STA 3021: Stochastic Processes
Midterm 1 (6:00 PM - 7:15 PM on Oct 8, 2020)

Pledge: *I have neither given nor received any unauthorized aid during this exam.*

Student ID & Full Name: ________________________________

Instructions: This test is a closed book exam, but you are allowed to use calculator. Clarity of your answer will also be a part of credit. When needed, use the notation $\Phi(z) = P(Z < z)$ for a standard normal distribution Z. Show your ALL work neatly.

1. (10 points) Let $X_1, \ldots, X_n$ be a sequence of IID random variables with mean μ and variance σ^2. State the following theorems as precisely as you can.

   (a) Weak law of large numbers.

   if $\mu < \infty$ (finite) then for $\forall \epsilon > 0$

   $\lim_{n\to\infty} P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \epsilon\right) = 0$ OR $\lim_{n\to\infty} P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| < \epsilon\right) = 1$

   (b) Central limit theorem.

   $Z_n = \dfrac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\cdot\sigma} \xrightarrow{d} N(0,1)$ OR $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0,1)$

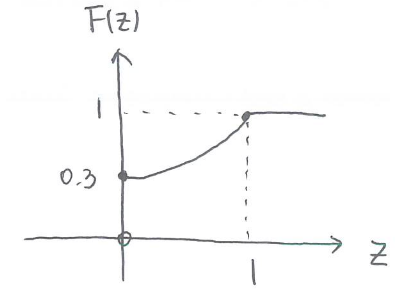
2. (10 points) There are three coins in a box. One is a two-headed coin, another is a fair coin, and the third is a biased coin that comes up head 75 percent of time. When one of the three coins is selected at random and flipped, it shows heads. What is the probability that it was the two-headed coin?

   $P(\text{two headed} \mid \text{head}) = \dfrac{\frac{1}{3}}{\frac{1}{3}\left(1+\frac{1}{2}+\frac{3}{4}\right)} = \dfrac{\frac{1}{3}}{\frac{1}{3}\times\frac{9}{4}} = \dfrac{4}{9}$

3. (10 points) For a random variable Z with cdf

$$F(z) = \begin{cases} 0, & z < 0, \\ .3, & z = 0, \\ .3 + .7z^2, & 0 < z < 1, \\ 1, & z \geq 1 \end{cases}$$

Sketch the cdf on a graph and find Var(Z).



$$E(Z) = 0 \times 0.3 + \int_0^1 1.4 z^2 \, dz$$

$$= \frac{1}{3} \times \frac{7}{5} \times 1 = \frac{7}{15}$$

$$f(z) = \begin{cases} 0.3 & , \text{ if } z = 0 \\ 1.4z & , \text{ if } 0 < z < 1 \\ 0 & , \text{ o.w} \end{cases}$$

$$E(Z^2) = 0^2 \times 0.3 + \int_0^1 1.4 z^3 \, dz$$

$$= \frac{1}{4} \times \frac{7}{5} = \frac{7}{20}$$

$$\text{Var}(Z) = \frac{7}{20} - \left(\frac{7}{15}\right)^2 = 0.132 \quad \left(= \frac{119}{900}\right)$$

4. (10 points) Consider 25 people and suppose that each of them has a birthday that is equally likely to be any of the 365 days of the year. Furthermore, assume that their birthdays are independent, and let A be the event that no two of them share the same birthday. Employ the Poisson paradigm to approximate $P(A)$.

X : # of pair share same birthday $\binom{25}{2}$

$$\rightarrow X \sim B\left(\binom{25}{2}, \binom{365}{1} \times \left(\frac{1}{365}\right)^2\right) \equiv B\left(\binom{25}{2}, \frac{1}{365}\right)$$

$$\approx \text{pois}\left(\binom{25}{2} \times \frac{1}{365}\right)$$

$$\rightarrow P(A) = P(X=0) = \exp\left(-\binom{25}{2} \times \frac{1}{365}\right) = 0.439$$

5. (20 points) Let X and Y follows the independent Geometric distribution with the same parameter p. Define

$$U = \min(X, Y), \quad V = X - Y.$$

Find the joint probability mass function of U and V.

$f(x,y) = p^2(1-p)^{x+y} \; I(x = 0, 1, 2 \cdots) \; I(y = 0, 1, 2, \cdots)$

i) $X < Y$

$u = X$, $V = X - Y$ $\rightarrow$ $X = u$, $Y = u - v$

$f(u,v) = p^2(1-p)^{2u-v} \; I(u < u - v)$

$= p^2(1-p)^{2u-v} \; I(v < 0)$

ii) $X \geq Y$

$u = Y$, $v = X - Y$ $\rightarrow$ $Y = u$, $X = v + u$

$f(u,v) = p^2(1-p)^{2u+v} \; I(u + v \geq v)$

$= p^2(1-p)^{2u+v} \; I(v \geq 0)$

$\therefore \; f(u,v) = p^2(1-p)^{2u+|v|}$

6. (20 points) Let X be a discrete random variable from a probability space (S, P) with expected value given by $EX = \sum_i x_i P_X(x_i)$. Show that

$$EX = \sum_{\omega \in S} X(\omega) P(\omega)$$

$$EX = \sum_i x_i P_X(x_i) = \sum_i x_i P(X = x_i)$$

$$= \sum_i x_i P(w \in S \mid X(w) = x_i)$$

$$= \Sigma x_i P(w \in S_i)$$

$$= \Sigma x_i \sum_{w \in S_i} P(w)$$

$$= \sum_i \sum_{w \in S_i} x_i P(w)$$

$$= \sum_i \sum_{w \in S_i} X(w) P(w)$$

$$= \sum_{w \in S_i} X(w) P(w)$$

7. (20 points) For a decreasing sequence of events $\{E_n\}$, show that

$$\lim_{n\to\infty} P(E_n) = P\left(\lim_{n\to\infty} E_n\right)$$

E_n is decreasing set $\rightarrow$ $\lim E_n = \bigcap_{i=1}^{\infty} E_i$

E_n^c is increasing set

let $A_1 = E_1^c$

$A_2 = E_2^c - E_1^c$

$\vdots$

$E_n^c = \bigcup_{i=1}^{n} E_i^c = \bigcup_{i=1}^{n} A_i$ $\quad$ $E_n = \left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c$

$1 - P(E_n) = P(E_n^c) = P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$ $\quad$ ($\because$ A_i's are disjoint)

$$1 - \lim_{n\to\infty} P(E_n) = \lim P(E_n^c) = \lim \sum_{i=1}^{n} P(A_i)$$

$$= \sum_{i=1}^{\infty} P(A_i)$$

$$= P\left(\bigcup^{\infty} A_i\right)$$

$$= 1 - P\left(\bigcap^{\infty} A_i^c\right)$$

$$= 1 - P(\lim E_n)$$

$\therefore \lim P(E_n) = P(\lim E_n)$