Chap. 20 Derivatives and Integrals

20.1 First fundamental theorem of calculus

Thm (1st FTC)

Assume that on $[a, b]$, $F(x)$ is diff & $F'(x) = f(x)$ ($f(x)$: a given ft) is integrable

$$\Rightarrow \quad \int_a^b f(x)\,dx = F(b) - F(a) \qquad \text{i.e.,} \quad \int_a^b F'(x)\,dx = F(b) - F(a)$$

(Any such $F(x)$ is called an antiderivative of a given integrand $f(x)$)

Pf. Let $\mathcal{P} : a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ be any partition of $[a, b]$.

Since $F(x)$ is diff on each subinterval $[\Delta x_i]$,

$$\begin{aligned} F(x_i) - F(x_{i-1}) &\overset{\text{MVT}}{=} F'(c_i)\Delta x_i, \quad c_i \in [\Delta x_i] \\ &= f(c_i)\Delta x_i \end{aligned}$$

$$\therefore \quad \sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n} \big(F(x_i) - F(x_{i-1})\big) \overset{\text{telescoping}}{=} F(x_n) - F(x_0)$$

$$\therefore \quad \sum_{i=1}^{n} f(c_i)\Delta x_i = F(b) - F(a) \quad \text{---}(\star)$$

Now, consider the sequence of standard n-partitions $\mathcal{P}^{(n)}$ of $[a, b]$.

Then, since f is integrable on $[a,b]$ and $|\,\mathcal{P}^{(n)}\,| = \dfrac{b-a}{n} \to 0$, we have

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(c_i)\Delta x_i}_{(*)} \qquad ((*) \text{ is a special Riemann sum constructed above})$$

$$\overset{(\star)}{=} \lim_{n\to\infty} \big(F(b) - F(a)\big) = F(b) - F(a)$$

Caution: Not every derivative is Riemann-integrable.

For example, take $F(x) = \begin{cases} x^2 \sin\dfrac{1}{x^2} & \text{if } x \in (0, 1] \\ 0 & \text{if } x = 0 \end{cases}$. Then we easily check that

$$F'(x) = \begin{cases} 2x \sin\dfrac{1}{x^2} - \dfrac{2}{x}\cos\dfrac{1}{x^2} & \text{if } x \in (0, 1] \\ 0 & \text{if } x = 0 \end{cases}$$

Thus F is differentiable on $[0, 1]$, but $F'(x) \notin \mathcal{R}[0, 1]$ since F' is not bounded on $[0, 1]$.

Remark. Evaluate $\int_0^1 f(x)\,dx$, where $f(x)=\begin{cases}2x\sin\dfrac{1}{x}-\cos\dfrac{1}{x} & \text{if } x\in(0,1]\\ 0 & \text{if } x=0\end{cases}$.

(Note that $f(x)$ is continuous on $(0,1]$, but not continuous at $x=0$ since $\lim\limits_{x\to 0} f(x)$ does not exist. However, $f(x)$ is clearly bounded on $[0,1]$. Therefore, $f\in\mathcal{R}[0,1]$)

Ans: $\int_0^1 f(x)\,dx=\sin 1$ --- seen in the last paragraph of the Chapter 19

20.2 Existence and "uniqueness" of antiderivatives

Problem: What is the corresponding statement for 1st FTC if we do not know $F(x)$ for which $F'(x)=f(x)$?

For example, we do not know $F(x)$ such that $F'(x)=\sin(x^2)$ or $F'(x)=\dfrac{\sin x}{x}$

Thm A (2nd FTC)

Let $f(x)$ be continuous on an interval I, and let $a\in I$.

Set $F(x)=\int_a^x f(t)\,dt$ for all $x\in I$. Then

$F(x)$ is diff on I, and $F'(x)=f(x)$ for all $x\in I$.

(Conclusion: every continuous function has an antiderivative)

Pf. Enough to prove: $\lim\limits_{\Delta x\to 0}\dfrac{\Delta F}{\Delta x}=f(x)$

$$\begin{aligned}\Delta F=F(x+\Delta x)-F(x)&=\int_a^{x+\Delta x} f(t)\,dt-\int_a^x f(t)\,dt\\ &=\left[\cancel{\int_a^x f(t)\,dt}+\int_x^{x+\Delta x} f(t)\,dt\right]-\cancel{\int_a^x f(t)\,dt}\\ &=\int_x^{x+\Delta x} f(t)\,dt\end{aligned}$$

Since $f(t)$ is continuous at x ,

$$\boxed{(*):\quad \text{given } \varepsilon>0,\quad f(x)-\varepsilon<f(t)<f(x)+\varepsilon\quad \text{for } t\approx x}$$

Case 1. $\Delta x>0$

By $(*)$, $\int_x^{x+\Delta x}(f(x)-\varepsilon)\,dt\le\int_x^{x+\Delta x} f(t)\,dt\le\int_x^{x+\Delta x}(f(x)+\varepsilon)\,dt$ for $\Delta x\approx 0^+$

$\therefore\quad (f(x)-\varepsilon)\Delta x\le\Delta F\le(f(x)+\varepsilon)\Delta x\quad$ for $\Delta x\approx 0^+$

$\therefore\quad (f(x)-\varepsilon)\le\dfrac{\Delta F}{\Delta x}\le(f(x)+\varepsilon)\quad$ for $\Delta x\approx 0^+$

Since $\varepsilon>0$ was arbitrary, $\lim\limits_{\Delta x\to 0^+}\dfrac{\Delta F}{\Delta x}=f(x)$

Case 2. $\Delta x < 0 \quad (\Rightarrow x + \Delta x < x)$

Recall: $\boxed{a > b,\ \ f(t) \le g(t) \quad \Rightarrow \quad \int_a^b f(t)\ dt \ge \int_a^b g(t)\ dt}$. Thus

$$\int_x^{x+\Delta x} (\, f(x) - \varepsilon \,)\ dt \ge \int_x^{x+\Delta x} f(t)\ dt \ge \int_x^{x+\Delta x} (\, f(x) + \varepsilon \,)\ dt \quad \text{for } \Delta x \approx 0^-$$

$$\therefore \qquad (\, f(x) - \varepsilon \,)\,\Delta x \ge \Delta F \ge (\, f(x) + \varepsilon \,)\,\Delta x \quad \text{for } \Delta x \approx 0^-$$

$$\overset{\Delta x < 0}{\Rightarrow} \qquad (\, f(x) - \varepsilon \,) \le \frac{\Delta F}{\Delta x} \le (\, f(x) + \varepsilon \,) \quad \text{for } \Delta x \approx 0^-$$

