Ch 8. Hypothesis testing: Part I

Motivating example

- Suppose that we have a coin and we want to test whether or not this coin is fair. To get sample, we flip a coin six times and get results.

  - Suppose we have three Heads and three Tails. Can we say a given coin is fair?

  - Suppose we have six Heads. Can we say a given coin is fair?

Terminologies

- Hypothesis: A statement about parameter
- Null hypothesis (H_0): A hypothesis that the parameter takes a particular value.
- Alternative hypothesis (H_1): A hypothesis that the parameter falls in some alternative range of values.
- Type I error: H_0 is rejected when H_0 is true
- Type II error: H_0 is not rejected when H_0 is false
- Significance level (α): Probability that Type I error occurs
- β: Probability that Type II error occurs
- Power$(1-\beta)$: Probability that H_0 is rejected when H_0 is false
- Critical region($\mathcal{C}$): A subset of sample space that results in rejection of H_0
- $\gamma_{\mathcal{C}}(\theta)$: Power function defined on a range of θ

Type I error and Type II error

- Type I error probability (α): The probability that we reject H_0 when H_0 is true.
- Type II error probability (β): The probability that we accept H_0 when H_0 is not true.

		True: H_0	True: H_1
Decision	H_0	No error	Type II error
	H_1	Type I error	No error

- Example 8.1

  - $X_1, X_2, X_3, X_4 \overset{iid}{\sim} b(1,\theta)$, $\theta \in \Omega = \{1/3, 2/3\}$
  - $H_0 : \theta = 1/3$ versus $H_1 : \theta = 2/3$
  - Critical region:

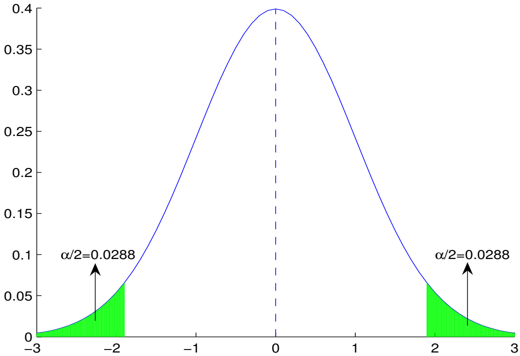
    $\mathcal{C} = \{(X_1, X_2, X_3, X_4) : X_1 + X_2 + X_3 + X_4 \geq 2\}$
  - Type I error probability(α):

  - Type II error probability(β):

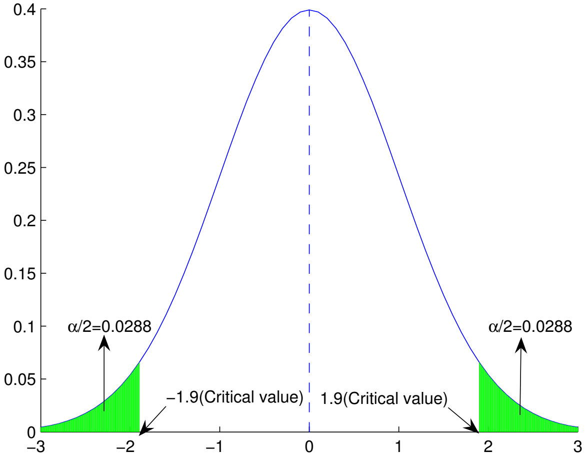
- Example 8.2
  - Suppose we have 10 random samples from $N(\mu, 2.5^2)$.
  - $H_0 : \mu = 50$ versus $H_1 : \mu \neq 50$
  - Critical (or rejection) region:

    $\mathcal{C} = \{X_1, \ldots, X_n | Z_0 < -1.9 \text{ or } Z_0 > 1.9\}$, where

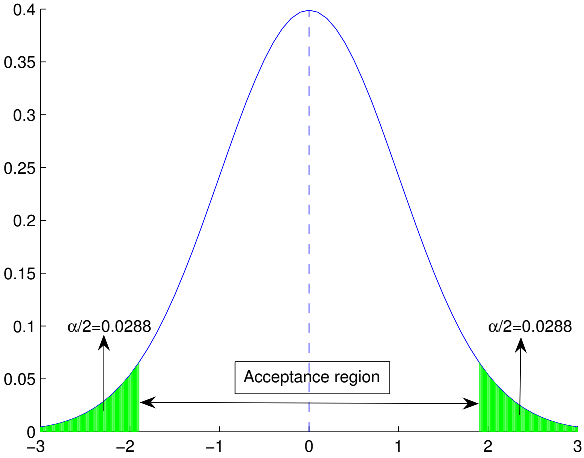
$$Z_0 = \frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} = \frac{\bar{X}_n - 50}{2.5/\sqrt{10}}$$



