

3.1 Definition of Limit

"the limit of $\{a_n\}$ is L " \equiv " a_n is a good approximation to L , when n is large"

- The number L is the limit of the sequence $\{a_n\}$ if, given $\epsilon > 0$, $a_n \approx_\epsilon L$ for $n \gg 1$

* convergent: $\lim_{n \rightarrow \infty} \{a_n\} = L$ Divergent: $\lim_{n \rightarrow \infty} \{a_n\} = \infty$

→ "the approximation can be made as close as desired, provided we go far enough out in the sequence - the smaller ϵ is, the farther out we must go, in general."

* Note that N depends on ϵ ; the smaller ϵ is, the bigger N is the further out you must go for the approximation to be valid within ϵ

3.2 The Uniqueness of Limits. The K - ϵ Principle

Theorem: Uniqueness Theorem for Limits

- A sequence a_n has at most one limit: $a_n \rightarrow L$ and $a_n \rightarrow L' \Rightarrow L = L'$

Theorem:

- $\{a_n\}$ increasing, $L = \lim_{n \rightarrow \infty} a_n \Rightarrow a_n \leq L$ for all n

- $\{a_n\}$ decreasing, $L = \lim_{n \rightarrow \infty} a_n \Rightarrow a_n \geq L$ for all n

The K - ϵ principle

- It often happens in analysis that arguments turn out to involve not just ϵ but a constant multiple of it. This may occur for instance when the limit involves a sum or several arithmetic processes

- If you come out in the end with 2ϵ , or 22ϵ , that's just as good as coming out with ϵ

→ - Suppose that $\{a_n\}$ is a given sequence, and you can prove that, given any $\epsilon > 0$, $a_n \approx_{K\epsilon} L$ for $n \gg 1$ where $K > 0$ is a fixed constant. Then $\lim_{n \rightarrow \infty} a_n = L$

3.3 Infinite Limits

- We say the sequence $\{a_n\}$ tends to infinity if, given any $M > 0$, $a_n > M$ for $n \gg 1$

$$\equiv \lim_{n \rightarrow \infty} a_n = \infty$$

3.4 An Important Limit

- The limit of a^n : $\lim_{n \rightarrow \infty} a^n = \begin{cases} \infty, & \text{if } a > 1 \\ 1, & \text{if } a = 1 \\ 0, & \text{if } |a| < 1 \end{cases}$

3.5 Writing Limit Proofs

3.6 Some Limits Involving Integrals

3.7 Another Limit Involving