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1) $f(0)=0,\ f(1)=2$

By Mean Value Theorem, $\dfrac{f(1)-f(0)}{1-0} = f'(x) = 2$ for at least one $x \in (0,1)$

2) $P(x) = (x-a)(x-b)(x-c)$, $a<b<c$

Given, $P(a)=0$, $P(b)=0$, where $a<b$

$\Rightarrow$ By Rolle's Theorem, $\exists z \in (a,b)$ s.t. $P'(z)=0$

$\Rightarrow$ $P(a)=0$, $P'(z)=0$, where $a<z$

$\Rightarrow$ $\exists y \in (a,z)$ s.t. $f''(y)=0$

$\Rightarrow$ $y \in (a,z)$, and $(a,z) \subset (a,c)$

Letting $x=y$, $P''(x)-6P'(x)+9P(x)=0$

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3) $\displaystyle \lim_{x\to 0} \frac{1}{\sin^2 x} - \frac{1}{x^2} = \lim_{x\to 0} \frac{x^2-\sin^2 x}{x^2\sin^2 x}$, using L'hospital's Rule

$\displaystyle = \lim_{x\to 0} \frac{2x - 2\sin x \cdot \cos x}{2x\sin^2 x + x^2 \cdot 2\sin x\cdot\cos x}$, using L'hospital's Rule

$\displaystyle = \lim_{x\to 0} \frac{2-[2\cos^2 x - \sin^2 x]}{2\sin^2 x + 4x\sin x\cos x + 4x\cdot\sin x\cdot\cos x + 2x^2[\cos^2 x - \sin^2 x]}$

4) $\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = f'(0)$ ~~and since~~

$\frac{d}{dx}\left[\frac{f(x)}{x}\right] = \frac{xf'(x) - f(x)}{x^2} = \frac{f'(x)}{x} - \frac{f(x)}{x^2}$

$\lim_{x\to 0} \frac{f'(x)}{x} - \frac{f(x)}{x^2} = \lim_{x\to 0} \frac{f'(x)}{x} = \lim_{x\to 0} f''(x)$, using L'hospital's Rule

$=$

5) $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

$\frac{f(x)-f(a)}{x-a} = f'(a) + \frac{f''(a)}{2}(x-a)$

$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$, since $f(x)$ is twice-differentiable, $f'(x)$ is continuous.

$\Rightarrow \lim_{x\to a} f(x) - f(a) = 0$, ~~which gives~~

Let $a \to \infty$, then $\lim_{x\to\infty} \frac{f(x)-f(a)}{x-a} = f'(a) = 0$ should hold.

6) $(a-b+c)^5 \leq (a-b)^5 + c^5$, since $a-b<0$, $c>0$

$\leq a^5 - b^5 + c^5$, since $a-b<0$

7) $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$\cos x - 1 = -\frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$|\cos x - 1| = \left| -\frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right|$

$= \frac{x^2}{2} \underbrace{- \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots}_{\leq 0}$

$\therefore \; |\cos x - 1| \leq \frac{x^2}{2}$ for all $x \in \mathbb{R}$

8. $\frac{k}{n^2+k^2} < 1$, $\sum_{k=0}^{2n} \frac{k}{n^2+k^2} = \frac{1}{1-\frac{k}{n^2+k^2}} = \frac{1}{\frac{n^2+k^2-k}{n^2+k^2}} = \frac{n^2+k^2}{n^2+k^2-k}$

$\lim_{n\to\infty} \frac{n^2+k^2}{n^2+k^2-k} = 1$

9) $|f(x) - f(y)| \leq |x-y|^{\frac{1}{3}}$

$\sup|f(x) - f(y)| \leq \sup|x-y|^{\frac{1}{3}}$

$\frac{1}{b-a}[U_f(P) - L_f(P)] \leq |P|^{\frac{1}{3}}$

$U_f(P) - L_f(P) \leq |P|^{\frac{1}{3}}(b-a)$