Time Series Analysis (STA 5015)
Chapter 5 Solution

1. Problem 5.1
   The Yule-Walker equation becomes (by multiplying X_{t-k} on both sides and taking expectation)

$$\gamma(k) = \phi_1\gamma(k-1) + \phi_2\gamma(k-2) + \mathrm{Cov}(Z_t, Z_{t-k} + \psi_1 Z_{t-k-1} + \psi_2 Z_{t-k-2} + \ldots)$$

   For

$$\begin{aligned} k = 0 &: \gamma(0) - \phi_1\gamma(1) - \phi_2\gamma(2) = \sigma^2 && (1) \\ k = 1 &: \gamma(1) - \phi_1\gamma(0) - \phi_2\gamma(1) = 0 && (2) \\ k = 2 &: \gamma(2) - \phi_1\gamma(1) - \phi_2\gamma(0) = 0 && (3) \end{aligned}$$

   Thus, by plug-in method of moment estimates of $\gamma(k)$ to (2) and (3), we have that

$$\hat{\phi}_1 = \frac{(\hat{\gamma}(0) - \hat{\gamma}(2))\hat{\gamma}(1)}{(\hat{\gamma}(0))^2 - (\hat{\gamma}(1))^2} = 1.317, \quad \hat{\phi}_2 = \frac{\hat{\gamma}(0)\hat{\gamma}(1) - (\hat{\gamma}(1))^2}{(\hat{\gamma}(0))^2 - (\hat{\gamma}(1))^2} = -.634$$

$$\hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\phi}_1\hat{\gamma}(1) - \hat{\phi}_2\hat{\gamma}(2) = 289.18.$$

   To calculate 95% CI for ϕ_1 and ϕ_2, note that for $\phi = (\phi_1, \phi_2)'$,

$$\sqrt{n}\left(\hat{\phi} - \phi\right) \approx \mathcal{N}\left(0, \hat{\sigma}^2\widehat{\Gamma}^{-1}\right), \quad \widehat{\Gamma} = \begin{pmatrix} \hat{\gamma}(0) & \hat{\gamma}(1) \\ \hat{\gamma}(1) & \hat{\gamma}(0) \end{pmatrix}$$

   Therefore, 95% CI for ϕ_1 and ϕ_2 becomes

$$\hat{\phi}_1 \pm 1.96\frac{\hat{\sigma}\sqrt{\hat{\gamma}(0)}}{\sqrt{n}\sqrt{(\hat{\gamma}(0))^2 - (\hat{\gamma}(1))^2}} = 1.317 \pm .152 = (1.165, 1.469)$$

$$\hat{\phi}_2 \pm 1.96\frac{\hat{\sigma}\sqrt{\hat{\gamma}(0)}}{\sqrt{n}\sqrt{(\hat{\gamma}(0))^2 - (\hat{\gamma}(1))^2}} = -.634 \pm .152 = (-.786, -.482)$$

2. Problem 5.3

   a. To become a causal process, the solution for

$$\phi(z) = 1 - \phi z - \phi^2 z^2 = 0$$

   must lie outside the unit-circle. Therefore,

$$|z| = \left|\frac{\phi \pm \sqrt{\phi^2 + 4\phi^2}}{2\phi^2}\right| > 1$$

   Thus, solving above inequality gives that

$$\left|\frac{1 \pm \sqrt{5}}{2\phi}\right| > 1 \iff |\phi| < \left|\frac{1 - \sqrt{5}}{2}\right| = .618$$

b. Yule-Walker equation gives that

$$k = 0 : \gamma(0) - \phi\gamma(1) - \phi^2\gamma(2) = \sigma^2 \tag{4}$$
$$k = 1 : \gamma(1) - \phi\gamma(0) - \phi^2\gamma(1) = 0 \tag{5}$$
$$k = 2 : \gamma(2) - \phi\gamma(1) - \phi^2\gamma(0) = 0 \tag{6}$$

From the condition in the problem, we have that $\hat{\gamma}(0) = 6.06$ $\hat{\gamma}(1) = \hat{\rho}(1)\hat{\gamma}(0) = 4.16$. Thus plug-in these values to equation (5) gives

$$\hat{\phi} = \frac{-\hat{\gamma}(0) \pm \sqrt{\hat{\gamma}(0)^2 + 4\hat{\gamma}(1)^2}}{2\hat{\gamma}(1)} = .728 \pm 1.237 = .509 \text{ or} - 1.965$$

However, we are interested ϕ for a causal process, hence $\hat{\phi} = .509$. Then, from equation (6), it leads to

