3.1 Definition of Limit

"the limit of $\{a_n\}$ is L" == "a_n is a good approximation to L, when n is large"

- The number L is the limit of the sequence $\{a_n\}$ if, given $\epsilon > 0$, $a_n \underset{\epsilon}{\approx} L$ for $n \gg 1$

☆ convergent : $\lim\limits_{n\to\infty} \{a_n\} = L$ Divergent : $\lim\limits_{n\to\infty} \{a_n\} = \infty$

→ "the approximation can be made as close as desired, provided we go far enough out in the sequence - the smaller ϵ is, the farther out we must go, in general."

☆ Note that N depends on ϵ ; the smaller ϵ is, the bigger N is the further out you must go for the approximation to be valid within ϵ

3.2 The Uniqueness of Limits. The K-ϵ Principle

Theorem : Uniqueness Theorem for Limits

- A sequence a_n has at most one limit : $a_n \to L$ and $a_n \to L'$ => $L = L'$

Theorem :

- $\{a_n\}$ increasing, $L = \lim\limits_{n\to\infty} a_n \Rightarrow a_n \le L$ for all n
- $\{a_n\}$ decreasing, $L = \lim\limits_{n\to\infty} a_n \Rightarrow a_n \ge L$ for all n

The K-ϵ principle

- It often happens in analysis that arguments turn out to involve not just ϵ but a constant multiple of it. This may occur for instance when the limit involves a sum or several arithmetic processes
- If you come out in the end with 2ϵ, or 22ϵ, that's just as good as coming out with ϵ

- Suppose that $\{a_n\}$ is a given sequence, and you can prove that, given any $\epsilon > 0$, $a_n \underset{K\epsilon}{\approx} L$ for $n \gg 1$ where $K > 0$ is a fixed constant. Then $\lim\limits_{n\to\infty} a_n = L$

3.3 Infinite Limits

- We say the sequence $\{a_n\}$ tends to infinity if, given any $M > 0$, $a_n > M$ for $n \gg 1$
  == $\lim\limits_{n\to\infty} a_n = \infty$

3.4 An Important Limit

- The limit of a^n : $\lim\limits_{n\to\infty} a^n = \begin{cases} \infty, & \text{if } a > 1 \\ 1, & \text{if } a = 1 \\ 0, & \text{if } |a| < 1 \end{cases}$