Chapter 23 Infinite sets and the Lebesgue integral

23.1 Omit (studied in Set Theory)

23.2 Sets of measure (or length) zero (= null sets)

Question (not easy) : If $S \subset \mathbb{R}$, how can we define the length of S?

Def A. Measure of intervals

- If I is an interval, we define the measure of I by its usual length.

Notation: $|\, I \,|$ (or $m(I)) =$ the measure of I = length of I. For example,

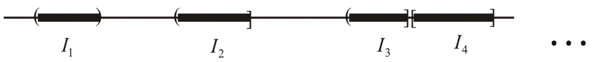
$$I = [a, b) \quad \Rightarrow \quad |\, I \,| = b - a$$

$$I = [a, \infty) \quad \Rightarrow \quad |\, I \,| = \infty$$

$$I = [a, a] \quad \Rightarrow \quad |\, I \,| = 0$$

- If S is a finite or countable union of intervals I_k, any of which overlap at most one endpoint, we define the measure $|\, S \,|$ of S to be the sum

$$|\, S \,| = \sum |\, I_k \,|$$



Remark. $\sum \underbrace{|\, I_k \,|}_{\geq 0}$ is an infinite series

Thus if the series converges, $|\, S \,| =$ its sum ; if the series diverges, $|\, S \,| = \infty$

※ Def. A set $S \subset \mathbb{R}$ has measure zero if, given $\varepsilon > 0$,

$\exists$ a finite or countable collection of intervals $I_1, I_2, \cdots$ (which may be overlap), such that

$$S \subset \bigcup_1 I_k \quad \text{and} \quad \sum_1 |\, I_k \,| \leq \varepsilon .$$

In other words, the set S can be covered by a finite or countable collection of intervals having arbitrarily small total length

Ex. ① one point in $\mathbb{R}$ has measure zero; because

$$\{x\} = [x, x] \quad \text{or} \quad \{x\} \subset [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}] \quad \forall \varepsilon > 0$$

② Any finite or countable set S has measure zero; because

$$S \overset{\text{let}}{=} \{x_1, x_2, \cdots, x_n, \cdots\} = [x_1, x_1] \cup [x_2, x_2] \cup \cdots \cup [x_n, x_n] \cup \cdots$$

$$\equiv I_1 \cup I_2 \cup \cdots \cup I_n \cup \cdots$$

$$(\Rightarrow \quad \sum_1 |\, I_k \,| = 0 \leq \varepsilon)$$

Or

$$S \overset{\text{let}}{=} \{x_1, x_2, \cdots, x_n, \cdots\} \subset [x_1 - \frac{\varepsilon}{4}, x_1 + \frac{\varepsilon}{4}] \cup [x_2 - \frac{\varepsilon}{8}, x_2 + \frac{\varepsilon}{8}] \cup \cdots \cup [x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}}] \cup \cdots$$

$$\equiv I_1 \cup I_2 \cup \cdots \cup I_n \cup \cdots$$

$$(\Rightarrow \quad \sum_1 |\, I_k \,| = \varepsilon / 2 + \varepsilon / 4 + \cdots = \varepsilon)$$

Theorem A

Let $S = \cup S_k$ be a finite or countable union of the sets S_k, where each S_k has measure zero.

Then S has measure zero.

Pf. Assume the union is countable.

Let $\varepsilon > 0$ be given. Then

S_1 has measure zero $\Rightarrow$ $\exists$ at most countable collection of intervals $I_{11}, I_{12}, \cdots$ s.t.

$$S_1 \subset \bigcup_{m=1}^{\infty} I_{1m} \quad \text{and} \quad \sum_{m=1}^{\infty} | I_{1m} | \leq \varepsilon / 2$$

S_2 has measure zero $\Rightarrow$ $\exists$ at most countable collection of intervals $I_{21}, I_{22}, \cdots$ s.t.

$$S_2 \subset \bigcup_{m=1}^{\infty} I_{2m} \quad \text{and} \quad \sum_{m=1}^{\infty} | I_{2m} | \leq \varepsilon / 2^2$$

$$\vdots$$

S_k has measure zero $\Rightarrow$ $\exists$ at most countable collection of intervals $I_{k1}, I_{k2}, \cdots$ s.t.

$$S_k \subset \bigcup_{m=1}^{\infty} I_{km} \quad \text{and} \quad \sum_{m=1}^{\infty} | I_{km} | \leq \varepsilon / 2^k$$

$$\vdots$$

Thus

$$S = \bigcup_{k \in \mathbb{N}} S_k \subset \underbrace{\bigcup_{k, m \in \mathbb{N}} I_{km}}_{\text{countable union}} \quad \text{and} \quad \sum_{k, m \in \mathbb{N}} | I_{km} | \leq \varepsilon / 2 + \varepsilon / 2^2 + \cdots + \varepsilon / 2^k + \cdots = \varepsilon$$

Ex(easy). $\quad | S | = 0 \quad \text{and} \quad T \subset S \quad \Rightarrow \quad | T | = 0$

Question: Does there exist a set which is uncountable but having measure zero?

Ans: It was first proved by Cantor that such a "strange set" exists.

We introduce a famous "uncountable" set of "measure zero"; the Cantor set C

Definition B The (triadic) Cantor set C (See the figures in our text)

Set $C_0 = [0, 1]$

From C_0, remove the open middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$; and get

$$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$$

From C_1, remove the two open middle thirds $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$; and get

$$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$$

$$\vdots$$

$$C \overset{\text{def}}{=} \bigcap_{n=0}^{\infty} C_n \quad \Big(\overset{\text{or}}{=} \bigcap_{n=1}^{\infty} C_n \overset{\text{or}}{=} \bigcap_{n=k}^{\infty} C_n \text{ for any } k \Big)$$ is called the (triadic) Cantor set

※ Theorem B (: a striking result)

$\underbrace{| C | = 0,}_{\text{easy to expect}}$ but C is uncountable

Pf. First, we will show $|C| = 0$

Since $C \subset C_n$ for any n, it suffices to show that

given $\varepsilon > 0$, $|C_n| < \varepsilon$ for $n \gg 1$

By construction (note that C_n is a finite union of disjoint intervals),

$$|C_n| = \frac{2}{3}|C_{n-1}| = \left(\frac{2}{3}\right)^2 |C_{n-2}| = \cdots = \left(\frac{2}{3}\right)^n |C_0| = \left(\frac{2}{3}\right)^n$$

$$\lim_{n\to\infty}\left(\frac{2}{3}\right)^n = 0 \quad \Rightarrow \quad \left(\frac{2}{3}\right)^n < \varepsilon \text{ for } n \gg 1 \quad \Rightarrow \quad |C_n| < \varepsilon \text{ for } n \gg 1$$

Next, we will prove C is uncountable.

