Lindley and Smith (JRSS-B, 1972).

$y_i \mid \theta_i, \mu \overset{indep}{\sim} N(\theta_i, \sigma^2)$, $\sigma^2 (>0)$ known

$\theta_i \mid \mu \overset{iid}{\sim} N(\mu, \tau^2)$, $\tau^2 (>0)$ known

$\mu \sim$ uniform$(-\infty, \infty)$

$$\Rightarrow p(\theta_1, \cdots, \theta_n, \mu \mid y_1, \cdots, y_n)$$

$$\propto \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\theta_i)^2\right\} \exp\Big\{-\frac{1}{2\tau^2}\underbrace{\sum_{i=1}^n (\theta_i-\mu)^2}_{\sum_{i=1}^n (\theta_i-\bar\theta)^2 + n(\mu-\bar\theta)^2}\Big\}$$

$$= \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\theta_i)^2\right\} \exp\left\{-\frac{1}{2\tau^2}\sum_{i=1}^n (\theta_i-\bar\theta)^2\right\} \underbrace{\exp\left\{-\frac{n}{2\tau^2}(\mu-\bar\theta)^2\right\}}_{\text{kernel of } N(\bar\theta, \frac{\tau^2}{n})} \quad -(*)$$

Then

$$p(\theta_1, \cdots, \theta_n \mid y_1, \cdots, y_n) = \int p(\theta_1, \cdots, \theta_n, \mu \mid y_1, \cdots, y_n)\, d\mu$$

$$\underset{\text{by } (*)}{\propto} \exp\Big\{-\frac{1}{2\sigma^2}\underbrace{\sum_{i=1}^n (y_i-\theta_i)^2}\Big\} \exp\Big\{-\frac{1}{2\tau^2}\underbrace{\sum_{i=1}^n (\theta_i-\bar\theta)^2}\Big\} \underbrace{\int \frac{1}{\sqrt{2\pi}\frac{\tau}{\sqrt n}} e^{-\frac{n}{2\tau^2}(\mu-\bar\theta)^2}\, d\mu}_{=1}$$

$$\sum_i y_i^2 - 2\sum_i y_i\theta_i + \sum \theta_i^2 = \underset{\sim}{y}^T\underset{\sim}{y} - 2\underset{\sim}{\theta}^T\underset{\sim}{y} + \underset{\sim}{\theta}^T\underset{\sim}{\theta}$$

$$= \sum_i \theta_i^2 - n\bar\theta^2$$
$$= \underset{\sim}{\theta}^T\underset{\sim}{\theta} - n\left(\frac{1}{n}\underset{\sim}{1}^T\underset{\sim}{\theta}\right)^2 \quad \text{where } \underset{\sim}{\theta} = \begin{pmatrix}\theta_1\\ \vdots \\ \theta_n\end{pmatrix}$$
$$= \underset{\sim}{\theta}^T\underset{\sim}{\theta} - \frac{1}{n}(\underset{\sim}{\theta}^T\underset{\sim}{1})(\underset{\sim}{1}^T\underset{\sim}{\theta})$$
$$\underset{J=\underset{\sim}{1}\underset{\sim}{1}^T}{=} \underset{\sim}{\theta}^T\underset{\sim}{\theta} - \underset{\sim}{\theta}^T\frac{1}{n}J\underset{\sim}{\theta}$$
$$= \underset{\sim}{\theta}\left(I - \frac{1}{n}J\right)\underset{\sim}{\theta}$$

$$\propto \exp\left\{-\frac{1}{2\sigma^2}(\underset{\sim}{\theta}^T\underset{\sim}{\theta} - 2\underset{\sim}{\theta}^T\underset{\sim}{y}) - \frac{1}{2\tau^2}\underset{\sim}{\theta}^T\left(I-\frac{1}{n}J\right)\underset{\sim}{\theta}\right\}$$

$$= \exp\left[-\frac{1}{2}\left\{\underset{\sim}{\theta}^T\underbrace{\left(\frac{1}{\sigma^2}I + \frac{1}{\tau^2}\left(I-\frac{1}{n}J\right)\right)}_{\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)I - \frac{1}{\tau^2 n}J}\underset{\sim}{\theta} - 2\frac{1}{\sigma^2}\underset{\sim}{\theta}^T\underset{\sim}{y}\right\}\right]$$

$$= \exp\left[-\frac{1}{2}\left\{\underset{\sim}{\theta}^T\left(\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)I - \frac{1}{n\tau^2}J\right)\underset{\sim}{\theta} - 2\frac{1}{\sigma^2}\underset{\sim}{\theta}^T\underset{\sim}{y}\right\}\right]$$

$$\propto \exp\left[-\frac{1}{2}\left\{\underset{\sim}{\theta} - \left(\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)I - \frac{1}{n\tau^2}J\right)^{-1}\frac{1}{\sigma^2}\underset{\sim}{y}\right\}\left(\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)I - \frac{1}{\tau^2 n}J\right)\left\{\underset{\sim}{\theta} - \left(\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)I - \frac{1}{n\tau^2}J\right)^{-1}\frac{1}{\sigma^2}\underset{\sim}{y}\right\}\right]$$

: kernel of multivariate normal dist. with

mean $\left(\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)I - \frac{1}{\tau^2}\left(\frac{1}{n}J\right)\right)^{-1}\frac{1}{\sigma^2}\underset{\sim}{y}$ and

Variance $\left(\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)I - \frac{1}{\tau^2 n}J\right)^{-1} = D^{-1}$

Let $D = \left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)I - \frac{1}{\tau^2}\left(\frac{1}{n}J\right)$.

