11.1 Sequences

- A sequence $\{a_n\}$ has the limit L and we write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$ if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence converges. Otherwise, we say the sequence diverges.
- If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

Squeeze Theorem :

- if $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$

- If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

- If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$

- The sequence $\{r^n\}$ is convergent if $-1 < r \le 1$ and divergent for all other values of r

$$\Rightarrow \lim_{n\to\infty} r^n = \begin{cases} 0, & \text{if } -1 < r < 1 \\ 1, & \text{if } r = 1 \end{cases}$$

- A sequence $\{a_n\}$ is bounded above if there is a number M such that $a_n \le M$ for all $n \ge 1$.
  It is bounded below if there is a number m such that $m \le a_n$ for all $n \ge 1$.
  If it is bounded above and below, then the sequence is a bounded sequence.

Monotonic Sequence Theorem :

- every bounded, monotonic sequence is convergent

11.2 Series

- Given a series $\sum_{i=1}^{\infty} a_n = a_1 + a_2 + \cdots$, let S_n denote its n^{th} partial sum :
  $S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$. If the sequence $\{S_n\}$ is convergent and $\lim_{n\to\infty} S_n = S$ exists as a real number, then the series $\sum a_n$ is convergent and we write $\sum_{n=1}^{\infty} a_n = S$. The number S is called the sum of the series. If the sequence $\{S_n\}$ is divergent, then the series is called divergent.

☆ the sum of a series is the limit of the sequence of partial sums

Geometric Series :

- the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ is convergent if $|r|<1$ and its sum is $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$, and if $|r| \geq 1$, the geometric series is divergent

Example of Proving the convergence of a series :

Q : Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

$$\Rightarrow S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$$
$$= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$
$$= 1 - \frac{1}{n+1}$$
$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$$

Q : Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is divergent.

$$S_1 = 1$$
$$S_2 = 1 + \frac{1}{2}$$
$$S_4 = 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right) > 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right) = 1+\frac{2}{2} = 2$$
$$S_8 = 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) > 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right) = 1+\frac{2}{2}+\frac{1}{2} = 1+\frac{3}{2}$$
$$\vdots$$
$$S_{2^n} > 1+\frac{n}{2}, \quad \text{since } \lim_{n\to\infty} 1+\frac{n}{2} = \infty, \ \sum_{n=1}^{\infty} \frac{1}{n} \text{ is divergent}$$

Theorem : if the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

✲ the converse of the theorem is not usually true

Test for Divergence :

- If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

✲ if we find that $\lim_{n\to\infty} a_n \neq 0$, we know that $\sum a_n$ is divergent. If we find that $\lim_{n\to\infty} a_n = 0$, we know nothing about the convergence or divergence of $\sum a_n$.

✲ a finite number of terms doesn't affect the convergence or divergence of a series.

11.3 The Integral Test and Estimates of Sums

Case of $\sum_{n=1}^{\infty} \frac{1}{n^2}$:

- if we convert the series as the function $y = \frac{1}{x^2}$, the series can be thought of adding the area of rectangles under the curve with an equal interval of 1. So the sum of the areas of the rectangles is $\sum_{x=1}^{\infty} \frac{1}{x^2}$, and if we exclude the first rectangle, the total area of the remaining rectangles is smaller than the area under the curve $y = \frac{1}{x^2}$, which is $\int_1^{\infty} \frac{1}{x^2}\,dx$.

$\Rightarrow$ $1 + \int_1^{\infty} \frac{1}{x^2}\,dx = 2$, we confirm the partial sum is bounded and increasing so we can conclude that the series is convergent.

