Chap. 19 The Riemann Integral (정적분의 구체적인 값에 관심)

Goal: is to define the "Riemann integral" of an integrable function $f(x)$, and establish its important properties.

19.1 Refinement of Partitions (세분)

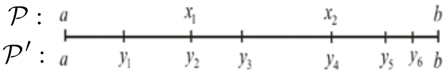
Def. Refinement

We say that the partition $\mathcal{P}'$ is a refinement of $\mathcal{P}$ if $\mathcal{P}'$ is formed by partitioning each subinterval of $\mathcal{P}$ (i.e., (each) $x_k \in \mathcal{P} \Rightarrow x_k \in \mathcal{P}'$; namely, $\mathcal{P} \subset \mathcal{P}'$).

Notation: We write $\mathcal{P}' \le \mathcal{P}$ if $\mathcal{P}'$ is a refinement of $\mathcal{P}$.

$\mathcal{P}$: $a < x_1 < x_2 < b$

$\mathcal{P}'$: $a < y_1 < y_2 = x_1 < y_3 < y_4 = x_2 < y_5 < y_6 < b$



Remark: refinement makes the mesh smaller: $\mathcal{P}' \le \mathcal{P} \quad \Rightarrow \quad |\, \mathcal{P}' \,| \le |\, \mathcal{P} \,|$

Exa A. Successive bisection

Repeated bisection of $[a, b]$ gives a sequence of standard partitions:

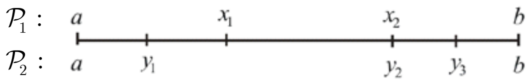
$$\mathcal{P}^{(1)} \ge \mathcal{P}^{(2)} \ge \mathcal{P}^{(4)} \ge \cdots \ge \mathcal{P}^{(2^i)} \ge \cdots; \quad \& \quad |\, \mathcal{P}^{(2^i)} \,| \to 0 \text{ as } i \to \infty$$

Exa B. Common refinement

Given two partitions $\mathcal{P}_1$ and $\mathcal{P}_2$, any partition $\mathcal{P}'$ satisfying $\mathcal{P}' \le \mathcal{P}_1$ and $\mathcal{P}' \le \mathcal{P}_2$ is called a "common refinement (공통세분)" of $\mathcal{P}_1$ and $\mathcal{P}_2$.

⦿ Any two partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ have a least common refinement $\mathcal{P}' = \mathcal{P}_1 \cup \mathcal{P}_2$;
Here "least" common refinement is the one whose total number of partition points is smallest among all common refinements.

$\mathcal{P}_1$: $a < x_1 < x_2 < b$; $\qquad \mathcal{P}_2$: $a < y_1 < y_2 = x_2 < y_3 < b$;



Their least common refinement $\mathcal{P}'$: $a < y_1 < x_1 < x_2 (= y_2) < y_3 < b$

Lemma 19.1 Upper and lower sum lemma

$$\boxed{\mathcal{P}' \le \mathcal{P} \quad \Rightarrow \quad U_f(\mathcal{P}') \le U_f(\mathcal{P}), \quad L_f(\mathcal{P}') \ge L_f(\mathcal{P})}$$

Pf. Key idea: Let $f(x)$ be bounded

$$\boxed{(*) : \quad I \subseteq J \quad \overset{\text{obvious}}{\Rightarrow} \quad \sup_I f(x) \le \sup_J f(x), \quad \inf_I f(x) \ge \inf_J f(x)}$$

To prove $U_f(\mathcal{P}') \le U_f(\mathcal{P})$, suppose that $\mathcal{P}'$ partitions the i -th interval $[\Delta x_i]$ of $\mathcal{P}$ into smaller intervals $I_1, I_2, \cdots, I_r$, of length $|\, I_k \,| (k = 1, 2, \cdots, r)$.

We have only to consider this part. Set (as before) $M_i = \sup_{[\Delta x_i]} f(x)$, $\quad m_i = \inf_{[\Delta x_i]} f(x)$. Then

$$I_k \subset [\Delta x_i] \ \ (\text{for each } k = 1, 2, \cdots, r) \quad \overset{(*)}{\Rightarrow} \quad \sup_{I_k} f(x) \le M_i \ \ \text{for each } k = 1, 2, \cdots, r$$

$$\therefore \quad \sum_{k=1}^{r} \sup_{I_k} f(x) \cdot | I_k | \le \sum_{k=1}^{r} M_i \cdot | I_k | = M_i \sum_{k=1}^{r} | I_k | = M_i \cdot \Delta x_i$$

$$\Downarrow$$

$$U_f(\mathcal{P}') \le U_f(\mathcal{P})$$

The proof of $L_f(\mathcal{P}') \ge L_f(\mathcal{P})$ is similar.

Corollary 19.1

Let f be a bounded function on $[a, b]$. Then

for any partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[a, b]$, $L_f(\mathcal{P}_1) \le U_f(\mathcal{P}_2)$

Pf. If $\mathcal{P}_1 = \mathcal{P}_2$, it is obvious since $m_i \le M_i$ for all i --- ($\odot$)

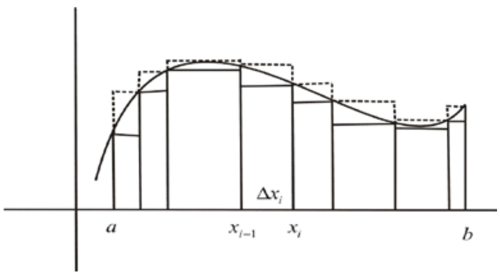
If $\mathcal{P}_1 \ne \mathcal{P}_2$, we let $\mathcal{P}' = \mathcal{P}_1 \cup \mathcal{P}_2$ (their least common refinement), then

$$\mathcal{P}' \le \mathcal{P}_1 \ \text{ and } \ \mathcal{P}' \le \mathcal{P}_2$$

$$\Rightarrow \quad L(\mathcal{P}_1) \overset{\text{lemma 19.1}}{\le} L(\mathcal{P}') \overset{\odot}{\le} U(\mathcal{P}') \overset{\text{lemma 19.1}}{\le} U(\mathcal{P}_2)$$

19.2 Definition of the Riemann integral

Basic idea:



		the "area" (문제점) under		
the total area of	$\le$	the graph of $f(x)$	$\le$	the total area of
any set of		$\|\|$ geometrically		any set of
inscribed rectangles		the Riemann integral of		circumscribed rectangles
		$f(x)$ over $[a, b]$		

