11.1 Continuity and Limits

Continuity of $x$ and $y$ :

- if $x$ vary a little, does $y$ change by a small amount?

- can $y$ be expressed in terms of $x$ ?

- We say $f(x)$ is continuous at $x_0$ if it is defined for $x \approx x_0$, and given any $\varepsilon > 0$, $f(x) \underset{\varepsilon}{\approx} f(x_0)$ for $x \approx x_0$

- We say $f(x)$ is continuous on the open interval $I$ if it is continuous at every point of $I$

- Assuming $f(x)$ is defined for the relevant $x$-values, we say,

  $f(x)$ is right-continuous at $x_0$ if, given $\varepsilon > 0$, $f(x) \underset{\varepsilon}{\approx} f(x_0)$ for $x \approx x_0^+$

  $f(x)$ is left-continuous at $x_0$ if, given $\varepsilon > 0$, $f(x) \underset{\varepsilon}{\approx} f(x_0)$ for $x \approx x_0^-$

  $f(x)$ is continuous on $[a, b]$ if $f(x)$ is $\begin{cases} \text{continuous on } (a,b) \\ \text{right-continuous at } a \\ \text{left-continuous at } b \end{cases}$

- We say $f(x)$ is continuous if its domain is an interval $I$ of positive or infinite length, and it is continuous on $I$

☆ Continuity at $x_0$ is an aspect of the local behavior of $f$ at $x_0$, since we verify it by looking at $f(x)$ in a neighborhood of $x_0$. Continuity on $I$ is a local property of $f(x)$ since the definitions say we verify it by checking that $f(x)$ is continuous at each point of $I$

☆ $\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx$

Discontinuities

- We call a point $x_0$ where $f$ is not continuous a point of discontinuity for the function $f(x)$ provided that it is also isolated, that is, $f(x)$ is continuous at all points near $x_0$ but different from it $(x \underset{\neq}{\approx} x_0)$

- We say $f(x)$ has a removable discontinuity if it is possible to define or change value at the point of discontinuity.

- We say $f(x)$ has an essential discontinuity if it is impossible to define at a point

11.2 Limits of Functions

**Definition 11.2A The limit of a function.**

Let $f(x)$ be defined for $x \approx x_0$, but not necessarily for $x = x_0$ (we abbreviate this by: for $x \underset{\neq}{\approx} x_0$ ).

We say $f(x)$ has the **limit** $L$ as $x \to x_0$ if there is a number $L$ such that

(5) $$\text{given } \epsilon > 0, \quad f(x) \underset{\epsilon}{\approx} L \quad \text{for } x \underset{\neq}{\approx} x_0.$$

If this is so, we write

$$\lim_{x \to x_0} f(x) = L, \qquad \text{or} \qquad f(x) \to L \text{ as } x \to x_0.$$

**Definition 11.2B** Assume $f(x)$ defined for $x \underset{\neq}{\approx} x_0^+$ or $x \underset{\neq}{\approx} x_0^-$, respectively.

(6) **right-hand limit** $\displaystyle\lim_{x \to x_0^+} f(x) = L:$ given $\epsilon > 0$, $f(x) \underset{\epsilon}{\approx} L$ for $x \underset{\neq}{\approx} x_0^+$

(7) **left-hand limit** $\displaystyle\lim_{x \to x_0^-} f(x) = L:$ given $\epsilon > 0$, $f(x) \underset{\epsilon}{\approx} L$ for $x \underset{\neq}{\approx} x_0^-$

**Theorem 11.2** *The limit on the left below exists if and only if both limits on the right exist and are equal; if this is so, all three limits are equal:*

(8) $$\lim_{x \to x_0} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to x_0^+} f(x) = L \text{ and } \lim_{x \to x_0^-} f(x) = L.$$

**Definition 11.2C** **Limits at infinity.** We define

(9) $\lim_{x\to\infty} f(x) = L:$ given $\epsilon > 0$, $f(x) \underset{\epsilon}{\approx} L$ for $x \gg 1$.

The corresponding definition for $-\infty$ substitutes $x \ll 1$ in (9).

**Definition 11.2D** **Infinite limits.** Let $f(x)$ be defined for $x \underset{\neq}{\approx} x_0$, etc.

(10) $\lim_{x\to x_0} f(x) = \infty:$ given any $b > 0$, $f(x) > b$ for $x \underset{\neq}{\approx} x_0$, etc.

The definition of $\lim_{x\to x_0} f(x) = -\infty$ substitutes $f(x) < -b$.

## 11.3 Limit Theorems for Functions

**Principle 11.3A** **Error form for limit.** Write $f(x) = L + e(x)$. Then

(11) $f(x) \to L \Leftrightarrow e(x) \to 0$, as $x \to x_0$, etc.

**Principle 11.3B** **The $K$-$\epsilon$ principle for limits of functions.**

If you can prove, for some $K$ *not depending on $x$ or $\epsilon$*, that

(12) given $\epsilon > 0$, $f(x) \underset{K\epsilon}{\approx} L$, for $x \underset{\neq}{\approx} x_0$, etc.,

then $f(x) \to L$ as $x \to x_0$.