Since $\varepsilon > 0$ was arbitrary, $\displaystyle \lim_{\Delta x \to 0^-} \frac{\Delta F}{\Delta x} = f(x)\,.$

Consequently, we have $\displaystyle \lim_{\Delta x \to 0} \frac{\Delta F}{\Delta x} = f(x).$

Alternative pf (without using $\varepsilon - \delta$ approach: for High-School Math. Teachers)

We first show $\displaystyle \lim_{\Delta x \to 0^+} \frac{\Delta F}{\Delta x} = f(x)\,.$

Assume $\Delta x > 0$, and let $m = f(x_1)$ and $M = f(x_2)$ be the min and max of $f(t)$ on $[x, x + \Delta x]$, where $x \le x_1, x_2 \le x + \Delta x$. (Existence of such x_1 and x_2 is guaranteed by the continuity of f)

Since $f(t)$ is continuous, $\lim_{\Delta x \to 0^+} f(x + \Delta x) = f(x)$, and thus

$$\lim_{\Delta x \to 0^+} f(x_1) = f(x) \quad \text{and} \quad \lim_{\Delta x \to 0^+} f(x_2) = f(x)$$

Clearly, we have $m = f(x_1) \le f(t) \le f(x_2) = M$ for $x \le t \le x + \Delta x$

Hence

$$\int_x^{x+\Delta x} m\ dt \le \int_x^{x+\Delta x} f(t)\ dt \le \int_x^{x+\Delta x} M\ dt \qquad \text{i.e.,} \quad m\Delta x \le \Delta F \le M\Delta x$$

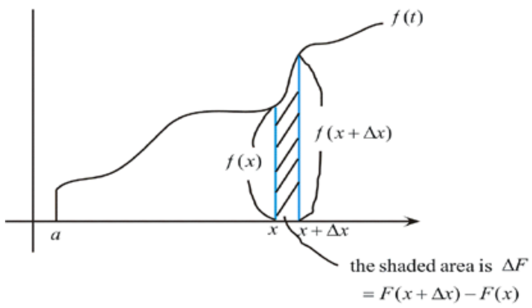
$$\begin{array}{cccccc} \therefore & m & \le & \dfrac{\Delta F}{\Delta x} & \le & M \\ & \downarrow & & & & \downarrow \quad (\text{as } \Delta x \to 0^+) \\ & f(x) & & & & f(x) \end{array}$$

$$\therefore \quad \lim_{\Delta x \to 0^+} \frac{\Delta F}{\Delta x} = f(x)$$

Similarly, we can show that $\displaystyle \lim_{\Delta x \to 0^-} \frac{\Delta F}{\Delta x} = f(x)$ (Ex)

Therefore, $\displaystyle F'(x) = \lim_{\Delta x \to 0} \frac{\Delta F}{\Delta x} = f(x).$

- Geometric meaning of the 2nd FTC



$F(x) = \int_a^x f(t)\,dt$: represents the cumulative area under f between a and x

$$\Delta F = F(x+\Delta x) - F(x) \approx f(x)\Delta x \quad \text{if} \quad \Delta x \approx 0$$

$$\therefore \quad \frac{\Delta F}{\Delta x} \approx f(x) \quad \text{if} \quad \Delta x \underset{\neq}{\approx} 0$$

$$\text{Indeed,} \quad \lim_{\Delta x \to 0} \frac{\Delta F}{\Delta x} = f(x)$$

Remark.

$$f(x) \text{ is continuous on } I = [a, b]$$

$$F(x) = \int_a^x f(t)\,dt \quad \text{for all } x \in I$$

$$\overset{\text{can check}}{\Rightarrow} \quad F'(a^+) = f(a) \quad \text{and} \quad F'(b^-) = f(b).$$

Thm B (Uniqueness thm for antiderivatives)

Let $F(x)$ and $G(x)$ be diff on an interval I. Then on I

$$G'(x) = F'(x) \quad \Rightarrow \quad G(x) = F(x) + c, \quad \text{for some constant } c$$

(That is, antiderivative of f is unique up to an additive constant)

Pf. $G'(x) - F'(x) = 0$ on an interval I

$$\Rightarrow \quad \big(G(x) - F(x)\big)' = 0 \quad \text{on } I$$

$$\Rightarrow \quad G(x) - F(x) = c \text{ (constant)} \quad \text{by Theorem 15.2 (5)}$$

Remark. The result is false if the domain is not connected (i.e., if the domain is the union of two or more disjoint intervals). For example,

$$F(x) := \begin{cases} 1, & x \in [0, 1] \\ 2, & x \in [2, 3] \end{cases}$$

$$\Rightarrow \quad F'(x) = 0 \quad \forall x \in [0, 1] \cup [2, 3]. \quad \text{However,} \quad F(x) \neq \text{constant}$$

Cor A (Existence and uniqueness thm for $y' = f(x)$)

Let $f(x)$ be continuous on an interval I, and let $a \in I$.

Then the differential equation with initial condition

$$\boxed{(\star): \quad y' = f(x), \quad y(a) = b}$$

has in I the unique solution $y = F(x)$, where $F(x) = b + \int_a^x f(t)\,dt$

Pf. $F(a) = b$ is obvious, and $F'(x) = f(x) \;\; \forall x \in I$ by 2nd FTC

$\therefore \;\; F(x)$ is a solution of $(\star)$.

If $F_1(x)$ is any other solution of $(\star)$, then

$$F_1'(x) = f(x) = F'(x) \qquad \forall x \in I$$

$$\therefore \;\; F_1(x) = F(x) + c \quad \text{by Thm B}$$

But, since $F_1(a) = b = F(a) + c = b + c$, we get $c = 0$.