For the purpose, we first represent the numbers in $[0, 1]$ to the base 3 :

$$x \in [0,1] \quad \Rightarrow \quad x = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \cdots + \frac{a_n}{3^n} + \cdots$$

$$\overset{\text{write}}{=} (0.a_1a_2a_3\cdots a_n\cdots)_3, \quad a_i = 0, 1, 2 \text{ for every } i$$

Caution: Such representation of $x \in [0, 1]$ is not unique; for example,

$$\frac{1}{3} = (0.100\cdots0\cdots)_3 = (0.022\cdots2\cdots)_3$$

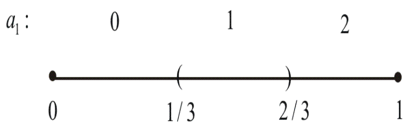
$$\frac{4}{9} = \frac{1}{3} + \frac{1}{3^2} + \cdots = (0.110\cdots0\cdots)_3 = (0.1022\cdots2\cdots)_3$$

$$\frac{2}{3} = (0.200\cdots0\cdots)_3 = (0.122\cdots2\cdots)_3$$

To "avoid the usage of 1" as far as possible, we take the equivalent non-terminating expansion if $x \in [0, 1]$ has a finite ternary expansion ending with 1.

Now if $x \in [0,1]$ and $x = (0.a_1a_2\cdots a_n\cdots)_3$ then

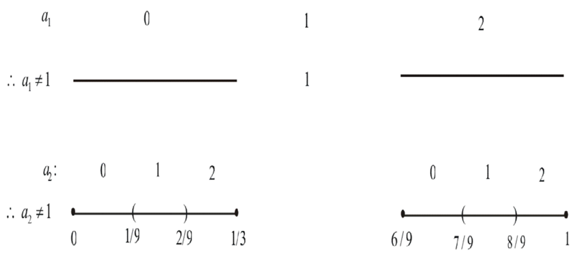
$$a_1 = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{3} \\ 1 & \text{if } \frac{1}{3} < x < \frac{2}{3} \\ 2 & \text{if } \frac{2}{3} \le x \le 1 \end{cases}$$



$$\therefore \quad C_1 = \{x \in [0,1] : a_1 \ne 1\}$$

Similarly,

$$a_2 = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{9} \text{ or } \frac{6}{9} \le x \le \frac{7}{9} \\ 1 & \text{if } \frac{1}{9} < x < \frac{2}{9} \text{ or } \frac{7}{9} < x < \frac{8}{9} \\ 2 & \text{if } \frac{2}{9} \le x \le \frac{1}{3} \text{ or } \frac{8}{9} \le x \le 1 \end{cases}$$



$$\therefore \quad C_2 = \{x \in [0,1] : a_1 \ne 1, \ a_2 \ne 1\}$$

$$\vdots$$

$$C_n = \{x \in [0,1] : a_1 \ne 1, \ a_2 \ne 1, \ \cdots, \ a_n \ne 1\}$$

$n \to \infty \quad \Rightarrow$

$$C = \bigcap_{n=0}^{\infty} C_n = \{x \in [0,1] : x = (0.a_1a_2\cdots a_n\cdots)_3, \quad a_i = 0 \text{ or } 2 \text{ for all } i\}$$

We will prove C is uncountable (by using the Cantor's diagonal argument)
Suppose C is countable and let $\{x_1, x_2, \cdots, x_n, \cdots\}$ be an enumeration of C.
Then

$$\begin{aligned} x_1 &= 0.a_{11}a_{12}a_{13}\cdots a_{1n}\cdots \\ x_2 &= 0.a_{21}a_{22}a_{23}\cdots a_{2n}\cdots \\ &\vdots \\ x_n &= 0.a_{n1}a_{n2}a_{n3}\cdots a_{nn}\cdots \\ &\vdots \end{aligned}$$

where each $a_{ij} = 0$ or 2

Define $x = 0.b_1b_2\cdots b_n\cdots$, where

$$b_1 = \begin{cases} 2 & \text{if } a_{11} = 0 \\ 0 & \text{if } a_{11} = 2 \end{cases}$$

$$b_2 = \begin{cases} 2 & \text{if } a_{22} = 0 \\ 0 & \text{if } a_{22} = 2 \end{cases}$$

$$\vdots$$

$$b_n = \begin{cases} 2 & \text{if } a_{nn} = 0 \\ 0 & \text{if } a_{nn} = 2 \end{cases}$$

$$\vdots$$

Then clearly $x \in C$ because each $b_i = 0$ or 2;
but $x \neq x_1, x_2, \cdots, x_n, \cdots$ so $x \notin C$; contradiction
Therefore, C is uncountable.

23.3 Measure zero and Riemann-integrability

Question: Which functions are Riemann integrable?

So far, we have proved that
continuous functions or monotone functions
&
(bounded and) p.w. continuous functions or p.w. monotone functions
[those have only a *finite* # of discontinuity points or changes of direction on any finite interval]
(on a compact interval) are Riemann-integrable.

