\}$$
$$\cup \{(z_1, \ldots, z_N) \in \mathbb{C}^N \mid |z_j| < 1, 1 \leqq j \leqq N; |z_k| < \varepsilon_k, 1 \leqq k \leqq n\}$$

which is equal to its n^{th} absolute gap-sheaf can always be extended to

$$Q = \{(z_1, \ldots, z_N) \in \mathbb{C}^N \mid |z_i| < 1, \ 1 \leqq i \leqq N\}.$$

This conjecture has not yet been proved. The closest result obtained so far is the following [15, p. 136, Th. I_n]:

(2) If $0 \leqq \varepsilon_j < 1$ $(n+1 \leqq j \leqq N)$, then a coherent analytic sheaf defined on

$$Q'' = \{(z_1, \ldots, z_N) \in \mathbb{C}^N \mid |z_i| < 1, \ 1 \leqq i \leqq N; \ |z_j| > \varepsilon_j \quad \text{for some} \quad n+1 \leqq j \leqq N\}$$

which is equal to its $(n+1)^{\text{th}}$ absolute gap-sheaf can always be extended to Q, where Q is the same as in (1).

In this paper we continue the investigation of extending coherent analytic subsheaves. The following Rothstein type and Osgood type theorems are proved:

(i) (Rothstein type). A coherent analytic subsheaf which is equal to its ϱ^{th} relative gap-sheaf can always be extended across a *-strongly ϱ-concave boundary.

(ii) (Osgood type). Suppose a coherent analytic subsheaf is defined on the complement of a subvariety of dimension n in a domain of $\mathbb{C}^N$. If the subsheaf is equal to its $(n-1)^{\text{th}}$ relative gap-sheaf, then it can be extended through the subvariety if and only if its tensorial restriction to any member of an open family of $(N-n+1)$-planes whose union covers the domain can be extended.

The Rothstein type theorem implies the conjecture (1) for the subsheaf case. Since the equality of a subsheaf and its ϱ^{th} relative gap-sheaf is equivalent

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to the dimension of every one of its associated subvarieties being $\geqq \varrho+1$, the Rothstein type theorem generalizes the corresponding statement for subvariety extension. The Osgood type theorem is slightly weaker than the corresponding statement for subvariety extension. However, it is the sharpest possible for subsheaf extension. The results in this paper illustrate the general principle that everything which can be done for subvariety extension can also be done for subsheaf extension with suitable modifications.

The Rothstein type theorem enables us to obtain a result for general sheaf extension which is stronger than (2) but weaker than (1).

For the proof of our results we use methods of normalization, meromorphic function extension, subvariety extension, and gap-sheaves.

§ 1. Gap-Sheaves

If $(X, \mathcal{O})$ is a complex space (not necessarily reduced) and $\mathscr{F}$ is a coherent analytic sheaf on X, we denote as usual by $S_\varrho(\mathscr{F})$ the set $\{x \in X \mid \operatorname{codh} \mathscr{F}_x \leqq \varrho\}$ $(\varrho \geqq 0)$. It is proved in [9] that $S_\varrho(\mathscr{F})$ is a subvariety and $\dim S_\varrho(\mathscr{F}) \leqq \varrho$.

If A is a subvariety in X, we denote by $\mathscr{H}_A^0 \mathscr{F}$ the (coherent) analytic subsheaf of $\mathscr{F}$ of all section-germs whose supports are contained in A. If ϱ is a non-negative integer, we introduce the analytic sheaf

$$\mathscr{H}_\varrho^0 \mathscr{F} = \varinjlim \{\mathscr{H}_A^0 \mathscr{F} \mid \dim A \leqq \varrho\}$$

which is the subsheaf of all section-germs of $\mathscr{F}$ whose supports have dimension $\leqq \varrho$. Applying [8, p. 359, Satz I], we obtain

$$\mathscr{H}_\varrho^0 \mathscr{F} = \mathscr{H}_{S_\varrho(\mathscr{F})}^0 \mathscr{F}\,.$$

Hence $\mathscr{H}_\varrho^0 \mathscr{F}$ is coherent. From [9, p. 81, Satz 5] we conclude easily the following (cf. [16, Lemma 4]):

(3) The ϱ-dimensional component of $\operatorname{Supp} \mathscr{H}_\varrho^0 \mathscr{F}$ agrees with the ϱ-dimensional component of $S_\varrho(\mathscr{F})$.

If $\mathscr{G}$ is a coherent analytic sheaf on X which contains $\mathscr{F}$ as a subsheaf and if $\lambda\colon \mathscr{G} \to \mathscr{G}/\mathscr{F}$ is the natural sheaf-epimorphism, we define

$$\mathscr{F}_\varrho = \lambda^{-1} \mathscr{H}_\varrho^0(\mathscr{G}/\mathscr{F})$$

and call $\mathscr{F}_\varrho$ the ϱ^{th} *(relative) gap-sheaf* of $\mathscr{F}$ in $\mathscr{G}$. If A is a subvariety of X, then we define

$$\mathscr{F}[A] = \lambda^{-1} \mathscr{H}_A^0(\mathscr{G}/\mathscr{F})$$

and call $\mathscr{F}[A]$ the *(relative) gap-sheaf* of $\mathscr{F}$ in $\mathscr{G}$ with exceptional subvariety A (cf. [18, 11]). $\mathscr{F}_\varrho$ and $\mathscr{F}[A]$ are coherent and $\dim \operatorname{Supp} \mathscr{F}_\varrho/\mathscr{F} \leqq \varrho$. If $\mathscr{F}'$ is a coherent analytic subsheaf of $\mathscr{G} \mid X - A$, we can still define $\mathscr{F}'[A]$ as the subsheaf of $\mathscr{G}$ generated by local sections $s \in \Gamma(U, \mathscr{G})$ satisfying $s \mid U - A \in \Gamma(U - A, \mathscr{F}')$. However, in general $\mathscr{F}'[A]$ is not coherent. In fact, $\mathscr{F}'[A]$

is coherent if and only if $\mathscr{F}'$ can be extended coherently to X as a subsheaf of $\mathscr{G}$. Hence, a coherent analytic subsheaf defined outside a subvariety can be extended through the subvariety if and only if it can be extended across each point of the subvariety.

We define the ϱ^{th} *absolute gap-sheaf* $\mathscr{R}^0_\varrho \mathscr{F}$ by the following presheaf

$$U \mapsto \varinjlim \{\Gamma(U-A, \mathscr{F}) \mid \dim A \leqq \varrho\}.$$

It is proved in [13] that $\mathscr{R}^0_\varrho \mathscr{F}$ is coherent if and only if

$$\dim \operatorname{Supp} \mathscr{H}^0_{\varrho+1} \mathscr{F} \leqq \varrho .$$

If $f \in \Gamma(X, \mathcal{O})$ and $Z(f) = \operatorname{Supp}(\mathcal{O}/f\mathcal{O})$, then (3) implies that for $x \in Z(f)$ the germ f_x is not a zero-divisor for $\mathscr{F}_x$ if and only if

$$\dim_x Z(f) \cap S_\varrho(\mathscr{F}) < \varrho$$

for every $\varrho \geqq 0$. Hence we have the following:

(4) If $f_1, \ldots, f_k \in \Gamma(X, \mathcal{O})$ and $(f_i)_x$ is not a unit of $\mathcal{O}_x (1 \leqq i \leqq k)$, then $(f_1)_x, \ldots, (f_k)_x$ form an $\mathscr{F}_x$-sequence if and only if

$$\dim_x \left(\bigcap_{i=1}^{k} Z(f_i) \right) \cap S_\varrho(\mathscr{F}) \leqq \varrho - k$$

for every $\varrho \geqq 0$ (cf. [24, p. 152, Satz (II. 2.1)] and [15, p. 141, Lemma 9]).

Suppose G is an open subset of $\mathbb{C}^n$ and $\mathscr{G}$ is a coherent analytic sheaf on G. Suppose E is an $(n-k)$-plane in $\mathbb{C}^n$. Let $\mathscr{I}$ be the ideal-sheaf on $\mathbb{C}^n$ for E. We denote by $\mathscr{G} \| E$ the coherent analytic sheaf $\mathscr{G}/\mathscr{I}\mathscr{G}$ on $E \cap G$. Suppose $\mathscr{F}$ is a coherent analytic subsheaf of $\mathscr{G}$ on G. In general, $\mathscr{F} \| E$ may fail to be a subsheaf of $\mathscr{G} \| E$, i.e. the sheaf-homomorphism

$$\mathscr{F} \| E \to \mathscr{G} \| E$$

induced by $\mathscr{F} \hookrightarrow \mathscr{G}$ may not be injective. However (4) implies that

(5) $\mathscr{F} \| E$ is a subsheaf of $\mathscr{G} \| E$ if

$$\dim E \cap S_\varrho(\mathscr{G}/\mathscr{F}) \leqq \varrho - k$$

for every $\varrho \geqq 0$.

From (3) and (4) we conclude the following:

(6) Suppose

$$\dim E \cap S_\varrho(\mathscr{G}/\mathscr{F}) \leqq \varrho - k$$

for every $\varrho \geqq 0$. If $\mathscr{F}_m = \mathscr{F}$ for some $m \geqq 0$, then

$$(\mathscr{F} \| E)_{m-k} = \mathscr{F} \| E .$$

§ 2. Subsheaf Extension and Meromorphic Functions

Suppose $G \subset \tilde{G}$ are connected open subsets of a normal reduced complex space $(X, \mathcal{O})$ of pure dimension n. Suppose $\mathcal{S}$ is a coherent analytic subsheaf of $\mathcal{O}^p | G$ satisfying $\mathcal{S}_{n-1} = \mathcal{S}$. Let $\mathcal{F} = \mathcal{O}^p / \mathcal{S}$. $\mathcal{F}$ is a torsion-free coherent analytic sheaf on G. Let $s_i \in \Gamma(G, \mathcal{F})$ be the image of $(0, \ldots, 0, 1, 0, \ldots, 0)$ under the natural sheaf-epimorphism $\mathcal{O}^p \to \mathcal{F}$, where 1 is in the i^{th} position. The following is clear.

(7) $\mathcal{S}$ can be extended coherently to $\tilde{G}$ as a subsheaf of $\mathcal{O}^p$ if and only if (i) $\mathcal{F}$ can be extended to a coherent analytic sheaf $\tilde{\mathcal{F}}$ on $\tilde{G}$ and (ii) s_i can be extended to some element of $\Gamma(\tilde{G}, \tilde{\mathcal{F}})$ $(1 \leqq i \leqq p)$.

Let $r = \operatorname{rank} \mathcal{F}$. Select $s_{i_1}, \ldots, s_{i_r}$ such that the sheaf-homomorphism

$$\varphi : \mathcal{O}^r \to \mathcal{F}$$

on G defined by these r sections is injective. Let $\mathcal{I}$ be the maximum ideal-sheaf on G such that $\mathcal{I}\mathcal{F} \subset \varphi(\mathcal{O}^r)$. $\mathcal{I}$ is coherent and the zero-set of $\mathcal{I}$ has dimension $\leqq n-1$. Since $\mathcal{I}\mathcal{F} \subset \varphi(\mathcal{O}^r)$, every local section of $\mathcal{F}$ can be lifted back through φ to a meromorphic section of the trivial vector bundle associated to $\mathcal{O}^r$. Therefore we have a sheaf-monomorphism

$$\psi : \mathcal{F} \to \mathcal{M}^r$$

on G such that

$$\begin{tikzcd} \mathcal{O}^r \arrow[r, "\varphi"] \arrow[d] & \mathcal{F} \arrow[dl, "\psi"] \\ \mathcal{M}^r & \end{tikzcd}$$

is commutative, where $\mathcal{M}$ is the sheaf of germs of meromorphic functions on X. Let $t_i = \psi(s_i)$. We call $t_1, \ldots, t_p$ a set of *associated meromorphic vector-functions* for $\mathcal{S}$.

If $\mathcal{S}$ can be extended to a coherent analytic subsheaf $\tilde{\mathcal{S}}$ of $\mathcal{O}^p | \tilde{G}$, then we can assume $\tilde{\mathcal{S}}_{n-1} = \tilde{\mathcal{S}}$. We repeat the preceding argument with $\tilde{\mathcal{F}} = \mathcal{O}^p / \tilde{\mathcal{S}}$ instead of $\mathcal{F}$ (using the same $i_1, \ldots, i_r$) and obtain associated meromorphic vector-functions $\tilde{t}_1, \ldots, \tilde{t}_p$ for $\tilde{\mathcal{S}}$. t_i is an r-tuple of meromorphic functions on $\tilde{G}$ extending $t_i (1 \leqq i \leqq p)$.

Conversely, if t_i can be extended to an r-tuple t_i^* of meromorphic functions on $\tilde{G}$ $(1 \leqq i \leqq p)$, then the subsheaf $\mathcal{F}^*$ of $\mathcal{M}^r | \tilde{G}$ generated by $t_1^*, \ldots, t_p^*$ is coherent. For, if f is a non-identically-zero holomorphic function on some connected open subset U of $\tilde{G}$ such that $f t_i^*$ is an r-tuple of holomorphic functions on U, then $\mathcal{F}^* \approx f \mathcal{F}^*$ on U and $f \mathcal{F}^*$ is a subsheaf of $\mathcal{O}^r | U$ generated by $f t_1^*, \ldots, f t_p^*$. If we identify $\mathcal{F}$ with $\psi(\mathcal{F})$, then $\mathcal{F}^*$ extends $\mathcal{F}$ and t_i^* extends $s_i (1 \leqq i \leqq p)$.

Hence by (7) we have the following.

Proposition 1. *$\mathcal{S}$ can be extended coherently to $\tilde{G}$ as a subsheaf of $\mathcal{O}^p$ if and only if $t_1, \ldots, t_p$ can be extended to r-tuples of meromorphic functions on $\tilde{G}$.*

§ 3. Rothstein Type Theorem on Subsheaf Extension

Suppose $(X, \mathcal{O})$ is a complex space. A real-valued function v on X is said to be **-strongly ϱ-convex* at $x \in X$ if there exist a nowhere degenerate holomorphic map π from some open neighborhood U of x in X to some open subset G of $\mathbb{C}^n$ and a real-valued C^2 function $\tilde{v}$ on G such that $v = \tilde{v} \circ \pi$ and at every point in G the Hermitian matrix $\left(\frac{\partial^2 \tilde{v}}{\partial z_i \partial \bar{z}_j}\right)$ has at least $n - \varrho + 1$ positive eigenvalues. v is said to be **-strongly ϱ-convex* on X if it is *-strongly ϱ-convex at every point of X.

Suppose D is an open subset of X and x is a boundary point of D. D is said to be **-strongly ϱ-concave* at x if there exists a *-strongly ϱ-convex function v on an open neighborhood U of x in X such that $D \cap U = \{y \in U \mid v(y) > v(x)\}$.

We assume that D is *-strongly ϱ-concave at x.

Suppose $\mathscr{F}$ is a coherent analytic sheaf on D. $\mathscr{F}$ is said to satisfy *condition* (E) if there exist an open neighborhood U of x in X and a coherent analytic sheaf $\mathscr{G}$ on $D \cup U$ such that

(i) $\mathscr{F}$ is a subsheaf of $\mathscr{G} \mid D$ and

(ii) for every open neighborhood W of x in U every element of $\Gamma(D \cap W, \mathscr{G})$ can be extended to an open neighborhood of x.

If $\mathscr{R}^0_{\varrho-1} \mathscr{G} = \mathscr{G}$, then (ii) is always satisfied according to [13, p. 373, Th. 3]. Suppose $\mathscr{F}$ is generated on D by a finite number of global sections.

Proposition 2. *Suppose X is reduced and normal and has pure dimension n. If $\varrho \leqq n-1$ and $\mathscr{F}$ is torsion-free, then $\mathscr{F}$ admits a coherent extension to some open neighborhood of x.*

Proof. By [2, p. 69, Cor. 5.4] there exists an open neighborhood U of x in X such that $D \cap U$ is connected. By replacing D by $D \cap U$, we can assume that D is connected. Since $\mathscr{F}$ is generated on D by a finite number of global sections, we have a sheaf-epimorphism $\lambda : \mathcal{O}^p \to \mathscr{F}$ on D. Let $\mathscr{S} = \operatorname{Ker} \lambda$ and let $t_1, \ldots, t_p$ be a set of associated meromorphic vector-functions for $\mathscr{S}$. By [2, p. 74, Prop. 6.9], t_i can be extended to a meromorphic vector-function $\tilde{t}_i$ on $D \cup V$ for some connected open neighborhood V of x $(1 \leqq i \leqq p)$. Since $\mathscr{F}$ is torsion-free, $\mathscr{S}_{n-1} = \mathscr{S}$. By Proposition 1, $\mathscr{S}$ can be extended to a coherent analytic subsheaf $\tilde{\mathscr{S}}$ of $\mathcal{O}^p \mid D \cup V$. $\mathcal{O}^p / \tilde{\mathscr{S}}$ is a coherent analytic sheaf on $D \cup V$ extending $\mathscr{F}$. Q.E.D.

Proposition 3. *Suppose X is reduced and has pure dimension n. If $\varrho \leqq n-1$ and $\mathscr{H}^0_{n-1} \mathscr{F} = 0$, then $\mathscr{F}$ satisfies condition (E).*

Proof. Consider the normalization $\pi : \tilde{X} \to X$ and define $\tilde{D} = \pi^{-1}(D)$. $\tilde{D}$ is *-strongly ϱ-concave at every point of $\pi^{-1}(x)$. The sheaf $\pi^* \mathscr{F}$ on $\tilde{D}$ (which is the inverse image of $\mathscr{F}$ under π) is coherent on $\tilde{D}$ and is generated by a finite number of global sections.

Let $\mathscr{T}$ be the torsion subsheaf of $\pi^* \mathscr{F}$ and let $\mathscr{G} = (\pi^* \mathscr{F}) / \mathscr{T}$. $\mathscr{G}$ is coherent and torsion-free and is generated by a finite number of global sections. By applying Proposition 2 to every point of $\pi^{-1}(x)$, we can extend $\mathscr{G}$ to a coherent analytic sheaf $\mathscr{G}'$ on $\tilde{D}\cup\pi^{-1}(U)$ for some open neighborhood U of x. We can assume that $\mathscr{G}'$ is torsion-free. The zero$^{\text{th}}$ direct image $\pi_*\mathscr{G}'$ of $\mathscr{G}'$ under π satisfies $\mathscr{H}^0_{n-1}\pi_*\mathscr{G}'=0$. Hence $\mathscr{R}^0_{\varrho-1}\pi_*\mathscr{G}'$ is coherent on $D\cup U$ and $\pi_*\mathscr{G}'$ can be regarded naturally as a subsheaf of $\mathscr{R}^0_{\varrho-1}\pi_*\mathscr{G}'\,|\,D$.

We have a natural sheaf-homomorphism $\lambda:\mathscr{F}\to\pi_*\pi^*\mathscr{F}$. The natural sheaf-epimorphism $\pi^*\mathscr{F}\to\mathscr{G}$ induces a sheaf-homomorphism $\mu\circ\lambda:\mathscr{F}\to\pi_*\mathscr{G}$. Since $\mathscr{H}^0_{n-1}\mathscr{F}=0$ and

$$\operatorname{Supp}\operatorname{Ker}\alpha\subset S\cup\pi(\operatorname{Supp}\mathscr{T}),$$

where S is the set of all singular points of X, we conclude that α is injective. $\mathscr{F}$ can be regarded as a subsheaf of $\mathscr{R}^0_{\varrho-1}\pi_*\mathscr{G}'\,|\,D$. Therefore $\mathscr{F}$ satisfies condition (E). Q.E.D.

Suppose $\mathscr{F}'$ and $\mathscr{F}''$ are coherent analytic sheaves on D and

$$0\to\mathscr{F}'\hookrightarrow\mathscr{F}\xrightarrow{\eta}\mathscr{F}''\to 0$$

is an exact sequence of sheaf-homomorphisms on D.

Proposition 4. *If $\mathscr{F}''$ satisfies condition* (E), *then there exists an open neighborhood U of x in X such that $\mathscr{F}'\,|\,D\cap U$ is generated by a finite number of global sections.*

Proof. Let $s_1,\dots,s_p\in\Gamma(D,\mathscr{F})$ generate $\mathscr{F}$. Let $\lambda:\mathscr{O}^p\to\mathscr{F}$ be the sheaf-epimorphism on D defined by $s_1,\dots,s_p$. Since $\mathscr{F}''$ satisfies condition (E), there exist an open neighborhood W of x in X and a coherent analytic sheaf $\mathscr{G}$ on $D\cup W$ such that

(i) $\mathscr{F}''$ is a subsheaf of $\mathscr{G}\,|\,D$ and

(ii) $\eta(s_i)$ can be extended to a section $\tilde{s}_i\in\Gamma(D\cup W,\mathscr{G})$ $(1\leqq i\leqq p)$.

Let $\mu:\mathscr{O}^p\to\mathscr{G}$ be the sheaf-homomorphism on $D\cup W$ defined by $\tilde{s}_1,\dots,\tilde{s}_p$. $\operatorname{Ker}\mu$ is coherent on $D\cup W$. On some open neighborhood U of x in W, $\operatorname{Ker}\mu$ is generated by section $t_1,\dots,t_q\in\Gamma(U,\operatorname{Ker}\mu)$. It is easily verified that $\mathscr{F}'$ is generated on $D\cap U$ by $\lambda(t_i)\,|\,D\cap U$, $1\leqq i\leqq q$. Q.E.D.

Proposition 5. *If $\mathscr{F}'$ and $\mathscr{F}''$ satisfy condition* (E), *then $\mathscr{F}$ can be extended coherently to some open neighborhood of x.*

Proof. Since $\mathscr{F}$ is generated on D by a finite number of global sections, there exists a sheaf-epimorphism $\lambda:\mathscr{O}^p\to\mathscr{F}$ on D. Since $\mathscr{F}''$ satisfies condition (E), there exist an open neighborhood U of x in X and a coherent analytic sheaf $\mathscr{G}''$ on $D\cup U$ such that $\mathscr{F}''$ is a subsheaf of $\mathscr{G}''\,|\,D$ and every element of $\Gamma(D,\mathscr{G}'')$ can be extended to some open neighborhood of x. After shrinking U, we conclude that $\eta\circ\lambda$ can be extended to a sheaf-homomorphism $\mu:\mathscr{O}^p\to\mathscr{G}''$ on $D\cup U$. After shrinking U further, we obtain a sheaf-homomorphism $\nu:\mathscr{O}^q\to\mathscr{O}^p$ on U such that

$$\mathscr{O}^q\xrightarrow{\nu}\mathscr{O}^p\xrightarrow{\mu}\mathscr{G}''$$

is exact on U.

Since $\mathscr{F}'$ satisfies condition (E), there exist an open neighborhood V of x in U and a coherent analytic sheaf $\mathscr{G}'$ on $D\cup V$ such that $\mathscr{F}'$ is a subsheaf of $\mathscr{G}'|D$ and every element of $\Gamma(D\cap U, \mathscr{G}')$ can be extended to some open neighborhood of x. Since $\operatorname{Im}\lambda v \subset \mathscr{F}'$ on $D\cap U$, after shrinking V, we obtain a sheaf-homomorphism $\sigma: \mathcal{O}^q \to \mathscr{G}'$ on V such that

$$\begin{tikzcd} \mathcal{O}^q \arrow[r, "v"] \arrow[d, "\sigma"'] & \mathcal{O}^p \arrow[d, "\lambda"] \\ \mathscr{G}' & \mathscr{F}' \arrow[l, hook'] \arrow[r, hook] & \mathscr{F} \end{tikzcd}$$

is commutative on $D\cap V$. The coherent analytic sheaf $\mathcal{O}^p/v(\operatorname{Ker}\sigma)$ on V extends $\mathscr{F}|D\cap V$. Q.E.D.

Proposition 6. *Suppose X has pure dimension n. If $\varrho \leqq n-1$ and $\mathscr{H}^0_{n-1}\mathscr{F}=0$, then $\mathscr{F}$ satisfies condition* (E).

Proof. Let $\mathscr{N}$ be the subsheaf of all nilpotent elements of $\mathcal{O}$ and let $\mathcal{O}_{\text{red}} = \mathcal{O}/\mathscr{N}$. Since only the local situation at x is involved, we can assume w.l.o.g. that $\mathscr{N}^k=0$ for some positive integer k.

Define $\mathscr{F}^{(0)}=\mathscr{F}$ and $\mathscr{F}^{(l)}=$ the gap-sheaf $(\mathscr{N}\mathscr{F}^{(l-1)})_{n-1}$ in $\mathscr{F}^{(l-1)}(1\leqq l\leqq k)$. Let $S=\bigcup_{l=1}^{k}\operatorname{Supp}\mathscr{F}^{(l)}/\mathscr{N}\mathscr{F}^{(l-1)}$. $\dim S\leqq n-1$. Since $\mathscr{F}^{(k)}=\mathscr{N}^k\mathscr{F}$ on $D-S$, $\mathscr{H}^0_{n-1}\mathscr{F}=0$ implies that $\mathscr{F}^{(k)}=0$. From the definition of $\mathscr{F}^{(l)}$, we have

$$\begin{cases} \mathscr{H}^0_{n-1}\mathscr{F}^{(l)}=0 \\ \mathscr{H}^0_{n-1}(\mathscr{F}^{(l)}/\mathscr{F}^{(l+1)})=0 \qquad (0\leqq l<k). \end{cases}$$

Since $\mathscr{N}\mathscr{F}^{(l)}\subset\mathscr{F}^{(l+1)}$, it follows that $\mathscr{F}^{(l)}/\mathscr{F}^{(l+1)}$ can be regarded as an $\mathcal{O}_{\text{red}}$-sheaf.

We are going to prove $(8)_l$ and $(9)_l$ for $0\leqq l<k$ by induction on l:

$(8)_l$ For some open neighborhood U_l of x, $\mathscr{F}^{(l)}$ is generated on $D\cap U_l$ by a finite number of global sections.

$(9)_l$ $\mathscr{F}^{(l)}/\mathscr{F}^{(l+1)}$ satisfies condition (E).

From Proposition 3, we conclude that $(8)_l$ implies $(9)_l$. $(8)_0$ follows from the assumption concerning $\mathscr{F}$. Assume that $(8)_l$ and $(9)_l$ are true for some $0\leqq l<k-1$. From the exact sequence

$$(10)\quad 0\to\mathscr{F}^{(l+1)}\to\mathscr{F}^{(l)}\to\mathscr{F}^{(l)}/\mathscr{F}^{(l+1)}\to 0\,,$$

we conclude that $(8)_{l+1}$ is a consequence of $(8)_l$, $(9)_l$ and Proposition 4.

From $(9)_{k-1}$ it follows that $\mathscr{F}^{(k-1)}$ satisfies condition (E). We are going to prove by descending induction on l that $\mathscr{F}^{(l)}$ satisfies condition (E) $(0\leqq l<k)$. Suppose $\mathscr{F}^{(l+1)}$ satisfies condition (E) for some $0\leqq l<k-1$. Consider the exact sequence (10). From $(9)_l$ and Proposition 5, it follows that $\mathscr{F}^{(l)}$ can be extended to a coherent analytic sheaf $\tilde{\mathscr{F}}$ on $D\cup U$ for some open neighborhood U of x. Since $\mathscr{H}^0_{n-1}\mathscr{F}^{(l)}=0$, we can assume that $\mathscr{H}^0_{n-1}\tilde{\mathscr{F}}=0$. $\mathscr{R}^0_{\varrho-1}\tilde{\mathscr{F}}$ is co-

herent on $D \cup U$ and $\mathscr{F}^{(l)}$ is a subsheaf of $\mathscr{R}^0_{\varrho-1}\tilde{\mathscr{F}}|D$. Hence $\mathscr{F}^{(l)}$ satisfies condition (E). The Proposition follows from $\mathscr{F} = \mathscr{F}^{(0)}$. Q.E.D.

Theorem 1. *If $\mathscr{H}^0_\varrho \mathscr{F} = 0$, then $\mathscr{F}$ satisfies condition* (E).

Proof. Since only the local situation at x is involved, we can assume w.l.o.g. that $\dim X = n < \infty$. For $\varrho \leqq l \leqq n$ define $\mathscr{G}^{(l)} = \mathscr{H}^0_l \mathscr{F}$ and for $\varrho < l \leqq n$ define $X_l = \operatorname{Supp} \mathscr{G}^{(l)}/\mathscr{G}^{(l-1)}$. X_l is of pure dimension l or empty, because $\mathscr{H}^0_{l-1}(\mathscr{G}^{(l)}/\mathscr{G}^{(l-1)}) = 0$ $(\varrho < l \leqq n)$.

By [2, p. 68, Th. 5.3], for some open neighborhood U of x in X, X_l can be extended to an empty or purely l-dimensional subvariety $\tilde{X}_l$ in $D \cup U$ $(\varrho < l \leqq n)$. By using Hilbert Nullstellensatz and [2, p. 69, Cor. 5.4], we can assume after shrinking U that, for some integer k, $(\mathscr{I}^{(l)})^k (\mathscr{G}^{(l)}/\mathscr{G}^{(l-1)}) = 0$ on U, where $\mathscr{I}^{(l)}$ is the ideal-sheaf on $D \cap U$ for $\tilde{X}_l (\varrho < l \leqq n)$. $\mathscr{G}^{(l)}/\mathscr{G}^{(l-1)}$ can be considered as a coherent analytic sheaf on $(\tilde{X}_l, \mathscr{O}/\mathscr{I}^{(l)})$ $(\varrho < l \leqq n)$.

We are going to prove $(11)_l$ and $(12)_l$ for $\varrho < l \leqq n$ by descending induction on l:

$(11)_l$ For some open neighborhood U_l of x, $\mathscr{G}^{(l)}$ is generated on $D \cap U_l$ by a finite number of global sections.

$(12)_l$ $\mathscr{G}^{(l)}/\mathscr{G}^{(l-1)}$ satisfies condition (E).

Since $X_l \cap D$ is *-strongly ϱ-concave at x whenever $x \in X_l$, it follows from Proposition 6 that $(11)_l$ implies $(12)_l$. Since $\mathscr{G}^{(n)} = \mathscr{F}$, $(11)_n$ follows from the assumption concerning $\mathscr{F}$. Assume that $(11)_{l+1}$ and $(12)_{l+1}$ are true for some $\varrho < l < n$. From the exact sequence

$$(13) \quad 0 \to \mathscr{G}^{(l)} \to \mathscr{G}^{(l+1)} \to \mathscr{G}^{(l+1)}/\mathscr{G}^{(l)} \to 0,$$

we conclude that $(11)_l$ is a consequence of $(11)_{l+1}$, $(12)_{l+1}$, and Proposition 4.

Since $\mathscr{G}^{(\varrho)} = \mathscr{H}^0_\varrho \mathscr{F} = 0$, according to $(12)_{\varrho+1}$, $\mathscr{G}^{(\varrho+1)}$ satisfies condition (E). We are going to prove by induction on l that $\mathscr{G}^{(l)}$ satisfies condition (E) $(\varrho < l \leqq n)$. Suppose $\mathscr{G}^{(l)}$ satisfies condition (E) for some $\varrho < l < n$. Consider the exact sequence (13). From $(12)_{l+1}$ and Proposition 5, it follows that $\mathscr{G}^{(l+1)}$ can be extended to a coherent analytic sheaf $\tilde{\mathscr{G}}$ on $D \cup V$ for some open neighborhood V of x. Since $\mathscr{H}^0_\varrho \mathscr{G}^{(l+1)} = 0$, we can assume that $\mathscr{H}^0_\varrho \tilde{\mathscr{G}} = 0$. $\mathscr{R}^0_{\varrho-1}\tilde{\mathscr{G}}$ is coherent on $D \cup V$ and $\mathscr{G}^{(l+1)}$ is a subsheaf of $\mathscr{R}^0_{\varrho-1}\tilde{\mathscr{G}}|D$. Hence $\mathscr{G}^{(l+1)}$ satisfies condition (E). The Proposition follows from $\mathscr{F} = \mathscr{G}^{(n)}$. Q.E.D.

Corollary. *If $\mathscr{H}^0_\varrho \mathscr{F} = 0$, then $\mathscr{F}$ can be extended coherently to an open neighborhood of x.*

Remark. It can be proved that Theorem 1 together with its corollary remain valid if, instead of assuming that $\mathscr{F}$ is generated on D by a finite number of global sections, we merely assume that $\mathscr{F}$ is generated by $\Gamma(D, \mathscr{F})$. The idea of the proof is to pull back the boundary of D to obtain an open subset $D' \subset\subset D$ and then extend $\mathscr{F}|D'$. If the pullback is small, the domain of the extension will contain x. The difference between $\mathscr{F}$ and the extension can be taken care of by using gap-sheaves.

Theorem 1a. *Suppose D is an open subset of a complex space $(X, \mathcal{O})$, x is a boundary point of D, and D is *-strongly ϱ-concave at x. Suppose $\mathcal{G}$ is a coherent analytic sheaf on X and $\mathcal{F}$ is a coherent analytic subsheaf of $\mathcal{G}|D$. If $\mathcal{F}_\varrho = \mathcal{F}$, then $\mathcal{F}$ can be extended coherently to an open neighborhood of x as a subsheaf of $\mathcal{G}$.*

Proof. Since only the local situation at x is involved, we can assume w.l.o.g. that we have a sheaf-epimorphism $\lambda: \mathcal{O}^p \to \mathcal{G}$ on X. Let $\mathcal{S} = \mathcal{O}^p/\lambda^{-1}(\mathcal{F})$. $\mathcal{H}^0_\varrho \mathcal{S} = 0$. By Theorem 1, there exist an open neighborhood U of x and a coherent analytic sheaf $\mathcal{T}$ on $D \cup U$ such that $\mathcal{S}$ is a subsheaf of $\mathcal{T}|D$ and every element of $\Gamma(D, \mathcal{T})$ can be extended to an open subset of x. After shrinking U, we obtain a sheaf-homomorphism $\mu: \mathcal{O}^p \to \mathcal{T}$ on $D \cup U$ such that

$$\begin{array}{ccc} \mathcal{O}^p & = & \mathcal{O}^p \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathcal{S} & \hookrightarrow & \mathcal{T} \end{array}$$

is commutative on D, where ν is the natural sheaf-epimorphism. $\lambda(\operatorname{Ker}\mu)$ is a coherent analytic subsheaf of $\mathcal{G}|D \cup U$ extending $\mathcal{F}$. Q.E.D.

Theorem 1 b. *Suppose V is a subvariety of dimension $\leq n$ in a complex space X and $\mathcal{G}$ is a coherent analytic sheaf on X. Suppose $\mathcal{F}$ is a coherent analytic subsheaf of $\mathcal{G}|X - V$ and $\mathcal{F}_n = \mathcal{F}$. If $\mathcal{F}$ can be extended across some point of every n-dimensional branch of V as a subsheaf of $\mathcal{G}$, then $\mathcal{F}$ can be extended through V as a subsheaf of $\mathcal{G}$.*

Proof. The conclusion of the Theorem is equivalent to the coherence of $\mathcal{F}[V]$. By using induction on $\dim V$, we can assume w.l.o.g. that V is non-singular and connected.

(a) Suppose $\dim V < n$. For every point $x \in V$ we can find an open subset D of $X - V$ such that x is a boundary point of D and D is *-strongly n-concave at x. By Theorem 1a, $\mathcal{F}|D$ can be extended to a coherent analytic subsheaf $\tilde{\mathcal{F}}$ of $\mathcal{G}|D \cup U$ for some open neighborhood U of x. We can assume that $\tilde{\mathcal{F}}_n = \tilde{\mathcal{F}}$. $\mathcal{F}[V]$ agrees with $\tilde{\mathcal{F}}$ on U. $\mathcal{F}[V]$ is coherent at x. Hence $\mathcal{F}[V]$ is coherent on X.

(b) Suppose $\dim V = n$. Let W be the set of points of V where $\mathcal{F}[V]$ is coherent. Obviously W is open in V. Since $\mathcal{F}$ can be extended coherently across some point of V as a subsheaf of $\mathcal{G}$, $W \neq \emptyset$. Suppose $x \in V$ is a boundary point of W. We can find an open subset D of $(X - V) \cup W$ such that x is a boundary point of D and D is *-strongly n-concave at x. As in (a), we conclude that $\mathcal{F}[V]$ is coherent at x. W is closed in V. Hence $W = V$. $\mathcal{F}[V]$ is coherent on X. Q.E.D.

By employing the exhaustion processes used in [7, § 8] and in the proof of [13, p. 374, Th. 4], we obtain readily from Theorem 1a the following two theorems.

Theorem 1c. *Suppose Q' and Q are as in* (1). *Suppose $\mathcal{G}$ is a coherent analytic sheaf on Q and $\mathcal{F}$ is a coherent analytic subsheaf of $\mathcal{G}|Q'$. If $\mathcal{F}_n = \mathcal{F}$, then $\mathcal{F}$*

can be extended uniquely to a coherent analytic subsheaf $\tilde{\mathscr{F}}$ *of* $\mathscr{G}$ *on* Q *satisfying* $\tilde{\mathscr{F}}_n = \tilde{\mathscr{F}}$.

Theorem 1d. *Suppose* v *is a *-strongly* ϱ*-convex function on a complex space* X *such that* $\{x \in X \mid \lambda < v(x) < \mu\}$ *is relatively compact in* X *for any two real numbers* $\lambda < \mu$. *Suppose* $\mathscr{G}$ *is a coherent analytic sheaf on* X *and* $\mathscr{F}$ *is a coherent analytic subsheaf of* $\mathscr{G} \mid X_\lambda$ *for some real number* λ, *where* $X_\lambda = \{x \in X \mid v(x) > \lambda\}$. *If* $\mathscr{F}_\varrho = \mathscr{F}$, *then* $\mathscr{F}$ *can be extended uniquely to a coherent analytic subsheaf* $\tilde{\mathscr{F}}$ *of* $\mathscr{G}$ *on* X *satisfying* $\tilde{\mathscr{F}}_n = \tilde{\mathscr{F}}$.

Theorem 1e. *Suppose* Q' *and* Q *are as in* (1) *and* $\mathscr{F}$ *is a coherent analytic sheaf on* Q'. *If* $\mathscr{H}^0_{n+2}\mathscr{F} = 0$ *and* $\mathscr{R}^0_n\mathscr{F} = \mathscr{F}$, *then* $\mathscr{F}$ *can be extended uniquely (up to isomorphism) to a coherent analytic sheaf* $\tilde{\mathscr{F}}$ *on* Q *satisfying* $\mathscr{R}^0_n\tilde{\mathscr{F}} = \tilde{\mathscr{F}}$.

Proof. Since $\mathscr{H}^0_{n+2}\mathscr{F} = 0$, $\mathscr{R}^0_{n+1}\mathscr{F}$ is coherent. By (2), $\mathscr{R}^0_{n+1}\mathscr{F} \mid Q''$ can be extended uniquely (up to isomorphism) to a coherent analytic sheaf $\hat{\mathscr{F}}$ on Q satisfying $\mathscr{R}^0_n\hat{\mathscr{F}} = \hat{\mathscr{F}}$, where Q'' is as in (2). By [15, p. 135, Prop. 20], $\mathscr{F}$ can be uniquely embedded as a subsheaf of $\hat{\mathscr{F}} \mid Q'$. By Theorem 1c, $\mathscr{F}$ can be extended uniquely to a coherent analytic subsheaf $\tilde{\mathscr{F}}$ of $\hat{\mathscr{F}}$ on Q satisfying $\tilde{\mathscr{F}}_n = \tilde{\mathscr{F}}$. It is clear that $\mathscr{R}^0_n\tilde{\mathscr{F}} = \tilde{\mathscr{F}}$ and $\tilde{\mathscr{F}}$ is uniquely determined up to isomorphism. Q.E.D.

§ 4. Osgood Type Theorem on Subsheaf Extension

The set $P_{n,k}$ of all k-planes in $\mathbb{C}^n$ can be regarded canonically as an open subset of the Grassmann manifold $G_{n+1,k+1}$. We give $P_{n,k}$ the topology induced from $G_{n+1,k+1}$. A subset $\mathscr{E}$ of $P_{n,k}$ is called an *open family of k-planes* if $\mathscr{E}$ is open in $P_{n,k}$. We denote $\cup\{E \mid E \in \mathscr{E}\}$ by $|\mathscr{E}|$.

The following lemma follows from [5, p. 84, Hauptsatz] and [3, p. 654, Satz 4].

Lemma 1. *Suppose* G *is an open subset of* $\mathbb{C}^n$, V *is a subvariety of dimension* $\leqq n-1$ *in* G, *and* f *is a meromorphic function on* $G - V$.

(a) *Suppose* $\mathscr{E}$ *is an open family of* 1*-planes in* $\mathbb{C}^n$ *such that* $G \subset |\mathscr{E}|$. *If for every* $E \in \mathscr{E}$ *the restriction of* f *to* $E \cap (G - V)$ *can be extended to a meromorphic function on* $E \cap G$, *then* f *can be extended to a meromorphic function on* G.

(b) *If as a meromorphic function* f *can be extended across some point of every* $(n-1)$*-dimensional branch of* V, *then* f *can be extended to a meromorphic function on* G.

The following lemma follows from [6, p. 340, Satz c 3].

Lemma 2. *Suppose* G *is an open subset of* $\mathbb{C}^n$, V *is a subvariety of dimension* $\leqq n-1$, *and* Z *is a subvariety of pure dimension* $n-1$ *in* $G - V$. *Suppose* $\mathscr{E}$ *is an open family of* 1*-planes in* $\mathbb{C}^n$ *such that* $G \subset |\mathscr{E}|$. *If for every* $E \in \mathscr{E}$, $Z \cap E$ *can be extended to a subvariety in* $E \cap G$, *then the closure of* Z *is a subvariety in* G.

The following lemma follows from results in [4].

Lemma 3. *Suppose $D \subset\subset \tilde{D}$ are open subsets of $\mathbb{C}^n$, V is a subvariety of dimension $\leqq k$ in $\tilde{D}$, and Z is a subvariety in $\tilde{D} - V$. Suppose $\mathscr{E}$ is an open family of $(n-k)$-planes in $\mathbb{C}^n$ such that $\tilde{D} \subset |\mathscr{E}|$. There there exists an open subfamily $\mathscr{E}'$ of $\mathscr{E}$ such that* (i) *$|\mathscr{E}'|$ intersects every k-dimensional branch of $V \cap D$ and* (ii) *for every branch Y of $Z \cap D$ and every $E \in \mathscr{E}'$,* $\dim E \cap Y \leqq \dim Y - k$.

We denote the structure sheaf of $\mathbb{C}^n$ by ${}_n\mathcal{O}$.

Proposition 7. *Suppose D is a domain in $\mathbb{C}^n$, V is a subvariety of dimension $n-1$ in D, and $\mathscr{R}$ is a coherent analytic subsheaf of ${}_n\mathcal{O}^p | D - V$. Suppose $\mathscr{E}$ is an open family of* 1*-planes in $\mathbb{C}^n$ such that $D \subset |\mathscr{E}|$. Suppose for every $E \in \mathscr{E}$,* $\operatorname{Im}(\mathscr{R} \,\|\, E \to {}_n\mathcal{O}^p \,\|\, E)$ *can be extended coherently to $E \cap D$ as a subsheaf of ${}_n\mathcal{O}^p \,\|\, E$. Then $\mathscr{R}$ can be extended coherently to D as a subsheaf of ${}_n\mathcal{O}^p$.*

Proof. Choose arbitrarily a relatively compact subdomain $\tilde{G}$ of D. Let $G = \tilde{G} - V$ and let $\mathscr{S} = \mathscr{R} | G$. We need only show that $\mathscr{S}$ can be extended to $\tilde{G}$ as a subsheaf of ${}_n\mathcal{O}^p$. We use the notations of § 2.

By Lemma 3, we can choose an open subfamily $\mathscr{E}'$ of $\mathscr{E}$ such that (i) $|\mathscr{E}'|$ intersects every $(n-1)$-dimensional branch of $V \cap \tilde{G}$ and (ii) $\dim E \cap (S_\varrho(\mathscr{F}) \cup S_\varrho(\operatorname{Coker} \varphi)) \leqq \varrho - n + 1$ for every $E \in \mathscr{E}'$ and every $\varrho \geqq 0$.

Fix arbitrarily $E \in \mathscr{E}'$. By using (5), we conclude the following:

(i) $\mathscr{S} \,\|\, E$ is a subsheaf of ${}_n\mathcal{O}^p \| E$ on $E \cap G$;

(ii) the sheaf-homomorphism ${}_n\mathcal{O}^r \,\|\, E \to \mathscr{F} \,\|\, E$ induced by φ is injective; and

(iii) $\operatorname{rank}(\mathscr{F} \,\|\, E) = r$.

Hence $t_i | E \cap G$, $1 \leqq i \leqq p$, form a set of associated meromorphic vector-functions for $\mathscr{S} \,\|\, E$. Since $\mathscr{S} \,\|\, E$ can be extended coherently to $E \cap \tilde{G}$ as a subsheaf of ${}_n\mathcal{O}^p \,\|\, E$, by Proposition 1, $t_i | E \cap G$ can be extended to an r-tuple of meromorphic functions on $E \cap \tilde{G} (1 \leqq i \leqq p)$.

Since E is an arbitrary element of $\mathscr{E}'$ and $|\mathscr{E}'|$ intersects every $(n-1)$-dimensional branch of $V \cap \tilde{G}$, by Lemma 1 (a), $t_1, \ldots, t_p$ as r-tuples of meromorphic functions can be extended across some point of every $(n-1)$-dimensional branch of $V \cap \tilde{G}$. By Lemma 1 (b), $t_1, \ldots, t_p$ can be extended to r-tuples of meromorphic functions on $\tilde{G}$. By Proposition 1, $\mathscr{S}$ can be extended coherently to $\tilde{G}$ as a subsheaf of ${}_n\mathcal{O}^p$. Q.E.D.

Proposition 8. *Suppose $\mathscr{G}$ is a coherent analytic sheaf on an open subset D of $\mathbb{C}^N$, V is a subvariety of dimension n in D, and $\mathscr{F}$ is a coherent analytic subsheaf of $\mathscr{G} | D - V$. Suppose $\mathscr{E}$ is an open family of $(N-n)$-planes in $\mathbb{C}^N$ such that $D \subset |\mathscr{E}|$. Suppose for every $E \in \mathscr{E}$,* $\operatorname{Im}(\mathscr{F} \,\|\, E \to \mathscr{G} \,\|\, E)$ *can be extended coherently to $E \cap D$ as a subsheaf of $\mathscr{G} \,\|\, E$.*

(a) *If $\mathscr{F}_n = \mathscr{F}$, then $\mathscr{F}$ can be extended coherently to D as a subsheaf of $\mathscr{G}$.*

(b) *If $\mathscr{F}_{n-1} = \mathscr{F}$, then* $\operatorname{Supp} \mathscr{F}_n / \mathscr{F}$ *can be extended to a subvariety of D.*

Proof. (a) By Theorem 1b, $\mathscr{F}_{n+1}$ can always be extended coherently to D as a subsheaf of $\mathscr{G}$. Hence, by replacing $\mathscr{G}$ by $\mathscr{F}_{n+1}$, we can assume w.l.o.g. that $\mathscr{G} = \mathscr{F}_{n+1}$. We can also assume that $\mathscr{F} \neq \mathscr{G}$.

Let $X = \operatorname{Supp} \mathscr{G} / \mathscr{F}$. X is a subvariety of pure dimension $n+1$ in $D - V$. By [4, p. 299, Satz 13], the closure X^- of X is a subvariety in D. Let $X^- = \bigcup_i X_i$

be the decomposition into branches. Let $\mathscr{I}_i$ be the ideal-sheaf on D for X_i. By Hilbert Nullstellensatz, there exists an integer k_i such that $\mathscr{I}_i^{k_i}\mathscr{G}\subset\mathscr{F}$ at some point of X_i which is a regular point of X. Let $\mathscr{I}=\Pi_i\mathscr{I}_i^{k_i}$. Since $\mathscr{F}_n=\mathscr{F}$, $\mathscr{I}\mathscr{G}\subset\mathscr{F}$. Let $\mathscr{O}=({}_N\mathscr{O}/\mathscr{I})|X^-$. $\mathscr{F}/\mathscr{I}\mathscr{G}$ can be regarded as a coherent analytic sheaf on $(X,\mathscr{O}|X)$ and $\mathscr{G}/\mathscr{I}\mathscr{G}$ can be regarded as a coherent analytic sheaf on $(X^-,\mathscr{O})$.

Take arbitrarily $x\in\bar{X}\cap V$. After a coordinates transformation, we can assume the following:

(i) $x=0$.

(ii) $\{z_1=\cdots=z_{n+1}=0\}$ intersects X^- in a subvariety of dimension 0.

(iii) $\{z_1=\cdots=z_n=0\}$ is an element of $\mathscr{E}$.

We can choose an open neighborhood U of x in X^- and a connected open neighborhood G of 0 in $\mathbb{C}^{n+1}$ such that the projection $\Pi:\mathbb{C}^N\to\mathbb{C}^{n+1}$ defined by $\Pi(z_1,\ldots,z_N)=(z_1,\ldots,z_{n+1})$ induces a nowhere degenerate proper holomorphic map π from U onto G.

Let $V'=\pi(V\cap U)$ and $V''=\pi^{-1}(V')$. Let $\mathscr{F}'$ be the zero$^{\text{th}}$ direct image of $(\mathscr{F}/\mathscr{I}\mathscr{G})|U-V''$ under $\pi|U-V''$ and let $\mathscr{G}'$ be the zero$^{\text{th}}$ direct image of $(\mathscr{G}/\mathscr{I}\mathscr{G})|U$ under π. $\mathscr{G}'$ is a coherent analytic sheaf on G and $\mathscr{F}'$ is a coherent analytic subsheaf of $\mathscr{G}'|G-V'$. Since $\mathscr{F}_n=\mathscr{F}$, $\mathscr{F}'_n=\mathscr{F}'$.

By shrinking G, we can assume that we have a sheaf-epimorphism η: ${}_{n+1}\mathscr{O}^p\to\mathscr{G}'$ on G. Let $\mathscr{S}=\eta^{-1}(\mathscr{F}')$. $\mathscr{S}$ is a torsion-free coherent analytic subsheaf of ${}_{n+1}\mathscr{O}^p|G-V'$.

Since

$$\{(z_1,\ldots,z_N)\in\mathbb{C}^N\,|\,z_1=\cdots=z_n=0\}$$

is an element of $\mathscr{E}$, there exists an open family $\mathscr{E}'$ of 1-planes in $\mathbb{C}^{n+1}$ such that

$$\{(z_1,\ldots,z_{n+1})\in\mathbb{C}^{n+1}\,|\,z_1=\cdots=z_n=0\}$$

belongs to $\mathscr{E}'$ and $\Pi^{-1}(E')\in\mathscr{E}$ for every $E'\in\mathscr{E}'$. By shrinking G, we can assume that $G\subset|\mathscr{E}'|$.

Since $\operatorname{Im}(\mathscr{F}\,\|\,E\to\mathscr{G}\,\|\,E)$ can be extended coherently to $E\cap D$ as a subsheaf of $\mathscr{G}\,\|\,E$ for every $E\in\mathscr{E}$, $\operatorname{Im}(\mathscr{S}\,\|\,E'\to{}_{n+1}\mathscr{O}^p\,\|\,E')$ can be extended coherently to $E'\cap G$ as a subsheaf of ${}_{n+1}\mathscr{O}^p\,\|\,E'$ for every $E'\in\mathscr{E}'$. By Proposition 7, $\mathscr{S}$ can be extended to a coherent analytic subsheaf $\tilde{\mathscr{S}}$ of ${}_{n+1}\mathscr{O}^p|G$.

Let $\mathscr{S}^*=\pi^*\eta(\tilde{\mathscr{S}})$. $\mathscr{S}^*$ is a coherent analytic subsheaf of $\mathscr{G}/\mathscr{I}\mathscr{G}$ on U. $\mathscr{S}^*$ agrees with $\mathscr{F}/\mathscr{I}\mathscr{G}$ on $U-V''$. Since $(\mathscr{F}/\mathscr{I}\mathscr{G})_n=\mathscr{F}/\mathscr{I}\mathscr{G}$ and $\dim V''\leqq n$, $\mathscr{S}^*[V'']$ is a coherent analytic subsheaf of $\mathscr{G}/\mathscr{I}\mathscr{G}$ on U extending $(\mathscr{F}/\mathscr{I}\mathscr{G})|U-V$. Let $\lambda:\mathscr{G}\to\mathscr{G}/\mathscr{I}\mathscr{G}$ be the natural sheaf-epimorphism. $\lambda^{-1}(\mathscr{S}^*[V''])$ is a coherent analytic subsheaf of $\mathscr{G}$ on $(G-X^-)\cup U$ extending $\mathscr{F}|(G-X^-)\cup U-V$. Since $(G-X^-)\cup U$ is an open neighborhood of x, $\mathscr{F}$ can be extended coherently across x as a subsheaf of $\mathscr{G}$. Since x is an arbitrary point of $X^-\cap V$, $\mathscr{F}$ can be extended coherently to D as a subsheaf of $\mathscr{G}$.

(b) Let $Z=\operatorname{Supp}\mathscr{F}_n/\mathscr{F}$. W.l.o.g. we assume that $Z\neq\emptyset$. Let Z^- be the closure of Z. Take arbitrarily a relatively compact open subset D' of D. We need only show that $Z^-\cap D'$ is a subvariety of D'.

Let $\mathscr{R} = \mathscr{G}/\mathscr{F}$. Since $\mathscr{F}_{n-1} = \mathscr{F}$, by (3), Z is the n-dimensional component of $S_n(\mathscr{R})$.

By Lemma 3, we can choose an open subfamily $\mathscr{E}'$ of $\mathscr{E}$ such that

(i) $|\mathscr{E}'|$ intersects every n-dimensional branch of $V \cap D'$ and

(ii) $\dim E \cap S_\varrho(\mathscr{R}) \cap D' \leqq \varrho - n$ for every $\varrho \geqq 0$ and every $E \in \mathscr{E}'$.

Take arbitrarily $E \in \mathscr{E}'$. By (4),

$$S_0(\mathscr{R} \parallel E) \cap D' = E \cap S_n(\mathscr{R}) \cap D' .$$

Since $\operatorname{Im}(\mathscr{F} \parallel E \to \mathscr{G} \parallel E)$ can be extended coherently to $E \cap D$ as a subsheaf of $\mathscr{G} \parallel E$, $\mathscr{R} \parallel E$ can be extended to a coherent analytic sheaf $\tilde{\mathscr{R}}$ on $E \cap D$. The subvariety $S_0(\tilde{\mathscr{R}}) \cap D'$ therefore extends $E \cap S_n(\mathscr{R}) \cap D'$.

Since E is an arbitrary element of $\mathscr{E}'$ and $|\mathscr{E}'|$ intersects every n-dimensional branch of $V \cap D'$, by Lemma 2, $Z^- \cap D'$ is a subvariety in D'. Q.E.D.

Theorem 2. *Suppose $\mathscr{G}$ is a coherent analytic sheaf on an open subset D of $\mathbb{C}^N$, V is a subvariety of dimension n in D, and $\mathscr{F}$ is a coherent analytic subsheaf of $\mathscr{G} | D - V$ satisfying $\mathscr{F}_{n-1} = \mathscr{F}$. Suppose $\mathscr{E}$ is an open family of $(N-n+1)$-planes in $\mathbb{C}^N$ such that $D \subset |\mathscr{E}|$. Then $\mathscr{F}$ can be extended coherently to D as a subsheaf of $\mathscr{G}$ if and only if, for every $E \in \mathscr{E}$, $\operatorname{Im}(\mathscr{F} \parallel E \to \mathscr{G} \parallel E)$ can be extended coherently to $E \cap D$ as a subsheaf of $\mathscr{G} \parallel E$.*

Proof. We need only prove the "if" part, because the "only if" part is trivial.

Let $\mathscr{E}' = \{E' \in P_{N,N-n} | E' \subset E$ for some $E \in \mathscr{E}\}$. $\mathscr{E}'$ is an open family of $(N-n)$-planes. For every $E' \in \mathscr{E}'$, $\operatorname{Im}(\mathscr{F} \parallel E' \to \mathscr{G} \parallel E')$ can be extended coherently to $E' \cap D$ as a subsheaf of $\mathscr{G} \parallel E'$.

Let $Z = \operatorname{Supp} \mathscr{F}_n/\mathscr{F}$. By applying Proposition 8 (b) to $\mathscr{E}'$, we conclude that the closure Z^- of Z is a subvariety of D.

$\mathscr{F}_n = \mathscr{F}$ on $D - Z^-$. By applying Proposition 8 (a) to $D - Z^-$ and $\mathscr{E}'$, we conclude that $\mathscr{F} | D - Z^- - V$ can be extended to a coherent analytic subsheaf $\mathscr{F}'$ of $\mathscr{G} | D - Z^-$. We can assume that $\mathscr{F}'_n = \mathscr{F}'$.

Let $V^* = V \cap Z^-$. $\dim V^* \leqq n-1$. Let $\mathscr{F}^*$ be the subsheaf of $\mathscr{G} | D - V^*$ which agrees with $\mathscr{F}$ on $D - V$ and agrees with $\mathscr{F}'$ on $D - Z^-$. $\mathscr{F}^*$ is a coherent extension of $\mathscr{F}$. $\mathscr{F}^*_n = \mathscr{F}^*$ on $V - V^*$.

Take arbitrarily a relatively compact open subset D^* of D. To finish the proof, it suffices to show that $\mathscr{F}^* | D^* - V^*$ can be extended coherently to D^* as a subsheaf of $\mathscr{G}$.

By Lemma 3, we can choose an open subfamily $\mathscr{E}^*$ of $\mathscr{E}$ such that

(i) $|\mathscr{E}^*|$ intersects every $(n-1)$-dimensional branch of $V^* \cap D^*$,

(ii) $\dim E \cap (V - V^*) \cap D^* \leqq 1$ for every $E \in \mathscr{E}^*$, and

(iii) $\dim E \cap S_\varrho(\mathscr{G}/\mathscr{F}^*) \cap D^* \leqq \varrho - n + 1$ for every $\varrho \geqq 0$ and every $E \in \mathscr{E}^*$.

Take arbitrarily $E \in \mathscr{E}^*$. $\mathscr{F} \parallel E$ can be extended to a coherent analytic subsheaf $\mathscr{S}$ of $\mathscr{G} \parallel E$ on $E \cap D$. We can assume that $\mathscr{S}[E \cap V] = \mathscr{S}$.

Let $W = E \cap (V - V^*) \cap D^*$. Since $\mathscr{F}^*_n = \mathscr{F}^*$ at $V - V^*$, by (6), $(\mathscr{F}^* \parallel E)_1 = (\mathscr{F}^* \parallel E)$ at W. Since $\dim W \leqq 1$, $(\mathscr{F}^* \parallel E)[W] = \mathscr{F}^* \parallel E$. Since $\mathscr{F}^* \parallel E$ and $\mathscr{S}$ agree on $E \cap (D - V)$, $\mathscr{F}^* \parallel E$ agrees with $\mathscr{S}$ on $E \cap (D^* - V^*)$. The coherent analytic subsheaf $\mathscr{S}$ of $\mathscr{G} \parallel E$ on $E \cap D^*$ therefore extends $\mathscr{F}^* \parallel E$.

Since E is an arbitrary element of $\mathscr{E}^*$ and $|\mathscr{E}^*|$ intersects every $(n-1)$-dimensional branch of $V^* \cap D^*$, by Proposition 8 (a), $\mathscr{F}^*$ can be extended coherently across some point of every $(n-1)$-dimensional branch of $V^* \cap D^*$ as a subsheaf of $\mathscr{G}$. Since $\mathscr{F}^*_{n-1} = \mathscr{F}^*$, by Theorem 1 b, $\mathscr{F}^*$ can be extended coherently to D^* as a subsheaf of $\mathscr{G}$. Q.E.D.

Remark. The theorem for subvariety extension corresponding to Theorem 2 requires only that $\mathscr{E}$ is an open family of $(N-n)$-planes. For subsheaf extension such an assumption is not enough. The following is a counter-example.

$$D = \mathbb{C}^2. \qquad V = \{z_1 = 0\}. \qquad \mathscr{G} = {}_2\mathscr{O}^2.$$

$\mathscr{F}$ is the subsheaf of $\mathscr{G} | D - V$ generated by $\left(1, \exp\frac{1}{z_1}\right)$, $(z_1 - z_2^2, 0)$, and $(0, z_1 - z_2^2)$. $\mathscr{E} = P_{2,1}$. For every $E \in \mathscr{E}$, $\operatorname{Im}(\mathscr{F} \| E \to \mathscr{G} \| E)$ differs from $\mathscr{G} \| E$ only at a finite number of points and hence can be extended coherently to $E \cap D$ as a subsheaf of $\mathscr{G} \| E$. $\mathscr{F}$ cannot be extended coherently to D as a subsheaf of $\mathscr{G}$, otherwise $\mathscr{F}/(z_1 - z_2^2)\mathscr{G}$ can be extended coherently to D as a subsheaf of $\mathscr{G}/(z_1 - z_2^2)\mathscr{G}$, which implies that the holomorphic function on $\{z_1 = z_2^2\} - \{0\}$ induced by $\exp\frac{1}{z_1}$ can be extended to a meromorphic function on $\{z_1 = z_2^2\}$.

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Remarks on some Variable Domain Problems in Abstract Evolution Equations

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1. Introduction

We will first establish some notation, describe briefly the problems under consideration, and indicate some of the results. Thus let H be a separable Hilbert space and let $V(t) \subset H$ be a family of Hilbert spaces, dense in H with continuous injections $i(t): V(t) \to H$ $(0 \leqq t \leqq T < \infty)$. Let $R(t): H \to V(t)$ be the operator defined by the formula $((R(t)x, y))_t = (x, y)$ for $x \in H$ and $y \in V(t)$ where $((\ ,\ ))_t$ (resp. $(\ ,\ )$) denotes the scalar product in $V(t)$ (resp. H). The operator $R(t)$ is completely "intrinsic" for $V(t)$, relative to H, and in fact, if $\tilde{\theta}: H' \to H$ is the canonical isomorphism between H and its antidual H', determined by the rule $(\tilde{\theta}h', h) = \langle h', h \rangle$, then $L(t) = R(t)\tilde{\theta}$ is the Schwartz kernel of $V(t)$ relative to H (cf. [10, 11, 33]). We consider $T(t) = R^{-1}(t)$ as a self-adjoint positive operator in H and it follows that $S(t) = T^{\frac{1}{2}}(t)$ maps its domain $V(t) = D(S(t))$ one to one onto H with $((x, y))_t = (S(t)x, S(t)y)$ for $x, y \in V(t)$ (see [10, 11]). We will refer to $S(t)$ as the standard or characteristic operator for $V(t)$ relative to H. As an example one notes that if $V(t) = D(B(t))$ with graph Hilbert structure then $T(t) = 1 + B^*(t)B(t)$ ($B(t)$ being a closed densely defined operator in H). Further we will make the customary identifications in writing $V(t) \subset H \subset V'(t)$, where $V'(t)$ is the antidual of $V(t)$, so that in particular $\langle x, y \rangle = (x, y)$ for $x \in H$ and $y \in V(t)$.

To begin let $W = L^2(V(t)) = \{u \in L^2(H);\ u(t) \in V(t) \text{ a.e.};\ Su \in L^2(H)\}$ with scalar product $(u, v)_W = \int_0^T ((u(t), v(t)))_t\, dt = \int_0^T (Su, Sv)\, dt$ and let $a(t, \cdot, \cdot)$ be a (suitable) continuous sesquilinear form on $V(t) \times V(t)$. The first question we deal with (see Section 3) concerns the uniqueness of solutions $u \in W$ of the weak problem of finding $u \in W$ such that given $f \in L^2(H)$ and $u_0 \in H$ one has

$$-\int_0^T (u, v')\, dt + \int_0^T a(t, u, v)\, dt = \int_0^T (f, v)\, dt + (u_0, v(0)) \tag{1.1}$$

for all $v \in W$ with $v' \in L^2(H)$ and $v(T) = 0$ ($'$ in $\mathscr{D}'(H)$). There are a number of results in this direction due to Lions [27, 28] and Baiocchi [1, 2] (cf. also Bardos [3]) in which one assumes in particular that the $V(t)$ are all closed subspaces of a fixed Hilbert space $K \subset H$. This is of course very natural in many applications but one can avoid such a constraint a priori in various ways which

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exploit the intrinsic characterization of $V(t)$ in terms of $S(t)$. Thus for example the variation of the $V(t)$ can be taken into account by means of hypotheses on $S^{-\alpha}(t) \in \mathscr{L}(H)$ and this was the basis of some regularity and uniqueness results of [12, 13] (with too strong hypotheses for uniqueness) which were subsequently refined and generalized in [11, 14, 16]. We mention in passing that use of the $S(t)$ lends itself to various notions of operator homotopy (see [15]). Section 3 develops this further and will deal with a large class of uniqueness and regularity theorems for weak linear problems using more general operators (instead of $S^{-\alpha}(t)$) to place oneself in $V(t)$. These results extend to the context of Remark 3.11 (as in [11, 16]) and apply then to various noncylindrical evolution problems (cf. [11, 16]). This point of view is also related to the technique of [28] and is further exploited in Section 4 to prove our main theorem (Theorem 4.8) which in particular leads to an extension of the results of [28] (see Section 5). We note explicitly however, that the results of Section 3 apply to some cases not covered by Theorem 4.8 (see Remark 3.12).

In Section 4 we will work with various nonlinear versions of (1.1) and will rephrase it slightly for this purpose by writing $a(t,x,y) = \langle A(t)x, y\rangle$ in the linear case where $A(t) \in \mathscr{L}(V(t), V'(t))$ ($V'(t)$ is the antidual of $V(t)$). Then for nonlinear situations one considers for example, divergence type differential expressions $\sum D^\alpha a_\alpha(t,\xi,u,Du,\ldots,D^m u)$, in say $H_0^m(\Omega)$, $\Omega \subset \mathbf{R}^n$, where $|\alpha| \leq m$ and $D^\beta = D_1^{\beta_1} \ldots D_n^{\beta_n}$ with $D_k = \frac{1}{i}\partial/\partial\xi_k$; this gives rise to a form $a(t,u,v) = \sum \int_\Omega a_\alpha(t,\xi,u,\ldots,D^m u)\,\overline{D^\alpha v}\,d\xi = \langle A(t)u, v\rangle$ with $A(t)$ nonlinear. Now set $W = L^p(V(t)) = \left\{u \in L^p(H); u(t) \in V(t) \text{ a.e.}; \int_0^T \|u(t)\|_t^p\,dt < \infty\right\}$, $1 < p < \infty$, and in our situation W will be reflexive with $W' = L^q(V'(t))$ where $\frac{1}{p} + \frac{1}{q} = 1$. Let $L: W \to W'$ be defined by $Lu = u'$ with $D(L) = \{u \in W; u' \in L^q(H); u(0) = 0\}$ and $L': W \to W'$ be determined by $L'u = -u'$ with $D(L') = \{u \in W;\ u' \in L^q(H); u(T) = 0\}$. We will use natural hypotheses to assure that L and $L' \subset L^*$ are densely defined so that in particular L is closable and we write $\bar{L} = L_s$. It is technically somewhat easier to take $u_0 = 0$ in (1.1) and we shall do this throughout in remarking that $u_0 \neq 0$ can be restored by various technical devices (see e.g. [6–8, 11, 27]). Then if $f \in W'$ and $A: W \to W'$ is defined by $\langle Au, v\rangle = \int_0^T \langle A(t)u, v\rangle\,dt$ one can rephrase a nonlinear version of (1.1) (with (f, v) replaced by the more general term $\langle f, v\rangle$) in the form

$$L_w u + Au = f, \tag{1.2}$$

where $L_w = L'^*$ makes good sense when L' is densely defined. On the other hand one is naturally interested in the strong problem (cf. Browder [7–9], Brezis [5, 6], Bardos-Brezis [4], and Lions-Magenes [29]).

$$L_s u + Au = f. \tag{1.3}$$

When $V(t)=V$ these two problems are easily seen to be equivalent (see [7, 8, 11]) and the question of equivalence for variable $V(t)$ was raised in [14]. Equivalence means that $L_s=L_w$ and this is also essentially the condition which yields strong existence under natural hypotheses of monotonicity and coercivity on A (strong uniqueness is immediate under such hypotheses). This condition can also be thought of more "generically" as a requirement of maximal monotonicity on L_s (cf. [5, 6, 9]). We give two equivalence theorems in Section 4 and although Theorem 4.8 is actually stronger thanTheorem 4.7 we believe there is enough interesting information in the proof of Theorem 4.7 to justify its inclusion. The main result is that if $S^{-2}(\cdot)$ is weakly C^1 and $p \geqq 2$ then $L_s=L_w$ and uniqueness holds in (1.2) for monotone nonlinear $A(t)$. The requirement $p \geqq 2$ seems to be a defect in the technique since we have no reason to suppose the situation is altered for $1<p<2$. We remark in passing that under suitable hypotheses one can take the $V(t)$ to be Banach spaces and obtain again $W'=L^q(V'(t))$ (cf. [14]) which leads to a similar equivalence problem for which however, the operators $S(t)$ are no longer available; we will deal only with Hilbert spaces $V(t)$ in the present paper.

Section 2 is devoted to some technical lemmas while Section 5 shows also how the uniqueness result of [28] can be derived from Theorem 4.8. Section 5 also contains two very elementary examples which illustrate some technical points. Interesting concrete examples to illustrate the theory of Section 3 and 4 can be found in the books [11, 27]. Thus, in the context of Section 5, the results of Section 4 apply to various concrete situations in [27] while the results of Section 3, phrased in the context of Remark 3.11, apply to concrete examples as in [11].

2. Technical Lemmas

First let us establish some basic framework (cf. [11] for further details). We will always assume, or make hypotheses which assure, that

$$S^{-1}(\cdot) \text{ is a measurable family on } [0,T] \text{ in the sense that } S^{-1}(\cdot)\,h \text{ is } H \text{ measurable for } h \in H, \tag{2.1}$$

$$\|S^{-1}(t)\| \leqq c. \tag{2.2}$$

In this event we can write $L^2(V(t))=\int^{\oplus} V(t)\,dt$, in the terminology of [25], relative to a suitable vector space $E \subset \Pi\, V(t)$ of measurable vectors (it suffices that $S^{-1}(\cdot)\,h \in E$ for $h \in H$ – see [11] for details) and $L^p(V(t))$ can be thought of in a similar manner following [21], where $S^{-1}\mathscr{D}(H)$ for example, can be taken as a fundamental family of continuous vector fields. In particular $L^p(V(t))$ is not empty and, as indicated in [21], $L^p(V(t))'=L^q(V'(t))$ since the maps $\theta(t) \in \mathscr{L}(V'(t), V(t))$ defined by $((\theta(t)\,x, y))=\langle x, y\rangle$ for $x \in V'(t)$ and $y \in V(t)$ induce an isometric isomorphism $L^q(V'(t)) \to L^q(V(t))$. One can also write, in an obvious notation, $L^p(V(t))=S^{-1}L^p(H)$.

We go now to some technical lemmas which will be useful later. First we recall that if $\Gamma(t) \in \mathscr{L}(H)$ is a family of operators which are weakly C^1 in H then there is an operator $\dot{\Gamma}(t) \in \mathscr{L}(H)$ such that for $h, k \in H$

$$\frac{d}{dt}(\Gamma(t)\,h, k) = (\dot{\Gamma}(t)\,h, k) \tag{2.3}$$

with $\|\dot{\Gamma}(t)\| \leqq c_1$ and $\|\Gamma(t)\| \leqq c_2$ on $[0, T]$ while $\Gamma(\cdot)$ is Lipschitz continuous in norm (cf. [11, Lemma 4.5.7]). Further if $R(\Gamma(t)) \subset V(t)$ then $S\Gamma \in \mathscr{L}(H)$ by a simple closure verification (cf. [11]). Now we generalize a remark in [14] (cf. also [23]).

Lemma 2.1. *Let $t \to \Gamma(t) \in \mathscr{L}(H)$ be weakly C^1 on $[0, T]$ with $R(\Gamma(t)) \subset V(t)$, $S\Gamma(\cdot)$ measurable with $\|S\Gamma(t)\| \leqq c_3$, and $(S\Gamma)^*(t)$ one to one. Assume* (2.1)–(2.2). *Then L and L′ are densely defined.*

Proof. We note first that $\Gamma\psi \in W$ for $\psi \in \mathscr{D}(H)$. Indeed $\|\Gamma(t)\| \leqq c_2$ since $\Gamma(\cdot)$ is weakly C^1 while $\Gamma(\cdot)$ is also trivially measurable so that $\Gamma\psi \in L^p(H)$. Similarly $S\Gamma\psi \in L^p(H)$ under our hypotheses so $\Gamma\psi \in W = L^p(V(t))$. Evidently $(\Gamma\psi)' = \dot{\Gamma}\psi + \Gamma\psi' \in L^q(H)$ ($'$ in $\mathscr{D}'(H)$) so $\Gamma\mathscr{D}(H) \subset D(L) \cap D(L')$. It suffices therefore to show that $\Gamma\mathscr{D}(H)$ is dense in W. Suppose $u \in W'$ with $\langle u, \Gamma\psi\rangle = 0$ for all $\psi \in \mathscr{D}(H)$. Then for all such ψ

$$\begin{aligned}\int_0^T \langle u, \Gamma\psi\rangle\, dt &= \int_0^T ((\theta u, \Gamma\psi))\, dt = \int_0^T ((S^{-1}\eta, \Gamma\psi))\, dt \\ &= \int_0^T (\eta, S\Gamma\psi)\, dt = \int_0^T ((S\Gamma)^*\eta, \psi)\, dt\end{aligned} \tag{2.4}$$

where $\theta u \in L^q(V(t))$ determines $\eta = S\theta u$ uniquely. Since $\mathscr{D}(H)$ is dense in $L^q(H)$ we have $(S\Gamma)^*\eta = 0$ whence by hypotheses $\eta = u = 0$. Q.E.D.

Lemma 2.2. *Let $t \to \Lambda(t) \in \mathscr{L}(H)$ be norm continuous on $[0, T]$ where $\Lambda(t)$ is self adjoint positive. Let $\sigma = \cup\, \sigma(\Lambda(t))$ be the union of spectra of the $\Lambda(t)$ (so $\bar{\sigma}$ is a bounded set in $\mathbf{R}^+$ since $\|\Lambda(t)\| \leqq c_4$ on $[0, T]$ by continuity) and let f be any continuous complex valued function on $\bar{\sigma}$. Then $t \to f(\Lambda(t))$ is norm continuous on $[0, T]$.*

Proof. If p is a complex polynomial then obviously $p(\Lambda(t))$ is norm continuous. By Stone-Weierstrass we can approximate any f as above uniformly on $\bar{\sigma}$ by polynomials p_n. Now for any fixed t we consider the uniformly closed *-algebra generated by $\Lambda(t)$ and the identity in $\mathscr{L}(H)$ and apply the Gelfand-Naimark theorem (cf. [11, 30, 31]) to conclude that

$$\begin{aligned}\|f(\Lambda(t)) - p_n(\Lambda(t))\| &\leqq \sup_{x \in \sigma(t)} |f(x) - p_n(x)| \\ &\leqq \sup_{x \in \bar{\sigma}} |f(x) - p_n(x)|\,.\end{aligned} \tag{2.5}$$

One can think of $f(\Lambda(t))$ as defined by the limiting process or by a spectral integral $\int f(x)\,dE_x(t)$ where the $dE_x(t)$ are a spectral resolution for $\Lambda(t)$ (cf. [11, 32]). In any case (2.5) shows that $f(\Lambda(t))$, being the uniform limit in norm of $p_n(\Lambda(t))$ on $[0, T]$, is continuous in norm. Q.E.D.

Remark 2.3. A case of particular importance in Lemma 2.1 occurs when $\Gamma(t)=S^{-\alpha}(t)$ for $\alpha \geqq 1$ (one can define $S^{-\alpha}(t)=\int \lambda^{\alpha} dE_{\lambda}(t)$, where $dE_{\lambda}(t)$ is a spectral resolution for $S^{-1}(t)$, for each t, without assuming (2.1) or (2.2)). Then $S\Gamma = S^{1-\alpha}$ and Lemma 2.2 applies since $S^{-\alpha}(\cdot)$ is norm continuous with $S^{1-\alpha} = (S^{-\alpha})^{1-1/\alpha}$ where $1-1/\alpha \geqq 0$. Thus $S\Gamma$ is automatically measurable with $\|S\Gamma(t)\| \leqq c_3$ and furthermore $(S\Gamma)^*(t)=S^{1-\alpha}(t)$ is one to one. Finally $S^{-1} = (S^{-\alpha})^{1/\alpha}$ so by Lemma 2.2 again S^{-1} is measurable with $\|S^{-1}(t)\| \leqq c$ on $[0, T]$ and this is (2.1)–(2.2). Hence in particular if $S^{-\alpha}(\cdot)$ is weakly C^1 for $\alpha \geqq 1$ it follows that L and L' are densely defined.

Lemma 2.4. *Let $t \to \Gamma(t) \in \mathscr{L}(H)$ be weakly C^1 on $[0, T]$ with $R(\Gamma(t)) \subset V(t)$, $S\Gamma(\cdot)$ measurable with $\|S\Gamma(t)\| \leqq c_3$, and assume* (2.1–(2.2). *Let L' be densely defined (e.g. let $(S\Gamma)^*(t)$ be 1—1) and suppose $x \in D(L_w)$ where $L_w = L'^*$ with $L_w x = y$. Then in $\mathscr{D}'(H)$*

$$(\Gamma^* x)' = (S\Gamma)^* S\theta y + \dot{\Gamma}^* x\,. \tag{2.6}$$

Proof. From $L_w x = y$ we have $\langle y, v\rangle = \langle x, L'v\rangle$ for all $v \in D(L')$. Hence

$$\begin{aligned}\int_0^T (S\theta y, Sv)\,dt &= \int_0^T ((\theta y, v))_t\,dt = \langle y, v\rangle = \langle x, L'v\rangle \\ &= -\int_0^T \langle x, v'\rangle\,dt = -\int_0^T (x, v')\,dt\,.\end{aligned} \tag{2.7}$$

Now take $v = \Gamma\psi$ for $\psi \in \mathscr{D}(H)$ to obtain

$$-\int_0^T (\Gamma^* x, \psi')\,dt = \int_0^T (S\theta y, S\Gamma\psi)\,dt + \int_0^T (x, \dot{\Gamma}\psi)\,dt\,. \tag{2.8}$$

This implies (2.6) by definitions (cf. [11]). Q.E.D.

Corollary 2.5. *Under the hypotheses of Lemma 2.4 if $p \geqq 2$ then $(\Gamma^* x)' \in L^q(H)$.*

Proof. Evidently $S\theta y \in L^q(H)$ (since $y \in W'$) with $\|(S\Gamma)^*(t)\| \leqq c_3$ when $\|S\Gamma(t)\| \leqq c_3$ so $(S\Gamma)^* S\theta y \in L^q(H)$. If $p \geqq 2$ then $q = 1 + \dfrac{1}{p-1} \leqq 2 \leqq p$ with $L^p \subset L^2 \subset L^q$ on $[0, T]$ and for $f \in L^p$ one has $c(T)\,|f|_q \leqq |f|_2 \leqq \hat{c}(T)\,|f|_p$ (cf. [25]). Hence, since $\|\dot{\Gamma}^*(t)\| \leqq c_1$ when $\|\dot{\Gamma}(t)\| \leqq c_1$, we have $\dot{\Gamma}^* x \in L^q(H)$. Q.E.D.

We will sketch now a version of a lemma from [18].

Lemma 2.6. *Let $G(t)$ be a family of (unbounded) self adjoint operators in H mapping $1-1$ onto H such that $t \to G^{-1}(t) \in \mathscr{L}(H)$ is weakly C^1. If $\lambda \in C$ with $\lambda \notin \sigma(G(t))$ for all $t \in [0, T]$ then $R_\lambda(t) = (\lambda I - G(t))^{-1}$ is weakly C^1 with $\dot{R}_\lambda = -G R_\lambda \dot{G}^{-1} G R_\lambda$.*

Proof. We note first that by a version of the spectral mapping theorem (cf. [20]) one knows that $\sigma(G^{-1}(t)) = (1/\sigma(G(t)) \cup \infty)$; thus if $\lambda \notin \sigma(G(t))$, $\lambda \neq \infty$ in C, then $\frac{1}{\lambda} \notin \sigma(G^{-1}(t))$. Now by previous remarks $G^{-1}(\cdot)$ is norm continuous as is $t \to G^{-1}(\cdot) - (I/\lambda)$. Since $G \to G^{-1}$ is continuous within the open set of invertible elements of $\mathscr{L}(H)$ it follows that $t \to (G^{-1}(t) - (I/\lambda))^{-1}$ is norm continuous (note that $(G^{-1}(t) - (I/\lambda))^{-1} = \lambda G(t) R_\lambda(t) = \lambda(\lambda R_\lambda(t) - I)$). Hence there exists $\varrho > 0$ such that $d\left(\frac{1}{\lambda}, \sigma(G^{-1}(t))\right) \geqq \varrho > 0$ for all $t \in [0, T]$ whenever $\lambda \notin \sigma(G(t))$ for $t \in [0, T]$. Let now $h(x) = 1/x\lambda - 1 = \frac{1}{\lambda}\left(x - \frac{1}{\lambda}\right)^{-1}$ for $x \in \mathbf{R}$. Then $h(x)$ is continuous on the bounded set $\sigma = \cup \sigma(G^{-1}(t))$ and in fact on $\bar{\sigma}$ by the above comments. From Lemma 2.2 it follows that $G(t) R_\lambda(t) = h(G^{-1}(t))$ is norm continuous. Now consider

$$
\begin{aligned}
R_\lambda(t) - R_\lambda(\xi) &= R_\lambda(\xi)(\lambda I - G(\xi)) R_\lambda(t) - R_\lambda(\xi)(\lambda I - G(t)) R_\lambda(t) \\
&= G(\xi) R_\lambda(\xi)(\lambda G^{-1}(\xi) G^{-1}(t) - G^{-1}(t)) G(t) R_\lambda(t) \\
&\quad - G(\xi) R_\lambda(\xi)(\lambda G^{-1}(\xi) G^{-1}(t) - G^{-1}(\xi)) G(t) R_\lambda(t) \\
&= G(\xi) R_\lambda(\xi)(G^{-1}(\xi) - G^{-1}(t)) G(t) R_\lambda(t)
\end{aligned}
\tag{2.9}
$$

then for $f, g \in H$ one has

$$
\begin{aligned}
&\frac{1}{t-\xi}((R_\lambda(t) - R_\lambda(\xi)) f, g) \\
&= \frac{1}{t-\xi}(G(\xi) R_\lambda(\xi)(G^{-1}(\xi) - G^{-1}(t)) G(t) R_\lambda(t) f, g) \\
&= \frac{1}{t-\xi}((G^{-1}(\xi) - G^{-1}(t)) G(t) R_\lambda(t) f, G(\xi) R_{\bar{\lambda}}(\xi) g).
\end{aligned}
\tag{2.10}
$$

Since $G(\cdot) R_\lambda(\cdot)$ is continuous in norm it follows easily that the limit exists in (2.10) when $t \to \xi$ and represents a continuous function $(\dot{R}_\lambda(\cdot) f, g) = (\dot{G}^{-1}(\cdot) G(\cdot) R_\lambda(\cdot) f, G(\cdot) R_{\bar{\lambda}}(\cdot) g)$. Q.E.D.

Next we observe that if $S^{-1}(\cdot)$ is weakly C^1 then so is $S^{-2}(\cdot)$. Indeed one need only write out the difference quotients and use the fact that $S^{-1}(\cdot)$ is then norm continuous. Similarly any polynomial in $S^{-1}(\cdot)$ will be weakly C^1. We want to extend this slightly in order to compare hypotheses in Sections 3 and 4. Now recall first that powers $S^{-\alpha}(t)$ can be defined for example by $S^{-\alpha}(t) = \int \lambda^\alpha dE_\lambda(t)$ as in Remark 2.3 where the $dE_\lambda(t)$ are a spectral resolution for $S^{-1}(t)$; this does not use (2.1)–(2.2). We prove now

Proposition 2.7. *Let $S^{-\beta}(\cdot)$ be weakly C^1 and $\alpha \geqq \beta \geqq 1$. Then $S^{-\alpha}(\cdot)$ is weakly C^1.*

Proof. Set $S^\beta = A$ and $\gamma = \alpha/\beta > 1$ so that $S^{-\alpha}(t) = A^{-\gamma}(t)$ and suppose $\gamma < n+1$ for some integer n but $\gamma \neq$ integer ($\gamma =$ integer is covered above). Now

(2.1)–(2.2) hold as in Remark 2.3 and, by Lemma 2.2, $S^{-\beta/2}(t) = (S^{-\beta}(t))^{1/2}$ is norm bounded by say c_5^{-1}. Hence $(S^{\beta}x, x) = |S^{\beta/2}x|^2 \geqq c_5^2|x|^2$ and by well known results (see e.g. [11, 24, 26])

$$S^{-\alpha}(t) = \frac{n!\,\mathrm{Sin}\,\pi\gamma}{\pi(-\gamma+1)\dots(-\gamma+n)} \int_0^\infty \lambda^{-\gamma+n} R_\lambda^{n+1}(-A(t))\,d\lambda \tag{2.11}$$

where $R_\lambda(-A) = (\lambda I + A)^{-1}$ (note that $\|R_\lambda(-A)\| \leqq 1/(\lambda + c_5^2)$ for $\lambda \geqq 0$). Now applying Lemma 2.6 to $G(t) = -A(t)$ we find that $R_\lambda(-A(\cdot))$ is weakly C^1 with $\dot{R}_\lambda = -AR_\lambda \dot{A}^{-1} AR_\lambda$. But then $R_\lambda^{n+1}(-A(\cdot)) = f^{n+1}(t)$ is weakly C^1 by remarks above and in order to differentiate under the integral sign in (2.11) it suffices to establish suitable bounds on the difference quotients $\Delta f/\Delta t$ for large values of λ (there is obviously no trouble near $\lambda = 0$). Thus writing $\Delta t = t - t_0$

$$\frac{\Delta f^{n+1}}{\Delta t} = \sum_{k=0}^{n} f^k(t)\left(\frac{f(t)-f(t_0)}{t-t_0}\right) f^{n-k}(t_0) \tag{2.12}$$

and since $((A+\lambda I)x, x) \geqq \lambda|x|^2$ one has $\|f(t)\| = \|R_\lambda(-A(t))\| \leqq 1/\lambda$ for λ large. Now $\|f^k(t)\| \leqq \|f(t)\|^k \leqq \lambda^{-k}$ and since $\|AR_\lambda(-A)\| \leqq 1$ it follows that $\|\dot{R}_\lambda\| \leqq \|\dot{A}^{-1}\| \leqq c_\beta$ which in turn implies that $\|\Delta f/\Delta t\| \leqq c_\beta$. Therefore, using (2.12), we obtain $\lambda^{-\gamma+n}\|\Delta f^{n+1}/\Delta t\| \leqq c_\beta \lambda^{-\gamma+1}$ which is integrable near infinity since $\gamma > 1$. Then, by the Lebesgue dominated convergence theorem for example, (2.11) can be differentiated under the integral sign (cf. [11]). Q.E.D.

The proof can be slightly simplified by noting that it suffices to consider $1 < \gamma < 2$ but we will not spell this out.

3.

In this section we will give a new class of regularity and uniqueness theorems for (1.1) which generalize results of [11, 14, 16]. The context in [11, 16] is somewhat more general in that the $V(t)$ are not required to be dense (and we will say something about this below); however, the present results can be extended to that context in a routine manner so we take advantage of a technically simpler situation. Upon specializing our hypotheses we will give some interpretation of the results and hypotheses of [11, 14, 16]. We will only deal here with linear problems and $p = 2$ (so that $W = L^2(V(t))$) but remark that the theorems have obvious nonlinear extensions as in Section 4; we continue with $u_0 = 0$ as indicated in Section 1.

Let $a(t, \cdot, \cdot)$ be a continuous sesquilinear form on $V(t) \times V(t)$ with $|a(t, x, y)| \leqq c_6 \|x\|_t \|y\|_t$ for $x, y \in V(t)$. One writes $a(t, x, y) = ((\mathscr{A}(t)x, y))_t$ where $\mathscr{A}(t) \in \mathscr{L}(V(t))$ and we suppose $\mathscr{A}(\cdot)$ is a measurable family of operators (cf. [11, 19]). In view of the estimate on $a(t, x, y)$ above this means here that $\mathscr{A}u \in W$ when $u \in W$ and then $\int_0^T a(t, u(t), v(t))\,dt$ makes sense for $u, v \in W$ (note $\|\mathscr{A}(t)\| \leqq c_6$ so $\mathscr{A}: W \to W$ is continuous). This situation is to be understood whenever (1.1) is mentioned. If $\mathrm{Re}\, a(t, x, x) \geqq \alpha\|x\|_t^2 - \lambda|x|^2$ for $x \in V(t)$ (coercivity) then one knows that there exist solutions of (1.1) (cf. [11, 27]). One should mention here that for special f and u_0 there will be noncoercive situations where existence also obtains (e.g. consider the backward heat equation with suitably analytic data). The uniqueness results of this section seem to apply to both coercive and noncoercive situations (see e.g. Remark 3.14) so one envisions some real applicability also in noncoercive cases.

