Final Solution (2022)

1. (a) From the property of exponential family, ΣX_i is a CSS for λ

$I(X_1 \le 1)$ is an obvious unbiased estimator for λ

$$\Rightarrow \hat{\eta}^{MVUE} = E[I(X_1 \le 1) \mid \Sigma X_i] = P(X_1 = 0 \mid \Sigma X_i = y) + P(X_1 = 1 \mid \Sigma X_i = y)$$

$$= \frac{P(X_1 = 0, \sum_{i=2}^{n} X_i = y)}{P(\sum_{1}^{n} X_i = y)} + \frac{P(X_1 = 1, \sum_{i=2}^{n} X_i = y-1)}{P(\Sigma X_i = y)}$$

$$= \left(\frac{n-1}{n}\right)^{y} + \frac{y}{n}\left(\frac{n-1}{n}\right)^{y-1}$$

$$\therefore \hat{\eta}^{MVUE} = \left(\left(1-\frac{1}{n}\right)^{n}\right)^{\bar{X}} + \bar{X}\left(\left(1-\frac{1}{n}\right)^{n}\right)^{\bar{X}-\frac{1}{n}}$$

(b) From $(1-\frac{1}{n})^{n} \longrightarrow e^{-1}$ & $\bar{X} \xrightarrow{P} \lambda$ by WLLN,

$$\hat{\eta}^{MVUE} = \left(\left(1-\frac{1}{n}\right)^{n}\right)^{\bar{X}} + \bar{X}\left(\left(1-\frac{1}{n}\right)^{n}\right)^{\bar{X}-\frac{1}{n}}$$

$$\xrightarrow{P} e^{-\lambda} + \lambda e^{-\lambda} = (1+\lambda)e^{-\lambda} = \eta$$

2. (a)

size $\alpha = 0.3$ Critical region	$\{5\}, \{3,4\}$	$\{2,4\}$	$\{0,1\}$
power	0.1 0.45	0.5	0.2

$\Rightarrow$ MP critical region is $\{2,4\}$

(b) $\text{size} = P(X_1X_2 \le 1 \mid \theta = 0) = P(X_1X_2 = 0 \mid \theta = 0) + P(X_1X_2 = 1 \mid \theta = 0)$

$= P(X_1 = 0 \mid \theta = 0) + P(X_2 = 0 \mid \theta = 0) - P(X_1 = 0, X_2 = 0 \mid \theta = 0) + P(X_1 = 1, X_2 = 1 \mid \theta = 0)$

$= 0.05 + 0.05 - 0.0025 + 0.0625 = 0.16$

$\text{power} = P(X_1X_2 \le 1 \mid \theta = 1) = P(X_1X_2 = 0 \mid \theta = 1) + P(X_1X_2 = 1 \mid \theta = 1)$

$= P(X_1 = 0 \mid \theta = 1) + P(X_2 = 0 \mid \theta = 1) - P(X_1 = 0, X_2 = 0 \mid \theta = 1) + P(X_1 = 1, X_2 = 1 \mid \theta = 1)$

$= 0.1 + 0.1 - 0.01 + 0.01 = 0.2$

3. (a) $\hat{\mu}_1^{\Omega} = \bar{X}$, $\hat{\mu}_2^{\Omega} = \bar{Y}$

Under $H_0: \mu_1 = 2\mu_2$,

$$L(\mu_2) = \left(\frac{1}{\sqrt{2\pi}}\right)^8 \exp\left(-\frac{\Sigma(X_i - 2\mu_2)^2}{2}\right)\left(\frac{1}{\sqrt{2\pi}}\right)^5 \exp\left(-\frac{\Sigma(Y_j - \mu_2)^2}{2}\right)$$

$$l(\mu_2) = -\frac{1}{2}\left(\Sigma(X_i - 2\mu_2)^2 + \Sigma(Y_j - \mu_2)^2\right)$$

