# HW II

1. (a)

$$y_{ij} \mid a_i \overset{\text{ind.}}{\sim} N(\mu + a_i + \beta_j, \sigma^2), \quad i = 1, \cdots, m, \quad j = 1, \cdots, n$$

$$a_i \overset{\text{iid}}{\sim} N(0, \sigma_a^2)$$

$$N = mn.$$

$$\Rightarrow y_{ij} = \mu + a_i + \beta_j + \varepsilon_{ij}$$

where

$$a_i \overset{\text{iid}}{\sim} N(0, \sigma_a^2)$$

$$\varepsilon_{ij} \overset{\text{iid}}{\sim} N(0, \sigma^2).$$

$$\begin{pmatrix} y_{11} \\ \vdots \\ y_{1n} \\ y_{21} \\ \vdots \\ y_{2n} \\ \vdots \\ y_{m1} \\ \vdots \\ y_{mn} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \\ \hline 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 0 & 0 & & 1 \\ \hline \vdots & \vdots & \vdots & & \vdots \\ \hline 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix}}_{(\mathbf{1}_N,\ \mathbf{1}_m \otimes I_n) \,=\, X} \begin{pmatrix} \mu \\ \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} + \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & \cdots & 0 \\ \hline 0 & 1 & & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 1 & & 0 \\ \hline \vdots & & & \\ \hline 0 & 0 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}}_{I_m \otimes \mathbf{1}_n \,=\, Z} \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} + \begin{pmatrix} \varepsilon_{11} \\ \vdots \\ \varepsilon_{1n} \\ \varepsilon_{21} \\ \vdots \\ \varepsilon_{2n} \\ \vdots \\ \varepsilon_{m1} \\ \vdots \\ \varepsilon_{mn} \end{pmatrix}$$

or $\underset{\sim}{y} = X\underset{\sim}{\beta} + Z\underset{\sim}{b} + \underset{\sim}{\varepsilon}$,

$$E(y_{ij}) = \mu + \beta_j$$

$$Var(y_{ij}) = \sigma_a^2 + \sigma^2$$

$$Cov(y_{ij}, y_{i'j'}) = \begin{cases} \sigma_a^2, & i = i', \ j \neq j' \\ 0, & i \neq i' \end{cases}$$

$$Var(\underset{\sim}{y}) = diag\{\Sigma_1, \Sigma_2, \cdots, \Sigma_i, \cdots, \Sigma_m\}$$

where $\Sigma_i = \sigma_a^2 J + \sigma^2 I$.

(b) $y_{ij} \mid a_i, b_j \overset{\text{indep}}{\sim} N(\mu + a_i + b_j, \sigma^2)$

$$\left.\begin{array}{l} a_i \overset{\text{iid}}{\sim} N(0, \sigma_a^2) \\ b_j \overset{\text{iid}}{\sim} N(0, \sigma_b^2) \end{array}\right\rangle \text{indep.}$$

$\Rightarrow y_{ij} = \mu + a_i + b_j + \varepsilon_{ij}$

where $$\left.\begin{array}{l} a_i \overset{\text{iid}}{\sim} N(0, \sigma_a^2) \\ b_j \overset{\text{iid}}{\sim} N(0, \sigma_b^2) \\ \varepsilon_{ij} \overset{\text{iid}}{\sim} N(0, \sigma^2) \end{array}\right) \text{indep.}$$

$$\underset{\sim}{y} = X\beta + Z\underset{\sim}{b} + \underset{\sim}{\varepsilon}$$

$$\begin{pmatrix} y_{11} \\ \vdots \\ y_{1n} \\ \hline y_{21} \\ \vdots \\ y_{2n} \\ \hline \vdots \\ \hline y_{m1} \\ \vdots \\ y_{mn} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ \vdots \\ 1 \\ \hline 1 \\ \vdots \\ 1 \\ \hline \vdots \\ \hline 1 \\ \vdots \\ 1 \end{pmatrix}}_{\underset{\sim}{1}_N} \mu + \underbrace{\left(\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ \hline 0 & 1 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \ddots & \vdots \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ \hline & \vdots & & & & \vdots & & \\ \hline 0 & 0 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 \end{array}\right)}_{(\underset{\sim}{I}_m \otimes \underset{\sim}{1}_n \mid \underset{\sim}{1}_m \otimes \underset{\sim}{I}_n)} \begin{pmatrix} a_1 \\ \vdots \\ a_m \\ b_1 \\ \vdots \\ b_n \end{pmatrix} + \begin{pmatrix} \varepsilon_{11} \\ \vdots \\ \varepsilon_{1n} \\ \hline \varepsilon_{21} \\ \vdots \\ \varepsilon_{2n} \\ \hline \vdots \\ \hline \varepsilon_{m1} \\ \vdots \\ \varepsilon_{mn} \end{pmatrix}$$

