Another (useful) tool for limits ($\limsup_{n\to\infty} a_n$ & $\liminf_{n\to\infty} a_n$)

⊙ For sequences which do **not** converge, there are two useful notions which sometimes can substitute for the non-existence of limit : the ***limit superior (or, supremum)*** and the ***limit inferior (or, infimum)***

For the **unified** treatment of **these** concepts (regardless of whether (a_n) is bounded or unbounded), we need to extend the definitions of $\sup S$ & $\inf S$ for a non-empty set $S(\subset \mathbb{R})$: Remind that $\sup S$ is defined (to be a **real** number) only when S has an upper bound.

Def. Let $S \subset \mathbb{R}$ with $S \neq \varnothing$. We say that $\sup S = \infty$ if S has no upper bound.
Similarly, we say that $\inf S = -\infty$ if S has no lower bound.

Eg. It is clear that $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ has no upper bound & has no lower bound.

$$\therefore \quad \sup \mathbb{Z} = \infty \quad \& \quad \inf \mathbb{Z} = -\infty$$

Warning: The statement that $\sup S = \infty$ does **not mean** that $\sup S$ is equal to the "extended number" ∞ [Note: ∞ is **not** a real number]. The notation $\sup S = \infty$ is just a shorthand way of saying that the non-empty set S has no upper bound.

Def. Let (a_n) be a sequence of real numbers. For each fixed $n (\geq 1)$,
we let $M_n := \sup\left\{a_n, \ a_{n+1}, \ a_{n+2}, \cdots\right\} = \sup_{k\geq n}\{a_k\} \overset{\text{shortly}}{=} \sup_{k\geq n} a_k$.

i.e., $M_1 = \sup\left\{a_1, \ a_2, \ a_3, \cdots\right\}, M_2 = \sup\left\{a_2, \ a_3, \ a_4, \cdots\right\}, M_3 = \sup\left\{a_3, \ a_4, \ a_5, \cdots\right\}, \cdots$

Obviously, $M_1 \geq M_2 \geq M_3 \geq \cdots$: i.e., $M_n \downarrow$ (as $n \to \infty$)

Hence,

either $\lim_{n\to\infty} M_n$ exists or $\lim_{n\to\infty} M_n = -\infty$ or ∞ (if (a_n) is unbounded above)

We define $\lim_{n\to\infty} M_n$ (always exists, but could be $\pm\infty$) as the limit superior:

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} M_n \left(= \lim_{n\to\infty} \sup_{k\geq n} a_k\right) \left(\overset{\text{see Cf below}}{=} \inf_{n\geq 1}\{\sup_{k\geq n} a_k\}\right)$$

Cf: Seen that: (a_n) is $\downarrow$ & bounded below $\Rightarrow$ $\lim_{n\to\infty} a_n$ exists & $\lim_{n\to\infty} a_n = \inf\{a_n : n \in N\}$

Alternative notations: $\limsup_{n\to\infty} a_n = \limsup a_n = \overline{\lim_{n\to\infty}} a_n = \overline{\lim}\, a_n$

Eg. (i) $a_n = (-1)^n$ (which is bounded above)

For every n, $M_n = 1$. So $\lim_{n\to\infty} M_n = 1$ $\quad\therefore\quad \limsup a_n = 1$

(ii) $(a_n) = \left\{1, \ -1, \ 1, \ -2, \ 1, \ -3, \ 1, \ -4, \ \cdots\right\}$ (which is bounded above)

For every n, $M_n = 1$. So $\lim_{n\to\infty} M_n = 1$ $\quad\therefore\quad \limsup a_n = 1$

(iii) $a_n = -n$ i.e., $(a_n)_1^\infty = \left\{-1, \ -2, \ -3, \ -4, \ \cdots\right\}$ (which is bounded above)

For every n, $M_n = \sup\left\{-n, \ -n-1, \ -n-2, \ \cdots\right\} = -n$

$$\therefore \quad \lim_{n\to\infty} M_n = \lim_{n\to\infty}(-n) = -\infty \qquad \therefore \quad \limsup a_n = -\infty$$

