Stochastic Processes (STA3021)
HW8 Solution

1. Exercise Chapter 5 #62

   This is an application of Bernoulli splitting of Poisson process. Suppose that the time $t = 1$ is fixed. Since the typographical error follows Poisson(λ), it follows that

   (a)
   $$X_1 \sim \text{Poisson}(p_1(1-p_2)\lambda), \quad X_2 \sim \text{Poisson}((1-p_1)p_2\lambda),$$
   $$X_3 \sim \text{Poisson}(p_1 p_2 \lambda), \quad X_4 \sim \text{Poisson}((1-p_1)(1-p_2)\lambda),$$
   and they are all **independent**. Thus the joint distribution of (X_1, X_2, X_3, X_4) becomes
   $$f(x_1, x_2, x_3, x_4) = f(x_1; p_1(1-p_2)\lambda) f(x_2; (1-p_1)p_2\lambda) f(x_3; p_1 p_2 \lambda) f(x_4; (1-p_1)(1-p_2)\lambda),$$
   where $f(x;\theta)$ denotes the pmf of Poisson distribution with rate θ.

   (b) Immediately follows from (a)

   (c) Note from (b) that
   $$\frac{1-p_2}{p_2} = \frac{E(X_1)}{E(X_3)} \quad \Rightarrow p_2 = \frac{E(X_3)}{E(X_1)+E(X_3)}$$
   $$\frac{1-p_1}{p_1} = \frac{E(X_2)}{E(X_3)} \quad \Rightarrow p_1 = \frac{E(X_3)}{E(X_2)+E(X_3)}$$
   Now, it is given that an estimator of $E(X_i)$ is x_i (I used small x to emphasize that this is a **sample** data), hence we have that
   $$\hat{p}_2 = \frac{x_3}{x_1+x_3}, \quad \hat{p}_1 = \frac{x_3}{x_2+x_3}.$$
   To estimate λ only by x_1, x_2 and x_3, observe that
   $$E(X_1+X_2+X_3) = \lambda(1-(1-p_1)(1-p)2))$$
   $$\hat{\lambda} = \frac{x_1+x_2+x_3}{1-x_1x_2/(x_2+x_3)(x_1+x_3)}.$$

   (d) Since $\lambda = E(X_1 + \ldots + X_4)$, we can estimate X_4 by
   $$\hat{X}_4 = \hat{\lambda} - x_1 - x_2 - x_3.$$

2. Exercise Chapter 5 #78

   Recall that for Nonhomogeneous Poisson Process (NPP)
   $$N(t) \sim \text{Poisson}\left(\int_0^t \lambda(u)du\right).$$

Since

$$\lambda(u) = \begin{cases} 4, & 0 \le u < 2, \\ 8, & 2 \le u < 4, \\ u+4, & 4 \le u < 6, \\ -2u+22, & 6 \le u \le 9. \end{cases}$$

the probability distribution of the number of customers that enter the store on a given day is Poisson distribution with parameter

$$\int_0^9 \lambda(u)du = 63.$$

3. Exercise Chapter 5 #80
   Recall the definition of NPP
   (a) Even though NPP can related to Bernoulli process with different success probability in the subintervals, the inter-arrival times are no longer **independent**. This is because the distribution of inter-arrivals are always depends on time t. For example, one can show that
   $$P(T_2 > t|T_1) = e^{m(T_1+t)-m(T_1)},$$
   where $m(s) = \int_0^s \lambda(u)du$ is the mean rate function. Hence T_1 and T_2 are not independent.
   (b) No
   (c) Note that
   $$P(T_1 > t) = P(N(t) = 0) = \exp\left(-\int_0^t \lambda(u)du\right).$$

4. Exercise Chapter 5 #86

   (a) Note that
   $$N(t) = \begin{cases} N_1(t), \text{with prob } .3 \\ N_2(t) \text{with prob } .7 \end{cases},$$
   where $N_1(t)$ represents the number of storms in good years and $N_1(t)$ are those in bad years. Thus,
   $$P(N(t) = n) = .3\frac{e^{-3t}(3t)^n}{n!} + .7\frac{e^{-5t}(5t)^n}{n!}.$$
   (b) No
   (c) Once the type of year is fixed, then it follows Poisson process. Since $N(t)$ denote the number of storms during the first t time units of next year, it has stationary increments.
   (d) No, because the number of event happen in a given interval depends on the type of year.

(e) Using Bayes rule, we have

$$P(good|N(1)=3) = \frac{P(N(1)=3|good)P(good)}{P(N(1)=3|good)P(good)+P(N(1)=3|bad)P(bad)}$$

$$= \frac{e^{-3}3^3/3!.3}{(e^{-3}3^3/3!).3+(e^{-5}5^3/3!).7}$$

5. Exercise Chapter 5 #53
Let $X(t)$ be the water level of the reservoir at t-th day. Then, it can be written as

$$X(t) = 5000 - 1000t + \sum_{i=1}^{N(t)} Y_i,$$

where $N(t) \sim PP(.2)$ and $Y_i = 5000$ with prob. .8 and $Y_i = 8000$ with prob. .2.

(a) Note that

$$P(X(5)=0) = P(\sum_{i=1}^{5} Y_i = 0) = P(N(5)=0) = e^{-1}$$

since $N(5)$ follows Poisson$(.2 \times 5)$.

(b) The problem is asking

$$P(X(5)=0) + P(X(6)=0) + \ldots + P(X(10)=0)$$

$$= e^{-1} + 0 + 0 + 0 + 0 + P(N(5)=1, N(10)-N(5)=0, Y_1=5000)$$

$$= e^{-1} + e^{-1}.8e^{-1}.$$

6. Exercise Chapter 5 #88 This is an example of Compound Poisson process. Let $X(t)$ be the amount of money withdrawn from the ATM machine. Then

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \quad \{N(t), t \geq 0\} \sim \text{PP}(12), E(Y_i) = 30, \text{Var}(Y_i) = 50^2.$$

The problem is asking to approximate $P(X(15) \leq 6000)$ (since it is a sum!). Note that

$$P(X(15) \leq 6000) = P\left(\frac{X(15)-5400}{\sqrt{612000}} \leq \frac{6000-5400}{\sqrt{612000}}\right)$$

$$\approx P(Z \leq .767) = .78$$

since $E(X(15)) = E(N(15))E(Y_1) = 12*15*30 = 5400$ and $\text{Var}(X(15)) = E(N(15))E(Y_1^2) = 12 * 15 * (30^2 + 50^2) = 612000$.