Therefore, $F_1(x) = F(x) \;\; \forall x \in I$

Cor B (2nd FTC $\Rightarrow$ 1st FTC if the integrand $f(x)$ is continuous on $[a, b]$)

Assume that on $[a, b]$, $F(x)$ is an antiderivative of a continuous ft $f(x)$

$$\Rightarrow \int_a^b f(t)\,dt = F(b) - F(a)$$

Pf. Let $G(x) = \int_a^x f(t)\,dt$. Then by 2nd FTC,

$$G'(x) = f(x) \overset{\text{Hypo}}{=} F'(x) \quad \text{on } [a, b]$$

$\therefore \;\; G(x) = F(x) + c$ for some constant c

i.e., $\int_a^x f(t)\,dt = F(x) + c$

Setting $x = a \;\; \Rightarrow \;\; c = -F(a)$

$\therefore \;\; \int_a^x f(t)\,dt = F(x) - F(a)$

Finally setting $x = b \;\; \Rightarrow \;\; \int_a^b f(t)\,dt = F(b) - F(a)$

Note.	1st FTC:	first differentiate and then integrate
	2nd FTC:	first integrate and then differentiate

20.3 Other relations between derivative and integrals

Notation: $F(x)\big]_a^b = F(b) - F(a)$

Thm A (Integration by parts)

If $u'(x)$ and $v'(x)$ are conti on $[a, b]$, then

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\big]_a^b - \int_a^b u'(x)v(x)\,dx$$

Pf. Ex

Thm B (Change of variable rule)

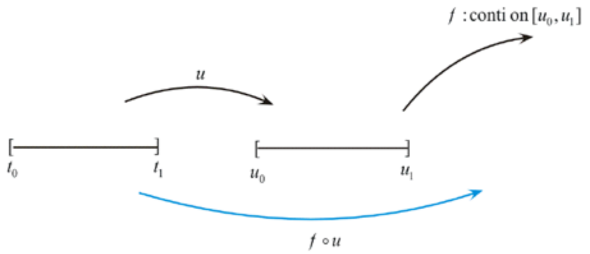
Suppose $u(t)$ is a continuously diff function which maps $[t_0, t_1]$ to $[u_0, u_1]$. That is,

(a) $u : [t_0, t_1] \to [u_0, u_1]$ and $u(t_0) = u_0,\ u(t_1) = u_1$

(b) $u'(t)$ exists and is conti on $[t_0, t_1]$.

Assume $f(u)$ is a conti function on $[u_0, u_1]$. Then

$$\int_{u_0}^{u_1} f(u)\,du = \int_{t_0}^{t_1} f(u(t))\,u'(t)\,dt$$



Pf. Let $F(u)$ be an antiderivative of $f(u)$ on $[u_0, u_1]$. Then

$$\begin{aligned}\int_{u_0}^{u_1} f(u)\,du &= F(u_1) - F(u_0) \quad \text{by the 1st FTC}\\ &= F(u(t_1)) - F(u(t_0))\\ &= \int_{t_0}^{t_1} \frac{d}{dt}F(u(t))\,dt \quad \text{by the 1st FTC}\\ &\overset{\frac{d}{dt}F(u(t))=\frac{d}{du}F(u)\frac{du}{dt}=f(u(t))\frac{du}{dt}}{=} \int_{t_0}^{t_1} f(u(t))\,\frac{du}{dt}\,dt\\ &= \int_{t_0}^{t_1} f(u(t))\,u'(t)\,dt\end{aligned}$$

Ex. Evaluate $\displaystyle\int_0^{1/2} \frac{tdt}{(1-t^2)^2}$

Sol. $u = 1-t^2 =: u(t) \quad\Rightarrow\quad u'(t) = -2t$: continuous on $[0,1/2]$

$$\int_0^{1/2} \frac{tdt}{(1-t^2)^2} = -\frac{1}{2}\int_0^{1/2} \frac{u'(t)dt}{u(t)^2} \overset{u=u(t)\text{ plus Theorem }B}{=} -\frac{1}{2}\int_1^{3/4} \frac{du}{u^2} = \frac{1}{2}\int_{3/4}^{1} \frac{du}{u^2} = \frac{1}{2}\left[-\frac{1}{u}\right]_{3/4}^{1} = \frac{1}{6}$$

Another form of change of variables formula [commonly used]

Let φ be diff on $[a,b]$ with $\varphi'(x)\neq 0\ \forall x\in[a,b]$, and let f be continuous on $I:=\varphi[a,b]$

$$\Rightarrow \int_a^b f(\varphi(x))dx=\int_{\varphi(a)}^{\varphi(b)} f(t)(\varphi^{-1})'(t)dt$$

Pf. Note that

$t:=\varphi(x)$ is diff on $[a,b]$ with $\varphi'(x)\neq 0\ \forall x\in[a,b] \overset{\text{Darboux}}{\Rightarrow}$ Either $\varphi'(x)>0$ or $\varphi'(x)<0$ on $[a,b]$

$$\therefore\ \varphi \text{ has an inverse } \varphi^{-1}(t)\ \ \&\ \ (\varphi^{-1})'(t)=\frac{1}{\varphi'(\varphi^{-1}(t))}=\frac{1}{\varphi'(x)}(\neq 0 \text{ by hypo})$$

Hence

$$\int_{\varphi(a)}^{\varphi(b)} f(t)(\varphi^{-1})'(t)dt \overset{t=\varphi(x)}{=} \int_a^b f(\varphi(x))\frac{1}{\varphi'(x)}\varphi'(x)dx=\int_a^b f(\varphi(x))dx$$

Remark: The above hypothesis '$\varphi'(x)\neq 0\ \forall x\in[a,b]$' may be slightly weakened as follows:

- Let φ be diff on $[a,b]$ with $\varphi'(x)\neq 0\ \forall x\in[a,b)$, and let f be conti on $I:=\varphi[a,b]$

  $$\Rightarrow \int_a^b f(\varphi(x))dx=\int_{\varphi(a)}^{\varphi(b)} f(t)(\varphi^{-1})'(t)dt$$

Idea: $\displaystyle\int_a^b f(\varphi(x))dx=\lim_{\varepsilon\to 0^+}\int_a^{b-\varepsilon} f(\varphi(x))dx$

