1) $p(y|\theta,r) = \frac{\sqrt{r}}{\sqrt{2\pi}} \exp\left(-\frac{r(y-\theta)^2}{2}\right) \quad \Rightarrow \quad p(y_1,\dots,y_n|\theta,r) = \left(\frac{r}{2\pi}\right)^{\frac{n}{2}} \exp\left(-r\sum_{i=1}^{n}\frac{(y_i-\theta)^2}{2}\right)$

$$p(\theta|r) = \frac{\sqrt{\lambda r}}{\sqrt{2\pi}} \exp\left(-\frac{\lambda r(\theta-\mu)^2}{2}\right)$$

$$p(r) = \frac{\left(\frac{b}{2}\right)^{\frac{a}{2}}}{\Gamma\left(\frac{a}{2}\right)} r^{\frac{a}{2}-1} e^{-\frac{b}{2}r}$$

$$\Rightarrow \quad p(\theta|r,y) = \frac{p(\theta,r,y)}{p(r,y)} \cdot \frac{p(\theta|r)}{p(\theta|r)}$$

$$= \frac{p(\theta,r,y)}{p(\theta,r)} \, p(r) \cdot \frac{p(\theta|r)}{p(r,y)}$$

$$= p(y|\theta,r) \cdot p(r) \cdot \frac{p(\theta|r)}{p(r,y)}$$

$$p(\theta|r,y) = p(y|\theta,r) \cdot p(r) \cdot \frac{p(\theta|r)}{p(r,y)}$$

$$\propto p(\theta,r,y) = p(y|\theta,r) \cdot p(r) \cdot p(\theta|r)$$

$$= \left(\frac{r}{2\pi}\right)^{-\frac{n}{2}} \exp\left(-r\sum_{i=1}^{n}\frac{(y_i-\theta)^2}{2}\right) \cdot \frac{\left(\frac{b}{2}\right)^{\frac{a}{2}}}{\Gamma\left(\frac{a}{2}\right)} r^{\frac{a}{2}-1} e^{-\frac{b}{2}r} \cdot \frac{\sqrt{\lambda r}}{\sqrt{2\pi}} \exp\left(-\frac{\lambda r(\theta-\mu)^2}{2}\right)$$

$$\propto \exp\left(-r\,\frac{n\theta^2-2n\bar{y}\theta}{2}\right) \exp\left(-\frac{\lambda r(\theta-\mu)^2}{2}\right)$$

$$= \exp\left(\frac{-r\,n(\theta^2-2\bar{y}\theta)-\lambda r(\theta^2-2\theta\mu+\mu^2)}{2}\right)$$

$$= \exp\left\{\frac{-rn\theta^2+2rn\bar{y}\theta-\lambda r\theta^2+2\lambda r\theta\mu-\lambda r\mu^2}{2}\right\}$$

$$\propto \exp\left\{\frac{-rn\theta^2+2rn\bar{y}\theta-\lambda r\theta^2+2\lambda r\theta\mu}{2}\right\}$$

$$= \exp\left\{-\frac{\theta^2 r(n+\lambda)-2\theta r(n\bar{y}+\lambda\mu)}{2}\right\}$$

$$\propto \exp\left\{-\frac{\left(\theta\sqrt{r(n+\lambda)}-\frac{\sqrt{r}(n\bar{y}+\lambda\mu)}{\sqrt{n+\lambda}}\right)^2}{2}\right\}$$

$$= \exp\left\{-\frac{r(n+\lambda)\left(\theta-\frac{(n\bar{y}+\lambda\mu)}{n+\lambda}\right)^2}{2}\right\} \sim N\left(\frac{n\bar{y}+\lambda\mu}{n+\lambda}, \frac{1}{r(n+\lambda)}\right) \qquad \therefore \quad \theta|r,y \sim N\left(\frac{n\bar{y}+\lambda\mu}{n+\lambda}, \frac{1}{r(n+\lambda)}\right)$$

$$p(\theta|r,y) = \frac{\sqrt{r(n+\lambda)}}{\sqrt{2\pi}} \exp\left[-\frac{r(n+\lambda)}{2}\left(\theta-\frac{n\bar{y}+\lambda\mu}{n+\lambda}\right)^2\right]$$

$$\Rightarrow \quad p(r|y) = \frac{p(r,y)}{p(y)}$$

$$= \frac{p(r,y)}{p(\theta,r,y)} \cdot \frac{p(\theta,r,y)}{p(\theta,r)} \cdot \frac{p(\theta,r)}{p(y)}$$

