⊙ **Abel's Limit Theorem**

Recall: Suppose $R \in (0,\infty)$ is the radius of convergence of the power series $f(x) := \sum_{n=0}^{\infty} a_n x^n$

Then we know that the convergence is uniform on any compact interval $[-r,r]$ with $0 < r < R$, and hence that f is continuous on $(-R,R)$

Remark. If $\sum_{n=0}^{\infty} a_n x^n$ converges **absolutely** at $x = R$, then M-test$\left(\text{with } M_n = |a_n| R^n\right)$ shows that $\sum_{n=0}^{\infty} a_n x^n$ converges (absolutely and) uniformly on $[-R,R]$, so its sum is continuous there

What happens if the convergence at $x = R$ is only **conditional**? [question below]

Question: Assume the p.s. $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges at one of the endpoints, say $x = R$

$$\Rightarrow \begin{cases} \text{the convergence is uniform on } (0,R]? \\ f(x) \text{ is continuous on } (0,R]? \end{cases}$$

Ans is Yes by Abel [The necessary tool is the summation-by-prats formula]

Abel's Limit Theorem:

If $\sum_{n=0}^{\infty} a_n$ converges (i.e., if $\sum_{n=0}^{\infty} a_n x^n$ converges at $x=1$), then

$$\boxed{\lim_{x\to 1^-} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n}$$

Similarly, if $\sum_{n=0}^{\infty} (-1)^n a_n$ converges (i.e., if $\sum_{n=0}^{\infty} a_n x^n$ converges at $x=-1$), then

$$\boxed{\lim_{x\to -1^+} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (-1)^n a_n}$$

Remark: If $\sum_{n=0}^{\infty} a_n x^n$ converges at $x=R$, then $\lim_{x\to R^-} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n R^n$

Similarly, if $\sum_{n=0}^{\infty} a_n x^n$ converges at $x=-R$, then $\lim_{x\to -R^+} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n (-R)^n$

Pf. (Optional) Since $\sum_{k=0}^{\infty} a_k$ converges, we know that $\sum_{k=0}^{\infty} a_k x^k$ converges for $|x|<1$ (by the Key property of the power series). Set $S_n = \sum_{k=0}^{n} a_k$. Then $S_n \to \sum_{k=0}^{\infty} a_k := S$, and so S_n is bounded; say $|S_n| \le M$ for $\forall n \ge 0$. So

$$\sum_{k=0}^{\infty} |S_k x^k| \le M \sum_{k=0}^{\infty} |x|^k \text{ converges if } |x|<1$$

$$\therefore \sum_{k=0}^{\infty} S_k x^k \text{ converges (absolutely) for } |x|<1$$

Write $f(x)=\sum_{k=0}^{\infty} a_k x^k$ for $|x|<1$. Then

$$f(x)=S_0+\sum_{k=1}^{\infty}(S_k-S_{k-1})x^k=\sum_{k=0}^{\infty} S_k x^k - x\sum_{k=1}^{\infty} S_{k-1}x^{k-1}=\sum_{k=0}^{\infty} S_k x^k - x\sum_{k=0}^{\infty} S_k x^k=(1-x)\sum_{k=0}^{\infty} S_k x^k$$

Notice that $(1-x)\sum_{k=0}^{\infty} x^k=1$ for $|x|<1$. So for $|x|<1$, we get

$$f(x)-S=(1-x)\sum_{k=0}^{\infty}(S_k-S)x^k$$

Since $S_n \to S$, given $\varepsilon>0$ we can find n_0 so that $|S_n-S|<\varepsilon$ for $n>n_0$. Then for $0<x<1$,

$$\begin{aligned}
\left|f(x)-S\right| &\le (1-x)\sum_{k=0}^{n_0}\left|S_k-S\right||x|^k+(1-x)\sum_{k=n_0+1}^{\infty}\varepsilon x^k \\
&\le (1-x)\sum_{k=0}^{n_0}\left|S_k-S\right|+(1-x)\cdot\varepsilon x^{n_0+1}(1-x)^{-1} \\
&\le (1-x)\underbrace{\sum_{k=0}^{n_0}\left|S_k-S\right|}_{\text{fixed \& indep of } x}+\varepsilon
\end{aligned}$$

Letting $x\to 1^-$ gives $\limsup_{x\to 1^-}|f(x)-S|\le\varepsilon$.

Since $\varepsilon>0$ was arbitrary, $\limsup_{x\to 1^-}|f(x)-S|=0$.

Therefore, $\lim_{x\to 1^-} f(x)=S$.