Theorem-Definition (The Riemann integral) (**without using** area)

$f(x)$: integrable on $[a, b]$ $\Rightarrow$ $\exists$ a unique real number I s.t. for any partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[a, b]$, $\boxed{(\circledast) \ : \ L_f(\mathcal{P}_1) \le I \le U_f(\mathcal{P}_2)}$

The number I is called the Riemann integral of $f(x)$ over $[a, b]$, and it is denoted by

$$I = \int_a^b f(x)\, dx \overset{\text{or}}{=} \int_a^b f$$

Pf. We will use the seq of standard partitions

$$\mathcal{P}^{(1)}, \mathcal{P}^{(2)}, \mathcal{P}^{(4)} \cdots, \quad \text{produced by successive bisections of } [a, b].$$

Lemma 19.1 & Corollary 19.1 $\Rightarrow$

$$\boxed{L(\mathcal{P}^{(1)}) \le L(\mathcal{P}^{(2)}) \le L(\mathcal{P}^{(4)}) \le \cdots \le U(\mathcal{P}^{(4)}) \le U(\mathcal{P}^{(2)}) \le U(\mathcal{P}^{(1)})}.$$

This shows the intervals $\left\{[L(\mathcal{P}^{(2^i)}),\ U(\mathcal{P}^{(2^i)})]\right\}_{i=1}^{\infty}$ form a sequence of nested intervals.

Since $f(x)$ is integrable on $[a, b]$ and $|\ \mathcal{P}^{(2^i)}\ | \to 0$ as $i \to \infty$,

$$\lim_{i\to\infty} \left(U(\mathcal{P}^{(2^i)} - L(\mathcal{P}^{(2^i)}) \right) = 0 \quad \text{(by the definition of integrability)}$$

Thus by NIT, $\exists$ a unique real number I s.t.

$$L(\mathcal{P}^{(2^i)}) \le I \le U(\mathcal{P}^{(2^i)}) \ \text{ for all } i$$

and

$$\lim_{i\to\infty} L(\mathcal{P}^{(2^i)}) = I = \lim_{i\to\infty} U(\mathcal{P}^{(2^i)})$$

Finally, we prove $(\circledast)$.
For any partition $\mathcal{P}$,

$$L(\mathcal{P}) \le U(\mathcal{P}^{(2^i)}) \ \text{ for all } i \ \text{ (by Cor 19.1)}$$

$$\overset{\text{LLT}}{\Rightarrow} \quad L(\mathcal{P}) \le \lim_{i\to\infty} U(\mathcal{P}^{(2^i)}) = I$$

Similarly, for any partition $\mathcal{P}$,

$$U(\mathcal{P}) \ge L(\mathcal{P}^{(2^i)}) \ \text{ for all } i \ \text{ (again by Cor 19.1)}$$

$$\overset{\text{LLT}}{\Rightarrow} \quad U(\mathcal{P}) \ge \lim_{i\to\infty} L(\mathcal{P}^{(2^i)}) = I$$

Thus $(\circledast)$ is proved.

※ Corollary 19.2 If $f(x)$ is integrable on $[a, b]$, then for any seq $\mathcal{P}_i$ of partitions of $[a, b]$ such that $|\ \mathcal{P}_i\ | \to 0$,

$$\lim_{i\to\infty} L(\mathcal{P}_i) = \int_a^b f(x)\,dx \qquad \& \qquad \lim_{i\to\infty} U(\mathcal{P}_i) = \int_a^b f(x)\,dx.$$

Pf. Let $I = \displaystyle\int_a^b f(x)\,dx$.

$f(x)$: integrable on $[a, b]$ and $|\ \mathcal{P}_i\ | \to 0 \quad \Rightarrow$

$$\text{given } \ \varepsilon > 0, \quad L_f(\mathcal{P}_i) \underset{\varepsilon}{\approx} U_f(\mathcal{P}_i) \ \text{ for } i \gg 1$$

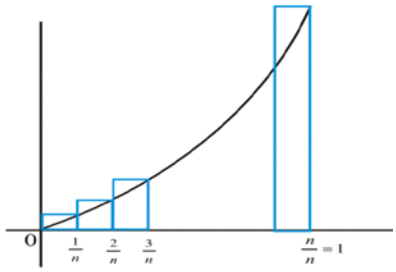
Also by $(\circledast)$, $L_f(\mathcal{P}_i) \le I \le U_f(\mathcal{P}_i)$ for all i

$$\therefore \quad L_f(\mathcal{P}_i) \underset{\varepsilon}{\approx} I \quad \text{and} \quad U_f(\mathcal{P}_i) \underset{\varepsilon}{\approx} I \quad \text{for } i \gg 1$$

Exa Calculate $\displaystyle\int_0^1 x^2\,dx$ directly from the definition.

(We know x^2 is integrable on $[0,1]$, since it is both monotone and continuous)

Sol. We use the seq $\mathcal{P}^{(n)}$ of the standard n-partitions of $[0,1]$.



$$\left|\mathcal{P}^{(n)}\right| = \frac{1}{n} \to 0 \ \text{ as } n \to \infty \ \overset{\text{Cor 19.2}}{\Rightarrow} \ \int_0^1 x^2\,dx = \lim_{n\to\infty} U(\mathcal{P}^{(n)})$$

$$\overset{x^2 \text{ is } \uparrow}{=} \lim_{n\to\infty} \frac{1}{n}\Big((\frac{1}{n})^2 + (\frac{2}{n})^2 + \cdots + (\frac{n}{n})^2 \Big)$$

$$= \lim_{n\to\infty} \frac{1^2 + 2^2 + \cdots + n^2}{n^3}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{6}n(n+1)(2n+1)}{n^3} = \frac{1}{3}$$

19.3 Riemann sums

Def. Let f be bounded on $[a,b]$ and let $\mathcal{P}$ be a partition of $[a,b]$. A Riemann sum for $f(x)$ over $\mathcal{P}$ is any sum of the form

$$S_f(\mathcal{P}) = \sum_1^n f(x_i')\Delta x_i, \ \text{ where } x_i' \in [\Delta x_i].$$

Note: There are infinitely many Riemann sums for $f(x)$ over a given partition $\mathcal{P}$, since there are infinitely many ways to choose the points x_i'.