$E(\underset{\sim}{y}) = X\beta = \underset{\sim}{1}_N \mu$

$Var(\underset{\sim}{y}) = Var(Z_1 \underset{\sim}{\alpha}_1 + Z_2 \underset{\sim}{\alpha}_2 + \underset{\sim}{\varepsilon})$

where $Z_1 = \underset{\sim}{I}_m \otimes \underset{\sim}{1}_n$, $Z_2 = \underset{\sim}{1}_m \otimes \underset{\sim}{I}_n$, $\underset{\sim}{\alpha}_1 = (a_1, \cdots, a_m)^T$

$\underset{\sim}{\alpha}_2 = (b_1, \cdots, b_n)^T$

$$Var(\underset{\sim}{y}) = \sigma_a^2 \underset{\sim}{Z_1}\underset{\sim}{Z_1}^T + \sigma_b^2 \underset{\sim}{Z_2}\underset{\sim}{Z_2}^T + \sigma^2 I_N$$

$$\underset{\sim}{Z_1}\underset{\sim}{Z_1}^T = (\underset{\sim}{I_m} \otimes \underset{\sim}{1_n})(\underset{\sim}{I_m} \otimes \underset{\sim}{1_n})^T = (\underset{\sim}{I_m} \otimes \underset{\sim}{1_n})(I_m^T \otimes \underset{\sim}{1}_n^T)$$

$$= (I_m \otimes \underset{\sim}{1}_n \underset{\sim}{1}_n^T)$$

$$= I_m \otimes J_n$$

$$Z_2 Z_2^T = (\underset{\sim}{1}_m \otimes I_n)(\underset{\sim}{1}_m \otimes I_n)^T = (\underset{\sim}{1}_m \otimes I_n)(\underset{\sim}{1}_m^T \otimes I_n^T)$$

$$= \underset{\sim}{1}_m \underset{\sim}{1}_m^T \otimes I_n I_n^T$$

$$= J_m \otimes I_n$$

$$= \sigma_a^2 (I_m \otimes J_n) + \sigma_b^2 (J_m \otimes I_n) + \sigma^2 I_N.$$

(c) $y_{ijk} \mid a_i, g_{ij} \overset{indep}{\sim} N(\mu + a_i + \beta_j + g_{ij}, \sigma^2)$

$$\left.\begin{array}{l} a_i \overset{iid}{\sim} N(0, \sigma_a^2), \\ g_{ij} \overset{iid}{\sim} N(0, \sigma_g^2) \end{array}\right\rangle \text{indep.}$$

for $i = 1, \cdots, m$; $j = 1, \cdots, n$; $k = 1, \cdots, r$.

$\Rightarrow \underset{\sim}{y} = X\beta + Z\underset{\sim}{b} + \underset{\sim}{\varepsilon}$

where $X = (\underset{\sim}{1}_N, \underset{\sim}{1}_m \otimes (I_n \otimes \underset{\sim}{1}_r))$

$\beta = (\mu, \beta_1, \cdots, \beta_n)^T$

$\underset{\sim}{b} = (\underset{\sim}{b}_1^T, \underset{\sim}{b}_2^T)^T$, $\underset{m\times 1}{\underset{\sim}{b}_1} = (a_1, \cdots, a_m)^T$, $\underset{mn\times 1}{\underset{\sim}{b}_2} = (g_{11}, \cdots, g_{mn})^T$

$Z = (\underset{\sim}{Z_1}, \underset{\sim}{Z_2})$

$Z_1 = I_m \otimes \underset{\sim}{1}_{nr}$, $Z_2 = I_{mn} \otimes \underset{\sim}{1}_r$

$$Var(\underset{\sim}{y}) = \sigma_a^2 Z_1 Z_1^T + \sigma_g^2 Z_2 Z_2^T + \sigma^2 I_N$$

$$Z_1 Z_1^T = (I_m \otimes \underset{\sim}{1}_{nr})(I_m^T \otimes \underset{\sim}{1}_{nr}^T)$$

$$= (I_m \otimes \underset{\sim}{1}_{nr} \underset{\sim}{1}_{nr}^T)$$

$$= I_m \otimes J_{nr}$$

$$Z_2 Z_2^T = (I_{mn} \otimes \underset{\sim}{1}_r)(I_{mn}^T \otimes \underset{\sim}{1}_r^T)$$

