1) For each $j$, $Y_j \sim Pois(\mu_j)$, so $f(y_j) = \frac{e^{-\mu_j}\cdot \mu_j^{y_j}}{y_j!}$.

By the properties of Poisson distribution, $\sum_{j=1}^{c} Y_j \sim Pois(\sum_{j=1}^{c}\lambda_j)$ and $f(y_1, y_2, \ldots, y_c) = \prod_{j=1}^{c} f(y_j) = \prod_{j=1}^{c} \frac{e^{-\mu_j}\cdot \mu_j^{y_j}}{y_j!} = \frac{e^{-\sum_{i=1}^{c}\mu_i}\cdot \prod_{j=1}^{c}\mu_j^{y_j}}{y_1!y_2!\cdots y_c!}$

$$P(y_1, y_2, \ldots, y_c \cap N=n) = \frac{e^{-\sum_{i=1}^{c}\mu_i}\cdot \prod_{j=1}^{c}\mu_j^{y_j}}{y_1!y_2!\cdots y_c!}$$

$$P(N=n) = \frac{e^{-\sum_{i=1}^{c}\mu_i}\left(\sum_{j=1}^{c}\mu_j\right)^n}{n!}$$

$$P(y_1, y_2, \ldots, y_c \mid N=n) = \frac{P(y_1, y_2, \ldots, y_c \cap N=n)}{P(N=n)}$$

$$= \frac{e^{-\sum_{i=1}^{c}\mu_i}\cdot \prod_{j=1}^{c}\mu_j^{y_j}}{y_1!y_2!\cdots y_c!} \cdot \frac{n!}{e^{-\sum_{i=1}^{c}\mu_i}\left(\sum_{j=1}^{c}\mu_j\right)^n}$$

$$= \frac{n!}{y_1!y_2!\cdots y_c!}\cdot\frac{\prod_{j=1}^{c}\mu_j^{y_j}}{\left(\sum_{j=1}^{c}\mu_j\right)^n}$$

$$= \binom{n}{y_1, y_2, \ldots, y_c}\prod_{j=1}^{c}\left(\frac{\mu_j}{\sum_{i=1}^{c}\mu_i}\right)^{y_j} \sim Multi\left(n; \frac{\mu_1}{\sum_{i=1}^{c}\mu_i}, \frac{\mu_2}{\sum_{i=1}^{c}\mu_i}, \ldots, \frac{\mu_3}{\sum_{i=1}^{c}\mu_i}\right) \text{ since } \frac{\mu_j}{\sum_{i=1}^{c}\mu_i} < 1 \text{ for } \forall j \text{ and } \sum_{j=1}^{c}\frac{\mu_j}{\sum_{i=1}^{c}\mu_i} = 1$$

2) Let $\theta = \frac{\pi_1/(1-\pi_1)}{\pi_2/(1-\pi_2)} = \frac{\pi_{11}/\pi_{12}}{\pi_{21}/\pi_{22}} = \frac{\pi_{11}\pi_{22}}{\pi_{21}\pi_{12}} = \frac{n_{11}n_{22}}{n_{21}n_{12}}$

$\Rightarrow$ Let $f(x,y) = \log\frac{x/(1-x)}{y/(1-y)}$, then since $\sqrt{n}(\hat\pi_i - \pi_i) \sim N(0, \hat\pi_i(1-\hat\pi_i))$, and $\left[\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial y}\right]$ exists and is non-zero value, we can apply the Delta method

$$\sqrt{n}\left(f(\hat\pi_1, \hat\pi_2) - f(\pi_1, \pi_2)\right) \sim N\left(0, \left[\frac{\partial f}{\partial \hat\pi_1}\ \frac{\partial f}{\partial \hat\pi_2}\right]\begin{bmatrix} Var(\hat\pi_1) & 0 \\ 0 & Var(\hat\pi_2)\end{bmatrix}\begin{bmatrix}\frac{\partial f}{\partial \hat\pi_1} \\ \frac{\partial f}{\partial \hat\pi_2}\end{bmatrix}\right)$$

$$\left[\frac{\partial f}{\partial \hat\pi_1}\ \frac{\partial f}{\partial \hat\pi_2}\right] = \left[\frac{1}{\pi_1} + \frac{1}{1-\pi_1} \quad -\left(\frac{1}{\pi_2} + \frac{1}{1-\pi_2}\right)\right]$$

$$\Sigma = \begin{bmatrix} \frac{\hat\pi_1(1-\hat\pi_1)}{n_{1+}} & 0 \\ 0 & \frac{\hat\pi_2(1-\hat\pi_2)}{n_{2+}}\end{bmatrix}$$

$$\Rightarrow \left[\frac{\partial f}{\partial \hat\pi_1}\ \frac{\partial f}{\partial \hat\pi_2}\right]\begin{bmatrix} Var(\hat\pi_1) & 0 \\ 0 & Var(\hat\pi_2)\end{bmatrix}\begin{bmatrix}\frac{\partial f}{\partial \hat\pi_1} \\ \frac{\partial f}{\partial \hat\pi_2}\end{bmatrix} = \left(\frac{1}{\pi_1} + \frac{1}{1-\pi_1}\right)^2\left(\frac{\hat\pi_1(1-\hat\pi_1)}{n_{1+}}\right) + \left(\frac{1}{\pi_2} + \frac{1}{1-\pi_2}\right)^2\left(\frac{\hat\pi_2(1-\hat\pi_2)}{n_{2+}}\right)$$

