베이즈통계입문 homework 2

1.

$$p(\theta,r|y) \propto p(y|\theta,r)\, p(\theta|r)\, p(r)$$

$$= r^{\frac{n}{2}} \exp\left\{-\frac{r}{2}\sum_{i=1}^{n}(y_i-\theta)^2\right\} \times r^{\frac{1}{2}} \exp\left\{-\frac{\lambda r}{2}(\theta-\mu)^2\right\} \times r^{\frac{a}{2}-1}\exp\left\{-\frac{b}{2}r\right\}$$

$$= r^{\frac{1}{2}(n+a-1)} \exp\left\{-\frac{r}{2}\left(\sum_{i=1}^{n}(y_i-\theta)^2+\lambda(\theta-\mu)^2+b\right)\right\}$$

$$= r^{\frac{1}{2}(n+a-1)} \exp\left\{-\frac{r}{2}\left(n(\bar{y}-\theta)^2+\sum_{i=1}^{n}(y_i-\bar{y})^2+\lambda(\theta^2-2\theta\mu+\mu^2)+b\right)\right\}$$

$$= r^{\frac{1}{2}(n+a-1)} \exp\left\{-\frac{r}{2}\left((n+\lambda)\theta^2-2(n\bar{y}+\lambda\mu)\theta+n\bar{y}^2+\sum_{i=1}^{n}(y_i-\bar{y})^2+\lambda\mu^2+b\right)\right\}$$

$$= r^{\frac{1}{2}(n+a-1)} \exp\left\{-\frac{r}{2}(n+\lambda)\left(\theta^2-\frac{2(n\bar{y}+\lambda\mu)}{n+\lambda}\theta+\left(\frac{n\bar{y}+\lambda\mu}{n+\lambda}\right)^2\right)\right\}$$

$$\times \exp\left\{-\frac{r}{2}\left(n\bar{y}^2+\lambda\mu^2-\left(\frac{n\bar{y}+\lambda\mu}{n+\lambda}\right)^2(n+\lambda)+\sum_{i=1}^{n}(y_i-\bar{y})^2+b\right)\right\}$$

$$= r^{\frac{1}{2}(n+a-1)} \exp\left\{-\frac{r}{2}(n+\lambda)\left(\theta-\frac{n\bar{y}+\lambda\mu}{n+\lambda}\right)^2\right\}$$

$$\times \exp\left\{-\frac{r}{2}\left(n\bar{y}^2+\lambda\mu^2-\frac{(n\bar{y}+\lambda\mu)^2}{n+\lambda}+\sum_{i=1}^{n}(y_i-\bar{y})^2+b\right)\right\}$$

$$= r^{\frac{1}{2}} \exp\left\{-\frac{1}{2\frac{1}{r(n+\lambda)}}\left(\theta-\frac{n\bar{y}+\lambda\mu}{n+\lambda}\right)^2\right\}$$

$$\times r^{\frac{n+a}{2}-1}\exp\left\{-\frac{r}{2}\left(\frac{(\cancel{n^2}+n\lambda)\bar{y}^2+(n\lambda+\cancel{\lambda^2})\mu^2-\cancel{n^2\bar{y}^2}-2n\lambda\bar{y}\mu-\cancel{\lambda^2\mu^2}}{n+\lambda}+\Sigma(y_i-\bar{y})^2+b\right)\right\}$$

$$= r^{\frac{1}{2}} \exp\left\{-\frac{1}{2\frac{1}{r(n+\lambda)}}\left(\theta-\frac{n\bar{y}+\lambda\mu}{n+\lambda}\right)^2\right\} \rightarrow N\left(\frac{n\bar{y}+\lambda\mu}{n+\lambda},\ \frac{1}{r(n+\lambda)}\right)$$

$$\times r^{\frac{n+a}{2}-1}\exp\left\{-\frac{r}{2}\left(\frac{n\lambda(\bar{y}-\mu)^2}{n+\lambda}+\Sigma(y_i-\bar{y})^2+b\right)\right\}$$

$$\hookrightarrow \text{Gamma}\left(\frac{n+a}{2},\ \frac{1}{2}\left(\Sigma(y_i-\bar{y})^2+\frac{n\lambda}{n+\lambda}(\bar{y}-\mu)^2+b\right)\right)$$

$$\boxed{\begin{aligned} &\therefore\ \theta|r,y \sim N\left(\frac{n\bar{y}+\lambda\mu}{n+\lambda},\ \frac{1}{r(n+\lambda)}\right) \\ &r|y \sim \text{Gamma}\left(\frac{n+a}{2},\ \frac{1}{2}\left(\Sigma(y_i-\bar{y})^2+\frac{n\lambda}{n+\lambda}(\bar{y}-\mu)^2+b\right)\right)\end{aligned}}$$

2.

The Poisson density function is

$$p(y|\theta) = \frac{e^{-\theta}\theta^{y}}{y!}$$

$$\log p(y|\theta) = -\theta + y\log\theta - \log(y!)$$

$$\frac{\partial \log p(y|\theta)}{\partial\theta} = -1 + \frac{y}{\theta}$$

$$\frac{\partial^2 \log p(y|\theta)}{\partial\theta^2} = -\frac{y}{\theta^2}$$

$$I(\theta) = E\left[-\frac{\partial^2 \log p(y|\theta)}{\partial\theta^2}\right] = \frac{\theta}{\theta^2} = \frac{1}{\theta}$$

Jeffreys' prior is

$$p(\theta) = \frac{1}{\sqrt{\theta}} = \theta^{-\frac{1}{2}} \propto \theta^{\frac{1}{2}-1} e^{-0\cdot\theta}$$: kernel of $Gamma(\frac{1}{2}, 0)$

It is improper prior.

