Experimental Design
Note 4-2
Graeco-Latin Square Design

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Graeco-Latin Square: An Example I

An experiment is conducted to compare four gasoline additives by testing them on four cars with four drivers over four days. Only four runs can be conducted in each day. The response is the amount of automobile emission.

- Treatment factor: gasoline additive, denoted by A, B, C, and D.
- Block factor 1: driver, denoted by 1, 2, 3, 4.
- Block factor 2: day, denoted by 1, 2, 3, 4.
- Block factor 3: car, denoted by α, β, γ, δ.

Graeco-Latin Square: An Example II

drivers	days 1	2	3	4
1	A	B	C	D
2	B	A	D	C
3	C	D	A	B
4	D	C	B	A

drivers	days 1	2	3	4
1	α	β	γ	δ
2	δ	γ	β	α
3	β	α	δ	γ
4	γ	δ	α	β

Graeco-Latin Square: An Example III

driver	day 1	2	3	4
1	$A\alpha = 32$	$B\beta = 25$	$C\gamma = 31$	$D\delta = 27$
2	$B\delta = 24$	$A\gamma = 36$	$D\beta = 20$	$C\alpha = 25$
3	$C\beta = 28$	$D\alpha = 30$	$A\delta = 23$	$B\gamma = 31$
4	$D\gamma = 34$	$C\delta = 35$	$B\alpha = 29$	$A\beta = 33$

Graeco-Latin Square I

- Consider a $p \times p$ Latin square, and superpose on it a second $p \times p$ Latin square in which the treatments are denoted by Greek/Latin letters.
- If the two squares when superimposed have the property that each Greek letter appears once and only once with each Latin letter, the two Latin squares are said to be orthogonal.
- the superimposed square is called Graeco-Latin square.
- Graceo-Latin squares exist for all $p > 3$.

Graeco-Latin Square II

Graeco-Latin Square Design Matrix:

driver	day	additive	car
1	1	A	α
1	2	B	β
1	3	C	γ
1	4	D	δ
$\vdots$	$\vdots$	$\vdots$	$\vdots$
4	1	D	γ
4	2	C	δ
4	3	B	α
4	4	A	β

Model and Assumptions

$$y_{ijkl} = \mu + \alpha_i + \tau_j + \beta_k + \zeta_l + \epsilon_{ijkl}$$

where

$$\mu = \text{grand mean}$$
$$\alpha_i = i\text{th block 1 effect (row effect)}, \quad \sum_i \alpha_i = 0$$
$$\tau_j = i\text{th treatment effect}, \quad \sum_j \tau_j = 0$$
$$\beta_k = k\text{th block 2 effect (column effect)}, \quad \sum_k \beta_k = 0$$
$$\zeta_l = l\text{th block 3 effect (Greek letter effect)}, \quad \sum_l \zeta_l = 0$$
$$\epsilon_{ijk} \sim N(0, \sigma^2) \text{ (independent)}$$

for $i = 1, 2, \cdots, p$; $j = 1, 2, \cdots, p$; $k = 1, 2, \cdots, p$; $l = 1, 2, \cdots, p$;

Completely additive model (no interaction)

Estimation and ANOVA I

- Rewrite observation as:

$$
\begin{aligned}
& y_{ijkl} \\
= & \ \bar{y}_{....} + (\bar{y}_{i...} - \bar{y}_{....}) + (\bar{y}_{.j..} - \bar{y}_{....}) + (\bar{y}_{..k.} - \bar{y}_{....}) + (\bar{y}_{...l} - \bar{y}_{....}) \\
+ & \ (y_{ijkl} - \bar{y}_{i...} - \bar{y}_{.j..} - \bar{y}_{..k.} - \bar{y}_{...l} + 3\bar{y}_{....}) \\
= & \ \hat{\mu} + \hat{\alpha}_i + \hat{\tau}_j + \hat{\beta}_k + \hat{\zeta}_l + \hat{\epsilon}_{ijkl}
\end{aligned}
$$

Estimation and ANOVA II

- Partition SS_T into:

$$
\begin{aligned}
& p\sum_i(\bar{y}_{i\cdots}-\bar{y}_{\cdots\cdot})^2+p\sum_j(\bar{y}_{\cdot j\cdot\cdot}-\bar{y}_{\cdots\cdot})^2+p\sum_k(\bar{y}_{\cdot\cdot k\cdot}-\bar{y}_{\cdots\cdot})^2 \\
& +p\sum_l(\bar{y}_{\cdots l}-\bar{y}_{\cdots\cdot})^2+\sum_i\sum_j\sum_k\sum_l\hat{\epsilon}_{ijkl}^2 \\
= \quad & SS_{Row}+SS_{Treatment}+SS_{Col}+SS_{Greek}+SSE
\end{aligned}
$$

with degree of freedom $p-1$, $p-1$, $p-1$, $p-1$, and $(p-3)(p-1)$, respectively.

ANOVA Table I

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F
Row	SS_{Row}	$p-1$	MS_{Row}	
Treatment	$SS_{Treatment}$	$p-1$	$MS_{Treatment}$	$MS_{Treatment}/MSE$
Column	SS_{Column}	$p-1$	MS_{Column}	
Greek	SS_{Greek}	$p-1$	MS_{Greek}	
Error	SSE	$(p-3)(p-1)$	MSE	
Total	SS_T	p^2-1		

$$SS_T = \sum_i \sum_j \sum_k \sum_l y_{ijkl}^2 - y_{....}^2/p^2; \quad SS_{Row} = \frac{1}{p}\sum_i y_{i...}^2 - y_{....}^2/p^2;$$

$$SS_{Treatment} = \frac{1}{p}\sum_j y_{.j..}^2 - y_{....}^2/p^2; \quad SS_{Column} = \frac{1}{p}\sum_k y_{..k.}^2 - y_{....}^2/p^2;$$

$$SS_{Greek} = \frac{1}{p}\sum_l y_{...l}^2 - y_{....}^2/p^2; \quad SSE = SS_T - SS_{Row} - SS_{Treatment} - SS_{Column} - SS_{Greek}$$

ANOVA Table II

Decision Rule: If $F_0 > F_{\alpha,p-1,(p-3)(p-1)}$, then reject H_0.
See Graeco-Latin.SAS.