A transcendence theorem for class-number problems (II)

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1. Introduction

Let d be the discriminant of a complex quadratic field of class-number $h(d)$. In [1] and [7] (the latter will be referred to as Part I), it was shown that it is possible to effectively find all those d with $h(d) = 2$. The main purpose of this paper is to do the numerical work necessary to prove the following theorem:

THEOREM 1. *If d is the discriminant of a complex quadratic field of class-number* 2, *then* $|d| < 10^{1030}$.

This theorem represents a minor numerical improvement of what was announced in Part I. In a later paper [8], we will consider the problem of lowering this bound to 427. Theorem 1 is deduced from a theorem on linear forms in logarithms of algebraic numbers. Such theorems have previously been used for the fields of class-number 1. Basing their work upon the first of Baker's results in this area, Bundschuh and Hock [6] showed that if $h(d) = 1$ then $|d| < \exp(1.6\cdot 10^5)$. Baker later used one of his more refined results to show [2] that if $h(d) = 1$ then $|d| < 10^{500}$. As a comparison with this result, our present theorem is able to prove

THEOREM 2. *If* $h(d) = 1$ *then* $|d| < 10^{130}$.

We can also get some partial results on the case that $h(d) = 4$ and there is one class per genus. In this case $d = d_1d_2d_3$ where the d_j are prime discriminants $\big(-4, \pm 8, (-1)^{(p-1)/2}p$ where p is an odd prime$\big)$ $|d_2| < |d_3|$. Using this notation, we can prove

THEOREM 3. *Suppose $h(d) = 4$ and there is one class per genus. Let δ be some fixed number in the range $0 < \delta < 1/6$. We may effectively find all those d in each of the following cases.*

(i) $d_1 = 8$.

(ii) $d_1 = -4$ *except when* $d_2 < 0$, $d_3 < 0$, $d_2 \equiv d_3 \equiv 5 \pmod 8$,

$$|d|^{1/2-\delta} \leqq |d_2| < |d_3| \leqq |d|^{1/2+\delta} .$$

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(iii) $d_1 = -8$ *except when* $d_2 > 0$, $d_3 > 0$, $d_2 \equiv d_3 \equiv 5 \pmod 8$,

$$|d|^{1/2-\delta} \leqq d_2 < d_3 \leqq |d|^{1/2+\delta} .$$

(iv) *d odd, $d_1 < 0$ is held fixed, except when $d_2 > 0$, $d_3 > 0$ and*

$$|d|^{1/2-\delta} \leqq d_2 < d_3 \leqq |d|^{1/2+\delta} .$$

In the remaining cases d is odd, $|d_1| < |d_2| < |d_3|$.

(v) $d_1 > 0$.

(vi) $d_1 < 0$ *except when* $d_2 > |d|^{1/3-\delta}$, $|d|^{(1-\delta)/2} \leqq d_3 \leqq |d|^{(1+\delta)/2}$ *or when* $d_2 < 0$, $d_3 < 0$, $|d_1| > |d|^{1/6-\delta}$, $|d|^{(1-\delta)/2} \leqq |d_3| \leqq |d|^{(1+\delta)/2}$.

All of these results are consequences of a theorem on transcendence. Let $\alpha_1 = -1$ and $\alpha_2, \cdots, \alpha_n$ ($n \geqq 2$) be the fundamental units of different real quadratic fields. Logarithms will be principal valued and if β and γ are complex then β^γ denotes $e^{\gamma \log \beta}$. We assume that $b_2, \cdots, b_n$ are rational integers, $b_n \neq 0$, and $b_1 = \sqrt{-D}$ where D is a non-negative rational integer. The simplest form of our transcendence theorem is (it is an improvement of Theorem 3′ of Part I which was announced but not proved there),

THEOREM 4. *Let* $\nu \geqq 1$ *and* $0 < \varepsilon = (8n^4 + 4n^3 + 4n^2)\Delta \leqq 1$. *Set* $A = \max_{1 \leqq j \leqq n-1} |\log \alpha_j|$. *If* $H > 0$ *satisfies,*

$$H^{(8n^2-8n-2)\Delta} > 2^{4n^3} A \left(\frac{\nu + 1}{n\Delta}\right)^{2n-2} \cdot \begin{cases} 3^5 & \text{if } n = 2 , \\ 1 & \text{if } n \geqq 3 , \end{cases}$$

$$|b_j| < H^\nu \qquad (j = 1, \cdots, n) ,$$

and

$$\log \alpha_n < H^{1-\varepsilon} ,$$

then

$$|b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n| \geqq e^{-H} .$$

One important aspect of Theorem 4 is that $\log \alpha_n$ is allowed to be much larger than A. This theorem is a consequence of the far more complicated Theorem 5 (to be given in the next section) which however gives better numerical results. (Theorem 4 would give 10^{3750} in Theorem 1.) In Section 6 we will use Theorem 5 to derive Theorems 1 and 2. The reason that we have derived Theorem 5 was a desire to get as low a bound as possible with this method so as to simplify the problem of covering the finite range that Theorem 1 leaves untouched. In a joint paper with A. Baker [5], Theorem 4 has been generalized but no explicit result for calculational purposes is provided. One can't get a better numerical result than Theorem 1 by following the method of [5] because everything that is done here is also done there but [5] has further complications (mainly the reduction argument that replaces α_n by $\alpha_n^{(1)}$ etc.; it is not necessary here thanks to Lemma 3). On the other hand, Baker's argument in [1] is different from our argument here and may give better results. It, too, will give better numerical results than the method of our joint paper.

Theorem 2 seems like a fair improvement of Baker's earlier result, but this is not entirely accurate. To explain why, we must go back to the theorem that Baker used to prove his result. The following special case of his result will do here.

THEOREM A (*Baker* [3]). *Suppose* $\alpha_1, \cdots, \alpha_n$ *are algebraic numbers,* $\alpha_j \neq 0$, *whose heights are less than* A' *and degrees are* $\leqq d$ (*where* A' *and* d *are* $\geqq 4$). *Assume that* $\log \alpha_1, \cdots, \log \alpha_n$ (*principal values*) *are linearly independent over* Q. *Suppose* $b_1, \cdots, b_n$ *are rational integers, not all zero,* $|b_j| \leqq H$ $(j = 1, \cdots, n)$ *such that*

$$|b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n| < e^{-\delta H}$$

where $0 < \delta \leqq 1$. *Then*

$$H < (4^{n^2}\delta^{-1}d^{2n} \log A')^{(2n+1)^2} .$$

However, for class-number problems, this theorem is inferior both numerically and theoretically to the following result which is a special case of what Baker actually proves in [3].

THEOREM B [3, p. 206]. *Under the same assumptions as Theorem A, except that the restriction* $|b_j| \leqq H$ *is relaxed to* $|b_j| \leqq H^\nu$ $(j = 1, \cdots, n)$ *where* ν *is an integer* $\geqq n$, *we have*

$$H < \left(4^{\nu^2}(2\delta)^{-1}d^{2\nu} \log A'\right)^{(2n+1)^2} .$$

For example if $h(d) = 2$, then $d = d_1 d_2$ where d_1 and d_2 are prime discriminants, $|d_1| < |d_2|$. Theorem A will give an effective determination of those d with $|d_1| < |d|^{1/50}$ (one is forced to take $\delta \approx |d_1|^{-1}$) whereas Theorem B allows this determination for all d with $|d_1| < d^{1/2-\varepsilon}$ ($\delta = 1$, $\nu \approx 1/\varepsilon$). As a numerical example, the result of Baker given earlier (if $h(d) = 1$ then $|d| < 10^{500}$) which he derived from Theorem A may be improved to $|d| < 10^{260}$ simply by using Theorem B (with $\delta = 1$, $\nu = 2$, $H = (\pi|d|^{1/2}/100)$) instead. Further in proving Theorem 2, we have used a different linear form which should give slightly better results than the linear form used by Baker. If we were to use Theorem 4 instead of Theorem 5, we would have only derived $|d| < 10^{435}$ in Theorem 2. Thus the main improvement that Theorem 2 represents over previous results is due to the optimum choice of variables that Theorem 5 allows rather than the theoretical improvements that Theorem 5 incorporates.

2. Theorem 5 and the deduction of Theorem 4 therefrom

The proof of Theorem 4 involves the introduction of many other variables whose values have been chosen. No choice of these variables can give an optimum result in all applications. The aim of Theorem 5 is to allow these variables to be chosen to fit the application. The proof of Theorem 5 involves the use of a great number of inequalities. A relatively small number of these imply the rest and have been made part of the hypotheses of Theorem 5.

The numbers α_j and b_j $(j = 1, \cdots, n)$ have the same meaning as in Theorem 4, in particular $n \geqq 2$ and $b_n \neq 0$. We again use the number A introduced earlier, $A = \max_{1 \leqq j \leqq n-1} |\log \alpha_j| \geqq \pi$.

THEOREM 5. *Suppose that* ε, ν, H *are positive real numbers,* $\varepsilon \leqq 1$, $\nu \geqq 1$ *such that*

$$|b_j| < H^{\nu} \qquad (1 \leqq j \leqq n)\,, \tag{2.1}$$

$$\log \alpha_n < H^{1-\varepsilon} \tag{2.2}$$

$$|b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n| < e^{-H}\,. \tag{2.3}$$

If H_0 *is picked so that it and the variables* δ, ζ, η, θ, h, c *satisfy* (2.4)-(2.12) *and either* (2.13) *or* (2.14) *below, then* $H < H_0$.

The variables δ, ζ, η, θ are variables in the exponent. In Section 3 we will see that (2.4)-(2.6) will guarantee the existence of H_0.

$$0 < \delta = \eta + \frac{\varepsilon - \eta - \zeta}{n}\,, \tag{2.4}$$

$$0 < n(\varepsilon - \delta) < \theta - \zeta < \eta\,, \tag{2.5}$$

$$0 < \zeta < \frac{\varepsilon - \eta}{2n + 1}\,. \tag{2.6}$$

The numbers h and c appear as coefficients. They satisfy

$$h^n \geqq \max\left[1, \frac{2^{n+1}}{(n-1)!}\left(1 + \frac{n-1}{H_0^{1-\eta}}\right)^{n-1}\right], \tag{2.7}$$

$$(2h)^n \geqq c \geqq (h + H_0^{\delta-1})^{n-1}(h + H_0^{\delta-\varepsilon})\,. \tag{2.8}$$

With these variables, H_0 should satisfy the further inequalities,

$$H_0 > 6\theta H_0^{\theta-\zeta} \log H_0 + 12\theta \cdot 2^{-\theta/\zeta} H_0^{1-\eta+\theta-\zeta} \log H_0\,, \tag{2.9}$$

$$\begin{aligned}\zeta H_0^{1+\zeta-\eta} \log H_0 > {} & 16 \log c + 16[(n-1)(1-\eta) + \zeta] \log H_0 \\ & + 8(2^n + 1)(n-1)AhH_0^{1+\zeta-\delta} + 8H_0^{1-\eta}[(2^{n+1} + 2)\log(2h) \\ & + \log A + (2^{n+1} + 2)(1 - \delta + \nu)\log H_0]\,,\end{aligned} \tag{2.10}$$

(2.11) $$\zeta H_0^{\delta-\eta-2\zeta} \log H_0 > 2(n+1)\cdot 2^{\theta/\zeta} A h\ ,$$

(2.12) $$\begin{aligned} H_0^{1+\theta-\zeta-\eta}(\zeta \log H_0 - \log 4) > 4.41\cdot 2^{2n+\theta/\zeta} h^n H_0^{1+n(\varepsilon-\delta)-\eta}[&\log(2h) \\ &+ (1-\delta+\nu)\log H_0]\ . \end{aligned}$$

Finally we require either

(2.13) $$6\theta H_0^{\eta-1} + 12\theta\cdot 2^{-\theta/\zeta} < e(\eta+\zeta-\theta)\ ,$$

or

(2.14) $$(\eta+\zeta-\theta)\log H_0 > 1\ .$$

We now deduce Theorem 4 from Theorem 5. With

$$0 < \varepsilon = (8n^4 + 4n^3 + 4n^2)\Delta \leqq 1\ ,$$

our choice of variables is given by

(2.15) $$\begin{aligned} \delta &= (8n^4 - 4n^3 + 8n^2 + 4n - 4)\Delta\ , \\ \theta &= (8n^4 - 4n^3 + 4n - 1)\Delta\ , \\ \eta &= (8n^4 - 4n^3)\Delta\ , \\ \zeta &= 4n\Delta\ , \\ h^n &= 9\ , \\ c &= (h+1)^n \leqq 4^n \\ H_0 &= \left(\frac{2^{4n^3}\cdot(\nu+1)^{2n-2}\cdot 3^5 A}{(n\Delta)^{2n-2}}\right)^{1/[(8n^2-8n-2)\Delta]}\ . \end{aligned}$$

The factor of 3^5 in the definition of H_0 is necessary only to verify (2.12) and then only for $n = 2$; it will not be used otherwise. Inequalities (2.4), (2.5), (2.6), and (2.8) are immediate. For the rest of the inequalities we need an estimate for H_0^{Δ}. For $n \geqq 2$,

$$2^{4n^3} > e^{2n^3} > e^{8n^2-8n-2}$$

and hence

(2.16) $$H_0 > e^{1/\Delta}\ .$$

Since $\eta + \zeta - \theta = \Delta$, this verifies (2.14). For (2.7), we note that

$$1 - \eta \geqq \varepsilon - \eta = (8n^3 + 4n^2)\Delta > [8(n-1)^3 + \log(n-1)]\Delta\ .$$

Thus

$$\left(1 + \frac{n-1}{H_0^{1-\eta}}\right)^{n-1} < (1 + e^{-8(n-1)^3})^{n-1} < \exp\left[(n-1)e^{-8(n-1)^3}\right] \leqq \exp(e^{-8}) < \frac{9}{8}$$

and (2.7) is immediate.

Of (2.9)-(2.12), (2.12) is the most difficult to prove with the choice of variables given in (2.15). By (2.16), $\zeta \log H_0 > 4n > 2n \log 4$ and hence

$$\zeta \log H_0 - \log 4 \geqq \frac{3}{4}\zeta \log H_0 .$$

Also

$$\begin{aligned}(\nu + 1) \log H_0 &> (\log 6 - \delta \log H_0) + (1 + \nu) \log H_0 \\ &\geqq \log(2h) + (1 - \delta + \nu) \log H_0 .\end{aligned}$$

Next

$$(1 + \theta - \zeta - \eta) - [1 + n(\varepsilon - \delta) - \eta] = (4n^2 - 4n - 1)\Delta ,$$

and

$$H_0^{(4n^2-4n-1)\Delta} \geqq 2^{2n^3}\left(\frac{\nu+1}{n\Delta}\right)\cdot\begin{Bmatrix}(3^5 A)^{1/2} & \text{if } n = 2 \\ \left(\frac{\nu+1}{n\Delta}\right) & \text{if } n \geqq 3\end{Bmatrix} > 27\cdot 2^{2n^3}\left(\frac{\nu+1}{n\Delta}\right) .$$

Thus

$$\begin{aligned}H_0^{(4n^2-4n-1)\Delta}(\zeta \log H_0 - \log 4) &\geqq 81\cdot 2^{2n^3}(\nu + 1) \log H_0 \\ &> 81\cdot 2^{2n+\theta/\zeta}\cdot 2^{n^2-2n-1}(\nu + 1) \log H_0 \\ &\geqq \frac{9}{2}\cdot 2^{2n+\theta/\zeta}h^n[\log (2h) + (1 - \delta + \nu) \log H_0] ,\end{aligned}$$

which verifies (2.12).

