Experimental Design
Note 13
Random Effects Models

Keunbaik Lee

Sungkyunkwan University

Design of Engineering Experiments - Experiments with Random Factors

- Previous chapters have considered **fixed** factors
  - A specific set of factor levels is chosen for the experiment
  - Inference confined to those levels
  - Often **quantitative** factors are fixed
- When factor levels are chosen at random from a larger population of potential levels, the factor is **random**
  - Inference is about the entire population of levels
  - Industrial applications include measurement system studies

Example 1 I

직기

A textile company weaves a fabric on a large number of looms. It would like the looms to be homogeneous so that it obtains a fabric of uniform strength. A process engineer suspects that, in addition to the usual variation in strength within samples of fabric from the same loom, there may also be significant variations in strength between looms. To investigate this, she selects four looms at random and makes four strength determinations on the fabric manufactured on each loom. The layout and data are given in the following.

pick n looms from N population looms

	Observations			
looms	1	2	3	4
1	98	97	99	96
2	91	90	93	92
3	96	95	97	95
4	95	96	99	98

Example 1 II

- Response variable is strength
- Interest focuses on determining if there is difference in strength due to the different looms
- However, the weave room contains many (100s) looms
- Solution . select a (random) **sample** of the looms, obtain fabric from each
- Consequently, “looms” is a **random factor**
- It looks like standard single-factor experiment with $a = 4$ and $n = 4$

Random Effects vs Fixed Effects

- Consider factor with numerous possible levels
- Want to draw inference on **population of levels**
- Not concerned with any specific levles
- Example of difference (1=fixed, 2=random)

fixed effects

  - Compare reading ability of 10 2nd grade classes in NY. Select $a = 10$ specific classes of interest. Randomly choose n students from each classroom. Want to compare τ_i (class-specific effects).

random effects

  - Study the availability **among all** 2nd grade classes in NY. **Randomly choose** $a = 10$ classes from large number of classes. Randomly choose n students from each classroom. Want to assess σ^2_τ (class to class variability).
- Inference broader in random effects case
- Levels chosen randomly $\rightarrow$ inference on population

Random Effects Model

- Similar model (as in the fixed case) with different assumptions

$E(y_{ij}) = E[E(y_{ij}|\tau_i)]$
$= E(\mu + \tau_i)$
$= \mu + E(\tau_i)$
$= \mu$

$$y_{ij} = \mu + \tau_i + \epsilon_{ij}, \quad i = 1, 2, \cdots, a; \; j = 1, 2, \cdots, n_i$$

where μ =grand mean; τ_i = ith treatment effect (random); $\epsilon_{ij} \sim N(0, \sigma^2)$.

- Instead of $\sum_i \tau_i = 0$, assume

$Var(y_{ij}) = Var[E(y_{ij}|\tau_i)] + E[Var(y_{ij}|\tau_i)]$
$= Var(\mu + \tau_i) + \sigma^2$
$= \sigma_\tau^2 + \sigma^2$

$$\tau_i \sim N(0, \sigma_\tau^2)$$ => random effect로 간주

$\{\tau_i\}$ and $\{\epsilon_{ij}\}$ are independent.

- $var(y_{ij}) = \sigma_\tau^2 + \sigma^2$
where σ_τ^2 and σ^2 are called variance components.

Relevant Hypotheses in the Random Effects (or Components of Variance) Model

- In the fixed effects model we test equality of treatment means
- This is no longer appropriate because the treatments are randomly selected
  - the individual ones we happen to have are not of specific interest
  - we are interested in the population of treatments
- The appropriate hypotheses are

$$H_0 : \sigma_\tau^2 = 0 \quad vs. \quad H_1 : \sigma_\tau^2 > 0$$

Statistical Analysis

- Same ANOVA table (as before)

Source	SS	DF	MS	F_0
Between	SS_{tr}	$a-1$	MS_{tr}	$F_0 = \frac{MS_{tr}}{MSE}$
Within	SSE	$N-a$	MSE	
Total	$SS_T = SS_{tr} + SSE$	$N-1$		

  - $E(MSE) = \sigma^2$ $\qquad \hat{\sigma}^2 = MS_E$
  - $E(MS_{tr}) = \sigma^2 + n\sigma_\tau^2$ $\qquad H_0: \sigma_\tau^2 = 0$ $\qquad \hat{\sigma}^2 + n\hat{\sigma}_\tau^2 = MS_{trt}$
- Under H_0, $F_0 \sim F_{a-1,N-a}$
- Same test as before
- Conclusions, however, pertain to entire population

$$y_{ij} = \mu + \tau_i + \varepsilon_{ij}, \quad \tau_i \sim N(0, \sigma_\tau^2)$$