(iv) $a_n = n$ i.e., $(a_n)_1^\infty = \left\{1, \ 2, \ 3, \ 4, \ \cdots\right\}$ (which is not bounded above)

$M_n = \sup\left\{n, \ n+1, \ \cdots\right\} = \infty \left(\leftarrow (a_n): \text{not bdd above}\right)$ $\quad\therefore\quad \lim_{n\to\infty} M_n = \infty$ $\quad\therefore\quad \limsup a_n = \infty$

Note: Changing a finite number of terms of the given sequence (a_n) *does not* change $\limsup a_n$

Eg. $(a_n) = \left\{10^{100},\ 10^{200},\ 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ \cdots\right\} \quad \rightarrow \quad \limsup a_n = 1 \quad \left(\text{cf.}\ \ \sup a_n = 10^{200}\right)$

Theorem

If (a_n) is convergent, then $\limsup a_n = \lim_{n\to\infty} a_n$

Pf. Let $L = \lim_{n\to\infty} a_n$. Then given $\varepsilon > 0$, $a_n \underset{\varepsilon}{\approx} L$ for $n \gg 1$.

i.e., given $\varepsilon > 0$, $L - \varepsilon < a_n < L + \varepsilon$ for every $n \geq$ (some) N

Thus if $n \geq N$ then

$L + \varepsilon$ is an upper bound for the set $\{a_n,\ a_{n+1},\ a_{n+2},\ \cdots\}$

But it is clear that $L - \varepsilon < \sup\{a_n,\ a_{n+1},\ a_{n+2},\ \cdots\} = M_n$ for $n \geq N$

So, given $\varepsilon > 0$, $L - \varepsilon < M_n = \sup\{a_n,\ a_{n+1},\ a_{n+2},\ \cdots\} \leq L + \varepsilon$ for $n \geq N$

Hence

$$\lim_{n\to\infty} M_n = L \qquad \therefore \quad \limsup a_n = L$$

Def. Let (a_n) be a sequence of real numbers. For each fixed $n(\geq 1)$,

we let $m_n := \inf\{a_n,\ a_{n+1},\ a_{n+2},\cdots\} = \inf_{k\geq n}\{a_k\} \overset{\text{shortly}}{=} \inf_{k\geq n} a_k$.

i.e., $m_1 = \inf\left\{a_1,\ a_2,\ a_3,\cdots\right\}, m_2 = \inf\left\{a_2,\ a_3,\ a_4,\cdots\right\}, m_3 = \inf\left\{a_3,\ a_4,\ a_5,\cdots\right\},\cdots$

Obviously, $m_1 \leq m_2 \leq m_3 \leq \cdots$: i.e., $m_n \uparrow$ (as $n \to \infty$)

Hence,

either $\lim_{n\to\infty} m_n$ exists or $\lim_{n\to\infty} m_n = \infty$ or $-\infty\left(\text{if } (a_n) \text{ is unbounded below}\right)$

We define $\lim_{n\to\infty} m_n$ (always exists, but could be $\pm\infty$) as the limit inferior:

$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} m_n \left(= \lim_{n\to\infty} \inf_{k\geq n} a_k\right) \left(\overset{\text{see Cf below}}{=} \sup_{n\geq 1}\{\inf_{k\geq n} a_k\}\right)$$

Cf: Seen that: (a_n) is $\uparrow$ & bounded above $\Rightarrow$ $\lim_{n\to\infty} a_n$ exists & $\lim_{n\to\infty} a_n = \sup\{a_n : n \in N\}$

Alternative notations: $\liminf_{n\to\infty} a_n = \liminf a_n = \underline{\lim}_{n\to\infty}\, a_n = \underline{\lim}\, a_n$

Eg. (i) $a_n = (-1)^n$ (which is bounded below)

For every n, $m_n = -1$. So $\lim_{n\to\infty} m_n = -1 \qquad \therefore \ \liminf a_n = -1$