$$= I_{mn} \otimes \underset{\sim}{1}_r \underset{\sim}{1}_r^T$$

$$= I_{mn} \otimes J_r$$

$$E(\underset{\sim}{y}) = X\beta \quad (E(y_{ijk}) = \mu).$$

2. (a) First consider $\text{var}(Y_{ij})$. If we write the first stage model as $Y_{ij} = (\beta_0 + \beta_1 t_{ij}) + b_{0i} + b_{1i}t_{ij} + e_{ij}$, because the term in parentheses is a constant, we only need find $\text{var}\{(b_{0i} + b_{1i}t_{i}j) + e_{ij}\}$. This is just the variance of the sum of the two random variables $(b_{0i} + b_{1i}t_{i}j)$ and $e_{ij}$, so that we get

$$\text{var}(b_{0i} + b_{1i}t_{ij}) + \text{var}(e_{ij}) + 2\text{cov}\{(b_{0i} + b_{1i}t_{i}j), e_{ij}\}.$$

Now, because $\boldsymbol{b}_i$ and $\boldsymbol{e}_i$ are independent, clearly the covariance in the last term must be equal to 0, as it is a covariance between a function of the elements of $\boldsymbol{b}_i$ and an element of $\boldsymbol{e}_i$. By the same result on the variance of a sum, the first term is equal to

$$\text{var}(b_{0i} + b_{1i}t_{ij}) = \text{var}(b_{0i}) + \text{var}(b_{1i}t_{i}j) + 2\text{cov}(b_{0i}, b_{1i}t_{ij}).$$

We have immediately that $\text{var}(b_{0i}) = D_{11}$, $\text{var}(b_{1i}t_{ij}) = t_{ij}^2 D_{22}$, and $\text{var}(e_{ij}) = \sigma^2$. So we only need to evaluate $\text{cov}(b_{0i}, b_{1i}t_{ij})$. Because both $b_{0i}$ and $b_{1i}$ have mean zero, we have $\text{cov}(b_{0i}, b_{1i}t_{ij}) = E(b_{0i}b_{1i}t_{ij})$, which, because $E(b_{0i}b_{1i} = \text{cov}(b_{0i}, b_{1i})$, is just equal to $t_{ij}D_{12}$. Putting all this together yields the result that

$$\text{var}(Y_{ij}) = D_1 + D_{22}t_{ij}^2 + 2D_{12}t_{ij} + \sigma^2.$$

To find $\text{cov}(Y_{ij}, Y_{ik})$, note that, as $E(Y_{ij}) = \beta_0 + \beta_1 t_{ij}$, and similarly for $Y_{ik}$, by the definition of covariance we can write this as

$$E\{(b_{0i} + b_{1i}t_{ij} + e_{ij})(b_{0i} + b_{1i}t_{ik} + e_{ik})\}.$$

Multiplying all this out and collecting terms gives

$$E(b_{0i}^2) + E(b_{1i}^2 t_{ij}t_{ik}) + E(e_{ij}e_{ik}) + E\{b_{0i} + b_{1i}t_{ij})e_{ik}\} + E\{b_{0i} + b_{1i}t_{ik})e_{ij}\} + E(b_{0i}b_{1i}t_{ij}) + E(b_{0i}b_{1i}t_{ik}).$$

Using the result that $E(be) = E(b)E(e)$ if $b$ and $e$ are independent and the fact that $b_{0i}$, $b_{1i}$, $e_{ij}$, and $e_{ik}$ all have mean zero, and that $\text{cov}(e_{ij}, e_{ik}) = 0$, it follows that the middle three expectations are zero. We are left with

$$E(b_{0i}^2) + E(b_{1i}^2 t_{ij}t_{ik}) + E(b_{0i}b_{1i}t_{ij}) + E(b_{0i}b_{1i}t_{ik}),$$

which is equal to $D_{11} + t_{ij}t_{ik}D_{22} + (t_{ij} + t_{ik})D_{12}$, the result.

(b) If $D_{12} = 0$, $\text{cov}(Y_{ij}, Y_{ik})$ becomes

$$D_{11} + D_{22}t_{ij}t_{ik} + \sigma^2.$$

This is clearly non-zero in general. Thus, even if the individual-specific intercepts and slopes are uncorrelated in the population, there is still non-zero correlation between elements of $\boldsymbol{Y}_i$. This makes intuitive sense. Whether $D_{12} = 0$ or not, the elements of $\boldsymbol{Y}_i$ all share a common dependence on the random effects $b_{0i}$ and $b_{1i}$. Thus, they would be expected to be associated with one another on this basis alone.