$$\varepsilon_{ij} \sim N(0, \sigma^2)$$

$$E(MS_E) = \sigma^2, \quad E(MS_{trt}) = \sigma^2 + n\sigma_\tau^2$$

$$H_0: \sigma_\tau^2 = 0$$

$$H_a: \sigma_\tau^2 > 0$$

$$\Rightarrow \quad SS_{trt} = n\sum_{i=1}^{a}(\bar{y}_{i.} - \bar{y}_{..})^2, \quad \text{where } \bar{y}_{i.} = \mu + \tau_i + \bar{\varepsilon}_{i.}$$

$$\bar{y}_{..} = \mu + \bar{\tau} + \bar{\varepsilon}_{..}$$

$$= n\sum_{i=1}^{a}(\tau_i - \bar{\tau} + \bar{\varepsilon}_{i.} - \bar{\varepsilon}_{..})^2$$

$$= n\sum_{i=1}^{a}(\tau_i - \bar{\tau})^2 + (\bar{\varepsilon}_{i.} - \bar{\varepsilon}_{..})^2 + 2(\tau_i - \bar{\tau})(\bar{\varepsilon}_{i.} - \bar{\varepsilon}_{..})$$

$$E(SS_{trt}) = n\sum_{i=1}^{a}E\left[(\tau_i - \bar{\tau})^2\right] + n\sum_{i=1}^{a}E\left[(\bar{\varepsilon}_{i.} - \bar{\varepsilon}_{..})^2\right] + 2n\sum_{i=1}^{a}E\left[(\tau_i - \bar{\tau})(\bar{\varepsilon}_{i.} - \bar{\varepsilon}_{..})\right],$$

$$\tau_i \sim N(0, \sigma_\tau^2)$$

$$\frac{\sum_{i=1}^{a}(\tau_i - \bar{\tau})^2}{\sigma_\tau^2} \sim \chi_{a-1}^2 \Rightarrow E(\chi_{a-1}^2) = a-1$$

$$\bar{\varepsilon}_{i.} \sim N\left(0, \frac{\sigma^2}{n}\right)$$

$$\frac{\sum_{i=1}^{a}(\bar{\varepsilon}_{i.} - \bar{\varepsilon}_{..})}{\sigma^2/n} \sim \chi_{a-1}^2 \Rightarrow E(\chi_{a-1}^2) = a-1$$

$$E\left[(\tau_i - \bar{\tau})(\bar{\varepsilon}_{i.} - \bar{\varepsilon}_{..})\right] = 0$$

$$\Rightarrow E(\tau_i) = E(\bar{\tau}) = E(\bar{\varepsilon}_{i.}) = E(\bar{\varepsilon}_{..}) = 0$$

$$\Rightarrow \operatorname{cov}\left[(\tau_i - \bar{\tau}), (\bar{\varepsilon}_{i.} - \bar{\varepsilon}_{..})\right] = 0$$

$$E(SS_{trt}) = n\sigma_\tau^2\sum_{i=1}^{a}E\left[\frac{(\tau_i - \bar{\tau})^2}{\sigma_\tau^2}\right] + n\frac{\sigma^2}{n}\sum_{i=1}^{a}E\left[\frac{(\bar{\varepsilon}_{i.} - \bar{\varepsilon}_{..})^2}{\sigma^2/n}\right]$$

$$= n\sigma_\tau^2(a-1) + \sigma^2(a-1)$$

$$E\left(\frac{SS_{trt}}{a-1}\right) = \sigma^2 + n\sigma_\tau^2$$

$$MS_{trt} = \sigma^2 + n\sigma_\tau^2$$

$$SS_E = \sum_{i=1}^{a}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{i.})^2 = \sum_{i=1}^{a}\sum_{j=1}^{n}(\varepsilon_{ij} - \bar{\varepsilon}_{i.})^2$$

$$= \sigma^2\sum_{i=1}^{a}\underbrace{\sum_{j=1}^{n}\frac{(\varepsilon_{ij} - \bar{\varepsilon}_{i.})^2}{\sigma^2}}_{\chi_{n-1}^2}$$

$$E(SS_E) = \sigma^2 a(n-1)$$

$$MS_E = \sigma^2$$

Estimation

- Usually interested in estimating variances
- Use mean squares (known as ANOVA method) or Method of Moments

$$\hat{\sigma}^2 = MSE$$
$$\hat{\sigma}^2_\tau = (MS_{tr} - MSE)/n$$

If unbalanced, replace n with

$$n_0 = \frac{1}{a-1}\left(\sum_{i=1}^{a} n_i - \frac{\sum_{i=1}^{a} n_i^2}{\sum_{i=1}^{a} n_i}\right)$$

- Estimate of σ^2_τ can be negative
  - Supports H_0? Use zero as estimate?
  - Validity of model? Nonlinear? maybe
  - Other approaches (MLE, Bayesian with nonnegative prior)

