Chap 3. The limit of a sequence

3.1 Definition of limit

In Chap 1, we discussed the limit of seqs that are monotone. But many important seqs are ***not*** monotone. For such sequences, the methods we used in Chap 1 won't work.

For instance, we can expect that the sequence

$$1.1,\quad 0.9,\quad 1.01,\quad 0.99,\quad 1.001,\quad 0.999,\cdots$$

has 1 as its limit, yet neither the integer part nor any of the decimal places of the numbers eventually constant.

So we need a more generally applicable definition of the limit.

We abandon the decimal expansions, and replace them by the approximation viewpoint, in which

"the limit of is a_n" means roughly
a_n is a good approximation to L, when n is large

※ Def 3.1 The number L is the limit of the seq (a_n) if

$$\text{given } \varepsilon > 0, \quad a_n \underset{\varepsilon}{\approx} L \quad \text{for } n \gg 1$$

i.e., given $\varepsilon > 0$, $a_n \underset{\varepsilon}{\approx} L$ for $n \geq$ (or $>$) some $N = N(\varepsilon)$

i.e., given $\varepsilon > 0$, $\exists$ a number $N = N(\varepsilon)$ s.t. $a_n \underset{\varepsilon}{\approx} L$ for $n \geq N$

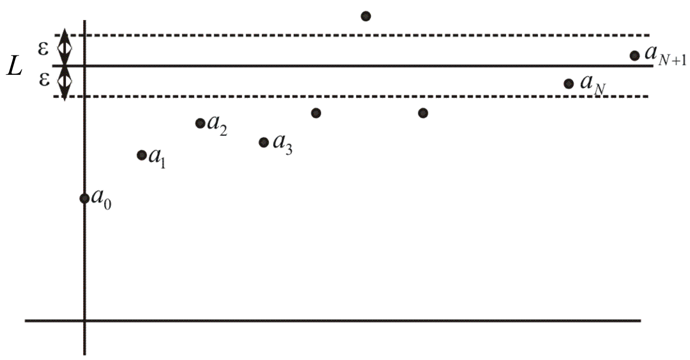
i.e., given $\varepsilon > 0$, $\exists$ a number $N = N(\varepsilon)$ s.t. $|\, a_n - L \,| < \varepsilon$ for $n \geq N$

If such an L exists, we say (a_n) converges (or is convergent) (to L)

and we write $\lim\limits_{n\to\infty} a_n = L$ or $a_n \to L$ (as $n \to \infty$)

If not, we say (a_n) diverges (or is divergent)

Geometrical meaning of $\lim a_n = L$:



Given $\varepsilon > 0$, $\exists$ some number $N = N(\varepsilon)$ s.t.
all terms $(a_n)_{n\geq N}$ lie in the strip above

Remember that

$$\boxed{\lim_{n\to\infty} a_n = L \text{ (a real number)} \quad \text{if} \quad \text{given } \varepsilon > 0, \quad a_n \underset{\varepsilon}{\approx} L \quad \text{for } n \gg 1}$$

Note:

(i) $a_n \underset{\varepsilon}{\approx} L$ (a_n approximates L to within ε)

(ii) $a_n \underset{\varepsilon}{\approx} L$ for $n \gg 1$ (the approximation holds for all a_n far enough out in the sequence)

(iii) given $\varepsilon > 0$, $a_n \underset{\varepsilon}{\approx} L$ for $n \gg 1$

(the approximation can be made as closed as described, provided we go far enough out in the sequence: the smaller ε is, the further out we must go, for the approximation to be valid within ε.)

Eg A. Show that $\displaystyle\lim_{n\to\infty} \frac{n-1}{n+1} = 1$

Pf. We must show:

$$\boxed{\text{given } \varepsilon > 0, \quad \frac{n-1}{n+1} \underset{\varepsilon}{\approx} 1 \quad \text{for } n \gg 1}$$

$$\left|\frac{n-1}{n+1} - 1\right| = \left|\frac{-2}{n+1}\right| = \frac{2}{n+1} < \varepsilon \quad \text{if} \quad n+1 > \frac{2}{\varepsilon}$$

$$i.e., \quad \text{if} \quad n > \frac{2}{\varepsilon} - 1 \overset{\text{take}}{=} : N(\text{depends on } \varepsilon)$$

Remarks on limit proofs:

