1) $X_i \sim NB(\mu, \alpha)$

a) $f(y;\mu,\alpha) = \frac{\Gamma(\alpha+y)}{y!\,\Gamma(\alpha)}\left(\frac{\alpha}{\alpha+\mu}\right)^{\alpha}\left(1-\frac{\alpha}{\alpha+\mu}\right)^{y}$, assuming α is known

$= \binom{\alpha+y-1}{y}\exp\left\{\alpha\log\left(\frac{\alpha}{\alpha+\mu}\right) + y\log\left(1-\frac{\alpha}{\alpha+\mu}\right)\right\}$, let $h(y) = \binom{\alpha+y-1}{y}$, $\eta(\mu) = \log\left(1-\frac{\alpha}{\alpha+\mu}\right)$,

$T(y) = y$, $A(\mu) = -\alpha\log\left(\frac{\alpha}{\alpha+\mu}\right)$

$\downarrow$

$= h(y)\exp\{\eta(\mu)T(y) - A(\mu)\}$, exponential dispersion family

$\therefore\ \eta(\mu) = \log\left(1-\frac{\alpha}{\alpha+\mu}\right)$, $A(\mu) = -\alpha\log\left(\frac{\alpha}{\alpha+\mu}\right)$

$= \log\left(\frac{\mu}{\alpha+\mu}\right)$ canonical link function

$:= \theta$

b) $L(\mu; y, \alpha) = \prod_{i=1}^{n}\frac{\Gamma(\alpha+y_i)}{y_i!\,\Gamma(\alpha)}\left(\frac{\alpha}{\alpha+\mu}\right)^{\alpha}\left(1-\frac{\alpha}{\alpha+\mu}\right)^{y_i}$

$\propto \prod_{i=1}^{n}\left(\frac{\alpha}{\alpha+\mu}\right)^{\alpha}\left(1-\frac{\alpha}{\alpha+\mu}\right)^{y_i}$

$= \left(\frac{\alpha}{\alpha+\mu}\right)^{n\alpha}\left(1-\frac{\alpha}{\alpha+\mu}\right)^{\sum_{i=1}^{n} y_i}$

$\Rightarrow\ \ell(\mu; y,\alpha) = n\alpha\log\left(\frac{\alpha}{\alpha+\mu}\right) + \left(\sum_{i=1}^{n} y_i\right)\log\left(1-\frac{\alpha}{\alpha+\mu}\right)$

$= n\alpha\log(\alpha) - n\alpha\log(\alpha+\mu) + \left(\sum_{i=1}^{n} y_i\right)\log(\mu) - \left(\sum_{i=1}^{n} y_i\right)\log(\alpha+\mu)$

$\Rightarrow\ \frac{\partial}{\partial\mu}\ell(\mu; y,\alpha) = -\frac{n\alpha}{\alpha+\mu} + \left(\sum_{i=1}^{n} y_i\right)/\mu - \left(\sum_{i=1}^{n} y_i\right)/(\alpha+\mu) = 0$

$\Rightarrow\ \left(\sum_{i=1}^{n} y_i\right)\left(\frac{\alpha}{\mu(\alpha+\mu)}\right) = \frac{n\alpha}{\alpha+\mu}$

$\therefore\ \hat{\mu}^{MLE} = \frac{1}{n}\sum_{i=1}^{n} y_i = \bar{y}$

By the invariant property of MLE, given α is known, $\hat{\theta}^{MLE} = \log\left(\frac{\bar{y}}{\alpha+\bar{y}}\right)$

c) $\frac{\partial}{\partial\alpha}\ell(\alpha; y,\mu) = n\log\alpha + n - n\log(\alpha+\mu) - \frac{n\alpha}{\alpha+\mu} - \left(\sum_{i=1}^{n} y_i\right)/(\alpha+\mu)$

$= n\log\left(\frac{\alpha}{\alpha+\mu}\right) + n - \frac{1}{\alpha+\mu}\left(n\alpha - \sum_{i=1}^{n} y_i\right)$

$= n\log\left(\frac{\alpha}{\alpha+\mu}\right) + n - \frac{1}{\alpha+\mu}\left(n\alpha - n\bar{y}\right)$

