5.1 Limits of sums, products, and quotients

Theorem: Assume that $a_n \to L$ and $b_n \to M$, as $n \to \infty$

- Linearity Theorem: $ra_n + sb_n \to rL + sM$, for any $r, s \in R$
- Product Theorem: $a_n \cdot b_n \to L \cdot M$
- Quotient Theorem: $\frac{b_n}{a_n} \to \frac{M}{L}$, any $L \neq 0$

- the limit theorems actually give the limit L of the sequence
- the limit theorems give a quick way of proving convergence without using ε-N arguments

Theorem: Algebraic Operations for Infinite Limits

- $a_n \to \infty$ $\quad \begin{cases} b_n \to \infty, \; b_n \to L > 0 \\ b_n \text{ bounded below} \end{cases} \Rightarrow a_n + b_n \to \infty$
- $a_n \to \infty$ $\quad \begin{cases} b_n \to \infty, \; b_n \to L > 0 \\ b_n \geq K > 0 \text{ for } n \gg 1 \end{cases} \Rightarrow a_n b_n \to \infty$
- $a_n \to \infty \Rightarrow \frac{1}{a_n} \to 0$
- $a_n \to 0$ and $a_n > 0$ for all $n \Rightarrow \frac{1}{a_n} \to \infty$

5.2 Comparison Theorems

Theorem: Squeeze Theorem for Limits of Sequences

- We are given three sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$, such that $a_n \leq b_n \leq c_n$ for $n \gg 1$.

  Suppose that $a_n \to L$ and $c_n \to L$, then $b_n \to L$

Theorem: Squeeze Theorem for Infinite Limits

- $a_n \to \infty$, $b_n \geq a_n \Rightarrow b_n \to \infty$

5.3 Location Theorems

- tell how the location of the terms of a convergent sequence are related to the location of their limit

Theorem: Limit Location Theorem

- $a_n \leq M$ for $n \gg 1 \Rightarrow \lim_{n \to \infty} a_n \leq M$
- $a_n \geq M$ for $n \gg 1 \Rightarrow \lim_{n \to \infty} a_n \geq M$

Theorem: Sequence Location Theorem

- Assuming $\{a_n\}$ converges, $\begin{cases} \lim_{n \to \infty} a_n < M \Rightarrow a_n < M \text{ for } n \gg 1 \\ \lim_{n \to \infty} a_n > M \Rightarrow a_n > M \text{ for } n \gg 1 \end{cases}$

5.4 Subsequences. Non-existance of Limits

Definition :

- A subsequence of $\{a_n\}$ is a sequence composed of terms of $\{a_n\}$ and having the form

  $a_{n_1}, a_{n_2}, \cdots, a_{n_i}, \cdots,$ where $n_1 < n_2 < \cdots$

Theorem : Subsequence Theorem

- If $\{a_n\}$ converges, every subsequence also converges, and to the same limit

  $\lim_{n \to \infty} a_n = L \Rightarrow \lim_{n \to \infty} a_{n_i} = L$ for every subsequence $\{a_n\}$

- Subsequences play an important role in showing limits do not exist

5.5 Two Common Mistakes