Question: If f is bounded and it has a countable number of discontinuities on $[a, b]$,
is f integrable?
Ans: Yes. Moreover, the following striking result is known

※※ Theorem (A famous characterization of Riemann-integrability; due to Lebesgue) [Remember]
Let f be defined and bounded on $[a, b]$, and let

$$D_f := \{x_0 \in [a, b] : f \text{ is discontinuous at } x_0\}$$

Then f is Riemann-integrable on $[a, b]$ $\Leftrightarrow$ $|D_f| = 0$

Pf. Its proof is not easy. So we omit it

Def. For $S \subset \mathbb{R}$, the characteristic function $\mathcal{X}_S(x)(\underset{\text{in our text}}{=} f_S(x))$ is defined by

$$\mathcal{X}_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

Ex. $$\mathcal{X}_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

$\mathcal{X}_{\mathbb{Q}}(x)$ is discontinuous at every point of $x \in (-\infty, \infty)$

$\therefore$ $\mathcal{X}_{\mathbb{Q}}(x)$ is not Riemann integrable on any finite interval $[a, b]$; because

$D_{\mathcal{X}_{\mathbb{Q}}} = \{x_0 \in [a, b] : \mathcal{X}_{\mathbb{Q}} \text{ is discontinuous at } x_0\} = [a, b], \text{ and hence } \left|D_{\mathcal{X}_{\mathbb{Q}}}\right| = b - a \neq 0$

※Ex A. Let C be the Cantor set. Then
$\mathcal{X}_C$ is Riemann integrable on $[0, 1]$

Pf. Suffices to show: "$\mathcal{X}_C$ is conti $[0,1] \setminus C$"(i.e., $D_{\mathcal{X}_C} \subset C$); because

if this is proved, we then have $\left|D_{\mathcal{X}_C}\right| = 0$ since $\left|D_{\mathcal{X}_C}\right| \le |C| = 0$.

Pf of the above " ":

Note that $\mathcal{X}_C = 0$ on the open set $[0,1] \setminus C$ ($\leftarrow$ $C = \cap_{n=0}^{\infty} C_n$ is a closed set in $[0,1]$)

Fix any $x_0 \in [0,1] \setminus C$. Then, since $[0,1] \setminus C$ is open (in $[0,1]$),

$$\exists(\text{small})\delta > 0 \text{ such that } \left(x_0 - \delta, x_0 + \delta\right) \subset [0,1] \setminus C.$$

This gives

$$\left.\begin{array}{l} |x - x_0| < \delta \\ x \in [0,1] \end{array}\right\} \Rightarrow |\mathcal{X}_C(x) - \mathcal{X}_C(x_0)| = |0 - 0| = 0 < \varepsilon, \quad \forall \varepsilon > 0$$

This shows $\mathcal{X}_C$ is conti at any point $x_0 \in [0,1] \setminus C$ Qed.

Alternative way:

Recall $C = \{x \in [0,1] : x = (0.a_1 a_2 \cdots a_n \cdots)_3, \quad a_i = 0 \text{ or } 2 \text{ for all } i\}$

Fix any $x_0 \notin C\left(\text{with } x_0 \in [0,1]\right)$. Then x_0 has an 1 in its "ternary" decimal expansion; that is,

$$x_0 = (0.\cdots\cdots \underset{n\text{th place}}{\underset{\smile}{1}} \cdots)_3$$

Thus all x satisfying $x \underset{1/3^{n+1}}{\approx} x_0$ also have the same 1^{st} nth places as x_0, hence all such x also have an 1 in its ternary decimal expansion. Therefore, $x \notin C$ whenever $x \underset{1/3^{n+1}}{\approx} x_0$

$$\therefore \quad \mathcal{X}_C(x) = \mathcal{X}_C(x_0) = 0 \text{ for all } x \approx x_0$$

$$\therefore \quad \mathcal{X}_C \text{ is continuous at } x_0$$

Comment. Actually, $\mathcal{X}_C$ is discontinuous on C (i.e., $D_{\mathcal{X}_C} = C$)

Pf. Fix any $x_0 \in C$. Then $\mathcal{X}_C(x_0) = 1$.

Note that for any interval $I := (x_0 - \delta, x_0 + \delta)(\delta > 0)$, $\exists x \in I$ such that $x \notin C$ $[\leftarrow |C| = 0]$;

and hence $\mathcal{X}_C(x) = 0$.

This shows that $\mathcal{X}_C$ is discontinuous at x_0

Theorem B (UCT: Uniform Convergence Theorem)
Assume that, on a compact interval $[a, b]$, every f_n is Riemann integrable and $f_n \rightrightarrows f$, then

(i) $f(x)$ is Riemann integrable on $[a, b]$, and

(ii) $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$

Pf. (i) We first show f is bounded on $[a, b]$.

Since $f_n \rightrightarrows f$ on $[a, b]$,

$$|f_n(x) - f(x)| < 1 \quad \text{for all } x \in [a, b] \text{ if } n \gg 1$$

In particular, $\exists$ a natural number N such that

$$|f_N(x) - f(x)| < 1 \quad \text{for all } x \in [a, b]$$

$$\therefore \quad |f(x)| \le |f_N(x)| + |f(x) - f_N(x)| \underset{f_N \text{ is bounded}}{\le} K + 1 \quad \forall x \in [a, b]$$

$\therefore$ $f(x)$ is bounded on $[a, b]$.

To prove $f(x)$ is Riemann integrable on $[a, b]$, it suffices to show that $|D_f| = 0$.

Since $f_n \rightrightarrows f$ on $[a, b]$, if every f_n is conti at $x_0 \in [a, b]$, then f should be conti at x_0.

In other words,

f is disconti at x_0 $\Rightarrow$ at least one of the f_n is disconti at x_0

i.e., $x_0 \in D_f \Rightarrow x_0 \in D_{f_n}$ for some n

$$\therefore \quad D_f \subset \cup_{n=1}^{\infty} D_{f_n}$$

By the way, since every f_n is Riemann integrable on $[a, b]$,

$|D_{f_n}| = 0$ for every n (by a famous Lebesgue's criterion of R-integrability)

$$\overset{\text{Thm 23.2 A}}{\Rightarrow} \quad |\cup_{n=1}^{\infty} D_{f_n}| = 0 \quad \overset{\text{Ex(seen): } |S|=0 \ \& \ T\subset S \Rightarrow |T|=0}{\Rightarrow} \quad |D_f| = 0$$

Another way of showing $f \in \mathcal{R}[a, b]$ (without using the concept of measure zero):

Let $\varepsilon > 0$ $\overset{f_n \rightrightarrows f \text{ on } [a,b]}{\Rightarrow}$ $\exists N = N(\varepsilon) \in \mathbb{N}$ such that if $n \ge N$, then

$$f_n(x) - \frac{\varepsilon}{b-a} < f(x) < f_n(x) + \frac{\varepsilon}{b-a} \qquad \forall x \in [a,b]$$