$= n\log\left(\frac{\alpha}{\alpha+\mu}\right) + \frac{1}{\alpha+\mu}\left(n\alpha + n\mu - n\alpha + n\bar{y}\right)$

$= n\log\left(\frac{\alpha}{\alpha+\mu}\right) + \frac{n(\mu-\bar{y})}{\alpha+\mu}$

$\Rightarrow\ n\log\left(\frac{\alpha}{\alpha+\mu}\right) = \frac{n(\bar{y}-\mu)}{\alpha+\mu}$

$\log\left(\frac{\alpha}{\alpha+\mu}\right) = \frac{\bar{y}-\mu}{\alpha+\mu}$

$\frac{\alpha}{\alpha+\mu} = \frac{\bar{y}-\mu}{\alpha+\mu}$

$\therefore\ \hat{\alpha}^{MLE} = \bar{y} - \mu$

c) $\theta = \log\left(\frac{\mu}{\alpha+\mu}\right)$

$e^{\theta} = \frac{\mu}{\alpha+\mu}$

$\alpha\cdot e^{\theta} + \mu\cdot e^{\theta} = \mu$ $\quad\therefore\ \hat{\alpha}^{ML} = \mu(1-e^{\theta})e^{-\theta}$, using the invariant property of MLE

d) (1) Define an initial value α_0, and calculate $\theta_{(1)} = \log\left(\frac{\bar{y}}{\alpha_0+\bar{y}}\right)$

(2) Calculate $\alpha_{(1)} = \mu(1-e^{\theta_{(1)}})e^{-\theta_{(1)}}$

(3) update α_i and θ_i and repeat for $i = 1, 2, \ldots$.

e)

```
data = read.csv('fish copy.csv')$count
sum((data - mean(data))^2/(mean(data)))
qchisq(0.975, 69)
```

```
> sum((data - mean(data))^2/(mean(data)))
[1] 3784.632
> qchisq(0.975, 69)
[1] 93.85647
```

$\therefore$ We can reject the null hypothesis.

2) a) $Var(y_i) = \phi\mu^2 \Rightarrow Var(\mu_i) = \frac{1}{\phi} Var(y_i)$

$Q(\mu_i; y_i) = \int_{y_i}^{\mu_i} \frac{y_i - t}{\sigma^2 Var(t)} dt$, $Var(t) = \frac{1}{\phi} Var(y_i)$

$=$

b) $Var(y_i) = \mu^2 = \sigma^2 \cdot Var(\mu_i)$

$\Rightarrow Var(\mu_i) = \left(\frac{\mu}{\sigma}\right)^2$

$\Rightarrow Q(\mu, y) = \sum_{i=1}^{n} \int_{y}^{\mu} \frac{y-t}{\sigma^2 \cdot Var(t)} dt$, $Var(t) = \left(\frac{t}{\sigma}\right)^2$

$= \sum_{i=1}^{n} \int_{y}^{\mu} \frac{y-t}{t^2} dt$

$= \sum_{i=1}^{n} \int_{y}^{\mu} \frac{y}{t^2} - \frac{1}{t} dt$

$= \sum_{i=1}^{n} \left(-\frac{y}{t} - \log|t| \Big|_{y}^{\mu}\right)$

$= \sum_{i=1}^{n} -\frac{y}{\mu} - \log|\mu| + 1 + \log|y|$

$\frac{\partial}{\partial\mu} Q(\mu, y) = \sum_{i=1}^{n} \frac{y}{\mu^2} - \frac{n}{\mu} = 0$

$\therefore \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} y_i$

c) $f(y) = \frac{1}{\mu} \exp\left(-\frac{y}{\mu}\right)$

$$L(y_1, \ldots, y_n ; \mu) = \frac{1}{\mu^n} \exp\left(-\sum_{i=1}^{n} \frac{y_i}{\mu}\right)$$

$$l(y_1, \ldots, y_n ; \mu) = -n \log \mu - \sum_{i=1}^{n} \frac{y_i}{\mu}$$

$$Q(\mu, y) = \sum_{i=1}^{n} -\frac{y_i}{\mu} - \log|\mu| + 1 + \log|y_i|$$

$$\propto -n \log \mu - \sum_{i=1}^{n} \frac{y_i}{\mu} \quad \text{w.r.t. } \mu$$

$$= l(y_1, \ldots, y_n ; \mu)$$

d) $f(y) = \frac{1}{\mu} \exp\left(-\frac{y}{\mu}\right)$

$$= \exp\left(-\log \mu - \frac{y}{\mu}\right) \sim h(y) \exp\{\eta(\mu) T(y) - A(\mu)\}$$

$$\Rightarrow h(y) = 1, \quad T(y) = -y, \quad \eta(\mu) = \frac{1}{\mu}, \quad A(\mu) = \log(\mu)$$

