4.1 Maximum and Minimum Values

Absolute Maximum :

- value of f on D if $f(c) \geq f(x)$ for all x in D

Absolute Minimum :

- value of f on D if $f(c) \leq f(x)$ for all x in D

☆ An absolute maximum or minimum is sometimes called a global maximum or minimum. The maximum and minimum values of f are called "extreme values of f."

Local Maximum :

- value of f if $f(c) \geq f(x)$ when x is near c

Local Minimum :

- value of f if $f(c) \leq f(x)$ when x is near c

☆ point at the tip of the given interval cannot be local maximum/minimum value. They can only be absolute values.

Extreme Value Theorem :

- If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$

☆ a discontinuous function could have maximum and minimum values.

Fermat's Theorem :

- If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$

☆ we can't always expect to locate extreme values simply by setting $f'(x) = 0$

Critical Number :

- a critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist

☆ If f has a local maximum or minimum at c, then c is a critical number of f

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

4.2 The Mean Value Theorem

Rolle's Theorem:

- Let f be a function that satisfies the following three hypothesis
  1. f is continuous on the closed interval $[a, b]$
  2. f is differentiable on the open interval (a, b)
  3. $f(a) = f(b)$

  Then there is a number c in (a, b) such that $f'(c) = 0$

CASE I $f(x) = k$, **a constant**
Then $f'(x) = 0$, so the number c can be taken to be *any* number in (a, b).

CASE II $f(x) > f(a)$ **for some** x **in** (a, b) [as in Figure 1(b) or (c)]
By the Extreme Value Theorem (which we can apply by hypothesis 1), f has a maximum value somewhere in $[a, b]$. Since $f(a) = f(b)$, it must attain this maximum value at a number c in the open interval (a, b). Then f has a *local* maximum at c and, by hypothesis 2, f is differentiable at c. Therefore $f'(c) = 0$ by Fermat's Theorem.

CASE III $f(x) < f(a)$ **for some** x **in** (a, b) [as in Figure 1(c) or (d)]
By the Extreme Value Theorem, f has a minimum value in $[a, b]$ and, since $f(a) = f(b)$, it attains this minimum value at a number c in (a, b). Again $f'(c) = 0$ by Fermat's Theorem.

The Mean Value Theorem:

- Let f be a function that satisfies the following hypothesis:
  1. f is continuous on the closed interval $[a, b]$
  2. f is differentiable on the open interval (a, b)

  Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \text{ or equivantly, } f(b) - f(a) = f'(c)(b - a)$$

☆ If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b)

☆ If $f'(x) = g'(x)$ for all x in an interval (a,b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

4.3 How Derivatives Affect the Shape of a Graph

Increasing / Decreasing Test:

a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

The First Derivative Test:

- Suppose that c is a critical number of a continuous function f.
  - a) If f' changes from positive to negative at c, then f has a local maximum at c.
  - b) If f' changes from negative to positive at c, then f has a local minimum at c.
  - c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

Concavity :

- If the graph of f lies above all of its tangents on an interval I, then it is called "concave - up" on I. If the graph of f lies below all of its tangents on I, it is called "concave - down" on I.

Concavity Test :

a) If $f''(x) > 0$ for all x in I, then the graph of f is concave up on I.
b) If $f''(x) < 0$ for all x in I, then the graph of f is concave down on I.

Inflection Point :

- A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from CU to CD or CD to CU at P.

The Second Derivative Test :

a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

4.4 Indeterminate Forms and l'Hospital's Rule

Indeterminate Forms of Type $\frac{0}{0}$:
- a limit of the form $\lim_{x\to a} \frac{f(x)}{g(x)}$ where both $f(x) \to 0$, $g(x) \to 0$

Indeterminate Forms of Type $\frac{\infty}{\infty}$:
- a limit of the form $\lim_{x\to a} \frac{f(x)}{g(x)}$ where both $f(x) \to \infty$, $g(x) \to \infty$

L'Hospital's Rule :
- Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a. If we have an indeterminate form of either $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

※ $\lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}} = \lim_{x\to a} \frac{\left\{\frac{f(x)-g(a)}{x-a}\right\}}{\left\{\frac{g(x)-g(a)}{x-a}\right\}} = \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a} \frac{f(x)}{g(x)}$

※ you may use l'Hospital's Rule as many as you want as long as the given conditions meet

※ $y = e^x$ grows more quickly than all the power functions $y = x^n$

Indeterminate Form of Type $0 \cdot \infty$:

- We can deal with it by writing the product fg as a quotient $\Rightarrow f \cdot g = \frac{f}{1/g}$, $f \cdot g = \frac{g}{1/f}$, and apply l'Hospital's Rule

Indeterminate Form of Type $\infty - \infty$:

- we try to convert the difference into a quotient

☆ by using a common denominator, rationalization, or factoring out a common factor

Indeterminate Powers $\lim_{x \to a} [f(x)]^{g(x)}$

1. $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ type 0^0
2. $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$ type ∞^0
3. $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \pm\infty$ type 1^∞

- Each of these cases can be treated either by taking natural logarithm :

$y = [f(x)]^{g(x)} \Rightarrow \ln y = g(x) \ln f(x)$, or $[f(x)]^{g(x)} = e^{g(x) \ln f(x)}$

4.5 Summary of Curve Sketching

Guidelines for Sketching a Curve :

1. Domain
2. Intercepts
3. Symmetry : even function / odd function / periodic function
4. Asymptotes : Horizontal / Vertical / slant
5. Intervals of increase / decrease
6. Local Maximum or minimum values
7. Concavity and Points of Inflection
8. Sketch the curve

Slant Asymptotes :

- If $\lim_{x \to \infty} [f(x) - (mx + b)] = 0$, where $m \neq 0$, then the line $y = mx + b$ is "Slant Asymptotes"

4.6 Graphing with Calculus and Calculators

4.7 Optimization Problems

4.8 Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- If the number x_n become closer and closer to r as n becomes large, then we say that the sequence converges to r and we write $\lim_{n \to \infty} x_n = r$

☆ there are circumstances that the sequence may not converge ; when $f'(x_1) \to 0$

☆ We can stop when successive approximations x_n and x_{n+1} agree to a certain decimal places

EX :

4.9 Antiderivatives

- A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I