

### Homework I (2022)

1. Suppose that if  $\theta = 1$ , then  $y$  has a normal distribution with mean 1 and standard deviation  $\sigma$ , and if  $\theta = 2$ , then  $y$  has a normal distribution with mean 2 and standard deviation  $\sigma$ . Also, suppose  $p(\theta = 1) = 0.5$  and  $p(\theta = 2) = 0.5$ .
  - (a) For  $\sigma = 2$ , write the formula for the marginal probability density for  $y$  and sketch it.
  - (b) What is  $p(\theta = 1|y = 1)$ , again supposing  $\sigma = 2$ ?
  - (c) Describe how the posterior density of  $\theta$  changes in shape as  $\sigma$  is increased and as it is decreased.
2. The example on Bayes' Theorem concerning the biology of twins was based on the assumption that birth of boys and girls occur equally frequently, and yet it has been known for a very long time that fewer girls are born than boys. Suppose that the probability a girl is  $p$ , so that

$$\begin{aligned}p(GG|M) &= p, & p(BB|M) &= 1 - p, & p(GB|M) &= 0 \\p(GG|F) &= p^2, & p(BB|F) &= (1 - p)^2, & p(GB|F) &= 2p(1 - p),\end{aligned}$$

where  $G$  and  $B$  respectively represent 'girl' and 'boy', and  $M$  and  $F$  respectively represent 'monozygotic twins' and 'fraternal twins'. Find the proportion of monozygotic twins in the whole population of twins in terms of  $p$  and the sex distribution among all twins.

3. For each of the following densities, provide a conjugate prior distribution for the unknown parameter(s), if one exists:
  - (a)  $X \sim \text{NegBin}(r, \theta)$ ,  $r$  is known.

$$p(x|\theta) = \frac{(x+r-1)!}{x!(r-1)!} \theta^r (1-\theta)^x,$$

for  $x = 0, 1, \dots$

- (b)  $X \sim \text{Gamma}(\alpha, \beta)$ ,  $\alpha$  is known.

$$p(x|\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x},$$

for  $x > 0$ .

- (c)  $X \sim \text{Gamma}(\alpha, \beta)$ ,  $\beta$  is known.

4. Setting parameters for a beta prior distribution: suppose your prior distribution for  $\theta$ , the proportion of Californians who support the death penalty is beta with mean 0.6 and standard deviation 0.3.
- (a) Determine the parameter  $\alpha$  and  $\beta$  of your prior distribution.
  - (b) A random sample of 1000 Californians is taken, and 65% support the death penalty. What are your posterior mean and variance for  $\theta$ ? Draw the posterior density function.