(c) In our derivation of $\text{var}(Y_{ij})$, everything stays the same except that now $\text{var}(e_{ij}) = \sigma_1^2 + \sigma_2^2$. So we find that

$$\text{var}(Y_{ij}) = D_1 + D_{22}t_{ij}^2 + 2D_{12}t_{ij} + \sigma_1^2 + \sigma_2^2.$$

For $\text{cov}(Y_{ij}, Y_{ik})$, everything is the same except that now $\text{cov}(e_{ij}, e_{ik}) = E(e_{ij}e_{ik}) \neq 0$. Rather, according to the Markov structure, $\text{cov}(e_{ij}, e_{ik}) = \sigma_2^2\rho^{d_{ijk}}$, say, where $d_{ijk} = |t_{ij} - t_{ik}|$. Thus, the covariance becomes

$$D_{11} + t_{ij}t_{ik}D_{22} + (t_{ij} + t_{ik})D_{12} + \sigma_2^2\rho^{d_{ijk}}.$$

3.

**(a) Random Intercepts**

$$Y_{ij} = \beta_0 + t_{ij} + b_{0i} + \epsilon_{ij} \quad ,$$
$$b_{0i} \sim N(0, \tau^2) \ (iid),$$
$$\epsilon_{ij} \sim N(0, \sigma^2) \ (iid),$$

where $b_{0i}$ and $\epsilon_{ij}$ are mutually independent.

· Give the general form for the covariance matrix $\Sigma = Cov(Y_i)$ .

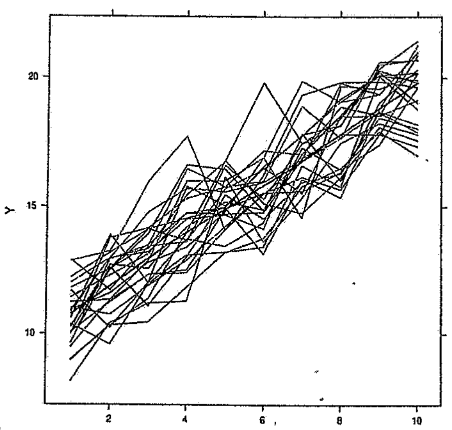
(solution) For a fixed $i$, $Var(Y_{ij}) = Var(b_{0i}) + Var(\epsilon_{ij}) = \tau^2 + \sigma^2$ . Also, the covariance of $Y_{ij}$ and $Y_{ik}$ for $j \neq k$ is $\tau^2$ because $Cov(b_{0i}, b_{0i}) = Var(b_{0i}) = \tau^2$ . Thus, the general form for $\Sigma$ is the following result.

$$\Sigma = \begin{bmatrix} \tau^2+\sigma^2 & \tau^2 & \cdots & \tau^2 \\ \tau^2 & \tau^2+\sigma^2 & \cdots & \tau^2 \\ \vdots & \vdots & \vdots & \vdots \\ \tau^2 & \tau^2 & \cdots & \tau^2+\sigma^2 \end{bmatrix}$$
, which is a $10 \times 10$ matrix.

· Give $Y_i$ and plot (lines) versus $t_i$ for $m$ = 25 using $\sigma$ = 1.0 and $\tau$ = 1.0.

(solution)

Plot of Y vs. Time with LOESS smoothing curve




In the above figure, the red line represents a LOESS smoothing curve for 25 subjects. I think there is an increasing trend overall.

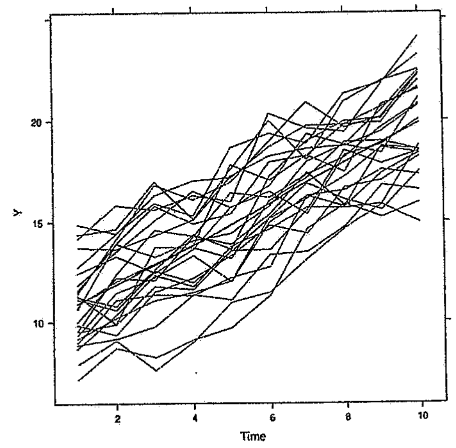




· Give $Y_i$ and plot (lines) versus $t_i$ for $m$ = 25 using $\sigma$ = 1.0 and $\tau$ = 2.0.

