Chap 13. Continuous functions on a compact intervals

13.1 Compact intervals

The principal goal is to prove three important **local-to-global** type theorems:

If $f(x)$ is **continuous** on any finite closed (= closed and bounded) interval I, then on I

(i) $f(x)$ is bounded

(ii) $f(x)$ has a maximum and minimum

(iii) $f(x)$ is **uniformly continuous** (the notion of uniform continuity will be introduced in section 13.5)

Note: Continuity (on I) is a local property but (i), (ii), & (iii) (i.e., boundedness, maximum (minimum) property, and uniform continuity on I) are global properties.

From now on, we call a finite closed (= closed and bounded) interval $[a,b]$ a **compact** interval.

Such intervals alone have the property called "sequential compactness"

Def. A set $S(\subseteq \mathbb{R})$ is said to be sequentially compact if every sequence of points in S has a subsequence converging to a point in S. (i.e., $\forall$ sequence (x_n) in S, $\exists$ a convergent subsequence (x_{n_i}) such that $\lim_{i\to\infty} x_{n_i} \in S$)

Theorem Sequential Compactness Theorem (SCT)

A compact interval $[a,b]$ is sequentially compact

Pf. Let $\{x_n\}$ be any seq in $[a,b]$. Then it is bounded since the interval is finite.

By BWT, it has a convergent subsequence $\{x_{n_i}\}$; set $\lim_{i\to\infty} x_{n_i} = c$

(--- boundedness of $[a,b]$ is used ---)

Since every $x_n \in [a,b]$, we have in particular $a \le x_{n_i} \le b$ for all i

Thus, by LLT (or by taking limits),

$$a \le \lim_{i\to\infty} x_{n_i} \le b \qquad \text{i.e., } c \in [a,b]$$

(--- closedness of $[a,b]$ is used ---)

Therefore, $[a,b]$ is sequentially compact

Remark. recall the different types of intervals:

$[a,b]$: finite closed i.e., compact

$$\left.\begin{array}{l} [a,\infty), (-\infty,a] : \text{ semi-infinite closed} \\ (a,b) : \text{ finite open} \\ (a,\infty), (-\infty,a) : \text{ semi-infinite open} \\ (a,b], [a,b) : \text{ finite half-open} \\ (-\infty,\infty) = \mathbb{R} : \text{ infinite open and closed} \end{array}\right\} \text{not compact}$$

For example,

$[a,\infty)$ (or (a,∞)) contains the sequence $\{n\}_{n_0}^{\infty}$ (with $n_0 > a$), which has no convergent subsequence

$\mathrm{I}=(a,b]$ (or (a,b)) contains a tail of the seq $\left\{a+\frac{1}{n}\right\}_{n_0}^{\infty}$, which converges to the point $a \notin \mathrm{I}$.

$\therefore$ any subsequence of $\left\{a+\frac{1}{n}\right\}_{n_0}^{\infty}$ also converges to $a \notin \mathrm{I}$, by the Subsequence Theorem.

$[a,b)$: consider $\left\{b-\frac{1}{n}\right\}_{n_0}^{\infty}$

13.2 Bounded continuous functions

Theorem (Boundedness Theorem)

If $f(x)$ is continuous on a compact interval I, then $f(x)$ is bounded on I

Pf. Suppose $f(x)$ is not bounded on I. Then

$f(x)$ is not bounded above on I or $f(x)$ is not bounded below on I.

Suppose first that $f(x)$ is not bounded above on I. Then

$$\begin{array}{lll} \exists\, x_1 \in \mathrm{I} & \text{s.t.} & f(x_1) > 1 \\ \exists\, x_2 \in \mathrm{I} & \text{s.t.} & f(x_2) > 2 \\ & \vdots & \\ \exists\, x_n \in \mathrm{I} & \text{s.t.} & f(x_n) > n \\ & \vdots & \end{array}$$

That is, $\exists$ a seq $\{x_n\}_1^{\infty}$ in I s.t. $f(x_n) > n$

$\{x_n\}_{n=1}^{\infty}$ is a seq in the compact interval I $\overset{\text{SCT}}{\Rightarrow}$ $\exists$ a subseq $\{x_{n_i}\}$ converging to a point $c \in \mathrm{I}$:

$$\lim_{i\to\infty} x_{n_i} = c, \quad \text{where} \quad c \in \mathrm{I}$$

We note first that, since $f(x_{n_i}) > n_i$,

$$\lim_{i\to\infty} f(x_{n_i}) \geq \lim_{i\to\infty} n_i = \infty \qquad i.e., \ \lim_{i\to\infty} f(x_{n_i}) = \infty$$

But since f is conti at $c \in \mathrm{I}$ and $\lim_{i\to\infty} x_{n_i} = c$,

$$\lim_{i\to\infty} f(x_{n_i}) = f(c) \quad \text{(by Sequential Continuity Theorem)}$$

This leads to a contradiction, since $c \in \mathrm{I}$ implies that $f(c)$ is definite and finite

$\therefore$ $f(x)$ must be *bounded above*

To show that $f(x)$ is also bounded below, we note that

$-f(x)$ is conti on the compact interval I

$$\overset{\text{the above result}}{\Rightarrow} \quad -f(x) \quad \text{is bounded above on I}$$

$$\text{i.e.,} \quad -f(x) < K \quad \text{for all } x \in \mathrm{I}$$

$$\Rightarrow \quad f(x) > -K \quad \text{for all } x \in \mathrm{I}$$

$\therefore$ $f(x)$ is *bounded below* on I

Remark. The conclusion in the Boundedness theorem would be false if "compact" were omitted.