(ii) $a_n = n$ (which is bounded below)

For every n, $m_n = \inf\{n,\ n+1,\ n+2,\ \cdots\} = n \qquad \therefore \ \lim_{n\to\infty} m_n = \lim_{n\to\infty} n = \infty$

$\therefore \ \liminf a_n = \infty$

(iii) $(a_n) = \{-1,\ -2,\ -3,\ -4,\ \cdots\}$ (which is not bounded below)

For $\forall n$, $m_n = -\infty\left(\leftarrow (a_n) : \text{not bounded below}\right) \quad \therefore \ \lim_{n\to\infty} m_n = -\infty \quad \therefore \ \liminf a_n = -\infty$

Note: Changing a finite number of terms of the given sequence (a_n) *does not* change $\liminf a_n$

Ex (easy) $a_n \leq b_n$ for $n \gg 1 \Rightarrow \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$ and $\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$

Theorem.

If (a_n) is convergent, then $\liminf a_n = \lim_{n\to\infty} a_n$

Pf. Essentially the same as the proof of $\limsup a_n = \lim_{n\to\infty} a_n$.

Theorem.

If (a_n) is a sequence of real numbers, then

$$\liminf a_n \le \limsup a_n$$

Pf. Recall that $\liminf a_n = \lim_{n\to\infty} m_n$ and $\limsup a_n = \lim_{n\to\infty} M_n$,

where $m_n := \inf\left\{a_n,\ a_{n+1},\ a_{n+2},\cdots\right\}$ and $M_n := \sup\left\{a_n,\ a_{n+1},\ a_{n+2},\cdots\right\}$.

Assume first that (a_n) is bounded.

Then, obviously, $m_n \le M_n$

So by LLT, $\lim_{n\to\infty} m_n \le \lim_{n\to\infty} M_n$ since both the (finite) limits exist.

That is, $\liminf a_n \le \limsup a_n$

On the other hand,

if (a_n) is not bounded above, then $\limsup a_n = \infty$;

so trivially $\liminf a_n \le \limsup a_n$ holds

If (a_n) is not bounded below, then $\liminf a_n = -\infty$;

so trivially $\liminf a_n \le \limsup a_n$ holds

※ Theorem

Let (a_n) be any sequence of real numbers.

If $\limsup a_n = \liminf a_n = L$ $(\&\ L \in \mathbb{R})$, then (a_n) converges (to L)

Pf. Suppose $\limsup a_n = \liminf a_n = L$ ---$(\odot)$

Observe that

$$\begin{array}{ccccc} \inf_{k\ge n}\{a_k\} & \le & a_n & \le \sup_{k\ge n}\{a_k\} & \text{for every } n \\ \downarrow & & & \downarrow & \text{as } n\to\infty \\ \lim\inf_{k\ge n}\{a_k\} & = L = & & \lim\sup_{k\ge n}\{a_k\} & (\leftarrow (\odot)) \end{array}$$

Thus $\lim_{n\to\infty} a_n = L$ by Sandwich Principle

An important consequence.

Let (a_n) be any sequence of real numbers.

(i) If $\limsup a_n = \liminf a_n (\in \mathbb{R})$, then (a_n) converges.

(ii) If $\limsup a_n \ne \liminf a_n$, then (a_n) diverges.

Pf. (i) Previous theorem

(ii) (대우증명)

If (a_n) converges, then we have seen in earlier two theorems that

$$\limsup a_n = \lim_{n\to\infty} a_n = \liminf a_n \qquad //$$

Summary: Let (a_n) be a (bounded or unbounded) sequence.