Lemma 3.1. *Let u satisfy* (1.1) *and let* (2.1)–(2.2) *hold. Let $t \to \Gamma(t) \in \mathscr{L}(H)$ be weakly C^1 on $[0, T]$ with $R(\Gamma(t)) \subset V(t)$ and assume $S\Gamma(\cdot)$ is measurable with $\|S\Gamma(t)\| \leqq c_3$ and $(S\Gamma)^*(t)$ one to one. Then in $\mathscr{D}'(H)$*

$$(\Gamma^* u)' = \dot{\Gamma}^* u + \Gamma^* f - (S\Gamma)^* S \mathscr{A} u. \tag{3.1}$$

Proof. Writing $a(t, u, v) = \langle A(t) u, v \rangle$ with $A(t) \in \mathscr{L}(V(t), V'(t))$ (1.1), with $u_0 = 0$, becomes (1.2) (note that $D(L')$ is dense under our hypotheses by Lemma 2.1). Observe here that we have the relation $\langle Au, v \rangle = ((\theta A u, v)) = ((\mathscr{A} u, v))$ which implies that $\mathscr{A} = \theta A$ and thus the $A(t)$ determine a continuous map $A: W \to W'$. Now apply Lemma 2.4 with $x = u$ and $y = f - Au$ to conclude that

$$(\Gamma^* u)' = (S\Gamma)^* S \theta (f - Au) + \dot{\Gamma}^* u. \tag{3.2}$$

We observe next that under our identifications $S^{-2}(t)$ is the restriction of $\theta(t)$ to H. Indeed $\langle w, v \rangle = (S\theta w, Sv)$ for $w \in H$ and $v \in V(t)$ so that $\langle w, v \rangle = (w, v)$ means $S\theta w \in D(S)$ with $S^2 \theta w = w$; thus $\theta w = S^{-2} w$ for $w \in H$. Now to obtain (3.1) from (3.2) we use the relation $\theta A = \mathscr{A}$ and the formula, for $g, h \in H$,

$$((S\Gamma)^* S^{-1} h, g) = (S^{-1} h, (S\Gamma) g) = (h, \Gamma g) = (\Gamma^* h, g) \tag{3.3}$$

which implies that $(S\Gamma)^* S^{-1} f = \Gamma^* f$ in (3.2). Q.E.D.

Corollary 3.2. *Under the hypotheses of Lemma* 3.1 *it follows that* $(\Gamma^* u)' \in L^2(H)$.

Proof. This is Corollary 2.5 applied to the present situation. Q.E.D.

Lemma 3.1 and Corollary 3.2 can be thought of as regularity results (cf. [11–14, 27] and see also Remark 3.4 – in [6, 7] related regularity holds in $\mathscr{D}'(V')$ where $V \subset V(t)$). Now we add a term $\lambda(u, v)$ to the right side of (1.1) in the standard manner (cf. [11, 27]); this corresponds to a change of variables and does not affect questions of existence or uniqueness. This gives rise to a term $-\lambda \Gamma^* u$ on the right side of (3.1). Then taking scalar products in (3.1) with $\Gamma^* u$ and integrating the real parts we obtain ($f = 0$)

$$\begin{aligned} &|\Gamma^* u|^2(T) + 2 \operatorname{Re} \int_0^T ((S\Gamma)^* S \mathscr{A} u, \Gamma^* u)\, dt \\ &\quad + 2\lambda \int_0^T |\Gamma^* u|^2\, dt - 2 \operatorname{Re} \int_0^T (\dot{\Gamma}^* u, \Gamma^* u)\, dt = 0. \end{aligned} \tag{3.4}$$

Now $((S\Gamma)^* S \mathscr{A} u, \Gamma^* u) = (S \mathscr{A} u, S\Gamma \Gamma^* u) = a(t, u, \Gamma \Gamma^* u)$ and we will phrase our hypotheses relevant to this term on the latter expression. The natural hypotheses extending those of [11, 14, 16] are then

$$\operatorname{Re} a(t, u, \Gamma\Gamma^* u) \geqq \delta|\Gamma^* S u|^2 - \gamma|\Gamma^* u|^2 , \tag{3.5}$$

$$|\dot{\Gamma}^* u| \leqq c_7|\Gamma^* S u| + c_8|\Gamma^* u| \tag{3.6}$$

for $u \in V(t)$ and this leads to

Theorem 3.3. *Under the hypotheses of Lemma 3.1 if* (3.5)–(3.6) *hold then solutions of* (1.1) *are unique.*

Proof. From (3.6) we have $|(\dot{\Gamma}^* u, \Gamma^* u)| \leqq c_7|\Gamma^* S u|\, |\Gamma^* u| + c_8|\Gamma^* u|^2 \leqq \varepsilon c_7|\Gamma^* S u|^2 + (c_8 + c_7/\varepsilon)\,|\Gamma^* u|^2$ and one takes ε small enough so that $\varepsilon c_7 < \delta$. Then pick $\lambda > \gamma + c_8 + c_7/\varepsilon$ and putting (3.5)–(3.6) in (3.4) yields

$$(\delta - c_7 \varepsilon) \int_0^T |\Gamma^* S u|^2\, dt + \left(\lambda - \gamma - c_8 - \frac{c_7}{\varepsilon}\right) \int_0^T |\Gamma^* u|^2\, dt \leqq 0 . \tag{3.7}$$

Consequently $\Gamma^*(t)$ one to one will imply that $u = 0$ and this is assured since $(\Gamma^* S)(t) \subset (S\Gamma)^*(t)$ which is $1-1$ (as is S) while $R(S(t)) = H$. Q.E.D.

Remark 3.4. One can obtain (3.1) directly, without using L_w in the proof, and in particular without the hypothesis that $(S\Gamma)^*(t)$ be $1-1$ (which guarantees the density of $D(L')$ and hence the good definition of L_w). To do this simply set $v = \Gamma\psi$, $\psi \in \mathscr{D}(H)$, and put this in (1.1) directly (cf. [11, 14]). In that case in Theorem 3.3 an additional hypothesis that $\Gamma^*(t)$ be $1-1$ is required. Since in practice one usually deals with L' densely defined we have chosen the present organization of the material in order to partially unify Sections 3 and 4. Let us also mention explicitly that to say $(S\Gamma)^*(t)$ is $1-1$ is equivalent to saying that $R((S\Gamma)(t))$ is dense in H or that $R(\Gamma(t))$ is dense in $V(t)$.

In the model case where $\Gamma(t) = S^{-\alpha}(t)$ for $\alpha \geqq 1$ the hypotheses (3.5)–(3.6) of Theorem 3.3 are often difficult to verify in practice and one is led to exploit the result (3.1) also in another way (see [11, 14, 16]). In the present more general framework this other technique automatically involves the operator $\Gamma + \Gamma^*$ and thus we will remove any asymmetry by working with self adjoint $\Lambda(t)$ in our hypotheses.

Theorem 3.5. *Let $\Lambda(t) \in \mathscr{L}(H)$ be a family of self adjoint positive operators such that $R(\Lambda(t)) \subset V(t)$, $S\Lambda(\cdot)$ is measurable with $\|S\Lambda(t)\| \leqq c_9$, $\Lambda(\cdot)$ is weakly C^1, $(S\Lambda)^*(t)$ is $1-1$, and $|u| \leqq c_{10}|\Lambda^{\frac{1}{2}} S u|$ for $u \in V(t)$. Assume* (2.1)–(2.2) *and also*

$$\operatorname{Re} a(t, u, \Lambda u) \geqq c_{11}|\Lambda^{\frac{1}{2}} S u|^2 - c_{12}|\Lambda^{\frac{1}{2}} u|^2 , \tag{3.8}$$

$$|\dot{\Lambda} u| \leqq c_{13}|\Lambda^{\frac{1}{2}} S u| + c_{14}|\Lambda^{\frac{1}{2}} u| \tag{3.9}$$

for $u \in V(t)$ where $c_{10} c_{13} < 2 c_{11}$. Then solutions of (1.1) *are unique.*

Proof. Let $\theta_n(t) = 1$ for $0 \leqq t \leqq T - \frac{2}{n}$, $\theta_n(t) = n\left(T - \frac{1}{n} - t\right)$ for $T - \frac{2}{n} \leqq t \leqq T - \frac{1}{n}$, and $\theta_n(t) = 0$ for $T - \frac{1}{n} \leqq t \leqq T$. Now Lemma 3.1 applies to give

$$(\Lambda u)' = \dot{\Lambda} u + \Lambda f - (S\Lambda)^* S \mathscr{A} u - \lambda \Lambda u \tag{3.10}$$

where we have added a $\lambda(u, v)$ term to (1.1) as before in deriving (3.4). Thus $v=\theta_n \Lambda u$ is admissible in (1.1) and we obtain ($f=0$)

$$\int_0^T \{-\theta_n'(u, \Lambda u)-\theta_n(u,(\Lambda u)')\}\, dt+\int_0^T \{\theta_n a(t, u, \Lambda u) + \lambda\theta_n(u, \Lambda u)\}\, dt=0. \tag{3.11}$$

Using (3.10) and taking limits we obtain

$$2\operatorname{Re}\int_0^T a(t, u, \Lambda u)\, dt+2\lambda\int_0^T |\Lambda^{\frac{1}{2}} u|^2\, dt-\int_0^T (u, \dot{\Lambda} u)\, dt \leqq 0. \tag{3.12}$$

Note that $\dot{\Lambda}(t)$ is self adjoint (see [11]) while $(u,(S\Lambda)^* S\mathscr{A} u)=(S\Lambda u, S\mathscr{A} u) = a(t, u, \Lambda u)$ and the limiting process is justified by the Lebesgue dominated convergence theorem in observing that $-\theta_n'(u, \Lambda u)=-\theta_n'|\Lambda^{\frac{1}{2}} u|^2 \geqq 0$ for all n. Now observe that $|(u, \dot{\Lambda} u)| \leqq |u|\,|\dot{\Lambda} u| \leqq c_{10}|\Lambda^{\frac{1}{2}} S u|\,(c_{13}|\Lambda^{\frac{1}{2}} S u|+c_{14}|\Lambda^{\frac{1}{2}} u|) \leqq c_{10} c_{13}|\Lambda^{\frac{1}{2}} S u|^2+c_{10}(\varepsilon|\Lambda^{\frac{1}{2}} S u|^2+(c_{14}^2/\varepsilon)\,|\Lambda^{\frac{1}{2}} u|^2)$. Then putting (3.8) and this estimate into (3.12) we obtain

$$\begin{aligned}
&[2c_{11}-c_{10}(c_{13}+\varepsilon)]\int_0^T |\Lambda^{\frac{1}{2}} S u|^2\, dt \\
&\quad+\left[2\lambda-2c_{12}-\frac{c_{10}c_{14}^2}{\varepsilon}\right]\int_0^T |\Lambda^{\frac{1}{2}} u|^2\, dt \leqq 0.
\end{aligned} \tag{3.13}$$

Since $c_{10}c_{13}<2c_{11}$ pick ε so that $c_{10}(c_{13}+\varepsilon)<2c_{11}$ and then pick λ so that $2\lambda>2c_{12}+c_{10}c_{14}^2/\varepsilon$. Since $\Lambda^{\frac{1}{2}}$ is $1-1$, by virtue of the estimate $|u| \leqq c_{10}|\Lambda^{\frac{1}{2}} S u|$ for example, the uniqueness is immediate. Q.E.D.

Remark 3.6. As indicated in Remark 3.4 we can obtain (3.10) directly using the assumption that $(S\Lambda)^*(t)$ be $1-1$ and then in the present case no additional hypothesis on $\Lambda(t)$ is needed since $\Lambda^{\frac{1}{2}}(t)$ is automatically $1-1$ as indicated.

We will now specialize the hypotheses to the case where $\Gamma(t)$ or $\Lambda(t)$ equals $S^{-\alpha}(t)$ for suitable $\alpha \geqq 1$. Letting $\Xi(t)$ denote either $\Gamma(t)$ or $\Lambda(t)$ we recall by Remark 2.3 that if $\Xi(\cdot)=S^{-\alpha}(\cdot)$ is weakly C^1 then $S\Xi(\cdot)$ is measurable with $\|S\Xi(t)\| \leqq c_3$, $(S\Xi)^*(t)$ is $1-1$, $R(\Xi(t)) \subset V(t)$, and (2.1)–(2.2) hold. Hence from Lemma 3.1 if u is a solution of (1.1) we have in this situation

$$(S^{-\alpha} u)'=\dot{S}^{-\alpha} u+S^{-\alpha} f-S^{2-\alpha} \mathscr{A} u. \tag{3.14}$$

We will write the resulting corollaries of Theorems 3.3 and 3.5 as separate theorems (cf. [11, 14, 16]) while rephrasing the hypotheses (3.5) and (3.8) for greater clarity. Thus Theorem 3.3 yields

Theorem 3.7. *Assume $S^{-\alpha}(\cdot)$ is weakly C^1 for some $\alpha \geqq 1$. Suppose*

$$\operatorname{Re}(S^{1-\alpha} \mathscr{A} u, S^{1-\alpha} u) \geqq \delta|S^{1-\alpha} u|^2-\gamma|S^{-\alpha} u|^2, \tag{3.15}$$

$$|\dot{S}^{-\alpha} u| \leqq c_7|S^{1-\alpha} u|+c_8|S^{-\alpha} u|. \tag{3.16}$$

Then the solutions of (1.1) are unique.

Now in working with Theorem 3.5 the hypothesis $|u| \leqq c_{10}|S^{1-\alpha/2}u|$ is true only when $\alpha \leqq 2$ so we obtain

Theorem 3.8. *Let $S^{-\alpha}(\cdot)$ be weakly C^1 for $1 \leqq \alpha \leqq 2$*

$$\mathrm{Re}\,(S^{1-\alpha/2}\mathscr{A}u, S^{1-\alpha/2}u) \geqq c_{11}|S^{1-\alpha/2}u|^2 - c_{12}|S^{-\alpha/2}u|^2, \tag{3.17}$$

$$|\dot{S}^{-\alpha}u| \leqq c_{13}|S^{1-\alpha/2}u| + c_{14}|S^{-\alpha/2}u| \tag{3.18}$$

with $c_{10}c_{13} < 2c_{11}$ where $|u| \leqq c_{10}|S^{1-\alpha/2}u|$. Then solutions of (1.1) *are unique.*

We will now examine various aspects of Theorems 3.7 and 3.8. Our comments are organized here into a sequence of remarks.

Remark 3.9. One can write (3.16) in the form $|\dot{S}^{-\alpha}S^{\alpha}x| \leqq c_7|Sx| + c_8|\dot{x}|$ which means that $\dot{S}^{-\alpha}S^{\alpha}(t)$ extends to be a bounded operator $\mathscr{S}(t) \in \mathscr{L}(V(t), H)$. This is equivalent to saying that $\mathscr{S}(t)\,S^{-1}(t) \in \mathscr{L}(H)$ and thus we have a situation where $\dot{S}^{-\alpha}S^{\alpha-1}(t)$ extends to be a bounded operator in H. Note here that if $c_7 = 0$ then $\dot{S}^{-\alpha}S^{\alpha}$ would extend to be a bounded operator in H. But it was pointed out by Kato (cf. [35]) that this would imply $D(\dot{S}^{\alpha}(t)) = D$ (constant). Indeed let $W(t) = \dot{S}^{-\alpha}S^{\alpha}(t) \in \mathscr{L}(H)$ and let $C(t) \in \mathscr{L}(H)$ be the solution of $C' = -CW$ with $C(0) = I$. Then $(CS^{-\alpha})' = C'S^{-\alpha} + C\dot{S}^{-\alpha} = -C\dot{S}^{-\alpha} + C\dot{S}^{-\alpha} = 0$ so that $CS^{-\alpha}(t) = CS^{-\alpha}(0) = S^{-\alpha}(0)$. Hence, taking adjoints, $S^{-\alpha}(0) = S^{-\alpha}(t)C^*(t)$ while $C^*(t)$ is one to one onto H by well known theorems (see e.g. [11]). This implies that $D(S^{\alpha}(0)) = R(S^{-\alpha}(0)) = R(S^{-\alpha}(t)) = D(S^{\alpha}(t))$. In particular we would like to point out that the hypotheses in Theorem 3 of [13] and Theorems 4.5.4 and 4.5.12 of [11] are too strong in one direction but too weak in another. They essentially involve $c_7 = 0$ (too strong since it implies $D(S^{\alpha}(t)) = \text{constant}$) and $\delta = 0$ (too weak since in practice one has $\delta > 0$ in the main cases of interest).

Remark 3.10. The condition (3.18) implies (as in Remark 3.9) that $\dot{S}^{-\alpha}S^{\alpha/2}(t)$ extends to be a bounded operator $V(t) \to H$ and thus that $\dot{S}^{-\alpha}S^{\alpha/2-1}(t)$ extends to $\mathscr{L}(H)$. However this is always true since $\alpha/2 - 1 \leqq 0$ and thus the "meaning" of (3.18) lies in the requirement that c_{13} be small. We remark also that (3.17) is equivalent to (3.15) but at a different "level" of action (cf. here Lemma 3.13).

Remark 3.11. The techniques and results of this section can be extended to a more general context where the $V(t)$ are not necessarily dense and moreover the hypotheses can be somewhat weakened in a technical way. This follows [11, 16] where some cases are worked out in detail and concrete examples are given there where the hypotheses are verified. The idea is to let $H(t)$ be the closure of $V(t)$ in H with $S(t)$ the standard operator of $V(t)$ relative to $H(t)$ and $P(t)$ the orthogonal projection $H \to H(t)$. Then $t \to E(t) = S^{-\alpha}(t)\,P(t)$ is assumed weakly C^1 (perhaps only on a dense subspace $K \subset H$ containing the $V(t)$) and estimates of the form (3.16) or (3.18) are invoked with $\dot{E}(t)$ on the left side. This allows one to treat various noncylindrical evolution problems readily and extends considerably the applicability of our results here. In fact, it is partly to this end that these results have been developed. Since the extension is straightforward however, following [11, 16], we will not pursue this further here.

Remark 3.12. We know there is a good existence theory for (1.1) where coercivity prevails, i.e., when $\operatorname{Re} a(t, x, x) \geqq \mu \|x\|^2 - \lambda |x|^2$ for $x \in V(t)$. However in this event $A(t) + \lambda$ is monotone and, while existence is unchanged, the uniqueness theory of part 4 is then relevant. In particular one has uniqueness when $S^{-2}(\cdot)$ is weakly C^1 by Theorem 4.8. Now by Proposition 2.7 if $S^{-\alpha}(\cdot)$ is weakly C' for $\alpha \leqq 2$ then $S^{-2}(\cdot)$ is weakly C^1. Therefore in a coercive situation Theorem 3.8 is weaker than Theorem 4.8, as is Theorem 3.7 for $\alpha \leqq 2$; however Theorem 3.7 for $\alpha > 2$ applies to coercive situations not covered by Theorem 4.8. Both Theorems 3.7 and 3.8 apply also to certain noncoercive situations as indicated below.

Lemma 3.13. *Coercivity in the form* $\operatorname{Re} a(t, x, x) \geqq \mu \| x \|^2 - \lambda |x|^2$ *for* $x \in V(t)$ *is equivalent to the stipulation that for* $x \in V(t)$, $\beta \geqq 1$,

$$\operatorname{Re} a(t, S^{-\beta}(t)\, x, S^{-\beta}(t)\, x) \geqq \mu |S^{1-\beta}(t)\, x|^2 - \lambda |S^{-\beta}(t)\, x|^2 . \tag{3.19}$$

Proof. First note that $S^{-\beta}(t)\, V(t)$ is dense in $V(t)$ since if $((S^{-\beta} x, y)) = 0$ for all $x \in V(t)$ we have $S^{1-\beta} y = 0$ in H. Then let $S^{-\beta}(t)\, x_n \to x$ in $V(t)$, i.e., $S^{1-\beta}(t)\, x_n \to Sx$ in H (and $S^{-\beta}(t)\, x_n \to x$ in H), so that taking limits in (3.19) we have coercivity. The other way is obvious. Q.E.D.

Remark 3.14. We will now compare (3.15) with (3.19). Since $a(t, S^{-\beta} x, S^{-\beta} y) = (S \mathscr{A} S^{-\beta} x, S^{1-\beta} y)$ we have

$$(S^{1-\beta} \mathscr{A} u, S^{1-\beta} u) = a(t, S^{-\beta} u, S^{-\beta} u) + (Qu, S^{1-\beta} u), \tag{3.20}$$

$$Q = S^{1-\beta} \mathscr{A} - S \mathscr{A} S^{-\beta} . \tag{3.21}$$

Suppose we had an estimate

$$|Qu| \leqq \eta |S^{1-\beta} u| + c_{15} |S^{-\beta} u| . \tag{3.22}$$

Then one could write

$$|(Qu, S^{1-\beta} u)| \leqq (\eta + \varepsilon)\, |S^{1-\beta} u|^2 + \frac{c_{15}^2}{\varepsilon}\, |S^{-\beta} u|^2 \tag{3.23}$$

with ε arbitrarily small. Hence if η were suitably small then (3.15) and (3.19) would be equivalent. With regard to the possibility of an estimate (3.22) we write $P = S^{-\beta} \mathscr{A} S^{\beta} - \mathscr{A}$, $Q_0 = SPS^{-1}$ and $Q_1 = SP$. Then $Q = Q_0 S^{1-\beta} = Q_1 S^{-\beta}$ so (3.22) depends in some sense on Q_0 belonging to $\mathscr{L}(H)$ with a small norm or on Q_1 belonging to $\mathscr{L}(H)$ (one means here $\bar{Q}_1 \in \mathscr{L}(H)$ etc.). The first alternative seems more promising and involves essentially $P(t) \in \mathscr{L}(V(t))$.

Example 3.15. Let $a(t, x, y) = (B(t)\, x, B^*(t)\, y)$ with $V(t) = D(B(t))$ where $B(t)$ is a closed densely defined operator in H satisfying $D(B(t)) \subset D(B^*(t))$ and say $|B^*(t)\, x|^2 \leqq c(|x|^2 + |B(t)\, x|^2)$ for $x \in V(t)$. This is a natural framework if for example $a(t,x,y)=(A(t)x,y)$ for $x\in D(A(t))$ and $B=A^{\frac{1}{2}}$. In any event $S(t)=(1+B^*(t)B(t))^{\frac{1}{2}}$ and $\mathscr{A}=S^{-1}K^*B$ where $K=B^*S^{-1}\in\mathscr{L}(H)$ (cf. [11]). Thus for example, in the terminology of Remark 3.14, $Q_0=S^{-\beta}(K^*BS^{-1})S^{\beta}-K^*BS^{-1}$ where $K^*BS^{-1}\in\mathscr{L}(H)$ and one is concerned with small perturbations of K^*BS^{-1} under the indicated action of $S^{-\beta}$. Similarly $Q_1=S^{-\beta}K^*BS^{\beta}-K^*B$ which involves perturbations of an unbounded operator K^*B. As a special case let $B=g(S)S$ where $g(S(t))\in\mathscr{L}(H)$ is defined for example by a spectral integral $g(S(t))=\int g(\lambda)\,dE_\lambda(t)$ (cf. [32]). Then $B^*=\bar{g}(S)S$ and $K=\bar{g}(S)$ with $\mathscr{A}=g^2(S)$. Therefore $P=0$ so by Remark 3.14 and Lemma 3.13 the condition (3.15) is equivalent to coercivity. In particular this holds when $\mathscr{A}=1$ which corresponds to $B=S$.

4.

We turn now to coercive-monotone nonlinear situations in the form (1.2). Our hypotheses in the equivalence Theorems 4.7 and 4.8 will always guarantee that (2.1)–(2.2) hold and (using Lemma 2.1) that L and L' are densely defined. In fact we will have $S^{-2}(\cdot)$ weakly C^1 in the main theorem and one recalls here Remark 2.3. Hence we assume in this section that L_s and L_w are well defined and that (2.1)–(2.2) hold. We only consider single valued operators in this paper. Let us begin with some standard definitions (recall that $W=L^p(V(t))$).

Definition 4.1. A (nonlinear) operator $Q: W\to W'$ is monotone if

$$\operatorname{Re}\langle Q(u)-Q(v),u-v\rangle\geqq 0 \tag{4.1}$$

for $u,v\in D(Q)$. When $D(Q)$ is dense with Q linear and monotone we say that Q is maximal monotone if it is not the restriction of another linear monotone operator.

Definition 4.2. A (nonlinear) operator $Q: W\to W'$ with $D(Q)=W$ is bounded if it takes bounded sets into bounded sets. Q is hemicontinuous if it is continuous from lines in W to the weak topology of W. Q is coercive if $\operatorname{Re}\langle Qx,x\rangle\geqq\varphi(\|x\|)\,\|x\|$ where $\varphi(r)\to\infty$ as $r\to\infty$. ($\varphi(r)$ may be negative for small r).

First we observe that $u\in D(L)$ or $u\in D(L')$ implies $u\in C^0(H)$ on $[0,T]$ (after possible modification on a set of measure zero). This follows immediately since then $u'\in L^1(H)$ and $u=\int_0^t u'dt+c$ (see [3] and cf. also [34]). The result extends as follows (cf. [7, 8]).

Lemma 4.1. *If $u\in D(L_s)$ then $u\in C^0(H)$ on $[0,T]$ (after possible modification on a set of measure zero).*

Proof. Let $u\in D(L)$ and let χ_t be the characteristic function of $[0,t]$. Then

$$\operatorname{Re}\langle Lu,\chi_t u\rangle=\operatorname{Re}\int_0^t(u',u)\,dt=\frac{1}{2}|u(t)|^2 \tag{4.2}$$

under our identifications and therefore

$$|u(t)|^2 \leqq 2 \, \|Lu\|_{W'} \, \|\chi_t u\|_W \leqq 2 \|Lu\|_{W'} \, \|u\|_W \tag{4.3}$$

so that $u \to u(t): D(L) \to H$ is continuous with a uniform bound ($D(L)$ having the graph topology). This extends to $D(\bar{L}) = \overline{D(L)}$ by continuity so that (except perhaps on a set of measure zero) every $u \in D(L_s)$ is the uniform limit of continuous functions. Q.E.D.

Now we recall the uniqueness theorem of [7, 8] for the strong problem (1.3) and the proof will be sketched for completeness (cf. also [11]).

Theorem 4.2. *Let $A(t): V(t) \to V'(t)$ be a family of monotone operators such that $Au \in W'$ for $u \in W$. Then solutions of* (1.3) *are unique (strong uniqueness).*

Proof. Let u and v be solutions of (1.3) so that $L_s(u-v) + Au - Av = 0$. Then take χ_t as in the proof of Lemma 4.1 to obtain (cf. (4.2))

$$\begin{aligned} &\operatorname{Re} \langle L_s(u-v), \chi_t(u-v)\rangle + \operatorname{Re} \langle Au - Av, \chi_t(u-v)\rangle \\ &\quad = \frac{1}{2} |u(t) - v(t)|^2 + \operatorname{Re} \int_0^t \langle A(\xi)\, u(\xi) - A(\xi)\, v(\xi), u(\xi) - v(\xi) \rangle \, d\xi = 0 \, . \end{aligned} \tag{4.4}$$

Consequently $|u(t) - v(t)|^2 \leqq 0$ for all t. Q.E.D.

We will be concerned here with proving uniqueness for the weak problem (1.2) by finding conditions under which it is equivalent to the strong problem (1.3) and then applying Theorem 4.2. Evidently this reduces to studying the linear operator L and finding when $L_s = L_w$. This question is also of interest in the existence theory for the strong problem and we will sketch this also for completeness (cf. [5–9]). First observe that if $L_s = L_w$ then $L_s^* = L_w^* = (L'^*)^* = \bar{L}'$. Now L is monotone by (4.2) and a similar calculation shows that L' is monotone. The monotonicity prevails upon passing to closures so L_s and L_s^* will be monotone. Then we apply the following recent theorem of Brezis [6] to conclude that L_s is maximal monotone.

Theorem 4.3. *Let W be a reflexive Banach space and $M: W \to W'$ a linear monotone operator defined on a linear subspace of W. The following are equivalent.*

1. *M is maximal monotone*
2. *M is a closed densely defined linear operator such that M^* is monotone*
3. *M is a closed densely defined linear operator such that M^* is maximal monotone.*

Combining this with Theorem 4.2 and other results of Browder (see [7–9] – cf. also [5, 11]) one has

Theorem 4.4. *Let $L_s = L_w$ and suppose $A: W \to W'$ is a monotone, hemicontinuous, bounded, coercive map. Then $L_s + A$ maps $D(L_s) \subset W$ onto W' and thus strong existence holds (for* (1.3)*). In this situation we also have weak uniqueness (for* (1.2)*).*

We go now to the equivalence theorems and recall first that $S^{-2}(t)$ is the restriction of $\theta(t)$ to H. Let $z_n \in H$ and $x \in V(t)$ so that $Sx = h \in H$ and

$$\langle z_n, x\rangle = (S\theta z_n, Sx) = (S^{-1} z_n, h)\,. \tag{4.5}$$

There results immediately

Lemma 4.5. *A sequence $z_n \in H$ converges weakly in $V'(t)$ if and only if $S^{-1}(t)\, z_n$ converges weakly in H.*

Corollary 4.6. *A sequence $u_n \in L^q(H)$ converges weakly in W' if and only if $S^{-1} u_n$ converges weakly in $L^q(H)$.*

The proof of the corollary is obvious and will be omitted. We go now to our first equivalence theorem.

Theorem 4.7. *If $S^{-1}(\cdot)$ is weakly C^1 and $p \geqq 2$ then $L_s = L_w$.*

Proof. We note first that by Remark 2.3 one has (2.1)–(2.2) while L and L' are densely defined. Now since $L_s \subset L_w$ the proof reduces to showing that if $u \in W$ satisfies $\langle u, L' v\rangle = -\langle u, v'\rangle = -(u, v') = \langle f, v\rangle$ for some $f \in W'$ and all $v \in D(L')$ then $u \in D(\bar{L})$. We will produce a sequence $u_n \in D(L)$ such that $u_n \to u$ in W and $S^{-1} u'_n$ is bounded in $L^q(H)$; then, using corollary 4.6, a weak compactness argument will show that $u \in D(L_s)$. Thus let $\varphi \in C_0^\infty(\mathbf{R})$, $\operatorname{supp} \varphi \subset [0, 1]$, $\varphi \geqq 0$, and $\int_{-\infty}^{\infty} \varphi(\xi)\, d\xi = 1$. Then set $\varphi_n(t) = n\varphi(nt)$ with

$$\begin{aligned} u_n(t) &= S^{-1}(t) \int_{-\infty}^{\infty} \varphi_n(t-\xi)\,(Su)\,(\xi)\, d\xi \\ &= S^{-1}(\varphi_n * Su) \end{aligned} \tag{4.6}$$

where we think of u to be extended as zero outside of $[0, T]$ while $S^{-1}(t) = S^{-1}(0)$ for $t \leqq 0$ and $S^{-1}(t) = S^{-1}(T)$ for $t \geqq T$ (thus the extended family $S^{-1}(\cdot)$ is piecewise C^1). Note that $\operatorname{supp} \varphi_n(t) \subset \left[o, \dfrac{1}{n}\right]$ and $u_n(0) = 0$ since the integral in (4.6) really extends only from $\xi = \max\left(0, t - \dfrac{1}{n}\right)$ to $\xi = t$ $\Bigl($thus $\operatorname{supp} u_n \subset \left[0, T + \dfrac{1}{n}\right]$ and $Su_n \in C_0^\infty(0, \hat{T}; H)$ for $\hat{T} > T + \dfrac{1}{n}\Bigr)$. Note that $u'_n = \dot{S}^{-1}(\varphi_n * Su) + S^{-1}(\varphi'_n * Su)$ $\in L^q(H)$ so that $u_n \in D(L)$ when restricted to $[0, T]$ (recall that $\|S^{-1}(t)\| \leqq c$ and $\|\dot{S}^{-1}(t)\| \leqq c_1$ when $S^{-1}(\cdot)$ is weakly C^1). By classical results (cf. [22]) $Su = \varphi_n * Su \to Su$ in $L^p(-\infty, \infty; H)$ and thus the restriction of u_n to $[0, T]$ tends to u in W. For notational convenience we will not distinguish between $L^p(-\infty, \infty; H)$ and $L^p(0, T; H)$ whenever the meaning is clear and will simply write $L^p(H)$; similarly we will not specify restrictions and will say $u_n \to u$ in W.

Next we note that Lemma 2.4 applies here (cf. (3.14)) and taking $S^{-1} = \Gamma$ in (2.6) we obtain $(S^{-1} u)' = S\theta f + \dot{S}^{-1} u$. Since $p \geqq 2$ it follows by Corollary 2.5

that $(S^{-1}u)' \in L^q(H)$. Now write

$$\begin{aligned} S^{-1}u_n' &= S^{-1}\frac{d}{dt}(S^{-1}(\varphi_n * Su)) \\ &= \frac{d}{dt}[S^{-2}(\varphi_n * Su)] - \dot{S}^{-1}S^{-1}(\varphi_n * Su) \\ &= \frac{d}{dt}\{S^{-2}(\varphi_n * Su) - (\varphi_n * S^{-2}Su)\} \\ &+ \varphi_n * (S^{-1}u)' - \dot{S}^{-1}S^{-1}(\varphi_n * Su) \end{aligned} \tag{4.7}$$

where we use the fact that $\frac{d}{dt}(\varphi_n * S^{-1}u) = \varphi_n * (S^{-1}u)'$. We know $(S^{-1}u)' \in L^q(H)$ so the terms $\varphi_n * (S^{-1}u)'$ remain bounded in $L^q(H)$ (note that if $h \in L^q$ then classical results give $|\varphi_n * h| \leqq |\varphi_n|_1 |h|_q \leqq |h|_q$). Similarly $\varphi_n * Su \in L^q(H)$ and remains bounded there (in fact in $L^p(H)$) so that the terms $\dot{S}^{-1}S^{-1}(\varphi_n * Su)$ are bounded in $L^q(H)$. Now as in Proposition 2.7 we know that $S^{-2}(\cdot)$ is (piecewise) weakly C^1. Therefore we can write the first term on the right side of (4.7) as

$$\Xi_n = \dot{S}^{-2}(\varphi_n * Su) + \int_0^t \varphi_n'(t-\xi)\,[S^{-2}(t) - S^{-2}(\xi)]\,(Su)(\xi)\,d\xi\,. \tag{4.8}$$

But $\varphi_n'(t-\xi) = n^2\varphi'(n(t-\xi))$ and since $\|\dot{S}^{-2}(t)\| \leqq \hat{c}$ we have (cf. Section 2)

$$\|S^{-2}(t) - S^{-2}(\xi)\| \leqq \hat{c}|t-\xi|\,. \tag{4.9}$$

Since $\operatorname{supp}\varphi_n' \subset \left[0, \frac{1}{n}\right]$ one has further $|t-\xi| \leqq \frac{1}{n}$ in the integral on the right whenever there is a contribution from the integrand $\left(\text{recall that the integral only extends from } \max\left(0, t-\frac{1}{n}\right) \text{ to } t\right)$. Consequently

$$\begin{aligned} &\left|\int_0^t \varphi_n'(t-\xi)\,[S^{-2}(t) - S^{-2}(\xi)]\,(Su)(\xi)\,d\xi\right| \\ &\leqq \hat{c}\int_0^t n|\varphi'(n(t-\xi))|\,|(Su)(\xi)|\,d\xi \end{aligned} \tag{4.10}$$

and using an inequality noted above this yields

$$\begin{aligned} &\left[\int_0^\infty \left|\int_0^t \varphi_n'(t-\xi)\,[S^{-2}(t) - S^{-2}(\xi)]\,(Su)(\xi)\,d\xi\right|^p dt\right]^{1/p} \\ &\leqq \hat{c}\left[\int_0^\infty n|\varphi'(n\xi)|\,d\xi\right]\left[\int_0^\infty |Su|^p(\xi)\,d\xi\right]^{1/p}. \end{aligned} \tag{4.11}$$

But $\int_0^\infty n|\varphi'(n\xi)|\,d\xi = \int_0^\infty |\varphi'(\eta)|\,d\eta \leqq \hat{c}$, and it follows that Ξ_n remains bounded in $L^p(H)$ and hence in $L^q(H)$.

We have produced a sequence $u_n \in D(L)$ such that $u_n \to u$ in W and $S^{-1}u_n'$ is bounded in $L^q(H)$. Since $L^q(H)$ is reflexive there exists a subsequence u_{n_k} such that $S^{-1}u_{n_k}'$ converges weakly in $L^q(H)$. By Corollary 4.6 this means that u_{n_k}' converges weakly in W'. But $G(\bar{L}) = \operatorname{graph}\bar{L}$ is a strongly closed linear subspace of $W \times W'$ and hence it is weakly closed by the Hahn-Banach theorem (see e.g. [11]). Consequently $u \in D(L_s)$. Q.E.D.

Now the hypotheses that $S^{-1}(\cdot)$ be weakly C^1 is hard to verify in practice and seems stronger than desired for certain applications (cf. [1, 2]); note here that of course we have dispensed with other hypotheses used in such applications. Therefore we give another theorem whose proof is related to that of Lions [28] but which eliminates in particular the requirement in [28] that $V(t)$ be a closed subspace of a fixed Hilbert space $K \subset H$. This theorem is actually stronger than Theorem 4.7 (since $S^{-1}(\cdot)$ weakly C^1 implies $S^{-2}(\cdot)$ is weakly C^1) but we have included Theorem 4.7 because the proof seems interesting in its own right. The requirment $p \geqq 2$ will appear again and this is quite possibly a fault of the technique only (cf. [7, 8]). Thus our main result is

Theorem 4.8. *If $S^{-2}(\cdot)$ is weakly C^1 and $p \geqq 2$ then $L_s = L_w$.*

Proof. First we observe again from Remark 2.3 that (2.1)–(2.2) hold with L and L' densely defined. We assume $L_w u = f$ so that $u \in W$ satisfies $-(u, v') = \langle f, v\rangle$ for $f \in W'$ and all $v \in D(L')$ (cf. the proof of Theorem 4.7). Let $\lambda > 0$ and set $M_\lambda(t) = (\lambda + S^2(t))^{-1}$. Then $M_\lambda(\cdot)$ is weakly C^1 with

$$\dot{M}_\lambda = -S^2 M_\lambda \dot{S}^{-2} S^2 M_\lambda \tag{4.12}$$

(see Lemma 2.6). Again Lemma 2.4 will apply with $\Gamma = M_\lambda$ in (2.6). To see this note first that in fact $\|(SM_\lambda)(t)\| \leqq \lambda^{-\frac{1}{2}}$ since, given for example $S^2(t) = \int_0^\infty \varrho\, dE_\varrho(t)$, $SM_\lambda(t)$ is determined by the formula

$$SM_\lambda(t) = \int_0^\infty \frac{\varrho^{\frac{1}{2}}}{\lambda + \varrho}\, dE_\varrho(t) \tag{4.13}$$

with $\varrho^{\frac{1}{2}}(\lambda+\varrho)^{-1} \leqq \lambda^{-\frac{1}{2}}$. Finally, taking $\Lambda(t) = S^{-2}(t)$ in Lemma 2.2, $SM_\lambda(t)$ corresponds to the function $f(x) = x^{\frac{1}{2}}(\lambda x + 1)^{-1}$ and hence is norm continuous (and measurable). Since M_λ, $\dot{M}_\lambda$, and SM_λ are self adjoint, (2.6), with $\Gamma = M_\lambda$, yields

$$(M_\lambda u)' = \dot{M}_\lambda u + (SM_\lambda)\, S\theta f. \tag{4.14}$$

As in Corollary 2.5 $(M_\lambda u)' \in L^q(H)$ since $p \geqq 2$. Now set $u_\lambda = \lambda M_\lambda u \in W$ and we will show that $u_\lambda(0) = 0$ so that $u_\lambda \in D(L)$. To see this we note first that by (4.14) and remarks before Lemma 4.1 it follows that $M_\lambda u$ can be identified with a continuous function and hence point values make sense. Now take $\psi \in C_0^\infty(-\infty, T; H)$ so that, restricted to $[0, T]$, $v = M_\lambda \psi \in D(L')$. Then write out $-(u, v') = \langle f, v\rangle$ for this v, integrating by parts and using (4.14), to obtain $((M_\lambda u)(0), \psi(0)) = 0$. Since $\psi(0) \in H$ may be arbitrary it follows that $u_\lambda(0) = \lambda(M_\lambda u)(0) = 0$.

The plan of the proof now follows that of Theorem 4.7, i.e., we will show that $u_\lambda \to u$ in W while $S^{-1}u'_\lambda$ remains bounded in $L^q(H)$. First we observe that $\lambda M_\lambda \to I$ strongly as $\lambda \to \infty$ (consider for example a representation $\lambda M_\lambda(t) = \int_0^\infty \lambda(\lambda+\varrho)^{-1} dE_\varrho(t)$ where $dE_\varrho(t)$ is a spectral resolution for $S^2(t)$ – cf. also [28]). Hence by Lebesgue dominated convergence $Su_\lambda = \lambda M_\lambda Su \to Su$ in $L^p(H)$ so that $u_\lambda \to u$ in W (note that $\|\lambda M_\lambda(t)\| \leqq 1$). Consider now from (4.14)

$$S^{-1}u'_\lambda = S^{-1}\lambda \dot{M}_\lambda u + \lambda M_\lambda S\theta f. \tag{4.15}$$

The second term on the right is bounded by $\|S\theta f\|$ in $L^q(H)$ while $S^{-1}\lambda\dot{M}_\lambda u = S^{-1}\lambda\dot{M}_\lambda S^{-1}Su$ with $S^{-1}\dot{M}_\lambda S^{-1} = -S^{-1}S^2M_\lambda \dot{S}^{-2}S^2M_\lambda S^{-1}$ from (4.12). Hence, using the estimate $\|SM_\lambda\| \leqq \lambda^{-\frac{1}{2}}$ obtained above, we have $\|(S^{-1}\dot{M}_\lambda S^{-1})(t)\| \leqq \lambda^{-1}\|\dot{S}^{-2}(t)\|$ and therefore

$$\begin{aligned} |S^{-1}(t)\,\lambda\dot{M}_\lambda(t)\,u(t)| &\leqq \|\dot{S}^{-2}(t)\|\;|S(t)\,u(t)| \\ &\leqq c_{16}|S(t)\,u(t)|\,. \end{aligned} \tag{4.16}$$

Thus the first term on the right in (4.15) remains bounded in $L^p(H)$, hence in $L^q(H)$, which proves that $S^{-1}u'_\lambda$ remains bounded in $L^q(H)$. Arguing now as in the proof of Theorem 4.7 it follows that $u \in D(L_s)$ as desired. Q.E.D.

For convenience we collect the results of theorems 4.2 and 4.8 as concerns the weak uniqueness question.

Theorem 4.9. *Let $S^{-2}(\cdot)$ be weakly C^1 and $p \geqq 2$ with $A(t): V(t) \to V'(t)$ a family of monotone operators such that $Au \in W'$ for $u \in W$. Then solutions of* (1.2) *are unique (weak uniqueness).*

5.

Given the context of [28] we now show that Theorem 4.8 implies the uniqueness result of [28]. Then some examples are given to illustrate our theory. Thus let K and H be separable Hilbert spaces with $K \subset H$ dense and the injection $K \to H$ continuous. Let $((,))$ (resp. $(,)$) denote the scalar product in K (resp. H) and let $V(t)$ be a closed subspace of K, dense in H, with the induced Hilbert structure from K. Let $P(t): K \to V(t)$ be the orthogonal projection and $S(t): V(t) \to H$ the standard operator as before. Now in [28] Lions proves a weak uniqueness theorem for (1.1), in the context just formulated, under the main assumption that $P(\cdot)$ is weakly C^1 in K. We will show that this can also be derived from Theorem 4.9 by proving

Theorem 5.1. *If $t \to P(t)$ is weakly C^1 in K then $t \to S^{-2}(t)$ is weakly C^1 in H.*

Proof. From remarks at the beginning of section 2 we know that $t \to P(t)$ is norm continuous in $\mathscr{L}(K)$. Now observe for $f \in H$, $k \in K$

$$((S^{-2}f, k))_K = ((S^{-2}f, Pk))_t = (S^{-1}f, SPk)_H = (f, Pk)_H . \tag{5.1}$$

Hence, writing $\Delta P = P(t+t_0) - P(t_0)$, etc., and noting that $|k| \leqq \tilde{c}\,\|k\|$ for $k \in K$, we have

$$\begin{aligned} \frac{|((\Delta S^{-2}f, k))_K|}{\|k\|} &= \frac{|(f, \Delta Pk)_H|}{\|k\|} \\ &\leqq \frac{\tilde{c}\,|f|\;\|\Delta Pk\|}{\|k\|} \leqq \tilde{c}\,|f|\;\|\Delta P\| \end{aligned} \tag{5.2}$$

from which it follows that $f \in H$ implies $t \to S^{-2}(t)f$ is continuous in K.

Next, following [5], one remarks that for $f, g \in H$

$$\begin{aligned} ([S^{-2}(t) - S^{-2}(\xi)]f, g) &= ([P(t) - P(\xi)]\,S^{-2}(t)f, g) \\ &\quad + (f, [P(t) - P(\xi)]\,S^{-2}(\xi)\,g) . \end{aligned} \tag{5.3}$$

Since for example $P(t)\,S^{-2}(t)f = S^{-2}(t)f$, to prove (5.3) it will suffice to observe that by (5.1)

$$\begin{aligned} (P(\xi)\,S^{-2}(t)f, g) &= ((S^{-2}(t)f, S^{-2}(\xi)\,g))_K \\ &= (f, P(t)\,S^{-2}(\xi)\,g) \end{aligned} \tag{5.4}$$

Dividing by $t - \xi$ in (5.3) and taking limits we obtain

$$\frac{d}{dt}(S^{-2}(t)f, g) = (\dot{P}(t)\,S^{-2}(t)f, g) + (f, \dot{P}(t)\,S^{-2}(t)\,g) \tag{5.5}$$

Note here that we use the continuity of $S^{-2}(\cdot)f$ in K for $f \in H$ as well as the fact that $(k, h) = ((k, \mathscr{K}h))_K$ where $\mathscr{K} \in \mathscr{L}(H, K)$ does not depend on t (which allows us to write for example $((\Delta P/\Delta t)\,S^{-2}(t)f, g) = ((\Delta P/\Delta t\,S^{-2}(t)f, \mathscr{K}g))_K$). Now finally $\dot{P}(t)\,S^{-2}(t)$ can be considered as a bounded operator in H since $\dot{P}(t) \in \mathscr{L}(K)$ and $V(t)$ has the same topology as K; hence we can say that $S^{-2}(\cdot)$ is weakly C^1 with

$$\dot{S}^{-2}(t) = \dot{P}(t)\,S^{-2}(t) + (\dot{P}(t)\,S^{-2}(t))^* \tag{5.6}$$

Q.E.D.

We consider next some situations in the context above where $S^{-1}(\cdot)$ will be weakly C^1. Thus let $\Lambda : K \to H$ be the standard operator for K with respect to H. Then we have

Theorem 5.2. *Let $P(\cdot)$ be weakly C^1 and suppose $K \subset D(S^{2\alpha}(t))$, $0 \leqq t \leqq T$, for some $\alpha > 0$, with $\|S^{2\alpha}(t)\,\Lambda^{-1}\|_{\mathscr{L}(H)} \leqq c_{17}$. Then $S^{-1}(\cdot)$ is weakly C^1.*

Proof. Let $\lambda > 0$ and set $M_\lambda(t) = (\lambda + S^2(t))^{-1}$. Then as in the proof of Proposition 2.7 we have

$$S^{-1}(t) = \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}}\, M_\lambda(t)\, d\lambda \tag{5.7}$$

Now by Theorem 5.1, $S^{-2}(\cdot)$ is weakly C^1 and by Lemma 2.6

$$\dot{M}_\lambda = -S^2 M_\lambda \dot{S}^{-2} S^2 M_\lambda$$

(cf. (4.12)). Using (5.6) we obtain

$$\dot{M}_\lambda = -S^2 M_\lambda \dot{P} S^{-2} S^2 M_\lambda - S^2 M_\lambda (\dot{P} S^{-2})^* S^2 M_\lambda . \tag{5.8}$$

Let us write the first term of (5.8) as $-S^{2(1-\alpha)} M_\lambda S^{2\alpha} \dot{P} S^{-1} S M_\lambda$. We recall then from the proof of Theorem 4.8 that $\|S M_\lambda\| \leqq^{-\frac{1}{2}}$ and furthermore

$$\|S^{2(1-\alpha)}(t)\, M_\lambda(t)\|_{\mathscr{L}(H)} \leqq \lambda^{-\alpha} . \tag{5.9}$$

We observe here that $\alpha > 0$ so that $2 > \beta = 2(1-\alpha) > -\infty$ and thus $0 < 1 - \beta/2 < \infty$. Therefore we can use Gelfand theory as in Lemma 2.2 with $\Lambda(t) = S^{-2}(t)$ and $S^\beta M_\lambda \sim f(x) = x^{1-\beta/2}(\lambda x + 1)^{-1}$ to obtain (5.9). Finally by hypotheses we have

$$\|S^{2\alpha}(t)\, \dot{P}(t)\, S^{-1}(t)\|_{\mathscr{L}(H)} \leqq c_{18} . \tag{5.10}$$

To see this we write out $S^{2\alpha} \dot{P} S^{-1}$ as $S^{2\alpha} \Lambda^{-1} \Lambda \dot{P} S^{-1}$ and note that since $\|x\|_t = \|x\|_K$ one has $\|S^{-1}(t)\|_{\mathscr{L}(H,K)} \leqq 1$, $\|\dot{P}(t)\|_{\mathscr{L}(K)} \leqq c_{19}$ since $P(\cdot)$ is weakly C^1, $\|\Lambda\|_{\mathscr{L}(K,H)} \leqq 1$ by definition, and, by hypothesis, $\|S^{2\alpha}(t)\, \Lambda^{-1}\|_{\mathscr{L}(H)} \leqq c_{17}$. Hence the first term of (5.8) is dominated in norm by $c_{18} \lambda^{-\alpha-\frac{1}{2}}$. A similar estimate holds for the second term in (5.8) (note that the adjoint of the second term is simply the first term). Hence

$$\|\dot{M}_\lambda(t)\|_{\mathscr{L}(H)} \leqq c_{20} \lambda^{-\alpha-\frac{1}{2}} . \tag{5.11}$$

We refer now to the proof of Proposition 2.7 for differentiation under the integral sign in (5.7). Given (5.11) and the associated development the difference quotients can be suitably bounded and the limiting procedure again justified by Lebesgue dominated convergence. Q.E.D.

Thus in certain natural situations Theorem 4.8 is no stronger than Theorem 4.7. However there are cases where one can prove that $S^{-2}(\cdot)$ is weakly C^1 but it is false or doubtful that $S^{-1}(\cdot)$ will be weakly C^1. As an example of this consider

Example 5.3. Let $H = L^2(0,1)$ and for $0 \leqq t < T$ let $V(t)$ be the Sobolev space $H^1(0,1)$ with inner product

$$((u,v))_t = ((u,v))_{H'} + \frac{t}{T-t}\,(u(1)\,\bar{v}(1) + u(o)\,\bar{v}(0)) \tag{5.12}$$

while $V(T) = H_0^1(0,1)$. It is easy to show that $S^2(t)\,u = -u_{xx} + u$ with $D(S^2(t)) = \{u \in H^2(0,1);\, tu(0) - (T-t)\,u_x(0) = 0;\, tu(1) + (T-t)\,u_x(1) = 0\}$ (cf. [3, 12, 24]). Thus at $t=0$ a Neumann condition is given while at $t=T$ one has a Dirichlet condition. For $f \in H$ an elementary calculation gives

$$(S^{-2}(t) f)(x) = C(t)\, e^x + D(t)\, e^{-x} - e^x \int_0^x \int_0^y e^{\xi - 2y} f(\xi)\, d\xi\, dy \tag{5.13}$$

where C and D are C^1 on $[0, T]$. Thus $S^{-2}(\cdot)$ is strongly C^1 so that Theorem 4.8 applies but because of the singular nature of (5.12) it seems doubtful that $S^{-1}(\cdot)$ will be weakly C^1 at $t=T$ (although it is norm continuous there).

As a final example (where $S^{-1}(\cdot)$ is definitely not weakly C^1) we indicate a situation where our theory applies but a restraint of Baiocchi [1, 2] does not hold.

Example 5.4. The technique of [1, 2] requires a constant subspace $V \subset V(t)$ dense in H (which is of course quite natural in many applications). Consider however $H = L^2(0, 1)$ with $V(t) = \{u \in L^2(0, 1); x \to |x-t|^{-\frac{1}{2}} u(x) \in L^2(0, 1)\}$ and let $S(t)$ be multiplication by $|x-t|^{-\frac{1}{2}}$. Then clearly $\cap V(t) = \{0\}$ and $S^{-2}(\cdot)$ is weakly C^1 while $S^{-1}(\cdot)$ is not weakly C^1. Note that [1,2] also requires $V(t) \subset K$ as in Section 5.

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Dichte, Archimedizität und Starrheit geordneter Körper

REINHOLD BAER *

Emil Artin und Otto Schreier haben in ihren klassischen Arbeiten gezeigt, daß sich die algebraische Substanz der Theorie der reellen Zahlen in ihrer Theorie der formal reellen Körper wiederfindet, ja erst richtig zeigt. Doch gibt es auszeichnende Eigenschaften der Körper reeller Zahlen, die wir zunächst diskutieren wollen, da sie den Hintergrund für unsere Überlegungen liefern werden.

Dichte und Archimedizität. Der algebraisch angeordnete Körper K ist archimedisch über seinem Unterkörper U, wenn es zu jedem $k \in K$ ein $u \in U$ mit $k < u$ gibt; und U ist dicht in K, wenn es zu jedem Paar verschiedener Elemente aus K ein zwischen ihnen gelegenes Element aus U gibt. Dann und nur dann ist der angeordnete Körper K im Körper $\boldsymbol{R}$ aller reellen Zahlen enthalten, wenn K dicht ist in jeder die Ordnung von K fortsetzenden, über K archimedischen Erweiterung von K. [Satz 1.2.]

Dichte im reellen Abschluß. Der angeordnete Körper K ist dann und nur dann in $\boldsymbol{R}$ enthalten, wenn jeder Unterkörper von K in seinem reellen [algebraischen] Abschluß dicht ist [Satz 5.6].

Dedekindsche Schnitte. Ein dedekindscher Schnitt im angeordneten Körper K ist eine Partition von K in zwei nicht leere Teilmengen S, D mit $s < d$ für jedes $s \in S$ und $d \in D$. Dieser dedekindsche Schnitt S, D heiße eigentlich, wenn es zu jedem positiven k in K Elemente $s \in S$ und $d \in D$ mit $0 < d - s < k$ gibt. In Unterkörpern von $\boldsymbol{R}$ ist bekanntlich jeder dedekindsche Schnitt eigentlich. Umgekehrt gilt: dann und nur dann ist jeder dedekindsche Schnitt im angeordneten Körper K eigentlich, wenn $K \subseteqq \boldsymbol{R}$ ist [Satz 1.2].

Starrheit. Der angeordnete Körper K heiße starr, wenn 1 der einzige ordnungerhaltende Automorphismus von K ist. Dann und nur dann ist der angeordnete Körper $K \subsetneqq \boldsymbol{R}$, wenn alle Unterkörper von K starr sind [Satz 1.2].

Wir wollen nun diese Überlegungen in einen allgemeineren Rahmen stellen, indem wir sie relativieren. Die Basis dafür bilden die verschiedenen möglichen Arten von Abschließungen angeordneter Körper.

Reeller Abschluß. Der Körper R ist der reelle Abschluß des angeordneten Körpers K, wenn R einmal ein reell abgeschlossener und als solcher ein angeordneter Körper ist, und wenn weiter R eine algebraische, die Ordnung von K

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fortsetzende Erweiterung von K ist. Es ist ein klassischer Satz, daß jeder angeordnete Körper sich auf eine und im wesentlichen nur eine Weise reell abschließen läßt.

Stetiger Abschluß. Der angeordnete Körper S ist stetiger Abschluß seines Unterkörpers K, wenn einmal K in S dicht ist, und wenn es weiter keine echte, die Ordnung von S fortsetzende Erweiterung von S gibt, in der S bzw. K dicht ist. Der stetige Abschluß eines angeordneten Körpers K ist im wesentlichen der in üblicher Weise konstruierte Körper der eigentlichen dedekindschen Schnitte in K [Satz 2.5 und Lemma 2.9].

Aus einer oben gemachten Bemerkung läßt sich erschließen, daß der stetige Abschluß eines angeordneten Körpers K nicht notwendig reell abgeschlossen ist; vgl. Satz 5.1. Andererseits ist der stetige Abschluß eines reell abgeschlossenen Körpers stets reell abgeschlossen [Satz 3.3]. Dies führt zu der folgenden Begriffsbildung.

Dedekindscher Abschluß. Dies ist der stetige Abschluß des reellen Abschlusses. Er ist im wesentlichen eindeutig bestimmt; für Kennzeichnungen vgl. Satz 3.9. Wichtig für uns sind Kriterien dafür, daß ein angeordneter Körper K und sein Unterkörper U denselben dedekindschen Abschluß haben: dies ist z. B. dann und nur dann der Fall, wenn der reelle Abschluß R von K einen U enthaltenden Unterkörper besitzt, der über U algebraisch und in R dicht ist [Zusatz 3.11]. – Es sei darauf hingewiesen, daß der reelle Abschluß des stetigen Abschlusses im allgemeinen ein echter Unterkörper des dedekindschen Abschlusses ist [Satz 5.4].

Starrheitskriterium [Satz 4.3]. Die folgenden Eigenschaften des geordneten Körpers K und seines Unterkörpers U sind äquivalent:

(i) K und U haben denselben dedekindschen Abschluß.

(ii) Ist Z ein Unterkörper mit $U \subseteqq Z \subseteqq K$, so ist 1 der einzige alle Elemente aus U fixierende, ordnungserhaltende Isomorphismus von Z in K.

(iii) Ist k ein über U transcendentes Element aus K, so ist 1 der einzige alle Elemente aus U fixierende, ordnungerhaltende Automorphismus von $U(k)$.

Im Abschnitt 0 stellen wir für die Bequemlichkeit des Lesers die benötigten Tatsachen aus der Artin-Schreierschen Theorie in einer für uns zweckmäßigen Form zusammen. Da die eine oder andere dieser Bemerkungen sich nicht in der Literatur zu finden scheint, werden wir außer Hinweisen auf Standardwerke auch hie und da Beweisandeutungen machen müssen.

Bezeichnungen

$\boldsymbol{Q}$ = Körper der rationalen Zahlen.

$\boldsymbol{R}$ = Körper aller [im klassischen Sinne] reellen Zahlen.

$\mathrm{Aut}_U K$ = Gruppe aller die Elemente aus U fixierenden Automorphismen von K.

$\mathrm{Aut}_U^{\circ} K$ = Gruppe aller Automorphismen aus $\mathrm{Aut}_U K$, die die Anordnung des angeordneten Körpers K erhalten.

$A \subset B$: = : A ist ein echter Unterkörper von B.

$U(M)$ = durch die Teilmenge M erzeugter Erweiterungskörper von U.

0. Zusammenstellung der benötigten Resultate aus der Artin-Schreierschen Theorie

Im folgenden werden wir die von uns benötigten Resultate aus der Artin-Schreierschen Theorie der [formal-] reellen Körper in einer für uns zweckmäßigen Form zusammenstellen. Im allgemeinen werden wir für die Beweise auf die geläufige Literatur verweisen können; und nur selten wird es nötig werden, eine Ableitung anzudeuten.

Die Grundbegriffe wie formal-reeller Körper, angeordneter Körper und reell-abgeschlossener Körper werden wir in der üblichen Form benutzen; man vergleiche etwa Jacobson [p. 270, Definition 1; p. 271, Definition 2; p. 273, Definition 3] oder van der Waerden [p. 235, 254]. Insbesondere werden wir unter einem Unterkörper des angeordneten Körpers K stets den in Übereinstimmung mit K angeordneten Körper verstehen.

Satz A. *Die folgenden Eigenschaften des Körpers K sind äquivalent:*

(i) *K ist reell abgeschlossen.*

(ii) *K ist ein angeordneter Körper; ist f ein Polynom über K, sind a, b Zahlen aus K mit $a<b$ und $f(a)f(b)<0$, so gibt es $c\in K$ mit $a<c<b$ und $f(c)=0$ [Weierstraß].*

(iii) *K ist ein angeordneter Körper; positive Zahlen aus K sind Quadrate in K; Polynome ungeraden Grades aus K haben Nullstellen in K.*

(iv) *K ist nicht algebraisch abgeschlossen; aber $K(\sqrt{-1})$ ist algebraisch abgeschlossen.*

(v) *K ist nicht algebraisch abgeschlossen und die Grade über K irreduzibler Polynome sind* 1 *und* 2 *[Euler].*

(vi) *K ist nicht algebraisch abgeschlossen und die Grade der über K irreduziblen Polynome sind beschränkt.*

(vii) *K ist nicht algebraisch abgeschlossen und der algebraische Abschluß von K ist endlich über K.*

(viii) *Jede endliche Teilmenge von K ist in einem reell abgeschlossenen Unterkörper von K enthalten.*

Beweis. Daß ein reell abgeschlossener Körper auf eine und nur eine Art geordnet werden kann, ist in van der Waerden [p. 254, Satz 1] enthalten. Daß in reell abgeschlossenen Körpern der Weierstraßsche Nullstellensatz [oder Zwischenwertsatz] gilt, findet sich bei van der Waerden [p. 257, Satz 5]. Also folgt (ii) aus (i).

Die einfache, in üblicher Art erfolgende Ableitung von (iii) aus (ii) sei dem Leser überlassen.

Daß (iv) aus (iii) folgt, ist etwa in van der Waerden [p. 256, Satz 3a] enthalten.

Die naheliegende Ableitung von (v) aus (iv) sei dem Leser überlassen; und (vi) ist nur eine abgeschwächte Form von (v), folgt also aus (v).

Gilt (vi), so gibt es eine positive ganze Zahl n derart, daß alle über K irreduziblen Polynome einen n nicht überschreitenden Grad haben.

Wir nehmen zunächst an, daß K nicht perfekt ist. Dann folgt – etwa aus Jacobson [p. 146, Theorem 3] – daß die Charakteristik von K eine Primzahl p und $K^p \neq K$ ist. Es gibt dann ein $k \in K$, das keine p-te Potenz in K ist; und hieraus folgert man die Irreduzibilität aller Polynome $x^{p^i} - a$. Dann wäre aber $p^i \leqq n$ für jedes positive ganze i, ein Widerspruch: K ist perfekt.

Ist E eine endliche Erweiterung von K, so ist $E = K(e)$ wegen der Perfektheit von K eine einfache algebraische Erweiterung von K; siehe etwa Jacobson [p. 54, Theorem 14]. Natürlich genügt e einer irreduziblen Gleichung über K, deren Grad [wegen (vi)] n nicht überschreitet. Also ist auch $[E:K] \leqq n$; und wir haben gezeigt, daß die endlichen Erweiterungen von K beschränkte [n nicht überschreitende] Grade über K haben.

Es gibt also insbesondere eine endliche algebraische Erweiterung M maximalen Grades $[M:K] = m$ über K. Sei A irgendeine algebraische, algebraisch abgeschlossene Erweiterung von M. Ist $a \in A$, so ist $M(a)$ eine endliche algebraische Erweiterung von K; und aus der Maximalität von M folgt $M = M(a)$. Also liegt a in M und es ist $M = A$ algebraisch abgeschlossen. Damit haben wir (vii) aus (vi) hergeleitet.

Daß schließlich (i) aus (vii) folgt, ergibt sich aus einem wunderschönen Satz von Artin-Schreier; siehe Jacobson [p. 316, Theorem 17]: Die Äquivalenz der Eigenschaften (i)–(vii) von K ist dargetan.

Natürlich ist (viii) nur eine abgeschwächte Form von (i). Gilt umgekehrt (viii), so bemerken wir zunächst, daß die Gleichung $x^2 + 1 = 0$ in keinem reell abgeschlossenen Unterkörper von K und also auch in K selbst keine Lösung hat: K ist nicht algebraisch abgeschlossen. Ist zweitens $f(x)$ ein irreduzibles Polynom über K, so liegen seine [endlich vielen] Coeffizienten wegen (viii) in einem reell abgeschlossenen Unterkörper U. Da f in K irreduzibel ist, ist f auch über U irreduzibel. In U gilt (v), so daß der Grad von f höchstens 2 ist. Damit haben wir (v) aus (viii) hergeleitet und die Äquivalenz von (i)–(viii) dargetan.

Zusatz B. *In reell abgeschlossenen Körpern gelten der Satz von Rolle, der Mittelwertsatz der Differentialrechnung und der Satz, daß Polynome in abgeschlossenen Intervallen größte und kleinste Werte haben und annehmen.*

Siehe Jacobson [p. 284, Exercises 1, 2, 3] oder van der Waerden [p. 251, Aufgaben 4, 5, 7].

Ist K ein angeordneter Körper, ist der reell abgeschlossene Körper R eine algebraische, die Ordnung von K fortsetzende Erweiterung von K, so werde R *als reeller Abschluß von* K bezeichnet.

Satz C. *Jeder angeordnete Körper besitzt einen und im wesentlichen nur einen reellen Abschluß.*

Siehe Jacobson [p. 285, Theorem 8] oder van der Waerden [p. 259, Satz 8].

Satz D. *Ist R_i für $i = 1, 2$ der reelle Abschluß des angeordneten Körpers K_i, ist σ ein ordnungserhaltender Isomorphismus von K_1 auf K_2, so gibt es einen und nur einen mit σ auf K_1 übereinstimmenden Isomorphismus von R_1 auf R_2.*

Siehe Jacobson [p. 285, Theorem 8].

Lemma E. (I) *Ist R ein reell abgeschlossener Unterkörper des angeordneten Körpers K, so enthält R jedes über R algebraische Element aus K.*

(II) *Ist U ein Unterkörper des reell abgeschlossenen Körpers R, so ist die Menge der über U algebraischen Elemente aus R ein reell abgeschlossener Unterkörper von R.*

(III) *Der Durchschnitt einer nicht leeren Menge reell abgeschlossener Unterkörper des angeordneten Körpers K ist ein reell abgeschlossener Unterkörper von K.*

(I) folgt unmittelbar aus der Definition der reellen Abgeschlossenheit; für (II) vgl. Jacobson [p. 278, Corollary]; und (III) ergibt sich durch Combination von (I) und (II).

Der angeordnete Körper K heißt *archimedisch über seinem Unterkörper U*, wenn es zu jedem $k \in K$ ein $u \in U$ mit $k < u$ gibt. Diese Eigenschaft ist offenbar gleichwertig mit der Forderung: Zu jedem $k \in K$ mit $0 < k$ gibt es ein $u \in U$ mit $0 < u < k$.

Lemma F. *Ist der angeordnete Körper K algebraisch über seinem Unterkörper U, so ist K archimedisch über U.*

Einer der möglichen Beweise dieser einfachen Tatsache ist etwa in van der Waerden [p. 238, Aufgabe 2] enthalten.

Das Compositum des angeordneten Körpers K mit dem reellen Abschluß seines Unterkörpers U erhält man folgendermaßen: Sei R der nach Satz C im wesentlichen eindeutig bestimmte reelle Abschluß von K. Wegen Lemma E, (II) ist die Menge A aller über U algebraischen Elemente aus R ein reell abgeschlossener Unterkörper von R, also der reelle Abschluß von U. Das Erzeugnis C der Unterkörper A und K von R ist dann im wesentlichen eindeutig durch $U \subseteqq K$ bestimmt und das gesuchte Compositum von K mit dem reellen Abschluß A von U.

1. Die Unterkörper des Körpers R aller reellen Zahlen

Wir erinnern zunächst an die Definition des *dedekindschen Schnitts* [im weiteren Sinne]. Dies ist ein Paar S, D von Teilmengen eines angeordneten Körpers K mit den folgenden Eigenschaften:

(a) Weder S noch D ist leer und K ist die Vereinigungsmenge von S und D.

(b) $s < d$ für jedes Paar $s \in S$ und $d \in D$.

Insbesondere liegt also jedes Element aus K in einer und nur einer der Mengen S und D; weiter enthält S mit einem Element alle kleineren Elemente aus K und D enthält mit einem Element alle größeren Elemente aus K.

Dann gilt der folgende Erweiterungssatz

Lemma 1.1. *Ist S, D ein dedekindscher Schnitt in dem angeordneten Körper K, so gibt es eine die Ordnung von K fortsetzende, über K archimedische Erweiterung E von K und ein Element $t \in E$ mit der Eigenschaft:*

$$\textit{Ist } k \in K \textit{ und } \begin{Bmatrix} k < t \\ t < k \end{Bmatrix}, \textit{ so ist } \begin{Bmatrix} k \in S \\ k \in D \end{Bmatrix}.$$

Kurz: $t \in E$ bestimmt den dedekindschen Schnitt S, D in K.

Beweis im Spezialfall eines reell abgeschlossenen Körpers K: Enthält zunächst S ein größtes oder D ein kleinstes Element, so können wir für t dieses extremale Element und $E=K$ wählen. Wir machen also im folgenden die Annahme:

(1) Es gibt kein größtes Element in S und kein kleinstes Element in D.

Sei E eine einfache transcendente Erweiterung von K. Dann ist $E=K(t)$ der Körper aller rationalen Funktionen in t mit Coefficienten aus K. Wir definieren: Das Polynom $f(t)$ aus E ist *positiv*, wenn f der folgenden Bedingung genügt:

(P) Es gibt $s\in S$ und $d\in D$ mit $0<f(z)$ für $s\leqq z\leqq d$.

Ist $f_i(t)$ für $i=1,2$ ein positives Polynom aus E, so gibt es wegen (P) Elemente $s_i\in S$, $d_i\in D$ mit $0<f_i(z)$ für $s_i\leqq z\leqq d_i$. Ist $s=\max[s_1,s_2]$ und $d=\min[d_1,d_2]$, so ist $s\in S$, $d\in D$ und $0<f_i(z)$ für $s\leqq z\leqq d$, so daß auch

$$0<f_1(z)+f_2(z) \quad \text{und} \quad 0<f_1(z)\,f_2(z) \quad \text{für} \quad s\leqq z\leqq d$$

gelten. Damit haben wir gezeigt:

(2) Summe und Produkt positiver Polynome sind positiv.

Klar ist:

(3) 0 ist kein positives Polynom.

Ist das Polynom f über K nicht das 0-Polynom, so hat f nur endlich viele Nullstellen in K. Da S kein maximales und D kein minimales Element besitzt [wegen (1)], erschließt man hieraus sofort die Existenz von $s\in S$ und $d\in D$ mit $f(z)\neq 0$ für $s\leqq z\leqq d$. Angenommen es gäbe a,b in K mit $s\leqq a<b\leqq d$ und $f(a)\,f(b)<0$. Aus der reellen Abgeschlossenheit von K folgt wegen des Weierstraßschen Nullstellensatzes – Satz A, (ii) – die Existenz von $k\in K$ mit $a<k<b$ und $f(k)=0$, ein Widerspruch. Damit haben wir gezeigt, daß entweder $0<f(z)$ oder $f(z)<0$ für alle $z\in K$ mit $s\leqq z\leqq d$ gilt; und hieraus folgt:

(4) Ist das Polynom f über K nicht das 0-Polynom, so ist genau eines der Polynome f und $-f$ positiv.

Aus (2), (3), (4) ergibt sich aber, daß unsere Definition (P) eine algebraische Anordnung im Ringe der Polynome $f(t)$ über K definiert. Da E der Quotientenkörper dieses Ringes ist, läßt sich die durch (P) im Ringe der Polynome aus E definierte Anordnung auf eine und nur eine Art auf E fortsetzen; siehe etwa van der Waerden [p. 237, Hilfssatz]. Wir werden diese Anordnung von E *die durch* (P) *definierte Anordnung von E* nennen. Es ist klar, daß sie die durch die reelle Abgeschlossenheit von K bestimmte Anordnung von K fortsetzt.

Ist $d \in D$, so ist d nicht das kleinste Element aus D und es folgt, daß $d-t$ ein positives Polynom aus E ist. Entsprechend ist $t-s$ für $s \in S$ ein positives Polynom aus E. Also wird

(5) $s<t<d$ für $s \in S$ und $d \in D$;

und dies ist gleichwertig damit, daß der dedekindsche Schnitt S, D durch t in K bestimmt wird.

Aus (5) folgert man mühelos:

(6) Ist $f(t)$ ein positives Polynom aus E, so gibt es $d \in D$ mit $f(t)<d$.

Ist $f(t)$ ein positives Polynom aus E, so folgt aus (P) die Existenz von $s \in S$ und $d \in D$ mit $0<f(z)$ für alle z aus K mit $s \leqq z \leqq d$. Da K ein reell abgeschlossener Körper ist, können wir Zusatz B anwenden: f nimmt im Intervall $[s, d]$ sein Minimum an. Hieraus ergibt sich die Existenz einer positiven Zahl $k \in K$ mit $k \leqq f(z)$ für $z \in K$ mit $s \leqq z \leqq d$. Dann ist aber $\frac{1}{2} k < f(z)$; und wir haben gezeigt:

(7) Ist $f(t)$ ein positives Polynom aus E, so gibt es ein $p \in K$ mit $0<p<f(t)$.

Ist e ein positives Element aus E, so gibt es positive Polynome $f(t)$, $g(t)$ aus E mit $e=f(t)\, g(t)^{-1}$. Aus (6) und (7) folgern wir die Existenz positiver Zahlen p, q aus K mit $f(t)<p$ und $q<g(t)$. Dann ist aber $e<pq^{-1}$, so daß E über K archimedisch ist: das Paar E, t leistet im vorliegenden Spezialfall das Verlangte.

Bemerkung. Robinson [p. 103, 4.2.18. Theorem] zeigt, daß die von uns angegebene Anordnung von E die einzige unsern Anforderungen genügende Anordnung von E ist.

Rückführung des allgemeinen Falles auf den Spezialfall. Nach dem fundamentalen Artin-Schreierschen Existenzsatz gibt es einen und im wesentlichen nur einen über K algebraischen, die Ordnung von K fortsetzenden, reell-abgeschlossenen Körper R; vgl. Satz C. Aus Lemma F folgt:

(8) R ist archimedisch über K.

Gibt es ein Element $r \in R$ mit $s \leqq r \leqq d$ für jedes $s \in S$ und jedes $d \in D$, so haben wir bereits das gewünschte Paar $t=r$, $E=R$ gefunden. Wir nehmen also im folgenden an:

(9) Ist $r \in R$, so ist entweder $r<s$ für ein $s \in S$ oder $d<r$ für ein $d \in D$.

Unter S^* wollen wir die Menge aller $r \in R$ mit $r \leqq s$ für ein $s \in S$ verstehen; und entsprechend sei D^* die Menge aller $r \in R$ mit $d \leqq r$ für ein $d \in D$. Wegen (8) und (9) gilt:

(10) S^*, R^* ist ein dedekindscher Schnitt in R mit $S \subseteqq S^*$ und $D \subseteqq D^*$.

Auf den dedekindschen Schnitt S^*, R^* im reell abgeschlossenen Körper R können wir den bereits bewiesenen Spezialfall anwenden: Es gibt eine die Ordnung von R fortsetzende, über R archimedische Erweiterung E von R und ein Element $t \in E$ mit der Eigenschaft:

$$\text{Ist } r \in R \text{ und } \begin{Bmatrix} r < t \\ t < r \end{Bmatrix}, \text{ so ist } \begin{Bmatrix} r \in S^* \\ r \in D^* \end{Bmatrix}.$$

Da R die Anordnung von K fortsetzt und E die Anordnung von R fortsetzt, setzt E die Anordnung von K fort. Da E archimedisch über R und R archimedisch über K ist, ist E auch archimedisch über K. Aus $S \subsetneqq S^*$ und $D \subsetneqq D^*$ folgt schließlich, daß t den dedekindischen Schnitt S, D in K bestimmt, womit alles bewiesen ist.

Im folgenden sei unter $\boldsymbol{Q}$ *der Körper aller rationalen Zahlen* und unter $\boldsymbol{R}$ *der Körper aller reellen Zahlen* [im klassischen Sinne] verstanden. Weiter wollen wir unter *einem eigentlichen dedekindschen Schnitt* einen dedekindschen Schnitt S, D im angeordneten Körper K verstehen, der noch der folgenden Bedingung genügt:

Zu jedem positiven $e \in K$ gibt es $s \in S$ und $d \in D$ mit $0 < d - s < e$.

Satz 1.2. *Die folgenden Eigenschaften des angeordneten Körpers K sind äquivalent:*

(i) $K \subseteqq \boldsymbol{R}$.

(ii) $\boldsymbol{Q}$ *ist dicht in K.*

(iii) *K ist archimedisch über $\boldsymbol{Q}$.*

(iv) *K ist in jeder die Ordnung von K fortsetzenden, über K archimedischen Erweiterung von K dicht.*

(v) *Jeder dedekindsche Schnitt in K ist eigentlich.*

(vi) *Ist U ein Unterkörper von K, so ist* 1 *der einzige ordnungerhaltende Automorphismus von U.*

(vii) *Die Mächtigkeiten der die Anordnung von K fortsetzenden, über K archimedischen Erweiterungen von K sind beschränkt.*

(viii) *Es gibt maximale, die Anordnung von K fortsetzende, über K archimedische Erweiterungen von K.*

Bemerkungen. **A.** Bedingung (i) besagt genauer, daß es einen ordnungerhaltenden Isomorphismus von K in $\boldsymbol{R}$ gibt; aber schließlich ist auch $\boldsymbol{R}$ nur bis auf Isomorphie eindeutig gegeben.

B. Die wohlbekannte Äquivalenz der Bedingungen (i)—(iii) wird nur aus Gründen der Beweisbequemlichkeit angegeben.

C. Eine weitere Kennzeichnung der Unterkörper von $\boldsymbol{R}$ findet sich unten in Satz 5.6.

Dem Beweis dieses Satzes sei der Beweis eines einfachen, wohlbekannten, von uns mehrfach benutzten Hilfssatzes vorausgeschickt.

(1.2.+) *Ist der angeordnete Körper K über seinem Unterkörper U nicht archimedisch, so gibt es einen Zwischenkörper Z mit $U \subset Z \subseteqq K$ und einen von* 1 *verschiedenen, die Elemente aus U fixierenden, ordnungerhaltenden Automorphismus von Z.*

Beweis. Da K über U nicht archimedisch ist, gibt es $w \in K$ mit $u < w$ für jedes $u \in U$. Aus Lemma F folgt, daß w transcendent über U ist. Also ist $U(w)$ der Körper aller rationalen Funktionen in w mit Coefficienten aus U. Natürlich ist auch $U \subset U(w) = Z \subseteqq K$. Es gibt einen und nur einen Automorphismus σ von Z, der alle Elemente aus U fixiert und w auf $2w$ abbildet. Ist $p = \sum_{i=0}^{n} u_i w^i$ mit $u_n \neq 0$ und $u_i \in U$, so ist dann und nur dann $0 < p$, wenn $0 < u_n$ ist. Das Element $p^\sigma = \sum_{i=0} 2^i u_i w^i$ ist offenbar dann und nur dann positiv, wenn $0 < 2^n u_n$ ist: $0 < p$ dann und nur dann, wenn $0 < p^\sigma$. Hieraus folgert man mühelos, daß σ ein von 1 verschiedener, ordnungerhaltender, die Elemente aus U fixierender Automorphismus von Z ist.

Beweis von Satz 1.2. Alle Definitionen von $\boldsymbol{R}$ schließen ein, daß der Primkörper $\boldsymbol{Q}$ von $\boldsymbol{R}$ in $\boldsymbol{R}$ dicht ist. Also folgt aus (i) auch, daß $\boldsymbol{Q}$ in $\boldsymbol{K}$ dicht ist: (ii) folgt aus (i); und es ist klar, daß (iii) aus (ii) folgt. Daß schließlich (i) aus (iii) folgt, ist wohlbekannt; siehe etwa van der Waerden [p. 245, 1.].

Wir nehmen als nächstes die Gültigkeit der äquivalenten Bedingungen (i)–(iii) an; und betrachten eine die Ordnung von K fortsetzende, über K archimedische Erweiterung E von K. Da K wegen (iii) über $\boldsymbol{Q}$ archimedisch ist, ist auch E über $\boldsymbol{Q}$ archimedisch; und aus der Äquivalenz von (ii) und (iii) ergibt sich, daß $\boldsymbol{Q}$ in E dicht ist. Also ist auch der Zwischenkörper K in E dicht: (iv) folgt aus den äquivalenten Bedingungen (i)–(iii).

Wir nehmen die Gültigkeit von (iv) an und betrachten einen dedekindschen Schnitt S, D im angeordneten Körper K. Aus Lemma 1.1 folgt die Existenz einer die Anordnung von K fortsetzenden Erweiterung E von K, die über K archimedisch ist, und eines Elementes $t \in E$, das den dedekindschen Schnitt S, D in K bestimmt. Aus (iv) folgt, daß K in E dicht ist. Ist $0 < k \in K$, so ist $t - 4^{-1}k < t - 5^{-1}k < t$; und aus der Dichte von K in E folgt die Existenz eines Elements $s \in K$ mit $t - 4^{-1}k \leqq s \leqq t - 5^{-1}k < t$, so daß insbesondere $s \in S$ ist. Ebenso folgt die Existenz eines Elements $d \in D$ mit $t < t + 5^{-1}k \leqq d \leqq t + 4^{-1}k$. Dann ist aber auch

$$0 < d - s \leqq \tfrac{1}{2}k < k,$$

so daß S, D ein eigentlicher dedekindscher Schnitt in K ist: (v) folgt aus (iv).

Wir nehmen die Gültigkeit von (v) an. Wäre K nicht archimedisch über $\boldsymbol{Q}$, so wäre die Menge D aller Elemente aus K, die größer als alle rationalen Zahlen sind, nicht leer. Das Complement S von D in K enthält alle rationalen Zahlen, ist also nicht leer, so daß S, D ein dedekindscher Schnitt in K ist. Dieser ist eigentlich wegen (v). Also gibt es $s \in S$ und $d \in D$ mit $0 < d - s < 1$. Da s im Complement von D in K liegt, gibt es eine rationale Zahl r mit $s \leqq r$. Hieraus folgt

$$d = (d - s) + s \leqq 1 + r.$$

Da $1+r$ rational ist und d zu D gehört, wird $1+r<d$, ein Widerspruch: K ist archimedisch über $\boldsymbol{Q}$. Damit haben wir (iii) aus (v) hergeleitet und die Äquivalenz von (i)–(v) dargetan.

Ist $\boldsymbol{Q}$ dicht in K, so ist $\boldsymbol{Q}$ auch dicht in jedem Unterkörper U von K. Ein ordnungerhaltender Automorphismus von U fixiert alle Elemente aus $\boldsymbol{Q}$, ist also 1. Ist umgekehrt K nicht archimedisch über $\boldsymbol{Q}$, so folgt aus Hilfssatz (1.2.+) die Ungültigkeit von (vi), so daß (iii) aus (vi) folgt, womit die Äquivalenz der Bedingungen (i)–(vi) dargetan ist.

Ist $K \subseteqq \boldsymbol{R}$ und E eine die Anordnung von K fortsetzende, über K archimedische Erweiterung von K, so ist K wegen (iii) archimedisch über $\boldsymbol{Q}$, so daß auch E archimedisch über $\boldsymbol{Q}$ ist und $E \subseteqq \boldsymbol{R}$ aus der Äquivalenz von (i) und (iii) folgt. Dann ist aber die Mächtigkeit von E nicht größer als die von $\boldsymbol{R}$, womit (vii) aus den äquivalenten Bedingungen (i) und (iii) abgeleitet ist.

Dem Nachweis, daß (viii) aus (vii) folgt, sei eine einfache Vorbemerkung vorausgeschickt. Sei $\boldsymbol{M}$ eine Menge angeordneter Körper mit folgenden Eigenschaften:

(a) Sind X, Y Körper aus $\boldsymbol{M}$ mit $X \subset Y$, so ist X ein angeordneter Unterkörper von Y.

(b) Sind X, Y Körper aus $\boldsymbol{M}$, so ist $X \subseteqq Y$ oder $Y \subset X$.

Man sieht dann sofort ein, daß sich die Vereinigungsmenge V der Elemente aus Körpern aus $\boldsymbol{M}$ auf eine und nur eine Weise derart zu einem angeordneten Körper machen läßt, daß jeder Körper $X \in \boldsymbol{M}$ ein angeordneter Unterkörper von V ist.

Gilt Bedingung (viii) nicht, so betrachten wir eine beliebige Ordinalzahl Σ und definieren durch vollständige transfinite Induktion für jede Ordinalzahl $\sigma \leqq \Sigma$ einen angeordneten Körper K_σ mit folgenden Eigenschaften:

(a) $K = K_0$.

(b) Ist $\sigma < \lambda$, so ist K_λ eine echte, die Ordnung von K_σ fortsetzende, über K_σ archimedische Erweiterung von K_σ.

Daß man solche Körper K_σ konstruieren kann, folgt beim Schritt von σ auf $\sigma+1$ aus der Ungültigkeit von (viii) und beim Limesschritt aus der Vorbemerkung.

Aus (a), (b) folgt sofort, daß die Mächtigkeit von K_σ und insbesondere die von K_Σ wenigstens so groß wie die der Ordinalzahl σ bzw. Σ ist. Also folgt aus der Ungültigkeit von (viii) auch die von (vii), so daß (viii) aus (vii) folgt.

Gilt (viii), so sei M eine maximale, die Anordnung von K fortsetzende, über K archimedische Erweiterung von K. Ist S, D ein dedekindscher Schnitt in M, so folgt aus Lemma 1.1 die Existenz einer die Ordnung von M fortsetzenden, über M archimedischen Erweiterung E von M und eines Elements $e \in E$ mit folgender Eigenschaft:

$$\text{Ist } m \in M \text{ und } \begin{Bmatrix} m < e \\ e < m \end{Bmatrix}, \text{ so ist } \begin{Bmatrix} m \in S \\ m \in D \end{Bmatrix}.$$

Da M über K und E über M archimedisch ist, ist auch E eine die Ordnung von K fortsetzende, über K archimedische Erweiterung von K, so daß $M = E$

aus der Maximalität von M folgt. Insbesondere liegt e in M, so daß S, D ein eigentlicher dedekindscher Schnitt in M ist. Also genügt M der Bedingung (v); und aus der Äquivalenz von (i) und (v) folgt $K \subseteqq M \subseteqq \boldsymbol{R}$, so daß (i) aus (viii) folgt: Die Bedingungen (i)–(viii) sind äquivalent.

Satz 1.3. *Die folgenden Eigenschaften des angeordneten Körpers K sind äquivalent:*

(i) *$K = \boldsymbol{R}$ ist der Körper aller reellen Zahlen.*

(ii) *Jeder dedekindsche Schnitt in K wird durch ein Element in K bestimmt.*

(iii) *Es gibt keine echte, die Ordnung von K fortsetzende, über K archimedische Erweiterung von K.*

(iv) (a) *Jeder dedekindsche Schnitt in K ist eigentlich.*
(b) *Es gibt keine echte, die Ordnung von K fortsetzende Erweiterung von K, in der K dicht ist.*

(v) (a) *Ist U ein Unterkörper von K, so ist* 1 *der einzige ordnungerhaltende Automorphismus von U.*
(b) *Ist E eine die Ordnung von K fortsetzende Erweiterung von K, so gibt es einen Zwischenkörper Z mit $K \subset Z \subseteqq E$ und einen von* 1 *verschiedenen, ordnungerhaltenden Automorphismus von Z.*

Beweis. Daß jeder dedekindsche Schnitt im Bereich der reellen Zahlen durch eine reelle Zahl bestimmt wird, ist eine der wohlbekannten, den Körper $\boldsymbol{R}$ definierenden Eigenschaften: (ii) folgt aus (i).

Wir nehmen die Gültigkeit von (ii) an und betrachten eine die Ordnung von K fortsetzende, über K archimedische Erweiterung E von K. Ist $e \in E$, so sei S die Menge aller $s \in K$ mit $s < e$; und es sei D die Menge aller $d \in K$ mit $e \leqq d$. Aus der Archimedizität von E über K folgt, daß S, D ein dedekindscher Schnitt in K ist; und aus (ii) folgt die Existenz eines Elements $k \in K$ mit folgender Eigenschaft:

Ist $s \in K$ und $s < k$, so ist $s \in S$;
ist $d \in K$ und $k < d$, so ist $d \in D$.

Wäre $e < k$, so gäbe es wegen der Archimedizität von K in E ein Element $h \in K$ mit $0 < h < k - e$. Aus $k - h < k$ folgt $k - h \in S$, da k und h und also auch $k - h$ zu K gehören; und hieraus folgt $k - h < e < k - h$, ein Widerspruch. Ebenso sieht man die Unmöglichkeit von $k < e$ ein, so daß $e = k \in K$ ist und also $E = K$ wird. Damit haben wir (iii) aus (ii) abgeleitet.