$$= \left(\frac{n_{1+}}{n_{11}} + \frac{n_{1+}}{n_{12}}\right)^2\left(\frac{\frac{n_{11}}{n_{1+}}\cdot\frac{n_{12}}{n_{1+}}}{n_{1+}}\right) + \left(\frac{n_{2+}}{n_{21}} + \frac{n_{2+}}{n_{22}}\right)^2\left(\frac{\frac{n_{21}}{n_{2+}}\cdot\frac{n_{22}}{n_{2+}}}{n_{2+}}\right)$$

$$= \frac{n_{1+}}{n_{11}n_{12}} + \frac{n_{2+}}{n_{21}n_{12}}$$

$$= \frac{1}{n_{11}} + \frac{1}{n_{12}} + \frac{1}{n_{21}} + \frac{1}{n_{22}}$$

$$\therefore\ SE(\log\hat\theta) = \sqrt{\frac{1}{n_{11}} + \frac{1}{n_{12}} + \frac{1}{n_{21}} + \frac{1}{n_{22}}}$$

3) a) $P(X=1 \mid Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{P(X=1)P(Y=1 \mid X=1)}{P(Y=1)}$, $P(Y=1) = P(Y=1, X=1) + P(Y=1, X=2)$

$= P(X=1)P(Y=1 \mid X=1) + P(X=2)P(Y=1 \mid X=2)$

$= \gamma\pi_1 + (1-\gamma)\pi_2$

$= \frac{\gamma\pi_1}{\gamma\pi_1 + (1-\gamma)\pi_2}$

$$\therefore\ P(X=1 \mid Y=1) = \frac{\gamma\pi_1}{\gamma\pi_1 + (1-\gamma)\pi_2}$$

b) Positive Predictive Value : $P(X=1 \mid Y=1) = \frac{\gamma\pi_1}{\gamma\pi_1 + (1-\gamma)\pi_2} = \frac{0.01(0.86)}{0.01(0.86) + (0.99)(0.12)} = 0.0675$

c) Given $P(X=1 \mid Y=1) = 0.0675$, $P(X=2 \mid Y=1) = 0.9325$.

$\Rightarrow P(Y=1) = \gamma\pi_1 + (1-\gamma)\pi_2 = 0.01(0.86) + (0.99)(0.12) = 0.1274$

$\therefore\ P(X=1, Y=1) = P(Y=1)P(X=1 \mid Y=1) = 0.1274(0.0675) = 0.0085995$

$P(X=2, Y=1) = (1-\gamma)\pi_2 = 0.99(0.12) = 0.1188$

$$P(X=1, Y=2) = P(X=1)\,P(Y=2 \mid X=1) = \gamma(1-\pi_1) = 0.01(0.14) = 0.0014$$

$$P(X=2, Y=2) = P(X=2)\,P(Y=2 \mid X=2) = (1-\gamma)(1-\pi_2) = 0.99(0.88) = 0.8712$$

$$\Rightarrow \frac{\hat{\pi}_{11}}{\hat{\pi}_{21}} = \frac{P(X=1, Y=1)}{P(X=2, Y=1)} = \frac{0.0085995}{0.1188} = 0.0724 \quad \left(\frac{1}{0.0724} \approx 14\right)$$

$\therefore$ It is roughly 14 times more likely to be tested positive for the people who are "not" actually infected than for the people who are actually infected

4)

| 2008 \ 2012 | Obama | McCain |
|---|---|---|
| Obama | $\pi_1 = \frac{802}{855} = 0.938$ | $1-\pi_1 = \frac{53}{855} = 0.062$ |
| McCain | $\pi_2 = \frac{34}{528} = 0.0644$ | $1-\pi_2 = \frac{494}{528} = 0.9356$ |

$$\hat{\theta} = \frac{\pi_1/(1-\pi_1)}{\pi_2/(1-\pi_2)} = \frac{0.938 \cdot 0.9356}{0.0644 \cdot 0.062} = 219.8 \Rightarrow \log\hat{\theta} = 5.4$$

$$SE(\hat{\theta}) = \sqrt{\frac{1}{802} + \frac{1}{34} + \frac{1}{53} + \frac{1}{494}} = 0.227$$

$$L(\log\hat{\theta}) = 5.4 - 1.96(0.227) = 4.95508$$

$$U(\log\hat{\theta}) = 5.4 + 1.96(0.227) = 5.84492$$

$$\Rightarrow L(\hat{\theta}) = e^{L(\log\hat{\theta})} = 141.9, \quad U(\hat{\theta}) = e^{U(\log\hat{\theta})} = 345.5$$

$\therefore$ 95% CI for $\hat{\theta}$ is $(141.9, 345.5)$.

Interpretation : $H_0: \hat{\theta} = 1$, $H_1: \hat{\theta} \neq 1$.

1 is not within the 95% confidence interval of the odds ratio so we can reject the null hypothesis.

Odds of people who voted for Obama in 2008 voted for Obama in 2012 were about 142 times odds for the people voted for McCain in 2008.