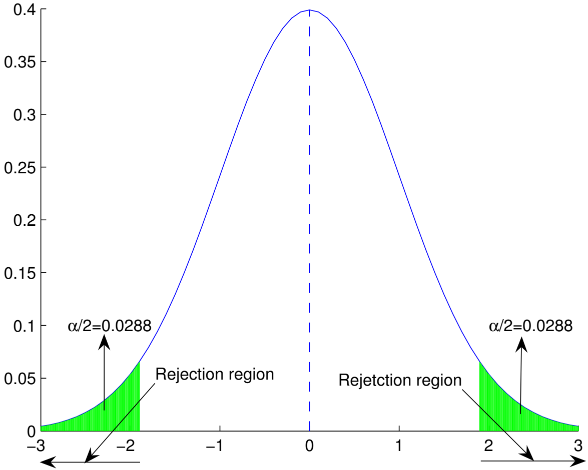
Critical value



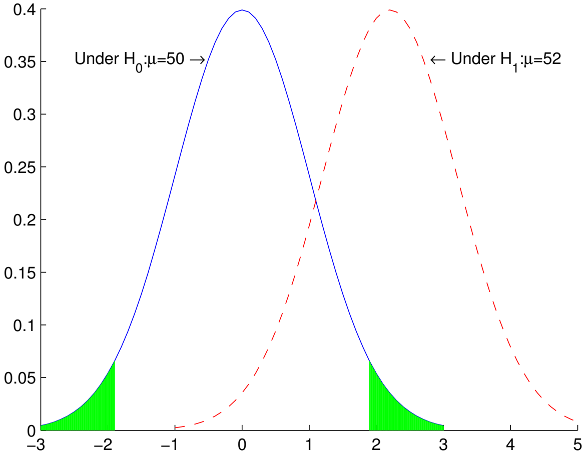
Acceptance region



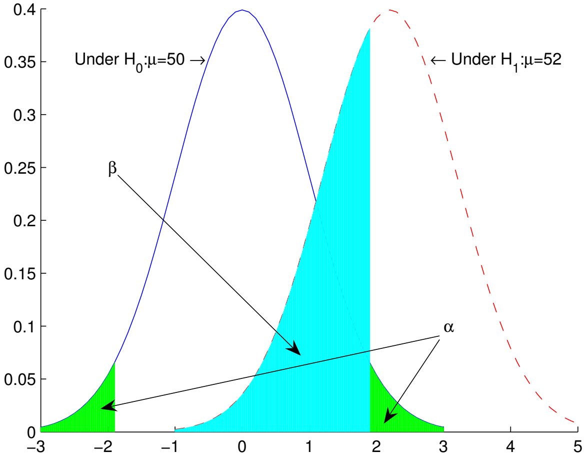
Critical (Rejection) region



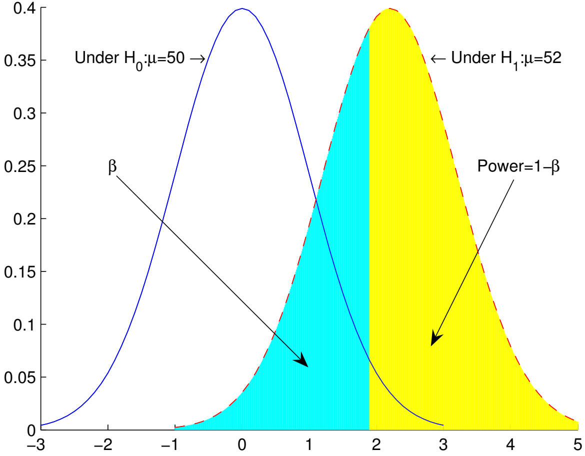
Under alternative hypothesis



Type I error and Type II error

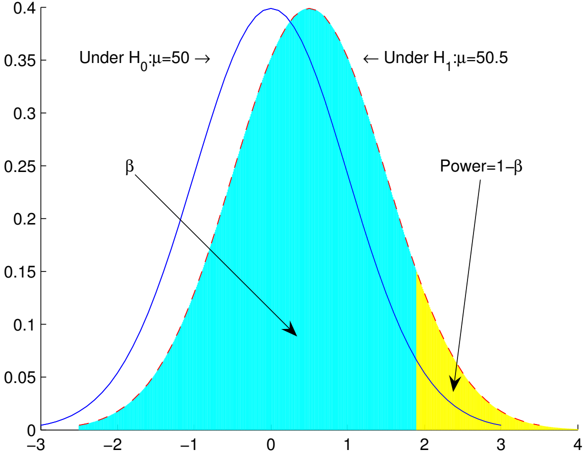


Power

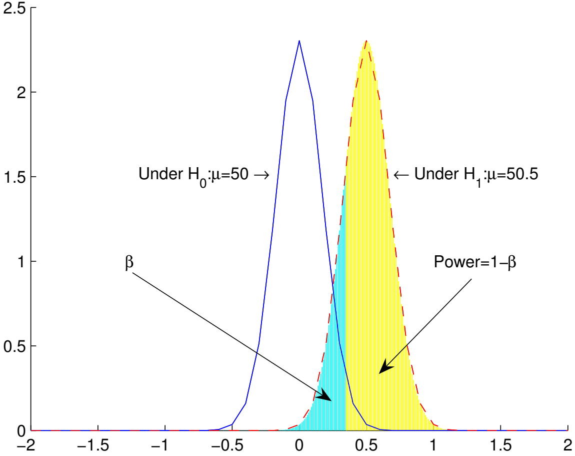


0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
–3
–2
–1
0
1
2
3
4
5
Under H_0:μ=50 →
← Under H_1:μ=52
β
Power=1–β

What if H_0 and H_1 are too close



Sample size and power



- Exercises: 4.5.3, 4.5.4, 4.5.8

How to find the critical region $\mathcal{C}$?

- Question 1: Can you construct a test with $\alpha = 0$ or $\beta = 0$?
- Question 2: Can you construct a "best" test?
- Question 3: What is "best"?

- In general, we cannot control both type I and type II erros for a given data. Our strategy to find a test is

(i) Fix α

(ii) Find a test that makes β as small as possible

Definition (p. 484)

For hypothesis testing $H_0 : \theta = \theta'$ versus $H_1 : \theta = \theta''$, we will say that $\mathcal{C}$ is a best critical region of size α if

(a) $P(\mathbf{X} \in \mathcal{C} | \theta = \theta') = \alpha$

(b) For any critical region $\mathcal{A}$ that satisfies $P(\mathbf{X} \in \mathcal{A} | \theta = \theta') = \alpha$, we have $P(\mathbf{X} \in \mathcal{C} | \theta = \theta'') \geq P(\mathbf{X} \in \mathcal{A} | \theta = \theta'')$