Integral Test:

- Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

☆ when we use the Integral Test, it is not necessary to start the series or the integral at $n=1$

☆ the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$

Estimating the Sum of a Series:

Remainder:

- the error made when S_n, the sum of the first n terms, is used as an approximation to the total sum

$R_n = S - S_n = a_{n+1} + a_{n+2} + \cdots$

Remainder Estimate for the Integral Test:

- Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = S - S_n$, then $\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)\,dx$

$\Rightarrow$ $S_n + \int_{n+1}^{\infty} f(x)\,dx \leq S \leq S_n + \int_n^{\infty} f(x)\,dx$

11.4 The Comparison Tests

- Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms

i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent

ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent

- In using the Comparison Test we must have some known series $\sum b_n$ for the purpose of comparison. Most of the time, we use one of the following series

i) p-series $\left[\sum \frac{1}{n^p}\right.$ converges if $p > 1$ and diverges if $\left.p \leq 1\right]$

ii) Geometric Series $\left[\sum ar^{n-1}\right.$ converges if $|r| < 1$ and diverges if $\left.|r| \geq 1\right]$

☆ although the condition $a_n \leq b_n$ or $a_n \geq b_n$ in the Comparison Test is given for all n, we need verify only that it holds for $n \geq N$, where N is some fixed integer, because the convergence of a series is not affected by a finite number of terms.

The Limit Comparison Test:

- suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is a finite number and $c > 0$, then either both series converge or both diverge

* notice that in testing many series we find a suitable comparison series $\sum b_n$ by keeping only the highest powers in the numerator and denominator

Estimating Sums:

- if we have used the Comparison Test to show that a series $\sum a_n$ converges by comparison with a series $\sum b_n$, then we may be able to estimate the sum $\sum a_n$ by comparing remainders,

  $R_n = S - S_n = a_{n+1} + a_{n+2} + \cdots$

  For the comparison series $\sum b_n$, we consider the corresponding remainder

  $T_n = t - t_n = b_{n+1} + b_{n+2} + \cdots$

  Since $a_n \leq b_n$ for all n, we have $R_n \leq T_n$. If $\sum b_n$ is a p-series, we can estimate its remainder T_n. If $\sum b_n$ is a geometric series, then T_n is the sum of a geometric series and we can sum it exactly

11.5 Alternating Series

Alternating Series Test:

- If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$ $b_n > 0$ satisfies i) $b_{n+1} \leq b_n$ for all n ii) $\lim_{n\to\infty} b_n = 0$, then the series is convergent

Estimating Sums

Alternating Series Estimation Theorem:

- If $s = \sum (-1)^{n-1} b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies i) $b_{n+1} \leq b_n$ and ii) $\lim_{n\to\infty} b_n = 0$, then $|R_n| = |s - s_n| \leq b_{n+1}$

* the rule that the error is smaller than the first neglected term is valid only for alternating series that satisfy the conditions of the Alternating Series Estimation Theorem. The rule does not apply to other types of series

11.6 Absolute Convergence and the Ratio and Root Tests

Absolute Convergence :

- a series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent

Conditionally Convergent :

- a series $\sum a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

If a series $\sum a_n$ is absolutely convergent, then it is convergent.

The Ratio Test :

i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and therefore convergent.

ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

The Root Test :

i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and therefore convergent.

ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent

iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

Rearrangements :

- if $\sum a_n$ is an absolutely convergent series with sum s, then any rearrangement of $\sum a_n$ has the same sum s.
- if $\sum a_n$ is a conditionally convergent series and r is any real number whatsoever, then there is a rearrangement of $\sum a_n$ that has a sum equal to r.

11.7 Strategy for Testing Series

1. If the series is of the form $\sum 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leqslant 1$.
2. If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geqslant 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.
3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series. Notice that most of the series in Exercises 11.4 have this form. (The value of p should be chosen as in Section 11.4 by keeping only the highest powers of n in the numerator and denominator.) The comparison tests apply only to series with positive terms, but if $\sum a_n$ has some negative terms, then we can apply the Comparison Test to $\sum |a_n|$ and test for absolute convergence.
4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.
5. If the series is of the form $\sum (-1)^{n-1} b_n$ or $\sum (-1)^n b_n$, then the Alternating Series Test is an obvious possibility.
6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.
7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.
8. If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

11.8 Power Series

- a power series is a series of the form $\sum_{n=0}^{\infty} C_n X^n = C_0 + C_1 X + C_2 X^2 + \cdots$, where X is a variable and the C_n's are constants called the coefficients of the series.

=> a series of the form $\sum_{n=0}^{\infty} C_n (X-a)^n$ is called a power series in $(X-a)$ or a power series centered at a or a power series about a.

