1.1 Introduction

Representations :

$\mathbb{R}$ or $(-\infty, \infty)$

- the set of real numbers

$\emptyset$

- the empty set

$a \in A$

- a is an element of A

$A \subset B$

- set A is a subset of set B

* proper subset

Relation on $X \times Y$

- any subset of $X \times Y$

- Function f and g are equal iff they have the same domain, and same values

== $f, g : X \to Y$, and $f(x) = g(x)$ for all $x \in X$

Real Functions :

- functions whose domains and ranges are subsets of $\mathbb{R}$

1.2 Ordered Field Axioms

Postulate 1 : Field Axioms

- There are functions $+$ and $\cdot$, defined on $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$, which satisfy the following properties for every $a, b, c \in \mathbb{R}$ :

**Closure Properties**. $a + b$ and $a \cdot b$ belong to $\mathbf{R}$.

**Associative Properties**. $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

**Commutative Properties**. $a + b = b + a$ and $a \cdot b = b \cdot a$.

**Distributive Law**. $a \cdot (b + c) = a \cdot b + a \cdot c$.

**Existence of the Additive Identity**. There is a unique element $0 \in \mathbf{R}$ such that $0 + a = a$ for all $a \in \mathbf{R}$.

**Existence of the Multiplicative Identity**. There is a unique element $1 \in \mathbf{R}$ such that $1 \neq 0$ and $1 \cdot a = a$ for all $a \in \mathbf{R}$.

**Existence of Additive Inverses**. For every $x \in \mathbf{R}$ there is a unique element $-x \in \mathbf{R}$ such that

$$x + (-x) = 0.$$

**Existence of Multiplicative Inverses**. For every $x \in \mathbf{R} \setminus \{0\}$ there is a unique element $x^{-1} \in \mathbf{R}$ such that

$$x \cdot (x^{-1}) = 1.$$

Postulate 2 : Order Axioms

- There is a relation $<$ on $R \times R$ that has the following properties :

**Trichotomy Property**. Given $a, b \in \mathbf{R}$, one and only one of the following statements holds:

$$a < b, \quad b < a, \quad \text{or} \quad a = b.$$

**Transitive Property**. For $a, b, c \in \mathbf{R}$,

$$a < b \quad \text{and} \quad b < c \quad \text{imply} \quad a < c.$$

**The Additive Property**. For $a, b, c \in \mathbf{R}$,

$$a < b \quad \text{and} \quad c \in \mathbf{R} \quad \text{imply} \quad a + c < b + c.$$

**The Multiplicative Properties**. For $a, b, c \in \mathbf{R}$,

$$a < b \quad \text{and} \quad c > 0 \quad \text{imply} \quad ac < bc$$

and

$$a < b \quad \text{and} \quad c < 0 \quad \text{imply} \quad bc < ac.$$

Natural Numbers – $N := \{1, 2, \dots\}$

Integers – $Z := \{\dots, -1, 0, 1, 2, \dots\}$

Rationals – $Q := \{\frac{m}{n} : m, n \in Z \text{ and } n \neq 0\}$

✱ $N \subset Z \subset Q \subset R$

- We will assume that the sets $N$ and $Z$ satisfy the following properties

i) If $n, m \in Z$, then $n+m$, $n-m$, and $mn$ belong to $Z$

ii) If $n \in Z$, then $n \in N$ iff $n \geq 1$

iii) There is no $n \in Z$ that satisfies $0 < n < 1$

We notice in passing that none of the other special subsets of **R** satisfies Postulate 1. **N** satisfies all but three of the properties in Postulate 1: **N** has no additive identity (since $0 \notin \mathbf{N}$), **N** has no additive inverses (e.g., $-1 \notin \mathbf{N}$), and only one of the nonzero elements of **N** (namely, 1) has a multiplicative inverse. **Z** satisfies all but one of the properties in Postulate 1: Only two nonzero elements of **Z** have multiplicative inverses (namely, 1 and $-1$). $\mathbf{Q}^c$ satisfies all but four of the properties in Postulate 1: $\mathbf{Q}^c$ does not have an additive identity (since $0 \notin \mathbf{R} \setminus \mathbf{Q}$), does not have a multiplicative identity (since $1 \notin \mathbf{R} \setminus \mathbf{Q}$), and does not satisfy either closure property. Indeed, since $\sqrt{2}$ is irrational, the sum of

3 ways of proof :

1) mathematical induction

2) direct deduction

3) contradiction

2 - we assume the hypotheses to be true and proceed step by step to the conclusion. Each step is justified by a hypothesis, a definition, a postulate, or a mathematical result that has already been proved.

3 - we assume the hypothesis to be true, the conclusion to be true, and work step by step deductively until a contradiction occurs

- Much of analysis deals with estimation in which inequalities and the concept of absolute values play a central role

Definition :

- The absolute value of a number $a \in R$ is the number $|a| := \begin{cases} a, & a \geq 0 \\ -a, & a < 0 \end{cases}$

- The following result is useful when solving inequalities involving absolute value signs

Theorem : Fundamental Theorem of Absolute values

- Let $a \in R$ and $M \geq 0$. Then $|a| \leq M$ iff $-M \leq a \leq M$

Theorem :

- The absolute value satisfies the following 3 properties

  i) [Positive Definite] For all $a \in R$, $|a| \geq 0$ with $|a| = 0$ iff $a = 0$

  ii) [Symmetric] For all $a, b \in R$, $|a-b| = |b-a|$

  iii) [Triangle Inequalities] For all $a, b \in R$, $|a+b| \leq |a| + |b|$ and $\big||a| - |b|\big| \leq |a-b|$

- A correct way to estimate using absolute value signs usually involves one of the triangle inequalities

**1.8 *EXAMPLE.***

Prove that if $-2 < x < 1$, then $|x^2 - x| < 6$.

*Proof.* By hypothesis, $|x| < 2$. Hence by the triangle inequality and Remark 1.5,

$$|x^2 - x| \leq |x|^2 + |x| < 4 + 2 = 6. \quad \blacksquare$$

Theorem :

- Let $x, y, a \in R$

  i) $x < y + \varepsilon$ for all $\varepsilon > 0$ iff $x \leq y$

  ii) $x > y - \varepsilon$ for all $\varepsilon > 0$ iff $x \geq y$

  iii) $|a| < \varepsilon$ for all $\varepsilon > 0$ iff $a = 0$

- An interval $I$ is said to be bounded iff it has the form $[a,b]$, $(a,b)$, $[a,b)$, or $(a,b]$ for some $-\infty < a \leq b < \infty$, in which case the numbers $a, b$ will be called the endpoints of $I$. All other intervals will be called unbounded. An interval with endpoints $a, b$ is called degenerate if $a = b$ and nondegenerate if $a < b$. Thus a degenerate open interval is the empty set, and a degenerate closed interval is a point.