Gilt (iii), so gilt Bedingung (viii) des Satzes 1.2, woraus $K \subseteqq \boldsymbol{R}$ folgt. Da aber jeder Unterkörper von $\boldsymbol{R}$ in $\boldsymbol{R}$ dicht ist [Satz 1.2, (i) + (ii)], so folgt $K = \boldsymbol{R}$ aus (iii): Die Bedingungen (i)–(iii) sind äquivalent.

Gelten die äquivalenten Bedingungen (i)–(iii), so folgt (iv.a) aus (ii) und (iv.b) aus (iii), da ja Dichte Archimedizität nach sich zieht. – Gilt umgekehrt (iv), so ergibt sich $K \subseteqq \boldsymbol{R}$ aus (iv.a) und Satz 1.2, (v). Mit $\boldsymbol{Q}$ ist also auch K in $\boldsymbol{R}$ dicht; und wir folgern $K = \boldsymbol{R}$ aus (iv.b): Die Bedingungen (i)–(iv) sind äquivalent.

Die Äquivalenz der untereinander äquivalenten Bedingungen (i)–(iv) mit (v) ergibt sich mühelos aus Satz 1.2 und Hilfssatz (1.2.+).

Bemerkung 1.4. A. Es gibt eine Fülle echter Unterkörper von $\boldsymbol{R}$; und diese genügen alle der Bedingung (iv.a), woraus die Unentbehrlichkeit von (iv.b) folgt.

B. Die Unentbehrlichkeit von (iv.a) wird in Bemerkung 2.8 bewiesen werden.

C. Unter Benutzung der üblichen transfiniten Schlüsse leitet man mühelos aus Lemma 1.1 die folgende Existenzaussage ab:

(1.4.**C**) Zu jedem angeordneten Körper K gibt es einen die Ordnung von K fortsetzenden, über K archimedischen Körper E derart, daß jeder dedekindsche Schnitt in K durch Elemente aus E bestimmt wird.

Daß man durch transfinite Iteration dieser Existenzaussage nur in Ausnahmefällen einen angeordneten Körper A erhalten wird, dessen sämtliche dedekindsche Schnitte durch Elemente aus A bestimmt werden, sieht man aus der Äquivalenz der Eigenschaften (i) und (ii) des Satzes 1.3.

(1.4.**C**) zeigt auch die Unmöglichkeit, die Unterkörper U von $\boldsymbol{R}$ durch ihre Einbettbarkeit in einen Körper zu charakterisieren, dessen Elemente die dedekindschen Schnitte in U bestimmen.

2. Dichte und stetiger Abschluß

Ist U ein Unterkörper des angeordneten Körpers K und S, D ein dedekindscher Schnitt in K, so setzen wir

$$\omega_{K/U}(S, D) = \omega_U(S, D) = \omega(S, D) = S \cap U, D \cap U;$$

hierbei werden wir von den möglichen Indices nur soviel benutzen, wie zur Vermeidung von Mißverständnissen nötig scheint. Ist insbesondere $k \in K$, so bestimmt k zwei im allgemeinen verschiedene dedekindsche Schnitte in K. Von diesen wählen wir

$$S(k) = \text{Menge aller } s \in K \text{ mit } s < k;$$

$$D(k) = \text{Menge aller } d \in K \text{ mit } k \leqq d.$$

Anstelle von $\omega_{K/U}(S(k), D(k))$ werden wir kürzer $\omega_{K/U}(k)$ schreiben.

Lemma 2.1. *Die folgenden Eigenschaften des Unterkörpers U des angeordneten Körpers K sind äquivalent:*

(i) *K ist archimedisch über U.*

(ii) *$\omega_{K/U}(S, D)$ ist für jeden dedekindschen Schnitt S, D in K ein dedekindscher Schnitt in U.*

(iii) *$\omega_{K/U}(k)$ ist für jedes $k \in K$ ein dedekindscher Schnitt in U.*

(iv) *Ist S, D ein dedekindscher Schnitt in K und $\omega_{K/U}(S, D)$ ein eigentlicher dedekindscher Schnitt in U, so ist S, D ein eigentlicher dedekindscher Schnitt in K.*

Beweis. Ist K archimedisch über U, so gibt es zu jedem $k \in K$ Elemente u, v in U mit $u \leqq k \leqq v$. Aus dieser Bemerkung folgt, daß für jeden dedekindschen

Schnitt S, D in K die Durchschnitte $S \cap U$, $D \cap U$ nicht leer sind, dieses Paar also ein dedekindscher Schnitt in U ist; (ii) folgt aus (i).

Es ist klar, daß der Spezialfall (iii) von (ii) aus (ii) folgt. Gilt (iii), so ist $D(k) \cap U$ für jedes $k \in K$ eine nicht leere Menge. Es gibt also zu jedem $k \in K$ ein $u \in U$ mit $k \leqq u$, so daß K archimedisch über U ist: (i)–(iii) sind äquivalent.

Sei K archimedisch über U und S, D ein dedekindscher Schnitt in K mit der Eigenschaft: $S \cap U$, $D \cap U$ ist ein eigentlicher dedekindscher Schnitt in U. Ist dann $0 < e \in K$, so folgt aus der Archimedizität von K über U die Existenz eines Elements $u \in U$ mit $0 < u < e$. Aus der Eigentlichkeit von $S \cap U$, $D \cap U$ ergibt sich die Existenz von Elementen $s \in S \cap U$ und $d \in D \cap U$ mit $0 < d - s < u < e$, woraus die Eigentlichkeit von S, D folgt. Also folgt (iv) aus (i).

Wir nehmen die Gültigkeit von (iv) an. Sei D die Menge aller $k \in K$ mit der Eigenschaft:

$$\text{Es gibt } u \in U \text{ mit } 0 < u \leqq k .$$

Weiter sei S das Komplement von D in K. Man überzeugt sich sofort davon, daß $D \cap U$ die Menge aller positiven Zahlen aus U und $S \cap U$ die Menge aller nicht positiven Zahlen aus U ist. Hieraus folgt sofort, daß S, D ein dedekindscher Schnitt in K und $\omega(S, D)$ ein eigentlicher dedekindscher Schnitt in U ist. Aus (iv) folgt dann, daß S, D ein eigentlicher dedekindscher Schnitt in K ist. Ist $0 < k \in K$, so gibt es folglich Elemente $d \in D$ und $s \in S$ mit $0 < d - s < k$. Aus der Definition des Paares S, D ergibt sich die Existenz eines Elementes $u \in U$ mit $0 < u \leqq d$. Da s im Komplement S von D liegt, ist $s < \frac{1}{2} u \in U$, so daß

$$0 < \tfrac{1}{2} u = u - \tfrac{1}{2} u < d - s < k$$

wird. Also ist K archimedisch über U; die Bedingungen (i) und (iv) und also auch die Bedingungen (i)–(iv) sind äquivalent.

Lemma 2.2. *Ist U ein Unterkörper des angeordneten Körpers K und S^*, D^* ein dedekindscher Schnitt in U, so gibt es einen dedekindschen Schnitt S, D in K mit $\omega_{K/U}(S, D) = S^*, D^*$.*

Beweis. Man wähle für D die Menge aller $k \in K$, zu denen es ein $t \in D^*$ mit $t \leqq k$ gibt; und S sei das Komplement von D in K. Dann ist offenbar S, D ein dedekindscher Schnitt in K mit

$$S^* = S \cap U, D^* = D \cap U \quad \text{und also} \quad \omega(S, D) = S^*, D^* .$$

Lemma 2.3. *Die folgenden Eigenschaften des Unterkörpers U des angeordneten Körpers K sind äquivalent:*

(i) *U ist dicht in K.*

(ii) *Für jedes $k \in K$ ist U dicht in $U(k)$.*

(iii) *Ist $k \in K$ und $0 < e \in K$, so gibt es Elemente u', u'' in U mit $u' < k < u''$ und $0 < u'' - u' < e$.*

(iv) *Ist S, D ein eigentlicher dedekindscher Schnitt in K, so ist $\omega_{K/U}(S, D)$ ein eigentlicher dedekindscher Schnitt in U.*

(v) *$\omega_{K/U}(k)$ ist für jedes $k \in K$ ein eigentlicher dedekindscher Schnitt in U.*

Beweis. Ist U dicht in K, so ist a fortiori U dicht in jedem Körper zwischen U und K, so daß (ii) aus (i) folgt. – Gilt (ii), ist $k \in K$ und $0 < e \in K$, so ist erstens U dicht in $U(e)$, so daß insbesondere $U(e)$ archimedisch über U ist. Also gibt es $u \in U$ mit $0 < u < \frac{1}{2}e$. Zweitens ist U dicht in $U(k)$. Also gibt es Elemente u', u'' in U mit

$$k - u \leqq u' \leqq k - \tfrac{1}{2}u < k < k + \tfrac{1}{2}u \leqq u'' \leqq k + u\,;$$

und es wird

$$0 < u'' - u' \leqq (k+u) - (k-u) = 2u < e\,.$$

Also folgt (iii) aus (ii).

Gilt (iii), ist S, D ein eigentlicher dedekindscher Schnitt in K und $0 < e \in U$, so gibt es Elemente $s \in S$ und $d \in D$ mit $0 < d - s < 4^{-1}e$. Aus (iii) folgern wir die Existenz von Elementen d', d'', s', s'' in U mit $s' < s < s''$, $d' < d < d''$ und

$$0 < s'' - s' < 4^{-1}e,\ 0 < d'' - d' < 4^{-1}e\,.$$

Aus $s \in S$ und $d \in D$ folgen dann $s' \in S \cap U$ und $d'' \in D \cap U$; und es wird

$$\begin{aligned} 0 < d'' - s' &= (d'' - d) + (d - s) + (s - s') \\ &< (d'' - d') + (d - s) + (s'' - s') < e\,. \end{aligned}$$

Also ist $S \cap U$, $D \cap U$ ein eigentlicher dedekindscher Schnitt in U; und wir haben (iv) aus (iii) abgeleitet.

Da $S(k)$, $D(k)$ offenbar ein eigentlicher dedekindscher Schnitt in K ist, folgt (v) aus (iv).

Gilt (v), so gilt auch Bedingung (iii) des Lemma 2.1, so daß K archimedisch über U ist. Sei weiter k', k'' ein Elementepaar aus K mit $k' < k''$. Aus $0 < k'' - k'$ und der Archimedizität von K über U folgt die Existenz von $u \in U$ mit $0 < u < 4^{-1}(k'' - k')$. Ist $k = \frac{1}{2}(k' + k'')$, so folgt aus (v) die Existenz von Elementen s, d in U mit

$$s < k \leqq d \quad \text{und} \quad 0 < d - s < u < 4^{-1}(k'' - k')\,.$$

Hieraus ergibt sich aber

$$\begin{aligned} \tfrac{1}{2}d &= \tfrac{1}{2}s + \tfrac{1}{2}(d-s) < \tfrac{1}{2}s + d - s < \tfrac{1}{2}k + 4^{-1}(k'' - k') = \tfrac{1}{2}k''\,, \\ \tfrac{1}{2}k' &= \tfrac{1}{2}k - 4^{-1}(k'' - k') < \tfrac{1}{2}d - (d-s) < \tfrac{1}{2}d - \tfrac{1}{2}(d-s) = \tfrac{1}{2}s\,, \end{aligned}$$

so daß $k' < s < d < k''$ wird: U ist dicht in K; und wir haben die Äquivalenz der Bedingungen (i)–(v) dargetan.

Bemerkung 2.4. Ist U ein Unterkörper des angeordneten Körpers K, ist K archimedisch über U, so bewirkt $\omega_{K/U}$ wegen Lemma 2.1 und 2.2 eine eindeutige Abbildung des Bereichs aller dedekindschen Schnitte in K auf die Menge aller dedekindschen Schnitte in U. Ist U dicht in K, so folgern wir aus Lemma 2.1 und 2.3, daß $\omega_{K/U}$ sogar eine eineindeutige Abbildung zwischen der Menge der eigentlichen dedekindschen Schnitte in K und der Menge der eigentlichen dedekindschen Schnitte in U induziert.

Satz 2.5. *Zu jedem angeordneten Körper K gibt es einen und im wesentlichen nur einen die Ordnung von K fortsetzenden, angeordneten Oberkörper $\tilde{K}$ von K mit folgenden Eigenschaften:*

(a) *K ist dicht in $\tilde{K}$.*

(b) *Ist Z ein die Anordnung von K fortsetzender, angeordneter Oberkörper von K, ist K dicht in Z, so gibt es einen und nur einen die Elemente aus K fixierenden, ordnungserhaltenden Isomorphismus von Z in $\tilde{K}$.*

(c) *Es gibt keine echte, die Ordnung von $\tilde{K}$ fortsetzende Erweiterung von $\tilde{K}$, in der $\tilde{K}$ dicht ist.*

(d) *Die Abbildung $t \to \omega_{\tilde{K}/K}(t)$ ist eine eineindeutige Abbildung von $\tilde{K}$ auf die Menge aller eigentlichen dedekindschen Schnitte in K.*

Wir werden $\tilde{K}$ *den stetigen Abschluß von K* nennen.

Beweis. Aus der Menge der eigentlichen dedekindschen Schnitte in K kann man in der üblichen Weise einen die Ordnung von K fortsetzenden, angeordneten Oberkörper $\tilde{K}$ von K machen; vgl. hierzu Baer [p. 217]. Dann ist $\omega_{\tilde{K}/K}(t)$ für jedes $t \in \tilde{K}$ ein eigentlicher dedekindscher Schnitt in K; und es folgt aus Lemma 2.3, daß K in $\tilde{K}$ dicht ist. – Ist Z ein die Anordnung von K fortsetzender, angeordneter Oberkörper von K, ist K dicht in Z, so ist $\omega_{Z/K}(z)$ für jedes $z \in Z$ wegen Lemma 2.3 ein eigentlicher dedekindscher Schnitt in K; und die Abbildung $z \to \omega_{Z/K}(z)$ induziert den gesuchten und offenbar eindeutig bestimmten, die Elemente aus K fixierenden, ordnungserhaltenden Isomorphismus von Z in $\tilde{K}$. – Ist E ein die Ordnung von $\tilde{K}$ fortsetzender, angeordneter Oberkörper von $\tilde{K}$, ist $\tilde{K}$ dicht in E, so folgt aus der Dichtheit von K in $\tilde{K}$, daß K in E dicht ist; und Anwendung der Eigenschaft (b) zeigt $E = \tilde{K}$. Also genügt $\tilde{K}$ den Bedingungen (a)–(d); und daß $\tilde{K}$ hierdurch wesentlich eindeutig bestimmt ist, erschließt man etwa aus (a), (b).

Zusatz 2.6. *Die folgenden Eigenschaften des angeordneten Körpers K sind äquivalent:*

(i) *$K = \tilde{K}$.*

(ii) *$K = \tilde{U}$ für wenigstens einen Unterkörper U von K.*

(iii) *Es gibt keine echte, die Ordnung von K fortsetzende Erweiterung von K, in der K dicht ist.*

(iv) *Ist der Unterkörper U von K dicht in K, so ist $K = \tilde{U}$.*

(v) *Jeder eigentliche dedekindsche Schnitt in K wird durch ein Element aus K bestimmt.*

Wir werden Körper K mit den äquivalenten Eigenschaften (i)–(v) *stetig abgeschlossene Körper* nennen.

Beweis. Es ist klar, daß (ii) aus (i) folgt. Daß (iii) aus (ii) folgt, ergibt sich aus Satz 2.5, (c). Gilt (iii), ist der Unterkörper U von K dicht in K, so folgt aus Satz 2.5, (b) die Existenz eines ordnungerhaltenden, die Elemente aus U fixierenden Isomorphismus von K in $\tilde{U}$. Wir können also o. B. d. A. annehmen, daß $K \subseteq \tilde{U}$ ist. Aus $U \subseteq K \subseteq \tilde{U}$ folgt wegen Satz 2.5, (a), daß K in $\tilde{U}$ dicht ist, so daß aus (iii) sich $K = \tilde{U}$ ergibt: (iv) folgt aus (iii). Da K in K dicht ist, folgt (i) aus (iv): Die Eigenschaften (i)–(iv) sind äquivalent.

Anwendung von Satz 2.5, (d) zeigt, daß (v) aus (i) folgt. Gilt umgekehrt (v), so ergibt Anwendung von Satz 2.5, (d) die Identität $K=\tilde{K}$: Die Bedingungen (i)–(v) sind äquivalent.

Folgerung 2.7. *Der Unterkörper U des angeordneten Körpers K ist dann und nur dann dicht in K, wenn $U \subseteqq K \subseteqq \tilde{U}$ gilt [wenn es also einen die Elemente in U fixierenden, ordnungerhaltenden Isomorphismus von K in $\tilde{U}$ gibt].*

Dies folgt mühelos aus Satz 2.5.

Bemerkung 2.8. Es gibt bekanntlich viele angeordnete Körper K, die über ihrem Primkörper $\boldsymbol{Q}$ nicht archimedisch sind. Die wegen Satz 2.5 existierende Erweiterung $\tilde{K}$ von K ist dann ebenfalls nicht über $\boldsymbol{Q}$ archimedisch und ist also kein Unterkörper von $\boldsymbol{R}$ [Satz 1.2]. Andererseits gibt es keine echte Erweiterung von $\tilde{K}$, in der $\tilde{K}$ dicht ist [Satz 2.5, (c)]. Hieraus folgt die Unentbehrlichkeit der Bedingung (iv. a) des Satzes 1.3.

Lemma 2.9. *Ist der angeordnete Körper K archimedisch über seinem Unterkörper U, so hat die Menge $E=E_{K/U}$ aller $k\in K$ mit eigentlichem $\omega_{K/U}(k)$ die folgenden Eigenschaften:*

(a) *E ist ein U enthaltender Unterkörper von K, in dem U dicht ist.*

(b) *Der Unterkörper V mit $U\subseteqq V\subseteqq K$ ist dann und nur dann in E enthalten, wenn U in V dicht ist.*

(c) *E ist das Compositum aller U enthaltenden Unterkörper von K, in denen U dicht ist.*

Beweis. Es ist klar, daß $\omega_{K/U}(u)$ für jedes $u\in U$ eigentlich ist, daß also U in E enthalten ist.

Sind a, b Elemente aus E, ist $0<e\in K$, so folgt aus der Eigentlichkeit der dedekindschen Schnitte $\omega_{K/U}(a)$ und $\omega_{K/U}(b)$ die Existenz von Elementen a', a'', b', b'' in U mit

$$a'<a<a'' \quad \text{und} \quad 0<a''-a'<\tfrac{1}{2}e,$$
$$b'<b<b'' \quad \text{und} \quad 0<b''-b'<\tfrac{1}{2}e.$$

Dann ist auch

$$a'+b'<a+b<a''+b'',$$
$$0<(a''+b'')-(a'+b')=(a''-a')+(b''-b')<e,$$
$$a'-b''<a-b<a''-b',$$
$$0<(a''-b')-(a'-b'')=(a''-a')+(b''-b')<e.$$

Damit haben wir gezeigt, daß $\omega_{K/U}(a+b)$ und $\omega_{K/U}(a-b)$ eigentliche dedekindsche Schnitte in U sind; und es folgt:

(1) Mit a und b gehören auch $a+b$ und $a-b$ zu E.

Wir betrachten positive Elemente a, b aus E und $0<e\in K$. Aus der Archimedizität von K über U folgt die Existenz von $u\in U$ mit $a<u$ und $b<u$.

Insbesondere ist $0<u$, so daß auch $f=\frac{1}{2}u^{-1}e$ ein positives Element aus K ist. Aus der Eigentlichkeit von $\omega_{K/U}(a)$ und $\omega_{K/U}(b)$ ergibt sich die Existenz von Elementen a^*, a^{**}, b^*, b^{**} in U mit

$$a^*<a<a^{**} \quad \text{und} \quad 0<a^{**}-a^*<f\,,$$
$$b^*<b<b^{**} \quad \text{und} \quad 0<b^{**}-b^*<f\,.$$

Da a und b positiv sind und $a<u$, $b<u$ ist, können wir o. B. d. A. annehmen, daß

$$0\leqq a^*<a^{**}\leqq u,\ 0\leqq b^*<b^{**}\leqq u$$

ist. Dann wird auch

$$a^*b^*<ab<a^{**}b^{**}\,,$$
$$0<a^{**}b^{**}-a^*b^*=a^{**}(b^{**}-b^*)+b^*(a^{**}-a^*)<$$
$$<(a^{**}+b^*)f\leqq 2uf=e\,.$$

Also ist $\omega_{K/U}(ab)$ eigentlich und $ab\in E$. Wir haben gezeigt:

(2) Sind a, b positive Elemente aus E, so ist auch $ab\in E$.

Sind a, b irgendwelche Elemente aus E, so gehört ab sicher dann zu E, wenn a oder b Null ist. Sei also $ab\neq 0$. Haben a und b gleiches Vorzeichen, so folgt aus $ab=(-a)(-b)$ und (2) die Zugehörigkeit von ab zu E. Haben a und b entgegengesetztes Vorzeichen, so folgt aus $a(-b)=-ab=(-a)b$ und (2) die Zugehörigkeit von $-ab$ zu E; und es folgt aus (1) die Zugehörigkeit von ab zu E. Damit haben wir gezeigt:

(3) E ist ein U umfassender Unterring von K.

Wir betrachten ein Element c mit $0<c\in E$ und ein Element e mit $0<e\in K$. Aus der Archimedizität von K über U folgt die Existenz eines Elements $u\in U$ mit $0<u<c$. Dann ist auch $f=eu^2$ ein positives Element aus K. Aus der Eigentlichkeit von $\omega_{K/U}(c)$ folgt die Existenz von Elementen c', c'' in U mit

$$c'<c<c'' \quad \text{und} \quad 0<c''-c'<f\,.$$

Wegen $0<u<c$ können wir o. B. d. A. annehmen, daß

$$0<u\leqq c'$$

ist. Dann wird

$$0<c''^{-1}<c^{-1}<c'^{-1}\leqq u^{-1}\,;$$

und es wird

$$0<c'^{-1}-c''^{-1}=(c''-c')c'^{-1}c''^{-1}<fu^{-2}=e\,.$$

Also ist $\omega_{K/U}(c^{-1})$ ein eigentlicher dedekindscher Schnitt in U; und wir haben gezeigt:

(4) Ist $0<c\in E$, so ist auch $c^{-1}\in E$.

Aus (3) und (4) folgt ohne weiteres, daß E ein U umfassender Unterkörper von K ist. Aus der Definition von E ergibt sich sofort, daß Bedingung (v) des Lemma 2.3 von dem angeordneten Körper E und seinem Unterkörper U erfüllt wird. Also ist U dicht in E.

Ist V ein Unterkörper von K mit $U \subseteqq V \subseteqq E$, so ist U dicht in V, da ja U dicht in E ist. – Ist umgekehrt V ein U enthaltender Unterkörper von K, in dem U dicht ist, so ist $\omega_{V/U}(v)$ für jedes $v \in V$ wegen Lemma 2.3, (v) eigentlich, so daß aus $\omega_{K/U}(v) = \omega_{V/U}(v)$ die Zugehörigkeit von v zu E und $V \subseteqq E$ folgen. Damit haben wir (a) und (b) bewiesen; und (c) ergibt sich durch Combination von (a) und (b).

Bemerkung 2.10. A. Wäre K nicht archimedisch über U, so gäbe es Elemente $j \in K$ mit $0 < j < u$ für jedes positive $u \in U$. Dann würde aber $\omega_{K/U}(j)$ aus der Klasse der nicht positiven und der Klasse der positiven Elemente aus U bestehen, wäre also ein eigentlicher dedekindscher Schnitt in U. Damit hätten wir die Zugehörigkeit von j zu E dargetan. Daraus würde aber folgen, daß U gewiß nicht dicht in E ist: Die Voraussetzung der Archimedizität von K über U ist unentbehrlich für die Gültigkeit von Lemma 2.9.

B. Den im Beweis des Satzes 2.5 enthaltenen, durch einen Hinweis auf die Literatur erledigten Beweis für die Existenz des Erweiterungskörpers $\tilde{K}$ von K könnte man auch folgendermaßen führen: Man bildet zunächst gemäß Bemerkung 1.4, **C** einen die Anordnung von K fortsetzenden, über K archimedischen Körper H mit der Eigenschaft, daß alle dedekindschen Schnitte aus K durch Elemente aus H definiert werden, bildet dann gemäß Satz 2.9 den Unterkörper $E_{H/K}$ und zeigt seine Identität mit $\tilde{K}$.

Folgerung 2.11. *Ist der angeordnete Körper $K = \tilde{K}$ archimedisch über seinem Unterkörper U, so ist das Compositum $E_{K/U}$ aller U enthaltenden Unterkörper von K, in denen U dicht ist, im wesentlichen mit $\tilde{U}$ identisch.*

Beweis. Es folgt aus Lemma 2.9, daß die Menge E aller $k \in K$ mit eigentlichem $\omega_{K/U}(k)$ ein U enthaltender Unterkörper von K ist, in dem U dicht ist, und der alle U enthaltenden Unterkörper von K, in denen U dicht ist, enthält.

Ist S, D ein eigentlicher dedekindscher Schnitt in U, so folgt aus Lemma 2.2 die Existenz eines dedekindschen Schnitts S^*, D^* in K mit $S = U \cap S^*$ und $D = U \cap D^*$. Anwendung von Lemma 2.1, (iv) zeigt, daß S^*, D^* ein eigentlicher dedekindscher Schnitt in K ist, da ja K archimedisch über U und S, D eigentlich ist. Aus $K = \tilde{K}$ folgt die Existenz eines S^*, D^* definierenden Elements k in K. Dann ist $\omega_{K/U}(k)$ eigentlich, so daß $k \in E$ ist. Damit haben wir $\tilde{U} \subseteqq E$ gezeigt. Da aber U in E dicht ist, folgt $E = \tilde{U}$ aus Satz 2.5, (c).

Bemerkung 2.12. Für die Unmöglichkeit $\tilde{U} = K$ zu beweisen vgl. etwa Satz 5.4. Es gibt auch einfachere Beispiele.

Folgerung 2.13. *Ist der angeordnete Körper K über seinem Unterkörper U archimedisch, ist $\mathbf{M}$ eine Menge von Unterkörpern von K, über denen K archimedisch ist, ist $U \cap X$ dicht in jedem $X \in \mathbf{M}$, so ist U dicht im Compositum von U und den Unterkörpern aus $\mathbf{M}$.*

Beweis. Da K archimedisch über U ist, ist die Menge $E=E_{K/U}$ aller $k\in K$ mit eigentlichem $\omega_{K/U}(k)$ wegen Lemma 2.9 ein U enthaltender Unterkörper von K, in dem U dicht ist. Sei $t\in X\in M$ und $0<u\in U$. Da K archimedisch über X ist, gibt es $x\in X$ mit $0<x<u$. Da $X\cap U$ dicht in X ist, folgt aus Lemma 2.3, (iii) die Existenz von Elementen t', t'' in $X\cap U$ mit $t'<t<t''$ und $0<t''-t'<x<u$. Hieraus folgt aber die Eigentlichkeit des dedekindschen Schnittes $\omega_{K/U}(t)$, so daß $t\in E$ ist. Damit haben wir aber $X\subseteqq E$ für jedes $X\in M$ dargetan. Da $U\subseteqq E$ ist, folgt, daß das Compositum C von U und den Unterkörpern $X\in M$ in E enthalten ist. Aus $U\subseteqq C\subseteqq E$ und der Dichte von U in E folgt, daß U in C dicht ist.

3. Dedekindscher Abschluß

Das folgende Resultat wird zwar im Verlauf unserer Überlegungen nicht benötigt werden, wird aber im Zusammenhang mit den Beispielen des Abschnitts 5 die hier behandelten Probleme ins rechte Licht setzen.

Satz 3.1. *Ist n eine positive ganze Zahl und jedes positive Element in dem angeordneten Körper K eine n-te Potenz eines positiven Elements aus K, so ist auch jedes positive Element aus $\tilde{K}$ eine n-te Potenz eines positiven Elements aus $\tilde{K}$.*

Beweis. Ist $0<p\in\tilde{K}$, so ist $\omega_{\tilde{K}/K}(p)=S, D$ der p definierende eigentliche dedekindsche Schnitt in K. Aus $0<p$ folgt, daß S positive Elemente enthält und D nur aus positiven Elementen besteht. Aus unserer Voraussetzung folgt, daß es für jedes k mit $0<k\in K$ ein und nur ein k' mit $0<k'\in K$ und $k'^n=k$ gibt. Da D nur aus positiven Elementen besteht, ist die Menge D' aller d' mit $d\in D$ wohlbestimmt; und das Complement S' von D' in K ist genau die Menge aller nicht positiven Elemente aus K zusammen mit allen s' mit $0<s\in S$. Man sieht sofort ein, daß S', D' ein dedekindscher Schnitt in K ist.

Aus $0<p$ folgt die Existenz eines r mit $0<r\in S$. Ist weiter e mit $0<e\in K$ gegeben, so setzen wir $e^*=nr'^{n-1}e$, so daß auch $0<e^*\in K$ ist. Da S, D ein eigentlicher dedekindscher Schnitt in K ist, gibt es $s\in S$ und $d\in D$ mit $0<d-s<e^*$; und wir können o. B. d. A. annehmen, daß $r\leqq s$ ist. Dann wird auch $0<r'\leqq s'<d'$; und hieraus folgt

$$nr'^{n-1}\leqq\sum_{i=0}^{n-1}d'^i s'^{n-1-i}.$$

Weiter ist

$$d-s=d'^n-s'^n=(d'-s')\sum_{i=0}^{n-1}d'^i s'^{n-1-i}$$

und also

$$d'-s'=(d-s)\left[\sum_{i=0}^{n-1}d'^i s'^{n-i-1}\right]^{-1}<e^* n^{-1}r'^{1-n}=e.$$

Damit haben wir die Eigentlichkeit des dedekindschen Schnitts S', D' in K dargetan. Das durch S', D' definierte Element p' in $\tilde{K}$ ist offenbar positiv und genügt der Gleichung $p'^n=p$.

Bemerkung 3.2. A. Die Unentbehrlichkeit der in Satz 3.1 gemachten Voraussetzung wird durch Satz 5.1 in Evidenz gesetzt.

B. Ist die multiplikative Gruppe der positiven Zahlen aus dem angeordneten Körper K radizierbar, so ist die Voraussetzung von Satz 3.1 für jedes ganze positive n erfüllt; und hieraus folgt, daß auch die multiplikative Gruppe der positiven Elemente aus $\tilde{K}$ radizierbar ist.

C. Ist der Körper R reell abgeschlossen, so folgt aus Satz A, (iii), daß jede positive Zahl aus R für jedes positive ganze n eine n-te Potenz ist; und hieraus folgt die Radizierbarkeit der multiplikativen Gruppe der positiven Zahlen aus R. Dies gestattet die Anwendung der Bemerkung **B** auf R. Das so erhaltene Resultat ist aber auch im folgenden Resultat enthalten.

Satz 3.3. *Mit R ist auch $\tilde{R}$ reell abgeschlossen.*

Beweis. Es gibt eine und im wesentlichen nur eine reell abgeschlossene, über $\tilde{R}$ algebraische, die Ordnung von $\tilde{R}$ fortsetzende Erweiterung E von $\tilde{R}$ [Satz C]. Da R wegen Satz 2.5, (a) in $\tilde{R}$ dicht ist, ist $\tilde{R}$ über R archimedisch. Da E über $\tilde{R}$ algebraisch ist, ist E wegen Lemma F über $\tilde{R}$ archimedisch. Also folgt:

(1) E ist über R archimedisch.

Ist $s \in \tilde{R}$ und $e \in E$ mit $s < e$, so folgt aus der Archimedizität von E über R die Existenz von $r \in R$ mit $0 < r < e - s$, so daß $s < r + s < e$ ist. Da R wegen Satz 2.5, (a) in $\tilde{R}$ dicht ist, gibt es ein $r' \in R$ mit $s < r' < r + s < e$. Damit haben wir [aus Symmetriegründen] gezeigt:

(2) Ist $a \in \tilde{R}$ und $a \neq b \in E$, so gibt es $c \in R$ [zwischen a und b, also] mit $0 < (b-c)(c-a)$.

Wäre unser Satz falsch, so wäre $\tilde{R} \subset E$. Dann gäbe es also Elemente in E, die nicht in $\tilde{R}$ liegen. Da E nach Konstruktion algebraisch über $\tilde{R}$ ist, genügt jedes dieser Elemente einer algebraischen Gleichung über $\tilde{R}$. Es gibt also unter diesen Elementen eines w, das Nullstelle eines Polynoms $f(x)$ minimalen Grades n über $\tilde{R}$ ist. Wir notieren die folgenden Eigenschaften von w, f, n:

(3) $w \in E$, $w \notin \tilde{R}$, $f(x)$ ist ein Polynom des Grades n über $\tilde{R}$ und $f(w) = 0$.

(4) Ist der Grad des Polynoms h über $\tilde{R}$ kleiner als n, ist $v \in E$ und $h(v) = 0$, so ist $v \in \tilde{R}$.

Wäre f reduzibel über $\tilde{R}$, so genügte w einer Gleichung über $\tilde{R}$, deren Grad kleiner als n wäre. Wegen (4) läge dann w in $\tilde{R}$, was (3) widerspricht. Also gilt:

(5) f ist irreduzibel über $\tilde{R}$.

Der Grad des g. g. T. (f, f') ist höchstens $n - 1$. Also folgt aus (5), daß der Grad von (f, f') Null ist:

(6) f und f' sind teilerfremd.

Hieraus folgt insbesondere:

(7) f und f' haben keine gemeinsamen Nullstellen [in E] und f hat keine „mehrfachen" Nullstellen.

Die in E liegenden Nullstellen von f wollen wir der Größe nach anordnen:

$$r_1 < \cdots < r_m \quad \text{und} \quad m \leqq n .$$

Läge eines der r_i in $\tilde{R}$, so folgte aus der Irreduzibilität von f, daß $n=1$ ist und also $w=r_1$ in $\tilde{R}$ läge. Dies widerspricht (3); und es gilt:

(8) $1<n$ und kein r_i liegt in $\tilde{R}$,

so daß wir o. B. d. A. annehmen können:

(9) $w=r_1$.

Wegen Zusatz B gilt in E der Satz von Rolle. Es folgt, daß f' in E zwischen r_i und r_{i+1} [mit Ausschluß der Gleichheit] eine Nullstelle besitzt; und diese gehört wegen (4) zu $\tilde{R}$. Wir notieren dies:

(10) Zu jedem i mit $0<i<m$ gibt es wenigstens ein $s \in \tilde{R}$ mit $r_i < s < r_{i+1}$ und $f'(s)=0$.

Da f' nur endlich viele Nullstellen in E besitzt, die wegen (7) sämtlich von den r_i verschieden sind, können wir aus (9), (10), (1) und (2) folgendes erschließen:

(11) Es gibt Elemente a, b in R mit folgenden Eigenschaften: $a<w<b$; aus $a \leqq r \leqq b$ und $r \in E$ folgt $f'(r) \neq 0$; aus $r \in E$, $a \leqq r \leqq b$ und $r \neq w$ folgt $f(r) \neq 0$.

Da f' im abgeschlossenen Intervall $[a, b]$ aller $c \in E$ mit $a \leqq c \leqq b$ keine Nullstellen besitzt, ergibt Zusatz B die Existenz eines positiven $e \in E$, so daß entweder $e \leqq f'(r)$ für alle $r \in [a, b]$ oder $f'(r) \leqq -e$ für alle $r \in [a, b]$ gilt. Wegen $(-f)' = -f'$ können wir o. B. d. A. die Gültigkeit der ersten dieser Alternativen annehmen. Aus der Archimedizität von E über R folgt weiter die Existenz eines $k \in R$ mit $0<k<e$. Folglich gilt:

(12) Es gibt ein $s \in R$ mit $0<s<f'(t)$ für alle $t \in [a, b]$.

Wegen Zusatz B gilt in E der Mittelwertsatz der Differentialrechnung. Folglich gibt es ein $c \in [a, b]$ mit

$$0<s<f'(c)=[f(b)-f(w)]\,[b-w]^{-1}=f(b)\,[b-w]^{-1},$$

woraus insbesondere $0<f(b)$ folgt. Entsprechend gilt $f(a)<0$. Wir notieren dieses Ergebnis:

(13) $f(a)<0<f(b)$.

Mit a, b gehört auch das Maximum d der Zahlen $|a^i|$, $|b^i|$ für $0 \leqq i \leqq n$ zu R; und es ist gewiß $1 \leqq d \in R$. Aus (13) folgt, daß das Minimum der Zahlen $-f(a)$,

$f(b)$ eine positive Zahl aus E ist; und aus der Archimedizität von E über R folgt also die Existenz einer Zahl $r \in R$ mit $0 < r < -f(a), f(b)$. Mit $n+1, d, r$ gehört auch $\frac{1}{2}(n+1)^{-1} d^{-1} r = j$ zu R; und es gilt

(14′) $0 < 2(n+1)\, dj = r < -f(a), f(b)$.

Ist p irgendeine positive Zahl aus E, so folgt aus der Archimedizität von E über R [wegen (1)] die Existenz eines $q \in R$ mit $0 < q < p$. Dann ist $(n+1)^{-1} d^{-1} sq$ [wegen (12)] eine positive Zahl aus R, so daß auch das Minimum e der Zahlen j und $(n+1)^{-1} d^{-1} sq$ zu R gehört. Damit haben wir gezeigt:

(14) Zu jeder positiven Zahl $p \in E$ gibt es eine positive Zahl $e = e(p) \in R$ mit $0 < 2(n+1)de < -f(a), f(b)$ und $(n+1)ds^{-1}e < p$. Hierbei ist $0 < s < f'(t)$ für $b \in [a, b]$ und d das Maximum der $|a^i|, |b^i|$ mit $0 \leqq i \leqq n$.

Wir betrachten im folgenden zu vorgegebenem positivem $p \in E$ eine den Bedingungen von (14) genügende positive Zahl $e \in R$. Wir erinnern daran, daß

$$f(x) = \sum_{i=0}^{n} f_i x^i \quad \text{mit} \quad f_i \in \tilde{R}$$

ist. Da R wegen Satz 2.5, (a) in $\tilde{R}$ dicht ist, gibt es Elemente $g_i \in R$ mit $|f_i - g_i| < e$ für $0 \leqq i \leqq n$. Setzen wir

$$g(x) = \sum_{i=0}^{n} g_i x^i,$$

so wird also für $y \in [a, b]$

$$|f(y) - g(y)| = \left| \sum_{i=0}^{n} (f_i - g_i) y^i \right| \leqq \sum_{i=0}^{n} |f_i - g_i|\, |y|^i < (n+1)ed.$$

Wir notieren dieses Resultat:

(15) $|f(y) - g(y)| < (n+1)ed$ für $y \in [a, b]$.

Ist $w \neq y \in [a, b]$, so folgt aus dem wegen Zusatz B in E geltenden Mittelwertsatz der Differentialrechnung die Existenz eines Elementes $y^* \in [a, b]$ mit

$$f(y) = f(y) - f(w) = (y - w) f'(y^*);$$

und hieraus ergibt sich wegen (12)

$$0 < f'(y^*)^{-1} = (y - w) f(y)^{-1} < s^{-1}.$$

Damit haben wir gezeigt:

(16) Ist $w \neq y \in [a, b]$, so ist $0 < (y - w) f(y)^{-1} < s^{-1}$.

Wäre $0 \leqq g(a)$, so würde aus (14), (15) folgen, daß

$$0 < 2(n+1)de < -f(a) \leqq -f(a) + g(a) = |f(a) - g(a)| < (n+1)ed$$

ist, ein Widerspruch. Also ist $g(a) < 0$; und ebenso sieht man $0 < g(b)$ ein. Da der Körper R reell abgeschlossen ist, gilt in R der Weierstraßsche Nullstellensatz [Satz A, (ii)]. Also gibt es ein $t \in R$ mit $a < t < b$ und $g(t) = 0$. Aus (15) folgt dann

$$|f(t)| = |f(t) - g(t)| < (n+1)ed. \qquad (+)$$

Da $t \in R$ und $w \notin \tilde{R}$ [wegen (3)] ist, ist $w \neq t \in [a, b]$; und Anwendung von (16) ergibt

$$0 < (t-w) f(t)^{-1} < s^{-1}.$$

Hieraus folgt wegen (+) und (14)

$$|t-w| < |f(t)| s^{-1} < (n+1)eds^{-1} < p.$$

Damit haben wir gezeigt:

(17) Ist $0 < p \in E$, so gibt es ein $t \in R$ mit [$a < t < b$ und] $|t-w| < p$.

Ist $0 < p \in E$, so gibt es wegen der Archimedizität von E über R [siehe (1)] ein $q \in R$ mit $0 < q < \frac{1}{2} p$. Wenden wir (17) auf q an, so folgt die Existenz eines Elements $t \in R$ mit $|t-w| < q$. Ist erstens $t < w$, so wird $0 < w - t < q$ und $w - q < t < w < t + q$ mit t und $t + q$ in R und $|t - (t+q)| < \frac{1}{2} p$. Ist aber $w \leqq t$, so wird sogar $w < t$, da w nicht in $\tilde{R}$ liegt; und es folgt $0 < t - w < q$ und $t - q < w < t$ mit $t - q$ und t in R und $|t - (t-q)| < \frac{1}{2} p$. Also gilt:

(18) Ist $0 < p \in E$, so gibt es Elemente t', t'' in R mit $t' < w < t''$ und $|t' - t''| < p$.

Hieraus folgt aber, daß der von w in R induzierte dedekindsche Schnitt $\omega_{E/R}(w)$ eigentlich ist. Dieser wird also durch ein Element $r \in \tilde{R}$ definiert. Aus (18) folgt aber

$$|w - r| < p \quad \text{für jedes positive} \quad p \in E,$$

so daß $w = r \in \tilde{R}$ ist im Widerspruch zu (3). Aus diesem Widerspruch folgt die Richtigkeit unseres Satzes.

Bemerkung 3.4. Ohne die Voraussetzung der reellen Abgeschlossenheit von R kann man im allgemeinen nicht die reelle Abgeschlossenheit von $\tilde{R}$ beweisen; siehe Satz 5.1, (b). Ist andererseits R der Körper der rationalen Zahlen, so ist $\tilde{R}$ der reell abgeschlossene Körper $\boldsymbol{R}$. Damit haben wir gezeigt, daß die Voraussetzung der reellen Abgeschlossenheit von R für die reelle Abgeschlossenheit von $\tilde{R}$ zwar unentbehrlich, aber nicht notwendig ist.

Zusatz 3.5. (A) *Ist der reell abgeschlossene Körper R archimedisch über seinem reell abgeschlossenen Unterkörper U, so ist die Menge $E = E_{R/U}$ aller $r \in R$ mit eigentlichem $\omega_{R/U}(r)$ ein reell abgeschlossener [U enthaltender] Unterkörper von R.*

(B) *Ist der reell abgeschlossene Körper $R = \tilde{R}$ archimedisch über seinem reell abgeschlossenen Unterkörper U, so ist $E_{R/U} = \tilde{E}_{R/U}$ reell abgeschlossen [und stetig abgeschlossen].*

(C) *Ist der angeordnete Körper $K = \tilde{K}$ archimedisch über seinem reell abgeschlossenen Unterkörper R, so gibt es einen reell abgeschlossenen, R ent-*

haltenden Unterkörper $M=\tilde{M}$ von K derart, daß M der einzige M enthaltende, reell abgeschlossene Unterkörper von K ist.

Beweis. Ist der reell abgeschlossene Körper R archimedisch über seinem reell abgeschlossenen Unterkörper U, so folgt aus Lemma 2.9, daß $E=E_{R/U}$ ein U enthaltender Unterkörper von R ist, daß U in E dicht ist, und daß E das Compositum aller U enthaltenden Unterkörper von R ist, in denen U dicht ist. Wir bilden den stetigen Abschluß $\tilde{E}$ von E; und bemerken, daß E wegen Lemma 2.5, (a) in $\tilde{E}$ dicht ist. Da U in E und E in $\tilde{E}$ dicht ist, ist U in $\tilde{E}$ dicht. Aus Folgerung 2.7 folgern wir $U \subseteqq \tilde{E} \subseteqq \tilde{U}$; und dies ergibt $\tilde{E}=\tilde{U}$ wegen Lemma 2,5, (a), (c). Aus der reellen Abgeschlossenheit von U folgt wegen Satz 3.3 die reelle Abgeschlossenheit von $\tilde{U}=\tilde{E}$. Sei A die Menge der über E algebraischen Elemente aus R und B die Menge der über E algebraischen Elemente aus $\tilde{E}$. Aus der reellen Abgeschlossenheit von R und $\tilde{E}$ folgt wegen Lemma E, (II), daß A und B reell abgeschlossene Körper sind. Also ist sowohl A wie B ein reeller Abschluß von E; und es folgt aus Satz C, daß A und B äquivalente Erweiterungen von E sind. Wegen Lemma 2.5, (a) ist E in $\tilde{E}$ dicht. Hieraus folgt, daß E in B und also auch in A dicht ist. Da aber U in E dicht ist, ist U auch in A dicht; und da E das Compositum aller U enthaltenden Unterkörper von R ist, in denen U dicht ist, folgt $A \subseteqq E \subseteqq A$, so daß $E=A$ reell abgeschlossen ist. Damit haben wir (A) bewiesen.

(B) ergibt sich sofort aus Folgerung 2.11 und (A).

Ist der angeordnete Körper $K=\tilde{K}$ archimedisch über seinem reell abgeschlossenen Unterkörper R, so ist die Menge $\boldsymbol{M}$ der R enthaltenden, reell abgeschlossenen Unterkörper von K nicht leer [$R \in \boldsymbol{M}$]. Da die Eigenschaft der reellen Abgeschlossenheit wegen Satz A, (viii) eine lokale Eigenschaft ist, können wir das Maximumprinzip der Mengenlehre auf $\boldsymbol{M}$ anwenden: Es gibt ein maximales Element $M \in \boldsymbol{M}$. Dann ist M ein R enthaltender, reell abgeschlossener Unterkörper von K; und M ist der einzige M enthaltende, reell abgeschlossene Unterkörper von K. Da K über R archimedisch ist, ist K auch über M archimedisch. Wegen Lemma 2.9, (c) ist $E_{K/M}$ das Compositum aller M enthaltenden Unterkörper von K, in denen M dicht ist; und aus $K=\tilde{K}$ folgt $E_{K/M}=\tilde{E}_{K/M}$ wegen Folgerung 2.11. Da M in $E_{K/M}$ dicht ist [Lemma 2.9, (a)], können wir aus $E_{K/M}=\tilde{E}_{K/M}$, Lemma 2.5 und Folgerung 2.7 erschließen, daß $\tilde{M}=E_{K/M}$ ist. Mit M ist wegen Satz 3.3 auch $\tilde{M}$ reell abgeschlossen; und aus $M \subseteqq \tilde{M} \subseteqq K$ und der Maximalität von M ergibt sich $M=\tilde{M}$.

Satz 3.6. *Die folgenden Eigenschaften des angeordneten Körpers K sind äquivalent:*

(i) *K ist reell abgeschlossen und $K=\tilde{K}$.*

(ii) *Ist E eine echte, die Ordnung von K fortsetzende Erweiterung von K, so gibt es keinen über K algebraischen und in E dichten Körper zwischen K und E.*

(iii) $\begin{cases} \text{(a) } \textit{Es gibt einen reell abgeschlossenen, in } K \textit{ dichten Unterkörper von } K. \\ \text{(b) } \textit{Es gibt keine echte, die Ordnung von } K \textit{ fortsetzende Erweiterung} \\ \textit{von } K\textit{, in der } K \textit{ dicht ist.} \end{cases}$

(iv) $\begin{cases} \text{(a) } \textit{Es gibt einen reell abgeschlossenen Unterkörper von } K\textit{, über dem } K \textit{ archimedisch ist.} \\ \text{(b) } \textit{Es gibt keine echte, die Ordnung von } K \textit{ fortsetzende Erweiterung von } K\textit{, in der } K \textit{ dicht ist.} \\ \text{(c) } \textit{Ist } K \textit{ über seinem reell abgeschlossenen Unterkörper } U = \tilde{U} \neq K \textit{ archimedisch, so gibt es einen reell abgeschlossenen Unterkörper } V \textit{ mit } U \subset V \subseteqq K. \end{cases}$

Körper K mit den äquivalenten Eigenschaften (i)–(iv) wollen wir *dedekind-abgeschlossen* nennen.

Beweis. Gilt (i), ist E eine die Ordnung von K fortsetzende Erweiterung von K und Z ein Körper zwischen K und E, der über K algebraisch und in E dicht ist, so ist $Z = K$, da K reell abgeschlossen ist: und aus Lemma 2.5, (c) folgt $K = E$, da $K = \tilde{K}$ und K in E dicht ist. Also ist (ii) eine Folge von (i). – Genügt weiter K der Bedingung (ii), so betrachten wir zunächst den wegen Satz C existierenden reellen Abschluß von K, der wegen (ii) gleich K sein muß: K ist reell abgeschlossen. Aus Satz 2.5, (a) folgt, daß K in $\tilde{K}$ dicht ist; und es folgt aus (ii), daß $K = \tilde{K}$ ist. Also gilt (i) und wir haben die Äquivalenz von (i) und (ii) gezeigt.

Es ist klar, daß die Bedingungen (iii. a) und (iv. a, c) aus (i) und die identischen Bedingungen (iii. b) = (iv. b) aus (ii) folgen; die Bedingungen (iii) und (iv) folgen aus den äquivalenten Bedingungen (i) und (ii).

Gilt schließlich eine der Bedingungen (iii) oder (iv), so folgt $K = \tilde{K}$ aus den identischen Bedingungen (iii. b) = (iv. b), da ja K wegen Satz 2.5, (a) in $\tilde{K}$ dicht ist. Aus (iii. a) folgt natürlich (iv. a), so daß Anwendung von Zusatz 3.5, (C) die Existenz eines reell abgeschlossenen Unterkörpers $M = \tilde{M} \subseteqq K$ ergibt, über dem K archimedisch ist, und der unter den reell abgeschlossenen Unterkörpern von K maximal ist. Natürlich ist M im Falle der Gültigkeit von (iii. a) sogar in K dicht. Gilt (iv. c), so folgt sofort $K = M$. Gilt (iii. a), so ist $M = \tilde{M}$ in K dicht; und $K = M$ folgt aus Satz 2.5, (c). Also ist in beiden Fällen $\tilde{K} = K = M$ reell abgeschlossen: (i) folgt aus (iii) und auch aus (iv), so daß die Bedingungen (i)–(iv) äquivalent sind.

Folgerung 3.7. *Ist ein reell abgeschlossener Unterkörper des angeordneten Körpers K dicht in K, so ist K dicht in seinem reellen Abschluß.*

Beweis. Wir bilden den stetigen Abschluß $\tilde{K}$. Dann folgt aus Satz 2.5, (c), daß $\tilde{K}$ der Bedingung (iii. b) des Satzes 3.6 genügt. Wegen Satz 2.5, (a) ist K dicht in $\tilde{K}$, so daß der nach Voraussetzung existierende reell abgeschlossene Unterkörper von K auch in $\tilde{K}$ dicht ist: $\tilde{K}$ genügt also der Bedingung (iii) des Satzes 3.6, so daß $\tilde{K}$ reell abgeschlossen ist. Die Menge R der über K algebraischen Elemente aus $\tilde{K}$ ist also wegen Lemma E, (II) ein reell abgeschlossener Unterkörper von $\tilde{K}$. Da R algebraisch über K ist und natürlich die Anordnung von K fortsetzt, ist R der reelle Abschluß von K. Da K dicht in $\tilde{K}$ ist [Satz 2.5, (a)], ist K auch dicht in dem Zwischenkörper R.

Bemerkung 3.8. A. Ist K reell abgeschlossen und algebraisch über seinem Unterkörper $U = \tilde{U}$, so braucht $K = \tilde{K}$ nicht zu gelten, wie aus Satz 5.4 hervorgeht. Die Voraussetzung von Folgerung 3.7 ist wegen Satz 5.1, (b) unentbehrlich.

B. Es ist wohlbekannt, daß der Körper $\boldsymbol{R}$ aller reellen Zahlen dedekind-abgeschlossen ist. Ein Überblick über die Constructionsmöglichkeiten dedekind-abgeschlossener Körper findet sich im folgenden Kriterium, zu dessen bequemer Formulierung wir den folgenden Begriff benötigen:

Der Unterkörper U des angeordneten Körpers K ist *dedekindsch in* K [und K ist dedekindsch über U], wenn es einen Körper zwischen U und K gibt, der algebraisch über U und dicht in K ist.

Naturgemäß kann man die Wahl des in dieser Definition auftretenden Zwischenkörpers dadurch eindeutig machen, daß man für ihn die Menge aller über U algebraischen Elemente aus K wählt; siehe Lemma E, (II).

Satz 3.9. *Die folgenden Eigenschaften des Unterkörpers U des angeordneten Körpers K sind äquivalent:*

(i) *U ist dedekindsch in K und K ist dedekind-abgeschlossen.*

(ii) *Der Körper A der über U algebraischen Elemente aus K ist reell abgeschlossen mit $K = \tilde{A}$.*

(iii) $\begin{cases} \text{(a) } K \text{ ist dedekind-abgeschlossen.} \\ \text{(b) } K \text{ ist archimedisch über } U. \\ \text{(c) Zwischenkörper } Z \text{ mit } U \subseteqq Z \subset K \text{ sind nicht dedekind-abgeschlossen.} \end{cases}$

(iv) $\begin{cases} \text{(a) } U \text{ ist dedekindsch in } K. \\ \text{(b) Es gibt keine echte, die Ordnung von } K \text{ fortsetzende, über } K \\ \text{dedekindsche Erweiterung von } K. \end{cases}$

Genügen U und K den äquivalenten Bedingungen (i)–(iv), so wollen wir K *den dedekindschen Abschluß von* U nennen. Die eindeutige Bestimmtheit des dedekindschen Abschlusses wird in Folgerung 3.10 erwiesen. – Für weitere Charakterisierungen des dedekindschen Abschlusses siehe unten Satz 4.2.

Beweis. Gilt (i), so ist K reell abgeschlossen; und es folgt aus Lemma E, (II), daß die Menge A der über U algebraischen Elemente aus K ein reell abgeschlossener Zwischenkörper ist. Da U dedekindsch in K ist, ist ein Unterkörper von A und also auch A selbst dicht in K. Da $K = \tilde{K}$ wegen (i) gilt, folgt nun $K = \tilde{A}$ aus Zusatz 2.6, (iv). Damit haben wir (ii) aus (i) hergeleitet. Gilt umgekehrt (ii), so ist A wegen Satz 2.5, (a) dicht in $K = \tilde{A}$, so daß U dedekindsch in K ist. Mit A ist wegen Satz 3.3 auch $\tilde{A} = K$ reell abgeschlossen; und aus $\tilde{A} = K$ folgt $K = \tilde{K}$ wegen Zusatz 2.6, (iv). Damit ist die Äquivalenz von (i) und (ii) dargetan.

Da (iv.b) wegen Satz 3.6 mit der Dedekind-Abgeschlossenheit von K äquivalent ist, sind auch die Bedingungen (i) und (iv) äquivalent.

Wir nehmen die Gültigkeit der äquivalenten Bedingungen (i), (ii) und (iv) an. Dann ist K wegen (i) dedekind-abgeschlossen. Weiter gibt es wegen (i) einen Zwischenkörper Z, der über U algebraisch und in K dicht ist. Es folgt aus Lemma F, daß Z über U archimedisch ist; und K ist selbstverständlich auch archimedisch über Z. Also ist K archimedisch über U. Sei weiter D ein dedekind-abgeschlossener Körper mit $U \subseteqq D \subseteqq K$. Da D reell abgeschlossen ist, ist auch $A \subseteqq D$, so daß A und also auch D wegen (i) in K dicht ist. Dann können wir aber auf den dedekind-abgeschlossenen Körper D Satz 3.6, (iii.b) anwenden, so daß $D = K$ aus der Dichte von D in K folgt. Damit haben wir (iii) aus (i) abgeleitet. – Gilt umgekehrt (iii), so ist die Menge A der über U algebraischen Elemente aus K wegen der reellen Abgeschlossenheit von K und Lemma E, (II) ein reell abgeschlossener Zwischenkörper. Da K archimedisch über U ist, ist $K = \tilde{K}$ auch archimedisch über A; und wir folgern $\tilde{A} \subseteqq K$ aus Folgerung 2.11. Anwendung von (iii.c) auf den dedekind-abgeschlossenen Zwischenkörper $\tilde{A}$ ergibt $K = \tilde{A}$. Wir haben (ii) aus (iii) hergeleitet und die Äquivalenz von (i)–(iv) bewiesen.

Folgerung 3.10. *Jeder angeordnete Körper besitzt einen und im wesentlichen nur einen dedekindschen Abschluß.*

Beweis. Ist K ein angeordneter Körper, so besitzt K wegen Satz C einen und im wesentlichen nur einen reellen Abschluß R; und es folgt aus Satz 3.9, (ii), daß $\tilde{R}$ ein dedekindscher Abschluß von K ist. – Ist D irgendein dedekindscher Abschluß von K, so folgt aus Satz 3.9, (ii), daß die Menge A der über K algebraischen Elemente aus D ein reell abgeschlossener Zwischenkörper mit $D = \tilde{A}$ ist. Da R und A beide reelle Abschlüsse von K sind, sind R und A im wesentlichen identisch [Satz C]. Also sind auch $\tilde{R}$ und $\tilde{A} = D$ im wesentlichen identisch.

Folgerung 3.11. *Ist D der dedekindsche Abschluß des angeordneten Körpers K, so ist der reelle Abschluß von $\tilde{K}$ in D enthalten.*

Beweis. Da D wegen Satz 3.9, (iii.b) archimedisch über K ist, ist die Menge $E = E_{D/K}$ aller Elemente $d \in D$ mit eigentlichem $\omega_{D/K}(d)$ ein K enthaltender Unterkörper von D, in dem K dicht ist [Lemma 2.9, (a)]. Da D wegen Satz 3.9, (i) dedekind-abgeschlossen ist, ist D wegen Satz 3.6, (i) reell abgeschlossen mit $D = \tilde{D}$. Anwendung von Lemma 2.9, (c) und Folgerung 2.11 zeigt nun, daß $E = \tilde{K}$ ist. Da D reell abgeschlossen ist, folgt aus Lemma E, (II), daß die Menge R aller über E algebraischen Elemente aus D ein reell abgeschlossener Zwischenkörper ist: R ist der reelle Abschluß von $E = \tilde{K}$.

Bemerkung 3.12. Daß der dedekindsche Abschluß D des angeordneten Körpers K im allgemeinen vom reellen Abschluß von $\tilde{K}$ verschieden ist, werden wir in Satz 5.4 zeigen.

Satz 3.13. *Die folgenden Eigenschaften des angeordneten Körpers K und seines Unterkörpers U sind äquivalent:*

(i) *K und U haben denselben dedekindschen Abschluß.*

(ii) *Es gibt eine die Ordnung von K fortsetzende, dedekind-abgeschlossene Erweiterung von K, in der U dedekindsch ist.*

(iii) *Der reelle Abschluß von K ist dedekindsch über U.*

(iv) *Es gibt eine Teilmenge M von K mit folgenden Eigenschaften:*

(a) *K ist algebraisch über $U(M)$ und archimedisch über U.*

(b) *Ist* $0 \neq u \in U$ *und* $t \in M$, *so existiert ein Element* a *in dem [im reellen Abschluß von* K *gebildeten] reellen Abschluß von* U *mit* $0 < (t+u-a)(a-t)$.

(v) *Der reelle Abschluß von* U *ist dicht in seinem [im reellen Abschluß von* K *gebildeten] Compositum mit* K.

(vi) *Es gibt eine die Ordnung von* K *fortsetzende, über* U *dedekindsche Erweiterung von* K.

(vii) *Ist* Z *ein Zwischenkörper mit* $U \subseteqq Z \subseteqq K$, *so haben* Z *und* K *den gleichen dedekindschen Abschluß.*

Beweis. Aussage (i) besagt, daß der dedekindsche Abschluß D von K gleichzeitig dedekindscher Abschluß von U ist. Aus Satz 3.9, (i) folgt dann, daß D dedekind-abgeschlossen und U dedekindsch in D ist: (ii) folgt aus (i).

Gilt (ii), so gibt es eine die Ordnung von K fortsetzende, dedekind-abgeschlossene Erweiterung E von K, in der U dedekindsch ist. Da E insbesondere reell abgeschlossen ist, folgt aus Lemma E, (II), daß die Menge R der über K algebraischen Elemente aus E ein reell abgeschlossener Zwischenkörper ist: R ist der reelle Abschluß von K. Aus $U \subseteqq K \subseteqq R \subseteqq E$ folgt, daß U dedekindsch in R ist. Denn es gibt ja einen Zwischenkörper Z, der über U algebraisch und in E dicht ist; und da die Elemente aus Z über U algebraisch sind, sind sie erst recht über K algebraisch, so daß $Z \subseteqq R$ und also Z in R dicht ist: (iii) folgt aus (ii).

Ist der reelle Abschluß R von K dedekindsch über U, so ist R archimedisch über U und es ist wegen Lemma E, (II) die Menge A der über U algebraischen Elemente aus R ein reell abgeschlossener Unterkörper von R. Natürlich ist dann A der reelle Abschluß von U. Da es einen über U algebraischen Zwischenkörper gibt, der in R dicht ist, ist auch A in R dicht. Ist also $t \in K$ und $0 < p \in U$, so gibt es a', a'' in A mit $t-p < a' < t < a'' < t+p$. Hieraus folgt:

Ist $t \in K$ und $0 \neq u \in U$, so gibt es $a \in A$ mit $0 < (t+u-a)(a-t)$.

Aus dieser Eigenschaft folgt aber (iv) [mit $M = K$].

Wir nehmen die Gültigkeit von (iv) an und bilden wie oben in dem reellen Abschluß R von K das Compositum C des reellen Abschlusses A von U mit K. Da K archimedisch über U und R wegen Lemma F archimedisch über K ist, ist R archimedisch über U, also auch über A. Wir können folglich Zusatz 3.5, (A) anwenden: Die Menge $E = E_{R/A}$ der $r \in R$ mit eigentlichem $\omega_{R/A}(r)$ ist ein reell abgeschlossener, A enthaltender Unterkörper von R, in dem A wegen Lemma 2.9, (a) dicht ist. Ist $t \in M$ und $0 < r \in R$, so folgt aus der Archimedizität von R über U die Existenz eines Elements $u \in U$ mit $0 < u < r$; und wir folgern aus Bedingung (iv.b) die Existenz von Elementen a', a'' in A mit

$$t-u < a' < t < a'' < t+u.$$

Also ist $\omega_{R/A}(t)$ ein eigentlicher dedekindscher Schnitt in A, so daß $t \in E$ gilt. Damit haben wir $U(M) \subseteqq A(M) \subseteqq E$ gezeigt. Da E reell abgeschlossen ist, enthält E alle über E, und besonders die über $U(M)$ algebraischen Elemente, so daß $K \subseteqq E$ aus (iv.a) folgt. Also ist sogar $A \subseteqq C \subseteqq E$; und da A in E dicht ist, ist A auch in C dicht: Wir haben (v) aus (iv) hergeleitet.

Es ist klar, daß (vi) aus (v) folgt.

Sei E gemäß (vi) eine die Ordnung von K fortsetzende Erweiterung von K, in der U dedekindsch ist. Es gibt also einen Zwischenkörper Z, der über U algebraisch und in E dicht ist. Insbesondere ist E über U archimedisch [Lemma F]. Wegen Folgerung 3.10 gibt es einen dedekindschen Abschluß D von E. Wegen Satz 3.9, (iii. b) ist D über E archimedisch. Also gilt:

D ist über U archimedisch; und D ist dedekind-abgeschlossen.

Also ist D reell abgeschlossen [Satz 3.6]. Anwendung von Lemma E, (II) ergibt, daß die Menge A der über U algebraischen Elemente aus D ein reell abgeschlossener Unterkörper ist. Natürlich ist $Z \subseteqq A$. Da D über U archimedisch und Z in E dicht ist, ergibt Folgerung 2.13, daß A dicht ist in dem [in D gebildeten] Compositum C von A und E. Da D dedekind-abgeschlossen ist, ist $D = \tilde{D}$; und da D über U archimedisch ist, folgt $A \subseteqq C \subseteqq \tilde{A}$ aus Folgerung 2.7 und $\tilde{A} \subseteqq D$ aus Folgerung 2.11. Aus Satz 3.3 und der reellen Abgeschlossenheit von A ergibt sich die Dedekind-Abgeschlossenheit von $\tilde{A}$. Aus $E \subseteqq C \subseteqq \tilde{A} \subseteqq D$ ergibt sich wegen Satz 3.9, (iii. c), daß der Dedekind-Abschluß D von E nicht von $\tilde{A}$ verschieden sein kann. Also ist $D = \tilde{A}$ dedekindsch über U, so daß D gleichzeitig der Dedekind-Abschluß von U ist: (i) folgt aus (vi) und (i)–(vi) sind äquivalent.

Gelten die äquivalenten Bedingungen (i)–(vi), so ist wegen (iii) der reelle Abschluß R von K dedekindsch über U. Wegen Lemma E, (II) ist die Menge A aller über U algebraischen Elemente aus R ein reell abgeschlossener Zwischenkörper. Es gibt einen über U algebraischen und in R dichten Zwischenkörper. Da dieser in A enthalten ist, ist A dicht in R. Ist nun Z ein Zwischenkörper mit $U \subseteqq Z \subseteqq K$, so ist das Compositum B von Z und A über Z algebraisch und in R dicht: also ist Z dedekindsch in R, so daß Z der Bedingung (iii) genügt und also den gleichen dedekindschen Abschluß wie K hat: (vii) folgt aus (iii). – Umgekehrt ist (i) eine Abschwächung von (vii): Wir haben die Äquivalenz von (i)–(vii) dargetan.

Folgerung 3.14. (A) *Ist der angeordnete Körper K dedekindsch über seinem Unterkörper U, so haben K und U den gleichen dedekindschen Abschluß.*

(B) *Der reell abgeschlossene Körper R und sein Unterkörper U haben dann und nur dann den gleichen dedekindschen Abschluß, wenn R über U dedekindsch ist.*

(C) *Der angeordnete Körper K und sein reell abgeschlossener Unterkörper A haben dann und nur dann den gleichen dedekindschen Abschluß, wenn A in K dicht ist.*

Beweis. (A) wie auch das Hinreichen der ad (B) angegebenen Bedingung sind in Satz 3.13, (vi) enthalten. – Haben der reell abgeschlossene Körper R und sein Unterkörper U den gleichen dedekindschen Abschluß, so folgt aus Satz 3.13, (iii), daß R dedekindsch über U ist. – (C) schließlich folgt unmittelbar aus Satz 3.13, (v).

Folgerung 3.15. *Ist $\aleph$ die Mächtigkeit des angeordneten Körpers K, so ist die Mächtigkeit des dedekindschen Abschlusses von K höchstens $2^{\aleph}$.*

Beweis. Aus Satz 3.9, (ii) folgt, daß man den dedekindschen Abschluß von K folgendermaßen erhält: Sei R der wegen Satz C existierende reelle Abschluß von K; dann ist $\tilde{R}$ der dedekindsche Abschluß von K. Nun haben K und R die gleiche Mächtigkeit $\aleph$; und die Mächtigkeit von $\tilde{R}$ ist wegen Satz 2.5, (d) nicht größer als die Mächtigkeit $2^{\aleph}$ der Menge aller Teilmengen von R.

Bemerkung 3.16. Der Körper $\boldsymbol{R}$ aller reellen Zahlen ist der dedekindsche Abschluß des Körpers $\boldsymbol{Q}$ aller rationalen Zahlen. Dies zeigt, daß die in Folgerung 3.15 angegebene obere Schranke wirklich erreicht werden kann. Eine Familie weiterer Beispiele für dieses Phänomen werden wir weiter unten angeben; vgl. Folgerung 5.3.

4. Starre Erweiterungen

Ist A ein Unterkörper des Körpers B, so verstehen wir wie üblich unter $\mathrm{Aut}_A B$ *die Gruppe aller die Elemente aus A fixierenden Automorphismen von B.* Ist B überdies ein angeordneter Körper, so sei $\mathrm{Aut}_A^\circ B$ *die Gruppe aller ordnungerhaltenden Automorphismen aus* $\mathrm{Aut}_A B$. Diese beiden Gruppen sind häufig identisch, vornehmlich wenn die Anordnung von B natürlich ist, etwa wenn die positiven Elemente aus B genau die von 0 verschiedenen Quadratsummen sind. *Starrheit* der Erweiterung B von A ist Trivialität von $\mathrm{Aut}_A B$ bzw. $\mathrm{Aut}_A^\circ B$.

Lemma 4.1. *Haben der angeordnete Körper K und sein Unterkörper U den gleichen dedekindschen Abschluß, ist der dedekind-abgeschlossene Körper A archimedisch über seinem Unterkörper B, so wird jeder ordnungerhaltende Isomorphismus von U auf B durch einen und nur einen ordnungerhaltenden Isomorphismus von K in A induziert.*

Dem Beweise dieses fundamentalen Lemmas schicken wir die Beweise einiger Hilfssätze voraus, die allerdings sämtlich als Spezialfälle in diesem Lemma enthalten sind.

(4.1.1) *Ist D der dedekindsche Abschluß seines Unterkörpers U, so ist* $\mathrm{Aut}_U D = 1$.

Beweis. Nach Voraussetzung ist die Menge A der über U algebraischen Elemente aus D ein reell abgeschlossener Zwischenkörper mit $D = \tilde{A}$ [Satz 3.9, (ii)]. Ist $\sigma \in \mathrm{Aut}_U D$, so induziert σ in A einen Automorphismus aus $\mathrm{Aut}_U A = 1$; vgl. Satz D. Da D wegen Satz 3.9, (i) dedekind-abgeschlossen und also wegen Satz 3.6, (i) reell abgeschlossen ist, ist σ ein ordnungerhaltender Automorphismus von D. Da aber A in $D = \tilde{A}$ wegen Satz 2.5, (a) dicht ist, und da σ alle Elemente aus A fixiert, ist $\sigma = 1$.

(4.1.2) *Ist D_i der dedekindsche Abschluß seines Unterkörpers U_i für $i = 1, 2$, so wird jeder ordnungerhaltende Isomorphismus vom U_1 auf U_2 durch einen und nur einen [ordnungerhaltenden] Isomorphismus von D_1 auf D_2 induziert.*

Beweis. Die Eindeutigkeitsaussage ist in (4.1.1) enthalten. – Nach Voraussetzung ist die Menge A_i der über U_i algebraischen Elemente aus D_i ein reell abgeschlossener Zwischenkörper mit $\tilde{A}_i = D_i$ [Satz 3.9, (ii)]. Ist σ ein ordnungerhaltender Isomorphismus von U_1 auf U_2, so folgt aus Satz D die Existenz von einem [und nur einem] σ in U_1 induzierenden Isomorphismus λ von A_1

auf A_2. Aus der reellen Abgeschlossenheit von A_i folgt, daß auch λ die Ordnung erhält; und hieraus ergibt sich natürlich, daß λ von einem [und nur einem] Isomorphismus ω von $\tilde{A}_1 = D_1$ auf $\tilde{A}_2 = D_2$ induziert wird [Satz 2.5, (d)]. Natürlich wird σ von ω in U_1 induziert.

Beweis von Lemma 4.1. Sei D der wegen Folgerung 3.10 existierende [und im wesentlichen eindeutig bestimmte] dedekindsche Abschluß von K, so daß D nach Voraussetzung auch der dedekindsche Abschluß von U ist.

Da A dedekind-abgeschlossen ist, ist A auch reell abgeschlossen [Satz 3.6, (i)]. Aus Lemma E, (II) folgt also, daß die Menge R der über B algebraischen Elemente aus A ein reell abgeschlossener Zwischenkörper ist. Da A über B archimedisch ist, ist A auch über R archimedisch. Aus der dedekindschen Abgeschlossenheit von A folgt $A = \tilde{A}$ [Satz 3.6, (i)]. Anwendung von Folgerung 2.11 ergibt $\tilde{R} \subseteqq A$. Da R der reelle Abschluß von B ist, ist $\tilde{R}$ wegen Satz 3.9, (ii) der dedekindsche Abschluß von B.

Ist σ ein ordnungerhaltender Isomorphismus von U auf B, so folgt aus (4.1.2) die Existenz von einem und nur einem Isomorphismus λ von D auf $\tilde{R}$, der σ in U induziert; und es ist klar, daß σ in K einen ordnungerhaltenden Isomorphismus ω von K in $\tilde{R} \subseteqq A$ induziert. Damit ist die Existenzaussage des Lemma 4.1 voll bewiesen.

Sei nun ω' ein weiterer ordnungerhaltender Isomorphismus von K in A, der ebenfalls σ in U induziert. Aus der bereits bewiesenen Existenzaussage des Lemma 4.1 ergibt sich dann die Existenz eines ω' in K induzierenden [ordnungerhaltenden] Isomorphismus λ' von D in A. Da D der dedekindsche Abschluß von K ist, ist $D^{\lambda'}$ ein dedekindscher Abschluß von $K^{\omega'}$ in A. Aus $B = U^\sigma = U^{\lambda'} \subseteqq D^{\lambda'} \subseteqq A$ folgern wir zunächst, daß der reelle Abschluß R von B in A auch in $D^{\lambda'}$ enthalten ist; und aus der Archimedizität von A über B ergibt sich dann $\tilde{R} = D^{\lambda'}$ [Satz 3.9, (ii)]. Also sind λ und λ' Isomorphismen von D auf $\tilde{R}$, die σ in U induzieren. Da aber D und $\tilde{R}$ dedekindsche Abschlüsse von U sind, folgt $\lambda = \lambda'$ aus (4.1.2); und hieraus folgt natürlich $\omega = \omega'$, womit auch die Eindeutigkeitsaussage unseres Lemma 4.1 bewiesen ist.

Satz 4.2. *Die folgenden Eigenschaften des Unterkörpers U des angeordneten Körpers K sind äquivalent:*

(i) *K ist der dedekindsche Abschluß von U.*

(ii) (a) *K ist dedekind-abgeschlossen und archimedisch über U.*
(b) *Ist E eine die Ordnung von U fortsetzende, dedekind-abgeschlossene, über U archimedische Erweiterung von U, so gibt es einen und nur einen die Elemente von U fixierenden [ordnungerhaltenden] Isomorphismus von K in E.*

(iii) (a) *U ist dedekindsch in K.*
(b) *Ist E eine die Ordnung von U fortsetzende, über U dedekindsche Erweiterung von U, so gibt es einen die Elemente von U fixierenden, ordnungerhaltenden Isomorphismus von E in K.*

Beweis. Ist K der dedekindsche Abschluß von U, so ist K wegen Satz 3.9 dedekind-abgeschlossen und archimedisch über U; und K und U haben den gleichen dedekindschen Abschluß [nämlich K]. Ist weiter E eine die Ordnung von U fortsetzende, dedekind-abgeschlossene, über U archimedische Erweiterung von U, so können wir Lemma 4.1 anwenden: Der 1-Automorphismus von U wird durch einen und nur einen ordnungerhaltenden Isomorphismus von K in E induziert, so daß (ii) aus (i) folgt.

Gilt (ii), so gilt sicher Bedingung (iii.a und b) des Satzes 3.9. Ist weiter Z ein dedekind-abgeschlossener Zwischenkörper: $U \subseteqq Z \subseteqq K$, so ist Z mit K über U archimedisch; und Anwendung von (ii.b) zeigt die Existenz eines die Elemente aus U fixierenden [ordnungerhaltenden] Isomorphismus σ von K in $Z \subseteqq K$. Eine zweite Anwendung von (ii.b) zeigt, daß 1 der einzige, die Elemente von U fixierende [ordnungerhaltende] Isomorphismus von K in K ist. Also ist $\sigma = 1$, woraus

$$K = K^{\sigma} \subseteqq Z \subseteqq K \quad \text{und} \quad K = Z$$

folgt. Damit haben wir auch Bedingung (iii.c) des Satzes 3.9 aus (ii) abgeleitet. Also ist K der dedekindsche Abschluß von U: Die Bedingungen (i) und (ii) sind äquivalent.

Ist wieder K der dedekindsche Abschluß von U, so folgern wir aus Satz 3.9, (iv.a), daß U dedekindsch in K ist. Ist weiter E eine die Ordnung von U fortsetzende, über U dedekindsche Erweiterung von U, so bilden wir den dedekindschen Abschluß D von E, der wegen Folgerung 3.10 existiert. Aus Folgerung 3.14, (A) ergibt sich, daß U und E den gleichen dedekindschen Abschluß D haben. Da der dedekindsche Abschluß von U wegen Folgerung 3.10 im wesentlichen eindeutig bestimmt ist, gibt es einen die Elemente von U fixierenden, ordnungerhaltenden Isomorphismus von D auf K; und dieser induziert einen die Elemente von U fixierenden, ordnungerhaltenden Isomorphismus von E in K, so daß (iii) aus (i) folgt.

Gilt (iii), so gilt auch Bedingung (iv.a) des Satzes 3.9. Ist E eine die Ordnung von K fortsetzende, über K dedekindsche Erweiterung von K, so ergibt sich aus Folgerung 3.14, (A), daß K und E denselben dedekindschen Abschluß haben; und aus (iii.a) folgt, daß auch U und K denselben dedekindschen Abschluß haben. Insbesondere ist U dedekindsch in dem dedekindschen Abschluß D von E; siehe Satz 3.9, (i). Anwendung von (iii.b) zeigt die Existenz eines die Elemente von U fixierenden, ordnungerhaltenden Isomorphismus σ von D in K. Also ist

$$U \subseteqq E^{\sigma} \subseteqq D^{\sigma} \subseteqq K \subseteqq E \subseteqq D\,.$$

Da D der dedekindsche Abschluß von U ist, ist wegen Lemma 4.1 der 1-Automorphismus der einzige die Elemente aus U fixierende, ordnungerhaltende Isomorphismus von D in sich. Es folgt, daß $\sigma = 1$ und also

$$E = E^{\sigma} \subseteqq K \subseteqq E$$

ist; es wird $K = E$ und wir haben auch Bedingung (iv.b) des Satzes 3.9 aus unserer Bedingung (iii) hergeleitet, womit die Äquivalenz der Bedingungen (i)–(iii) dargetan ist.

Satz 4.3. *Die folgenden Eigenschaften des Unterkörpers U des angeordneten Körpers K sind äquivalent:*

(i) *K und U haben denselben dedekindschen Abschluß.*

(ii) *Ist Z ein Zwischenkörper mit $U \subseteqq Z \subseteqq K$, so ist 1 der einzige, die Elemente aus U fixierende, ordnungerhaltende Isomorphismus von Z in K.*

(iii) *Ist k ein über U transcendentes Element aus K, so ist $\mathrm{Aut}_U^\circ U(k) = 1$.*

(iv) *Es gibt eine Teilmenge M von K mit folgenden Eigenschaften:*

(a) *K ist algebraisch über $U(M)$ und archimedisch über U.*

(b) *$\mathrm{Aut}_U^\circ U(t) = 1$ für jedes $t \in M$.*

(v) $\begin{cases} \text{(a) } \mathrm{Aut}_U^\circ K = 1. \\ \text{(b) } \textit{Ist } Z \textit{ ein Zwischenkörper mit } U \subseteqq Z \subseteqq K, \textit{ so wird jeder Automorphismus aus } \mathrm{Aut}_U^\circ Z \textit{ durch einen Automorphismus aus } \mathrm{Aut}_U^\circ K \textit{ induziert.} \end{cases}$

Beweis. Gilt (i), so ist der [wegen Folgerung 3.10 existierende] dedekindsche Abschluß D von K gleichzeitig der dedekindsche Abschluß von U. Ist Z ein Zwischenkörper mit $U \subseteqq Z \subseteqq K$ und σ eine die Elemente aus U fixierender, ordnungerhaltender Isomorphismus von Z in K, so gilt auch $U \subseteqq Z^\sigma \subseteqq K$; und wir erschließen aus Satz 3.13, (vii), daß D der dedekindsche Abschluß von Z wie auch von Z^σ ist. Aus Satz 3.9 ergibt sich die Anwendbarkeit von Lemma 4.1. Also gibt es einen und nur einen ordnungerhaltenden Automorphismus λ von D, der σ in Z induziert. Da D der dedekindsche Abschluß von U ist, folgt $\mathrm{Aut}_U D = 1$ aus (4.1.1) [oder Lemma 4.1]. Also ist $\lambda = 1$, woraus $\sigma = 1$ folgt. Damit haben wir gezeigt, daß (ii) aus (i) folgt; und es ist klar, daß (iii) ein Spezialfall von (ii) ist.

Gilt (iii), so folgt aus Hilfssatz (1.2.+), daß K über U archimedisch ist, und daß also (iv) aus (iii) [mit $M = K$] folgt.

Wir nehmen weiter die Existenz einer den Bedingungen (iv.a + b) genügenden Teilmenge M von K an. Im reellen Abschluß R von K bilden wir den reellen Abschluß A von U; siehe Satz C und Lemma E, (II). Wir können o. B. d. A. annehmen, daß alle Elemente aus M über U transcendent sind, da ja die Teilmenge der über U transcendenten Elemente aus M immer noch den Bedingungen (iv.a + b) genügt. Wir bemerken, daß unsere Bedingung (iv.a) mit der Bedingung (iv.a) des Satz 3.13 übereinstimmt. Wir betrachten ein Element u mit $0 \neq u \in U$ und ein Element $t \in M$. Da t über U transcendent ist, gibt es genau einen die Elemente aus U fixierenden und t auf $t+u$ abbildenden Automorphismus von $U(t)$. Da $\sigma \neq 1$ und $\mathrm{Aut}_U^\circ U(t) = 1$ wegen (iv.b) ist, wird die Anordnung in $U(t)$ von σ nicht erhalten. Da $U(t)$ wegen der Transcendenz von t über U der Körper der rationalen Funktionen in t mit Coefficienten aus U ist, folgt hieraus die Existenz eines Polynoms $f(x)$ über U mit $f(t)\,f(t+u) < 0$. Wir wenden den Weierstraßschen Nullstellensatz [Satz A, (ii)] auf das Polynom f über $U \subseteqq A$ an: Es gibt ein $a \in A$ mit $f(a) = 0$ und $0 < (t+u-a)(a-t)$. Damit haben wir auch Bedingung (iv.b) des Satz 3.13 hergeleitet: K und U haben den gleichen dedekindschen Abschluß. Damit haben wir (i) aus (iv) hergeleitet und die Äquivalenz der Bedingungen (i)–(iv) bewiesen.

Schließlich sieht man mühelos ein, daß die Eigenschaften (ii) und (v) äquivalent sind.

Folgerung 4.4. *Dann und nur dann ist der reell abgeschlossene Körper R dedekindsch über seinem Unterkörper U, wenn* $\mathrm{Aut}_U Z = 1$ *für jeden reell abgeschlossene Zwischenkörper Z mit $U \subset Z \subseteqq R$ und Transcendenzgrad* 1 *von Z über U.*

Beweis. Ist R dedekindsch über U, so haben R und U denselben dedekindschen Abschluß [Folgerung 3.14, (A)]. Anwendung von Satz 4.3, (ii) zeigt dann die Notwendigkeit unserer Bedingung, da ja alle Automorphismen reell abgeschlossener Körper die [natürliche] Anordnung erhalten.

Wir nehmen umgekehrt die Gültigkeit unserer Bedingung an und betrachten ein über U transcendentes Element $t \in R$. Anwendung von Lemma E, (II) zeigt, daß die Menge Z aller über $U(t)$ algebraischen Elemente aus dem reell abgeschlossenen Körper R ein reell abgeschlossener Zwischenkörper: $U \subset U(t) \subsetneqq Z \subsetneqq R$ mit Transcendenzgrad 1 von Z über U ist. Ist $\sigma \in \mathrm{Aut}_U^\circ U(t)$, so folgt aus Satz D die Existenz eines eindeutig bestimmten Automorphismus λ von Z, der σ in $U(t)$ induziert.

Es folgt $\lambda \in \mathrm{Aut}_U Z = 1$ aus unserer Bedingung, so daß $\lambda = 1$ und also auch $\sigma = 1$ wird. Damit haben wir die Gültigkeit der Bedingung (iii) des Satzes 4.3 bewiesen: R und U haben denselben dedekindschen Abschluß. Anwendung von Folgerung 3.14, (B), zeigt, daß U dedekindsch in dem reell abgeschlossenen Körper R ist.

Folgerung 4.5. *Der reell abgeschlossene Unterkörper A des angeordneten Körpers K ist dann und nur dann dicht in K, wenn* $\mathrm{Aut}_A^\circ A(k) = 1$ *für jedes [nicht in A enthaltene] Element $k \in K$.*

Beweis. Aus Folgerung 3.14, (C) folgt, daß A dann und nur dann dicht in K ist, wenn K und A den gleichen dedekindschen Abschluß haben. Nicht in A enthaltene Elemente aus K sind transcendent über A. Anwendung der Äquivalenz der Bedingungen (i) und (iii) des Satzes 4.3 zeigt jetzt die Gültigkeit unserer Behauptung.

Bemerkung 4.6. Ist R ein reell abgeschlossener Unterkörper des reell abgeschlossenen Körpers F, ist weiter der Transcendenzgrad von F über R genau 1, so ist R wegen Folgerung 4.4 dann und nur dann dicht in F, wenn $\mathrm{Aut}_R F = 1$. Dr. Geyer [Heidelberg] hat mir ein Beispiel mitgeteilt, aus dem hervorgeht, daß die Transcendenzgradvoraussetzung unentbehrlich ist: In diesem Beispiel ist der Transcendenzgrad von F über R gleich 2; es ist $\mathrm{Aut}_R F = 1$; aber R ist nicht dicht in F.

Satz 4.7. *Der angeordnete Körper K ist dann und nur dann dedekind-abgeschlossen, wenn* $\mathrm{Aut}_K^\circ E \neq 1$ *für jede echte, die Ordnung von K fortsetzende, einfache Erweiterung E von K gilt.*

Beweis. Ist erstens K dedekind-abgeschlossen und E eine echte, die Ordnung von K fortsetzende, einfache Erweiterung von K, so ist K reell abgeschlossen

[Satz 3.6, (i)] und also E nicht algebraisch über K. Aus Satz 3.6, (iv. b) folgt, daß K nicht dicht in E ist; und Anwendung von Folgerung 4.5 ergibt $\mathrm{Aut}_K^\circ E \neq 1$.

Ist zweitens K nicht dedekind-abgeschlossen, so gibt es eine echte, die Ordnung von K fortsetzende Erweiterung E von K und einen Zwischenkörper Z, der über K algebraisch und in E dicht ist [Satz 3.6, (ii)]. Da E eine dedekindsche Erweiterung von K ist, ergibt Satz 3.13, daß K und E denselben dedekindschen Abschluß haben. Es gibt ein Element $e \in E$ mit $e \notin K$; und aus Satz 4.3, (ii) folgt $\mathrm{Aut}_K^\circ K(e) = 1$: unsere Bedingung ist notwendig und hinreichend für Dedekind-Abgeschlossenheit von K.

Bemerkung 4.8. Kennzeichnungen des dedekindschen Abschlusses eines angeordneten Körpers ergeben sich durch naheliegende Kombination der Sätze 4.3 und 4.7.

5. Einfache nicht archimedische Erweiterungen

Ist der angeordnete Körper L einfach über seinem Unterkörper K, ist weiter L nicht archimedisch über K, so ergibt sich aus Lemma E, (II), daß L eine reine transcendente Erweiterung von K ist. Wir werden deshalb im folgenden einen angeordneten Körper K und eine einfache transcendente Erweiterung $L = K(j)$ von K betrachten. Dann ist L der Körper der rationalen Funktionen in j mit Coefficienten aus K; und jedes von 0 verschiedene Element aus L kann auf die folgende „Normalform“ (N) gebracht werden:

$$\text{(N)} \qquad kj^n[1 + jf(j)]\,[1 + jg(j)]^{-1};$$

hierbei ist $0 \neq k \in K$ und n eine ganze Zahl [positiv, 0 oder negativ], während $f(j)$, $g(j)$ Polynome in j mit Coefficienten aus K sind. Man bemerke, daß zwar k und n, aber nicht f und g Invarianten des gegebenen Elements sind.

Wir werden das gemäß (N) normalisierte Element

$$kj^n[1 + jf(j)]\,[1 + jg(j)]^{-1}$$

dann und nur dann positiv nennen, wenn $0 < k$ in der vorgegebenen Anordnung von K gilt. Man erhält hierdurch eine [algebraische] Anordnung von L, die offenbar durch die beiden folgenden Eigenschaften eindeutig bestimmt ist:

(a) Die Anordnung von $L = K(j)$ setzt die vorgegebene Anordnung von K fort.

(b) $0 < j < k$ für jedes $0 < k \in K$.

Aus (b) folgt natürlich, daß L über K nicht archimedisch ist; und wegen (b) können wir L *als Erweiterung von K durch das [positive] infinitesimale Element j* bezeichnen, besonders da ja, wie oben vermerkt, die Erweiterung wesentlich durch die Eigenschaften (a) und (b) bestimmt ist.