Confidence Intervals

$$\star\; MS_E = \frac{SS_E}{N-a}, \quad \frac{(N-a)MS_E}{\sigma^2} = \frac{SS_E}{\sigma^2} = \frac{\sum\sum(y_{ij}-\bar{y}_{i\cdot})^2}{\sigma^2} = \underbrace{\frac{\sum_i^a\sum_j^b(\varepsilon_{ij}-\bar{\varepsilon}_{i\cdot})^2}{\sigma^2}}_{\sim\chi^2_{n-1}} = \chi^2_{N-a}$$

$$\Rightarrow \chi^2_{1-\frac{\alpha}{2},N-a} < \frac{(N-a)MS_E}{\sigma^2} < \chi^2_{\frac{\alpha}{2},N-a}$$

- σ^2: Same as fixed case

$$\frac{(N-a)MSE}{\sigma^2} \sim \chi^2_{N-a}$$

$$\frac{(N-a)MSE}{\chi^2_{\alpha/2,N-a}} \le \sigma^2 \le \frac{(N-a)MSE}{\chi^2_{1-\alpha/2,N-a}}$$

- For $\sigma^2_\tau/(\sigma^2+\sigma^2_\tau)$: intraclass correlation coefficient

$$\frac{L}{L+1} \le \frac{\sigma^2_\tau}{\sigma^2+\sigma^2_\tau} \le \frac{U}{U+1}$$

or

$$\frac{F_0 - F_{\alpha/2,a-1,N-a}}{F_0+(n-1)F_{\alpha/2,a-1,N-a}} \le \frac{\sigma^2_\tau}{\sigma^2+\sigma^2_\tau} \le \frac{F_0 - F_{1-\alpha/2,a-1,N-a}}{F_0+(n-1)F_{1-\alpha/2,a-1,N-a}}$$

$$\frac{MS_{trt}/(n\sigma_\tau^2+\sigma^2)}{MS_E/\sigma^2} \sim F_{a-1,N-a}$$

$$\Rightarrow F_{1-\frac{\alpha}{2},a-1,N-a} < \frac{\underbrace{\frac{MS_{trt}}{MS_E}}_{F_0}/(n\sigma_\tau^2+\sigma^2)}{/\sigma^2} < F_{\frac{\alpha}{2},a-1,N-a}$$

Loom Experiment (continued) I

ANOVA Table is

Source	SS	DF	MS	F_0
Between	89.19	3	29.73	15.68
Within	22.75	12	1.90	
Total	111.94	15		

Highly significant result ($F_{.05,3,12} = 3.49$)
$\hat{\sigma}_\tau^2 = (29.73 - 1.90)/\overset{n}{4} = 6.98$
78.6% ($= 6.98/(6.98 + 1.90)$) is attributable to loom differences
Time to improve consistency of the looms

Loom Experiment (continued) II

- 95% CI for σ^2

$$\frac{SSE}{\chi^2_{.025,12}} \leq \sigma^2 \leq \frac{SSE}{\chi^2_{.975,12}} \quad => \quad (0.97, 5.17)$$

- 95% CI for $\sigma_\tau^2/(\sigma_\tau^2 + \sigma^2)$

$$\left(\frac{15.68 - 4.47}{15.68 + (4-1)4.47}, \frac{15.68 - (1/14.34)}{15.68 + (4-1)(1/14.34)}\right) = (0.385, 0.982)$$

$F_{0.025,3,12} = 4.47$, $F_{.975,3,12} = 1/14.34$ using property that

$$F_{1-\alpha/2,v_1,v_2} = 1/F_{\alpha/2,v_2,v_1}$$

See random_effect_1.SAS.

A Measurement Systems Capability Study I

A typical gauge R&R experiment is shown below. An instrument or gauge is used to measure a critical dimension of certain part. Twenty parts have been selected from the production process, and three randomly selected operators measure each part twice with this gauge. The order in which the measurements are made is completely randomized, so this is a two-factor factorial experiment with design factors parts and operators, with two replications. Both parts and operators are random factors.

A Measurement Systems Capability Study II

Parts	Operator 1		Operator 2		Operator 3	
1	21	20	20	20	19	21
2	24	23	24	24	23	24
3	20	21	19	21	20	22
.	.	.	.	.	.	.
19	25	26	25	24	25	25
20	19	19	18	17	19	17

Variance components equation: $\sigma_y^2 = \sigma_\tau^2 + \sigma_\beta^2 + \sigma_{\tau\beta}^2 + \sigma^2$

Total variability=Parts+Operators+Interaction+Experimental Error

Statistical Model with Two Random Factors I

$\overset{iid}{\sim} N(0, \sigma^2)$

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk}$$

random effects 4개

random

where $\tau_i \sim^{iid} N(0, \sigma^2_\tau)$, $\beta_j \sim^{iid} N(0, \sigma^2_\beta)$, $(\tau\beta)_{ij} \sim^{iid} N(0, \sigma^2_{\tau\beta})$ for $i = 1, \cdots, a$; $j = 1, \cdots, b$; $k = 1, \cdots, n$.

