Chap 12. Intermediate value theorem (IVT)

12.1 The existence of zeros

Continuity is a **local** property of functions. But we will prove it implies certain **global** properties, basic to analysis and its applications.

- an application: root-finding (i.e., want to solve equations $f(x) = 0$)

"Solving" means finding the real zeros of $f(x) = 0$ (i.e., those real c for which $f(c) = 0$)

In detail, we can ask about

(i) existence: are there any zeros?

(ii) number: are there finitely many? how many, or approximately how many?

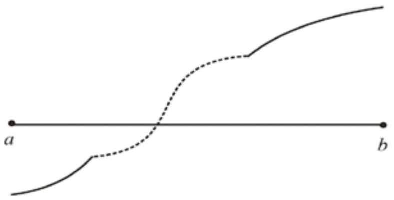
(iii) approximate location: find small intervals containing only one zero.

(iv) calculation: determine the zero "exactly", or to a given accuracy.

- Bolzano's theorem (it is **about continuous functions which change sign**) is applicable to **all** of these questions

Def. We say $f(x)$ changes sign on $[a,b]$ if it is defined on this interval and has opposite signs at a and b :

$$f(a) < 0,\ \ f(b) > 0 \quad \text{or} \quad f(a) > 0,\ \ f(b) < 0 \quad (\text{equivalently,} \quad f(a)f(b) < 0)$$

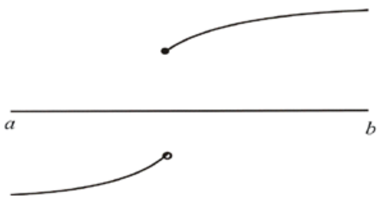


Theorem (Bolzano's theorem)

Let $f(x)$ be continuous on $[a,b]$. Then

$$\boxed{f(x) \text{ changes sign on } [a,b] \text{ (i.e., } f(a)f(b) < 0) \quad \Rightarrow \quad f(x) \text{ has a zero on } [a,b] \text{ (} \therefore \text{ on } (a,b) \text{)}}$$

Note. The property "$\boxed{}$" is not true in general when $f(x)$ is discontinuous on $[a,b]$; for example,



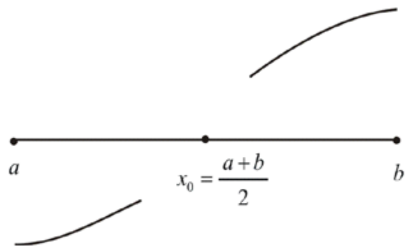
Remark1: "The conclusion is geometrically "obvious"

Remark2. The conclusion asserts that a certain real number exists, so we can expect that the proof will require the "**Completeness property**". Actually, we will use "NIT" for the proof.

Proof of the Bolzano's theorem.

We consider the case where $f(x)$ changes from $-$ to $+$. We shall prove:

$$\boxed{f(a) < 0, \quad f(b) > 0 \quad \Rightarrow \quad f(c) = 0 \text{ for some } c \in [a, b] \ \left(\text{actually}, \ c \in (a, b)\right)}$$



Let $a_0 = a$ and $b_0 = b$. Divide $[a_0, b_0]$ into two by its midpoint x_0.

If $f(x_0) > 0$, let $[a_1, b_1] = [a, x_0]$.

If $f(x_0) < 0$, let $[a_1, b_1] = [x_0, b]$.

If $f(x_0) = 0$, we have found a zero $(= x_0)$.

In each of the first two cases, we have

$$f(a_1) < 0, \quad f(b_1) > 0$$

& on the interval $[a_1, b_1]$, $f(x)$ still changes from $-$ to $+$.

We continue this process with $[a_1, b_1]$, bisecting it and choosing as $[a_2, b_2]$ the half on which $f(x)$ changes from $-$ to $+$.

If at any stage the midpoint is a zero of $f(x)$, we are done.

If not, we get an infinite sequence of nested intervals

$$[a, b] \supset [a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n] \supset \cdots$$

such that

$$\boxed{(*) \quad f(a_n) < 0, \ f(b_n) > 0, \ \text{ and } \ b_n - a_n \ (= \frac{b-a}{2^n}) \to 0 \ \text{ as } n \to \infty}$$

By NIT, $\exists\, c \in \bigcap_{n=0}^{\infty} [a_n, b_n] \ \left(\therefore c \in [a, b]\right)$ such that $\lim_{n\to\infty} a_n = c = \lim_{n\to\infty} b_n$.