Use a simple fact: $\boxed{f \le g \text{ on } [a,b] \Rightarrow \underline{\int_a^b} f \le \underline{\int_a^b} g \ \& \ \overline{\int_a^b} f \le \overline{\int_a^b} g}$ to see that

$$\underline{\int_a^b} \left(f_n(x) - \frac{\varepsilon}{b-a}\right) dx \le \underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx \le \overline{\int_a^b} \left(f_n(x) + \frac{\varepsilon}{b-a}\right) dx$$

$$\| \qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$\underline{\int_a^b} f_n(x)dx - \varepsilon \qquad\qquad\qquad\qquad \overline{\int_a^b} f_n(x)dx + \varepsilon$$

$$\| \leftarrow f_n \in \mathcal{R}[a,b] \qquad\qquad f_n \in \mathcal{R}[a,b] \rightarrow \|$$

$$\int_a^b f_n(x)dx - \varepsilon \qquad\qquad\qquad\qquad \int_a^b f_n(x)dx + \varepsilon$$

That is, $\int_a^b f_n(x)dx - \varepsilon \le \underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx \le \int_a^b f_n(x)dx + \varepsilon$

This implies that $0 \le \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \le 2\varepsilon$

Since $\varepsilon > 0$ was arbitrary, we get

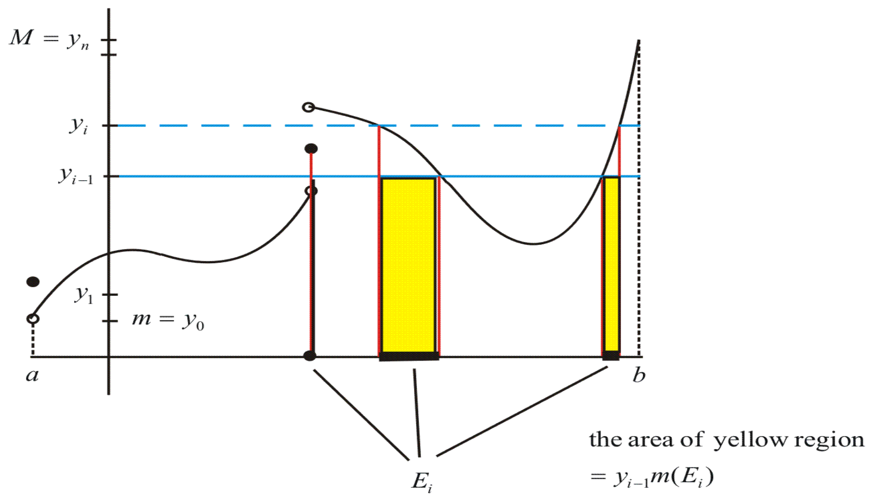
$$\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx; \text{ so } f \in \mathcal{R}[a,b]$$

(ii) $$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| = \left| \int_a^b \big(f_n(x) - f(x)\big)\,dx \right|$$
$$\le \int_a^b \left| f_n(x) - f(x) \right| dx$$
$$\le \left\| f_n - f \right\|_{[a,\,b]} \cdot (b-a) \to 0 \quad \text{since } f_n \rightrightarrows f \text{ on } [a,\,b]$$

23.4 Lebesgue integration

difference in approach	
Riemann	Lebesgue
partition the domain of f	partition the range of f
	(partition some interval containing the range of f)

Assume $f(x)$ is bounded on $[a,\,b]$.



Let $m = \inf_{x\in[a,\,b]} f(x)$, $\quad M = \sup_{x\in[a,\,b]} f(x)$ and let

$\mathcal{Q} : m = y_0 < y_1 < y_2 < \cdots < y_n = M$ be a partition of $[m,\,M]$

$E_i = \{x \in [a,\,b] : y_{i-1} \le f(x) < y_i\} \quad (i = 1, 2, \cdots, n-1)$

$E_n = \{x \in [a,\,b] : y_{n-1} \le f(x) \le y_n\}$

$$L_f(\mathcal{Q}) = \sum_{i=1}^{n} y_{i-1} m(E_i) : \text{ the lower Lebesgue sum (w.r.t. the partition } \mathcal{Q})$$

$$U_f(\mathcal{Q}) = \sum_{i=1}^{n} y_i m(E_i) : \text{ the upper Lebesgue sum (w.r.t. the partition } \mathcal{Q})$$

Notation:

$$|\mathcal{Q}| = \max_{1 \le i \le n}(y_i - y_{i-1}) : \text{ the mesh of the partition } \mathcal{Q}$$

Def. We say that a bounded function $f : [a, b] \to \mathbb{R}$ is Lebesgue integrable on $[a, b]$ if

$$\boxed{\text{given } \varepsilon > 0, \ \exists \text{ partition } \mathcal{Q} = \mathcal{Q}_\varepsilon \text{ of } [m, M] \text{ s.t. } U_f(\mathcal{Q}) \underset{\varepsilon}{\approx} L_f(\mathcal{Q})}$$

Or equivalently,

$$\boxed{\text{given } \varepsilon > 0, \quad U_f(\mathcal{Q}) \underset{\varepsilon}{\approx} L_f(\mathcal{Q}) \quad \text{for all } \mathcal{Q} \text{ with } |\mathcal{Q}| \approx 0}$$

Again equivalently,

$$\boxed{\text{given } \varepsilon > 0, \quad \exists\, \delta = \delta(\varepsilon) > 0 \text{ s.t. } U_f(\mathcal{Q}) \underset{\varepsilon}{\approx} L_f(\mathcal{Q}) \quad \text{for all } \mathcal{Q} \text{ with } |\mathcal{Q}| < \delta}$$

In short, $\boxed{\lim\limits_{|\mathcal{Q}| \to 0} \Big(U_f(\mathcal{Q}) - L_f(\mathcal{Q})\Big) = 0}$ or $\lim\limits_{\delta \to 0} \Big\{U_f(\mathcal{Q}) - L_f(\mathcal{Q}) : \big|\mathcal{Q}\big| \le \delta\Big\} = 0$

Def. We say that $f : [a, b] \to \mathbb{R}$ is Lebesgue measurable if for every $\alpha, \beta \in \mathbb{R}$,

$\{x \in [a, b] : \alpha \le f(x) < \beta\}$ is a "Lebesgue measurable set"