For example,

$$f(x) = \frac{1}{x} \quad \text{is conti on } (0,1] \text{ but it is not bounded there}$$

Or

$$f(x) = x \quad \text{is conti on } [0,\infty) \text{ but it is not bounded there}$$

13.3 Extremal points of continuous functions

Theorem Maximum-Minimum theorem (최대-최소 정리)

Let $f(x)$ be continuous on the compact interval I. Then $\exists$ $\overline{x}$, $\underline{x} \in \mathrm{I}$ such that

$$f(\overline{x}) = \sup_{x\in \mathrm{I}} f(x), \qquad f(\underline{x}) = \inf_{x\in \mathrm{I}} f(x)$$

i.e., every conti ft $f(x)$ has a maximum and minimum on the compact interval I.

(Recall $M \overset{\text{let}}{=} \sup_{x\in \mathrm{I}} f(x) \quad \Rightarrow \quad f(x) \le M \quad \forall x \in \mathrm{I}$

Thus if $\exists\, \overline{x} \in \mathrm{I}$ s.t. $f(\overline{x}) = M$, then M becomes the maximum of $f(x)$ on I)

Pf. Since $f(x)$ is continuous on a compact interval I,

$f(x)$ is bounded on I (by the Boundedness Theorem)

$\therefore$ $M = \sup_{x\in \mathrm{I}} f(x)$ exists (by the Completeness Property for sets)

Then by the definition of the supremum, $f(x) \le M \quad \forall x \in \mathrm{I}$

We have to show that $\exists$ $\overline{x} \in \mathrm{I}$ s.t. $f(\overline{x}) = M$

To do this, for each integer $n > 0$, we can choose a point $x_n \in \mathrm{I}$ s.t.

$$(M \ge)\ f(x_n) \ge M - \frac{1}{n}$$

This is possible, since $M-\dfrac{1}{n}$ is not an upper bound for $f(x)$ on I

By the SCT, $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ converging to a point of I:

$$x_{n_i} \;\to\; \overline{x}, \quad \overline{x}\in \mathrm{I}$$

By the Squeeze theorem, we now have

$$\underbrace{M-\frac{1}{n_i}}_{\substack{\downarrow\\ M}} \le \underbrace{f(x_{n_i})}_{\substack{\therefore\,\downarrow\\ M}} \le \underbrace{M}_{\substack{\downarrow\\ M}}$$

This shows

$$\lim_{i\to\infty} f(x_{n_i}) = M \qquad ---(*)$$

On the other hand, since $f(x)$ is conti at $\overline{x}\in \mathrm{I}$ & $x_{n_i}\to \overline{x}$ $(as\ i\to\infty)$,

$$\lim_{i\to\infty} f(x_{n_i}) = f(\overline{x}) \qquad ---(**)$$ (by the Sequential Continuity Theorem)

$(*)$ & $(**)$ $\Rightarrow$ $f(\overline{x}) = M$.

To see that $f(x)$ also attains its minimum on I, we apply the above to $-f(x)$

Note that $-f(x)$ is continuous on the compact interval I

$$\overset{\text{the above}}{\Rightarrow} \quad -f(x) \text{ has a maximum point } \underline{x}\in \mathrm{I}$$

$$\Rightarrow \quad f(x) \text{ has a minimum point } \underline{x}\in \mathrm{I}$$

Remark.

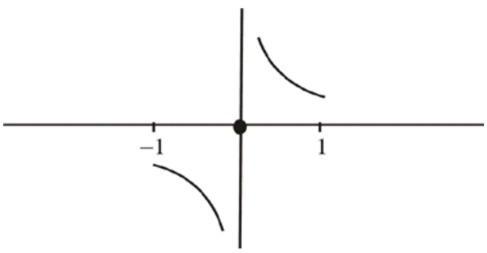
(i) The conclusion in the Max-Min theorem would be false if "compact" were omitted; for example,

$$f(x) = x \quad \text{has no max \& no min on } (0,1)$$

$$\text{has no max on } [0,\infty)$$

(ii) The conclusion in the Max-Min theorem would be false if "continuity" were omitted; for example,

$$f(x) = \begin{cases} 1/x & x\neq 0 \\ 0 & x=0 \end{cases} \quad \forall x\in \underbrace{[-1,1]}_{\text{cpt interval}}$$



Obviously $f(x)$ is discontinuous at 0, and it has no max or min on $[-1,1]$

13.4 The mapping view point (about conti functions)

Q: Suppose f is continuous. Is it true that

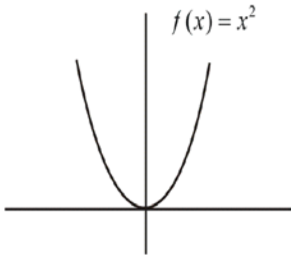
$$(i) \quad \text{open interval} \overset{?}{\longrightarrow} \text{open interval}$$

$$(ii) \quad \text{closed interval} \overset{?}{\longrightarrow} \text{closed interval}$$

$$(iii) \quad \text{bounded interval} \overset{?}{\longrightarrow} \text{bounded interval}$$

$$(iv) \quad \text{compact interval} \overset{?}{\longrightarrow} \text{compact interval}$$

$$(v) \quad \text{interval} \overset{?}{\longrightarrow} \text{interval}$$



$$f(x) = x^2 \quad \Rightarrow \quad f\{(-1,1)\} = [0,1) \qquad \therefore \ (i) \text{ is false}$$

$$f(x) = \frac{1}{1+x^2} \Rightarrow \quad f\{(-\infty,\infty)\} = (0,1] \quad \therefore \ (ii) \text{ is false}$$

$$f(x) = \tan x \quad (x \in (-\pi/2, \pi/2)) \quad \Rightarrow \quad f\{(-\pi/2,\pi/2)\} = (-\infty,\infty) \quad \therefore (iii) \text{ is false}$$

$$f(x) \equiv 1 \ \ (\forall x \in (-\infty,\infty)) \quad \Rightarrow \quad f\{\text{any interval}\} = \{1\}(\text{single point})$$

