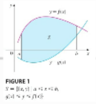
6.1 Areas Between Curves



$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x = \int_a^b f(x) - g(x)\, dx$$

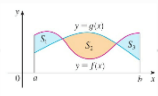


FIGURE 11

- If we are asked to find the area between the curves $y = f(x)$ and $y = g(x)$ where $f(x) \ge g(x)$ for some values of x but $g(x) \ge f(x)$ for other values of x, then

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{when } f(x) \ge g(x) \\ g(x) - f(x) & \text{when } g(x) \ge f(x) \end{cases}$$

$\Rightarrow$ the area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)|\, dx$$

6.2 Volumes

- if the area of the base is A and the height of the cylinder is defined as $V = Ah$
- Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the volume of S is $V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\, dx$

Volume of a Sphere

$$V = \frac{4}{3}\pi r^3$$

Solids of Revolution

$$V = \int_a^b A(x)\, dx, \quad \text{where } A = \pi(\text{Outer Radius})^2 - \pi(\text{Inner Radius})^2$$

We now find the volumes of three solids that are *not* solids of revolution.

EXAMPLE 7 Figure 12 shows a solid with a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.

SOLUTION Let's take the circle to be $x^2 + y^2 = 1$. The solid, its base, and a typical cross-section at a distance x from the origin are shown in Figure 13.

Since B lies on the circle, we have $y = \sqrt{1 - x^2}$ and so the base of the triangle ABC is $|AB| = 2y = 2\sqrt{1 - x^2}$. Since the triangle is equilateral, we see from Figure 13(c)

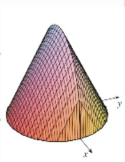
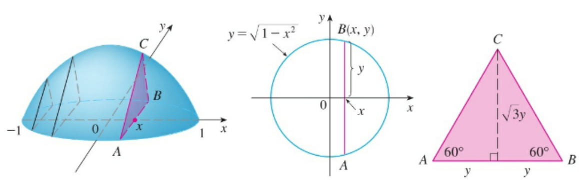


FIGURE 12 Computer-generated picture of the solid in Example 7



(a) The solid (b) Its base (c) A cross-section

FIGURE 13

that its height is $\sqrt{3}\,y = \sqrt{3}\sqrt{1 - x^2}$. The cross-sectional area is therefore

$$A(x) = \tfrac{1}{2} \cdot \underbrace{2\sqrt{1 - x^2}}_{|A-B|} \cdot \underbrace{\sqrt{3}\sqrt{1 - x^2}}_{\text{height of triangle}} = \sqrt{3}\,(1 - x^2)$$

and the volume of the solid is

$$V = \int_{-1}^{1} A(x)\, dx = \int_{-1}^{1} \sqrt{3}\,(1 - x^2)\, dx$$

$$= 2\int_0^1 \sqrt{3}\,(1 - x^2)\, dx = 2\sqrt{3}\left[x - \frac{x^3}{3}\right]_0^1 = \frac{4\sqrt{3}}{3}$$ ■

EXAMPLE 8 Find the volume of a pyramid whose base is a square with side L and whose height is h.

SOLUTION We place the origin O at the vertex of the pyramid and the x-axis along its central axis as in Figure 14. Any plane P_x that passes through x and is perpendicular to the x-axis intersects the pyramid in a square with side of length s, say. We can express s in terms of x by observing from the similar triangles in Figure 15 that

$$\frac{x}{h} = \frac{s/2}{L/2} = \frac{s}{L}$$

and so $s = Lx/h$. [Another method is to observe that the line OP has slope $L/(2h)$ and so its equation is $y = Lx/(2h)$.] Therefore the cross-sectional area is

$$A(x) = s^2 = \frac{L^2}{h^2}x^2$$

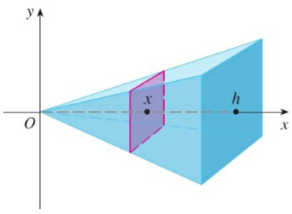


FIGURE 14

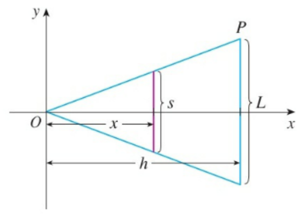


FIGURE 15

The pyramid lies between $x = 0$ and $x = h$, so its volume is

$$V = \int_0^h A(x)\,dx = \int_0^h \frac{L^2}{h^2}x^2\,dx = \frac{L^2}{h^2}\frac{x^3}{3}\Big]_0^h = \frac{L^2h}{3}$$

EXAMPLE 9 A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge.