Theorem. Let f be a bounded function on $[a, b]$. Then

f is Lebesgue measurable on $[a, b]$ $\Rightarrow$ f is Lebesgue integrable on $[a, b]$

Pf (sketch)

$$\big|U_f(\mathcal{Q}) - L_f(\mathcal{Q})\big| = \sum_{i=1}^{n} (y_i - y_{i-1}) m(E_i)$$

$$\le |\mathcal{Q}| \sum_{i=1}^{n} m(E_i) \overset{\text{expect}}{=} |\mathcal{Q}| \cdot (b - a) \to 0 \text{ as } |\mathcal{Q}| \to 0$$

$$\therefore \ \forall \varepsilon > 0, \quad U_f(\mathcal{Q}) \underset{\varepsilon}{\approx} L_f(\mathcal{Q}) \quad \text{for all } \mathcal{Q} \text{ with } |\mathcal{Q}| \approx 0\,.$$

Def. Let f be a bounded function on $[a, b]$.

If f is Lebesgue integrable on $[a, b]$, we define

$$\underbrace{\int_a^b f(x)\, dx}_{\text{Lebesgue}} = \lim_{|\mathcal{Q}| \to 0} L_f(\mathcal{Q}) \overset{\text{or}}{=} \lim_{|\mathcal{Q}| \to 0} U_f(\mathcal{Q})$$

Remark.

$$S_f(\mathcal{Q}) \overset{\text{def}}{=} \sum_{i=1}^{n} y_i^{*} m(E_i), \text{ where } y_i^{*} \text{ is any point in } [y_{i-1}, y_i)$$

: is called the Lebesgue sum of f w.r.t. the partition $\mathcal{Q}$

Easy fact:

(i) $L_f(\mathcal{Q}) \le \forall S_f(\mathcal{Q}) \le U_f(\mathcal{Q})$

(ii) f : Lebesgue integrable on $[a, b]$ $\Rightarrow$ $\underbrace{\int_a^b f(x)\,dx}_{\text{Lebesgue}} = \lim_{|\mathcal{Q}|\to 0} S_f(\mathcal{Q})$

Notation: $\underbrace{\int_a^b f(x)\,dx}_{\text{Lebesgue}} = \underbrace{\int_{[a,\,b]} f(x)\,dx \overset{\text{or}}{=} \int_{[a,\,b]} f\,dm}_{\text{most common notations}} \overset{\text{or}}{=} \int_{[a,\,b]} f(x)\,dm(x)$

: called the Lebesgue integral of f over $[a, b]$.

Ex. $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q}^c \end{cases}$

We know that f is not Riemann integrable on $[0, 1]$.

How about the Lebesgue-integrability?

Ans. Range of $f = \{0, 1\}$. $\Rightarrow$ $\inf_{x\in[0,\,1]} f(x) = 0,\quad \sup_{x\in[0,\,1]} f(x) = 1$.

Let $\mathcal{Q} : y_0 = 0 < y_1 < y_2 < \cdots < y_{n-1} < y_n = 1$ be a partition of $[0, 1]$

Then

$$E_1 = \{x \in [0,1] : 0 = y_0 \le f(x) < y_1\} = \mathbb{Q}^c \cap [0,1]$$
$$E_2 = \{x \in [0,1] : y_1 \le f(x) < y_2\} = \varnothing$$
$$\vdots$$
$$E_{n-1} = \{x \in [0,1] : y_{n-2} \le f(x) < y_{n-1}\} = \varnothing$$
$$E_n = \{x \in [0,1] : y_{n-1} \le f(x) \le y_n = 1\} = \mathbb{Q} \cap [0,1]$$

$$\therefore\quad U_f(\mathcal{Q}) = \sum_{i=1}^{n} y_i m(E_i) = y_1 m(E_1) + y_2 \underbrace{m(E_2)}_{=0} + \cdots + y_{n-1} \underbrace{m(E_{n-1})}_{=0} + y_n m(E_n)$$
$$= y_1 \cdot m\Big(\mathbb{Q}^c \cap [0,1]\Big) + 1 \cdot \underbrace{m\Big(\mathbb{Q} \cap [0,1]\Big)}_{=0}$$
$$= y_1 \cdot m\Big(\mathbb{Q}^c \cap [0,1]\Big) \le y_1 m([0,1]) = y_1 \cdot 1$$

$\therefore\quad \lim_{|\mathcal{Q}|\to 0} U_f(\mathcal{Q}) = 0$ ($\leftarrow y_1 \to 0$ as $|\mathcal{Q}| \to 0$), so, $\lim_{|\mathcal{Q}|\to 0} L_f(\mathcal{Q}) = 0$

$\therefore\quad f$ is Lebesgue integrable on $[0, 1]$ and $\underbrace{\int_a^b f(x)\,dx}_{\text{Lebesgue}} = \lim_{|\mathcal{Q}|\to 0} U_f(\mathcal{Q}) = 0$

※ Note:

⦿ Riemann integral is considered only for bounded functions defined on a compact interval. (Improper (Riemann) integral is separately designed for handling unbounded functions or functions defined on an unbounded interval)

◈ Lebesgue integral can be defined, in a unified way, even for unbounded functions or for functions defined on an unbounded interval. However, for handling such functions, we allow the integral to have ∞ as a "value". Precise definition of general Lebesgue integral is introduced in "Measure Theory" course.

※ Riemann vs Lebesgue (**only state without proof, but remember and freely use the results**)

Theorem A. (easy to expect)

$f(x)$: Riemann integrable on $[a, b]$ $\Rightarrow$ $f(x)$: Lebesgue integrable on $[a, b]$, and

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Riemann}} = \underbrace{\int_a^b f\,dm}_{\text{Lebesgue}}$$

Def. A statement $P(x)$ is said to hold almost everywhere (for short, a.e.) on an interval I if $|\{x \in I : P(x) \text{ is false}\}| = 0$ i.e., $\{x \in I : P(x) \text{ is false}\}$ is a null set

Ex

(a) $\tan x$ is continous a.e. on $\mathbb{R}$; because

$\{x \in \mathbb{R} : \tan x \text{ is not continuous at } x\} = \{\cdots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \cdots\}$ has measure zero