Note: IVT (사이값 정리): continuous fct maps interval $\longrightarrow$ an interval or a single point

(Ex: Prove this)

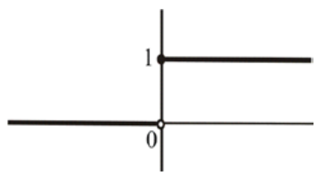
$$\therefore \quad (v) \text{ is true if we regard single point as an (trivial)interval}$$

Expect: any connected set in $\mathbb{R}$ = an interval or single point (trivial interval)

(Easy to expect)

Thus, **continuous function maps "connected sets"** $\longrightarrow$ **"connected sets"**

Remark.



$$f(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases} \quad (\text{unit step function}) \quad \Rightarrow f\{[-1,1]\} = \{0,1\} \ (\text{two points})$$

This example shows that if f is discontinuous, then the image of an interval under the map f need not be an interval.

The next theorem shows that (iv) is true.

Theorem **Continuity Mapping Theorem**

If $f(x)$ is defined and continuous on the compact interval I, then $f(\mathrm{I})$ is a compact interval; that is, **the continuous image of a compact interval is a compact interval.**

Pf. By the Max-Min theorem, $\exists \ \underline{x}, \overline{x} \in \mathrm{I}$ s.t.

$$f(\underline{x}) = m = \inf_{x\in \mathrm{I}} f(x), \qquad f(\overline{x}) = M = \sup_{x\in \mathrm{I}} f(x)$$

We shall prove $f(\mathrm{I}) = [m,\, M]$

$f(\mathrm{I}) \subset [m,\, M]$ is easy $\left(\because \ x_0 \in \mathrm{I} \Rightarrow m \le f(x_0) \le M \Rightarrow f(x_0) \in [m,\, M] \right)$

To prove $f(\mathrm{I}) \supset [m,\, M]$, we must show that

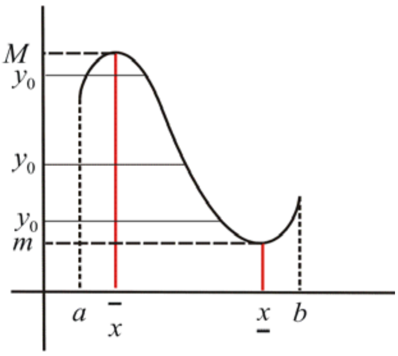
$$y_0 \in [m,\, M] \ \Rightarrow \ \exists\, x_0 \in \mathrm{I} \ \text{ s.t. } y_0 = f(x_0)$$

Note that

$$y_0 \in [m,\, M] \ \Leftrightarrow \ f(\underline{x}) \le y_0 \le f(\overline{x}), \quad \text{where } \ \underline{x}, \overline{x} \in \mathrm{I}$$

Since f is conti on I, f is conti on $[\underline{x}, \overline{x}]$ if $\underline{x} < \overline{x}$ or on $[\overline{x}, \underline{x}]$ if $\overline{x} < \underline{x}$

$\overset{\text{IVT}}{\Rightarrow}$ $\exists\, x_0$ between $\underline{x}$ and $\overline{x}$ $(\therefore x_0 \in \mathrm{I})$ s.t. $y_0 = f(x_0)$. Thus we are done



Often useful to remember:

$$f : \text{conti on } [a,b] \quad \Rightarrow \quad f\{[a,b]\} = [m,\, M],$$

where $m = \min\limits_{x\in[a,b]} f(x) (= \inf\limits_{x\in[a,b]} f(x)), \quad M = \max\limits_{x\in[a,b]} f(x) (= \sup\limits_{x\in[a,b]} f(x))$

A comment on the IVT:

A subset I of $\mathbb{R}$ is called an ***interval*** if whenever $a<c<b$ & $a,b\in I$, then $c\in I$

Every interval is one of the following forms:

$(a,b),\quad (a,b],\quad [a,b),\quad [a,b]\big(\text{where } a<b\big),\quad (a,\infty),\quad [a,\infty),\quad (-\infty,b),\quad (-\infty,b]$

Singleton sets are often regarded as ***degenerate*** intervals

Notice that if I_1 and I_2 are intervals with $I_1\cap I_2\neq\varnothing$ then $I_1\cup I_2$ is an interval.

Ex. Show that if f is continuous on an interval I, then $f(I)$ is an interval

Pf. Notice that IVT can be stated as follows:

Suppose that f is continuous on an interval I, and $a,b\in I$ with $a<b$, and that $f(a)<k<f(b)$

Then $a<\exists c<b$ such that $f(c)=k$ --- ♦

To show that $f(I)$ is an interval, we have to show that

whenever $r<k<s$ with $r,s\in f(I)$, then $k\in f(I)$

Obviously, $\exists a,b\in I$ s.t. $f(a)=r,\ f(b)=s$. May assume $a<b$

That is, $\exists a,b\in I$ with $a<b$ such that $f(a)<k<f(b)$

By IVT$[=$♦$]$, $\exists c\in(a,b)\subset I$ s.t. $f(c)=k$ --- this is what we wanted

Ex [optional].

Show that if f is continuous and strictly monotone on an **open** interval I, then $f(I)$ is an **open** interval.