Satz 5.1. *Ist K ein angeordneter Körper und $L = K(j)$ Erweiterung von K durch das [positive] infinitesimale Element j, so gilt:*

(a) *Ist i eine ganze Zahl mit $1 < i$, so hat die Gleichung $x^i = j$ keine Lösung in $\tilde{L}$.*

(b) *L ist nicht dicht in seinem reellen Abschluß.*

(c) *$\tilde{L}$ ist nicht reell abgeschlossen.*

Dem eigentlichen Beweis schicken wir den Beweis zweier Hilfssätze voraus.

(5.1.1) *Der alle Elemente aus K fixierende und j auf j^i mit $1<i$ abbildende Isomorphismus von $K(j)$ auf $K(j^i)$ erhält die Anordnung.*

Beweis. Das gemäß (N) normierte Element

$$kj^n[1+jf(j)]\,[1+jg(j)]^{-1}$$

wird durch unsern Isomorphismus auf das ebenfalls gemäß (N) normierte Element

$$kj^{in}[1+j\{j^{i-1}f(j^i)\}]\,[1+j\{j^{i-1}g(j^i)\}]^{-1}$$

abgebildet; und hieraus folgt sofort, daß unser Isomorphismus die Anordnung erhält.

(5.1.2) *Ist $1<i$, so liegt zwischen den Elementen kj mit $0<k\in K$ kein Element aus $K(j^i)$.*

Beweis. Sei $0<k\in K$ und $0\neq r\in K(j^i)$. Dieses Element r hat dann die Normalform (N) in $K(j^i)$:

$$r=sj^{in}[1+j^if(j^i)]\,[1+j^ig(j^i)]^{-1}.$$

Wir betrachten das Element $t=kj-r$.

Fall 1. $0<n$.

Dann wird $0\leqq in-2$, so daß

$$h(j)=j^{i-1}g(j^i)-sk^{-1}j^{in-2}[1+f(j^i)]$$

ein Polynom in j mit Coefficienten aus K wird. Die Normalform (N) in $K(j)$ von t wird dann:

$$t=kj[1+jh(j)]\,[1+j\{j^{i-1}g(j^i)\}]^{-1}.$$

Aus $0<k$ folgt also $0<t$; und dies ist gleichwertig mit

$$r<kj.$$

Fall 2. $n\leqq 0$.

Es wird $0<i-1$ und $0\leqq -in$, so daß

$$H(j)=j^{i-1}f(j^i)-ks^{-1}j^{-in}[1+j^ig(j^i)]$$

ein Polynom in j mit Coefficienten aus K wird. Die Normalform (N) von t in $K(j)$ wird dann

$$t=(-s)j^{in}[1+jH(j)]\,[1+j\{j^{i-1}g(j^i)\}]^{-1}.$$

Also ist dann und nur dann $0<t$, wenn $s<0$ ist; und es ist dann und nur dann $t<0$, wenn $0<s$ ist. Dies ist aber gleichwertig mit

$$r<kj \quad \text{dann und nur dann, wenn} \quad s<0;$$

$$kj<r \quad \text{dann und nur dann, wenn} \quad 0<s.$$

Zusammenfassend ergibt sich, daß ein Element aus $K(j^i)$ entweder links von allen kj mit $0<k$ liegt – dies ist der Fall, wenn entweder $0<n$ oder $n \leqq 0$ und $s<0$ ist – oder rechts von allen kj liegt – dies ist der Fall, wenn $n \leqq 0$ und $0<s$ ist. Die Null liegt natürlich links von allen kj mit $0<k \in K$, da ja $0<j$. Damit ist alles bewiesen.

Beweis des Satzes 5.1. Wäre die Aussage (a) falsch, so gäbe es eine ganze Zahl i mit $1<i$ und ein Element $w \in \tilde{L}$ mit $w^i = j$. Natürlich ist

$$K(j^i) \subsetneqq K(j) = L = K(w^i) \subsetneqq K(w) = L(w) \subseteqq \tilde{L}\,.$$

Ist σ der alle Elemente aus K fixierende und j auf j^i abbildende Isomorphismus von L auf $K(j^i)$, so folgt aus (5.1.1), daß σ ein ordnungerhaltender Isomorphismus ist. Dieser wird in L durch einen ebenfalls ordnungerhaltenden, die Elemente aus K fixierenden und w auf $w^i = j$ abbildenden Isomorphismus λ von $K(w)$ auf L induziert. Da L wegen Lemma 2.5, (a) in $\tilde{L}$ dicht ist, ist L auch in $K(w)$ dicht. Also ist $L^\lambda = K(j^i)$ in $K(w)^\lambda = K(w^i) = K(j)$ dicht; und dies widerspricht (5.1.2), womit (a) bewiesen ist.

Wäre L dicht in seinem reellen Abschluß R, so folgte $R \subseteqq \tilde{L}$ aus Folgerung 2.7. Wegen Satz A, (iii) ist aber für jede ganze Zahl i mit $1<i$ die Gleichung $x^i = j$ [wegen $0<j$] in R lösbar. Dies widerspricht $R \subseteqq \tilde{L}$ und (a), womit (b) bewiesen ist.

(c) folgt sofort aus (a) und Satz A, (iii).

Ist K ein angeordneter Körper, so sei $K((j))$ *der Körper aller formalen Potenzreihen in j mit Coefficienten aus K*. Die von 0 verschiedenen Elemente aus $K(j)$ können wir folgendermaßen normieren:

$$kj^n\left[1+\sum_{i=1}^{\infty} k_i j^i\right]; \qquad \text{(N*)}$$

hierbei sind k und die k_i Elemente aus K und n ist eine ganze Zahl [positiv, 0 oder negativ]. Daß dies ein Körper ist, entnimmt man etwa Fuchs [p. 137, Theorem 10]. Es sei darauf hingewiesen, daß dieses Mal nicht nur n und k, sondern auch alle andern k_i Invarianten des betrachteten Elements sind, und daß $K(j)$ ein Unterkörper von $K((j))$ ist. Wir werden das gemäß (N*) normalisierte Element

$$kj^n\left[1+\sum_{i=1}^{\infty} k_i j^i\right]$$

dann und nur dann positiv nennen, wenn $0<k$ [in K] gilt. Man erhält hierdurch eine [algebraische] Anordnung von $K((j))$, die offenbar die von uns am Anfang dieses Abschnitts eingeführte Anordnung von $K(j)$ – gemäß (a) und (b) – fortsetzt; siehe Fuchs [p. 137, Corollary 11].

Satz 5.2. *Ist $L = K(j)$ die Erweiterung des angeordneten Körpers K durch das [positive] infinitesimale Element j, so ist $\tilde{L}$ im wesentlichen mit dem formalen Potenzreihenkörper $K((j))$ identisch.*

Beweis. Sei zunächst $p \in K((j))$. Wir wollen p einen eigentlichen dedekindschen Schnitt in L zuordnen. Hierbei wollen wir zunächst annehmen, daß $p \notin L$ und also insbesondere p kein Polynom in j mit Coefficienten aus K ist; und wir werden später sehen, daß wir diese Annahme o. B. d. A. machen können. Dann hat also p wegen (N*) die Form:

$$p = kj^n \left[1 + \sum_{i=1}^{\infty} k_i j^i \right];$$

Hierbei ist n eine ganze Zahl [positiv, 0 oder negativ]; k und die k_i sind Elemente aus K; und es ist $k \neq 0$ und unendlich viele k_i sind von 0 verschieden.

Wir setzen

$$p_m = kj^n \left[1 + \sum_{i=1}^{m} k_i j^i \right] \quad \text{für} \quad 0 < m.$$

Dann sei

Dp die Menge aller $y \in L$ mit $p_m \leqq y$ für unendlich viele m,

Sp die Menge aller $y \in L$ mit $y \leqq p_m$ für unendlich viele m,

Angenommen nun, es gäbe $d \in Dp$ und $s \in Sp$ mit $d \leqq s$. Ist dann u eine beliebige positive ganze Zahl, so gibt es ganze Zahlen u', u'' mit $u < u'$, $u < u''$ und $p_{u'} \leqq d$ und $s \leqq p_{u''}$. Also wird

$$\begin{aligned} 0 \leqq s - d \leqq p_{u''} - p_{u'} &= kj^n \left[\sum_{i=u+1}^{u''} k_i j^i - \sum_{i=u+1}^{u'} k_i j^i \right] \\ &= kj^{n+u+1} f(j), \end{aligned}$$

wobei $f(j)$ ein Polynom in j mit Coefficienten aus K ist. Da diese Abschätzung für alle positiven ganzen Zahlen u gilt, ergibt die Definition der Anordnung in $L = K(j)$, daß $s = d$ ist; und hieraus folgt dann insbesondere, daß $p = s = d \in L$ ist. Dies haben wir aber ausgeschlossen; und nun sieht man mühelos ein, daß Sp, Dp ein dedekindscher Schnitt ist. Daß Sp, Dp ein eigentlicher dedekindscher Schnitt in L ist, folgt aus $p_m - kj^{n+m} \in Sp$ und $p_m + kj^{n+m} \in Dp$. Es ist nun leicht, diese Abbildung $p \to Sp, Dp$ auf ganz $K((j))$ auszudehnen, indem man jedem $p \in L$ irgendwie einen der beiden durch p in L bestimmten dedekindschen Schnitte zuordnet; und man überzeugt sich leicht davon, daß man auf diese Weise einen ordnungerhaltenden Isomorphismus λ von $K((j))$ in $\tilde{L}$ erhält, der [im wesentlichen] die Elemente aus L fixiert.

Ist

$$p = kj^n \left[1 + \sum_{i=1}^{\infty} k_i j^i \right]$$

ein normalisiertes, von 0 verschiedenes Element aus $K((j))$, so ist

$$p = kj^n \left[1 + \sum_{i=1}^{m} k_i j^i \right] + kj^{n+m+1} \sum_{i=m+1}^{\infty} k_i j^{i-m-1};$$

und wir können durch hinreichend große Wahl von m erreichen, daß das zweite Glied absolut genommen kleiner wird als eine beliebig vorgegebene positive Potenz von j. Hieraus folgt weiter:

Ist S, D ein eigentlicher dedekindscher Schnitt in $K((j))$, so gibt es zu jeder positiven ganzen Zahl n Polynome $s(j)$, $d(j)$ in j mit Coefficienten aus K derart, daß $s(j) \in S$, $d(j) \in D$ und $0 < d(j) - s(j) < j^n$.

Überlegt man sich weiter, daß zwei Polynome, deren Differenz absolut genommen j^n für positives ganzes n nicht überschreitet, in ihren ersten n Gliedern übereinstimmen müssen, so ist es nicht schwer einzusehen, daß jeder eigentliche dedekindsche Schnitt in $K((j))$ durch ein Element aus $K((j))$ definiert wird, so daß also insbesondere $\tilde{L} \subseteq K((j))$ wird, woraus die gewünschte wesentliche Identität von $\tilde{L}$ und $K((j))$ folgt.

Folgerung 5.3. *Ist $\aleph$ die Mächtigkeit des angeordneten Körpers K, ist $L = K(j)$ die Erweiterung von K durch das [positive] infinitesimale Element j, so ist $2^{\aleph}$ die Mächtigkeit von $\tilde{L}$ und von dem dedekindschen Abschluß von L.*

Beweis. Es ist klar, daß K und $K(j) = L$ die gleiche Mächtigkeit $\aleph$ haben; und hieraus folgt, daß der Körper $K((j))$ der formalen Potenzreihen die Mächtigkeit $2^{\aleph}$ hat. Aus Satz 5.2 folgt dann, daß auch $\tilde{L}$ die Mächtigkeit $2^{\aleph}$ hat.

Ist D der dedekindsche Abschluß von L, so ergibt sich aus Folgerung 3.15, daß D höchstens die Mächtigkeit $2^{\aleph}$ hat. Aus Folgerung 3.14, (a) und Satz 2.5, (a) ergibt sich, daß D im wesentlichen der dedekindsche Abschluß von $\tilde{L}$ ist, so daß die Mächtigkeit von D mindestens $2^{\aleph}$ und also gleich $2^{\aleph}$ ist.

Satz 5.4. *Ist $L = K(j)$ die Erweiterung des angeordneten Körpers K durch das [positive] infinitesimale Element j, so ist der reelle Abschluß von $\tilde{L}$ nicht der dedekindsche Abschluß von L.*

Beweis. Wegen Satz 5.2, ist $\tilde{L}$ im wesentlichen mit dem formalen Potenzreihenkörper $K((j))$ identisch. Sei R der reelle Abschluß von $\tilde{L} = K((j))$. Man folgert aus Satz A, (iii), daß jedes positive Element aus R für jede positive ganze Zahl die n-te Potenz eines und nur eines positiven Elements aus R ist. Dies gilt insbesondere für j, so daß wir für jede positive ganze Zahl n unter $j^{n^{-1}}$ die eindeutig bestimmte, in R enthaltene, positive Lösung der Gleichung $x^n = j$ verstehen können.

Sei D der dedekindsche Abschluß von L. Wegen Folgerung 3.11 ist $R \subseteq D$. Da D reell abgeschlossen ist, ist die Menge A aller über L algebraischen Elemente aus D ein reell abgeschlossener Zwischenkörper [Lemma E, (II)]; und natürlich ist $A \subseteq R$. Aus Satz 3.9, (ii) folgern wir $D = \tilde{A}$. Insbesondere ist A dicht in D [Satz 2.5, (a)], so daß auch R dicht in D ist und $D = \tilde{D} = \tilde{R}$ aus Folgerung 2.11 sich ergibt.

Sei $r(n)$ irgendeine mit n gegen unendlich strebende Folge rationaler Zahlen. Wir setzen

$$f(n) = \sum_{i=1}^{n} j^{r(i)}.$$

Da die $j^{n^{-1}}$ wohlbestimmte Elemente aus R sind, sind auch die $j^{r(i)}$ wohlbestimmte Elemente aus R, so daß auch jedes $f(n) \in R$ ist. Sei

T die Menge aller $t \in R$ mit $f(n) < t$ für alle n;

$S =$ Complement von T in R.

Man sieht sofort, daß $1 \in T$ und $0 \in S$ und daß also S, T ein dedekindscher Schnitt in R ist. Für jede positive ganze Zahl h gilt $h \leqq r(i)$ für fast alle i und also $j^{r(i)} \leqq j^h$ für fast alle i; und hieraus folgt leicht, daß S, T ein eigentlicher dedekindscher Schnitt in R ist, der also durch ein Element $d \in D = \tilde{R}$ definiert wird.

Wir wollen jetzt die Folge $r(n)$ rationaler Zahlen spezialisieren. Sei $r(n) = N(n) Z(n)^{-1}$ mit teilerfremden ganzen $N(n)$ und $Z(n)$. Wir fordern nun, daß die Menge der verschiedenen Primteiler aller $Z(n)$ unendlich ist. Wäre dann $d \in R$, so wäre d algebraisch über $K((j))$; und wir wollen mit g den Grad von d über $K((j))$ bezeichnen. Anwendung von Schilling [p. 56, Corollary 2] zeigt dann, daß die Primteiler der $Z(n)$ sämtlich Teiler von g sind – im Widerspruch zur Unendlichkeit der Menge der Primteiler der $Z(n)$. Also liegt d zwar in D, aber nicht in R; und damit haben wir $R \subset D$ dargetan.

Bemerkung 5.5. Man kann zeigen, daß der dedekindsche Abschluß D von L [in der Bezeichnung von Satz 5.4] im wesentlichen mit dem Körper aller formalen Potenzreihen der Form

$$\sum_{n=0}^{\infty} k_n j^{r(n)}$$

identisch ist; hierbei sind die $k_n \in K$ und die $r(n)$ bilden eine mit n gegen unendlich strebende Folge rationaler Zahlen.

Wir geben eine Anwendung der vorhergehenden Resultate.

Satz 5.6. *Der angeordnete Körper K ist dann und nur dann ein Unterkörper des Körpers $\boldsymbol{R}$ aller reellen Zahlen, wenn gilt: Jeder Unterkörper von K ist in seinem reellen Abschluß dicht.*

Beweis. Ist $K \subseteqq \boldsymbol{R}$ und $U \subseteqq K$, so ist auch $U \subseteqq \boldsymbol{R}$. Sei A der nach Satz C existierende reelle Abschluß von U. Dann ist A wegen Lemma E, (II) über U archimedisch; und es folgt aus Satz 1.2, (iv), daß U in A dicht ist: Unsere Bedingung ist notwendig.

Ist umgekehrt K nicht in $\boldsymbol{R}$ enthalten, so folgt aus Satz 1.2, (iii), daß K nicht archimedisch über seinem Primkörper $\boldsymbol{Q}$ ist. Also gibt es ein Element $j \in K$ mit $0 < j < r$ für jede positive rationale Zahl $r \in \boldsymbol{Q}$. Es ist $U = \boldsymbol{Q}(j) \subseteqq K$; und Anwendung von Satz 5.1, (b) zeigt, daß U in seinem reellen Abschluß nicht dicht ist: Unsere Bedingung ist hinreichend.

Zusatz bei der Korrektur [28. Mai 1970]: Vor kurzem wurde ich auf die folgenden beiden Arbeiten aufmerksam gemacht, die mit unseren Untersuchungen verwandt sind:

Kurt Hauschild: Cauchyfolgen höheren Typus in angeordneten Körpern. Zeitschrift für math. Logik und Grundlagen der Math. **13**, 55—66 (1967).

Dana Scott: On completing ordered fields. Applications of Model Theory. Proc. International Symposium Cal. Tech. (1967), 274—278. Holt, Rinehart & Winston; New York 1969.

Insbesondere ist unser Satz 3.3 als Spezialfall in Hauschilds Satz 14 [p. 62] enthalten; und Scott's Theorem 2 [p. 275] hängt mit unserer Folgerung 3.11 zusammen.

Auch finden einige der von uns behandelten bzw. benutzten Beispiele bei diesen beiden Autoren wie natürlich auch bei Artin-Schreier Erwähnung.

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A Class of Topological Semigroups

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Introduction

If x and y are elements of a commutative semigroup G, then we say that x *divides* y, and write $x|y$ if $x=y$, or if $x+a=y$ for some element a in G. A commutative semigroup G is said to be *naturally pseudometrizable* – abbreviated n.p. – if whenever m, n are natural numbers, and x is an element of G such that $mx|nx$, then $m \leqq n$.

As the terminology is meant to suggest, and as we shall demonstrate, there is a natural way of equipping any n.p. semigroup with a pseudometric. As our main result, we shall prove that such a semigroup is a topological semigroup when endowed with the topology determined by its natural pseudometric.

A mapping f of one semigroup G into another is called *divisibility preserving* if $x|y$ implies $f(x)|f(y)$; it is called *homogeneous* if $f(nx)=nf(x)$ for each element x in G, and each natural number n; it is called a *quasi-homomorphism* if it is both divisibility preserving and homogeneous. Clearly, any homomorphism is a quasi-homomorphism. The set of all quasi-homomorphisms of G into the additive semigroup R^+ of positive real numbers is called the *quasi-dual* of G, and is denoted by G'.

The latter part of the paper is devoted to an investigation of an n.p. archimedean semigroup in terms of its quasi-dual. We obtain an expression for the natural pseudometric of such a semigroup G in terms of the elements of G'. This leads easily to the result that the natural pseudometric of G is a metric if and only if G' separates points of G. In the concluding section, it is shown that the natural topology of G coincides with the weak topology determined by G'.

1. Archimedean Semigroups

All semigroups considered in this paper are assumed to be commutative, and the semigroup operation of such semigroups is written as addition. A semigroup is called *archimedean* if given any pair of its elements, each divides some multiple of the other.

If x and y are elements of an archimedean semigroup, then define $p(x, y)$ to be the infimum of all rationals m/n, where m, n are natural numbers such that $nx|my$.

The function p was introduced in [4], where some of its properties were developed; for the convenience of the reader, those of its properties which will be invoked in this paper are recorded below.

1.1. Lemma. *If x, y and z are elements of an archimedean semigroup, then*
(i) $p(x, y) \leqq p(x, z)\, p(z, y)$;
(ii) *if* $x \mid y$, *then* $p(x, z) \leqq p(y, z)$, *and* $p(z, x) \geqq p(z, y)$;
(iii) $p(kx, y) = kp(x, y)$ *for each natural number* k;
(iv) $p(x, ky) = p(x, y)/k$ *for each natural number* k; *and*
(v) $p(kx, ky) = p(x, y)$ *for each natural number* k.

Proof. Properties (i), (iii) and the first half of (ii) were proved in [4]. It is easy to prove the second half of (ii).

To prove (iv), let m and n be natural numbers such that $nx \mid my$. Then $k(nx) \mid k(my)$, i.e., $(kn)x \mid m(ky)$, and hence $p(x, ky) \leqq m/kn$. It follows that $p(x, ky) \leqq p(x, y)/k$. If m, n are natural numbers such that $nx \mid m(ky)$ then $p(x, y) \leqq mk/n$. We conclude that $p(x, y) \leqq kp(x, ky)$, and hence (iv) is proved.

Property (v) is an easy consequence of (iii) and (iv).

If x and y are elements of an archimedean semigroup G, then put $s(x, y) = \max\{p(x, y), p(y, x)\}$. Clearly, the function s is symmetric, i.e., $s(x, y) = s(y, x)$. If z is another element of G, then $p(x, y) \leqq p(x, z)\, p(z, y) \leqq s(x, z)\, s(z, y)$, and $p(y, x) \leqq p(y, z)\, p(z, x) \leqq s(y, z)\, s(z, x)$, so the function s satisfies the same fundamental inequality (Lemma 1.1 (i)) as does the function p.

Now if the archimedean semigroup G is naturally pseudometrizable, then by Lemma 2.2 of [4], $s(x, x) = 1$ for all x. From this, and the fundamental inequality, we conclude that $s(x, y) \geqq 1$ for all x, y in G. It follows that the function $d = \log s$ is a pseudometric on an n.p. archimedean semigroup. We call it the *natural pseudometric* of an n.p. archimedean semigroup.

Remark. The pseudometric topology on an n.p. archimedean semigroup G can't be indiscrete. For since G is n.p., Lemma 2.2 of [4] guarantees that $p(x, x) = 1$ for all x. If m, n are natural numbers, then by (v) of Lemma 1.1, $p(mx, nx) = m/n$. Thus, $s(mx, nx) = \max\{m/n, n/m\}$, and hence $d(mx, nx) = |\log m - \log n|$. It follows easily from this that the topology of G can never be indiscrete. There are n.p. archimedean semigroups whose pseudometric topology is discrete, e.g., the additive semigroup of positive integers. A large class of examples of n.p. archimedean semigroups whose pseudometric topology is not discrete is given in Section 4.

We wish to show that the pseudometric d can, in effect, be put on any n.p. semigroup. The results of the next two sections are crucial for such a program.

2. Archimedean Components

This brief section is devoted to a sketch of some results of Tamura and Kimura [7] which will be needed later. The exposition given here is derived from [3]. No proofs are given – the interested reader can find them in the reference just cited.

Throughout this section, G will denote a (commutative) semigroup. It is not assumed to be archimedean, or naturally pseudometrizable.

If x and y are elements of G, then write $x \eta y$ provided that x divides some multiple of y, and y divides some multiple of x. The relation η is a congruence on G, and each congruence class is an archimedean subsemigroup of G. Thus, G can be represented as the union of pairwise disjoint archimedean subsemigroups $\{G_\alpha\}$. This representation is called the decomposition of G into its archimedean components. (Loonstra [6] has defined and studied the relation η in the case where the semigroup in question is the cone of positive elements of an ordered (i.e., partially ordered) group. See also [4, pp. 44–45].)

Since η is a congruence on G, the set $G/\eta = \{G_\alpha\}$ of congruence classes can be made into a semigroup in the usual way: if $x \in G_\alpha$, and $y \in G_\beta$, then define $G_\alpha + G_\beta = G_\gamma$, where G_γ is the congruence class of $x+y$.

3. More on Archimedean Components

Let G_α be an archimedean component of the semigroup G. Since G_α is an archimedean semigroup, we may compute the function p_α with respect to divisibility $|_\alpha$ in G_α. But if x and y are elements of G_α, then it also makes sense to compute $p(x, y) = \inf\{m/n : nx \mid my\}$, where $|$ denotes the divisibility relation in G. Clearly, the relation $|_\alpha$ on G_α implies the relation $|$ on G_α and so $p \leqq p_\alpha$. Even though the divisibility relation in G, when restricted to G_α, needn't be the same as $|_\alpha$, it turns out that the functions p and p_α are equal on G_α. The following lemma, which is taken from [3], will be used to prove this.

3.1. Lemma. *If x, y are elements of G_α, and if $x \mid ky$ for some natural number k, then $x|_\alpha(k+1)y$.*

Proof. If $x = ky$, then $x + y = (k+1)y$, and so $x|_\alpha(k+1)y$ since y is a member of G_α. If $x + a = ky$ for some element a in G, then $x + (a+y) = (k+1)y$. The latter equality means $(a+y) \mid (k+1)y$. But clearly, $y \mid (a+y)$, and hence we conclude that $y \eta (a+y)$, i.e., $a+y$ is in the archimedean component G_α of y. The relation $x|_\alpha(k+1)y$ is now immediate.

3.2. Lemma. *If G_α is an archimedean component of the semigroup G, then $p_\alpha(x, y) = p(x, y)$ for each pair x, y in G_α.*

Proof. We have already observed that $p \leqq p_\alpha$.

Assume that $p(x, y) < p_\alpha(x, y)$ for some pair x, y in G_α. Put $\varepsilon = p_\alpha(x, y) - p(x, y)$, and choose natural numbers m, n for which $nx \mid my$, and $m/n < p(x, y) + \varepsilon/2$. Pick a natural number k so that $kn > 2/\varepsilon$. Since $nx \mid my$, we conclude that $knx \mid kmy$. By Lemma 3.1, $knx|_\alpha(km+1)y$. Therefore, $p_\alpha(x, y) \leqq (km+1)/kn = m/n + 1/kn < p(x, y) + \varepsilon = p_\alpha(x, y)$, which is a contradiction. This completes the proof of the lemma.

Now let G be an n.p. semigroup, and let $\{G_\alpha\}$ denote its archimedean components. Each G_α is an n.p. archimedean semigroup, so it can be equipped with the pseudometric $d_\alpha(x, y) = \log\max\{p_\alpha(x, y), p_\alpha(y, x)\}$. Define a pseudometric d^* on G by putting

$$d^*(x, y) = \begin{cases} 1 & \text{if } x \text{ and } y \text{ are in different archimedean components;} \\ \dfrac{d_\alpha(x, y)}{1 + d_\alpha(x, y)} & \text{if } x \text{ and } y \text{ are in the archimedean component } G_\alpha. \end{cases}$$

The pseudometric d^* is called the *natural pseudometric* on an n.p. semigroup G. Note that this is at variance with our earlier use of the term. But if G is an n.p. archimedean semigroup, then the pseudometrics d and d^* are equivalent, so no confusion should be possible.

We intend to show that an n.p. semigroup is a topological semigroup when it is equipped with the topology determined by its natural pseudometric. Before doing so, we shall illustrate the concepts considered up to this point with an important class of examples of n.p. semigroups. This is done in the next section.

4. Cones

A nonempty subset C of a real vector space E is said to be a *cone* if it is closed under addition and multiplication by positive scalars; we also assume that the origin of E is never a member of C.

For each cone C in a real vector space E, we can define an order $\leqq$ on E by putting $x \leqq y$ if $x = y$, or if $y - x \in C$. It is clear that this order, when restricted to C, is exactly the relation of divisibility on the semigroup C. Moreover, if m, n are natural numbers, and if x is an element of C such that $mx \mid nx$, then either $mx = nx$, in which case $m = n$, or $nx - mx = (n - m)x \in C$, in which case $m < n$. (If $n - m < 0$, then $(m - n)^{-1} > 0$, so $(m - n)^{-1}(n - m)x = -x \in C$; thus $x + (-x) = 0 \in C$, which is impossible.) Hence, a cone C in a real vector space is an n.p. semigroup.

We shall now determine, for a given cone C, its archimedean components $\{C_\alpha\}$, the functional p_α, and the natural pseudometric d_α on each component C_α.

If x and y are members of C, then we assert that $x \,\eta\, y$ if and only if $\alpha^{-1}x \leqq y \leqq \alpha x$ for some positive real number α. For if $x \,\eta\, y$, i.e., if there are natural numbers m, n such that $x \leqq my$ and $y \leqq nx$, then $x \leqq py$ and $y \leqq px$, where p is the maximum of m and n; thus, $\alpha^{-1}x \leqq y \leqq \alpha x$, with $\alpha = p$. Conversely, suppose α is a positive real number for which $\alpha^{-1}x \leqq y \leqq \alpha x$, i.e., for which $x \leqq \alpha y$ and $y \leqq \alpha x$. If n is a natural number such that $\alpha \leqq n$, then $x \leqq ny$, and $y \leqq nx$. This proves that $x \,\eta\, y$.

Easy computations show that each archimedean component C_α of C is a cone, that if x, y are elements of C_α, then $p_\alpha(x, y) = \inf\{\alpha > 0 : x \leqq \alpha y\}$, and that $s_\alpha(x, y) = \inf\{\alpha > 0 : \alpha^{-1}x \leqq y \leqq \alpha x\}$. (The latter equality is also a consequence of the above characterization of C_α, and Lemma 5.1.) Therefore, $d_\alpha(x, y) = \log \inf\{\alpha > 0 : \alpha^{-1}x \leqq y \leqq \alpha x\}$.

A comparison of what we have done so far in this section with work of Bauer and Bear [1] will show that the archimedean components $\{C_\alpha\}$ of C, are just the *parts* of C, and that d_α is the *part metric* on the part C_α. Thus, the natural pseudometric d^* on a cone is a metric. For a related pseudometric on a cone, see [2].

The simplest example of a cone is the set of positive real numbers with its usual addition and multiplication. This cone is archimedean, and we have $p(x, y) = x/y$; hence, $d(x, y) = |\log x - \log y|$. The metric d is equivalent to the usual metric on the set of positive real numbers.

5. Topological Semigroups

The next two lemmas will be used to prove our main result. The first of these gives another expression for the function s on an n.p. archimedean semigroup.

5.1. Lemma. *If G is an n.p. archimedean semigroup, and if x, y are elements of G, then $s(x, y) = \inf\{m/n : nx \mid my, ny \mid mx\}$, where m and n are natural numbers.*

Proof. Let α denote the term on the right side of the equality. If m, n are natural numbers such that $nx \mid my$ and $ny \mid mx$, then $p(x, y) \leqq m/n$, $p(y, x) \leqq m/n$, and so $s(x, y) = \max\{p(x, y), p(y, x)\} \leqq m/n$. Consequently, $s(x, y) \leqq \alpha$.

Let $\varepsilon > 0$ be given. Choose a pair of natural numbers m_1, n_1 such that $n_1 x \mid m_1 y$, and $m_1/n_1 < p(x, y) + \varepsilon$. Similarly, choose m_2, n_2 so that $n_2 y \mid m_2 x$, and $m_2/n_2 < p(y, x) + \varepsilon$. Now $n_2 n_1 x \mid n_2 m_1 y \mid \max\{n_2 m_1, n_1 m_2\} y$, and $n_1 n_2 y \mid n_1 m_2 x \mid \max\{n_2 m_1, n_1 m_2\} x$. Thus, $\alpha \leqq (\max\{n_2 m_1, n_1 m_2\})/n_1 n_2$. But $m_1 < n_1(p(x, y) + \varepsilon)$, and $m_2 < n_2(p(y, x) + \varepsilon)$; consequently, $n_2 m_1 < n_2 n_1(p(x, y) + \varepsilon)$, and $n_1 m_2 < n_1 n_2(p(y, x) + \varepsilon)$. It follows that $\max\{n_2 m_1, n_1 m_2\}/n_1 n_2 < \max\{p(x, y) + \varepsilon, p(y, x) + \varepsilon\} = s(x, y) + \varepsilon$.

We have shown that $\alpha < s(x, y) + \varepsilon$ for each $\varepsilon > 0$. Thus, $\alpha \leqq s(x, y)$, and hence $s(x, y) = \alpha$.

5.2. Lemma. *Let $G = \{G_\alpha\}$ be an n.p. semigroup. Assume that x_1, y_1 are members of the archimedean component G_α, and that x_2, y_2 are members of the component G_β. If G_γ is the component to which the pair $x_1 + x_2, y_1 + y_2$ belongs, then $d_\gamma(x_1 + x_2, y_1 + y_2) \leqq d_\alpha(x_1, y_1) + d_\beta(x_2, y_2)$.*

Proof. Let m_i, n_i, $i = 1, 2$, be pairs of natural numbers such that $n_i x_i \mid m_i y_i$, and $n_i y_i \mid m_i x_i$. We have already observed that $s(x, y) \geqq 1$ for each pair x, y of an n.p. archimedean semigroup. From this, and Lemma 5.1, we conclude that $n_i \leqq m_i$.

Since $n_1 x_1 \mid m_1 y_1$ and $n_2 x_2 \mid m_2 y_2$, it follows that $n_2 n_1 x_1 \mid n_2 m_1 y_1$ and $n_1 n_2 x_2 \mid n_1 m_2 y_2$. Now $n_i \leqq m_i$ implies $n_2 m_1 y_1 \mid m_2 m_1 y_1$, and $n_1 m_2 y_2 \mid m_1 m_2 y_2$. Therefore, $n_2 n_1 x_1 \mid m_2 m_1 y_1$, and $n_1 n_2 x_2 \mid m_1 m_2 y_2$. From this, we find that $n_1 n_2(x_1 + x_2) \mid m_1 m_2(y_1 + y_2)$. Similarly, $n_1 n_2(y_1 + y_2) \mid m_1 m_2(x_1 + x_2)$, and so $s_\gamma(x_1 + x_2, y_1 + y_2) \leqq m_1 m_2/n_1 n_2$. Therefore, $s_\gamma(x_1 + x_2, y_1 + y_2) \leqq s_\alpha(x_1, y_1) \times s_\beta(x_2, y_2)$. The desired inequality follows by taking logarithms.

Our main result is next.

5.3. Theorem. *An n.p. semigroup, equipped with the topology induced by the pseudometric d^*, is a topological semigroup.*

Proof. Let x_1 and x_2 be elements of the n.p. semigroup $G = \{G_\alpha\}$. Assume that $x_1 \in G_\alpha$, $x_2 \in G_\beta$, and that $x_1 + x_2 \in G_\gamma$. Let $W = \{z \in G : d^*(x_1 + x_2, z) < \varepsilon\}$, where, without loss of generality, we assume $0 < \varepsilon < 1$. We must find neighborhoods U of x_1, and V of x_2 such that $U + V \subseteq W$. Put $\delta = \varepsilon/(2+\varepsilon)$; it will be shown that appropriate candidates for U and V are $\{z : d^*(x_1, z) < \delta\}$ and $\{z : d^*(x_2, z) < \delta\}$ respectively.

Since $\delta < 1$, if $y_1 \in U$, it must be the case that $y_1 \in G_\alpha$ so $d^*(x_1, y_1) = d_\alpha(x_1, y_1)/(1 + d_\alpha(x_1, y_1)) < \varepsilon/(2+\varepsilon)$, and hence $d_\alpha(x_1, y_1) < \varepsilon/2$. Similarly, if $y_2 \in V$, then $y_2 \in G_\beta$, and $d_\beta(x_2, y_2) < \varepsilon/2$. By Lemma 5.2, $d_\gamma(x_1 + x_2, y_1 + y_2) \leqq d_\alpha(x_1, y_1) + d_\beta(x_2, y_2) < \varepsilon$. Therefore, $d^*(x_1 + x_2, y_1 + y_2) \leqq d_\alpha(x_1 + x_2, y_1 + y_2) < \varepsilon$. We have shown that $U + V \subseteq W$, and thus addition is continuous.

6. The Quasi-dual

Suppose G is an n.p. archimedean semigroup, and that f is a quasi-homomorphism of G into the additive semigroup P of nonnegative real numbers. Let x, y be elements of G, and let m, n be natural numbers for which $nx \mid my$. Then $nf(x) \leqq mf(y)$ since divisibility in P is the same as the usual ordering in P. If $f(y) = 0$, then $f(x) = 0$; if $f(y) \neq 0$, then $f(x) \leqq f(y)\, p(x, y)$. Thus, in either case, the latter inequality holds for all x, y in G. The inequality shows that if f vanishes at one point of G, then it vanishes identically, in which case it is of no interest to us. We therefore define the *quasi-dual* G' of an n.p. archimedean semigroup G to be the set of all quasi-homomorphisms of G into the additive semigroup R^+ of positive real numbers. Note that for each element z in G, the mappings p_z and p^z defined by $p_z(x) = p(x, z)$, and $p^z(x) = 1/p(z, x)$ are members of G'. This follows from (ii), (iii), and (iv) of Lemma 1.1 (recall that the functional p can never vanish on an n.p. archimedean semigroup).

The next result is the main one of the section.

6.1. Theorem. *Let d be the natural pseudometric of an n.p. archimedean semigroup G. Then*

$$\begin{aligned} d(x, y) &= \sup\{|\log f(x) - \log f(y)| : f \in G'\} \\ &= \sup\{|\log p(x, z) - \log p(y, z)| : z \in G\} \\ &= \sup\{|\log p(z, x) - \log p(z, y)| : z \in G\} \end{aligned}$$

for each pair x, y in G.

Proof. If $f \in G'$, then, as we proved at the beginning of this section, $f(x)/f(y) \leqq p(x, y)$. Interchange the roles of x and y in this inequality to obtain $f(y)/f(x) \leqq p(y, x)$. From these two inequalities, we conclude that $|\log f(x) - \log f(y)| \leqq \max\{\log p(x, y), \log p(y, x)\} = \log \max\{p(x, y), p(y, x)\} = d(x, y)$.

For $z \in G$, the mapping p_z is in G', so the previous inequality implies that $|\log p(x, z) - \log p(y, z)| \leqq d(x, y)$. We may suppose $p(y, x) \leqq p(x, y)$; then the last inequality becomes an equality when z is replaced by y. Hence, $d(x, y) = \sup\{|\log p(x, z) - \log p(y, z)| : z \in G\}$. The last number is no larger than

$\sup\{|\log f(x) - \log f(y)| : f \in G'\}$, and consequently the first two expressions for $d(x, y)$ in the statement of the theorem are valid.

To demonstrate the validity of the third expression for d, argue as in the previous paragraph, using the fact that the mapping p^z is in G' for each element z in G.

6.2. Corollary. *The natural pseudometric on an n.p. archimedean semigroup G is a metric if and only if G' separates points of G. (To say that G' separates points means that whenever x, y are distinct elements of G then there is a function f in G' for which $f(x) \neq f(y)$.)*

Proof. This is an easy consequence of the first expression for d in the previous result.

Remark. It is also a consequence of Theorem 6.1 that the pseudometric of an n.p. archimedean semigroup is a metric if and only if $x \neq y$ implies $p(x, z) \neq p(y, z)$ for some z; and if and only if $x \neq y$ implies $p(z, x) \neq p(z, y)$ for some z.

7. Weak Topologies

If z is an element of a pseudometric space $(X; \varrho)$, then define a mapping ϱ_z by $\varrho_z(x) = \varrho(x, z)$. It is easy to see that the pseudometric topology of X is the same as the weak topology generated by the family of functions $\{\varrho_z : z \in X\}$. The next result is a refinement of the previous assertion in the case where the pseudometric space in question is an n.p. archimedean semigroup.

The elements p_z and p^z of G', for z an element of an n.p. archimedean semigroup G, were defined in the previous section. Denote by Q the totality of functions $\{p_z, p^z : z \in G\}$.

7.1. Theorem. *If G is an n.p. archimedean semigroup, then the natural pseudometric topology T on G, the weak topology $w(G, Q)$ determined by the family Q, and the weak topology $w(G, G')$ determined by the quasi-dual G' of G are all the same.*

Proof. If f is a member of G', then as shown in the proof of Theorem 6.1, $|\log f(x) - \log f(y)| \leq d(x, y)$ for each pair x, y in G. This inequality implies that the mapping $x \to \log f(x)$ of G into the space of real numbers with its usual topology is continuous. Consequently, the composition $x \to \log f(x) \to e^{\log f(x)} = f(x)$ is continuous, i.e., each member f of G' is a continuous function on G. From this, and since Q is a subset of G', it follows that $w(G, Q)$ is coarser than $w(G, G')$, and that the latter topology is coarser than T.

We shall establish the theorem by proving that T is coarser than $w(G, Q)$. To do this, let y be a member of $B(z, \varepsilon) = \{x \in G : d(z, x) < \varepsilon\}$, where $\varepsilon > 0$. Choose a natural number k which satisfies $\log(1 + 1/k) < \varepsilon$. Both of the mappings p_y and p^y are $w(G, Q)$-continuous, $p_y(y) = p^y(y) = 1$, and so there is a $w(G, Q)$-neighborhood U of y such that $|p_y(x) - 1| < 1/k$, and $|p^y(x) - 1| < 1/(k+1)$ for

x in U. Consequently $p(x, y) < 1 + 1/k$; and $1 - 1/(k+1) < 1/p(y, x)$, i.e., $p(y, x) < 1 + 1/k$ for x in U. Therefore, $s(x, y) < 1 + 1/k$ if x is in U, and hence $d(x, y) < \varepsilon$ for such x. We have shown that $U \subseteqq B(z, \varepsilon)$, and so T is coarser than $w(G, Q)$. This completes the proof.

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Additive Semigroups of Positive Real Numbers

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Introduction

Hölder [4] showed that any cancellative archimedean naturally fully ordered semigroup is order isomorphic to a subsemigroup of the additive semigroup of nonnegative real numbers. A proof of this result using present day terminology and notation can be found in [3].

The object of this article is to present a variation of Hölder's theorem: we determine conditions on a commutative semigroup which guarantee that it be isomorphic to a subsemigroup of the additive semigroup R^+ of positive real numbers. (All semigroups considered in this paper are assumed to be commutative. By the theorem on p. 164 of [3], archimedean naturally fully ordered semigroups are automatically commutative.)

The technique we use in our approach to characterizing additive subsemigroups of positive real numbers is not radically different from the technique used by Hölder. His main tool, as is ours, is a function which generalizes the classical logarithmic function. The novelty in our approach is that we shall be concerned with order relations only indirectly (i.e., via divisibility), and, more importantly, we shall replace Hölder's cancellative condition by a condition which we call naturally pseudometrizable. (These conditions are not equivalent.)

We make no claim that our method of characterizing subsemigroups of R^+ is more efficient than Hölder's. However, our method does have an interesting byproduct: it suggests a way of endowing a certain type of commutative semigroup with a nontrivial topology which renders the semigroup a topological semigroup. In the concluding section of the paper, we shall briefly indicate how this is done, leaving the details for another paper.

1. Archimedean Semigroups

As stipulated in the introduction, all semigroups considered here are assumed to be commutative; unless otherwise stated, the semigroup operation of all such semigroups will be written as addition.

Given two elements x and y of a semigroup G, we say that x *divides* y, and write $x|y$ if $x=y$, or if there is an element a in G such that $x+a=y$. A semigroup is said to be *archimedean* if given any pair of its elements, each divides some multiple of the other[1].

[1] This definition is taken from [1]. The meaning of the word *archimedean* as it was used in the introduction (for arbitrary fully ordered semigroups), and its meaning, as just given, are, in general, different. This discrepancy needn't concern us here, because in the setting of our main theorem (Theorem 3.1), the concepts turn out to be the same. Hence, from now on, our use of the term *archimedean* corresponds to its use in [1].

Suppose now that G is an archimedean semigroup. Define a nonnegative real-valued function p on $G \times G$ by putting

$$p(x, y) = \inf\{m/n : nx \mid my\},$$

where m and n are natural numbers.

We shall list some examples of archimedean semigroups and their corresponding functions p.

Example 1. Any group G is archimedean. For if x, y are elements of G, then $x \mid y$ since $x + (y - x) = y$. In this case, the function p is identically zero.

Example 2. Let α be a nonnegative real number, and let β be a positive real number. Then the intervals (α, ∞) and $[\beta, \infty)$, under the usual addition of real numbers, are archimedean semigroups. In all cases, $p(x, y) = x/y$.

Example 3. The interval $(1, \infty)$, under the usual multiplication of real numbers, is an archimedean semigroup. In this case, $p(x, y) = \log_y x$. For details, consult [2].

The last example shows that the function p is a generalization of the classical logarithmic function.

The following result is basic for the entire paper.

1.1. Lemma. *If x, y and z are any elements of an archimedean semigroup, then $p(x, y) \leqq p(x, z)\, p(z, y)$.*

Proof. Let m_1, n_1 and m_2, n_2 be natural numbers such that $n_1 x \mid m_1 z$ and $n_2 z \mid m_2 y$. Then $n_2 n_1 x \mid n_2 m_1 z$ and $m_1 n_2 z \mid m_1 m_2 y$. Since divisibility is a transitive relation, $n_1 n_2 x \mid m_1 m_2 y$, and hence $p(x, y) \leqq (m_1/n_1)(m_2/n_2)$. The desired inequality is an immediate consequence of this.

Since $x \mid x$ for any element x of an archimedean semigroup G, it follows that $p(x, x) \leqq 1$. It so happens that if $p(z, z) < 1$ for one point z in G, then the function p is identically zero. To see this, assume that $p(z, z) < 1$. Apply the fundamental inequality of Lemma 1.1 to conclude that $p(z, z) \leqq p(z, z)\, p(z, z) = p(z, z)^2$. By repeated application of Lemma 1.1, we obtain $p(z, z) \leqq p(z, z)^n$ for any natural number n. From this, and the fact that $p(z, z) < 1$, it must be the case that $p(z, z) = 0$. Now if x and y are any elements of G, then, by two more applications of the fundamental inequality, we conclude that $p(x, y) \leqq p(x, z)\, p(z, y) \leqq p(x, z)\, p(z, z)\, p(z, y) = 0$.

It now becomes necessary to find a condition on the archimedean semigroup G which guarantees that the function p is nontrivial, which is to say, not identically zero. The next result will enable us to determine such a condition.

1.2. Lemma. *If x, y and z are any elements of an archimedean semigroup, then*

(i) *$p(kx, y) = kp(x, y)$ for all natural numbers k; and*

(ii) *if $x \mid y$ then $p(x, z) \leqq p(y, z)$.*

Proof. (i) Suppose m and n are natural numbers for which $nx \mid my$. Then $n(kx) \mid kmy$, and so $p(kx, y) \leqq km/n$, i.e., $p(kx, y) \leqq kp(x, y)$. If m and n are natural numbers such that $n(kx) \mid my$, then $(nk)x \mid my$, and thus $p(x, y) \leqq m/kn$, i.e., $kp(x, y) \leqq p(kx, y)$.

(ii) Suppose m and n are natural numbers such that $ny \mid mz$. Since $x \mid y$, we have $nx \mid ny \mid mz$, and so $p(x, z) \leqq m/n$. The desired inequality follows.

2. Naturally Pseudometrizable Semigroups

2.1. Definition. A (not necessarily archimedean) semigroup G will be called *naturally pseudometrizable* – abbreviated *n.p.* – if whenever $mx|nx$, where m, n are natural numbers, and x is any element of G, then $m \leqq n$. (We shall subsequently justify our use of the term *naturally pseudometrizable*.)

Note that an n.p. semigroup must be infinite. In fact, if x is any element of such a semigroup, then the elements nx, n a natural number, are distinct for distinct n. Thus, an n.p. semigroup can't contain idempotents (i.e., elements e such that $2e = e$); in particular, it can't contain an identity.

The function p can't be trivial on an n.p. archimedean semigroup; this is a consequence of the next result.

2.2. Lemma. *An archimedean semigroup G is naturally pseudometrizable if and only if $p(x, x) = 1$ for each element x in G.*

Proof. As observed before, we always have $p(x, x) \leqq 1$. If $nx|mx$, then by the n.p. property, $n \leqq m$, and hence $p(x, x) \geqq 1$.

Conversely, assume that $p(x, x) = 1$ for each element x in G. If $mx|nx$, then by Lemma 1.2, $mp(x, x) = p(mx, x) \leqq p(nx, x) = np(x, x)$, and hence $m \leqq n$.

Remarks. 1. The previous lemma will be adequate for our purposes, but we note here that a stronger form of it is valid. Namely, it is true that an archimedean semigroup G is n.p. if and only if $p(x, x) = 1$ for *some* x in G. To see this, assume that $p(x, x) = 1$ for an element x of G. If y is any element of G, then by Lemma 1.1, $p(x, x) \leqq p(x, y)\, p(y, x)$, and consequently $p(y, x) \neq 0$. By another application of Lemma 1.1, we get $p(y, x) \leqq p(y, y)\, p(y, x)$. It follows that $p(y, y) \geqq 1$, and hence $p(y, y) = 1$ for each element y in G.

2. The function p can never vanish on an n.p. archimedean semigroup. For if $p(x, y) = 0$, then $p(x, x) \leqq p(x, y)\, p(y, x) = 0$. Thus, $p(x, x) = 0$, which contradicts the previous lemma.

3. In an n.p. archimedean semigroup, divisibility is an order, i.e., it is a reflexive, transitive, and antisymmetric relation. We have already made use of the fact that divisibility is a preorder on any semigroup; this means that it is both reflexive $(x|x)$, and transitive ($x|y$, and $y|z$ imply $x|z$). To see that it is antisymmetric (i.e., $x|y$ and $y|x$ imply $x = y$) in an n.p. archimedean semigroup G, assume that x and y are distinct elements of G, each of which divides the other. Then $x + a = y$, and $y + b = x$ for some pair a, b in G. Hence, $x + c = x$, where $c = a + b$. Thus, $x + c + c = x + c = x$, and, in general, $x + nc = x$ for any natural number n. Therefore, $nc|x$ for each n, and so $p(c, x) \leqq 1/n$ for all natural numbers n; consequently, $p(c, x) = 0$. But, as pointed out in the previous remark, this can't happen in an n.p. archimedean semigroup.

2.3. Lemma. *If x, y and z are elements of an archimedean semigroup, then*

$$p(x + y, z) \leqq p(x, z) + p(y, z).$$

Proof. Let m_1, n_1 and m_2, n_2 be pairs of natural numbers such that $n_1 x \mid m_1 z$ and $n_2 y \mid m_2 z$. Then $n_2 n_1 x \mid n_2 m_1 z$, and $n_1 n_2 y \mid n_1 m_2 z$. It follows that $n_1 n_2(x+y) = n_1 n_2 x + n_1 n_2 y$ divides $n_2 m_1 z + n_1 m_2 z = (n_2 m_1 + n_1 m_2)z$, and hence $p(x+y, z) \leqq (n_2 m_1 + n_1 m_2)/n_1 n_2 = m_1/n_1 + m_2/n_2$. Consequently, $p(x+y, z) \leqq p(x, z) + p(y, z)$.

3. Valuation Semigroups

A semigroup is called a *valuation semigroup* if given any pair of its elements, x, y, either $x \mid y$ or $y \mid x$.

We can now state and prove our main result.

3.1. Theorem. *If z is any element of an n.p. archimedean valuation semigroup G, then the mapping $x \to p(x, z)$ is an isomorphism of G upon a subsemigroup of the additive semigroup of positive real numbers.*

Proof. By the previous lemma, we have, for any elements x, y and z of G, $p(x+y, z) \leqq p(x, z) + p(y, z)$.

Suppose $p(x+y, z) < p(x, z) + p(y, z)$; then we can find natural numbers m, n such that $p(x+y, z) < m/n < p(x, z) + p(y, z)$, and $n(x+y) \mid mz$.

Put $\varepsilon = p(x, z) + p(y, z) - m/n$, and choose pairs of natural numbers m_1, n_1 and m_2, n_2 such that $p(x, z) - \varepsilon/2 < m_1/n_1 < p(x, z)$, and $p(y, z) - \varepsilon/2 < m_2/n_2 < p(y, z)$. Hence, $m/n < m_1/n_1 + m_2/n_2$. Moreover, $n_1 x$ can't divide $m_1 z$, and $n_2 y$ can't divide $m_2 z$. Since G is a valuation semigroup, it must be the case that $m_1 z \mid n_1 x$ and $m_2 z \mid n_2 y$. Therefore, $n_2 m_1 z \mid n_2 n_1 x$, and $n_1 m_2 z \mid n_1 n_2 y$. It follows that $(n_2 m_1 + n_1 m_2) z \mid n_1 n_2 (x+y)$. We now conclude that $n(n_2 m_1 + n_1 m_2) z \mid n n_1 n_2 (x+y) \mid m n_1 n_2 z$. But G is an n.p. semigroup, so $n(n_2 m_1 + n_1 m_2) \leqq m n_1 n_2$, i.e., $m_1/n_1 + m_2/n_2 \leqq m/n$, which is a contradiction.

Thus, $p(x+y, z) = p(x, z) + p(y, z)$, so the mapping in question is a homomorphism of G into the additive semigroup of positive real numbers. To prove that this mapping is an isomorphism, let x and y be distinct elements of G; we may assume that $x \mid y$ so that $x + a = y$ for some element a in G. Then $p(y, z) = p(x+a, z) = p(x, z) + p(a, z) \neq p(x, z)$ since $p(a, z) > 0$. This completes the proof of the theorem.

Observe that the only isomorphisms of an n.p. archimedean valuation semigroup G into R^+ are, except for positive multiples, of the form $x \to p(x, z)$ for some element z in G. For a proof of this, assume that θ is merely a homomorphism of such a semigroup G into R^+. If x, z are elements of G, then let m, n be natural numbers for which $nx \mid mz$. Then $n\theta(x) \leqq m\theta(z)$, and so $\theta(x)/\theta(z) \leqq p(x, z)$. (The first inequality follows since G is a homomorphism, and since the usual order is the same as divisibility in R^+.) Suppose there are natural numbers m, n such that $\theta(x)/\theta(z) < m/n < p(x, z)$. Then nx can't divide mz, from which we conclude that $mz \mid nx$, and hence $\theta(x)/\theta(z) \geqq m/n$, a contradiction. We have shown that $\theta(x) = \theta(z)\, p(x, z)$. Thus, θ is a positive multiple of the isomorphism $x \to p(x, z)$, and consequently is itself an isomorphism.

4. Topological Semigroups

If G is an n.p. archimedean semigroup, then put $s(x, y) = \max\{p(x, y), p(y, x)\}$. It is easy to see that s satisfies the same fundamental inequality that p does, i.e., $s(x, y) \leqq s(x, z)\, s(z, y)$. Moreover, by Lemma 2.2, $s(x, x) = 1$ for each x, and thus $1 = s(x, x) \leqq s(x, y)\, s(y, x) = s(x, y)^2$. Consequently, the function $d = \log s$ is a pseudometric on G. We shall prove elsewhere [5] that G is a topological semigroup when equipped with the topology induced by the pseudometric d; in fact, using the known result that any semigroup can be split into disjoint archimedean subsemigroups, we shall prove that the natural topologies on the archimedean components of an n.p. semigroup G can be combined in such a way as to make G a topological semigroup.

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Adjoining a Unit to a Biregular Ring

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1. Introduction, Notations, Preliminaries

We will use Dauns' and Hofmann's representation of biregular rings by sheaves [2] to show that every biregular ring can be embedded, in a canonical way, as an ideal into a biregular ring with unit (Theorem 2.8). The method used is a generalization of the well-known one for embedding a Boolean ring into a Boolean ring with unit (e.g. see [4], Example 1.1). The author thanks to I. Vidav for helpful discussions and to R. A. Wiegand for his reading the manuscript.

First we introduce some notations. P will denote the set of prime natural numbers; it will be often understood as a discrete topological space and $\bar{P} = P \cup \{\infty\}$ will be its one-point compactification. Z will always mean the ring of integers and Z_n the ring of integers modulo n. Q will be the field of rational numbers. The word *ideal* will always mean a two-sided ideal.

Let $(\mathscr{A}, \pi, X)$ be a sheaf of rings on a space X with projection π. By a *section* of $\mathscr{A}$ we will always understand a continuous section. The support of a section s will be denoted by $|s|$. For any subset $Y \subset X$ let $\Gamma(\mathscr{A}, Y)$ be the ring of all sections of $\mathscr{A}$ over Y; let $\Gamma_c(\mathscr{A}, Y)$ be the ideal of $\Gamma(\mathscr{A}, Y)$ consisting of sections of $\mathscr{A}$ over Y with compact supports. Notations $\Gamma(\mathscr{A}, X)$ and $\Gamma_c(\mathscr{A}, X)$ will be shortened to $\Gamma(\mathscr{A}), \Gamma_c(\mathscr{A})$, respectively. The stalk of $\mathscr{A}$ over a point $x \in X$ will be denoted by $\mathscr{A}_x$. If each stalk of $\mathscr{A}$ has a unit and if the mapping $X \to \mathscr{A}$ which assigns to each $x \in X$ the unit of $\mathscr{A}_x$ belongs to $\Gamma(\mathscr{A})$, we will say that $\mathscr{A}$ is a *sheaf of rings with unit*. A *sheaf of division rings* is a sheaf of rings with unit whose stalks are division rings.

A ring R is *biregular* if each principal ideal of R is generated by a central idempotent. R is *strongly regular* if for each $r \in R$ there is an $x \in R$ such that $r^2 x = r$. Every strongly regular ring is biregular ([1], Theorem 3.2). Throughout this paper, R will be an arbitrary biregular ring if not explicitly stated otherwise.

Let R be a biregular ring. The set of all maximal (= primitive – see [5], IX.4.4) ideals of R will be denoted by $\mathscr{M}(R)$ or simply by $\mathscr{M}$ if there is no danger of confusion. For each ideal of R let $F(I) \subset \mathscr{M}$ be the set of all maximal ideals of R which contain I, and let $G(I) = \mathscr{M} - F(I)$. If I is a principal ideal (r), $r \in R$, we will write $F(r)$, $G(r)$ instead of $F((r))$, $G((r))$.

It will be understood throughout that $\mathscr{M}(R)$ has the *hull-kernel topology*. The open sets in $\mathscr{M}$ in this topology are exactly the sets $G(I)$ for all possible ideals I of R. The set $\mathscr{M}$ topologized in this way is called the *structure space* of R. It can be proved (see [5], IX.5.2) that $\mathscr{M}(R)$ is a Hausdorff, locally compact,

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totally disconnected space and that a subset $A \subset \mathcal{M}(R)$ is compact and open in $\mathcal{M}$ if and only if $A = G(r)$ for some $r \in R$.

If M is any maximal ideal of R, the quotient ring R/M is a biregular simple ring, thus a simple ring with unit. Dauns and Hofmann [2] showed that there is a sheaf of rings $\mathcal{R}(R)$ on $\mathcal{M}(R)$ *(the sheaf of simple quotient rings of R)* such that $\mathcal{R}(R)_M$, the stalk of $\mathcal{R}(R)$ over $M \in \mathcal{M}$, is equal to R/M for each $M \in \mathcal{M}$. We will write $\mathcal{R}$ instead of $\mathcal{R}(R)$ if no confusion is likely. The unit and the zero element of $\mathcal{R}_M$ will usually be denoted by $1_M, 0_M$, respectively.

For each $r \in R$ let $\hat{r}: \mathcal{M} \to \mathcal{R}$ be the mapping defined by $\hat{r}(M) = r + M \in R/M = \mathcal{R}_M$ $(M \in \mathcal{M})$. The topology in $\mathcal{R}$ is defined by requiring that all mappings $\hat{r}$ $(r \in R)$ are global sections of $\mathcal{R}$. Then we have:

Lemma 1.1 ([2], 2.16). *The mapping* $\lambda: R \to \Gamma(\mathcal{R}(R))$, *defined by* $\lambda(r) = \hat{r}$ $(r \in R)$, *is an isomorphism of* R *onto* $\Gamma_c(\mathcal{R}(R))$.

This is the representation theorem for biregular rings. We list now some properties of the sheaf $\mathcal{R}$.

Lemma 1.2 ([2], 2.5). $\mathcal{R}(R)$ *is a sheaf of simple rings with unit, i.e. the mapping* $M \to 1_M$ *is a global section of* $\mathcal{R}(R)$.

Lemma 1.3 ([2], 2.8). $\mathcal{R}(R)$ *is a Hausdorff space. Therefore, for each global section* s *of* $\mathcal{R}$, *the support* $|s|$ *is open and closed in* $\mathcal{M}(R)$.

Lemma 1.4. *If* I *is an ideal of* R, *then* $\lambda(I) = \{s \in \Gamma_c(\mathcal{R}): |s| \subset G(I)\} \cong \Gamma_c(\mathcal{R}, G(I))$, *where* λ *is the isomorphism of* 1.1.

Proof. It follows from the definition of λ that for any $r \in I$ the section $\hat{r}$ vanishes on $F(I)$, thus $|\hat{r}| \subset G(I)$. Conversely, if $s \in \Gamma_c(\mathcal{R})$ is such that $s(M) = 0_M$ for every $M \in F(I)$, then $s = \hat{r}$ for some $r \in R$ which belongs to every $M \in F(I)$, i.e. $r \in I$ ([1], Corollary 3 in Sec. 2).

The representation theorem 1.1 can be inverted:

Lemma 1.5 ([2], Theorem I). *Let* X *be a locally compact and totally disconnected Hausdorff space and let* $\mathcal{A}$ *be a sheaf of simple rings with unit on* X *such that no stalk* $\mathcal{A}_x$ *is the zero ring. Then*

(1) *the ring* $R = \Gamma_c(\mathcal{A})$ *is biregular;*

(2) *there is a natural homeomorphism* $f: \mathcal{M}(R) \to X$ *and a natural isomorphism* $\tilde{f}: \mathcal{R}(R) \to \mathcal{A}$ *compatible with* f;

(3) $\tilde{f} \circ \hat{r} = r \circ f$ *for each* $r \in R$.

The next lemma is an immediate consequence of Theorem 3.2 in [1].

Lemma 1.6. *If* R *is a strongly regular ring, then the stalks of* $\mathcal{R}(R)$ *are division rings.*

Conversely we have:

Lemma 1.7. *Let* X *be a locally compact and totally disconnected Hausdorff space and let* $\mathcal{A}$ *be a sheaf of division rings on* X. *Then the ring* $R = \Gamma_c(\mathcal{A})$ *is strongly regular.*

Proof. Take any $r \in R$. Define a mapping $s: X \to \mathscr{A}$ by: $s(x) = [r(x)]^{-1}$ for $x \in |r|$, $s(x) = 0_x$ for $x \in X - |r|$. It is not difficult to prove that s is a global section of $\mathscr{A}$. Clearly $s \in R$ and $r^2 s = r$.

We will often identify a biregular ring R with $\Gamma_c(\mathscr{R}(R))$ by the isomorphism λ.

Lemma 1.8. *Let I be an ideal of a biregular ring R. The following statements are equivalent:*

(1) *The set $G(I)$ is dense in $\mathscr{M}(R)$.*

(2) *For each $r \in R$ $(r \neq 0)$, $Ir \neq 0 \neq rI$.*

(3) *For each ideal J of R $(J \neq 0)$, $IJ \neq 0 \neq JI$.*

(4) *For each ideal J of R $(J \neq 0)$, $I \cap J \neq 0$.*

Proof. The ring R will be identified with $\Gamma_c(\mathscr{R})$. We will prove the following chain of implications: (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4)$\Rightarrow$(1).

(1)$\Rightarrow$(2). Take any $r \in R, r \neq 0$. If $G(I)$ is dense in $\mathscr{M}$, then $|r| \cap G(I)$ is a nonempty open set. Thus there is a nonempty compact open set $U \subset |r| \cap G(I)$. Let $e: \mathscr{M} \to \mathscr{R}$ be the section which is equal to 1 on U and to 0 outside U. Then $re \neq 0$. By 1.4, $e \in I$.

Obviously (2)$\Rightarrow$(3)$\Rightarrow$(4).

(4)$\Rightarrow$(1). Suppose that $G(I)$ is not dense in $\mathscr{M}$. Then there exists a nonempty open set $U \subset \mathscr{M} - G(I)$. If J is the ideal $\{r \in R: |r| \subset U\}$ of R, then $J \neq 0$ and, by 1.4, $I \cap J = 0$. The lemma is proved.

If the statements listed in 1.8 are true, we will say that I is a *large ideal* in R.

Lemma 1.9. *Let I be an ideal of a biregular ring R. There is one and only one ideal J of R such that the restriction $q|I$ of the quotient map $q: R \to R/J$ is a monomorphism and that the ideal $q(I) \cong I$ is large in the biregular ring R/J.*

Proof. Let J be the intersection of all $M \in G(I)$. Then $G(J) = \mathscr{M} - \overline{G(I)}$. It follows from 1.4 that $I \cap J = 0$ and that J contains every ideal J' which intersects I trivially. If $q: R \to R/J$ is the quotient homomorphism, then $q|I$ is a monomorphism. Let K be any ideal of R/J such that $q(I) \cap K = 0$. Then

$$I \cap q^{-1}(K) \subset q^{-1}(q(I)) \cap q^{-1}(K) \subset J .$$

It follows that $I \cap q^{-1}(K) = 0$, thus $q^{-1}(K) \subset J$ and $K = 0$. We see that $q(I)$ is large in R/J.

Take now any ideal J' such that the restriction $q'|I$ of the quotient map $q': R \to R/J'$ is a monomorphism and that $q'(I)$ is large in R/J'. From the first assumption it follows that $I \cap J' = 0$, thus $J' \subset J$. Denote $K = q'(I) \cap q'(J)$. Since

$$q'^{-1}(K) = (I + J') \cap (J + J') = (I + J') \cap J = J' \cap J = J' ,$$

K is trivial. This implies that $q'(J)$ is also trivial, for $q'(I)$ is large in R/J'. Hence $J \subset J'$, and therefore $J = J'$.

2. Embedding a Biregular Ring into a Biregular Ring with Unit

Throughout this section let R be an arbitrary biregular ring (without unit), $\mathscr{M} = \mathscr{M}(R)$ its structure space, and $\mathscr{R} = \mathscr{R}(R)$ the sheaf of simple quotient rings of R. The ring R will be identified with $\Gamma_c(\mathscr{R})$. We wish to embed R into a biregular ring $\bar{R}$ with unit. The ring $\Gamma(\mathscr{R})$ has a unit, by 1.2, and we already have a natural embedding of R into this ring (which is naturally isomorphic to the ring of endomorphisms of R as a right R-module [2]). So our first idea might be to take $\Gamma(\mathscr{R})$ as a candidate for $\bar{R}$. However, this ring is not always biregular, as shown in [3], 4.22, Example 2.

The ring $\bar{R}$ is obtained as follows. We construct a Hausdorff, totally disconnected compactification $\bar{\mathscr{M}}$ of the structure space $\mathscr{M}$. Then we "extend" the sheaf $\mathscr{R}$ to a sheaf $\bar{\mathscr{R}}$ on $\bar{\mathscr{M}}$ by adding appropriate simple rings (in fact fields) as stalks over points of $\bar{\mathscr{M}}-\mathscr{M}$. The ring $\bar{R}=\Gamma_c(\bar{\mathscr{R}})=\Gamma(\bar{\mathscr{R}})$ is biregular by 1.5 and it has a unit. We can identify the elements of R in an obvious way with those global sections of $\bar{\mathscr{R}}$ which vanish on $\bar{\mathscr{M}}-\mathscr{M}$.

We will first prove a few lemmas, In the additive group of a simple ring L, all elements have the same order. This common order, or the *characteristic* of L, is either a prime number or ∞. We will show that the characteristic of $\mathscr{R}_M$ is a continuous function of M.

Theorem 2.1. *Let R be a biregular ring and $\bar{P}$ the one-point compactification of the discrete space P of prime numbers. The mapping $\chi: \mathscr{M}(R)\to\bar{P}$, where $\chi(M)$ is the characteristic of $\mathscr{R}(R)_M$ $(M\in\mathscr{M}(R))$, is continuous.*

Corollary 2.2. *Denote by $\mathscr{M}_p$ the subset $\chi^{-1}(p)\subset\mathscr{M}$ $(p\in\bar{P})$. The sets $\mathscr{M}_p$ $(p\in P)$ are open and closed in $\mathscr{M}$ and $\mathscr{M}_\infty$ is closed in $\mathscr{M}$.*

Obviously 2.1 and 2.2 are equivalent. We will prove 2.2.

Proof of 2.2. Denote by $i:\mathscr{M}\to\mathscr{R}$ the unit section $M\to 1_M$ of $\mathscr{R}$. Take any $p\in P$. The set $\mathscr{M}_p$ contains exactly those $M\in\mathscr{M}$ for which $p\,1_M=0_M$. Thus $\mathscr{M}-\mathscr{M}_p$ is the support of the section pi. This support is open and closed, by 1.3, therefore $\mathscr{M}_p$ is open and closed. It follows immediately that $\mathscr{M}_\infty$ is closed.

The next example shows that $\mathscr{M}_\infty$ is not open, in general. We will also use this example in the proof of the main theorem.

Example 2.3. Consider the following sheaf $\mathscr{A}$ on the space $\bar{P}$: for each $p\in P$ let $\mathscr{A}_p=Z_p$ and let $\mathscr{A}_\infty=Q$. Take any rational number m/n $(m,n\in Z$, $n\neq 0)$. Denote by $m_p, n_p\in Z_p$ the congruence classes of m and n modulo p. If $p>n$, then m_p/n_p is defined and is an element of $Z_p=\mathscr{A}_p$. We take the family $\{U_k: k\in Z, k>n\}$, where U_k is the set $\{m_p/n_p\in\mathscr{A}_p: p>k\}\subset\mathscr{A}$, to be a basis of neighborhoods of the point $m/n\in\mathscr{A}_\infty$. The points of $\mathscr{A}-\mathscr{A}_\infty$ are assumed to be isolated. It is easy to show that $\mathscr{A}$ is a sheaf of fields. It follows from 1.5 that the ring $R=\Gamma(\mathscr{A})$ is biregular and that $\mathscr{M}(R)=\bar{P}$, $\mathscr{R}(R)=\mathscr{A}$. The set $\mathscr{M}_\infty=\{\infty\}$ is not open in $\mathscr{M}$.

The following lemma is very easy to prove.

Lemma 2.4. *Let L be a simple ring with unit. There is one and only one prime subfield of L which contains the unit of L.*

We will refer to this subfield as *the* prime subfield of L. (Note that L can contain several subfields isomorphic to a prime field, at least if the characteristic of L is finite.)

Lemma 2.5. *For each $M \in \mathcal{M}$ let $\mathcal{P}_M$ be the prime subfield of $\mathcal{R}_M$ and let $\mathcal{P}$ be the union of all $\mathcal{P}_M$ $(M \in \mathcal{M})$. Then $\mathcal{P}$ is a subsheaf of $\mathcal{R}$.*

Proof. We have to show that $\mathcal{P}$ is open in $\mathcal{R}$. Let $i: \mathcal{M} \to \mathcal{R}$ be the unit section $M \to 1_M$ of $\mathcal{R}$. Obviously $i(\mathcal{M}) \subset \mathcal{P}$. Choose any point $b \in \mathcal{P}$, e.g. $b \in \mathcal{P}_{M_0}$. If $M_0 \in \mathcal{M}_p$ for some $p \in P$, then $\mathcal{P}_{M_0} \cong Z_p$. Thus $b = ni(M_0)$ for some $n \in Z$. The set $ni(\mathcal{M}_p) \subset \mathcal{P}$ is then, by 2.2, an open neighborhood of b in $\mathcal{R}$, therefore b is an interior point of $\mathcal{P}$ in $\mathcal{R}$.

If $M_0 \in \mathcal{M}_\infty$, then $\mathcal{P}_{M_0} \cong Q$. Therefore there are integers $m, n (n \neq 0)$ such that $b = [mi(M_0)]\,[ni(M_0)]^{-1}$. By continuity of multiplication in sheaves, it follows from the equation $ni(M_0)\,b = mi(M_0)$ that there is an open neighborhood U of M_0 in $\mathcal{M}$ and a section $s: U \to \mathcal{R}$ through b such that $(ni|U)\,s = mi|U$. Because of 1.3 we may assume that $U \subset |ni|$. Since all the stalks of $\mathcal{P}$ are fields and since $mi(M) \in \mathcal{P}$ and $0_M \neq ni(M) \in \mathcal{P}$ for each $M \in U$, we have $s(M) = [mi(M)]\,[ni(M)]^{-1} \in \mathcal{P}$ for $M \in U$. The set $s(U) \subset \mathcal{P}$ is an open neighborhood of b in $\mathcal{R}$, thus b is an interior point of $\mathcal{P}$ in $\mathcal{R}$.

It is clear that for each $p \in P$ the restriction $\mathcal{P}|\mathcal{M}_p$ is isomorphic to the constant sheaf $\mathcal{M}_p \times Z_p$ and that $\mathcal{P}|\mathcal{M}_\infty \cong \mathcal{M}_\infty \times Q$.

Now we are going to construct the compactification $\overline{\mathcal{M}}$ of $\mathcal{M}$. Since $\overline{\mathcal{M}}$ will be the structure space of the biregular ring $\bar{R}$, it must have a structure implied by 2.2, i.e. it has to be a union of disjoint subsets, $\overline{\mathcal{M}}_p$ say, where $p \in \bar{P}$ and $\overline{\mathcal{M}}_p$ is open and closed for $p \in P$.

Lemma 2.6. *There is a totally disconnected compact Hausdorff space $\overline{\mathcal{M}}$ and an embedding $\alpha: \mathcal{M} \to \overline{\mathcal{M}}$ with the properties:*

(1) *there exist disjoint compact subsets $\overline{\mathcal{M}}_p \subset \overline{\mathcal{M}}$ $(p \in \bar{P})$ (some of them may be empty) such that their union is $\overline{\mathcal{M}}$;*

(2) *for $p \in P$, $\overline{\mathcal{M}}_p$ is the closure of $\alpha(\mathcal{M}_p)$ in $\overline{\mathcal{M}}$;*

(3) *for $p \in P$, $\overline{\mathcal{M}}_p$ is open in $\overline{\mathcal{M}}$;*

(4) *$\alpha(\mathcal{M})$ is an open dense subset of $\overline{\mathcal{M}}$;*

(5) *for each $p \in \bar{P}$, $\overline{\mathcal{M}}_p - \alpha(\mathcal{M}_p)$ contains at most one point.*

Proof. Take an arbitrary $p \in P$. If $\mathcal{M}_p$ is not compact, let X_p be its one-point compactification; if $\mathcal{M}_p$ is compact, let $X_p = \mathcal{M}_p$. For each $p \in P$ we thus have a natural embedding $f_p: \mathcal{M}_p \to X_p$. It follows from 2.2 that each $\mathcal{M}_p$ is locally compact, therefore each X_p is a totally disconnected Hausdorff compactum. Denote by X the disjoint sum $\mathcal{M} + \sum_{p \in P} X_p$.

Identify each point $M \in \mathcal{M}_p \subset \mathcal{M} \subset X$ $(p \in P)$ with $f_p(M) \in X_p \subset X$; let Y be the corresponding identification space of X and $q: X \to Y$ the identification map. For each $p \in P$ let $Y_p = q(X_p)$ and let $Y_\infty = q(M_\infty)$. It is easy to show that q is an open map. Hence it follows that the restrictions $q|\mathcal{M}, q|X_p$ $(p \in P)$ are embeddings, that the sets $q(\mathcal{M})$, Y_p $(p \in P)$ are open in Y, and that Y has a basis consisting of open compact sets. We are not going to prove all the details; the final results are:

(a) Y is a totally disconnected, locally compact Hausdorff space;

(b) the sets Y_p $(p \in \bar{P})$ are mutually disjoint and their union is Y; for $p \in P$, Y_p is compact;

(c) for each $p \in \bar{P}$, Y_p is the closure of $q(\mathscr{M}_p)$ in Y;
(d) for $p \in P$, Y_p is open in Y;
(e) $q(\mathscr{M})$ is open and dense in Y;
(f) for each $p \in \bar{P}$, $Y_p - q(\mathscr{M}_p)$ contains at most one point.

If Y is compact, then we can take $\bar{\mathscr{M}} = Y, \alpha = q|\mathscr{M}, \bar{\mathscr{M}}_p = Y_p$. If Y is not compact, then let $\bar{\mathscr{M}}$ be its one-point compactification and $j: Y \to \bar{\mathscr{M}}$ the natural embedding. It is easy to prove that for this space $\bar{\mathscr{M}}$, for the map $\alpha = j \circ (q|\mathscr{M})$, and for the sets $\bar{\mathscr{M}}_p = j(Y_p)$ $(p \in P)$, $\bar{\mathscr{M}}_\infty = j(Y_\infty) \cup (\bar{\mathscr{M}} - j(Y))$ all statements of 2.6 are true.

Lemma 2.7. *Let $\mathscr{N}$ be a totally disconnected, compact Hausdorff space and $\beta: \mathscr{M} \to \mathscr{N}$ an embedding such that the statements* (1)–(4) *of* 2.6 *(with $\bar{\mathscr{M}}, \alpha, \bar{\mathscr{M}}_p$ replaced by $\mathscr{N}, \beta, \mathscr{N}_p$, respectively) are true. Then there is one and only one map $\varphi: \mathscr{N} \to \bar{\mathscr{M}}$ of $\mathscr{N}$ onto $\bar{\mathscr{M}}$ such that $\varphi \circ \beta = \alpha$.*

Proof. It is easy to show that $\mathscr{N}' = \mathscr{N} - [\mathscr{N}_\infty - \beta(\mathscr{M})]$ is an open dense subset of $\mathscr{N}$. Similarly, $\bar{\mathscr{M}}' = \bar{\mathscr{M}} - [\bar{\mathscr{M}}_\infty - \alpha(\mathscr{M})]$ is an open dense subset of $\bar{\mathscr{M}}$. We will first construct a surjective map $\psi: \mathscr{N}' \to \bar{\mathscr{M}}'$.

Take a number $p \in P$. Note that $\mathscr{N}_p = \beta(\mathscr{M}_p)$ if and only if $\bar{\mathscr{M}}_p = \alpha(\mathscr{M}_p)$ (and this is true if and only if $\mathscr{M}_p$ is compact). If this is the case, let $\psi_p = (\alpha \circ \beta^{-1} | \mathscr{N}_p)$: $\mathscr{N}_p \to \bar{\mathscr{M}}_p$. If $\mathscr{M}_p$ is not compact, then $\mathscr{N}_p$ is a compactification of $\mathscr{M}_p$ and $\bar{\mathscr{M}}_p$ is the one-point compactification of $\mathscr{M}_p$. The map $\psi_p: \mathscr{N}_p \to \bar{\mathscr{M}}_p$ which is equal to $\alpha \circ \beta^{-1}$ on $\beta(\mathscr{M}_p)$ and which maps $\mathscr{N}_p - \beta(\mathscr{M}_p)$ into the point $\bar{\mathscr{M}}_p - \alpha(\mathscr{M}_p)$ is obviously continuous. The family $\{\beta(\mathscr{M})\} \cup \{\mathscr{N}_p : p \in P\}$ is an open covering of $\mathscr{N}'$. Therefore, the maps $\alpha \circ \beta^{-1}: \beta(\mathscr{M}) \to \bar{\mathscr{M}}'$ and $\psi_p: \mathscr{N}_p \to \bar{\mathscr{M}}'$ $(p \in P)$ compose a continuous map $\psi: \mathscr{N}' \to \bar{\mathscr{M}}'$. Obviously $\psi \circ \beta = \alpha$.

Again, $\mathscr{N}' = \mathscr{N}$ if and only if $\bar{\mathscr{M}}' = \bar{\mathscr{M}}$ (and this happens if and only if $\mathscr{M}_\infty$ and all but finitely many $\mathscr{M}_p, p \in P$, are compact). If $\mathscr{N}' = \mathscr{N}$, then we can take $\varphi = \psi$ and we are through. Suppose that $\mathscr{N} - \mathscr{N}' \neq \emptyset$. Then $\bar{\mathscr{M}} - \bar{\mathscr{M}}'$ is a point. Let $\varphi: \mathscr{N} \to \bar{\mathscr{M}}$ be defined by $\varphi|\mathscr{N}' = \psi$, $\varphi(\mathscr{N} - \mathscr{N}') = \bar{\mathscr{M}} - \bar{\mathscr{M}}'$. Then $\varphi(\mathscr{N}) = \bar{\mathscr{M}}$ and $\varphi \circ \beta = \alpha$. Choose any open neighborhood U of the point $\bar{\mathscr{M}} - \bar{\mathscr{M}}'$ in $\bar{\mathscr{M}}$. We will show that the set $\mathscr{N} - \varphi^{-1}(U) = \psi^{-1}(\bar{\mathscr{M}} - U)$ is compact and this will prove that φ is continuous.

The set $\bar{\mathscr{M}} - U - \alpha(\mathscr{M})$ is compact and covered by the family $\{\bar{\mathscr{M}}_p : p \in P\}$ of disjoint open sets. Hence there is a natural number n such that $[\bar{\mathscr{M}} - U - \alpha(\mathscr{M})] \cap \bar{\mathscr{M}}_p = \emptyset$ for $p > n$. Denote $A = \bigcup_{p \leqq n} \bar{\mathscr{M}}_p$, $B = \bar{\mathscr{M}} - A$. Since $A - U$ is compact, the set $\psi^{-1}(A - U)$ is closed in the compact set $\psi^{-1}(A) = \bigcup_{p \leqq n} \mathscr{N}_p$, thus it is itself compact. We chose n so that $B - U = [B \cap \alpha(\mathscr{M})] - U \subset \alpha(\mathscr{M})$. Since $B - U$ is compact and since $\beta \circ \alpha^{-1}: \alpha(\mathscr{M}) \to \beta(\mathscr{M})$ is a homeomorphism, the set $\psi^{-1}(B - U) = \beta[\alpha^{-1}(B - U)]$ is compact, too. Thus $\psi^{-1}(\bar{\mathscr{M}} - U) = \psi^{-1}(A - U) \cup \psi^{-1}(\mathrm{B} - \mathrm{U})$ is compact.

It is very easy to show that φ is the only map of $\mathscr{N}$ into $\bar{\mathscr{M}}$ such that $\varphi \circ \beta = \alpha$. Thus Lemma 2.7 is proved.

Now we state the main theorem.

Theorem 2.8. *For every biregular ring R there is a biregular ring $\bar{R}$ with unit and a monomorphism $f: R \to \bar{R}$ such that*

(1) *$f(R)$ is a large ideal in $\bar{R}$;*

(2) *if S is any biregular ring with unit and $g: R \to S$ any monomorphism such that $g(R)$ is a large ideal in S, then there is a unique monomorphism $h: \bar{R} \to S$ such that $h \circ f = g$;*

(3) *if R is strongly regular, then so is $\bar{R}$.*

We can easily deduce from 1.9 how the statement (2) of the Theorem can be extended to the case that $g(R)$ is any ideal of S, not necessarily large. It is clear that $\bar{R}$ is determined uniquely up to isomorphism.

Note that $h(\bar{R})$ is never an ideal of S (unless $h(\bar{R}) = S$). Indeed, if $h(\bar{R})$ is an ideal of S, then $G(h(\bar{R}))$ is a dense open subset of $\mathscr{M}(S)$. Moreover, $G(h(\bar{R}))$ is compact since it is homeomorphic to $\mathscr{M}(\bar{R})$ ([5], IX.2.2). Thus $G(h(\bar{R})) = \mathscr{M}(S)$ and $h(\bar{R}) = S$.

Proof of 2.8. Let $\mathscr{M}$ be the structure space of R and $\mathscr{R}$ the sheaf of simple quotient rings of R. Let $\mathscr{P}$ be the subsheaf of $\mathscr{R}$ as in 2.5 and $\bar{\mathscr{M}} \supset \alpha(\mathscr{M})$ be the compactification of $\mathscr{M}$ as in 2.6. It follows from (1) and (3) of 2.6 that there is a map $\bar{\chi}: \bar{\mathscr{M}} \to \bar{P}$ which is an extension of the map χ of 2.1: $\bar{\chi}(\bar{\mathscr{M}}_p) = p$ for each $p \in \bar{P}$. Let $\mathscr{A}$ be the sheaf on $\bar{P}$ as constructed in 2.3. Denote by $\bar{P}$ the inverse image $\bar{\chi}^* \mathscr{A}$. We have: $\bar{\mathscr{P}} | \bar{\mathscr{M}}_p \cong \bar{\mathscr{M}}_p \times Z_p$ for each $p \in P$ and $\bar{\mathscr{P}} | \bar{\mathscr{M}}_\infty \cong \bar{\mathscr{M}}_\infty \times Q$.

Let us identify for a moment the spaces $\mathscr{M}$ and $\alpha(\mathscr{M})$. Then $\mathscr{R}$ is a sheaf on $\alpha(\mathscr{M})$ and $\bar{\mathscr{P}}$ is a natural extension of $\mathscr{P} \subset \mathscr{R}$ over the whole $\bar{\mathscr{M}}$, i.e. $\bar{\mathscr{P}} | \alpha(\mathscr{M}) = \mathscr{P}$. Since $\alpha(\mathscr{M})$ is open in $\bar{\mathscr{M}}$ there is obviously a sheaf $\bar{\mathscr{R}}$ on $\bar{\mathscr{M}}$ such that $\bar{\mathscr{R}} | \alpha(\mathscr{M}) = \mathscr{R}$, $\bar{\mathscr{R}} | (\bar{\mathscr{M}} - \alpha(\mathscr{M})) = \bar{\mathscr{P}} | (\bar{\mathscr{M}} - \alpha(\mathscr{M}))$. Let $\tilde{\alpha}: \mathscr{R} \to \bar{\mathscr{R}}$ be the homomorphism, compatible with α, which establishes the natural isomorphism between $\mathscr{R}$ and $\bar{\mathscr{R}} | \alpha(\mathscr{M})$.

By 1.5, $\bar{R} = \Gamma(\bar{\mathscr{R}}) = \Gamma_c(\bar{\mathscr{R}})$ is a biregular ring with unit. If R is strongly regular, then the stalks of $\mathscr{R}$ and thus all stalks of $\bar{\mathscr{R}}$ are division rings. By 1.7, the ring $\bar{R}$ is then strongly regular.

For any $r \in R$ let $f(r): \bar{\mathscr{M}} \to \bar{\mathscr{R}}$ be the mapping which is equal to $\tilde{\alpha} \circ r \circ \alpha^{-1}$ on $\alpha(\mathscr{M})$ and to the zero-section on $\bar{\mathscr{M}} - \alpha(\mathscr{M})$. Obviously $f(r) \in \bar{R}$. Moreover, $f: R \to \bar{R}$ is a monomorphism and $f(R)$ is the ideal $\{s \in \bar{R}: |s| \subset \alpha(\mathscr{M})\} \subset \bar{R}$. Since $G(f(R)) = \alpha(\mathscr{M})$ (see 1.4) is dense in $\bar{\mathscr{M}}$, the ideal $f(R)$ is large in $\bar{R}$.

Suppose now that S is a biregular ring with unit and that $g: R \to S$ is a monomorphism such that $g(R)$ is a large ideal of S. Denote: $\mathscr{N} = \mathscr{M}(S)$, $\mathscr{S} = \mathscr{R}(S)$. The ring S will be identified with $\Gamma(\mathscr{S})$ in the usual way. The ideal $g(\mathscr{R})$ gets then identified with $\{s \in S: |s| \subset G(g(R))\} \cong \Gamma_c(\mathscr{S} | G(g(R)))$. By 1.5 there is a homeomorphism $\beta: \mathscr{M} \to G(g(R))$ and an isomorphism $\tilde{\beta}: \mathscr{R} \to \mathscr{S} | G(g(R))$ compatible with β. We will understand β as an embedding of $\mathscr{M}$ into $\mathscr{N}$ and $\tilde{\beta}$ as a homomorphism of $\mathscr{R}$ into $\mathscr{S}$, compatible with β. The assertion (3) of 1.5 indicates that $g: R \to S$ coincides with the mapping that assigns to each $r \in R = \Gamma_c(\mathscr{R})$ the global section of $\mathscr{S}$ which is equal to $\tilde{\beta} \circ r \circ \beta^{-1}$ on $\beta(\mathscr{M})$ and vanishes on $\mathscr{N} - \beta(\mathscr{M})$.

Let $\mathcal{N}_p = \mathcal{M}(S)_p \subset \mathcal{N}$ for $p \in \bar{P}$ (see 2.2). Clearly, $\mathcal{N}, \mathcal{N}_p$ and β satisfy the conclusions (1)–(4) of 2.6. By 2.7 there is a surjection $\varphi: \mathcal{N} \to \bar{\mathcal{M}}$ such that $\varphi \circ \beta = \alpha$. Let $\mathcal{Q}$ be the subsheaf of $\mathcal{S}$ whose stalks are the prime subfields of stalks of $\mathcal{S}$ (cf. 2.5). Since $\varphi(\mathcal{N}_p) = \bar{\mathcal{M}}_p$ for each $p \in \bar{P}$, there is obviously a unique φ-cohomomorphism $\mathcal{P} \to \mathcal{Q}$ which is an isomorphism on each stalk. The sheaf isomorphism $\tilde{\beta} \circ \tilde{\alpha}^{-1} : \tilde{\alpha}(\mathcal{R}) \to \tilde{\beta}(\mathcal{R})$, compatible with $\beta \circ \alpha^{-1}$, coincides with this cohomomorphism on $\bar{\mathcal{P}} \cap \tilde{\alpha}(\mathcal{R}) = \tilde{\alpha}(\mathcal{P})$. Therefore they together determine a φ-cohomomorphism $k = \{k_N : \bar{\mathcal{R}}_{\varphi(N)} \to \mathcal{S}_N : N \in \mathcal{N}\}$. The mapping k_N is a monomorphism for each $N \in \mathcal{N}$.

