Time Series Analysis (STA 5015)
Chapter 3 Solution

1. Problem 3.1

   a. Let $X_t + 0.2X_{t-1} - 0.48X_{t-2} = Z_t$. Since

   $$\phi(z) = 1 + 0.2z - 0.48z^2 = 0$$

   has two solutions $z_1 = 5/3$ and $z_2 = -5/4$ outside the unit circle, $\{X_t\}$ is causal. Since we have $\theta(z) = 1$ for all z, $\{X_t\}$ is invertible.

   b. Let $X_t + 1.9X_{t-1} + 0.88X_{t-2} = Z_t + 0.2Z_{t-1} + 0.7Z_{t-2}$. Since

   $$\phi(z) = 1 + 1.9z + 0.88z^2 = 0$$

   has two solutions $z_1 = -10/11$ and $z_2 = -5/4$ with $|z_1| < 1$ inside the unit circle, $\{X_t\}$ is not causal. Since

   $$\theta(z) = 1 + 0.2z + 0.7z^2 = 0$$

   has two solutions $z_1 = -(1 - i\sqrt{69})/7$, $z_2 = -(1 + i\sqrt{69})/7$ and $|z_1| = |z_2| = \sqrt{70}/7 > 1$ outside the unit circle, $\{X_t\}$ is invertible.

   c. Let $X_t + 0.6X_{t-1} = Z_t + 1.2Z_{t-1}$. Since

   $$\phi(z) = 1 + 0.6z = 0$$

   has a solution $z = -5/3$ outside the unit circle, $\{X_t\}$ is causal. Since

   $$\theta(z) = 1 + 1.2z = 0$$

   has a solution $z = -5/6$ inside the unit circle, $\{X_t\}$ is not invertible.

   d. Let $X_t + 1.8X_{t-1} + 0.81X_{t-2} = Z_t$. Since

   $$\phi(z) = 1 + 1.8z + 0.81z^2 = 0$$

   has a solution $z_1 = z_2 = -10/9$ outside the unit circle, $\{X_t\}$ is causal. Since $\theta(z) = 1$ for all z, $\{X_t\}$ is invertible.

   e. Let $X_t + 1.6X_{t-1} = Z_t - 0.4Z_{t-1} + 0.04Z_{t-2}$. Since

   $$\phi(z) = 1 + 1.6z = 0$$

   has a solution $z = -5/8$ inside the unit circle, $\{X_t\}$ is not causal. Since

   $$\theta(z) = 1 - 0.4z + 0.04z^2 = 0$$

   has a solution $z_1 = z_2 = 5$ outside the unit circle, $\{X_t\}$ is invertible.

2. Problem 3.3

   In Problem 3.1, we have seen that (a), (c) and (d) are causal. The first 6 coefficients are calculated by solving

$$\boxed{\psi(B)\phi(B) = \theta(B)}$$

   a. Solving

$$(1 + \psi_1 B + \psi_2 B^2 + \ldots)(1 + .2B - .48B^2) = 1$$

$$\iff 1 + (.2 + \psi_1)B + (\psi_2 + .2\psi_1 - .48)B^2 + (\psi_3 + .2\psi_2 - .48\psi_1)B^3 + \ldots = 1$$

   gives ($\psi_0 = 1$), $\psi_1 = -.2$, $\psi_2 = .52$, $\psi_3 = -.2$, $\psi_4 = .2896$, $\psi_5 = -.1539$.

   c. Solving

$$(1 + \psi_1 B + \psi_2 B^2 + \ldots)(1 + .6B) = 1 + 1.2B.$$

   gives ($\psi_0 = 1$), $\psi_1 = .6$, $\psi_2 = -.6^2$, $\psi_3 = .6^3$, $\psi_4 = -.6^4$, $\psi_5 = .6^5$.

   d. Solving

$$(1 + \psi_1 B + \psi_2 B^2 + \ldots)(1 + 1.8B + .81B^2) = 1$$

   gives ($\psi_0 = 1$), $\psi_1 = -1.8$, $\psi_2 = 2.43$, $\psi_3 = -2.916$, $\psi_4 = 3.285$, $\psi_5 = -3.542$.

3. Problem 3.6

   Note first that for $X_t = Z_t + \theta Z_{t-1}$ the covariance is given by

$$\gamma_X(h) = \mathrm{Cov}(X_{t+h}, X_t) = \mathrm{Cov}(Z_{t+h} + \theta Z_{t+h-1}, Z_t + \theta Z_{t-1})$$

$$= \gamma_Z(h) + \theta\gamma_Z(h+1) + \theta\gamma_Z(h-1) + \theta^2\gamma_Z(h) = (\theta^2+1)\gamma_Z(h) + \theta\gamma_Z(h+1) + \theta\gamma_Z(h-1)$$

$$= \begin{cases} (\theta^2+1)\sigma^2 & , \quad h = 0, \\ \theta\sigma^2 & , \quad h = \pm 1, \\ 0 & , \quad \text{o.w.} \end{cases}$$