3) $E(Y_i) = \mu_i, \quad Var(Y_i) = \sigma^2 Var(\mu_i), \quad Var(\mu) = \mu + \alpha \mu^2$

a) For a single y_i :

$$QS: \quad U(\mu_i ; y_i) = \frac{y_i - \mu_i}{\sigma^2 Var(\mu_i)} = \frac{y_i - \mu_i}{\sigma^2(\mu + \alpha \mu^2)}$$

$$QL: \quad Q(\mu_i ; y_i) = \int_{y_i}^{\mu_i} \frac{y_i - t}{\sigma^2 Var(t)} dt, \quad Var(t) = t + \alpha t^2$$

$$= \int_{y_i}^{\mu_i} \frac{y_i - t}{\sigma^2(t + \alpha t^2)} dt$$

$$= \int_{y_i}^{\mu_i} \frac{y_i}{\sigma^2(t + \alpha t^2)} dt - \int_{y_i}^{\mu_i} \frac{t}{\sigma^2(t + \alpha t^2)} dt$$

$$= \frac{y_i}{\sigma^2}\left(\log|t| - \log|\alpha t + 1| \Big|_{y_i}^{\mu_i} \right) - \frac{1}{\sigma^2}\left(\frac{1}{\alpha} \log|1 + \alpha t| \Big|_{y_i}^{\mu_i} \right)$$

$$= \frac{y_i}{\sigma^2}\left[\log|\mu_i| - \log|\alpha \mu_i + 1| - \log|y_i| + \log|\alpha y_i + 1|\right] - \frac{1}{\sigma^2}\left[\frac{1}{\alpha} \log|1 + \alpha \mu_i| - \frac{1}{\alpha} \log|1 + \alpha y_i|\right]$$

$$= \frac{1}{\sigma^2}\left\{ y_i \log|\mu_i| - \left(y_i + \frac{1}{\alpha}\right) \log|1 + \alpha \mu_i| - y_i \log|y_i| + \left(y_i + \frac{1}{\alpha}\right) \log|1 + \alpha y_i| \right\}$$

$$= \frac{1}{\sigma^2}\left\{ y_i \log\left|\frac{\mu_i}{y_i}\right| + \left(y_i + \frac{1}{\alpha}\right) \log\left|\frac{1 + \alpha y_i}{1 + \alpha \mu_i}\right| \right\}$$

$$QD: \ D(\mu_i ; y_i) = -2\sigma^2\left(Q(\mu_i ; y_i)\right) = -2\left\{ y_i \log\left|\frac{\mu_i}{y_i}\right| + \left(y_i + \frac{1}{\alpha}\right)\log\left|\frac{1+\alpha y_i}{1+\alpha\mu_i}\right| \right\}$$

For $y_1, \dots, y_n$:

$$QS: \ \sum_{i=1}^{n} \frac{y_i - \mu_i}{\sigma^2(\mu + \alpha\mu^2)}$$

$$QL: \ \sum_{i=1}^{n} \frac{1}{\sigma^2}\left\{ y_i \log\left|\frac{\mu_i}{y_i}\right| + \left(y_i + \frac{1}{\alpha}\right)\log\left|\frac{1+\alpha y_i}{1+\alpha\mu_i}\right| \right\}$$

$$QD: \ -2\sum_{i=1}^{n}\left\{ y_i \log\left|\frac{\mu_i}{y_i}\right| + \left(y_i + \frac{1}{\alpha}\right)\log\left|\frac{1+\alpha y_i}{1+\alpha\mu_i}\right| \right\}$$

b) $QS: \ \sum_{i=1}^{n} \frac{y_i - \mu_i}{\sigma^2(\mu - \mu^2)}$

$$QL: \ \sum_{i=1}^{n} \frac{1}{\sigma^2}\left\{ y_i \log\left|\frac{\mu_i}{y_i}\right| + (y_i - 1)\log\left|\frac{1-y_i}{1-\mu_i}\right| \right\}$$

$$QD: \ -2\sum_{i=1}^{n}\left\{ y_i \log\left|\frac{\mu_i}{y_i}\right| + (y_i - 1)\log\left|\frac{1-y_i}{1-\mu_i}\right| \right\}$$