(solution)

Plot of Y vs. Time with LOESS smoothing curve

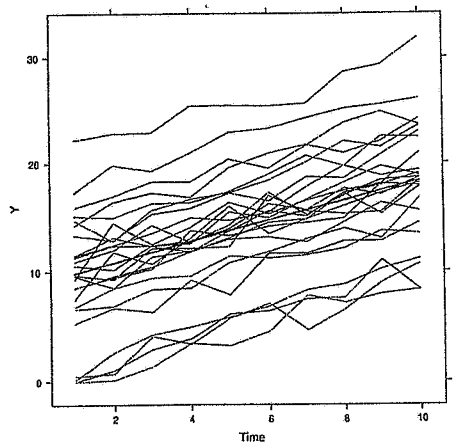



Like the preceding, we can see that there is an increasing trend. When comparing with the figure in the above question($\tau$ = 1.0), although two $\tau$'s are different, two figures look very similar.

· Give $Y_i$ and plot (lines) versus $t_i$ for $m$ = 25 using $\sigma$ = 1.0 and $\tau$ = 5.0.

(solution)

Plot of Y vs. Time with LOESS smoothing curve



Also, in this figure, we can conclude that there is an increasing trend. However, unlike above two figures, this figure has greater variation than others. The difference comes from the fact that this $b_{0i}$ has variance 5.0. It is greater than two values of above $b_{0i}$s' variances.

(b) **Random Intercepts and Slopes**

$$Y_{ij} = \beta_0 + \beta_1 t_{ij} + b_{0i} + b_{1i} t_{ij} + \epsilon_{ij} \ ,$$
$$b \sim N(0,\ D)\ ,$$
$$\epsilon_{ij} \sim N(0,\ \sigma^2)\ (iid),$$
where $b = (b_{0i},\ b_{1i})$ and $\epsilon_{ij}$ are mutually independent.

· Give the general form for the covariance matrix $\Sigma = Cov(Y_i)$ .

(solution) Let $D = \begin{bmatrix} d_{00} & d_{01} \\ d_{10} & d_{11} \end{bmatrix}$ , where $d_{00} = Var(b_{0i})$ , $d_{01} = d_{10} = Cov(b_{0i},\ b_{1i})$ , and $d_{11} = Var(b_{1i})$ . First, let's find the variance of $Y_{ij}$.

$$\begin{aligned} Var(Y_{ij}) &= Var(\beta_0 + \beta_1 t_{ij} + b_{0i} + b_{1i} t_{ij} + \epsilon_{ij}) \\ &= Var(b_{0i}) + Var(b_{1i} t_{ij}) + Var(\epsilon_{ij}) + 2Cov(b_{0i},\ b_{1i} t_{ij}) \\ &= d_{00} + t_{ij}^2 d_{11} + \sigma^2 + 2t_{ij} d_{01} \\ &= d_{00} + j^2 d_{11} + \sigma^2 + 2j d_{01} \quad \text{(because } t_{ij} = j \text{ in the problem.)} \end{aligned}$$

Also, the covariance of $Y_{ij}$ and $Y_{ik}$ for $j \neq k$ is as follows.

$$\begin{aligned} Cov(Y_{ij},\ Y_{ik}) &= Cov(b_{0i} + b_{1i} t_{ij} + \epsilon_{ij},\ b_{0i} + b_{1i} t_{ik} + \epsilon_{ik}) \\ &= Var(b_{0i}) + Cov(b_{1i} t_{ij},\ b_{1i} t_{jk}) + Cov(b_{0i},\ b_{1i} t_{ik}) + Cov(b_{1i} t_{ij},\ b_{0i}) \\ &= d_{00} + t_{ij} t_{ik} d_{11} + t_{ik} d_{01} + t_{ij} d_{01} \\ &= d_{00} + jk d_{11} + (k+j) d_{01} \quad \text{(because } t_{ij} = j \text{ and } t_{ik} = k.) \end{aligned}$$

Thus, the general form for $\Sigma$ can be constructed as follows.

$$\Sigma = \begin{bmatrix} d_{00} + d_{11} + \sigma^2 + 2d_{01} & d_{00} + 2d_{11} + 3d_{01} & \cdots & d_{00} + 10d_{11} + 11d_{01} \\ d_{00} + 2d_{11} + 3d_{01} & d_{00} + 4d_{11} + \sigma^2 + 4d_{01} & \cdots & d_{00} + 20d_{11} + 12d_{01} \\ \vdots & \vdots & \vdots & \vdots \\ d_{00} + 10d_{11} + 11d_{01} & d_{00} + 20d_{11} + 12d_{01} & \cdots & d_{00} + 10d_{11} + \sigma^2 + 20d_{01} \end{bmatrix}$$