Consider the mapping $h: \bar{R} \to S$ which sends each $r \in \bar{R}$ to the section $N \to k_N \circ r \circ \varphi(N)$ $(N \in \mathcal{N})$. It is clear that h is a monomorphism. For each $r \in R$, $h[f(r)]$ is the section which vanishes on $\mathcal{N} - \beta(\mathcal{M})$ and is equal to the mapping $N \to k_N \circ f(r) \circ \varphi(N) = (\tilde{\beta} \circ \tilde{\alpha}^{-1}) \circ (\tilde{\alpha} \circ r \circ \alpha^{-1}) \circ (\alpha \circ \beta^{-1})(N) = \tilde{\beta} \circ r \circ \beta^{-1}(N) = g(r)(N)$ for $N \in \beta(\mathcal{M})$. Thus $h \circ f = g$.

We have only else to prove that h is unique. Let $h': \bar{R} \to S$ be any homomorphism such that $h' \circ f = g$. Suppose that there is an $r \in \bar{R}$ such that the sections $s = h(r)$ and $s' = h'(r)$ are different. Since $\beta(\mathcal{M})$ is open and dense in $\mathcal{N}$ and since the support of $s - s'$ is open, there exists a compact open set $V \subset \beta(\mathcal{M}) \cap |s - s'|$. The set $U = \varphi(V)$ is also compact and open and lies in $\alpha(\mathcal{M})$. Denote by e the global section of $\bar{\mathcal{R}}$ which is equal to 1 on U and to 0 on $\bar{\mathcal{M}} - U$. Then $e \in f(R)$. Therefore, $h'(e) = g \circ f^{-1}(e) = h(e)$ is the section which is equal to 1 on V and to 0 on $\mathcal{N} - V$. Now we have a contradiction: on the one hand $h'(re)$ is different from $h(re)$ because $h(re)|V = [h(r)h(e)]|V = s|V \neq s'|V = h'(re)|V$. On the other hand $h(re)$ and $h'(re)$ should be equal since $re \in f(R)$. This contradiction shows that h is unique.

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On Some Classes of Operators

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1. The purpose of the present Note is to give results concerning conditions implying normality on the one hand and conditions under which an operator is in the class $\bar{R}_1$ of Halmos and an invariant subspace theorem, on the other hand.

All operators are understood to be bounded and the underlying space is always an infinite dimensional Hilbert space H; $\sigma(T)$ denotes the spectrum of the operator T, $\varrho(T)$ resolvent set, spectral radius $r(T)$, numerical range $W(T)=\{\langle Tx, x\rangle, \|x\|=1\}$, numerical radius $w(T)=\sup\{|\lambda|, \lambda\in W(T)\}$ and norm of T, $\|T\|$, T is called normaloid if $w(T)=\|T\|$.

2. In this section we give results concerning conditions implying normality. The following is a simple result and we omit the proof.

Lemma. *If T is an operator on a Hilbert space and $\lambda\in\sigma(T)$ $|\lambda|=\|T\|$ then λ is an approximate proper value for T (and $\bar{\lambda}$ is for T^* with the same sequence $\{x_n\}$).*

Our main result is the following:

Theorem 1. *If Λ is a subset of complex plane and the operator $T_\lambda=T-\lambda I$ is normaloid for all $\lambda\in\Lambda$, then T is normal*

1° *trivially if* $\dim H=1$,

2° *if* $\dim H=2$ *and Λ is any point,*

3° *if* $\dim H=3$ *and Λ is any set of two distinct points,*

4° *if* $\dim H=4$ *and Λ is any set of three distinct points.*

Proof. We consider the case 2°. Since T_λ is normaloid, we find $\mu\in\sigma(T)$ such that

$$\|T_\lambda\|=|\mu-\lambda|$$

and by the lemma, there exists a sequence $\{x_n\}$ such that

$$T_\lambda-(\mu-\lambda)\,x_n\to 0\,,$$

$$T_\lambda^*-(\bar{\mu}-\bar{\lambda})\,x_n\to 0$$

(the bar denotes the complex conjugate). Since, in all cases 1°–4° the operator is compact, we have that the sequence $\{x_n\}$ is convergent (or some of its subsequence)

$$\lim x_n=x_0$$

and thus

$$Tx_0=\mu\,x_0\,,$$

$$T^*x_0=\bar{\mu}\,x_0\,.$$

Now, if $\Lambda = \{\lambda_0\}$, we obtain, the existence of an element x_0 with the properties

$$Tx_0 = \mu_0 x_0,$$
$$T^* x_0 = \bar{\mu}_0 x_0$$

with some $\mu_0 \in \sigma(T)$. Since the space generated by x_0 is reducing, the space of all elements orthogonal to x_0 (which is one-dimensional) is also reducing. This clear implies the normality of T.

We consider now the case 3°. If $\Lambda = \{\lambda_0, \lambda_1\}$ we find μ_0, μ_1 in $\sigma(T)$ and the elements x_0, y_0 such that

$$Tx_0 = \mu_0 x_0, \qquad Ty_0 = \mu_1 y_0,$$
$$T^* x_0 = \bar{\mu}_0 x_0, \qquad T^* y_0 = \bar{\mu}_1 y_0.$$

If $\mu_0 \neq \mu_1$ the subspaces generated by x_0, y_0 respectively are orthogonal, and on the orthogonal complement of their orthogonal sum (which is one dimensional) T is normal. Thus for this possibility on μ_0, μ_1, T is normal. Now, if $\mu_0 = \mu_1$,

$$(T - \lambda_0)\, x_0 = (\mu_0 - \lambda_0)\, x_0, \qquad \|T - \lambda_0\| = |\mu_0 - \lambda_0|,$$
$$(T - \lambda_1)\, y_0 = (\mu_1 - \lambda_1)\, y_0, \qquad \|T - \lambda_1\| = |\mu_1 - \lambda_1|$$

gives, easy that x_0 and y_0 are orthogonal elements. Let $\{x_0, y_0\}$ be the closed subspace generated by x_0 and y_0. It is clear that $\{x_0, y_0\}^\perp$ is one-dimensional and reducing. This yields that T is normal.

For 4° if $\Lambda = (\lambda_0, \lambda_1, \lambda_2)$, the discussion is similar to 2° and 3°, and we omit it. Also we remark that the dimension of H plays an essential role (for example if $\Lambda = C$, the complexe plane, $\dim H = 5$, the theorem is false).

Remark. Perhaps the theorem is valid under weder weaker condition, namely T to be spectraloid; but this for us is an open question.

Theorem 2. *An operator T is normal if and only if every quadratic polynomial in T and T has the G_1-property (a quadratic polynomial in T and T^* is of the form*

$$aT^2 + bTT^* + CT^*T + dT^{*2} + a_1 T + b_1 T^* + C_1$$

where $a, b, \ldots c_1$ are complex numbers).

Proof. The proof is simply and follows from the fact that an operator with G, property is connexoid and theorem 1 of [1].

Suppose now that the operator T is an invertible operator, then

$$T = UR$$

where $U = T(T^*T)^{-\frac{1}{2}}$, $R = (T^*T)^{-\frac{1}{2}}$, is the polar factorization of T, Further, if U is a cramped operator, i.e. with the spectrum in an open arc $\{e^{i\theta}, \theta_0 < \theta < \theta_0 + \pi\}$, then S. Berberian [2] raised the question; does it follows that $0 \notin \operatorname{cl} W(T)$? For the case of normal operators, he showed that the answer is negative. Results, extending this, were obtained in [4], [10]. The following is more general and is with simple proof.

Theorem 3. *Let T be an invertible operator such that $U = T(T^*T)^{-\frac{1}{2}}$ is cramped, then $0 \notin \operatorname{conv}\sigma(T)$.*

Proof. It is clear that the above theorem is equivalent with to the following.

Lemma. *If T is an operator such that*
1. *T is invertible,*
2. *$0 \in \operatorname{conv}\sigma(T)$*

then the unitary operator $T(T^*T)^{-\frac{1}{2}}$ is not cramped.

Proof. Since $T = UR$, $R = (T^*T)^{\frac{1}{2}}$ and T is invertible

$$\sigma(T) = \sigma(UR) = \sigma(UR^{\frac{1}{2}}R^{\frac{1}{2}}) = \sigma(R^{\frac{1}{2}}UR^{\frac{1}{2}})$$

[3, Problem 61]. By our hypothesis

$$0 \in \operatorname{conv}\sigma(T) = \operatorname{conv}\sigma(R^{\frac{1}{2}}UR^{\frac{1}{2}}) \subseteq \operatorname{cl} W(R^{\frac{1}{2}}UR^{\frac{1}{2}})$$

and thus we find a sequence of unit vectors $\{x_n\}$ such that

$$\langle R^{\frac{1}{2}}UR^{\frac{1}{2}}X_n, X_n\rangle \to 0$$

or

$$\langle UR^{\frac{1}{2}}X_n, R^{\frac{1}{2}}X_n\rangle \to 0.$$

Since $\|R^{\frac{1}{2}}X_n\|$ is bounded below, setting $y_n = \|RX_n\|^{-1}.\ RX_n$ we have

$$\langle Uy_n, y_n\rangle \to 0$$

and this implies that U is not cramped. The lemma is proved and also the theorem.

3. If $\mathscr{G}_1$ denotes the class of operators with G_1-property and R_1 consists of all operators with a one-dimensional reducing reducing subspace then our result about the connection of these classe is

Theorem 4. *Norm closure $\{\mathscr{G}_1 + \mathscr{K}\} \subset \overline{R}_1$ (where $\mathscr{K}$ is the set of compact operators, $\overline{R}_1$ denotes the closure of R_1 in the norm topology).*

Proof. Let us consider $\lambda_0 \in \varrho(T)$. Since

$$\|(T - \lambda_0 I)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda_0, \sigma(T))} = \frac{1}{|\mu_0 - \lambda_0|}$$

where $\mu_0 \in \sigma(T)$, we find a sequence of unit vectors, converging weakly to zero

$$(T - \lambda_0)^{-1}x_n - \frac{1}{\mu_0 - \lambda_0}x_n \to 0,$$

$$(T^* - \lambda_0)^{-1}x_n - \frac{1}{\bar{\mu}_0 - \bar{\lambda}_0}x_n \to 0$$

which leads to

$$\mu_0 x_n - T x_n \to 0,$$
$$\bar{\mu}_0 x_n - T^* x_n \to 0,$$

and if K is a compact operator we have

$$\lim \|(T+K)\, x_n - \mu_0 x_0\| = 0,$$
$$\lim \|(T^*+K^*)\, x_n - \bar{\mu}_0 x_n\| = 0.$$

Thus the Theorem 4 is proved.

Remark. For the case of hyponormal operators the theorem was proved in [9, Th. 2].

In [6] a class generalizing the class of hyponormal operators was considered: an operator is of class N if

$$\|x\| = 1 \quad \|T^2 x\| \geqq \|Tx\|^2$$

and in [7] some new classes are considered, an operator is in the class (N, k) if

$$\|x\| = 1, \|T^k x\| \geqq \|Tx\|^k$$

$(k \geqq 2)$. Our results is the following

Theorem 5. *Let T be an operator of class (N, k) with a maximal vector x_0. Then T has a proper invariant subspace.*

Proof. We consider the set

$$M = \{x, \|Tx\| = \|T\| \; \|x\|\}$$

which is a subspace. Now, we can prove that M is invariant under T.

Let $x \in M$ and we suppose, without loss of generality that $\|x\| = 1$. Thus, since T is of class (N, k)

$$\|T^k x\| \geqq \|Tx\|^k = \|T\|^k \tag{*}$$

which gives

$$\|T^k(Tx)\| = \left\| T^k \frac{Tx}{\|Tx\|} \right\| \; \|Tx\| \geqq \|T^2 x\|^k \frac{1}{\|Tx\|^{k-1}}.$$

But $\|T^2 x\| = \|Tx\|^2$ (which follows from (*)) and thus

$$\|T^k(Tx)\| \geqq \|Tx\|^{k+1}$$

which shows T is invariant under T. Since T has a maximal vector $M \neq 0$. If $M = H$ then T^k is a on isometry.

The theorem is proved.

The classes (N, k) suggests a generalization of the notion of maximal vector as follows: a vector xH is maximal of order k for an arbitrar operator T if $\|T^k x\| = \|T^k\| \|x\|$. Using a result of P. Rosenthal we have

Theorem 6. *If T^k is an unicellular operator with a maximal vector of order k then T has an invariant subspace if T^k is of class (N, m).*

Remark. In [9] a generalization of a theorem of Deckard, Douglas and Pearcy is given. Also, the theorem may be proved for operators on Banach spaces such that T has a decomposition $T = H + iJ$ where H, J are hermitian elements with $T^* = H - iJ$, the condition 1 in that theorem may be replaced as

$$H^n J^m - J^m H^n = C_{n,m}$$

where $C_{n,m}$ are compact operators for all integers n, m. This result will be appear in [5].

In the following we present a proof of Th. 7 Sect. 4 [9] which is independent of essential spectrum.

Theorem 7. *Let Γ be a Jordan curve which consists of a finite number of rectifiable smooth arcs and T be a hyponormal such that $T = A + C$ where $\sigma(A) \subset \Gamma$ and C is compact. The T is normal.*

Proof. We know [11] that $T = T_1 \oplus T_2$ where $T_1 \oplus T_2$ is defined on $H_1 \oplus H_2$ and H_1 is spaneed by all proper vectors of T and such that

1. $\sigma(T_2) = \sigma_c(T_2)$,
2. T_1 is normal and T is normal if and only if T_2 is normal.

From the Lemma 2.2 of [8] we have that $\sigma_c(T_1) \subset \sigma(A)$ and from the construction in [11] of spaces H_1 and H_2 it follows that

$$\sigma(T_2) = \sigma_c(T) \subset \sigma(A) \subset \Gamma.$$

But T_2 is also hyponormal and from Theorem 9 of [11] T_2 is normal.

Remark. Perhaps the theorem is valid in a more generale case, e.g. T is hyponormal and for some integer m, $T^m = A + C$ where $\sigma(A) \subset \Gamma$ and C is compact.

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Die abzählbare Topologie und die Existenz von Orthogonalbasen in unendlichdimensionalen Räumen

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Einleitung

Vektorräume E von höchstens abzählbarer Dimension über einem Körper k mit Charakteristik $\neq 2$, versehen mit einer symmetrischen Bilinearform $\Phi : E \times E \rightarrow k$, besitzen bezüglich Φ immer orthogonale Basen. Ist die Dimension von E hingegen überabzählbar, dann ist diese Aussage falsch. Sie ist sogar dann noch falsch, wenn (E, Φ) als Teilraum eines Raumes mit Orthogonalbasis vorausgesetzt wird. Beispiele dazu findet man auf S. 105 in [2]. Es sei an dieser Stelle betont, daß wir immer von rein algebraischen oder sog. Hamelbasen sprechen, d. h. es werden nur endliche Summen zugelassen. Besitzt E bezüglich Φ eine orthogonale Basis und ist Φ nicht ausgeartet, dann heißt (E, Φ) euklidisch.

Das Hauptziel dieser Arbeit ist es, Bedingungen anzugeben, unter denen Teilräume von euklidischen Räumen orthogonale Basen besitzen. Es zeigt sich, daß ein enger Zusammenhang besteht zwischen der Existenz von Orthogonalbasen in solchen Teilräumen und gewissen Eigenschaften einer linearen Topologie, die wir die abzählbare nennen wollen und welche die orthogonalen Komplemente aller abzählbardimensionalen Teilräume als Nullumgebungsbasis hat (Sätze 1, 5, 6). Ferner studieren wir das Verhalten von (E, Φ) bei Körpererweiterungen und beantworten im negativen Sinne die in [2] aufgeworfene Frage, ob durch bloße Erweiterung des Skalenbereiches ein Raum ohne Orthogonalbasis in einen solchen mit Orthogonalbasis übergeführt werden kann (Korollar 1 zu Satz 3). Als Anwendungen der allgemeinen Sätze erhalten wir Resultate im Zusammenhang mit dem in [2] eingeführten euklidischen Defekt und beantworten insbesondere eine weitere offene Frage aus [2] (Korollare 2 und 4 zu Satz 5, Satz 9 und Korollar). Schließlich geben wir für die Räume von abzählbarem euklidischem Defekt eine notwendige und hinreichende Bedingung für die isometrische Einbettbarkeit in euklidische Räume (Satz 10). Die vorliegenden Untersuchungen werden sogleich für reflexive Sesquilinearformen mit beliebigem Schiefkörper als Skalarenbereich durchgeführt. Man hat nur an Stelle einer orthogonalen Basis eine orthogonale Zerlegung des Raumes in endlichdimensionale Teilräume zu betrachten. Unter einem euklidischen Raum hat man dann einen Raum zu verstehen, der eine solche Zerlegung besitzt und dessen Sesquilinearform nicht ausgeartet ist.

I. Bezeichnungen und Resultate

I.1

Alle betrachteten Räume sind k-Linksvektorräume, wo k ein Schiefkörper ist. Mit Ausnahme von Korollar 4 zu Satz 3 setzen wir immer voraus, daß die Form $\Phi: E\times E\to k$ eine beliebige reflexive Sesquilinearform bezüglich eines Antiautomorphismus $\alpha\to\alpha^*$ ist, d. h. Φ ist additiv in beiden Argumenten, $\Phi(\lambda x, y)=\lambda\Phi(x,y)$, $\Phi(x,\lambda y)=\Phi(x,y)\lambda^*$ und $\Phi(x,y)=0$ genau dann, wenn $\Phi(y,x)=0$ (Reflexivität). Zu jeder reflexiven Sesquilinearform Φ gibt es ein $\varepsilon\in k$ mit $\Phi(x,y)=\varepsilon\Phi(y,x)^*$ für alle $x,y\in E$, und der Antiautomorphismus $*$ ist eine Involution. Man nennt Φ deshalb auch eine ε-hermitesche Form. Eine solche Form heißt spurwertig, falls für jedes $x\in E$ ein $\alpha\in k$ existiert, so daß $\Phi(x,x)=\alpha+\varepsilon\alpha^*$. Diese Eigenschaft ist immer erfüllt, wenn $\operatorname{char} k\neq 2$ (man setze $\alpha=\frac{1}{2}\Phi(x,x)$). Für eine detaillierte Behandlung der Sesquilinearformen verweisen wir auf [1].

Teilräume F von (E,Φ) werden gewöhnlich mit der von Φ induzierten Form (Restriktion) versehen. $F\cap F^\perp$ heißt das Radikal von F und wird mit $\operatorname{rad} F$ bezeichnet. Ist Φ nicht ausgeartet, so sagt man auch, daß E halbeinfach sei, was gleichbedeutend ist mit $\operatorname{rad} E=(0)$. Ein Vektor x, für den $\Phi(x,x)=0$, heißt isotrop. Ein Teilraum F mit $F\subset F^\perp$ heißt totalisotrop. Besitzt F keinen isotropen Vektor, so nennt man F anisotrop. F heißt orthogonal abgeschlossen, falls $F^{\perp\perp}=F$. Sind A und B Teilmengen von E, dann bezeichnen wir mit $\Phi(A,B)$ die Menge $\{\Phi(x,y)\in k/x\in A, y\in B\}$. Stehen in einer direkten Summe $\bigoplus_{\iota\in J} F_\iota$ die Teilräume F_ι paarweise orthogonal aufeinander, d. h. ist $\Phi(F_\iota, F_\kappa)=\{0\}$ für alle $\iota,\kappa\in J$ und $\iota\neq\kappa$, so schreiben wir oft $\overset{\perp}{\bigoplus_{\iota\in J}} F_\iota$. (E,Φ) heißt äußere Summe einer Familie $\{(E_\iota,\Phi_\iota)\}_{\iota\in J}$ von Räumen (E_ι,Φ_ι), falls $E=\overset{\perp}{\bigoplus_{\iota\in J}} E_\iota$ und $\Phi|_{E_\iota\times E_\iota}=\Phi_\iota$ für alle $\iota\in J$. Eine Isometrie $\varphi:(E,\Phi)\to(G,\Psi)$ ist ein Vektorraumisomorphismus $\varphi: E\to G$ mit $\Phi(x,y)=\Psi(\varphi(x),\varphi(y))$ für alle $x\in E$, $y\in E$. Ein halbeinfacher Raum (E,Φ) heißt präeuklidisch, wenn er isometrisch ist zu einem Teilraum eines euklidischen Raumes. Der euklidische Defekt $d(E)$ eines halbeinfachen Raumes E ist das Minimum von $\dim E/F$, wo F alle euklidischen Teilräume von E durchläuft (vgl. [2]).

Eine lineare Topologie τ auf einem Vektorraum E ist eine Topologie, welche eine Nullumgebungsbasis aus linearen Teilräumen besitzt und deren Umgebungen in jedem Punkt durch Translation der Nullumgebungen entstehen. Versieht man noch den Grundkörper k mit der diskreten Topologie, dann heißt (E,τ) ein lineartopologischer Raum. Ist $\mathscr{B}$ eine Basis des Nullumgebungsfilters bezüglich τ, so sagen wir auch, daß $\mathscr{B}$ „die lineare Topologie τ auf E erzeugt".

Zu einer beliebigen unendlichen Kardinalzahl $\mathfrak{a}$ bilden wir das System $\mathscr{B}=\{F^\perp\mid F \text{ Teilraum von } E \text{ mit } \dim F<\mathfrak{a}\}$. $\mathscr{B}$ bildet wegen $F^\perp\cap G^\perp=(F+G)^\perp$ eine Nullumgebungsbasis einer linearen Topologie. Die Topologie mit $\mathfrak{a}=\aleph_0$

heißt schwache oder endliche Topologie und wird im folgenden mit $\sigma(\Phi)$ bezeichnet. Die Topologie mit $\mathfrak{a}=\aleph_1$ nennen wir die abzählbare Topologie. Wir bezeichnen sie mit $\tau(\Phi)$. Offenbar gilt $\sigma(\Phi)\leqq\tau(\Phi)$. Diese Topologien sind separiert genau dann, wenn Φ nicht ausgeartet ist. Ist F ein Teilraum von E, dann sei $\bar{F}$ die $\tau(\Phi)$-abgeschlossene Hülle von F in E. Für alle Details betreffend lineare Topologien auf Vektorräumen verweisen wir auf [5].

I.2

Wir werden folgende Sätze benötigen:

Satz 1. ([5], § 10, 4.) *Ein Teilraum F eines Raumes (E, Φ) ist genau dann orthogonal abgeschlossen, wenn er schwach abgeschlossen ist, d. h. abgeschlossen bezüglich $\sigma(\Phi)$.*

Eine Witt-Zerlegung von (E, Φ) ist eine direkte Zerlegung von E in drei Teilräume R, R' und E_0, wobei R und R' totalisotrop sind und aufgespannt von den beiden Hälften einer symplektischen Basis $\{r_\iota, r'_\iota\}_{\iota\in J}$, $R=k(r_\iota)_{\iota\in J}$, $R'=k(r'_\iota)_{\iota\in J}$, $\Phi(r_\iota, r'_\kappa)=\delta_{\iota\kappa}$ (Kronecker) und $R\oplus R'\perp E_0$. Es gilt der

Satz 2. ([4], Satz 7). *Sei E ein Raum von höchstens abzählbarer Dimension, versehen mit einer nicht ausgearteten, spurwertigen ε-hermiteschen Form Φ und R ein totalisotroper, orthogonal abgeschlossener Teilraum von E. Dann gibt es zu R eine Wittsche Zerlegung $E=(R\oplus R')\overset{\perp}{\bigoplus} E_0$.*

Der Beweis wurde in [4] nur für alternierende Formen durchgeführt. Er kann aber bei geringfügiger Modifikation auf spurwertige ε-hermitesche Formen erweitert werden.

Bemerkung. Ist Φ eine beliebige reflexive Sesquilinearform, E und R bezüglich Φ wie in Satz 2, dann gilt etwas schwächer:

Es gibt zu R eine Zerlegung $E=(R\oplus R')\overset{\perp}{\bigoplus} E_0$ und eine Basis $\{r_\iota\}_{\iota\in J}$ von R sowie eine Basis $\{r'_\iota\}_{\iota\in J}$ von R' mit $\Phi(r_\iota, r'_\kappa)=\delta_{\iota\kappa}$ und $\Phi(r'_\nu, r'_\mu)=0$ für alle $\nu\neq\mu$, wo $\nu, \mu\in J$.

R' braucht dabei nicht totalisotrop zu sein. Auch den Beweis dieses Sachverhaltes erhält man durch leichte Modifikation des Beweises von Satz 7 in [4].

Satz 3. ([2], Korollar 3 zu Satz 11)[1]. *Sei E ein halbeinfacher Teilraum eines euklidischen Raumes (L, Φ) mit $d(E)\leqq\aleph_0$. Es ist $d(E)\leqq\dim L/E$.*

Satz 4. *Ist (E, Φ) ein präeuklidischer Raum und F ein Teilraum von E, dann gilt* $\dim E/F^\perp=\dim F=\dim F^{\perp\perp}$.

Wir geben den folgenden, aus [3] stammenden

Beweis. Sei (H, Ψ) ein euklidischer Oberraum von (E, Φ) und $H=\overset{\perp}{\bigoplus\limits_{\iota\in J}} S_\iota$ eine orthogonale Zerlegung von H in endlichdimensionale Teilräume S_ι.

[1] Dieses Korollar stützt sich auf Satz 4 in [2]. In der Formulierung jenes Satzes ist nur von präeuklidischen Räumen die Rede, ohne daß dies ausdrücklich gesagt wird. Der Satz 4 ist für präeuklidische Räume richtig, obschon der Beweis in [2] einen Fehler enthält.

Wir können annehmen, daß $\dim F$ unendlich ist, denn für endlichdimensionales F sind die obenstehenden Beziehungen bekannt. Dann gibt es zu F einen Oberraum F_0, der die Summe einiger S_ι ist und für den $\dim F = \dim F_0$ ist. Es gilt $\dim E/F^\perp = \dim[E/(F^{\perp_H} \cap E)] = \dim[(E+F^{\perp_H})/F^{\perp_H}] \leqq \dim H/F^{\perp_H} \leqq \dim H/F_0^{\perp_H} = \dim F_0 = \dim F$. Also ist $\dim E/F^\perp \leqq \dim F$ für jeden Teilraum F von E.

Sei nun G ein linearer Teilraum von F und L ein lineares Komplement von G in F, d.h. $F = G \oplus L$. Dann ist $\dim G^\perp/F^\perp = \dim[G^\perp/(G^\perp \cap L^\perp)] = \dim[(G^\perp + L^\perp)/L^\perp] \leqq \dim E/L^\perp \leqq \dim L = \dim F/G$. Ebenso gilt $\dim F^{\perp\perp}/G^{\perp\perp} \leqq \dim G^\perp/F^\perp \leqq \dim F/G$. Setzt man $G=(0)$, so erhält man die gesuchten Beziehungen.

II. Vorbereitungen

II.1

Sei $\{e_\iota\}_{\iota \in J}$ eine Basis von (E, Φ). Jeder abzählbardimensionale Teilraum V von E ist in einem Teilraum W enthalten, der von abzählbar vielen e_ι aufgespannt wird. Da $W^\perp \subset V^\perp$, bildet auch $\mathscr{V} = \{[k(e_\iota)_{\iota \in P}]^\perp / P \subset J \text{ mit } \operatorname{card} P = \aleph_0\}$ eine Nullumgebungsbasis von $\tau(\Phi)$. *k sei im folgenden immer versehen mit der diskreten Topologie.*

Ist $\mathscr{F}$ ein Cauchyfilter in E bezüglich $\tau(\Phi)$, dann gibt es zu jedem $P \subset J$ mit $\operatorname{card} P = \aleph_0$ ein $F \in \mathscr{F}$ mit $F - F \subset [k(e_\iota)_{\iota \in P}]^\perp$, was gleichbedeutend damit ist, daß für jedes $\iota \in P$ $\Phi(F, e_\iota)$ nur aus einem Element besteht. Da jedes $x \in E$ Linearkombination von endlich vielen e_ι ist, folgt daraus insbesondere, daß $\lim\limits_{F \in \mathscr{F}} \Phi(F, x)$ existiert (bezüglich der diskreten Topologie von k).

Lemma 1. *$\mathscr{F}$ bzw. $\mathscr{G}$ seien Filter in (E, Φ), die bezüglich $\tau(\Phi)$ oder $\sigma(\Phi)$ gegen f bzw. gegen g konvergieren. Dann existieren $\lim\limits_{F \in \mathscr{F}} \lim\limits_{G \in \mathscr{G}} \Phi(F, G)$ und $\lim\limits_{G \in \mathscr{G}} \lim\limits_{F \in \mathscr{F}} \Phi(F, G)$, und es ist $\lim\limits_{F \in \mathscr{F}} \lim\limits_{G \in \mathscr{G}} \Phi(F, G) = \lim\limits_{G \in \mathscr{G}} \lim\limits_{F \in \mathscr{F}} \Phi(F, G) = \Phi(f, g)$.*

Beweis. Sei $x \in E$. Da $\mathscr{G}$ gegen g konvergiert, gibt es ein $G_0 \in \mathscr{G}$ mit $G_0 \subset g + k(x)^\perp$, d.h. $G_0 - g \subset k(x)^\perp$. Für jedes $y \in G_0$ ist $\Phi(x, y) = \Phi(x, y-g) + \Phi(x, g) = \Phi(x, g)$. Also ist $\lim\limits_{G \in \mathscr{G}} \Phi(x, G) = \Phi(x, g)$. Da $\mathscr{F}$ gegen f konvergiert, gibt es ein $F_0 \in \mathscr{F}$ mit $F_0 \subset f + k(g)^\perp$, d.h. $F_0 - f \subset k(g)^\perp$. Für jedes $y \in F_0$ ist $\lim\limits_{G \in \mathscr{G}} \Phi(y, G) = \lim\limits_{G \in \mathscr{G}} \Phi(y - f, G) + \lim\limits_{G \in \mathscr{G}} \Phi(f, G) = \Phi(y - f, g) + \Phi(f, g) = \Phi(f, g)$. Also ist $\lim\limits_{F \in \mathscr{F}} \lim\limits_{G \in \mathscr{G}} \Phi(F, G) = \Phi(f, g)$. In analoger Weise zeigt man, daß $\lim\limits_{G \in \mathscr{G}} \lim\limits_{F \in \mathscr{F}} \Phi(F, G) = \Phi(f, g)$.

II.2. Erweiterung des Grundkörpers

Sei nun k' ein Erweiterungskörper von k. Wir nehmen an, daß sich der Antiautomorphismus * auf k zu einem solchen auf k' fortsetzen läßt. Der

Vektorraum $E' = k' \otimes_k E$ über k kann als Vektorraum über k' aufgefaßt werden. Mit $\Phi' : E' \times E' \to k'$, definiert durch $\Phi'\left(\sum_i \lambda_i \otimes x_i, \sum_j \mu_j \otimes y_j\right) = \sum_{i,j} \lambda_i \Phi(x_i, y_j) \mu_j^*$ für $\lambda_i, \mu_j \in k'$, ist eine Sesquilinearform auf E' gegeben. Φ' ist nicht ausgeartet genau dann, wenn Φ nicht ausgeartet ist. Da Φ nach Voraussetzung eine ε-hermitesche Form ist, ist auch Φ' ε-hermitesch, also reflexiv.

Wir betrachten die Abbildung $i : E \to E'$, definiert durch $i(x) = 1 \otimes x$ für alle $x \in E$. i ist injektiv, k-linear und isometrisch. Sei F ein Teilraum von E. Die k'-lineare Hülle von $i(F)$ in E' ist $F' = k' \otimes_k F$. Jedem Teilraum F von E können wir also auf diese Weise einen Teilraum F' von E' zuordnen. Es gilt:

$$\dim_k F = \dim_{k'} F' \tag{1}$$

Für eine Familie $\{F_\iota\}_{\iota \in I}$ von Teilräumen von E ist

$$\left(\bigcap_{\iota \in I} F_\iota\right)' = \bigcap_{\iota \in I} F'_\iota \tag{2}$$

Wenn wir mit $\perp'$ das orthogonale Komplement in E' bezeichnen, dann gilt

$$(F^\perp)' = F'^{\perp'} \tag{3}$$

Für den Beweis von (3) siehe [2], III.4.

Lemma 2. *E' sei versehen mit der Topologie $\tau(\Phi')$. Dann ist $\tau(\Phi)$ gerade die durch i von $(E', \tau(\Phi'))$ auf E induzierte Topologie.*

Beweis. Sei $\{e_\iota\}_{\iota \in J}$ eine Basis von E. Dann ist $\{1 \otimes e_\iota\}_{\iota \in J}$ eine Basis von E'. Wir betrachten zu $S \subset J$ die Teilräume $U = k(e_\iota)_{\iota \in S}$ von E und $U' = k'(1 \otimes e_\iota)_{\iota \in S}$ von E'. Durchläuft S alle abzählbaren Teilmengen von J, dann durchläuft $U^\perp$ eine Nullumgebungsbasis von E bezüglich $\tau(\Phi)$ und $U'^{\perp'}$ eine Nullumgebungsbasis von E' bezüglich $\tau(\Phi')$. $i(U^\perp) \subset (U^\perp)' = U'^{\perp'}$ nach (3), also $U^\perp \subset i^{-1}(U'^{\perp'})$. Sei $x \in i^{-1}(U'^{\perp'})$ und $y \in U$ beliebig. Dann ist $i(x) = 1 \otimes x \in U'^{\perp'}$, $1 \otimes y \in U'$ und $\Phi(x, y) = \Phi'(1 \otimes x, 1 \otimes y) = 0$, also $i^{-1}(U'^{\perp'}) \subset U^\perp$, woraus folgt $i^{-1}(U'^{\perp'}) = U^\perp$.

Lemma 3. *(E, Φ) sei halbeinfach und E' $\tau(\Phi')$-vollständig. Dann ist E $\tau(\Phi)$-vollständig.*

Beweis. $\mathscr{F}$ sei ein Cauchyfilter auf E bezüglich $\tau(\Phi)$. Da $i : (E, \tau(\Phi)) \to (E', \tau(\Phi'))$ nach Lemma 2 gleichmäßig stetig ist, ist $\mathscr{F}' = i(\mathscr{F})$ ein Cauchyfilter in E', konvergiert also gegen ein Element f'. Sei $\{e_\iota\}_{\iota \in J}$ eine Basis von E. $f' = \sum_{\iota \in M} \alpha_\iota 1 \otimes e_\iota$, wo M eine endliche Teilmenge von J ist. Wir zeigen, daß $\alpha_\iota \in k$ für alle $\iota \in M$ und damit $f' \in i(E)$. Für $x \in E$ ist $\Phi'(f', i(x)) = \lim_{C' \in \mathscr{F}'} \Phi'(C', i(x)) = \lim_{C \in \mathscr{F}} \Phi(C, x) \in k$. Zu jedem $\iota \in M$ gibt es ein $x_\iota \in [k(e_\kappa)_{\kappa \in M \setminus \{\iota\}}]^\perp$ mit $\Phi(x_\iota, e_\iota) = 1$, da Φ nicht ausgeartet und daher jeder endlichdimensionale Teilraum orthogonal abgeschlossen ist. Dann gilt $\Phi'(f', i(x_\kappa)) = \Phi'\left(\sum_{\iota \in M} \alpha_\iota 1 \otimes e_\iota, 1 \otimes x_\kappa\right) = \sum_{\iota \in M} \alpha_\iota \Phi(e_\iota, x_\kappa) = \alpha_\kappa \in k$. Sei $f = \sum_{\iota \in M} \alpha_\iota e_\iota$. $f' = i(f)$.

Zu jedem abzählbardimensionalen Teilraum U von E gibt es ein $C \in \mathscr{F}$ mit $i(C) = C' \subset f' + U'^{\perp'}$. $C' - f' \subset U'^{\perp'}$. Für ein beliebiges $x \in U$ ist $i(x) \in i(U) \subset U'$ und damit $\Phi(x, C - f) = \Phi'(i(x), C' - f') = \{0\}$, d. h. $C - f \subset U^{\perp}$, $C \subset f + U^{\perp}$, also $f = \lim \mathscr{F}$. Q.E.D.

III. Euklidische Räume

III.1

Definition 1. Ein Raum (E, Φ) heißt euklidisch genau dann, wenn (E, Φ) halbeinfach und orthogonale Summe endlichdimensionaler Teilräume ist.

Definition 2. Ein Raum (E, Φ) heißt präeuklidisch genau dann, wenn er halbeinfach und isometrisch zu einem Teilraum eines euklidischen Raumes ist.

Ist ein Raum (E, Φ) orthogonale Summe endlichdimensionaler Teilräume, dann ist (E, Φ) sogar direkte, orthogonale Summe solcher Räume. Nach Definition 1 sind alle höchstens abzählbardimensionalen, halbeinfachen Räume euklidisch (siehe [4]).

Satz 1. *Besitzt (E, Φ) eine orthogonale Zerlegung in endlichdimensionale Teilräume, dann ist E $\tau(\Phi)$-vollständig.*

Beweis. Nach Voraussetzung ist $E = \overset{\perp}{\bigoplus\limits_{\iota \in J}} S_\iota$, wo die S_ι endlichdimensionale Teilräume sind. Für jedes $x \in E$ ist $M(x) = \{\iota \in J / \Phi(x, S_\iota) \neq \{0\}\}$ eine endliche Menge. Sei $\mathscr{F}$ ein Cauchyfilter in E bezüglich $\tau(\Phi)$. Dann gibt es zu jedem $\iota \in J$ ein $F_\iota \in \mathscr{F}$ mit $F_\iota - F_\iota \subset S_\iota^{\perp}$. Ist $x \in S_\iota$, so ist $\Phi(x, F_\iota)$ einelementig.

Wir zeigen zunächst, daß die Menge $M = \{\iota \in J / \Phi(S_\iota, F_\iota) \neq \{0\}\}$ endlich ist. Gesetzt M sei unendlich, so gibt es zu einer abzählbaren Teilmenge P von M ein $F \in \mathscr{F}$ mit $F - F \subset \left[\bigoplus\limits_{\iota \in P} S_\iota\right]^{\perp}$. Ferner existiert zu jedem $\iota \in P$ ein $a_\iota \in S_\iota$ mit $\Phi(a_\iota, F_\iota) = \{\alpha_\iota\}$ und $\alpha_\iota \neq 0$. Da $\Phi(a_\iota, F \cap F_\iota) = \{\alpha_\iota\}$, ist auch $\Phi(a_\iota, F) = \{\alpha_\iota\}$. Für ein beliebiges $x \in F$ und alle $\iota \in P$ ist dann $\Phi(a_\iota, x) = \alpha_\iota \neq 0$, also $M(x)$ unendlich. Widerspruch. Der Raum $U = \bigoplus\limits_{\iota \in M} S_\iota$ ist somit endlichdimensional. Es gibt ein $H \in \mathscr{F}$ mit $H - H \subset U^{\perp}$. Sei $h \in H$. $h = f + g$, wo $f \in U$ und $g \in U^{\perp}$. Wir zeigen nun, daß $\mathscr{F}$ gegen f konvergiert. R sei eine beliebige abzählbare Teilmenge von J und $V = \left[\bigoplus\limits_{\iota \in R} S_\iota\right]^{\perp}$. Es gibt ein $G \in \mathscr{F}$ mit $G - G \subset V$. Für alle $\iota \in R \cap M$, $x \in S_\iota$, $y \in G$ und $l \in G \cap H$ ist $\Phi(x, y - f) = \Phi(x, y - l) + \Phi(x, l - f) = \Phi(x, l - h) + \Phi(x, g) = 0$, d. h. $\Phi(S_\iota, G - f) = \{0\}$ für alle $\iota \in R \cap M$. Für $\iota \in R \setminus M$, $x \in S_\iota$, $y \in G$ und $l \in G \cap F_\iota$ ist $\Phi(x, y - f) = \Phi(x, y) = \Phi(x, y - l) + \Phi(x, l) = 0$, d. h. $\Phi(S_\iota, G - f) = \{0\}$ für alle $\iota \in R \setminus M$. Daraus folgt $G - f \subset V$, $G \subset f + V$. $\mathscr{F}$ konvergiert also gegen f.

Lemma 4. *J sei eine Menge und R eine symmetrische Relation auf J. Ferner gebe es eine unendliche Kardinalzahl $\mathfrak{a}$ derart, daß für jedes $\iota \in J$ $\operatorname{card}\{\kappa \in J / \kappa R \iota\} \leqq \mathfrak{a}$ ist. Dann ist J Vereinigung von disjunkten Mengen L_α, $\alpha \in A$, mit $\operatorname{card} L_\alpha \leqq \mathfrak{a}$, so daß aus $\iota R \kappa$ und $\iota \in L_\alpha$ folgt $\kappa \in L_\alpha$.*

Beweis. Wir definieren eine neue Relation S auf J mit den folgenden Eigenschaften:

1) $\iota S \iota$ für alle $\iota \in J$,
2) $\iota S \kappa$, falls es endlich viele $\mu_1, \mu_2, \ldots, \mu_n \in J$ gibt, so daß

$$\iota R \mu_1 R \mu_2 R \mu_3 R \ldots R \mu_n R \kappa .$$

Dann ist S eine Äquivalenzrelation auf J. Es ist daher in J eine Einteilung in Äquivalenzklassen bezüglich S gegeben. Diese können folgendermaßen erhalten werden: Sei $L_\iota^1 = \{\iota\}$ für jedes $\iota \in J$, $L_\iota^2 = \{\kappa \in J / \kappa R \iota\}$. Ist L_ι^{k-1} bereits definiert, dann sei $L_\iota^k = \{\kappa \in J / \kappa R \mu$ für ein $\mu \in L_\iota^{k-1}\}$. $L_\iota = \bigcup_{k \in \mathbb{N}} L_\iota^k$ ist die zu ι gehörige Äquivalenzklasse. $\operatorname{card} L_\iota^2 \leqq \mathfrak{a}$ für jedes $\iota \in J$. Dabei ist $\mathfrak{a} \geqq \aleph_0$. Hat man für $k-1$ gezeigt, daß $\operatorname{card} L_\iota^{k-1} \leqq \mathfrak{a}$ ist, dann ist auch $\operatorname{card} L_\iota^k \leqq \operatorname{card} L_\iota^{k-1} \cdot \mathfrak{a} = \mathfrak{a}$ und $\operatorname{card} L_\iota \leqq \aleph_0 \cdot \mathfrak{a} = \mathfrak{a}$. Nach dem Auswahlaxiom können wir jeder Äquivalenzklasse eines seiner Elemente zuordnen. Bezeichnen wir die Menge dieser Elemente mit A, so erfüllt die Familie $\{L_\alpha\}_{\alpha \in A}$ die gewünschten Eigenschaften.

Als Anwendung dieses Lemmas erhalten wir den

Satz 2. *Sei $\{a_\iota\}_{\iota \in J}$ eine Basis des Raumes (E, Φ) derart, daß jedes a_ι auf höchstens abzählbar vielen a_κ nicht senkrecht steht. E ist direkte, orthogonale Summe von Teilräumen höchstens abzählbarer Dimension, die von Teilfamilien der a_ι aufgespannt werden. Insbesondere besitzt E eine orthogonale Zerlegung in endlichdimensionale Teilräume.*

Beweis. Wir definieren die symmetrische Relation R auf J durch $\iota R \kappa \Leftrightarrow \Phi(a_\iota, a_\kappa) \neq 0$. $\operatorname{card}\{\kappa \in J / \kappa R \iota\} \leqq \aleph_0$ für alle $\iota \in J$ nach Voraussetzung. Nach Lemma 4 zerfällt also J in disjunkte, höchstens abzählbare Mengen L_α, und zu dieser Zerlegung gehört eine orthogonale Zerlegung von E in höchstens abzählbardimensionale Teilräume.

Für einen anderen Beweis siehe [2], II.

Sei F ein Teilraum des euklidischen Raumes (E, Φ) und $\Phi|_{F \times F} = \Psi$. Die Topologie $\tau(\Psi)$ auf F ist gröber als die von $\tau(\Phi)$ auf F induzierte Topologie. Im allgemeinen sind sie verschieden. Addiert man zu jeder Nullumgebung U von $(E, \tau(\Phi))$ das Radikal $R = \operatorname{rad}(F^\perp)$, so erhält man wegen $(U \cap V) + R \subset (U + R) \cap (V + R)$ eine Filterbasis. Diese erzeugt auf E eine lineare Topologie, welche gröber als $\tau(\Phi)$ ist. Für die von ihr auf F induzierte Topologie χ gilt nun

Lemma 5. $\chi = \tau(\Psi)$.

Beweis. 1) χ ist feiner als $\tau(\Psi)$:

$E = \overset{\perp}{\bigoplus_{\iota \in J}} S_\iota$, wo S_ι endlichdimensionale Teilräume sind. Sei $V \subset F$ mit $\dim V = \aleph_0$. Dann gibt es eine abzählbare Teilmenge P von J, so daß $V \subset U = \bigoplus_{\iota \in P} S_\iota$. Man hat $U^\perp \subset V^\perp$, $R \subset F^\perp \subset V^\perp$, $U^\perp + R \subset V^\perp$, also $(U^\perp + R) \cap F \subset V^\perp \cap F = V^{\perp_F}$.

2) $\tau(\Psi)$ ist feiner als χ:

Sei $U_1 = \bigoplus_{\iota \in P} S_\iota$, wo P eine abzählbare Teilmenge von J ist. Wir konstruieren auf induktive Weise Folgen $\{U_i\}_{i\in\mathbb{N}}$, $\{V_i\}_{i\in\mathbb{N}}$, $\{W_i\}_{i\in\mathbb{N}}$ von Teilräumen von E.

a) p_1 sei die orthogonale Projektion von E auf U_1 und A_1 eine Teilmenge von F, so daß $p_1(A_1)$ eine Basis von $p_1(F)$ ist. Wir setzen $V_1 = k(A_1)$ und $W_1 = F \cap U_1^\perp$. Dann ist $\dim V_1 \leqq \aleph_0$ und $F = V_1 \oplus W_1$.

b) Seien U_i, V_i, W_i bereits konstruiert für $1 \leqq i < n$ mit den Eigenschaften $F = V_i \oplus W_i$, wo $W_i = F \cap U_i^\perp$, $\dim V_i \leqq \aleph_0$, $V_{i-1} \subset V_i$, $U_{i-1} \subset U_i$. Dann sei U_n der kleinste Teilraum von E, der die Summe einiger S_ι ist und der $U_1 + V_{n-1}$ umfaßt. $\dim U_n = \aleph_0$. Sei p_n die orthogonale Projektion von E auf U_n und A_n eine Teilmenge von W_{n-1}, so daß $p_n(A_n)$ eine Basis von $p_n(W_{n-1})$ ist. Wir setzen $V_n = V_{n-1} \oplus k(A_n)$ und $W_n = F \cap U_n^\perp$. Es folgt $F = V_n \oplus W_n$, $U_{n-1} \subset U_n$, $V_{n-1} \subset V_n$, $\dim V_n \leqq \aleph_0$.

Man bekommt aufsteigende Folgen $U_1 \subset U_2 \subset U_3 \subset \cdots$ und $V_1 \subset V_2 \subset V_3 \subset \cdots$, wobei $\dim U_i = \aleph_0$, $\dim V_i \leqq \aleph_0$ für alle $i \in \mathbb{N}$. Setzen wir $U = \bigcup_{i\in\mathbb{N}} U_i$, $V = \bigcup_{i\in\mathbb{N}} V_i$, so ist $\dim U = \aleph_0$ und $\dim V \leqq \aleph_0$. Da für jedes i $V_i \subset U_{i+1}$, ist $V \subset U$. Ferner ist $E = U \oplus U^\perp$. Wir zeigen nun, daß $V^{\perp_F} \subset (U_1^\perp + R) \cap F$. Sei p die orthogonale Projektion von E auf U, p' die orthogonale Projektion von E auf $U^\perp$. Zunächst beweisen wir, daß

$$p(V^\perp) \subset F^\perp . \tag{1}$$

Für ein $x \in V^\perp$ hat man $x = p(x) + p'(x)$, wo $p'(x) \in U^\perp \subset V^\perp$, also auch $p(x) \in V^\perp$. $p(x) \in U$ und damit $p(x) \in U_n$ für ein gewisses n. Zerlegt man ein beliebiges $y \in F$ bezüglich $F = V_n \oplus W_n$ in ein $y_1 \in V_n$ und ein $y_2 \in W_n$, dann ist $\Phi(y, p(x)) = \Phi(y_1, p(x)) + \Phi(y_2, p(x))$. $\Phi(y_1, p(x)) = 0$ wegen $p(x) \in V^\perp \subset V_n^\perp$. $\Phi(y_2, p(x)) = 0$ wegen $y_2 \in W_n \subset U_n^\perp$, also $\Phi(y, p(x)) = 0$, woraus folgt $p(x) \in F^\perp$. Ferner gilt

$$p(F) \subset F^{\perp\perp} . \tag{2}$$

Zum Beweis seien $v \in F$ und $w \in F^\perp$ beliebig. $\Phi(p(v), w) = \Phi(v, p(w))$. Da $w \in V^\perp$, ist nach (1) $p(w) \in F^\perp$, also $\Phi(v, p(w)) = 0$, woraus folgt $p(v) \in F^{\perp\perp}$.

Aus (1) und (2) ergibt sich $p(V^\perp \cap F) \subset R$. Wegen $p'(V^\perp \cap F) \subset U^\perp \subset U_1^\perp$ ist $V^{\perp_F} \subset U_1^\perp + R$. Q.E.D.

Bemerkung. Ist also $\operatorname{rad} F^\perp = (0)$, dann fällt auf F die Topologie $\tau(\Psi)$ mit der von $\tau(\Phi)$ induzierten zusammen.

Satz 3. *k' sei ein Erweiterungskörper von k, (E, Φ) ein halbeinfacher k-Raum und F ein $\tau(\Phi)$-vollständiger Teilraum von E. Der zu (E, Φ) gehörige k'-Raum (E', Φ') sei Teilraum eines euklidischen Raumes (H, Ψ), wobei $E'^\perp = (0)$ (orthogonales Komplement in H). Es gibt eine orthogonale Zerlegung $H = \overset{\perp}{\bigoplus}_{\kappa \in K} H_\kappa$ mit $\dim H_\kappa \leqq \aleph_0$ und zu dieser eine Zerlegung $F = \overset{\perp}{\bigoplus}_{\kappa \in K} F_\kappa$ mit $F'_\kappa = k' \otimes_k F_\kappa \subset H_\kappa$.*

Beweis. Sei $H = \overset{\perp}{\bigoplus}_{\iota \in J} S_\iota$, wo die S_ι endlichdimensionale Teilräume sind. Für jedes $x \in H$ ist $M(x) = \{\iota \in J / \Psi(x, S_\iota) \neq \{0\}\}$ eine endliche Menge. Wir betrachten den k-Raum $i(F)$. Sei $x \in i(F)$. Ist $x = \sum_{i=1}^{n} x_i$, wo $x_i \in i(F)$ und $M(x_i)$ echte Teilmengen von $M(x)$ für alle i, $1 \leqq i \leqq n$, dann nennen wir x „zerlegbar", im andern Fall „unzerlegbar". Durch Induktion nach $\operatorname{card} M(x)$ zeigt man, daß jedes $x \in i(F)$ Summe von unzerlegbaren Elementen ist. Die unzerlegbaren Elemente von $i(F)$ bilden also ein Erzeugendensystem, aus dem eine Basis $\{f'_\nu\}_{\nu \in I}$ ausgewählt werden kann. Zu jedem $\nu \in I$ gibt es ein $f_\nu \in F$ mit $f'_\nu = 1 \otimes f_\nu = i(f_\nu)$. $\{f_\nu\}_{\nu \in I}$ ist eine Basis von F. Sei $A_\mu = \{\nu \in I / \Psi(f'_\nu, S_\mu) \neq \{0\}\}$. Wir zeigen, daß $\operatorname{card} A_\mu \leqq \aleph_0$ ist für jedes $\mu \in J$.

Beweis indirekt: Angenommen es sei $\operatorname{card} A_\kappa > \aleph_0$ für ein gewisses $\kappa \in J$. Dann gibt es eine überabzählbare Teilmenge B von A_κ mit $\operatorname{card} M(f'_\nu) = n$ für ein gewisses $n \in \mathbb{N}$ und alle $\nu \in B$. M sei eine maximale Teilmenge von J, so daß $M \subset M(f'_\nu)$ für überabzählbar viele $\nu \in B$, d. h. ist $M \subset M'$ und $M \neq M'$, dann gibt es höchstens abzählbar viele $\nu \in B$ mit $M' \subset M(f'_\nu)$. Wegen $\Psi(f'_\nu, S_\kappa) \neq \{0\}$ für alle $\nu \in B$ ist $\kappa \in M$, also $1 \leqq \operatorname{card} M < n$. Sei $C = \{\nu \in B / M \subset M(f'_\nu)\}$, $U = \bigoplus_{\iota \in M} S_\iota$ und p die orthogonale Projektion von H auf U. Ferner sei $\{h_\kappa\}_{\kappa \in K}$ eine Basis von E. Setzt man $h'_\kappa = 1 \otimes h_\kappa$, so ist $\{h'_\kappa\}_{\kappa \in K}$ eine Basis von E'. Nach Voraussetzung ist $E'^\perp = (0)$, also auch $p(E')^\perp \cap U = E'^\perp \cap U = (0)$. Da $p(E') \subset U$ und U endlichdimensional ist, muß $p(E') = U$ sein. Es gibt somit eine endliche Teilmenge N von K, so daß $\{p(h'_\kappa)\}_{\kappa \in N}$ eine Basis von U ist. Setzen wir $p(h'_\kappa) = x_\kappa$ und $D = \left\{\nu \in C / [M(f'_\nu) \setminus M] \cap \left[\bigcup_{\kappa \in N} M(h'_\kappa)\right] = \emptyset\right\}$. Da $\bigcup_{\kappa \in N} M(h'_\kappa)$ endlich ist, ergibt sich aus der Konstruktion von C, daß D ebenfalls überabzählbar ist. Für jedes $\nu \in D$ und jedes $\kappa \in N$ ist $\Psi(p(f'_\nu), x_\kappa) = \Psi(p(f'_\nu), p(h'_\kappa)) = \Phi'(f'_\nu, h'_\kappa) = \Phi(f_\nu, h_\kappa) \in k$. Es ist $p(f'_\mu) = \sum_{\nu \in N} \xi_{\mu\nu} x_\nu$, $\Psi(p(f'_\mu), x_\kappa) = \sum_{\nu \in N} \xi_{\mu\nu} \Psi(x_\nu, x_\kappa) = \alpha_{\mu\kappa} \in k$. Dieses Gleichungssystem kann nach $\xi_{\mu\nu}$ aufgelöst werden, da Ψ auf U nicht ausgeartet und daher die Matrix $(\Psi(x_\nu, x_\kappa))_{\nu \in N, \kappa \in N}$ regulär ist. Dabei ist $\xi_{\mu\nu} = \sum_{\kappa \in N} \alpha_{\mu\kappa} \varepsilon_{\kappa\nu}$, wo $\varepsilon_{\kappa\nu} \in k'$. $p(f'_\mu) = \sum_{\nu \in N} \sum_{\kappa \in N} \alpha_{\mu\kappa} \varepsilon_{\kappa\nu} x_\nu = \sum_{\kappa \in N} \alpha_{\mu\kappa} \sum_{\nu \in N} \varepsilon_{\kappa\nu} x_\nu$. Setzen wir $v_\kappa = \sum_{\nu \in N} \varepsilon_{\kappa\nu} x_\nu$ und $V = k(v_\kappa)_{\kappa \in N}$, so ist $V \subset U$ und $p(f'_\mu) \in V$ für alle $\mu \in D$. Wir betrachten auf D eine Wohlordnung und definieren eine Abbildung $S : D \to \mathscr{P}(D)$ (Potenzmenge von D) durch transfinite Induktion:

1) Ist ι_0 der erste Index in D, dann bestehe $S(\iota_0)$ aus den ersten Indizes in D, für welche die orthogonalen Projektionen der zugehörigen Basisvektoren f'_ν auf U eine Basis von $k(p(f'_\nu))_{\nu \in D} \subset V$ bilden.

2) Ist $S(\nu)$ für alle $\nu < \kappa$ definiert, dann bestehe $S(\kappa)$ aus den ersten Indizes in $D_\kappa = D \setminus \left[\bigcup_{\nu < \kappa} S(\nu)\right]$, für welche die orthogonalen Projektionen der zugehörigen Basisvektoren f'_ν auf U eine Basis von $k(p(f'_\nu))_{\nu \in D_\kappa}$ bilden.

Sei $G_\nu = k(f'_\mu)_{\mu \in S(\nu)}$. Man bekommt eine absteigende Kette von k-Räumen $p(G_{\iota_0}) \supset p(G_{\iota_1}) \supset \cdots \supset p(G_\nu) \supset \cdots$, wobei immer $p(G_\kappa) \supset p(G_\nu)$ für $\kappa < \nu$. Diese Kette enthält überabzählbar viele k-Räume, welche nicht aus dem Nullelement allein bestehen. Da $\dim p(G_{\iota_0}) \leqq \dim V < \aleph_0$ ist, müssen von einem Index κ_0

an überabzählbar viele $p(G_\nu)$ zusammenfallen, welche nicht aus der Null allein bestehen. Es ist also $X=\{\nu\in D/p(G_\nu)=p(G_{\kappa_0})\}$ überabzählbar.

Sei $\nu_0\in S(\sigma)$, wo $\sigma\in X$. Dann ist $p(f'_{\nu_0})\in p(G_\mu)$ für alle $\mu\in X$. Zu jedem $\mu\in X$ gibt es ein eindeutig bestimmtes $g'_\mu\in G_\mu$, so daß $p(f'_{\nu_0})=p(g'_\mu)$ ist. G sei die Menge aller dieser g'_μ. Die Komplemente aller höchstens abzählbaren Teilmengen von G in G erzeugen einen Filter $\mathscr{F}'$ in H. Wir zeigen, daß $\mathscr{F}'$ gegen $p(f'_{\nu_0})$ konvergiert. Sei L eine abzählbare Teilmenge von J. Für $\iota\in M\cap L$ und $x\in S_\iota$ ist $\Psi(x,g'_\mu)=\Psi(x,p(g'_\mu))=\Psi(x,p(f'_{\nu_0}))$ für jedes $\mu\in X$, d.h. $\Psi(S_\iota,F-p(f'_{\nu_0}))=\{0\}$ für jedes $F\in\mathscr{F}'$. Für $\iota\in L\backslash M$ ist $\Psi(S_\iota,f'_\kappa)=\{0\}$ für alle $\kappa\in C$, ausgenommen höchstens abzählbar viele. Da die $S(\kappa)$ alle disjunkt sind, gibt es daher eine höchstens abzählbare Teilmenge P_ι von X, so daß $\Psi(S_\iota,g'_\mu)=\{0\}$ für alle $\mu\in X\backslash P_\iota$. $P=\bigcup_{\iota\in L\backslash M}P_\iota$ ist ebenfalls höchstens abzählbar. Also ist $Y=\{g'_\kappa\}_{\kappa\in X\backslash P}\in\mathscr{F}'$ und $\Psi(S_\iota,Y-p(f'_{\nu_0}))=\{0\}$ für alle $\iota\in L$, da $\Psi(S_\iota,p(f'_{\nu_0}))=\{0\}$, wenn $\iota\notin M$. Es läßt sich also zu jedem abzählbaren $L\subset J$ ein $Y\in\mathscr{F}'$ finden, für das $Y-p(f'_{\nu_0})\subset\left[\bigoplus_{\iota\in L}S_\iota\right]^\perp$, d.h. $\mathscr{F}'$ konvergiert gegen $p(f'_{\nu_0})$.

Da auf E' $\tau(\Phi')$ gröber ist als $\tau(\Psi)$, ist $\mathscr{F}'$ auch Cauchyfilter auf E' bezüglich $\tau(\Phi')$. Nun existiert zu jedem $\kappa\in X$ $i^{-1}(g'_\kappa)=g_\kappa\in F$ und damit $i^{-1}(\mathscr{F}')=\mathscr{F}$. $\mathscr{F}$ ist nach Lemma 2 Cauchyfilter in F bezüglich $\tau(\Phi)$. Wegen der Vollständigkeit von F bezüglich $\tau(\Phi)$ konvergiert also $\mathscr{F}$ gegen ein Element $f\in F$. $i(\mathscr{F})=\mathscr{F}'$, also $\lim_{\tau(\Phi')}\mathscr{F}'=i(f)=f'$. Da $E'^\perp=(0)$, ist $\operatorname{rad}E'^\perp=(0)$. Die Topologien $\tau(\Psi)$ und $\tau(\Phi')$ fallen somit auf E' zusammen. Dann ist $f'=\lim_{\tau(\Psi)}\mathscr{F}'=p(f'_{\nu_0})\in i(F)$. $M(p(f'_{\nu_0}))=M$ ist aber eine nicht leere, echte Teilmenge von $M(f'_{\nu_0})$, woraus folgt, daß f'_{ν_0} zerlegbar ist, entgegen unserer Voraussetzung. Also ist $\operatorname{card}A_\mu\leqq\aleph_0$ für alle $\mu\in J$.

Wir definieren eine symmetrische Relation R auf J durch $\iota R\kappa\Leftrightarrow$ es gibt ein $\nu\in I$, so daß $\Psi(f'_\nu,S_\iota)\neq\{0\}$ und $\Psi(f'_\nu,S_\kappa)\neq\{0\}$. Aus $\operatorname{card}A_\iota\leqq\aleph_0$ folgt $\operatorname{card}\{\kappa\in J/\Psi(f'_\nu,S_\kappa)\neq\{0\}$ für ein $\nu\in A_\iota\}\leqq\aleph_0$, d.h. $\operatorname{card}\{\kappa\in J/\kappa R\iota\}\leqq\aleph_0$ für alle $\iota\in J$. Nach Lemma 4 zerfällt J in disjunkte, höchstens abzählbare Mengen. Zu dieser Zerlegung gehört eine orthogonale Zerlegung von H in Teilräume H_κ von höchstens abzählbarer Dimension: $H=\overset{\perp}{\bigoplus_{\kappa\in K}}H_\kappa$. Jedes f'_ν liegt in einem der Teilräume H_κ. Für jedes $\kappa\in K$ sei $L_\kappa=\{\nu\in I/f'_\nu\in H_\kappa\}$ und $F_\kappa=k(f_\nu)_{\nu\in L_\kappa}$. Dann ist $F=\overset{\perp}{\bigoplus_{\kappa\in K}}F_\kappa$, $i(F_\kappa)\subset H_\kappa$, also auch $F'_\kappa=k'\otimes_k F_\kappa\subset H_\kappa$. Q.E.D.

Zusatz. *Man kann die H_κ in der Zerlegung von H speziell so wählen, daß jedes H_κ Summe einiger S_ι ist, wenn $H=\overset{\perp}{\bigoplus_{\iota\in J}}S_\iota$ eine vorgelegte Zerlegung von H in endlichdimensionale Teilräume ist.*

Ist speziell (E',Φ') euklidisch und somit $\tau(\Phi')$-vollständig, dann ist E nach Lemma 3 $\tau(\Phi)$-vollständig. Setzt man in diesem Fall in Satz 3 $F=E$ und $E'=H$, so folgt

Korollar 1. *Sei k' ein Erweiterungskörper von k. Der k-Raum (E,Φ) ist euklidisch genau dann, wenn der zugehörige k'-Raum $E'=k'\otimes_k E$ euklidisch ist.*

Bemerkung: Nach Korollar 1 folgt also aus $d(k'\otimes_k E)=0$ immer, daß $d(E)=0$ ist. In scharfem Kontrast dazu steht die folgende Tatsache: Es gibt zu einer beliebig vorgeschriebenen Kardinalzahl $\mathfrak{a}$ Beispiele von Räumen (E, Φ) und Körpererweiterungen $k\subset k'$ derart, daß $d(E)=\mathfrak{a}$ und $d(k'\otimes_k E)=1$ ist ([2], p. 126).

Korollar 2. *Ist (E, Φ) ein euklidischer Raum und F ein $\tau(\Phi)$-abgeschlossener Teilraum von E, dann ist F orthogonale Summe endlichdimensionaler Teilräume.*

Beweis. Da ein abgeschlossener Teilraum eines vollständigen Raumes vollständig ist, ergibt sich das Korollar 2 aus Satz 3, indem man $k=k'$ und $E'=H$ setzt.

Korollar 3. *Ein orthogonal abgeschlossener Teilraum F eines euklidischen Raumes (E, Φ) ist orthogonale Summe endlichdimensionaler Teilräume.*

Beweis. Ist F orthogonal abgeschlossen, dann ist F nach I.2 abgeschlossen bezüglich der schwachen Topologie $\sigma(\Phi)$ und damit auch abgeschlossen bezüglich $\tau(\Phi)$. Dann folgt die Behauptung aus Korollar 2.

Eine Erweiterung des Satzes 2 in I.2 auf euklidische Räume von beliebiger Dimension ist das

Korollar 4. *Sei (E, Φ) euklidisch bezüglich der spurwertigen ε-hermiteschen Form Φ und F ein totalisotroper, orthogonal abgeschlossener Teilraum von E. Es gibt zu F eine Wittsche Zerlegung $E=(F\oplus F')\overset{\perp}{\bigoplus} G$.*

Entsprechende Behauptung für präeuklidische Räume ist falsch ([2], VII.5).

Beweis. Nach dem Vorangehenden ist F als orthogonal abgeschlossener Teilraum $\tau(\Phi)$-vollständig. Also gibt es nach Satz 3 eine Zerlegung $E=\overset{\perp}{\bigoplus\limits_{\kappa\in K}} E_\kappa$ mit $\dim E_\kappa\leqq\aleph_0$ und zu dieser eine Zerlegung $F=\overset{\perp}{\bigoplus\limits_{\kappa\in K}} F_\kappa$ mit $F_\kappa\subset E_\kappa$. Da F orthogonal abgeschlossen, ist auch jedes F_κ orthogonal abgeschlossen in E_κ. Jedes F_κ ist zudem totalisotrop. Nach I.2 gibt es eine Wittsche Zerlegung $E_\kappa=(F_\kappa\oplus F'_\kappa)\overset{\perp}{\bigoplus} G_\kappa$ für jedes $\kappa\in K$. Setzt man $F'=\overset{\perp}{\bigoplus\limits_{\kappa\in K}} F'_\kappa$, $G=\overset{\perp}{\bigoplus\limits_{\kappa\in K}} G_\kappa$, so ist $E=(F\oplus F')\overset{\perp}{\bigoplus} G$ eine Wittsche Zerlegung von E.

Bemerkung. Ist Φ eine beliebige reflexive Sesquilinearform, E und F bezüglich Φ wie in Korollar 4, dann gilt etwas schwächer:

Es gibt zu F eine Zerlegung $E=(F\oplus F')\overset{\perp}{\bigoplus} G$ und eine Basis $\{f_\iota\}_{\iota\in I}$ von F sowie eine Basis $\{f'_\iota\}_{\iota\in I}$ von F' mit $\Phi(f_\iota, f'_\kappa)=\delta_{\iota\kappa}$ und $\Phi(f'_\nu, f'_\mu)=0$ für alle $\nu\neq\mu$, wo $\nu, \mu\in I$.

Wie beim Beweis von Korollar 4 kann der überabzählbare Fall auf den abzählbaren zurückgeführt werden.

Satz 4. *Sei (E, Φ) ein euklidischer Raum, $E=\overset{\perp}{\bigoplus\limits_{\iota\in J}} S_\iota$ eine orthogonale Zerlegung in endlichdimensionale Teilräume. Ein Teilraum F von E ist genau dann*

$\tau(\Phi)$-abgeschlossen, wenn F eine Basis $\{f_\nu\}_{\nu\in I}$ besitzt, so daß für jedes $\iota\in J$ $\Phi(f_\nu, S_\iota)\neq\{0\}$ für höchstens abzählbar viele $\nu\in I$.

Beweis. 1) F sei $\tau(\Phi)$-abgeschlossen. Nach Satz 3 gibt es Zerlegungen $E=\overset{\perp}{\bigoplus\limits_{\kappa\in L}} E_\kappa$ und $F=\overset{\perp}{\bigoplus\limits_{\kappa\in L}} F_\kappa$ mit $F_\kappa\subset E_\kappa$ und $\dim E_\kappa\leqq\aleph_0$, wobei jedes E_κ Summe einiger S_ι ist. Setzt man beliebige Basen der F_κ zu einer Basis von F zusammen, so hat diese die gewünschte Eigenschaft.

2) F besitze eine Basis $\{f_\nu\}_{\nu\in I}$, so daß für jedes $\iota\in J$ $\Phi(f_\nu, S_\iota)\neq\{0\}$ für höchstens abzählbar viele $\nu\in I$. Dann gibt es Zerlegungen $E=\overset{\perp}{\bigoplus\limits_{\kappa\in L}} E_\kappa$ und $F=\overset{\perp}{\bigoplus\limits_{\kappa\in L}} F_\kappa$ wie in Satz 3 (siehe Überlegung am Schluß des Beweises von Satz 3). Sei $a\in\bar{F}$ beliebig. Die Menge $M=\{\kappa\in L/\Phi(a, E_\kappa)\neq\{0\}\}$ ist endlich, die Dimension von $U=\bigoplus\limits_{\kappa\in M} E_\kappa$ also höchstens abzählbar. Zudem ist $a\in U$. Es gibt ein $b\in F$ mit $b\in a+U^\perp$. $b=a+c$, wo $c\in U^\perp$. Zerlegen wir b in ein $b_1\in\bigoplus\limits_{\kappa\in M} F_\kappa\subset U$ und ein $b_2\in\bigoplus\limits_{\kappa\in L\setminus M} F_\kappa\subset U^\perp$, dann ist $a-b_1=b_2-c$, wobei $a-b_1\in U, b_2-c\in U^\perp$, also $a-b_1\in U\cap U^\perp=(0)$, d. h. $a=b_1\in F$. Da a in $\bar{F}$ beliebig war, ist $\bar{F}=F$.

Korollar. *Ist $\{F_i\}_{i\in S}$ eine höchstens abzählbare Familie von linearen Unterräumen des euklidischen Raumes (E,Φ), so gilt*

$$\overline{\sum_{i\in S} F_i}=\sum_{i\in S}\bar{F}_i\,.$$

Jeder höchstens abzählbardimensionale Teilraum von E ist $\tau(\Phi)$-abgeschlossen.

Beweis. $\bar{F}_i\subset\overline{\sum_{i\in S} F_i}$, also $\sum_{i\in S}\bar{F}_i\subset\overline{\sum_{i\in S} F_i}$. Andererseits ist $\sum_{i\in S} F_i\subset\sum_{i\in S}\bar{F}_i$ und damit $\overline{\sum_{i\in S} F_i}\subset\overline{\sum_{i\in S}\bar{F}_i}$. Ist $E=\overset{\perp}{\bigoplus\limits_{\iota\in J}} S_\iota$ eine orthogonale Zerlegung von E in endlichdimensionale Teilräume, so besitzen die $\tau(\Phi)$-abgeschlossenen Teilräume $\bar{F}_i$ bezüglich dieser Zerlegung eine Basis mit der im Satz angegebenen Eigenschaft. Diese Basen bilden zusammen ein Erzeugendensystem von $\sum_{i\in S}\bar{F}_i$, in dem ein maximales linear unabhängiges Teilsystem als Basis ausgewählt werden kann. Das Erzeugendensystem und damit auch die Basis besitzt wiederum dieselbe Eigenschaft, da die Summe höchstens abzählbar ist. Dann folgt nach dem Satz $\overline{\sum_{i\in S}\bar{F}_i}=\sum_{i\in S}\bar{F}_i$ und daraus $\overline{\sum_{i\in S} F_i}=\sum_{i\in S}\bar{F}_i$.

Satz 5. *Ein Unterraum F eines euklidischen Raumes (E,Φ) ist genau dann orthogonale Summe endlichdimensionaler Teilräume, wenn $\bar{F}\subset F+\operatorname{rad}F^\perp$.*

Beweis. 1) Sei $\bar{F}\subset F+\operatorname{rad}F^\perp$ und G ein lineares Komplement von F in $F+\operatorname{rad}F^\perp$ mit $G\subset\operatorname{rad}F^\perp$. Nach Korollar 2 zu Satz 3 ist $\bar{F}=\overset{\perp}{\bigoplus\limits_{\iota\in I}} S_\iota$, wo S_ι

endlichdimensionale Teilräume sind. Wir betrachten bezüglich der Zerlegung $F \oplus G$ von $F + \operatorname{rad} F^{\perp}$ die Projektion p von $\bar{F}$ auf F. Dann ist $F = p(\bar{F}) = \sum_{\iota \in I} p(S_\iota)$. Seien $x, y \in \bar{F}$, $x = p(x) + x'$, $y = p(y) + y'$, wo $x', y' \in G$. $\Phi(x, y) = \Phi(p(x), p(y))$, da $G \subset \operatorname{rad} F^{\perp}$. Also ist $\sum_{\iota \in I} p(S_\iota)$ eine orthogonale Summe.

2) Sei nun F orthogonale Summe endlichdimensionaler Teilräume. Ferner sei H ein lineares Komplement von F in $\bar{F}$, $\{h_\nu\}_{\nu \in S}$ eine Basis von H und $\Phi|_{F \times F} = \Psi$. Zu jedem $\nu \in S$ gibt es bezüglich $\tau(\Phi)$ einen Cauchyfilter $\mathscr{H}_\nu$ in F mit $h_\nu = \lim_{\tau(\Phi)} \mathscr{H}_\nu$. $\mathscr{H}_\nu$ ist aber auch Cauchyfilter in F bezüglich $\tau(\Psi)$. Da F orthogonale Summe endlichdimensionaler Teilräume ist, konvergiert $\mathscr{H}_\nu$ nach Satz 1 bezüglich $\tau(\Psi)$ gegen ein Element $f_\nu \in F$. Sei $g_\nu = h_\nu - f_\nu$ und $G = k(g_\nu)_{\nu \in S}$. Dann ist $\bar{F} = F \oplus G$. Für ein beliebiges $x \in F$ ist $\Phi(x, h_\nu) = \lim_{C \in \mathscr{H}_\nu} \Phi(x, C)$ $= \lim_{C \in \mathscr{H}_\nu} \Psi(x, C)$. $\Phi(x, f_\nu) = \Psi(x, f_\nu) = \lim_{C \in \mathscr{H}_\nu} \Psi(x, C)$, da $x \in F$. Also ist $\Phi(x, g_\nu) = 0$ für jedes $\nu \in S$, d. h. $G \subset F^{\perp}$. Da $F^{\perp\perp}$ schwach abgeschlossen und damit $\tau(\Phi)$-abgeschlossen ist, ist $G \subset \bar{F} \subset F^{\perp\perp}$, also $G \subset \operatorname{rad} F^{\perp}$. Q.E.D.

Korollar 1. *Ein Teilraum F eines anisotropen euklidischen Raumes (E, Φ) ist genau dann euklidisch, wenn er $\tau(\Phi)$-abgeschlossen ist.*

Korollar 2. *(E, Φ) sei euklidisch und anisotrop. Dann besitzt E keine Unterräume F mit $1 \leqq d(F) \leqq \aleph_0$.*

Beweis. Sei $F = G \oplus H$, wo G euklidisch und $\dim H = d(F)$. $\bar{F} = \bar{G} + \bar{H}$ $= G + \bar{H}$ nach Korollar 1 und dem Korollar zu Satz 4. Ist $d(F) \neq 0$, dann ist $\bar{F} \neq F$ und damit $\bar{H} \neq H$, also $\dim H > \aleph_0$. Siehe auch [2], III. Satz 5.

Korollar 3. *(E, Φ) sei euklidisch; F sei ein Teilraum mit* $\dim [E/(F + F^{\perp})] \leqq \aleph_0$ *und F besitze eine orthogonale Zerlegung in endlichdimensionale Teilräume. Jeder Teilraum von E, der $F + F^{\perp}$ umfaßt, besitzt eine orthogonale Zerlegung in endlichdimensionale Teilräume.*

Beweis. Sei $G = (F + F^{\perp}) \oplus H$. Da $\dim H \leqq \aleph_0$ und $F^{\perp}$ schwach abgeschlossen und damit auch $\tau(\Phi)$-abgeschlossen ist, hat man $\bar{G} = \bar{F} + \overline{F^{\perp}} + \bar{H} \subset F + \operatorname{rad} F^{\perp}$ $+ F^{\perp} + H = G$, also $\bar{G} = G$. Korollar 3 folgt dann aus Korollar 2 zu Satz 3. (Siehe auch [2], Korollar 1 zu Satz 11.)

Korollar 4. *(L, Ψ) sei ein euklidischer Raum und E ein $\tau(\Psi)$-dichter Teilraum von L mit $d(E) \leqq \aleph_0$. Ist F ein euklidischer Teilraum von E mit* $\dim E/F = d(E)$ *und $F^{\perp} \cap E = (0)$, dann ist $F^{\perp}$ ein totalisotropes lineares Komplement von E in L.*

Beweis. Sei H ein lineares Komplement von F in E. Da $\dim H = d(E) \leqq \aleph_0$, ist $L = \bar{E} = \bar{F} + \bar{H} \subset F + \operatorname{rad} F^{\perp} + H$. $(\operatorname{rad} F^{\perp}) \cap E \subset F^{\perp} \cap E = (0)$, also $L = E \oplus \operatorname{rad} F^{\perp} = E \oplus F^{\perp}$ und $F^{\perp} = \operatorname{rad} F^{\perp}$. (Vgl. auch Korollar 4 zu Satz 11 in [2].)

III.2. Die euklidische Hülle

Satz 6. *Zu jedem präeuklidischen Raum (E, Φ) gibt es einen bis auf Isometrie eindeutig bestimmten, kleinsten euklidischen Raum, der E umfaßt: die euklidische*

Hülle. Sie ist gleich der vollständigen Hülle $\tilde{E}$ bezüglich der Topologie $\tau(\Phi)$, versehen mit der in natürlicher Weise existierenden Fortsetzung $\tilde{\Phi}$ der Sesquilinearform Φ auf $\tilde{E} \times \tilde{E}$. Dabei ist die Topologie $\widetilde{\tau(\Phi)}$ auf $\tilde{E}$ gleich $\tau(\tilde{\Phi})$.

Beweis. Sei (H, Ψ) ein euklidischer Oberraum von (E, Φ). Mit $\perp$ bezeichnen wir das orthogonale Komplement in H. Sei ferner $R = \operatorname{rad} E^{\perp}$. Da $R^{\perp\perp} = R$, gibt es nach der Bemerkung zu Korollar 4 von Satz 3 eine Zerlegung $H = (R \oplus R') \overset{\perp}{\bigoplus} S$ mit $R^{\perp} \cap R' = (0)$. Für ein beliebiges $x \in E$ bilden wir die Zerlegung $x = r + r' + s$ mit $r \in R$, $r' \in R'$, $s \in S$. Wegen $R \subset E^{\perp}$ ist für jedes $u \in R$ $\Psi(u, x) = \Psi(u, r') = 0$, also $r' = 0$, d. h. $E \subset R \overset{\perp}{\bigoplus} S$. $\{e_\nu\}_{\nu \in I}$ sei eine Basis von E. $e_\nu = r_\nu + s_\nu$, wo $r_\nu \in R$, $s_\nu \in S$. Nun ist $E \cap R = E \cap E^{\perp} \cap E^{\perp\perp} = \operatorname{rad} E = (0)$, da E halbeinfach ist. Daraus folgt, daß $\{s_\nu\}_{\nu \in I}$ linear unabhängig ist. Sei $E_0 = k(s_\nu)_{\nu \in I}$. $\Psi(e_\nu, e_\mu) = \Psi(s_\nu, s_\mu)$ für alle $\nu, \mu \in I$, d. h. E und E_0 sind isometrisch.

Wir zeigen nun, daß $\operatorname{rad} E_0^{\perp} = (0)$ ist. Es ist $E_0 \subset S$, $E_0^{\perp\perp} \subset S$, also $\operatorname{rad} E_0^{\perp} \subset S$. Für $x \in \operatorname{rad} E_0^{\perp}$ ist $\Psi(x, e_\nu) = \Psi(x, r_\nu) + \Psi(x, s_\nu) = 0$ für alle $\nu \in I$, also $x \in E^{\perp}$ und $\operatorname{rad} E_0^{\perp} \subset E^{\perp}$. $E_0 \subset E + R \subset E^{\perp\perp}$, $\operatorname{rad} E_0^{\perp} \subset E_0^{\perp\perp} \subset E^{\perp\perp}$. Daraus folgt, daß $\operatorname{rad} E_0^{\perp} \subset R \cap S = (0)$.

Ohne Beschränkung der Allgemeinheit können wir also annehmen, daß $\operatorname{rad} E^{\perp} = (0)$ ist. $E^{\perp\perp}$ ist $\tau(\Psi)$-abgeschlossen und daher $\bar{E} \subset E^{\perp\perp}$, also $\bar{E}^{\perp\perp} = E^{\perp\perp}$, $\bar{E}^{\perp} = E^{\perp}$, $E^{\perp} \cap \bar{E} = \operatorname{rad} \bar{E} \subset \operatorname{rad} \bar{E}^{\perp} = \operatorname{rad} E^{\perp} = (0)$. Daraus und aus Korollar 2 zu Satz 3 ersieht man zunächst, daß $\bar{E}$ euklidisch ist. Sei $\Psi|_{\bar{E} \times \bar{E}} = \bar{\Phi}$. Wegen $E^{\perp} \cap \bar{E} = (0)$ und Lemma 5 fallen die Topologien $\tau(\Psi)$, $\tau(\bar{\Phi})$ und $\tau(\Phi)$ auf E alle zusammen. Ferner ist E dicht in $\bar{E}$ bezüglich $\tau(\bar{\Phi})$, $(\bar{E}, \tau(\bar{\Phi}))$ somit eine „Realisierung" der $\tau(\Phi)$-vollständigen Hülle von E in H. Die Sesquilinearform $\bar{\Phi}$ auf $\bar{E}$ ist nach Lemma 1 in eindeutiger Weise durch die Sesquilinearform Φ auf E bestimmt. Da H ein beliebiger euklidischer Oberraum war, ist $\bar{E}$ der kleinste euklidische Oberraum von E.

Bemerkung. Es soll schon noch hervorgehoben werden, daß bezüglich $\tau(\Phi)$ eine getrennt stetige Erweiterung $\tilde{\Phi}$ von Φ auf $\tilde{E}$ überhaupt existiert. Bei der schwachen Topologie $\sigma(\Phi)$ ist das nicht der Fall. (Sonst würde $(\tilde{E}, \tilde{\Phi})$ linear kompakt sein, also von endlicher Dimension.)

Korollar. *Ein präeuklidischer Raum (E, Φ) ist genau dann euklidisch, wenn er $\tau(\Phi)$-vollständig ist.*

Satz 7. *Sei k' ein Erweiterungskörper von k und (E, Φ) ein halbeinfacher k-Raum. (E, Φ) ist genau dann präeuklidisch, wenn der zugehörige k'-Raum $E' = k' \otimes_k E$ präeuklidisch ist.*

Beweis. Ist E präeuklidisch, dann ist es auch E'. Sei umgekehrt E' präeuklidisch. Dann läßt sich die Sesquilinearform von E in natürlicher Weise auf die $\tau(\Phi)$-vollständige Hülle $\tilde{E}$ von E fortsetzen. Denn sind $\mathscr{F}$ und $\mathscr{G}$ zwei Cauchyfilter in E und $\mathscr{F}' = i(\mathscr{F})$, $\mathscr{G}' = i(\mathscr{G})$ ihre Bilder in E', so existiert $\lim_{F \in \mathscr{F}} \lim_{G \in \mathscr{G}} \Phi(F, G) = \lim_{F' \in \mathscr{F}'} \lim_{G' \in \mathscr{G}'} \Phi'(F', G')$, da E' präeuklidisch ist. Die $\tau(\Phi')$-

vollständige Hülle $\widetilde{E'}$ von E' ist euklidisch nach Satz 6 und man hat $E' \subset (\tilde{E})' = k' \otimes_k \tilde{E} \subset \widetilde{E'}$. Sei $\tilde{\Phi}$ die Sesquilinearform auf $\tilde{E}$, $\tilde{\Phi}'$ diejenige auf $(\tilde{E})'$ und $\widetilde{\Phi'}$ diejenige auf $\widetilde{E'}$. Wir zeigen zunächst, daß die durch die Vervollständigung auf $\tilde{E}$ gegebene Topologie gleich $\tau(\tilde{\Phi})$ ist. Betrachten wir zu diesem Zweck das folgende Diagramm:

$$\begin{array}{ccc} (E, \tau(\Phi)) & \xrightarrow{i} & (E', \tau(\Phi')) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j'} \\ (\tilde{E}, \tau(\tilde{\Phi})) & \xrightarrow{\tilde{i}} & ((\tilde{E})', \tau(\tilde{\Phi}')) \end{array}$$

j und j' seien die natürlichen Injektionen, i und $\tilde{i}$ diejenigen Abbildungen, welche jedem x das Element $1 \otimes x$ zuordnen. i und $\tilde{i}$ sind nach Lemma 2 stetig. Aus $(\tilde{E})'^{\perp} \subset E'^{\perp} = (0)$ folgt nach Lemma 5, daß j' stetig ist. Da $\tau(\tilde{\Phi})$ die von $\tilde{i}$ induzierte Topologie ist, ist wegen der Stetigkeit von $j' \circ i = \tilde{i} \circ j$ auch j stetig. Die von $\tau(\tilde{\Phi})$ auf E induzierte Topologie ist deshalb einerseits gröber als $\tau(\Phi)$; andererseits ist sie aber auch feiner als $\tau(\Phi)$. Also fallen sie zusammen.

Sei nun $x \in \tilde{E}$ und U ein abzählbardimensionaler Teilraum von $\tilde{E}$. Es gibt einen Cauchyfilter $\mathscr{F}$ in $(E, \tau(\Phi))$, der gegen x konvergiert. $\mathscr{F}$ ist nach dem Vorhergehenden auch Cauchyfilter bezüglich $\tau(\tilde{\Phi})$. Es existiert also ein $F \in \mathscr{F}$ mit $F - F \subset U^{\perp}$. Sei $u \in U$ und $\mathscr{U}$ ein Cauchyfilter in $(E, \tau(\Phi))$, der gegen u konvergiert. Dann ist $\lim_{V \in \mathscr{U}} \Phi(F - F, V) = \tilde{\Phi}(F - F, u) = \{0\}$ und daher $\tilde{\Phi}(F - x, u) = \lim_{X \in \mathscr{F}} \lim_{V \in \mathscr{U}} \Phi(F - X, V) = \{0\}$, woraus folgt, daß $F - x \subset U^{\perp}$, $F \subset x + U^{\perp}$. $\mathscr{F}$ konvergiert also auch gegen x bezüglich $\tau(\tilde{\Phi})$. Somit ist E dicht in $\tilde{E}$ bezüglich $\tau(\tilde{\Phi})$, und $\tau(\tilde{\Phi})$ ist gleich der durch die Vervollständigung auf $\tilde{E}$ gegebenen Topologie. Insbesondere ist $\tilde{E}$ $\tau(\tilde{\Phi})$-vollständig. Da $(\tilde{E})'$ und damit auch $\tilde{E}$ halbeinfach ist, folgt dann aus Satz 3, daß $\tilde{E}$ euklidisch ist. Also ist E prä-euklidisch.

Man sieht ferner, daß $(\tilde{E})' = \widetilde{E'}$ sein muß, da $(\tilde{E})'$ ebenfalls euklidisch ist und nach Satz 6 $\widetilde{E'}$ der kleinste euklidische Raum ist, der E' umfaßt.

Satz 8. (E, Φ) *sei die äußere Summe einer Familie* $\{(E_\iota, \Phi_\iota)\}_{\iota \in J}$ *von prä-euklidischen Räumen. Die euklidische Hülle* $\tilde{E}$ *von* E *ist gleich der äußeren Summe der euklidischen Hüllen* $\tilde{E}_\iota$ *von* E_ι.

Beweis. Seien $\iota, \kappa \in J$, $\iota \neq \kappa$, $\mathscr{F}$ ein Cauchyfilter in E_ι bezüglich $\tau(\Phi_\iota)$ und $\mathscr{G}$ ein Cauchyfilter in E_κ bezüglich $\tau(\Phi_\kappa)$. $\mathscr{F}$ und $\mathscr{G}$ sind dann auch Cauchyfilter in E bezüglich $\tau(\Phi)$. Ist $f = \lim \mathscr{F} \in \tilde{E}$, $g = \lim \mathscr{G} \in \tilde{E}$ und $\tilde{\Phi}$ die Sesquilinearform auf $\tilde{E}$, dann ist $\tilde{\Phi}(f, g) = \lim_{C \in \mathscr{F}} \lim_{D \in \mathscr{G}} \Phi(C, D) = 0$, also $\overset{\perp}{\bigoplus_{\iota \in J}} \tilde{E}_\iota \subset \tilde{E}$. Da $\overset{\perp}{\bigoplus_{\iota \in J}} \tilde{E}_\iota$ euklidisch und $\tilde{E}$ nach Satz 6 der kleinste euklidische Oberraum von E ist, muß $\overset{\perp}{\bigoplus_{\iota \in J}} \tilde{E}_\iota = \tilde{E}$ sein.

