1) a) $y \mid \theta=1 \sim N(1, \sigma)$, $y \mid \theta=2 \sim N(2, \sigma)$

$P(\theta=1) = 0.5$, $P(\theta=2) = 0.5$

$\Rightarrow$ Let θ be a random variable following a $B(1, P)$, where $\theta \in \{1, 2\}$

$\Rightarrow$ $P(\theta) = p^{2-\theta}(1-p)^{\theta-1}$

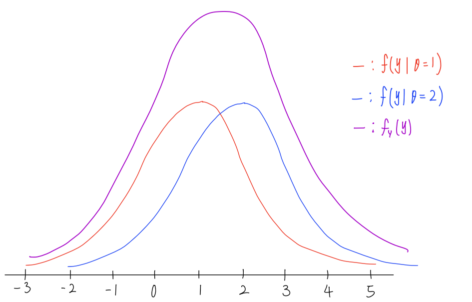
$\Rightarrow$ $f(y \mid \theta=1) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-1)^2}{2\sigma^2}\right)$

$f(y \mid \theta=2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-2)^2}{2\sigma^2}\right)$, where $\sigma = 2$

$\Rightarrow$ $f(y, \theta=1) = f(y \mid \theta=1) P(\theta=1) = \frac{1}{2\sqrt{8\pi}} \exp\left(-\frac{(y-1)^2}{8}\right)$, given $\sigma=2$

$f(y, \theta=2) = f(y \mid \theta=2) P(\theta=2) = \frac{1}{2\sqrt{8\pi}} \exp\left(-\frac{(y-2)^2}{8}\right)$, given $\sigma=2$

$\Rightarrow$ $f_Y(y) = f(y, \theta=1) + f(y, \theta=2) = \frac{1}{2\sqrt{8\pi}} \exp\left(-\frac{(y-1)^2}{8}\right) + \frac{1}{2\sqrt{8\pi}} \exp\left(-\frac{(y-2)^2}{8}\right)$, mixture model



b) $P(y, \theta) = P(y \mid \theta) P(\theta)$, where $P(y \mid \theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-1)^2}{2\sigma^2}\right) I(\theta=1) + \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-2)^2}{2\sigma^2}\right) I(\theta=2)$ and $P(\theta) = p^{2-\theta}(1-p)^{\theta-1}$

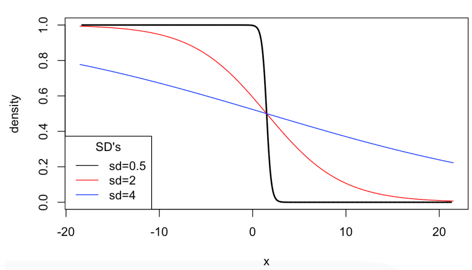
$= p^{2-\theta}(1-p)^{\theta-1}\left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-1)^2}{2\sigma^2}\right) I(\theta=1) + \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-2)^2}{2\sigma^2}\right) I(\theta=2)\right]$

$\Rightarrow$ $P(\theta \mid y) = p^{2-\theta}(1-p)^{\theta-1}\left[\frac{1}{\sqrt{8\pi}} \exp\left(-\frac{(y-1)^2}{8}\right) I(\theta=1) + \frac{1}{\sqrt{8\pi}} \exp\left(-\frac{(y-2)^2}{8}\right) I(\theta=2)\right] \Big/ \left[\frac{1}{2\sqrt{8\pi}} \exp\left(-\frac{(y-1)^2}{8}\right) + \frac{1}{2\sqrt{8\pi}} \exp\left(-\frac{(y-2)^2}{8}\right)\right]$

$P(\theta=1 \mid y=1) = P\left[\frac{1}{\sqrt{8\pi}}\right] \Big/ \left[\frac{1}{2\sqrt{8\pi}} + \frac{1}{2\sqrt{8\pi}} \exp\left(-\frac{1}{8}\right)\right]$, $p = \frac{1}{2}$

$= 0.53121$

c)



$\Rightarrow$ As the standard deviation increases, the graph becomes flatter, which notes that the data has less information.

2) Find $P(M)$ given $P(G\cap G|M)=P$, $P(B\cap B|M)=1-P$, $P(G\cap B|M)=0$, $P(G\cap G|F)=P^2$, $P(B\cap B|F)=(1-P)^2$, $P(G\cap B|F)=2P(1-P)$

$$P(G\cap G) = P(G\cap G\cap M) + P(G\cap G\cap F)$$

$$= P(G\cap G|M)P(M) + P(G\cap G|F)P(F)$$

$$\Rightarrow P(M) = \frac{P(G\cap G) - P(G\cap G|F)P(F)}{P(G\cap G|M)}$$

$\Rightarrow$ $P(G\cap G)$ is unknown since we do not have any information about how associated they are.

$$\Rightarrow P(G\cap G|F)P(F) = P^2, \quad P(G\cap G|M) = P$$

$$\therefore P(M) = \frac{P(G\cap G) - P^2}{P}$$

3) a) Suppose $\theta \sim Beta(\alpha,\beta)$, then $P(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$

$$\Rightarrow P(X,\theta) = P(X|\theta)P(\theta) = \frac{(X+r-1)!}{X!(r-1)!}\theta^{r}(1-\theta)^{X}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$$= \frac{1}{\beta(X,r-1)}\cdot\frac{1}{\beta(\alpha,\beta)}\theta^{r+\alpha-1}(1-\theta)^{X+\beta-1} \propto \theta^{r+\alpha-1}(1-\theta)^{X+\beta-1}$$