Remark. $\quad L_f(\mathcal{P}) \le \forall\ S_f(\mathcal{P}) \le U_f(\mathcal{P}), \quad$ for each $\mathcal{P}$

$$\Big(\because \quad m_i \le f(x_i') \le M_i \quad \Rightarrow \quad \sum_1^n m_i \Delta x_i \le \sum_1^n f(x_i')\Delta x_i \le \sum_1^n M_i \Delta x_i \ \Big)$$

※ Theorem 19.3 Let $f(x)$ be integrable on $[a, b]$, and let $|\mathcal{P}_k| \to 0$. For each k, let $S_f(\mathcal{P}_k)$ be a Riemann sum for $f(x)$ over $\mathcal{P}_k$. Then

$$\int_a^b f(x)\,dx = \lim_{k\to\infty} S_f(\mathcal{P}_k).$$

In particular,

$$\bullet \int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}) = \lim_{n\to\infty}\sum_{k=1}^{n} f\left(a+\frac{b-a}{n}k\right)\frac{b-a}{n}$$

$$\bullet \int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1}) = \lim_{n\to\infty}\sum_{k=0}^{n-1} f(x_k)(x_{k+1} - x_k) = \lim_{n\to\infty}\sum_{k=0}^{n-1} f\left(a+\frac{b-a}{n}k\right)\frac{b-a}{n}$$

Pf. For each k,

$$\begin{array}{ccccc} L_f(\mathcal{P}_k) & \le & S_f(\mathcal{P}_k) & \le & U_f(\mathcal{P}_k) \\ \downarrow\leftarrow & & \text{Cor 19.2} & & \rightarrow\downarrow \quad (\,|\mathcal{P}_k| \to 0\,)\ (\Leftarrow)\ \text{as } k\to\infty \\ \int_a^b f(x)\,dx & & & & \int_a^b f(x)\,dx \end{array}$$

Thus by the squeeze principle,

$$\lim_{k\to\infty} S_f(\mathcal{P}_k) = \int_a^b f(x)\,dx.$$

Exa $\displaystyle \lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{n\pi}{n}\right) = ?$

Sol. Let $\mathcal{P}^{(n)}: 0 = \frac{0\pi}{n} < \frac{\pi}{n} < \frac{2\pi}{n} < \cdots < \frac{(i-1)\pi}{n} < \frac{i\pi}{n} < \cdots < \frac{(n-1)\pi}{n} < \frac{n\pi}{n} = \pi$

be the standard n-partition of $[0, \pi]$. Thus $[\Delta x_i] = [\frac{(i-1)\pi}{n}, \frac{i\pi}{n}]$ & $\Delta x_i = \frac{\pi}{n}$ $(i = 1, 2, \cdots, n)$.

Take $x_i' = \frac{i\pi}{n}$ (the right hand endpoint of $[\Delta x_i]$. Then the corresponding Riemann sum (for $f(x) = \sin x$) is

$$S_f(\mathcal{P}^{(n)}) = \sum_{i=1}^{n} f(x_i')\Delta x_i = \sum_{i=1}^{n}\sin\frac{i\pi}{n}\cdot\frac{\pi}{n} = \frac{\pi}{n}\sum_{i=1}^{n}\sin\frac{i\pi}{n}$$

$$\therefore \lim_{n\to\infty} S_f(\mathcal{P}^{(n)}) = \lim_{n\to\infty}\frac{\pi}{n}\sum_{i=1}^{n}\sin\frac{i\pi}{n} = \int_0^\pi \sin x\,dx$$

$$\therefore \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\sin\frac{i\pi}{n}\right) = \frac{1}{\pi}\int_0^\pi \sin x\,dx = \frac{2}{\pi}$$

19.4 Basic properties of integrals

Theorem A (Linearity theorem for integrals)

Suppose that f, g are integrable on $[a, b]$ and c_1, c_2 are constants

$$\Rightarrow \int_a^b [c_1 f(x) + c_2 g(x)]\,dx = c_1\int_a^b f(x)\,dx + c_2\int_a^b g(x)\,dx$$

Pf. Already seen (in Chap. 18) that

$$c_1 f(x) + c_2 g(x) \text{ is integrable on } [a, b].$$

Take a sequence $\mathcal{P}_k$ of partitions s.t. $|\mathcal{P}_k| \to 0$. Then

$$\begin{aligned} S_{c_1 f + c_2 g}(\mathcal{P}_k) &= \sum_{i=1}^{n} [\, c_1 f(x_i') + c_2 g(x_i')\,] \Delta x_i \\ &= c_1 \sum_{i=1}^{n} f(x_i') \Delta x_i + c_2 \sum_{i=1}^{n} g(x_i') \Delta x_i = c_1 S_f(\mathcal{P}_k) + c_2 S_g(\mathcal{P}_k) \end{aligned}$$

Let $k \to \infty \quad \Rightarrow$

$$\int_a^b [\, c_1 f(x) + c_2 g(x)\,]\, dx = c_1 \int_a^b f(x)\, dx + c_2 \int_a^b g(x)\, dx$$

Theorem B (Comparison theorem for integrals)

If f, g are integrable on $[a, b]$, then

$$f(x) \le g(x) \text{ on } [a, b] \quad \Rightarrow \quad \int_a^b f(x)\, dx \le \int_a^b g(x)\, dx$$

Pf. Take a sequence $\mathcal{P}_k$ of partitions s.t. $|\mathcal{P}_k| \to 0$. Let

$$S_f(\mathcal{P}_k) = \sum_{i=1}^{n} f(x_i') \Delta x_i, \quad \text{each } x_i' \text{ is a point of } [\Delta x_i]$$

&

$$S_g(\mathcal{P}_k) = \sum_{i=1}^{n} g(x_i') \Delta x_i, \quad \text{each } x_i' \text{ is the same chosen point of } [\Delta x_i]$$

Then by hypo $\quad f(x_i') \le g(x_i')$

$$\begin{aligned} &\Rightarrow S_f(\mathcal{P}_k) \le S_g(\mathcal{P}_k) \\ &\Rightarrow \lim_{k\to\infty} S_f(\mathcal{P}_k) \le \lim_{k\to\infty} S_g(\mathcal{P}_k) \\ &\qquad\quad \| \qquad\qquad\qquad \| \\ &\quad \int_a^b f(x)\, dx \qquad \int_a^b g(x)\, dx \end{aligned}$$