1. The heart of a limit proof is in getting *a small upper estimate* for $|\, a_n - L \,|$.
   Often most of the work will consist in showing how to rewrite this difference so that a good upper estimate can be made.
2. In giving the proof, you must exhibit a value N concealed in "for $n \gg 1$"
   You need not give the smallest possible N (if we could find one candidate for N, any bigger number would be a candidate)
   N depends on ε : In general, the smaller ε is, the bigger N is.
3. The phrase "given $\varepsilon > 0$" has equivalent forms:

$$\text{for all } \varepsilon > 0, \quad \text{for any } \varepsilon > 0, \quad \text{for every } \varepsilon > 0, \quad \text{for each } \varepsilon > 0,$$
$$\text{given any } \varepsilon > 0$$

4. It is not hard to show that if a monotone sequence (a_n) has the limit L in the sense of Chap 1 then L is also its limit in the sense of Def 3.1 [**See Problem 3.3**] (The converse is also true, but more trouble to show because of the difficulties with decimal notation)

Thus the limit results of Chap 1, the **Completeness Property** in particular, **are still valid when our new definition of limit is used**.

From now on, "limit" will always refer to Def 3.1

Eg B. Show $\lim\limits_{n\to\infty}(\sqrt{n+1}-\sqrt{n}) = 0$

Sol. $$\left|\sqrt{n+1}-\sqrt{n}\right| = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}};$$

$$\therefore \quad \text{given any } \varepsilon > 0, \quad \frac{1}{2\sqrt{n}} < \varepsilon \quad \text{if} \quad \frac{1}{4n} < \varepsilon^2$$

$$i.e., \quad \text{if} \quad n > \frac{1}{4\varepsilon^2} \equiv N$$

3.2 The uniqueness of limits. The $K-\varepsilon$ Principle.

Can a sequence (a_n) have more than one limit?

Common sense says ***no***:

If there were two different limits L and L', the a_n could not be arbitrarily close to both, since L and L' are at a fixed distance from each other. This is the idea behind the proof of uniqueness theorem for limits. The theorem shows that if (a_n) is convergent, the notation $\lim\limits_{n\to\infty} a_n$ makes sense.

Theorem A (Uniqueness theorem for limits)

A seq (a_n) has at most one limit:

$$a_n \to L \quad \text{and} \quad a_n \to L' \quad \Rightarrow \quad L = L'$$

Pf. By hypothesis,

$$\text{given } \varepsilon > 0, \qquad a_n \underset{\varepsilon}{\approx} L \quad \text{for} \quad n \gg 1$$

$$\&$$

$$a_n \underset{\varepsilon}{\approx} L' \quad \text{for} \quad n \gg 1$$

Therefore, given $\varepsilon > 0$, we can choose some large number k such that

$$L \underset{\varepsilon}{\approx} a_k \underset{\varepsilon}{\approx} L'$$

By the transitive law of "$\approx$", it follows that

$$\text{given } \varepsilon > 0, \qquad L \underset{2\varepsilon}{\approx} L' \quad ---(*)$$

Seen earlier that $(*)$ implies $L = L'$.

Indeed, if $L \neq L'$, choose $\varepsilon = \dfrac{|L - L'|}{2} (> 0)$. Then

$$|L - L'| < 2\varepsilon \quad \text{by } (*)$$

$$\textit{i.e.}, \quad |L - L'| < |L - L'| \quad \otimes$$

Theorem B

$$(a_n) \text{ is inc } \& \ L = \lim_{n\to\infty} a_n \quad \Rightarrow \quad a_n \leq L \quad \text{for all } n$$

$$(a_n) \text{ is dec } \& \ L = \lim_{n\to\infty} a_n \quad \Rightarrow \quad a_n \geq L \quad \text{for all } n$$

Pf. We will prove the 1st assertion only.

Suppose not.

i.e., $\exists$ a term a_N such that $a_N > L$

$$\varepsilon \overset{\text{let}}{=} \frac{a_N - L}{2} (> 0)$$

Since (a_n) is $\uparrow$, we have $a_n - L \geq a_N - L > \varepsilon$ for all $n \geq N$;

$$\therefore \quad |a_n - L| \overset{a_n - L > \varepsilon > 0}{=} a_n - L > \varepsilon \quad \text{for all } n \geq N$$

contradicting the def of $\lim\limits_{n\to\infty} a_n = L$

◈ The $K - \varepsilon$ principle.