Suffices to show: $f(c) = 0$.

Since f is conti on $[a, b]$, the **Sequential Continuity Theorem** implies that

$$\lim_{n\to\infty} f(a_n) = f(c) = \lim_{n\to\infty} f(b_n)$$

According to $(*)$, we have

$$f(a_n) < 0 \ \text{ and } \ f(b_n) > 0 \ \text{ for all } n$$

Thus by LLT (for sequences), we have

$$\lim_{n\to\infty} f(a_n) \le 0 \quad \text{and} \quad \lim_{n\to\infty} f(b_n) \ge 0. \qquad \therefore\ f(c) = 0$$

If $f(x)$ changes from $+$ to $-$, then $-f(x)$ changes from $-$ to $+$.

$\Rightarrow \quad \exists c \in [a, b]$ such that $(-f)(c) = 0$

$\Rightarrow \quad f(c) = 0.$

Corollary. Intermediate Value Theorem (IVT)

Assume $f(x)$ is continuous on $[a, b]$, and $f(a) \le f(b)$ $\quad (f(a) \ge f(b)$ resp.).

Then for $k \in \mathbb{R}$,

$$f(a) \le k \le f(b) \quad (f(a) \ge k \ge f(b) \text{ resp.}) \quad \Rightarrow \quad \exists c \in [a,b] \text{ such that } f(c) = k\,.$$

i.e., if f is conti on $[a, b]$, it takes on all values between $f(a)$ and $f(b)$ as x varies over $[a, b]$.

More common statement: Assume $f(x)$ is continuous on $[a, b]$. Then

whenever $f(a) < k < f(b)$ or $f(a) > k > f(b) \quad \Rightarrow \quad \exists c \in (a,b)$ such that $f(c) = k$

Pf of Corollary. If $k = f(a)$ or $k = f(b)$, we are done

If not (i.e., $f(a) < k < f(b)$), we consider

$f(x) - k$: it is conti on $[a,b]$ and $f(a) - k < 0$ and $f(b) - k > 0$

$\overset{\text{Bolzano's thm}}{\Rightarrow} \quad f(x) - k$ has a zero c on $[a, b]$ (Actually, has a zero c on (a, b))

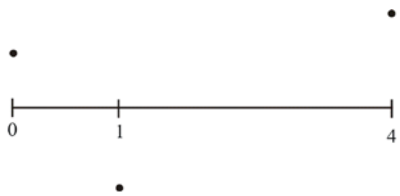
i.e., $\quad f(c) = k$ for some $c \in [a, b]$ $\big($actually, $c \in (a, b)\big)$.

Consequently, in any case, $\exists c \in [a,b]$ such that $f(c) = k$

Exa. $e^x - 3x$ has at least two positive zeros.

Sol. $f(x) = e^x - 3x$: conti on $(-\infty, \infty)$

$f(0) = 1 > 0, \quad f(1) = e - 3 < 0, \quad f(4) = e^4 - 12 > 0$



Thus, by Bolzano's theorem, $f(x)$ has at least two positive zeros.

HS1 [Fixed-Point Theorem] If $f : [a,b] \to [a,b]$ is continuous, then prove that f has at least one fixed point; that is, $\exists x_0 \in [a,b]$ such that $f(x_0) = x_0$

HS2. Prove that a continuous function **whose values are always rational numbers** is a constant function

12.2 Applications of Bolzano's theorem

Exa A (existence of a zero): A poly of odd degree has at least one (real) zero.

Pf. Let $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$, where n is odd.