Alternative statement of Abel's Limit Theorem:

Assume that the power series $f(x)=\sum_{k=0}^{\infty} a_k x^k$ converges at $x=1$ [$\Rightarrow$ converges for each $|x|<1$]

[i.e., assume that $\sum_{k=0}^{\infty} a_k$ converges]

Then $\sum_{k=0}^{n} a_k x^k \rightrightarrows f(x)$ on the compact interval $[0,1]$, i.e., $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly on $[0,1]$

Thus, f is continuous on $[0,1]$. In particular, f is (left) continuous at $x=1$,

$$\text{so } \lim_{x\to 1^-} f(x)=\sum_{k=0}^{\infty} a_k \left(=f(1)\right) \qquad \text{i.e., } \lim_{x\to 1^-}\sum_{k=0}^{\infty} a_k x^k=\sum_{k=0}^{\infty} a_k$$

Pf. Hypo: $\sum_{k=0}^{\infty} a_k$ converges

Goal: $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly on $[0,1]$

Need to show: $\sup_{0\le x\le 1}\left|\sum_{k=0}^{n-1} a_k x^k-\sum_{k=0}^{\infty} a_k x^k\right|=\sup_{0\le x\le 1}\left|\sum_{k=n}^{\infty} a_k x^k\right|\to 0$ as $n\to\infty$

Hypo$\left(: \sum_{k=0}^{\infty} a_k \text{ converges}\right)$ says $\sum_{k=0}^{n-1} a_k \to \sum_{k=0}^{\infty} a_k$; i.e., $\sum_{k=n}^{\infty} a_k = \sum_{k=0}^{\infty} a_k - \sum_{k=0}^{n-1} a_k \to 0$

For $n \geq 1$, put $B_n = \sum_{k=n}^{\infty} a_k = a_n + a_{n+1} + \cdots \left[= n\text{th tail end of } \sum_{k=0}^{\infty} a_k\right]$, so that $a_n = B_n - B_{n+1}$

$\Rightarrow \quad B_n \to 0 \left[\leftrightarrow \lim_{n\to\infty} B_n = 0\right]$

Now, for any x, $0 \leq x < 1$, we have

$$
\begin{aligned}
\sum_{k=n}^{\infty} a_k x^k &= a_n x^n + a_{n+1} x^{n+1} + \cdots \\
&= (B_n - B_{n+1}) x^n + (B_{n+1} - B_{n+2}) x^{n+1} + \cdots \\
&= B_n x^n + B_{n+1}(x^{n+1} - x^n) + B_{n+2}(x^{n+2} - x^{n+1}) + \cdots : \text{summation-by parts formula} \\
&= B_n x^n + (x-1)x^n \left\{B_{n+1} + B_{n+2} x + \cdots\right\}
\end{aligned}
$$

Given $\varepsilon > 0$, choose N so that $|B_j| < \varepsilon$ whenever $j \geq N$ $\left[\leftarrow \lim_{n\to\infty} B_n = 0\right]$

Then, for $0 \leq x < 1$, and $n \geq N$,

$$
\begin{aligned}
\left|\sum_{k=n}^{\infty} a_k x^k\right| &\leq |B_n| x^n + |x-1| x^n \left\{|B_{n+1}| + |B_{n+2}| x + \cdots\right\} \\
&\leq \varepsilon x^n + (1-x)x^n \left\{\varepsilon + \varepsilon x + \varepsilon x^2 + \cdots\right\} \\
&= \varepsilon x^n + \varepsilon(1-x)x^n \left\{1 + x + x^2 + \cdots\right\} = \varepsilon x^n + \varepsilon(1-x)x^n \frac{1}{1-x} \\
&= 2\varepsilon x^n < 2\varepsilon
\end{aligned}
$$

This also holds when $x = 1$, $\left[\leftarrow \left|\sum_{k=n}^{\infty} a_k\right| = |B_n| < \varepsilon < 2\varepsilon \text{ if } n \geq N\right]$

$$\therefore \quad \sup_{0\leq x\leq 1} \left|\sum_{k=n}^{\infty} a_k x^k\right| < 2\varepsilon \quad \text{for all } n \geq N$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\sup_{0\leq x\leq 1} \left|\sum_{k=n}^{\infty} a_k x^k\right| \to 0 \quad \text{as } n \to \infty$$

$$\therefore \quad \sum_{k=0}^{\infty} a_k x^k \quad \text{converges uniformly on } 0 \leq x \leq 1$$

Applications of Abel's Limit Theorem

Exa 1. Use Abel's Limit Theorem to show that $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} \quad (=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots) \quad = \quad \ln 2$

Sol. Notice that $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots \quad = \quad x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\bigg|_{x=1}$

We start with an obvious fact:

$$1-x+x^2-x^3+\cdots=\frac{1}{1+x} \quad \text{for} \quad |x|<1 \quad \text{(\& the radius of convergence of LHS = 1)}$$

Recall that any power series can be integrated term-by-term within its radius of convergence.