τ_i, β_j, and $(\tau\beta)_{ij}$ are mutually independent.

- $var(y_{ijk}) = \sigma^2 + \sigma^2_\tau + \sigma^2_\beta + \sigma^2_{\tau\beta}$
- Expected MS's similar to one-factor random model

$$E(MSE) = \sigma^2; \quad E(MS_A) = \sigma^2 + bn\sigma^2_\tau + n\sigma^2_{\tau\beta}$$
$$E(MS_B) = \sigma^2 + an\sigma^2_\beta + n\sigma^2_{\tau\beta}; \quad E(MS_{AB}) = \sigma^2 + n\sigma^2_{\tau\beta}$$

Statistical Model with Two Random Factors II

- EMS determine what MS to use in denominator

$$
\begin{aligned}
H_0: \sigma_\tau^2 = 0 &\quad \rightarrow \quad MS_A/MS_{AB} \\
H_0: \sigma_\beta^2 = 0 &\quad \rightarrow \quad MS_B/MS_{AB} \\
H_0: \sigma_{\tau\beta}^2 = 0 &\quad \rightarrow \quad MS_{AB}/MSE
\end{aligned}
$$

- Using ANOVA method

$$
\begin{aligned}
\hat{\sigma}^2 &= MSE \\
\hat{\sigma}_\tau^2 &= (MS_A - MS_{AB})/bn \\
\hat{\sigma}_\beta^2 &= (MS_B - MS_{AB})/an \\
\hat{\sigma}_{\tau\beta}^2 &= (MS_{AB} - MSE)/n
\end{aligned}
$$

Statistical Model with Two Random Factors III

- Sometimes results in negative estimates
- Proc Varcomp and Proc Mixed compute estimates
- Can use different estimation procedures
  ANOVA method - Method=type1
  RMLE method - Method=reml
- Proc Mixed
  Variance component estimates
  Hypothesis tests and confidence intervals

Statistical Model with Two Random Factors IV

Example 13.2: See Gauge_Capability.SAS.

Confidence Intervals for Variance Components

- Can use asymptotic variance estimates to form CI
- Known as Wald's approximate CI
- Mixed: option CL=WALD or METHOD=TYPE1
  Use standard normal $\rightarrow$ 95% CI uses 1.96

$$\hat{\sigma}^2_\beta \pm 1.96(0.0330) = (-0.05, 0.08)$$
$$\hat{\sigma}^2_\tau \pm 1.96(3.3738) = (3.67, 16.89)$$

Two-Factor Mixed Effects Model I

- Unrestricted mixed effects model:

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk}$$

- Assume A fixed and B random
  $\sum_i \tau_i = 0$, $\beta_j \sim^{iid} N(0, \sigma^2_\beta)$, and $(\tau\beta)_{ij} \sim^{iid} N(0, \sigma^2_{\tau\beta})$
- SAS uses **unrestricted mixed model** in analysis

Two-Factor Mixed Effects Model II

- Expected mean squares:

$$
\begin{aligned}
E(MS_A) &= \sigma^2 + bn\sum_i \tau_i^2/(a-1) + n\sigma_{\tau\beta}^2 \\
E(MS_B) &= \sigma^2 + an\sigma_{\beta}^2 + n\sigma_{\tau\beta}^2 \\
E(MS_{AB}) &= \sigma^2 + n\sigma_{\tau\beta}^2 \\
E(MSE) &= \sigma^2
\end{aligned}
$$

- random statement in SAS also gives these results

Two-Factor Mixed Effects Model III

- Diagnostics
  - Histogram or QQ plot
    Normality or Unusual Obervations
  - Residual Plots
    Constant variance or Unusual Observations

See Gauge_Capability.SAS.

Restricted mixed effects model I

- Assume A fixed and B random

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk}$$

1. $\sum_i \tau_i = 0$ and $\beta_j \sim N(0, \sigma^2_\beta)$ usual assumptions
2. $(\tau\beta)_{ij} \sim N(0, (a-1)\sigma^2_{\tau\beta}/a)$ $(a-1)/a$ simplifies EMS
3. $\sum_i (\tau\beta)_{ij} = 0$ for β level j added restriction

- Due to added restriction
  - Not all $(\tau\beta)_{ij}$ indep. $cov((\tau\beta)_{ij}, (\tau\beta)_{i'j}) = -\frac{1}{a}\sigma^2_{\tau\beta}$
  - $cov(y_{ijk}, y_{i'jk'}) = \sigma^2_\beta - \frac{1}{a}\sigma^2_{\tau\beta}$, $i \neq i'$.
- Known as **restricted** mixed effects model