$$= \frac{1}{p(\theta|r,y)} \cdot p(y|\theta,r) \cdot \frac{p(\theta,r)}{p(y)}$$

$$\propto \frac{1}{p(\theta|r,y)} \cdot p(y|\theta,r) \cdot p(\theta,r) \quad , \quad p(\theta,r) = p(\theta|r)\,p(r) = \frac{\sqrt{\lambda r}}{\sqrt{2\pi}} \exp\left(-\frac{\lambda r(\theta-\mu)^2}{2}\right) \frac{\left(\frac{b}{2}\right)^{\frac{a}{2}}}{\Gamma\left(\frac{a}{2}\right)} r^{\frac{a}{2}-1} e^{-\frac{b}{2}r}$$

$$= \frac{\sqrt{2\pi}}{\sqrt{r(n+\lambda)}} \exp\left[\frac{r(n+\lambda)}{2}\left(\theta-\frac{n\bar{y}+\lambda\mu}{n+\lambda}\right)^2\right] \left(\frac{r}{2\pi}\right)^{-\frac{n}{2}} \exp\left(-r\sum_{i=1}^{n}\frac{(y_i-\theta)^2}{2}\right) \frac{\sqrt{\lambda r}}{\sqrt{2\pi}} \exp\left(-\frac{\lambda r(\theta-\mu)^2}{2}\right) \frac{\left(\frac{b}{2}\right)^{\frac{a}{2}}}{\Gamma\left(\frac{a}{2}\right)} r^{\frac{a}{2}-1} e^{-\frac{b}{2}r}$$

$$\propto \frac{1}{\cancel{\sqrt{r}}} \exp\left[\frac{r(n+\lambda)}{2}\left(\theta-\frac{n\bar{y}+\lambda\mu}{n+\lambda}\right)^2\right] (r)^{\frac{n}{2}} \exp\left(-r\sum_{i=1}^{n}\frac{(y_i-\theta)^2}{2}\right) \cancel{\sqrt{r}} \exp\left(-\frac{\lambda r(\theta-\mu)^2}{2}\right) r^{\frac{a}{2}-1} \exp\left(-r\frac{b}{2}\right)$$

$$= r^{\frac{n+a}{2}-1} \exp\left[\frac{r(n+\lambda)}{2}\left(\theta-\frac{n\bar{y}+\lambda\mu}{n+\lambda}\right)^2\right] \exp\left(-r\sum_{i=1}^{n}\frac{(y_i-\theta)^2}{2}\right) \exp\left(-\frac{\lambda r(\theta-\mu)^2}{2}\right) \exp\left(-r\frac{b}{2}\right)$$

$$= r^{\frac{n+a}{2}-1} \exp\left\{\frac{r(n+\lambda)}{2}\left(\frac{\theta(n+\lambda)-(n\bar{y}+\lambda\mu)}{n+\lambda}\right)^2 - r\sum_{i=1}^{n}\frac{(y_i-\theta)^2}{2} - \frac{\lambda r(\theta-\mu)^2}{2} - r\frac{b}{2}\right\}$$

$$= r^{\frac{n+a}{2}-1} \exp\left\{-\frac{r}{2}\left[(n+\lambda)\left(\frac{(n\bar{y}+\lambda\mu)-\theta(n+\lambda)}{n+\lambda}\right)^2 + \sum_{i=1}^{n}(y_i-\theta)^2 + \lambda(\theta-\mu)^2 + b\right]\right\}$$

$$= r^{\frac{n+a}{2}-1} \exp\left\{-\frac{r}{2}\left[(n+\lambda)\left(\frac{(n\bar{y}+\lambda\mu)-\theta(n+\lambda)}{n+\lambda}\right)^2 + \sum_{i=1}^{n}(y_i-\theta)^2 + \lambda(\theta-\mu)^2 + b\right]\right\}$$

$$= r^{\frac{n+a}{2}-1} \exp\left\{-\frac{r}{2}\left[\underbrace{\frac{\left((n\bar{y}+\lambda\mu)-\theta(n+\lambda)\right)^2 + (n+\lambda)\sum_{i=1}^{n}(y_i-\theta)^2 + (n+\lambda)\lambda(\theta-\mu)^2 + (n+\lambda)b}{n+\lambda}}\right]\right\}$$