(b) $f_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q}^c \end{cases}$

$f_{\mathbb{Q}}$ is zero a.e. on $\mathbb{R}$; because $\{x \in \mathbb{R} : f_{\mathbb{Q}}(x) \neq 0\} = \mathbb{Q}$ has measure zero

(c) f_C (C is the Cantor set) is zero a.e. on $[0, 1]$; because

$\{x \in [0, 1] : f_C(x) \neq 0\} = C$ has measure zero

※(d) (A reformulation of the Riemann-integrability)

Assume $f(x)$ is bounded on $[a, b]$. Then

$f(x)$ is Riemann integrable on $[a, b]$ $\Leftrightarrow$ $f(x)$ is conti a.e. on $[a, b]$ ($\Leftrightarrow$ $|D_f| = 0$)

※Theorem B

Suppose $f(x) = g(x)$ a.e. on $[a, b]$. Then

$g(x)$ is Lebesgue integrable on $[a, b]$ $\Rightarrow$ $f(x)$ is also Lebesgue integrable on $[a, b]$, and

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Lebesgue}} = \underbrace{\int_a^b g(x)\,dx}_{\text{Lebesgue}}$$

Applications of **Theorem A** and **Theorem B**

Ex 1. Show $f_{\mathbb{Q}}$ is Lebesgue integrable on $[a, b]$, and evaluate $\underbrace{\int_a^b f_{\mathbb{Q}}(x)\,dx}_{\text{Lebesgue}}$

Pf. $f_{\mathbb{Q}} = 0$ a.e. on $[a, b]$

Since the RHS $\underbrace{(= 0)}_{\text{conti}}$ is Riemann integrable on $[a, b]$,

it is also Lebesgue integrable on $[a, b]$ by Thm A, and

$$\underbrace{\int_a^b f_{\mathbb{Q}}(x)\,dx}_{\text{Lebesgue}} \overset{\text{Thm B}}{=} \underbrace{\int_a^b 0\,dx}_{\text{Lebesgue}} \overset{\text{Thm A}}{=} \underbrace{\int_a^b 0\,dx}_{\text{Riemann}} = 0$$

Ex 2. Let C be the Cantor set. We know that f_C is Riemann integrable on $[0, 1]$.

Evaluate $\int_0^1 f_C(x)\,dx$ (Riemann integral)

Sol. f_C is Riemann integrable on $[0, 1]$ --- already seen

$\overset{\text{Thm A}}{\Rightarrow}$ f_C is Lebesgue integrable on $[0, 1]$, and

$$\int_0^1 f_C(x)\,dx = \underbrace{\int_0^1 f_C(x)\,dx}_{\text{Lebesgue}}$$

On the other hand,

$f_C = 0$ a.e. on $[0, 1]$ & RHS $(= 0)$ is (clearly) Lebesgue integrable on $[0, 1]$

Thus

$$\underbrace{\int_0^1 f_C(x)\,dx}_{\text{Lebesgue}} \overset{\text{Thm B}}{=} \underbrace{\int_0^1 0\,dx}_{\text{Lebesgue}} \overset{\text{Thm A}}{=} \underbrace{\int_0^1 0\,dx}_{\text{Riemann}} = 0$$

Therefore, $\int_0^1 f_C(x)\,dx = 0$

Theorem (Basic properties of Lebesgue integrals)
Suppose $f(x)$ and $g(x)$ are Lebesgue integrable functions on I, and c_1, c_2 are constants.

Then

(i) $c_1 f(x) + c_2 g(x)$ is Lebesgue integrable on I, and

$$\underbrace{\int_I (c_1 f(x) + c_2 g(x))\,dx}_{\text{Lebesgue}} = c_1 \underbrace{\int_I f(x)\,dx}_{\text{Lebesgue}} + c_2 \underbrace{\int_I g(x)\,dx}_{\text{Lebesgue}}$$

(ii) If $f(x) \le g(x)$ a.e. on I, then as Lebesgue integrals,

$$\int_I f(x)\,dx \le \int_I g(x)\,dx$$

Theorem C (Absolute value property)

$f(x)$: L - integrable on an interval I

$\Rightarrow$ $|f(x)|$ is also L - integrable on an interval I, and

$$\left| \underbrace{\int_I f(x)dx}_{\text{Lebesgue}} \right| \le \underbrace{\int_I | f(x) |\,dx}_{\text{Lebesgue}}$$

⦿ Two (or three) important convergence theorems about L -integral

Thm 1 (MCT: Monotone Convergence Theorem)
Suppose on an interval I,

each f_n is L -integrable and $0 \le f_1 \le f_2 \le \cdots \le f_n \le \cdots$ and $f_n \to f$

(i.e., each $f_n \ge 0$, L -integrable and $f_n \uparrow f$)

Then f is also L -integrable (allowing the value ∞ for the integral) and

$$\lim_{n\to\infty} \underbrace{\int_I f_n(x)\,dx}_{\text{Lebesgue}} = \underbrace{\int_I f(x)\,dx}_{\text{Lebesgue}}$$

Corollary. Suppose on an interval I,

(a) every $u_n(x)$ is L-integrable and $u_n(x) \geq 0$

(b) $\sum u_n(x)$ converges (pointwise)

Then

$\sum u_n(x)$ is L-integrable and

$$\underbrace{\int_I \sum u_n(x)\,dx}_{\text{Lebesgue}} = \sum \underbrace{\int_I u_n(x)\,dx}_{\text{Lebesgue}}$$

Thm 2 (DCT: Dominated Convergence Theorem)

Suppose on an interval I,

(a) every $f_n(x)$ is L-integrable and $f_n(x) \to f(x)$ a.e.

(b) $\underbrace{|f_n(x)| \leq g(x) \text{ for all } n}_{\left(\text{i.e., } \sup_n |f_n(x)| \leq g(x)\right)}$, and $\underbrace{\int_I g(x)\,dx}_{\text{Lebesgue}}$ exists and is finite

Then

$f(x)$ is L-integrable on I, and

$$\lim_{n\to\infty} \underbrace{\int_I f_n(x)\,dx}_{\text{Lebesgue}} = \underbrace{\int_I f(x)\,dx}_{\text{Lebesgue}}$$

Cor. BCT (Bounded Convergence Theorem)

Suppose on a finite interval I,

(a) every $f_n(x)$ is L-integrable and $f_n(x) \to f(x)$ a.e.