Ex (HS) [remember the result]

$f, \; g :$ continuous on $[a, b]$

$f(x) \le g(x)$ on $[a, b]$, & $f(x_0) < g(x_0)$ at some point $x_0 \in [a, b]$

$$\Rightarrow \qquad \int_a^b f(x)\, dx < \int_a^b g(x)\, dx$$

Equivalently,

$h :$ conti on $[a, b]$

$h(x) \ge 0$ on $[a, b]$, & $h(x_0) > 0$ at some point $x_0 \in [a, b]$

$$\Rightarrow \qquad \int_a^b h(x)\, dx > 0$$

Theorem C (Absolute value theorem for integrals)

If f is integrable on $[a, b]$, then $\displaystyle \left| \int_a^b f(x)\, dx \right| \le \int_a^b |f(x)|\, dx$

Pf. Already seen (in Chap. 18) that $|\, f(x)\,|$ is integrable on $[a,\, b]$.

Clearly

$$-\,|\, f(x)\,| \le f(x) \le |\, f(x)\,| \quad \forall x \in [a,\, b]$$

$$\overset{\text{Theorem B}}{\Rightarrow} \underbrace{\int_a^b -\,|\, f(x)\,|\, dx}_{\|\leftarrow \text{Thm A}} \le \int_a^b f(x)\, dx \le \int_a^b |\, f(x)\,|\, dx$$

$$-\int_a^b |\, f(x)\,|\, dx$$

$$\therefore \quad \left|\int_a^b f(x)\, dx\right| \le \int_a^b |f(x)|\, dx$$

19.5 The interval addition property

Theorem 19.5 (Interval addition for integrals)

Suppose $a < b < c$.

$$f \text{ is integrable on } [a,\, b] \text{ and on } [b,\, c] \quad \Rightarrow \quad f \text{ is integrable on } [a,\, c]$$

&

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx$$

Pf. Omit (The proof is not difficult but lengthy): 생략해도 무방

Def. $\displaystyle\int_a^a f(x)\, dx \overset{\text{def}}{=} 0$ for all a; $\quad \displaystyle\int_a^b f(x)\, dx \overset{\text{def}}{=} -\int_b^a f(x)\, dx$ if $a > b$

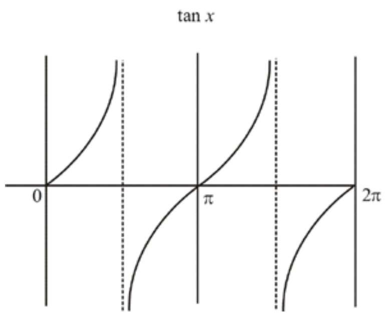
19.6 Piecewise continuous and monotone functions

Def. A ft $f(x)$ is said to be piecewise continuous (monotone, resp) on $[a,\, b]$ if $\exists$ a partition $\mathcal{P} : a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ such that $f(x)$ is continuous (monotone, resp) on each open subinterval $(x_{i-1},\, x_i)$.

Remark. A p.w. continuous (or p.w. monotone) ft $f(x)$ need not be defined at the points of partition, including two endpoints.

Examples

(a) $\tan x$ on $[0,\, 2\pi]$



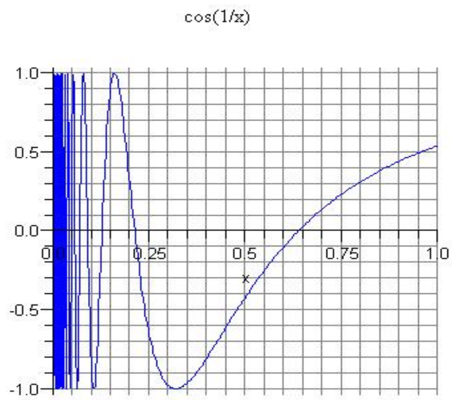
$\tan x$ is conti and monotone on each of the open intervals $(0, \pi/2), (\pi/2, 3\pi/2), (3\pi/2, 2\pi)$.

(It is not defined at $\pi/2$ or $3\pi/2$)

Therefore, $\tan x$ is p.w. continuous & p.w. monotone on $[0, 2\pi]$

(or w.r.t. the partition $\mathcal{P} : 0 < \pi/2 < 3\pi/2 < 2\pi$)

(b) $\cos\dfrac{1}{x}$ on $[0, 1]$



$\cos\dfrac{1}{x}$ is conti on $(0, 1]$. Thus it is p.w. conti on $(0, 1]$.

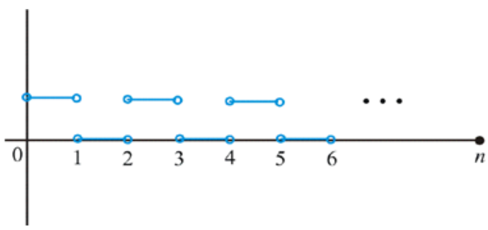
Note that $\cos t$ is monotone on the intervals $[n\pi, (n+1)\pi]$ $(n = 0, 1, 2, \cdots)$

$\therefore$ $\cos\dfrac{1}{x}$ is monotone on the intervals $[\dfrac{1}{(n+1)\pi}, \dfrac{1}{n\pi}]$ $(n = 1, 2, \cdots)$

However, $\cos\dfrac{1}{x}$ is not p.w. monotone on $[0, 1]$ since the intervals $[\dfrac{1}{(n+1)\pi}, \dfrac{1}{n\pi}]_{n=1}^{\infty}$ on which $\cos\dfrac{1}{x}$ is monotone form an infinite partition of $[0, 1]$.