Restricted mixed effects model II

- This model coincides with EMS algorithm

$$
\begin{aligned}
E(MSE) &= \sigma^2 \\
E(MS_A) &= \sigma^2 + bn\sum_i \tau_i^2/(a-1) + n\sigma^2_{\tau\beta} \\
E(MS_B) &= \sigma^2 + an\sigma^2_\beta \\
E(MS_{AB}) &= \sigma^2 + n\sigma^2_{\tau\beta}
\end{aligned}
$$

- Testing hypotheses:

$$
\begin{aligned}
&H_0 : \tau_1 = \tau_2 = \cdots = 0 \quad \rightarrow \quad MS_A/MS_{AB} \\
&H_0 : \sigma^2_\beta = 0 \quad \rightarrow \quad MS_B/MSE \\
&H_0 : \sigma^2_{\tau\beta} = 0 \quad \rightarrow \quad MS_{AB}/MSE
\end{aligned}
$$

Restricted mixed effects model III

- Variance Estimates (Using ANOVA method)

$$\hat{\sigma}^2 = MSE$$
$$\hat{\sigma}_\beta^2 = (MS_B - MSE)/an$$
$$\hat{\sigma}_{\tau\beta} = (MS_{AB} - MSE)/n$$

- Differences between restricted and unrestricted models
  - $E(MS_B)$
  - Test $H_0 : \sigma_{\tau\beta}^2 = 0$ using MS_{AB} in denominator
  - $cov(y_{ijk}, y_{i'jk'}) = \sigma_\beta^2,\ i \neq i'$.

Restricted mixed effects model IV

- Connection between restricted and unrestricted random effects mdoels

$$(\bar{\tau\beta})_{.j} = \sum_i (\tau\beta)_{ij}/a$$

$$y_{ijk} = \mu + \tau_i + (\beta_j - (\bar{\tau\beta})_{.j}) + ((\tau\beta)_{ij} - (\bar{\tau\beta})_{.j}) + \epsilon_{ijk}$$

  Check the model above satisfies the conditions of restricted mixed model
- Restricted model is slightly more general.

Restricted mixed effects model V

Revisit Measurement Systems Capability Study

From ANOVA Table the results from Gauge_Capability.SAS,

- $H_0 : \tau_1 = \tau_2 = \tau_3 = 0$

$$F_0 = \frac{MS_A}{MS_{AB}} = \frac{1.308}{0.712} = 1.84$$

  P-value based on $F_{2,38}$ is 0.173.

- $H_0 : \sigma^2_{\tau\beta} = 0$

$$F_0 = \frac{MS_{AB}}{MSE} = \frac{0.712}{0.992} = 0.72.$$

  P-value based on $F_{38,60}$ is 0.86.

Restricted mixed effects model VI

- Variance components estimates:

$$\hat{\sigma}^2_{\beta} = \frac{62.39 - 0.99}{(3)(2)} = 10.23$$
$$\hat{\sigma}^2_{\tau\beta} = \frac{0.71 - 0.99}{2} = -.14(\approx 0)$$
$$\hat{\sigma}^2 = 0.99$$

- Pairwise comparison for τ_1, τ_2, and τ_3.

Rules for Expected Mean Squares I

- In models so far, EMS fairly straightforward
- Could calculate EMS using brute force method
- For mixed models, good to have formal procedure
- Montgomery describes procedure for restricted model
  1. Write the error term in the models as $\epsilon_{ij..m}$ where m represents the replication subscript.
  2. Write each variable term in model as a row heading in a two-way table.
  3. Write the subscripts in the model as column headings. Over each subscript write F if factor fixed and R in random. Over this, write down the levels of each subscript.

Rules for Expected Mean Squares II

4. For each row, copy the number of observations under each subscript, providing the subscript does not appear in the row variable term.
5. For any bracketed subscripts in the model, place a 1 under those subscripts that ar inside the brackets.
6. Fill in remaining cells with a 0 (if subscript represents a fixed factor) or a 1 (if random factor).
7. To find the expected mean square of any term (row), cover the entries in the columns that contain non-bracketed subscript letters in this term in the model. For those rows with a least the same subscripts, multiply the remaining numbers to get coefficient for corresponding term in the model.

Expected Mean Squares for Two-Factor Model I

Two-Factor Model (A: fixed, B: fixed):

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk}$$

	F	F	R	
	a	b	n	
term	i	j	k	EMS
τ_i	0	b	n	$\sigma^2 + \frac{bn\sum_i \tau_j^2}{a-1}$
β_j	a	0	n	$\sigma^2 + \frac{an\sum_j \beta_j^2}{b-1}$
$(\tau\beta)_{ij}$	0	0	n	$\sigma^2 + \frac{n\sum_i\sum_j(\tau\beta)_{ij}^2}{(a-1)(b-1)}$
$\epsilon_{(ij)k}$	1	1	1	σ^2

Expected Mean Squares for Two-Factor Model II

Two-Factor Model (A: random, B: random):

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk}$$

	R	R	R	
	a	b	n	
term	i	j	k	EMS
τ_i	1	b	n	$\sigma^2 + n\sigma^2_{\tau\beta} + bn\sigma^2_{\tau}$
β_j	a	1	n	$\sigma^2 + n\sigma^2_{\tau\beta} + an\sigma^2_{\beta}$
$(\tau\beta)_{ij}$	1	1	n	$\sigma^2 + n\sigma^2_{\tau\beta}$
$\epsilon_{(ij)k}$	1	1	1	σ^2

Expected Mean Squares for Two-Factor Model III

Two-Factor Mixed Model (A: fixed, B: random):
We consider **unrestricted mixed effects model**.

