Chapter 4. Error term analysis

Goal: Convergence & its speed (= rate of convergence); attack by one shot

4.1 The error term

It is an important practical (and often theoretical) matter to know not just that a sequence (a_n) converges to a limit L, but also to have some idea of **how rapidly** it converges to L.

For example, it can be proved that

$$a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} \quad \to \quad \ln 2 \quad \text{(very slowly)}$$

$$b_n = \frac{2}{1\cdot 3} + \frac{2}{3\cdot 3^3} + \frac{2}{5\cdot 3^5} + \cdots + \frac{2}{(2n-1)\cdot 3^{2n-1}} \quad \to \quad \ln 2 \quad \text{(rapidly)}$$

"Expect" for the limit: ln2

(i) $\ln(1+x) \xleftarrow{\text{integrate}} \dfrac{1}{1+x}$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \quad \text{for } |x|<1$$

Take $\int_0^x$: where $0<x<1$ $\Rightarrow$

$$\ln(1+x) \overset{\text{expect}}{=} x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for } 0<x<1$$ (true: will be proved in section 4.2)

$$\therefore \lim_{x\to 1^-} \ln(1+x) = \ln 2 = \lim_{x\to 1^-}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right]$$

$$\overset{\text{expect}}{=} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} + \cdots$$ (true; there are 'two' ways to verify this)

(An elementary proof will be given soon)

(ii) $$f(x) \overset{\text{let}}{=} \sum_{n=1}^{\infty} \frac{2}{2n-1}\left(\frac{x}{3}\right)^{2n-1} \quad \left(\text{assume } \left|\frac{x}{3}\right|<1\right)$$

$$f'(x) \overset{\text{expect}}{=} \frac{2}{3}\sum_{n=1}^{\infty}\left(\frac{x}{3}\right)^{2n-2}$$ (true for $|x|<3$: will be proved in Chap22)

$$= \frac{2}{3}\left[1 + \left(\frac{x}{3}\right)^2 + \left(\frac{x}{3}\right)^4 + \cdots\right] = \frac{2}{3}\,\frac{1}{1-\left(\frac{x}{3}\right)^2} \quad \text{for } |x|<3$$

$$= \frac{6}{9-x^2} = \frac{1}{3-x} + \frac{1}{3+x} \quad \text{for } |x|<3$$

Thus for $|x| < 3$,

$$\underbrace{\int_0^x f'(x)\,dx}_{\substack{\| \leftarrow f(0)=0 \\ f(x)}} = \int_0^x \left[\frac{1}{3-x} + \frac{1}{3+x} \right] dx = \ln\left(\frac{3+x}{3-x} \right)$$

$$\therefore \quad \sum_{n=1}^{\infty} \frac{2}{2n-1} \left(\frac{x}{3} \right)^{2n-1} = \ln\left(\frac{3+x}{3-x} \right) \qquad \text{for } |x| < 3$$

Take $x = 1 \Rightarrow$ $\ln 2 = \sum_{n=1}^{\infty} \frac{2}{(2n-1)} \left(\frac{1}{3} \right)^{2n-1} = \frac{2}{1 \cdot 3} + \frac{2}{3 \cdot 3^3} + \frac{2}{5 \cdot 3^5} + \cdots$

The first sequence ($= a_n$) is useless for computing ln2, because it converges too slowly

(Since $a_{100} = a_{99} - \frac{1}{100}$, at the 100-th term of the sequence, the second decimal place is still changing)

By contrast, (b_n) converges rapidly; the term b_3 gives ln2 to three decimal places.