(b) $\left|f_n(x)\right| \leq \underbrace{K}_{\text{indep of } x\in I \ \&\ n}$ for all $x \in I$ & all n (i.e., $\sup_n \sup_{x\in I} \left|f_n(x)\right| \leq K$)

(Such $\{f_n\}$ satisfying (b) is called a uniformly bounded sequence on I)

Then

$f(x)$ is L-integrable on I, and

$$\lim_{n\to\infty} \underbrace{\int_I f_n(x)\,dx}_{\text{Lebesgue}} = \underbrace{\int_I f(x)\,dx}_{\text{Lebesgue}}$$

Applications of **three important convergence theorem**: MCT, DCT, BCT

Ex 1. Evaluate $\displaystyle \lim_{n\to\infty} \int_0^1 \frac{2nx}{(1+n^2x^2)^2}\,dx$

Sol. $\displaystyle f_n(x) \overset{\text{let}}{=} \frac{2nx}{(1+n^2x^2)^2} \in C[0,1] \quad (n = 1, 2, \cdots)$

$f_n(x) \to 0$ (pointwise) on $[0,1]$ (easy)

Question: $f_n(x) \rightrightarrows 0$ on $[0,1]$?

$$f_n'(x) = \frac{2n(1-3n^2x^2)}{(1+n^2x^2)^2} = 0 \quad \Leftrightarrow \quad x = \frac{1}{n\sqrt{3}}$$

$$f_n(0) = 0, \quad f_n(1) = \frac{2n}{(1+n^2)^2}, \qquad f_n(\frac{1}{n\sqrt{3}}) = \frac{3\sqrt{3}}{8}$$

$$\therefore \quad \sup_{x\in[0,\,1]} |f_n(x) - 0| = \frac{3\sqrt{3}}{8} \not\to 0 \qquad \therefore \quad f_n(x) \not\rightrightarrows 0 \ \text{ on } \ [0,1].$$

Thus we cannot apply UCT(uniform convergence theorem). However,

every $f_n \in C[0,1]$, so ev f_n is R-integrable on $[0,1]$

$\therefore$ ev f_n is L-integrable on $[0,1]$

Also, $f_n \to 0$ pointwise on $[0,1]$ and

$$|f_n(x)| \le \frac{1}{1+n^2x^2} \le 1 \quad \text{for all } x \in [0,1] \text{ and } \forall n$$

Thus by BCT,

$$\lim_{n\to\infty} \underbrace{\int_0^1 f_n(x)\,dx}_{\text{Lebesgue}} = \underbrace{\int_0^1 0\,dx}_{\text{Lebesgue}} = 0$$

$$\| \leftarrow \text{ ev } f_n \text{ is } R\text{-integrable}$$

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$$

Ex 2. Evaluate $\displaystyle \lim_{n\to\infty} \int_0^{\pi/2} \sin^n x\,dx$

Sol. $f_n(x) \overset{\text{let}}{=} \sin^n x \in C[0,\pi/2] \ (\forall n)$

$$f_n(x) \quad \to \quad \underbrace{\begin{cases} 0, & \text{if } 0 \le x < \pi/2 \\ 1, & \text{if } x = \pi/2 \end{cases}}_{f(x)}$$

If $f_n(x) \rightrightarrows f(x)$ on $[0,\pi/2]$, then since ev $f_n \in C[0,\pi/2]$, f must be conti on $[0,\pi/2]$.
However since f is not conti at $x = \pi/2$, $f_n(x) \not\rightrightarrows f(x)$ on $[0,\pi/2]$.
Thus we cannot apply UCT. But

$$f_n(x) \to 0 \ \text{ a.e. on } [0,\pi/2] \ \text{ and}$$
$$|f_n(x)| \le 1 \qquad \forall x \in [0,\pi/2] \text{ and } \forall n$$

Hence by BCT,

$$\lim_{n\to\infty} \underbrace{\int_0^{\pi/2} \sin^n x\,dx}_{\text{Lebesgue}} = \underbrace{\int_0^{\pi/2} 0\,dx}_{\text{Lebesgue}} = 0$$

$$\| \leftarrow \text{ ev } \sin^n x \text{ is } R\text{-integrable on } [0,\pi/2]$$

$$\lim_{n\to\infty} \int_0^{\pi/2} \sin^n x\,dx$$

Ex3. Evaluate $\displaystyle\lim_{n\to\infty}\int_0^1 nx(1-x)^n\,dx$ (without using integration by parts)

Sol. Easy to check that $f_n(x)=nx(1-x)^n \ \to\ 0$ pointwise on $[0,1]$,

but $f_n(x)=nx(1-x)^n \ \not\rightrightarrows\ 0$ on $[0,1]$; Thus we cannot apply UCT.

It is easy to see that $f_n(x)$ has its maximum at $x=1/(n+1)$, whence $|f_n(x)|\le 1\ \ \forall x\in[0,1]$.

Hence by BCT,

$$\lim_{n\to\infty}\int_0^1 nx(1-x)^n\,dx \ = \ \int_0^1 0\,dx = 0$$

Remark. This limit is easily calculated by using integration by parts:

$$\int_0^1 nx(1-x)^n\,dx = \underbrace{nx\cdot\frac{-(1-x)^{n+1}}{n+1}\bigg]_0^1}_{=0} + \frac{n}{n+1}\int_0^1 (1-x)^{n+1}\,dx = \frac{n}{(n+1)(n+2)}$$

$$\therefore\quad \lim_{n\to\infty}\int_0^1 nx(1-x)^n\,dx = 0$$

Ex4. Evaluate $\displaystyle\lim_{n\to\infty}\int_0^1\frac{n\cos x}{1+n^2x^2}\,dx$

Sol. $\displaystyle\lim_{n\to\infty}\int_0^1\frac{n\cos x}{1+n^2x^2}\,dx \overset{nx=t}{=} \lim_{n\to\infty}\int_0^n\frac{\cos(t/n)}{1+t^2}\,dt = \lim_{n\to\infty}\int_{[0,\infty)}\frac{\chi_{[0,n]}(t)\cos(t/n)}{1+t^2}\,dt$