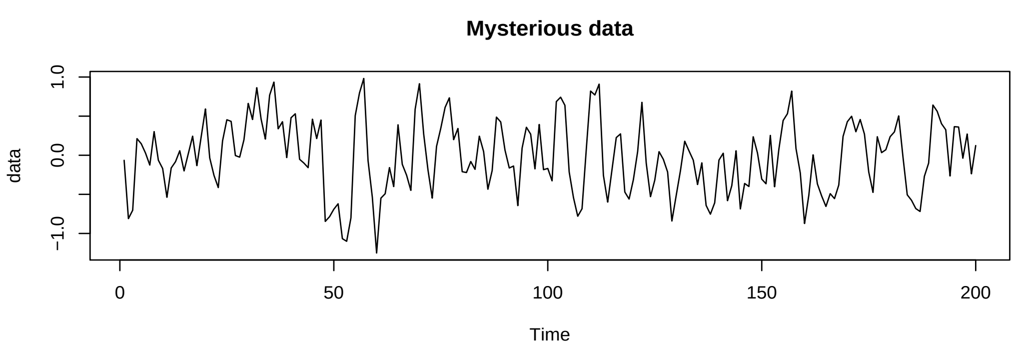
$$\hat{\gamma}(2) = \hat{\phi}\hat{\gamma}(1) - \hat{\phi}^2\hat{\gamma}(0) = 3.687.$$

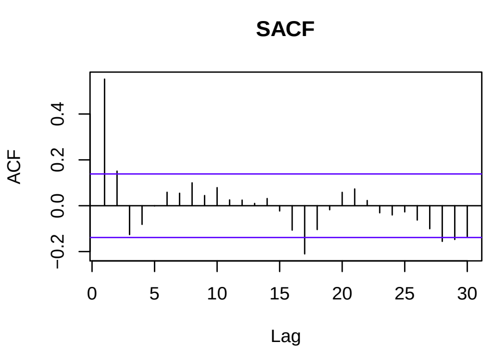
Finally, from equation (4)

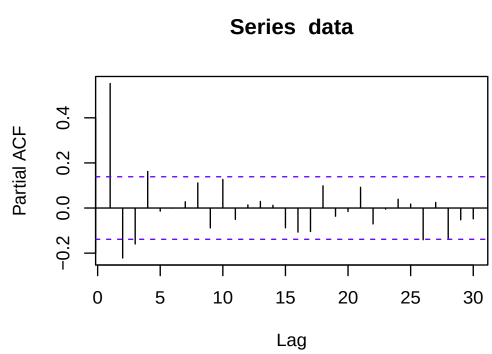
$$\hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\phi}\hat{\gamma}(1) - \hat{\phi}^2\hat{\gamma}(2) = 2.985.$$

3. `mysterious.txt`

(a) Time, ACF and PACF plot follows:







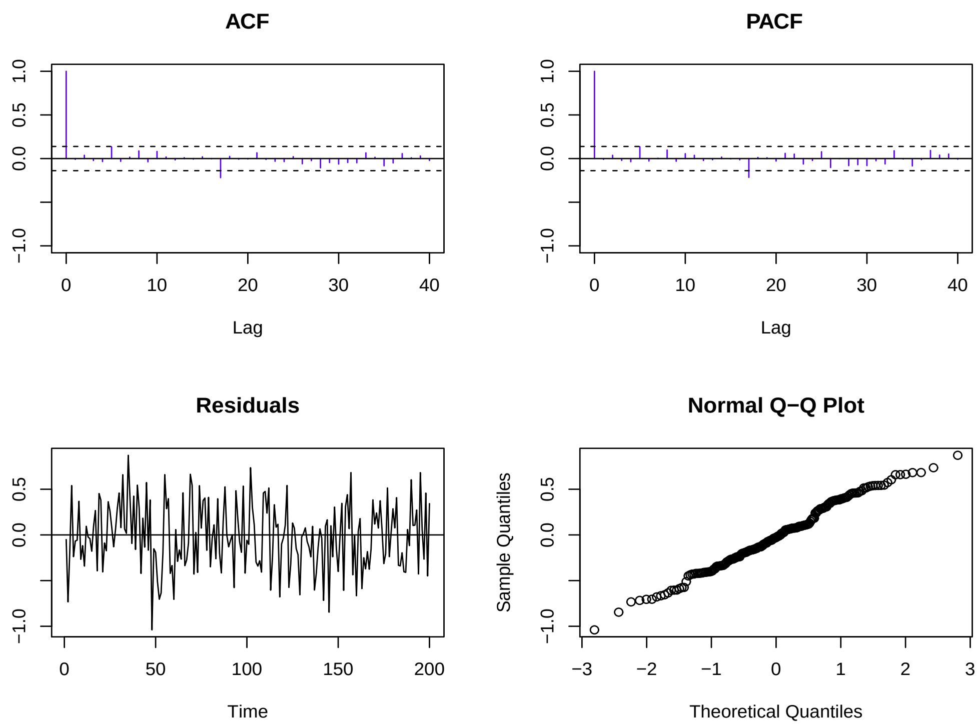
No obvious trend and seasonality is observed. From PACF, AR(4) seems plausible.