☆ the ratio test gives no information when $|x-c| = 1$ so we must consider $x=a$ and $x=b$ separately.

Theorem :

- For a given power series $\sum_{n=0}^{\infty} C_n (x-a)^n$, there are only three possibilities :

i) the series converges only when $X = a$

ii) the series converges for all X

iii) there is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$ (radius of convergence)

11.9 Representations of Functions as Power Series

$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$, $|x| < 1$ (Geometric Series)

☆ Regard series as geometric series of X and obtain the radius of convergence

Differentiation and Integration of Power Series

Theorem :

- If the power series $\sum C_n (x-a)^n$ has radius of convergence $R > 0$, then the function f defined by $f(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} C_n (x-a)^n$ is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

i) $f'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} n C_n (x-a)^{n-1}$

ii) $\int f(x)\,dx = C + C_0(x-a) + C_1 \frac{(x-a)^2}{2} + C_2 \frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} C_n \frac{(x-a)^{n+1}}{n+1}$

=> the radii of convergence of the power series in Equation (i) and (ii) are both R.

☆ Although the above theorem says that the radius of convergence remains the same when a power series is differentiated or integrated, this does not mean that the interval of convergence remains the same. It may happen that the original series converges at an endpoint, whereas the differentiated series diverges there.

☆ To determine the value of C, we put any value of x within the radius of convergence (usually $x=0$) to eliminate all polynomials and solve for the constant C.

11.10 Taylor and Maclaurin Series

- For a power series of the form $f(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots \quad |x-a| < R$

$\Rightarrow f^{(n)}(a) = n! C_n \quad \rightsquigarrow \quad C_n = \frac{f^{(n)}(a)}{n!}$

Theorem :

- If f has a power series representation at a, $f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$, $|x-a| < R$, then its coefficients are given by the formula $C_n = \frac{f^{(n)}(a)}{n!}$

Taylor Series of the function f at a : (or Maclaurin Series)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

※ Maclaurin Series is a special form of Taylor series that $a = 0$

※ We have shown that if f can be represented as a power series about a, then f is equal to the sum of its Taylor series. But there exist functions that are not equal to the sum of their Taylor series.

n^{th}-degree Taylor Polynomial of f at a : (or partial sums of a Taylor series)

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x-a)^i$$
$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Remainder of Taylor Series :

$$R_n(x) = f(x) - T_n(x)$$

Theorem :

- If $f(x) = T_n(x) + R_n(x)$, where $T_n(x)$ is the n^{th}-degree Taylor polynomial of f at a and $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$

Taylor's Inequality :

- If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \le d$

※ $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for every real number x

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

- the Maclaurin series of $f(x) = (1+x)^k$ is $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n = \sum_{n=0}^{\infty} \binom{k}{n} x^n$, and this is called the binomial series. Notice that if k is a nonnegative integer, then the terms are eventually 0 and so the series is finite. For other values of k, none of the term is 0 and so we can try the Ratio Test.

The binomial series always converges when $|x| < 1$, but the question of whether or not it converges at the endpoints, ± 1, depends on the value of k. The series converges at 1 if $-1 < k \le 0$ and at both endpoints if $k \ge 0$

Some Important Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \cdots \qquad R = 1$$

Multiplication and Division of Power Series

- If power series are added or subtracted, they behave like polynomials. In fact, they can also be multiplied and divided like polynomials.
- If both $f(x) = \sum c_n x^n$ and $g(x) = \sum b_n x^n$ converge for $|x| < R$ and the series are multiplied as if they were polynomials, then the resulting series also converges for $|x| < R$ and represents $f(x)g(x)$. For division we require $b_0 \neq 0$; the resulting series converges for sufficiently small $|x|$.

11.11 Applications of Taylor Polynomials

Approximating Functions by Polynomials :

- In general, it can be shown that the derivatives have the same values at a that f and f' have.

3 possible methods for estimating the size of the error :

1. If a graphing device is available, we can use it to graph $|R_n(x)|$ and thereby estimate the error.
2. If the series happens to be an alternating series, we can use the Alternating Series Estimation Theorem.
3. In all cases, we can use Taylor's Inequality ; if $|f^{(n+1)}(x)| \le M$, then $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$