$$l'(\mu_2) = \Sigma X_i - 32\mu_2 + \Sigma Y_j - 5\mu_2 = 0$$

$$\Rightarrow \hat{\mu}_2^{\Omega_0} = \frac{2\Sigma X_i + \Sigma Y_j}{37} = \frac{16}{37}\bar{X} + \frac{5}{37}\bar{Y}, \quad \hat{\mu}_1^{\Omega_0} = 2\hat{\mu}_2^{\Omega_0}$$

$$\Lambda = \frac{L(\hat{\mu}_1^{\Omega_0}, \hat{\mu}_2^{\Omega_0})}{L(\hat{\mu}_1^{\Omega}, \hat{\mu}_2^{\Omega})} = \frac{\exp\left(-\Sigma(X_i - 2\hat{\mu}_2^{\Omega_0})^2 - \Sigma(Y_j - \hat{\mu}_2^{\Omega_0})^2\right)}{\exp\left(-\Sigma(X_i - \bar{X})^2 - \Sigma(Y_j - \bar{Y})^2\right)} < k$$

$$\Leftrightarrow \Sigma(X_i - \bar{X})^2 + \Sigma(Y_j - \bar{Y})^2 - \left(\Sigma(X_i - \bar{X})^2 + 8(\bar{X} - 2\hat{\mu}_2^{\Omega_0})^2 + \Sigma(Y_j - \bar{Y})^2 + 5(\bar{Y} - \hat{\mu}_2^{\Omega_0})^2\right) < k'$$

$$\Leftrightarrow 8\left(\bar{X} - \frac{32}{37}\bar{X} - \frac{10}{37}\bar{Y}\right)^2 + 5\left(\bar{Y} - \frac{16}{37}\bar{X} - \frac{5}{37}\bar{Y}\right)^2 > k''$$

$$\Leftrightarrow (\bar{X} - 2\bar{Y})^2 > C^2$$

$$\Leftrightarrow \bar{X} - 2\bar{Y} < -C \quad \text{or} \quad \bar{X} - 2\bar{Y} > C$$

$$\bar{X} - 2\bar{Y} \sim N\left(0, \frac{37}{40}\right)$$

$$\Rightarrow \text{If } \left|\frac{\bar{X} - 2\bar{Y}}{\sqrt{\frac{37}{40}}}\right| > 1.96, \text{ then reject } H_0$$

(b) $\bar{X} - 2\bar{Y} = 6 - 4 = 2$

Because $\dfrac{\bar{X} - 2\bar{Y}}{\sqrt{\frac{37}{40}}} = \sqrt{\dfrac{160}{37}} > 2$, we reject H_0

4.

(a) $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1 > \theta_0$

$$\frac{L(\theta_0)}{L(\theta_1)} = \frac{\left(\frac{1}{\theta_0}\right)^n \prod_{i=1}^n x_i^{\frac{1}{\theta_0}-1}}{\left(\frac{1}{\theta_1}\right)^n \prod_{i=1}^n x_i^{\frac{1}{\theta_1}-1}} = \left(\frac{\theta_1}{\theta_0}\right)^n \left(\prod_{i=1}^n x_i\right)^{\frac{1}{\theta_0}-\frac{1}{\theta_1}} < k$$

$$\Longleftrightarrow \prod_{i=1}^n x_i < k' \Longleftrightarrow \sum \log X_i < C \quad : \text{UMP test}$$

(b) $-\log X_i \sim Gamma(1, \theta) \Rightarrow -\sum_{i=1}^3 \log X_i \sim Gamma(3, \theta)$

Under H_0, $-\sum_{i=1}^3 \log X_i \sim Gamma(3, 1)$

$\Rightarrow -2\sum_{i=1}^3 \log X_i \sim Gamma(3, 2) = \chi^2_6$

$\Rightarrow$ UMP size $\alpha = 0.05$ test rejects H_0

if $-2\sum_{i=1}^3 \log X_i > \chi^2_{6, 0.05}$

Because, $-2\sum_1^3 \log X_i = 2(2 + \frac{1}{2} + 1) = 7 < \chi^2_{6, 0.05} = 12.592$,

we cannot reject H_0.