$\therefore \sim$ Binomial Distribution

c) $QD: \ -2\sum_{i=1}^{n}\left\{ y_i \log\left|\frac{\mu_i}{y_i}\right| \right\}$

Deviance for poisson-distributed data:

$$\Rightarrow \ l(\mu) = \sum_{i=1}^{n} y_i \log \mu_i - \sum_{i=1}^{n} \mu_i - \sum_{i=1}^{n} \log(y_i!)$$

$$l(\hat{\mu}) = \sum_{i=1}^{n} y_i \log y_i - \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \log(y_i!)$$

$$\text{Deviance} = -2\left(l(\mu) - l(\hat{\mu})\right), \ \hat{\mu} = y_i$$

$$= -2\sum_{i=1}^{n}\left\{ y_i \log\left|\frac{\mu_i}{y_i}\right| \right\}$$

d) $QD: \ -2\sum_{i=1}^{n}\left\{ y_i \log\left|\frac{\mu_i}{y_i}\right| + \left(y_i + \frac{1}{\alpha}\right)\log\left|\frac{1+\alpha y_i}{1+\alpha\mu_i}\right| \right\}, \quad \alpha = \frac{1}{r}$

$$= -2\sum_{i=1}^{n}\left\{ y_i \log\left|\frac{\mu_i}{y_i}\right| + (y_i + r)\log\left|\frac{1+\frac{1}{r}y_i}{1+\frac{1}{r}\mu_i}\right| \right\}$$

$$= -2\sum_{i=1}^{n}\left\{ y_i \log\left|\frac{\mu_i}{y_i}\right| + (y_i + r)\log\left|\frac{r+y_i}{r+\mu_i}\right| \right\}$$

Deviance for NB-distributed data:

$$P(y; r, \mu) = \frac{\Gamma(y+r-1)}{\Gamma(r)\Gamma(y+1)}\left(\frac{r}{\mu+r}\right)^r\left(1 - \frac{r}{\mu+r}\right)^y$$

$$\log P(y_i; r, \mu) = \log\left[\frac{\Gamma(y_i+r-1)}{\Gamma(r)\Gamma(y_i+1)}\right] + r\log\left(\frac{r}{\mu+r}\right) + y_i \log\left(\frac{\mu}{\mu+r}\right)$$

$$\sum_{i=1}^{n} \log P(y_i; r, \mu) = \sum_{i=1}^{n} \log\left[\frac{\Gamma(y_i+r-1)}{\Gamma(r)\Gamma(y_i+1)}\right] + r\log\left(\frac{r}{\mu+r}\right) + y_i \log\left(1 - \frac{r}{\mu+r}\right)$$

$$\text{Deviance} = -2\left(l(\mu) - l(\hat{\mu})\right), \ \hat{\mu} = y_i$$

$$= -2\sum_{i=1}^{n}\left\{ r\log\left(\frac{r}{\mu+r}\right) + y_i \log\left(\frac{\mu}{\mu+r}\right) - r\log\left(\frac{r}{y_i+r}\right) - y_i \log\left(\frac{y_i}{y_i+r}\right) \right\}$$

$$= -2\sum_{i=1}^{n}\left\{ r\log\left(\frac{y_i+r}{\mu+r}\right) + y_i \log\left(\frac{y_i+r}{\mu+r}\right) + y_i\left(\frac{\mu}{y_i}\right) \right\}$$

$$= -2\sum_{i=1}^{n}\left\{ y_i \log\left|\frac{\mu}{y_i}\right| + (y_i + r)\log\left|\frac{r+y_i}{r+\mu}\right| \right\}$$