STA 3021: Stochastic Processes
Midterm 2 (6:00 PM - 7:15 PM on Nov 12, 2020)

Pledge: *I have neither given nor received any unauthorized aid during this exam.*

Student ID & Full Name: ____

Instructions: This test is a closed book exam, but you are allowed to use calculator. Clarity of your answer will also be a part of credit. When needed, use the notation $\Phi(z) = P(Z < z)$ for a standard normal distribution Z. Show your ALL work neatly.

1. (15 points) Suppose that the conditional distribution of N, given $Y = y$, is a Poisson distribution mean y. That is

$$P(N = n|Y = y) = \frac{e^{-y}y^n}{n!}.$$

Further, we assume that Y is a Gamma random variable with density

$$f_Y(y) = \frac{\lambda e^{-\lambda y}(\lambda y)^{r-1}}{(r-1)!}, \quad y > 0$$

(a) $E(N)$ $= E(E(N|Y)) = E(Y) = \frac{r}{\lambda}$

(b) Var(N). $= E(Var(N|Y)) + Var(E(N|Y))$

$= E(Y) + Var(Y) = \frac{r}{\lambda} + \frac{r}{\lambda^2}$

(c) Find $P(N = n)$.

$= \int_0^\infty p(N|Y)\, p(Y)\, dy$

$= \int_0^\infty \frac{e^{-y} y^n}{n!} \cdot \frac{\lambda^r y^{r-1}}{(r-1)!} e^{-\lambda y}\, dy$

$= \frac{\lambda^r}{n!(r-1)!} \int_0^\infty y^{n+r-1} e^{-(\lambda+1)y}\, dy$

$= \frac{\lambda^r}{n!(r-1)!} \cdot \frac{(n+r-1)!}{(\lambda+1)^{n+r}} = \frac{(n+r-1)!}{n!(r-1)!} \cdot \left(\frac{\lambda}{\lambda+1}\right)^r \cdot \left(\frac{1}{\lambda+1}\right)^n$

$\sim neg(r, \frac{\lambda}{\lambda+1})$

2. (10 points) Find he MGF of a compound random variable

$$S = \sum_{i=1}^{N} X_i,$$

where X_i's are IID random variables with MGF $M_X(\cdot)$ and independent of N with MGF of $M_N(\cdot)$.

$$M_S(t) = E(e^{St}) = E_N(E(e^{St} | N))$$

$$= E_N(E(e^{(X_1 + \cdots + X_N)t} | N))$$

$$= E_N(E(e^{X_1 t}) \cdots E(e^{X_N t}))$$

$$= E_N(E(e^{X_1 t})^N)$$

$$= E_N(M_X(t)^N)$$

$$= E_N[\exp(N \cdot \log M_X(t))]$$

$$= M_N(\log M_X(t))$$

3. (10 points) Suppose three players 1,2,3 play a (everlasting) tournament as follows: Initially player 1 plays against player 2. The winner of the nth game plays against the player who was not involved in the nth game. Suppose p_{ij} is the probability that in any given play between i and j, player i beats player j. Obviously $p_{ij} + p_{ji} = 1$. Suppose that the outcome successive games are independent. Let X_n be the pair that played nth game. Show that $\{X_n, n \geq 0\}$ is a DTMC. What is the transition probability?

i) $S = \{ (1,2), (1,3), (3,2) \}$

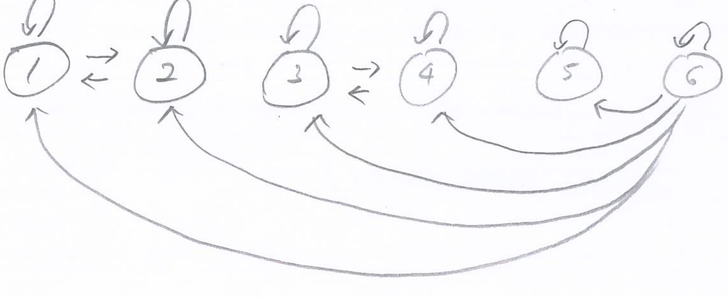
ii) X_{n+1} only depends on X_n

$\Rightarrow$ Markov Chain

$$P = \begin{array}{c|ccc} & (1,2) & (1,3) & (2,3) \\ \hline (1,2) & 0 & p_{12} & p_{21} \\ (1,3) & p_{13} & 0 & p_{31} \\ (2,3) & p_{23} & p_{32} & 0 \end{array}$$

4. (10 points) For the transition probability matrix with state space $E = \{1, 2, 3, 4, 5, 6\}$, do a complete classification of states, that is, identify communicating classes, periodic/aperiodic, postive/null recurrent or transient.

$$\begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 2/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix}$$



$C_1 = \{1,2\}$ closed, aperiodic, positive recurren

$C_2 = \{3,4\}$ closed, aperiodic, positive recurrent

$C_3 = \{5\}$ closed, aperiodic, positive recurrent

$C_4 = \{6\}$ open, aperiodic, transient

5. (15 points) Consider three urns, one colored red, one white, and one blue. The red urn contains 1 red and 4 blue balls; the white urn contains 3 white balls, 2 red balls, and 2 blue balls; the blue urn contains 4 white balls, 3 red balls, and 2 blue balls. At the initial stage, a ball is randomly selected from the red urn and then returned to that urn. At every subsequent stage, a ball is randomly selected from the urn whose color is the same as that of the ball previously selected and is then returned to that urn. In the long run, what proportion of the selected balls are red? What proportion are white? What proportion are blue?