Hint:

- I : an open interval and $x\in I \quad\Rightarrow\quad \exists a,b\in I$ with $a<x<b$
- •• $\forall x\in I[=\text{an interval}],\quad \exists a,b\in I$ with $a<x<b \quad\Rightarrow\quad I=\text{open interval}$

Pf. If I is an open interval and $x\in I$, then $\exists a,b\in I$ with $a<x<b$

Hence

either $f(x)\in\big(f(a),f(b)\big)\subset f(I)$ or $f(x)\in\big(f(b),f(a)\big)\subset f(I)$ $\quad[\leftarrow f$ is strictly monotone$]$

This shows that $f(I)$ is an open interval by ••

13.5 Uniform continuity

Uniform continuity is stronger than continuity

- **Continuity is a local property**
- **Uniform continuity is a global property**, formulated only for a function on an interval; "uniform continuity at a point" makes no sense

Def. We say that f is uniformly conti on the interval I (on the set $E(\neq \varnothing) \subset \mathbb{R}$) if:

given $\varepsilon > 0$, $\exists\, \delta > 0$ (depending only on ε) such that

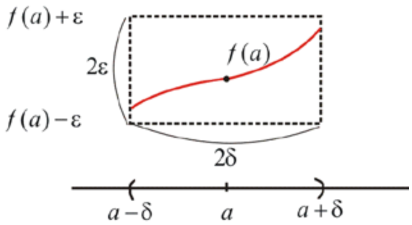
$$f(x') \underset{\substack{\varepsilon \\ \text{높이}}}{\approx} f(x'') \quad \textit{for} \quad x' \underset{\substack{\delta \\ \text{밑변}}}{\approx} x'', \quad x', x'' \in \mathrm{I} \ \ (\,E\,)$$

Recall: f is continuous on the interval I ($\overset{def}{\Leftrightarrow}$ f is continuous at every point $a \in \mathrm{I}$)

$\Leftrightarrow$ Given $a \in \mathrm{I}$ and given $\varepsilon > 0$, $\exists\, \delta = \delta(a, \varepsilon) > 0$ (may depending on ε & a) s.t.

$$f(x) \underset{\varepsilon}{\approx} f(a) \quad \text{for} \quad x \underset{\delta}{\approx} a, \quad x \in \mathrm{I}$$

(점을 고정할 때 마다 연속이라는 것을 의미함; 점을 먼저 고정하고 조사함)

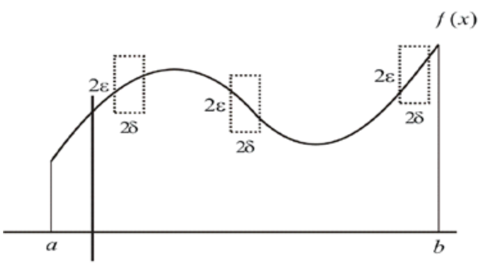


Meaning of the (pointwise) continuity : The curve $y = f(x)$ does <u>not touch</u> the <u>top</u> or <u>bottom</u> of the rectangle $(= 2\varepsilon \times 2\delta)$ which is centered at $f(a)$

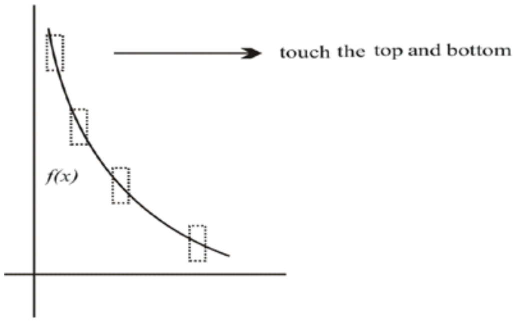
주의: δ (밑변의 길이)는 ε (세로의 길이) 뿐만 아니라 점 a 의 위치에 따라 변할 수 있다

※ 세로의 길이 $(2\varepsilon > 0)$ 가 주어졌을때, 곡선의 기울기의 절대값이 큰 부분일수록 위 조건을 만족하는 직사각형의 밑변의 길이 (2δ) 는 작다

(Rough) Meaning of the the uniform continuity: 점에 영향을 받지 않는 밑변의 길이 [즉, 세로의 길이에만 영향을 받는 직사각형]가 존재한다



$f(x)$ is uniformly continuous on $[a, b]$



Expect $f(x)$ is not uniformly conti

- f is uniformly conti on I $\Leftrightarrow$ $\sup\limits_{\substack{|x'-x''|<\delta \\ x',x''\in \mathrm{I}}} |f(x')-f(x'')| \to 0$ as $\delta \to 0$

- f is conti on I $\Leftrightarrow$ For each $a \in \mathrm{I}$, $\sup\limits_{\substack{|x-a|<\delta \\ x\in \mathrm{I}}} |f(x)-f(a)| \to 0$ as $\delta \to 0$

Exa 1. $f(x)=x^2$ is uniformly conti on $[-a, a]$, $a>0$.

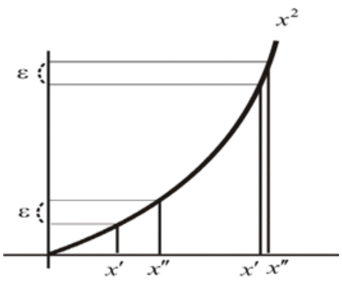
Pf. Let $\varepsilon>0$ be given. Then for $x', x'' \in [-a, a]$,

$$\begin{aligned} |f(x')-f(x'')| = |x'^2 - x''^2| &= |x'-x''||x'+x''| \\ &\le |x'-x''|(|x'|+|x''|) \\ &\le 2a|x'-x''| < \varepsilon \quad \text{if} \quad |x'-x''| < \frac{\varepsilon}{2a} (=\delta) \end{aligned}$$

That is, $f(x') \underset{\varepsilon}{\approx} f(x'')$ for $x' \underset{\varepsilon/2a}{\approx} x''$, $x', x'' \in [-a, a]$

$\therefore$ $f(x)=x^2$ is uniformly conti on $[-a, a]$.