To think about questions of this kind, we slightly change our point of view about limits;
Instead of looking at the approximation itself, $a_n \underset{\varepsilon}{\approx} L$, we focus our attention on the error term $e_n = a_n - L$

a_n이 L로 수렴한다는 것은
알려주지만 수렴하는 속도를
알진 못한다

Theorem (Error-form Principle)

Let $a_n = L + e_n$. Then $a_n \to L \Leftrightarrow e_n \to 0$

4.2 The error in the geometric series

Proposition (geometric sum limit)

$$a_n \overset{\text{let}}{=} 1 + a + a^2 + \cdots + a^n \Rightarrow \lim_{n \to \infty} a_n = \frac{1}{1-a} \quad \text{if } |a| < 1$$

Pf. $\boxed{1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1-a}} = \frac{1}{1-a} - \frac{a^{n+1}}{1-a}$ if $a \neq 1$

i.e., $e_n = -\frac{a^{n+1}}{1-a}$ if $a \neq 1$

Since $|a| < 1$, we have $a^n \to 0$ as $n \to \infty$ $\quad \therefore \ e_n \to 0$

Ex. Let $a_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{(-1)^{n-1}}{n}$

Show $\lim_{n\to\infty} a_n = \ln 2$

Pf. Idea: $\left. x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+(-1)^{n-1}\frac{x^n}{n}\right|_{x=1} = a_n$

$$\uparrow \;\leftarrow\; \left.\int_0^x (*)\right|_{x=1}$$

$$(*):\ 1-x+x^2-x^3+\cdots+(-1)^{n-1}x^{n-1}$$

Based on this, we consider

$$1-x+x^2-x^3+\cdots+(-1)^{n-1}x^{n-1} = \frac{1}{1+x}-(-1)^n\frac{x^n}{1+x}:\quad x\neq -1$$

Take $\int_0^1 \Rightarrow$ $\qquad = \frac{1-(-1)^n x^n}{1+x}$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{(-1)^{n-1}}{n} = \ln 2 \pm \int_0^1 \frac{x^n}{1+x}dx$$

Suffices to show: $e_n = \pm\int_0^1 \frac{x^n}{1+x}dx \to 0$

Clearly, $|e_n| = \int_0^1 \frac{x^n}{1+x}dx \underset{(1/2\le)\frac{1}{1+x}\le 1 \text{ for } 0\le x\le 1}{\le} \int_0^1 x^n\,dx = \frac{1}{n+1} \to 0$

$$\therefore\ \lim_{n\to\infty} a_n = \ln 2$$

4.3 A sequence converging to $\sqrt{2}$: Newton's method

In the rest of this chapter (sections 4.3 & 4.4), we illustrate the use of the error form on sequences whose general term a_n is not given explicitly in terms of n, but instead is given recursively by a formula involving a_{n-1} and previous terms as well

(Such sequences are the normal thing one encounters in numerical analysis and computation)
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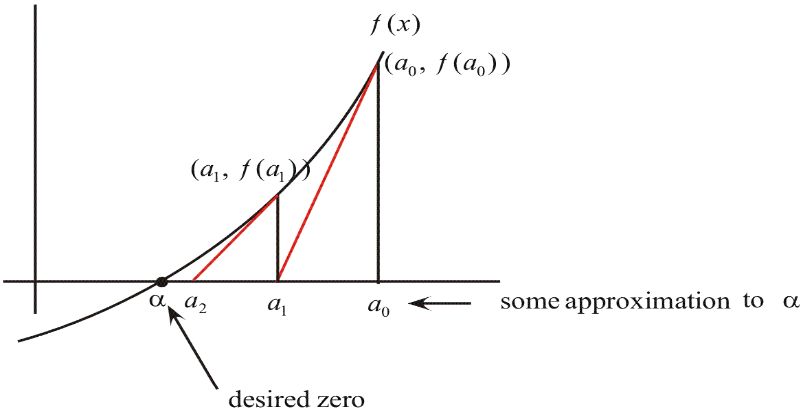
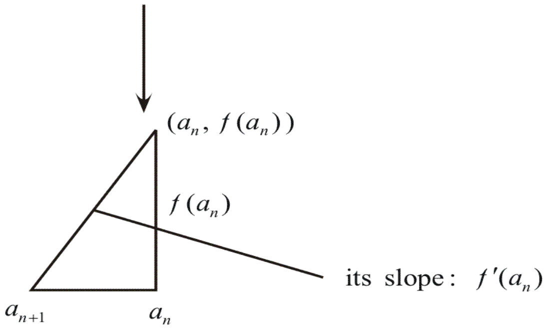
Newton's method (a numerical method for locating a zero α of a given function $f(x)$ to any accuracy desired)