For (2.11), $\delta - \eta - 2\zeta = (8n^2 - 4n - 4)\Delta > (8n^2 - 8n - 2)\Delta$ and this with (2.16) gives

$$\zeta H_0^{\delta-\eta-2\zeta} \log H_0 > 4n\cdot 2^{4n^3}A > 2n\cdot 2^{\theta/\zeta}\cdot 2^{2n^3+n^2}A > 6(n + 1)\cdot 2^{\theta/\zeta}A$$

and (2.11) follows.

We will break (2.10) into three parts. Note that

$$\zeta/[(8n^2 - 8n - 2)\Delta] > 1/(2n - 2) .$$

Thus

$$\begin{aligned}\frac{1}{2}\zeta H_0^{\zeta} \log H_0 &> \frac{1}{4}\zeta H_0^{\zeta} \log H_0 + \frac{1}{4}H_0^{\zeta}\cdot\frac{\zeta}{(8n^2 - 8n - 2)\Delta} \log A \\ &> \frac{\zeta}{4}\cdot 2^{2n^2}(\nu + 1)(n\Delta)^{-1} \log H_0 + \frac{ne^{4n}}{8n^2 - 8n - 2} \log A \\ &> 8(2^{n+1} + 2)(\nu + 1) \log H_0 + 8 \log A \\ &> 8[(2^{n+1} + 2) \log (2h) + \log A + (2^{n+1} + 2)(1 - \delta \\ &\quad + \nu) \log H_0] .\end{aligned} \tag{2.17}$$

Since $(8n^2 + 4n - 4)\Delta > (8n^2 - 8n - 2)\Delta$,

$$\frac{1}{4}\zeta H_0^{(8n^2+4n-4)\Delta} \log H_0 > n\cdot 2^{4n^3}A > 2^{n+4}\cdot 3nA . \tag{2.18}$$

Next, by using what was done in (2.17),

$$\begin{aligned}(2.19)\qquad \frac{1}{4}\zeta H_0^{1+\zeta-\eta}\log H_0 &> \frac{1}{4}\zeta H_0^{\zeta}\log H_0\\ &> 8(2^{n+1}+2)(\nu+1)\log H_0\\ &> 32n\log H_0\\ &> 16n\Delta^{-1}+16n\log H_0\\ &> 16\log(4^n)+16[(n-1)(1-\eta)+\zeta]\log H_0\,.\end{aligned}$$

But $(1+\zeta-\eta)-(1-\eta)=\zeta$, $(1+\zeta-\eta)-(1+\zeta-\delta)=(8n^2+4n-4)\Delta$ and thus (2.17), (2.18), and (2.19) imply (2.10).

We will split (2.9) into two parts. First

$$1-\theta+\zeta \geqq \varepsilon-\theta+\zeta > \theta/n+(8n^2-8n-2)\Delta$$

and hence

$$(2.20)\qquad \frac{1}{2}H_0^{1-\theta+\zeta} > \frac{1}{2}H_0^{\theta/n}\cdot H_0^{(8n^2-8n-2)\Delta} > \frac{1}{2}\log(H_0^{\theta/n})\cdot 2^{4n^3} > 6\theta\log H_0\,.$$

Second, $\theta/\zeta > 2n^3-n^2 \geqq 6n \geqq 12$ so that

$$96n\theta/\zeta < 16(\theta/\zeta)^2 < 2^{\theta/\zeta}\,.$$

Thus

$$\begin{aligned}(2.21)\qquad \frac{1}{2}H_0^{1-(1-\eta+\theta-\zeta)} &= \frac{1}{2}H_0^{\Delta} > \frac{\Delta}{2}\log H_0\\ &= (96n\theta/\zeta)^{-1}\cdot 12\theta\log H_0 > 2^{-\theta/\zeta}\cdot 12\theta\log H_0\,.\end{aligned}$$

Inequality (2.9) follows from (2.20) and (2.21).

3. A mass of inequalities

In order to make the proof of Theorem 5 somewhat smoother, we assemble in this section all the inequalities between the variables that are actually used in the proof. This section is devoted to proving that if (2.4)–(2.12) and either (2.13) or (2.14) are satisfied then for $H \geqq H_0$, (3.1)–(3.23) below are true. We will deduce these inequalities in the order given; this is not the order of application. Inequalities (2.4)–(2.6) guarantee that these inequalities all hold for sufficiently large H; the difficulty consists in showing that $H \geqq H_0$ is sufficiently large. The reader may wish to skip this section in a first reading. The 23 inequalities are

$$(3.1)\qquad \theta H^{\theta-\zeta}\log H+\theta(H^{\zeta}/2)^{\theta/\zeta-1}H^{1-\eta}\log H < H/6\,,$$

$$\begin{aligned}(3.2)\qquad & 2.1(4hH^{\varepsilon-\delta})^nH^{1-\eta}[\log(2h)+(1-\delta+\nu)\log H]\\ & < (2.1\cdot 2^{\theta/\zeta})^{-1}H^{1+\theta-\zeta-\eta}(\zeta\log H-\log 4)\,,\end{aligned}$$

(3.3) $$\text{for } 1 \leqq K \leqq \theta/\zeta - 1\,,$$

$$2^{-K-1}\zeta H^{K\zeta+1-\eta}\log H > \log(c^2) + 2[(n-1)(1-\eta)+\zeta]\log H + (2^n+1)(n-1)AhH^{1-\delta+\zeta} + H^{1-\eta}[(2^{n+1}+2)\log(2h) + \log A + (2^{n+1}+2)(1-\delta+\nu)\log H] + nAhH^{1-\delta+2\zeta+K\zeta}[1 + n^{-1}H^{-\zeta}(n-1)\cdot 2^n]\,,$$

(3.4) $$H^{\zeta} > 2\,,$$

(3.5) $$\frac{1}{2}H^{1-\eta} + 1 < H^{1-\eta}\,,$$

(3.6) $$hH^{\varepsilon-\delta} > 2\,,$$

(3.7) $$hH^{\varepsilon-\delta+\theta} < e^H\,,$$

(3.8) $$-\frac{1}{2}H + (\theta+1-\eta)\log H + H^{1-\eta}\log(8n-8) < -\frac{1}{3}H\,,$$

(3.9) $$\text{for } K \geqq 1,\quad H^{K\zeta} - 1 > 2^{-1/3}H^{K\zeta}\,,$$

(3.10) $$\zeta\log H - \log 2 > 2^{-1/3}\zeta\log H\,,$$

(3.11) $$\text{for } K \geqq 1,\ 2^{-K-2/3}\zeta H^{K\zeta+1-\eta}\log H - \log 4 > 2^{-K-1}\zeta H^{K\zeta+1-\eta}\log H\,,$$

(3.12) $$nAhH^{1-\delta+\theta} + H^{1-\eta}[\log(4Ah^2) + 2(1-\delta+\nu)\log H] < \frac{1}{2}H\,,$$

(3.13) $$-2^n(n-1)AhH^{1-\delta}(H^{\zeta}+H^{\theta}) - 2^{n+1}H^{1-\eta}[\log(2h) + (1-\delta+\nu)\log H] > \log 2 - \frac{1}{6}H\,,$$

(3.14) $$(2-2^{n+1})\log c + (2-2^{n+1})[(n-1)(1-\eta)+\zeta]\log H + (1-2^n)(n-1)AhH^{1-\delta+\zeta} + H^{1-\eta}[(2-2^{n+1})\log(2h) + \log A_0 + \{(2-2^{n+1})(1-\delta+\nu)-\nu\}\log H] + (1-2^n)(n-1)AhH^{1-\delta+\theta} > \log 2 - H + nAhH^{1-\delta+\theta} + H^{1-\eta}[\log(4Ah^2) + 2(1-\delta+\nu)\log H]\,,$$

(3.15) $$\text{for } l \geqq 0\,,$$

$$\log\left(\frac{1}{2}c^{2-2^{n+1}}\right) + (2-2^{n+1})[(n-1)(1-\eta)+\zeta]\log H + (1-2^n)(n-1)AhH^{1-\delta+\zeta} + H^{1-\eta}[(2-2^{n+1})\log(2h) + \log A_0 + \{(2-2^{n+1})(1-\delta+\nu)-\nu\}\log H] + (1-2^n)(n-1)AhH^{1-\delta}l > -2^n(n-1)AhH^{1-\delta+\zeta} - 2^{n+1}H^{1-\eta}[\log(2h) + (1-\delta+\nu)\log H] - 2^n(n-1)AhH^{1-\delta}l\,,$$

$$(n-1)hH^{1-\delta} < 2^{-\theta/\zeta}H^{1-\eta}\,, \tag{3.16}$$

$$2^{-\theta/\zeta}H^{1-\eta} + 1 < 2^{-\theta/\zeta+1}H^{1-\eta}\,, \tag{3.17}$$

$$\begin{aligned} &-\frac{1}{2}H + (1+\theta-\zeta-\eta)\log H - (\theta/\zeta - 1)\log 2 \\ &\quad + 2^{-\theta/\zeta+1}H^{1-\eta}\log(8n-8) \\ &\quad + H^{\theta-\zeta}(2^{-\theta/\zeta}H^{1-\eta}+1)(\theta\log H + \log 2) \\ &< -\log 2 - 2^{-\theta/\zeta-1}H^{1+\theta-\zeta-\eta}(\zeta\log H - \log 4)\,, \end{aligned} \tag{3.18}$$

$$\begin{aligned} &-H + nAhH^{1-\delta+\theta} + 2H^{1-\eta}[\log(2h) + (1-\delta+\nu)\log H] \\ &< -\log 2 - (2.1\cdot 2^{\theta/\zeta})^{-1}H^{1+\theta-\zeta-\eta}(\zeta\log H - \log 4)\,, \end{aligned} \tag{3.19}$$

$$\begin{aligned} &-2^{-\theta/\zeta-1}H^{1+\theta-\zeta-\eta}(\zeta\log H - \log 4) + H^{1-\eta}(\nu\log H - \log A_0) \\ &< -\log 2 - (2.1\cdot 2^{\theta/\zeta})^{-1}H^{1+\theta-\zeta-\eta}(\zeta\log H - \log 4)\,, \end{aligned} \tag{3.20}$$

$$\begin{aligned} &\log(c^2) + 2[(n-1)(1-\eta)+\zeta]\log H + (n-1)AhH^{1-\delta+\zeta} \\ &\quad + (n-1)AhH^{1-\delta} < \frac{1}{10}H^{1-\eta}[\log(2h) + (1-\delta+\nu)\log H]\,, \end{aligned} \tag{3.21}$$

$$\begin{aligned} &\log(2c^2) - 2^{-\theta/\zeta}(1-H^{\zeta-\theta})H^{1+\theta-\zeta-\eta}(\zeta\log H - \log 4) \\ &\quad + 2[(n-1)(1-\eta)+\zeta]\log H + (n-1)AhH^{1-\delta+\zeta} \\ &\quad + H^{1-\eta}[\log(4Ah^2) + 2(1-\delta+\nu)\log H] + nAhH^{1-\delta+\theta+\zeta} \\ &< -\log 2 - 2^{-\theta/\zeta-1}H^{1+\theta-\zeta-\eta}(\zeta\log H - \log 4)\,, \end{aligned} \tag{3.22}$$

$$\begin{aligned} &\log(ec^2h) + [2(n-1)(1-\eta) + 2\zeta + \varepsilon - \delta + \theta]\log H \\ &\quad + (n-1)AhH^{1-\delta+\zeta} < AhH^{1-\delta+\theta}\,. \end{aligned} \tag{3.23}$$

In several of the above inequalities we have introduced

$$A_0 = \min_{1\leqq j\leqq n-1}(1, |\log a_j|)\,. \tag{3.24}$$

Our first task is to show that if $H \geqq H_0$, then (2.7)–(2.14) remain valid when H_0 is replaced by H. Now $\varepsilon \leqq 1$ and (2.6) and (2.5) say that $\eta<\varepsilon$, $\delta<\varepsilon$. Hence (2.7) and (2.8) clearly are good if H_0 increases. Inequalities (2.13) and (2.14) clearly continue to hold if H_0 increases. (Note $\eta+\zeta-\theta>0$ by (2.5).) Inequalities (2.10)–(2.12) offer no difficulties thanks to the following fact:

LEMMA 1. *If* $H_0^a \log H_0 > bH_0^a + \sum_{j=1}^k b_jH_0^{a_j} + \sum_{j=1}^k c_jH_0^{a_j}\log H_0$ *with* $H_0>1$, $a>a_j$ $(j=1,\cdots,k)$, $a>0$, $b\geqq 0$, $b_j\geqq 0$, $c_j\geqq 0$, $j=1,\cdots,k$ *then this inequality continues to hold for larger* H_0.

Proof. Differentiate both sides with respect to H_0 and multiply by H_0. The left side has been multiplied by $a+1/\log H_0$ while each term on the right is multiplied by a or a_j or $a_j+1/\log H_0$ and thus the derivative of the difference is positive at $H=H_0$. The lemma is now clear.

The coefficients in (2.10)–(2.12) satisfy the hypotheses here (in (2.12) we take $b = \log 4$), we need only show that the exponents do also ($H_0 > 1$ by (2.11)). In (2.10) $1 + \zeta - \eta > 0$, $1 + \zeta - \eta > 1 - \eta$ are clear. That $1 + \zeta - \eta > 1 + \zeta - \delta$ follows since $\delta > \eta$. In fact $\delta > \eta + 2\zeta$ since by (2.4) and (2.6)

$$\delta - \eta = \frac{\varepsilon - \eta - \zeta}{n} > \frac{(2n+1)\zeta - \zeta}{n} = 2\zeta \, .$$

This settles (2.10) and also (2.11). For (2.12), by (2.5) $1 + \theta - \zeta - \eta > 1 - \eta > 0$ and $1 + \theta - \zeta - \eta > 1 + n(\varepsilon - \delta) - \eta$ also by (2.5). This settles (2.12). If we assume (2.14) then for (2.9) we also differentiate both sides with respect to H_0 and multiply through by H_0. Here the left side is multiplied by 1 and the terms on the right by $\theta - \zeta + (\log H_0)^{-1}$ and $1 - \eta + \theta - \zeta + (\log H_0)^{-1}$; the second is larger and < 1 by (2.14). Hence (2.9) is true for $H > H_0$ if (2.14) holds. If on the other hand we assume (2.13) then the elementary inequality $x \geqq e \log x$ if $x > 0$ shows us that for $H \geqq H_0$,

$$H \geqq H^{1-\eta+\theta-\zeta} \cdot e \log H^{\eta+\zeta-\theta} > H^{1-\eta+\theta-\zeta}(6\theta H^{\eta-1} + 12\theta \cdot 2^{-\theta/\zeta}) \log H$$

which again shows that (2.9) is valid for $H \geqq H_0$. We will not see (2.13) and (2.14) again; their only use is to show that (2.9) holds for all $H \geqq H_0$. For convenience, we will state what we have proved as

LEMMA 2. *If* (2.4)–(2.12) *and either* (2.13) *or* (2.14) *are valid then* (2.7)–(2.12) *remain valid if* H_0 *is replaced by any* $H \geqq H_0$.