Exa 2. Show that $f(x)=x^2$ is not uniformly conti on $[0,\infty)$



Pf. Suppose to the contrary that f is uniformly conti on $[0,\infty)$. Then

$$\exists\, \delta>0 \ \text{ s.t. } \ |x'^2 - x''^2| < 1 \ \text{ if } \ |x'-x''|<\delta, \quad x', x'' \in [0,\infty)$$

By the A.P., $\exists$ a natural number n so large that $n\delta > 1$.

$$\text{Set } x' = n \quad \text{and} \quad x'' = n + \frac{\delta}{2}. \text{ Then } |x' - x''| = \frac{\delta}{2} < \delta \quad \text{but}$$

$$1 > |x'^2 - x''^2| = \left|n^2 - (n + \frac{\delta}{2})^2\right| = n\delta + \frac{\delta^2}{4} > n\delta > 1, \text{ is a contradiction}$$

Remember: f is uniformly conti on I $\Rightarrow$ f is conti on I.

※ **Standard examples of uniformly continuous functions**

1. **Lipschitz functions** (often called Lipschitz continuous functions)

Suppose $f : \mathrm{I} \to \mathbb{R}$ is a Lipschitz function, that is,

$$\exists \ M > 0 \quad \text{s.t.} \quad |f(x) - f(y)| \leq M|x - y| \quad \text{for all } x, y \in \mathrm{I}$$

Then f is uniformly continuous on I

Pf. Given $\varepsilon > 0$,

$$|f(x) - f(y)| \leq M|x - y| < \varepsilon \quad \text{if} \quad |x - y| < \underbrace{\frac{\varepsilon}{M}(= \delta)}_{\text{(depends only on } \varepsilon)} \quad \text{and} \quad x, y \in \mathrm{I}$$

$$i.e., \quad f(x) \underset{\varepsilon}{\approx} f(y) \quad \text{for} \quad x \underset{\varepsilon/M}{\approx} y, \quad x, y \in \mathrm{I}$$

Therefore, f is uniformly continuous on I

Examples: $ax \ (a : real)$, $\sin x$, $\cos x$, $\sin^2 x$, $\cos^2 x$, $\dfrac{1}{1 + x^2}$ are Lipschitz fcts

For instance, if $f(x) = \dfrac{1}{1 + x^2}$, then $\exists \ \xi$ between x and y such that

$$\begin{aligned} f(x) - f(y) &= f'(\xi)(x - y) \quad (\text{by MVT}) \\ &= -\frac{2\xi}{(1 + \xi^2)^2}(x - y) \end{aligned}$$

$$\therefore \quad |f(x) - f(y)| = \frac{2\,|\,\xi\,|}{1 + \xi^2} \cdot \frac{1}{1 + \xi^2}\,|\,x - y\,| \leq 1 \cdot 1 \cdot |\,x - y\,| \quad \text{for all } x, y \in \mathbb{R}$$

Remark: f is diff on I and $|f'(x)| \leq M \ \ \forall x \in \mathrm{I} \ \Rightarrow \ f$ is Lipschitz on I

Ex (easy). **Give a geometric interpretation of Lipschitz function**

Remark.

① $f : \mathrm{I} \to \mathbb{R}$ is such that

$$\exists \ M > 0 : \quad |f(x) - f(y)| \leq M|x - y|^{\alpha} \quad (0 < \alpha < 1)$$

$\Rightarrow$ f is uniformly conti on I

② $f : \mathrm{I} \to \mathbb{R}$ is such that

$$\exists \ M > 0 : \quad |f(x) - f(y)| \le M|x - y|^{\alpha} \quad (\alpha > 1)$$

$\Rightarrow$ f is constant on I

Pf. ① Given $\varepsilon > 0$,

$$|f(x) - f(y)| \le M|x - y|^{\alpha} < \varepsilon \quad \text{if} \quad |x - y| < \underbrace{\left(\frac{\varepsilon}{M}\right)^{1/\alpha}}_{\equiv \delta (\text{depends only on } \varepsilon)} \quad \& \quad x, y \in \mathrm{I}$$

$$\therefore \quad f(x) \underset{\varepsilon}{\approx} f(y) \quad \text{for} \quad x \underset{\delta}{\approx} y, \ \ x, y \in \mathrm{I}$$

② Suppose $\alpha > 1$ and let $y \in \mathrm{I}$ be fixed. Then the hypo $\Rightarrow$

$$\left| \frac{f(x) - f(y)}{x - y} \right| \le M|x - y|^{\alpha - 1} \qquad \forall x, y \in \mathrm{I} \ \text{ with } \ x \ne y$$

$$\therefore \quad \lim_{x \to y} \left| \frac{f(x) - f(y)}{x - y} \right| \le M \lim_{x \to y} |x - y|^{\alpha - 1} = 0 \quad (\because \ \alpha > 1)$$

$$\text{LHS} = \left| \lim_{x \to y} \frac{f(x) - f(y)}{x - y} \right| \qquad (\text{by the continuity of } | \ \ |)$$

i.e., $f'(y) = 0 \quad \forall y \in \mathrm{I}. \qquad \therefore \ f = \text{constant on I}$

Note. In general, f is Lipschitz on I $\not\Leftarrow$ f is diff on I

For example, $f(x) = |\ x\ |$ is Lipschitz on $[-1, 1]$ (easy to check), but clearly

the function $|\ x\ |$ is not diff at the point 0.

