Stochastic Processes (STA3021)
HW2 Solution

1. Straightforward calculation.

$$\int_0^3 \left(\int_1^2 x^2 y dy \right) dx = \int_1^2 \left(\int_0^3 x^2 y dx \right) dy = \frac{27}{2},$$

   and can verify Fubini's theorem.

2. Two methods are possible. First, use that the density function is given by

$$f(t) = 0.012e^{-0.02t} + 0.012e^{-0.03t},\ t \geq 0,$$

   and calculate EX and $\mathrm{Var}X$. Note that

$$EX = \int_0^\infty t(0.012e^{-0.02t} + 0.02e^{-0.03t})dt = \int_0^\infty 0.012te^{-0.02t}dt + \int_0^\infty 0.02te^{-0.03t}dt.$$

   Using integration by parts or from that Gamma(α, β) random variable integrates to 1, it equals to

$$0.012\Gamma(2)(\frac{1}{0.02^2}) + 0.012\Gamma(2)(\frac{1}{0.03^2}) = 43.3333.$$

   Similarly,

$$EX^2 = \int_0^\infty t^2(0.012e^{-0.02t} + 0.02e^{-0.03t})dt = \int_0^\infty 0.012t^2e^{-0.02t}dt + \int_0^\infty 0.02t^2e^{-0.03t}dt$$

$$= 0.012\Gamma(3)(\frac{1}{0.02^3}) + 0.012\Gamma(3)(\frac{1}{0.03^3}) = 3888.9.$$

   Thus, $\mathrm{Var}X = EX^2 - (EX)^2 = 2011.13$.

   The second calculation is based on the expected formula for a non-negative random variable and its extension to differentiable function g with $g(0) = 0$ (See Exercise 2.48 on page 85)

$$E(g(X)) = \int_0^\infty P(X > t)g'(t)dt.$$

   If you use this formula, we have that

$$EX = \int_0^\infty .6e^{-.02t} + .4e^{-.03t}dt = \frac{.6}{.02} + \frac{.4}{.03} = 43.33$$

$$EX^2 = \int_o^\infty 2t\left(.6e^{-.02t} + .4e^{-.03t}\right)dt = 3888.9,$$

   which is identical to previous calculation.

3. (a) Jump occurs at $x = 1$

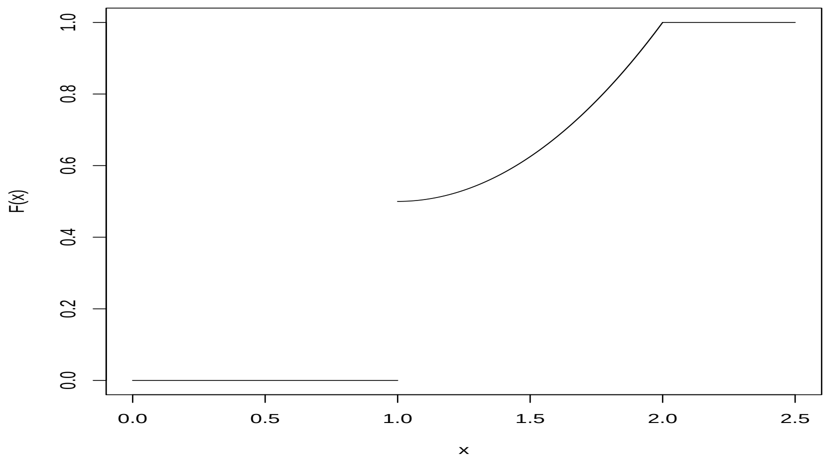

Figure 1: Distribution function

(b) Since,

$$P(X = 1) = \frac{1}{2},$$

$$f_C(x) = \begin{cases} x - 1 & \text{, if } 1 < x < 2; \\ 0 & \text{, otherwise.} \end{cases}$$

$$EX = 1 * P(X = 1) + \int_1^2 x f_C(x) dx = 1 * \frac{1}{2} + \int_1^2 x(x-1) dx = \frac{4}{3}$$

$$EX^2 = 1^2 * P(X = 1) + \int_1^2 x^2 f_C(x) dx = 1 * \frac{1}{2} + \int_1^2 x^2(x-1) dx = \frac{23}{12}$$

so,

$$\mathrm{Var}X = EX^2 - (EX)^2 = \frac{5}{36}.$$

4. Chapter 2 Exercise # 34

(a) The density function should add up to 1, that is $\int_{\mathbb{R}} f(x)dx = 1$, thus

$$\int_0^2 c(4x - 2x^2)dx = 4c\,\frac{1}{2}x^2\Big|_0^2 - 2c\,\frac{1}{3}x^3\Big|_2^0 = \frac{8}{3}c = 1$$

gives $c = \frac{3}{8}$.

(b) The probability is the area under density curve, hence

$$P\left(\frac{1}{2} < X < \frac{3}{2}\right) = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{3}{8}(4x - 2x^2)dx$$

$$= \frac{3}{2}\frac{1}{2}x^2\Big|_{\frac{1}{2}}^{\frac{3}{2}} - \frac{3}{4}\frac{1}{3}x^3\Big|_{\frac{1}{2}}^{\frac{3}{2}} = \frac{11}{16}$$

5. Chapter 2 Exercise # 68

   (a) Straightforward calculation using

   $$f_X(x) = \int f_{X,Y}(x,y)dy = \int_x^\infty \lambda^2 e^{-\lambda y}dy = \lambda^2 \frac{-1}{\lambda}e^{-\lambda y}\Big|_x^\infty = \lambda e^{-\lambda x}\ , \quad 0 < x < \infty.$$

   (b) Similarly,

   $$f_Y(y) = \int f_{X,Y}(x,y)dx = \int_0^y \lambda^2 e^{-\lambda y}dx = \lambda^2 e^{-\lambda y}y\ , 0 < y < \infty.$$

   (c) We can use the transformation method to derive the joint pdf of X and W. Let $W = Y - X, X = X$, then the Jacobian is

   $$|J| = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = 1.$$

   Therefore,

   $$f_{X,W}(x,w) = |J|^{-1} f_{X,Y}(x, x+w) = \lambda^2 e^{-\lambda(x+w)}, \quad 0 < x < \infty,\ 0 < w < \infty.$$

   (d) In (c), the joint pdf of X and W can be factorized as a function of $g(x)$ and $h(w)$. It means that X and W are independent, hence the density of W is given by

   $$f_W(w) = \lambda e^{-\lambda w}, \quad 0 < w < \infty.$$