$$X_n = \begin{cases} 0 & \text{red at } n\text{-th stage} \\ 1 & \text{white} \quad " \\ 2 & \text{blue} \quad " \end{cases}$$

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} \frac{1}{5} & 0 & \frac{4}{5} \\ \frac{2}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{3}{9} & \frac{4}{9} & \frac{2}{9} \end{pmatrix} \end{array} \Rightarrow \text{aperiodic}$$

& positive recurrent

Solve $\pi P = \pi$, $\pi_0 + \pi_1 + \pi_2 = 1$

$$\begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{5}\pi_0 + \frac{2}{7}\pi_1 + \frac{3}{9}\pi_2 \\ \frac{3}{7}\pi_1 + \frac{4}{9}\pi_2 \\ \frac{4}{5}\pi_0 + \frac{4}{9}\pi_1 + \frac{2}{9}\pi_2 \end{pmatrix}$$

$\Rightarrow \pi_0 = \frac{25}{28}\pi_1$, $\pi_2 = \frac{9}{7}\pi_1$

$\Rightarrow \frac{25}{28}\pi_1 + \pi_1 + \frac{9}{7}\pi_1 = 1$

$\Rightarrow 25\pi_1 + 28\pi_1 + 36\pi_1 = 28$, $\therefore \pi_1 = \frac{28}{89}$

$\therefore (\pi_0, \pi_1, \pi_2) = \left(\frac{25}{89}, \frac{28}{89}, \frac{36}{89}\right)$

6. (5 points) For a branching process, calculate π_0 when

$$P_0 = 1/2, \quad P_1 = 1/4, \quad P_3 = 1/4.$$

$\pi_0 = \frac{1}{2} + \frac{1}{4}\pi_0 + \frac{1}{4}\pi_0^3$

$4\pi_0 = 2 + \pi_0 + \pi_0^3$

$\pi_0^3 - 3\pi_0 + 2 = 0$

$(\pi_0 - 1)^2(\pi_0 + 2) = 0$

$\therefore \pi_0 = 1$ ($\because \pi_0$ is the smallest positive number)

7. (15 points) For the transition matrix given in the below, find the stationary distribution in terms of p and q such that $p + q = 1$.

$$P = \begin{pmatrix} q & p & 0 & 0 \\ 0 & 0 & q & p \\ q & p & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

P is aperiodic & positive recurrent

Solve $\pi P = \pi$, $\Sigma \pi_i = 1$

$$\Rightarrow \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} q\pi_0 + q\pi_2 \\ p\pi_0 + p\pi_2 + \pi_3 \\ q\pi_1 \\ p\pi_1 \end{pmatrix}, \quad \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$$

$$\Rightarrow \pi_0 = \frac{q}{1-q}\pi_2, \quad \pi_1 = \frac{1}{q}\pi_2, \quad \pi_3 = \frac{p}{q}\pi_2$$

$$\Rightarrow \pi_2\left(\frac{q}{1-q} + \frac{1}{q} + 1 + \frac{p}{q}\right) = 1$$

$$\therefore \pi_2 = \frac{q \cdot p}{q^2 + 2p}$$

$$(\pi_0, \pi_1, \pi_2, \pi_3) = \left(\frac{q^2}{q^2+2p}, \frac{p}{q^2+2p}, \frac{pq}{q^2+2p}, \frac{p^2}{q^2+2p}\right)$$

More rigorosly, if $p=1$, stationary distribution does not exist.

8. (20 points) Let $\{X_n, n \geq 0\}$ be a DTMC with the state space $S = \{1, 2, 3, 4\}$ and following transition probability matrix

$$P = \begin{pmatrix} .4 & .3 & .2 & .1 \\ .5 & 0 & 0 & .5 \\ .5 & .0 & 0 & .5 \\ .4 & .3 & .2 & .1 \end{pmatrix}.$$

Suppose the initial distribution is given by $P(X_0 = 1) = 1$. Compute

(a) $P(X_2 = 4)$

$$= P(X_2=4 \mid X_0=1) \cdot P(X_0=1)$$

$$= P^2_{14} = 0.3$$

$$P^2 = \begin{pmatrix} 0.45 & 0.15 & 0.1 & 0.3 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ 0.45 & 0.15 & 0.1 & 0.3 \end{pmatrix}$$

(b) $P(X_1 = 2, X_2 = 4, X_3 = 1)$

$$= P(X_3=1 \mid X_2=4) \cdot P(X_2=4 \mid X_1=2) \cdot P(X_1=2 \mid X_0=1)$$

$$= P_{41} \cdot P_{24} \cdot P_{12} = 0.4 \times 0.5 \times 0.3 = 0.06$$

(c) $P(X_7 = 4 \mid X_5 = 2)$

$$= P^2_{24}$$

$$= 0.1$$

$$P^3 = \begin{pmatrix} 0.425 & 0.225 & 0.15 & 0.2 \\ 0.450 & 0.150 & 0.1 & 0.3 \\ 0.450 & 0.150 & 0.1 & 0.3 \\ 0.425 & 0.225 & 0.15 & 0.2 \end{pmatrix}$$

(d) $E(X_3)$

$$= 1 \cdot P(X_3=1) + 2 \cdot P(X_3=2) + 3 \cdot P(X_3=3) + 4 \cdot P(X_3=4)$$

$$= P^3_{11} + 2 \cdot P^3_{12} + 3 \cdot P^3_{13} + 4 \cdot P^3_{14}$$

$$= 0.425 + 2 \times 0.025 + 3 \times 0.15 + 4 \times 0.2$$

$$= 2.125$$