Ex. Already seen that if f is diff & has bounded derivative on I, then f is Lipschitz on I.
However, in general, f is diff on I $\not\Rightarrow$ f is Lipschitz on I : Give such an example

Ex. Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

2. Uniform Continuity Theorem (= UCT)

If I is a compact interval, then
f is conti on I $\Rightarrow$ f is uniformly conti on I

Pf. Suppose to the contrary that f is not uniformly continuous on I.

$$\boxed{\begin{array}{c} \forall \delta > 0, \quad \exists \ \text{ a pair of points } \ x', x'' \in \mathrm{I} \ \text{ s.t.} \\ |x' - x''| < \delta, \quad \text{but} \quad |f(x') - f(x'')| \ge \varepsilon_0 \quad \text{for some} \ \ \varepsilon_0 > 0 \end{array}}$$

In particular, the above property holds for $\delta = \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \cdots, \dfrac{1}{n}, \cdots.$

In other words, for every positive integer $n \geq 2$, $\exists$ a pair of points $x_n', x_n'' \in \mathrm{I}$ s.t.

$$\begin{aligned} &(1) \quad \left|x_n' - x_n''\right| < \frac{1}{n}, \text{ but} \\ &(2) \quad \left|f(x_n') - f(x_n'')\right| \geq \varepsilon_0 \end{aligned}$$

Since I is compact, the Sequential Compactness Theorem says the sequence $\{x_n'\}$ has a convergent subsequence $\{x_{n_i}'\}$ converging to a point $c \in \mathrm{I}$:

$$(3) \quad \lim_{i\to\infty} x_{n_i}' = c, \quad c \in \mathrm{I}$$

$$\text{Also,} \ (4) \quad \lim_{i\to\infty} \left(x_{n_i}' - x_{n_i}''\right) = 0 \quad (\text{by } (1))$$

Then we also have

$$\lim_{i\to\infty} x_{n_i}'' = c \quad \left(\because \ x_{n_i}'' = (x_{n_i}'' - x_{n_i}') + x_{n_i}' \to 0 + c = c\right)$$

We now show $f(x)$ is not continuous at $c \, [\in I]$.

If f were continuous at c, then the Sequential Continuity Theorem, together with (3) & (4), would imply that

$$f(x_{n_i}') - f(x_{n_i}'') \quad \to \quad f(c) - f(c) = 0 \quad \text{as} \quad i \to \infty$$

Therefore

$$\left|f(x_{n_i}') - f(x_{n_i}'')\right| < \varepsilon_0 \quad \text{for} \quad i \gg 1, \text{ which contradicts } (2).$$

Thus $f(x)$ is not continuous at c. This completes the proof by contraposition

Remark.

Theorem (A useful criterion for non-uniform continuity)

Let $f : \mathrm{I} \to \mathbb{R}$ be a function. Then

f is not uniformly conti on I if and only if

$\exists$ $\varepsilon_0 > 0$ and a pair of sequences $\{x_n'\}$ and $\{x_n''\}$ in I such that

$$x_n' - x_n'' \to 0 \text{ as } n \to \infty, \quad \text{but} \quad \left|f(x_n') - f(x_n'')\right| \geq \varepsilon_0 \quad \text{for every } n$$

Pf. $(\Rightarrow)$ Already seen

$(\Leftarrow)$ Assume that the latter holds. Then, by the first part, given $\delta > 0$,

$$x_n', x_n'' \in \mathrm{I} \quad \text{and} \quad x_n' \underset{\delta}{\approx} x_n'' \quad \text{for } n \gg 1, \quad \text{say for } n \ge N$$

$$\left(\text{In particular, } \quad x_N', x_N'' \in \mathrm{I} \quad \text{and} \quad x_N' \underset{\delta}{\approx} x_N'' \right)$$

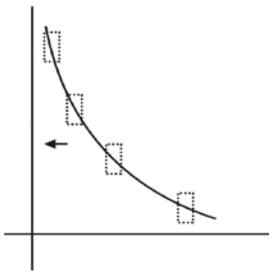
Consequently,

$$\forall \delta > 0, \quad \exists \text{ a pair of points } x_N', x_N'' \in \mathrm{I} \quad \text{such that}$$

$$x_N' \underset{\delta}{\approx} x_N'', \quad \textit{but} \quad \left| f(x_N') - f(x_N'') \right| \ge \varepsilon_0 \quad (\text{for some } \varepsilon_0 > 0)$$

$\therefore \ f$ is not uniformly conti on I

Exa A. $f(x) = \dfrac{1}{x}$ is conti (already seen) but not uniformly conti on $(0, \infty)$



Pf. Choose the sequences $\{x_n'\}$ and $\{x_n''\}$ in $(0, \infty)$ as

$$x_n' = \frac{1}{n} \quad \text{and} \quad x_n'' = \frac{1}{n+1} \quad (n = 1, 2, \cdots)$$

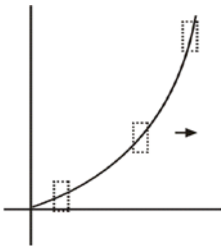
Then $\quad x_n' - x_n'' = \dfrac{1}{n} - \dfrac{1}{n+1} = \dfrac{1}{n(n+1)} \to 0 \quad \text{as} \quad n \to \infty$

But

$$\left| f(x_n') - f(x_n'') \right| = |\, n - (n+1) \,| = 1 \ge \varepsilon_0 (\equiv 1/2) \quad \text{for every } n$$

$\therefore \ f$ is not uniformly conti on $(0, \infty)$

Exa B. $f(x) = x^2$ is not uniformly conti on $[0, \infty)$



First pf. An indirected proof was previously given in Exa 2

Second pf. Let $x_n' = n + \dfrac{1}{n}$, $x_n'' = n$ $(n = 1, 2, \cdots)$

Then $\{x_n'\}$ and $\{x_n''\}$ are two sequences in $[0, \infty)$ such that

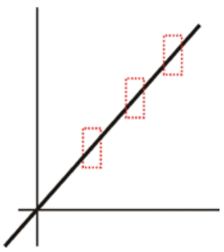
$$x_n' - x_n'' = \frac{1}{n} \to 0 \quad \text{as} \quad n \to \infty,$$

but

$$\left| f(x_n') - f(x_n'') \right| = (n + \frac{1}{n})^2 - n^2 = 2 + \frac{1}{n^2} \geq 2 (\equiv \varepsilon_0) \quad \text{for every } n$$