Until the end of Section 5, whenever we refer to (2.7)–(2.12) it will be with H_0 replaced by an $H \geqq H_0$. We have now proved (3.1) and (3.2) since they are other forms of (2.9) and (2.12) respectively. It follows from (2.10) that

$$\zeta H^{1+\zeta-\eta} \log H > 8H^{1-\eta}(2^{n+1} + 2)(1 - \delta + \nu) \log H$$

and hence

$$H^{\zeta} > \frac{2^{n+4} + 16}{\zeta}(1 - \delta + \nu) > \frac{2^{n+4}\nu}{\zeta} \, . \tag{3.25}$$

By (2.5) and (2.4),

$$\eta > n(\varepsilon - \delta) = (n-1)(\varepsilon - \eta) + \zeta$$

and hence

$$\varepsilon - \zeta > n(\varepsilon - \eta) \, .$$

Thus from (2.6)

$$\varepsilon - \zeta > n(2n+1)\zeta$$

and hence

$$\zeta < (2n^2 + n + 1)^{-1}\varepsilon \leqq (2n^2 + n + 1)^{-1} . \tag{3.26}$$

It follows from (3.25) that

$$H^\zeta > (2n^2 + n + 1)\cdot 2^{n+4}\nu \geqq (2n^2 + n + 1)\cdot 2^{n+4} . \tag{3.27}$$

In particular $H^\zeta > (n-1)\cdot 2^n$ and thus from (2.11), if $1 \leqq K \leqq \theta/\zeta - 1$,

$$\begin{aligned}\frac{1}{2}\,\frac{\zeta}{2^{K+1}}H^{K\zeta+1-\eta}\log H &\geqq \frac{1}{2}\,\frac{\zeta}{2^{\theta/\zeta}}H^{K\zeta+1-\eta}\log H \\ &> (n+1)AhH^{1-\delta+2\zeta+K\zeta} \\ &> nAhH^{1-\delta+2\zeta+K\zeta}\left(1 + \frac{(n-1)2^n}{nH^\zeta}\right) .\end{aligned} \tag{3.28}$$

Also, by (3.27), $H^\zeta > 2$ and hence from (2.10) if $1 \leqq K$,

$$\begin{aligned}\frac{1}{2}\,\frac{\zeta}{2^{K+1}}H^{K\zeta+1-\eta}\log H &= \frac{\zeta}{4}H^{1-\eta}\left(\frac{H^\zeta}{2}\right)^K\log H \\ &\geqq \frac{\zeta}{8}H^{1-\eta+\zeta}\log H \\ &> \log(c^2) + 2[(n-1)(1-\eta) + \zeta]\log H \\ &\quad + (2^n + 1)(n-1)AhH^{1-\delta+\zeta} \\ &\quad + H^{1-\eta}[(2^{n+1} + 2)\log(2h) + \log A \\ &\quad + (2^{n+1} + 2)(1 - \delta + \nu)\log H] .\end{aligned} \tag{3.29}$$

When we add (3.28) and (3.29), we get (3.3).

Incidentally, the statement $H^\zeta > 2$ used above is (3.4). To prove (3.5), we note that by (2.6) $1 - \eta \geqq \varepsilon - \eta > \zeta$ and thus (3.5) follows from (3.4). For (3.6), we see from (2.4) and (2.6) that

$$\varepsilon - \delta = \frac{(n-1)(\varepsilon - \eta) + \zeta}{n} > \frac{(n-1)(2n+1)\zeta + \zeta}{n} = (2n-1)\zeta > \zeta \tag{3.30}$$

so that (3.6) follows from (3.4) since $h \geqq 1$ by (2.7). From (3.30) and (2.5),

$$\theta > n(\varepsilon - \delta) + \zeta > (2n^2 - n + 1)\zeta . \tag{3.31}$$

Now from (2.10),

$$\zeta H^{1+\zeta-\eta}\log H > 8(2^{n+1} + 2)H^{1-\eta}\log(2h)$$

and hence

$$H^{2\zeta} > H^\zeta\log(H^\zeta) > 2^{n+4}\log(2h) . \tag{3.32}$$

It now follows from (2.9), (3.31), (3.32) that

$$H > 6\theta H^{\theta-\zeta}\log H > 6\theta\log H + H^{\theta-\zeta}$$

($\log H^{6\theta} > 2$ and $H^{\theta-\zeta} > 2$ and if $a > 2$, $b > 2$ then $ab > a + b$ since

$(a-1)(b-1)>1\big)$,

$$H>(\theta+\varepsilon-\delta)\log H+\log h$$

which proves (3.7).

From (3.1)

$$\frac{1}{6}H>\theta H^{\theta-\zeta}\log H+H^{1-\eta}\cdot\left(\frac{H^{\zeta}}{2}\right)^{\theta/\zeta-1}\cdot\frac{\theta}{\zeta}\log(H^{\zeta})\,. \tag{3.33}$$

By (3.31) and (3.25),

$$\theta H^{\theta-\zeta}>\zeta H^{\zeta}>2^{n+4}>2>\theta+1-\eta$$

while by (3.31) and (3.27),

$$\left(\frac{H^{\zeta}}{2}\right)^{\theta/\zeta-1}\cdot\frac{\theta}{\zeta}\log(H^{\zeta})>\log(8n-8)\,.$$

Setting these in (3.33) proves (3.8).

Now we come to three similar inequalities. From $H^{\zeta}>5$ $\big($see (3.27)$\big)$ and $2^{1/3}>5/4$, we get for $K\geqq 1$,

$$(1-2^{-1/3})H^{K\zeta}>\frac{1}{5}H^{\zeta}>1$$

which proves (3.9). Since $H^{\zeta}>2^{5}$ $\big($see (3.27)$\big)$,

$$(1-2^{-1/3})\log H^{\zeta}>\frac{1}{5}\log(2^{5})=\log 2$$

which proves (3.10). Since $H^{\zeta}>2^{4}$ $\big($(3.27) again$\big)$, we have for $K\geqq 1$,

$$\frac{1}{2}\left(\frac{H^{\zeta}}{2}\right)^{K}H^{1-\eta}(2^{1/3}-1)\log H^{\zeta}>\frac{1}{2}\cdot(2^{3})\cdot 1\cdot\frac{1}{4}\log 16>\log 4$$

which proves (3.11).

By (2.5), $\delta-\theta>\delta-\eta-\zeta$ and thus by (2.11) and (3.31)

$$\begin{aligned}\frac{1}{4}H^{\delta-\theta}&>\frac{1}{4}H^{\delta-\eta-2\zeta}H^{\zeta}>\frac{1}{4}H^{\delta-\eta-2\zeta}\log H^{\zeta}\\&>\frac{1}{2}(n+1)Ah\cdot 2^{\theta/\zeta}>nAh\cdot 2^{2n^2-n}\\&\geqq 2^{2(n^2-n)}\cdot 2^{n}nAh>12\cdot 2^{n}nAh\,.\end{aligned}\tag{3.34}$$

By (2.9), (3.31), and (3.25)

$$\begin{aligned}\frac{1}{4}H&>\frac{3}{2}\theta H^{\theta-\zeta}\log H>\frac{3}{2}\frac{\theta}{\zeta}\cdot\zeta H^{\zeta}\log H>n\cdot 2^{n+4}\log H\\&>\log 2^{6}+n\cdot 2^{n+1}\log H>6\log 2+2^{n+1}[(n-1)(1-\eta)+\zeta]\log H\,.\end{aligned}\tag{3.35}$$

By (2.5) and (3.30),

$$\eta > (2n^2 - n)\zeta \tag{3.36}$$

and so by (3.32) and (3.27)

$$\frac{1}{8}H^{\eta} > \frac{1}{8}H^{2\zeta}\cdot H^{\zeta} > 6\cdot 2^{n+1}\log(2h)\ . \tag{3.37}$$

Also by (2.5) and (2.4)

$$\begin{aligned}\eta - \delta + \theta &= [\theta - \zeta - n(\varepsilon - \delta)] + n(\varepsilon - \delta) + \eta - \delta + \zeta > n(\varepsilon - \delta) + \eta - \delta \\ &= (n-1)(\varepsilon - \eta) + \zeta - \frac{\varepsilon - \eta - \zeta}{n} > 0\end{aligned}$$

so that by (3.34)

$$\frac{1}{8}H^{\eta} > \frac{1}{2}\cdot\frac{1}{4}H^{\delta-\theta} > 12A > 12\log A > 6\,(\log A - \log A_0); \tag{3.38}$$

the last part since by its definition in (3.24),

$$\log A_0 \geqq \log\log\frac{1+\sqrt{5}}{2} > \log \pi^{-1} \geqq -\log A\ . \tag{3.39}$$

Next by (3.36), (3.25), and (3.27),

$$\begin{aligned}\frac{1}{4}H^{\eta} > \frac{1}{4}H^{3\zeta} > \frac{H^{\zeta}}{4}\zeta H^{\zeta}\log H &> 6\cdot 2^{n+2}(1 - \delta + \nu)\log H \\ &> 6\cdot[2^{n+1}(1 - \delta + \nu) + \nu]\log H\ .\end{aligned} \tag{3.40}$$

Adding (3.37), (3.38), and (3.40) gives

$$\begin{aligned}\frac{1}{2}H^{\eta} > 6[2^{n+1}&\log(2h) + \log A - \log A_0 \\ &+ \{2^{n+1}(1 - \delta + \nu) + \nu\}\log H]\ .\end{aligned} \tag{3.41}$$

Now if we add (3.35), (3.34) multiplied by $H^{1-\delta+\theta}$ (note that by (3.31) $H^{1-\delta+\zeta} < H^{1-\delta+\theta}$) and (3.41) multiplied by $H^{1-\eta}$, we get

$$\begin{aligned}H > 6\log 2 &+ 2^{n+1}[(n-1)(1-\eta) + \zeta]\log H \\ &+ 6\cdot 2^{n}(n-1)AhH^{1-\delta+\zeta} + 6\cdot 2^{n}nAhH^{1-\delta+\theta} + 6H^{1-\eta}[2^{n+1}\log(2h) \\ &+ \log A - \log A_0 + \{2^{n+1}(1 - \delta + \nu) + \nu\}\log H]\ .\end{aligned} \tag{3.42}$$

This contains both (3.12) and (3.13). As another special case of (3.42), we have

$$\begin{aligned}(3.43)\qquad H > \log 2 &+ (2^{n+1}-2)[(n-1)(1-\eta)+\zeta]\log H \\ &+ 2^n(n-1)AhH^{1-\delta+\zeta} + 2^n\cdot nAhH^{1-\delta+\theta} \\ &+ H^{1-\eta}[2^{n+1}\log(2h) + \log A - \log A_0 \\ &+ \{2^{n+1}(1-\delta+\nu)+\nu\}\log H]\,.\end{aligned}$$

We see from (2.8) that

$$n\log(2h) \geqq \log c\,. \tag{3.44}$$

By (2.5), $1-\delta+\zeta \geqq \varepsilon-\delta+\zeta > \zeta$ and hence by (3.27) and (3.44)

$$(n-1)AhH^{1-\delta-\zeta} > hH^{\zeta} > 2h\cdot 2^{n+1}\cdot n > 2^{n+1}\cdot n\log(2h) \geqq 2^{n+1}\log c\,.$$

Therefore certainly

$$(n-1)AhH^{1-\delta+\zeta} > (2^{n+1}-2)\log c\,. \tag{3.45}$$

If we add this to (3.43), we get a slightly stronger version of (3.14) (the $H^{1-\delta+\theta}$ term is slightly better).

From (2.11),

$$H^{1-\eta} = H^{1-\eta-\zeta}H^{\zeta} > H^{1-\eta-\zeta}\log H^{\zeta} > 2(n+1)Ah\cdot 2^{\theta/\zeta}H^{1-\delta+\zeta}\,. \tag{3.46}$$

In particular, by (3.30) and (3.25),

$$H^{1-\delta+\zeta} \geqq H^{\varepsilon-\delta+\zeta} > H^{2\zeta} > H^{\zeta}\log H^{\zeta} > 2^{n+4}\log H$$

and hence

$$H^{1-\eta} > 2^{n+5+\theta/\zeta}(n+1)Ah\log H\,.$$

Therefore clearly $\big(\log A_0 > -2\log(2h)$ by (3.39) and (2.7)$\big)$,

$$\begin{aligned}&H^{1-\eta}[2\log(2h) + \log A_0 + \{\nu + 2(1-\delta)\}\log H] > H^{1-\eta} \\ &> \log 2 + 2^{n+1}\cdot n\log H > \log 2 + (2^{n+1}-2)[(n-1)(1-\eta)+\zeta]\log H\,.\end{aligned}$$

If we add this to (3.45), we get

$$\begin{aligned}&(n-1)AhH^{1-\delta+\zeta} + H^{1-\eta}[2\log(2h) + \log A_0 + \{\nu+2(1-\delta)\}\log H] \\ &> \log(2c^{2^{n+1}-2}) + (2^{n+1}-2)[(n-1)(1-\eta)+\zeta]\log H\,.\end{aligned}$$

This is equivalent to (3.15) with $l=0$. Inequality (3.15) with $l>0$ follows from (3.15) with $l=0$.

A simple consequence of (3.46) is

$$H^{1-\eta} > (n-1)h\cdot 2^{\theta/\zeta}H^{1-\delta}$$

which proves (3.16). If we weaken this still further, we get

$$H^{1-\eta} > 2^{\theta/\zeta}$$

which is equivalent to (3.17).