Satz 9. *Sei (E, Φ) ein präeuklidischer Raum, $\tilde{E}$ seine euklidische Hülle. Dann gilt* $\dim \tilde{E}/E \leqq d(E)$. *Ist* $d(E) \leqq \aleph_0$, *dann gilt sogar das Gleichheitszeichen.*

Einen weiteren Fall, für den das Gleichheitszeichen gilt, liefert der Satz 13 in [2].

Beweis. Sei F ein beliebiger euklidischer Teilraum von E, $\tilde{\Phi}$ die Sesquilinearform auf $\tilde{E}$ und H ein lineares Komplement von F in E. Es ist zu zeigen, daß $\dim \tilde{E}/E \leqq \dim E/F = \dim H$ ist. Angenommen es sei $\dim \tilde{E}/E > \dim H$. Da $\tilde{E} = \overline{F+H} = \bar{F} + \bar{H}$, gilt also $\dim H < \dim [(\bar{F} + \bar{H})/(F \oplus H)] \leqq \dim \bar{F}/F + \dim \bar{H}/H$, wie man leicht einsieht. Ist $\dim H \leqq \aleph_0$, dann ist $\bar{H} = H$ und damit $\dim H < \dim \bar{F}/F$. Ist $\dim H > \aleph_0$, so hat man wegen $\bar{H} \subset H^{\perp\perp}$ und $\dim H^{\perp\perp} = \dim H$ (I.2, Satz 4) $\dim \bar{H}/H \leqq \dim H$. Mit $\perp$ haben wir dabei das orthogonale Komplement in $\tilde{E}$ bezeichnet. Dann gilt $\dim H < \dim \bar{F}/F + \dim H$, also ebenfalls $\dim H < \dim \bar{F}/F$. Nach Satz 5 gibt es ein lineares Komplement G von F in $\bar{F}$ mit $G \subset \operatorname{rad} F^{\perp} \subset F^{\perp}$. Es ist $G \cap H^{\perp} = G \cap F^{\perp} \cap H^{\perp} = G \cap E^{\perp} = (0)$ und damit $\dim G = \dim [G/(G \cap H^{\perp})] = \dim [(G + H^{\perp})/H^{\perp}] \leqq \dim \tilde{E}/H^{\perp} = \dim H$ nach I.2, Satz 4. Da andererseits $\dim H < \dim G$ ist, hat man einen Widerspruch. Die zweite Aussage folgt dann aus Satz 3 in I.2.

Korollar 1. *Sei (E, Φ) die äußere Summe einer Familie $\{(E_\iota, \Phi_\iota)\}_{\iota \in J}$ von präeuklidischen Räumen E_ι mit $d(E_\iota) \leqq \aleph_0$ für alle $\iota \in J$. Dann ist $d(E) = \sum_{\iota \in J} d(E_\iota)$.*

Beweis. Trivialerweise ist $d(E) \leqq \sum_{\iota \in J} d(E_\iota)$. $\tilde{E}$ sei die euklidische Hülle von E, $\tilde{E}_\iota$ diejenige von E_ι. Nun ist nach Satz 8 $\tilde{E} = \overset{\perp}{\bigoplus_{\iota \in J}} \tilde{E}_\iota$ und damit $\dim \tilde{E}/E = \sum_{\iota \in J} \dim \tilde{E}_\iota/E_\iota$. Da wegen $d(E_\iota) \leqq \aleph_0$ $\dim \tilde{E}_\iota/E_\iota = d(E_\iota)$ ist, gilt $\sum_{\iota \in J} d(E_\iota) = \dim \tilde{E}/E \leqq d(E)$.

Bemerkung zu Korollar 1: Die Voraussetzung $d(E_\iota) \leqq \aleph_0$ ist wesentlich, wie das folgende Beispiel zeigt: Sei L aufgespannt von der überabzählbaren, orthonormierten Basis $\{e_\iota\}_{\iota \in I}$. Der Grundkörper k sei so, daß L anisotrop ist. Für die Hyperebene $E_1 = k(e_\iota - e_0)_{\iota \in I}$ ist $d(E_1) > \aleph_0$ (Satz III.1, [2] und Korollar 2 zu Satz 5). E_2 sei eine hyperbolische Ebene. Es gibt ein $a \in E_2$ mit $\Phi_2(a, a) = -1$. $F = k(a + e_0 - e_\iota)_{\iota \in I}$ ist ein euklidischer Teilraum von $E_1 \overset{\perp}{\oplus} E_2$ mit $\dim [(E_1 \overset{\perp}{\oplus} E_2)/F] = 2$. Also ist $d(E_1 \overset{\perp}{\oplus} E_2) \leqq 2 < d(E_1) + d(E_2)$.

$(\tilde{E})' = \tilde{E'}$ für präeuklidische Räume E liefert das

Korollar 2. *Ist E präeuklidisch, dann ist $d(E') \geqq \dim \tilde{E}/E$ für jede Körpererweiterung $k' \supset k$. Ist insbesondere $d(E) = \dim \tilde{E}/E$, dann ist $d(E) = d(E')$.*

Beweis. $d(E') \geqq \dim \tilde{E'}/E' = \dim (\tilde{E})'/E' = \dim \tilde{E}/E$.

Da insbesondere immer $d(E) = \dim \tilde{E}/E$ für $d(E) \leqq \aleph_0$ ist, hat man das

Korollar 3. *Aus $d(E) = \aleph_0$ folgt $d(E') = \aleph_0$ für beliebige (E, Φ) (präeuklidisch oder nicht).*

Beweis. Es ist $d(E') \leqq \aleph_0$. Wäre $d(E') < \aleph_0$, dann wäre E' präeuklidisch (Satz 10), also auch E, also nach Korollar 2 $d(E') = d(E) = \aleph_0$. Widerspruch.

III.3. Ein Einbettungssatz

Satz 10. (E, Φ) *sei ein halbeinfacher Raum mit* $d(E) \leqq \aleph_0$. *Folgende Aussagen sind äquivalent:*

(i) *Es gibt einen euklidischen Teilraum F von E mit* $\dim E/F = d(E)$ *und ein lineares Komplement G von* $F + F^\perp$ *mit* $\dim E/G^\perp \leqq \aleph_0$.

(ii) (E, Φ) *ist präeuklidisch.*

Beweis. 1) Sei E präeuklidisch, F ein euklidischer Teilraum mit $\dim E/F = d(E)$ und G ein lineares Komplement von $F + F^\perp$. Da $\dim G \leqq \aleph_0$, ist nach I.2, Satz 4 $\dim E/G^\perp \leqq \aleph_0$.

2) Sei die Eigenschaft (i) erfüllt für die Teilräume F und G. Es ist $F \cap F^\perp = (0)$, da F halbeinfach ist. Sei V ein lineares Komplement von $\operatorname{rad} F^\perp$ in $F^\perp$ und $F' = F \overset{\perp}{\bigoplus} V$. Wegen der Halbeinfachheit von V und $\dim V \leqq \aleph_0$ ist F' euklidisch und $F'^\perp = \operatorname{rad} F^\perp$ totalisotrop. Ferner ist $F' + F'^\perp = F + F^\perp$. Wir können also annehmen, daß $F^\perp$ totalisotrop ist.

Sei nun G^* der Raum aller Semilinearformen auf G bezüglich des Körperantiautomorphismus $*$ und $U = F + F^\perp$. Wir betrachten die Abbildung $\varphi : U \to G^*$, definiert durch $\langle \varphi(x), y \rangle = \Phi(x, y)$ für alle $y \in G$. φ ist linear. Sei $L = \varphi(U)$. Wegen $\operatorname{Ker} \varphi = U \cap G^\perp$ ist $\dim L = \dim [U/\operatorname{Ker} \varphi] = \dim [U/(U \cap G^\perp)] = \dim [(U + G^\perp)/G^\perp] \leqq \dim E/G^\perp \leqq \aleph_0$. H sei ein lineares Komplement von $\varphi(F^\perp)$ in L. Wir bilden $E' = E \oplus H$ und definieren auf E' eine reflexive Sesquilinearform Φ' wie folgt: $\Phi'|_{E \times E} = \Phi$, $\Phi'(H, H) = \{0\}$, $\Phi'(H, U) = \{0\}$, $\Phi'(h, g) = \langle h, g \rangle$ für alle $h \in H$, $g \in G$. Durch die letzte Beziehung ist auch $\Phi'(g, h)$ gegeben, da es zu der reflexiven Sesquilinearform Φ ein $\varepsilon \in k$ gibt mit $\Phi(x, y) = \varepsilon \Phi(y, x)^*$ für alle $x, y \in E$. Dann muß auch $\Phi'(x, y) = \varepsilon \Phi'(y, x)^*$ sein für alle $x, y \in E'$. Wegen der Halbeinfachheit von E ist $G^\perp \cap F^\perp = (0)$. φ ist somit auf $F^\perp$ eineindeutig, und es gilt $\Phi'(r, g) = \Phi(r, g) = \langle \varphi(r), g \rangle$ für alle $r \in F^\perp$ und $g \in G$. Wir können also $F^\perp$ und $\varphi(F^\perp)$ identifizieren.

Nun zeigen wir, daß $E^{\perp'} = (0)$ ist, wobei wir mit $\perp'$ das orthogonale Komplement in E' bezeichnen. Man hat $E' = F \oplus F^\perp \oplus H \oplus G$. Sei $x \in E^{\perp'}$, $x = y + z$, wo $y \in F \oplus G$ und $z \in F^\perp \oplus H = L$. Aus $x \perp F$ folgt $y = 0$, da $F^\perp \cap (F \oplus G) = (0)$ und $H \perp F$, d. h. $x = z \in L \subset G^*$. $z \in L \cap G^{\perp'} = (0)$, also $x = 0$, $E^{\perp'} = (0)$, d. h. E' ist halbeinfach.

Wir definieren eine Abbildung $\Psi : F \to E'$ durch $\Psi(x) = x - \varphi(x)$. Ψ ist linear und isometrisch bezüglich Φ', da $L = F^\perp \oplus H \perp F$ und totalisotrop ist. $F' = \Psi(F)$ ist also ebenfalls euklidisch. Für $g \in G$ und $x \in F$ gilt $\Phi'(\Psi(x), g) = \Phi'(x, g) - \Phi'(\varphi(x), g) = \Phi(x, g) - \langle \varphi(x), g \rangle = 0$. Also ist $F' \perp G$ und $E' = F' \overset{\perp}{\bigoplus} (G \oplus L)$. Da $\dim (G \oplus L) \leqq \aleph_0$, ist $G \oplus L$ und damit auch E' euklidisch. Q.E.D.

Für halbeinfache Räume von endlichem euklidischen Defekt ist die Eigenschaft (i) immer erfüllt. Also sind alle diese Räume präeuklidisch, wie schon in [2] gezeigt wurde.

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Compactness Criteria for Köthe Spaces

SIGRUN GOES and ROBERT WELLAND

Many authors have extended Kolmogorov's L_p-compactness criterion to other function spaces. We give in this paper the Köthe space version of this criterion and by so doing, we obtain virtually all previous results on the subject as special cases of quite general theorems.

The paper contains four compactness criteria – the first and most general is a variant of Mazur's result [2, p. 205]; it is used to obtain a generalization of the Phillips theorem [7, p. 297]. We then use the Mazur theorem to obtain the main theorem of this paper – our Köthe space version of Kolmogorov's result [13], which in its turn is then used to obtain a generalization of the compactness theorem by M. Riesz [19].

Let us recall Kolmogorov's theorem and its generalizations up to this time.

Let $L_p([0,1],\mu)$ be the space of L_p functions defined on the whole real line whose supports are contained in $[0,1]$; here μ is the Lebesgue measure. If χ_E is the characteristic function of a measurable set E and $\varepsilon>0$ we let

$$T_\varepsilon f(x)=\chi_{[0,1]}(x)\left(\frac{1}{\varepsilon}\int\limits_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} f\,d\mu\right).$$

Theorem (Kolmogorov). *A subset $\mathfrak{A}$ of $L_p([0,1])$ $(1<p<\infty)$ is relatively compact if and only if*

1) *$\mathfrak{A}$ is a bounded subset of L_p, and*
2) *$T_\varepsilon f$ converges to f in the L_p norm, uniformly on $\mathfrak{A}$.*

By adding a growth condition at infinity Tamarkin [23] obtained a similar result for $L_p(R^n,\mu)$ where $1<p<\infty$ and R^n is the n-dimensional Euclidean space. Tulajkov [24] obtained Tamarkin's result for the case $p=1$ and Takahashi [22] obtained the result for Orlicz space satisfying the Δ_2 condition.

Conditions of this type were used by Izumi [12] to obtain a compactness criterion for $C([0,1])$; by Phillips [7, p. 297] and Aleksandrov [1] for L_p spaces over totally σ-finite measure spaces; by Veress [25] for $L_\infty([a,b])$; by L. W. Cohen [4] for ℓ_p spaces and by Cohen and Dunford [5] for Banach spaces with Schauder basis.

M. Riesz [19] proved a compactness theorem for subsets of $L_p(R^n,\mu)$ $(1\leqq p<\infty)$ similar to the one by Kolmogorov-Tamarkin-Tulajkov. Instead of the operators T_ε he used the translation operators $f\to f_a$ where $f_a(x)=f(x-a)$. Weil [26] extended Riesz' result to L_p spaces over locally compact abelian topological groups with Haar measure, and Gribanov [10] proved it for Orlicz spaces $L_\Phi(R^n,\mu)$.

These results are corollaries of our Theorems 1, 2, 3, and 4.

1. Partitions of the Identity and Mazur's Theorem

Throughout this section X denotes a locally convex topological vector space and $\mathscr{U}_X$ denotes the set of closed, circled, convex symmetric neighborhoods of 0. If $U \in \mathscr{U}_X$, let $P_U = \inf\{\lambda : x \in \lambda U\}$ be the Minkowski function of the set U.

Let $L(X, X)$ be the set of linear mappings from X to X and let I be the identity linear mapping. Let S be a subset of X, B a directed system [8] and $\{T_\beta\}_{\beta \in B} \subset L(X, X)$ a net. We will say that the net $\{T_\beta\}$ is a *partition of the identity on S* if

$$\lim_{\beta} T_\beta(x) = x \tag{1}$$

for every x in S and that it is a *partition of the identity* if (1) holds for every x in X.

We will say that a partition of the identity $\{T_\beta\}$ is a *strong partition of the identity* if for every compact subset C of X and for every U in $\mathscr{U}_X$ there exists β_0 in B and $\delta > 0$ such that $T_\beta(x) \in U$ for every $\beta > \beta_0$ and $x \in C \cap \delta U$.

If $\{T_\beta\}$ is a net of operators on X and A a subset of X we will say that "*$T_\beta(x)$ converges uniformly to x on A*", or equivalently "*$\{T_\beta\}$ converges uniformly to the identity on A*" if (1) holds uniformly for x in A, i.e. if for any U in $\mathscr{U}_X$ there exists β_0 in B such that $T_\beta(x) - x$ belongs to U for every x in A and for every $\beta > \beta_0$.

A partition of the identity $\{T_\beta\}$ will be said to *converge uniformly on A* if it converges uniformly to the identity on A.

Lemma 1. *A partition of the identity $\{T_\beta\}_{\beta \in B} \subset L(X, X)$ converges uniformly on every compact set if and only if it is a strong partition of the identity.*

Proof. Suppose $\{T_\beta\}_{\beta \in B}$ converges uniformly on every compact set C and let C be a fixed compact subset of X. Let $U \in \mathscr{U}_X$ and choose β_0 such that $(T_\beta(x) - x) \in \frac{1}{2} U$ for every $\beta \geqq \beta_0$ and x in C. Clearly $T_\beta(x) \in U$ for every $\beta \geqq \beta_0$ and x in $C \cap \frac{1}{2} U$.

Suppose $\{T_\beta\}_{\beta \in B}$ is a strong partition of the identity and C is an arbitrary compact subset of X. The set $\{x - y : y, x \in C\} = \bar{C}$ is also compact. If U is an arbitrary element of $\mathscr{U}_X$ there exists a β_0 and $1 > \delta > 0$ such that $T_\beta(\bar{x}) \in \frac{1}{3} U$ for $\beta \geqq \beta_0$ and $\bar{x}$ in $\bar{C} \cap \delta U$. As C is compact there exists a finite subset $\{x_i\}_{1 \leqq i \leqq m} \subset C$ such that $C \subset \bigcup_{1 \leqq i \leqq m} \left(\frac{\delta}{3} U + x_i\right)$. Because $\{T_\beta\}_{\beta \in B}$ is a partition of the identity for each x_i there exists β_i such that $T_\beta(x_i) - x_i \in \frac{1}{3} U$. Let γ be an element of B such that $\gamma \geqq \beta_i$ for $0 \leqq i \leqq m$.

If x is an arbitrary element in C there exists x_i with $x_i - x$ in $\frac{\delta}{3} U$, $T_\beta(x_i) - x_i$ in $\frac{1}{3} U$ for all $\beta \geqq \gamma$ and $T_\beta(x_i - x) \in \frac{1}{3} U$ $(x_i - x = \bar{x} \in \bar{C})$. Consequently,

$$T_\beta(x) - x = T_\beta(x - x_i) + T_\beta(x_i) - x_i + x_i - x$$

is an element of U. We have shown that for sufficiently large β and all x in C we have $T_\beta(x) - x \in U$. Since U was an arbitrary element of $\mathscr{U}_X$ this completes the proof.

Lemma 2. *Suppose X is a locally convex topological vector space, S is a dense subset of X and $\{T_\beta\}_{\beta\in B}$ is a partition of the identity on S; then $\{T_\beta\}_{\beta\in B}$ is a partition of the identity on X if for every W in $\mathscr{U}_X$ there exists $\beta_0\in B$ and V in $\mathscr{U}_X$ such that $T_\beta(x)\in W$ whenever $x\in V$ and $\beta>\beta_0$.*

Proof. Let x in X and W in $\mathscr{U}_X$ be given; we must show there exists β_0 such that $T_\beta(x)-x\in W$ for all $\beta>\beta_0$. Choose V in $\mathscr{U}_X$ such that $V+V+V\subset W$. Then choose $\bar\beta\in B$ and U in $\mathscr{U}_X$ such that $U\subset V$ and $T_\beta(z)\in V$ if $z\in U$ and $\beta>\bar\beta$. Choose s in S such that $x-s\in U$ and $\bar{\bar\beta}$ such that $T_\beta(s)-s\in V$ for $\beta>\bar{\bar\beta}$. Let $\beta_0\in B$ be such that $\beta_0>\bar\beta$ and $\beta_0>\bar{\bar\beta}$. Now if $\beta>\beta_0$, then

$$T_\beta(x)-x=T_\beta(x-s)+T_\beta(s)-s+s-x\in V+V+U\subset W.$$

Mazur's Theorem

The following result, due to Mazur, appeared in [2, p. 205]:

Theorem. *A bounded subset S of a Banach space X is relatively compact if and only if there exists a partition of the identity $\{U_n\}_{n=1,2,\dots}$ which converges uniformly on S and such that $U_n(S)$ is relatively compact for each $n=1,2,\dots$.*

The boundedness assumption was omitted in [2], an omission which makes the theorem incorrect. Sudakov [21] made this observation and was able to show that the boundedness of S could be omitted provided the equation $U_n(x)=x$ has only the trivial solution for at least one U_n.

If S is a relatively compact set then the partition of the identity $\{U_n\}_{n=1,2,\dots}$ with $U_n=I$ for every n satisfies the conditions of the theorem. Consequently, this half of the theorem is trivial. However, in general partitions of the identity occur in some natural way. Then we can ask the question – are the necessary conditions satisfied by this particular partition of the identity.

These observations with the fact that our work is in locally convex spaces motivates the following generalization of Mazur's theorem.

Theorem 1. *Let X be a complete locally convex topological vector space; let $\mathfrak{A}$ be a bounded subset of X and let $\{T_\beta\}_{\beta\in B}$ be a net in $L(X,X)$.*

A) *Then $\mathfrak{A}$ is relatively compact if $\{T_\beta\}_{\beta\in B}$ converges uniformly to the identity on $\mathfrak{A}$ and $T_\beta(\mathfrak{A})$ is relatively compact for each β in B.*

B) *$\{T_\beta\}_{\beta\in B}$ converges uniformly to the identity on each relatively compact set exactly if $\{T_\beta\}_{\beta\in B}$ is a strong partition of the identity.*

Proof of part A). It is well known that a subset of a complete metric space is relatively compact if and only if it is totally bounded. This criterion also applies to complete locally convex spaces [8, p. 279]. Thus, to show that a subset K of X is relatively compact it will suffice to show there exists for each U in $\mathscr{U}_X$ a set $\{x_1,x_2,\dots,x_n\}\subset X$ such that

$$K\subset\bigcup_{1\leq i\leq n}(U+x_i).$$

Suppose that $T_\beta(\mathfrak{A})$ is relatively compact for each β in B and that U in $\mathscr{U}_X$ is given. Choose β such that $(T_\beta(x)-x)\in\frac{1}{2}U$ for every x in $\mathfrak{A}$ and choose $\{x_1, x_2, \ldots, x_n\}\subset X$ such that

$$T_\beta(\mathfrak{A})\subset \bigcup_{1\leqq i\leqq n} (\tfrac{1}{2}U+x_i).$$

We now have

$$\mathfrak{A}\subset \bigcup_{1\leqq i\leqq n} (U+x_i)$$

which completes the proof.

Part B) is a restatement of Lemma 1.

It is clear, in this theorem, that it would suffice to assume that $T_\beta(\mathfrak{A})$ is relatively compact for all β in a cofinal subset B' of B.

We can use this result to obtain a slight extension of a result of Cohen and Dunford [5].

Corollary. *Suppose X is a Banach space of real or complex sequences which contains all finite sequences. Let*

$$T_n(x)=\{x_1, x_2, \ldots, x_n, 0, 0, \ldots\}$$

for each $x=\{x_k\}_{k=1}^\infty$.

A) *If the net $\{T_n\}_{n=1}^\infty$ is a partition of the identity on X then a bounded subset $\mathfrak{A}$ of X is relatively compact if and only if $T_n(x)$ converges to x uniformly on $\mathfrak{A}$.*

B) *If $\{T_n\}_{n=1}^\infty$ is not a partition of the identity on X then the condition that $T_n(x)$ converge to x uniformly on $\mathfrak{A}$ is sufficient but not necessary for the relative compactness of a bounded subset $\mathfrak{A}$ of X.*

Proof. Assume that $\mathfrak{A}$ is a bounded subset of X. Since each transformation T_n is continuous [5], the sets $T_n(\mathfrak{A})=\{T_n(x): x\in\mathfrak{A}\}$ are bounded subsets of the finite dimensional subspace $T_n(X)$ in X. Such subspaces are linearly homeomorphic to R^n [14, p. 154], and therefore each set $T_n(\mathfrak{A})$ is relatively compact in X. Hence Theorem 1 implies that $\mathfrak{A}$ is relatively compact.

Conversely suppose that $\mathfrak{A}$ is relatively compact and $\{T_n\}$ is a partition of the identity on X. By the Banach-Steinhaus theorem we know that

$$\sup_{n=1,2,\ldots} \|T_n\| < \infty .$$

This implies that $\{T_n\}$ is a strong partition of the identity and the proof is complete using Theorem 1.

2. Köthe Spaces and the Phillips Compactness Criterion

Let us recall the definition of a Köthe space and look at the Phillips net of operators. Let E be a set and μ a non-negative totally σ-finite measure defined on a σ-algebra of subsets of E. Let $E_1\subset E_2\subset\cdots$ be a sequence of measurable sets each of finite measure such that $E=\bigcup E_i$. If $\mu(E)<\infty$ we put $E=E_1=E_2=\cdots$.

A real valued measurable function f on E is said to be *locally integrable* if

$$N_n(f) = \int_{E_n} |f| \, d\mu < \infty$$

for each $n = 1, 2, \ldots$. The set of equivalence classes of these functions is denoted by Ω; two functions being in the same equivalence class provided they differ on a set of measure zero only. As usual we identify functions with the equivalence classes. The functions N_n define seminorms on Ω and with these, Ω becomes a complete locally convex space whose dual Φ is the set of essentially bounded functions on E which vanish outside of some set E_n.

If Γ is a subset of Ω let

$$\Lambda = \Lambda(\Gamma) = \{g \in \Omega : f g \; L_1(E, \mu) \quad \text{for all} \quad f \in \Gamma\}$$

and set $\Lambda^\times = \Lambda(\Lambda)$. The spaces Λ and $\Lambda^\times$ are called *Köthe spuces;* $\Lambda(\Lambda^{\times\times}) = \Lambda$, and they are put in weak duality by the bilinear form

$$\langle f, g \rangle = \int f g \, d\mu \quad (f \in \Lambda, \; g \in \Lambda^\times).$$

With this duality the spaces Λ and $\Lambda^\times$ can be given weak topologies. Let $\mathscr{H}$ be a set of weakly bounded subsets of $\Lambda^\times$ whose union is the whole of $\Lambda^\times$ and let $\mathscr{K}_{\mathscr{H}}$ be the topology on Λ given by the seminorms

$$p_H(f) = \sup_{h \in H} \int |f h| \, d\mu \qquad (H \in \mathscr{H}).$$

We call such a topology a *Köthe topology* and we will assume from now on that Λ is endowed with a Köthe topology. With any Köthe topology, Λ becomes a complete locally convex topological vector lattice which is moreover conditionally complete ([9] and [28]).

With Phillips [7, p. 297] we define a net $\{P_\pi\}_{\pi \in \Pi}$ in $L(\Lambda, \Lambda)$. Let $\pi = \{A_1, \ldots, A_n\}$ be a set of pairwise disjoint measurable subsets of E, each having positive measure and such that each A_i is contained in some E_n $(n = 1, 2, \ldots)$. Let Π be the set of all π and say that $\pi_1 \geqq \pi_2$ if each set in π_2 is a union of sets in π_1. Here we do everything modulo sets of measure zero. For $\pi = \{A_1, \ldots, A_n\}$ in Π and f in Ω we set

$$P_\pi(f) = \sum_{i=1}^{n} \left(\frac{1}{\mu(A_i)} \int_{Ai} f \, d\mu \right) \chi_{Ai}$$

where as usual χ_{A_i} is the charactersitic function of the set A_i.

Proposition 2.1. *For any $\pi \in \Pi$, the mapping P_π is a continuous linear operator on any Köthe space Λ.*

Proof. Let $\pi = \{A_1, \ldots, A_n\}$ in Π be given and let E_m be such that $\bigcup A_i \subset E_m$. Then

$$\|P_\pi(f)\|_\infty \leqq \left[\sup_{1 \leqq i \leqq n} \frac{1}{\mu(A_i)} \right] \int_{E_m} |f| \, d\mu$$

for every f in Ω. Hence P_π is a continuous mapping from Ω into the space Φ of essentially bounded functions having support in some E_n. But on Λ, the Köthe topology $\mathcal{K}_{\mathcal{H}}$ is stronger than the topology of Ω, and on Φ, the sup norm topology is stronger than the Köthe topology $\mathcal{K}_{\mathcal{H}}$. Hence P_π is a continuous mapping from Λ into Λ, considered under the Köthe topology $\mathcal{K}_{\mathcal{H}}$.

Proposition 2.2. *The net $\{P_\pi\}$ in $L(\Omega, \Omega)$ is a strong partition of the identity.*

Proof. Let $f \in \Omega$ and let a natural number n and an $\varepsilon > 0$ be given. Choose $s \geqq 0$ such that

$$\int_{A_0} |f| \, d\mu < \frac{\varepsilon}{3},$$

where $A_0 = \{x \in E_n : f(x) \geqq s \text{ or } f(x) < -s\}$. Choose a natural number m such that

$$\mu(E_n) \frac{s}{m} < \frac{\varepsilon}{3}.$$

Let

$$A_i = \left\{x \in E_n : -s + (i-1)\frac{s}{m} \leqq f(x) < -s + i\frac{s}{m}\right\}$$

for $i = 1, 2, \ldots, 2m$. If $\pi_0 = \{A_0, A_1, \ldots, A_{2m}\}$, then a simple calculation shows that $N_n(P_{\pi_0}(f) - f) < \varepsilon$. Because n and $\varepsilon > 0$ were arbitrary, this implies that

$$\lim_\pi P_\pi(f) = f$$

for every f in Ω.

A further simple computation shows that $N_n(P_\pi(f)) \leqq N_n(f)$ for every f in Ω and every $\pi \in \Pi$. This proves that $\{P_\pi\}$ is a strong partition of the identity.

While the net $\{P_\pi\}$ is defined on every Köthe space Λ, i.e. $\{P_\pi\} \subset L(\Lambda, \Lambda)$, it is not true, despite Proposition 2.2, that $\{P_\pi\}$ is always a partition of the identity. In fact, some Köthe spaces have the annoying property that the simple functions do not form a dense subspace. In particular, any Orlicz space L_Φ defined on $E = [0, 1]$ and for which Φ does not satisfy the Δ_2 condition is such a Köthe space [16]. Therefore, in such a space the net $\{P_\pi\}$ is not even a partition of the identity.

To single out those Köthe spaces for which $\{P_\pi\}$ is a strong partition of the identity, we say that a Köthe space is *set regular* provided $\{P_\pi\}$ is a strong partition of the identity on Λ.

We have already seen that Ω is a set regular Köthe space; the next proposition shows that the set of such spaces is quite large.

Proposition 2.3. *Let L_Φ be the Orlicz space defined on (E, μ) with Young's function Φ. Then the space L_Φ is a set regular Köthe space if and only if Φ satisfies the Δ_2 condition (i.e. there exists $M > 1$ and $u_0 > 0$ such that $\Phi(2u) \leqq M\,\Phi(u)$ for all $u > u_0$).*

Proof. Assume that Φ satisfies the Δ_2 condition. Let S be the set of simple functions s in L_Φ such that s vanishes outside E_n for some n. This set S is dense in L_Φ [16]. Let s in S be given by

$$s = \sum_{i=1}^{m} a_i \chi_{A_i}$$

where the A_i are pairwise disjoint. Let $\pi_0 = \{A_1, \ldots, A_m\}$. If $\pi \geqq \pi_0$ then $P_\pi(s) = s$, from which we conclude that $\{P_\pi\}$ is a partition of the identity on S.

Using Jensen's inequality, we have for any π in Π and f in L_Φ

$$\Phi(|P_\pi(\chi_{A_i} f)|) \leqq \frac{1}{\mu(A_i)} \int_{Ai} \Phi(|f|)\, d\mu\, \chi_{A_i}.$$

Integrating over E and summing over i gives

$$\int \Phi(|P_\pi(f)|)\, d\mu \leqq \int \Phi(|f|)\, d\mu. \tag{1}$$

If E is a bounded open subset of the Euclidean space R^n and μ is the Lebesgue measure, Krasnoselskiĭ and Rutickiĭ [15] have shown that $\|f\|_{L_\Phi} \leqq 1$ implies that

$$\int \Phi(|f|)\, d\mu \leqq \|f\|_{L_\Phi}. \tag{2}$$

A very slight change in their arguments shows that this inequality holds for a general totally σ-finite measure space (E, μ).

Zaanen [29] shows that when Φ satisfies the Δ_2 condition then

$$\int \Phi(|f|)\, d\mu \leqq M^{-p} \quad \text{implies} \quad \|f\|_{L_\Phi} \leqq 2^{-p+1} \quad \text{for any} \quad p = 1, 2, \ldots. \tag{3}$$

Let $\|f\|_{L_\Phi} = \dfrac{1}{M}$. Then (2) implies that $\int \Phi(|f|)\, d\mu \leqq \dfrac{1}{M}$, and hence by (1) it follows that $\int \Phi(|P_\pi(f)|)\, d\mu \leqq \dfrac{1}{M}$ for every $\pi \in \Pi$. Taking $p = 1$ in (3) we get $\|P_\pi(f)\|_{L_\Phi} \leqq 1$ for all $\pi \in \Pi$. Thus

$$\sup_{\pi} \sup_{\|f\|_{L_\Phi} = M^{-1}} \|P_\pi(f)\|_{L_\Phi} \leqq 1,$$

i.e. for every $\pi \in \Pi$ the mapping P_π has norm $\leqq M$. This shows first, with Lemma 2, that $\{P_\pi\}$ is a partition of the identity, and second, directly from the definition, that $\{P_\pi\}$ is a strong partition of the identity.

If Φ does not satisfy the Δ_2 condition then by [16] the simple functions are not dense in L_Φ and hence $\{P_\pi\}$ is not even a partition of the identity.

Proposition 2.4. *$L_\infty(E, \mu)$ is a set regular Köthe space if and only if $\mu(E) < \infty$.*

Proof. This follows trivially from the fact that P_π is a norm decreasing linear operator on $L_\infty(E, \mu)$ for every $\pi \in \Pi$.

The next result is the Köthe space version of the Phillips theorem [7, p. 297].

Theorem 2. A) *If Λ is a set regular Köthe space then a bounded subset $\mathfrak{A}$ of Λ is relatively compact if and only if $\{P_\pi\}$ converges uniformly to the identity on $\mathfrak{A}$.*

B) *If the Köthe space Λ is not set regular then the condition that $\{P_\pi\}$ converge uniformly to the identity on $\mathfrak{A}$ is sufficient but not necessary for the relative compactness of a bounded subset $\mathfrak{A}$ of Λ.*

Proof. Assume that $\mathfrak{A}$ is a bounded subset of Λ and that $\{P_\pi\}$ converges uniformly to the identity on $\mathfrak{A}$. Let $\pi = \{A_1, \ldots, A_n\}$ be in Π. By Proposition 2.1 the mapping P_π is a linear continuous operator on Λ. Consequently, $P_\pi(\mathfrak{A})$ is a bounded subset of Λ. However, $P_\pi(\mathfrak{A})$ is contained in the finite dimensional subspace of Λ spanned by the functions $\chi_{A_1}, \ldots, \chi_{A_n}$. This space is linearly homeomorphic to R^n [14, p. 154], so the bounded set $P_\pi(\mathfrak{A})$ is relatively compact. Now applying Theorem 1, we see that $\mathfrak{A}$ itself is relatively compact.

The remaining part of the theorem is a direct consequence of Theorem 1.

Corollary 1. A) *Let L_Φ be an Orlicz space such that Φ satisfies the Δ_2 condition or let $L_\Phi = L_\infty$ with $\mu(E) < \infty$. Then a bounded subset $\mathfrak{A}$ of L_Φ is relatively compact if and only if $\{P_\pi\}$ converges uniformly to the identity on $\mathfrak{A}$.*

B) *Let L_Φ be an Orlicz space such that Φ does not satisfy the Δ_2 condition or let $L_\Phi = L_\infty$ with $\mu(E) = \infty$. Then the condition that $\{P_\pi\}$ converge uniformly to the identity on $\mathfrak{A}$ is sufficient but not necessary for the relative compactness of a bounded subset $\mathfrak{A}$ of L_Φ.*

Proof. This follows from Proposition 2.3 and 2.4 and from Theorem 2.

The next result follows directly from Corollary 1.

Corollary 2. (Veress [25], see also Poppe [18]). *A closed bounded subset $\mathfrak{A}$ of $L_\infty([a, b])$ is compact if and only if for every $\varepsilon > 0$ there exists a partition of $[a, b]$ into measurable subsets A_i $(i = 1, 2, \ldots, n)$ such that*

$$\sup_{f \in \mathfrak{A}} \sup_{x_i, y_i \in A_i} |f(x_i) - f(y_i)| < \varepsilon$$

for $i = 1, 2, \ldots, n$.

3. Köthe Spaces and Kolmogorov's Compactness Criterion

Let E be a paracompact locally compact uniform space which is covered by a nested sequence of compact neighborhoods $E_1 \subset E_2 \subset \cdots$. Let $\mathscr{E}$ be a uniform structure subbase consisting of closed symmetric sets [8] and let μ be a non-negative Radon measure on E. We suppose that $\mathscr{E}$ and μ satisfy:

a) for each E_n and for each U in $\mathscr{E}$ the set

$$E_n + U = \{y \in E : (x, y) \in U \quad \text{for some} \quad x \in E_n\}$$

is relatively compact;

b) if $x + U = \{y \in E : (x, y) \in U\}$, then $0 < \mu(x + U) < \infty$ for every $x \in E$ and for every $U \in \mathscr{E}$;

c) for $U \in \mathscr{E}$ fixed we have $\lim\limits_{x \to x_0} \int |\chi_{x_0 + U} - \chi_{x + U}| \, d\mu = 0$ for every $x_0 \in E$;

d) for every E_m there exist a constant M_m and a $U_m \in \mathscr{E}$ such that

$$\int_{E_m} \frac{\chi_{x+U}(t)}{\mu(t+U)}\, d\mu(t) < M_m$$

for every $x \in E$ and $U \in \mathscr{E}$ satisfying $U \subset U_m$.

There is a great variety of spaces which posses these properties – the simplest are open subsets of Lie groups.

We are going to define a partition of the identity in terms of $\mathscr{E}$. Let B be the directed system made up of pairs (n, U) with n a natural number and U an element of $\mathscr{E}$; we define $(n, U) \leqq (n', U')$ if $n \leqq n'$ and $U' \subset U$. If $\beta = (n, U)$ is in B and f is in Ω, then we define

$$T_\beta(f)(x) = \frac{1}{\mu(x+U)} \int \chi_{x+U}\, \chi_{E_n} f\, d\mu .$$

We will say that a Köthe space Λ is *continuously regular* if $\{T_\beta\}_{\beta \in B}$ is a strong partition of the identity in $L(\Lambda, \Lambda)$. The next set of propositions are interesting individually as they provide insight into the nature of the net of operators $\{T_\beta\}_{\beta \in B}$. However, they are more correctly viewed as lemmas in the proof of Theorem 3, our form of Kolmogorov's compactness criterion.

Proposition 3.1. *If $f \in \Omega$, then $T_\beta(f)$ is a continuous function whose support is contained in E_m for some m which depends only on β.*

Proof. Let $\beta = (n, U)$. Because $E_n + U$ is relatively compact there exists $m = m(\beta)$ such that $E_n + U \subset E_m$. It follows directly from the definition that the support of $T_\beta(f)$ is contained in E_m.

Let x_0 be an arbitrary point in E and consider

$$\begin{aligned} &\lim_{x \to x_0} |T_\beta(f)(x) - T_\beta(f)(x_0)| \\ &\qquad \leqq \lim_{x \to x_0} \frac{1}{\mu(x+U)} \int |\chi_{E_n} f|\, |\chi_{x+U} - \chi_{x_0+U}|\, d\mu \qquad \text{(i)} \\ &\qquad + \lim_{x \to x_0} \left| \frac{1}{\mu(x+U)} - \frac{1}{\mu(x_0+U)} \right| \int |\chi_{E_n} f|\, d\mu . \qquad \text{(ii)} \end{aligned}$$

We consider limit (i) first. By hypothesis we have that

$$\lim_{x \to x_0} \int |\chi_{x+U} - \chi_{x_0+U}|\, d\mu = 0 .$$

Next, because $\chi_{E_n} f$ is an element of $L_1(E, \mu)$, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mu(A) < \delta$ implies

$$\int_A |\chi_{En} f|\, d\mu < \varepsilon .$$

These two remarks show that

$$\lim_{x \to x_0} \int |\chi_{E_n} f|\, |\chi_{x+U} - \chi_{x_0+U}|\, d\mu = 0 .$$

Because $\mu(x+U)$ is bounded away from zero on E_m we see that the limit (i) is 0.

Because $\chi_{E_n} f$ is in $L_1(E, \mu)$ and because for $U \in \mathscr{E}$ fixed, $\mu(x+U)$ is a continuous function of x, the second limit, limit (ii) is also 0.

Corollary. *T_β is a linear mapping from Λ to Λ for any Köthe space Λ.*

Proof. As Λ is contained in Ω, T_β is defined on Λ, and because $T_\beta(f)$ is a continuous function with compact support whenever $f \in \Omega$, $T_\beta(f)$ is an element of Φ, the set of all essentially bounded functions on E with compact support. Now $\Phi \subset \Lambda$ for every Λ, so that $T_\beta(f)$ is an element of Λ for every $f \in \Lambda$.

Proposition 3.2. *Let Λ be a Köthe space and let $\beta = (n, U)$ be an element of B. Then T_β is a continuous linear operator on Λ.*

Proof. Choose m such that $E_n + U \subset E_m$. Then

$$\|T_\beta(f)\|_\infty \leqq \left[\sup_{x \in E_m} \frac{1}{\mu(x+U)}\right] N_n(f)$$

and hence T_β is a continuous mapping from Ω into the space $C_0(E)$ of all continuous functions on E with compact support. But since Λ is a subset of Ω on which the topology of Ω is weaker than the Köthe topology $\mathscr{K}_{\mathscr{H}}$, and since $C_0(E)$ is a subset of Λ on which the sup norm topology is stronger than the Köthe topology $\mathscr{K}_{\mathscr{H}}$, it follows that T_β is also a continuous mapping from Λ into Λ, endowed with the Köthe topology $\mathscr{K}_{\mathscr{H}}$.

Proposition 3.3. *Given m, there exist constants M, m' and β_0 in B such that for every $f \in \Omega$, $\beta > \beta_0$ implies*

$$N_m(T_\beta(f)) \leqq M N_{m'}(f).$$

Proof. Choose U_0 in $\mathscr{E}$ and M so that

$$K(t) \leqq \int_{E_m} \frac{\chi_{x+U}(t)}{\mu(x+U)} \, d\mu(x) < M$$

for every $t \in E$ and for every $U \in \mathscr{E}$ satisfying $U \subset U_0$. Choose m' so that $E_m + U_0 \subset E_{m'}$; it is clear that $E_m + U \subset E_{m'}$ for every $U \subset U_0$. Let $\beta_0 = (m', U_0)$ and let $\beta = (n, U)$ with $n \geqq m'$ and $U \subset U_0$. Because $K(t)$ is a bounded measurable function

$$\begin{aligned}
N_m(T_\beta(f)) &\leqq \iint_{E_m} \frac{\chi_{x+U}(t)}{\mu(x+U)} \chi_{E_n}(t) \, |f(t)| \, d\mu(t) \, d\mu(x) \\
&= \int \left[\int_{E_m} \frac{\chi_{x+U}(t)}{\mu(x+U)} \, d\mu(x)\right] \chi_{E_n}(t) \, |f(t)| \, d\mu(t) \\
&\leqq M \int_{E_{m'}} |f(t)| \, d\mu(t) \\
&= M N_{m'}(f).
\end{aligned}$$

The next proposition is suggested by the work of Tulajkov [24].

Proposition 3.4. *Let $\mathfrak{A}$ be a bounded subset of a Köthe space Λ. Then $T_\beta^2(\mathfrak{A}) = \{T_\beta(T_\beta(f)) : f \in \mathfrak{A}\}$ is a uniformly bounded set of equicontinuous functions all of whose supports are contained in some E_m, where m depends on β only.*

Proof. Let $\beta = (n, U)$ and let m be such that $E_n + U \subset E_m$. By Proposition 3.1 $T_\beta(f)$ is a continuous function with support in E_m for any f in Ω. This implies that $T_\beta^2(f)$ has its support in E_m. Because $\mathfrak{A}$ is bounded in Λ it is also bounded in Ω (the identity map $\Lambda \to \Omega$ is continuous). Let k be positive and less than $\mu(x+U)$ for every x in E_m. A computation shows that $|T_\beta(f)(x)| \leqq N_n(f)\, k^{-1}$. Because $\mathfrak{A}$ is bounded in Ω there exists a positive constant M such that $N_n(f) \leqq M$ for every f in $\mathfrak{A}$. This implies that $|T_\beta(f)(x)| \leqq M k^{-1}$ for every f in $\mathfrak{A}$ and for every x in E. Furthermore $\|T_\beta^2(f)\|_\infty \leqq \|T_\beta(f)\|_\infty$, and hence $T_\beta^2(\mathfrak{A})$ is uniformly bounded.

Next for every $f \in \mathfrak{A}$ we have

$$\begin{aligned}
&|T_\beta^2(f)(x_0) - T_\beta^2(f)(x)| \\
&= \left| \int \left[\frac{\chi_{x_0+U}}{\mu(x_0+U)} \chi_{E_n} T_\beta(f) - \frac{\chi_{x+U}}{\mu(x+U)} \chi_{E_n} T_\beta(f) \right] d\mu \right| \\
&\leqq M k^{-1} \int_{E_n} \left| \frac{\chi_{x_0+U}}{\mu(x_0+U)} - \frac{\chi_{x+U}}{\mu(x+U)} \right| d\mu \\
&\leqq M k^{-1} \int_{E_n} \frac{1}{\mu(x_0+U)} |\chi_{x_0+U} - \chi_{x+U}|\, d\mu \\
&\quad + M k^{-1} \int_{E_n} \left| \frac{1}{\mu(x_0+U)} - \frac{1}{\mu(x+U)} \right| \chi_{x+U}\, d\mu .
\end{aligned}$$

Each of the two terms in the last line tends to 0 as x tends to x_0, independently of f in $\mathfrak{A}$. Hence $T_\beta^2(\mathfrak{A})$ consists of equicontinuous functions.

Corollary. *T_β^2 is a compact operator from Λ to Λ.*

Proof. The preceeding proposition shows that T_β^2 takes bounded subsets of Λ into uniformly bounded sets of equicontinuous functions all of whose supports are contained in the compact set E_m. By the Ascoli-Arzelà theorem, such a set is a relatively compact subset of $C_0(E)$ (the space of continuous functions which vanish outside of compact sets, endowed with the sup norm topology). Now $C_0(E)$ is a subspace of Φ and the identity map of $C_0(E)$ into Φ is continuous; therefore, the identity map of $C_0(E)$ into Λ is continuous. This implies that T_β^2 takes bounded subsets of Λ into relatively compact subsets of Λ.

Theorem 3. A) *If Λ is a continuously regular Köthe space then a bounded subset $\mathfrak{A}$ of Λ is relatively compact if and only if $\{T_\beta\}$ converges uniformly to the identity on $\mathfrak{A}$.*

B) *If Λ is a Köthe space that is not continuously regular then the condition that $\{T_\beta\}$ converge uniformly to the identity on $\mathfrak{A}$ is sufficient but not necessary for the relative compactness of a bounded subset $\mathfrak{A}$ of Λ.*

Proof. Assume that $\mathfrak{A}$ is a bounded subset of a Köthe space Λ and that $\{T_\beta\}$ converges uniformly to the identity on $\mathfrak{A}$.

To begin with, we show that $\mathfrak{A}$ is a relatively compact subset of Ω. Let $n=1,2,\ldots$. We will show that $N_n(T_\beta^2(f)-f)$ tends uniformly to 0 on $\mathfrak{A}$. Since by the corollary to proposition 3.4, $T_\beta^2(\mathfrak{A})$ is relatively compact in Λ and hence also in Ω, this then shows, together with Theorem 1, that $\mathfrak{A}$ is a relatively compact subset of Ω.

Now consider

$$\begin{aligned} N_n(T_\beta^2(f)-f) &= N_n(T_\beta^2(f)-T_\beta(f)+T_\beta(f)-f) \\ &\leqq N_n(T_\beta(T_\beta(f)-f))+N_n(T_\beta(f)-f). \end{aligned} \tag{1}$$

By Proposition 3.3 there exist constants M, $m'\geqq n$ and $\beta_0\in B$ such that

$$N_n(T_\beta(T_\beta(f)-f))\leqq M\, N_{m'}(T_\beta(f)-f)$$

for $\beta>\beta_0$ and for every f in Ω. Hence by (1)

$$N_n(T_\beta^2(f)-f)\leqq (M+1)\, N_{m'}(T_\beta(f)-f) \tag{2}$$

for all $\beta>\beta_0$ and for all f in Ω. But $T_\beta(f)$ tends by hypothesis uniformly to f on $\mathfrak{A}$ in Λ, and hence also in Ω. Hence (2) implies that $N_n(T_\beta^2(f)-f)$ tends uniformly to 0 on $\mathfrak{A}$. Thus, as we noted before, it follows that $\mathfrak{A}$ is a relatively compact subset of Ω.

By [6, Proposition 5] this implies that

for every compact subset K of E and for every $\varepsilon>0$ there exists $\delta>0$ (3)
such that $\sup\limits_{f\in\mathfrak{A}}\int\limits_A |f|\,d\mu<\varepsilon$ whenever A is a measurable subset of K with $\mu(A)<\delta$.

We will use this to show that every $T_\beta(\mathfrak{A})$ is a set of equicontinuous functions. Let $\beta=(n,U)$ be given and $x_0\in E$. Then

$$\begin{aligned} &\sup_{f\in\mathfrak{A}}|T_\beta(f)(x_0)-T_\beta(f)(x)| \qquad (4)\\ &=\sup_{f\in\mathfrak{A}}\left|\int \chi_{E_n} f\left[\frac{\chi_{x_0+U}}{\mu(x_0+U)}-\frac{\chi_{x+U}}{\mu(x+U)}\right]d\mu\right| \\ &\leqq \frac{1}{\mu(x_0+U)}\sup_{f\in\mathfrak{A}}\int|\chi_{E_n}f|\,|\chi_{x_0+U}-\chi_{x+U}|\,d\mu \\ &\quad+\left|\frac{1}{\mu(x_0+U)}-\frac{1}{\mu(x+U)}\right|\sup_{f\in\mathfrak{A}}\int\limits_{E_n}|\chi_{x+U}f|\,d\mu. \end{aligned}$$

But by hypothesis, $\mu(x+U)$ is a continuous function of x, and hence

$$\lim_{x\to x_0}\left|\frac{1}{\mu(x_0+U)}-\frac{1}{\mu(x+U)}\right|=0.$$

Since furthermore $\mathfrak{A}$ is bounded in Λ and hence also in Ω, it follows that the second term in the right hand side of (4) tends to 0 as x tends to x_0.

In order to show that also the first term of the right hand side of (4) tends to 0 as x tends to x_0, we note that for the symmetric difference $(x_0+U)\,\Delta(x+U)$ we have by hypothesis

$$\mu((x_0+U)\,\Delta(x+U)) = \int |\chi_{x_0+U} - \chi_{x+U}|\,d\mu \to 0 \quad \text{as} \quad x\to x_0\,.$$

Hence from (3) we know that

$$\sup_{f\in\mathfrak{A}} \int_{E_n} |f|\,|\chi_{x_0+U} - \chi_{x+U}|\,d\mu = \sup_{f\in\mathfrak{A}} \int_{[(x_0+U)\,\Delta(x+U)]\cap E_n} |f|\,d\mu \to 0$$

as $x\to x_0$. Thus also the first term in the right hand side of (4) tends to 0 as x tends to x_0, which implies that $T_\beta(\mathfrak{A})$ is a set of equicontinuous functions.

Furthermore, if we let m be such that $E_n+U\subset E_m$ then all functions in $T_\beta(\mathfrak{A})$ have their supports in E_m and

$$\sup_{f\in\mathfrak{A}} \|T_\beta(f)\|_\infty \leqq \sup_{f\in\mathfrak{A}} \frac{1}{\mu(x+U)} \sup_{f\in\mathfrak{A}} \int_{E_n} |f|\,d\mu < \infty\,.$$

Exactly as in the proof of the corollary of Proposition 3.4 this implies that $T_\beta(\mathfrak{A})$ is a relatively compact subset of Λ. Hence by Theorem 1, $\mathfrak{A}$ is relatively compact in Λ.

The remainder of the theorem is a direct consequence of Theorem 1.

Our first corollary is the analogue for Köthe sequence spaces of a theorem proved by Cohen and Dunford [5] for Banach spaces of sequences.

Corollary 1. *Suppose Λ is a Köthe space of real or complex sequences which is endowed with a Köthe topology. Let $T_n(x)=\{x_1,\ldots,x_n,0,0,\ldots\}$ for each $x=\{x_k\}_{k=1}^\infty$.*

A) *If $\{T_n\}_{n=1}^\infty$ forms a partition of the identity on Λ, then a bounded subset $\mathfrak{A}$ of Λ is relatively compact if and only if $\{T_n\}$ converges uniformly to the identity on $\mathfrak{A}$.*

B) *If $\{T_n\}_{n=1}^\infty$ is not a partition of the identity on Λ, then the condition that $\{T_n\}$ converge uniformly to the identity on $\mathfrak{A}$ is sufficient but not necessary for the relative compactness of a bounded subset $\mathfrak{A}$ of Λ.*

Proof. Let μ be the Haar measure on the group of all integers under addition, and let $E=\{1,2,\ldots\}$. Let $\mathscr{E}$ consist of the diagonal $U=\{(x,x): x\in E\}$ only, and let $E_n=\{1,2,\ldots,n\}$. Then $T_\beta(x)=T_n(x)$ for $\beta=(n,U)$ and for every $x\in E$. Hence the corollary follows from Theorem 3.

The next proposition characterizes the class of continuously regular Orlicz spaces. This then will establish our Corollaries 2 and 3, an extension of the Kolmogorov theorem to Orlicz spaces.

Proposition 3.5. *Let E be a locally compact abelian topological group, μ the Haar measure on E and L_Φ an Orlicz space on (E,μ). Then L_Φ is a continuously regular Köthe space if and only if Φ satsifies the Δ_2 condition.*

Moreover if Φ does not satisfy the Δ_2 condition then $\{T_\beta\}$ is not even a partition of the identity on L_Φ.

Proof. Assume that Φ satisfies the Δ_2 condition. Then by [16] the simple functions with compact support are dense in L_Φ. But any such simple function can be approximated by continuous functions with compact supports, in the L_Φ norm. For let χ_A be the characteristic function of a relatively compact subset A of E. Since μ is a regular measure there exist compact subsets $K_1 \subset K_2 \subset \cdots$ of A and open, relatively compact sets $O_1 \supset O_2 \supset \cdots$ containing A such that $\mu(O_n \backslash K_n) < \frac{1}{n}$. By Urysohn's lemma there exist continuous functions g_n on E such that $0 \leqq g_n \leqq 1$, $g_n = 1$ on K_n and $g_n = 0$ outside O_n. Then

$$\int \Phi(|\chi_A - g_n|)\, d\mu \leqq \frac{\Phi(2)}{n} \to 0 \qquad (n \to \infty),$$

and by [29] this implies that $\|\chi_A - g_n\|_{L_\Phi}$ tends to 0 as $n \to \infty$. Thus the space $C_0(E)$ of continuous functions with compact support is dense in L_Φ.

Now let g be a continuous real valued function on E with compact support K and let m be such that $E_m \supset K$. Let $\varepsilon > 0$. Since g is uniformly continuous there exists $U_0 \in \mathscr{E}$ such that $|g(x) - g(y)| < \varepsilon$ whenever $y \in x + U_0$. If $\beta_0 = (m, U_0)$ then

$$|g(x) - T_\beta(g)(x)| \leqq \frac{1}{\mu(x+U)} \int_{x+U} |g(x) - g(y)|\, d\mu(y) < \varepsilon$$

for every $x \in E$ and for every $\beta \geqq \beta_0$. Hence $\|g - T_\beta(g)\|_\infty \to 0$, which implies that $\|g - T_\beta(g)\|_{L_\Phi} \to 0$. Thus $\{T_\beta\}$ is a partition of the identity on the dense subset $C_0(E) \subset L_\Phi$. Now by Jensen's inequality we have

$$\int \Phi(|T_\beta(f)|)\, d\mu \leqq \int \Phi(|f|)\, d\mu,$$

and hence by the same argument as in the proof of Proposition 2.3, it follows that

$$\|T_\beta(f)\|_{L_\Phi} \leqq M \|f\|_{L_\Phi} \tag{1}$$

for every $f \in L_\Phi$ and every $\beta \in B$, and that hence L_Φ is a continuously regular Köthe space.

Conversely assume that Φ does not satisfy the Δ_2 condition. Every continuous function with compact support can be uniformly approximated by simple functions with compact support. Hence if $C_0(E)$ were dense in L_Φ then also the set of simple functions with compact support would be dense in L_Φ. But by [16] this is not the case. Thus $C_0(E)$ is not dense in L_Φ. This implies in particular (see Proposition 3.1) that the functions $T_\beta(f)$ cannot be dense in L_Φ, i.e. $\{T_\beta\}$ is not a partition of the identity on L_Φ.

The next two corollaries follow directly from Theorem 3 and the last proposition.

Corollary 2. *Let E be a locally compact abelian topological group with Haar measure μ, and let L_Φ be an Orlicz space on (E, μ).*

A) *If Φ satisfies the Δ_2 condition then a bounded subset $\mathfrak{A}$ of L_Φ is relatively compact if and only if $\{T_\beta\}$ converges uniformly to the identity on $\mathfrak{A}$.*

B) *If Φ does not satsify the Δ_2 condition or if $L_\Phi = L_\infty$ and $L_\infty \neq C_0(E)$, then the condition that $\{T_\beta\}$ converge uniformly to the identity on $\mathfrak{A}$ is sufficient but not necessary for the relative compactness of a bounded subset $\mathfrak{A}$ of L_Φ.*

Corollary 3. *Let E be a locally compact abelian topological group with Haar measure μ and let L_Φ be an Orlicz space on (E, μ) such that Φ satisfies the Δ_2 condition. For every $U \in \mathscr{E}$ and every $f \in L_\Phi$ we define*

$$\tau_U(f)(x) = \frac{1}{\mu(x+U)} \int_{x+U} f \, d\mu .$$

A subset $\mathfrak{A}$ of L_Φ is relatively compact if and only if the following three conditions hold:

1) *$\mathfrak{A}$ is bounded;*
2) *$\{\tau_U\}$ converges uniformly to the identity on $\mathfrak{A}$;*
3) *$\chi_{E_n} f$ converges uniformly to f on $\mathfrak{A}$.*

Proof. Assume that $\mathfrak{A}$ is a subset of L_Φ for which conditions 1), 2) and 3) hold. By Jensen's inequality it follows that

$$\int \Phi(|\tau_U(f)|)\, d\mu \leqq \int \Phi(|f|)\, d\mu$$

for every $U \in \mathscr{E}$ and every $f \in L_\Phi$. As in the proof of Proposition 2.3 this implies that

$$\|\tau_U(f)\|_{L_\Phi} \leqq M \|f\|_{L_\Phi} . \tag{1}$$

Furthermore for every $\beta = (n, U) \in B$ and every $f \in L_\Phi$ we have

$$\|T_\beta(f) - f\|_{L_\Phi} \leqq \|\tau_U(\chi_{E_n} f - f)\|_{L_\Phi} + \|\tau_U(f) - f\|_{L_\Phi} . \tag{2}$$

Hence (1) together with conditions 3) and 2) implies that $\|T_\beta(f) - f\|_{L_\Phi}$ converges uniformly to 0 on $\mathfrak{A}$. By the last corollary this implies that $\mathfrak{A}$ is relatively compact.

Conversely assume that $\mathfrak{A}$ is a relatively compact subset of L_Φ. Since $\chi_{E_n} f$ converges to f for every function f in the subset $C_0(E)$ of L_Φ consisting of all continuous functions with compact support, and since furthermore $\|\chi_{E_n} f\|_{L_\Phi} \leqq \|f\|_{L_\Phi}$ for every $f \in L_\Phi$, it follows from Lemma 2 that $\chi_{E_n} f$ converges to f for every $f \in L_\Phi$. Hence Theorem 1 implies that condition 3) holds.

Furthermore

$$\begin{aligned} \sup_{f \in \mathfrak{A}} \|\tau_U(f) - f\|_{L_\Phi} &\leqq \sup_{f \in \mathfrak{A}} \|\tau_U(f) - T_\beta(f)\|_{L_\Phi} + \sup_{f \in \mathfrak{A}} \|T_\beta(f) - f\|_{L_\Phi} \\ &\leqq \sup_{f \in \mathfrak{A}} \|\tau_U(\chi_{E_n} f - f)\|_{L_\Phi} + \sup_{f \in \mathfrak{A}} \|T_\beta(f) - f\|_{L_\Phi} . \end{aligned} \tag{3}$$

Since $\mathfrak{A}$ is relatively compact, the last corollary implies that

$$\lim_{\beta \in B} \sup_{f \in \mathfrak{A}} \|T_\beta(f) - f\|_{L_\Phi} = 0 . \tag{4}$$

Hence (3) together with condition 3), (1) and (4) implies that $\tau_U(f)$ converges to f uniformly on $\mathfrak{A}$, i.e. condition 2) holds. Condition 1) holds trivially.

If $L_\Phi = L_\infty$ then conditions 1), 2) and 3) in the last corollary are sufficient for the relative compactness of a subset $\mathfrak{A}$ of L_∞. However, as in Corollary 2 of Theorem 3, they are not necessary unless $L_\infty = C_0(E)$.

The last corollary was proved by Kolmogorov [13] for subsets of $L_p([0, 1], \mu)$ $(1 < p < \infty)$, where μ is the Lebesgue measure on R^n. It was extended to $L_p(R^n, \mu)$ by Tamarkin [23]. Tulajkov [24] showed that the theorem of Kolmogorov and Tamarkin is also true if $p = 1$. Takahashi [22] proved our last corollary for the case $E = R^n$.

Tamarkin claimed in [23] that for $L_\Phi = L_p(R^n, \mu)$ $(1 \leqq p < \infty)$ the conditions 1), 2) and 3) in our last corollary are independent of each other. However, Natanson [17] noted that Tamarkin's proof contained an error. Later Sudakov [21] showed that Tamarkin's claim was false. He proved that for $L_\Phi = L_p(R^n, \mu)$ condition 1) in our last corollary follows from condition 2) together with 3). His proof can easily be generalized to show that in Theorems 3 (and hence in its corollaries), the condition that $\mathfrak{A}$ be bounded can be omitted if E is in addition connected, Λ does not contain the non zero constant functions and the Köthe topology on Λ is a norm topology (conditions under which Köthe topologies are norm topologies are given in [28]). However in general the boundedness condition cannot be omitted in Theorem 3 and its corollaries. For if $\Lambda = \ell_1$ and $\mathfrak{A}$ consists of all sequences $\{n, 0, 0, \ldots\}$ $(n = 1, 2, \ldots)$ then $\{T_\beta\}$ converges to the identity uniformly on $\mathfrak{A}$ but $\mathfrak{A}$ is unbounded and hence not relatively compact.

4. Köthe Spaces and the M. Riesz Theorem

We will now show that a compactness theorem of the Riesz type for translation invariant Köthe spaces follows from Theorem 3.

Let E be a locally compact abelian topological group, μ the Haar measure on E and Λ a *translation invariant Köthe space* on (E, μ), i.e. whenever $f \in \Lambda$ and $a \in E$ then the translate f_a defined by $f_a(x) = f(x - a)$, belongs to Λ too. If $n = 1, 2, \ldots$ and $a \in E$, we define the operator $S_{n,a}$ on Λ by

$$S_{n,a}(f) = (\chi_{E_n} f)_a \,.$$

We say that *the functions $S_{n,a}(f)$ converge to a function g in Λ* if for every neighborhood $\mathcal{N}$ of g there exist a natural number N and a $U \in \mathscr{E}$ such that $S_{n,a}(f) \in \mathcal{N}$ for every $n \geqq N$ and for every $a \in U$.

We will call a translation invariant Köthe space *translation regular* if $\{S_{n,a}\}$ is a strong partition of the identity.

Theorem 4. *Let Λ be a translation invariant Köthe space endowed with a Köthe topology.*

A) *If Λ is translation regular then a bounded subset $\mathfrak{A}$ of Λ is relatively compact if and only if $\{S_{n,a}\}$ converges uniformly to the identity on $\mathfrak{A}$.*

B) *If Λ is not translation regular then the condition that $\{S_{n,a}\}$ converge uniformly to the identity on $\mathfrak{A}$ is sufficient but not necessary for the relative compactness of a bounded subset $\mathfrak{A}$ of Λ.*

Proof. Let $\mathcal{K}_{\mathcal{H}}$ be a Köthe topology on Λ. Assume that $\mathfrak{A}$ is a bounded subset of Λ on which $\{S_{n,a}\}$ converges uniformly to the identity. Let $n = 1, 2, \ldots$, $U \in \mathscr{E}$ and $\beta = (n, U)$. If T_β is defined as in the previous section and $H \in \mathcal{H}$ then

$$
\begin{aligned}
&p_H(T_\beta(f) - f) \\
&\quad \leqq \sup_{h\in H} \iint \frac{\chi_{0+U}(y-x)}{\mu(x+U)} |f(x) - \chi_{E_n}(y) f(y)| \, d\mu(y) \, |h(x)| \, d\mu(x) \\
&\quad = \sup_{h \in H} \int \frac{\chi_{0+U}(t)}{\mu(0+U)} \int |f(x) - \chi_{E_n}(x+t) f(x+t)| \, |h(x)| \, d\mu(x) \, d\mu(t) \qquad (1) \\
&\quad \leqq \int \frac{\chi_{0+U}(t)}{\mu(0+U)} \sup_{h\in H} \int |f(x) - (\chi_{E_n} f)_{-t}(x)| \, |h(x)| \, d\mu(x) \, d\mu(t) \\
&\quad \leqq \int \frac{\chi_{0+U}(t)}{\mu(0+U)} \sup_{t \in U} p_H(f - (\chi_{E_n} f)_{-t}) \, d\mu(t) \\
&\quad = \sup_{t\in U} p_H(f - S_{n,t}(f)),
\end{aligned}
$$

which by 1) tends uniformly to 0 on $\mathfrak{A}$ as $\beta \in B$. Thus $\{T_\beta\}$ converges uniformly to the identity on $\mathfrak{A}$, and by Theorem 3 this implies that $\mathfrak{A}$ is relatively compact.

The remaining part of the theorem follows directly from Theorem 1.

The next corollary contains the theorem of M. Riesz [19].

Corollary. A) *If the function Φ satisfies the Δ_2 condition then a bounded subset $\mathfrak{A}$ of $L_\Phi \dotplus L_\infty$ is relatively compact if and only if the following two conditions hold:*

1) *f_a converges uniformly to f on $\mathfrak{A}$;*
2) *$\chi_{E_n} f$ converges uniformly to f on $\mathfrak{A}$.*

B) *If Φ does not satisfy the Δ_2 condition or if $L_\Phi = L_\infty \dotplus C_0(E)$ then conditions 1) and 2) are sufficient but not necessary for the relative compactness of a bounded subset $\mathfrak{A}$ of L_Φ.*

Proof. Assume that $\mathfrak{A}$ is a bounded subset of L_Φ for which conditions 1) and 2) hold. As before let

$$\tau_U(f)(x) = \frac{1}{\mu(x+U)} \int \chi_{x+U} f \, d\mu$$

for every $U \in \mathscr{E}$ and every $f \in L_\Phi$. Exactly as in the proof of the last theorem it follows that

$$\|\tau_U(f) - f\|_{L_\Phi} \leqq \sup_{t \in U} \|f - f_t\|_{L_\Phi}.$$

Hence by Corollary 3 of Theorem 3, $\mathfrak{A}$ is relatively compact.

As we saw in the proof of Proportion 3.5, $C_0(E)$ is dense in $L_\Phi \dotplus L_\infty$ if and only if Φ satisfies the Δ_2 condition. Since on $C_0(E) \subset L_\Phi$, the translation operators $S_a : f \to f_a$ form obviously a partition of the identity, and since furthermore $\|f_a\|_{L_\Phi} = \|f\|_{L_\Phi}$ for every $f \in L_\Phi$, the remaining part of the corollary follows exactly as in the proof of Corollary 3 of Theorem 3.

The last corollary was proved by M. Riesz [19] for $L_p(R^n, \mu)$ where μ is the Lebesgue measure on R^n and $1 \leqq p < \infty$. A. Weil [26] extended the theorem to $L_p(E, \mu)$ where μ is the Haar measure on the locally compact abelian group E. Gribanov [10] extended Riesz' theorem to Orlicz spaces $L_\Phi(R^n, \mu)$ where Φ satisfies the Δ_2 condition.

Since Theorem 4 is a consequence of Theorem 3, it follows that the condition that $\mathfrak{A}$ be bounded can be omitted in Theorem 4 and in its corollary whenever it can be omitted in Theorem 3, i.e. whenever E is in addition connected, Λ does not contain the non-zero constant functions and the Köthe topology on Λ is a norm topology. It can, however, not be omitted in general.

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The Torsion Submodule Splits Off

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A classical question for modules over an integral domain is, "When is the torsion submodule $t(A)$ of a module A a direct summand of A?" A module is said to split when its torsion submodule is a direct summand. Kaplansky has shown [14] that if R is a Dedekind domain, then every module whose torsion submodule is of bounded order splits. The converse of this result has been shown by Chase [5]. Results of [15] and [3] show that every finitely generated module splits if and only if R is a Prüfer domain. Finally, if every R-module splits, then Rotman has shown [19] that R must be a field.

Recently, many concepts of torsion have been proposed for modules over arbitrary associative rings with identity. Almost all of these are special cases of "torsion theories" in the sense of Dickson [6]. Moreover, most of these torsion theories are hereditary (i.e., the submodule of a torsion module is torsion); and hereditary torsion classes are classes of negligible modules associated with a topologizing and idempotent filter of left ideals in the sense of Gabriel [12] (also see [17]). Some recent papers ([4], [7], and [11]) have dealt with splitting results for specific hereditary torsion theories over certain commutative rings.

The main purpose of this paper is to continue the investigation of the splitting properties of a torsion theory of modules over a commutative ring. Some characterizations for the splitting of modules, whose torsion submodules have bounded order, are obtained (see definition of bounded order below and Theorems 2.2 and 4.6). In particular, these results generalize the above-mentioned theorems of Chase [5] and Kaplansky [14]. Our results show that the splitting of modules whose torsion submodules have bounded order frequently forces non-zero torsion modules to have non-zero socles. This increases our interest in the smallest hereditary torsion class $\mathscr{S}$ containing the simple modules. The class $\mathscr{S}$ has previously been used in the study of commutative Noetherian rings and (left) perfect rings (e.g., see [7] and [10] and their references). For a commutative ring R, we show that $\mathscr{S}(A)$ is a summand of each R-module A if and only if non-zero R-modules have non-zero socles. This generalizes the main results of [7] and [11]; moreover, in case R is a Dedekind domain, our result coincides with the above result of Rotman.

We also examine the properties of a splitting hereditary torsion theory of modules over a local ring R (unique maximal left ideal). We show that if R possesses a non-trivial torsion theory $(\mathscr{T}, \mathscr{F})$ such that every finitely generated module splits, then R is an integral domain, and $(\mathscr{T}, \mathscr{F})$ is Goldie's torsion theory. In case R is also commutative, then R is a Prüfer domain. We also prove that a commutative local ring does not admit a non-trivial, hereditary torsion theory $(\mathscr{T}, \mathscr{F})$ such that $\mathscr{T}(A)$ is a summand of every R-module A.

1. Terminology and Notation

In this paper all rings R are associative with identity. Unless otherwise stated all modules will be unitary left R-modules, and the category of all left R-modules will be denoted by ${}_R\mathscr{M}$.

A *torsion theory* of modules is a pair of subclasses $(\mathscr{T}, \mathscr{F})$ of ${}_R\mathscr{M}$ satisfying:

(1) $\mathscr{T} \cap \mathscr{F} = 0$.

(2) $B \subseteqq A$ and $A \in \mathscr{T}$ implies $A/B \in \mathscr{T}$.

(3) $B \subseteqq A$ and $A \in \mathscr{F}$ implies $B \in \mathscr{F}$.

(4) For each $A \in {}_R\mathscr{M}$, there exists an exact sequence

$$0 \to T \to A \to F \to 0$$

such that $T \in \mathscr{T}$ and $F \in \mathscr{F}$.

Then $\mathscr{T}$ is called the *torsion class*, and $\mathscr{F}$ is called the *torsionfree class*. $\mathscr{T}$ is closed under homomorphic images, direct sums, and extensions of one torsion module by another. $\mathscr{F}$ is closed under submodules, direct products, and extensions of one torsionfree module by another. A class is called *hereditary* if it is closed under submodules; and a torsion theory $(\mathscr{T}, \mathscr{F})$ is called *hereditary* if $\mathscr{T}$ is a hereditary class. A hereditary torsion theory $(\mathscr{T}, \mathscr{F})$ is uniquely associated with a topologizing and idempotent *filter* of left ideals $F(\mathscr{T}) = \{I \mid R/I \in \mathscr{T}\}$. We shall always use $F(\mathscr{T})$ as our standard notation for the filter associated with $\mathscr{T}$. $F(\mathscr{T})$ is said to have a *cofinal subset of finitely generated left ideals* if, for each $I \in F(\mathscr{T})$, there exists a finitely generated left ideal $K \in F(\mathscr{T})$ such that $K \subseteqq I$. For further properties of $(\mathscr{T}, \mathscr{F})$ the reader is referred to [1], [6], [11], and [23]. Properties of filters may be found in [12], [17], and [21].

A torsion theory $(\mathscr{T}, \mathscr{F})$ is called *trivial* if either $\mathscr{T} = {}_R\mathscr{M}$ or $\mathscr{F} = {}_R\mathscr{M}$. For any $A \in {}_R\mathscr{M}$, we let $\mathscr{T}(A)$ denote the (unique) largest submodule of A in $\mathscr{T}$.

A module $A \in {}_R\mathscr{M}$ is said to have *bounded order* (relative to a torsion theory $(\mathscr{T}, \mathscr{F})$) if $IA = 0$ for some $I \in F(\mathscr{T})$. In the case where R is not commutative the concept of uniformly negligible [21] seems to be more relevant than the concept of bounded order. A module $A \in \mathscr{T}$ is called *uniformly negligible* if there exists $I \in F(\mathscr{T})$ such that every $x \in A$ is annihilated by a left ideal of the form

$$(I \cap r_1) \cap (I : r_2) \cap \cdots \cap (I : r_n),$$

where n and $r_i \in R$ depend on x. Equivalently, $A \in \mathscr{T}$ is uniformly negligible if the annihilator of every element of A is in the smallest filter of left ideals containing some fixed $I \in F(\mathscr{T})$. In case R is a commutative ring, the concepts of "bounded order" and "uniformly negligible" coincide.

Let $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory. A module A is said to *split* (relative to $(\mathscr{T}, \mathscr{F})$) if $\mathscr{T}(A)$ is a direct summand of A. We say $(\mathscr{T}, \mathscr{F})$ has the *splitting property* (SP) if every $A \in {}_R\mathcal{M}$ splits. A torsion theory $(\mathcal{T}, \mathcal{F})$ is said to have the *bounded splitting property* (BSP) if every module, whose torsion submodule has bounded order, splits. A torsion theory $(\mathcal{T}, \mathcal{F})$ has the *uniformly negligible splitting property* (UNSP) if every module, whose torsion submodule is uniformly negligible, splits. Finally, we say that a torsion theory $(\mathcal{T}, \mathcal{F})$ has the *finitely generated splitting property* (FGSP) if every finitely generated module splits.

When a torsion theory $(\mathcal{T}, \mathcal{F})$ for ${}_R\mathcal{M}$ has BSP, consider the ring $R' = R/\mathcal{T}(R)$. Then $F(\mathcal{T})$ induces a topologizing and idempotent filter of left ideals

$$F(\mathcal{T}') = \{K/\mathcal{T}(R) \mid K \supseteqq \mathcal{T}(R) \quad \text{and} \quad K \in F(\mathcal{T})\}$$

in R' such that the torsion theory $(\mathcal{T}', \mathcal{F}')$ for ${}_{R'}\mathcal{M}$ corresponding to $F(\mathcal{T}')$ has BSP and $\mathcal{T}'(R') = 0$. In light of these comments, it is sufficient to consider only torsion theories $(\mathcal{T}, \mathcal{F})$ such that $\mathcal{T}(R) = 0$ whenever we are studying BSP. Similar remarks hold for torsion theories with SP or FGSP. Note that whenever $\mathcal{T}(R) = 0$, then every left ideal in $F(\mathcal{T})$ is essential in R.

Next, we mention two special torsion theories which occur frequently in the proofs of our results:

(1) $(\mathcal{G}, \mathcal{N})$ denotes Goldie's torsion theory (see [1], [13], or [24]). $\mathcal{N}$ is the class of non-singular modules, and $F(\mathcal{G})$ is the smallest topologizing and idempotent filter of left ideals containing all the essential left ideals of R. If $\mathcal{G}(R) = 0$, then $\mathcal{G}(M)$ is the singular submodule of M. When R is an integral domain, then $\mathcal{G}$ coincides with the usual torsion class. The filter $F(\mathcal{G})$ possesses a cofinal subset of finitely generated left ideals if and only if R contains no infinite direct sums of left ideals in $\mathcal{N}$ (see [24] for other equivalent conditions).

(2) $(\mathcal{S}, \mathcal{F})$ denotes the simple torsion theory (see [6], [7], [8], or [11]). $\mathcal{S}$ is the smallest torsion class containing the simple modules, and $\mathcal{F}$ is the class of modules with zero socles. In case R is a Dedekind domain, then $\mathcal{S}$ coincides with the usual torsion class. $\mathcal{S}(R) = 0$ if and only if R has zero socle.

Finally, we mention some notational conventions. When R is commutative, we can make $A \in {}_R\mathcal{M}$ into a right R-module in the natural way and can form $A \otimes_R B$ for $B \in {}_R\mathcal{M}$. For brevity all such tensor products over R will simply be written as $A \otimes B$. We denote by $\operatorname{Soc}(A)$ the sum of all simple submodules of A. $E(A)$ denotes the injective envelope of A. For homological notations and results, the reader is referred to [3] and [18]. For notation concerning localization by a maximal ideal of a commutative ring, the reader is directed to [25].

2. Torsion Theories with BSP

In this section we give a characterization of the bounded splitting property (BSP) for hereditary torsion theories $(\mathcal{T}, \mathcal{F})$ for modules over a commutative ring R satisfying: (i) $\mathcal{T}(R) = 0$ and (ii) the filter $F(\mathcal{T})$ possesses a cofinal subset of finitely generated ideals. Note that the hypotheses (i) and (ii) are not very restrictive; indeed from the remarks of Section 1, (i) is no real restriction at all. Both (i) and (ii) hold for the usual torsion theory over an integral domain and for the torsion theory of Levy [16] over any commutative ring. Conditions implying (ii) are studied in [23] and [24]; obviously, (ii) holds for any torsion theory over a commutative Noetherian ring.

We begin with an elementary lemma:

Lemma 2.1. *Let R be a commutative ring, and let $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory for ${}_R\mathcal{M}$ such that $\mathcal{T}(R)=0$. Then each statement implies the next:*

(1) *$(\mathcal{T}, \mathcal{F})$ has BSP.*

(2) *$\mathrm{Tor}_1^R(A, R/I)=0$ for all $A \in \mathcal{F}$, for all $I \in F(\mathcal{T})$.*

(3) *For each integer $n \geqq 1$, $\mathrm{Tor}_n^R(A, R/I)=0$ for all $A \in \mathcal{F}$, for all $I \in F(\mathcal{T})$.*

(4) *I is flat for all $I \in F(\mathcal{T})$.*

Proof. (1)$\Rightarrow$(2): Since $I \in F(\mathcal{T})$, it is easily verified that $\mathrm{Hom}_R(R/I, E) \in \mathcal{T}$ is of bounded order for any (injective) module E. So by [3] VI 5.1 and (1), we have $0=\mathrm{Ext}_R(A, \mathrm{Hom}_R(R/I, E)) \cong \mathrm{Hom}_R(\mathrm{Tor}_1^R(A, R/I), E)$ for any $A \in \mathcal{F}$. Since E can be any injective module, $\mathrm{Tor}_1^R(A, R/I)=0$ as desired.

(2)$\Rightarrow$(3): Note (2) serves as a first step for induction. So suppose that for all $k<n$, $I \in F(\mathcal{T})$, and $B \in \mathcal{F}$ we have $\mathrm{Tor}_k^R(B, R/I)=0$. Given $A \in \mathcal{F}$ choose an exact sequence

$$0 \rightarrow K \rightarrow F \rightarrow A \rightarrow 0$$

with F free. Since $\mathcal{T}(R)=0$, then $K \in \mathcal{F}$ by the fact that $\mathcal{F}$ is closed under direct products and submodules. Hence the induced exact sequence

$$0=\mathrm{Tor}_n^R(F, R/I) \rightarrow \mathrm{Tor}_n^R(A, R/I) \rightarrow \mathrm{Tor}_{n-1}^R(K, R/I)$$

has 0 as its right end term by our induction hypothesis. Therefore, $\mathrm{Tor}_n^R(A, R/I)=0$ by exactness, and thus (3) follows from (2) by induction.

(3)$\Rightarrow$(4): Since $\mathcal{T}(R)=0$, every ideal K is in $\mathcal{F}$. Thus for each $K \subseteqq R$ and $I \in F(\mathcal{T})$, the sequence

$$0=\mathrm{Tor}_{n+1}^R(R, R/I) \rightarrow \mathrm{Tor}_{n+1}^R(R/K, R/I) \rightarrow \mathrm{Tor}_n^R(K, R/I)=0$$

is exact for each $n \geqq 1$ by (3). Hence $\mathrm{Tor}_n^R(R/K, R/I)=0$ for all $n \geqq 2$ by exactness. But then the exact sequence

$$0=\mathrm{Tor}_{n+1}^R(R/K, R/I) \rightarrow \mathrm{Tor}_n^R(R/K, I) \rightarrow \mathrm{Tor}_n^R(R/K, R)=0$$

yields $\mathrm{Tor}_n^R(R/K, I)=0$ for all $n \geqq 1$, all $K \subseteqq R$, and all $I \in F(\mathcal{T})$. Thus I is flat by an exercise of [3] (p. 123).

Now we are ready to give a characterization of BSP for a hereditary theory $(\mathcal{T}, \mathcal{F})$ of modules over a commutative ring such that $\mathcal{T}(R)=0$ and $F(\mathcal{T})$ has a cofinal subset of finitely generated ideals.

Theorem 2.2. *Let R be a commutative ring. Suppose $(\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory for ${}_R\mathcal{M}$ such that $\mathcal{T}(R)=0$ and $F(\mathcal{T})$ has a cofinal subset of finitely generated ideals. Then the following are equivalent:*

(1) *$(\mathcal{T}, \mathcal{F})$ has BSP.*

(2) *For each* $I \in F(\mathscr{T})$, R/I *is a perfect ring and*

$$\mathrm{Tor}_1^R(A, R/I) = 0 \quad \text{for all} \quad A \in \mathscr{F}.$$

Proof. (1)⇒(2): Let $I \in F(\mathscr{T})$. By Lemma 2.1, $\mathrm{Tor}_n^R(A, R/K) = 0$ for all integers $n \geqq 1$ and all $A \in \mathscr{F}$, where $K \subsetneqq I$ is a finitely generated ideal in $F(\mathscr{T})$. Then for any R/K-module B, we have

$$\mathrm{Ext}_{R/K}(A \otimes R/K, B) \cong \mathrm{Ext}_R(A, B) = 0$$

by (1) and [3] VI 4.1.3. Hence $A \otimes R/K \cong A/KA$ is projective as an R/K-module whenever $A \in \mathscr{F}$.

Now for any given index set J of cardinality $\xi \geqq |R|$, choose an exact sequence

$$0 \to \ker \sigma \to F \xrightarrow{\sigma} \prod_{\alpha \in J} R^{(\alpha)} \to 0$$

with F free and $R^{(\alpha)} \cong R$. Tensoring by R/K, we obtain the induced exact sequence

$$0 \to \ker \sigma' \to F \otimes R/K \xrightarrow{\sigma'} \prod_{\alpha \in J} R^{(\alpha)} \otimes R/K \to 0$$

which splits since $\prod_{\alpha \in J} R^{(\alpha)} \otimes R/K$ is R/K-projective by the preceeding paragraph. Hence $\prod_{\alpha \in J} R^{(\alpha)}/K \prod_{\alpha \in J} R^{(\alpha)}$ can be embedded as an R/K-direct summand in $F \otimes R/K \cong \oplus \sum R/K$. But since K is finitely generated, it is easily verified that

$$\prod_{\alpha \in J} R^{(\alpha)}/K \prod_{\alpha \in J} R^{(\alpha)} \cong \prod_{\alpha \in J} [R^{(\alpha)}/K^{(\alpha)}]$$

where $K^{(\alpha)} \cong K$. Hence by [5] Theorem 3.1 and [2] Theorem P, R/K is a perfect ring. Since R/I is a homomorphic image of R/K, then R/I is also perfect.

(2)⇒(1): It is sufficient to show that $\mathrm{Ext}_R(A, B) = 0$ for all $A \in \mathscr{F}$ and all $B \in \mathscr{T}$ of bounded order. By (2), Lemma 2.1, and [3] VI 4.1.3, we obtain

$$\mathrm{Ext}_R(A, B) \cong \mathrm{Ext}_{R/I}(A \otimes R/I, B)$$

where $IB = 0$ and $I \in F(\mathscr{T})$. Since R/I is perfect by (2), it follows from [2] Theorem P that it is sufficient to show $A \otimes R/I$ is flat as an R/I-module. But by (2) and Lemma 2.1, $\mathrm{Tor}_n^R(A, R/I) = 0$ for each integer $n \geqq 1$. Thus [3] VI 4.1.2 yields

$$\mathrm{Tor}_n^{R/I}(B, A \otimes R/I) \cong \mathrm{Tor}_n^R(B, A) \tag{$*$}$$

for any R/I-module B and any $n \geqq 1$.

As in [8], define $T^1(B) = \mathrm{Soc}(B)$. If α is not a limit ordinal, let $T^\alpha(B)$ be the inverse image (under the natural map) in B of $\mathrm{Soc}(B/T^{\alpha-1}(B))$. If α is a limit ordinal, define $T^\alpha(B) = \bigcup_{\gamma < \alpha} T^\gamma(B)$. Since R/I is perfect by (2), it follows from [2] Theorem P that $T^\sigma(B) = B$ for some ordinal σ. But from (2) and Lemma 2.1

it is easy to see that $\mathrm{Tor}_n^R(T^\alpha(B), A)=0$ for all $\alpha \leqq \sigma$ by transfinite induction. So from (*) it follows that $A \otimes R/I$ is a flat R/I-module as desired.

An easy consequence of Theorem 2.2 is the following result of Kaplansky ([14], p. 334):

Corollary 2.3. *Let R be a Dedekind domain. Then the usual torsion theory $(\mathscr{T}, \mathscr{F})$ for ${}_R\mathscr{M}$ has BSP.*

Proof. If $I \in F(\mathscr{T})$, then non-zero R/I-modules have non-zero socles by the remarks on $(\mathscr{S}, \mathscr{F})$ in Section 1. Hence the radical of R/I is T-nilpotent. Moreover, since I is a (unique) product of finitely many maximal ideals, R is semi-local. Thus R/I is perfect by [2] Theorem P. Since $\mathrm{Tor}_1^R(\_\,, F)=0$ for $F \in \mathscr{F}$ by [3] VII 4.2, the corollary follows from Theorem 2.2.

Remarks. (1) [5] Theorem 4.3 is also a special case of Theorem 2.2. However, we shall prove this as a corollary to Theorem 4.6, as Lemma 3.1 is very useful in obtaining a short proof of Chase's result.

(2) If Theorem 2.2 applies to a torsion theory $(\mathscr{T}, \mathscr{F})$, then R/I is perfect for each $I \in F(\mathscr{T})$. Hence non-zero modules in $\mathscr{T}$ have non-zero socles. When a torsion theory has this latter property, then $\mathscr{T}$ is said to be a *torsion theory of simple type.* (This terminology is due to Alin.) Torsion theories of simple type have been used in several papers to study perfect and semi-perfect rings.

(3) Also when R/I is perfect for each $I \in F(\mathscr{T})$, then R/I has a primary decomposition in the sense of [8], i.e., R/I is a T-ring. So it is not difficult to see that each module in $\mathscr{T}$ decomposes into a direct sum of its primary submodules.

(4) The "Tor condition" in (2) of Theorem 2.2 is not "nice" in the sense that it is not a completely ring-theoretical condition. In Section 4 we shall see that the "Tor condition" can be replaced by ring-theoretical conditions when we add the hypothesis that $\mathscr{T}$ is stable (i.e., closed under injective envelopes).

3. Local Rings and Splitting

By a *local* ring, we mean only a ring with unique maximal left ideal. No chain conditions are assumed.

The main purpose of this section is to examine the splitting properties of modules relative to a non-trivial hereditary torsion theory $(\mathscr{T}, \mathscr{F})$ for a local ring R. If such a $(\mathscr{T}, \mathscr{F})$ has FGSP, then R is an integral domain and $(\mathscr{T}, \mathscr{F})$ is Goldie's torsion theory; moreover, if R is also commutative, then R is a Prüfer domain (and hence a valuation ring). Analogously, if such a torsion theory has UNSP and if $\mathscr{T}$ is closed under injective envelopes, then R is a principal ideal domain whenever R is commutative. As a consequence of this, we show that a commutative local ring does not admit a non-trivial, hereditary torsion theory with SP.

Before establishing these theorems, we prove two preliminary results which do not require R to be a local ring.