(b) From YW and MLE, the estimated values are given in the below. Both methods produce similar estimates.

```
> ar4.yw = ar.yw(data, aic=FALSE, order.max=4)
Call:
ar.yw.default(x = data, aic = FALSE, order.max = 4)

Coefficients:
     1       2       3       4
 0.665  -0.096  -0.263   0.162

Order selected 4  sigma^2 estimated as  0.134

> ar4.mle = arima(data, order=c(4,0,0), method = c("CSS-ML"), include.mean = TRUE)
Series: data
ARIMA(4,0,0) with non-zero mean

Coefficients:
        ar1     ar2     ar3    ar4  intercept
      0.666  -0.097  -0.271  0.170     -0.024
s.e.  0.070   0.082   0.083  0.071      0.048

sigma^2 estimated as 0.13:  log likelihood=-80.4
AIC=172.81   AICc=173.24   BIC=192.6
```

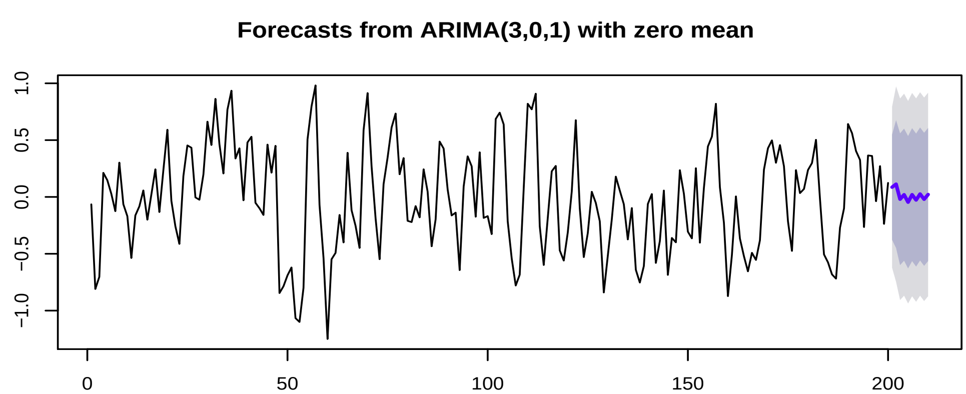
(c) The best model is ARMA(3,1), and all coefficients are away from zero. Also, test of randomness for residuals seems OK.

```
> library(forecast)
> fit = auto.arima(data, d=0, ic=c("aicc"))
Series: data
ARIMA(3,0,1) with zero mean

Coefficients:
         ar1    ar2     ar3    ma1
      -0.130  0.395  -0.317  0.822
s.e.   0.108  0.074   0.068  0.097

sigma^2 estimated as 0.13:  log likelihood=-80.37
AIC=170.74   AICc=171.05   BIC=187.23

> pval=2*(1-pnorm(abs(fit$coef/(sqrt(diag(fit$var.coef))))))
  ar1        ar2        ar3        ma1
2.2608e-01 9.8505e-08 3.4035e-06 0.0000e+00
```



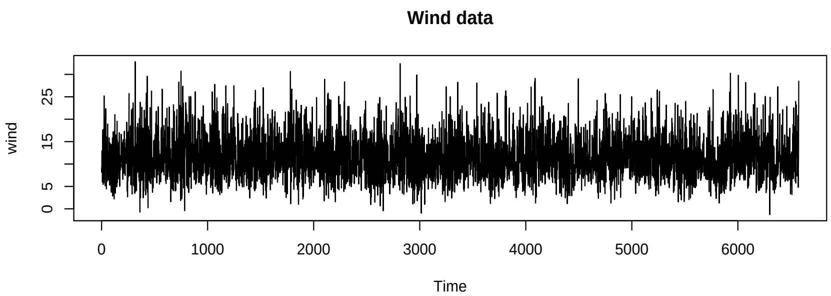
(d) 10 step ahead forecasting and 95% CI is given in the below.

