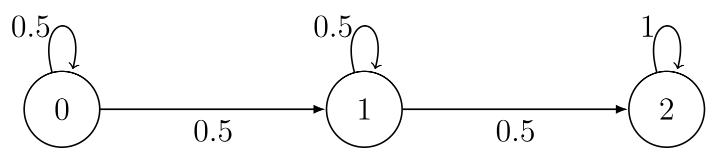
Stochastic Processes (STA3021)
HW5 Solution

1. (a) We have the following classification of states:

$$C_1 = \{0\}, \quad C_2 = \{1\}, \quad C_3 = \{2\}.$$

Since C_1 is open and finite communicating class, the state 0 is transient and aperiodic since $p_{0,0} > 0$. Similarly, C_2 is also open comm. class, hence state 1 is transient. Since $p_{1,1} > 0$, it is aperiodic. For C_3, it is a closed communicating class, hence state 2 is positive recurrent and aperiodic since $p_{2,2} > 0$. The transition diagram is represented as

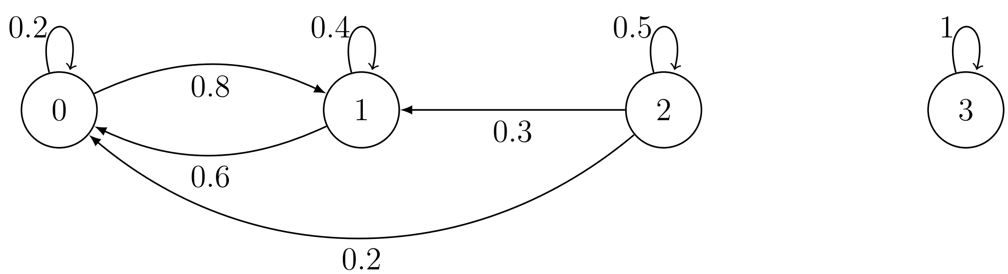


(b) Similar to part (a), we have that

$$C_1 = \{0, 1\} \quad \text{closed / aperiodic / positive recurrent}$$

$$C_2 = \{2\} \quad \text{open / aperiodic / transient}$$

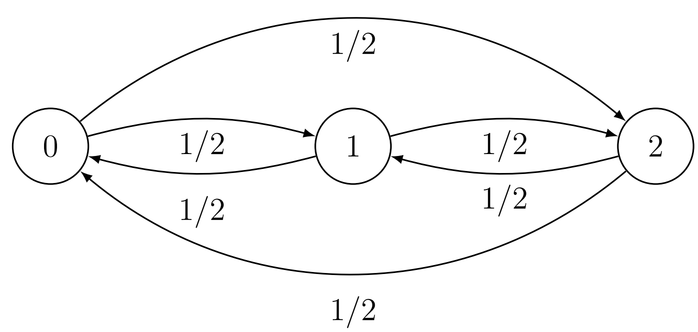
$$C_3 = \{3\} \quad \text{closed / aperiodic / positive recurrent}$$



2. Chapter 4 #14.

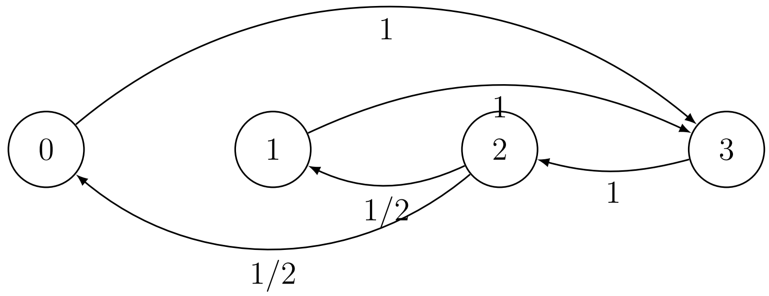
(a)

$$C_1 = \{0, 1, 2\} \quad \text{closed / aperiodic / positive recurrent}$$



(b)

$$C_1 = \{0,1,2,3\} \quad \text{closed / periodic : } d(0)=d(1)=d(2)=d(3)=3 \text{ / positive recurrent}$$

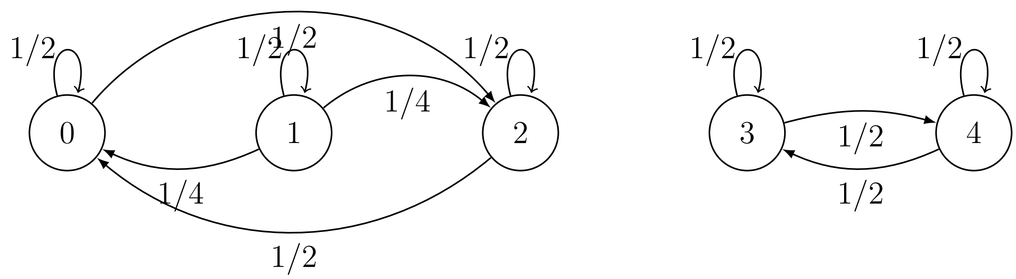


(c)