Lemma 3.1. *If R is a commutative semi-hereditary ring, then finitely generated ideals of R/Rx are finitely related for each $x \in R$.*

Proof. Let $x = x_0 \in R$ and $\left(\sum_{k=0}^{n} Rx_k\right)\Big/Rx_0$ be a finitely generated ideal of R/Rx_0. Now consider the commutative diagram:

$$\begin{array}{ccccccc}
0 \to K \xrightarrow{\xi} & \oplus \sum_{k=0}^{n} R & \xrightarrow{\sigma} & \sum_{k=0}^{n} Rx_k \to 0 \\
& v\downarrow & & \downarrow \\
& \oplus \sum_{k=0}^{n} \dfrac{R}{Rx_0} & \xrightarrow{\sigma'} & \left(\sum_{k=0}^{n} Rx_k\right)\Big/Rx_0 \to 0
\end{array}$$

where $\sigma'\left(\sum_{k=0}^{n} (r_k + Rx_0)\right) = \left(\sum_{k=0}^{n} r_k x_k\right) + Rx_0$. Since R is semi-hereditary, we can write $\sum_{k=0}^{n} R = \xi K \oplus D$, where $t: \sum_{k=0}^{n} Rx_k \to D$ is an isomorphism such that $\sigma t = 1$. Then if we let t' be the restriction of t to Rx_0, an easy diagram chase shows that $\ker \sigma' = v\xi K + vt' Rx_0$, and hence $\ker \sigma'$ is finitely generated.

Let $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory for ${}_R\mathscr{M}$. As in [21] a module $A \in \mathscr{T}$ will be called *uniformly negligible* if there exists $I \in F(\mathscr{T})$ such that every element x of A is annihilated by a left ideal of the form

$$(I : r_1) \cap (I : r_2) \cap \cdots \cap (I : r_n),$$

where n depends on x and where each $r_i \in R$. In case R is a commutative ring, the concepts of "bounded order" and "uniformly negligible" coincide.

In [12] a torsion class $\mathscr{T}$ is called *stable* if $\mathscr{T}$ is closed under injective envelopes (i.e., $A \in \mathscr{T} \Rightarrow E(A) \in \mathscr{T}$). A torsion theory $(\mathscr{T}, \mathscr{F})$ is called stable if $\mathscr{T}$ is stable. As examples of stable torsion theories, we note the following: Goldie's torsion theory [1] is always stable. Any hereditary torsion theory over a commutative Noetherian ring or a commutative perfect ring is stable (see [12] and [8]). Any torsion theory with SP is stable (see [22]).

Lemma 3.2. *Let $(\mathscr{T}, \mathscr{F})$ be a stable, hereditary torsion theory for ${}_R\mathscr{M}$. If $A \in {}_R\mathscr{M}$ is finitely generated, then $\mathscr{T}(A)$ is uniformly negligible.*

Proof. If $\mathscr{T}(A) = A$ for a finitely generated module A, then the result is obvious.

If $\mathscr{T}(A) \neq A$ and A is finitely generated, choose $B \subsetneqq A$ maximal with respect to $B \cap \mathscr{T}(A) = 0$. Then $\mathscr{T}(A) \oplus B/B$ is essential in A/B. Since $(\mathscr{T}, \mathscr{F})$ is stable, it follows that $\mathscr{T}(A/B) = A/B$. Hence A/B is uniformly negligible. But since $\mathscr{T}(A)$ is isomorphic to a submodule of A/B, it follows that $\mathscr{T}(A)$ is also uniformly negligible.

Recall that a hereditary torsion theory $(\mathscr{T}, \mathscr{F})$ is said to have the *uniformly negligible splitting property* (UNSP) if each module A, such that $\mathscr{T}(A)$ is uniformly negligible, splits. We now examine local rings which possess a non-trivial, stable, hereditary torsion theory with UNSP.

Theorem 3.3. *Let R be a local ring. If there exists a non-trivial, stable, hereditary torsion theory $(\mathscr{T}, \mathscr{F})$ with UNSP for ${}_R\mathscr{M}$, then R is an integral domain with simple left classical quotient ring, and $(\mathscr{T}, \mathscr{F})$ is Goldie's torsion theory. In case R is also commutative, then R is a principal ideal domain.*

Proof. Since $(\mathscr{T}, \mathscr{F})$ is non-trivial, then $M \in F(\mathscr{T})$, where M is the unique maximal (left) ideal of R. Hence any simple R-module is in $\mathscr{T}$. Now we claim that every non-zero left ideal of R is in $F(\mathscr{T})$. For suppose I is a left ideal. Then by Lemma 3.2 and UNSP, we have $R/I = \mathscr{T}(R/I) \oplus L/I$ for some L. But this leads to at least two maximal left ideals in R unless $\mathscr{T}(R/I) = 0$ or $L/I = 0$. Since R is local, it follows that $R/I \in \mathscr{T}$ or else $R/I \in \mathscr{F}$. Now if $I \neq 0$, then choose $0 \neq x \in I$ and $K \subsetneqq I$ maximal with respect to $x \notin K$. Then $Rx + K/K$ is an essential simple submodule of I/K. Since $R/K \in \mathscr{T}$ or $R/K \in \mathscr{F}$, it follows from $0 \neq Rx + K/K \in \mathscr{T}$ that $R/K \in \mathscr{T}$. But $R/K \to R/I \to 0$ is exact, and hence $R/I \in \mathscr{T}$ also.

Since $(\mathscr{T}, \mathscr{F})$ is non-trivial, it also follows from the preceeding paragraph that $\mathscr{T}(R) = 0$. Hence $(0:x) \notin F(\mathscr{T})$ for each $x \in R$; so $0 \neq x \in R$ implies $(0:x) = 0$. Therefore R is an integral domain. Moreover, $\mathscr{T}(R) = 0$ implies every left ideal in $F(\mathscr{T})$ is essential, and hence R has a left classical quotient ring. Since $F(\mathscr{T})$ consists of all non-zero left ideals, we have $F(\mathscr{T}) = F(\mathscr{G})$. Since every hereditary torsion theory is completely determined by its filter, we have $\mathscr{T} = \mathscr{G}$.

Now assume that R is commutative. Then every ideal of R is flat by Lemma 2.1. Hence the global weak dimension of R is $\leqq 1$. So by [9] Theorem 4, R is a valuation ring. In particular, R is semi-hereditary. So for each $x \neq 0$, finitely generated ideals of R/Rx are finitely related by Lemma 3.1. Since R/Rx is perfect by Theorem 2.2, then R/Rx is Artinian by [5] Theorem 3.4. It follows that every ideal of R is finitely generated. But a Noetherian valuation ring is a principal ideal domain.

Remark. By Corollary 2.3, we know that a commutative principal ideal domain, which is not a field, possesses a non-trivial torsion theory with BSP.

The first part of the following theorem can be proved by exactly the same method as we used for Theorem 3.3. The second part of the theorem is immediate from the theorem of [15].

Theorem 3.4. *Let R be a local ring. If there exists a non-trivial, hereditary torsion theory $(\mathscr{T}, \mathscr{F})$ with FGSP for ${}_R\mathscr{M}$, then R is an integral domain with simple left classical quotient ring, and $(\mathscr{T}, \mathscr{F})$ is Goldie's torsion theory. In case R is also commutative, then R is a Prüfer domain (and hence a valuation ring).*

Remark. In case R is a Prüfer domain, which is not a field, then R possesses a non-trivial torsion theory with FGSP by [3] VII 4.1. The reader should also compare Theorem 3.1 with [16] Theorem 6.1, since Levy's torsion theory coincides with Goldie's torsion theory whenever R possesses a semi-simple two-sided classical quotient ring.

Theorem 3.5. *A commutative local ring R does not admit a non-trivial, hereditary torsion theory with SP.*

Proof. Suppose not, i.e., assume that $(\mathcal{T}, \mathcal{F})$ is a non-trivial, hereditary torsion theory with SP. Then by [22] Lemma 1.1, $(\mathcal{T}, \mathcal{F})$ is stable. Hence R is a principal ideal domain and $(\mathcal{T}, \mathcal{F})$ is the usual torsion theory by Theorem 3.3. But then Rotman's theorem [19] implies that R is a field. This contradicts the fact that $(\mathcal{T}, \mathcal{F})$ is non-trivial.

4. Stable Torsion Theories with BSP

In this section we use the results of § 3 to sharpen the characterization of BSP (Theorem 2.2) in the case where $(\mathcal{T}, \mathcal{F})$ is stable. In particular, we characterize BSP for the stable, hereditary torsion theories $(\mathcal{T}, \mathcal{F})$ of modules over a commutative ring R satisfying: (1) $\mathcal{T}(R)=0$ and (2) the filter $F(\mathcal{T})$ possesses a cofinal subset of finitely generated ideals.

Recall that $(\mathcal{T}, \mathcal{F})$ is stable if $\mathcal{T}$ is closed under injective envelopes.

We now prove a sequence of technical lemmas which will lead us to the desired characterization of BSP (Theorem 4.6).

Lemma 4.1. *Let R be a commutative ring, and let $(\mathcal{T}, \mathcal{F})$ be a torsion theory of simple type for ${}_R\mathcal{M}$ such that $\mathcal{T}(R)=0$. Suppose that $M \in F(\mathcal{T})$ is a maximal ideal and that $(\mathcal{S}_M, \mathcal{F}_M)$ is the simple theory for the local ring R_M. Then for all $A \in {}_{R_M}\mathcal{M}$, $\mathcal{S}_M(A) \cong \mathcal{S}_M(A) \otimes R_M = \mathcal{T}(A) \otimes R_M$.*

Proof. From the exact sequence

$$0 \to \mathcal{T}_s(A) \to \mathcal{T}(A) \to \mathcal{T}(A)/\mathcal{T}_s(A) \to 0$$

we obtain the exact sequence

$$0 \to \mathcal{T}_s(A) \otimes R_M \to \mathcal{T}(A) \otimes R_M \to [\mathcal{T}(A)/\mathcal{T}_s(A)] \otimes R_M \to 0, \tag{$*$}$$

where $\mathcal{T}_s$ is the smallest torsion class containing S (see [8]). Now $\mathcal{S}_M(A) = \mathcal{T}_s(A)$ as sets by [11] Lemma 2.7, and thus $\mathcal{S}_M(A) \otimes R_M = \mathcal{T}_s(A) \otimes R_M$. Let $C = \mathcal{T}(A)/\mathcal{T}_s(A)$. Then it is sufficient to show that $C \otimes R_M = 0$ by $(*)$.

Note that

$$C \otimes R_M = [\mathcal{T}(A)/\mathcal{T}_s(A)] \otimes R_M \cong \frac{\mathcal{T}(A) \otimes R_M}{\mathcal{T}_s(A) \otimes R_M} \subseteq \frac{A}{\mathcal{S}_M(A)}$$

since R_M is R-flat. Hence $\mathcal{T}_s(C \otimes R_M) = 0$. Consider the exact sequence

$$0 \to T^1(C) \to C \to C/T^1(C) \to 0,$$

where T^1 is defined as in the proof of Theorem 2.2. This induces the exact sequence

$$0 = T^1(C) \otimes R_M \to C \otimes R_M \to [C/T^1(C)] \otimes R_M \to 0,$$

since $S' \otimes R_M = 0$ for any simple S' not isomorphic to R/M. It follows that $[C/T^1(C)] \otimes R_M \cong C \otimes R_M$. We proceed by transfinite induction.

Let α be a non-limit ordinal, and assume that $C\otimes R_M \cong [C/T^{\alpha-1}(C)]\otimes R_M$. If $S\cong R/M$ and $S\subsetneq T^{\alpha}(C)/T^{\alpha-1}(C)$, then

$$S\cong S\otimes R_M \subsetneq [T^{\alpha}(C)/T^{\alpha-1}(C)]\otimes R_M \subsetneq [C/T^{\alpha-1}(C)]\otimes R_M .$$

But then by induction hypothesis, S is isomorphic to a submodule of $C\otimes R_M \in \mathscr{F}_M$, which is a contradiction. Hence S' simple and $S'\subsetneq T^{\alpha}(C)/T^{\alpha-1}(C)$ implies $S'\ncong S$. Therefore $[T^{\alpha}(C)/T^{\alpha-1}(C)]\otimes R_M = 0$, giving the exact sequence

$$0=[T^{\alpha}(C)/T^{\alpha-1}(C)]\otimes R_M \to [C/T^{\alpha-1}(C)]\otimes R_M \to [C/T^{\alpha}(C)]\otimes R_M \to 0 .$$

So by exactness and the induction hypothesis, we have

$$[C/T^{\alpha}(C)]\otimes R_M \cong [C/T^{\alpha-1}(C)]\otimes R_M \cong C\otimes R_M .$$

Let α be a limit ordinal, and suppose that $[C/T^{\beta}(C)]\otimes R_M \cong C\otimes R_M$ for each ordinal $\beta<\alpha$. Then the collection $\{T^{\beta}(C)\}_{\beta<\alpha}$ forms a directed system whose direct limit is $T^{\alpha}(C)$. Hence $C/T^{\alpha}(C)$ is a direct limit of the directed system $\{C/T^{\beta}(C)\}_{\beta<\alpha}$. Since "$\otimes$" commutes with direct limits, we have

$$C/T^{\alpha}(C)\otimes R_M \cong \varinjlim [C/T^{\beta}(C)]\otimes R_M \cong C\otimes R_M .$$

By transfinite induction it follows that $C\otimes R_M \cong [C/T^{\alpha}(C)]\otimes R_M$ for each ordinal α. But since $\mathscr{T}$ is of simple type and C is a homomorphic image of $\mathscr{T}(A)$, then $T^{\gamma}(C)=C$ for some ordinal γ, whence $C\otimes R_M \cong [C/T^{\gamma}(C)]\otimes R_M = 0$ as desired.

Lemma 4.2. *Let R be a commutative ring, and let $(\mathscr{T}, \mathscr{F})$ be a torsion theory of simple type for ${}_R\mathscr{M}$ such that $\mathscr{T}(R)=0$. Let M be a maximal ideal in $F(\mathscr{T})$. If $B\in\mathscr{F}$ can be embedded in $R_M\otimes B$ via $b\to 1\otimes b$, then $\mathscr{T}(R_M\otimes B)=0$.*

Proof. Consider B as an R-submodule of $R_M\otimes B$, and note that

$$v: R_M\otimes B \to R_M\otimes B/\mathscr{T}(R_M\otimes B)$$

is a natural epimorphism of R-modules. Since $B\cap\mathscr{T}(R_M\otimes B)=0$ by $\mathscr{T}(B)=0$, then $\bar{v}=v\,|\,B$ induces the exact sequence

$$0\to B \xrightarrow{\bar{v}} R_M\otimes B/\mathscr{T}(R_M\otimes B) .$$

Hence the sequence

$$0\to R_M\otimes B \to R_M\otimes (R_M\otimes B/\mathscr{T}(R_M\otimes B))$$

is exact since R_M is R-flat. But using Lemma 4.1, we have

$$R_M\otimes\left[\frac{R_M\otimes B}{\mathscr{T}(R_M\otimes B)}\right] \cong \frac{R_M\otimes(R_M\otimes B)}{R_M\otimes\mathscr{T}(R_M\otimes B)} \cong \frac{R_M\otimes B}{\mathscr{S}_M(R_M\otimes B)} \in \mathscr{F}_M ,$$

where $(\mathscr{S}_M, \mathscr{F}_M)$ denotes the simple torsion theory for R_M. Hence $\mathscr{S}_M(R_M\otimes B)=0$ follows from $\mathscr{F}_M$ closed under submodules. So by [11] Lemma 2.7, $\mathscr{T}_s(R_M\otimes B)=0$, where $\mathscr{T}_s$ is the smallest torsion class containing $S\cong R/M$.

We now claim $\mathrm{Soc}_R(R_M \otimes B)=0$. For if $R/M' \cong S' \subseteq \mathrm{Soc}_R(R_M \otimes B)$, let $0 \neq [r/t] \otimes x \in S'$. Then there exists $v \in (0:[r/t]\otimes x)=M'$ such that $v \notin M$. Hence $[r/t]\otimes x = [\frac{1}{v}](v\{[r/t]\otimes x\})=0$, which is a contradiction.

But from $\mathrm{Soc}_R(R_M \otimes B)=0$, it follows that $\mathscr{T}(R_M \otimes B)=0$, because $\mathscr{T}$ is of simple type.

Lemma 4.3. *Let R be a commutative ring, and let $(\mathscr{T}, \mathscr{F})$ be a torsion theory of simple type for ${}_R\mathscr{M}$ such that $\mathscr{T}(R)=0$. For a maximal ideal M of R, let $(\mathscr{S}_M, \mathscr{F}_M)$ denote the simple torsion theory for the local ring R_M. If $(\mathscr{T}, \mathscr{F})$ has BSP, then $(\mathscr{S}_M, \mathscr{F}_M)$ has BSP for each maximal ideal M in $F(\mathscr{T})$. Moreover, if $(\mathscr{T}, \mathscr{F})$ has SP, then $(\mathscr{S}_M, \mathscr{F}_M)$ also has SP for each maximal ideal M in $F(\mathscr{T})$.*

Proof. First, suppose that $(\mathscr{T}, \mathscr{F})$ has BSP. Let $A \in {}_{R_M}\mathscr{M}$ such that $\mathscr{S}_M(A)$ has bounded order (relative to $(\mathscr{S}_M, \mathscr{F}_M)$). Then by [11] Theorem 2.6, $\mathscr{S}_M(A)$ is annihilated by a MR_M-primary ideal KR_M of R_M, where $K \in F(\mathscr{T}_s) \subseteq F(\mathscr{T})$. Hence as an R-module, $\mathscr{S}_M(A)$ is annihilated by $KR_M \cap R = K$, since K is M-primary (see [25] for notation). Hence as a R-module, $\mathscr{S}_M(A)$ is of bounded order relative to $(\mathscr{T}, \mathscr{F})$. Since $A \cong A \otimes R_M$ via $\theta: a \to a \otimes 1$, then the restriction of θ to $\mathscr{S}_M(A)$ and $\mathscr{T}(A)$ produce isomorphisms of these modules into $A \otimes R_M$. But $\mathscr{S}_M(A)=\mathscr{T}_s(A) \subseteq \mathscr{T}(A)$; and hence $t \in \mathscr{T}(A)-\mathscr{S}_M(A)$ implies $t \otimes 1 \in \mathscr{T}(A) \otimes R_M - \mathscr{S}_M(A) \otimes R_M$, which contradicts Lemma 4.1. Therefore, $\mathscr{S}_M(A)=\mathscr{T}(A)$ is of bounded order, so by BSP, there is a R-module B such that

$$A=\mathscr{T}(A)\oplus B=\mathscr{S}_M(A)\oplus B.$$

Tensoring by R_M, we obtain

$$A \cong [\mathscr{S}_M(A)\otimes R_M]\oplus[B\otimes R_M] \cong \mathscr{S}_M(A)\oplus[B\otimes R_M].$$

But by Lemma 4.2, $B \otimes R_M \in \mathscr{F}$; whence

$$\mathscr{S}_M(B\otimes R_M)=\mathscr{T}_s(B\otimes R_M)\subseteq \mathscr{T}(B\otimes R_M)=0.$$

Therefore, $(\mathscr{S}_M, \mathscr{F}_M)$ has BSP.

The second part of the lemma can be proved similarly.

Lemma 4.4. *Let R be a commutative ring, and let $(\mathscr{T}, \mathscr{F})$ be a stable torsion theory of simple type such that $\mathscr{T}(R)=0$. If $(\mathscr{T}, \mathscr{F})$ has BSP and M is a maximal ideal in $F(\mathscr{T})$, then the simple theory $(\mathscr{S}_M, \mathscr{F}_M)$ for R_M is stable.*

Proof. By Skornjakov's theorem [20], it is sufficient to show that cyclic R_M-modules split. Let I be an ideal of R_M. If $\mathscr{S}_M(R_M/I)=0$, then there is nothing to prove. So assume $T/I=\mathscr{S}_M(R_M/I)\neq 0$. Choose an R-submodule L/I of R_M/I maximal with respect to $T/I \cap L/I=0$. Then $T+L/L$ is essential as an R-submodule of R_M/L. By [11] Lemma 2.7 we have $T+L/L \in \mathscr{T}$, and hence $R_M/L \in \mathscr{T}$ by the stability of $\mathscr{T}$. Since $-\!\!-\otimes R_M$ is an exact functor from ${}_R\mathscr{M}$ to ${}_{R_M}\mathscr{M}$, it is not hard to see that T/I can be embedded (as an R_M-submodule) in

$$(R_M/L)\otimes R_M \cong R_M \otimes R_M/L \otimes R_M \cong R_M/(L\otimes R_M).$$

But $R_M/L \in \mathscr{T}$ implies $(R_M/L)\otimes R_M \in \mathscr{S}_M$; and hence T/I is a submodule of $R_M/(L\otimes R_M) \in \mathscr{S}_M$, which has bounded order relative to $(\mathscr{S}_M, \mathscr{F}_M)$. But $(\mathscr{S}_M, \mathscr{F}_M)$ has BSP for ${}_{R_M}\mathscr{M}$ by Lemma 4.3; whence $R_M/I = T/I \oplus Q/I$ for some $Q \subseteqq R_M$. Since R_M is local, it follows that $Q = I$, i.e. $R_M/I \in \mathscr{S}_M$. Thus we have shown every cyclic R_M-module is either in $\mathscr{S}_M$ or $\mathscr{F}_M$, and hence cyclic R_M-modules split trivially.

Lemma 4.5. *Let R be a commutative ring, and let $(\mathscr{T}, \mathscr{F})$ be a stable torsion theory of simple type such that $\mathscr{T}(R) = 0$. If $(\mathscr{T}, \mathscr{F})$ has BSP and M is a maximal ideal in $F(\mathscr{T})$, then R_M is a principal ideal domain.*

Proof. By Lemma 4.3, $(\mathscr{S}_M, \mathscr{F}_M)$ has BSP for R_M. Also $(\mathscr{S}_M, \mathscr{F}_M)$ is stable by Lemma 4.4. If $(\mathscr{S}_M, \mathscr{F}_M)$ is non-trivial, then Theorem 3.3 implies R_M is a principal ideal domain.

On the other hand, if $(\mathscr{S}_M, \mathscr{F}_M)$ is trivial, then non-zero R_M-modules have non-zero socles. Since R_M is local, it follows from [2] Theorem P that R_M is perfect. Now by Lemma 2.1, I is flat for each $I \in F(\mathscr{T})$. Hence $\mathrm{Tor}_n^R(T, \text{—}) = 0$ for each $n \geqq 2$ and $T \in \mathscr{T}$. Now let $A, B \in {}_{R_M}\mathscr{M}$. Then $A, B \in \mathscr{S}_M$, and hence $A, B \in \mathscr{T}$ as R-modules. Thus for $n \geqq 2$, we have

$$\mathrm{Tor}_n^{R_M}(A, B) \cong \mathrm{Tor}_n^R(A, B) \otimes R_M = 0$$

by [18] Theorem 7 (p. 171). Hence the global weak dimension of R_M is $\leqq 1$. Since R_M is perfect, [2] Theorem P and the result of Kaplansky [2] imply every R_M-module is projective. Hence R_M is a field whenever $\mathscr{S}_M(R_M) \neq 0$.

Let M be a maximal ideal of R. An ideal K of R is called *M-special* if $K \subseteqq M$ and $(M/K)\otimes R_M = 0$.

If $(\mathscr{T}, \mathscr{F})$ is a hereditary torsion theory for ${}_R\mathscr{M}$, then we call an ideal I an *$\mathscr{F}$-ideal* if $R/I \in \mathscr{F}$.

We are now ready for our characterization of BSP for stable, hereditary torsion theories $(\mathscr{T}, \mathscr{F})$ of modules over a commutative ring satisfying: (i) $\mathscr{T}(R) = 0$ and (ii) $F(\mathscr{T})$ contains a cofinal subset of finitely generated ideals.

Theorem 4.6. *Let R be a commutative ring, and let $(\mathscr{T}, \mathscr{F})$ be a stable, hereditary torsion theory for ${}_R\mathscr{M}$ such that $\mathscr{T}(R) = 0$. Suppose that $F(\mathscr{T})$ has a cofinal subset of finitely generated ideals. Then $(\mathscr{T}, \mathscr{F})$ has BSP if and only if the following three properties hold:*

(1) R/I is a perfect ring for each $I \in F(\mathscr{T})$.

(2) For each maximal ideal M in $F(\mathscr{T})$, R_M is a principal ideal domain.

(3) If there exists an M-special $\mathscr{F}$-ideal for a maximal ideal M in $F(\mathscr{T})$, then R_M is a field.

Proof. "only if": (1) holds by Theorem 2.2, and hence $(\mathscr{T}, \mathscr{F})$ is of simple type. Hence (2) holds by Lemma 4.5.

Now suppose K is an M-special $\mathscr{F}$-ideal, where M is a maximal ideal in $F(\mathscr{T})$. Consider the exact sequence

$$0 = M/K \otimes R_M \to R/K \otimes R_M \to R/M \otimes R_M \to 0\,.$$

This yields $R/K \otimes R_M \cong R/M \otimes R_M \cong R/M$, which is simple and in $\mathcal{T}$. By Theorem 2.2 and BSP, we have

$$\begin{aligned} 0 &= \operatorname{Tor}_1^R(R/K, R/M) \cong \operatorname{Tor}_1^R(R/K, R/M) \otimes R_M \\ &\cong \operatorname{Tor}_1^{R_M}(R/K \otimes R_M, R/M \otimes R_M) \cong \operatorname{Tor}_1^{R_M}(R/M, R/M). \end{aligned}$$

But $\mathcal{T}(R/M) = \mathcal{S}_M(R/M)$, so that it follows from (2) and a direct limit argument that every R_M-module is flat, whence R_M is von Neumann regular. Since R_M is also an integral domain, it follows that R_M is a field. Thus (3) holds.

"if": By (1), $(\mathcal{T}, \mathcal{F})$ is of simple type and has primary decomposition for modules in $\mathcal{T}$ (see Remark (2) following Corolllary 2.3). Let $A \in \mathcal{F}$, and let S be a simple module in $\mathcal{T}$. By [10] Theorem 4, $\operatorname{Tor}_1^R(A, S) \in \mathcal{T}_s \subsetneq \mathcal{T}$. So it follows from the primary decomposition for $\mathcal{T}$ and properties of Tor that for $B \in \mathcal{T}$ of bounded order, we have $\operatorname{Tor}_1^R(A, B) = 0$ if and only if $\operatorname{Tor}_1^R(A, B) \otimes R_M = 0$ for every maximal ideal M in $F(\mathcal{T})$. From this fact and Theorem 2.2, it follows that it is sufficient to show $\operatorname{Tor}_1^R(A, B) \otimes R_M = 0$ for each maximal ideal M in $F(\mathcal{T})$. We divide our argument into two cases:

Case I: R_M is a field. Then every R_M-module is flat, and hence

$$0 = \operatorname{Tor}_1^{R_M}(A \otimes R_M, B \otimes R_M) \cong \operatorname{Tor}_1^R(A, B) \otimes R_M.$$

Case II: R_M is not a field. First, we claim that $A \otimes R_M \in \mathcal{F}$. For if not, let $x \otimes 1 \in S$, where S is a simple module contained in the socle of $A \otimes R_M$ (so $S \cong R/M$). Let $\varphi: A \to A \otimes R_M : a \to a \otimes 1$. Then

$$0 \to \ker\varphi \cap Rx \to Rx \to \frac{Rx}{Rx \cap \ker\varphi} \to 0$$

is exact and $Rx/Rx \cap \ker\varphi \cong S$. From the Fundamental Theorem of Homomorphisms, it follows that $\ker\varphi \cap Rx \cong M/(0:x)$ and hence $(0:x)$ is M-special. Moreover, $Rx \subsetneq A \in \mathcal{F}$ implies $(0:x)$ is also an $\mathcal{F}$-ideal, whence R_M is a field by (3). This contradicts the hypothesis of Case II, and hence the claim is proved.

Since $A \otimes R_M \in \mathcal{F}$, then $\mathcal{S}_M(A \otimes R_M) = 0$ by [11] Lemma 2.7. But from (2) and comments in [6], $\mathcal{S}_M$ is the usual torsion class for the integral domain R_M. Hence $A \otimes R_M$ is torsionfree in the usual sense. So by [3] VII 4.2, we have

$$0 = \operatorname{Tor}_1^{R_M}(A \otimes R_M, B \otimes R_M) \cong \operatorname{Tor}_1^R(A, B) \otimes R_M.$$

Since the usual torsion theory for an integral domain is stable, we can now obtain [5] Theorem 4.2 as a corollary.

Corollary 4.7. *Let R be an integral domain, and let $(\mathcal{T}, \mathcal{F})$ be the usual torsion theory. If $(\mathcal{T}, \mathcal{F})$ has BSP, then R is a Dedekind domain.*

Proof. By BSP and Theorem 4.6(2) R_M is a principal ideal domain for each maximal ideal M. So R is semihereditary by [9] Theorem 2. Let $0 \neq x \in R$. Applying Theorem 4.6 (1), Lemma 3.1, and [5] Theorem 4.3, R/Rx must be Artinian. It follows that R is Noetherian, and hence R is a Dedekind domain.

5. The Simple Theory

In [7] Dickson has conjectured that the simple theory $(\mathscr{S}, \mathscr{F})$ for ${}_R\mathscr{M}$ has SP if and only if non-zero modules have non-zero socles. This conjecture has been verified for the following types of commutative rings:

(a) Noetherian [7] Theorem 1.

(b) von Neumann regular [11] Theorem 3.9.

(c) local rings [11] Theorem 4.6.

We now apply a lemma of Section 4 to verify Dickson's conjecture for all commutative rings, and hence we obtain a generalization of (a), (b), and (c).

Theorem 5.1. *Let R be a commutative ring, and let $(\mathscr{S}, \mathscr{F})$ denote the simple theory for ${}_R\mathscr{M}$. Then the following are equivalent:*

(1) *$(\mathscr{S}, \mathscr{F})$ has SP.*

(2) *Non-zero modules have non-zero socles.*

Proof. (1)$\Rightarrow$(2): By (1) and [22] Lemma 1.1, $(\mathscr{S}, \mathscr{F})$ is stable. If (2) is false, we may assume $\mathscr{S}(R)=0$. By Lemma 4.5, the local ring R_M is a principal ideal domain for each maximal ideal M. By Lemma 4.3 and (1) the simple theory for ${}_{R_M}\mathscr{M}$ splits, and hence non-zero R_M-modules have non-zero socles by [11] Theorem 4.6. Since R_M is a principal ideal domain with non-zero socle, then R_M is a field. Hence [9] Theorem 1 implies R is regular. But then [11] Theorem 3.9 yields $R \in \mathscr{S}$, which is a contradiction to $\mathscr{S}(R)=0$.

(2)$\Rightarrow$(1): $\mathscr{S} = {}_R\mathscr{M}$ is immediate from (2), and hence $(\mathscr{S}, \mathscr{F})$ splits trivially.

Corollary 5.2. *Let R be a commutative ring. Then the socle of each R-module splits off if and only if R is a direct sum of finitely many fields.*

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A Spectral Theory for Direct Integrals of Operators

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A spectral theory for certain operators on a finite direct sum of Hilbert spaces was established in 1966 by N. Dunford [4]. In this paper, we shall show by using the approach of reduction theory of von Neumann (cf. [2, 7]) that his Theorem 2.7, [4] is true for a wider class of operators. In fact, we proved that any closed psectral operator in a Hilbert space can be decomposed into a direct integral of closed irreducible spectral operators.

In Sections 1 and 3, we shall include some already developed results about the reduction theory of bounded and unbounded operators. Our main results are in Sections 2 and 4. A method of decomposition of closed spectral operator into irreducible part is included in the last section, in which we also indicate a problem of structure theory that we shall investigate in the future. The various applications of our results will appear elsewhere. Throughout this paper, the Hilbert space will be over the complex field.

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1. Direct Integrals of Bounded Operators in Hilbert Spaces

Let Σ be a locally compact Hausdorff space; μ, a regular Borel measure on Σ. Let

$$H = \int_{\Sigma}^{\oplus} H(s)\,\mu(ds)$$

be a direct integral Hilbert space (cf. [2, 7]).

Definition 1.1. *A bounded operator in H is said to be decomposable if it can be represented by the form*

$$A = \int_{\Sigma}^{\oplus} A(s)\,\mu(ds)$$

in the sense that $s \to A(s)$ is a function whose value is a bounded operator in $H(s)$ such that

$$Ax = \int_{\Sigma}^{\oplus} A(s)\,x(s)\,\mu(ds)$$

for every $x = \int_{\Sigma}^{\oplus} x(s)\,\mu(ds)$ in H.

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In this case, we say that A is defined by the μ-measurable field of bounded operators $s \to A(s)$. In particular, if $A(s) = c(s)I(s)$, c denoting a μ-essentially bounded scalar-valued function and $I(s)$ denoting the identity operator in $H(s)$, then A is called a bounded diagonalizable operator.

We shall give several elementary properties of direct integral decompositions without the proofs. For details, we refer to [2] and [7].

Lemma 1.2. *If A and B are decomposable operators such that*

$$A = \int_{\Sigma}^{\oplus} A(s)\,\mu(ds)$$

and

$$B = \int_{\Sigma}^{\oplus} B(s)\,\mu(ds),$$

then for any complex number α, β

(a) *A^* is decomposable and*

$$A^* = \int_{\Sigma}^{\oplus} A(s)^*\,\mu(ds).$$

(b) *$\alpha A + \beta B$ is decomposable and*

$$\alpha A + \beta B = \int_{\Sigma}^{\oplus} (\alpha A(s) + \beta B(s))\,\mu(ds).$$

(c) *AB is decomposable and*

$$AB = \int_{\Sigma}^{\oplus} A(s)B(s)\,\mu(ds).$$

(d) *The norm of the operator A is given by*

$$\|A\| = \mu\text{-ess.}\sup \|A(s)\|.$$

(e) *The operator A is decomposable if and only if it commutes with every bounded diagonalizable operator.*

Lemma 1.3. *A decomposable operator A is invertible if and only if the operator $A(s)$ is invertible for μ-a.a. s and the mapping $s \to A(s)^{-1}$ is a μ-essentially bounded operator-valued function. In this case*

$$A^{-1} = \int_{\Sigma}^{\oplus} A(s)^{-1}\,\mu(ds).$$

Proof. Suppose that A^{-1} exists as a bounded operator. For every diagonal operator T, $TA^{-1} = A^{-1}T$. It follows from Lemma 1.2e. that $B = A^{-1}$ is decomposable. Let

$$B = \int_{\Sigma}^{\oplus} B(s)\,\mu(ds).$$

Since $AB = BA = I$, and

$$I = \int_{\Sigma}^{\oplus} I(s)\mu(ds),$$

we have, for μ-a.a. s,

$$A(s)B(s) = B(s)A(s) = I(s); \quad \text{i.e.}$$

$$B(s) = A(s)^{-1} \quad \mu\text{-a.e.}$$

$s \to B(s)$ is μ-essentially bounded. Hence, $s \to A(s)^{-1}$ is μ-essentially bounded.

Conversely, it is obvious if $(A(s))^{-1}$ exists for μ-a.a. s then A is one-to-one. The fact that $s \to A(s)^{-1}$ is μ-essentially bounded implies A^{-1} is bounded.

2. Spectral Theory for Bounded Direct Integrals

As in section 1, let

$$H = \int_{\Sigma}^{\oplus} H(s)\mu(ds)$$

be the direct integral Hilbert spaces. Throughout this section the operators are all bounded unless otherwise stated.

Lemma 2.1. *Let* $A = \int_{\Sigma}^{\oplus} A(s)\mu(ds)$. *If* $\sigma(T)$ *denotes the spectrum of some operator* T *and* δ^c *denotes the complement of set* δ, *then*

$$\cap \left\{ \overline{\bigcup_{s \in \delta} \sigma(A(s))} : \delta \in \mathscr{B}(\Sigma) \quad \text{and} \quad \mu(\delta^c) = 0 \right\} \subset \sigma(A) \tag{1}$$

where $\mathscr{B}(\Sigma)$ *is the Borel field on* Σ.

If, in addition, the operator-valued function $s \to (\lambda I(s) - A(s))^{-1}$ *is* μ*-essentially bounded for every* λ *in the complement of* $\overline{\bigcup_{s \in \delta} \sigma(A(s))}$ *for every Borel subset* δ *of* Σ *with* $\mu(\delta^c) = 0$, *the relation in the formula* (1) *is an equality.*

Proof. Denote the resolvent by $\varrho(.)$. Let $\lambda_0 \in \varrho(A)$. Then $\lambda_0 \in \varrho(A(s))$ for μ-a.a. s and $s \to (\lambda_0 I(s) - A(s))^{-1}$ is μ-essentially bounded as an operator-valued function. Let δ be a Borel subset of Σ with $\mu(\delta^c) = 0$ such that

$$\sup_{s \in \delta} \|(\lambda_0 I(s) - A(s))^{-1}\| = K < \infty .$$

If we denote by $\mathscr{L}(H(s))$ the space of all continuous linear operators in $H(s)$ with the uniform norm, then it is clear that, for every $s \in \delta$, the set

$$\left\{ T(s) \in \mathscr{L}(H(s)) \,\middle|\, \|T(s) - (\lambda_0 I(s) - A(s))\| < \frac{1}{k} \right\}$$

is open in $\mathscr{L}(H(s))$. This implies λ_0 is in the interior of the set $\bigcap_{s\in\delta} \varrho(A(s))$. Hence

$$\varrho(A) \subset \cup \left\{ \left(\bigcap_{s\in\delta} \varrho(A(s)) \right)^0 : \mu(\delta^c) = 0 \right\}$$

By taking the complement, we have the formula (1):

To show the converse inclusion, we let δ be an arbitrary Borel subset in Σ with $\mu(\delta^c)=0$ and let λ_0 be a complex number with $\lambda_0 \notin \overline{\bigcup_{s\in\delta} \sigma(A(s))}$. Then $(\lambda_0 I(s) - A(s))^{-1}$ exists for every $s \in \delta$ and $s \to (\lambda_0 I(s) - A(s))^{-1}$ is bounded for every such δ by assumptions. It follows that $(\lambda_0 I - A)^{-1}$ exists as a bounded operator in H. This implies

$$\sigma(A) \subset \overline{\bigcup_{s\in\delta} \sigma(A(s))}$$

Thus $\sigma(A) \subset \cap \left\{ \overline{\bigcup_{s\in\delta} \sigma(A(s))} : \mu(\delta^c) = 0 \right\}$. q.e.d.

We now introduce the notion of a spectral operator developed by Dunford [3].

Definition 2.2. *Let $\mathscr{B}$ be a Boolean algebra of subsets of a set Λ. A spectral measure in H is a homomorphism E of $\mathscr{B}$ into a Boolean algebra of projections in a Hilbert space H such that it is bounded and $E(\Lambda) = I$.*

Definition 2.3. *An operator $A \in \mathscr{L}(H)$ is called a spectral operator if*

(i) *there exists a spectral measure E in H with the Borel field $\mathscr{B}$ of subsets of complex plane as its domain,*

(ii) *for every arbitrary ξ, η in H, the set function $\alpha \to (E(\alpha)\xi, \eta)$ is σ-additive*[1]*; and*

(iii) *for every $\alpha \in \mathscr{B}$,*

$$A E(\alpha) = E(\alpha) A$$

and

$$\sigma(T, E(\alpha)H) \subset \bar{\alpha},$$

where $\sigma(T, E(\alpha)H)$ is the spectrum of T considered as an operator in $E(\alpha)H$.

The spectral measure E is then called the resolution of the identity for A.

The resolution of the identity for a spectral operator is unique. Dunford has shown a characteristic theorem for the spectral operator that every spectral operator A is the sum of two operators S and N such that

(i) N is a generalized nilpotent,
(ii) $SN = NS$; and
(iii) $S = \int_{\sigma(S)} \lambda E(d\lambda)$,

where the integral exists as a Riemann integral in the uniform topology of operators. A and S have the same spectrum and the same resolution of the identity (cf. [3]).

[1] E is then said to be σ-additive.

Definition 2.4. *In the above decomposition, S is called the scalar part of A while N is called the radical part. In case $N=0$, we call A a scalar operator.*

Lemma 2.5. *Let $\mathscr{B}$ be the Borel field of subsets of complex plane. If $E_s(.)$ is a σ-additive spectral measure in μ-a.a. $H(s)$ and if for every Borel subset α in $\mathscr{B}$, $s \to E_s(\alpha)$ is μ-measurable and μ-essentially bounded, then the formula*

$$\alpha \to \overset{\oplus}{\int_{\Sigma}} \int_{\Lambda} \chi_\alpha E_s(d\lambda)\, \mu(ds) \tag{2}$$

defines a σ-additive spectral measure of $\mathscr{B}$ in H where χ_α denotes the characteristic function of α; and for every bounded measurable function φ

$$\overset{\oplus}{\int_{\Sigma}} \int_{\Lambda} \varphi(\lambda)\, E_s(d\lambda)\, \mu(ds) = \int_{\Lambda} \varphi(\lambda) \overset{\oplus}{\int_{\Sigma}} E_s(d\lambda)\, \mu(ds)\,. \tag{3}$$

Proof. It is clear that $\int_{\Lambda} \varphi(\lambda)\, E_s(d\lambda)$ is bounded on μ-a.a. $H(s)$. Since $s \to (E_s(\alpha)\,\xi(s), \eta(s))$ is μ-measurable for every Borel subset α and arbitrary $\xi, \eta \in H$, it follows that $s \to \int_{\Lambda} \varphi(\lambda) E_s(d\lambda)$ is μ-measurable for every Borel measurable function φ. The μ-essential boundedness of $s \to E_s(\alpha)$ implies that $s \to \int_{\Lambda} \varphi(\lambda) E_s(d\lambda)$ is μ-essentially bounded. Hence (2) is a well-defined decomposable operator in H. Let ξ, η be a pair of arbitrary vectors in H. Then, by Fubini's theorem,

$$\begin{aligned}
&\left(\left\{\overset{\oplus}{\int_{\Sigma}} \int_{\Lambda} \varphi(\lambda)\, E_s(d\lambda)\, \mu(ds)\right\} \xi, \eta\right) \\
&\quad = \int_{\Sigma} \left(\left\{\int_{\Lambda} \varphi(\lambda)\, E_s(d\lambda)\right\} \xi(s), \eta(s)\right) \mu(ds) \\
&\quad = \int_{\Sigma} \int_{\Lambda} \varphi(\lambda)\, (E_s(d\lambda)\, \xi(s), \eta(s))\, \mu(ds) \\
&\quad = \int_{\Lambda} \varphi(\lambda) \int_{\Sigma} (E_s(d\lambda)\, \xi(s), \eta(s))\, \mu(ds) \\
&\quad = \left(\left\{\int_{\Lambda} \varphi(\lambda) \overset{\oplus}{\int_{\Sigma}} E_s(d\lambda)\, \mu(ds)\right\} \xi, \eta\right).
\end{aligned}$$

This gives the Eq. (3).

It is easy to see that the formula (2) does define a spectral measure in H. Moreover, the spectral measure defined by (2) is σ-additive in the sense that

$$\alpha \to \left(\overset{\oplus}{\int_{\Sigma}} \int_{\Lambda} \chi_\alpha E_s(d\lambda)\, \mu(ds)\, \xi, \eta\right)$$

is σ-additive for every pair ξ, η in H. q.e.d.

Lemma 2.6. *Let* $A = \int_{\Sigma}^{\oplus} A(s)\,\mu(ds)$ *and let*

$$\Lambda = \cap \left\{ \overline{\bigcup_{s\in\delta} \sigma(A(s))} : \mu(\delta^c) = 0 \right\} \tag{4}$$

Suppose that

(i) $A(s)$ *is a spectral operator in* $H(s)$ *for* μ*-a.a.* s *in* Σ;

(ii) $s \to E_s(.)$, *the field of the resolutions of the identity corresponding to* $s \to A(s)$, *is* μ*-measurable and* μ*-essentially bounded; and*

(iii) *if* $N(s)$ *denotes the radical part of* $A(s)$, *then* $\mu\text{-ess.}\sup_s \|N(s)^n\|^{1/n} \to 0$ *as* $n \to \infty$.

Then

$$\sigma(A) = \Lambda \tag{5}$$

Proof. By Lemma 2.1, it suffices to show under the hypotheses the map $s \to \underline{(\lambda I(s) - A(s))^{-1}}$ is μ-essentially bounded on Σ for every λ in the complement of $\overline{\bigcup_{s\in\delta} \sigma(A(s))}$ for every Borel subset δ with $\mu(\delta^c) = 0$. It follows from the representation theorem of Dunford [3] that, for every such δ, $\lambda \notin \overline{\bigcup_{s\in\delta} \sigma(A(s))}$ implies

$$(\lambda I(s) - A(s))^{-1} = \sum_{m=0}^{\infty} N(s)^m \int_{\sigma(A(s))} \frac{E_s(d\lambda')}{(\lambda - \lambda')^{m+1}} \tag{6}$$

where $N(s) = A(s) - \int_{\sigma(A(s))} \lambda E_s(d\lambda)$.

Now, $s \to A(s)$ is μ-essentially bounded. It follows from the fact that $s \to E_s(\alpha)$ is μ-essentially bounded by assumption, $s \to N(s)$ is μ-essentially bounded. By condition (iii) there is a sequence $\{M_n\}$ of real numbers such that, for μ-a.a. s in δ

$$\|N^n(s)\| \leqq M_n \qquad n = 1, 2, 3, \ldots$$

and

$$(M_n)^{1/n} \to 0$$

as $n \to \infty$.

Since

$$\sum_{m=0}^{\infty} \|N(s)^m\| \left\| \int_{\sigma(A(s))} \frac{E_s(d\lambda')}{(\lambda - \lambda')^{m+1}} \right\| \leqq \Sigma M_n \left\| \int_{\sigma(A(s))} \frac{E_s(d\lambda')}{(\lambda - \lambda')^{m+1}} \right\|,$$

we conclude that $s \to (\lambda I(s) - A(s))^{-1}$ is μ-essentially bounded on δ for every $\lambda \notin \overline{\bigcup_{s\in\delta} \sigma(A(s))}$, thus is μ-essentially bounded on Σ. q.e.d.

We shall now state our main theorem:

Theorem 2.7. *Let*

$$H = \int_{\Sigma}^{\oplus} H(s)\,\mu(ds)$$

be a direct integral Hilbert spaces and

$$A = \int_{\Sigma}^{\oplus} A(s)\mu(ds).$$

Denote by Λ the spectrum of A. Then A is a spectral operator in Dunford's sense [Definition 2.3] if and only if

(i) *$A(s)$ is a spectral operator in $H(s)$ for μ-a.a. s in Σ;*

(ii) *for every Borel subset α of Λ, the field of resolutions of the identity $s \to E_s(\alpha)$ corresponding to $s \to A(s)$ is μ-measurable and μ-essentially bounded; and*

(iii) *μ-ess. sup $\|N(s)^n\|^{1/n} \to 0$ as $n \to \infty$ while $N(s)$ denotes the radical part of $A(s)$.*

Proof. Suppose A is a spectral operator in H. Let E be the resolution of identity for A. Since A is decomposable, it commutes with every diagonalizable operator. By theorem 5 [3], E commutes with every diagonalizable operator; hence is decomposable. For every $\alpha \in \mathscr{B}(\Lambda)$, the Borel field of subsets of Λ,

$$E(\alpha) = \int_{\Sigma}^{\oplus} E_s(\alpha)\mu(ds). \tag{7}$$

For μ-a.a. s, E_s is a σ-additive spectral measure in $H(s)$ by a standard argument in measure theory. Since $E(\alpha)A = AE(\alpha)$, it follows that for μ-a.a. s $E_s(\alpha)A(s) = A(s)E_s(\alpha)$. We shall show that

$$\sigma(A(s), E_s(\alpha)H(s)) \subset \bar{\alpha}. \tag{8}$$

Since $\sigma(A, E(\alpha)H) \subset \bar{\alpha}$, it is clear that if $\lambda \notin \bar{\alpha}$, then $(\lambda I - A)$ restricted to $E(\alpha)H$ has an inverse. Thus $\lambda \notin \sigma(A(s), E_s(\alpha)H)$ for μ-a.a. s. We conclude that E_s is the resolution of the identity for μ-a.a. $A(s)$. For every Borel subset α of Λ, it is clear that $s \to E_s(\alpha)$ is a μ-measurable field of resolutions of the identity. Since $E(\alpha)$ has decomposition (7), $s \to E_s(\alpha)$ is μ-essentially bounded for every Borel subset α of Λ.

Finally, the measurability of the field of the resolutions of the identity $s \to E_s(.)$ implies the measurability of $s \to \int_{\sigma(A(s))} \lambda E_s(d\lambda)$. Hence $s \to S(s)$ is a μ-measurable and μ-essentially bounded field of operators where $S(s)$ is the scalar part of $A(s)$. By Lemma 2.5, Lemma 2.6 and the compactness of $\sigma(A)$,

$$\begin{aligned} \int_{\sigma(A)} \lambda E(d\lambda) &= \int_{\sigma(A)} \lambda \int_{\Sigma}^{\oplus} E_s(d\lambda)\mu(ds) \\ &= \int_{\Sigma}^{\oplus} \int_{\sigma(A)} \lambda E_s(d\lambda)\mu(ds) \\ &= \int_{\Sigma}^{\oplus} S(s)\mu(ds). \end{aligned}$$

By Lemma 1.2 b,

$$N = A - S = \int_{\Sigma}^{\oplus} (A(s) - S(s))\, \mu(ds) = \int_{\Sigma}^{\oplus} N(s)\, \mu(ds).$$

It is clear that $NS = SN$. Thus $N^n = \int_{\Sigma}^{\oplus} N(s)^n \mu(ds)$; condition (iii) follows immediately. This completes the proof of the necessary part.

For the converse, we consider

$$E(.) = \int_{\Sigma}^{\oplus} E_s(.)\, \mu(ds). \tag{9}$$

We shall show that under the hypotheses, formula (9) provides us the resolution of the identity for the operator A. By Lemma 2.6, the spectrum Λ of A is given by

$$\Lambda = \cap \left\{ \overline{\bigcup_{s \in \delta} \sigma(A(s))} : \mu(\delta^c) = 0 \right\}.$$

Let $E(\alpha) = \int_{\Sigma}^{\oplus} \int_{\Lambda} \chi_\alpha E_s(d\lambda)\, \mu(ds)$. It is readily seen that

$$\begin{aligned} AE(\alpha) &= \int_{\Sigma}^{\oplus} A(s) \int_{\Lambda} \chi_\alpha E_s(d\lambda)\, \mu(ds) \\ &= \int_{\Sigma}^{\oplus} \left(\int_{\Lambda} \chi_\alpha E_s(d\lambda) \right) A(s)\, \mu(ds) \\ &= E(\alpha) A. \end{aligned}$$

It is also clear that E is a σ-additive spectral measure in H. We shall show that

$$\sigma(A, E(\alpha)H) \subset \bar{\alpha}. \tag{*}$$

Again by Lemma 2.6,

$$\sigma(A, E(\alpha)H) = \cap \left\{ \overline{\bigcup_{s \in \delta} \sigma(A(s), E_s(\alpha) H(s))} : \mu(\delta^c) = 0 \right\}$$

But for every Borel subset δ of Σ with $\mu(\delta^c) = 0$,

$$\sigma(A(s) \mid E_s(\alpha) H(s)) \subset \bar{\alpha}$$

for every $s \in \delta$. It follows that (*) holds. A is therefore a spectral operator. q.e.d.

In the process of proving theorem 2.7, we obtained the following corollary.

Corollary 2.8. *Let* $A = \int_{\Sigma}^{\oplus} A(s)\, \mu(ds)$ *be a spectral operator in* H. *Then the scalar part* S *and the radical part* N *of* A *are given by the formulas*

$$S = \int_{\Sigma}^{\oplus} S(s)\, \mu(ds) \tag{10}$$

and

$$N = \int_{\Sigma}^{\oplus} N(s)\,\mu(ds) \tag{11}$$

where $S(s)$ and $N(s)$ are respectively the scalar and the radical parts of the spectral operator $A(s)$.

We actually have more. Eq. (3) of Lemma 2.5 gives, for every bounded μ-measurable function φ,

$$\varphi(S) = \int_{\sigma(A)} \varphi(\lambda)\,E(d\lambda) = \int_{\Sigma}^{\oplus} \varphi(S(s))\,\mu(ds)\,. \tag{12}$$

If f is a scalar function analytic and single-valued on the spectrum $\sigma(A)$, we have (cf. [3])

$$f(A) = \sum_{m=0}^{\infty} \frac{N^m}{m!} \int_{\sigma(A)} f^{(m)}(\lambda)E(d\lambda)\,.$$

By Lemma 1.2 e), for every positive integer m,

$$N^m = \int_{\Sigma}^{\oplus} N(s)^m \mu(ds) \tag{13}$$

We have

$$f(A) = \sum_{m=0}^{\infty} \int_{\Sigma}^{\oplus} \left(\frac{N(s)^m}{m!} \int_{\sigma(A(s))} f^{(m)}(\lambda)E_s(d\lambda)\right)\mu(ds)\,.$$

A similar argument as employed in proving Lemma 2.5 gives us a dominate convergence theorem. It follows that

$$\begin{aligned} f(A) &= \int_{\Sigma}^{\oplus} \left(\sum_{m=0}^{\infty} \frac{N(s)^m}{m!} \int_{\sigma(A(s))} f^{(m)}(\lambda)\,E_s(d\lambda)\right)\mu(ds) \\ &= \int_{\Sigma}^{\oplus} f(A(s))\,(ds)\,. \end{aligned}$$

We thus proved the following theorem:

Theorem 2.9. *Let $A = \int_{\Sigma}^{\oplus} A(s)\,\mu(ds)$ be a spectral operator in H. Then for every scalar function f analytic and single-valued on the spectrum $\sigma(A)$, we have*

$$f(A) = \int_{\Sigma}^{\oplus} f(A(s))\,\mu(ds) \tag{14}$$

where, in the uniform topology of operators,

$$f(A(s)) = \sum_{m=0}^{\infty} \frac{N(s)^m}{m!} \int_{\sigma(A(s))} f^{(m)}(\lambda)\,E_s(d\lambda) \tag{15}$$

with $N(s)$ denoting the radical part of $A(s)$.

Definition 2.10. *Let N be the radical part of the spectral operator A; then A is of type p if and only if $N^{p+1}=0$.*

As a direct consequence of Eq. (13), we obtain:

Corollary 2.11. $A=\int_{\Sigma}^{\oplus} A(s)\,\mu(ds)$ *is of type p if and only if $A(s)$ is of type p μ-a.a. s.*

If our Hilbert space H is a finite direct sum of n-copies of Hilbert space $\mathscr{H}$, $H=\Sigma^{\oplus}\mathscr{H}$, we can always decomposed H as a direct integral of Hilbert spaces. In fact, there is a locally compact space Σ, a positive Radon measure μ, and a constant μ-measurable field $s\to H(s)$ of complex n-dimensional Euclidean space $\mathbb{E}^n$ such that

$$H=\int_{\Sigma}^{\oplus} H(s)\,\mu(ds)$$

Let $\mathscr{A}$ be a commutative C^*-algebra of operators in $\mathscr{H}$. Then the algebra $\mathscr{A}\otimes\mathscr{L}(\mathbb{E}^n)$ can be considered as a subalgebra of $\int_{\Sigma}^{\oplus}\mathscr{L}(\mathbb{E}^n)\,\mu(ds)$. We now can state Dunford's theorem (Theorem 2.7, [4]) in our setting:

Corollary 2.12. *Let $H=\int_{\Sigma}^{\oplus}\mathbb{E}^n\mu(ds)$. Then an element A in $\mathscr{A}\otimes\mathscr{L}(\mathbb{E}^n)$ is a spectral operator if and only if the field of resolutions of the identity corresponding to $A(s)$, which is always spectral for μ-a.a. s, is μ-essentially bounded.*

Proof. In finite dimensional space, the resolution of the identity depends on the operator continuously. Thus the measurability of $s\to A(s)$ implies that of $s\to E_s(\alpha)$ for every Borel measurable set α. By Theorem 2.7, A is a spectral operator if $s\to E_s(.)$ is μ-essentially bounded.

The converse of the theorem is a special case of the necessary part of Theorem 2.7.

Remark: It appears that in Theorem 2.7 the condition of the measurability of $s\to E_s(.)$ is redundant; however, we have not been able to prove it.

3. Reduction Theory for Closed Operators

We shall extend our results in Sections 2 to closed but not necessarily bounded operators. Before we can do this, we have to develope a reduction theory for a closed operator. A. E. Nussbaum established a reduction theory for such operators [6] by using the characteristic matrix considered by M. H. Stone [8]. We shall recall part of Nussbaum's work which is essential to us. For details, readers should confer with his original paper [6].

Definition 3.1. *Let A be a closed operator in a Hilbert space H, and let $G(A)$ be the graph of A in $H\times H$. Then the matrix (P_{ij}) of the projection P of $H\times H$*

onto the closed subspace generated by the graph $G(A)$ is called the characteristic matrix of the operator A.

It was proved by Stone [8] that, if P is the characteristic matrix of a closed operator A, then

(i) the null space $N(I-P_{22})$ of $I-P_{22}$ is $\{0\}$.
(ii) $P_{21}=AP_{11}$, $P_{22}=AP_{12}$; and
(iii) A is uniquely determined by

$$A\colon P_{11}x_1+P_{12}x_2 \to P_{21}x_1+P_{22}x_2 .$$

If, in addition, A is densely defined, the null space $N(P_{11})$ of P_{11} is also $\{0\}$. Conversely, if $N(I-P_{22})=\{0\}$ then the projection P is the characteristic of some closed operator. We shall say that the field of characteristic matrices of the field of closed operators $s\to A(s)$ is μ-measurable if the fields of bounded operators $s\to P_{ij}(s)$ $i,j=1,2$ are μ-measurable. It can be shown that a field of bounded operators $s\to A(s)$ is μ-measurable in the usual sense if and only if the field of characteristic matrices of $s\to A(s)$ is μ-measurable. This inspires the following definitions:

Definition 3.2. *A field of closed operators $s\to A(s)$ is said to be μ-measurable if the field of characteristic matrices of $s\to A(s)$ is μ-measurable.*

Definition 3.3. *A closed operator A in*

$$H=\int_{\Sigma}^{\oplus} H(s)\,\mu(ds)$$

is said to be decomposable if it is defined by a μ-measurable field of closed operator $s\to A(s)$. This is denoted by

$$A=\int_{\Sigma}^{\oplus} A(s)\,\mu(ds) \tag{16}$$

with $\mathscr{D}(A)=\left\{x\in H \mid x(s)\in\mathscr{D}(A(s)) \text{ and } \int_{\Sigma}\|A(s)\,x(s)\|^2\,\mu(ds)<\infty\right\}.$

Theorem 3.4. *A closed operator A in*

$$H=\int_{\Sigma}^{\oplus} H(s)\,\mu(ds)$$

is decomposable if and only if A permutes with every bounded diagonalizable operator B (by "permutes" we mean $BA\subset AB$ or equivalently $BAx=ABx$ for each x in $\mathscr{D}(A)$).

Proof. Let B be a bounded diagonalizable operator and let B^* be its adjoint. If A permutes with all bounded diagonalizable operators, then by Theorem 5, [8] $BA\subset AB$ and $B^*A\subset AB^*$ imply $P_{ij}B=BP_{ij}$ $i,j=1,2$. Thus every P_{ij},

$i, j = 1, 2$ commutes with every bounded diagonalizable operator. Hence

$$P_{ij} = \int_{\Sigma}^{\oplus} P_{ij}(s)\, \mu(ds) \qquad i, j = 1, 2\,.$$

Now since $P = (P_{ij})$ is the characteristic matrix of closed operator A, it follows that $I - P_{22}$ has an inverse (not necessarily bounded). Hence, for μ-a.a. s, $(I - P_{22})(s)$ has an inverse. Together with the fact that $(P_{ij}(s))$ is a projection for μ-a.a. s, it follows that $(P_{ij}(s))$ is the characteristic matrix of a closed operator $A(s)$ for every s in some Borel set δ with $\mu(\delta^c) = 0$. It is clear that $s \to A(s)$ is μ-measurable and

$$A = \int_{\Sigma}^{\oplus} A(s)\, \mu(ds)\,. \tag{+}$$

Conversely, if A has the representation as $(+)$, then

$$P_{ij} = \int_{\Sigma}^{\oplus} P_{ij}(s)\, \mu(ds) \qquad i, j = 1, 2\,.$$

Hence every P_{ij}, $i, j = 1, 2$ commutes with every bounded diagonalizable operator in H. It is clear that for every $x \in \mathscr{D}(A)$, $BAx = ABx$. Thus $BA \subset AB$.

q.e.d.

Theorem 3.4 shows the notion of decomposability defined by Definition 3.3 does generalize that of the usual decomposability of the bounded operators. We may easily prove the following proposition.

Proposition 3.5. *Suppose* $A = \int_{\Sigma}^{\oplus} A(s)\, \mu(ds)$ *is a closed decomposable operator. Then*

(i) *if* A^* *exists,* A^* *is decomposable and*

$$A^* = \int_{\Sigma}^{\oplus} A(s)\, \mu(ds) \tag{17}$$

where $B(s) = A(s)^*$ μ*-a.e.;*

(ii) *if* A^{-1} *exists,* A^{-1} *is decomposable and*

$$A^{-1} = \int_{\Sigma}^{\oplus} C(s)\, \mu(ds) \tag{18}$$

where $C(s) = A(s)^{-1}$ μ*-a.e.*

The following proposition is ours.

Proposition 3.6. *Let*

$$A = \int_{\Sigma}^{\oplus} A(s)\, \mu(ds)$$

$$B = \int_{\Sigma}^{\oplus} B(s)\, \mu(ds)$$

be two decomposable closed operators in H. Then

(i) *if $A+B$ is closed and if $\mathscr{D}(A) \subset \mathscr{D}(B)$, $A+B$ is decomposable and*

$$A+B = \int_{\Sigma}^{\oplus} (A(s)+B(s))\,\mu(ds) \tag{19}$$

(ii) *if AB is closed, AB is decomposable and*

$$AB = \int_{\Sigma}^{\oplus} A(s)\,B(s)\,\mu(ds)\,. \tag{20}$$

Proof. (i) We shall show that

$$D(A+B) \subset (A+B)D$$

for every bounded diagonalizable operator D. If $\mathscr{D}(A) \subset \mathscr{D}(B)$, then $\mathscr{D}(A+B) = \mathscr{D}(A)$. Hence for every $x \in \mathscr{D}(A)$,

$$D(A+B)x = D(Ax+Bx) = DAx + DBx = ADx + BDx = (A+B)Dx\,.$$

(ii) $\mathscr{D}(AB) = \{x \in \mathscr{D}(B) \mid Bx \in \mathscr{D}(A)\}$. If $x \in \mathscr{D}(AB)$, $(DA)(Bx) = ADBx = ABDx$. Thus $D(AB) \subset ABD$.

Then a standard argument will provide us formulas (19) and (20) q.e.d.

Definition 3.7. *A closed operator $A = \int_{\Sigma}^{\oplus} A(s)\,\mu(ds)$ is said to be boundedly decomposable of $A(s)$ is a bounded operator (with domain $H(s)$) μ-a.e.*

It is easy to see that if A is boundedly decomposable, then $s \to \|A(s)\|$ is μ-measurable, and $s \to \|A(s)\|$ is μ-essentially bounded if and only if A is bounded. Let g be a μ-measurable and μ-essentially bounded scalar-valued function on Σ. Let

$$A_g = \int_{\Sigma}^{\oplus} g(s)\,I(s)\,\mu(ds)$$

and let

$$A = \int_{\Sigma}^{\oplus} A(s)\,\mu(ds)$$

It was proved by Nussbaum (Th. 6 [6]) that A is boundedly decomposable if and only if there exists such A_g such that

(i) A_g has an inverse, and
(ii) $AA_g = B$ is a bounded operator (with domain H).

4. Spectral Theory for A Closed Decomposable Operator

Definition 4.1. *Let $\mathscr{B}$ be the Borel field of subsets of complex plane. A closed operator A is called a spectral operator if there is a σ-additive spectral measure E in H such that*

(i) *the domain $\mathscr{D}(A)$ of A contains the dense subspace $H_0 = \{x \mid E(\alpha)x = x,$ $\alpha \in \mathscr{B}$ α bounded$\}$,*

(ii) *if* $\alpha \in \mathscr{B}$, $E(\alpha)A \subset AE(\alpha)$; *and*

(iii) $\sigma(A, E(\alpha)H) \subset \bar{\alpha}$ *where* $\sigma(A, E(\alpha)H)$ *is the spectrum of* A *considered as an operator in subspace* $E(\alpha)H$.

It is easy to see that if A is a spectral operator, then A is a bounded spectral operator in the subspace $E(\alpha)H$ for bounded Borel subset α. We now proceed to prove the following theorem:

Theorem 4.2. *Let* $H = \int_{\Sigma}^{\oplus} H(s)\,\mu(ds)$. *A closed operator* $A = \int_{\Sigma}^{\oplus} A(s)\,\mu(ds)$ *is a spectral operator if and only if*

(i) $A(s)$ *is a spectral operator for* μ-*a.a.* s

(ii) *the field* $s \to E_s(\alpha)$ *of the resolutions of the identity corresponding to* $s \to A(s)$ *is* μ-*measurable and* μ-*essentially bounded for every Borel subset* α *in* $\mathscr{B}$; *and*

(iii) $\mu\text{-ess.}\sup_{s} \|N_\alpha(s)^n\|^{1/n} \to 0$ *as* $n \to \infty$ *for every bounded* $\alpha \in \mathscr{B}$ *where* $N_\alpha(s)$ *denotes the radical part of* $A(s)$ *restricted to* $E_s(\alpha)\,H(s)$.

Proof. Suppose that A is a spectral operator. Then the resolution E of the identity for A is bounded. If A is decomposable, E is also decomposable such that

$$E(\alpha) = \int_{\Sigma}^{\oplus} E_s(\alpha)\,\mu(ds) \tag{21}$$

for every $\alpha \in \mathscr{B}$. Hence $s \to E_s(\alpha)$ is μ-measurable and μ-essentially bounded for $\alpha \in \mathscr{B}$. We shall show that, for μ-a.a. s, E_s is the resolution of the identity for $A(s)$. For any arbitrary ξ, η in H, the set function $(E(\cdot)\,\xi, \eta)$ is σ-additive and

$$(E(\cdot)\,\xi, \eta) = \int_{\Sigma} (E_s(\cdot)\,\xi(s), \eta(s))\,\mu(ds)$$

It follows by a standard argument in measure theory that $E_s(\cdot)$ is also σ-additive.

Let $H_0(s) = \{x(s) \mid x \in H_0\}$. Then $H_0(s) \subset \mathscr{D}(A(s))$ for μ-a.a. s. Since $E(\alpha)A \subset AE(\alpha)$, we have for every $x \in \mathscr{D}(A)$,

$$\int_{\Sigma}^{\oplus} E_s(\alpha)\,A(s)\,x(s)\,\mu(ds) = \int_{\Sigma}^{\oplus} A(s)\,E_s(\alpha)\,x(s)\,\mu(ds) \tag{22}$$

Hence $E_s(\alpha)\,A(s) \subset A(s)\,E_s(\alpha)$ for μ-a.a. s.

Finally, we shall show that $\sigma(A(s), E_s(\alpha)\,H(s)) \subset \bar{\alpha}$ for every $\alpha \in \mathscr{B}$. If a is a bounded Borel subset, then A is a bounded spectral operator on $E(\alpha)H$. It follows from Theorem 2.7 and Lemma 2.6, $\sigma(A(s), E_s(\alpha)\,H(s)) \subset \bar{\alpha}$. Let e_n be an increasing sequence of bounded Borel subsets such that

$$E(\cup e_n) = I\,,$$

and denote $A \mid E(\alpha)H$ be A_α. It was proved by W. G. Bade (Lemma 3.1 [1]) that

$$\sigma(A_\alpha) = \overline{\bigcup_{n=1}^{\infty} \sigma(A_\alpha, E(\alpha)\,E(e_n)H)}$$

It is obvious that for μ-a.a. s

$$E_s(\cup e_n) = I(s).$$

Thus we see that

$$\sigma(A_\alpha(s)) = \overline{\bigcup_{n=1}^{\infty} \sigma(A_\alpha(s), E_s(\alpha)\, E_s(e_n)\, H(s))} \subset \overline{\bigcup_{n=1}^{\infty} \overline{(\alpha \cap e_n)}} \subset \bar{\alpha}.$$

This completes the proof of the necessary part.

Conversely, if conditions (i) (ii), and (iii) are satisfied, we shall show that E given by formula (21) is the resolution of the identity for A. It is obvious E is σ-additive and bounded. It is also clear that if $\mathscr{D}(A(s))$ contains $H_0(s)$, which is dense in $H(s)$, $\mathscr{D}(A)$ also contains H_0 as a dense subspace in H. Eq. (22) implies that $E(\alpha)A \subset AE(\alpha)$ for every Borel subset α. Finally, for every $\alpha \in \mathscr{B}$,

$$\begin{aligned} \sigma(A, E(\alpha)H) &= \overline{\bigcup_{n=1}^{\infty} \sigma(A, E(\alpha)\, E(e_n) H)} \\ &= \overline{\bigcup_{n=1}^{\infty} \cap \left\{ \overline{\bigcup_{s \in \delta} \sigma(A(s), E_s(\alpha)\, E_s(e_n) H)} : \mu(\delta^c) = 0 \right\}} \subset \bar{\alpha}. \end{aligned}$$

This completes the proof of the theorem.

Let e_n be an increasing sequence of bounded Borel subsets of complex plane with $E(\subset e_n) = I$, and let f be a bounded measurable function. Then for each n, the integral

$$\int_{e_n} f(\lambda)\, E(d\lambda)$$

exists in the uniform topology of operators in $E(e_n)H$. It was proved by Bade (cf. [1]) that

$$f(S) = \lim_{n\to\infty} \int_{e_n} f(\lambda)\, E(d\lambda) x \tag{23}$$

defines a closed operator in H, and $\mathscr{D}(f(S))$ is defined to be the set of all x such that the limit exists in (23).

Definition 4.3. *We shall denote by S the operator obtained by taking $f(\lambda) = \lambda$ in formula* (23) *and call it the scalar operator associated with E. In case of E is the resolution of the identity for a spectral operator A, we simply call S the scalar operator associated with A.*

If A is a bounded spectral operator, then it is a unique sum of a scalar operator and a generalized nilpotent (cf. Th. 8 [3]). This characterization theorem is no longer valid in the unbounded case; however, we do have the following theorem:

Theorem 4.4. *Let $A = \int_{\Sigma}^{\oplus} A(s)\, \mu(ds)$ be a decomposable spectral operator. Then $A = S + N$, where S is a scalar operator N, a bounded generalized nilpotent* *and $NS \subset SN$, if and only if $A(s)$ has the similar representation for μ-a.e. s, $s \to N(s)$ is μ-essentially bounded, and* $\mu\text{-ess.}\sup_s \|N(s)^n\|^{1/n} \to 0$ *as* $n \to \infty$.

Proof. Necessity: Suppose A has the representation in the theorem. Then $\mathscr{D}(A) = \mathscr{D}(S)$. Let $s \to E_s(\cdot)$ be the field of the resolutions of the identity corresponding to $s \to A(s)$ such that

$$E(\cdot) = \int_{\Sigma}^{\oplus} E_s(\cdot)\,\mu(ds)\,.$$

For every $x \in \mathscr{D}(A)$, then

$$\begin{aligned}
Sx &= \lim_{n\to\infty} \int_{e_n} \lambda E(d\lambda)x \\
&= \lim_{n\to\infty} \left(\int_{e_n} \lambda \int_{\Sigma}^{\oplus} E_s(d\lambda)\,\mu(ds)x \right) \\
&= \lim_{n\to\infty} \left(\int_{\Sigma}^{\oplus} \int_{e_n} \lambda E_s(d\lambda)\,\mu(ds)x \right) \\
&= \lim_{n\to\infty} \int_{\Sigma}^{\oplus} \int_{e_n} \lambda E_s(d\lambda)\,x(s)\,\mu(ds) \\
&= \int_{\Sigma}^{\oplus} \lim_{n\to\infty} \int_{e_n} \lambda E_s(d\lambda)\,x(s)\,\mu(ds) \\
&= \int_{\Sigma}^{\oplus} S(s)\,x(s)\,\mu(ds) \\
&= \left(\int_{\Sigma}^{\oplus} S(s)\,\mu(ds) \right) x\,.
\end{aligned}$$

Thus $N = \int_{\Sigma}^{\oplus} N(s)\,\mu(ds)$ in the sense that

$$Nx = \int_{\Sigma}^{\oplus} N(s)\,\mu(ds)x$$

for every $x \in \mathscr{D}(A)$. N is bounded if and only if $s \to N(s)$ is μ-essentially bounded, and N is a generalized nilpotent if and only if so is $N(s)$ for μ-a.a. s, and $\mu\text{-ess.}\sup_s \|N(s)^n\|^{1/n} \to 0$.

Sufficiently is immediate. q.e.d.

Given an unbounded spectral operator A, let f be a scalar-valued function analytic and single-valued in the complement of a closed set θ_f for which $E(\theta_f) = 0$. Let

$$e_n = \left\{ \lambda \,\middle|\, |\lambda| \leqq n,\ \operatorname{dist}(\lambda, \theta_f) \geqq \frac{1}{n} \right\}. \tag{24}$$

Then $\{e_n\}$ is an increasing sequence of closed sets with $E(\cup e_n)=I$, and on each of which f is analytic. On the subspace $E(e_n)H$, A is actually a bounded spectral operator; therefore has the form $A=S+N$. Define

$$f(A)x=\lim_{n\to\infty}\sum_{i=0}^{\infty}\frac{N^i}{i!}\int_{e_n} f^{(i)}(\lambda)\,E(d\lambda)x\,, \tag{25}$$

$$\mathscr{D}(f(A))=\{x \mid \text{limit in (25) exists}\}\,. \tag{26}$$

It was shown in Section 5 [1] by example that $f(A)$ defined by (25) need not be a spectral operator; however, it was also shown there that if f is analytic on $\sigma(A)$ with the exception of a finite set $\theta=(p_1, p_2, \ldots, p_k)$ of pole s for which $E(\theta)=0$, and if f is either analytic at infinite or have a pole there, then $f(A)$ is a spectral operator with the resolution of the identity

$$E_f(\alpha)=E(f^{-1}(\alpha)) \tag{27}$$

We have the following theorem

Theorem 4.5. *Let the condition in Theorem* 4.2 *be satisfied , and f be as before with $E_s(\theta)=0$ for μ-a.a. s. Then*

$$f(A)=\int_{\Sigma}^{\oplus} f(A(s))\,\mu(ds) \tag{28}$$

is a spectral operator.

Proof. By Theorem 4.2, A is a spectral operator. $f(A)$ is thus a spectral operator as discussed above. By Theorem 2.10

$$\begin{aligned}
f(A)x &= \lim_{n\to\infty}\sum_{i=0}^{\infty}\frac{N^i}{i!}\int_{e_n} f^{i}(\lambda)\,E(d\lambda)x\\
&= \lim_{n\to\infty}\int_{\Sigma}^{\oplus}\sum_{i=0}^{\infty}\frac{N(s)^i}{i!}\int_{e_n} f^{i}(\lambda)\,E_s(d\lambda)\,x(s)\,\mu(ds)\\
&= \int_{\Sigma}^{\oplus}\lim_{n\to\infty}\sum_{i=0}^{\infty}\frac{N(s)^i}{i!}\int_{e_n} f^{i}(\lambda)\,E_s(d\lambda)\,x(s)\,\mu(ds)\\
&= \int_{\Sigma}^{\oplus} f(A(s))x(s)\,\mu(ds)\\
&= \left(\int_{\Sigma}^{\oplus} f(A(s))\,\mu(ds)\right)x\,.
\end{aligned}$$

q.e.d.

Corollary 4.6. *Let A be a spectral operator and let p be a polynomial. Then*

$$p(A)=\int_{\Sigma}^{\oplus} p(A(s))\,\mu(ds) \tag{29}$$

is a spectral operator.

5. Decompositions of Spectral Operators

Definition 5.1. *A closed linear operator A is said to be irreducible if there is no non-trivial closed subspaces reducing A.*

Let A be a linear closed operator. Denote the set of all bounded everywhere defined operators in H, which permute with A by A'. Suppose that H is separable. Then the von Neumann algebra $R(A)=A''$ is separable. By the decomposition theory (cf. Theorem 1.2 [5]) there is a direct integral decomposition of Hilbert spaces

$$H=\int_{\Sigma}^{\oplus} H(s)\,\mu(ds)$$

with respect to the maximal abelian von Neumann subalgebra of A' so that

$$R(A)=\int_{\Sigma}^{\oplus} R(A)(s)\,\mu(ds) \tag{30}$$

with $R(A)(s)$ irreducible for μ-a.a. s. By Theorem 3.4, A is thus decomposable. Let

$$A=\int_{\Sigma}^{\oplus} A(s)\,\mu(ds)$$

and let (P_{ij}) and $(P_{ij}(s))$ be the characteristic matrices of A and $A(s)$ respectively. It is clear that

$$R(A)=(P_{ij})''=A'' \tag{31}$$

and

$$R(A)(s)=(P_{ij}(s))''=A(s)''\,. \tag{32}$$

Hence $P_{ij}(s)$ is irreducible for μ-a.a. s. Consequently, $A(s)$ is irreducible for μ-a.a. s.

This is a theorem proved by Nussbaum in [6].

Theorem 5.2. *Let A be a closed linear operator in H. Then there exists a direct integral decomposition*

$$A=\int_{\Sigma}^{\oplus} A(s)\,\mu(ds)$$

such that $A(s)$ is irreducible for μ-a.a. s.

If A is boundedly decomposable into irreducible part, then we have reduced the structure problem of a closed spectral operator into the one of bounded irreducible operators. In [9], using this ideal of reduction N. Suzuki has established the following structure theorem:

Theorem [Suzuki]. *Let A be a bounded spectral operator on a Hilbert space H whose imaginary part is compact. Then H is decomposable into an algebraic direct sum $H=H_0+H_1+H_2+\cdots$ of a countable family of invariant subspaces of A such that $A|H_0$ is a scalar operator with real spectrum and each*

$A|H_i$ ($i=1,2,3,\ldots$) has the form $\lambda_i I_i + N_i$, where I_i is the identity operator in H_i, $\{N_i\}$ is a sequence of generalized nilpotent compact operators approaching zero in norm and $Im\,\lambda_i \to 0$ as $n \to \infty$.

We shall investigate in the future that if a similar type of structure theorem is possible for the unbounded spectral operators.

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Hamiltonian Systems: Generic Properties of Closed Orbits and Local Perturbations

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§ 1. Introduction, Definitions and Theorem

The aim of this paper is to extend parts of R. C. Robinsons work on generic properties of Hamiltonian systems [7]. We first give some definitions.

Let M be a $2n$-dimensional C^∞-manifold with *symplectic form* ω (i.e. ω is a closed 2-form and $\omega \wedge \cdots \wedge \omega$ (n-times) is a nowhere zero volume form). A *Hamiltonian system* on M is a vectorfield X such that the 1-form $\iota_X \omega$ is exact (i.e. $\iota_X \omega$ is the differential of a function). In general a Hamiltonian system is given by a function $H: M \to \mathbb{R}$, the *Hamiltonian*; the *corresponding vectorfield* X_H on M is then determined by the equation $\iota_{X_H}\omega = dH$. We say that the *Hamiltonian system*, defined by $H: M \to \mathbb{R}$, *is* C^k *if* X_H is C^k, or equivalently if H is C^{k+1}. The *flow* $\mathscr{D}_H: U_H \to M$, U_H an open subset of $M \times \mathbb{R}$, of the Hamiltonian H is the map defined by:

(i) $\mathscr{D}_H(m, 0) = m$ for each $m \in M$ ($M \times \{0\} \subset U_H$).

(ii) $t \mapsto \mathscr{D}_H(m, t)$ is the integral curve of X_H through m.

(iii) U_H is the largest connected open subset of $M \times \mathbb{R}$ on which $\mathscr{D}_H$ can be defined.

If H is C^{k+1}, $\mathscr{D}_H$ is C^k; for more details see S. Lang [4].

A *closed orbit* γ of a Hamiltonian system is a subset of M which is of the form $\mathscr{D}_H(m, [0, t])$ where $\mathscr{D}_H(m, t) = \mathscr{D}_H(m, 0) = m$ and $\mathscr{D}_H(m, t') \neq m$ for all $t' \in (0, t)$; t is called the period of γ. In order to study the properties of the flow $\mathscr{D}_H$ near a closed orbit γ, we introduce a flowbox with local section and a Poincaré map for γ.

For $m \in \gamma$ $(dH)_m \neq 0$. Then, by the flowbox theorem [2], there is a neighbourhood U of m and a map $A: U \to \mathbb{R}^{2n}$ such that:

1. $A^*\left(\sum_{i=1}^{n} dx_i \wedge dy_i\right) = \omega \mid U$; $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ are the coordinate functions on $\mathbb{R}^{2n}$.

2. $H \mid U = y_1 \circ A$.

A is called a *flowbox at* m. The *local section* $\Sigma_{m,A}$ is defined by $\Sigma_{m,A} = A^{-1}(\{x_1 = x_1(A(m))\})$. Σ_m is called a local section at m if it can be obtained in the above way from some flowbox at m. Notice that $A_*(X_H) = \partial/\partial x_1$; so X_H is nowhere tangent to Σ_m.

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For a sufficiently small section Σ_m there is a function T, defined on a small neighbourhood W of m in Σ_m such that $\mathscr{D}_H(m', T(m')) \in \Sigma_m$ and $\mathscr{D}_H(m', (0, T(m')))$ $\cap \Sigma_m = \emptyset$ for all $m' \in W$. The existence of T follows from the implicit function theorem; $T(m) = t$ = period of γ. A Poincaré map $P: W \to \Sigma_m$ can now be defined by

$$P(m') = \mathscr{D}_H(m', T(m')).$$

For a C^k Hamiltonian system, the Poincaré map is C^k. m is clearly a fixed point of P; closed orbits, near γ and with period near t, correspond to fixed points of P.

A Poincaré map as defined above is not just a local diffeomorphism, but has some extra properties which are due to the fact that X_H is a Hamiltonian vectorfield. The extra properties are:

(a) $H(m') = H(P(m'))$ for $m' \in W$; this is due to the fact that m' and $P(m')$ are on the same integral curve; see also [2].

(b) For each $h \in \mathbb{R}$, ω, restricted to $\Sigma_m \cap H^{-1}(h)$, defines a symplectic structure on $\Sigma_m \cap H^{-1}(h)$; P, restricted to $W \cap H^{-1}(h)$ preserves this symplectic structure; see also [3, Appendice 31].

The following definitions are motivated by the above properties:

A *P.S. (parameter symplectic) structure* on an odd dimensional manifold Σ is a structure given by a function $h: \Sigma \to \mathbb{R}$ without critical points and a closed 2-form $\tilde{\omega}$ with the property that for each $c \in \mathbb{R}$ $\tilde{\omega} | h^{-1}(c)$ defines a symplectic structure on $h^{-1}(c)$. Two P.S. structures $(h, \tilde{\omega})$ and $(h', \tilde{\omega}')$ on Σ are equivalent if $h = h'$ and $(\tilde{\omega} - \tilde{\omega}')$, restricted to any level of h, is zero (i.e. $(\tilde{\omega} - \tilde{\omega}')$ is of the form $dh \wedge \ldots$).

A P.S. map between two manifolds Σ_1 and Σ_2 with P.S. structures $(h_1, \tilde{\omega}_1)$ and $(h_2, \tilde{\omega}_2)$ is a map $f: \Sigma_1 \to \Sigma_2$ such that $(h_1, \tilde{\omega}_1)$ and $(h_2 \circ f, f^* \tilde{\omega}_2)$ are equivalent.

From the definitions it follows that every local section, with its P.S. structure induced by H and ω, is P.S. equivalent with an open set of $\mathbb{R}^{2n-1}$, equiped with the standard P.S. structure. This standard P.S. structure on $\mathbb{R}^{2n-1}$ is defined as follows: let $y_1, x_2, \ldots, x_n, y_2, \ldots, y_n$ be the coordinate functions on $\mathbb{R}^{2n-1}$, take $h \equiv y_1$ and $\tilde{\omega} = \sum_{i=2}^{n} dx_i \wedge dy_i$. Notice that we can replace the Poincaré map P by $\tilde{P} = A P A^{-1}$ which maps a neighbourhood $\tilde{W} = A(W) \subset \mathbb{R}^{2n-1}$ $= \{x_1 = x_1(A(m))\}$ to $\mathbb{R}^{2n-1}$; $\tilde{P}$ is P.S. One can see as follows that the fact that $\tilde{P}$ is P.S. implies that "in general" the fixed points of $\tilde{P}$, and hence the closed orbits of X_H, occurs in one-parameter families:

Let $J^0_{\text{P.S.}}(\tilde{W}, \mathbb{R}^{2n-1})$ be the set of zero-jets of P.S. maps of $\tilde{W}$ to $\mathbb{R}^{2n-1}$; i.e.

$$J^0_{\text{P.S.}}(\tilde{W}, \mathbb{R}^{2n-1}) \cong \{(z_1, z_2) \mid z_1 \in \tilde{W}, z_2 \in \mathbb{R}^{2n-1}, y_1(z_1) = y_1(z_2)\}.$$

If the 0-jet extension $\tilde{P}^{(0)}: \tilde{W} \to J^0_{\text{P.S.}}(\tilde{W}, \mathbb{R}^{2n-1})$ of $\tilde{P}$ is transversal with respect to $\text{Fix} \subset J^0_{\text{P.S.}}(\tilde{W}, \mathbb{R}^{2n-1})$ ($\text{Fix} = \{(z_1, z_2) \mid z_1 \in \tilde{W}, z_2 \in \mathbb{R}^{2n-1}$ and $z_1 = z_2\}$) then $(\tilde{P}^{(0)})^{-1}(\text{Fix})$ is 1-dimensional, because the codimension of Fix is $2n-2$.

$(\tilde{P}^{(0)})^{-1}$ (Fix) consists of the fixed points of $\tilde{P}$. This is a special case of transversality of jet extensions of a Poincaré map. In general we consider k-jet extensions

$$\tilde{P}^{(k)}: \tilde{W} \to J^k_{\text{P.S.}}(\tilde{W}, \mathbb{R}^{2n-1}) = J^0_{\text{P.S.}}(\tilde{W}, \mathbb{R}^{2n-1}) \times J^k_{\text{P.S.}}(2n-1),$$

where $J^k_{\text{P.S.}}(2n-1)$ is the Lie group of k-jets of P.S. maps $\alpha: (\mathbb{R}^{2n-1}, 0) \to (\mathbb{R}^{2n-1}, 0)$. We then look for transversality with respect to a given "normal subset" Q. A subset $Q \subset J^k_{\text{P.S.}}(2n-1)$ is a *normal subset* if:

1. Q is an analytic or semi-analytic subset,
2. for every $\alpha \in J^k_{\text{P.S.}}(2n-1)$, $\alpha^{-1} Q \alpha = Q$.

$\tilde{P}$ is said to be Q transversal if $\tilde{P}^{(k)}$ is transversal with respect to Fix $\times\, Q \subset J^k_{\text{P.S.}}(\tilde{W}, \mathbb{R}^{2n-1})$. Notice that in general Q and Fix $\times\, Q$ is not a manifold but a W-object [8]: transversality with respect to W-objects however works just as transversality with respect to a proper submanifold. In the definition of a normal subset, condition 1 may be weakened to: Q is a W-object in $J^k_{\text{P.S.}}(2n-1)$. Our main result can now be stated as follows:

Theorem A. *Let $Q_1, Q_2, \ldots$ be a countable set of normal subsets of $J^k_{\text{P.S.}}(2n-1)$. Then there is a residual set $R \subset C^{k+2}(M)$, $C^{k+2}(M)$ is the space of C^{k+2} functions on M with the strong topology, such that if $H \in R$, then any closed orbit of X_H has a Poincaré map which is Q_i-transversal for every i.*

For the case $k=1$ the above theorem is essentially contained in [7]. Extension to the case $k>1$ was motivated by the fact that certain important properties of closed orbits, like "Moser stability" or "Meyer bifurcation" depend on the higher order jet of the corresponding Poincaré map.

In fact Theorem A has the following two corollaries which follow immediately from the quoted papers:

Corollary A, 1 [6]. *Let M be a 4-dimensional symplectic manifold. Then there is a residual set $R \in C^{334}(M)$ such that for each $H \in R$ there is a residual subset $K_H \subset \mathbb{R}$ with the property that the elliptic closed orbits of X_H, contained in $H^{-1}(K_H)$ are orbital stable.*

Corollary A, 2 [5]. *Let M be a 4-dimensional symplectic manifold and let k be an integer, greater than 2. Then there is a residual set $R_k \subset C^k(M)$ such that for each $H \in R_k$ and each closed orbit γ of X_H, for which the differential of the Poincaré map, restricted to the H level of γ, is a rotation over $\frac{n}{m} 2\pi$ with $m \leqq k$, there is a generic bifurcation of periodic points for the Poincaré map as described in* [5] *and hence in M a generic bifurcation of closed orbits.*

The proof of Theorem A is obtained by combining Robinson's methods with a new perturbation theorem. This perturbation theorem deals with the following situation:

On $\mathbb{R}^{2n}$, with symplectic form $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$, we consider the Hamiltonian $H = y_1$ (i.e. $X_H = \partial/\partial x_1$). V_+ resp. $V_- = \{(x_1, \ldots, y_n) \mid x_1 = +1 \text{ resp. } x_1 = -1\}$; V_+ and V_- have a natural P.S. structure.