- Example 8.3: $X \sim b(5, \theta)$, $H_0 : \theta = \frac{1}{2}$ versus $H_1 : \theta = \frac{3}{4}$

x	0	1	2	3	4	5
$P\left(X = x \vert \theta = \frac{1}{2}\right)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$
$P\left(X = x \vert \theta = \frac{3}{4}\right)$	$\frac{1}{1024}$	$\frac{15}{1024}$	$\frac{90}{1024}$	$\frac{270}{1024}$	$\frac{405}{1024}$	$\frac{243}{1024}$

- What is the best critical region of size $\alpha = \frac{1}{32}$?

- Example 8.3: $X \sim b(5, \theta)$, $H_0 : \theta = \frac{1}{2}$ versus $H_1 : \theta = \frac{3}{4}$

x	0	1	2	3	4	5
$P\left(X = x \mid \theta = \frac{1}{2}\right)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$
$P\left(X = x \mid \theta = \frac{3}{4}\right)$	$\frac{1}{1024}$	$\frac{15}{1024}$	$\frac{90}{1024}$	$\frac{270}{1024}$	$\frac{405}{1024}$	$\frac{243}{1024}$

- What is the best critical region of size $\alpha = \frac{6}{32}$?

Neyman-Pearson Theorem

Theorem (Neyman-Pearson Theorem, p.486)

$X_1, \ldots, X_n \overset{iid}{\sim} f(x;\theta)$, $\Omega = \{\theta', \theta''\}$

For hypothesis testing $H_0 : \theta = \theta'$ *versus* $H_1 : \theta = \theta''$, $\mathcal{C}$ *is a best critical region of size* α *if*

$$(a) \frac{L(\theta'; \mathbf{X})}{L(\theta''; \mathbf{X})} \leq k \textit{ for each } \mathbf{X} \in \mathcal{C}.$$

$$(b) \frac{L(\theta'; \mathbf{X})}{L(\theta''; \mathbf{X})} \geq k \textit{ for each } \mathbf{X} \in \mathcal{C}^c.$$

$$(c) P(\mathbf{X} \in \mathcal{C} | \theta = \theta') = \alpha.$$

- Example 8.4 (p. 489)