Eg Let $a_n = \dfrac{1}{n} + \dfrac{\sin n}{n+1}$. Show $a_n \to 0$

Sol. $$\left|\frac{1}{n} + \frac{\sin n}{n+1}\right| \leq \frac{1}{n} + \frac{|\sin n|}{n+1} \leq \frac{1}{n} + \frac{1}{n+1}$$

So, given $\varepsilon > 0$,

$$\frac{1}{n} < \varepsilon \quad \text{for } n > \frac{1}{\varepsilon} \quad \& \quad \frac{1}{n+1} < \varepsilon \quad \text{for } n > \frac{1}{\varepsilon} - 1$$

$$\therefore \quad \frac{1}{n} + \frac{1}{n+1} < 2\varepsilon \quad \textit{for} \quad n > \frac{1}{\varepsilon} \quad (\text{note here that 2 does not depend on } n \ \& \ \varepsilon)$$

Set $\varepsilon' = \varepsilon/2$ & work with ε' instead of ε $\Rightarrow$

$$\left|\frac{1}{n} + \frac{\sin n}{n+1}\right| < 2\varepsilon', \quad \text{for } n > \frac{1}{\varepsilon'}$$

Since $2\varepsilon' = \varepsilon$,

$$\left|\frac{1}{n} + \frac{\sin n}{n+1}\right| < \varepsilon, \quad \text{for } n > \frac{1}{\varepsilon/2} = \frac{2}{\varepsilon} \qquad \text{---}//$$

"Conclusion"

The $K-\varepsilon$ principle

Suppose that (a_n) is a given seq, and we can prove that

$$\text{given any } \varepsilon > 0, \qquad a_n \underset{K\varepsilon}{\approx} L \quad \text{for} \quad n \gg 1,$$

where $K > 0$ is a fixed constant (i.e., a number not depending on n or ε).

Then $\lim_{n\to\infty} a_n = L$.

3.3 Infinite limits

Even though ∞ is not a number, it is convenient to allow it as a sort of **"limit"** in describing sequences which become and remain arbitrarily large as n increases.

Def. We say that the seq (a_n) tends to infinity (∞) if

$$\text{given any } M > 0, \qquad a_n > M \quad \text{for} \quad n \gg 1.$$

$$(\text{The } N \text{ concealed in "for } n \gg 1\text{" depends on } M)$$

In symbols, $\lim_{n\to\infty} a_n = \infty$ or $a_n \to \infty$ as $n \to \infty$

Eg A. Do the following seqs tend to ∞ ?

(i) $(a_n)_1^\infty = 1,\ 10,\ 2,\ 20,\ 3,\ 30,\ \cdots, k,\ 10k,\ \cdots$

(ii) $(a_n) = 1,\ 2,\ 1,\ 3,\ 1,\ 4, \cdots, 1,\ k,\ \cdots$

Sol (i)

$$a_n = \begin{cases} 5n & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore \quad \text{given } M > 0, \quad a_n > M \quad \text{if} \quad \frac{n+1}{2} > M \quad (\text{i.e., if } n > 2M - 1)$$

$$\therefore \quad \lim_{n\to\infty} a_n = \infty$$

(ii) Take $M = 10$. It is clear that $a_n < 10 = M$ for any odd integer $n (\geq 1)$

$$\therefore \quad \lim_{n\to\infty} a_n \neq \infty$$

Eg B. Show that $\lim_{n\to\infty} \ln n = \infty$

Pf. $\ln x$ is strictly $\uparrow$ for $x > 0$

i.e., $\ln a > \ln b$ if $a > b$

$\therefore$ given $M > 0$, $\ln n > \ln(e^M) = M$ if $n > e^M$

<HS> (i) Formulate a definition for $\lim_{n\to\infty} a_n = -\infty$

(ii) Prove $\lim_{n\to\infty} \ln\left(\frac{1}{n}\right) = -\infty$

Ans to (i): Given $M > 0$, $a_n < -M$ for $n \gg 1$

3.4 An important limit

Theorem

$$\lim_{n\to\infty} a^n = \begin{cases} \infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } |a| < 1 \end{cases}$$

Pf. The case $a > 1$;

$a > 1 \Rightarrow$ we can write $a = 1 + k (k > 0)$

$$\therefore \quad a^n = (1+k)^n$$

$$\overset{\text{Binomial thm}}{=} 1 + nk + \underbrace{\frac{n(n-1)}{2!}k^2 + \frac{n(n-1)(n-2)}{3!}k^3 + \cdots + k^n}_{\geq 0}$$

$$\geq 1 + nk > nk > (\text{given any})M \quad \text{if } n > \frac{M}{k} (= \text{depends on } M)$$

$$\therefore \quad \lim_{n\to\infty} a^n = \infty \quad \text{if } a > 1$$

The case $a = 1$; Obviously, $\lim_{n\to\infty} a^n = \lim_{n\to\infty} 1 = 1$

The case $|a| < 1$;

$$|a| < 1 \Rightarrow \frac{1}{|a|} > 1 \overset{\text{case1}}{\Rightarrow} \left(\frac{1}{|a|}\right)^n \to \infty$$