By (3.35),

$$\frac{1}{4}H > 2\log H . \tag{3.47}$$

By (3.34),

$$\frac{1}{8}H > nAhH^{1-\delta+\theta} . \tag{3.48}$$

By (2.9), the fact that $1-\eta > \delta-\eta > \delta-\eta-\zeta$, and (2.11)

$$\begin{aligned} \frac{1}{8}H &> \frac{1}{2}\cdot 2^{-\theta/\zeta}H^{1-\eta}\cdot 3\theta H^{\theta-\zeta}\log H \\ &> \left(\frac{1}{2}\cdot 2^{-\theta/\zeta}H^{\delta-\eta-2\zeta}\cdot H^{\zeta}\right)(3\theta H^{\theta-\zeta}\log H) \\ &> \left(\frac{1}{2}\cdot 2^{-\theta/\zeta}H^{\delta-\eta-2\zeta}\log H^{\zeta}\right)(3\theta H^{\theta-\zeta}\log H) \\ &> (n+1)Ah\cdot 3\theta H^{\theta-\zeta}\log H \\ &> 3\theta H^{\theta-\zeta}\cdot\log H \\ &> 2H^{\theta-\zeta}(\theta\log H+\log 2) \end{aligned} \tag{3.49}$$

since $\theta\log H > \zeta\log H > 2\log 2$ by (3.31) and (3.27). By (3.37), (2.7), and (3.31)

$$\begin{aligned} \frac{1}{8}H^{\eta} &> 6\cdot 2^{n+1}\log(2h) > 2\log(2h) + 4\cdot 2^{n}\log 4 \\ &> 2\log(2h) + 4\cdot 2^{-\theta/\zeta}\log(8n-8) . \end{aligned}$$

Hence

$$\frac{1}{8}H > [2\log(2h) + 4\cdot 2^{-\theta/\zeta}\log(8n-8)]H^{1-\eta} . \tag{3.50}$$

By (3.40)

$$\frac{1}{8}H = 2H^{1-\eta}\cdot\frac{1}{16}H^{\eta} > 2H^{1-\eta}(1-\delta+\nu)\log H . \tag{3.51}$$

Next from (2.9) and the fact that $\theta > \zeta$ (see (3.31)),

$$\frac{1}{4}H > 3\theta\cdot 2^{-\theta/\zeta}H^{1-\eta+\theta-\zeta}\log H > 2^{-\theta/\zeta}(2\theta+\zeta)H^{1-\eta+\theta-\zeta}\log H . \tag{3.52}$$

When we add (3.47), (3.48), (3.49), (3.50), (3.51), and (3.52), we get

$$\begin{aligned} H > 2\log H &+ nAhH^{1-\delta+\theta} + 2H^{\theta-\zeta}(\theta\log H+\log 2) \\ &+ H^{1-\eta}[2\log(2h) + 4\cdot 2^{-\theta/\zeta}\log(8n-8) + 2(1-\delta+\nu)\log H] \\ &+ 2^{-\theta/\zeta}(2\theta+\zeta)H^{1-\eta+\theta-\zeta}\log H . \end{aligned}$$

This implies (3.18) since $2(1+\theta-\zeta-\eta) < 2$ by (2.5) and $(\theta/\zeta-2)\log 2 > 0$ by (3.31), and it implies (3.19) since $2\log H > \log 2$.

By (2.12) and the fact that $h > 1$ by (2.7),

$$H^{1+\theta-\zeta-\eta}(\zeta \log H - \log 4) > 32\cdot 2^{\theta/\zeta} H^{n(\varepsilon-\delta)}\cdot H^{1-\eta}[\log(4h^2) + 2(1-\delta+\nu)\log H]. \tag{3.53}$$

By (3.30), $n(\varepsilon-\delta) > n(2n-1)\zeta \geqq 6\zeta$ and so by (2.10) and (3.27),

$$H^{n(\varepsilon-\delta)} > H^{3\zeta}\cdot H^{\zeta}\cdot H^{\zeta}\log H^{\zeta} > 2^6\cdot nH^{3\zeta}\cdot 8\log A.$$

Since $2\log A > 2$ and $\log(4h^2) + 2(1-\delta+\nu)\log H > 2$ and $ab > a+b$ if $a > 2$, $b > 2$, we see from (3.53) that

$$2^{-\theta/\zeta}H^{1+\theta-\zeta-\eta}(\zeta\log H - \log 4) > 2^{11}nH^{3\zeta}\cdot H^{1-\eta}[\log(4Ah^2) + \log A + 2(1-\delta+\nu)\log H]. \tag{3.54}$$

Since $\log A > -\log A_0$ by (3.39), this more than implies (3.20).

By (2.11) and (3.31),

$$\begin{aligned} \frac{1}{10}H^{1-\eta}[\log(2h) + (1-\delta+\nu)\log H] &> (10\zeta)^{-1}H^{1-\delta+2\zeta}\cdot\zeta H^{\delta-\eta-2\zeta}\log H \\ &> (10\zeta)^{-1}H^{1-\delta+2\zeta}\cdot 2(n+1)\cdot 2^{\theta/\zeta}Ah \\ &> 5\zeta^{-1}nAhH^{1-\delta+2\zeta}. \end{aligned} \tag{3.55}$$

Further

$$2\zeta^{-1}nAhH^{1-\delta+2\zeta} > (n-1)AhH^{1-\delta+\zeta} + (n-1)AhH^{1-\delta}, \tag{3.56}$$

$$\begin{aligned} 2\zeta^{-1}nAhH^{1-\delta+2\zeta} &> 2n\zeta^{-1}\log(H^{\zeta}) \geqq 2n\log H \\ &> 2[(n-1)(1-\eta)+\zeta]\log H, \end{aligned} \tag{3.57}$$

and by (3.27) and (3.44),

$$\zeta^{-1}nAhH^{1-\delta+2\zeta} > hH^{\zeta} > 4\cdot 2^n h > 2\cdot n\log(2h) \geqq 2\log c. \tag{3.58}$$

When we combine (3.55)-(3.58), we get (3.21).

By (2.4) and (2.6), $\delta-\eta-2\zeta = n^{-1}[\varepsilon-\eta-(2n+1)\zeta] > 0$ and thus by (2.5) and (2.4),

$$\begin{aligned} \frac{\theta}{\delta-\eta-2\zeta} &> \frac{n(\varepsilon-\delta)+\zeta}{n^{-1}[\varepsilon-\eta-(2n+1)\zeta]} > \frac{n[(n-1)(\varepsilon-\eta)+\zeta]}{\varepsilon-\eta-(2n+1)\zeta} \\ &> n(n-1), \end{aligned} \tag{3.59}$$

the last part being clear after multiplying through by the denominator. By (2.11),

$$H^{\delta-\eta-2\zeta}\log(H^{\zeta}) > 2^{\theta/\zeta}$$

and hence if $\delta-\eta-2\zeta \leqq \zeta$, then by (3.31)

$$H^{\zeta}\log(H^{\zeta}) > 2^{2n^2-n+1} > 2^{n(n-1)},$$

while if $\delta-\eta-2\zeta > \zeta$ then by the fact that $H^{\zeta} > e$ and (3.59)

$$H^{\zeta}\log(H^{\zeta}) > [H^{\delta-\eta-2\zeta}\log(H^{\zeta})]^{\zeta/(\delta-\eta-2\zeta)} > 2^{\theta/(\delta-\eta-2\zeta)} > 2^{n(n-1)},$$

so that $H^{\zeta}\log(H^{\zeta}) > 2^{n(n-1)}$ is true in any case. For $x\log x > y > e$, we have $x > y/\log y$ and hence for $n \geqq 2$,

$$H^{\zeta} > \frac{2^{n(n-1)}}{n(n-1)\log 2} > n^{-2}\cdot 2^{n(n-1)} . \tag{3.60}$$

For $n \geqq 5$, $2^n > n^2$ and $n^2 - (9/2)n - 2 > 0$ and thus for $n \geqq 5$,

$$H^{\zeta} > 2^{n(n-2)} > 2^{(5/2)n+2} \geqq 4^{(5/4)n+1} ,$$

while for $n = 2, 3, 4$ we have by (3.27),

$$H^{\zeta} > 2n^2\cdot 2^{n+4} \geqq 2^{2n+5} > 4^{n(1+1/n)+1} \geqq 4^{(5/4)n+1} .$$

It follows that for all $n \geqq 2$,

$$\zeta \log H - \log 4 > n(n + 4/5)^{-1}\zeta \log H . \tag{3.61}$$

By (3.31) and (3.27),

$$H^{\theta-\zeta} > H^{\zeta} > 2^6 n > 20n + 18 ,$$

and hence

$$1 - 2H^{\zeta-\theta} > \left(n + \frac{4}{5}\right)\Big/\left(n + \frac{9}{10}\right) . \tag{3.62}$$

Therefore by (3.61) and (3.62) ,

$$\begin{aligned} &2^{-\theta/\zeta}(1 - 2H^{\zeta-\theta})H^{1+\theta-\zeta-\eta}(\zeta \log H - \log 4) \\ &> 2^{-\theta/\zeta}n(n + 9/10)^{-1}\zeta H^{1+\theta-\zeta-\eta} \log H . \end{aligned} \tag{3.63}$$

But (2.11) is equivalent to

$$2^{-\theta/\zeta}n(n + 1)^{-1}\zeta H^{1+\theta-\zeta-\eta} \log H > 2nAhH^{1-\delta+\theta+\zeta} ,$$

and since

$$\frac{n}{n + 9/10} = \frac{n}{n+1} + \frac{n/10}{(n + 9/10)(n+1)} > \frac{n}{n+1} + \frac{1}{40n} ,$$

we get from (3.63) followed by (3.54) and (3.21),

$$\begin{aligned} &2^{-\theta/\zeta}(1 - 2H^{\zeta-\theta})H^{1+\theta-\zeta-\eta}(\zeta \log H - \log 4) - 2nAhH^{1-\delta+\theta+\zeta} \\ &> 2^{-\theta/\zeta}(40n)^{-1}H^{1+\theta-\zeta-\eta}(\zeta \log H - \log 4) \\ &> 2^5 H^{1-\eta}[\log (4Ah^2) + 2(1 - \delta + \nu) \log H] \\ &> 2 \log (4c^2) + 4[(n - 1)(1 - \eta) + \zeta] \log H + 2(n - 1)AhH^{1-\delta+\zeta} \\ &\quad + 2H^{1-\eta}[\log (4Ah^2) + 2(1 - \delta + \nu) \log H] . \end{aligned}$$

This is equivalent to (3.22).

By (3.31) and (3.27),

$$AhH^{1-\delta+\theta} > AhH^{1-\delta+3\zeta}\cdot H^{\zeta} > 3nAhH^{1-\delta+3\zeta} . \tag{3.64}$$

Now

$$nAhH^{1-\delta+3\zeta} > (n - 1)AhH^{1-\delta+\zeta} , \tag{3.65}$$

and by (3.25) and the fact that $A > 1$, $h \geqq 1$,

$$\text{(3.66)} \qquad nAhH^{1-\delta+3\zeta} > nH^{2\zeta} > n\zeta H^{\zeta}\log H > 4n\log H$$
$$> [2(n-1)(1-\eta)+2\zeta+\varepsilon+\theta-\delta]\log H\,.$$

Next, by (3.27) and (3.44),

$$\text{(3.67)} \qquad nAhH^{1-\delta+3\zeta} > 8nh > (2n+1)\log(2h)+1 > \log(ec^2h)\,.$$

When we add (3.65)-(3.67) and compare the result with (3.64), we get (3.23).

4. Some preparatory lemmas

In this section we assemble those lemmas of Part I which may be used without change. This includes the key algebraic lemma which we state first.

LEMMA 3. *Let $\varepsilon_1, \cdots, \varepsilon_r$ be the fundamental units of r different real quadratic fields. Let D be a non-negative rational integer and let p be an odd rational prime. Then the field $Q(\sqrt{-D}, e^{2\pi i/p}, \varepsilon_1^{1/p}, \cdots, \varepsilon_r^{1/p})$ is an extension of degree p of $Q(\sqrt{-D}, e^{2\pi i/p}, \varepsilon_1^{1/p}, \cdots, \varepsilon_{r-1}^{1/p}, \varepsilon_r)$.*

Proof. This is Lemma 13 of Part I.

LEMMA 4. *Let M and N be integers, $N > M > 0$ and let u_{jk} be rational integers $(1 \leqq j \leqq M, 1 \leqq k \leqq N)$ with absolute values at most U. Then there exist rational integers $x_1, \cdots, x_N$ not all zero with absolute values at most $(NU)^{M/(N-M)}$ such that*

$$\sum_{k=1}^{N} u_{jk}x_k = 0 \qquad (1 \leqq j \leqq M)\,.$$

Proof. This is Lemma 3 of Part I.

LEMMA 5. *Let $f(z)$ be an entire function,*

$$F(z) = \{(z-1)\cdots(z-R)\}^{S+1}$$

where R and S are integers, $R \geqq 1$, $S \geqq 0$. Suppose $\beta \neq r$ $(r = 1, \cdots, R)$. Let Γ and Γ_r $(r = 1, \cdots, R)$ be circles oriented in the positive direction such that Γ contains $1, \cdots, R$ and β, while Γ_r contains r but no other integral point nor β. Put $f_m(z) = (d^m/dz^m)f(z)$. Then

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f(z)}{(z-\beta)F(z)}dz = \frac{f(\beta)}{F(\beta)} + \frac{1}{2\pi i}\sum_{r=1}^{R}\sum_{m=0}^{S}\frac{f_m(r)}{m!}\int_{\Gamma_r}\frac{(z-r)^m}{(z-\beta)F(z)}dz\,.$$

Proof. This is Lemma 4 of Part I.