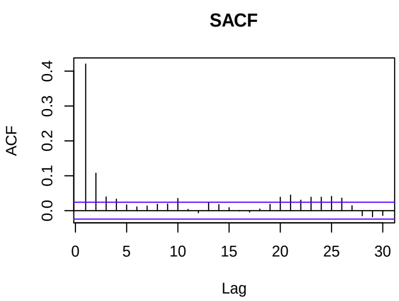
```
> library(forecast)
> forecast(fit, 10)
    Point Forecast     Lo 80    Hi 80     Lo 95   Hi 95
201       0.085658 -0.37694 0.54826 -0.62183 0.79315
202       0.113016 -0.44939 0.67542 -0.74711 0.97314
203      -0.020283 -0.60004 0.55947 -0.90694 0.86638
204       0.020043 -0.56103 0.60111 -0.86863 0.90871
205      -0.046493 -0.62900 0.53601 -0.93736 0.84437
206       0.020413 -0.56466 0.60549 -0.87438 0.91521
207      -0.027369 -0.61245 0.55772 -0.92218 0.86744
208       0.026383 -0.55877 0.61154 -0.86853 0.92130
209      -0.020720 -0.60621 0.56477 -0.91615 0.87471
210       0.021801 -0.56374 0.60734 -0.87370 0.91730
```

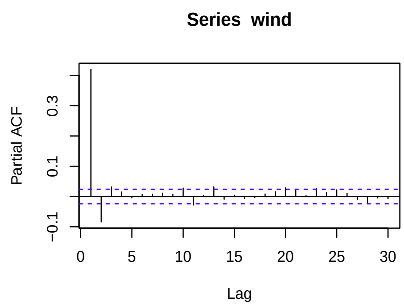


4. `wind.txt`

(a) Time, SACF and SPACF plots are in the below. It seems that there is no obvious trend or seasonality. Also variance seems to be stable over time. However, it seems that more spikes are observed on upward rather than downward. It indicates that the marginal distribution is not symmetric, but skewed.







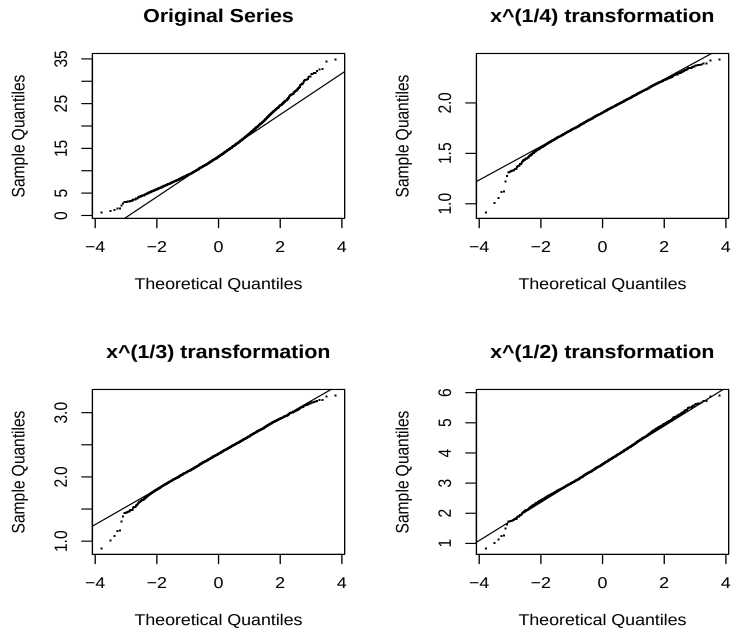
(b) Before taking power-transformation, observe that some of wind data is actually negative. So, we make the data positive by adding some constant on it.

$$Y_t = (X_t + 2)^\lambda$$

Then, square root transformation seems to be the best. This example also shows you that applying Box-Cox transformation can make the data as Normally distributed.

(c) The best model is ARMA(1,1), and all coefficients are away from zero.

```
> auto.arima(wind.s, d=0, ic=c("bic"))
Series: wind.s
ARIMA(1,0,1) with non-zero mean

Coefficients:
        ar1    ma1  intercept
      0.257  0.200      3.641
s.e.  0.028  0.028      0.011

sigma^2 estimated as 0.332:  log likelihood=-5699.9
AIC=11408   AICc=11408   BIC=11435
```




```
> pval=2*(1-pnorm(abs(fit$coef/(sqrt(diag(fit$var.coef))))))
       ar1          ma1   intercept
0.0000e+00 1.0407e-12 0.0000e+00
```