$\therefore$ f is not uniformly conti on $[0, \infty)$

Exa C. $f(x) = x$ is uniformly conti on $(-\infty, \infty)$



Pf. $| f(x) - f(y) | = | x - y |$ for all $x, y \in (-\infty, \infty)$

Thus, given $\varepsilon > 0$,

$$| f(x) - f(y) | = | x - y | < \varepsilon \quad \text{whenever} \quad | x - y | < \varepsilon (\equiv \delta)$$

$\therefore$ f is uniformly conti on $(-\infty, \infty)$

In fact, f is Lipschitz continuous on $(-\infty, \infty)$

Exa D. $f(x) = x^2$ is uniformly conti on $[0, b]$, where $b > 0$

Pf. f is conti on $[0, b]$ & $[0, b]$ is a compact interval

$\overset{\text{UCT}}{\Rightarrow}$ f is uniformly conti on $[0, b]$

"Another pf"

$$\begin{aligned} | f(x) - f(y) | = | x^2 - y^2 | &= | x - y || x + y | \\ &\leq | x - y | (| x | + | y |) \\ &\leq 2b | x - y | \quad \forall x, y \in [0, b] \end{aligned}$$

$\therefore$ f is Lipschitz conti on $[0, b]$

$\therefore$ f is uniformly conti on $[0, b]$

Remark. $f(x) = x^2$ is uniformly conti on $(0, b)$, where $b > 0$

Pf 1. f is uniformly conti on $[0, b]$ by UCT

$\therefore$ f is uniformly conti on the smaller interval $(0, b)$

Pf 2.

$$\begin{aligned} | f(x) - f(y) | = | x^2 - y^2 | &= | x - y || x + y | \\ &\leq | x - y | (| x | + | y |) \\ &< 2b | x - y | \quad \forall x, y \in (0, b) \end{aligned}$$

$\therefore$ f is Lipschitz conti on $(0, b)$

$\therefore$ f is uniformly conti on $(0, b)$

Exa E. $f(x) = \sqrt{x}$ is uniformly conti on $[1, \infty)$

Pf. $| f(x) - f(y) | = | \sqrt{x} - \sqrt{y} | = \dfrac{| x - y |}{\sqrt{x} + \sqrt{y}} \leq \dfrac{1}{2} | x - y | \quad \forall x, y \in [1, \infty)$

$\therefore$ f is Lipschitz conti on $[1, \infty)$

$\therefore$ f is uniformly conti on $[1, \infty)$

Exa F. Not every uniformly continuous function is Lipschitz

Sol. $f(x) = \sqrt{x}$ is uniformly conti on $[0, 2]$ (by UCT)

Claim: f is not Lipschitz conti on $[0, 2]$

Pf of Claim: Suppose f were Lipschitz conti on $[0, 2]$. Then

$$\exists\, M > 0 \text{ such that } | f(x) - f(y) | \leq M | x - y | \quad \forall x, y \in [0, 2]$$

In particular (by taking $y = 0$), we have

$$| f(x) | \leq M | x | \quad \forall x \in [0, 2]$$

$$\therefore \quad \frac{| f(x) |}{| x |} \leq M \quad \forall x \in (0, 2]$$

Recall that M is independent of $x \in (0,2]$

Letting $x \to 0^+$ $\Rightarrow$ $\displaystyle\lim_{x \to 0^+} \frac{| f(x) |}{| x |} \leq M$

This contradicts the fact that

$$\frac{| f(x) |}{| x |} = \frac{\sqrt{x}}{| x |} = \frac{1}{\sqrt{x}} \to \infty \quad \text{as} \quad x \to 0^+$$

Therefore, f is not Lipschitz conti on $[0, 2]$

Exa. Show that $f(x) = x \sin \dfrac{1}{x}$ is u.c. on $(0,1)$

Pf. $$F(x) \overset{\text{def}}{=} \begin{cases} x \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

It is obvious that $\displaystyle\lim_{x \to 0} F(x) = 0 = F(0)$.

It follows that $F(x)$ is continuous $\forall x \in (-\infty, \infty)$

In particular, $F(x)$ is conti on the compact interval $[0, 1]$

Thus $F(x)$ is u.c. on $[0, 1]$ (by UCT), and so $F(x)$ is u.c. on $(0, 1)$

But, since $F(x) = x \sin \frac{1}{x} = f(x)$ on $(0, 1)$, $f(x)$ is u.c. on $(0, 1)$

Remark. Assume f is continuous on (a, b).

If, in addition, $\lim_{x \to a^+} f(x)$ and $\lim_{x \to b^-} f(x)$ both exist, then f is uniformly continuous on (a, b)

Question. f is **conti & bounded** on an interval I $\overset{?}{\Rightarrow}$ f is **u.c.** on I

Ans

Yes if I is any compact interval (by UCT). In fact, in that case, the boundedness of f on I is not necessary (automatically satisfied by Boundedness theorem)

but no in general if I is *not* a compact interval.

For example, $f(x) \overset{\text{let}}{=} \sin \frac{1}{x}$ on the open interval $(0, \infty)$

Then $f(x)$ is conti & bounded (by 1) on $(0, \infty)$

However, $f(x)$ is not u.c. on $(0, \infty)$ (roughly) because it is too oscillates near 0

(Draw the picture of $f(x)$)

To give a rigorous pf, take $x_n' = \frac{1}{n\pi}$, $x_n'' = \frac{1}{2n\pi + \pi/2}$ $(n = 1, 2, \cdots)$.

Then $\{x_n'\}$ and $\{x_n''\}$ are two sequences in $(0, \infty)$ such that

$$x_n' - x_n'' \to 0 \quad \text{as} \quad n \to \infty,$$

But $\left| f(x_n') - f(x_n'') \right| = 1 \geq \frac{1}{2} (\equiv \varepsilon_0)$ for every n

$$\therefore \ f(x) \text{ is not u.c. on } (0, \infty)$$

Home-Study problems.