- Let φ be diff on $[a,b]$ with $\varphi'(x)\neq 0\ \forall x\in[a,b]\setminus\{c\}\ (\text{with } a<c<b)$, and let f be conti on $I:=\varphi[a,b]$

  $$\Rightarrow \int_a^b f(\varphi(x))dx=\int_{\varphi(a)}^{\varphi(b)} f(t)(\varphi^{-1})'(t)dt$$

Idea: $\displaystyle\int_a^b f(\varphi(x))dx=\lim_{\varepsilon\to 0^+}\int_a^{c-\varepsilon} f(\varphi(x))dx+\lim_{\varepsilon\to 0^+}\int_{c+\varepsilon}^{b} f(\varphi(x))dx$

Ex. Evaluate $\displaystyle\int_1^4\frac{dx}{1+\sqrt{x}}$

Sol. Set $\varphi(x)=\sqrt{x} \Rightarrow \varphi'(x)=\dfrac{1}{2\sqrt{x}}>0\ (\therefore\ \neq 0)$ for $x\in[1,4]$ and $\varphi([1,4])=[1,2]$

Note also that $f(t):=\dfrac{1}{1+t}$ is continuous on $[1,2]=\varphi([1,4])$. Hence

$$\int_1^4\frac{dx}{1+\sqrt{x}}=\int_1^4\frac{dx}{1+\varphi(x)}\left[=\int_1^4 f(\varphi(x))dx\right]\overset{\varphi(x)=\sqrt{x}=t}{=}\int_1^2\frac{2tdt}{1+t}$$

$$=2\int_1^2\frac{t+1-1dt}{1+t}=2\int_1^2\left[1-\frac{1}{1+t}\right]dt=2\left[t-\ln(1+t)\right]_1^2=2\left[1-\ln\frac{3}{2}\right]$$

Note: $\varphi(x)=\sqrt{x}=t \Rightarrow x=t^2=(\varphi^{-1})(t)$ and $(\varphi^{-1})'(t)=2t\neq 0$ on $[1,2]=\varphi([1,4])$

HS: Evaluate $\displaystyle\int_{-1}^1\frac{1}{1+x^2}dx$ and $\displaystyle\int_{-1}^1\sqrt{1-x^2}\,dx$

20.4 Another look at $\ln x$ and e^x **(Home study: Carefully read each definition)**

Key (a geometric definition of $\ln x$):

$$\ln x \overset{\text{def}}{=} \int_1^x\frac{1}{t}dt\ \ (x>0)\ =\ \text{area uner the graph of } \frac{1}{t} \text{ over } [1,x] \text{ if } x>1\ \ (\ ?\ \text{ if } 0<x\le 1)$$

20.5 Stirling's formula

(A famous formula which estimates $n!$ for $n \gg 1$)

Recall the notation:

We write $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ (iff $\lim_{n\to\infty} \frac{a_n - b_n}{b_n} = 0$)

(i.e., a_n and b_n are relatively close as $n \to \infty$)

Warning: $a_n \sim b_n$ does not mean that $a_n - b_n \to 0$ as $n \to \infty$; for example,

$$n^2 + n \sim n^2 \quad \text{but} \quad (n^2+n) - n^2 \not\to 0 \ \text{ as } n \to \infty$$

Theorem (Stirling's formula) [**Remember the result**]

$$\boxed{n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \quad \left(\text{i.e., } \lim_{n\to\infty} \frac{n!}{n^{n+1/2}\, e^{-n}} = \sqrt{2\pi}\right)}$$

(It is not easy to prove the formula)

We are going to just prove

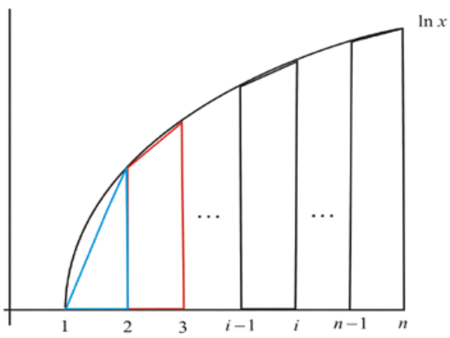
$$\boxed{(*) \quad n! \sim \left(\frac{n}{e}\right)^n \sqrt{n}\cdot K}, \quad \text{where } K \text{ is a positive constant}$$

Pf of $(*)$ (Idea: 적분과 비교)

Let $S_n = \ln n! = \underbrace{\ln 1}_{=0} + \ln 2 + \ln 3 + \cdots + \ln n$

Note that $(\ln x)' > 0$ & $(\ln x)'' < 0$

$\Rightarrow$ $\ln x$ is strictly $\uparrow$ and strictly concave for $x \geq 1$



The total area of the crescent-like regions

$$= \int_1^n \ln x \, dx - \underbrace{\frac{1}{2}\ln 2}_{\text{삼각형의 면적}} - \underbrace{\frac{1}{2}\left(\overbrace{\ln 2}^{1\text{개}} + \overbrace{\ln 3 + \ln 3}^{2\text{개}} + \overbrace{\ln 4 + \ln 4}^{2\text{개}} + \cdots + \overbrace{\ln(n-1) + \ln(n-1)}^{2\text{개}} + \overbrace{\ln n}^{1\text{개}}\right)}_{\text{사다리꼴의 면적}}$$

