STA 3021: Stochastic Processes
Final (6:15 PM - 7:30 PM on Dec 6, 2021)

Instructions:

- This test is a closed book exam, but you are allowed to use calculator. Clarity of your answer will also be a part of credit. When needed, use the notation $\Phi(z) = P(Z < z)$ for a standard normal distribution Z. Show your ALL work neatly.
- Your answer sheets must be written in English.
- Remind that you can submit your answer sheets over icampus in the **pdf** file format ONLY.
- By submitting your report online, it is assumed that you agree with the following pledge;

  Pledge: *I have neither given nor received any unauthorized aid during this exam.*
- Don't forget to write down your name and student ID on your answer sheet.

1. (10 points) Suppose that you arrive at a single-teller bank to find five other customers in the bank, one being served and the other four waiting in line. You join the end of the line. If the service times are all exponential with rate μ, what is the expected amount of time you will spend in the bank?

2. (10 points) If X, Y_1, Y_2 be independent exponential random variables; X having rate λ and Y_i having rate μ. Find $P(X > \max(Y_1, Y_2))$.

3. (15 points) Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ.
   (a) Write down axiomatic definition of the Poisson process.

   (b) Showt that $\{N_s(t), t \geq 0\}$ satisfies the axioms for being a Poisson process with rate λ where

$$N_s(t) = N(s + t) - N(s).$$

4. (20 points) Suppose that customers arrive at a bank according to a Poisson process with $\lambda = 8$ per hour. Find
   (a) The mean and variance of the number of customers who enter the bank during an 8-hour working day.
   (b) Probability that more than for customers enter the bank during the hour-long lunch break.
   (c) Probability that nobody enters the bank during the last 15 min of the working day.
   (d) Correlation coefficient between the number of customers who enter the bank between 9:00 AM to 11:00 AM and those who enter between 10:00 AM to 12:00 Noon.

5. (10 points) An insurance company pays out claims on its life insurance policies in accordance with a Poisson process having rate $\lambda = 5$ per week. If the amount of money paid on each policy is exponentially distributed with mean \$2000, what is the mean and variance of the amount of money paid by the insurance company in a four-week span?

6. (15 points) Let $\{N(t), t \geq 0\}$ be a PP(λ) and independent of the sequence $X_1, \ldots,$ of IID random variables with mean μ and variance σ^2. Find

$$\mathrm{Cov}\left(N(t), \sum_{i=1}^{N(t)} X_i\right)$$

7. (20 points) For a standard Brownian motion $\{B(t), t \geq 0\}$, and $0 \leq s \leq t$, find

   (a) $E(B(t)|B(s) = y)$.

   (b) $E(B(s)|B(t) = x)$.

   (c) $E(B^2(t)|B(s))$

   (d) Define the Brownian bridge as

   $$B^0(t) = B(t) - tB(1), \quad t \in [0, 1].$$

   Find the covariance between $B^0(s)$ and $B^0(t)$, $s < t$.

1.

C_i : serving time for ith customer

S : serving time for me

T : total waiting time in the bank

$C_i, S \sim \mathrm{Exp}(\mu)$

$$\mathbb{E}(T) = \sum_{i=1}^{5} \mathbb{E}(C_i) + \mathbb{E}(S) = \frac{6}{\mu}$$

2.

$X \sim \mathrm{Exp}(\lambda)$, $Y_1, Y_2 \sim \mathrm{Exp}(\mu)$, $P(X > \max(Y_1, Y_2))$?

$$
\begin{aligned}
&P(X > \max(Y_1, Y_2)) \\
&= P(X > \max(Y_1, Y_2) \mid Y_1 > Y_2)\, P(Y_1 > Y_2) \\
&\quad + P(X > \max(Y_1, Y_2) \mid Y_2 > Y_1)\, P(Y_2 > Y_1) \\
&= P(X > Y_1) P(Y_1 > Y_2) + P(X > Y_2) P(Y_2 > Y_1) \\
&= \frac{\mu}{\mu+\lambda} \cdot \frac{\mu}{2\mu} + \frac{\mu}{\mu+\lambda} \cdot \frac{\mu}{2\mu} = \frac{\mu}{\mu+\lambda}
\end{aligned}
$$

3.

(a)

$N(t)$: counting process, rate $\lambda > 0$

$N(t)$ is $PP(\lambda)$, if

i) $N(0) = 0$ (starting point 0)

ii) $\{N(t)\}$ has independent increments

iii) $P(N(t+h) - N(t) = 1) = \lambda h + o(h)$, $\forall\, t, h > 0$

iv) $P(N(t+h) - N(t) \geq 2) = o(h)$, $\forall\, t, h > 0$

(b)

$\{N(t), t \geq 0\}$: $PP(\lambda) \Rightarrow N(t) \sim \text{Poisson}(\lambda t)$

i) $N_s(0) = N(s+0) - N(s) = 0$ (Axiom i))

ii) $\{N_s(t)\}$ has independent increments

($\because$ $N(t)$ is $PP(\lambda)$) (Axiom ii))

iii) $P(N_s(t+h) - N_s(t) = 1)$

$= P(N(t+h+s) - N(s) - N(s+t) + N(s) = 1)$

$= P(N((s+t)+h) - N(s+t) = 1)$

$= \lambda t + o(h)$, $\forall s, t, h > 0$

($\because$ $N(t)$ is $PP(\lambda)$) (Axiom iii))

iv) $P(N_s(t+h) - N_s(t) \geq 2)$

$= P(N((s+t)+h) - N(s+t) \geq 2)$

$= o(h)$, $\forall s, t, h > 0$

($\because$ $N(t)$ is $PP(\lambda)$) (Axiom iv))

4.

