해석학2 기말고사 (타전공생용) 2021. 12. 07

배점: 5,7번 각 15점; 나머지 각 문항 10점

주의사항: 문제의 증명 또는 풀이과정을 상세히 기술하시오

1. For what values of p is the improper integral $\int_0^\infty \frac{x^p}{1+x^p}dx$ convergent?

2. Is the sequence of functions $f_n(x) = \begin{cases} \frac{1}{x}\sin\frac{x}{n} & \text{if } x \neq 0 \\ \frac{1}{n} & \text{if } x = 0 \end{cases}$ uniformly convergent on $[0,\infty)$?

3. Find the set of all points where the series $\sum_{n=1}^\infty \frac{2^n}{2n^2+n}\left(\frac{x}{1-x}\right)^n$ is uniformly convergent

4. Find the sum of the power series $f(x) = \sum_{n=1}^\infty (-1)^{n+1}\frac{x^{n+1}}{n(n+1)}$ for any $x \in [-1,1]$

5. Let $f(x) = \sum_{n=0}^\infty 5^{-n}\cos(4^n x)$.

(i) Show that $f(x)$ is differentiable on $[0,2\pi]$ and $f'(x)$ is Riemann integrable on $[0,2\pi]$ ---8pt

(ii) Evaluate the Riemann integral $\int_0^{2\pi} f'(x)\,dx$ --- 7pt

6. Is $f(x) = \begin{cases} \sin x & \text{if } x \text{ is a rational number} \\ \cos x & \text{if } x \text{ is a irrational number} \end{cases}$ Riemann-integrable on $[0, \pi / 2]$?

7. Let $f(x) = \begin{cases} \frac{e^{-x}-1}{x}, & x \neq 0 \\ -1, & x = 0 \end{cases}$

(i) Find the n-th order Taylor polynomial $T_n(x)$ at $x=0$ of the function $f(x)$ --- 8pt

(ii) Does $T_n(x)$ converge uniformly to $f(x)$ on $\mathbb{R} = (-\infty,\infty)$? ---7pt

8. Determine each statement is **true** or **false**: 답만 쓰시오 [각 2점, 틀리면 2점씩 **감점**함]

① If f is integrable on $[a,b]$ and $a < x_0 < b$, then $F(x) := \int_a^x f(t)dt$ is differentiable at x_0

② Suppose that $f, g \in \mathcal{R}[a,b]$ and $f = g$ almost everywhere on $[a,b]$

$\Rightarrow \int_a^b f(x)dx = \int_a^b g(x)dx$ (where both are Riemann integrals)

③ The improper integral $\int_0^\infty \sqrt{x}\sin(x^2)\,dx$ is convergent

④ The series $\sum_{n=1}^\infty (-1)^{n+1}\frac{1}{n+x^2}$ does not converge uniformly on $(-\infty, \infty)$

⑤ If every f_n is Riemann- integrable on any compact interval of $[0,\infty)$ and $f_n \rightrightarrows f$ on $[0,\infty)$, then

$$\lim_{n\to\infty}\int_0^\infty f_n(x)e^{-x}\,dx = \int_0^\infty f(x)e^{-x}\,dx$$