3. (a) Since $\tau = \dfrac{\theta}{1-\theta}$, $\theta = \dfrac{\tau}{\tau+1}$.

10 $$f(\underset{\sim}{x}|\theta) \propto \theta^{\sum_i x_i}(1-\theta)^{n-\sum_i x_i}$$

$$= \theta^{y}(1-\theta)^{n-y} \quad \text{where } y = \sum_{i=1}^{n} x_i$$

$$\Rightarrow f(\underset{\sim}{x}|\tau) \propto \left(\frac{\tau}{1+\tau}\right)^{y}\left(1-\frac{\tau}{1+\tau}\right)^{n-y}$$

$$= \tau^{y}(1+\tau)^{-y}(1+\tau)^{y-n}$$

$$= \tau^{y}(1+\tau)^{-n}$$

10 (b) $$\log f(\underset{\sim}{x}|\tau) = y\log\tau - n\log(1+\tau)$$

$$\frac{d}{d\tau}\log f(\underset{\sim}{x}|\tau) = \frac{y}{\tau} - \frac{n}{1+\tau}$$

$$\frac{d^2}{d\tau^2}\log f(\underset{\sim}{x}|\tau) = -\frac{y}{\tau^2} + \frac{n}{(1+\tau)^2}$$

$$E\left(-\frac{d^2\log f(\underset{\sim}{x}|\tau)}{d\tau^2}\right) = \frac{E(y)}{\tau^2} - \frac{n}{(1+\tau)^2} = \frac{n\theta}{\tau^2} - \frac{n}{(1+\tau)^2}$$

$$= \frac{n\frac{\tau}{1+\tau}}{\tau^2} - \frac{n}{(1+\tau)^2} = n\left\{\frac{1+\tau-\tau}{\tau(1+\tau)^2}\right\}$$

$$= \frac{n}{\tau(1+\tau)^2}$$

So $$P_J(\tau) \propto \left[\frac{n}{\tau(1+\tau)^2}\right]^{\frac{1}{2}}$$

$$\propto \tau^{-\frac{1}{2}}(1+\tau)^{-1}$$

(c) From class, $p_J(\theta) \propto \theta^{-\frac{1}{2}}(1-\theta)^{-\frac{1}{2}}$.

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$$\frac{d\tau}{d\theta} = \frac{d}{d\theta}\left(\frac{\theta}{1-\theta}\right) = \frac{1}{(1-\theta)^2}$$

$$p_J(\tau) \propto \tau^{-\frac{1}{2}}(1+\tau)^{-1}$$

$$= \left(\frac{\theta}{1-\theta}\right)^{-\frac{1}{2}}\left(\frac{1}{1-\theta}\right)^{-1}$$

$\uparrow$

$\tau = \frac{\theta}{1-\theta}$

$$= \frac{(1-\theta)^{\frac{3}{2}}}{\theta^{\frac{1}{2}}}$$

So, $$p_J(\tau)\left|\frac{d\tau}{d\theta}\right| = \frac{(1-\theta)^{\frac{3}{2}}}{\theta^{\frac{1}{2}}}\,\frac{1}{(1-\theta)^2} = \frac{1}{\theta^{\frac{1}{2}}(1-\theta)^{\frac{1}{2}}} = p_J(\theta).$$

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(a). $p(y|\theta) = \binom{n}{y} \theta^{y} (1-\theta)^{n-y}$

$p(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} \cdot \theta^{\alpha-1} (1-\theta)^{\beta-1}$

$$\Rightarrow p(y) = \int p(y|\theta)\cdot p(\theta)\, d\theta$$

$$= \int_0^1 \binom{n}{y}\theta^{y}(1-\theta)^{n-y} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)}\, \theta^{\alpha-1}\cdot(1-\theta)^{\beta-1}\, d\theta$$

$$= \binom{n}{y} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} \int_0^1 \theta^{\alpha+y-1} (1-\theta)^{n-y+\beta-1}\, d\theta$$

$$= \frac{\Gamma(n+1)}{\Gamma(y+1)\cdot\Gamma(n-y+1)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+y)\cdot\Gamma(n-y+\beta)}{\Gamma(n+\alpha+\beta)}$$

which is the beta-binomial density.

(b). To make $p(y)$ to be a constant in y, we only need to look at $\Gamma(\alpha+y)\cdot\Gamma(n-y+\beta) / \Gamma(y+1)\cdot\Gamma(n-y+1)$.

Obviously, if $\alpha=\beta=1$, $p(y)$ is a constant in y.

On the other hand, since $p(y)$ is constant in y,

$p(0) = p(n)$, $p(0) = p(1)$.

$p(0) = p(n) \Rightarrow \Gamma(a)\cdot\Gamma(b+n) = \Gamma(a+n)\cdot\Gamma(b)$.

$\Leftrightarrow \Gamma(a)\cdot\Gamma(b)\cdot(b+n-1)\cdots(b+1)\cdot b = \Gamma(a)\cdot\Gamma(b)\cdot(a+n-1)\cdots(a+1)\cdot b$

$\Leftrightarrow (b+n-1)\cdots(b+1)\cdot b = (a+n-1)\cdots(a+1)\cdot a$

$\Leftrightarrow a=b$. ①

$p(0) = p(1) \Rightarrow \Gamma(a)\cdot\Gamma(b+n)/\Gamma(1)\cdot\Gamma(n+1) = \Gamma(a+1)\cdot\Gamma(b+n-1)/\Gamma(2)\cdot\Gamma(n)$

$\Leftrightarrow b+n-1 = na$ ②

From ① ②, we have $a=b=1$.

So $a=b=1$ is necessary and sufficient condition for $p(y)$ to be constant in y.