(a)

$N(t) \sim \text{Poisson}(\lambda t)$

$\mathbb{E}(N(8)) = 8 \cdot 8 = 64$

$\text{Var}(N(8)) = 8 \cdot 8 = 64$

(b)

$N(1) \sim \text{Poisson}(8)$

$P(N(1) > 4) = 1 - P(N(1) \leq 4)$

$= 1 - P(N(1)=0) - P(N(1)=1) - P(N(1)=2) - P(N(1)=3) - P(N(1)=4)$

$= 1 - e^{-8} - 8 \cdot e^{-8} - \dfrac{8^2 e^{-8}}{2!} - \dfrac{8^3 e^{-8}}{3!} - \dfrac{8^4 e^{-8}}{4!}$

$= 1 - 297 e^{-8}$

(c)

$N(1/4) \sim \text{Poisson}(2)$

$P(N(1/4) = 0) = e^{-2}$

(d)

$$\operatorname{cov}(N(11)-N(9),\ N(12)-N(10))$$

$$= \operatorname{cov}(N(11)-N(10)+N(10)-N(9),\ N(12)-N(11)+N(11)-N(10))$$

$$= \operatorname{Var}(N(11)-N(10)) = \operatorname{Var}(N(1)) = 8$$

$$\operatorname{Var}(N(11)-N(9)) = \operatorname{Var}(N(2)) = 16$$

$$\operatorname{Var}(N(12)-N(10)) = \operatorname{Var}(N(2)) = 16$$

$$\therefore \operatorname{corr}(N(11)-N(9),\ N(12)-N(10)) = \frac{1}{2}$$

5.

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \quad Y_i \sim Exp(2000), \quad N(t) \sim Poisson(4.5)$$

$$\mathbb{E}(X(4)) = \mathbb{E}(N(4))\,\mathbb{E}(Y_1) = 4.5 \cdot 2000 = 40000$$

$$Var(X(4)) = 4.5\ \mathbb{E}(Y_1^2) = 160000000$$

$$\because\ \mathbb{E}(Y_1^2) = \mathbb{E}(Y_1)^2 + Var(Y_1) = 8000000$$

6.

$$Cov\left(N(t), \sum_{i=1}^{N(t)} X_i\right) = \mathbb{E}\left(N(t)\sum_{i=1}^{N(t)} X_i\right) - \mathbb{E}(N(t))\,\mathbb{E}\left(\sum_{i=1}^{N(t)} X_i\right)$$

$$\mathbb{E}\left(N(t)\sum_{i=1}^{N(t)} X_i\right) = \mathbb{E}\left[\mathbb{E}\left(N(t)\sum_{i=1}^{N(t)} X_i \,\middle|\, N(t)\right)\right]$$

$$= \mathbb{E}\left[(N(t))^2 \sum_{i=1}^{N(t)} X_i\right]$$

$$= \mu(\lambda t + (\lambda t)^2)$$

$$\mathbb{E}(N(t))\,\mathbb{E}\left(\sum_{i=1}^{N(t)} X_i\right) = (\lambda t)^2 \mu$$

$$\therefore\ Cov\left(N(t), \sum_{i=1}^{N(t)} X_i\right) = \lambda t \mu + (\lambda t)^2 \mu - (\lambda t)^2 \mu = \lambda t \mu$$

7.

(a)

$$\mathbb{E}(B(t) \mid B(s)=y) = \mathbb{E}(B(t)-B(s)+B(s) \mid B(s)=y)$$

$$= \mathbb{E}(B(t)-B(s)) + y = y$$

(b)

$$\begin{pmatrix} B(s) \\ B(t) \end{pmatrix} \sim MVN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} s & s \\ s & t \end{pmatrix} \right),$$

$$B(s) \mid B(t)=x \sim N\left(0+\frac{s}{t}(x-0),\ s-\frac{s^2}{t}\right)$$

$$\Rightarrow \mathbb{E}(B(s) \mid B(t)=x) = \frac{sx}{t}$$

(c)

$$\mathbb{E}(B^2(t) \mid B(s)) = Var(B(t) \mid B(s)) + [\mathbb{E}(B(t) \mid B(s))]^2$$

$$Var(B(t) \mid B(s)) = t-s$$

$$\Rightarrow \mathbb{E}(B^2(t) \mid B(s)) = t-s+(B(s))^2$$

(d)

$$Cov(B^\circ(s), B^\circ(t))$$

$$= Cov(B(s)-sB(1),\ B(t)-tB(1))$$

$$= Cov(B(s), B(t)) - t\,Cov(B(s), B(1)) - s\,Cov(B(1), B(t)) + st\,Var(B(1))$$

$$= s - ts - ts + ts = s(1-t)$$