STA: 3021 Stochastic Processes

Final (1:30 PM - 2:45 PM on Dec 19, 2017)

Student ID & Full Name: ______________________

Instructions: This test is a closed book exam, but you are allowed to use calculator. Clarity of your answer will also be a part of credit. Show your ALL work neatly.

1. (10 points) For the transition probability matrix with state space $E = \{1, 2, 3, 4, 5\}$, do a complete classification of states, that is, identify communicating classes, periodic/aperiodic, postive/null recurrent or transient.

$$P_1 = \begin{pmatrix} .8 & 0 & .2 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ .3 & .4 & 0 & .3 \end{pmatrix}.$$

1, 3 : closed / positive recurrent / aperiodic

2 : open / transient / periodic

4 : open / transient / aperiodic

2. (10 points) Compute $\lim_{n\to\infty} P^n$ for the following transition matrix:

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{2}{5} & 0 & \frac{3}{5} \end{pmatrix}.$$

$$\begin{cases} \pi_0 = \frac{1}{4}\pi_1 + \frac{2}{5}\pi_2 \\ \pi_1 = \frac{1}{3}\pi_0 + \frac{3}{4}\pi_1 \\ \pi_2 = \frac{2}{3}\pi_0 + \frac{3}{5}\pi_2 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

$\therefore \quad \pi_0 = \frac{1}{4} \qquad \pi_1 = \frac{1}{3} \qquad \pi_2 = \frac{5}{12}$

$$\therefore \lim_{n\to\infty} P_n = \begin{pmatrix} \frac{1}{4} & \frac{1}{3} & \frac{5}{12} \\ \frac{1}{4} & \frac{1}{3} & \frac{5}{12} \\ \frac{1}{4} & \frac{1}{3} & \frac{5}{12} \end{pmatrix}$$

3. (20 points) A professor continually gives exams to her students. She can give three possible types of exams, and her class is graded as either having done **well** or **badly**. Let p_i denote the probability that the class does well on a type i exam, and suppose that $p_1 = .3$, $p_2 = .6$ and $p_3 = .9$. If the class does well on an exam, then the next exam is equally likely to be any of the three types. If the class does badly, then the next exam is always type 1.

(a) Let $X_n, n \geq 0$ be the type of exam at time n. Find the transition probability matrix of $\{X_n, n \geq 0\}$.

$$P = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.6 & 0.2 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{pmatrix}$$

where $P_{ij} = \begin{cases} \frac{1}{3}P_i + (1-P_i) & , j=1 \\ \frac{1}{3}P_i & , j \neq 1 \end{cases}$

(b) What proportion of exams are type i, $i = 1, 2, 3$?

$$\begin{cases} \pi_1 = 0.8\pi_1 + 0.6\pi_2 + 0.4\pi_3 \\ \pi_2 = 0.1\pi_1 + 0.2\pi_2 + 0.3\pi_3 \\ \pi_3 = 0.1\pi_1 + 0.2\pi_2 + 0.3\pi_3 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

$\therefore \quad \pi_1 = \frac{5}{7} \quad \pi_2 = \frac{1}{7} \quad \pi_3 = \frac{1}{7}$

4. (10 points) Let $\{N(t), t \geq 0\}$ is a PP(λ). Compute

$$P\{N(t) = k | N(t+s) = k+m\}$$

for $t \geq 0$, $s \geq 0$, $k \geq 0$ and $m \geq 0$.

$$P(N(t) = k \mid N(t+s) = k+m) = \frac{P(N(t+s) = k+m \mid N(t) = k) \cdot P(N(t) = k)}{P(N(t+s) = k+m)}$$

$$= \frac{P(N(s) = m)\, P(N(t) = k)}{P(N(t+s) = k+m)}$$

$$= \frac{e^{-\lambda s}(\lambda s)^m / m! \cdot e^{-\lambda t}(\lambda t)^k / k!}{e^{-\lambda(t+s)}\{\lambda(t+s)\}^{k+m} / (k+m)!}$$

$$= \frac{(k+m)!}{k!\,m!}\left(\frac{t}{t+s}\right)^k \left(\frac{s}{t+s}\right)^m$$

5. (10 points) Suppose that customers arrive at a bank according to a PP(λ) with $\lambda = 8$ per hour. Compute the correlation coefficient between the number of customers who enter the bank between 9:00 and 11:00 AM and those who enter between 10:00 AM and 12:00 noon. (Recall that correlation coefficient between X and Y is given by $\mathrm{Cov}(X,Y)/\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}$).