   On the other hand, for $Y_t = \widetilde{Z}_t + 1/\theta\widetilde{Z}_{t-1}$, observe that

$$\gamma_Y(h) = \mathrm{Cov}(Y_{t+h}, Y_t) = \mathrm{Cov}(\widetilde{Z}_{t+h} + 1/\theta\widetilde{Z}_{t+h-1}, \widetilde{Z}_t + 1/\theta\widetilde{Z}_{t-1})$$

$$= \gamma_{\widetilde{Z}}(h) + 1/\theta\gamma_{\widetilde{Z}}(h+1) + 1/\theta\gamma_{\widetilde{Z}}(h-1) + 1/\theta^2\gamma_{\widetilde{Z}}(h)$$

$$= \left(1 + \frac{1}{\theta^2}\right)\gamma_{\widetilde{Z}}(h) + \frac{1}{\theta}\gamma_{\widetilde{Z}}(h+1) + \frac{1}{\theta}\gamma_{\widetilde{Z}}(h-1)$$

$$= \begin{cases} \left(\frac{1}{\theta^2}+1\right)\theta^2\sigma^2 = (\theta^2+1)\sigma^2 & , \quad h = 0, \\ \frac{1}{\theta}\theta^2\sigma^2 = \theta\sigma^2 & , \quad h = \pm 1, \\ 0 & , \quad \text{o.w.} \end{cases}$$

   Therefore, the ACVF of $\{X_t\}$ and $\{Y_t\}$ are the same.

4. For the mean of a given process, note that

$$E(X_t) = 2 + 1.3E(X_{t-1}) - .4E(X_{t-2}) + E(Z_t) + E(Z_{t-1}).$$

Let $\mu = E(X_t)$, then stationarity gives that

$$\mu = 2 + 1.3\mu - .4\mu$$

Hence, the mean is given by $\mu = 20$. Rewrite the given process as

$$(X_t - 20) - 1.3(X_{t-1} - 20) + .4(X_{t-2} - 20) = Z_t + Z_{t-1}.$$

Denote $X^* = X_t - 20$, then it becomes

$$X_t^* - 1.3X_{t-1}^* + .4X_{t-2}^* = Z_t + Z_{t-1}. \tag{1}$$

Since

$$\phi(z) = 1 - 1.3z + .4z^2 = 0$$

gives two solutions $z_1 = 2$ and $z_2 = 5/4$, both outside the unit circle, hence $\{X_t^*\}$ is a causal process. For invertibility note that

$$\theta(z) = 1 + z = 0$$

has one solution $z_1 = -1$ at unit circle, hence $\{X_t^*\}$ is not invertible. To find the ACVF, multiplying X_{t-k}^* on both sides of (1) and taking expectation gives

$$E\left(X_{t-k}^*\left(X_t^* - 1.3X_{t-1}^* + .4X_{t-2}^*\right)\right) = E\left(\left(Z_t + Z_{t-1}\right)X_{t-k}^*\right)$$

$$\gamma(k) - 1.3\gamma(k-1) + .4\gamma(k-2) = \mathrm{Cov}(Z_t + Z_{t-1}, X_{t-k}^*)$$

$$\gamma(k) - 1.3\gamma(k-1) + .4\gamma(k-2) = \mathrm{Cov}(Z_t + Z_{t-1}, Z_{t-k} + \psi_1 Z_{t-k-1} + \psi_2 Z_{t-k-2} + \ldots)$$

For

$$k = 0 : \gamma(0) - 1.3\gamma(1) + .4\gamma(2) = (1 + \psi_1)\sigma^2 = 3.3\sigma^2. \tag{2}$$
$$k = 1 : \gamma(1) - 1.3\gamma(0) + .4\gamma(1) = \sigma^2. \tag{3}$$
$$k = 2 : \gamma(2) - 1.3\gamma(1) + .4\gamma(0) = 0. \tag{4}$$
$$k \geq 3 : \gamma(k) = 1.3\gamma(k-1) - .4\gamma(k-2). \tag{5}$$

Solving equations (2)-(4) gives initial values $\gamma(0), \gamma(1), \gamma(2)$ and equation (5) gives the solution for all integers $k \geq 3$ iteratively. Remark that the coefficient ψ_1 is calculated from the equation

$$(1 - 1.3B + .4B^2)(1 + \psi_1 B + \psi_2 B^2 + \ldots) = 1 + B.$$

5. Multiplying X_{t-k} on both sides and taking expectation gives

$$E\left(X_{t-k}\left(X_t - X_{t-1} + .29X_{t-2} - .02X_{t-3}\right)\right) = E\left(Z_t X_{t-k}\right)$$

$$\gamma(k) - \gamma(k-1) + .2\gamma(k-2) - \gamma(k-3) = \mathrm{Cov}(Z_t, Z_{t-k} + \psi_1 Z_{t-k-1} + \psi_2 Z_{t-k-2} + \ldots)$$

For

$$
\begin{aligned}
k = 0 &: \gamma(0) - \gamma(1) + .29\gamma(2) - .02\gamma(3) = \sigma^2. && (6)\\
k = 1 &: \gamma(1) - \gamma(0) + .29\gamma(1) - .02\gamma(2) = 0 && (7)\\
k = 2 &: \gamma(2) - \gamma(1) + .29\gamma(0) - .02\gamma(1) = 0 && (8)\\
k = 3 &: \gamma(3) - \gamma(2) + .29\gamma(1) - .02\gamma(0) = 0 && (9)\\
k \geq 4 &: \gamma(k) = \gamma(k-1) - .2\gamma(k-2) + \gamma(k-3). && (10)
\end{aligned}
$$

Thus, solving equations (6)-(9) gives initial values $\gamma(0), \gamma(1), \gamma(2), \gamma(3)$ and equation (10) gives the solution for all integers $k \geq 4$ iteratively.