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk}$$

	F	R	R	
	a	b	n	
term	i	j	k	EMS
τ_i	0	b	n	$\sigma^2 + n\sigma^2_{\tau\beta} + \frac{bn\sum\tau_i^2}{a-1}$
β_j	a	1	n	$\sigma^2 + n\sigma^2_{\tau\beta} + an\sigma^2_{\beta}$
$(\tau\beta)_{ij}$	1	1	n	$\sigma^2 + n\sigma^2_{\tau\beta}$
$\epsilon_{(ij)k}$	1	1	1	σ^2

13주차 끝

Approximate F Tests I

- For some models, no exact F-test exits
- Possible approaches:
  - Could assume some variances are negligible, not recommended without "conclusive" evidence
  - Pool (insignificant) means squares with error, also risky, not recommended when *df* for error is already big.
- Recall 3 Factor Mixed Model (F-fixed)
- No exact test for A based on EMS
  Assume $\overset{\text{Fixed}}{a} = 3$, $\overset{\text{random}}{b} = 2$, $\overset{\text{random}}{c} = 3$, $n = 2$ and following MS were obtained (using restricted mixed effects model)

Approximate F Tests II

$\frac{1}{a-1}\sum_{i=1}^{a}\tau_i^2$

Source	DF	MS	EMS	F	P	
A	2	0.7866	$12\phi_A + 6\sigma^2_{AB} + 4\sigma^2_{AC}$ $+2\sigma^2_{ABC} + \sigma^2$	?	?	$= E(MS_{AB} + MS_{AC} - MS_{ABC})$
B	1	0.0010	$18\sigma^2_B + 6\sigma^2_{BC} + \sigma^2$	0.33	.622	
AB	2	0.0056	$6\sigma^2_{AB} + 2\sigma^2_{ABC} + \sigma^2$	2.24	.222	
C	2	0.0560	$12\sigma^2_C + 6\sigma^2_{BC} + \sigma^2$	18.87	.051	
AC	4	0.0107	$4\sigma^2_{AC} + 2\sigma^2_{ABC} + \sigma^2$	4.28	.094	$H_0: \sigma^2_{AC} = 0$
BC	2	0.0030	$6\sigma^2_{BC} + \sigma^2$	10.00	.001	$H_0: \sigma^2_{BC} = 0$
ABC	4	0.0025	$2\sigma^2_{ABC} + \sigma^2$	8.33	.001	$H_0: \sigma^2_{ABC} = 0$
Error	18	0.0003	σ^2			

Satterthwaite's Approximate I

- H_0 : effect $= 0$, e.g., $H_0 : \tau_1 = \cdots = \tau_a = 0$ or equivalently $H_0 : \sum_i \tau_i^2 = 0$.
  No exact test exists.
- Get two linear combinations of mean squares

$$MS^{'} = MS_r \pm \cdots \pm MS_s$$
$$MS^{''} = MS_u \pm \cdots \pm MS_v$$

such that 1) $MS^{'}$ and $MS^{''}$ do not share common mean squares; 2) $E(MS^{'}) - E(MS^{''})$ is a multiple of the effect.

Satterthwaite's Approximate II

- approximate test statistic

$$F = \frac{MS'}{MS''} = \frac{MS_r \pm \cdots \pm MS_s}{MS_u \pm \cdots \pm MS_v} \approx F_{p,q}$$

  where $p = \frac{(MS_r \pm \cdots \pm MS_s)^2}{MS_r^2/f_r + \cdots + MS_s^2/f_s}$ and $q = \frac{(MS_u \pm \cdots \pm MS_v)^2}{MS_u^2/f_u + \cdots + MS_v^2/f_v}$
- f_i is the degrees of freedom associated with MS_i.
- p and q may not be integers, interpolation is needed. SAS can handle noninteger dfs.
- Caution when subtraction is used.