We may assume that $a_0 > 0$ (otherwise use $-f$)

$$f(x) = x^n (a_0 + a_1 \frac{1}{x} + \cdots + a_n \frac{1}{x^n}) \equiv x^n g(x)$$

$$\begin{aligned} |\, x \,| \to \infty \quad &\Rightarrow \quad \frac{1}{x}, \cdots, \frac{1}{x^n} \quad \to \quad 0 \\ &\Rightarrow \quad g(x) \to a_0 \end{aligned}$$

Since $a_0 > 0$, by **FLT** we get $g(x) > 0$ for $|\, x \,| \gg 1$

$$\therefore \quad x^n g(x) < 0 \text{ for } x \ll -1 \quad \& \quad x^n g(x) > 0 \text{ for } x \gg 1 \text{ since } n \text{ is odd.}$$

$$\text{i.e., } \quad f(x) < 0 \text{ for } x \ll -1 \quad \& \quad f(x) > 0 \text{ for } x \gg 1$$

Since $f(x)$ is conti on $(-\infty, \infty)$, $f(x)$ has a zero (by Bolzano's theorem).

Remark. In computer searches for zeros of a polynomial, one looks for intervals on which $f(x)$ changes sign, and then uses the **bisection process** or **Newton's method** to find a zero inside. (Newton's method is faster; the **bisection method** is more reliable and doesn't require calculation of derivatives)

⦿ The isolation of the zeros (want to have each interval on which $f(x)$ changes sign so small there is only one zero of $f(x)$ inside it)

Exa B. Approximate location of a zero.

Let $f(x) = x^3 + hx - 1$, $h \approx 0^+$

Think of $f(x)$ as a small perturbation of $x^3 - 1$.

Since $x^3 - 1$ has a zero at 1, $f(x)$ should have a zero close to 1: call it $z(h)$.

Give the approximate value of $z(h)$, and prove it is **right-continuous** at 0.

Sol. Write $z = z(h) = 1 + \varepsilon$, where $\varepsilon = \varepsilon(h) \approx 0$.

$$f(z) = 0 \quad \Rightarrow \quad (1+\varepsilon)^3 + h(1+\varepsilon) - 1 = 0$$

$$\therefore \quad 1 + 3\varepsilon + 3\varepsilon^2 + \varepsilon^3 + h + h\varepsilon - 1 = 0$$

Since $h(>0)$ is small (i.e., $h \approx 0^+$ by hypo) & $\varepsilon \approx 0$, we get $1 + 3\varepsilon + h - 1 \approx 0$

$$\therefore \quad \varepsilon \approx -\frac{h}{3} \qquad \therefore \quad z \approx 1 - \frac{h}{3} \ (\text{for } h \approx 0^+)$$

To prove $z(h)$ is right-continuous at 0 (i.e., $\lim_{h \to 0^+} z(h) = 1$; which seems plausible by $z(h) \approx 1 - \frac{h}{3}$), it suffices to prove

$$z = z(h) \in [1-h, 1] \text{ for small } h > 0. \quad \text{i.e.,} \quad 1 - h \le z = z(h) \le 1 \text{ for small } h > 0$$

Note that

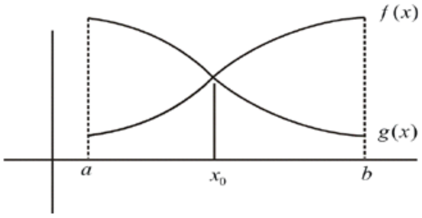
$$f(1-h) = (1-h)^3 + h(1-h) - 1 = -2h + 2h^2 - h^3 < 0 \text{ for } h \approx 0^+$$
$$\& \quad f(1) = h > 0$$

Thus by Bolzano's theorem, $1 - h \le z \le 1$ for small $h > 0$.

◎ In trying to establish the existence of a zero, it may be better to rewrite the equation in the form $f(x) = g(x)$ since the graph of $f(x)$ and $g(x)$ may be easier to plot or visualize than the graph of $f(x) - g(x)$.

Intersection Principle

(a) The roots of $f(x) = g(x)$ are the x −coordinates of the points where the graph of $f(x)$ and $g(x)$ intersect.



(it is evident from the picture)

(b) If $f(x)$ and $g(x)$ are continuous on $[a, b]$, and

$$\boxed{f(a) < g(a) \text{ and } f(b) > g(b)} \quad \text{or} \quad \boxed{f(a) > g(a) \text{ and } f(b) < g(b)},$$

then the two graphs intersect over some point $c \in [a, b]$.

