Stochastic Processes (STA3021)
HW3 Solution

1. Chapter 2 Exercise # 73

   (a) The probability of i, j th person have same birthday at a specific day in the year is $\frac{1}{365}\frac{1}{365}$ and since there are 365 days,

   $$P(S_{i,j}) = \binom{365}{1}\frac{1}{365^2} = \frac{1}{365}$$

   (b) The probability of i, j have the same birthday, and k, r have the same birthday is given by

   $$P(S_{i,j} \cap S_{k,r}) = \binom{365}{1}\frac{1}{365^2} * \binom{365}{1}\frac{1}{365^2} = \frac{1}{365^2}.$$

   From (a), $P(S_{i,j}) * P(S_{k,r}) = \frac{1}{365^2}$. Therefore it satisfies

   $$P(S_{i,j} \cap S_{k,r}) = P(S_{i,j}) * P(S_{k,r}),$$

   so they are independent.

   (c) The probability of i, j, k have the same birthday is

   $$P(S_{i,j,k}) = \frac{365}{365^3} = \frac{1}{365^2}.$$

   From (a),

   $$P(S_{i,j}) * P(S_{k,j}) = \frac{1}{365^2},$$

   thus they are independent.

   (d) $S_{1,2} \cap S_{2,3} \cap S_{1,3}$ means that 1,2 and 3rd person have the same birthday, and the probability is

   $$P(S_{1,2} \cap S_{2,3} \cap S_{1,3}) = \frac{365}{365^3} = \frac{1}{365^2}.$$

   However,

   $$P(S_{1,2}) * P(S_{2,3}) * P(S_{1,3}) = \frac{1}{365^3},$$

   hence

   $$P(S_{1,2} \cap S_{2,3} \cap S_{1,3}) \neq P(S_{1,2}) * P(S_{2,3}) * P(S_{1,3})$$

   show that $S_{1,2}$, $S_{1,3}$ and $S_{2,3}$ are not independent.

   (e) Let X be the number of the pairs of people sharing the same birthday amongst n people. Then, we have $\binom{n}{2}$ of pairs and each pair has the probability of $\frac{1}{365}$ to have the same birthday. Thus,

   $$X \sim B\left(\binom{n}{2}, \frac{1}{365}\right) \approx \text{Poisson}\left(\binom{n}{2}\frac{1}{365}\right)$$

by the Poisson paradigm. Since the event A represents no two of them share the same birthday,

$$P(A) = P(X = 0) \approx \frac{e^{-\binom{n}{2}\frac{1}{365}}}{0!}.$$

(f) When $n = 23$,

$$P(A) \approx \frac{e^{-\binom{23}{2}\frac{1}{365}}}{0!} = 0.4999,$$

so it fairly close to .5. It means that if there are more than 23 people in the room, there are more than 50% of chance to observe pairs of people sharing the same birthday.

(g) Let Y be the number of triplet of people, say i, j and k, sharing the same birthday out of n people in the room. Then, from the calculation in (b) and using the Poisson paradigm,

$$Y \sim B\left(\binom{n}{3}, \frac{1}{365^2}\right) \approx \text{Poisson}\left(\binom{n}{3}\frac{1}{365^2}\right).$$

Thus, the probability of no three people have the same birthday is approximated as

$$P(B) = P(Y = 0) \approx \frac{e^{-\binom{n}{3}\frac{1}{365^2}}}{0!}.$$

To find the n with $P(B) \approx .5$, solving

$$e^{-\binom{n}{3}\frac{1}{365^2}} = 0.5 \iff n(n-1)(n-2) - 6(265)^2 \log 2 = 0$$

gives that $n \approx 83$ as shown in the next Figure.

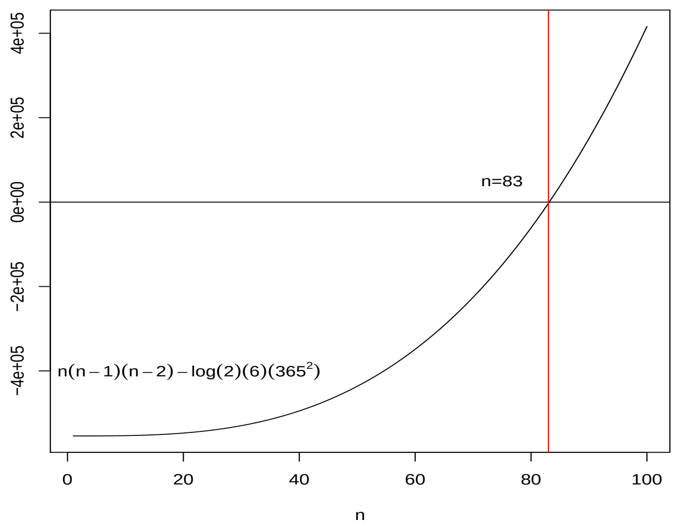

Figure 1: Solution of $n(n-1)(n-2) - 6(265)^2 \log 2 = 0$.