$$\limsup a_n = \overline{\lim_{n\to\infty}} a_n \left(=\overline{\lim}\, a_n\right) \overset{\text{def}}{=} \lim_{n\to\infty}\left(\sup_{k\ge n} a_k\right) \overset{\text{or}}{=} \inf_{n\ge1}\{\sup_{k\ge n} a_k\} = \lim_{n\to\infty} M_n$$

$$\liminf a_n = \underline{\lim}_{n\to\infty}\, a_n \left(=\underline{\lim}\, a_n\right) \overset{\text{def}}{=} \lim_{n\to\infty}\left(\inf_{k\ge n} a_k\right) \overset{\text{or}}{=} \sup_{n\ge1}\{\inf_{k\ge n} a_k\} = \lim_{n\to\infty} m_n$$

where $M_n := \sup\{a_n,\ a_{n+1},\ a_{n+2},\cdots\}$ and $m_n := \inf\{a_n,\ a_{n+1},\ a_{n+2},\cdots\}$.

$$\odot \quad -\infty \le \underline{\lim}_{n\to\infty}\, a_n \le \overline{\lim_{n\to\infty}}\, a_n \le \infty$$

Theorem (important) [key properties of $\overline{\lim_{n\to\infty}}\, a_n = M$]

Let (a_n) be a **bounded** sequence. Then

(i) $\overline{\lim_{n\to\infty}}\, a_n = M \ \Rightarrow\ \exists$ a subsequence (a_{n_k}) of (a_n) such that $\lim_{k\to\infty} a_{n_k} = M$

(ii) $M' > \overline{\lim_{n\to\infty}}\, a_n \ \Rightarrow\ M' > a_n$ for $n \gg 1$ (say for $\forall n \ge (\text{some})N$)

$\left(\text{i.e., } M' \le a_n \ (\&\ M' < a_n) \text{ for only (at most) finitely many } n\right)$

(iii) $M'' < \overline{\lim_{n\to\infty}}\, a_n \ \Rightarrow\ M'' < a_n$ for infinitely many n [$\Leftarrow$ (i): why?]

Pf . (i) Recall that $\overline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty} M_n$, where $M_n = \sup_{k\ge n} a_k$

In particular, $M_1 = \sup_{k\ge1} a_k \overset{\text{i.e.}}{=} \sup\{a_1, a_2, \cdots\}$. Then by the definition of supremum,

$$\exists n_1 \in \mathbb{N} \text{ such that } M_1 - 1 < a_{n_1} \le M_1$$

Applying the same argument to $M_{n_1+1} = \sup\{a_{n_1+1}, a_{n_1+2}, \cdots\}$ shows

$$\exists n_2\,(\ge n_1 + 1 > n_1) \text{ such that } M_{n_1+1} - \frac{1}{2} < a_{n_2} \le M_{n_1+1}$$

Continue this process to get a sequence of integers $n_1 < n_2 < n_3 < \cdots$ such that

$$M_{n_{k-1}+1} - \frac{1}{k} < a_{n_k} \le M_{n_{k-1}+1} \qquad \text{for every } k \in \mathbb{N}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$M \qquad\qquad\qquad M \qquad \text{(by } \lim_{n\to\infty} M_n = M \text{ plus Subsequence theorem)}$$

By Squeeze principle, we conclude that $\lim_{k\to\infty} a_{n_k} = M$.

(ii) Let $\overline{\lim_{n\to\infty}}\, a_n < M'$ i.e., $\lim_{n\to\infty}\left(\sup_{k\ge n} a_k\right) < M'$

$$\overset{\text{SLT}}{\Rightarrow} \quad \sup_{k\ge n} a_k < M' \quad \text{for } n \gg 1 \text{ (say for } n \ge N\text{)}$$

$$\Rightarrow \quad \sup_{k\ge N} a_k < M' \quad \Rightarrow \quad a_k < M' \text{ for } \forall k \ge N$$

(iii) [another indirect proof] Let $\limsup_{n\to\infty} a_n = M > M''$.

If the conclusion were false, then $a_n > M''$ holds for at most finitely many n;

which clearly says that $\exists N \in \mathbb{N}$ such that $a_n \le M''$ for all $n \ge N$.