5. Proof of Theorem 5

The proof of Theorem 5 is the same as the proof of Theorem 1 of Part I except that here we know what "sufficiently large" is and many of the inequalities are sharper here so as to get better numerical results. The choice of variables used here to prove Theorem 4 is the choice of variables used to prove Theorem 1 of Part I (except that $h = 3^{2/n}$ here is replaced by $h = 3$ in

Part I; this partially accounts for the improvement of Theorem 4 over Theorem 1′ of Part I). We will assume throughout this section that the hypotheses of Theorem 5 are satisfied but with a value of $H \geqq H_0$ and ultimately derive a contradiction. By Lemma 2, (2.7)–(2.12) are valid for $H \geqq H_0$ and whenever we refer to these inequalities, we assume it is with H in place of H_0. Further, (3.1)–(3.23) are valid where in some of these we have used

$$A_0 = \min_{1 \leqq j \leqq n-1} (1, |\log \alpha_j|) .$$

We begin with some further notation. Let

$$\beta_j = -b_j/b_n \qquad (j = 1, \cdots, n-1) . \tag{5.1}$$

Since by hypothesis b_n is a non-zero rational integer, it follows from (2.3) that

$$|\beta_1 \log \alpha_1 + \cdots + \beta_{n-1} \log \alpha_{n-1} - \log \alpha_n| < e^{-H} . \tag{5.2}$$

Set

$$L_1 = \cdots = L_{n-1} = hH^{1-\delta}, \quad L_n = hH^{\varepsilon-\delta} . \tag{5.3}$$

In particular $L_n \leqq L_1$ and by (2.2) even $L_n \log \alpha_n \leqq L_1$. By (2.4),

$$\prod_{j=1}^{n} L_j = h^n H^{(n-1)(1-\delta)+(\varepsilon-\delta)} = h^n \exp\{[(n-1)(1-\eta) + \zeta] \log H\} \tag{5.4}$$

and by (2.8)

$$\begin{aligned} \prod_{j=1}^{n} (L_j + 1) &= (h + H^{\delta-1})^{n-1}(h + H^{\delta-\varepsilon})H^{(n-1)(1-\delta)+(\varepsilon-\delta)} \\ &\leqq c \exp\{[(n-1)(1-\eta) + \zeta] \log H\} . \end{aligned} \tag{5.5}$$

Further by the definition of A and (2.2), for any real $x \geqq 0$,

$$\begin{aligned} \left|\prod_{j=1}^{n} \alpha_j^{L_j x}\right| = \prod_{j=2}^{n} \alpha_j^{L_j x} &\leqq \exp[(n-2)AhH^{1-\delta}x + hH^{1-\delta}x] \\ &\leqq \exp[(n-1)AhH^{1-\delta}x] , \end{aligned} \tag{5.6}$$

while for any complex z and real λ_j, $0 \leqq \lambda_j \leqq L_j (j = 1, \cdots, n)$,

$$\left|\prod_{j=1}^{n} \alpha_j^{\lambda_j z}\right| \leqq \exp[(n-1)AhH^{1-\delta}|z| + hH^{1-\delta}|z|] \leqq \exp(nAhH^{1-\delta}|z|) . \tag{5.7}$$

By $\sum_{\lambda=0}^{L}$ we will always mean $\sum_{\lambda_1=0}^{[L_1]} \sum_{\lambda_2=0}^{[L_2]} \cdots \sum_{\lambda_n=0}^{[L_n]}$. Let

$$\psi(z) = \psi(z, m_1, \cdots, m_{n-1}) = \prod_{j=1}^{n-1} b_n^{m_j} \sum_{\lambda=0}^{L} p(\lambda) \prod_{j=1}^{n} \alpha_j^{\lambda_j z} \prod_{j=1}^{n-1} \gamma_j^{m_j} , \tag{5.8}$$

$$\phi_{m_1}, \cdots, {}_{m_{n-1}}(z_1, \cdots, z_{n-1}) = \prod_{j=1}^{n-1} (\log \alpha_j)^{m_j} \sum_{\lambda=0}^{L} p(\lambda) \prod_{j=1}^{n-1} (\alpha_j^{\gamma_j z_j} \gamma_j^{m_j}) , \tag{5.9}$$

where we have written

$$\gamma_j = \lambda_j + \lambda_n \beta_j \qquad (j = 1, \cdots, n-1) . \tag{5.10}$$

The numbers $p(\lambda) = p(\lambda_1, \cdots, \lambda_n)$ are rational integers, *not all zero*, to be determined in Lemma 6 and we let

$$P = \max_{\lambda_1, \dots, \lambda_n} |p(\lambda_1, \cdots, \lambda_n)| . \tag{5.11}$$

Note that

$$\phi_{m_1, \cdots, m_{n-1}}(z_1, \cdots, z_{n-1}) = \frac{\partial^{m_1+\cdots+m_{n-1}}}{\partial z_1^{m_1} \cdots \partial z_{n-1}^{m_{n-1}}} \phi_{0,\ldots,0}(z_1, \cdots, z_{n-1});$$

this makes sense as we shall assume throughout that the m_j are integers which are $\geqq 0$.

The following four estimates will be used repeatedly: for $0 \leqq \lambda_j \leqq L_j$ $(1 \leqq j \leqq n)$ and $\sum_{j=1}^{n-1} m_j \leqq H^{1-\eta}$,

$$|\prod_{j=1}^{n-1} (b_n\gamma_j)^{m_j}| < \exp\{H^{1-\eta}[\log(2h) + (1-\delta+\nu)\log H]\}, \tag{5.12}$$

$$|\prod_{j=1}^{n-1} \gamma_j^{m_j}| < \exp\{H^{1-\eta}[\log(2h) + (1-\delta+\nu)\log H]\}, \tag{5.13}$$

$$|\prod_{j=1}^{n-1} (b_n^{-1}\log\alpha_j)^{m_j}| < \exp(H^{1-\eta}\log A), \tag{5.14}$$

$$|\prod_{j=1}^{n-1} (b_n/\log\alpha_j)^{m_j}| < \exp[H^{1-\eta}(\nu\log H - \log A_0)]; \tag{5.15}$$

all follow from (2.1) and the definitions in (5.1), (5.3), and (5.10).

LEMMA 6. *The rational integers $p(\lambda)$ can be chosen, not all zero, such that*

$$\begin{aligned} P < c\exp\{&[(n-1)(1-\eta)+\zeta]\log H + (n-1)AhH^{1-\delta+\zeta} \\ &+ H^{1-\eta}[\log(2h) + (1-\delta+\nu)\log H]\}, \end{aligned} \tag{5.16}$$

and for all integers $l, m_1, \cdots, m_{n-1}$ with $1 \leqq l \leqq H^\zeta$, and $\sum_{j=1}^{n-1} m_j \leqq H^{1-\eta}$,

$$\psi(l, m_1, \cdots, m_{n-1}) = 0. \tag{5.17}$$

Proof. Suppose α_j is in $Q(\sqrt{k_j})$ where k_j is square-free $(j = 2, \cdots, n)$ and set $\delta_j = \sqrt{k_j}$ if $k_j \not\equiv 1 \pmod 4$ and $\delta_j = (1+\sqrt{k_j})/2$ if $k_j \equiv 1 \pmod 4$. Then we may write

$$\alpha_j^l = a_{j0}^{(l)} + a_{j1}^{(l)}\delta_j \qquad (j = 2, \cdots, n),$$

and

$$(b_n\gamma_1)^{m_1} = b_{10}^{(m_1)} + b_{11}^{(m_1)}\sqrt{-D}.$$

We then have the following two estimates which are equations (21) and (23) respectively of Part I where they are proved:

$$|a_{js}^{(l)}| \leqq \alpha_j^l \qquad (2 \leqq j < n;\ s = 0,1;\ l \geqq 0), \tag{5.18}$$

$$|b_{1s}^{(m_1)}| \leqq |b_n\gamma_1|^{m_1} \qquad (s = 0,1). \tag{5.19}$$

The equation $\psi(l) = 0$ is

$$\sum_{\lambda=0}^{L} p(\lambda)\alpha_1^{\lambda_1 l}\prod_{j=2}^{n}(a_{j0}^{(\lambda_j l)} + a_{j1}^{(\lambda_j l)}\delta_j)\cdot(b_{10}^{(m_1)} + b_{11}^{(m_1)}\sqrt{-D})\cdot\prod_{j=2}^{n-1}(b_n\gamma_j)^{m_j} = 0.$$

It will be satisfied if the following 2^n equations are satisfied ($s_j = 0$ or 1, $2 \leqq j \leqq n$, $t = 0$ or 1),

$$\sum_{\lambda=0}^{L} p(\lambda)\alpha_1^{\lambda_1 l}\prod_{j=2}^{n} a_{js_j}^{(\lambda_j l)}\cdot b_{1t}^{(m_1)}\prod_{j=2}^{n-1}(b_n\gamma_j)^{m_j} = 0. \tag{5.20}$$

This gives us a total of M equations,

$$M \leqq 2^n H^\zeta \cdot \frac{(H^{1-\eta} + n - 1)^{n-1}}{(n-1)!}$$

for the N unknowns $p(\lambda)$,

$$N = \prod_{j=1}^{n}([L_j] + 1) > h^n H^{(n-1)(1-\eta)+\zeta} \geqq 2M\,, \tag{5.21}$$

the estimate following from (5.4) and (2.7). The coefficients of the $p(\lambda)$ in (5.20) are all in absolute value $\leqq U$ with

$$\begin{aligned} U &< 1\cdot\prod_{j=2}^{n}\alpha_j^{L_j H^{\zeta}}\cdot\exp\{H^{1-\eta}[\log(2h) + (1-\delta+\nu)\log H]\} \\ &< \exp\{(n-1)AhH^{1-\delta+\zeta} \\ &\quad + H^{1-\eta}[\log(2h) + (1-\delta+\nu)\log H]\}\,, \end{aligned} \tag{5.22}$$

by (5.18), (5.19), (5.12), and (5.6). Hence by Lemma 4, there are rational integers $p(\lambda_1, \cdots, \lambda_n)$, not all zero, satisfying (5.17) with

$$P < NU \leqq U\prod_{j=1}^{n}(L_j + 1)$$

which gives (5.16) thanks to (5.5) and (5.22).

LEMMA 7. *For real x, $0 \leqq x \leqq H^{\theta}$, $\sum_{j=1}^{n-1} m_j \leqq H^{1-\eta}$, we have*

$$\begin{aligned} &|\psi(x, m_1, \cdots, m_{n-1}) - \prod_{j=1}^{n-1}(b_n/\log\alpha_j)^{m_j}\phi_{m_1,\cdots,m_{n-1}}(x, \cdots, x)| \\ &< \exp\{-H + nAhH^{1-\delta+\theta} + 2H^{1-\eta}[\log(2h) + (1-\delta+\nu)\log H]\}\,, \end{aligned} \tag{5.23}$$

and

$$\begin{aligned} &|\prod_{j=1}^{n-1}(b_n^{-1}\log\alpha_j)^{m_j}\psi(x, m_1, \cdots, m_{n-1}) - \phi_{m_1,\cdots,m_{n-1}}(x, \cdots, x)| \\ &< \exp\{-H + nAhH^{1-\delta+\theta} + H^{1-\eta}[\log(4Ah^2) + 2(1-\delta+\nu)\log H]\}\,. \end{aligned} \tag{5.24}$$

Proof. For $0 \leqq y \leqq e^H$, we have from (5.2),

$$|y(\beta_1\log\alpha_1 + \cdots + \beta_{n-1}\log\alpha_{n-1}) - y\log\alpha_n| \leqq ye^{-H} \leqq 1$$

and hence by the mean value theorem,

$$\begin{aligned} |\prod_{j=1}^{n-1}\alpha_j^{\beta_j y} - \alpha_n^y| &= |\exp[y(\beta_1\log\alpha_1 + \cdots + \beta_{n-1}\log\alpha_{n-1})] - \exp(y\log\alpha_n)| \\ &\leqq |y(\beta_1\log\alpha_1 + \cdots + \beta_{n-1}\log\alpha_{n-1}) \\ &\quad - y\log\alpha_n|\exp(1 + y\log\alpha_n) \leqq y\alpha_n^y\exp(-H+1)\,. \end{aligned}$$

Next, we note that

$$\prod_{j=1}^{n}\alpha_j^{\lambda_j x} - \prod_{j=1}^{n-1}\alpha_j^{\gamma_j x} = \prod_{j=1}^{n-1}\alpha_j^{\lambda_j x}(\alpha_n^{\lambda_n x} - \prod_{j=1}^{n-1}\alpha_j^{\beta_j\lambda_n x})\,.$$

For $0 \leqq x \leqq H^{\theta}$, we have $xL_n \leqq hH^{\varepsilon-\delta+\theta} < e^H$ by (3.7). Hence for $0 \leqq x \leqq H^{\theta}$, it follows from the definitions in (5.8) and (5.9) as well as (5.12) that the difference on the left side of (5.23) is less than or equal to

$$\begin{aligned} &\prod_{j=1}^{n}(L_j + 1)P\cdot 1\prod_{j=2}^{n-1}\alpha_j^{L_j x}\cdot xL_n\alpha_n^{xL_n}\exp\{-H + 1 \\ &\quad + H^{1-\eta}[\log(2h) + (1-\delta+\nu)\log H]\} \end{aligned}$$

which, thanks to (5.5), (5.6), and (5.16) is in turn less than or equal to

$$ec^2h\exp\{-H+[2(n-1)(1-\eta)+2\zeta+(\varepsilon-\delta+\theta)]\log H$$
$$+(n-1)AhH^{1-\delta}(H^{\zeta}+H^{\theta})$$
$$+2H^{1-\eta}[\log(2h)+(1-\delta+\nu)\log H]\}.$$

This proves (5.23) thanks to (3.23) and (5.24) then follows from (5.23) and (5.14).

When $x \geqq 0$ is rational, $\psi(x)$ is an algebraic integer and we need an estimate for the size of its conjugates.

LEMMA 8. *If $x \geqq 0$ is rational and $\sum_{j=1}^{n-1} m_j \leqq H^{1-\eta}$ then*

$$\begin{aligned}&|\text{any conjugate of } \psi(x, m_1, \cdots, m_{n-1})|\\ &< c^2\exp\{2[(n-1)(1-\eta)+\zeta]\log H+(n-1)AhH^{1-\delta+\zeta}\\ &\quad +2H^{1-\eta}[\log(2h)+(1-\delta+\nu)\log H]+(n-1)AhH^{1-\delta}x\}\,.\end{aligned} \tag{5.25}$$

Proof. By (5.12), the left side is less than

$$\prod_{j=1}^{n}(L_j+1)P\cdot\left|\prod_{j=1}^{n}\alpha_j^{L_jx}\right|\cdot\exp\{H^{1-\eta}[\log(2h)+(1-\delta+\nu)\log H]\}$$

which is less than the right side thanks to (5.5), (5.6), and (5.16).

LEMMA 9. *For any rational integer l, $1 \leqq l \leqq H^{\theta}$, $\sum_{j=1}^{n-1} m_j \leqq H^{1-\eta}$, either*

$$|\phi_{m_1\cdots m_{n-1}}(l, \cdots, l)| < \exp\left(-\frac{1}{2}H\right) \tag{5.26}$$

or

$$\begin{aligned}&|\phi_{m_1,\cdots,m_{n-1}}(l, \cdots, l)| > \exp\{-2^n(n-1)AhH^{1-\delta+\zeta}\\ &\quad -2^{n+1}H^{1-\eta}[\log(2h)+(1-\delta+\nu)\log H]\\ &\quad -2^n(n-1)AhH^{1-\delta}l\}\,.\end{aligned} \tag{5.27}$$

Proof. If $\psi(l, m_1 \cdots, m_{n-1}) = 0$ then (5.24) and (3.12) imply (5.26). So suppose that $\psi(l, m_1, \cdots, m_{n-1}) \neq 0$. It is then an algebraic integer of degree $\leqq 2^n$ and norm $\geqq 1$. Hence by Lemma 8 and (5.15),

$$\begin{aligned}&\left|\prod_{j=1}^{n-1}(b_n^{-1}\log\alpha_j)^{m_j}\psi(l, m_1, \cdots, m_{n-1})\right|\\ &> c^{2-2^{n+1}}\exp\{(2-2^{n+1})[(n-1)(1-\eta)+\zeta]\log H\\ &\quad +(1-2^n)(n-1)AhH^{1-\delta+\zeta}+H^{1-\eta}[(2-2^{n+1})\log(2h)+\log A_0\\ &\quad +\{(2-2^{n+1})(1-\delta+\nu)-\nu\}\log H]+(1-2^n)(n-1)AhH^{1-\delta}l\}\,.\end{aligned} \tag{5.28}$$

Now (3.14) says precisely that the right side of (5.28) is greater than twice the right side of (5.24) and hence $|\phi_{m_1,\cdots,m_{n-1}}(l, \cdots, l)|$ is greater than 1/2 the right side of (5.28). Inequality (5.27) follows from this and (3.15).