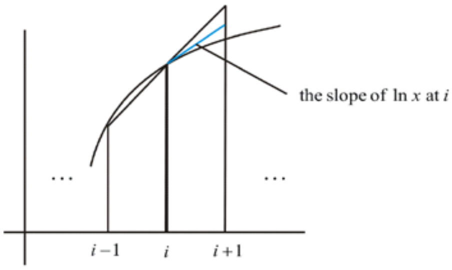
$$= \int_1^n \ln x \, dx - (\ln 2 + \ln 3 + \cdots + \ln(n-1) + \ln n) + \frac{1}{2}\ln n$$

$$=: \int_1^n \ln x \, dx - S_n + \frac{1}{2}\ln n$$

$$= \int_1^n \ln x \, dx + \frac{1}{2}\ln n - S_n = n\ln n - n + 1 + \frac{1}{2}\ln n - S_n$$

$$= (n + \frac{1}{2})\ln n - n + 1 - S_n \overset{\text{let}}{=} A_n - S_n$$

Claim: the slope of the chord over $[i-1,\ i] >$ the slope of $\ln x$ at i



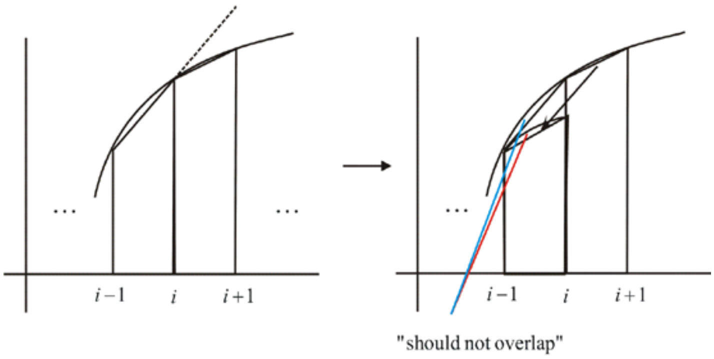
Pf of Claim.

The slope of the chord over $[i-1,\ i]$

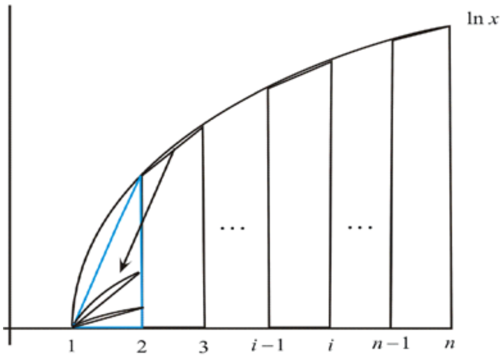
$$= \ln i - \ln(i-1) \overset{\text{MVT}}{=} (\ln x)'\Big|_c \cdot 1, \quad \text{where } i-1 < c < i$$

$$= \frac{1}{c} > \frac{1}{i} = \text{the slope of } \ln x \text{ at } i$$

Claim gives



Therefore, we have the following picture



The total area of crescent-like regions over $[1, 2]$ $= A_n - S_n$

$\therefore$ $\{A_n - S_n\}$ is $\uparrow$ and bounded by $\displaystyle\int_1^2 \ln x\, dx$

Thus by the Completeness Property, $\lim\limits_{n\to\infty}(A_n - S_n)$ exists, call it L

Since e^x is conti on $\mathbb{R}$,

$$e^{A_n - S_n} \longrightarrow e^L \quad \text{by the SCT}$$

That is, $\dfrac{e^{A_n}}{e^{S_n}} = \dfrac{e^{(n+\frac{1}{2})\ln n - n + 1}}{n\,!} = \dfrac{n^{n+\frac{1}{2}}\, e^{-n}\, e}{n\,!} \longrightarrow e^L$

Equivalently, $\dfrac{\left(\dfrac{n}{e}\right)^n \sqrt{n}\, e}{n\,!} \longrightarrow e^L \qquad$ i.e., $\dfrac{\left(\dfrac{n}{e}\right)^n \sqrt{n}\, e^{1-L}}{n\,!} \longrightarrow 1$

Finally letting $e^{1-L} = K$ gives $n\,! \sim \left(\dfrac{n}{e}\right)^n \sqrt{n} \cdot K \quad (\text{as } n \to \infty)$

An application of Stirling's formula

Ex. Find the radius R of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,!}\left(\frac{nx}{e}\right)^n$

Sol. $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{1}{n\,!}\left(\frac{n}{e}\right)^n}_{\equiv a_n} x^n$

$$R = \frac{1}{\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|} = \lim_{n\to\infty} \frac{\dfrac{1}{n\,!}\left(\dfrac{n}{e}\right)^n}{\dfrac{1}{(n+1)\,!}\left(\dfrac{n+1}{e}\right)^{n+1}}$$

$$\overset{\substack{\text{Stirling's formula} \\ n\,! \,\sim\, \left(\frac{n}{e}\right)^n \sqrt{2\pi n}}}{=} \quad \lim_{n\to\infty} \frac{\dfrac{1}{\sqrt{2\pi n}}}{\dfrac{1}{\sqrt{2\pi(n+1)}}} = 1$$

20.6 Growth rate of functions

In many cases, it is necessary to estimate the relative size of functions, when $x \gg 1$ or when $x \approx a$

Terminology:

Assume $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to \infty$

$\dfrac{f(x)}{g(x)} \to \infty \quad \overset{\text{means}}{\Leftrightarrow} \quad f$ tends to ∞ faster than g or f grows faster than g

$\dfrac{f(x)}{g(x)} \to 0 \quad \overset{\text{means}}{\Leftrightarrow} \quad f$ tends to ∞ more slowly than g

$\dfrac{f(x)}{g(x)} \to 1 \ (\text{or } f(x) \sim g(x)) \quad \overset{\text{means}}{\Leftrightarrow} \quad f$ grows at the same rate as g or f is asymptotic to g

Caution: $f(x) \sim g(x)$ (as $x \to \infty$) does not mean that $f(x)$ and $g(x)$ are close for $x \gg 1$.

$$f(x) \sim g(x) \ (\text{as } x \to \infty) \quad \Leftrightarrow \quad \frac{f(x)}{g(x)} \to 1 \ (\text{as } x \to \infty)$$

$$\Leftrightarrow \quad \text{given } \varepsilon > 0, \ \left|\frac{f(x)}{g(x)} - 1\right| < \varepsilon \quad \text{for } x \gg 1$$

$$\Leftrightarrow \quad \text{given } \varepsilon > 0, \ \left|\frac{f(x) - g(x)}{g(x)}\right| < \varepsilon \quad \text{for } x \gg 1$$

Exa A. As $x \to \infty$, show that

(a) $x^3 - 2x^2 - 1 \sim x^3$

(b) $\sqrt{x^5 + 5x^3 + 2}$ grows more slowly than x^3

(c) e^{ax} grows more rapidly than e^{bx} if $a > b > 0$

Sol.