This would imply $\sup_{n\ge N}\{a_n\} := M_N \le M''$. Taking limits gives

$\lim_{N\to\infty} M_N \left(= \lim_{N\to\infty} \sup a_N\right) \le M''$ (by LLT) ; contradiction to the hypothesis $\limsup_{n\to\infty} a_n = M > M''$

Proposition $(\Leftarrow \ \text{(ii)})$.

$$\overline{\lim_{n\to\infty}}\, a_n < M' \quad \Rightarrow \quad M' \text{ is not a cluster point of } (a_n)$$

Pf. Choose K so that $\overline{\lim_{n\to\infty}}\, a_n < K < M'$. Then by (ii)

$$a_n < K \quad \text{for } n \gg 1 \quad (\text{i.e., } \exists N \in \mathbb{N} \text{ such that } a_n < K \text{ for } n \geq N)$$

Let $\varepsilon = M' - K (> 0)$. Then $a_n \underset{\varepsilon}{\approx} M'$ for only (at most) finitely many n

$\therefore$ M' is not a cluster point of (a_n)

Cor. (important) [another way of understanding $\overline{\lim_{n\to\infty}}\, a_n$]

Let (a_n) be a bounded sequence. Then

$$\overline{\lim_{n\to\infty}}\, a_n = \textbf{the largest cluster point of } (a_n)$$

That is,

(i) $\exists$ a subsequence (a_{n_k}) of (a_n) such that $\lim_{k\to\infty} a_{n_k} = \overline{\lim_{n\to\infty}}\, a_n$ (by Theorem –(i))

(ii) C is any cluster point of (a_n) $\Rightarrow$ $C \leq \overline{\lim_{n\to\infty}}\, a_n$ (by Proposition)

[$(\text{ii})'$ C is the limit of any convergent subsequence of (a_n) $\Rightarrow$ $C \leq \overline{\lim_{n\to\infty}}\, a_n$]

Another pf of Cor:

(i) $\exists$ a subsequence (a_{n_k}) of (a_n) such that $\lim_{k\to\infty} a_{n_k} = \overline{\lim_{n\to\infty}}\, a_n$ (by Theorem –(i))

$(\text{ii})'$ C is the limit of any convergent subsequence of (a_n) $\Rightarrow$ $C \leq \overline{\lim_{n\to\infty}}\, a_n$

To show $(\text{ii})'$ holds, we let $\left(a_{n_i}\right)$ be a convergent subsequence of (a_n), with $\lim_{i\to\infty} a_{n_i} = C$

Note that $\quad a_{n_i} \leq M_{n_i} = \sup\left\{a_{n_i}, a_{n_{i+1}}, a_{n_{i+2}}, \cdots\right\}$

so that by LLT

$$C = \lim_{i\to\infty} a_{n_i} \leq \lim_{i\to\infty} M_{n_i} = \overline{\lim_{n\to\infty}}\, a_n \left(\leftarrow \lim_{n\to\infty} M_n = \overline{\lim_{n\to\infty}}\, a_n \text{ plus Subsequence theorem}\right)$$

Theorem. Let (a_n) be a bounded sequence. Then

(i) $\underline{\lim_{n\to\infty}}\, a_n = m \quad \Rightarrow \quad \exists$ a subsequence (a_{n_k}) of (a_n) such that $\lim_{k\to\infty} a_{n_k} = m$

(ii) $\underline{\lim_{n\to\infty}}\, a_n > m' \quad \Rightarrow \quad a_n > m'$ for $n \gg 1$ (say for $\forall n \geq (\text{some}) N$)

(iii) $\underline{\lim_{n\to\infty}}\, a_n < m'' \quad \Rightarrow \quad a_n < m''$ for infinitely many n

Pf. Apply previous theorem to $(-a_n)$

Cor. (important) Let (a_n) be a bounded sequence. Then

$$\underline{\lim_{n\to\infty}}\, a_n = \textbf{the smallest cluster point of } (a_n)$$

That is,

(i) $\exists$ a subsequence (a_{n_k}) of (a_n) such that $\lim_{k\to\infty} a_{n_k} = \underline{\lim_{n\to\infty}}\, a_n$