We have one more simple estimate before the main two lemmas.

LEMMA 10. *For any complex z and $\sum_{j=1}^{n-1} m_j \leqq H^{1-\eta}$,*

$$|\phi_{m_1,\cdots,m_{n-1}}(z,\cdots,z)| < c^2 \exp\{2[(n-1)(1-\eta)+\zeta]\log H + (n-1)AhH^{1-\delta+\zeta} + H^{1-\eta}[\log(4Ah^2) + 2(1-\delta+\nu)\log H] + nAhH^{1-\delta}|z|\} . \tag{5.29}$$

Proof. We note that

$$\prod_{j=1}^{n-1} \alpha_j^{\gamma_j z} = \prod_{j=1}^{n} \alpha_j^{\lambda_j z} \exp[\lambda_n z(\beta_1 \log\alpha_1 + \cdots + \beta_{n-1}\log\alpha_{n-1} - \log\alpha_n)] .$$

Using this, (5.9), (5.12), (5.14), (5.2), (5.3), and (5.7),

$$|\phi_{m_1,\cdots,m_{n-1}}(z,\cdots,z)| < \exp(H^{1-\eta}\log A)\cdot\prod_{j=1}^{n}(L_j+1)\cdot P \cdot\exp\{(n-1)AhH^{1-\delta}|z| + hH^{1-\delta}|z| + hH^{\varepsilon-\delta}|z|e^{-H} + H^{1-\eta}[\log(2h) + (1-\delta+\nu)\log H]\} .$$

This plus (5.5), (5.16), and the fact that $A \geqq \pi$ gives (5.29).

LEMMA 11. *If J and l are rational integers, $1 \leqq J \leqq \theta/\zeta$, $1 \leqq l \leqq H^{\zeta J}$, $\sum_{j=1}^{n-1} m_j \leqq 2^{1-J}H^{1-\eta}$, then*

$$|\phi_{m_1,\cdots,m_{n-1}}(l,\cdots,l)| < \exp\left(-\frac{1}{2}H\right) . \tag{5.30}$$

Proof. We note that for $J=1$, this is true because of (5.24) of Lemma 7, (5.17) of Lemma 6, and (3.12). So we shall assume the truth of the lemma for $J = K \leqq \theta/\zeta - 1$ and prove it for $J = K+1$. Let

$$f(z) = \phi_{m_1,\cdots,m_{n-1}}(z,\cdots,z),\ f_m(z) = \frac{d^m}{dz^m}f(z) , \tag{5.31}$$

$$R_J = [H^{J\zeta}],\ S_J = [2^{1-J}H^{1-\eta}] , \tag{5.32}$$

$$F(z) = \{(z-1)\cdots(z-R_K)\}^{S_{K+1}+1} . \tag{5.33}$$

Since we are assuming that (5.30) is true for $1 \leqq l \leqq R_K$, $\sum_{j=1}^{n-1} m_j \leqq S_K$, and since $S_{K+1} \leqq S_K$, we see that with the restriction $1 \leqq l \leqq R_K$, the lemma is true for $J = K+1$. Hence in proving the lemma for $J = K+1$, it will suffice to consider only those l with

$$R_K < l \leqq R_{K+1} . \tag{5.34}$$

Since $m \leqq S_{K+1}$ and $\sum_{j=1}^{n-1} m_j \leqq S_{K+1}$ implies $m + \sum_{j=1}^{n-1} m_j \leqq S_K$, we see that the lemma for $J = K$ gives for $1 \leqq r \leqq R_K$, $0 \leqq m \leqq S_{K+1}$, $\sum_{j=1}^{n-1} m_j \leqq S_{K+1}$,

$$(5.35)\quad \begin{aligned} |f_m(r)| &= \left| \sum_{\substack{j_1=0 \\ j_1+\cdots+j_{n-1}=m}}^{m} \cdots \sum_{j_{n-1}=0}^{\mathfrak{z}} \frac{m!}{j_1!\cdots j_{n-1}!} \phi_{m_1+j_1,\cdots,m_{n-1}+j_{n-1}}(r,\cdots,r) \right| \\ &< (n-1)^m \exp\left(-\frac{1}{2}H\right) < \exp\left[-\frac{1}{2}H + H^{1-\eta}\log(n-1)\right]. \end{aligned}$$

We apply Lemma 5 with $\beta = l$, $R = R_K$, $S = S_{K+1}$, Γ centered at 0 with radius $H^{(K+2)\zeta}$, Γ_r centered at r with radius 1/2;

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-l)F(z)}dz - \frac{f(l)}{F(l)} = \frac{1}{2\pi i}\sum_{r=1}^{R_K}\sum_{m=0}^{S_{K+1}} \frac{f_m(r)}{m!}\int_{\Gamma_r}\frac{(z-r)^m}{(z-l)F(z)}dz\,.$$

For z on Γ_r,

$$\left|\frac{1}{F(z)}\right| < \frac{1}{\left(1\cdots 1\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot 1\cdots 1\right)^{S_{K+1}+1}} = 8^{S_{K+1}+1}$$

and hence by (5.35),

$$\begin{aligned} \left|\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-l)F(z)}dz - \frac{f(l)}{F(l)}\right| &< R_K(S_{K+1}+1)8^{S_{K+1}+1}\,\cdot \\ &\cdot\exp\left[-\frac{1}{2}H + H^{1-\eta}\log(n-1)\right]. \end{aligned}$$

By (3.5) $S_{K+1} + 1 \leqq S_2 + 1 < H^{1-\eta}$ and hence from (3.8),

$$(5.36)\quad \begin{aligned} &\left|\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-l)F(z)}dz - \frac{f(l)}{F(l)}\right| \\ &< \exp\left[-\frac{1}{2}H + (\theta+1-\eta)\log H + H^{1-\eta}\log(8n-8)\right] \\ &< \exp\left(-\frac{1}{3}H\right). \end{aligned}$$

If $|f(l)| < \exp(-H/2)$, there is nothing to prove and so we make the last assumption that $|f(l)| \geqq \exp(-H/2)$. The rest of the proof of the lemma is devoted to proving that this gives a contradiction. Thanks to (5.27) of Lemma 9, this assumption may be given the stronger form,

$$(5.37)\quad \begin{aligned} |f(l)| > \exp\{&-2^n(n-1)AhH^{1-\delta+\zeta} \\ &- 2^{n+1}H^{1-\eta}[\log(2h) + (1-\delta+\nu)\log H] \\ &- 2^n(n-1)AhH^{1-\delta}l\}\,. \end{aligned}$$

In particular, since $l \leqq R_{K+1} \leqq H^\theta$, we see from (3.13) that

$$(5.38)\quad |f(l)| > 2\exp\left(-\frac{1}{6}H\right).$$

Also

$$|F(l)| < R_{K+1}^{R_K(S_{K+1}+1)} \leqq \exp\left[(K+1)\zeta H^{K\zeta}(2^{-K}H^{1-\eta}+1)\log H\right]$$

$$\leqq \exp\left[\theta H^{\theta-\zeta}\log H + \theta(H^{\zeta}/2)^K H^{1-\eta}\log H\right] \tag{5.39}$$

since $K+1 \leqq \theta/\zeta$. In particular, by (3.4) and (3.1),

$$|F(l)| < \exp\left(\frac{1}{6}H\right). \tag{5.40}$$

From (5.38) and (5.40)

$$\left|\frac{f(l)}{F(l)}\right| > 2\exp\left(-\frac{1}{3}H\right)$$

and thus it follows from (5.36) that

$$\left|\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-l)F(z)}dz\right| > \frac{1}{2}\left|\frac{f(l)}{F(l)}\right|. \tag{5.41}$$

Let

$$\Phi = \min_{z\in\Gamma}|F(z)|,\quad \phi = \max_{z\in\Gamma}|f(z)|, \tag{5.42}$$

where $z \in \Gamma$ means $|z| = H^{(K+2)\zeta}$. Since by (3.4),

$$H^{(K+2)\zeta} \geqq 2H^{(K+1)\zeta} \geqq 2R_{K+1},$$

we see from (5.41) that

$$2\frac{\phi}{\Phi} > \frac{1}{2}\left|\frac{f(l)}{F(l)}\right|.$$

Therefore by the definitions in (5.33) and (5.32),

$$\begin{aligned}\frac{\phi}{|f(l)|} > \frac{1}{4}\frac{\Phi}{|F(l)|} &> \frac{1}{4}\frac{\left(\frac{1}{2}H^{(K+2)\zeta}\right)^{R_K(S_{K+1}+1)}}{(H^{(K+1)\zeta})^{R_K(S_{K+1}+1)}}\\ &\geqq \frac{1}{4}\left(\frac{1}{2}H^{\zeta}\right)^{R_K(S_{K+1}+1)}\\ &\geqq \exp\left[(H^{K\zeta}-1)\cdot 2^{-K}H^{1-\eta}(\zeta\log H\right.\\ &\qquad \left.-\log 2)-\log 4\right],\end{aligned}$$

which simplifies by (3.9), (3.10), and (3.11) to

$$\frac{\phi}{|f(l)|} > \exp\left(2^{-K-1}\zeta H^{K\zeta+1-\eta}\log H\right). \tag{5.43}$$

On the other hand, by Lemma 10 and the definitions in (5.42) and (5.31),

$$\begin{aligned}\phi < c^2\exp\{&2[(n-1)(1-\eta)+\zeta]\log H + (n-1)AhH^{1-\delta+\zeta}\\ &+ H^{1-\eta}[\log(4Ah^2) + 2(1-\delta+\nu)\log H]\\ &+ nAhH^{1-\delta+(K+2)\zeta}\}.\end{aligned}$$

This, along with (5.37) and the fact that $l \leqq R_{K+1} \leqq H^{(K+1)\zeta}$ by (5.34), gives

$$\frac{\phi}{|f(l)|} < c^2 \exp\{2[(n-1)(1-\eta)+\zeta]\log H + (2^n+1)(n-1)AhH^{1-\delta+\zeta} + H^{1-\eta}[(2^{n+1}+2)\log(2h) + \log A + (2^{n+1}+2)(1-\delta+\nu)\log H] + nAhH^{1-\delta+2\zeta+K\zeta}[1+n^{-1}H^{-\zeta}(n-1)\cdot 2^n]\}\,.$$

But this contradicts (5.43) thanks to (3.3) and completes the proof of the lemma for $J = K+1$. The lemma now follows by induction.

LEMMA 12. *For* $|z| \leqq H^{\theta}$ *and* $\sum_{j=1}^{n-1} m_j \leqq 2^{-\theta/\zeta} H^{1-\eta}$,

$$(5.44)\qquad |\phi_{m_1,\cdots,m_{n-1}}(z,\cdots,z)| < \exp[-2^{-\theta/\zeta-1}H^{1+\theta-\zeta-\eta}(\zeta\log H - \log 4)]\,.$$

Proof. We continue to use the $f(z)$ defined in (5.31) and the R_J and S_J defined in (5.32) for integral J. In addition for $K = \theta/\zeta - 1$ (which may not be an integer), we set

$$(5.45)\qquad R = [H^{K\zeta}],\ S = [2^{-1-K}H^{1-\eta}]\,,$$

$$(5.46)\qquad F(z) = \{(z-1)\cdots(z-R)\}^{S+1}\,.$$

Note that

$$(5.47)\qquad R \leqq R_{[K+1]},\ 2S \leqq S_{[K+1]}\,.$$

We will first prove the lemma for $z = \xi$, where $|\xi| = H^{\theta} = H^{(K+1)\zeta}$. We apply Lemma 5 with $\beta = \xi$, the R and S of Lemma 5 being the R and S of (5.45), Γ centered at 0 with radius $H^{(K+2)\zeta}$, Γ_r centered at r with radius 1/2:

$$(5.48)\qquad \frac{f(\xi)}{F(\xi)} = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(z)}{(z-\xi)F(z)}dz - \frac{1}{2\pi i}\sum_{r=1}^{R}\sum_{m=0}^{S}\frac{f_m(r)}{m!}\int_{\Gamma_r}\frac{(z-r)^m}{(z-\xi)F(z)}dz\,.$$

We apply Lemma 11 with $J = [K+1]$. Because of (5.47) we see that the argument of (5.35) is true for $1 \leqq r \leqq R$, $0 \leqq m \leqq S$ and $\sum_{j=1}^{n-1} m_j \leqq [2^{-\theta/\zeta}H^{1-\eta}] \leqq S$. Hence in this range

$$|f_m(r)| < (n-1)^m \exp\left(-\frac{1}{2}H\right) \leqq \exp\left[-\frac{1}{2}H + 2^{-1-K}H^{1-\eta}\log(n-1)\right].$$

Further, by (3.17)

$$S + 1 \leqq 2^{-1-K}H^{1-\eta} + 1 < 2^{-K}H^{1-\eta}\,,$$

and by (3.4), $|z-\xi| > 1/2$ if z is on Γ_r. Therefore, for $\sum_{j=1}^{n-1} m_j \leqq S$,

$$(5.49)\qquad \begin{aligned} &\left|\frac{1}{2\pi i}\sum_{r=1}^{R}\sum_{m=0}^{S}\frac{f_m(r)}{m!}\int_{\Gamma_r}\frac{(z-r)^m}{(z-\xi)F(z)}dz\right| \\ &< R(S+1)8^{S+1}\cdot\exp\left[-\frac{1}{2}H + 2^{-1-K}H^{1-\eta}\log(n-1)\right] \\ &< \exp\left[-\frac{1}{2}H + (K\zeta+1-\eta)\log H \right.\\ &\qquad \left. - K\log 2 + 2^{-K}H^{1-\eta}\log(8n-8)\right]. \end{aligned}$$

Further, if we use the ϕ of (5.42), by (5.46) we have

$$\left|\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-\xi)F(z)}dz\right| < 2\phi\left(\frac{1}{2}H^{(K+2)\zeta}\right)^{-R(S+1)}$$

and

$$|F(\xi)| < (2H^{(K+1)\zeta})^{R(S+1)} . \tag{5.50}$$

Therefore

$$\begin{aligned}|F(\xi)|\cdot\left|\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-\xi)F(z)}dz\right| &< 2\phi\left(\frac{1}{4}H^{\zeta}\right)^{-R(S+1)} \\ &< 2\phi\exp[-2^{-1-K}(H^{K\zeta} \\ &\quad -1)H^{1-\eta}(\zeta\log H-\log 4)] .\end{aligned} \tag{5.51}$$

Now, if we multiply (5.48) through by $F(\xi)$ and use (5.49), (5.50), (5.51), and Lemma 10 with $|z| = H^{(K+2)\zeta}$, we get

$$\begin{aligned}|f(\xi)| < \exp\Big[&-\frac{1}{2}H + (K\zeta+1-\eta)\log H - K\log 2 \\ &+ 2^{-K}H^{1-\eta}\log(8n-8) \\ &+ H^{K\zeta}(2^{-K-1}H^{1-\eta}+1)(\theta\log H+\log 2)\Big] \\ &+ 2c^2\exp\{-2^{-1-K}(1-H^{-K\zeta})H^{1+K\zeta-\eta}(\zeta\log H-\log 4) \\ &+ 2[(n-1)(1-\eta)+\zeta]\log H + (n-1)AhH^{1-\delta+\zeta} \\ &+ H^{1-\eta}[\log(4Ah^2) + 2(1-\delta+\nu)\log H] + nAhH^{1-\delta+(K+2)\zeta}\} .\end{aligned}$$

By (3.18) and (3.22), this proves the lemma for $z = \xi$ and the lemma now follows from the maximum modulus theorem.