Then posterior variance is D^{-1}

and posterior mean is $\frac{1}{\sigma^2}D^{-1}\underset{\sim}{y}$.

Thus

$$\underset{\sim}{\theta} \mid \underset{\sim}{y} \sim N\left(\frac{1}{\sigma^2}D^{-1}\underset{\sim}{y},\ D^{-1}\right).$$

Note

$$D = \left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)\left(I - \frac{\sigma^2}{\sigma^2+\tau^2}\left(\frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T\right)\right)$$

$$= \frac{1}{\sigma^2(1-B)}\left(I - B\left(\frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T\right)\right)$$

$B = \frac{\sigma^2}{\sigma^2+\tau^2}$

(**Remark** If A and $A + \underset{\sim}{u}\underset{\sim}{v}^T$ are both invertible, then

$$(A+\underset{\sim}{u}\underset{\sim}{v}^T)^{-1} = A^{-1} - \frac{A^{-1}\underset{\sim}{u}\underset{\sim}{v}^T A^{-1}}{1+\underset{\sim}{v}^T A^{-1}\underset{\sim}{u}}$$
)

$$D^{-1} = \sigma^2(1-B)\left(I - B\left(\tfrac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T\right)\right)^{-1}$$

$$= \sigma^2(1-B)(I + \underset{\sim}{u}\,\underset{\sim}{v}^T)$$

$\uparrow$

$\underset{\sim}{u} = \sqrt{B}\frac{1}{\sqrt{n}}\underset{\sim}{1}$

$\underset{\sim}{v} = -\underset{\sim}{u}$

$$= \sigma^2(1-B)\left[I^{-1} - \frac{I^{-1}\sqrt{B}\frac{1}{\sqrt{n}}\underset{\sim}{1}(-\underset{\sim}{u})^T I^{-1}}{1 + (-\underset{\sim}{u})^T I^{-1}\sqrt{B}\frac{1}{\sqrt{n}}\underset{\sim}{1}}\right]$$

$\uparrow$
by Remark

$$= \sigma^2(1-B)\left[I + \frac{\sqrt{B}\frac{1}{\sqrt{n}}\underset{\sim}{1}\left(\sqrt{B}\frac{1}{\sqrt{n}}\underset{\sim}{1}\right)^T}{1 - \left(\sqrt{B}\frac{1}{\sqrt{n}}\underset{\sim}{1}\right)^T\sqrt{B}\frac{1}{\sqrt{n}}\underset{\sim}{1}}\right]$$

$$= \sigma^2(1-B)\left[I + \frac{\frac{B}{n}\underset{\sim}{1}\underset{\sim}{1}^T}{1 - \frac{B}{n}\underbrace{\underset{\sim}{1}^T\underset{\sim}{1}}_{=n}}\right]$$

$$= \sigma^2(1-B)\left(I + \frac{\frac{B}{n}J}{1-B}\right)$$

$$= \sigma^2\left[(1-B)I + B\left(\tfrac{1}{n}J\right)\right].$$

Thus,

$$\frac{1}{\sigma^2}D^{-1}\underset{\sim}{y} = \frac{1}{\sigma^2}\left[\sigma^2\left\{(1-B)I + B\left(\tfrac{1}{n}J\right)\right\}\right]\underset{\sim}{y}$$

$$= (1-B)\underset{\sim}{y} + B\left(\tfrac{1}{n}\underbrace{J\underset{\sim}{y}}_{\underset{\sim}{1}\underbrace{\underset{\sim}{1}^T\underset{\sim}{y}}_{=\Sigma y_i}}\right)$$

$$= (1-B)\underset{\sim}{y} + B\frac{1}{n}\Sigma y_i\,\underset{\sim}{1}$$

$$= (1-B)\underset{\sim}{y} + B\,\bar{y}\,\underset{\sim}{1}$$

Rewrite the posterior dist. of $\underset{\sim}{\theta}$ given $\underset{\sim}{y}$ is

$$\Rightarrow \underset{\sim}{\theta}\,|\,\underset{\sim}{y} \sim N\left((1-B)\underset{\sim}{y} + B\bar{y}\underset{\sim}{1},\ \sigma^2\left[(1-B)I + B\left(\tfrac{1}{n}J\right)\right]\right).$$