$\therefore$ given $\varepsilon > 0$, $\left(\frac{1}{|a|}\right)^n > \frac{1}{\varepsilon}$ for $n \gg 1$

i.e., given $\varepsilon > 0$, $|a^n| < \varepsilon$ for $n \gg 1$

$$\therefore \quad a^n \to 0$$

3.5 Writing limit proofs

Wrong	Right
$a_n \to 0 \quad \text{for } n \gg 1$	$a_n \to 0 \quad as \quad n \to \infty$
$\lim\limits_{n\to\infty} 2^n = \infty \quad \text{for } n \gg 1$	$\lim\limits_{n\to\infty} 2^n = \infty$
$\lim\limits_{n\to\infty} \frac{1}{n} = 0 \quad \text{if } n > \frac{1}{\varepsilon}$	$\frac{1}{n} \underset{\varepsilon}{\approx} 0 \quad \text{if} \quad n > \frac{1}{\varepsilon}$

⦿ 'given $\varepsilon > 0$' or 'given $M > 0$' must come first.

3.6 Some limits involving integrals

Eg A. Let $a_n = \int_0^1 (x^2+2)^n \, dx$. Show that $\lim\limits_{n\to\infty} a_n = \infty$

(참고: 적분을 직접 계산하는 것은 복잡하므로 다른 전략이 요구됨)

Sol. $x^2 + 2 \ge 2 \quad \text{for all } x$

$\therefore \quad (x^2+2)^n \ge 2^n \quad \text{for all } x \quad \& \quad \text{all } n \ge 0$

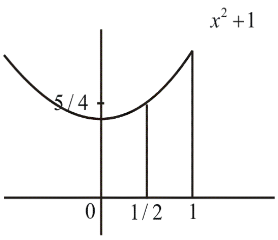
$\therefore \quad \int_0^1 (x^2+2)^n \, dx \ge \int_0^1 2^n \, dx = 2^n \quad \to \infty \quad \text{as} \quad n \to \infty$

$\therefore \quad \lim\limits_{n\to\infty} \int_0^1 (x^2+2)^n \, dx = \infty$

Or, given any $M > 0$, $\int_0^1 (x^2+2)^n \, dx \ge 2^n > M$, for $n > \log_2 M$

Eg B. Show $\lim\limits_{n\to\infty} \int_0^1 (x^2+1)^n \, dx = \infty$

Pf. The previous argument gives the estimate $(x^2+1)^n \ge 1^n = 1$ for all x, which is useless. It may be modified as follows.



$x^2 + 1$ has the value $\frac{5}{4} \equiv A$ at the point $x_0 = \frac{1}{2}$

Since x^2+1 is $\uparrow$ on $[0, 1]$, $x^2+1 \ge A > 1$ for $\frac{1}{2} \le x \le 1$

$\therefore \quad (x^2+1)^n \ge A^n \quad \text{for } \frac{1}{2} \le x \le 1$

$\therefore \quad \int_0^1 (x^2+1)^n \, dx \ge \int_{1/2}^1 (x^2+1)^n \, dx \ge \int_{1/2}^1 A^n \, dx = \frac{A^n}{2} \xrightarrow{\text{as } n\to\infty} \infty \quad \text{since } A > 1$

$\therefore \quad \lim\limits_{n\to\infty} \int_0^1 (x^2+1)^n \, dx = \infty$

Home Study. Determine $\lim\limits_{n\to\infty} \int_0^1 (x^2+0.1)^n \, dx$. Ans ∞

3.7 Another limit involving an integral

※ Eg. Let $a_n = \int_0^{\pi/2} \sin^n x \, dx$. Determine $\lim_{n\to\infty} a_n$

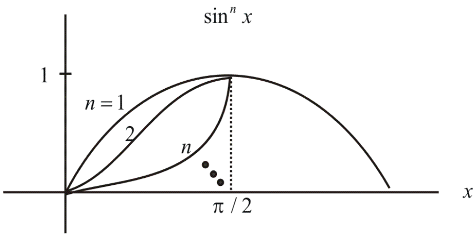
Observations:

① It is not so easy to calculate the integral explicitly.

