Some of basic problems in Mathematical Analysis

Question 1

Assume that every $f_k(x)$ is continuous on $[a,b]$ $(k = 0,1,2,\cdots)$.

Is $\sum_{k=0}^{\infty} f_k(x)$ continuous on $[a,b]$?

Everybody knows: f & g : conti on $[a,b]$ $\Rightarrow$ $f+g$: conti on $[a,b]$

Thus by Mathematical Induction, we conclude that

$$\text{if } f_0, f_1, f_2, \cdots, f_n \text{ are all conti on } [a,b], \text{ then } \sum_{k=0}^{n} f_k \text{ is conti on } [a,b]$$

⊙ What about if $\sum_{k=0}^{n} f_k$ (finite sum) is replaced by $\sum_{k=0}^{\infty} f_k$ (infinite sum) ?

Ans (to Question 1) is ***No*** in general.

Example 1.

Obviously, $1, x, x^2, \cdots, x^n, \cdots$ are all conti on $[0,1]$

But

$$f(x) \overset{\text{let}}{=} 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{k=0}^{\infty} x^k$$: conti on $[0,1)$, and *not* conti at $x = 1$

In fact, $f(x) = \dfrac{1}{1-x}$ for $0 \le x < 1$ [so $f(x)$ is conti on $[0,1)$]

& $f(x)$ is not conti at $x = 1$ because $f(1) = \infty$ (i.e., $f(1)$ is not a finite value)

Example 2.

Obviously, $x, \dfrac{x^2}{2}, \dfrac{x^3}{3} \cdots, \dfrac{x^n}{n}, \cdots$ are all conti on $[0,1]$

Is $f(x) := \sum_{k=1}^{\infty} \dfrac{x^k}{k}$ conti on $[0, 1]$?

Ans is No

Note that $f'(x) = \left(\sum_{k=1}^{\infty} \dfrac{x^k}{k} \right)' = \sum_{k=1}^{\infty} x^{k-1} = 1 + x + x^2 + \cdots + x^n + \cdots = \dfrac{1}{1-x}$ for $0 \le x < 1$

Taking $\int_0^x ()\ dt$ gives

$$f(x) = \ln\frac{1}{1-x} = -\ln(1-x) \quad \text{for } 0 \le x < 1$$

But $f(x)$ is not conti at $x=1$ because $f(1) = \sum_{k=1}^{\infty}\frac{1}{k} = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots \overset{p\text{-급수 판정법}}{=} \infty$

Example 3. Is $f(x) := \sum_{k=1}^{\infty}\frac{x^k}{k^2}$ conti on $[0,1]$?

Ans is Yes: An evidence: $f(1) = \sum_{k=1}^{\infty}\frac{1}{k^2}\ (=\frac{\pi^2}{6})$: converges $(p-\text{급수 판정법})$

How can we prove that $f(x) := \sum_{k=1}^{\infty}\frac{x^k}{k^2}$ is conti on $[0,1]$?

A *natural* approach: $f'(x) = \left(\sum_{k=1}^{\infty}\frac{x^k}{k^2}\right)' = \sum_{k=1}^{\infty}\frac{x^{k-1}}{k} = \frac{1}{x}\sum_{k=1}^{\infty}\frac{x^k}{k} = -\frac{\ln(1-x)}{x}$ for $0<x<1$

Notice that $\lim_{x\to 0^+}\frac{-\ln(1-x)}{x} \overset{\text{로피탈 (L'Hospital rule)}}{=} \lim_{x\to 0^+}\frac{1}{1-x} = 1\ (\text{exists})$. Thus, we may write

$$f'(x) = -\frac{\ln(1-x)}{x} \quad \text{for } 0 \le x < 1$$

Taking $\int_0^x ()\ dt$ gives $f(x) = -\int_0^x \frac{\ln(1-t)}{t}\,dt = ??$ (impossible to find a closed form)

This approach [for finding a simple closed form of $f(x)$] is **not** good for our goal.

Do you have any good idea? (will be back shortly later)

Question 2 (not easy): Is there a continuous function $f : [0,1] \to \mathbb{R}$ such that

f is nowhere differentiable on $[0,1]$?

Expect (roughly): In geometrical viewpoint, we may guess there is no such a function

Ans (to Question 2) is unexpectedly ***Yes*** (settled by Van der Waerden, Bolzano(1830), Weierstrass)

Example [famous]. $f(x) = \sum_{n=0}^{\infty}\frac{1}{2^n}\cos(5^n\pi x)$ or $\sum_{n=0}^{\infty}\frac{1}{2^n}\sin(5^n\pi x)$

Question 3 Is there a function $f : \mathbb{R} \to [0,1]$ for which f is

discontinuous at every rational number & continuous at every irrational number ?

Ans: Yes

To construct such kind of functions, we need the following important result in Analysis

Good Series Theorem (it is a corollary of the famous **Weierstrass M-test**: see below)

Suppose that

(i) every $f_k(x) (k = 0, 1, 2, \cdots)$ is conti on the interval I

(ii) $| f_k(x) | \leq M_k$ for all $x \in I$ (note: M_k is independent of $x \in I$)

(iii) $\sum_{k=0}^{\infty} M_k$: converges (or, equivalently, $\sum_{k=0}^{\infty} M_k < \infty$)

Then $\sum_{k=0}^{\infty} f_k(x)$ is conti on I

Cf: **Continuity is a local property (later)**

$$f \text{ is conti on } I \overset{\text{def}}{\Leftrightarrow} f \text{ is conti at each point } x_0 \in I$$

Good Series Theorem-L (Localization of the above theorem)