$f(x)=0$ 을 성립하는 x값을 찾고 싶은 것





$$\therefore \quad f'(a_n) = \frac{f(a_n)}{a_n - a_{n+1}}$$, and rephrase it in terms of a_{n+1}

This gives the formula $\boxed{a_{n+1} = a_n - \dfrac{f(a_n)}{f'(a_n)}}$

$$\lim_{n\to\infty} a_n \overset{\text{expect}}{=} \alpha$$

That is, start with $a_0 \to a_1 \to a_2 \to \cdots \overset{\text{hope}}{\to} \alpha$

Ex. Find a sequence (a_n) s.t. $a_n \to \sqrt{2}$, by using Newton's method

(& investigate its rate of convergence)

Sol. $\sqrt{2}$: the positive zero of $f(x) = x^2 - 2$

Recall

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)} = a_n - \frac{a_n^{\,2} - 2}{2a_n}$$

$$= \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \quad --- \quad (*)$$

Expect: any starting value "a_0 close enough to $\sqrt{2}$" will generate a sequence converging to $\sqrt{2}$

$$\left(a_0 \underset{\text{should determine}}{\approx} \sqrt{2} \right)$$

we have $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$

$$e_n \overset{\text{let}}{=} a_n - \sqrt{2} \quad \& \quad \text{show } e_n \to 0$$

(Notice that we have no explicit formula for a_n)

Key idea: $\boxed{\text{Use } (*) \text{ to relate } e_{n+1} \text{ to } e_n}$

$$e_{n+1} = a_{n+1} - \sqrt{2} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) - \sqrt{2}$$

$$\therefore \quad e_{n+1} = \frac{1}{2}\left(\left(\sqrt{2}+e_n\right) + \frac{2}{\sqrt{2}+e_n}\right) - \sqrt{2} \quad (\leftarrow a_n = \sqrt{2}+e_n)$$

$$= \frac{{e_n}^2}{2\left(\sqrt{2}+e_n\right)}$$

To show e_{n+1} gets small, must show the denominator is *not* small

$$\left|\sqrt{2}+e_n\right| \ge \sqrt{2} - |e_n| > 1.4 - 0.9 = 0.5 \quad \text{provided } |e_n| < 0.9$$

$|e_n|$이 0.9보다 작으면 $|\sqrt{2}+e_n|$이 최소 0.5보다 크고, 이는 곧 $2(\sqrt{2}+e_n)$이 1보다 크다는 말이다.

So if $|e_n| < 0.9$, then $|e_{n+1}| < {e_n}^2$

Thus if we choose a starting value a_0 satisfying $|e_0| < 0.9$, we see that

$$|e_1| < {e_0}^2, \quad |e_2| < e_1^2 < {e_0}^4, \quad \cdots, \quad |e_n| < {e_0}^{2^n} \to 0 \text{ very rapidly as } n \to \infty$$

$$\therefore \quad e_n \to 0 \text{ very rapidly}$$

Remark:

If we take a_0 such that $|e_0| < 0.1$, then

$$|e_1| < 0.01 = 10^{-2}$$
$$|e_2| < 0.0001 = 10^{-4}$$
$$\vdots$$

Home Study: Let $a > 0$.

(i) Find a sequence (a_n) converging to $\sqrt{a}$

(ii) Find a sequence (a_n) converging to $\sqrt[3]{a}$

Remark: $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$ with $a_0 \approx \sqrt{2}$ (or $a_0 > 0$)

$\Rightarrow$ (a_n) is convergent

Pf. $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \overset{AG}{\geq} \sqrt{a_n \cdot \frac{2}{a_n}} = \sqrt{2} \quad \forall n \geq 0$

$\therefore$ $(a_n)_1^\infty$ is bounded below by $\sqrt{2}$ (even if $0 < a_0 < \sqrt{2}$)