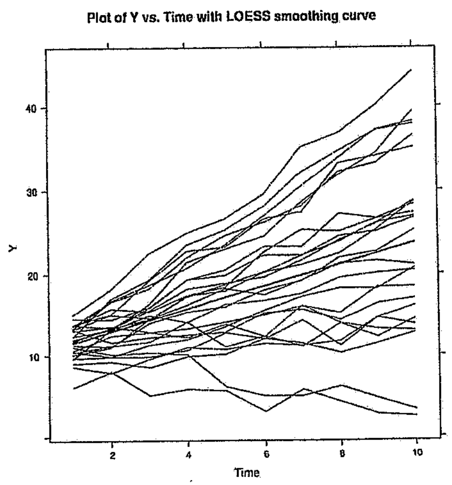
, which is a $10 \times 10$ matrix. Only diagonal elements have $\sigma^2$.

· Generate $Y_i$ and plot (lines) versus $t_i$ for $m$ = 25 using $\sigma$ = 1.0, $D = \begin{bmatrix} 2.0 & 0 \\ 0 & 2.0 \end{bmatrix}$ .

(solution) We can see that the variation of $Y_i$ over time is not constant from this figure. The red line means a LOESS curve and it shows an increasing trend of $Y_i$. But since each subject has its own intercept and slope, a LOESS curve doesn't have many meanings.

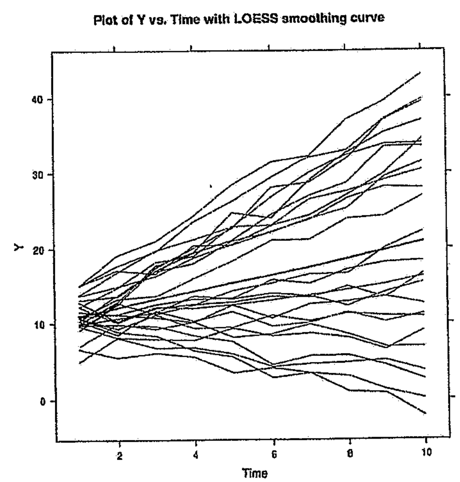







· Generate $Y_i$ and plot (lines) versus $t_i$ for $m$ = 25 using $\sigma$ = 1.0, $D = \begin{bmatrix} 2.0 & -0.2 \\ -0.2 & 2.0 \end{bmatrix}$ .

(solution)

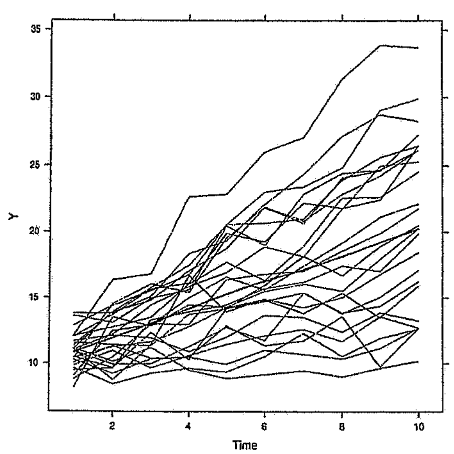



When comparing with the above figure, it seems that there is no difference between two figures even though $b_{0i}$ and $b_{1i}$ are correlated(but, the correlation is almost zero.). A LOESS curve shows still an increase of $Y_i$ over time and $Y_i$ has heteroscedasticity over time. This figure looks like a right-opening megaphone.

• Generate $Y_i$ and plot (lines) versus $t_i$ for $m$ = 25 using $\sigma$ = 1.0, $D=\begin{bmatrix}0.5 & 0\\ 0 & 0.4\end{bmatrix}$ .

(solution)

Plot of Y vs. Time with LOESS smoothing curve




Since $Var(b_{0i})$ and $Var(b_{1i})$ are different(but, they are almost same!), the shape of dispersion of $Y_i$ over time is a little bit different from above two figures. But, still $Y_i$ has different variation as time passes.

**(c) Serial Correlation**

$$Y_{ij} = \beta_0 + \beta_1 t_{ij} + W_i(t_{ij}) + \epsilon_{ij} \ ,$$
$$W_i \sim N(0,\ D)\ ,$$
$$Var[\,W_i(t_{ij})] = \tau^2\ ,$$
$$Cov[\,W_i(t_{ij}),\ W_i(t_{ik})] = \tau^2\rho^{|t_{ij}-t_{ik}|}\ ,$$
$$\epsilon_{ij} \sim N(0,\ \sigma^2)\ (iid),$$

where $W_i$ and $\epsilon_{ij}$ are mutually independent.