$X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, 1)$, $\theta \in \Omega = \{0, 1\}$

$H_0 : \theta = 0$ versus $H_1 : \theta = 1$

- Example 8.4 (continued)

$X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, 1)$, $\theta \in \Omega = [0, \infty)$

$H_0 : \theta = 0$ versus $H_1 : \theta > 0$

Exercises: 8.1.2, 8.1.5, 8.1.9

UMP test

Definition (p. 493)

For hypothesis testing $H_0 : \theta = \theta'$ versus $H_1 : \theta \in \Omega_1$, where $\Omega = \Omega_1 \cup \{\theta'\}$, we will say that C^{UMP} is a uniformly most powerful (UMP) critical region of size α if

(a) $P(\mathbf{X} \in \mathcal{C}^{UMP} | \theta = \theta') = \alpha$

(b) $P(\mathbf{X} \in \mathcal{C}^{UMP} | \theta = \theta'') \geq P(\mathbf{X} \in \mathcal{C} | \theta = \theta'')$ for all $\theta'' \in \Omega_1$ and all size α critical regison $\mathcal{C}$.

- Example 8.4 (continued)

$X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, 1)$, $\theta \in \Omega = (-\infty, \infty)$

$H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$

Definition

$L(\theta; \mathbf{X})$ is said to have monotone likelihood ratio(MLR) property in the statistic $Y = u(\mathbf{X})$ if $\frac{L(\theta_1, \mathbf{X})}{L(\theta_2; \mathbf{X})}$ is a monotone function of $Y = u(\mathbf{X})$ for $\theta_1 < \theta_2$.

- Example 8.4 (continued)

$X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, 1)$, $\theta \in \Omega = [0, \infty)$
$H_0 : \theta = 0$ versus $H_1 : \theta > 0$

- Example 8.5

$X_1, \ldots, X_n \overset{iid}{\sim} N(0, \theta)$, $H_0 : \theta = \theta'$ versus $H_1 : \theta > \theta'$

- UMP test for $H_0 : \theta = \theta'$ versus $H_1 : \theta > \theta'$ with a regular exponential family

- $X_1, \ldots, X_n \overset{iid}{\sim} f(x;\theta) = \exp\{p(\theta)K(x) + H(x) + q(\theta)\}$

  - If $p(\theta)$ is monotone increasing,

$$\mathcal{C}^{UMP} = \left\{ \sum_{i=1}^{n} K(X_i) \geq c \right\} \text{ for some } c \text{ satisfying } P(\mathbf{X} \in \mathcal{C}^{UMP}) = \alpha$$

  - If $p(\theta)$ is monotone decreasing,

$$\mathcal{C}^{UMP} = \left\{ \sum_{i=1}^{n} K(X_i) \leq c \right\} \text{ for some } c \text{ satisfying } P(\mathbf{X} \in \mathcal{C}^{UMP}) = \alpha$$

- Example 8.4 (revisited)

$X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, 1)$, $\theta \in \Omega = [0, \infty)$

$H_0 : \theta = 0$ versus $H_1 : \theta > 0$

- Example 8.5 (revisited)

$X_1, \ldots, X_n \overset{iid}{\sim} N(0, \theta)$, $H_0 : \theta = \theta'$ versus $H_1 : \theta > \theta'$

- In fact, if $X_1, \dots, X_n$ is a random sample from a regular exponential family $f(x;\theta) = \exp\{p(\theta)K(x) + H(x) + q(\theta)\}$, then

- The MLE is a fuction of $\sum K(X_i)$
- $\sum K(X_i)$ is a complete sufficient statistic for θ
- If $p(\theta)$ is monotone function, it has MLR property
- If $p(\theta)$ is monotone function, $\mathcal{C}^{UMP}$ for one-sided test should be either $\{\sum_{i=1}^n K(X_i) \geq c\}$ or $\{\sum_{i=1}^n K(X_i) \leq c\}$

- Example 8.6

$X_1, \ldots, X_n \overset{iid}{\sim} Exp(\theta)$, $H_0 : \theta = 2$ versus $H_1 : \theta > 2$

- Example 8.7

$X_1, \ldots, X_n \overset{iid}{\sim} Gamma(3, \theta)$, $H_0 : \theta = \theta_0$ versus $H_1 : \theta < \theta_0$

- Example 8.8

$X_1, \ldots, X_n \overset{iid}{\sim} U[0, \theta]$, $H_0 : \theta = \theta_0$ versus $H_1 : \theta < \theta_0$

Exercises: 8.2.5, 8.2.6, 8.2.11, 8.2.12 except (d)