Quiz 5

#1. If $X_1, \cdots, X_n \overset{iid}{\sim}$ Poisson (θ), find the MVUE of $P(X=0) = e^{-\theta}$.

$\rightarrow$ $P(X=x) = \frac{\theta^x}{x!} e^{-\theta}$

$= \exp(\underbrace{x \log \theta}_{P(\theta)\, k(x)} - \log x! - \theta)$

$\therefore \sum_{i=1}^{n} x_i$ is a CSS.

Let $Y = \sum_{i=1}^{n} X_i$. Then $Y \sim$ Poisson $(n\theta)$.

Let $\eta = e^{-\theta}$.

$P(X=0) = E(\underbrace{I(X_1=0)}_{\text{unbiased estimator of } \eta}) = e^{-\theta}$

$\hat{\eta} = E(I(X_1=0) \mid \sum_{i=1}^{n} X_i = y) = P(X_1 = 0 \mid \sum_{i=1}^{n} X_i = y)$

$= \frac{P(X_1=0, \sum_{i=1}^{n} X_i = y)}{P(\sum X_i = y)} = \frac{P(X_1 = 0, \sum_{i=2}^{n} X_i = y-0)}{P(\sum X_i = y)}$

$= \frac{e^{-\theta} \cdot \frac{((n-1)\theta)^y}{y!} e^{-(n-1)\theta}}{\frac{(n\theta)^y}{y!} e^{-n\theta}}$ $\quad (\because \sum_{i=2}^{n} X_i \sim$ Poisson $((n-1)\theta)$, $Y \sim$ Poisson $(n\theta)$)

$= \left(\frac{n-1}{n}\right)^y$

$= \left(1 - \frac{1}{n}\right)^{\sum_{i=1}^{n} x_i}$

$\therefore \left(1 - \frac{1}{n}\right)^{\sum X_i}$ is MVUE of $P(X=0) = e^{-\theta}$ by Rao-Blackwell & Lehmann scheffe.

#2. Suppose that $X \sim f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, $\theta > 0$.

For hypothesis testing $H_0: \theta = 2$ versus $H_1: \theta = 3$, if we use the critical region $C = \{X > \frac{2}{3}\}$, find the size of this test.

$\rightarrow$ Size of Test : the probability of incorrectly rejecting the null hypothesis

(= prob of committing a Type I error)

$\Rightarrow$ Size of Test $= \int_{\frac{2}{3}}^{1} 2x \, dx = [x^2]_{\frac{2}{3}}^{1} = 1 - \frac{4}{9} = \frac{5}{9}$

$\therefore$ Size of Test is $\frac{5}{9}$.