(a) $\displaystyle\lim_{x\to\infty} \frac{x^3 - 2x^2 - 1}{x^3} = \lim_{x\to\infty} \left(1 - \frac{2}{x} - \frac{1}{x^3}\right) = 1$

(b) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x^5 + 5x^3 + 2}}{x^3} = \lim_{x\to\infty} \sqrt{\frac{1}{x} + \frac{5}{x^3} + \frac{2}{x^6}} = 0$

(c) $\displaystyle\lim_{x\to\infty} \frac{e^{ax}}{e^{bx}} = \lim_{x\to\infty} e^{(a-b)x} = \infty$ if $a > b > 0$

Remark. Assume $f(x)$ and $g(x) \to 0$ as $x \to \infty$

$$\frac{f(x)}{g(x)} \to 0 \overset{\text{means}}{\Leftrightarrow} f \text{ tends to } 0 \text{ more rapidly than } g \quad \text{or} \quad g \text{ tends to } 0 \text{ more slowly than } f$$

Show (easy)

(d) $\displaystyle\frac{1}{x^2 + 3x} \sim \frac{1}{x^2}$ as $x \to \infty$

(e) $\displaystyle\frac{1}{x^2}$ tends to 0 more rapidly than $\displaystyle\frac{1}{x}$ as $x \to \infty$,

but $\displaystyle\frac{1}{x^2}$ tends to 0 more slowly than $\displaystyle\frac{1}{x^3}$ as $x \to \infty$

Remark. The same terminology extends to limits as $x \to a$, $x \to a^+$ (a^-), $x \to -\infty$ etc

⦿ Another proof of L'Hospital's rule for ∞ / ∞ (optional)

Theorem (L'Hospital's rule for ∞ / ∞)

Suppose $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to \infty$ (as $x \to a^+$, resp, etc), &
assume that

$f'(x)$ and $g'(x)$ are conti, and $g'(x) \neq 0$ for $x \gg 1$ (for $x \approx a^+$, resp, etc)

Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} \quad \text{if the limit on the right exists}$$

Pf. Let $\displaystyle\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L(=$ a finite real number$)$. Then

$$\boxed{(*):\quad \text{given } \varepsilon > 0, \quad \left|\frac{f'(x)}{g'(x)} - L\right| < \varepsilon, \quad \text{for } x \gg 1}$$

Recall

Bolzano's theorem
Let $f(x)$ be conti on $[a, b]$. Then
$f(x)$ changes sign on $[a, b]$ (i.e., $f(a)f(b) < 0$) $\Rightarrow$ $f(x)$ has a zero on $[a, b]$

Since $g'(x)$ is continuous and $g'(x) \neq 0$ for $x \gg 1$,

either $g'(x) > 0$ for $x \gg 1$ or $g'(x) < 0$ for $x \gg 1$ (by Bolzano's theorem)

[Cf: Let g be diff & $g'(x) \neq 0$ for $x \gg 1$. Then we still have

either $g'(x) > 0$ for $x \gg 1$ or $g'(x) < 0$ for $x \gg 1$ (by Darboux's IVT for derivative) --- Ch15]

But, since $g(x) \to \infty$ as $x \to \infty$, we actually have

$$g'(x) > 0 \quad \text{for} \quad x \gg 1$$

(To prove this, suppose $g'(x) < 0$ for $x \gg 1$.

Since $g(x) \to \infty$ as $x \to \infty$, we get for a fixed y with $y \gg 1$,

$$g(x) > g(y) \quad \text{if} \quad x > y (\gg 1)$$

Then $\underbrace{\underbrace{g(x) - g(y)}_{>0} \overset{\text{MVT}}{=} \underbrace{g'(c)}_{<0}\underbrace{(x-y)}_{>0}}_{\text{contradiction}}, \quad x > c > y(\gg 1)$)

Now rewrite $(*)$ as

$$-\varepsilon < \frac{f'(x)}{g'(x)} - L < \varepsilon \quad \text{for } x \gg 1 \quad --- \oplus$$

Since $g'(x) > 0$ for $x \gg 1$,

$$\oplus \quad \Leftrightarrow \quad -\varepsilon g'(x) < f'(x) - Lg'(x) < \varepsilon g'(x) \quad \text{for } x \gg 1$$

Fix a large x-value a and let $u > a$. Then

$$\int_a^u -\varepsilon g'(x)\, dx < \int_a^u \big(f'(x) - Lg'(x)\big)\, dx < \int_a^u \varepsilon g'(x)\, dx$$

$$\therefore \quad -\varepsilon g(u) + \varepsilon g(a) < f(u) - Lg(u) - f(a) + Lg(a) < \varepsilon g(u) - \varepsilon g(a)$$

i.e., $\exists$ constants b and c such that

$$-\varepsilon g(u) + b < f(u) - Lg(u) < \varepsilon g(u) + c \quad \text{for} \quad u \gg 1$$

Since $g(u) \to \infty$ as $u \to \infty$,

$$\frac{|\,b\,|}{g(u)} < \varepsilon \quad \text{and} \quad \frac{|\,c\,|}{g(u)} < \varepsilon \quad \text{for} \quad u \gg 1$$

Accordingly,

$$-\varepsilon - \varepsilon < -\varepsilon + \frac{b}{g(u)} < \frac{f(u)}{g(u)} - L < \varepsilon + \frac{c}{g(u)} < \varepsilon + \varepsilon \quad \text{for} \quad u \gg 1$$

That is, $\left|\frac{f(u)}{g(u)} - L\right| < 2\varepsilon$ for $u \gg 1$. Equivalently, $\lim\limits_{u\to\infty} \frac{f(u)}{g(u)} = L.$

Rk. The hypo "$f'(x)$ and $g'(x)$ are conti", and $g'(x) \neq 0$ for $x \gg 1$ (for $x \approx a^+$, etc) can be slightly weakened as

"$f(x)$ and $g(x)$ are diff", and $g'(x) \neq 0$ for $x \gg 1$ (for $x \approx a^+$, etc)]

Remark. The L'Hospital's rule can be applied even if $\lim_{x\to\infty}\frac{f'(x)}{g'(x)}=\infty$