For any $\tilde{H}$, close to H and with support $(H-\tilde{H})\subset\{|x_1|\leqq 1\}$, we define a P.S. map $\varphi_{\tilde{H}}: V_-\to V_+$ by "$\varphi_{\tilde{H}}(v)$ and v are on the same integral curve".

Theorem B. *Let U_1 and U_2 be bounded open subsets of V_+ with $\bar{U}_1\subset U_2$. Then for any integer $k\geqq 0$ there is a smooth function $\mathscr{H}$ on $\mathcal{O}\times\mathbb{R}^{2n}$, $\mathcal{O}$ is an open neighbourhood of the origin 0 in some finite dimensional vectorspace whose dimension depends on k and n, such that:*

(a) *$\mathscr{H}_0=H$ (for $w\in\mathcal{O}$, $\mathscr{H}_w:\mathbb{R}^{2n}\to\mathbb{R}$ is the function $\mathscr{H}_w(x_1,\ldots,y_n)=\mathscr{H}(w;x_1,\ldots,y_n)$).*

(b) *For any $w\in\mathcal{O}$, support $(\mathscr{H}_w-H)\subset\{|x_1|\leqq 1\}\times\bar{U}_2$.*

(c) *The map $\mathscr{H}^{(k)}:\mathcal{O}\times U_2\to J^k_{\mathrm{P.S.}}(U_2,V_+)$, which assigns to (w,v) the k-jet of $\varphi_{\mathscr{H}_w}\circ\varphi_H^{-1}$ at v, induces a codimension zero embedding of $\mathcal{O}\times\bar{U}_1$ in $J^k_{\mathrm{P.S.}}(U_2,V_+)$.*

Remark. Let Q be a proper submanifold (or a W-object) in $J^k_{\mathrm{P.S.}}(U_2,V_+)$. By transversality theory [1] (and Theorem B) there is an open and dense subset $R\subset\mathcal{O}$ such that if $w\in R$ the k-jet extension of $\varphi_{\mathscr{H}_w}\circ\varphi_H^{-1}$, restricted to a small neighbourhood of $\bar{U}_1$, is Q-transversal.

§ 2. Local Perturbations, Theorem B

We shall first state and prove a preliminary form of Theorem B. Consider the case where a P.S. map $\Psi: V_-\to V_+$ is given which is of the form $\tilde{\Psi}\circ\varphi_H$ where $\tilde{\Psi}$ is obtained by integrating a "P.S. vectorfield" on V_+ (P.S. vectorfields are defined below). Roughly speaking the preliminary perturbation theorem (Theorem B′) states that for maps like Ψ there is a Hamiltonian $\tilde{H}$ such that $\varphi_{\tilde{H}}=\Psi$.

A *P.S. vectorfield* on a manifold N, with parameter symplectic structure, is a vectorfield Y such that for every t $\mathscr{D}_{Y,t}: N\to N$ is a P.S. map; $\mathscr{D}_{Y,t}$ is defined by $\mathscr{D}_{Y,t}(n)=\mathscr{D}_Y(n,t)$, $\mathscr{D}_Y$ is the integral of Y.

Let Y be a P.S. vectorfield on $\mathbb{R}^{2n-1}$ (with its standard P.S. structure). Then Y is tangent to the levels of y_1 and on each y_1 level, Y is a Hamiltonian vectorfield. Just as in the case of Hamiltonian vectorfields there is a function $L:\mathbb{R}^{2n-1}\to\mathbb{R}$ such that

$$Y=\sum_{i=2}^{n}\left(\frac{\partial L}{\partial y_i}\,\frac{\partial}{\partial x_i}-\frac{\partial L}{\partial x_i}\,\frac{\partial}{\partial y_i}\right);$$

for a given P.S. vectorfield Y, the function L is unique up to a function which depends on y_1 only. We shall use the following notation:

For a function L on $\mathbb{R}^{2n-1}$, the corresponding P.S. vectorfield will be denoted by $\tilde{X}_L$ and its integral by $\tilde{\mathscr{D}}_L:\mathbb{R}^{2n-1}\times\mathbb{R}\to\mathbb{R}^{2n-1}$; $\tilde{\mathscr{D}}_{L,t}:\mathbb{R}^{2n-1}\to\mathbb{R}^{2n-1}$ is defined by $\tilde{\mathscr{D}}_{L,t}(p)=\tilde{\mathscr{D}}_L(p,t)$.

Theorem B′. *Let $L:\mathbb{R}^{2n-1}=V_+\to\mathbb{R}$ be a smooth function with compact support and such that $\left|\frac{\partial L}{\partial y_1}\right|<1$. Then there is a Hamiltonian $\tilde{H}$ on $\mathbb{R}^{2n}$ such that:*

(i) *support* $(\tilde{H}-H)\subset\{|x_1|\leqq 1\}\times\{\textit{support}\ (L)\}$,

(ii) $\varphi_{\tilde{H}}=\tilde{\mathscr{D}}_{L,1}\circ\varphi_H$.

Proof. Let $\mu:\mathbb{R}\to\mathbb{R}$ be a smooth non-negative function, zero on $(-\infty,-1]$ and one on $[0,+\infty)$. The function $K:\mathbb{R}^{2n}\to\mathbb{R}$ is defined by

$$K(x_1,\ldots,y_n)=\mu(x_1)\,.\,L(y_1,x_2,\ldots,y_n)\,.$$

Consider $\mathscr{D}_{K,1}$:

An integral curve ℓ of $X_H=\partial/\partial x_1$ is transformed by $\mathscr{D}_{K,1}$ into a line ℓ' with the following properties:

$$\ell'\cap V_-=\ell\cap V_-=(-1,x_2(\ell),\ldots,x_n(\ell),y_1(\ell),\ldots,y_n(\ell))\,,$$

$$\ell'\cap V_+=(+1,\bar{x}_2(\ell),\ldots,\bar{y}_n(\ell))$$

with

$$(\bar{y}_1(\ell),\bar{x}_2(\ell),\ldots,\bar{y}_n(\ell))=\tilde{\mathscr{D}}_{L,1}(y_1(\ell),x_2(\ell)\ldots y_n(\ell))\,.$$

This follows from the following observations:

1. All derivatives of K are zero in V_-, so each point of V_- is fixed under $\mathscr{D}_{K,1}$.

2. $\left|\frac{\partial L}{\partial y_1}\right|<1$, so $\left|\frac{\partial K}{\partial y_1}\right|<1$ and so the x_1 component of X_K is, in absolute value, smaller than 1. So for any p the x_1 coordinates of p and $\mathscr{D}_{K,1}(p)$ differ by less than 1. In particular, if $\mathscr{D}_{K,1}(p)\in V_+$ then, for any $t\in[0,1]$ $\mathscr{D}_{K,t}(p)\in\{x_1>0\}$.

3. For a point $p\in\mathbb{R}^{2n}$, such that $\mathscr{D}_K(p,[0,1])\subset\{x_1>0\}$, $\mathscr{D}_{K,t}(p)=(x_1(t),\ldots,y_n(t))$ satisfies

$$(y_1(t),x_2(t),\ldots,y_n(t))=\tilde{\mathscr{D}}_{L,t}(y_1(0),x_2(0),\ldots,y_n(0))\,.$$

This follows from the fact that on $\{x_1>0\}$ we have:

(a) the y_1 component of X_K is zero because $\frac{\partial K}{\partial x_1}=0$,

(b) the x_i and y_i components $(i=2,\ldots,n)$ of X_K in $(x_1,\ldots,y_n)$ are equal to those of $\tilde{X}_L$ in $(y_1,x_2,\ldots,x_n)$ because $\frac{\partial K}{\partial y_i}$ resp. $\frac{\partial K}{\partial x_i}$ and $\frac{\partial L}{\partial y_i}$ resp. $\frac{\partial L}{\partial x_i}$ are so.

Now we define $\tilde{H}=H\circ(\mathscr{D}_{K,1})^{-1}$. Because $\mathscr{D}_{K,1}$ is symplectic, it maps integral curves of X_H on integral curves of $X_{\tilde{H}}$ so $\varphi_{\tilde{H}}=\tilde{\mathscr{D}}_{L,1}\varphi_H\circ(H-\tilde{H})$ is clearly zero in points $(x_1,\ldots,y_n)$ for which $x_1\leqq -1$ or $(y_1,x_2,\ldots,y_n)\notin$ support (L). Because $\tilde{\mathscr{D}}_{L,1}$, being a P.S. automorphism, does not change the y_1 coordinate, $H-\tilde{H}$ is also zero on $\{x_1\geqq 1\}$. This proves Theorem B′.

Proof of Theorem B. We take for $\mathcal{O}$ a neighbourhood of the origin in the vectorspace of those polynomials of degree $\leqq k+1$ on V_+ which vanish along the line $\{x_2=\cdots=x_n=y_2=\cdots=y_n=0\}$. Let λ be a smooth function which is one on U_1 and zero outside U_2. To each $w\in\mathcal{O}$ we assign the function L_w on V_+ which is obtained by multiplying the polynomial w with λ. We assume that $\left|\frac{\partial L_w}{\partial y_1}\right|<1$ for all $w\in\mathcal{O}$; this is the case if $\mathcal{O}$ is small enough. For each L_w we construct a function H_w as in the proof of Theorem B′. The only free choice which was made in the proof of Theorem B′ was the choice of the function μ; we assume that the same function μ is used in the construction of each H_w. These functions H_w determine the function $\mathscr{H}:\mathcal{O}\times\mathbb{R}^{2n}\to\mathbb{R}$ by $\mathscr{H}_w=H_w$; $\mathscr{H}$ is smooth.

$\mathscr{H}$ clearly satisfies the conditions (a) and (b). For $k=0$ also condition (c) is clear; for $k>0$ we need the following:

Lemma. *Let $J_*^{k+1}(V_+,p)$ be the set of those $(k+1)$-jets in p of functions on V_+ whose first derivative in p is zero. Then the natural map α from $J_*^{k+1}(V_+,p)$ to the Lie algebra of $J^k_{\text{P.S.}}((V_+,p),(V_+,p))$ is surjective and its kernel consists of those functions which can be represented by a function of y_1 only.*

The map α is defined as follows: let $[f]_{k+1}$ be a $(k+1)$-jet of $J_*^{k+1}(V_+,p)$. $[f]_{k+1}$ defines a k-jet $[\tilde{X}_f]_k$ of a P.S. vectorfield which is zero in p. On the other hand the elements of the Lie algebra of $J^k_{\text{P.S.}}((V_+,p),(V_+,p))$ are just the k-jets of vectorfields at p, so we can define α by $\alpha([f]_{k+1})=[\tilde{X}_f]_k$.

The proof of the lemma is straightforward and left to the reader.

Using the lemma it follows that in points $(0,p)$ of $\mathcal{O}\times\bar{U}_1$, the differential $D(\mathscr{H}^{(k)})$ is an isomorphism. (The kernel, occuring in the lemma is not present here because all functions L_w, $w\in\mathcal{O}$, are zero along the y_1 axis, so there are no two functions L_{w_1} and L_{w_2} such that at some point $p\in\bar{U}_1$ the $(k+1)$-jet of their difference can be represented by a function of y_1 only.)

$\mathscr{H}^{(k)}\mid\{0\}\times\bar{U}_1$ is an embedding in $J^k_{\text{P.S.}}(U_2,V_+)$, so for $\mathcal{O}$ sufficiently small $\mathscr{H}^{(k)}$ induces a co-dimension zero embedding of $\mathcal{O}\times\bar{U}_1$ in $J^k_{\text{P.S.}}(U_2,V_+)$.

§ 3. Generic Properties of Closed Orbits, Theorem A

We shall prove the following proposition:

Proposition. *Let H be a Hamiltonian on a symplectic manifold M^{2n}, γ a closed orbit of X_H with period t, $m\in\gamma$, and let Q be a normal subset of $J^k_{\text{P.S.}}(2n-1)$. Then there is a compact neighbourhood V of m in M, $\varepsilon>0$, a neighbourhood W of H in $C^{k+2}(M)$ and an open and dense subset R of W, such that for every $H'\in R$ and every closed orbit γ' of $X_{H'}$ with period $\in(t-\varepsilon,t+\varepsilon)$ and passing through V, there is a Poincaré map for γ' which is Q-transversal.*

From this proposition and Robinson's method for globalizing [7; § 12] Theorem A follows immediately in case we have only one normal subset. For a countable number of normal subsets the proof then follows from the fact that the intersection of a countable number of residual sets is residual. The rest of this paragraph is devoted to the proof of the above proposition.

First we show that it does not matter which local section we take for a given closed orbit, or more precise:

Statement 1. *Let γ be a closed orbit of X_H and $m, m' \in \gamma$; let A, A' be flowboxes at m and m' and $\Sigma_{m,A}$, $\Sigma_{m',A'}$ local sections with Poincaré map P resp. P'. Then there is a P.S. map φ from a neighbourhood of m in $\Sigma_{m,A}$ to a neighbourhood of m' in $\Sigma_{m',A'}$ such that*

$$P = \varphi^{-1} \circ P' \circ \varphi$$

holds in a neighbourhood of m in $\Sigma_{m,A}$.

Proof of Statement 1. Let U, U' be the domains of definition of A, A'. There is a $t' < t$, t is the period of γ, such that $\mathscr{D}_H(m, t') = m'$. A small neighbourhood of m is then mapped by $\mathscr{D}_{H,t'}$ to a small neighbourhood of m' contained in U'. The map φ is defined as follows:

for $p \in \Sigma_{m,A}$ $\varphi(p)$ is obtained by first taking $\mathscr{D}_{H,t'}(p)$ and then, in U', going along a trajectory of X_H to a (unique) point in $\Sigma_{m',A'}$.

With the above definition φ is only well defined in a sufficiently small neighbourhood of m. The equation $P = \varphi^{-1} \circ P \circ \varphi$ is evident. The fact that φ is P.S. follows from [3; appendice 31].

Remark. Because for every $\alpha \in J^k_{\text{P.S.}}(2n-1)$ $\alpha^{-1}\, Q\alpha = Q$, the property of being Q-transversal is preserved under conjugation with a P.S. map, and, by Statement 1, also independent of the choice of the local section.

Statement 2. *Let A be a flowbox at m for a given Hamiltonian H; the domain of A is assumed to be sufficiently small. Then, if H' is C^{k+2} close to H, there is a flowbox A', C^{k+1} close to A, at m for H'.*

Proof of Statement 2. A is a map from a neighbourhood U of m to $\mathbb{R}^{2n}$. Let t_0 be such that $\mathscr{D}_{H,t_0}(\bar{U}) \cap \bar{U} = \emptyset$; such a t_0 exists if U is small enough. Choose a partition function $\varphi : M \to [0, 1]$ with $\varphi = 1$ on a neighbourhood of $\bar{U}$ and $\varphi = 0$ on $\mathscr{D}_{H,t_0}(\bar{U})$. For H' close to H, we define H'_φ by $H'_\varphi = \varphi . H' + (1-\varphi) . H$. $\mathscr{D}_{H,-t_0} \circ A$ is a flowbox for H and for H'_φ, defined on $\mathscr{D}_{H,t_0}(U)$. $A' = A \circ \mathscr{D}_{H,-t_0} \circ \mathscr{D}_{H'_\varphi,t_0}$ is, for H' sufficiently close to H a flowbox at m for H'_φ; again for H' close enough to H, the domain U' of A' is contained in $\varphi^{-1}(1)$ so A' is also a flowbox for H'. $\mathscr{D}_H$ and $\mathscr{D}_{H'_\varphi}$ are C^{k+1} close so A' is C^{k+1} close to A.

Fixing the neighbourhood V of m, $\varepsilon > 0$ and the neighbourhood W of H in $C^{k+2}(M)$.

First we choose a flowbox A at m for H such that A maps U, a neighbourhood of m, onto $(-\delta, +\delta) \times \tilde{U} \subset \mathbb{R}^{2n}$, where $\tilde{U} \subset \mathbb{R}^{2n-1} = \{x_1 = 0\}$ and such that $7\delta < t$ and for any point $p \in U$ $\mathscr{D}_H(p, (2\delta, t-3\delta]) \cap \bar{U} = \emptyset$.

Now we choose a compact neighbourhood V of m and an $\varepsilon > 0$ such that $V \subset U$ and $\mathscr{D}_{H,t'}(V) \subset U$ for every $|t'-t| \leqq \varepsilon$. From the construction in the proof of statement 2 it follows that there is a neighbourhood W' of H in $C^{k+2}(M)$ with, for each $H' \in W'$, a C^{k+1} flowbox $A(H')$ depending continuously on H', with domain $U(H')$ and with image $(-\delta, +\delta) \times \tilde{U}$. We choose $W \subset W'$ so small that for each $H' \in W$, $\mathscr{D}_{H',t'}(V)$ and V are contained in the domain of $A(H')$ for $|t-t'| \leq \varepsilon$, and such that for any $p \in U(H')$, $\mathscr{D}_{H'}(p, (2\delta, t-3\delta]) \cap \overline{U(H')} = \emptyset$.

The Poincaré Maps. For each $H' \in W$ we define a Poincaré map $P(H') : S(H') \to \Sigma_{m, A(H')}$, $S(H')$ an open neighbourhood of m in $\Sigma_{m, A(H')}$. Let $V(H')$ be the projection of V along the trajectories of $X_{H'}$ on $\Sigma_{m, A(H')}$. From our assumptions on W it follows that we may assume that $S(H') \supset V(H')$ for all $H' \in W$.

Definition of $R \subset W$. We define $R \subset W$ as the subset of those $H' \in W$ for which the Poincaré map $P(H')$, restricted to some neighbourhood of $V(H')$, is Q-transversal.

Remark. It follows directly that, for $H' \in R$, every closed orbit of $X_{H'}$, passing throufh V and with period $\in (t-\varepsilon, t+\varepsilon)$ has a Q-transversal Poincaré map. We still have to prove that R is open and dense in W.

R is open in W. Let $H' \in R$ and let $H'' \in W$ be very close to H' in the C^{k+2} sense. Then, by statement 2 $A(H')$, $\Sigma_{m, A(H')}$ and $A(H'')$, $\Sigma_{m, A(H'')}$ are very close in the C^{k+1} sense; also $V(H')$ and $V(H'')$ are very close. Because $H' \in R$, $P(H')$ restricted to some neighbourhood of $V(H')$ is Q-transversal. $P(H'')$ is C^{k+1} close to $P(H')$, so their k-jet extensions are C^1 close. Because transversality is preserved under small C^1 changes, $P(H'')$ will be Q-transversal on a neighbourhood of $V(H'')$ if H'' is C^{k+2} close enough to H'. This proves that R is open in W.

R is dense in W. Take $H' \in W$. We have to show that H' can be approximated in the C^{k+2} sense by elements in R. We need a flowbox for H' which is disjoint from $A(H')$. For that we take the flowbox, defined by the map $\tilde{A} = A(H') \circ \mathscr{D}_{H', -2\delta}$. From the assumptions that $\mathscr{D}_{H'}(U(H'), (2\delta, t-3\delta]) \cap \overline{U(H')} = \emptyset$ and $7\delta < t$, it follows that the domain $\tilde{U}$ of $\tilde{A}$ is disjoint from $U(H')$. In $\tilde{U}$ we are going to construct perturbations of the Hamiltonian H' as in § 2.

Let T_1 and T_2 be open neighbourhoods of $V(H')$ in $\Sigma_{m, A(H')}$ with $\bar{T}_1 \subset T_2 \subset \bar{T}_2 \subset S(H')$. Applying Theorem B to the flowbox $\tilde{A}$ we obtain:

There is a C^{k+2} function $\mathscr{H} : \mathscr{O} \times M \to \mathbb{R}$, $\mathscr{O}$ is a neighbourhood of the origin 0 in a finite dimensional vectorspace whose dimension depends on n and k, such that:

(i) $\mathscr{H}_0 = H'$,

(ii) support $(H' - \mathscr{H}_w) \subset \tilde{U}$, the domain of $\tilde{A}$,

(iii) $\mathscr{H}_w \in W$ for all $w \in \mathscr{O}$,

(iv) the Poincaré map for $\mathscr{H}_w$ is defined on $S(H')$ and equals $P(H')$ on $S(H') \backslash T_2$.

(v) The map $\mathscr{O} \times T_2 \to J^k_{\text{P.S.}}(T_2, \Sigma_{m, A(H')})$, which assigns to (w, p) the k-jet of the Poincaré map for the Hamiltonian $\mathscr{H}_w$ at p, induces a codimension zero embedding of $\mathscr{O} \times \bar{T}_1$ in $J^k_{\text{P.S.}}(T_2, \Sigma_{m, A(H')})$.

According to the remark at the end of § 1 it follows that for almost all $w \in \mathscr{O}$, $\mathscr{H}_w$ is an element of R. This proves that R is dense in W.

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Totally Geodesic Foliations

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1. Introduction

In this article we consider foliations with certain properties common to several geometrically defined ones. We show that their codimension is either large or zero. As an application we obtain characterisations of the totally geodesic, complete submanifolds of S^N and $\mathbf{C}P^N$ which for most dimensions are sharper than those given by Nomizu [6], Abe [1], and the author [3]. Using this characterisation and a slightly generalized formula from [2], we prove that the standard imbedding of S^n into S^{n+p} is rigid for certain codimensions p, see also [7].

2. Statement of Results

Let $\varrho(t)$ denote the largest integer such that the fibration

$$V'_{t,\varrho(t)} \to V'_{t,1}$$

of Stiefel manifolds has a global cross section. (The points in $V'_{t,r}$ are the ordered r-tuples of linearly independent vectors in $\mathbf{R}^t$.) For every integer n define v_n to be the largest integer such that $\varrho(n-v_n) \geqq v_n + 1$. Numerical values of v_n will be given below.

Theorem 1. *Let M^n be an n-dimensional riemannian manifold, and T_0 a v-dimensional, integrable distribution on M^n with the following properties:*

(1) *the maximal integral manifolds of T_0 are totally geodesic and complete,*

(2) *the sectional curvature of M has the same positive value k on all planes spanned by tangent vectors X and Y with $X \in T_0$ and $Y \in T_0^\perp$.*

Then $v > v_n$ implies $v = n$.

Examples of T_0 with the properties (1), (2) include the riemannian (resp. holomorphic) k-nullity distribution of a riemannian (resp. kählerian) manifold of constant (holomorphic) k-nullity, *if* $k > 0$, and the integral manifolds are complete, see [4]. They also include the minimum relative nullity distribution of a complete submanifold of S^N, or of a complete kählerian submanifold of $\mathbf{C}P^N$, see [1]. We recall that for an isometric immersion $f: M \to M'$ of riemannian manifolds the *index of relative nullity at $x \in M$* is defined to be $v(x) :=$ dimension of

$$T_0^f(x) := \{X \in T_x M \mid X \text{ in the kernel of the second fundamental tensor } A_U \text{ for all normal vectors } U \in (T_x M)^\perp\}.$$

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The restriction of T_0^f to the (open) submanifold of M where $\nu(x)$ attains its minimum is called the *minimum relative nullity distribution of f*. Also, if k is a real number and R denotes the riemannian curvature tensor of M, the integer $\mu_k(x) :=$ dimension of

$$T_k(x) := \{X \in T_x M \mid R(X, Y)Z = k(\langle Y, Z\rangle X - \langle X, Z\rangle Y) \quad \text{for all} \quad Y, Z \in T_x M\}$$

is called the *index of k-nullity of M*, and T_k the *k-nullity distribution of M*. – For more explicite examples see Paragraph 4 below.

We can now state the following consequence of Theorem 1:

Theorem 2. *Let $f: M^n \to S^N$ be a complete, immersed submanifold of the standard sphere S^N. If for the index of relative nullity of f*

$$\nu(x) > \nu_n \quad \textit{for all} \quad x \in M^n,$$

then

$$\nu(x) = n \quad \textit{for all} \quad x \in M^n,$$

and f is totally geodesic. The same is true for complete holomorphic submanifolds of real dimension n of $\mathbf{C}\,P^N$.

Chern and Kuiper [2] proved

$$\nu(x) \leqq \mu_0(x) \leqq \nu(x) + p$$

for isometric immersions $f: M^n \to \mathbf{R}^{n+p}$. This formula is easily expanded to immersions into manifolds of constant curvature $k \neq 0$, we only have to replace μ_0 by μ_k. Since $\mu_1 = n$ for the standard n-sphere S^n, we find $n - p \leqq \nu(x)$ for any isometric immersion $f: S^n \to S^{n+p}$. Combining this with Theorem 2 we obtain

Theorem 3. *Let $f: S^n \to S^{n+p}$ be an isometric immersion of spheres (both of curvature 1). If*

$$p < n - \nu_n,$$

then f is totally geodesic, i.e. it imbeds S^n as a great sphere.

Before proving Theorem 1, we finally give some values and estimates for ν_n which can be obtained from the well-known formula $\varrho(\text{odd} \cdot 2^{c+4d}) = 2^c + 8d$, where $0 \leqq c \leqq 3$.

We have

$$\nu_n = n - (\text{highest power of } 2 \leqq n) \text{ for } n \leqq 24,$$

and

$$\nu_n = 0 \quad \text{for } n \text{ a power of 2, or} \quad n = 16q,$$

where $q \leqq 16^3$, and $q \not\equiv 1 \bmod 16$. (If $q > 16^3$, certain additional assumptions on its congruence classes mod 16^p for higher values of p are necessary.) Furthermore

$$\nu_n \leqq \tfrac{1}{2}(n-1),$$

and

$$\nu_n \leqq 8d - 1 \quad \text{for} \quad n < 16^d, d \quad \text{any positive integer.}$$

3. Proof of Theorem 1

Let us assume $0 < \nu < n$. Considering T_0 and $T_0^\perp$ as subbundles of the tangent bundle of M we define a cross section C in $\mathrm{Hom}(T_0, \mathrm{End}(T_0^\perp))$ by

$$C_X Y := -P(\nabla_Y X)$$

for vector fields X, Y with X in T_0, Y in $T_0^\perp$. Here $P: TM \to T_0^\perp$ is the orthogonal projection, and ∇ denotes the canonical riemannian connection on M. C is called the *co-nullity operator*, [8].

Lemma 1. *Let $\gamma: \mathbf{R} \to M$ be a unit-speed geodesic in one of the integral manifolds of T_0. Let X be the tangent field of γ, and let $\nabla^\perp$ denote the connection of $T_0^\perp$ induced by ∇. Then*

$$\nabla_X^\perp(C_X) = (C_X)^2 + k(\mathrm{Id} \circ \gamma), \tag{3}$$

where k is the constant specified in (2).

Proof. Since T_0 is totally geodesic,

$$P(\nabla_U V) = P(\nabla_U P V) = \nabla_U P V \tag{4}$$

for any vector field V and any U in T_0. Now let Y be any vector field in $T_0^\perp$. Then, locally extending X to a section in T_0 with open domain of definition and of unit-length, we get

$$\begin{aligned}(\nabla_X^\perp C_X)(Y) &= \nabla_X^\perp(C_X(Y)) - C_X(\nabla_X^\perp Y)\\ &= -\nabla_X^\perp P(\nabla_Y X) + P(\nabla_{(\nabla_X^\perp Y)} X)\\ &= -P(\nabla_X \nabla_Y X - \nabla_{\nabla_X Y} X)\\ &= -P(R(X,Y)X + \nabla_Y \nabla_X X + \nabla_{[X,Y]} X - \nabla_{\nabla_X Y} X)\\ &= -P(R(X,Y)X + \nabla_Y \nabla_X X - \nabla_{\nabla_Y X} X).\end{aligned} \tag{5}$$

From (2) we conclude $P(R(X,Y)X) = -kY$. Now, if Z is a vector field in $T_0^\perp$,

$$\langle \nabla_Y \nabla_X X, Z\rangle = Y \cdot \langle \nabla_X X, Z\rangle - \langle \nabla_X X, \nabla_Y Z\rangle .$$

The first term vanishes according to (4), while the second is zero along γ by definition of X. Finally (4) shows $P(\nabla_{\nabla_Y X} X) = C_X^2 Y$. Hence, restricting (5) to γ, we obtain (3).

Lemma 2. *Let $x \in M$ and $X \in T_0(x)$. If C_X has a real eigenvalue, then $X = 0$.*

Proof. This follows easily from the fact (proven in [3]) that a solution of (3), which is regular on the whole real line, cannot have real eigenvalues at any point.

We can now easily complete the proof of Theorem 1. We choose $x \in M$ and a basis $X_1, \dots, X_\nu$ of $T_0(x)$. Writing $C_i := C_{X_i}$ we define a map from $T_0^\perp(x) - \{0\}$ into $(T_0^\perp(x))^{\nu+1}$ by $Y \mapsto (Y, C_1 Y, \dots, C_\nu Y)$. The $\nu + 1$ vectors in the brackets are linearly independent, because

$$0=\alpha Y+\sum \alpha_i C_i Y=\alpha Y+C_{\Sigma\alpha_i X_i} Y$$

implies $\alpha=\alpha_1=\cdots=\alpha_\nu=0$ according to Lemma 2. Hence $V'_{n-\nu,\nu+1}\to V'_{n-\nu,1}$ has a cross section, and therefore $\nu\leqq\nu_n$ by definition of ν_n.

4. Examples

Finally we give some examples that shed light on the question up to what extent the estimates in our theorems possibly might be improved.

a) The Hopf-fibrations $S^{2n-1}\to \mathbf{C}P^{n-1}$, $S^{4n-1}\to \mathbf{H}P^{n-1}$, and $S^{15}\to S^8$ induce totally geodesic foliations of the total spaces (endowed with their standard metric), which satisfy the conditions of Theorem 1.

b) The action of $SU(3)$ on S^7 induced by the adjoint representation of this group has 6-dimensional principal orbits. For the one of maximal volume computation shows $\nu(x)=\mu_1(x)=2$. See also [5].

From the values given for ν_n, and from a) and b) we conclude that Theorem 1 is a best possible result for dimensions $\leqq 9$. On the other hand we want to mention that, at least in some dimensions, Theorem 2, if restricted to hypersurfaces of S^N, can be improved. With respect to Theorem 3 we give the following example:

c) The equations $|z_0|^2=\cdots=|z_n|^2=(n+1)^{-1}$ describe a flat $(n+1)$-torus in $S^{2n+1}\subset \mathbf{C}^{n+1}$. The (isometric) covering projection of $\mathbf{R}^{n+1}$ onto this torus induces a non-standard isometric immersion of S^n into S^{2n+1}.

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Locally Compact Principal Ideal Domains

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Locally compact fields and the (compact) valuation rings of totally disconnected locally compact fields are examples of metrizable, locally compact principal ideal domains, the subject of investigation here. To exclude locally compact fields and principal ideal domains topologized by the discrete topology, we shall say that a topological integral domain A is *proper* if A is not a field and if the topology of A is not the discrete topology. In § 1, certain general theorems about locally compact integral domains are established. They are applied in § 2 to show that a proper, metrizable, locally compact principal ideal domain A is local (a *local* ring is a commutative ring with identity that has only one maximal ideal) and hence is a valuation ring, that its topology is stronger than the topology defined by its valuation, and that its maximal ideal is topologically nil (an ideal $\mathfrak{a}$ of a topological ring is *topologically nil* if $\lim_{n\to\infty} a^n = 0$ for all $a \in \mathfrak{a}$, and $\mathfrak{a}$ is *topologically nilpotent* if the powers of $\mathfrak{a}$ form a filter base converging to zero). In § 3, the residue field of A is exhibited as an absolutely algebraic field of prime characteristic (a field is *absolutely algebraic* if it is an algebraic extension of its prime subfield); one consequence of this fact is the existence of a compact open subdomain (i.e., subring containing 1) that contains a generator of the maximal ideal of A. In § 4, we show the existence and uniqueness of a multiplicative "coefficient group" (analogous to a coefficient field) which, together with zero, forms a representative set for the residue field of A. This enables us to show, for example, that A is compact if and only if its residue field is finite and to discuss in § 5 properties of compact open subdomains. In § 6 we establish the existence of proper, metrizable, locally compact principal ideal domains of prime and zero characteristics whose residue fields are isomorphic to a given absolutely algebraic field of prime characteristic.

1. Locally Compact Integral Domains

We begin by showing that, with two obvious exceptions, a locally compact integral domain is totally disconnected. A *Cohen* algebra over a field is a local algebra whose maximal ideal has codimension one. We shall denote the (Jacobson) radical of a ring A by $\mathrm{Rad}(A)$.

Theorem 1. *If C is a nonzero, commutative, connected, locally compact ring whose two-sided annihilator is the zero ideal, then C is the topological direct product of finitely many (necessarily finite-dimensional) locally compact Cohen algebras over either the real field $\mathbf{R}$ or the complex field $\mathbf{C}$.*

Proof. By [4, Theorem 3; 14, Theorem 5], C is a finite-dimensional topological algebra over $\boldsymbol{R}$ and hence is artinian, and C has only finitely many regular maximal ideals. If C is a radical ring, let $e=0$; otherwise, the finite-dimensional semisimple algebra $C/\mathrm{Rad}(C)$ has an identity element 1 by Wedderburn's theorem, and C possesses an idempotent e that is canonically mapped onto 1 as C is suitable for building idempotents [5, Propositions 3 and 4, p. 54]. Thus C is the direct product of Ce and $D=\{x-ex: x\in C\}$. By the definition of e, $D\subseteqq \mathrm{Rad}(C)$, and hence D is nilpotent. If $D\neq(0)$, then $D^n\neq(0)$ but $D^{n+1}=(0)$ for some $n\geqq 1$, and consequently D^n is a nonzero ideal contained in the two-sided annihilator of C, a contradiction. Thus $D=(0)$, so e is the identity element of C. The assertion now follows from [13, Theorem 3].

Theorem 2. *If A is a locally compact integral domain, then either A is topologically isomorphic to $\boldsymbol{R}$ or $\mathbf{C}$, or A is totally disconnected.*

Proof. Assume that A is not totally disconnected. Then the connected component C of zero is a nonzero locally compact ring. By Theorem 1, C is a field and hence is topologically isomorphic to $\boldsymbol{R}$ or $\boldsymbol{C}$, as a local artinian ring with no proper zero-divisors is a field. In particular, as C has an identity element, $C=A$ as the identity element of A is its only nonzero idempotent.

Theorem 3. *If A is a totally disconnected, locally compact integral domain and if B is a compact open subring of A, then* $\mathrm{Rad}(B)$ *is a compact open subring contained in every open prime ideal of A.*

Proof. By Kaplansky's theorem [6, Theorem 17], B is either a radical ring or a local ring. In the latter case, $\mathrm{Rad}(B)$ is a closed maximal ideal of B [6, Corollary, p. 163], so $B/\mathrm{Rad}(B)$ is a compact field and hence is finite, whence $\mathrm{Rad}(B)$, being closed, is open. Thus in either case, $\mathrm{Rad}(B)$ is open. Let $\mathfrak{p}$ be an open prime ideal; then $B/(B\cap\mathfrak{p})$ is compact and discrete and hence is finite; as $B/(B\cap\mathfrak{p})$ is canonically isomorphic to the subring $(B+\mathfrak{p})/\mathfrak{p}$ of $A/\mathfrak{p}$, $B/(B\cap\mathfrak{p})$ has no proper zero-divisors; therefore $B/(B\cap\mathfrak{p})$ is either (0) or a field, that is, either $B\subseteqq\mathfrak{p}$ or $B\cap\mathfrak{p}$ is a regular maximal ideal of B; in either case, $\mathrm{Rad}(B)\subseteqq\mathfrak{p}$.

Theorem 4. *If A is a proper, locally compact noetherian domain every nonzero prime ideal of which is maximal (in particular, if A is a Dedekind domain, or a principal ideal domain), then A contains only finitely many open maximal ideals.*

Proof. By Theorem 2, A is totally disconnected and hence contains a compact open subring B [7, Lemma 4]. By Theorem 3, the intersection $\mathfrak{o}$ of all the open maximal ideals is a nonzero open ideal. The maximal ideals containing $\mathfrak{o}$ are precisely the open maximal ideals; but by [15, Theorem 7, p. 211], the only maximal ideals containing $\mathfrak{o}$ are the prime ideals of $\mathfrak{o}$, which are finite in number.

Theorem 5. *Let A be a complete, metrizable integral domain, let $\mathscr{T}$ be its topology, and let Ac be a nonzero, closed principal ideal. The topology $\mathscr{T}_c$ on A*

making $R_c : x \mapsto xc$ a homeomorphism from A onto Ac, equipped with the topology induced from $\mathscr{T}$, is compatible with the ring structure of A. Moreover, $\mathscr{T}_c$ is weaker than $\mathscr{T}$, and if A is locally compact for $\mathscr{T}$, then A is also locally compact for $\mathscr{T}_c$.

Proof. As R_c is an isomorphism from the additive group A onto the additive group Ac, $\mathscr{T}_c$ is a Hausdorff topology compatible with addition. If $\mathscr{V}$ is a fundamental system of neighborhoods of zero for $\mathscr{T}$, then $\{(V:c): V \in \mathscr{V}\}$ is a fundamental system of neighborhoods of zero for $\mathscr{T}_c$, where $(V:c) = \{x \in A : xc \in V\} = R_c^{-1}(V) = R_c^{-1}(V \cap Ac)$. For each $a \in A$, $L_a : x \mapsto ax$ is continuous at zero for $\mathscr{T}_c$, for if $aW \subseteq V$, then $a(W:c) \subseteq (V:c)$; consequently as A is a topological group under addition for $\mathscr{T}_c$, L_a is continuous on A for $\mathscr{T}_c$. Thus $\mathscr{T}_c$ is a topology on A compatible with addition for which multiplication is separately continuous in each variable; moreover, A is complete and metrizable for $\mathscr{T}_c$ since Ac is closed in A for topology $\mathscr{T}$. By a slight generalization of a theorem of Montgomery [9, Theorem 1; 1, Exercise 22a), p. 83], $\mathscr{T}_c$ is compatible with the ring structure of A. If $V \in \mathscr{V}$, $(V:c)$ is clearly a neighborhood of zero for $\mathscr{T}$, so $\mathscr{T}_c$ is weaker than $\mathscr{T}$. If A is locally compact for $\mathscr{T}$, then Ac is also locally compact as it is closed, whence A is locally compact for $\mathscr{T}_c$.

2. The Locality Theorem

Theorem 6. *Let A be a proper, metrizable, locally compact principal ideal domain. Then A has only one maximal ideal Ac, all the nonzero ideals of A are open, and Ac is topologically nil. Moreover, A contains a compact open subring; if B is any compact open subring, $(B \cap Ac^n)_{n \geqq 1}$ is a fundamental system of neighborhoods of zero.*

Proof. Let $\mathscr{T}$ be the topology of A, and let P be a representative set of irreducible elements of A; we shall call elements of P "primes". By Theorem 2, A is totally disconnected and hence contains a compact open subring B [7, Lemma 4]. By Theorem 4, A has only finitely many open maximal ideals. We shall show that A contains infinitely many open ideals. Suppose not. Then the intersection of all the open ideals is a nonzero open ideal Ac, where $c = p_1^{r_1} \dots p_n^{r_n}$ is a product of primes. Thus $Ap_1, \dots, Ap_n$ are the open maximal ideals of A. As A is locally compact, A is complete. Hence, with the notation of Theorem 5, A, equipped with topology $\mathscr{T}_c$, is a locally compact ring. As $\mathscr{T}_c$ is weaker than $\mathscr{T}$, if q is a prime not among $p_1, \dots, p_n$, then Aq is not open for $\mathscr{T}_c$. Suppose that Ap_i were open for $\mathscr{T}_c$. Then as Ac is open for $\mathscr{T}$, there would exist a neighborhood V of zero for $\mathscr{T}$ such that $V \subseteq Ac$ and $(V:c) \subseteq Ap_i$. If $x \in V$, then $x = yc$ for some $y \in A$, so $y \in (V:c) \subseteq Ap_i$, whence $x \in Acp_i$; thus $V \subseteq Acp_i$, so the ideal Acp_i, which is properly contained in Ac, would be open for $\mathscr{T}$, a contradiction of the fact that Ac is the smallest open ideal for $\mathscr{T}$. Hence no Ap_i is open for $\mathscr{T}_c$. Therefore A, equipped with $\mathscr{T}_c$, is a locally compact integral domain having no proper open ideals. But by [3, Lemma 4.5], a locally compact integral domain having no proper open ideals is a field, in contradiction to our hypothesis. Consequently, A contains infinitely many open ideals for $\mathscr{T}$.

Since there are infinitely many open ideals but only finitely many open maximal ideals, there exists a prime c such that Ac^n is open for all $n \geqq 1$. As $\bigcap_{n=1}^{\infty} Ac^n = (0)$, $\{B \cap Ac^n : n \geqq 1\}$ is a fundamental system of neighborhoods of zero for a Hausdorff topology on B weaker than its given compact topology and hence identical with its given topology. Thus $\{B \cap Ac^n : n \geqq 1\}$ is a fundamental system of neighborhoods of zero for $\mathscr{T}$.

By Theorem 3, Rad(B) is open, so there is a nonzero element a in Rad(B). Let $a = up_1^{r_1} \dots p_m^{r_m}$ be its prime decomposition, where u is invertible. By Theorem 3, Rad$(B) \subseteqq Ac$, so we may assume that $p_1 = c$. Let $b = p_1 \dots p_m$. We shall show that Ab is a topologically nil ideal. As $\{B \cap Ac^n : n \geqq 1\}$ is a fundamental system of neighborhoods of zero, there exists $k \geqq 1$ such that

$$u(B \cap Ac^k) \subseteqq B, \quad p_i(B \cap Ac^k) \subseteqq B, \quad 1 \leqq i \leqq m,$$

whence

$$u(B \cap Ac^k) \subseteqq B \cap Ac^k, \quad p_i(B \cap Ac^k) \subseteqq B \cap Ac^k, \quad 1 \leqq i \leqq m.$$

But $\{x \in A : x(B \cap Ac^k) \subseteqq B \cap Ac^k\}$ is a ring. Thus for all $t, t_1, \dots, t_m \geqq 0$,

$$u^t p_1^{t_1} \dots p_m^{t_m} (B \cap Ac^k) \subseteqq B \cap Ac^k.$$

Let $s = \max\{r_1 k, \dots, r_m k\}$. Then

$$b^s = (u^{s-k} p_1^{s-r_1 k} \dots p_m^{s-r_m k})(up_1^{r_1} \dots p_m^{r_m})^k \in B$$

as $a^k = (up_1^{r_1} \dots p_m^{r_m})^k \in B \cap Ap_1^{r_1 k} \subseteqq B \cap Ac^k$. Let $x \in Ab$, and let $y \in A$ be such that $x = yb$. For some $q \geqq 0$, $y(B \cap Ac^{qs}) \subseteqq B$, whence $y(B \cap Ac^{qs}) \subseteqq B \cap Ac^{qs}$ and therefore $y^n(B \cap Ac^{qs}) \subseteqq B \cap Ac^{qs}$ for all $n \geqq 0$. Consequently, $y^n b^{qs} \in B$ for all $n \geqq 0$, as $b^{qs} \in B \cap Ac^{qs}$. Therefore if $i > q$,

$$x^{is} = y^{is} b^{is} = y^{is} b^{qs} b^{is-qs} \in Bb^{s(i-q)} \subseteqq B \cap Ac^{i-q}.$$

Thus $\lim_{i \to \infty} x^{is} = 0$. Hence for each $r \in [0, s]$, $\lim_{i \to \infty} x^{is+r} = 0 \cdot x^r = 0$. Consequently, $\lim_{n \to \infty} x^n = 0$. Therefore Ab is a topologically nil ideal.

By [6, Corollary, p. 163], $Ab \subseteqq$ Rad(A). If q is a prime, then $Ab \subseteqq$ Rad$(A) \subseteqq Aq$, so q is one of $p_1, \dots, p_m$. Hence A has only finitely many maximal ideals. As Ab is the intersection of maximal ideals and hence contains Rad(A), $Ab =$ Rad(A). Thus Rad(A) is a closed and topologically nil ideal. By [12, Lemma 4], A is suitable for building idempotents. As $A/\text{Rad}(A)$ is isomorphic to $\prod_{i=1}^{m} (A/Ap_i)$, the cartesian product of finitely many fields, we therefore conclude from [5, Proposition 5, p. 54] that $m = 1$ as A is an integral domain. Hence $c = p_1 = b$, so Ac is a topologically nil ideal, and the only nonzero ideals of A are the open ideals Ac^n, $n \geqq 0$.

Theorem 7. *If A is an indiscrete locally compact principal ideal domain, then A is separable and metrizable if and only if A is either a locally compact field or the (compact) valuation ring of an indiscrete, totally disconnected, locally compact field.*

Proof. The assertion is a consequence of Theorem 6 and [12, Theorem 7], since a compact principal ideal domain is the valuation ring of a locally compact field [11, p. 163].

A local principal ideal domain A that is not a field is the valuation ring of a nontrivial discrete valuation defined on its quotient field, namely, the valuation v satisfying $v(uc^n) = n$ for all units u of A and all integers n, where c is a generator of the maximal ideal of A [2, Proposition 2, p. 93]. We shall call the restriction of v to A the *valuation* of A and the topology it defines on A, for which $(Ac^n)_{n \geqq 0}$ is a fundamental system of neighborhoods of zero, the *valuation topology* of A. A paraphrase of Theorem 6 is that if A is a proper, metrizable, locally compact principal ideal domain, then A is local, its maximal ideal is topologically nil, its given topology is stronger than its valuation topology, and the topology inherited by any compact open subring of A is identical with that induced by the valuation topology.

3. The Residue Field

Here we shall determine the nature of the residue field of a proper, metrizable, locally compact principal ideal domain A and infer that if c is a generator of the maximal ideal of A, then there is a compact open subdomain (i.e., subring that contains 1) containing c.

Theorem 8. *The residue field of a proper, metrizable, locally compact principal ideal domain A is an absolutely algebraic field of prime characteristic.*

Proof. Let B be a compact open subring of A, let Ac be the maximal ideal of A, and let φ be the canonical epimorphism from A onto A/Ac. Let $x \notin Ac$. As $\lim_{n\to\infty} c^n = 0$ and as $(B \cap Ac^n)_{n \geqq 0}$ is a fundamental system of neighborhoods of zero, there exists $s \geqq 0$ such that $x(B \cap Ac^s) \subseteqq B$ and $c^s \in B$. Then $x(B \cap Ac^s) \subseteqq B \cap Ac^s$; therefore as $\{t \in A : t(B \cap Ac^s) \subseteqq B \cap Ac^s\}$ is a ring, $x^n(B \cap Ac^s) \subseteqq B \cap Ac^s$ for all $n \geqq 1$. In particular, $x^n c^s \in B \cap Ac^s$ for all $n \geqq 1$, so as $B \cap Ac^s$ is compact, there is a strictly increasing sequence $(n_k)_{k \geqq 1}$ of natural numbers such that $\lim_{k\to\infty} x^{n_k} c^s = ac^s$ for some $a \in A$. Consequently, with the notation of Theorem 5, $\lim_{k\to\infty} x^{n_k} = a$ for topology $\mathscr{T}_{c^s}$ on A. By Theorems 5 and 6, Ac is open for $\mathscr{T}_{c^s}$, so A/Ac is discrete for the associated quotient topology. Therefore as $\lim_{k\to\infty} \varphi(x^{n_k}) = \varphi(a)$ for the discrete topology, there exists k such that $\varphi(x)^{n_k} = \varphi(x)^{n_{k+1}} = \varphi(a)$, whence $\varphi(x)^{n_{k+1} - n_k} = \varphi(1)$, i.e., $\varphi(x)$ is a root of unity. Thus every nonzero element of A/Ac is a root of unity, so A/Ac is an absolutely algebraic field of prime characteristic.

Theorem 9. *If A is a proper, metrizable, locally compact principal ideal domain of characteristic zero and if p is the characteristic of the residue field of A, then the closed subdomain generated by 1 is the compact ring of p-adic integers.*

Proof. Let Z be the subdomain generated by 1. By Theorem 8, the prime integer p belongs to the maximal ideal of A. Therefore $\lim_{n\to\infty} p^n = 0$ by Theorem 6, so the topology induced on Z is not the discrete topology. As A is totally disconnected, the open additive subgroups and hence the open ideals of Z form a fundamental system of neighborhood of zero for the topology induced on Z. If q is a prime integer distinct from p, then no power p^n of p belongs to Zq; hence Zq is not open in Z as $\lim_{n\to\infty} p^n = 0$. Therefore as the topology of Z is Hausdorff, the ideals Zp^n, $n \geqq 1$, form a fundamental system of neighborhoods of zero in Z, that is, the topology induced on Z is the p-adic topology.

Theorem 10. *If A is a proper, metrizable, locally compact principal ideal domain and if c is a generator of the maximal ideal of A, then there is a compact open subdomain B of A that contains c.*

Proof. Let B_1 be a compact open subring. The closure D of the smallest subdomain containing 1 is compact by Theorem 9. Since $\{x \in A : xB_1 \subseteqq B_1\}$ is a closed subring containing 1, $DB_1 \subseteqq B_1$; hence $B_2 = D + B_1$ is a compact open subdomain of A. As $\lim_{n\to\infty} c^n = 0$, there exists $s \geqq 0$ such that $c^r \in B_2$ for all $r \geqq s$. Consequently, $B = B_2 + B_2 c + \cdots + B_2 c^{s-1}$ is a compact open subdomain of A containing c.

4. The Coefficient Group

If A is an equicharacteristic local ring, a *coefficient field* of A is, of course, any subfield of A mapped isomorphically onto the residue field by the canonical epimorphism. An analogous definition is the following:

Definition. If A is a local ring, a *coefficient group* of A is any subgroup of the multiplicative group of invertible elements of A mapped isomorphically onto the multiplicative group of nonzero elements of the residue field of A by the canonical epimorphism from A onto its residue field.

Throughout, A is a local ring, $\mathfrak{m}$ its maximal ideal, c a generator of $\mathfrak{m}$ if $\mathfrak{m}$ is principal, φ the canonical epimorphism from A onto $A/\mathfrak{m}$.

Theorem 11. *If A is a local integral domain whose residue field $A/\mathfrak{m}$ is absolutely algebraic of prime characteristic and if S^* is a coefficient group of A, then $S^* = \{x \in A : x$ is a root of unity, and the multiplicative order of x = the multiplicative order of its coset $x + \mathfrak{m}$ in $A/\mathfrak{m}\}$.*

Proof. As φ maps S^* isomorphically onto the multiplicative group $(A/\mathfrak{m})^*$ of nonzero elements of $A/\mathfrak{m}$, every element of S^* is a root of unity and has the same multiplicative order as its image in $A/\mathfrak{m}$. Conversely, let x be a root of unity whose multiplicative order r is the same as that of $\varphi(x)$. There exists $a \in S^*$ such that $\varphi(a)=\varphi(x)$, whence the order of a is r. Thus $1, a, a^2, \ldots, a^{r-1}$ are r roots of X^r-1; x is also a root of X^r-1, which has only r roots as A is an integral domain, so $x=a^s \in S^*$ for some s (actually, it is clear that $x=a$).

Theorem 12. *If A is a proper, metrizable, locally compact principal ideal domain, then A contains a unique coefficient group.*

Proof. Let p be the characteristic of $A/\mathfrak{m}$, and let S^* be the set described in the statement of Theorem 11. We shall first show that $\varphi(S^*)=(A/\mathfrak{m})^*$. Given $x \notin \mathfrak{m}$, let $q=p^f$ be such that $q-1$ is the multiplicative order of $\varphi(x)$. Then $x^q - x \in \mathfrak{m}$, so as in the proof of [2, Proposition 3, p. 157], $x^{q^{n+1}} - x^{q^n} \in \mathfrak{m}^{1+fn}$ for all $n \geqq 0$. Let B be a compact open subring of A. As in the proof of Theorem 8, there exists $s \geqq 0$ such that $x^n c^s \in B \cap Ac^s$ for all $n \geqq 1$. Hence $x^{q^{n+1}}c^s - x^{q^n}c^s \in B \cap Ac^{1+fn}$ for all $n \geqq 0$, so $(x^{q^n}c^s)_{n\geqq 0}$ is a Cauchy sequence in compact $B \cap Ac^s$ and hence converges to some $ac^s \in Ac^s$. Consequently, $\lim_{n\to\infty} x^{q^n} = a$ for topology $\mathscr{T}_{c^s}$ (with the terminology of Theorem 5), whence

$$a^q = \lim_{n\to\infty} x^{q^{n+1}} = \lim_{n\to\infty} x^{q^n} = a$$

for that topology, and therefore $a^{q-1}=1$. Moreover, $\varphi(a)=\lim_{n\to\infty}\varphi(x)^{q^n}=\varphi(x)$, so the multiplicative order of $\varphi(a)$ is $q-1$, and therefore the multiplicative order of a must be $q-1$. Hence $a \in S^*$. Consequently, $\varphi(S^*)=(A/\mathfrak{m})^*$.

We observe next that if $a \in S^*$, then the multiplicative cyclic subgroup $[a]$ generated by a is contained in S^*. Indeed, φ maps $[a]$ epimorphically onto the cyclic subgroup $[\varphi(a)]$ generated by $\varphi(a)$, but both cyclic subgroups have the same order as $a \in S^*$, so the restriction of φ to $[a]$ is an isomorphism onto $[\varphi(a)]$. In particular, the multiplicative order of a^m is the same as that of $\varphi(a^m)$ for all $m \geqq 1$, so $[a] \subseteq S^*$.

If $a, b \in S^*$ and if $\varphi(a)=\varphi(b)$, then, as in the proof of Theorem 11, b is one of the roots $1, a, a^2, \ldots, a^{r-1}$ of X^r-1, where r is the order of $\varphi(a)$, whence $b=a^s$ for some $s \in [0, r-1]$. But φ maps $[a]$ isomorphically onto $[\varphi(a)]$, so as $\varphi(b)=\varphi(a)$, we conclude that $b=a$. Therefore φ is a bijection from S^* onto $(A/m)^*$.

It remains for us to show that S^* is a multiplicative group, that is, if $a, b \in S^*$, then $ab^{-1} \in S^*$. The subfield L of $A/\mathfrak{m}$ generated by $\varphi(a)$ and $\varphi(b)$ is finite. As $\varphi(S^*)=(A/\mathfrak{m})^*$, there exists $d \in S^*$ such that $\varphi(d)$ generates the multiplicative group L^* of nonzero elements of L. In particular, $\varphi(a)=\varphi(d)^s$ and $\varphi(b)=\varphi(d)^t$ for some s, t. As $[d] \subseteq S^*$ and as the restriction of φ to S^* is bijective, therefore, we conclude that $a=d^s$, $b=d^t$, whence $ab^{-1}=d^{s-t} \in [d] \subseteq S^*$.

The following theorem is a special case of a theorem of Teichmüller [10, Lemma 5]:

Theorem 13. *If K is a complete topological field whose topology is given by a nontrivial discrete valuation and if the residue field of the valuation ring A*

of K is absolutely algebraic of prime characteristic, then A contains a unique coefficient group.

The proof is entirely similar to that of Theorem 12. One simplification is that if $x \notin \mathfrak{m}$ and if $q-1$ is the multiplicative order of $\varphi(x)$, then $(x^{q^n})_{n \geqq 0}$ converges to an element a of S^* satisfying $\varphi(a) = \varphi(x)$.

In general, the coefficient group is not identical with the group of all roots of unity. For example, -1 is a root of unity in the compact ring of 2-adic integers, but the coefficient group of that ring is $\{1\}$. However, if we add to the hypotheses of either Theorem 12 or 13 that A have prime characteristic, then the coefficient group is the group of all roots of unity, a consequence of the following theorem:

Theorem 14. *Let A be an indiscrete, complete, local topological ring of prime characteristic p whose open additive subgroups form a fundamental system of neighborhoods of zero, let $\mathfrak{m}$ be the maximal ideal of A, and let $S = \{a \in A : a^{p^r} = a$ for some $r \geqq 1\}$. If $\mathfrak{m}$ is a topologically nil ideal and if $A/\mathfrak{m}$ is absolutely algebraic, then S is a coefficient field of A, and $S^* = S - \{0\}$ is the set of all roots of unity in A.*

Proof. Clearly S is a subring of A that contains 1. If $a \in S^*$, then $a^{q^k} = a$ for all $k \geqq 1$, where q is a power of p, so $a \notin \mathfrak{m}$ as $\mathfrak{m}$ is topologically nil, whence a is invertible; clearly $a^{-1} \in S^*$. Thus S is a subfield of A. Given $x \in A$, let q be a power of p such that $\varphi(x)^q = \varphi(x)$. Then $x^q - x \in \mathfrak{m}$, so as $\mathfrak{m}$ is topologically nil,

$$\lim_{k \to \infty} (x^{q^{k+1}} - x^{q^k}) = \lim_{k \to \infty} (x^q - x)^k = 0,$$

whence $(x^{q^k})_{k \geqq 0}$ is a Cauchy sequence. Let $a = \lim_{k \to \infty} x^{q^k}$. Then

$$a^q = \lim_{k \to \infty} x^{q^{k+1}} = \lim_{k \to \infty} x^{q^k} = a,$$

so $a \in S$ and $\varphi(a) = \lim_{k \to \infty} \varphi(x)^{q^k} = \varphi(x)$. Thus S is a coefficient field. Every element of S^* is clearly a root of unity. Conversely, let a be a primitive nth root of unity in A. Then $p \nmid n$, so $n \mid p^r - 1$ for some $r \geqq 0$ (for as $p \nmid n$, there is a finite field of characteristic p that contains a primitive nth root of unity ζ, so $\zeta^{p^r} = \zeta$ for some r, whence $n \mid p^r - 1$), and consequently $a^{p^r} = a$.

From Theorems 6 and 8 we obtain the following corollary:

Corollary. *Let A be a proper, topological principal ideal domain of prime characteristic p. If A is metrizable and locally compact, or if A is the valuation ring of a complete topological field whose topology is given by a proper discrete valuation and whose residue field is absolutely algebraic, then A contains a unique coefficient field S, namely,*

$$S = \{a \in A : a^{p^r} = a \text{ for some } r \geqq 1\} = \{a \in A : a \text{ is a root of unity}\} \cup \{0\}.$$

We recall that if S is any representative set of elements for the residue field of a local principal ideal domain (i.e., if the canonical epimorphism maps S bijectively onto the residue field), then for each $x \in A$ there is a unique sequence $(s_k)_{k \geqq 0}$ of elements of S such that $x = \sum_{k=0}^{\infty} s_k c^k$ (where Ac is the maximal ideal of A), the sum converging for the valuation topology (the sequence is defined recursively so that $x \equiv \sum_{k=0}^{n} s_k c^k \pmod{Ac^{n+1}}$ for all $n \geqq 0$).

Theorem 15. *If A is a proper, metrizable, locally compact principal ideal domain, then A is compact if and only if its residue field is finite.*

Proof. If A is compact, its residue field is compact and discrete and hence finite. Conversely, assume that A/Ac is finite, where Ac is the maximal ideal of A. Let $S = S^* \cup \{0\}$ where S^* is the coefficient group of A; by Theorem 10 there is a compact open subdomain B that contains c. For each $k \geqq 0$, let $S_k = \{s \in S: sc^k \in B\}$. As $c \in B$, $(S_k)_{k \geqq 0}$ is an increasing sequence of sets. Moreover, $\bigcup_{k=0}^{\infty} S_k = S$, for if $s \in S$, then $sc^n \in B$ for some $n \geqq 0$ as $\lim_{n \to \infty} c^n = 0$. Since A/Ac is finite, S is also, and therefore $S = S_m$ for some m. Given $x \in Ac^m$, let $(s_k)_{k \geqq 0}$ be the sequence in S such that $x = \sum_{k=0}^{\infty} s_k c^k$ (the sum converging for the valuation topology). Clearly $s_k = 0 \in S_k$ for all $k < m$, and $s_k \in S = S_k$ for all $k \geqq m$. Thus each $s_k c^k \in B$; therefore, as the topology of compact B is the valuation topology, $x \in B$. Hence $B \supseteq Ac^m$. Since A/Ac is finite, so is A/Ac^m, whence $(A : B)$ is also. Thus A is the union of finitely many cosets of B and hence is compact.

5. Compact Open Subdomains

A compact open subdomain B of a proper, metrizable, locally compact principal ideal domain A completely determines A, as the following theorem shows:

Theorem 16. *Let B be a compact open subdomain of a proper, metrizable, locally compact principal ideal domain A. Then:*

1° *A is the valuation ring of the quotient field of B, equipped with the unique valuation extending that B inherits from A.*

2° *A is the integral closure of B.*

3° *If K is any valued field containing A whose valuation v extends that of A, and if c is any nonzero element of B satisfying $v(c) > 0$, then $A = \{z \in K : v(z) \geqq 0$ and $c^m z \in B$ for some $m \geqq 0\}$.*

Proof. To show 1°, it suffices to show that A is contained in the quotient field of B. Let $z \in A$. If Ac is the maximal ideal of A, then there exists $m \geqq 0$

such that $c^m, c^m z \in B$ as $\lim_{n\to\infty} c^n = 0$, so $z = (c^m z)\, c^{-m}$, an element of the quotient field of B. A similar argument establishes 3°.

As A is integrally closed, to show 2° it suffices by 1° to show that every element of A is integral over B. Let S^* be the coefficient group of A. If $a \in S^*$, then a is a root of unity and hence is integral over B. If b belongs to the maximal ideal of A, then $\lim_{n\to\infty} b^n = 0$ by Theorem 6, so $b^m \in B$ for some $m \geqq 1$, whence b is a root of the polynomial $X^m - b^m$ over B. Every element of A is the sum of an element of $S^* \cup \{0\}$ and an element of the maximal ideal; therefore A is integral over B.

By Theorem 6, the topology inherited by a compact open subdomain B of a proper, metrizable, locally compact principal ideal domain A is given by a valuation whose value semigroup is a subsemigroup of the additive semigroup N of natural numbers. Consequently, B inherits from A some arithmetic properties:

Theorem 17. *Let B be a compact integral domain whose topology is given by a valuation v whose value semigroup is a subsemigroup of the additive semigroup N.*

1° *For each $a \in B$, $v(a) = 0$ if and only if a is invertible.*

2° *Each nonzero element of B that is not invertible is the product of irreducible elements.*

Proof. By [6, Theorem 17], B is local; hence if a is not invertible, then a belongs to the maximal ideal of A, which is topologically nilpotent [6, Theorem 14], so $\lim_{n\to\infty} a^n = 0$, whence $v(a) > 0$. Conversely, if a is invertible, clearly $v(a) = 0$. Assertion 2° follows by an inductive argument from 1°.

However, the possession by B of any one of several natural properties forces A to be compact and hence the compact valuation ring of a totally disconnected, locally compact field [11, p. 163].

Theorem 18. *Let A be a proper, metrizable, locally compact principal ideal domain. The following statements are equivalent:*

1° *A is compact.*

2° *A has a countable dense set.*

3° *The residue field of A is finite.*

4° *A contains a compact open ideal.*

5° *$R_c : x \mapsto xc$ is an open mapping, where c is a generator of the maximal ideal of A.*

6° *The topology of A is minimal in the class of all locally compact, metrizable topologies on A compatible with its ring structure.*

7° Rad(A) *is topologically nilpotent.*

8° *A is complete for its valuation topology.*

9° *Every compact open subdomain of A is noetherian.*

10° *A contains a compact open subdomain that has a finitely generated maximal ideal.*

11° *A contains a compact open unique factorization subdomain.*

12° *A contains a compact open subdomain that is integrally closed.*

Proof. By Theorem 7, 15, and 16, 1°, 2°, 3°, and 12° are equivalent. Clearly 1° implies 4°, 10°, and 11°, and by [6, Theorem 14], 1° implies 7°. Also, if 1° holds, then 6° clearly holds, and the topology of A must coincide with its valuation topology, so 5° and 8° also hold. To show that 1° implies 9°, let $\mathfrak{a}$ be a nonzero ideal of a compact open subdomain B of A. As the topology of A is the valuation topology, $B \supseteq Ac^m$ for some $m \geqq 0$. Let $uc^r \in \mathfrak{a}$, where u is a unit of A; then $\mathfrak{a} \supseteq Ac^{m+r}$, since for any $x \in A$, $xc^{m+r} = (xu^{-1}c^m)(uc^r) \in \mathfrak{a}$. But again, as A is a compact valuation ring, A/Ac^{m+r} is compact and discrete and hence finite, so $B/\mathfrak{a}$ is also finite. Thus every nonzero ideal of B is contained in only finitely many ideals of B, so B is noetherian. Thus 1° implies each of the remaining assertions.

If Ac^n is compact, then (with the notation of Theorem 5), A is compact for topology $\mathcal{T}_{c^n}$, so its residue field is finite; thus 4° implies 3°. Since the composite of open mappings is an open mapping, if 5° holds, then $R_a : x \mapsto xa$ is an open mapping for every nonzero $a \in A$, whence 1° holds by [7, Theorem 8] and [12, Theorem 5 and 7]. Moreover, 6° implies 5°, for if 6° holds, the given topology $\mathcal{T}$ of A coincides with the topology $\mathcal{T}_c$.

Let S^* be the coefficient group of A, let $S = S^* \cup \{0\}$, and let B be a compact open subdomain of A. Then S is discrete, for if $s \in S$, $(s + Ac) \cap S = \{s\}$. For each $n \geqq 0$ let p_n be the "projection" function from A into S defined by

$$p_n\left(\sum_{k=0}^{\infty} s_k c^k\right) = s_n .$$

If $x_0 = \sum_{k=0}^{\infty} s_k c^k$, then $p_n(x) = p_n(x_0)$ for all $x \in x_0 + Ac^{n+1}$, so p_n is continuous at x_0. Consequently, $p_n(B)$ is a compact subset of discrete S and hence is finite. Let $m \geqq 0$ be such that $c^k \in B$ for all $k \geqq m$. Then clearly $(p_n(B))_{n \geqq m}$ is an increasing sequence of subsets. Assume that S is infinite. Then there exists $t_k \in S$ such that $t_k \notin p_{2k}(B)$ for all $k \geqq m$. In particular, $t_k \notin p_k(B)$, so $t_k c^k \notin B$. Thus $Ac = \mathrm{Rad}(A)$ is not topologically nilpotent. The sequence $(x_n)_{n \geqq m}$, where $x_n = \sum_{k=m}^{n} t_k c^k$, is a Cauchy sequence for the valuation topology. If 8° were true, then $z = \sum_{k=m}^{\infty} t_k c^k$ would belong to A, and consequently $c^r z \in B$ for some $r \geqq m$, whence $t_r \in p_{2r}(B)$, a contradiction. Thus each of 7° and 8° implies 3°.

Certainly 9° implies 10°. Assume that B is a compact open subdomain possessing a finitely generated maximal ideal. Then B is a local domain [6, Theorem 17]. By Theorem 3, the maximal ideal $\mathfrak{m}$ of B is contained in Ac, and hence $\mathfrak{m} = B \cap Ac$. By [11, Theorem 2], $(\mathfrak{m}^n)_{n \geqq 1}$ is a fundamental system of neighborhoods of zero in B; consequently, the product of open ideals of B is an open ideal. Let $B \cap Ac = Ba_1 + \cdots + Ba_n$, where $a_1, \ldots, a_n \in B \cap Ac$. Let $a_i = u_i c$ where $u_i \in A$. As $(B \cap Ac^k)_{k \geqq 1}$ is a fundamental system of neighborhoods of zero, there exists $s \geqq 1$ such that $c(B \cap Ac^s) \subseteqq B$ and $u_i(B \cap Ac^s) \subseteqq B$, $1 \leqq i \leqq n$. Let $W = B \cap Ac^s$. Then $cW \subseteqq W$ and $u_i W \subseteqq W$, $1 \leqq i \leqq n$. As $\{x \in A : xW \subseteqq W\}$ is a ring, $a_i a_j W = u_i u_j c Wc \subseteqq Wc$ for all $i, j \in [1, n]$, whence, as W is an ideal of B, $(B \cap Ac)^2 W \subseteqq Wc$. As the product of open ideals of B is open, we conclude that Wc is open, and hence Bc is open. But the restriction of R_c to compact B is a homeomorphism from B onto Bc; therefore if U is any neighborhood of zero contained in B, Uc is open in Bc and hence in A. Thus R_c is an open mapping. Consequently, 10° implies 5°.

Finally, assume that B is a compact open unique factorization subdomain of A. Let v be the valuation of A, and let $p \in B$ be such that $v(p)$ is the smallest of the strictly positive integers belonging to $v(B)$. It follows easily from Theorem 17 that p is a prime of B. Suppose that S is not contained in B, and let $s \in S$ be such that $s \notin B$. As every nonzero ideal of A is open and as $Ap \subseteqq Ac$, $(B \cap Ap^n)_{n \geqq 0}$ is a fundamental system of neighborhoods of zero; let n be the smallest natural number such that $s(B \cap Ap^n) \subseteqq B$. Then $n \geqq 1$ as $s \notin B$. Moreover, $s(B \cap Ap^n) \subseteqq B \cap Ap^n$, so $s^2(B \cap Ap^n) \subseteqq s(B \cap Ap^n) \subseteqq B \cap Ap^n$. By the definition of n, there exists $a \in A$ such that $ap^{n-1} \in B$ but $sap^{n-1} \notin B$. Then $ap^n = (ap^{n-1})\,p \in B \cap Ap^n$, so sap^n and s^2ap^n belong to B; in B, therefore, $p \mid (sap^n)^2$ since $s^2a^2p^{2n-1} = (s^2ap^n)\,(ap^{n-1}) \in B$, but $p \nmid sap^n$ since $sap^{n-1} \notin B$. Consequently, B is not a unique factorization domain, a contradiction. Therefore $S \subseteqq B$. Let m be such that $c^k \in B$ for all $k \geqq m$. Then $\sum_{k=m}^{\infty} s_k c^k \in B$ for every sequence $(s_k)_{k \geqq m}$ in S as B is compact and its topology is that induced by v. Thus $Ac^m \subseteqq B$, so Ac^m is compact. Hence 11° implies 4°.

6. Existence of Locally Compact Principal Ideal Domains

If B is a topological ring that algebraically is a subring of a ring A, there is a unique topology $\mathscr{T}$ on A compatible with the additive group structure of A such that B is open and the topology induced on B by $\mathscr{T}$ is its given topology, namely, that obtained by declaring the filter of neighborhoods of zero in B a fundamental system of neighborhoods of zero in A; this topology we shall call the *B-topology* on A. Multiplication is continuous at (0, 0) for the B-topology, and for each $b \in B$, $R_b : x \mapsto xb$ and $L_b : x \mapsto bx$ are continuous for the B-topology. However, A may fail to be a topological ring with the B-topology since R_b or L_b may fail to be continuous for some b belonging to A but not to B.

In view of Theorems 10 and 16, it is natural to ask: If B is a compact integral domain whose topology is given by a valuation v whose value semigroup is N, and if A is the valuation ring of the quotient field of B (furnished with the unique valuation extending that of B), is A, equipped with the B-topology, a topological ring (and hence a metrizable, locally compact, principal ideal domain)? As we shall see, the answer is not always yes, and it is an open problem to classify those B's for which the answer is yes. We show next that the problem may be reformulated in terms of the topology of B and its valuation, and hence is a problem of valuation theory.

Definition. Let B be a topological integral domain. Nonzero elements a and b of B are *topological associates* if for every neighborhood V of zero there exists a neighborhood U of zero such that $Ua \subseteqq Vb$ and $Ub \subseteqq Va$.

If a and b are associates, then they are topological associates: Indeed, $a = ub$ for some unit u; given neighborhood V of zero, $U = Vu^{-1} \cap Vu$ is a neighborhood of zero that satisfies the conditions of the definition. The relation of being topological associates is clearly an equivalence relation. The equivalence class determined by 1, elements of which might properly be called "topological units", are those elements b for which $L_b : x \to bx$ is an open mapping.

Theorem 19. *Let B be a topological integral domain whose topology is given by a valuation v whose value semigroup is $\mathbf{N}$, and let A be the valuation ring of the quotient field of B (whose valuation is the unique extension of v). Topologized with the B-topology, A is a topological ring if and only if for all nonzero elements a, b of B, if $v(a) = v(b)$, then a and b are topological associates.*

Proof. Necessity: Let $v(a) = v(b)$. Then $a^{-1}b$ and $b^{-1}a$ are elements of A, so $x \mapsto a^{-1}bx$ and $x \mapsto b^{-1}ax$ are continuous functions; consequently, if V is a neighborhood of zero in B, there is a neighborhood U of zero in B such that $a^{-1}bU \subseteqq V$ and $b^{-1}aU \subseteqq V$, whence $bU \subseteqq aV$ and $aU \subseteqq bV$.

Sufficiency: It suffices to show that if $d \in A$ and if V is a neighborhood of zero in B, then there is a neighborhood U of zero in B such that $dU \subseteqq V$. Let $d = ab^{-1}$ where a, $b \in B$; then $v(a) \geqq v(b)$. Let $r = v(a) - v(b)$, and let $c \in B$ be such that $v(c) = 1$. As $c \in B$, there exists a neighborhood W of zero in B such that $c^r W \subseteqq V$. As $v(a) = v(bc^r)$, there exists a neighborhood U of zero in B such that $aU \subseteqq bc^r W$, whence $aU \subseteqq bV$, and therefore $dU \subseteqq V$.

The following theorem show that compact subdomains of proper, metrizable, locally compact principal ideal domains of prime characteristic p are topologically isomorphic to subdomains of $F[[X]]$, the ring of formal power series (equipped with the usual X-adic topology) over an absolutely algebraic field F of characteristic p.

Theorem 20. *Let A be a proper, metrizable, locally compact principal ideal domain of prime characteristic, let F be the coefficient field of A, and let B be a compact open subdomain of A containing a generator c of the maximal ideal of A. There is an algebraic isomorphism Φ from A onto a subring A' of $F[[X]]$ such that $\Phi(c) = X$ and the restriction of Φ to B is a topological isomorphism from B onto a subring B' of $F[[X]]$, equipped with the topology induced by the natural X-adic topology of $F[[X]]$.*

Proof. We define Φ in the obvious way: If $a=\sum_{k=0}^{\infty} s_k c^k$ where $s_k \in F$ for all $k \geqq 0$, then $\Phi(a)=\sum_{k=0}^{\infty} s_k X^k$. It is easy to verify that Φ is a monomorphism from A into $F[[X]]$ satisfying $v(\Phi(a))=v_A(a)$ for all $a \in A$, where v is the X-adic valuation of $F[[X]]$ and v_A the valuation of A.

Theorem 20 suggests how to construct locally compact principal ideal domains having a given absolutely algebraic field F of prime characteristic as coefficient field. Let $(F_i)_{i \geqq 0}$ be an exhaustive filtration of finite additive subgroups of F that is compatible with its ring structure; thus $F_i F_j \subseteq F_{i+j}$ for all $i, j \geqq 0$, $1 \in F_0$, and $\bigcup_{k=0}^{\infty} F_k = F$. Then $B=\left\{\sum_{k=0}^{\infty} s_k X^k : s_k \in F_k \text{ for all } k \geqq 0\right\}$ is a subdomain of $F[[X]]$ containing X. Clearly B is compact for its induced topology, since it is homeomorphic to $\prod_{k=0}^{\infty} F_k$. Let A be the valuation ring of the quotient field of B contained in the field $F((X))$ of formal power series over F, let v be the X-adic valuation on $F((X))$, and let

$$C=\{g \in F[[X]] : X^m g \in B \quad \text{for some} \quad m \geqq 0\}.$$

Clearly C is a subdomain of $F[[X]]$ satisfying $B \subseteq C \subseteq A$.

Equipped with the B-topology, C is a topological ring if and only if for all $g \in F[[X]]$,

$$\text{if } X^m g \in B \text{ for some } m \geqq 0, \quad \text{then } (B \cap (X^r))\, g \subseteq B \text{ for some } r \geqq 0. \tag{1}$$

The continuity of $f \mapsto fg$ at zero insures the necessity of (1). Conversely, (1) implies that for every $s \geqq r$, $(B \cap (X^s))\, g \subseteq B \cap (X^s)$, whence $f \mapsto fg$ is continuous at zero.

Moreover, $C=A$ if and only if for all $g \in F[[X]]$,

$$\text{if } X^m g \in B \text{ for some } m \geqq 0 \text{ and if } v(g)=0, \text{ then } X^n g^{-1} \in B \text{ for some } n \geqq 0. \tag{2}$$

The condition is clearly necessary, for if the hypothesis of (2) holds, then g is a unit of A as $X \in A$, so $g^{-1} \in A = C$ and thus $X^n g^{-1} \in B$ for some $n \geqq 0$. Conversely, let $g \in A$, and let $g = f h^{-1}$ where $f, h \in B$. Let $h = X^s h_0$ where $s = v(h) \geqq 0$ and $v(h_0)=0$. Then by (2), $X^t h_0^{-1} \in B$ for some $t \geqq 0$, and therefore $X^{s+t} g = X^t h_0^{-1} f \in B$, whence $g \in C$.

Since F is absolutely algebraic of prime characteristic, F is the union of an increasing sequence $(F_i)_{i \geqq 0}$ of finite subfields. Clearly $(F_i)_{i \geqq 0}$ is a filtration compatible with the ring structure of F; moreover, (1) and (2) hold, for easy calculations show that if $X^m g \in B$, then $(B \cap (X^m))\, g \subseteq B$ and, if $v(g)=0$, $X^m g^{-1} \in B$. As $s X^i \in B$ for all $s \in F_i$, F is clearly the coefficient field of A.

More generally, let $s \geqq 0$, and let $(F_i)_{i \geqq 0}$ be an exhaustive filtration of finite additive subgroups of F compatible with the ring structure of F such that $F_i F_j \subseteq F_j$ whenever $j \geqq i + s$. (For example, let $(F_{ks})_{k \geqq 0}$ be an increasing sequence of finite subfields whose union is F, let $c_k \in F_{(k+1)s}$ be such that $F_{(k+1)s} = F_{ks}(c_k)$, and let $F_{ks+j}=F_{ks}+F_{ks}c_k+F_{ks}c_k^2+\cdots+F_{ks}c_k^j$ for each $j\in[1,s-1]$.) An easy calculation shows that if $X^m g\in B$, then $(B\cap(X^{m+s}))_g\subseteq B$ and, if $v(g)=0$ and m is so large that $s_0^{-1}\in F_m$, where s_0 is the constant coefficient of g, $X^{m+s}g^{-1}\in B$. Thus we have shown:

Theorem 21. *If F is an absolutely algebraic field of prime characteristic p, there is a proper, metrizable, locally compact principal ideal domain of characteristic p whose residue field is isomorphic to F.*

For an example of a compact subdomain of $F[[X]]$ that is not the compact open subdomain of a locally compact principal ideal domain, assume that F is infinite, and let $(K_n)_{n\geqq 0}$ be an increasing sequence of finite subfields whose union is F such that $[K_{n+1}:K_n]\geqq 9^n$ for all $n\geqq 0$. Let $c_n\in K_{n+1}$ be such that $K_{n+1}=K_n(c_n)$. Let F_0 be the prime subfield of F, and for all $n\geqq 0$, let

$$\begin{aligned} F_{3^n}&=K_n\,,\\ F_{3^n+j}&=K_n+K_nc_n+K_nc_n^2+\cdots+K_nc_n^{3^n} \quad \text{if} \quad 1\leqq j\leqq 3^n\,,\\ F_{2\cdot 3^n+k}&=K_n+K_nc_n+K_nc_n^2+\cdots+K_nc_n^{3^nk} \quad \text{if} \quad 1\leqq k<3^n\,. \end{aligned}$$

If $1\leqq j\leqq 3^n$, $1\leqq k\leqq 3^n$, then $F_{3^n+j}F_{3^n+k}\subseteq F_{2\cdot 3^n+2}$ and $F_{3^n+j}F_{2\cdot 3^n+k}\subseteqq F_{3^{n+1}}$; consequently, $F_rF_s\subseteqq F_{r+s}$ for all $r,s\geqq 0$. Let

$$g=\sum_{n=0}^{\infty}c_n^{3^n}X^{3^n}\,.$$

As $c_n^{3^n}\in F_{3^n+1}$ for all $n\geqq 0$, $Xg\in B$. We shall show that (1) fails. Given $r\geqq 0$, let n be such that $3^n\geqq r$, and let $h=c_n^{3^n}X^{3^n+1}\in B\cap(X^r)$. The coefficient of $X^{2\cdot 3^n+1}$ in hg is $c_n^{2\cdot 3^n}$, which does not belong to $F_{2\cdot 3^n+1}=K_n+K_nc_n+\cdots$ $\cdots+K_nc_n^{3^n}$, so $hg\notin B$. Therefore C and *a fortiori* A are not topological rings for the B-topology.

We turn next to the construction of a proper, metrizable, locally compact principal ideal domain of characteristic zero whose residue field is isomorphic to a given absolutely algebraic field F of prime characteristic p. By [8, Theorem 2] there exists a complete topological field K of characteristic zero whose topology is given by a discrete valuation and whose residue field is isomorphic to F. Let c be a generator of the maximal ideal of its valuation ring A. By Theorem 13 there is a unique coefficient group S^* for A; let $S=S^*\cup\{0\}$.

We note that if $(a_k)_{k\geqq 0}$ is any sequence of elements of A and if $(s_k)_{k\geqq 0}$ is the unique sequence of elements of S such that $\sum_{k=0}^{\infty}s_kc^k=\sum_{k=0}^{\infty}a_kc^k$, than there is a unique sequence $(u_k)_{k\geqq 0}$ of elements of A such that $u_0=0$ and

$$s_k=a_k+u_k-u_{k+1}c \tag{3}$$

for all $k\geqq 0$.

As A/Ac is absolutely algebraic of prime characteristic, it is the union of an increasing sequence $(F_i')_{i\geqq 0}$ of finite subfields. Let φ be the canonical epimorphism from A onto A/Ac, and let φ_S be the bijection from S onto A/Ac obtained by restricting φ to S. We define sequences $(H_k)_{k\geqq 0}$, $(F_k)_{k\geqq 0}$, $(S_k)_{k\geqq 0}$ of finite subsets of A, A/Ac, and A respectively as follows:

$$H_0=\emptyset,\quad F_0=F'_0,\quad S_0=\varphi_S^{-1}(F_0),$$

$$H_k=\{u\in A: uc \quad \text{is the sum of } k+2 \text{ elements of the set } S_{k-1}\cup H_{k-1}\cup(-S_{k-1})\cup(-H_{k-1})\},$$

$$F_k=F'_k[\varphi(H_k)],\quad \text{the subfield of } A/Ac \text{ generated by } F'_k\cup\varphi(H_k),$$

$$S_k=\varphi_S^{-1}(F_k).$$

An inductive argument shows that H_k is finite, $(F_k)_{k\geqq 0}$ is an increasing sequence of finite subfields of A/Ac whose union is A/Ac, and $S_k^*=S_k-\{0\}$ is a finite subgroup of S^*.

Let $B=\left\{\sum_{k=0}^{\infty} s_k c^k : s_k\in S_k \text{ for all } k\geqq 0\right\}$. Clearly $\sum_{k=0}^{\infty} s_k c^k \mapsto (s_k)$ is a homeomorphism from B, with its induced topology, onto $\prod_{k=0}^{\infty} S_k$; hence B is compact. We shall first show that B is a subdomain of A. Let $x=\sum_{k=0}^{\infty} s'_k c^k$, $y=\sum_{k=0}^{\infty} s''_k c^k$ where $s'_k, s''_k\in S_k$ for all $k\geqq 0$. Let $x-y=\sum_{k=0}^{\infty} s_k c^k$, $xy=\sum_{k=0}^{\infty} t_k c^k$ where $s_k, t_k\in S$ for all $k\geqq 0$. By (3) there exist unique sequences $(u_k)_{k\geqq 0}$, $(v_k)_{k\geqq 0}$ of elements of A such that $u_0=v_0=0$,

$$s_k=s'_k-s''_k+u_k-u_{k+1}c,$$

$$t_k=\sum_{i=0}^{k} s'_i s''_{k-i}+v_k-v_{k+1}c.$$

Now $\varphi(s_0)=\varphi(s'_0)-\varphi(s''_0)\in F_0$, so $s_0\in S_0$, whence $u_1\in H_1$ as $u_1c=s_0-s'_0+s''_0$. Assume that $s_{k-1}\in S_{k-1}$, $u_k\in H_k$. Then $\varphi(s_k)=\varphi(s'_k)-\varphi(s''_k)+\varphi(u_k)\in F_k$, so $s_k\in S_k$, whence $u_{k+1}\in H_{k+1}$ as $u_{k+1}c=s'_k-s''_k+u_k-s_k$. By induction, therefore, $s_k\in S_k$ for all $k\geqq 0$, so $x-y\in B$. Moreover, $t_0=s'_0s''_0$ since $s'_0s''_0\in S$, so $t_0\in S_0$ and $v_1=0\in H_1$. Assume that $t_{k-1}\in S_{k-1}$, $v_k\in H_k$. Then $\varphi(t_k)=\sum_{i=0}^{k}\varphi(s'_i)\,\varphi(s''_{k-i})+\varphi(v_k)\in F_k$, so $t_k\in S_k$, whence $v_{k+1}\in H_{k+1}$ as $v_{k+1}c=s'_0s''_k+\cdots+s'_ks''_0-t_k+v_k$. By induction, therefore, $t_k\in S_k$ for all $k\geqq 0$, so $xy\in B$.

Next, we shall show that if $xc^m\in B$ and if x is invertible in A, then $x^{-1}c^m\in B$. Let $x=\sum_{k=0}^{\infty} s_k c^k$ where $s_k\in S$ for all $k\geqq 0$; then $s_k\in S_{k+m}$ and $s_0\neq 0$; let $x^{-1}=\sum_{k=0}^{\infty} t_k c^k$ where $t_k\in S$ for all $k\geqq 0$. By (3) there is a unique sequence $(w_k)_{k\geqq 0}$ such that $w_0 = 0$,

$$s_0 t_0 = 1 + w_1 c,$$
$$s_0 t_k + \cdots + s_k t_0 = -w_k + w_{k+1} c.$$

As $s_0^{-1} \in S_m$, $t_0 = s_0^{-1}$ and $w_1 = 0 \in H_{1+m}$. Assume that $t_i \in S_{i+m}$ for all $i < k$ and that $w_k \in H_{k+m}$. Then $\varphi(t_k) = -\varphi(s_0)^{-1}[\varphi(s_1)\varphi(t_{k-1}) + \cdots + \varphi(s_k)\varphi(t_0) - \varphi(w_k)] \in F_{k+m}$, so $t_k \in S_{k+m}$ and hence $w_{k+1} \in H_{k+1+m}$ as S_{k+m}^* is a subgroup. Therefore by induction, $t_k \in S_{k+m}$ for all $k \geqq 0$, so $x^{-1}c^m \in B$.

Let A_1 be the valuation ring of the quotient field of B; thus A_1 is the intersection of A and the quotient field of B. It follows from the preceding paragraph, just as in the prime characteristic case, that $A_1 = \{z \in A : zc^m \in B$ for some $m \geqq 0\}$. Let φ_1 be the restriction to A_1 of the canonical epimorphism from A onto A/Ac. Then φ_1 is an epimorphism, since $S_k \subseteqq A_1$ for all $k \geqq 0$. If $ac \in A_1$ where $a \in A$, then $ac^{m+1} \in B$ for some $m \geqq 0$, whence $a \in A_1$, and therefore $ac \in A_1 c$. Thus the kernel of φ_1 is $Ac \cap A_1 = A_1 c$, so $A_1/A_1 c$ is isomorphic to A/Ac and hence to F.

To show that A_1 is a topological ring for the B-topology, it suffices to show that if $xc^m \in B$, then $(B \cap Ac^n) x \subseteqq B \cap Ac^n$ for all $n \geqq m$. Let $x = \sum_{k=0}^{\infty} r_k c^k$, $y = \sum_{k=n}^{\infty} s_k c^k$, where $r_k \in S_{k+m}$, $s_k \in S_k$, and let $yx = \sum_{k=n}^{\infty} t_k c^k$ where $t_k \in S$ for all k. Then by (3) there is a unique sequence $(u_k)_{k \geqq n}$ in A such that $u_n = 0$,

$$t_k = \sum_{i=n}^{k} s_i r_{k-i} + u_k - u_{k+1} c.$$

Since $s_n r_0 \in S_n S_m \subseteqq S_n$, $t_n = s_n r_0$ and so $u_{n+1} = 0 \in H_{n+1}$. Assume that $t_{k-1} \in S_{k-1}$, $u_k \in H_k$. Then

$$\varphi(t_k) = \sum_{i=n}^{k} \varphi(s_i)\varphi(r_{k-i}) + \varphi(u_k) \in F_k,$$

so $t_k \in S_k$, whence $u_{k+1} \in H_{k+1}$. Therefore, by induction, $t_k \in S_k$ for all $k \geqq n$, so $yx \in B \cap Ac^n$. In sum, we have proved:

Theorem 22. *If F is an absolutely algebraic field of prime characteristic, there is a proper metrizable, locally compact principal ideal domain of characteristic zero whose residue field is isomorphic to F.*

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