Pf. (b) $\quad h(x) \overset{\text{let}}{=} f(x) - g(x) \quad \overset{\text{easy}}{\Rightarrow} \quad \text{conclusion}$

Exa C. Counting zeros

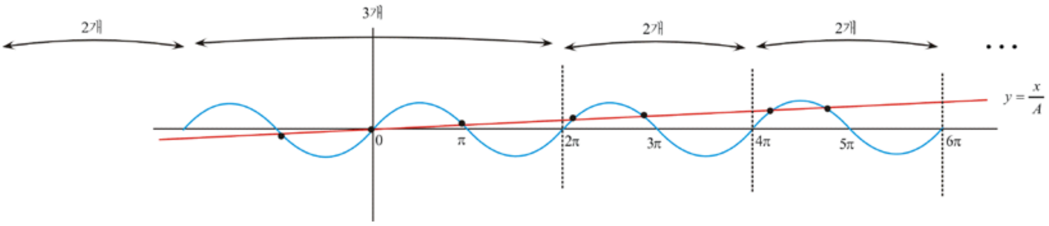
Approximately how many zeros has $x - A\sin x$, if A is large?

Sol. Note that $x - A\sin x = 0 \quad \Leftrightarrow \quad \sin x = \dfrac{x}{A}$.

Hence the number of zeros of $x - A\sin x$ is the same as that of the roots of $\sin x = \dfrac{x}{A}$.

Since $|\sin x| \le 1$, the roots of $\sin x = \dfrac{x}{A}$ satisfy $\left|\dfrac{x}{A}\right| \le 1$ (i.e., $|x| \le A$).

That is , all the roots lie in $[-A,\ A]$.



From the figure, we see that

"over each interval $[2n\pi,\ (2n+2)\pi]$ lying inside $[-A,\ A]$, there will be two intersections"

(this isn't quite right at the ends of $[-A,\ A]$ and $0(=\text{origin})$ is in two intervals, but we don't need to worry about it since we are assuming that A is large)

Since there are about $2A/2\pi$ numbers of these intervals of length 2π inside $[-A,\ A]$, each with two intersections, there are in all approximately $2A/\pi$.

Conclusion: Approximately $x - A\sin x$ has $\dfrac{2A}{\pi}$ zeros.

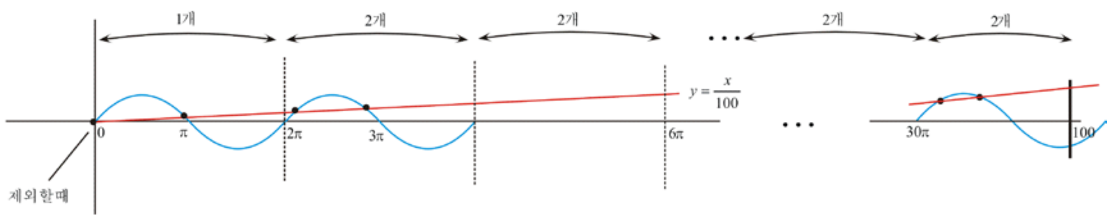
Ex. (Exactly) How many zeros of $\sin x = \dfrac{x}{100}$?

Sol. Note that, since $\sin x$ and $\dfrac{x}{100}$ are odd functions, if x_0 is a root of the equation, then $-x_0$ is also a root. Thus "the number of negative roots = the number of positive roots".

Also, 0 is clearly a root of the equation.

It suffices to find how many positive roots there are.

If $\sin x = \dfrac{x}{100}$, then $\left|\dfrac{x}{100}\right| \le 1$ since $|\sin x| \le 1$. i.e., $|x| \le 100$ (i.e., $|\text{every root}| \le 100$)



$$2\pi = 6.28\cdots \quad \Rightarrow \quad \frac{100}{6.28\cdots} = 15.** \quad \Rightarrow \quad 15 < \frac{100}{2\pi} < 16 \quad \Rightarrow \quad \left[\frac{100}{2\pi}\right] = 15$$

$$\left[\frac{100}{2\pi}\right] \cdot 2\pi = 30\pi \doteq 94.2 \quad \Rightarrow \quad \pi < 5 < 100 - \left[\frac{100}{2\pi}\right] \cdot 2\pi < 2\pi$$