$\Rightarrow P(\theta|X) \sim Gamma(r+\alpha, X+\beta)$ $\therefore$ $P(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$ is a conjugate prior

b) Let $\beta \sim \Gamma(d,C)$, then $P(\beta) = \frac{c^d}{\Gamma(d)}\beta^{d-1}e^{-d\beta}$

$$\Rightarrow P(X,\beta) = P(X|\beta)P(\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}X^{\alpha-1}e^{-\beta X}\frac{c^d}{\Gamma(d)}\beta^{d-1}e^{-d\beta}$$

$$\Rightarrow P(X) = \int_0^\infty P(X|\beta)P(\beta)\,d\beta$$

$$= \int_0^\infty \frac{\beta^{\alpha}}{\Gamma(\alpha)}X^{\alpha-1}e^{-\beta X}\frac{c^d}{\Gamma(d)}\beta^{d-1}e^{-d\beta}$$

$$= \frac{X^{\alpha-1}c^d}{\Gamma(\alpha)\Gamma(d)}\frac{\Gamma(\alpha+d)}{(X+d)^{\alpha+d}}\int_0^\infty \frac{(X+d)^{\alpha+d}}{\Gamma(\alpha+d)}\beta^{\alpha+d-1}e^{-\beta(X+d)}\,d\beta$$

$$= \frac{\Gamma(\alpha+d)}{\Gamma(\alpha)\Gamma(d)}\frac{1}{(X+d)^{\alpha+d}}X^{\alpha-1}c^d$$

$$\Rightarrow P(\beta|X) = \frac{P(X,\theta)}{P(X)} = \frac{\beta^{\alpha}}{\cancel{\Gamma(\alpha)}}\cancel{X^{\alpha-1}}e^{-\beta X}\frac{\cancel{c^d}}{\cancel{\Gamma(d)}}\beta^{d-1}e^{-d\beta}\Big/\frac{\Gamma(\alpha+d)}{\cancel{\Gamma(\alpha)}\cancel{\Gamma(d)}}\frac{1}{(X+d)^{\alpha+d}}\cancel{X^{\alpha-1}}\cancel{c^d}$$

$$= \frac{\beta^{\alpha+d-1}}{\Gamma(\alpha+d)}(X+d)^{\alpha+d}e^{-(X+d)\beta} \propto \Gamma(\alpha+d, X+d)$$

$$\therefore P(\beta) = \frac{c^d}{\Gamma(d)}\beta^{d-1}e^{-d\beta} \text{ is a conjugate prior}$$

c) 포아송 분포, 지수 분포, 이항분포, 감마분포, 베타분포 등등 모두 해보았으나 나오지 않음.

4) a) Let X be the number of Californians supporting the death penalty, then $P(X|\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x} I(0 \le x \le n)$.

$\Rightarrow$ By the given conditions, $\theta \sim Beta(\alpha, \beta)$, so $P(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$

Given that $E(\theta) = 0.6$, $Var(\theta) = 0.09$, it is notable $\frac{\alpha}{\alpha+\beta} = 0.6$ and $\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2} = 0.09$

$\Rightarrow$ $\alpha = \frac{3}{2}\beta$ $\Rightarrow$ $\frac{\frac{3}{2}\beta^2}{(\frac{3}{2}\beta+\beta+1)(\frac{3}{2}\beta+\beta)^2} = \frac{\beta^2}{\left(\frac{5\beta+2}{3}\right)\left(\frac{25\beta^2}{4}\right)} = \frac{12\beta^2}{125\beta^3+50\beta^2} = \frac{9}{100}$

$$1200\beta^2 = 1125\beta^3 + 450\beta^2$$

$$0 = 1125\beta^3 - 750\beta^2$$

$$= \beta^2(9\beta - 6) \quad \therefore \beta = \frac{2}{3}, \alpha = 1$$

b) $P(X|\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x} I(0 \le x \le n)$, $P(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$

$\Rightarrow$ $P(\theta|X) \propto \theta^{x+\alpha-1}(1-\theta)^{n-x+\beta-1}$

$\Rightarrow$ $\theta|X \sim Beta(x+\alpha, n-x+\beta)$, $E(\theta|X) = \frac{x+\alpha}{n+\alpha+\beta}$, $Var(\theta|X) = \frac{(x+\alpha)(n-x+\beta)}{(n+\alpha+\beta)^2(n+\alpha+\beta+1)}$

It is given that $n = 1000$, and $\hat{\theta} = 0.65$ implies $x = 650$, and using the results of α, β from (a),

$$E(\theta|X) = \frac{x+\alpha}{n+\alpha+\beta} = \frac{650+1}{1000+1+\frac{2}{3}} = 0.64992$$

$$Var(\theta|X) = \frac{(x+\alpha)(n-x+\beta)}{(n+\alpha+\beta)^2(n+\alpha+\beta+1)} = \frac{(650+1)(1000-650+\frac{2}{3})}{(1000+1+\frac{2}{3})^2(1000+1+\frac{2}{3}+1)} = 0.000227$$