STA 3021: Stochastic Processes
Final (6:15 PM - 7:30 PM on Dec 17, 2020)

Instructions:

- This test is a closed book exam, but you are allowed to use calculator. Clarity of your answer will also be a part of credit. When needed, use the notation $\Phi(z) = P(Z < z)$ for a standard normal distribution Z. Show your ALL work neatly.
- Your answer sheets must be written in English.
- Remind that you can submit your answer sheets over icampus in a **pdf** file format ONLY.
- By submitting your report online, it is assumed that you agree with the following pledge;

  Pledge: *I have neither given nor received any unauthorized aid during this exam.*
- Don't forget to write down your name and student ID on your answer sheet.

1. (20 points) If X_i, $i = 1, 2, 3$ are independent exponential random variables with rates λ_i, $i = 1, 2, 3$, find
   (a) $P(X_1 < X_2 < X_3)$
   (b) $P(X_1 < X_2 | \max(X_1, X_2, X_3) = X_3)$
   (c) $E(\min(X_1, X_2, X_3))$
   (d) $\mathrm{Var}(\min(X_1, X_2, X_3))$

2. (10 points) A doctor has scheduled two appointments, one at 1:00 P.M. and the other at 1:30 P.M. The amounts of time that appointments last are independent exponential random variables with mean 30 minutes. Assuming that both patients are on time, find the expected amount of time that the 1:30 appointment spends at the doctor's office.

3. (10 points) A businessman parks his car illegally in the streets for a period of exactly two hours. Parking surveillances occur according to a Poisson process with an average of λ passes per hour. What is the probability of the businessman getting a fine on a given day?

4. (15 points) Suppose that people arrive at a bus stop in accordance with a Poisson process with rate λ. The bus departs at time t. Let X denote the total amount of waiting time of all those who get on the bus at time t. Let $N(t)$ denote the number of arrivals by time t.
   (a) $E(X|N(t))$
   (b) $\mathrm{Var}(X|N(t))$
   (c) $\mathrm{Var}(X)$

5. (10 points) Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ that is independent of the nonnegative random variable T with mean μ and variance σ^2. Find

   (a) $\text{Cov}(T, N(T))$

   (b) $\text{Var}(N(T))$

6. (10 points) For a standard Brownian motion $\{B(t), t \geq 0\}$, and $0 \leq s \leq t$, find

   (a) $E(B(t)|B(s) = y)$

   (b) Variance of $B(t) - tB(1)$, $t \in [0, 1]$.

7. (10 points) Consider a random walk

$$X_t = \sum_{k=1}^{t} Z_k, \quad X_0 = 0, \quad t = 1, 2, \ldots,$$

   and $\{Z_i\}$'s are i.i.d. with $P(Z_k = 1) = p$, $P(Z_k = -1) = 1 - p$, $p \in (0, 1)$. Find

   (a) $P(X_4 = 0)$

   (b) $P(Z_2 = 1|X_3 = 1)$

8. (15 points) Consider a random walk

$$X_t = \sum_{k=1}^{t} Z_k, \quad X_0 = 0, \quad t = 1, 2, \ldots,$$

   and $\{Z_i\}$'s are i.i.d. and **symmetric** random variables. Show that

$$P\left(\max_{0 \leq i \leq t} |X_i| \geq a\right) \leq 2P(|X_t| > a).$$