LEMMA 13. *If q is an integer, $0 < q \leqq 2hH^{\varepsilon-\delta}$, and $\sum_{j=1}^{n-1} m_j \leqq 2^{-\theta/\zeta}H^{1-\eta}$, then*

$$\psi(1/q, m_1, \cdots, m_{n-1}) = 0 .$$

Proof. If we apply Lemma 12 and (5.15) to (5.23) of Lemma 7, we get

$$\begin{aligned}|\psi(1/q, m_1, \cdots, m_{n-1})| &< \exp[-2^{-\theta/\zeta-1}H^{1+\theta-\zeta-\eta}(\zeta\log H-\log 4) \\ &\quad + H^{1-\eta}(\nu\log H - \log A_0)] \\ &\quad + \exp\{-H + nAhH^{1-\delta+\theta} \\ &\quad + 2H^{1-\eta}[\log(2h) + (1-\delta+\nu)\log H]\} \\ &< \exp[-(2.1\cdot 2^{\theta/\zeta})^{-1}H^{1+\theta-\zeta-\eta}(\zeta\log H - \log 4)]\end{aligned} \tag{5.52}$$

by (3.19) and (3.20). By Lemma 8 and (3.21)

$$\begin{aligned}(5.53)\qquad &|\text{any conjugate of } \psi(1/q, m_1, \cdots, m_{n-1})| \\ &\quad < c^2 \exp\{2[(n-1)(1-\eta)+\zeta]\log H + (n-1)AhH^{1-\delta+\zeta} \\ &\qquad + 2H^{1-\eta}[\log(2h) + (1-\delta+\nu)\log H] + (n-1)AhH^{1-\delta}\} \\ &\quad < \exp\{2.1H^{1-\eta}[\log(2h)+(1-\delta+\nu)\log H]\}\,.\end{aligned}$$

Now $\psi(1/q)$ is in the field $Q(\sqrt{-D}, \alpha_1^{1/q}, \cdots, \alpha_n^{1/q})$ which has degree $\leqq 2^n q^n$ over Q. Thus $\psi(1/q)$ has at most $(2q)^n$ conjugates and hence by (5.52) and (5.53), the norm of $\psi(1/q)$ satisfies

$$\begin{aligned}|N\psi(1/q, m_1, \cdots, m_{n-1})| &< \exp\{-(2.1\cdot 2^{\theta/\zeta})^{-1}H^{1+\theta-\zeta-\eta}(\zeta\log H - \log 4) \\ &\qquad + 2.1(2q)^n H^{1-\eta}[\log(2h)+(1-\delta+\nu)\log H]\} < 1\end{aligned}$$

by (3.2). But $\psi(1/q)$ is an algebraic integer and therefore equals 0.

We can now show that all the $p(\lambda)=0$ thereby contradicting their definition and proving Theorem 5. Since $L_n > 2$ by (5.3) and (3.6), there is an odd prime p such that

$$L_n < p < 2L_n = 2hH^{\varepsilon-\delta}\,.$$

By Lemma 13, when $\sum_{j=1}^{n-1} m_j \leqq 2^{-\theta/\zeta}H^{1-\eta}$,

$$(5.54)\qquad \psi(1/p, m_1, \cdots, m_{n-1}) = 0\,.$$

In particular, if

$$(5.55)\qquad 0 \leqq m_j \leqq [L_j] \qquad (j = 1, \cdots, n-1)$$

then (5.54) holds by (5.3) and (3.16). By the definition of $\psi(1/p)$ in (5.8), we have

$$(5.56)\qquad \sum_{\lambda_n=0}^{[L_n]}\{\sum_{\lambda_1=0}^{[L_1]}\cdots\sum_{\lambda_{n-1}=0}^{[L_{n-1}]} p(\lambda_1, \cdots, \lambda_n)\prod_{j=1}^{n-1}(\alpha_j^{\lambda_j/p}\gamma_j^{m_j})\}(\alpha_n^{1/p})^{\lambda_n} = 0$$

for all m_j satisfying (5.55).

We apply Lemma 3 to (5.56) with $r = n-1$, $\varepsilon_j = \alpha_{j+1}$, $j = 1, \cdots, n-1$ and note that $Q(\alpha_1^{1/p}) = Q(e^{2\pi i/p})$. Since $p > [L_n]$, we see from Lemma 3 that for all λ_n,

$$(5.57)\qquad \sum_{\lambda_1=0}^{[L_1]}\{\sum_{\lambda_2=0}^{[L_2]}\cdots\sum_{\lambda_{n-1}=0}^{[L_{n-1}]} p(\lambda_1, \cdots, \lambda_n)\alpha_1^{\lambda_1/p}\prod_{j=2}^{n-1}(\alpha_j^{\lambda_j/p}\gamma_j^{m_j})\}\gamma_1^{m_1} = 0\,.$$

But by (5.10), we see that the determinant, $\det(\gamma_1^{m_1}) = \det\{(\lambda_1 + \lambda_n\beta_1)^{m_1}\}$; $0 \leqq \lambda_1 \leqq [L_1]$, $0 \leqq m_1 \leqq [L_1]$ is a Vandermonde determinant and thus not zero. It follows from (5.57) that for all λ_1 and λ_n,

$$(5.58)\qquad \sum_{\lambda_2=0}^{[L_2]}\{\sum_{\lambda_3=0}^{[L_3]}\cdots\sum_{\lambda_{n-1}=0}^{[L_{n-1}]} p(\lambda_1, \cdots, \lambda_n)\alpha_2^{\lambda_2/p}\prod_{j=3}^{n-1}(\alpha_j^{\lambda_j/p}\gamma_j^{m_j})\}\gamma_2^{m_2} = 0\,.$$

Again we have a Vandermonde determinant and if we continue this process, we arrive at

$$p(\lambda_1, \cdots, \lambda_n) = 0$$

for all $\lambda_1, \lambda_2, \cdots, \lambda_n$. This contradicts their definition in Lemma 6 and completes the proof of Theorem 5.

6. Proof of Theorems 1, 2, 3

In this section $d<0$ is the discriminant of a complex quadratic field, k is the discriminant of a quadratic field such that $(k, d) = 1$, $h(d)$ and $h(k)$ are the class-numbers of these fields, and if $k>0$, ε_k is the fundamental unit of $Q(\sqrt{k})$. We set $\omega_d = 3, 2,$ or 1 according as $d = -3$, $d = -4$, or $d < -4$ and we set $\chi_k(n) = (k/n)$ (Kronecker symbol). Let us write $d = g \cdot t$ where g and t are discriminants of quadratic fields (except that $g=1$, $t=d$ or $g=d, t=1$ is allowed) such that $kg>0$. If d has r distinct prime factors then there are 2^{r-1} such decompositions of d. Let

$$Q(x, y) = ax^2 + bxy + cy^2, \quad b^2 - 4ac = d, a > 0$$

be a positive definite binary quadratic form of discriminant d. By means of L-functions for quadratic forms, we obtained in Part I,

LEMMA 14. *Suppose the quadratic forms of discriminant d have exactly one class per genus. Assuming that $(k, d) = 1$ and that k has at least two distinct prime factors, we have*

$$\left|\frac{\chi_k(a)\,|k|}{6a}\prod\nolimits_{p\,|\,|k|}(1-p^{-2})\cdot\sqrt{d}\log(-1)+2^{2-r}\omega_d\sum\nolimits_{g,t}\chi_{g,t}(Q)h(kg)h(kt)\omega_{kt}^{-1}\log\varepsilon_{kg}\right|$$
$$< 4|k|\{1-\exp[-\pi\sqrt{|d|}/(a|k|)]\}^{-2}\exp[-\pi\sqrt{|d|}/(a|k|)]\,.$$

Here the product is extended over the distinct prime factors of $|k|$ and the $\chi_{g,t}$ are genus characters corresponding to the decomposition $d = gt$ above.

Proof. This is a special case of Lemma 15 of Part I.

Now suppose that $h(d) = 2$. Then there are two quadratic forms of discriminant d and two genera. One quadratic form is given by

$$Q_1(x, y) = \begin{cases} x^2 + \dfrac{|d|}{4}y^2 & \text{if } d \equiv 0 \pmod 4 \\ x^2 + xy + \dfrac{|d|+1}{4}y^2 & \text{if } d \equiv 1 \pmod 4 \end{cases}$$

and an inequivalent form may be taken to be

$$Q_2(x, y) = \begin{cases} 2x^2 + 2xy + \dfrac{p_2+1}{2}y^2 & \text{if } d = -p_1p_2,\ p_1 = 4,\ p_2 \text{ odd} \\ 2x^2 + p_2y^2 & \text{if } d = -p_1p_2,\ p_1 = 8,\ p_2 \text{ odd} \\ p_1x^2 + p_1xy + \dfrac{p_1+p_2}{4}y^2 & \text{if } d = -p_1p_2,\ p_1 < p_2 \text{ both odd.} \end{cases}$$

Let a denote the leading coefficient of $Q_2(m, n)$ so that $a \leqq p_1$. We choose k to be 21 if d is even, 28 if $p_1 = 3$ and 12 otherwise. Then for $|d| > 56$, we have $(k, d) = 1$. One of p_2 and $-p_2$ is the discriminant of a quadratic field; since it could be either, we set $d = gt$ where g and t are discriminants of quadratic fields, $g > 1$, $t < 0$. By Lemma 14, we have for $|d| > 56$,

$$\begin{aligned}(6.1)\qquad &\left|\frac{k}{6}\prod_{p|k}(1 - p^{-2})\sqrt{d}\log(-1) + h(k)h(kd)\log\varepsilon_k + h(kg)h(kt)\log\varepsilon_{kg}\right| \\ &< 4k[1 - \exp(-\pi\sqrt{|d|}/k)]^{-2}\exp(-\pi\sqrt{|d|}/k)\end{aligned}$$

and

$$\begin{aligned}(6.2)\qquad &\left|\frac{k}{6}\left(1 + \frac{\chi_k(a)}{a}\right)\prod_{p|k}(1 - p^{-2})\sqrt{d}\log(-1) + 2h(k)h(kd)\log\varepsilon_k\right| \\ &< 8k\{1 - \exp[-\pi\sqrt{|d|}/(ak)]\}^{-2}\exp[-\pi\sqrt{|d|}/(ak)]\,.\end{aligned}$$

In the case that d is even, we take $k = 21$, $a = 2$ in (6.2) and get the following linear form in two logarithms:

$$\begin{aligned}(6.3)\qquad &\left|16\sqrt{d}\log(-1) + 21h(21d)\log\left(\frac{5 + \sqrt{21}}{2}\right)\right| \\ &< 1764\,[1 - \exp(-\pi\sqrt{|d|}/42)]^{-2}\exp(-\pi\sqrt{|d|}/42)\,.\end{aligned}$$

Take $H = 3\sqrt{|d|}/42 = \sqrt{|d|}/14$. For $|d| > 10^{100}$, clearly the right side of (6.3) is $< e^{-H}$ and we may now apply Theorem 4 (we get a good estimate on the size of $21h(21d)$ from (6.3) also). Taking $n = 2$, $\nu = 1.1$, $\varepsilon = .99$, $\Delta = \varepsilon/176$, $A = \pi$ in Theorem 4, we see that if (6.3) is satisfied with $|d| > 10^{100}$ then

$$\sqrt{|d|}/14 = H \leqq \left(\frac{2^{32}\cdot 3^5\cdot 2.1^2\cdot\pi}{(.99/88)^2}\right)^{176/(14\cdot .99)} < 10^{216.7}$$

so that $|d| < 10^{440}$ and this takes care of even d. The case of even d has been included for completeness since Baker has already done this case in [4] getting $|d| < 10^{500}$ by use of Theorem A quoted in Section 1. Theorem B applied to his linear form would reduce his result to $|d| < 10^{260}$.

In the case that $3|d$, we take $k = 28$, $a = 3$ in (6.2) and get

$$\begin{aligned}(6.4)\qquad &|16\sqrt{d}\log(-1) + 7h(28d)\log(8 + 3\sqrt{7})| \\ &< 784\,[1 - \exp(-\pi\sqrt{|d|}/84)]^{-2}\exp(-\pi\sqrt{|d|}/84)\,.\end{aligned}$$

With $H = \sqrt{|d|}/28$ we find that for $|d| > 10^{100}$, the right side of (6.4) is $< e^{-H}$ and we may again apply Theorem 4 with $n = 2$, $\nu = 1.1$, $\varepsilon = .99$, $\Delta = \varepsilon/176$, $A = \pi$. As before we get $H < 10^{216.7}$ from wich we see that $|d| < 10^{440}$.

We now come to the main case that neither 2 nor 3 divides d and here we may choose $k = 12$, $a = p_1$, $g \leqq p_2$. In this case, we have

$$\begin{aligned}(6.5)\qquad &|4\sqrt{d}\log(-1) + 3h(12d)\log(2+\sqrt{3}) + 3h(12g)h(12t)\log\varepsilon_{12g}| \\ &< 144\,[1-\exp(-\pi\sqrt{|d|}/12)]^{-2}\exp(-\pi\sqrt{|d|}/12)\end{aligned}$$

and

$$\begin{aligned}(6.6)\qquad &|2[a+\chi_{12}(a)]\sqrt{d}\log(-1) + 3ah(12d)\log(2+\sqrt{3})| \\ &< 144a\{1-\exp[-\pi\sqrt{|d|}/(12a)]\}^{-2}\exp[-\pi\sqrt{|d|}/(12a)]\,.\end{aligned}$$

We first consider those d such that $a = p_1 \leqq |d|^{3/7}$. Let

$$H = \frac{\pi|d|^{1/14}}{12.01}\,.$$

If $|d| > 10^{980}$, then the right side of (6.6) is clearly less than e^{-H}. We now apply Theorem 5 with $n = 2$, $\alpha_2 = 2+\sqrt{3}$, $\varepsilon = .99$, $\nu = 14$, $A = \pi$. We pick the variables in Theorem 5 to be $H_0 = \exp(168)$, $\delta = .7575$, $\zeta = .059$, $\eta = .584$, $\theta = .6412$, $h = 3$, $c = 10$. Here (2.4)–(2.13) are satisfied and hence by Theorem 5, $H < H_0$ which means

$$|d| < [12.01\,\pi^{-1}\exp(168)]^{14} < 10^{1030}\,.$$

Incidentally, Theorem 4 applied to the same linear form gives

$$H \leqq \left(\frac{2^{32}\cdot 3^5\cdot 15^2\cdot\pi}{(.99/88)^2}\right)^{176/(14\cdot .99)} < 10^{238.3}$$

from which we obtain $|d| < 10^{3350}$.