(c)

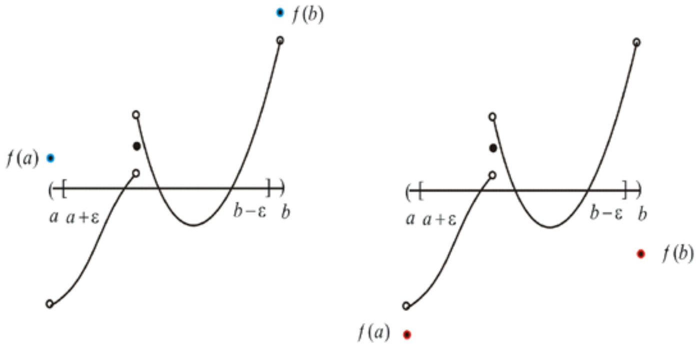
$$w(x) := \begin{cases} 1, & x \in (0,1), (2,3), (4,5), \cdots \\ 0, & x \in (1,2), (3,4), (5,6), \cdots \end{cases} \quad \text{on } [0, n] \quad (w(x) \text{ is called a square wave})$$



Clearly $w(x)$ is p.w. conti and p.w. monotone w.r.t. $\mathcal{P} : 0 < 1 < 2 < \cdots < n$ (or on $[0, n]$).

※ Endpoint Lemma

If $f(x)$ is bounded on (a, b) and integrable on any closed subinterval $I \subset (a, b)$, then for any choice of $f(a)$ and $f(b)$ the function $f(x)$ is integrable on $[a, b]$, and the value of the integral will be the same.



Pf. Since $f(a)$ and $f(b)$ are finite, $f(x)$ is bounded on $[a, b]$; let

$$K = \sup_{x \in [a,\, b]} |\, f(x) \,| \quad (< \infty).$$

Let $\varepsilon > 0$ be given; we may assume $\varepsilon \ll (b - a)$.

By hypo, $f(x)$ is integrable on $[a + \varepsilon, b - \varepsilon]$.

$\therefore\ \exists\, \delta = \delta(\varepsilon)$ with $0 < \delta < \varepsilon$ s.t. for any partition $\mathcal{P}'$ of $[a + \varepsilon, b - \varepsilon]$ with $|\, \mathcal{P}' \,| < \delta$,

$$U_f(\mathcal{P}') - L_f(\mathcal{P}') < \varepsilon$$

Now, let $\mathcal{P}$ be any partition of $[a, b]$ having $|\, \mathcal{P} \,| < \delta$. Then $\mathcal{P}$ induces a partition $\mathcal{P}'$ of $[a + \varepsilon, b - \varepsilon]$ having $|\, \mathcal{P}' \,| < \delta$. Note that

$$\boxed{(*): \quad \sup_I f(x) - \inf_I f(x) \le 2K \qquad \forall I \subset [a, b]}$$

Let $I_1, I_2, \cdots, I_m$ be the subintervals of $\mathcal{P}$ overlapping $[a, a + \varepsilon]$.

Since $|\, \mathcal{P} \,| < \delta < \varepsilon$, we have $I_k \subset [a, a + 2\varepsilon]$ $(k = 1, 2, \cdots, m)$. Moreover, $\sum_{k=1}^{m} |\, I_k \,| \le 2\varepsilon$. Hence

$$\left| \text{The part of } U_f(\mathcal{P}) - L_f(\mathcal{P}) \text{ involving } I_k \right|$$
$$\le \sum_{k=1}^{m} \left(\sup_{I_k} f(x) - \inf_{I_k} f(x) \right) |\, I_k \,| \underset{(*)}{\le} 2K \sum_{k=1}^{m} |\, I_k \,| \le 2K \cdot 2\varepsilon = 4K\varepsilon$$

Similar estimate holds for the interval $[b - \varepsilon, b]$.

Thus $U_f(\mathcal{P}) - L_f(\mathcal{P}) < U_f(\mathcal{P}') - L_f(\mathcal{P}') + 8K\varepsilon < \varepsilon + 8K\varepsilon$.

$$\therefore \quad U_f(\mathcal{P}) - L_f(\mathcal{P}) < (1 + 8K)\varepsilon$$

(Note that K depends on the choice of $f(a)$ and $f(b)$, but not on ε)

By K-ε principle, $f(x)$ is integrable on $[a, b]$.

On the other hand, since $f(x)$ is integrable on $[a, b]$, $\int_a^b f(x)\, dx = \lim_{k \to \infty} S_f(\mathcal{P}_k)$ for any Riemann sums over a seq of partitions $\mathcal{P}_k$ s.t. $|\, \mathcal{P}_k \,| \to 0$.

Choose the Riemann sums so that they never use the endpoints a and b. Then the values $f(a)$ and $f(b)$ never into the sums, and therefore $\int_a^b f(x)\,dx$ does not depend on $f(a)$ and $f(b)$. Thus the Endpoint Lemma is proved.

Theorem 19.6 (Integration of p.w. conti or p.w. monotone functions) [Remember the result]
If $f(x)$ is bounded and p.w. conti or p.w. monotone on $[a,\,b]$, with respect to the partition $\mathcal{P}: a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$, then for any assigned values $f(x_i)$ $(i = 0, 1, 2, \cdots, n)$, $f(x)$ is integrable on $[a,\,b]$, and the integral does not depend on the choice of $f(x_i)$ and

$$\int_a^b f(x)\,dx = \int_a^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

Pf. Actually it follows from "Endpoint lemma + Theorem 19.5" (Check)

Theorem [will be frequently used].

① f is bounded on $[a,\,b]$ and continuous on $[a,\,b]$ except at a single point $c \in [a, b]$

$\Rightarrow$ $f(x)$ is integrable on $[a,\,b]$

② f is bounded on $[a,\,b]$ and continuous at all except finitely many points in $[a,\,b]$

$\Rightarrow$ $f(x)$ is integrable on $[a,\,b]$

③ f is bounded on $[a,\,b]$ and continuous on $(a,\,b)$ $\Rightarrow$ $f(x)$ is integrable on $[a,\,b]$ and

the value of the integral, $\int_a^b f(x)\,dx$, does not depend on $f(a)$ and $f(b)$

For example, $f(x) = \begin{cases} \sin(1/x) & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}$ is integrable on $[0,1]$

Pf. Each follows from Theorem 19.6

Ex(!!!) Give a direct proof of ① & ② --- see the **last paragraph** of this chapter

⊙ 19장에서 공부한 내용 중 핵심적인 결과 요약

- Cor 19.1: Let f be bounded on $[a,\,b]$. Then

  $L_f(\mathcal{P}_1) \le U_f(\mathcal{P}_2)$ for every partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[a,\,b]$,

- Definition of Riemann integral: $f(x)$: integrable on $[a,\,b]$ $\Rightarrow$

$\exists$ a unique real number I s.t. $\boxed{L_f(\mathcal{P}_1) \le I \le U_f(\mathcal{P}_2)}$ for any partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[a,\,b]$