1. Find an example of a continuous function $f : \mathbb{R} \to [-1, 1]$ such that f is not uniformly continuous.

Answer. $f(x) := \cos(x^2)$ $\left[\text{or } f(x) := \sin(x^2)\right]$ is the desired example --- verify this

2. Let $f(x) = 2\sqrt{x} - 3\sin x + \ln(x^2 + 1)$, $I = [1, \infty)$

   Is the function f uniformly continuous on I?

Ex. Show that $f(x) = \sqrt{x}$ is uniformly conti on $[0, \infty)$.

Pf. We know that

$f(x) = \sqrt{x}$ is uniformly conti on $[0, 1]$ (by **UCT**)

and

$f(x) = \sqrt{x}$ is uniformly conti on $[1, \infty)$. $[\leftarrow f(x) = \sqrt{x}$ is Lipschitz conti on $[1, \infty)]$

Hence, given any $\varepsilon > 0$, there is a $\delta_1 = \delta_1(\varepsilon) > 0$ such that

$$x, y \in [0, 1], \ |x - y| < \delta_1 \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon .$$

There is also a $\delta_2 = \delta_2(\varepsilon) > 0$ such that

$$x, y \in [1, \infty), \quad |x - y| < \delta_2 \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon .$$

Now define $\delta := \min\{\delta_1(\varepsilon), \delta_2(\varepsilon)\}$ and let $x, y \in [0, \infty)$ be such that $|x - y| < \delta$.

If both x & $y \in [0,1]$, or if both x & $y \in [1, \infty)$, then it is clear that $\left|f(x) - f(y)\right| < \varepsilon$

For the remaining case, we may suppose without essential loss of generality that $x < 1 < y$. Then

$$|1 - x| < |y - x| < \delta \le \delta_1 \quad \text{and so} \quad \left|f(1) - f(x)\right| < \varepsilon$$

Similarly,

$$|y - 1| < |y - x| < \delta \le \delta_2 \quad \text{and so} \quad \left|f(y) - f(1)\right| < \varepsilon$$

Therefore,

$$\left|f(x) - f(y)\right| \le \left|f(x) - f(1)\right| + \left|f(1) - f(y)\right| < \varepsilon + \varepsilon = 2\varepsilon$$

Another (lucky) pf. For any $x, y \in [0, \infty)$, we have

$$\left|f(x) - f(y)\right|^2 = \left|\sqrt{x} - \sqrt{y}\right|^2 \le \left|\sqrt{x} - \sqrt{y}\right|\left|\sqrt{x} + \sqrt{y}\right| = |x - y|$$

$$\therefore \quad \left|f(x) - f(y)\right| \le |x - y|^{1/2}$$

Let $\varepsilon > 0$ be given. Take $\delta = \varepsilon^2$. Then

$$|x - y| < \delta \quad \Rightarrow \quad \left|f(x) - f(y)\right| \le |x - y|^{1/2} < \sqrt{\delta} = \varepsilon$$

Proposition [**A criterion for non-uniform continuity: essentially proved earlier**]

--- Remember the result ---

Let $f : I[\subset \mathbb{R}] \to \mathbb{R}$ be the function. Then

$$f \text{ is uniformly continuous on } I \quad \Leftrightarrow \quad \begin{cases} \forall \text{ two sequences } \{u_n\} \ \& \ \{v_n\} \text{ such that} \\ \lim\limits_{n\to\infty}(u_n - v_n) = 0 \quad \Rightarrow \quad \lim\limits_{n\to\infty}[f(u_n) - f(v_n)] = 0 \end{cases}$$

Pf. $(\Rightarrow)$ Let $\varepsilon > 0$. Since f is u.c. on I, $\exists \delta > 0$ such that

$$x, y \in I \ \& \ |x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon \ \text{---}\blacksquare$$

Suppose $\{u_n\}$ & $\{v_n\}$ are two sequences in I such that $\lim(u_n - v_n) = 0$

$\Rightarrow$ $\exists N[= N(\delta) = N(\varepsilon)] \in \mathbb{N}$ such that $|u_n - v_n| < \delta$ for $\forall n \ge N$

$$\therefore \quad |f(u_n) - f(v_n)| < \varepsilon \ \text{ for every } n \ge N \quad [\leftarrow\blacksquare]$$

Therefore, $\lim\limits_{n\to\infty}[f(u_n) - f(v_n)] = 0$

$(\Leftarrow)$ Suppose f is not uniformly continuous on I.

$\Rightarrow$ $\exists \varepsilon_0 > 0$ such that $\forall \delta > 0$, $\exists x_\delta, y_\delta \in I$ for which $|x_\delta - y_\delta| < \delta$ & $|f(x_\delta) - f(y_\delta)| \ge \varepsilon_0$

Set $\delta = 1$ $\Rightarrow$ $\exists x_1, y_1 \in I$ for which $|x_1 - y_1| < 1$ & $|f(x_1) - f(y_1)| \ge \varepsilon_0$

Set $\delta = 1/2$ $\Rightarrow$ $\exists x_2, y_2 \in I$ for which $|x_2 - y_2| < 1/2$ & $|f(x_2) - f(y_2)| \ge \varepsilon_0$

In general, set $\delta = 1/n$ $\Rightarrow$ $\exists x_n, y_n \in I$ for which $|x_n - y_n| < 1/n$ & $|f(x_n) - f(y_n)| \ge \varepsilon_0$

Consequently, we have two sequences $\{x_n\}$ & $\{y_n\}$ in I s.t.

$$(x_n - y_n) \to 0 \ \text{ but } \ \left(f(x_n) - f(y_n)\right) \not\to 0 \ \text{ as } n \to \infty$$