Example: 3-Factor Mixed Model (A fixed) I

$$
\begin{aligned}
&H_0 : \tau_1 = \tau_2 = \tau_3 = 0 \\
&MS' = MS_A \\
&MS'' = MS_{AB} + MS_{AC} - MS_{ABC} \\
&E(MS' - MS'') = 12\phi_A = 12\frac{\sum_i \tau_i^2}{3-1} \\
&F = \frac{MS_A}{MS_{AB} + MS_{AC} - MS_{ABC}} = \frac{.7866}{.0107 + .0056 - .0025} = 57.0 \\
&p = 2, \quad q = \frac{.0138^2}{.0107^2/4 + .0056^2/2 + .0025^2/4} = 4.15
\end{aligned}
$$

Example: 3-Factor Mixed Model (A fixed) II

- Interpolation needed

$$P(F_{2,4} > 57) = .0011 \qquad P(F_{2,5} > 57) = .0004$$
$$P = .85(.0011) + .15(.0004) = .001$$

- SAS can be used to compute P-values and quantile values for F and χ^2 values with noninteger degrees of freedom.
  Upper Tail Probability: probf(x,df1, df2) and probchi(x,df)
  Quantiles: finv(p,df1,df2) and cinv(p,df)
  See Approximate-F.SAS.

Example: 3-Factor Mixed Model (A fixed) III

Another Approach to Testing $H_0 : \tau_1 = \tau_2 = \tau_3 = 0$

$$MS^{'} = MS_A + MS_{ABC}$$
$$MS^{''} = MS_{AB} + MS_{AC}$$
$$E(MS^{'} - MS^{''}) = ?$$
$$F = \frac{MS_A + MS_{ABC}}{MS_{AB} + MS_{AC}} = \frac{.7866 + .0025}{.0107 + .0056} = 48.41$$
$$p = \frac{.7891^2}{.7866^2/2 + .0025^2/4} = 2.01 \quad q = \frac{.0163^2}{.0107^2/4 + .0056^2/2} = 6.00$$
$$\text{P-value} = P(F > 48.41) = 0.002$$

Example: 3-Factor Mixed Model (A fixed) IV

- This is again found significant
- Avoid subtraction, summation should be preferred.

Approximate Confidence Intervals I

Suppose we are interested in σ_x^2

- Case 1: there exists a mean square MS_x with df_x such that $E(MS_x) = \sigma_x^2$. Then $\hat{\sigma}_x^2 = MS_x$, and

$$\frac{df_x MS_x}{\sigma_x^2} \sim \chi^2(df_x)$$

  Exact $100(1-\alpha)\%$ CI:

$$\frac{df_x MS_x}{\chi^2_{\alpha/2,df}} \le \sigma_x^2 \le \frac{df_x MS_x}{\chi^2_{1-\alpha/2,df}}$$

Approximate Confidence Intervals II

- Case 2: there exist $MS^{'} = MS_r + \cdots + MS_s$ and $MS^{''} = MS_u + \cdots + MS_v$ such that $E(MS^{'} - MS^{''}) = k\sigma_x^2$. Then

$$\hat{\sigma}_x^2 = \frac{MS^{'} - MS^{''}}{k}, \text{ and } \frac{df_x\hat{\sigma_x^2}}{\sigma_x^2} \approx \chi^2(df_x)$$

where

$$df_x = \frac{(\hat{\sigma_x^2})^2}{\sum_i \frac{MS_i}{k^2 f_i}} = \frac{(MS_r + \cdots + MS_s - MS_u - \cdots - MS_v)^2}{MS_r^2/f_r + \cdots + MS_s^2/f_s + MS_u^2/f_u + \cdots + MS_v^2/f_v}$$

Approximate $100(1-\alpha)\%$ CI:

$$\frac{df_x\hat{\sigma}_x^2}{\chi^2_{\alpha/2,df_x}} \le \sigma_x^2 \le \frac{df_x\hat{\sigma}_x^2}{\chi^2_{1-\alpha/2,df_x}}$$

Approximate Confidence Intervals III

From SAS result for Gauge_Capability.SAS,
$MS_A = 62.39$, $df_A = 19$; $MS_B = 1.31$, $df_B = 2$; $MS_{AB} = 0.71$, $df_{AB} = 38$

$$\hat{\sigma}_\tau^2 = \frac{MS_A - MS_{AB}}{bn} = (62.39 - 0.71)/6 = 10.28$$

$$df = \frac{(62.39 - 0.71)^2}{62.39^2/19 + 0.71^2/38} = 18.57$$

CI: $(18.57(10.28)/32.28, 18.57(10.28)/8.61) = (5.91, 22.17)$

Approximate Confidence Intervals IV

$$\hat{\sigma}^2_\beta = \frac{MS_B - MS_{AB}}{an} = (1.31 - 0.71)/40 = 0.015$$

$$df = \frac{(1.31 - 0.71)^2}{1.31^2/2 + 0.71^2/38} = .413$$

CI: $\left(.413(.015)/3.079, .413(.015)/2.29 \times 10^{-8}\right) = (.002, 270781)$

PROC GLM vs. PROC MIXED I

- PROC GLM
  Uses the method of least squares to fit general linear models. No other parameter estimation method can be specified.
- PROC MIXED
  Fits a variety of mixed linear models to data and allows specification of the parameter estimation method to be used.