In this case, $I = \int_a^b f(x)\,dx$

- Cor 19.2 + Theorem 19.3

  $f(x)$: integrable on $[a,\,b]$ & let $|\mathcal{P}_k| \to 0$ $\Rightarrow$

$$\lim_{k\to\infty} L(\mathcal{P}_k) = \int_a^b f(x)\,dx\,, \quad \lim_{k\to\infty} U(\mathcal{P}_k) = \int_a^b f(x)\,dx \quad \& \quad \lim_{k\to\infty} S_f(\mathcal{P}_k) = \int_a^b f(x)\,dx.$$

- **Endpoint Lemma**: please state it
- Theorem 19.6: f is bounded and $\begin{cases} \text{p.w. conti} \\ \text{or} \\ \text{p.w. monotone} \end{cases}$ on $[a, b]$ $\Rightarrow$ f is integrable on $[a, b]$
- **Last theorem**: f is bounded on $[a, b]$ and continuous at all except finitely many points in $[a, b]$

  $\Rightarrow$ $f(x)$ is integrable on $[a, b]$

Theorem (A popular criterion for integrability) Let f be bounded on $[a,b]$. Then

$\forall \varepsilon > 0$, $\exists\ \mathcal{P} = \mathcal{P}_\varepsilon$ of $[a, b]$ such that $U_f(\mathcal{P}) - L_f(\mathcal{P}) < \varepsilon$ (def of integrability in most texts)

$\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ such that $U_f(\mathcal{P}) - L_f(\mathcal{P}) < \varepsilon$ for $\forall \mathcal{P}$ with $|\mathcal{P}| < \delta$ (in our text)

Pf. $\Leftarrow$: clear

$\Rightarrow$: Let $\varepsilon > 0$ and choose a partition $\mathcal{P}_0 = \{a = t_0 < t_1 < \cdots < t_\ell = b\}$ of $[a, b]$ such that

$$U_f(\mathcal{P}_0) - L_f(\mathcal{P}_0) < \varepsilon / 2 .$$

Since f is bounded, $\exists M > 0$ such that $|f(x)| \le M$ for $\forall x \in [a, b]$.

Let $\delta = \dfrac{\varepsilon}{8\ell M}$ $\left(cf:\ \ell \text{ is the number of the "natural" sub-intervals in } \mathcal{P}_0\right)$

Now we let $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be any partition of $[a, b]$ with $|\mathcal{P}| < \delta$.

Let $Q = \mathcal{P} \cup \mathcal{P}_0$ (= a common refinement of $\mathcal{P}$ and $\mathcal{P}_0$).

Assume first that Q has one more element than $\mathcal{P}$, and call the element t^*.

Then $t^* \in (x_{i-1}, x_i)$ for some i (x_{i-1} t^* x_i)(점선을 실선으로 생각), and we have

$$
\begin{aligned}
&(0 \le) U_f(\mathcal{P}) - U_f(Q) \\
&= \sup_{x \in [x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1}) - \sup_{x \in [x_{i-1}, t^*]} f(x) \cdot (t^* - x_{i-1}) - \sup_{x \in [t^*, x_i]} f(x) \cdot (x_i - t^*) \\
&= \Big(\sup_{x \in [x_{i-1}, x_i]} f(x) - \sup_{x \in [x_{i-1}, t^*]} f(x) \Big) \cdot (x_i - t^*) + \Big(\sup_{x \in [x_{i-1}, x_i]} f(x) - \sup_{x \in [x_{i-1}, t^*]} f(x) \Big) \cdot (t^* - x_{i-1}) \\
&\qquad \left(\text{Here we used } (x_i - x_{i-1}) = (x_i - t^*) + (t^* - x_{i-1}) \right) \\
&\le 2M \left[(x_i - t^*) + (t^* - x_{i-1}) \right] = 2M(x_i - x_{i-1}) \le 2M |\mathcal{P}|
\end{aligned}
$$

Since Q has at most ℓ elements that are not in $\mathcal{P}$, we see (by an inductive argument) that

$$U_f(\mathcal{P}) - U_f(Q) \le \ell \cdot 2M |\mathcal{P}| = 2\ell M |\mathcal{P}| < 2\ell M \delta = \frac{\varepsilon}{4}$$

$$\therefore\ U_f(\mathcal{P}) < U_f(Q) + \frac{\varepsilon}{4} < U_f(\mathcal{P}_0) + \frac{\varepsilon}{4} \quad \left(\leftarrow Q \text{ is a refinement of } \mathcal{P}_0\right) \quad \text{--- (i)}$$

Similarly,

$$L_f(Q) - L_f(\mathcal{P}) \le 2\ell M |\mathcal{P}| < 2\ell M \delta = \frac{\varepsilon}{4}$$

$$\therefore\ L_f(\mathcal{P}_0) \le L_f(Q) < L_f(\mathcal{P}) + \frac{\varepsilon}{4} \quad \left(\leftarrow Q \text{ is a refinement of } \mathcal{P}_0\right) \quad \text{--- (ii)}$$

(i) and (ii) give the picture: $L_f(\mathcal{P}_0) - \frac{\varepsilon}{4}$ $L_f(\mathcal{P})$ $U_f(\mathcal{P})$ $U_f(\mathcal{P}_0) + \frac{\varepsilon}{4}$

$$\therefore\ U_f(\mathcal{P}) - L_f(\mathcal{P}) < U_f(\mathcal{P}_0) - L_f(\mathcal{P}_0) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad (\text{for } \forall \mathcal{P} \text{ with } |\mathcal{P}| < \delta)$$

Ex.

(i) f is bounded on $[a, b]$ and continuous on $[a, b]$ **except at** the left endpoint a

$\Rightarrow$ $f(x)$ is integrable on $[a, b]$

(ii) f is bounded on $[a, b]$ and continuous on $[a, b]$ **except at** a single point $c \in (a, b)$

$\Rightarrow$ $f(x)$ is integrable on $[a, b]$

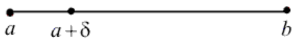
(iii) f is bounded on $[a, b]$ and continuous at all except finitely many points $c_1, c_2, \cdots, c_k$ in $[a, b]$

$\Rightarrow$ $f(x)$ is integrable on $[a, b]$

Pf. (i) Write $\sup\limits_{x \in [a,\, b]} |\, f(x)\,| = K\ (< \infty)$.