Goal: $(a_n)_1^\infty$ is $\downarrow$

$$a_n - a_{n+1} = a_n - \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) = \frac{1}{2}\left(a_n - \frac{2}{a_n}\right)$$

$$= \frac{1}{2}\frac{a_n^{\,2} - 2}{a_n} \geq \frac{1}{2}\cdot\frac{0}{a_n} = 0 \quad \forall n \geq 1 \text{ (note } a_n \geq \sqrt{2} > 0 \text{ for } n \geq 1)$$

$\therefore$ $(a_n)_1^\infty$ is $\downarrow$

Thus (a_n) is convergent (by the Completeness Property of $\mathbb{R}$)

Now, we let $\lim_{n\to\infty} a_n = \alpha$ $(\Rightarrow \alpha \geq \sqrt{2})$

Since $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$, we have (by taking limits)

$$\alpha = \frac{1}{2}\left(\alpha + \frac{2}{\alpha}\right) \qquad \therefore \quad \alpha^2 = 2 \qquad \therefore \quad \alpha = \sqrt{2}$$

4.4 The sequence of Fibonacci fractions

The Fibonacci sequence is given by

1 1 2 3 5 8 $\cdots$: often written as $F_0, F_1, F_2, \cdots$

Let $a_n = \frac{F_n}{F_{n+1}} (n \geq 0)$, the ratios of successive terms of the Fibonacci sequence:

$$\frac{1}{1}(=1) \quad \frac{1}{2}(=0.5) \quad \frac{2}{3}(\doteq 0.667) \quad \frac{3}{5}(=0.6) \quad \frac{5}{8}(=0.625) \quad \frac{8}{13} \quad \frac{13}{21} \quad \cdots$$

Question: $a_n \quad \to \quad ?$

Note that if $a_{n+1} = \frac{F_{n+1}}{F_{n+2}} = \frac{F_{n+1}}{F_n + F_{n+1}} = \frac{1}{\frac{F_n}{F_{n+1}} + 1} = \frac{1}{a_n + 1} (n \geq 0)$ and $a_0 = 1$

$$\therefore \quad a_{n+1} = \frac{1}{a_n + 1} \qquad (a_0 = 1, \quad a_1 = 0.5, \quad a_2 \doteq 0.667)$$

If we **assume** $\lim_{n\to\infty} a_n \equiv M$ exists, it is easy to find M

Indeed, if $\lim_{n\to\infty} a_n \equiv M$ exists, then

$$\lim_{n\to\infty} a_{n+1} = \frac{1}{\lim_{n\to\infty} a_n + 1} \qquad \text{i.e.,} \quad M = \frac{1}{M+1}$$

$$\text{i.e.,} \quad M^2 + M - 1 = 0 \qquad \therefore \quad M = \frac{\sqrt{5}-1}{2} \quad (\because \quad M > 0)$$

Target: $\lim_{n\to\infty} a_n = \dfrac{\sqrt{5}-1}{2} \overset{\text{let}}{\equiv} M$

Must examine $e_n = a_n - M$ & try to show $e_n \to 0$

$$e_{n+1} = a_{n+1} - M = \frac{1}{a_n+1} - M$$

$$= \frac{1}{e_n + M + 1} - M = \frac{1 - M - M^2 - Me_n}{e_n + M + 1}$$

$$= -\frac{M}{e_n + M + 1} e_n \quad (\leftarrow \quad M^2 + M - 1 = 0)$$

$$= -\frac{\sqrt{5}-1}{2e_n + \sqrt{5} + 1} e_n \quad (\leftarrow \quad M = \frac{\sqrt{5}-1}{2}) \quad (\text{note that } \sqrt{5} - 1 < 2.3 - 1 = 1.3)$$

$$\left|\sqrt{5} + 1 + 2e_n\right| \ge 2.2 + 1 - 2|e_n|$$

$$\ge 2.2 + 1 - 2(0.2) = 2.8 \quad \text{if} \quad |e_n| \le 0.2$$

Using $\sqrt{5} < 2.3$ $(\to \sqrt{5} - 1 < 1.3)$, we get

$$(*): \quad |e_{n+1}| < \frac{1.3}{2.8}|e_n| < \frac{1}{2}|e_n| \quad \text{if} \quad |e_n| \le 0.2$$

