13.1 Compact Intervals

- A set $S \subseteq \mathbb{R}$ is said to be sequentially compact if every sequence of points in S has a subsequence converging to a point in S

Sequential Compactness Theorem :

- A compact interval $[a, b]$ is sequentially compact

13.2 Bounded Continuous Functions

Boundedness Theorem :

- If $f(x)$ is continuous on a compact interval I, then $f(x)$ is bounded on I

13.3 External Points of Continuous Functions

Maximum Theorem :

- Let $f(x)$ be continuous on the compact interval I. Then $f(x)$ has a maximum and minimum on I, that is, there exist points $\bar{x}, x \in I$ such that $f(\bar{x}) = \sup_{x \in I} f(x)$, $f(\underline{x}) = \inf_{x \in I} f(x)$

13.4 The Mapping Viewpoint

Continuous Mapping Theorem :

- If $f(x)$ is defined and continuous on the compact interval I, then $f(I)$ is a compact interval.

13.5 Uniform Continuity

- We say $f(x)$ is uniformly continuous on the interval I if, given $\varepsilon > 0$, there is a $\delta > 0$ such that

$f(x') \underset{\varepsilon}{\approx} f(x'')$ if $x' \underset{\delta}{\approx} x''$, $x', x'' \in I$

Uniform continuity on I

Given $\epsilon > 0$, there is a $\delta > 0$ (depending only on ϵ) such that

$$f(x) \underset{\epsilon}{\approx} f(a) \quad \text{for } x \underset{\delta}{\approx} a, \quad x, a \in I .$$

Ordinary continuity on I

Given $\epsilon > 0$, there is a $\delta > 0$ (depending on ϵ and a) such that

$$f(x) \underset{\epsilon}{\approx} f(a) \quad \text{for } x \underset{\delta}{\approx} a, \quad x, a \in I .$$

Uniform Continuity Theorem :

- If I is a compact interval, $f(x)$ continuous on $I \Rightarrow f(x)$ uniformly continuous on I