Note that

$$\frac{\chi_{[0,n]}(t)\cos(t/n)}{1+t^2}\ \to\ \frac{1}{1+t^2}\quad \text{(everywhere) on } [0,\infty)$$

and

$$\left|\frac{\chi_{[0,n]}(t)\cos(t/n)}{1+t^2}\right|\le\frac{1}{1+t^2}\quad\text{on } [0,\infty) \quad\text{and}\quad \int_{[0,\infty)}\frac{1}{1+t^2}\,dm=\int_0^\infty\frac{1}{1+t^2}\,dt=\frac{\pi}{2}<\infty$$

Thus by DCT,

$$\lim_{n\to\infty}\int_{[0,\infty)}\frac{\chi_{[0,n]}(t)\cos(t/n)}{1+t^2}\,dt=\int_{[0,\infty)}\frac{1}{1+t^2}\,dm=\frac{\pi}{2}$$

Ex5. Evaluate $\displaystyle\lim_{n\to\infty}\int_0^1\frac{n^{3/2}x}{1+n^2x^2}\,dx$

Sol. Easy to check that $f_n(x)=\dfrac{n^{3/2}x}{1+n^2x^2}\ \to\ 0$ pointwise on $[0,1]$,

but $\displaystyle\sup_{x\in[0,1]}|f_n(x)|=\sup_{x\in[0,1]}\frac{n^{3/2}x}{1+n^2x^2}\ \underset{x=\frac{1}{n}}{\ge}\ \frac{\sqrt{n}}{2}\to\infty$; $f_n(x)=\dfrac{n^{3/2}x}{1+n^2x^2}\ \not\rightrightarrows\ 0$ on $[0,1]$

Thus we cannot apply UCT.

However, the function $t(>0)\mapsto\dfrac{t^{3/2}x}{1+t^2x^2}$ is bounded by $\dfrac{3^{3/4}}{4\sqrt{x}}$ (not hard to check). Hence

$$|f_n(x)|=\frac{n^{3/2}x}{1+n^2x^2}\le\frac{3^{3/4}}{4\sqrt{x}}\quad\text{for every } n(\ge 1)\quad \&\quad \int_0^1\frac{3^{3/4}}{4\sqrt{x}}\,dx=\frac{3^{3/4}}{4}\int_0^1\frac{1}{\sqrt{x}}\,dx<\infty.$$

$$\therefore\quad \lim_{n\to\infty}\int_0^1\frac{n^{3/2}x}{1+n^2x^2}\,dx=\int_0^1 0\,dx = 0\quad\text{(by DCT)}$$

Another easy way:

$$\int_0^1\frac{n^{3/2}x}{1+n^2x^2}\,dx=\frac{1}{2\sqrt{n}}\int_0^1\frac{2n^2x}{1+n^2x^2}\,dx=\frac{1}{2\sqrt{n}}\ln\left(1+n^2x^2\right)\Big]_{x=0}^{x=1}=\frac{1}{2\sqrt{n}}\ln(1+n^2)\overset{\text{L'Hospital}}{\longrightarrow}0$$

⊙ An advanced convergence theorem on Riemann integral (not introduced in most elementary texts):

- Arzela's Bounded Convergence Theorem (**Arzela's BCT,** for short)

--- "Bartle and Sherbert's book "Introduction to Real Analysis", Second edition, p. 297" ---

Let $\{f_n\}$ be a sequence of Riemann-integrable functions on $[a, b]$. Assume that

(i) $f_n(x) \to$ (some function) $f(x)$ **pointwise** on $[a,b]$ & $f \in \mathcal{R}[a,b]$

[more generally, $f_n(x) \to$ (some function) $f(x)$ **almost everywhere** on $[a,b]$ & $f \in \mathcal{R}[a,b]$]

(ii) $\exists$ a constant $K > 0$ such that $|f_n(x)| \le K$ for all $x \in [a,b]$ and for all n.

$$\text{(i.e.,}\quad \sup_n \sup_{x\in[a,b]} |f_n(x)| \le K)$$

Then

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

Remark1. Its pf (without using L-integral) of the above theorem is quite delicate and will be omitted.

Remark2. Arzela's BCT is a special case of the corresponding BCT for L-integral (Check).

Ex1. Using only Riemann integration theory, evaluate $\lim_{n\to\infty} \int_0^{\pi/2} \sin^n x\,dx$

[Already settled by using BCT for L-integral]

Sol. $f_n(x) \overset{\text{let}}{=} \sin^n x \in C[0, \pi/2]$ $(\forall n)$ $\quad\therefore$ every $f_n \in \mathcal{R}[0, \pi/2]$

$$f_n(x) \quad\to\quad \underbrace{\begin{cases} 0, & \text{if } 0 \le x < \pi/2 \\ 1, & \text{if } x = \pi/2 \end{cases}}_{f(x)} \qquad \text{and} \quad f \in \mathcal{R}[0, \pi/2]$$

Moreover, $|f_n(x)| \le 1 \quad \forall x \in [0, \pi/2]$ and $\forall n$

Hence by Arzela's BCT, $\quad \lim_{n\to\infty} \int_0^{\pi/2} \sin^n x\,dx \;=\; \int_0^{\pi/2} f(x)\,dx \overset{\text{easy}}{=} 0$

Ex2. Evaluate $\lim_{n\to\infty} \int_0^1 nx(1-x)^n\,dx$ (without using integration by parts)

Sol. Easy to check that $f_n(x) = nx(1-x)^n \to 0$ pointwise on $[0,1]$,

but $f_n(x) = nx(1-x)^n \not\rightrightarrows 0$ on $[0,1]$; Thus we cannot apply UCT.

It is easy to see that $f_n(x)$ has its maximum at $x = 1/(n+1)$, whence $|f_n(x)| \le 1$ $\forall x \in [0,1]$.

Hence by Arzela's BCT,

$$\lim_{n\to\infty} \int_0^1 nx(1-x)^n\,dx \;=\; \int_0^1 0\,dx = 0\,.$$

HS: Let $f_n(x) = \dfrac{nx}{1+nx}$ for $x \in [0,1]$ $(n = 1, 2, \cdots)$

(i) Show that (f_n) converges **non**-uniformly to an integrable function f

(ii) Show that $\int_0^1 f(x)dx = \lim_{n\to\infty} \int_0^1 f_n(x)dx$ and determine the value $\int_0^1 f(x)dx$