Homework 4 (STA 3021) Solutions

1. Chapter 4 Exercise #1
   Let $\{X_n, n \geq 0\}$ be the number of white balls in the first urn. Then, according to the scheme that a ball is exchanges each other, it is deduced that only the number of white balls at n-th step determines the number of white balls at $(n+1)$-th step. Hence it is a DTMC with the state space $S = \{0, 1, 2, 3\}$ and the transition probability

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/9 & 4/9 & 4/9 & 0 \\ 0 & 4/9 & 4/9 & 1/9 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

   For example,

   $P_{1,0} = P(X_{n+1} = 0 | X_n = 1) = P(\text{white selected from the first urn} \cap \text{black selected from the secon}$

   $= P(\text{white selected from the first urn})P(\text{black selected from the second urn}) = 1/3 \times 1/3 = 1/9.$

2. Chapter 4 Exercise #3
   Let the state space be the whether spell for the last three days. Then,

$$S = \{RRR, RRD, RDR, RDD, DRR, DRD, DDR, DDD\},$$

   where R is rain and D is dry. From the condition in the problem, the one-step transition probability is given by

$$P = \begin{array}{c} \\ RRR \\ RRD \\ RDR \\ RDD \\ DRR \\ DRD \\ DDR \\ DDD \end{array} \begin{array}{cccccccc} RRR & RRD & RDR & RDD & DRR & DRD & DDR & DDD \\ \hline 0.8 & 0.2 & & & & & & \\ & & 0.4 & 0.6 & & & & \\ & & & & 0.6 & 0.4 & & \\ & & & & & & 0.4 & 0.6 \\ 0.6 & 0.4 & & & & & & \\ & & 0.4 & 0.6 & & & & \\ & & & & 0.6 & 0.4 & & \\ & & & & & & 0.2 & 0.8 \end{array}$$

3. Chapter 4 Exercise #5
   Since the state space $S = \{0, 1, 2\}$,

$$EX_3 = 0 \times P(X_3 = 0) + 1 \times P(X_3 = 1) + 2 \times P(X_3 = 2).$$

   The marginal probability becomes

$$P(X_3 = j) = \sum_{i=0}^{2} P(X_3 = j | X_0 = i) P(X_0 = i)$$

$$= \frac{1}{4}P(X_3 = j|X_0 = 0) + \frac{1}{4}P(X_3 = j|X_0 = 1) + \frac{1}{2}P(X_3 = j|X_0 = 2).$$

Three step transition probability comes from the Chapman-Kolmogorov equations, namely,

$$P^{(3)} = P^3 = \begin{pmatrix} 13/36 & 11/54 & 47/108 \\ 4/9 & 4/27 & 11/27 \\ 5/12 & 2/9 & 13/36 \end{pmatrix},$$

Thus, it gives that

$$P(X_3 = 1) = 1/4 * 11/54 + 1/4 * 4/27 + 1/2 * 2/9 = .199$$

$$P(X_3 = 2) = 1/4 * 47/108 + 1/4 * 11/27 + 1/2 * 13/36 = .391$$

and finally $EX_3 = .199 + 2 * .391 = .981$.

4. Chapter 4 Exercise #7
Recall the Example 4.4, where X_n is the whether spell for two consecutive days. Hence $S = \{RR, NR, RN, NN\}$ and

$$P = \begin{pmatrix} .7 & 0 & .3 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & .4 & 0 & .6 \\ 0 & .2 & 0 & .8 \end{pmatrix}$$

The probability in the problem is given by

$$P(X_{n+1} = NR \text{ or } RR|X_{n-1} = NN)$$

$$= P(X_{n+1} = NR|X_{n-1} = NN) + P(X_{n+1} = RR|X_{n-1} = NN)$$

$$= P^{(2)}_{NN,NR} + P^{(2)}_{NN,RR}.$$

Since

$$P^{(2)} = P^2 = \left(\begin{array}{ccccc} & RR & NR & RN & NN \\ RR & .49 & .12 & .21 & .18 \\ NR & .35 & .2 & .15 & .3 \\ RN & .2 & .12 & .2 & .48 \\ NN & .1 & .16 & .1 & .64 \end{array}\right),$$

the probability becomes $.1 + .16 = .26$.

5. Chapter 4 Exercise #8
Let $\{X_n, n \geq 0\}$ denote the coin number flipped on the n-th day. Then, the state space becomes $S = \{1, 2\}$ and transition probability becomes

$$P = \begin{pmatrix} .7 & .3 \\ .6 & .4 \end{pmatrix}$$

The initial distribution is given by $P(X_0 = 1) = P(X_1 = 2) = 1/2$. First question is $P(X_3 = 1)$. Since

$$P^3 = \begin{pmatrix} .667 & .333 \\ .666 & .334 \end{pmatrix}$$

$P(X_3 = 1) = P(X_3 = 1|X_0 = 1)P(X_0 = 1) + P(X_3 = 1|X_0 = 2)P(X_0 = 2) = .6665.$

For the second question, now we are interested in the outcome of coin, so define MC $\{Y_n, n \geq 0\}$ be the outcome of coin flip on the n-th day. Then, $S = \{H, T\}$ with transition probability

$$Q = \begin{pmatrix} .7 & .3 \\ .6 & .4 \end{pmatrix}$$

For example

$$P(Y_{n+1} = H|Y_n = H) = P(\text{Head appears from coin 1}) = .7$$

By calculating Q^4, $P(Friday = H|Monday = H) = .6667$.