Pf. Claim:

$$\lim_{x\to\infty}\frac{f'(x)}{g'(x)}=\infty \text{ and } g'(x)\neq 0 \text{ for } x\gg 1 \quad\Rightarrow\quad f'(x)\neq 0 \text{ for } x\gg 1$$

Pf of Claim.

$$\lim_{x\to\infty}\frac{f'(x)}{g'(x)}=\infty \quad\Rightarrow\quad \frac{f'(x)}{g'(x)}>0 \text{ for } x\gg 1$$

$$\therefore\ \frac{f'(x)}{g'(x)}\neq 0 \text{ for } x\gg 1$$

$$\Rightarrow\quad f'(x)=\underbrace{\underbrace{\frac{f'(x)}{g'(x)}}_{\neq 0}\underbrace{g'(x)}_{\neq 0}}_{\neq 0} \text{ for } x\gg 1$$

Now

$$\lim_{x\to\infty}\frac{f'(x)}{g'(x)}=\infty\ (\&\ g'(x)\neq 0 \text{ for } x\gg 1)$$

$$\Rightarrow\quad \lim_{x\to\infty}\frac{g'(x)}{f'(x)}=0\ \ \&\ f'(x)\neq 0 \text{ for } x\gg 1$$

$$\overset{\text{L'Hospital}}{\Rightarrow}\quad \lim_{x\to\infty}\frac{g(x)}{f(x)}=0$$

$$\underset{(f(x)\ \&\ g(x)\to\infty\ \Rightarrow\ f(x)\ \&\ g(x)>0 \text{ for } x\gg 1)}{\Rightarrow}\quad \lim_{x\to\infty}\frac{f(x)}{g(x)}=\infty$$

Exa B. Verify that

(a) $\lim_{x\to\infty}\frac{\ln x}{x^k}=0$ for all $k>0$

(b) $\lim_{x\to\infty}\frac{e^{ax}}{x^m}=\infty$ for all $a,m>0$

Pf

(a) $\lim_{x\to\infty}\frac{\ln x}{x^k}\overset{\text{L'Hospital}}{=}\lim_{x\to\infty}\frac{1/x}{kx^{k-1}}=\lim_{x\to\infty}\frac{1}{kx^k}\underset{k>0}{=}0$

(b) $\lim_{x\to\infty}\frac{e^{ax}}{x^m}=\lim_{x\to\infty}\left(\frac{e^{ax/m}}{x}\right)^m=\left(\lim_{x\to\infty}\frac{e^{ax/m}}{x}\right)^m\overset{\text{L'Hospital}}{=}\left(\lim_{x\to\infty}\frac{a}{m}e^{ax/m}\right)^m=\infty$

Ex. Find $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}$

Sol. By Stirling's formula $\left[n!\sim\sqrt{2\pi n}(n/e)^n \text{ for } n\gg 1\right]$, we have

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\frac{1}{n}\frac{n}{e}(2\pi n)^{\frac{1}{2n}}=\frac{1}{e}\lim_{n\to\infty}(2\pi n)^{\frac{1}{2n}}=\frac{1}{e}\left[\leftarrow n^{\frac{1}{n}}\to 1\ \&\ a^{\frac{1}{n}}(a>0)\to 1, \text{ as } n\to\infty\right]$$

$\odot$ MVT for integrals

1. (The first MVT for integrals)

Let $f(x)$ be conti on $[a, b]$. Then $\exists\, c \in (a, b)$ such that

$$\int_a^b f(x)\, dx = f(c)(b-a)$$

Pf. Method 1

If f is constant on $[a, b]$, we can take any point in $[a, b]$ as c.

Thus we may assume $f(x)$ is not constant on $[a, b]$.

Since $f \in C[a, b]$, it has its max and its min on $[a, b]$.

Let $m = \min\limits_{x\in[a,\, b]} f(x) = f(\underline{x})$, $\quad \underline{x} \in [a, b] \quad \& \quad M = \max\limits_{x\in[a,\, b]} f(x) = f(\overline{x})$, $\quad \overline{x} \in [a, b]$

Note that $\underline{x} \neq \overline{x}$ since f is not constant on $[a, b]$.

Clearly,

$$f(\underline{x}) \le f(x) \le f(\overline{x}) \qquad \forall x \in [a, b]$$

$$\therefore \quad \underbrace{\int_a^b f(\underline{x})\, dx}_{=f(\underline{x})(b-a)} \le \int_a^b f(x)\, dx \le \underbrace{\int_a^b f(\overline{x})\, dx}_{=f(\overline{x})(b-a)}$$

$$\therefore \qquad f(\underline{x}) \le \frac{\int_a^b f(x)\, dx}{b-a} \le f(\overline{x})$$

Thus by (the usual) IVT, $\exists\, c \in [\underline{x}, \overline{x}]$ or $[\overline{x}, \underline{x}]$ $(\, \therefore c \in [a, b]\,)$ such that

$$f(c) = \frac{\int_a^b f(x)\, dx}{b-a}$$

To prove $c \in (a, b)$, we need only show that $c \neq \underline{x}$ and $c \neq \overline{x}$.

If $c = \underline{x}$, then $m(b-a) = \int_a^b f(x)\, dx$

$$\Rightarrow \int_a^b \underbrace{(f(x) - m)}_{\ge 0 \text{ and conti on } [a,\, b]} dx = 0$$

$\Rightarrow \quad f(x) - m = 0$ on $[a, b]$ $\qquad$ i.e., $f(x) = m \quad \forall x \in [a, b]$

$\Rightarrow \quad f(x)$ is constant on $[a, b]$; $\quad$ contradiction

$\therefore \quad c \neq \underline{x}$

Similarly, we can see that $c \neq \overline{x}$

Method 2

Let $F(x) = \int_a^x f(t)\, dt, \quad x \in [a, b]$

$\overset{2\text{nd FTC}}{\Rightarrow} \quad F(x)$ is diff on $[a, b]$ and $F'(x) = \underbrace{f(x)}_{:\text{ conti on } [a,\, b]} \quad \forall x \in [a, b]$

$$\overset{\text{ordinary MVT}}{\Rightarrow} \quad \underbrace{F(b) - F(a)}_{\int_a^b f(t)\,dt} = \underbrace{F'(c)(b-a)}_{f(c)(b-a)} \quad \text{for some } c \in (a, b)$$

