6.1 Introduction. Nested Intervals.

- If the sequence itself is really new, unrelated to other sequences whose limits we already know, the only tool we have for showing it has a limit is the Completeness Property

  "A bounded monotone sequence converges to a limit"

Definition : Suppose we have a sequence of closed intervals $I_n = [a_n, b_n]$, $n = 0, 1, 2, \dots$, having the property that each interval lies inside the previous one $a_n \le a_{n+1} \le b_{n+1} \le b_n$.

Such a sequence of intervals is said to be nested

Theorem : The Nested Intervals Theorem

- Suppose $[a_n, b_n]$ is an infinite sequence of nested intervals, whose lengths tend to 0. $\left(\lim_{n\to\infty} b_n - a_n = 0\right)$

  Then there is one and only one number L in all the intervals

6.2 Cluster Points of Sequences

- numbers that the sequence gets arbitrarily close to, infinitely often
- a sequence can have many cluster points

Definition : Cluster Points (or point of accumulation, or limit point)

- K is a cluster point of the sequence $\{a_n\}$ if, given $\varepsilon > 0$, $a_n \underset{\varepsilon}{\approx} K$ for infinitely many n
- For both a limit L and a cluster point K of a sequence $\{a_n\}$, the a_n must get arbitrarily close. But the a_n must stay close to a limit L, whereas they need only visit the vicinity of a cluster point K infinitely often. Every limit L is automatically a cluster point

Theorem : Cluster Point Theorem

- K is a cluster point of $\{a_n\}$ $\Leftrightarrow$ K is the limit of some subsequence $\{a_n\}$

6.3 The Bolzano - Weiestrass Theorem

Theorem : Bolzano - Weierstrass

- A bounded sequence $\{X_n\}$ has a convergent subsequence

6.4 Cauchy Sequence

- given $\varepsilon > 0$, $a_m \underset{\varepsilon}{\approx} a_n$ for $m, n \gg 1$

Theorem: The Cauchy Criterion for Convergence

- If $\{a_n\}$ is a Cauchy sequence, then $\{a_n\}$ converges
  - i) $\{a_n\}$ is bounded
  - ii) $\{a_n\}$ has a convergent subsequence $\{a_n\}$
  - iii) Let $L = \lim_{n\to\infty} \{a_n\}$, then $\{a_n\} \to L$

6.5 The Completeness Property for sets

Definitions

- An upper bound for S is a number b such that $x \le b$ for all $x \in S$
- S is said to be bounded above if S has an upper bound
- A number m is the maximum of S if m is an upper bound for S and $m \in S$

Definition: Supremum

- Let $S \subseteq \mathbb{R}$. The supremum of S is a number $\bar{m}$ satisfying:
  - sup-1: $\bar{m}$ is an upper bound for S: $x \le \bar{m}$ for all $x \in S$
  - sup-2: $\bar{m}$ is the least upper bound for S, that is $x \le b$ for all $x \in S \Rightarrow \bar{m} \le b$

Proposition:

- If $\max S$ exists, then $\sup S$ exists, and $\sup S = \max S$. The numbers $\sup S$ and $\max S$ are unique, if they exist.

Theorem: Completeness Property for Sets

- If S is non-empty and bounded above, $\sup S$ exists

Definitions

- A lower bound for S is a number b such that $x \ge b$ for all $x \in S$
- S is said to be bounded below if S has a lower bound
- A number m is the minimum of S if m is a lower bound for S and $m \in S$

Definition: Infimum

- Let $S \subseteq \mathbb{R}$. The infimum of S is a number $\bar{m}$ satisfying:
  - inf-1: $\bar{m}$ is a lower bound for S: $x \ge \bar{m}$ for all $x \in S$
  - inf-2: $\bar{m}$ is the greatest lower bound for S, that is $x \ge b$ for all $x \in S \Rightarrow \bar{m} \ge b$

Proposition :

- If $\min S$ exists, then $\inf S$ exists, and $\inf S = \inf S$. The numbers $\inf S$ and $\inf S$ are unique, if they exist.

Theorem : Completeness Property for Sets

- If S is non-empty and bounded below, $\inf S$ exists