(ii) c is any cluster point of (a_n) $\Rightarrow$ $c \geq \underline{\lim_{n\to\infty}}\, a_n$

[$(\text{ii})'$ c is the limit of any convergent subsequence of (a_n) $\Rightarrow$ $c \geq \underline{\lim_{n\to\infty}}\, a_n$]

Two widely used popular results:

Ex 1. Let $\{a_n\}$ & $\{b_n\}$ be two bounded sequences in $\mathbb{R}$. Show that

$$\overline{\lim_{n\to\infty}}(a_n+b_n)\le\overline{\lim_{n\to\infty}}a_n+\overline{\lim_{n\to\infty}}b_n$$

Pf. For each $m\ge n$, we have

$$a_m+b_m\le\sup\{a_k:k\ge n\}+\sup\{b_k:k\ge n\}$$

$\therefore$ $\sup\{a_m+b_m:m\ge n\}\left[=\sup\{a_k+b_k:k\ge n\}\right]\le\sup\{a_k:k\ge n\}+\sup\{b_k:k\ge n\}$

Taking $n\to\infty$ $\Rightarrow$

$$\lim_{n\to\infty}\left(\sup\{a_k+b_k:k\ge n\}\right)\le\lim_{n\to\infty}\left(\sup\{a_k:k\ge n\}\right)+\lim_{n\to\infty}\left(\sup\{b_k:k\ge n\}\right)$$

Therefore, $\overline{\lim_{n\to\infty}}(a_n+b_n)\le\overline{\lim_{n\to\infty}}a_n+\overline{\lim_{n\to\infty}}b_n$

Note: Obviously,

$\overline{\lim_{n\to\infty}}\left(Cb_n\right)=C\,\overline{\lim_{n\to\infty}}b_n$ for any sequence $\{b_n\}$ and any real number C

Ex 2. Let $\{a_n\}$ & $\{b_n\}$ be two sequences with $b_n>0$ for all $n\in\mathbb{N}$.
Assume further that $\lim_{n\to\infty}a_n$ exists with $\lim_{n\to\infty}a_n\ne0$. Then show that

$$\overline{\lim_{n\to\infty}}\left(a_nb_n\right)=\left(\lim_{n\to\infty}a_n\right)\cdot\overline{\lim_{n\to\infty}}b_n$$

Pf. Let $\lim a_n=:A\ne0$ and let $\varepsilon>0$ be given. Then

$\exists N\in\mathbb{N}$ such that $A-\varepsilon<a_n<A+\varepsilon$ for $n\ge N$

Since $b_n>0\left(\forall n\right)$, it follows that

$$(A-\varepsilon)b_n<a_nb_n<(A+\varepsilon)b_n \quad \text{for } n\ge N$$

Taking $\overline{\lim_{n\to\infty}}$ $\Rightarrow$

$$(A-\varepsilon)\overline{\lim_{n\to\infty}}b_n\le\overline{\lim_{n\to\infty}}\left(a_nb_n\right)\le(A+\varepsilon)\overline{\lim_{n\to\infty}}b_n \quad [\leftarrow \text{LLT}]$$

Finally letting $\varepsilon\to0$ $\Rightarrow$

$$A\,\overline{\lim_{n\to\infty}}b_n\le\overline{\lim_{n\to\infty}}\left(a_nb_n\right)\le A\,\overline{\lim_{n\to\infty}}b_n$$

$$\therefore\ \overline{\lim_{n\to\infty}}\left(a_nb_n\right)=A\,\overline{\lim_{n\to\infty}}b_n=\lim_{n\to\infty}a_n\cdot\overline{\lim_{n\to\infty}}b_n$$

Application: Prove $\overline{\lim_{n\to\infty}}\sqrt[n]{n\,|\,a_n\,|}=\overline{\lim_{n\to\infty}}\sqrt[n]{|\,a_n\,|}$