• Give the general form for the covariance matrix $\Sigma = Cov(Y_i)$ .

(solution) For a specific subject $i$, let's find the variance of $Y_{ij}$.

$$Var(Y_{ij}) = Var[\beta_0 + \beta_1 t_{ij} + W_i(t_{ij}) + \epsilon_{ij}]$$
$$= Var[\,W_i(t_{ij})] + Var(\epsilon_{ij})$$
$$= \tau^2 + \sigma^2$$

And, $Cov(Y_{ij},\ Y_{ik})$ could be obtained like the following for $j \neq k$.

$$Cov(Y_{ij},\ Y_{ik}) = Cov[\beta_0 + \beta_1 t_{ij} + W_i(t_{ij}) + \epsilon_{ij},\ \beta_0 + \beta_1 t_{ik} + W_i(t_{ik}) + \epsilon_{ik}]$$
$$= Cov[\,W_i(t_{ij}),\ W_i(t_{ik})]$$





$$= \tau^2\rho^{|t_{ij}-t_{ik}|}$$
$$= \tau^2\rho^{|j-k|} \quad (\text{because } t_{ij} = j \text{ and } t_{ik} = k.)$$

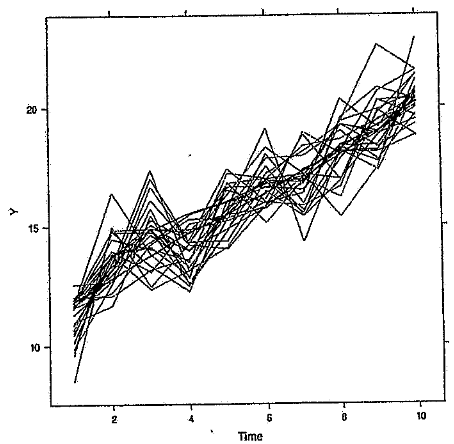
Therefore, the covariance matrix $\Sigma = Cov(Y_i)$ has the following form.

$$\Sigma = \begin{bmatrix} \tau^2+\sigma^2 & \tau^2\rho & \cdots & \tau^2\rho^9 \\ \tau^2\rho & \tau^2+\sigma^2 & \cdots & \tau^2\rho^8 \\ \vdots & \vdots & \vdots & \vdots \\ \tau^2\rho^9 & \tau^2\rho^8 & \cdots & \tau^2+\sigma^2 \end{bmatrix}$$ is a 10×10 matrix. Off-diagonal elements doesn't have $\sigma^2$ but have $\rho$.

• Generate $Y_i$ and plot (lines) versus $t_i$ for $m$ = 25 using $\sigma$ = 1.0, $\tau$ = 2.0, and $\rho$ = 0.7.

(solution)

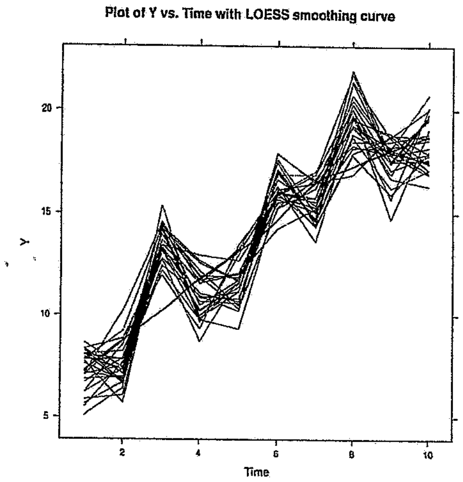
Plot of Y vs. Time with LOESS smoothing curve



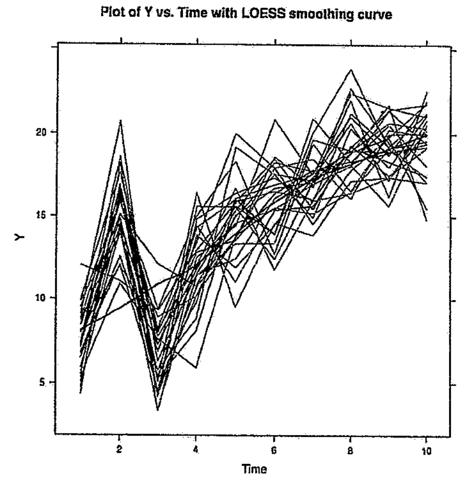

As we saw before, the covariance matrix of $Y_i$ is very similar to the one in the first question of (a). The only different part is whether $Cov(Y_{ij},\ Y_{ik})$ has the correlation $\rho^{|j-k|}$ or not. Due to the effect of this correlation, this figure has denser than the one in the first question of (a). Note that the red line is a LOESS curve.