Suppose that

(i) every $f_k(x)$ is conti at the point $x_0 \in I$ $(k = 0, 1, 2, \cdots)$

(ii) $| f_k(x) | \leq M_k$ for all $x \in I$ (note: M_k is independent of $x \in I$)

(iii) $\sum_{k=0}^{\infty} M_k$: converges (or, equivalently, $\sum_{k=0}^{\infty} M_k < \infty$)

Then $\sum_{k=0}^{\infty} f_k(x)$ is conti at $x_0 \in I$

Weierstrass M- test (A sufficient condition for the *uniform* convergence of Series of functions)

If $| f_k(x) | \leq M_k$ for all $x \in I$ $(k = 0, 1, 2, \cdots)$ (note M_k is indep of $x \in I$)

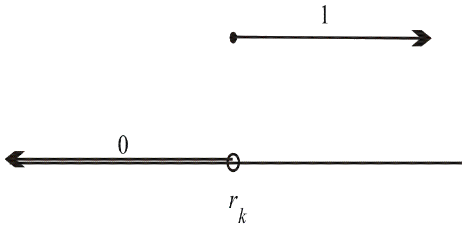
& $\sum_{k=0}^{\infty} M_k$: converges (or, equivalently, $\sum_{k=0}^{\infty} M_k < \infty$)

Then $\sum_{k=0}^{\infty} f_k(x)$ converges *uniformly* on I

Construction of a function $f:\mathbb{R}\to[0,1]$ s.t. $\begin{cases} f \text{ is discontinuous at every rational number} \\ f \text{ is continuous at every irrational number} \end{cases}$

Let $r_1, r_2, \cdots, r_n, \cdots$ be an enumeration of the rational numbers & let

$$f_k(x) = \begin{cases} 1 & \text{if } x \ge r_k \\ 0 & \text{if } x < r_k \end{cases} \qquad (k = 1, 2, \cdots)$$



It is clear that each $f_k(x)$ is conti at every point except r_k

Now we define $f(x) = \sum_{k=1}^{\infty} 2^{-k} f_k(x)$

Claim:

① $f(x)$ is continuous at every irrational number

② $f(x)$ is discontinuous at every rational number

③ $0 \le f(x) \le 1$ for every $x \in \mathbb{R}$

④ $f(x)$ is $\uparrow$ (increasing) on $\mathbb{R}$

Pf.

③ $$0 \le f(x) = \sum_{k=1}^{\infty} 2^{-k} f_k(x) = \sum_{k=1}^{\infty} 2^{-k} \mid f_k(x) \mid \le \sum_{k=1}^{\infty} 2^{-k} = 1$$

④

$$\begin{aligned} x \ge y \quad &\Rightarrow \quad f_k(x) \ge f_k(y) \quad (\because\ f_k \text{ is } \uparrow\ ) \\ &\Rightarrow \quad 2^{-k} f_k(x) \ge 2^{-k} f_k(y) \\ &\Rightarrow \quad \sum_{k=1}^{\infty} 2^{-k} f_k(x) \ge \sum_{k=1}^{\infty} 2^{-k} f_k(y) \\ &\quad\ \therefore \quad f(x) \ge f(y) \end{aligned}$$

$\therefore \quad f(x)$ is $\uparrow$ (increasing)

① Choose an arbitrary irrational number x_0 and fix it.

We will show that $f(x)$ is continuous at x_0

Note that every $2^{-k} f_k(x)$ is continuous at x_0 .

We have seen that

$$| 2^{-k} f_k(x) | \leq 2^{-k} \quad \forall x \in \mathbb{R} \quad \& \quad \sum_{k=1}^{\infty} 2^{-k} \text{ : converges (in fact, } \sum_{k=1}^{\infty} 2^{-k} = 1 < \infty)$$

Therefore by **Good Series Theorem-L**,

$$\sum_{k=1}^{\infty} 2^{-k} f_k(x) \quad \text{is conti at } x_0$$

② Choose any rational number x_0 and fix it. Then $x_0 = r_m$ for some m

We will show that $f(x)$ is *not* continuous at r_m

Write $f(x) = \dfrac{1}{2^m} f_m(x) + \displaystyle\sum_{k \neq m}^{\infty} \frac{1}{2^k} f_k(x)$

Recall that each $\dfrac{1}{2^k} f_k(x)$ is conti at every point x *if* $x \neq r_k$

& disconti at $x = r_k$

Thus if $k \neq m$, then $\dfrac{1}{2^k} f_k(x)$ is conti at r_m

So $\displaystyle\sum_{k \neq m}^{\infty} \frac{1}{2^k} f_k(x)$ is conti at the point r_m (by **Good Series Theorem-L**)

& clearly $\dfrac{1}{2^m} f_m(x)$ is disconti at r_m

If $f(x)$ is conti at r_m , then $f(x) - \displaystyle\sum_{k \neq m}^{\infty} \frac{1}{2^k} f_k(x)$ should be conti at r_m .

Then $\dfrac{1}{2^m} f_m(x)$ is conti at r_m . This is a contradiction.

Therefore, $f(x)$ is not continuous at r_m

Return to Example 3: Prove $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k^2}$ is conti on $[0, 1]$

Pf. Every $\dfrac{x^k}{k^2}$ $(k \geq 1)$ is continuous on $[0, 1]$. Also

$$\sum_{k=1}^{\infty} \left| \frac{x^k}{k^2} \right| \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \text{ for } \forall x \in [0, 1] \quad \& \quad \sum_{k=1}^{\infty} \frac{1}{k^2} \text{ converges}$$

Thus by **Good Series Theorem** (or, Weierstrass M-test), $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k^2}$ is conti on $[0, 1]$

Ex. Show that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} \cos(5^n \pi x)$ is continuous on $\mathbb{R}$