Ch 8. Hypothesis testing: Part II

Likelihood Ratio Tests (Ch 6.3 & 8.3)

$H_0 : \theta \in \Omega_0$ versus $H_1 : \theta \in \Omega \cap \Omega_0^c$

- Likelihood Ratio Test (LRT) of size α rejects H_0 if

$$\frac{\max_{\theta \in \Omega_0} L(\theta; \mathbf{X})}{\max_{\theta \in \Omega} L(\theta; \mathbf{X})} \leq k,$$

with k satisfying

$$\max_{\theta \in \Omega_0} P\left(\frac{\max_{\theta \in \Omega_0} L(\theta; \mathbf{X})}{\max_{\theta \in \Omega} L(\theta; \mathbf{X})} \leq k\right) = \alpha$$

$\rightarrow$ If $\Omega_0 = \{\theta_0\}$ (i.e. $H_0 : \theta = \theta_0$ versus $H_1 : \theta \in \Omega \cap \Omega_0^c$), LRT of size α reject H_0 if

- Likelihood Ratio Test (LRT) of size α rejects H_0 if

$$\frac{L(\theta_0; \mathbf{X})}{L\left(\hat{\theta}^{MLE}; \mathbf{X}\right)} \leq k,$$

with k satisfying

$$P_{\theta_0}\left(\frac{L(\theta_0; \mathbf{X})}{L(\hat{\theta}^{(MLE)}; \mathbf{X})} \leq k\right) = \alpha$$

$\rightarrow$ LRT is not the "best" in general, but it is known that LRT is asymptotically "best".

- Equivalent form of LRT:

Reject H_0 if

$$2\left(\ell\left(\hat{\theta}^{\Omega_0};\mathbf{X}\right)-\ell\left(\hat{\theta}^{\Omega};\mathbf{X}\right)\right)\leq c$$

with

$$\max_{\theta\in\Omega_0} P\left[2\left(\ell\left(\hat{\theta}^{\Omega_0};\mathbf{X}\right)-\ell\left(\hat{\theta}^{\Omega};\mathbf{X}\right)\right)\leq c\right]=\alpha$$

where

$$\hat{\theta}^{\Omega}=\arg\max_{\theta\in\Omega} L(\theta;\mathbf{X})$$

$$\hat{\theta}^{\Omega_0}=\arg\max_{\theta\in\Omega_0} L(\theta;\mathbf{X})$$

- If $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$, the equivalent LRT is

Reject H_0 if

$$2\left(\ell\left(\theta_0; \mathbf{X}\right) - \ell\left(\hat{\theta}^{MLE}; \mathbf{X}\right)\right) \leq c$$

with

$$P_{\theta_0}\left[2\left(\ell\left(\theta_0; \mathbf{X}\right) - \ell\left(\hat{\theta}^{MLE}; \mathbf{X}\right)\right) \leq c\right] = \alpha$$

- Example 8.9 (two-sided test)

$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, σ^2 is known and $\mu \in \Omega = (-\infty, \infty)$

$H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$

- Example 8.9 (One-sided test)

$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, σ^2 is known and $\mu \in \Omega = (-\infty, \infty)$

$H_0 : \mu \leq \mu_0$ versus $H_1 : \mu > \mu_0$

- Example 8.10

$X_1, \ldots, X_n \overset{iid}{\sim} Exp(\theta)$, $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$

- Example 8.11

$X_1, \ldots, X_n \stackrel{iid}{\sim} N(\theta_1, \theta_2)$, θ_2 is unknown and $-\infty < \theta_1 < \infty$, $\theta_2 > 0$

$H_0 : \theta_1 = \theta_0$ versus $H_1 : \theta_1 \neq \theta_0$

- Example 8.12

$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \theta)$, $-\infty < \mu < \infty$, $\theta > 0$

$H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$

- Example 8.13

$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu_1, \sigma^2)$, $Y_1, \ldots, Y_m \overset{iid}{\sim} N(\mu_2, \sigma^2)$, $-\infty < \mu_1, \mu_2 < \infty$, $\sigma^2 > 0$ is known, $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$

- Example 8.14

$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu_1, \theta_1)$, $Y_1, \ldots, Y_m \overset{iid}{\sim} N(\mu_2, \theta_2)$, $H_0 : \theta_1 = \theta_2$ versus $H_1 : \theta_1 \neq \theta_2$

Exercises: 6.3.6, 6.3.10, 6.3.11, 8.3.11, 8.3.12, 8.3.13, 8.3.14,

Other tests

Let $\hat{\theta}$ be the MLE of θ.

- Likelihood Ratio Test (LRT):
  Reject H_0 if $-2(\ell(\theta_0) - \ell(\hat{\theta})) > k$

- Wald test:
  Reject H_0 if $\left(\sqrt{nI(\hat{\theta})}(\hat{\theta} - \theta_0)\right)^2 > k$

- Score test:
  Reject H_0 if

$$\left(\frac{\ell'(\theta_0)}{\sqrt{nI(\theta_0)}}\right)^2 > k$$

Theorem (Asymptotic likelihood ratio test)

Under regularity conditions (R0)∼(R4), if $H_0 : \theta = \theta_0 \in \mathbb{R}^p$ *is true, we have*

$$-2\log\Lambda \xrightarrow{d} \chi^2_p \text{ where } \Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} \text{ and } \hat{\theta} \text{ is the MLE of } \theta$$

That is, $-2(\ell(\theta_0) - \ell(\hat{\theta})) \xrightarrow{d} \chi^2_p$

- Example 8.10 (revisited)

$X_1, \ldots, X_n \overset{iid}{\sim} Exp(\theta)$, $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$

Exercises: 6.3.16, 6.3.18, 6.3.19