Let $\varepsilon > 0$ be given, and choose $\delta > 0$ such that $2K\delta < \varepsilon$ & $a + \delta < b$

i.e., $(0 <)\ \delta < \min\left\{\dfrac{\varepsilon}{2K}, b - a\right\}$



Notice that $\left(\sup\limits_{x \in [a, a+\delta]} f(x) - \inf\limits_{x \in [a, a+\delta]} f(x)\right) \cdot \delta \le 2K\delta < \varepsilon$

Since $f(x)$ is continuous on $[a + \delta, b]$, $f(x)$ is integrable on $[a + \delta, b]$. Thus

$\exists\ \mathcal{P}' =: \{a + \delta = x_1 < x_2 < \cdots < x_n = b\}$ (= a partition of $[a + \delta, b]$) such that

$U_f(\mathcal{P}') - L_f(\mathcal{P}') < \varepsilon$.

Set $\mathcal{P} = \{a =: x_0 < a + \delta = x_1 < x_2 < \cdots < x_n = b\} (= \mathcal{P}(\delta) = \mathcal{P}(\varepsilon))$. Then $\mathcal{P}$ becomes a partition of $[a, b]$. Moreover,

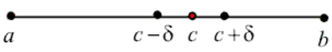
$$U_f(\mathcal{P}) - L_f(\mathcal{P}) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) = (M_1 - m_1)(x_1 - x_0) + \sum_{i=2}^{n} (M_i - m_i)(x_i - x_{i-1})$$
$$< 2K\delta + U_f(\mathcal{P}') - L_f(\mathcal{P}') < \varepsilon + \varepsilon = 2\varepsilon$$

Therefore, $f(x)$ is integrable on $[a, b]$.

(ii) Write $\sup\limits_{x \in [a,\, b]} |\, f(x)\,| = K\ (< \infty)$.

Let $\varepsilon > 0$ be given, and choose $\delta > 0$ such that $4K\delta < \varepsilon$ & $a < c - \delta$ and $c + \delta < b$

i.e., $(0 <)\ \delta < \min\left\{\dfrac{\varepsilon}{4K}, c - a, b - c\right\}$



Notice that $\left(\sup\limits_{x \in [c-\delta, c+\delta]} f(x) - \inf\limits_{x \in [c-\delta, c+\delta]} f(x)\right) \cdot 2\delta \le 2K \cdot 2\delta < \varepsilon$

Since $f(x)$ is continuous on each of the intervals $[a, c - \delta]$ & $[c + \delta, b]$, it follows that

$f(x)$ is integrable on each of the intervals $[a, c - \delta]$ & $[c + \delta, b]$.

Thus

$\exists\ \mathcal{P}' =: \{a = x_0 < x_1 < \cdots < x_\ell = c - \delta\}$ (= a partition of $[a, c - \delta]$) such that

$U_f(\mathcal{P}') - L_f(\mathcal{P}') < \varepsilon$.

& $\exists\ \mathcal{P}'' =: \{c + \delta = x_{\ell+1} < x_{\ell+2} < \cdots < x_n = b\}$ (= a partition of $[c + \delta, b]$) such that

$U_f(\mathcal{P}'') - L_f(\mathcal{P}'') < \varepsilon$

Set $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_\ell < x_{\ell+1} < x_{\ell+2} < \cdots < x_n = b\} = \mathcal{P}' \cup \mathcal{P}'' (= \mathcal{P}(\delta) = \mathcal{P}(\varepsilon))$.

Then $\mathcal{P}$ becomes a partition of $[a, b]$. Moreover,

$$\begin{aligned} U_f(\mathcal{P}) - L_f(\mathcal{P}) &= U_f(\mathcal{P}') - L_f(\mathcal{P}') + \left(\sup_{x\in[c-\delta,c+\delta]} f(x) - \inf_{x\in[c-\delta,c+\delta]} f(x) \right) \cdot 2\delta + U_f(\mathcal{P}'') - L_f(\mathcal{P}'') \\ &< U_f(\mathcal{P}') - L_f(\mathcal{P}') + 4K\delta + U_f(\mathcal{P}'') - L_f(\mathcal{P}'') < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon \end{aligned}$$

Therefore, $f(x)$ is integrable on $[a, b]$.

(iii) Prove only the case $a < c_1 < c_2 < \cdots < c_k < b$.
Write $\sup_{x\in[a,\, b]} |\, f(x) \,| = K \ (< \infty)$.
Let $\varepsilon > 0$ be given, and choose $\delta > 0$ such that $4kK\delta < \varepsilon$ & $a < c_1 - \delta$ and $c_k + \delta < b$.

i.e., $(0 <)\ \delta < \min\left\{ \dfrac{\varepsilon}{4kK}, c_1 - a, b - c_k \right\}$

Notice that

$$\left(\sup_{x\in[c_1-\delta,c_1+\delta]} f(x) - \inf_{x\in[c_1-\delta,c_1+\delta]} f(x) \right) \cdot 2\delta + \cdots + \left(\sup_{x\in[c_k-\delta,c_k+\delta]} f(x) - \inf_{x\in[c_k-\delta,c_k+\delta]} f(x) \right) \cdot 2\delta \le 2K \cdot 2\delta \cdot k = 4kK\delta < \varepsilon .$$

In each of the intervals $[a, c_1 - \delta], [c_1 + \delta, c_2 - \delta], \cdots, [c_{k-1} + \delta, c_k - \delta], [c_k + \delta, b]$, the function $f(x)$ is continuous, so $f(x)$ is integrable in each of them. Thus

$\exists$ a partition $\mathcal{P}_1$ of $[a, c_1 - \delta]$ such that $U_f(\mathcal{P}_1) - L_f(\mathcal{P}_1) < \varepsilon$

$\vdots$

$\exists$ a partition $\mathcal{P}_{k+1}$ of $[c_k + \delta, b]$ such that $U_f(\mathcal{P}_{k+1}) - L_f(\mathcal{P}_{k+1}) < \varepsilon$