2. (The second MVT for integrals)

Let $f(x)$ and $g(x)$ be conti and $g(x) \geq 0$ on $[a, b]$. Then $\exists\, c \in (a, b)$ such that

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$$

Caution. The hypo $g(x) \geq 0$ on $[a, b]$ is essential: Take $f(x) = g(x) = x$ on $[-1, 1]$. Then

$$\int_{-1}^{1} f(x)g(x)dx = \int_{-1}^{1} x^2 dx > 0; \quad \text{but} \quad \int_{-1}^{1} g(x)dx = 0; \ \text{so } f(c)\int_{-1}^{1} g(x)dx = 0$$

Pf. Method 1

Since $f \in C[a, b]$, we can let

$$m = \min_{x\in[a,\, b]} f(x) = f(\underline{x}), \quad \underline{x} \in [a, b] \quad \& \quad M = \max_{x\in[a,\, b]} f(x) = f(\overline{x}), \quad \overline{x} \in [a, b]$$

Then, since $g(x) \geq 0$ on $[a, b]$, we have

$$mg(x) \leq f(x)g(x) \leq Mg(x) \quad \forall x \in [a, b]$$

$$\therefore \quad m\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M\int_a^b g(x)\,dx$$

If $\displaystyle\int_a^b \underbrace{g(x)}_{\geq 0 \ \& \ \text{conti}}\,dx = 0$, then $g(x) = 0 \ \ \forall x \in [a, b]$.

$$\therefore \quad \int_a^b f(x)g(x)\,dx = \int_a^b 0\,dx = 0$$

$$\therefore \quad \underbrace{\int_a^b f(x)g(x)\,dx}_{=0} = f(c)\underbrace{\int_a^b g(x)\,dx}_{=0} \quad \text{for any choice } c \in (a, b)$$

Assume $\displaystyle\int_a^b g(x)\,dx > 0$.

If $f(x)$ is constant on $[a, b]$, then the assertion is trivially OK

If $f(x)$ is not constant on $[a, b]$, then $\underline{x} \neq \overline{x}$ &

$$f(\underline{x}) \leq \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \leq f(\overline{x})$$

Thus by (the usual) IVT, $\exists\, c \in [a, b]$ $\big(\leftarrow \exists c \in [\underline{x}, \overline{x}] \text{ or } [\overline{x}, \underline{x}]\big)$ such that

$$f(c) = \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx}$$

As in the proof of the first MVT for integrals, we can check that $c \neq \underline{x}$ and $c \neq \overline{x}$.

$$\therefore \quad c \in (a, b)$$

Method 2 Will prove only the case $\int_a^b g(x)\,dx > 0$.

(A proof of the easy case $\int_a^b g(x)\,dx = 0$ is left as an exercise)

$$H(x) :\overset{\text{let}}{=} f(x)\int_a^b g(x)\,dx \in C[a,b]$$

$f \in C[a,b] \;\Rightarrow\; f$ takes its max and min on $[a,b]$

i.e., $\exists\ \overline{x}\ \&\ \underline{x} \in [a,b]$ s.t. $f(\underline{x}) \le f(x) \le f(\overline{x}) \quad \forall x \in [a,b]$

$$\therefore\quad \underbrace{f(\underline{x})\int_a^b g(x)\,dx}_{\substack{\|\\ H(\underline{x})}} \le \int_a^b f(x)g(x)\,dx \le \underbrace{f(\overline{x})\int_a^b g(x)\,dx}_{\substack{\|\\ H(\overline{x})}}$$

By (the usual) IVT, $\exists\ c \in [a,b]$ $\big(\leftarrow \exists c \in [\underline{x}, \overline{x}]$ or $[\overline{x}, \underline{x}]\big)$ such that

$$\int_a^b f(x)g(x)\,dx = H(c) = f(c)\int_a^b g(x)\,dx$$

As in the proof of the first MVT for integrals, we can check that $c \ne \underline{x}$ and $c \ne \overline{x}$.

$$\therefore\quad c \in (a,b)$$

Method 3. We **further assume** $g(x) > 0\ \ \forall x \in [a,b]$ (stronger than $\int_a^b g(x)\,dx > 0$)

Let $F(x) = \int_a^x f(t)g(t)\,dt, \quad G(x) = \int_a^x g(t)\,dt; \quad x \in [a,b]$

$$\overset{\text{2nd FTC}}{\Rightarrow}\quad F(x) \text{ is diff on } [a,b] \text{ and } F'(x) = \underbrace{f(x)g(x)}_{:\ \text{conti on } [a,b]} \quad \forall x \in [a,b]$$

$$G(x) \text{ is diff on } [a,b] \text{ and } G'(x) = \underbrace{g(x)}_{:\ \text{conti on } [a,b]} \quad \forall x \in [a,b]$$

$$\overset{\text{Cauchy MVT}}{\Rightarrow}\quad \frac{F(b)-F(a)}{G(b)-G(a)} = \frac{F'(c)}{G'(c)} \quad \text{for some } c \in (a,b)$$

$$\| \qquad\qquad \|$$

$$\frac{\int_a^b f(t)g(t)\,dt}{\int_a^b g(t)\,dt} \qquad \frac{f(c)\cancel{g(c)}}{\cancel{g(c)}}$$

This means

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx, \quad \text{for some } c \in (a,b)$$

Exa. Suppose f is continuous on $[1,5]$. Prove that

$$\int_1^5 2xe^{x^2} f(x)\,dx = (e^{25} - e) f(c) \ \text{ for some } c \in (1,5).$$

Sol. Note that $g(x) = 2xe^{x^2} > 0$ and continuous on $[1,5]$. Thus 2nd MVT for integrals gives that

$$1 < {}^{\exists}c < 5 \ \text{ s.t. } \int_1^5 2xe^{x^2} f(x)\,dx = f(c)\left(\int_1^5 2xe^{x^2}\,dx\right) = f(c)\left[e^{x^2}\right]_1^5 = \left(e^{25} - e\right) f(c)$$