# Stochastic Processes (STA3021)
# HW6 Solution

1. Exercise Chapter 4 #53

   Let $\pi_i, \pi_i'$ denote the proportion of time a policyholder whose yearly number of accidents follows Poisson distribution with mean $\frac{1}{2}, \frac{1}{4}$ in state $i$ respectively.

   To calculate the average premium when the clients accidents follow Poisson(1/4), solve the equation to get $\pi_i'$'s.

$$\begin{cases} \pi_1' = \pi_1' * .7788 + \pi_2' * .7788 \\ \pi_2' = \pi_1' * .1947 + \pi_3' * .7788 \\ \pi_3' = \pi_1' * .0243 + \pi_2' * .1947 + \pi_4' * .7788 \\ \pi_1' + \pi_2' + \pi_3' + \pi_4' = 1 \end{cases}.$$

   The solutions are

$$\pi_1' = 0.6926, \pi_2' = 0.1967, \pi_3' = 0.0794, \pi_3' = 0.0312.$$

   Hence,

   $E$[Premium|Clients with Pois(1/4)] $= 200*0.6926+250*0.1967+400*0.0794+600*0.0312.$

   Since we have calculated the average premium when clients accidents follows Poisson(1/2) as 326.375 in class (see the class note),

$$E[\text{Premium}] = E[\text{Premium}|\text{Clients with Pois}(1/2)]\frac{2}{3} + E[\text{Premium}|\text{Clients with Pois}(1/4)]\frac{1}{3}$$

$$= \frac{2}{3}326.375 + \frac{1}{3}238.175 = 296.975.$$

2. Exercise Chapter 4 #66

   Recall for the branching process,

$$\pi_0 = P\left(\text{population dies out}\right)$$

$$= \sum_{j=0}^{\infty} P\left(\text{population dies out}|X_1 = j\right) P\left(X_1 = j\right)$$

$$= \sum_{j=0}^{\infty} \pi^j P_j \text{ , } \pi_0 \text{ is the smallest positive solution.}$$

(a) Straightforward calculation gives

$$\pi_0 = \frac{1}{4} + \pi_0 * 0 + \pi_0^2 * \frac{3}{4}$$

$$\pi_0 = \frac{1}{3}.$$

(b)

$$\pi_0 = \frac{1}{4} + \pi_0 * \frac{1}{2} + \pi_0^2 * \frac{1}{4}$$
$$\pi_0 = 1.$$

(c)

$$\pi_0 = \frac{1}{6} + \pi_0 * \frac{1}{2} + \pi_0^3 * \frac{1}{3}.$$
$$\pi_0 = \frac{1}{2}(\sqrt{3} - 1).$$

3. Exercise Chapter 4 #31
Let the state on day $n$ be 0 if sunny, 1 if cloudy, and 2 if rainy. This gives a three state Markov chain with transition probability matrix

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}.$$

Since the Markov chain is irreducible and recurrent, the long-run proportions are the solutions of following equation.

$$\begin{cases} \pi_0 = \frac{1}{4}\pi_1 * + \frac{1}{4}\pi_2 \\ \pi_1 = \frac{1}{2}\pi_0 + \frac{1}{2}\pi_1 + \frac{1}{4}\pi_2 \\ \pi_2 = \frac{1}{2}\pi_0 + \frac{1}{4}\pi_1 + \frac{1}{2}\pi_2 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}.$$

The solutions are

$$\pi_0 = \frac{1}{5}, \pi_1 = \frac{2}{5}, \pi_2 = \frac{2}{5}.$$