$$C_1 = \{0,2\} \quad \text{closed / aperiodic / positive recurrent}$$

$$C_2 = \{1\} \quad \text{open / aperiodic / transient}$$

$$C_3 = \{3,4\} \quad \text{closed / aperiodic / positive recurrent}$$



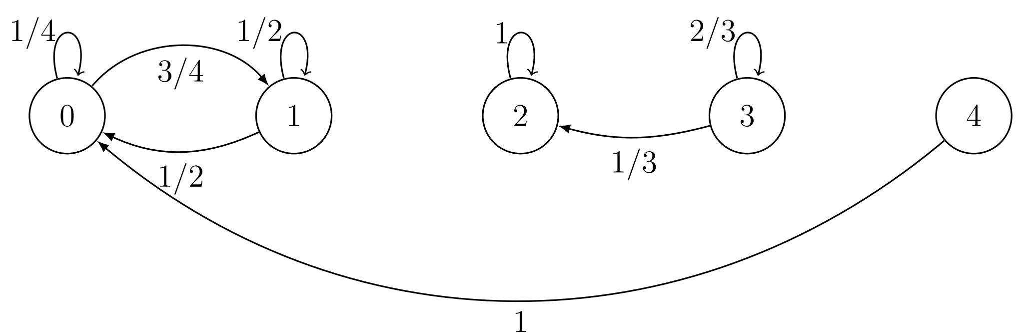
(d)

$$C_1 = \{0,1\} \quad \text{closed / aperiodic / positive recurrent}$$

$$C_2 = \{2\} \quad \text{closed / aperiodic / positive recurrent}$$

$$C_3 = \{3\} \quad \text{open / aperiodic / transient}$$

$$C_4 = \{4\} \quad \text{open / periodic : } d(4)=\infty \text{ / transient}$$



3. Chapter 4 #18

   (a) Define $\{X_n\}$ be the coin number flipped at the n-th toss. Then, $\{X_n\}$ is a Markov chain with state space $E = \{1, 2\}$ and transition probability

   $$P = \begin{pmatrix} .6 & .4 \\ .5 & .5 \end{pmatrix}.$$

   Since the Markov chain is irreducible and recurrent, the long-run proportion of time the chain spends in state1 is equal to π_j. Hence, by solving

   $$(\pi_1, \pi_2) = (\pi_1, \pi_2)P, \quad \pi_1 + \pi_2 = 1,$$

   we have $\pi_1 = 5/9$ and $\pi_2 = 4/9$. Therefore, the proportion of flips using coin 1 is $5/9$.

   (b) The desired probability is calculated by $P(X_4 = 2|X_1 = 1) = P^4(1, 2) = .44$.

4. Chapter 4 #23
   Suppose that 0 denotes a good year and 1 a bad year. Define $\{X_n\}$ be the state of either good or bad year, then $\{X_n\}$ is a Markov chain with the state space $E = \{0, 1\}$ and transition probability

   $$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

   We have initial condition that $X_0 = 0$ with probability 1.

   (a) If we let S_i denote the number of storms in the next i year, then

   $$E[S_i] = E[S_i|X_i = 0]P(X_i = 0) + E[S_i|X_i = 1]P(X_i = 1).$$

   For example,

   $$E[S_1] = E[S_1|X_1 = 0]P(X_1 = 0) + E[S_1|X_1 = 1]P(X_1 = 1).$$

   First note that $P(X_0 = 0) = 1$ implies that $P(X_1 = 0) = P(0, 0) = 1/2$ and $P(X_1 = 1) = P(0, 1) = 1/2$. Once the state of year is determined, it follows Poisson distribution, so

   $$E[S_1] = 1 \cdot 1/2P(X_1 = 0) + 3 \cdot 1/2 = 2.$$

   For the second year, similar calculation gives

   $$\begin{aligned} E[S_2] &= E[S_2|X_2 = 0]P(X_2 = 0) + E[S_2|X_2 = 1]P(X_2 = 1) \\ &= 1 \cdot 5/12 + 3 \cdot 7/12 = 26/12 \end{aligned}$$

   from $P(X_2 = 0) = P^2(0, 0)$ and $P(X_2 = 1) = P^2(0, 1)$ with

   $$P^2 = \begin{pmatrix} 5/12 & 7/12 \\ 7/18 & 11/18 \end{pmatrix}.$$

   Thus, $E[S_1] + E[S_2] = 25/6 = 4.17$.

(b) The problem is asking $P(S_3 = 0)$. Observe that

$$P(S_3 = 0) = P(S_3 = 0|X_3 = 0)\,P(X_3 = 0) + P(S_3 = 0|X_3 = 1)\,P(X_3 = 1)$$
$$= e^{-1}P^3(0,0) + e^{-3}P^3(0,1) = 29/72e^{-1} + 43/72e^{-3}$$

since $S_i|X_i = 0 \sim Poisson(1)$, $S_i|X_i = 1 \sim Poisson(3)$.