Since $|e_2| = a_2 - \dfrac{\sqrt{5}-1}{2} \doteq 0.667 - 0.618 \approx 0.05,$ we have

$$|e_n| < 0.2 \quad \text{for all} \quad n \ge 2 \quad \text{by} \quad (*)$$

Therefore

$$|e_3| < \frac{1}{2}|e_2|, \quad |e_4| < \frac{1}{2}|e_3| < \left(\frac{1}{2}\right)^2 |e_2|, \quad \cdots, \quad |e_n| < \underbrace{\left(\frac{1}{2}\right)^{n-2} |e_2|}_{\to 0 \text{ as } n\to\infty}$$

$$\therefore \quad a_n \to M$$

Home Study: Let $a_{n+1} = \dfrac{1}{a_n + 1}$ with $a_0 = A$ & $A \neq -1$

For what values of A, is (a_n) convergent

(Hint: Draw a graph suggested by the recursive formula)

$f(x) = \frac{1}{x+1}$

Return to a rigorous but elementary pf of: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad -1 < x \le 1$

Idea: $\frac{d}{dx}\ln(1+x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \quad$ for $|x| < 1$

Integrating ($\int_0^x \; dt$) term by term $\Rightarrow$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad -1 < x < 1$$

(Later, we will prove that every power series can be integrated term by term "within the (open) interval of convergence"; the radius of convergence R of the RHS = 1) --- not studied in high school math

Remember that we already proved (by only using high school math) that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2$$

Thus, it suffices to verify the following:

Claim: Using only "High School-Math" (the same idea as seen in Example of section 4.2), prove that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad -1 < x < 1$$

Pf. Start with $\quad 1 - x + x^2 - x^3 + \cdots + (-1)^{n-1}x^{n-1} = \frac{1}{1+x} - (-1)^n \frac{x^n}{1+x}: \quad x \ne -1$

Case 1: $\quad 0 \le x < 1 \qquad$ Take $\int_0^x () \; dt \;\Rightarrow$

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} = \ln(1+x) \pm \int_0^x \frac{t^n}{1+t}\,dt$$

Suffices to show: $\quad e_n(x) := \pm \int_0^x \frac{t^n}{1+t}\,dt \quad \to 0 \;$ as $\; n \to \infty$

Clearly, $\; |e_n(x)| = \int_0^x \frac{t^n}{1+t}\,dt \underset{\substack{\uparrow \\ (1/2\le)\frac{1}{1+t}\le 1 \text{ for } 0\le t\le x<1}}{\le} \int_0^x t^n \; dt = \frac{x^{n+1}}{n+1} \le \frac{1}{n+1} \;\to 0$

Case 2: $\quad -1 < x < 0 \qquad$ Want: $\; \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad -1 < x < 0$

By letting $\; x = -y \;\; (0 < y < 1)$, we need to show

$$-\ln(1-y) = y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \cdots \quad 0 < y < 1$$

Start with $\quad 1 + y + y^2 + y^3 + \cdots + y^{n-1} = \frac{1}{1-y} - \frac{y^n}{1-y}: \quad y \ne 1$

Take $\; \int_0^y () \; dt \;\; (0 < y < 1) \;\Rightarrow$

$$y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \cdots + \frac{y^n}{n} = -\ln(1-y) - \int_0^y \frac{t^n}{1-t}\,dt$$

Suffices to show: $\; \int_0^y \frac{t^n}{1-t}\,dt \;\; (0 < y < 1) \;$ as $\; n \to \infty$

$$\int_0^y \frac{t^n}{1-t}\,dt \le \int_0^y \frac{t^n}{1-y}\,dt = \frac{1}{1-y}\int_0^y t^n \; dt = \frac{1}{1-y}\frac{y^{n+1}}{n+1} \le \frac{1}{1-y}\frac{1}{n+1} \to 0 \;\text{ as } n \to \infty$$