Thus the number of positive roots:

$$\underbrace{1}_{\text{in } (0,2\pi)} + \underbrace{\underbrace{2}_{\text{in } [2\pi,4\pi)} + \underbrace{2}_{\text{in } [4\pi,6\pi)} + \cdots + \underbrace{2}_{\text{in } [28\pi,30\pi)}}_{14 \text{ intervals}} + \underbrace{2}_{\text{in } [30\pi,100]} = 1+2\cdot 14+2=31(\text{개})$$

Therefore, the number of all roots: $\underbrace{1}_{x=0} + 31\cdot 2 = 63(\text{개})$

12.3 Graphical continuity

IVP (Intermediate Value Property): A function $f(x)$ defined on $[a,b]$ **is said to have the IVP** on $[a,b]$ if for each k between $f(a)$ and $f(b)$, $\exists c \in [a,b]$ such that $f(c)=k$.

[or, if whenever $f(a)<k<f(b)$ or $f(a)>k>f(b)$, then $\exists c\in(a,b)$ such that $f(c)=k$]

Recall: IVT (Intermediate Value Theorem) says " Any continuous function on $[a,b]$ has the IVP on $[a,b]$"

We now **prove the converse** of this, **for strictly monotone functions**.

Theorem (**Continuity theorem for monotone functions**)

If $f(x)$ is strictly monotone and has the IVP on $[a,b]$, then it is continuous on $[a,b]$.

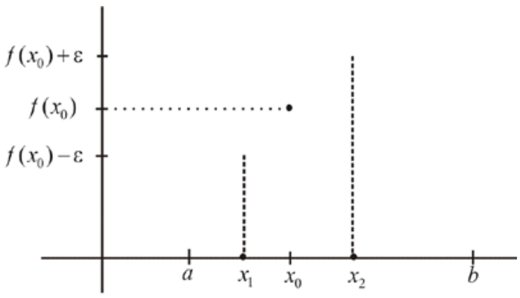
Pf. Suppose $f(x)$ is strictly inc on $[a,b]$ ($\therefore$ $f(a)<f(b)$). Let x_0 be a point in (a,b)

(i.e., assume x_0 is not an endpoint of $[a,b]$). Then $f(a)<f(x_0)<f(b)$.

We show f is conti at x_0.

Let $\varepsilon>0$ be small. Then by IVP on $[a,b]$, $\exists\, x_1,\ x_2 \in [a,b]$ such that

$$f(x_1)=f(x_0)-\varepsilon, \qquad f(x_2)=f(x_0)+\varepsilon$$



Indeed, they are unique and $x_1<x_2$ $(\because x_1<x_0<x_2)$ since $f(x)$ is strictly inc.

$$x_1<x<x_2 \overset{f \text{ is strictly inc}}{\Rightarrow} f(x_1)<f(x)<f(x_2)$$

$$\therefore \quad f(x_0)-\varepsilon<f(x)<f(x_0)+\varepsilon \quad \text{for } x\in(x_1,x_2)$$

$$\therefore \quad f(x) \underset{\varepsilon}{\approx} f(x_0) \quad \text{for} \quad x\approx x_0$$

If x_0 is an endpoint, for example say $x_0=a$, then by the preceding argument

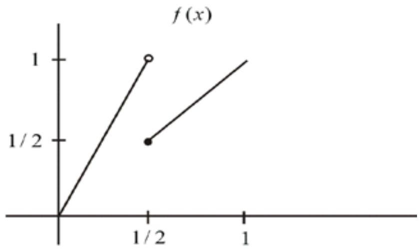
$$(f(a)-\varepsilon <)\ f(a) < f(x) < f(a)+\varepsilon \quad \text{for} \quad a < x < x_2$$
$$\therefore \quad f(x) \underset{\varepsilon}{\approx} f(a) \quad \text{for} \quad x \approx a^+$$

This implies that $f(x) \longrightarrow f(a)$ as $x \longrightarrow a^+$ which means $f(x)$ is right-conti at $x = a$.

Conclusion: Suppose $f(x)$ is strictly monotone on $[a,b]$. Then
f has the IVP on $[a,b]$ $\Leftrightarrow$ f is continuous on $[a,b]$.

Ex. Give an example of a ft $f(x)$ having the IVP on $[a,b]$ but which is not continuous on $[a,b]$.