• Generate $Y_i$ and plot (lines) versus $t_i$ for $m$ = 25 using $\sigma$ = 1.0, $\tau$ = 2.0, and $\rho$ = 0.9.

(solution) When Comparing with the above figure, we can see a more fluctuation of $Y_{ij}$. It has a deeper valley than the above one.




· Generate $Y_i$ and plot (lines) versus $t_i$ for $m = 25$ using $\sigma = 2.0$, $\tau = 2.0$, and $\rho = 0.9$.
(solution)




Since the standard deviation of a measurement error is bigger than the above two cases, this figure shows the greatest fluctuation of $Y_{ij}$ over time. The variation of $Y_{ij}$ tends to decrease as time passes because $\rho^{|j-k|}$ decreases.

4 (a) From the spaghetti plots, , each child's lead level profile looks like it reasonably come from an overall "inherent" straight-line trend, with some "jitter." So it seems like this is a reasonable model.

(b) As in the notes, we may think of $\boldsymbol{e}_i$ as being the sum of two components, $\boldsymbol{e}_i = \boldsymbol{e}_{1i} + \boldsymbol{e}_{2i}$. Model assumptions (i) and (ii) mean the following:

(i) implies that we believe that $\boldsymbol{b}_i$ and $\boldsymbol{e}_{2i}$ are independent and that the variance of measurement errors is constant regardless of the lead level for the child

(ii) Implies that the elements of $\boldsymbol{e}_{1i}$ are uncorrelated with constant variance over time and are independent of $\boldsymbol{b}_i$ and $\boldsymbol{e}_{2i}$.

Under (i) and (ii), we thus have that $\boldsymbol{R}_i$ is of the general form on the bottom of page ~~315~~, 318

$$\boldsymbol{R}_i = \sigma_1^2 \boldsymbol{I}_{n_i} + \sigma_2^2 \boldsymbol{I}_{n_i}.$$

Under the first assumption on (iii) and (iv), $\boldsymbol{D}$ is the same for all strategies and $\sigma_1^2$ is the same for all treatments, so that the total within-child variance $\sigma_1^2 + \sigma_2^2 = \sigma^2$ is the same for all treatments.

Under the second assumption on (iii) and (iv), $\boldsymbol{D}$ is the same for all treatments but now $\sigma_1^2$ might be different, so that total within-child variance $\sigma_1^2 + \sigma_2^2$ is possibly different.

Under the third assumption on (iii) and (iv), $\boldsymbol{D}$ is possibly different for all treatments but $\sigma_1^2$ is the same for all strategies, so that the total within-mouse variance $\sigma_1^2 + \sigma_2^2 = \sigma^2$ is the same for all treatments.

Under the fourth assumption on (iii) and (iv), $\boldsymbol{D}$ is possibly different for all treatments *and* $\sigma_1^2$ might be different.

The attached SAS program fits the model under these three different assumptions.

(c) From the output, we have

| Model | *AIC* | *BIC* |
|---|---|---|
| Same $\boldsymbol{D}$ and same within-child variance | 3106.9 | 3118.0 |
| Same $\boldsymbol{D}$ and different within-child variance | 3108.2 | 3124.9 |
| Different $\boldsymbol{D}$ and same within-child variance | 3113.1 | 3140.9 |
| Both $\boldsymbol{D}$ and within-child variance different | 3115.8 | 3149.2 |

Both criteria prefer that model with the same $\boldsymbol{D}$ (variation/covariation of intercepts/slopes) same in each treatment and same within-child variance (reflecting that magnitude of within-child fluctuations is similar in each treatment). This is the "usual" model that is generally (but not always correctly!) assumed in mixed model analyses.

(d) We have included in each fit a `contrast` statement for the comparison of the three "typical" mean slopes. From the `contrast` statement for the fit of the model in (c), $T_L = 7.64$ with an associated p-value of 0.02. At level of significance 0.05, there is evidence to support the contention that there is a difference in lead level patterns, as represented by mean slope for each treatment. From the `Solution for Fixed Effects` for this model, informally, it appears that this is because both low- and high-dose succimer seem to lead to more dramatic rates of (decreasing) lead levels than placebo (so seem to be having an effect relative to placebo).