$N(t) \sim \mathrm{Poisson}(\lambda t)$

$$\mathrm{Cov}(N(11) - N(9),\ N(12) - N(10))$$
$$= \mathrm{Cov}(N(11) - N(10) + N(10) - N(9),\ N(12) - N(11) + N(11) - N(10))$$
$$= \mathrm{Cov}(N(11) - N(10),\ N(11) - N(10))$$
$$= \mathrm{Var}(N(1)) = \lambda$$

$$\mathrm{Var}(N(11) - N(9)) = \mathrm{Var}(N(2)) = 2\lambda$$
$$\mathrm{Var}(N(12) - N(10)) = \mathrm{Var}(N(2)) = 2\lambda$$

$$\therefore \quad \mathrm{Corr} = \frac{\lambda}{\sqrt{2\lambda}\sqrt{2\lambda}} = \frac{1}{2}$$

6. (10 points) Motor vehicles arrive at a bridge toll gate according to a Poisson process with rate $\lambda = 2$ per minute. The drivers pay tolls of \$1 (class 1), \$2 (class 2) or \$5 (class 3) depending on which of three weight classes their vehicles belong. Assuming that the vehicles arriving at the gate belong to classes 1, 2 and 3 with probabilities 1/2, 1/3 and 1/6, respectively. Find the mean and variance of the amount in dollars collected in any given hour.

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \qquad N(t) \sim \mathrm{Poisson}(2t)$$

$$E(Y) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{3} + 5 \cdot \frac{1}{6} = 2$$
$$E(Y^2) = 1 \cdot \frac{1}{2} + 4 \cdot \frac{1}{3} + 25 \cdot \frac{1}{6} = 6$$
$$E(X(t)) = E(N(t))\,E(Y_1) = 4t$$
$$\mathrm{Var}(X(t)) = E(N(t))\,\mathrm{Var}(Y_1) + \mathrm{Var}(N(t))\,E(Y_1)^2$$
$$= 2t\,E(Y_1^2) = 12t$$

$$\therefore \quad E(X(60)) = 240$$
$$\mathrm{Var}(X(60)) = 720$$

7. (15 points) For the compound Poisson process given by

$$X(t) = \sum_{i=1}^{N(t)} Y_i,$$

where $\{Y_i\}$ are i.i.d random variables, find the Laplace transformation of $X(t)$.

$$\begin{aligned}
M_{X(t)}(-s) &= E\left(e^{-sX(t)}\right) \\
&= E\left(e^{-s\sum_{i=1}^{N(t)} Y_i}\right) \\
&= \sum_{k=0}^{\infty} E\left(e^{-s\sum_{i=1}^{N(t)} Y_i} \,\middle|\, N(t)=k\right) P(N(t)=k) \\
&= \sum_{k=0}^{\infty} E\left(e^{-s\sum_{i=1}^{k} Y_i}\right) \frac{e^{-\lambda t}(\lambda t)^k}{k!} \\
&= \sum_{k=0}^{\infty} \left\{E\left(e^{-sY_1}\right)\right\}^k \frac{e^{-\lambda t}(\lambda t)^k}{k!} \\
&= e^{-\lambda t} \sum_{k=0}^{\infty} \frac{\left(E(e^{-sY_1})\lambda t\right)^k}{k!} \\
&= e^{-\lambda t} \exp\left\{E\left(e^{-sY_1}\right)(\lambda t)\right\} \\
&= \exp\left\{-\lambda t\left(1 - E\left(e^{-sY_1}\right)\right)\right\}
\end{aligned}$$

8. (15 points) The Brownian bridge is defined by

$$B^0(t) = B(t) - tB(1), \quad t \in [0, 1],$$

where $B(t)$ is a standard Brownian motion.

(a) Find the mean of $B^0(t)$.

$B(t) \sim N(0, t)$, $tB(1) \sim N(0, t^2)$

$\therefore B^\circ(t) = E[B(t) - tB(1)]$

$= E(B(t)) - E(tB(1))$

$= 0$

(b) Find the covariance between $B^0(s)$ and $B^0(t)$, $s < t$.

$s < t \leq 1$

$Cov(B^\circ(s), B^\circ(t))$

$= Cov(B(s) - sB(1), B(t) - tB(1))$

$= Cov(B(s), B(t)) - t\, Cov(B(s), B(1))$

$- s\, Cov(B(1), B(t)) + st\, Var(B(1))$

$= s - ts - ts + ts$

$= s(1-t)$