1.

(a) $P(y) = P(\theta=1)\, P(y|\theta=1) + P(\theta=2)\, P(y|\theta=2)$

$$= 0.5\, N(y|1, 2^2) + 0.5\, N(y|2, 2^2)$$

See R code

(b)
$$P(\theta=1|y=1) = \frac{P(\theta=1, y=1)}{P(\theta=1, y=1) + P(\theta=2, y=1)}$$

$$= \frac{P(\theta=1) P(y=1|\theta=1)}{P(\theta=1) P(y=1|\theta=1) + P(\theta=2)\, P(y=1|\theta=2)}$$

$$= \frac{0.5\, N(1|1, 2^2)}{0.5\, N(1|1, 2^2) + 0.5\, N(1|2, 2^2)}$$

$$= 0.53$$

(c) As $\sigma \to \infty$, the posterior density for θ approaches the prior (the data contain no information):

$$P(\theta=1|y=1) = \frac{0.5\, N(1|1, \sigma^2)}{0.5\, N(1|1, \sigma^2) + 0.5\, N(1|2, \sigma^2)}$$

$$\longrightarrow \frac{1}{2} \quad (\sigma \to \infty)$$

As $\sigma \to 0$, the posterior density for θ becomes concentrated at 1:

$$P(\theta=1|y=1) \longrightarrow 1$$

2.

Use $P(GG) = P(M)P(GG|M) + P(D)P(GG|D)$

$$= P(M)\,p + (1-P(M))\,p^2$$

$$\Rightarrow P(M) = \frac{P(GG) - p^2}{p(1-p)}$$

3.

(a) $X \sim NegBin(r, \theta)$

Let $\theta \sim \beta eta(\alpha, \beta)$. Then

$$p(\theta|x) \propto \theta^r (1-\theta)^x \times \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$$= \theta^{r+\alpha-1}(1-\theta)^{x+\beta-1}$$

which is $\beta eta(r+\alpha, x+\beta)$.

Thus, the conjugate prior for θ in $NegBin(r,\theta)$ is $\beta eta(\alpha, \beta)$.

(b) $X \sim Gamma(\alpha, \beta)$, α is known

Let $\beta \sim Gamma(a, b)$.

Then

$$p(\beta|x) \propto \beta^{\alpha} e^{-\beta x} \times \beta^{a-1} e^{-b\beta}$$

$$= \beta^{\alpha+a-1} e^{-(x+b)\beta}$$

Which is $Gamma(\alpha+a, x+b)$.

(c) $X \sim Gamma(\alpha, \beta)$, β is known.

In this case, we cannot write the distribution of α.

So none is available.

4. $\theta \sim Beta(\alpha, \beta)$ with $E(\theta) = 0.6$

$Var(\theta) = 0.3^2$

(a) $E(\theta) = \dfrac{\alpha}{\alpha+\beta} = 0.6$ — ①

$Var(\theta) = \dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = 0.3^2$ — ②

In ②,

$$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{\alpha}{\alpha+\beta}\,\frac{\beta}{\alpha+\beta}\,\frac{1}{\alpha+\beta+1}$$

$$= \left(\frac{\alpha}{\alpha+\beta}\right)\left(1-\frac{\alpha}{\alpha+\beta}\right)\left(\frac{1}{\alpha+\beta+1}\right)$$

$$= (0.6)(1-0.6)\frac{1}{\alpha+\beta+1} = 0.09$$

$$\Rightarrow \alpha+\beta+1 = \frac{24}{9}$$

$$\Rightarrow \alpha+\beta = \frac{5}{3} \quad \text{— ②}'$$

In ① $\alpha = 0.6\alpha + 0.6\beta$

$$\Rightarrow 0.4\alpha = 0.6\beta$$

$$\Rightarrow \alpha = \frac{3}{2}\beta \quad \text{— ①}'$$

From ①', ②'

$\alpha = 1$, $\beta = \frac{2}{3}$

Thus, $\theta \sim Beta(\alpha = 1, \beta = \frac{2}{3})$.

(b). $y_1, \cdots, y_{1000} \overset{iid}{\sim}$ Bernoulli(θ).

$\Rightarrow \bar{y} = 0.65$.

$\theta \sim Beta(1, \frac{2}{3})$.

$$P(\theta | y_1, \cdots, y_{1000}) \propto \theta^{\sum_1^{1000} y_i} (1-\theta)^{1000 - \sum_1^{1000} y_i} \quad \theta^{1-1} (1-\theta)^{\frac{2}{3}-1}$$

$$= \theta^{\sum_1^{1000} y_i} (1-\theta)^{1000 + \frac{2}{3} - \sum_1^{1000} y_i - 1}$$

$$= \theta^{1000\bar{y}} (1-\theta)^{1000 + \frac{2}{3} - 1000\bar{y} - 1}$$

$$= \theta^{650} (1-\theta)^{350 + \frac{2}{3} - 1}$$

which is $Beta(651, 350+\frac{2}{3})$ distribution.

The posterior mean and variance are given by

$$E(\theta | y_1, \cdots, y_{1000}) = \frac{651}{651+350+\frac{2}{3}} = 0.6499.$$

$$Var(\theta | y_1, \cdots, y_{1000}) = \frac{(651)(350+\frac{2}{3})}{(651+350+\frac{2}{3})^2 (651+350+\frac{2}{3}+1)} = 0.00023$$