(c) The stationary probabilities are the solution of

$$\pi_0 = \pi_0\frac{1}{2} + \pi_1\frac{1}{3}, \quad \pi_0 + \pi_1 = 1.$$

It gives

$$\pi_0 = \frac{2}{5}, \pi_1 = \frac{3}{5},$$

hence the long-run average number of storms is $1 \cdot 2/5 + 3 \cdot 3/5 = 11/5$.

5. Chapter 4 #28
Consider the Markov chain indicating the result of a game in the team. Then the transition matrix of winning or losing the game is,

$$P = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}.$$

Then, it is an irreducible, positive recurrent, aperiodic chain, hence limiting probability is calculated by solving

$$\pi_w = \pi_w * .8 + \pi_l * .3, \quad \pi_w + \pi_l = 1,$$

which gives $\pi_w = 3/5$ and $\pi_l = 2/5$. Therefore, the proportion of having team dinner is

$$\pi_w * .7 + \pi_l * .2 = .5.$$

6. Chapter 4 #29
Each employee moves according to a Markov chain whose limiting probabilities are the solution of

$$\begin{cases} \pi_1 = \pi_1 * .7 + \pi_2 * .2 + \pi_3 * .1 \\ \pi_2 = \pi_1 * .2 + \pi_2 * .6 + \pi_3 * .4 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}.$$

The solution is

$$\pi_1 = \frac{6}{17}, \pi_2 = \frac{7}{17}, \pi_3 = \frac{4}{17}.$$

Hence, if N is large, it follows from the law of large numbers that approximately 35.2% (6/17) are in job category 1, 41.1% are in category 2, and 23.5% are in category 3.

7. Chapter 4 #36

(a) Straightforward calculation gives

$$p_o * P_{0,0} + p_1 * P_{0,1} = p_o * .4 + p_1 * .6.$$

(b) Since Monday to Friday takes 4 steps of transition,

$$P^4 = \begin{pmatrix} 0.2512 & 0.7488 \\ 0.2496 & 0.7504 \end{pmatrix}$$

$$p_o * P^4_{0,0} + p_1 * P^4_{0,1} = p_o * .2512 + p_1 * .7488.$$

(c) First calculate the limiting distribution on the state of process. Solving

$$\begin{cases} \pi_1 = \pi_1 * .4 + \pi_2 * .2 \\ \pi_1 + \pi_2 = 1 \end{cases}$$

gives $\pi_1 = \frac{1}{4}, \pi_2 = \frac{3}{4}$. Hence, the long-run proportion of good messages is calculated by

$$p_0\pi_0 + p_1\pi_1 = p_0\frac{1}{4} + p_1\frac{3}{4}.$$

(d) This is not a Markov Chain in general. For example, we can check whether it satisfies Markov property

$$P(Y_2 = 1|Y_1 = 1) = P(Y_2 = 1|Y_1 = 1, Y_0 = 1)$$

Note that

$$P(Y_2 = 1, Y_1 = 1, Y_0 = 1)$$

$$= \sum_{i,j,k=0}^{1} P(Y_2 = 1, Y_1 = 1, Y_0 = 1|X_0 = i, X_1 = j, X_2 = k) P(X_0 = i, X_1 = j, X_2 = k)$$

$$= \sum_{i,j,k=0}^{1} p_i p_j p_k P(X_0 = i, X_1 = j, X_2 = k)$$

$$= \sum_{i,j,k=0}^{1} p_i p_j p_k P_{j,k} P_{i,j} P(X_0 = i)$$

$$P(Y_1 = 1, Y_0 = 1) = \sum_{i,j=0}^{1} P(Y_1 = 1, Y_0 = 1|X_0 = i, X_1 = j) P(X_0 = i, X_1 = j)$$

$$= \sum_{i,j=0}^{1} p_i p_j P_{i,j} P(X_0 = i)$$

Hence,

$$P\left(Y_2=1|Y_1=1,Y_0=1\right)=\frac{\sum_{i,j,k=0}^{1}p_ip_jp_kP_{j,k}P_{i,j}P\left(X_0=i\right)}{\sum_{i,j=0}^{1}p_ip_jP_{i,j}P\left(X_0=i\right)}$$

Similarly, it can be shown that

$$P\left(Y_2=1|Y_1=1\right)=\frac{\sum_{i,j,k=0}^{1}p_jp_kP_{j,k}P_{i,j}P\left(X_0=i\right)}{\sum_{i,j=0}^{1}p_jP_{i,j}P\left(X_0=i\right)}$$

Therefore,

$$P\left(Y_2=1|Y_1=1,Y_0=1\right)\neq P\left(Y_2=1|Y_1=1\right)$$

implies that $\{Y_n,n\geq 1\}$ is not a Markov chain unless $p_i=1$ for $i=0,1$.