Ans:



Another example: $f(x)=\begin{cases}\sin(1/x), & 0<x\le 1\\ 0, & x=0\end{cases}$

Cf: More interesting example --- see the Corollary below: [needs Darboux's theorem]

Darboux's theorem [Intermediate Value Theorem for derivative] (will be proved in chap 15)

If f is diff on $[a,b]$, and if k between $f'(a)$ and $f'(b)$, then $\exists c \in [a,b]$ such that $f'(c)=k$.

[If f is diff on $[a,b]$, and if $f'(a)<k<f'(b)$ or $f'(a)>k>f'(b)$, then $\exists c\in(a,b)$ s.t. $f'(c)=k$]

Remark. In the theorem above, the continuity of f' is not necessary.

Cor (of Darboux)**:** There is a discontinuous function $g(x)$ having IVP on $[a,b]$.

Pf. Take an f that is diff on $[a,b]$ but whose derivative f' is not conti at some point $x_0 \in (a,b)$.

(e.g., $f(x) := \begin{cases} x^2\sin(1/x) & \text{if } x \neq 0\\ 0 & \text{if } x=0\end{cases}$: diff on $[-1,1]$ but $f'(x)$ is not conti at $x=0\in(-1,1)$)

Set $g=f'$. Then g has the IVP on $[a,b]$ (by Darboux's theorem), but g is not conti at $x_0\in(a,b)$.

12.4 Inverse functions

Theorem (Inverse function theorem **for continuity**).

If $y=f(x)$ is continuous & strictly inc on $[a,b]$, then it has an inverse function $x=g(y)$ on $[f(a),f(b)]$ which is continuous & strictly inc.

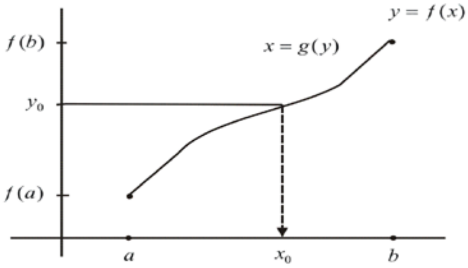
(Remark. The theorem is also true for strictly dec functions; in that case $[f(a),f(b)]$ should be replaced by $[f(b),f(a)]$)

Pf. There are three things to prove.

(A) The inverse function is defined on $[f(a), f(b)]$

For, if $y_0 \in [f(a), f(b)]$, then by IVT

$\exists\, x_0 \in [a,b]$ such that $f(x_0) = y_0$; it is unique because $f(x)$ is strictly inc.



Therefore one can define g at y_0 by $g(y_0) = x_0$.

(B) The function $g(y)$ is strictly inc (i.e., $y_0 < y_1 \quad \Rightarrow \quad g(y_0) < g(y_1)$)

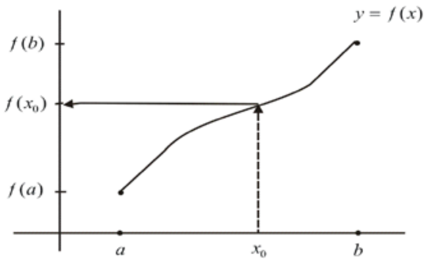
To prove this, set $x_0 = g(y_0)$ and $x_1 = g(y_1)$ where $y_0 < y_1$. Then

$$x_0 \geq x_1 \overset{f \text{ is strictly } \uparrow}{\Rightarrow} f(x_0) \geq f(x_1) \qquad \text{i.e., } f(g(y_0)) \geq f(g(y_1)) \quad i.e., y_0 \geq y_1$$

Therefore (B) holds.

(C) The function $g(y)$ is continuous.

By (B), $g(y)$ is strictly inc on $[f(a), f(b)]$ & $g(y)$ has the IVP on this interval, since if $a \leq x_0 \leq b$, then $g(f(x_0)) = x_0$ where $f(x_0) \in [f(a), f(b)]$.



$$\overset{\text{Continuity thm for monotone fts}}{\Rightarrow} \quad g \text{ is conti on } [f(a), f(b)].$$

An application.

$$f(x) = x^n \ (n \in \mathbb{N}) \text{: conti \& strictly inc on } [0, \infty)$$

$$\therefore \overset{\text{Inverse ft thm}}{\Rightarrow} \sqrt[n]{x} (= x^{1/n}) \text{ is also conti \& strictly inc on } [0, \infty).$$