1)

$$\int_0^\infty \frac{x^p}{1+x^p}\,dx \le \int_0^\infty x^p\,dx$$, converges when $p<1$, diverges $p\ge 1$

$\therefore$ By the comparison theorem, $\int_0^\infty \frac{x^p}{1+x^p}\,dx$ converges for $p<1$.

2)

$\lim_{n\to\infty} f_n(x) = 0$ for $x\in[0,\infty)$ $\therefore$ The sequence converges pointwise.

$\|f_n(x)-0\| = \|f_n(x)\| = \|\frac{1}{x}\sin\frac{x}{n}\| \le \|\frac{1}{x}\|$, since $\frac{1}{x}$ increases as $x\to 0^+$

it does not converge uniformly on $[0,\infty)$

3)

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{2^{n+1}}{2(n+1)^2+(n+1)}\left(\dfrac{x}{1-x}\right)^{n+1}}{\dfrac{2^n}{2n^2+n}\left(\dfrac{x}{1+x}\right)^n} = \frac{2^{n+1}(2n^2+n)}{2^n[2(n+1)^2+(n+1)]}\left(\frac{x}{1-x}\right)$$

$$= \frac{(2n^2+n)}{[2(n+1)^2+(n+1)]}\left(\frac{2x}{1-x}\right)$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{2x}{1-x}, \quad R = \frac{1-x}{2x}$$

$\therefore$ the series uniformly converges for $x\in(-R, R)$, where $R<\frac{1}{2}$

4) $f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n+1}}{n(n+1)}$, $x \in [-1, 1]$

$= \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{x^{n+1}}{n} - \frac{x^{n+1}}{n+1} \right)$, telescoping series

$= x^2 - \frac{x^2}{2}$, $x \in [-1, 1]$

5) i) $f(x) = \sum_{n=0}^{\infty} \frac{\cos(4^n x)}{5^n}$, continuous for all $x \in [0, 2\pi]$

$|\Delta f| = \frac{2}{5^n}$, $\lim_{n \to \infty} \frac{2}{5^n} = 0$

$\therefore \exists \varepsilon > 0$ s.t. $|f(x) - f(x_0)| < \varepsilon$

for any $\delta > 0$ s.t. $|x - x_0| < \delta$ ($f(x)$ is differentiable)

ii) $\lim_{n \to \infty} f(x) = 0$, pointwise convergence ✓

$\Rightarrow \| f(x) - 0 \| \le \sum_{n=0}^{\infty} \frac{1}{5^n}$, converges uniformly ✓

$\Rightarrow$ We can apply term-by-term differentiation.

$\Rightarrow f'(x) = \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n \cos(4^n x)$, which is continuous on $x \in [0, 2\pi]$
so it is integrable,

$\Rightarrow \lim_{n \to \infty} f'(x) = 0$, $\| f'(x) - 0 \| = \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n$, $f'(x)$ is point-wise and uniformly conv.
$\Rightarrow$ integrable term-by-term.

$\therefore$ By the First Fundamental Theorem of Calculus,

$\int_0^{2\pi} f'(x)\,dx = f(2\pi) - f(0) = \sum_{n=0}^{\infty} \frac{[\cos(4^n(2\pi)) - 1]}{5^n}$

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6) $M_i = \sup_{x \in [0, 2\pi\}} f(x) = 1$

$m_i = \inf f(x) = 0$

$\| U_f(P) - L_f(P) \| \leq 2\pi$, where P: partitions of $[0, 2\pi]$

$> \varepsilon$

$\therefore$ $f(x)$ is not Riemann-Integrable on $[0, \frac{\pi}{2}]$

7) ~~$f(x) = f'(x) = \cdots = f^{(n)}(x)$~~ ~~$f(0) = f'(0) = \cdots = f^{(n)}(a) = 0$~~

ii) $\lim_{x \to \infty} f(x) = 0$, pointwise convergent

$\| f(x) - 0 \| < \infty$, as $x \to 0^+$

$\therefore$ ~~Since $T_n(x)$ and $f(x)$ should have the same,~~

$T_n(X)$ does not converge uniformly since $f(x)$ does not.

8) 1) False

2) True

3) False

4) False

5) True

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7) i) $f'(0) = f''(0) = \cdots = f^{(n)}(0) = 0$

$T_n(x) = -1$