For computational side, you can visit `www.wolframalpha.com` and type

`solve n*(n-1)*(n-2) = 6*365^2*log(2)`

to find the numerical answer.

2. Chapter 2 Exercise #78.

   (a) Let $S = X_1 + X_2 + \cdots + X_{10}$, then $ES = E\sum_{i=1}^{10} X_i = \sum_{i=1}^{10} EX_i = 10$. Recall the Markov inequality:
   $$P\left(|x| > a\right) \le \frac{E|X|^r}{a^r}.$$
   Since S is a non-negative random variable, applying the Markov inequality to our problem with $r = 1$ gives
   $$P\left(S > 15\right) \le \frac{ES}{15} = \frac{2}{3}.$$

   (b) From the CLT,
   $$P\left(S \ge 15\right) = P\left(\frac{S - ES}{\sqrt{\mathrm{Var} S}} \ge \frac{15 - 10}{\sqrt{10}}\right)$$
   $$\approx P\left(Z \ge 1.58\right) = 1 - P\left(Z \ge 1.58\right) = 1 - .9429 = .0571.$$

3. Chapter 2 Exercise #86.

(a) The probability of taking more than 420 minutes to process 40 books can be represented as $P\left(\sum_{i=1}^{40} X_i > 420\right)$. Hence, CLT gives

$$P\left(\sum_{i=1}^{40} X_i > 420\right) = P\left(\frac{\sum_{i=1}^{40} X_i - E\sum_{i=1}^{40} X_i}{\sqrt{\mathrm{Var}\sum_{i=1}^{40} X_i}} > \frac{420-400}{\sqrt{360}}\right)$$

$$\approx P\left(Z > 1.05\right) = 1 - P\left(Z < 1.05\right) = 0.1469.$$

(b) The probability in the question can be stated as $P\left(\sum_{i=1}^{25} X_i < 240\right)$. Approximating this by CLT gives

$$P\left(\sum_{i=1}^{25} X_i < 240\right) = P\left(\frac{\sum_{i=1}^{25} X_i - E\sum_{i=1}^{25} X_i}{\sqrt{Var\sum_{i=1}^{25} X_i}} < \frac{240-250}{\sqrt{225}}\right)$$

$$\approx P\left(Z < -0.67\right) = P\left(Z > 0.67\right) = 1 - P\left(Z < 0.67\right) = 0.2514.$$

4. Chapter 3 Exercise #15

To compute $E\left(X^2|Y=y\right)$, we first calculate $f(x|y)$. Note that

$$f(y) = \int f(x,y)dx = \int_0^y \frac{e^{-y}}{y}dx = \frac{e^{-y}y}{y} = e^{-y}, \quad 0 < y < \infty$$

gives that

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{\frac{e^{-y}y}{y}}{e^{-y}} = \frac{1}{y}, \quad 0 < x < y.$$

Therefore,

$$E(X^2|Y=y) = \int x^2 f(x|y)dx = \int_0^y x^2\frac{1}{y}dx = \frac{1}{y}\frac{1}{3}x^3\Big|_0^y = \frac{1}{3}y^2, \quad 0 < y < \infty.$$

5. Chapter 3 Exercise #37

(a) Let X denote the number of errors. Then we have $X|A \sim Poisson(2.6)$, $X|B \sim Poisson(3)$, $X|C \sim Poisson(3.4)$. Therefore

$$EX = E(X|A)P(A) + E(X|B)P(B) + E(X|C)P(C) = \frac{1}{3}\left(2.6+3+3.4\right) = 3.$$

(b) Similarly,

$$EX^2 = E(X^2|A)P(A) + E(X^2|B)P(B) + E(X^2|C)P(C)$$

$$= \frac{1}{3}\left(2.6 + 2.6^2 + 3 + 3^2 + 3.4 + 3.4^2\right) = 12.1067$$

from $E(X^2|A) = \text{Var}(X|A) + (E(X|A))^2$. Thus,

$$\text{Var}X = EX^2 - (EX)^2 = 12.106667 - 3^2 = 3.1067.$$

6. Chapter 3 Exercise #56

(a) Let X denote the number of traffic accidents. Then we have

$$X|Rain \sim Poisson(9),\ P\left(Rain\right) = 0.6$$

$$X|Dry \sim Poisson(3),\ P\left(Dry\right) = 0.4$$

Therefore,

$$EX = E(X|Rain).6 + E(X|Dry).4 = 6.6.$$

(b) Straightforward calculation using conditional probability gives that

$$P\left(X=0\right) = P\left(X=0|Rain\right)P\left(Rain\right) + P\left(X=0|Dry\right)P\left(Dry\right)$$