Secondly, we consider those d such that $a = p_1 > |d|^{3/7}$. This means that both g and $|t|$ are between $|d|^{3/7}$ and $|d|^{4/7}$. Let

$$(6.7)\qquad H = \frac{\pi|d|^{1/2}}{12.01}\,.$$

If $|d| > 10^{980}$ then the right side of (6.5) is clearly less than e^{-H}. We now want to estimate the coefficients of the linear form in (6.5). We can, of course, use L-series but we can also do without. If we divide (6.6) through by a and note $a < |d|^{1/2}$, we get

$$3h(12d)\log(2+\sqrt{3}) < 4\pi|d|^{1/2} + 144\,[1-\exp(-\pi/12)]^{-2}\exp(-\pi/12)\,,$$

so that for $|d| > 10^{980}$,

$$h(12d) < 4\,|d|^{1/2}\,.$$

Now $\varepsilon_{12g} > \sqrt{12g} > e$ and thus if $|d| > 10^{980}$, we see from (6.5) that

$$3h(12g)h(12t) < 4\pi|d|^{1/2} + 12|d|^{1/2}\log(2+\sqrt{3}) + 1 < 30|d|^{1/2}\,.$$

Thus the coefficients of the logarithms in the linear form in (6.5) are, for $|d| > 10^{980}$, all less than $H^{1.01}$.

We also need to estimate ε_{12g} and here we need $L(s, \chi_{12g})$. As is well known,

$$\log \varepsilon_{12g} = \frac{(12g)^{1/2} L(1, \chi_{12g})}{2h(12g)} < (3g)^{1/2} \cdot 3 \log(12g) < 6g^{1/2} \log(12g) .$$

Since $g < |d|^{4/7}$, we find for $|d| > 10^{980}$ that

$$\log \varepsilon_{12g} < 6|d|^{2/7} \log(12|d|^{4/7}) < \left(\frac{\pi}{12.01} |d|^{1/2}\right)^{.582} .$$

Thus we may apply Theorem 5 to (6.5) with $n = 3$, $\alpha_2 = 2 + \sqrt{3}$, $\alpha_3 = \varepsilon_{12g}$, $\varepsilon = .418$, $\nu = 1.01$, $A = \pi$. We pick the variables in Theorem 5 to be $H_0 = \exp(1183)$, $\delta = .33118$, $\zeta = .00866$, $\eta = .2921$, $\theta = .30075$, $h = 2.1$, $c = 10$. Here (2.4)–(2.13) are satisfied and hence by Theorem 5, $H < H_0$ which means

$$|d| < [12.01\, \pi^{-1} \exp(1183)]^2 < 10^{1030} .$$

This completes the proof of Theorem 1.

Again, for comparison purposes, we apply Theorem 4 to the last linear form. The result is

$$H \leqq \left(\frac{2^{108} \cdot 2.01^4 \pi}{(.418/264)^4}\right)^{792/(46 \cdot .418)} < 10^{1874}$$

from which we get $|d| < 10^{3750}$. Thus from Theorem 4 we would obtain $|d| < 10^{3750}$ in place of the 10^{1030} in Theorem 1.

For Theorem 2, we assume $h(d) = 1$, $|d| > 8$. Then $(12, d) = 1$ and by Lemma 14, we have (using a form $Q(x, y)$ with $a = 1$),

$$\begin{aligned} &|2\sqrt{|d|} \log(-1) + 3h(12d) \log(2 + \sqrt{3})| \\ &< 72[1 - \exp(-\pi\sqrt{|d|}/12)]^{-2} \exp(-\pi\sqrt{|d|}/12) . \end{aligned} \tag{6.8}$$

Let H be given again by (6.7); even for $|d| > 10^{10}$, we see that the right side of (6.8) is less than e^{-H}. Further for $|d| > 10^{100}$, the coefficients of the logarithms in (6.8) are less than $H^{1.1}$ and $\log(2 + \sqrt{3}) < H^{.01}$. We thus apply Theorem 5 with $n = 2$, $\alpha_2 = 2 + \sqrt{3}$, $\varepsilon = .99$, $\nu = 1.1$, $A = \pi$. We pick the variables in Theorem 5 to be $H_0 = \exp(146)$, $\delta = .7612$, $\zeta = .0524$, $\eta = .5848$, $\theta = .636$, $h = 3$, $c = 10$. Here (2.4)–(2.13) are satisfied and hence by Theorem 5, $H < H_0$ which means

$$|d| < [12.01\pi^{-1} \exp(146)]^2 < 10^{130} .$$

Theorem 4 would yield

$$H < \left(\frac{2^{32} \cdot 3^5 \cdot 2.1^2 \cdot \pi}{(.99/88)^2}\right)^{176/(14 \cdot .99)} < 10^{216.7}$$

from which we obtain $|d| < 10^{435}$.

For Theorem 3, we again set $d = d_1d_2d_3$ where d_1, d_2, d_3 are prime discriminants, d_1 fixed and $|d_2| < |d_3|$ [cases (i)–(iv)] or $|d_1| < |d_2| < |d_3|$ [cases (v) and (vi)]. We see that for one of the choices $k = 12 = (-3)(-4)$, $k = 65 = 5\cdot 13$, $k = 77 = (-7)(-11)$, we have $(k, d_1d_2) = 1$ and hence $(k, d) = 1$ if $|d| > 13\cdot 11\cdot 8$ (since then $|d_3| > 13$). To begin with, we suppose none of the d_j is -4. In this case there are four quadratic forms

$$\begin{array}{lll} Q_{1,|d|} & \text{which represents } 1 & \text{and } |d|, \\ Q_{|d_1|,|d_2d_3|} & \text{which represents } |d_1| & \text{and } |d_2d_3|, \\ Q_{|d_2|,|d_1d_3|} & \text{which represents } |d_2| & \text{and } |d_1d_3|, \\ Q_{|d_3|,|d_1d_2|} & \text{which represents } |d_3| & \text{and } |d_1d_2|, \end{array}$$

and these forms are inequivalent. We apply Lemma 14 to the four forms. If $|d_3| < |d|^{(1-\delta)/2}$ or $|d_1d_2| < |d|^{(1-\delta)/2}$ we add the four results so as to get a linear form in $\log(-1)$ and $\log \varepsilon_k$. If $d_1 > 0$, $|d|^{(1-\delta)/2} \leqq |d_3| \leqq |d|^{(1+\delta)/2}$ but $|d_2| < |d|^{1/2-\delta}$, we may combine the results from the first three quadratic forms so as to get a linear form in three logarithms; $\log(-1)$, $\log \varepsilon_k$ and ε_{kd_1}. If $d_1 > 0$ and $|d|^{1/2-\delta} \leqq |d_2| < |d_3| \leqq |d|^{1/2+\delta}$ then we may use the first two forms to get a linear form in the four logarithms: $\log(-1)$, $\log \varepsilon_k$, $\log \varepsilon_{kd_1}$ and either $\log \varepsilon_{kd_3}$ $(d_3 > 0)$ or $\log \varepsilon_{kd_1d_2}$ $(d_3 < 0)$. In each of these cases Theorem 4 applies although we must use very small values of δ in the last case since A may be as large as $3|d_1|^{1/2} \log |d_1|$ (the ε of Theorem 4 in this case being slightly smaller than 1/2). Note that just the existence theorem, Theorem 1, of Part I would not suffice to show that this case can be done since A varies with d. This completes cases (i) and (v) of Theorem 3.

Suppose now that $d_1 < 0$, $0 < d_2 \leqq |d|^{1/3-\delta}$ and $|d|^{(1-\delta)/2} \leqq d_3 \leqq |d|^{(1+\delta)/2}$. We may use the first three quadratic forms to get a linear form in the three logarithms: $\log(-1)$, $\log \varepsilon_k$, and $\log \varepsilon_{kd_2}$. Theorem 4 applies to this linear form. This gives us the $d_2 > 0$ part of case (vi). If $d_1 < 0$ and $d_2 < 0$ then $d_3 < 0$ also and we now consider this situation.

The key to further progress is that when we combine the results for two different quadratic forms from Lemma 14 so as to eliminate one logarithm, we actually eliminate two logarithms (the second logarithm is not free to be chosen as it is when we use three quadratic forms). Consider the values of the $\chi_{g,t}$ shown in Table 1 $\big(+$ means $\chi_{g,t}(Q) = 1$ and $-$ means $\chi_{g,t}(Q) = -1\big)$. The unfilled rows and columns all have two $+'^s$ and two $-'^s$. Because of this, the diagonal of the table cannot have exactly $3\ +'^s$. Further if the diagonal has two $+'^s$ or four $+'^s$, the matrix of $+'^s$ and $-'^s$ is symmetric while if the diagonal has one $+$ and three $-'^s$, the matrix is not symmetric. If $d_1 < 0$, $d_2 < 0$, $d_3 < 0$ then at least two of them are odd, say d_2 and d_3, and

	$\chi_{1,d}$	χ_{d_1,d_2d_3}	χ_{d_2,d_1d_3}	χ_{d_3,d_1d_2}
$Q_{1,\|d\|}$	+	+	+	+
$Q_{\|d_1\|,\|d_2d_3\|}$	+			
$Q_{\|d_2\|,\|d_1d_3\|}$	+			
$Q_{\|d_3\|,\|d_1d_2\|}$	+			

TABLE 1

by the quadratic reciprocity law,

$$\chi_{d_3,d_1d_2}(Q_{|d_2|,|d_1d_3|}) = \left(\frac{d_3}{|d_2|}\right) = -\left(\frac{d_2}{|d_3|}\right) = -\chi_{d_2,d_1d_3}(Q_{|d_3|,|d_1d_2|}) \,.$$

Thus if $d_1 < 0$, $d_2 < 0$, $d_3 < 0$ then Table 1 is not symmetric and the remaining diagonal elements are all $-$. In particular

$$\chi_{d_1,d_2d_3}(Q_{|d_1|,|d_2d_3|}) = -1 \,. \tag{6.9}$$

The remaining two elements of the second row consist of a $+$ and a $-$ but not necessarily in that order. In any event when we add the results of Lemma 14 for the first two forms, we get a linear form in three logarithms: $\log(-1)$, $\log \varepsilon_k$, and either $\log \varepsilon_{kd_1d_3}$ or $\log \varepsilon_{kd_1d_2}$. When $|d|^{(1-\delta)/2} \leqq |d_3| \leqq |d|^{(1+\delta)/2}$, this may be handled whenever $|d_1| \leqq |d|^{1/6-\delta}$. This completes case (vi).

The condition $|d_1| \leqq |d|^{1/6-\delta}$ is automatic if we are considering the problem for fixed $d_1 < 0$, e.g. $d_1 = -3$ or $d_1 = -8$. Thus from case (vi) we see that for fixed d_1, the difficulties come with $d_3 \approx |d|^{1/2}$ and thus also $d_2 \approx |d|^{1/2}$; in other words $d_2 > 0$, $d_3 > 0$ and $|d|^{1/2-\delta} \leqq d_2 < d_3 \leqq |d|^{1/2+\delta}$. Here if (6.9) holds, we get a linear form with three logarithms: $\log(-1)$, $\log \varepsilon_k$, and either $\log \varepsilon_{kd_2}$ or $\log \varepsilon_{kd_3}$. This case can be covered by Theorem 4. We are left with

$$\chi_{d_1,d_2d_3}(Q_{|d_1|,d_2d_3}) = 1 \,. \tag{6.10}$$

For example if $d_1 = -8$ we must have $d_2d_3 \equiv 1 \pmod 8$ and then the rest of the second row of Table 1 gives $d_2 \equiv d_3 \equiv 5 \pmod 8$. All other cases of $d_1 = -8$ can be settled. We have now completed cases (iii) and (iv).

Now suppose that $d_1 = -4$ so that $d = -4d_2d_3$ where $|d_2| < |d_3|$ and d_2 and d_3 are both odd and both positive or both negative. In this case we have four quadratic forms,

$Q_{1,\|d\|}$	which represents 1	and $\|d\|$,
$Q_{2,2d_2d_3}$	which represents 2	and $2d_2d_3$,
$Q_{\|d_2\|,4\|d_3\|}$	which represents d_2	and $4d_3$,
$Q_{2\|d_2\|,2\|d_3\|}$	which represents $2d_2$	and $2d_3$,

and these forms are inequivalent. Table 1 is now replaced by Table 2. When $|d_2| < |d|^{1/2-\delta}$, we may use all four quadratic

	$\chi_{1,d}$	χ_{-4,d_2d_3}	$\chi_{d_2,-4d_3}$	$\chi_{d_3,-4d_2}$
$Q_{1,\|d\|}$	+	+	+	+
$Q_{2,2d_2d_3}$	+			
$Q_{\|d_2\|,4\|d_3\|}$	+			
$Q_{2\|d_2\|,2\|d_3\|}$	+			

TABLE 2

forms to get a linear form in $\log(-1)$ and $\log \varepsilon_k$. The problem of $|d|^{1/2-\delta} \leqq |d_2| < |d_3| \leqq |d|^{1/2+\delta}$ remains. Here again,

$$\chi_{-4,d_2d_3}(Q_{2,2d_2d_3}) = 1 \tag{6.11}$$

is the hard case. But when (6.11) holds, the last two entries of the second column of Table 2 are $-$'s and in particular

$$\left(\frac{-4}{|d_2|}\right) = -1 .$$

Thus d_2 and d_3 are negative. Using the second row of Table 2, which is now $+ + - -$, we find that the problem of $d = -4d_2d_3$ can be completely settled except when

$$d_2 < 0,\ d_3 < 0,\ d_2 \equiv d_3 \equiv 5 \pmod 8,\ |d|^{1/2-\delta} \leqq |d_2| < d_3 \leqq |d|^{1/2+\delta} .$$

This completes case (ii) and the proof of Theorem 3.

We close by noting that negative values of k do not allow Theorems 4 or 5 to be applied to any of the remaining cases since ε_k is replaced by the much worse ε_{kd} and this more than compensates for the improvement that we get in $\varepsilon_{kd_2d_3}$ being replaced by ε_{kd_1} when $d_1 < 0$. However, negative values of k may be of use if sufficiently strong theorems on linear forms with two large logarithms could be proved.

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