**Theorem 11.3A** **Algebraic limit theorems.**

*If $a, b$ are constants, and $f(x) \to L$, $g(x) \to M$ as $x \to x_0$, etc.,*

(13) **Linearity theorem** $af(x) + bg(x) \to aL + bM$ as $x \to x_0$;

(14) **Product theorem** $f(x) \cdot g(x) \to L \cdot M$ as $x \to x_0$;

(15) **Quotient theorem** $f(x)/g(x) \to L/M$ as $x \to x_0$;

(for the Quotient Theorem, assume $g(x) \neq 0$ for $x \underset{\neq}{\approx} x_0$, and $M \neq 0$.)

**Theorem 11.3A∞** **Infinite limit theorems.**

In the statements below, the limits are taken as $x \to x_0$, etc., while the properties are assumed to hold for $x \underset{\neq}{\approx} x_0$, etc.

(17) $f(x) \to \infty$, $\begin{cases} g(x) \to \infty, \text{ or} \\ g(x) \text{ bounded below} \end{cases}$ $\Rightarrow f(x) + g(x) \to \infty$.

(18) $f(x) \to \infty$, $\begin{cases} g(x) \to L > 0, \text{ or} \\ g(x) > k > 0 \text{ for some } k \end{cases}$ $\Rightarrow f(x)g(x) \to \infty$.

(19) $f(x) \to \infty \Rightarrow \dfrac{1}{f(x)} \to 0$; if $f(x) > 0$, the converse is true.

**Theorem 11.3B** **Squeeze theorem.**

*Suppose $f(x) \le g(x) \le h(x)$ for $x \underset{\neq}{\approx} x_0$, etc. Then:*

$f(x) \to L$ and $h(x) \to L$ as $x \to x_0 \Rightarrow g(x) \to L$ as $x \to x_0$.

**Theorem 11.3B∞** **Squeeze theorem for infinite limits.**

*Suppose $f(x) \ge g(x)$ for $x \underset{\neq}{\approx} x_0$, etc. Then*

$$\lim_{x\to x_0} g(x) = \infty \Rightarrow \lim_{x\to x_0} f(x) = \infty.$$

**Theorem 11.3C** **Limit location (for functions).** *If the limits exist,*

(20) $f(x) \le M$ for $x \underset{\neq}{\approx} x_0 \Rightarrow \lim_{x\to x_0} f(x) \le M$;

(21) $f(x) \le g(x)$ for $x \underset{\neq}{\approx} x_0 \Rightarrow \lim_{x\to x_0} f(x) \le \lim_{x\to x_0} g(x)$.

**Theorem 11.3D** **Function location theorem.** *If the limit exists,*

(22) $\lim_{x\to x_0} f(x) < M \Rightarrow f(x) < M$ for $x \underset{\neq}{\approx} x_0$.

## 11.4 Limits and Continuous Functions

Limit form of Continuity :

- Let $f(x)$ be defined for $x \approx x_0$.

$f(x)$ is continuous at $x_0 \iff \lim_{x\to x_0} f(x) = f(x_0)$

Positivity Theorem for Continuous Functions :

- $f(x)$ continuous at $x_0$, $f(x_0) > 0 \Rightarrow f(x) > 0$ for $x \approx x_0$

Algebraic Operations on Continuous Functions :

- If $f$ and $g$ are continuous at $x_0$, and $a, b$ are constants, then the following functions are also continuous at $x_0$

$$af + bg \quad , \quad f \cdot g \quad , \quad f/g \quad (\text{given } g \neq 0)$$

Types of Discontinuity :

- Removable Discontinuity
- Jump Discontinuity
- Infinite Discontinuity
- Essential Discontinuity

Composition Theorem :

- Let $x = g(t)$, $x_0 = g(t_0)$, then $\left.\begin{array}{l} g(t) \text{ continuous at } t_0 \\ f(x) \text{ continuous at } x_0 \end{array}\right\} \Rightarrow f(g(t))$ continuous at $x_0$

- Let $x = g(t)$, and suppose $g(a) \leq g(t) \leq g(b)$ for $a \leq t \leq b$.

$$\left.\begin{array}{l} g(t) \text{ continuous on } [a,b] \\ f(x) \text{ continuous on } [g(a), g(b)] \end{array}\right\} \Rightarrow f(g(t)) \text{ continuous on } [a,b]$$

11.5 Continuity and Sequences

Sequential Continuity Theorem :

- $x_n \to a$, $f(x)$ continuous at $a \Rightarrow f(x_n) \to f(a)$

Limit Form of Sequential Continuity :

- If $x_n \to a$, $x_n \neq a$, and $\lim_{x \to a} f(x) = L$, then $\lim_{n \to \infty} f(x_n) = L$

- Let $f(x)$ be defined for $x \approx a$, and suppose that for all $\{x_n\}$ such that $x_n \to a$, $x_n \neq a$, we have $\lim_{n \to \infty} f(x_n) = L$. Then $\lim_{x \to a} f(x) = L$