$$\rightarrow \; -(n\bar{y}+\lambda\mu)^2 + (2\theta n+2\theta\lambda)(n\bar{y}+\lambda\mu) - \theta^2(n+\lambda)^2 + (n+\lambda)\left[\sum_{i=1}^{n}(y_i-\bar{y})^2 + n(\bar{y}-\theta)^2\right] + (\lambda n+\lambda^2)(\theta^2-2\theta\mu+\mu^2) + (n+\lambda)b$$

$$= \cancel{-n^2\bar{y}^2} - 2n\bar{y}\lambda\mu \cancel{-\lambda^2\mu^2} \cancel{+2\theta n^2\bar{y}} \cancel{+2\theta n\lambda\mu} \cancel{+2\theta\lambda n\bar{y}} \cancel{+2\theta\lambda^2\mu} \cancel{-\theta^2 n^2} \cancel{-2n\theta^2\lambda} \cancel{-\theta^2\lambda^2} + (n+\lambda)\sum_{i=1}^{n}(y_i-\bar{y})^2 \cancel{+n^2\bar{y}^2} + \lambda n\bar{y}^2 \cancel{-2n^2\bar{y}\theta} \cancel{-2\lambda n\bar{y}\theta} \cancel{+n^2\theta^2} \cancel{+\lambda n\theta^2} \cancel{+\lambda n\theta^2} \cancel{-2\lambda n\theta\mu} + \mu^2\lambda n \cancel{+\lambda^2\theta^2} \cancel{-2\lambda^2\theta\mu} \cancel{+\lambda^2\mu^2} + (n+\lambda)b$$

$$= (n+\lambda)\sum_{i=1}^{n}(y_i-\bar{y})^2 + n\lambda\left(\bar{y}^2-2\bar{y}\mu+\mu^2\right) + (n+\lambda)b \;=\; (n+\lambda)\sum_{i=1}^{n}(y_i-\bar{y})^2 + n\lambda\left(\bar{y}-\mu\right)^2 + (n+\lambda)b$$

$$= r^{\frac{n+a}{2}-1} \exp\left\{-\frac{r}{2}\left(\sum_{i=1}^{n}(y_i-\bar{y})^2 + \frac{n\lambda}{n+\lambda}(\bar{y}-\mu)^2 + b\right)\right\}$$

$$\sim \text{Gamma}\left(\frac{n+a}{2}, \; \sum_{i=1}^{n}(y_i-\bar{y})^2 + \frac{n\lambda}{n+\lambda}(\bar{y}-\mu)^2 + b\right)$$

2) $p(y|\theta) = \dfrac{e^{-\theta}\theta^{y}}{y!}$

$L(\theta|\underset{\sim}{y}) = \dfrac{e^{-n\theta}\theta^{\sum_{i=1}^{n} y_i}}{\prod_{i=1}^{n}(y_i!)}$

$l(\theta|\underset{\sim}{y}) = -n\theta + (\log\theta)\left(\sum_{i=1}^{n} y_i\right) - \sum_{i=1}^{n}\log(y_i!)$

$\frac{\partial}{\partial\theta} l(\theta|\underset{\sim}{y}) = -n + \frac{1}{\theta}\sum_{i=1}^{n} y_i$

$\frac{\partial^2}{\partial\theta^2} l(\theta|\underset{\sim}{y}) = -\frac{1}{\theta^2}\sum_{i=1}^{n} y_i \qquad \therefore E\left[-\frac{\partial^2}{\partial\theta^2} l(\theta|\underset{\sim}{y})\right] = \frac{n}{\theta} = I(\theta)$

$\therefore p(\theta) \propto I^{\frac{1}{2}}(\theta) = \sqrt{\frac{n}{\theta}} \propto \theta^{-\frac{1}{2}} = \theta^{\frac{1}{2}-1}\exp(-0\cdot\theta)$, it appears to follow a Gamma distribution with $\alpha = \frac{1}{2}$, but it does not have a corresponding β. Hence, it is an improper prior.

3) a) $f(x|\theta) = \theta^{x}(1-\theta)^{1-x}$, $T = \frac{\theta}{1-\theta} \Leftrightarrow \theta = \frac{T}{1+T}$

$f(y|\theta) = \binom{n}{y}\theta^{y}(1-\theta)^{n-y} = \binom{n}{y}\left(\frac{\theta}{1-\theta}\right)^{y}(1-\theta)^{n}$

$\Rightarrow f(y|T) = \binom{n}{y}T^{y}(1+T)^{-n}$

$\propto T^{y}(1+T)^{-n}$

b) $l(\theta|y) = \log\binom{n}{y} + y\log\theta + (n-y)\log(1-\theta)$

$\frac{\partial}{\partial\theta} l(\theta|y) = \frac{y}{\theta} - \frac{n-y}{1-\theta}$