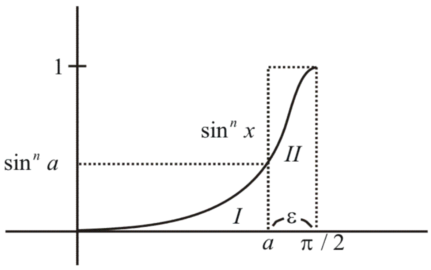
(Hint: show first that $a_n = \dfrac{n-1}{n} a_{n-2}$)

② The integral represents an area, so it helps to have some idea of how the curves $\sin^n x$ look.

Since $0 \le \sin x < 1$ on the interval $0 \le x < \pi/2$, we get $\sin^n x \to 0$.
Thus as n increases, the successive curves get closer and closer to the $x-$axis, except at the right point $x = \pi/2$



The area under the curve seems to get small, as n increases, so the limit should be 0.
How do we prove this?



Given (small) $\varepsilon > 0$, let $a = \pi/2 - \varepsilon$.

Obviously, $\text{area}(II) = \int_a^{\pi/2} \sin^n x \, dx < \varepsilon$ (for every $n \ge 1$)

On the other hand, $\sin^n a < \varepsilon$ for $n \gg 1$ since $|\sin a| < 1$ ($\because a < \pi/2$)

$$\therefore \quad \text{area}(I) < a \cdot \sin^n a \le \frac{\pi}{2} \sin^n a < \frac{\pi}{2}\varepsilon < 2\varepsilon \quad \text{for } n \gg 1$$

$$\therefore \quad \int_0^{\pi/2} \sin^n x \, dx = \text{total area under the curve}$$

$$= \text{area}(I) + \text{area}(II) < \varepsilon + 2\varepsilon = 3\varepsilon \quad \text{for } n \gg 1$$

Thus by $K-\varepsilon$ principle, $\lim_{n\to\infty} \int_0^{\pi/2} \sin^n x \, dx = 0$

Ex. Let $a_n = \int_0^{\pi/2} e^{-n\sin\theta}\, d\theta$. Show that $\lim_{n\to\infty} a_n = 0$.

Observation:

Troublesome: We do not know the primitive of the integrand $e^{-n\sin\theta}$

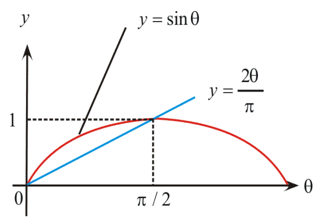
Expect:

$$\lim_{n\to\infty}\int_0^{\pi/2} e^{-n\sin\theta}\, d\theta \overset{??}{=} \int_0^{\pi/2} \lim_{n\to\infty} e^{-n\sin\theta}\, d\theta = \int_0^{\pi/2} \begin{Bmatrix} e^{-\infty}=0 & \text{if } 0<\theta\le \pi/2 \\ 1 & \text{if } \theta=0 \end{Bmatrix} d\theta = 0$$

To attack this problem, we need

Jordan's inequality: $\dfrac{2\theta}{\pi} \le \sin\theta \;(\le \theta)$ for $0 \le \theta \le \pi/2$

(Geometric) Pf of the Jordan's inequality: See the figure below



Pf of Ex.

$$0 \le \int_0^{\pi/2} e^{-n\sin\theta}\, d\theta \overset{\text{Jordan's inequality}}{\le} \int_0^{\pi/2} e^{-\frac{2n\theta}{\pi}}\, d\theta = \left[\frac{e^{-\frac{2n\theta}{\pi}}}{-\frac{2n}{\pi}}\right]_{\theta=0}^{\theta=\pi/2}$$

$$= \frac{1}{\frac{2n}{\pi}}\left(1-e^{-n}\right) = \frac{\pi}{2n}\left(1-e^{-n}\right) \le \frac{\pi}{2n} \to 0 \quad \text{as } n\to\infty$$

Consequently,

$$\lim_{n\to\infty}\int_0^{\pi/2} e^{-n\sin\theta}\, d\theta = 0.$$

Remark. The same argument shows that

$$\lim_{R\to\infty}\int_0^{\pi/2} e^{-aR\sin\theta}\, d\theta \;(a>0) = 0.$$

Home Study. Give an alternative proof of $\lim_{n\to\infty}\int_0^{\pi/2} \sin^n x\, dx = 0$

Hint: $a_n := \int_0^{\pi/2} \sin^n x\, dx \quad\Rightarrow\quad a_n = \dfrac{n-1}{n} a_{n-2}$