Set $\mathcal{P} = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_{k+1} \left(= \mathcal{P}(\delta) = \mathcal{P}(\varepsilon) \right)$. Then $\mathcal{P}$ becomes a partition of $[a, b]$. Moreover,

$$\begin{aligned} &U_f(\mathcal{P}) - L_f(\mathcal{P}) = U_f(\mathcal{P}_1) - L_f(\mathcal{P}_1) + \cdots + U_f(\mathcal{P}_{k+1}) - L_f(\mathcal{P}_{k+1}) \\ &\quad + \left(\sup_{x\in[c_1-\delta,c_1+\delta]} f(x) - \inf_{x\in[c_1-\delta,c_1+\delta]} f(x) \right) \cdot 2\delta + \cdots + \left(\sup_{x\in[c_k-\delta,c_k+\delta]} f(x) - \inf_{x\in[c_k-\delta,c_k+\delta]} f(x) \right) \cdot 2\delta \\ &\quad < (k+1)\varepsilon + 4kK\delta < (k+1)\varepsilon + \varepsilon = (k+2)\varepsilon \end{aligned}$$

Therefore, $f(x)$ is integrable on $[a, b]$.

Another proof of (iii) (using Theorem 19.5):

Assume $a \le c_1 < c_2 < \cdots < c_k \le b$. Choose $k-1$ points $d_1, d_2, \cdots, d_{k-1}$ so that

$$a \le c_1 < d_1 < c_2 < d_2 < \cdots < d_{k-1} < c_k \le b$$

Notice that f is discontinuous at exactly a single point on the subinterval $[a, d_1]$.

It follows from (i) or (ii) that

$f(x)$ is integrable on $[a, c_1]$ & $[c_1, d_1]$

This, combined with Theorem 19.5, shows

$f(x)$ is integrable on $[a, d_1]$, and $\displaystyle\int_a^{d_1} f(x)\, dx = \int_a^{c_1} f(x)\, dx + \int_{c_1}^{d_1} f(x)\, dx$

Same reasoning shows

$f(x)$ is integrable on $[d_1, d_2]$, and $\displaystyle\int_{d_1}^{d_2} f(x)\, dx = \int_{d_1}^{c_2} f(x)\, dx + \int_{c_2}^{d_2} f(x)\, dx$

$\vdots$

$f(x)$ is integrable on $[d_{k-1}, b]$, and $\displaystyle\int_{d_{k-1}}^{b} f(x)\, dx = \int_{d_{k-1}}^{c_k} f(x)\, dx + \int_{c_k}^{b} f(x)\, dx$

Thus $f(x)$ is integrable on each of the intervals $[a, d_1]$, $[d_1, d_2]$, $\cdots$, and $[d_{k-1}, b]$.

Again using Theorem 19.5, we finally get that

$f(x)$ is integrable on the entire interval $[a, b]$ &

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{d_1} f(x)\,dx + \int_{d_1}^{c_2} f(x)\,dx + \int_{c_2}^{d_2} f(x)\,dx + \cdots + \int_{d_{k-1}}^{c_k} f(x)\,dx + \int_{c_k}^{b} f(x)\,dx$$
$$= \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \cdots + \int_{c_k}^{b} f(x)\,dx$$

Notation: $\mathcal{R}[a,b] := \{ f \text{ is (Riemann-) integrable on } [a,b] \} \big[\subset \{ f \text{ is bounded on } [a,b] \} \big]$

Final comment: Evaluate $\int_0^1 f(x)\,dx$, where $f(x) = \begin{cases} 2x\sin\dfrac{1}{x} - \cos\dfrac{1}{x} & \text{if } x \in (0, 1] \\ 0 & \text{if } x = 0 \end{cases}$.

(Note that $f(x)$ is continuous on $(0, 1]$, but not continuous at $x = 0$ since $\lim_{x\to 0} f(x)$ does not exist.

But, $f(x)$ is clearly bounded on $[0, 1]$. Therefore, $f \in \mathcal{R}[0,1]$)

Sol. Let $F(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & x \in (0, 1] \\ 0, & x = 0 \end{cases}$.

Then it is easy to see that F: diff on $[0, 1]$ and $F'(x) = f(x) \quad \forall x \in [0, 1]$

Since $F'(= f)$ is integrable on $[0, 1]$, we have by First FTC below

$$\int_0^1 F'(x)\,dx = F(1) - F(0) = \sin 1$$

Alternative way: Let $F(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & x \in (0, 1] \\ 0, & x = 0 \end{cases}$. Seen that $F'(= f)$ is integrable on $[0, 1]$.

Moreover, F: diff on $(0,1]$ and $\lim_{x\to 0^+} F(x) [= \lim_{x\to 0} F(x) = 0]$ exists. Hence

$$\int_0^1 F'(x)\,dx \overset{\text{Corollary below}}{=} F(1) - \lim_{\varepsilon\to 0^+} F(\varepsilon) = F(1) = \sin 1$$

First FTC [First Fundamental Theorem of Calculus] --- will be proved in next chapter

Assume that $F(x)$ is diff on $[a, b]$ & $F'(x) = f(x) \in \mathcal{R}[a, b]$ ($f(x)$: a given ft)

$$\Rightarrow \quad \int_a^b f(x)\,dx = F(b) - F(a) \quad \text{i.e.,} \quad \int_a^b F'(x)\,dx = F(b) - F(a)$$

A variant of First FTC. Assume that $f \in \mathcal{R}[a,b]$. Suppose also that

$F \in C[a,b]$, F is diff on (a,b), and $F'(x) = f(x)$ for all $x \in (a,b)$

$$\Rightarrow \quad \int_a^b f(x)\,dx = F(b) - F(a) \quad \text{i.e.,} \quad \int_a^b F'(x)\,dx = F(b) - F(a)$$

Corollary [$\leftarrow$ A variant of First FTC**]**: Assume that $f \in \mathcal{R}[a,b]$. Suppose also that

F is diff on (a,b), $F'(x) = f(x) \ \forall x \in (a,b)$, and that $\lim_{x\to a^+} F(x)$ & $\lim_{x\to b^-} F(x)$ exist

$$\Rightarrow \quad \int_a^b f(x)dx = \lim_{x\to b^-} F(x) - \lim_{x\to a^+} F(x) \left[\leftarrow F \in C[a,b] \text{ by } F(a) \overset{\text{def}}{=} \lim_{x\to a^+} F(x),\ F(b) \overset{\text{def}}{=} \lim_{x\to b^-} F(x) \right]$$