$$= \frac{e^{-9}9^0}{0!}0.6 + \frac{e^{-3}3^0}{0!}0.4 = 0.01998887.$$

(c) Simiar to (a),

$$EX^2 = 0.6E(X^2|Rain) + 0.4E(X^2|Dry)$$

$$= 0.6 * (9 + 9^2) + 0.4 * (3 + 3^2) = 58.8.$$

Thus,

$$\text{Var}X = EX^2 - (EX)^2 = 58.8 - 6.6^2 = 15.24.$$

7. Chapter 3 Exercise #92

(a) Let N denote the number of coins Josh spots, and X be the amount of money Josh picks up at each spot. Then, $S = \sum_{i=1}^{N} X_i$ is a compound random variable. Note for a compound random variable S that $ES = (EN)(EX)$. Since $N \sim Poisson(6)$, and each coin has equal probability,

$$EN = 6,$$

$$EX = \sum xP\left(X=x\right) = 0 * \frac{1}{4} + 5 * \frac{1}{4} + 10 * \frac{1}{4} + 25 * \frac{1}{4} = 10.$$

Therefore,

$$ES = 6 * 10 = 60.$$

(b) Note that, $\text{Var}S = \mu^2\text{Var}N + \sigma^2 EN$, $(\mu = EX,\ \sigma^2 = \text{Var}X)$. Since, $N \sim Poisson(6)$

$$\text{Var}N = 6.$$

$$EX^2 = \sum xP(X = x) = 0^2 * \frac{1}{4} + 5^2 * \frac{1}{4} + 10^2 * \frac{1}{4} + 25^2 * \frac{1}{4} = 187.5.$$

$$\text{Var}X = EX^2 - (EX)^2 = 187.5 - 10^2 = 87.5.$$

Therefore,

$$\text{Var}S = 10^2 * 6 + 87.5 * 6 = 1125.$$

(c) Let N^* denote the number of coins Josh "picked up". Then $N^* \sim Poisson(6 * \frac{3}{4})$. (We will see this in Chapter 5 again. This is called the Bernoulli splitting of Poisson process.) Since there are 4 cases to pick up exactly 25cents, namely,

$$5cents \times 5,$$

$$5cents \times 3 + 10cents \times 1,$$

$$5cents \times 1 + 10cents \times 2,$$

$$25cents \times 1,$$

it leads to

$$\begin{aligned} P(S = 25) &= P(5cents \times 5 | N^* = 5)\, P(N^* = 5) \\ &+ P(5cents \times 3 + 10cents \times 1 | N^* = 4)\, P(N^* = 4) \\ &+ P(5cents \times 1 + 10cents \times 2 | N^* = 3)\, P(N^* = 3) \\ &+ P(25cents \times 1 | N^* = 1)\, P(N^* = 1). \end{aligned}$$

Given $N^* = n^*$ the probability of "picked up coins are 25cents" can be considered as multinomial event. Hence,

$$P(5cents \times 5 | N^* = 5) = \frac{5!}{5!0!0!}\left(\frac{1}{3}\right)^5$$

$$P(5cents \times 3 + 10cents \times 1 | N^* = 4) = \frac{4!}{3!1!0!}\left(\frac{1}{3}\right)^4$$

$$P(5cents \times 1 + 10cents \times 2 | N^* = 3) = \frac{3!}{1!2!0!}\left(\frac{1}{3}\right)^3$$

$$P(25cents \times 1 | N^* = 1) = \frac{1!}{0!0!1!}\left(\frac{1}{3}\right).$$

Therefore,

$$P(S = 25) = \left(\frac{1}{3}\right)^5 \frac{e^{-4.5}4.5^5}{5!} + 4\left(\frac{1}{3}\right)^4 \frac{e^{-4.5}4.5^4}{4!} + 3\left(\frac{1}{3}\right)^3 \frac{e^{-4.5}4.5^3}{3!} + \left(\frac{1}{3}\right)\frac{e^{-4.5}4.5^1}{1!}.$$

8. Chapter 3 Exercise #98.

From the definition of covariance,

$$\mathrm{Cov}(N, S) = E(NS) - E(N)E(S) = E\left(N\sum_{i=1}^{N} X_i\right) - E(N)E\left(\sum_{i=1}^{N} X_i\right).$$

First note that

$$E\left(N\sum_{i=1}^{N} X_i \middle| N = n\right) = E\left(n\sum_{i=1}^{n} X_i \middle| N = n\right) = n\sum_{i=1}^{n} E(X_i)$$

by the independence of X and N, and furthermore it reduces to

$$n^2 E(X_1)$$

if X_i's are IID. Thus, we have that

$$E\left(N\sum_{i=1}^{N} X_i\right) = E_N\left(E\left(N\sum_{i=1}^{N} X_i|N\right)\right) = E(N^2)E(X_1).$$

Similarly,

$$E\left(\sum_{i=1}^{N} X_i\right) = E_N\left(E\left(\sum_{i=1}^{N} X_i|N\right)\right) = E(N)E(X_1).$$

Therefore,

$$\begin{aligned}\mathrm{Cov}(N, S) &= E(N^2)E(X_1) - E(N)E(N)E(X_1)\\ &= (E(N^2) - (E(N))^2)E(X_1) = \mathrm{Var}(N)E(X_1).\end{aligned}$$