$\frac{\partial^2}{\partial\theta^2} l(\theta|y) = -\frac{y}{\theta^2} - \frac{n-y}{(1-\theta)^2}$

$E\left[-\frac{\partial^2}{\partial\theta^2} l(\theta|y)\right] = \frac{n}{\theta} + \frac{n}{1-\theta} = n\left(\frac{1}{\theta(1-\theta)}\right) \propto \frac{1}{\theta(1-\theta)} = I(\theta)$

$\Rightarrow p_J(\theta) \propto \theta^{\frac{1}{2}-1}(1-\theta)^{\frac{1}{2}-1}$, $\theta = \frac{T}{1+T}$. $d\theta = \frac{1}{(1+T)^2}dT$

$\Rightarrow I^{\frac{1}{2}}(T) = I^{\frac{1}{2}}\left(\frac{T}{1+T}\right)\left|\frac{d\theta}{dT}\right|$

$= \left(\frac{T}{1+T}\right)^{\frac{1}{2}-1}\left(1-\frac{T}{1+T}\right)^{\frac{1}{2}-1}\frac{1}{(1+T)^2}$

$= \frac{\sqrt{1+T}}{\sqrt{T}}\sqrt{1+T}\,\frac{1}{(1+T)^2}$

$= T^{\frac{1}{2}-1}(1+T)^{2} \propto p_J(T)$

c) $\theta = \frac{T}{1+T}$ is a one-to-one transformation, then

$I(\theta) = E\left[\left(\frac{\partial\log(y|\theta)}{\partial\theta}\right)^2 \middle|\, \theta\right]$, where $\frac{\partial\log(y|\theta)}{\partial\theta} = \frac{\partial\log(y|\frac{T}{1+T})}{\partial T}\cdot\frac{\partial T}{\partial\theta}$

$= \left(\frac{\partial T}{\partial\theta}\right)^2 E\left[\left(\frac{\partial\log p(y|T)}{\partial T}\right)^2 \middle|\, T\right]$

$\Rightarrow I^{\frac{1}{2}}(\theta) = I^{\frac{1}{2}}(T)\left|\frac{\partial T}{\partial\theta}\right|$

$p_J(\theta) = p_J(T)\left|\frac{\partial T}{\partial\theta}\right|$

4) a) $y \sim B(n, \theta)$, $\theta \sim Beta(\alpha, \beta)$

$\Rightarrow$ $p(y|\theta) = \binom{n}{y} \theta^{y} (1-\theta)^{n-y}$

$p(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}$

$\Rightarrow$ $p(y) = \int_0^1 p(y|\theta)p(\theta)\,d\theta$

$$= \binom{n}{y} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \theta^{\alpha+y-1}(1-\theta)^{n+\beta-y-1}\,d\theta$$

$$= \binom{n}{y} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+y)\Gamma(n+\beta-y)}{\Gamma(n+\alpha+\beta)} \int_0^1 \frac{\Gamma(n+\alpha+\beta)}{\Gamma(\alpha+y)\Gamma(n+\beta-y)} \theta^{\alpha+y-1}(1-\theta)^{n+\beta-y-1}\,d\theta$$

$$= \binom{n}{y} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+y)\Gamma(n+\beta-y)}{\Gamma(n+\alpha+\beta)}$$

b) For a fixed constant C, $p(y) = C$ if and only if $\alpha = \beta = 1$.

$\Rightarrow$ If $\alpha, \beta \neq 0$ for at least one, $p(y_i) \neq p(y_j)$ $\forall i \neq j$

$$p(y|\alpha', \beta') = \binom{n}{y} \frac{\Gamma(\alpha'+\beta')}{\Gamma(\alpha')\Gamma(\beta')} \frac{\Gamma(y+\alpha')\Gamma(n-y+\beta')}{\Gamma(n+\alpha'+\beta')}$$

$$= \frac{n!}{y!(n-y)!} \frac{(\alpha'+\beta'-1)!}{(\alpha'-1)!(\beta'-1)!} \frac{(y+\alpha'-1)!(n-y+\beta'-1)!}{(n+\alpha'+\beta'-1)!}$$

$$\propto \frac{(y+\alpha'-1)!(n-y+\beta'-1)!}{y!(n-y)!} = 1 \quad \text{iff} \quad \alpha' = \beta' = 1$$

$\therefore$ If the beta-binomial probability is constant in y, then the prior distribution has to have $\alpha = \beta = 1$.