SOLUTION If we place the x-axis along the diameter where the planes meet, then the base of the solid is a semicircle with equation $y = \sqrt{16 - x^2}$, $-4 \leqslant x \leqslant 4$. A cross-section perpendicular to the x-axis at a distance x from the origin is a triangle ABC, as shown in Figure 17, whose base is $y = \sqrt{16 - x^2}$ and whose height is $|BC| = y\tan 30° = \sqrt{16 - x^2}/\sqrt{3}$. So the cross-sectional area is

$y^2 + x^2 = r^2$

* equation of radius

$$A(x) = \tfrac{1}{2}\sqrt{16 - x^2} \cdot \frac{1}{\sqrt{3}}\sqrt{16 - x^2} = \frac{16 - x^2}{2\sqrt{3}}$$

and the volume is

$$V = \int_{-4}^{4} A(x)\,dx = \int_{-4}^{4} \frac{16 - x^2}{2\sqrt{3}}\,dx$$

$$= \frac{1}{\sqrt{3}}\int_0^4 (16 - x^2)\,dx = \frac{1}{\sqrt{3}}\left[16x - \frac{x^3}{3}\right]_0^4$$

$$= \frac{128}{3\sqrt{3}}$$

For another method see Exercise 64.

6.3 Volumes by Cylindrical Shells

$V = 2\pi \cdot r \cdot h \cdot \Delta r$, $r = \frac{r_2 + r_1}{2}$, $\Delta r = r_2 - r_1$

$= [\text{Circumference}][\text{height}][\text{Thickness}]$

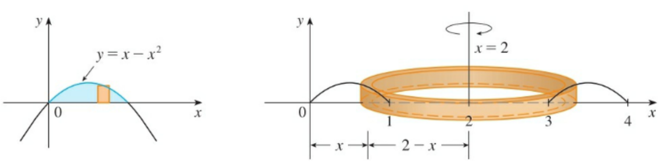
$\Rightarrow V_i = (2\pi \bar{x}_i)[f(\bar{x}_i)]\Delta x$

$\approx V = \lim_{n\to\infty} \sum_{i=1}^{n} V_i = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i)\Delta x$

$= \int_a^b 2\pi x f(x)\,dx$

EXAMPLE 4 Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

SOLUTION Figure 10 shows the region and a cylindrical shell formed by rotation about the line $x = 2$. It has radius $2 - x$, circumference $2\pi(2 - x)$, and height $x - x^2$.



The volume of the given solid is

$$V = \int_0^1 2\pi(2 - x)(x - x^2)\,dx$$

$$= 2\pi\int_0^1 (x^3 - 3x^2 + 2x)\,dx$$

$$= 2\pi\left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 = \frac{\pi}{2}$$

6.4 Work

$F = ma = m\frac{d^2s}{dt^2}$, m = mass , a = acceleration , F = force

or, in newtons,

$N = kg \cdot m/s^2$

$\Rightarrow W = Fd \approx$ Work = Force · distance

If F is measured in newtons and d in meters, then the unit for W is a newton-meter, which is called a joule (J). If F is measured in pounds and d in feet, then the unit for W is a foot-pound (ft-lb), which is about 1.36 J.

- the above equation defines work as long as the force is constant, but what happens if the force is variable? Suppose the object moves along the x-axis in the positive direction in the interval $[a,b]$ with equal width Δx,

$$\Rightarrow W_i \approx f(x_i^*)\Delta x$$

$$\Rightarrow W \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx$$

6.5 Average Value of a Function

Mean Value Theorem for Integrals:

- if f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ such that

$$f(c) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

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