PROC GLM vs. PROC MIXED II

- The default fitting method used in PROC MIXED maximizes the restricted likelihood of the data under the assumption that the data are normally distributed and any missing data are missing at random. This general framework accommodates many common correlated-data methods, including variance component models and repeated measures analyses.
- PROC GLM fits some random-effects and repeated-measures models, although its methods are based on method-of-moments estimation and a portion of the output applies only to the fixed-effects model. The effects specified in the RANDOM statement are still treated as fixed as far as the model fit is concerned, and they serve only to produce corresponding expected mean squares.

PROC GLM vs. PROC MIXED III

Comparison

- PROC GLM
  1. Designed for "fixed effects" models with allowance for some adjustments when random effects are present.
  2. Estimation of fixed effects: Estimates are based on Ordinary Least Squares.
  3. Estimation of variance components via the RANDOM statement: Uses method of moments estimation of solving expected mean squares for variance components.
  4. Both fixed and random effects are listed in the model statement.

PROC GLM vs. PROC MIXED IV

5. The LSMEANS statement interprets all effects as fixed, even the random effects. While the least square means are correct, their standard errors are not necessarily correct. This is true even if a RANDOM statement is used.
6. The RANDOM statement under PROC GLM invokes the calculation of expected mean squares for the listed effects and the appropriate test using the "test" option. The randomness of the effect is not incorporated into the tests of main effects, lsmeans, contrasts etc.
7. The TEST statement is an important statement to use correct error terms in testing model effects (e.g. Subsampling designs, split-plot design).

PROC GLM vs. PROC MIXED V

8. The REPEATED statement in PROC GLM is used to specify various transformations with which to conduct the traditional univariate or multivariate tests.
9. PROC GLM is not efficient in handling missing values.

- PROC MIXED
  1. Designed for "random" and "mixed effects" models.
  2. Estimation of fixed effects: Based on Generalized Least Squares in normal error model (estimates are maximum likelihood estimates under normality).
  3. Estimation of variance components: Uses maximum likelihood or restricted maximum likelihood estimation methods for variance components (unless type3 is requested).
  4. Only the fixed effects are listed in the model statement. The random effects are listed in the random statement.

PROC GLM vs. PROC MIXED VI

5. Effects in the MODEL statement are assumed fixed. The standard error estimates for least square means account for the random effects.
6. Signals incorporation of the listed effects in all aspects of inference. Various correlation structures, describing the dependencies of multiple random effects can be selected using different options.
7. No TEST statement under PROC MIXED
8. The REPEATED statement in PROC MIXED is used to specify covariance structures for repeated measurements on subjects. The approach used in PROC MIXED is more flexible and more widely applicable than either the univariate or multivariate approaches.

PROC GLM vs. PROC MIXED VII

9. PROC MIXED has a better mechanism for handling missing values.

General Mixed Effect Model I

- In terms of linear model

$$Y = X\beta + Z\delta + \epsilon$$

  where β is a vector of fixed-effect parameters, δ is a vector of random-effect parameters, and ϵ is the error vector.
- δ and ϵ assumed uncorrected
  mean 0; covariance matrices G and R (allows correlation)
- $cov(Y) = ZGZ^T + R$
- If $R = \sigma^2 I$ and $Z = 0$, back to standard linear model
- SAS PROC MIXED allows one to specify G and R.
- G through RANDOM, R through REPEATED
- Unrestricted linear mixed model is default

Sample Size Calculations I

Use Charts V and VI

Random Effects Model

Factor	λ	DF_{Num}	DF_{Den}
A	$\sqrt{1+\frac{bn\sigma_\tau^2}{\sigma^2+n\sigma_{\tau\beta}^2}}$	$a-1$	$(a-1)(b-1)$
B	$\sqrt{1+\frac{an\sigma_\beta^2}{\sigma^2+n\sigma_{\tau\beta}^2}}$	$b-1$	$(a-1)(b-1)$
AB	$\sqrt{1+\frac{n\sigma_{\tau\beta}^2}{\sigma^2}}$	$(a-1)(b-1)$	$ab(n-1)$

Mixed Effects Model

Factor	λ or Φ	DF_{Num}	DF_{Den}
A	$\sqrt{\frac{bn\sum\tau_i^2}{a(\sigma^2+n\sigma_{\tau\beta}^2)}}$	$a-1$	$(a-1)(b-1)$
B	$\sqrt{1+\frac{an\sigma_\beta^2}{\sigma^2}}$	$b-1$	$ab(n-1)$
AB	$\sqrt{1+\frac{n\sigma_{\tau\beta}^2}{\sigma^2}}$	$(a-1)(b-1)$	$ab(n-1)$