Pf. It is well-known that $\lim_{n\to\infty}\sqrt[n]{n}=1$

$$\therefore\ \overline{\lim_{n\to\infty}}\sqrt[n]{n\,|\,a_n\,|}=\lim_{n\to\infty}\sqrt[n]{n}\cdot\overline{\lim_{n\to\infty}}\sqrt[n]{|\,a_n\,|}=\overline{\lim_{n\to\infty}}\sqrt[n]{|\,a_n\,|}$$

- **Alternative way of understanding the radius of convergence of a given power series:**

Proposition [seen before]. ($\limsup_{n\to\infty}$ - version of SLT)

Let $\{a_n\}$ be a bounded sequence. Then

$$\limsup_{n\to\infty}a_n=M>M' \quad \Rightarrow \quad a_n>M' \ \text{for infinitely many } n$$

Theorem. (Generalized n-th root test; often called **n-th root test**)

Suppose $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = M$. Then

$$M < 1 \quad \Rightarrow \quad \sum a_n \quad \text{conv} \quad \text{(absolutely)}$$

$$M > 1 \quad \Rightarrow \quad \sum a_n \quad \text{diverges}$$

If $M = 1$, the test fails and there is no conclusion

Pf. Case1. $M < 1$

Choose a number M' so that $M < M' < 1$. Then

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|\, a_n \,|} \left(= \lim_{n\to\infty} \sup \left\{ \sqrt[n]{|\, a_n \,|}, \sqrt[n+1]{|\, a_{n+1} \,|}, \cdots \right\} \right) = M < M'$$

$$\overset{\text{SLT}}{\Rightarrow} \quad \sup \left\{ \sqrt[n]{|\, a_n \,|}, \sqrt[n+1]{|\, a_{n+1} \,|}, \cdots \right\} < M' \quad \text{for } n \gg 1, \text{ say for } n \geq N$$

$$\Rightarrow \quad |\, a_n \,| < (M')^n \quad \text{for } n \geq N$$

$\sum_N^\infty (M')^n$ converges since $M' < 1$ $\quad \therefore \sum_N^\infty |\, a_n \,|$ conv (by the Comparison thm)

$\therefore \sum_0^\infty |\, a_n \,|$ converges (by Tail-convergence Thm)

Case2. $M > 1$

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = M > 1 \overset{\text{Proposition}}{\Rightarrow} \sqrt[n]{|a_n|} > 1 \quad \text{for infinitely many } n$$

$$\Rightarrow \quad |a_n| > 1 \quad \text{for infinitely many } n$$

$$\Rightarrow \quad |a_n| \not\to 0 \text{ as } n \to \infty \quad (\text{i.e., } \{a_n\} \text{ does not conv to } 0)$$

$$\Rightarrow \quad \sum a_n \quad \text{diverges}$$

Theorem (Cauchy-Hadamard theorem: a consequence of the Generalized n-th root test)

Let $\sum_0^\infty a_n x^n$ be a given power series, and let $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = M$ $(0 \leq M \leq \infty \text{ is possible})$. Then

$$\sum_0^\infty a_n x^n \begin{cases} \text{conv (absolutely)} & \text{if } |\, x \,| < \dfrac{1}{M} \\ \text{div} & \text{if } |\, x \,| > \dfrac{1}{M} \end{cases}$$

As a consequence,

$$R \ (= \text{the radius of convergence of } \sum_0^\infty a_n x^n) = \frac{1}{M} = \frac{1}{\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}}$$

Pf Assume $0 < M < \infty$ (The case $M = 0$ or ∞: Home Study)

Since $\sqrt[n]{|a_n x^n|} = |\, x \,| \sqrt[n]{|a_n|}$, we have $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n x^n|} = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} \cdot |\, x \,| = M \,|\, x \,|$

Applying the Generalized n-th root test to $\sum_0^\infty a_n x^n$ gives

$$\sum_0^\infty a_n x^n \begin{cases} \text{conv (absolutely)} & \text{if } M \,|\, x \,| < 1 \\ \text{div} & \text{if } M \,|\, x \,| > 1 \end{cases}$$