Hence integrating ($\int_0^x$) both sides gives

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots \quad = \quad \ln(1+x) \quad \text{for} \quad |x|<1$$

Everybody knows that $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ is convergent by the Alternating series test.

This means $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$ converges at $x=1$. Thus, by Abel's limit theorem

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots \quad = \quad \lim_{x\to 1^-}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right) \quad = \quad \lim_{x\to 1^-}\ln(1+x) \quad = \quad \ln 2$$

Exa 2. Use Abel's limit theorem to show that $\sum_{n=0}^{\infty}\frac{(-1)^{n}}{2n+1} \quad (=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots) \quad = \quad \pi/4$

Sol. Notice that $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots \quad = \quad x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots\bigg|_{x=1}$

We start with an obvious fact:

$$1-x^2+x^4-x^6+\cdots=\frac{1}{1+x^2} \quad \text{for} \quad |x|<1 \quad \text{(\& the radius of convergence = 1)}$$

Integrating ($\int_0^x$) both sides gives

$$x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots \quad = \quad \tan^{-1}x \quad \text{for} \quad |x|<1$$

Note that $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$ is convergent by the Alternating series test.

This means $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$ converges at $x=1$. Thus, by Abel's limit theorem

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots \quad = \quad \lim_{x\to 1^-}\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots\right) \quad = \quad \lim_{x\to 1^-}\tan^{-1}x \;=\; \tan^{-1}1 \;=\; \frac{\pi}{4}$$

Home Study. Show that $\sum_{n=0}^{\infty}\frac{(-1)^{n}}{3n+1} \quad (=1-\frac{1}{4}+\frac{1}{7}-\frac{1}{10}+\cdots) \quad = \quad \frac{1}{3}\left(\ln 2+\frac{\pi}{\sqrt{3}}\right)$

Exa 3(tricky). Show that $1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\cdots+\frac{1}{6n-5}-\frac{1}{6n-1}+\cdots \quad = \quad \frac{\pi}{2\sqrt{3}}$

Sol. Let $f(x):=x-\frac{x^5}{5}+\frac{x^7}{7}-\cdots+\frac{x^{6n-5}}{6n-5}-\frac{x^{6n-1}}{6n-1}+\cdots$.

Note that RHS converges at $x=1$ by the Alternating series test.

So the power series converges absolutely for $|x|<1$, and hence any rearrangement of the series converges to the same sum. In particular, we have

$$f(x)=\left(x+\frac{x^7}{7}+\cdots+\frac{x^{6n-5}}{6n-5}+\cdots\right)-\left(\frac{x^5}{5}+\frac{x^{11}}{11}-\cdots+\frac{x^{6n-1}}{6n-1}+\cdots\right) \quad \text{for} \quad |x|<1$$

Recall that any power series can be differentiated term-by-term within its interval of convergence.

Hence, for $|x|<1$, we have

$$\begin{aligned} f'(x) &= \left(1+x^6+\cdots+x^{6n-6}+\cdots\right)-\left(x^4+x^{10}-\cdots+x^{6n-2}+\cdots\right) \\ &= \frac{1}{1-x^6}-\frac{x^4}{1-x^6} = \frac{1-x^4}{1-x^6} = \frac{1+x^2}{1+x^2+x^4} \\ &= \frac{1}{2}\left(\frac{1}{1+x+x^2}+\frac{1}{1-x+x^2}\right) \end{aligned}$$

Integrating both sides over $[0,x]$ $(0<x<1)$, together with $f(0)=0$, gives

$$f(x) \overset{\text{check}}{=} \frac{1}{\sqrt{3}}\left(\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\right) \quad (0<x<1)$$

Now applying Abel's limit theorem shows

$$\begin{aligned} 1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\cdots+\frac{1}{6n-5}-\frac{1}{6n-1}+\cdots &= \lim_{x\to 1^-} f(x) \\ = \frac{1}{\sqrt{3}}\left(\tan^{-1}(1/\sqrt{3})+\tan^{-1}(\sqrt{3})\right) &\overset{\text{easy}}{=} \frac{\pi}{2\sqrt{3}} \end{aligned}$$