14.1 Functions of Several Variables

Functions of Two Variables

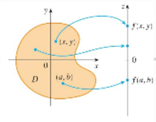


FIGURE 1

- A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the domain of f and its range is the set of values that f takes on, that is, $\{f(x,y) \mid (x,y) \in D\}$
- A function of two variables is just a function whose domain is a subset of R^2 and whose range is a subset of R

☆ Not all functions can be represented by explicit formulas

Cobb-Douglass Production Function : $P(L, K) = 1.01 \cdot L^{0.75} K^{0.25}$, $L \geq 0$, $K \geq 0$
- P is the total production, L is the amount of labor, and K is the amount of capital invested

Graphs :

Level Curves :

- The level curves of a function f of two variables are the curves with equations $f(x, y) = K$, where K is a constant
- A level curve $f(x,y) = K$ is the set of all points in the domain of f at which f takes on a given value K

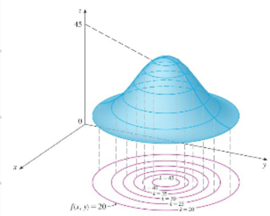


FIGURE 11

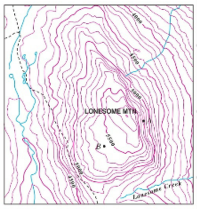


FIGURE 12

☆ getting the x, y, z-intercepts helps draw the plane of $f(x, y) = z$

Functions of Three or More Variables :

- A function of three variables is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset R^3$ a unique real number denoted by $f(x, y, z)$
- A function of n variables is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers

14.2 Limits and Continuity

- Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). Then we say that the limit of $f(x,y)$ as (x,y) approaches (a,b) is L and we write $\lim_{(x,y)\to(0,0)} f(x,y) = L$ if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x,y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x,y) - L| < \varepsilon$
- If any small interval $(L-\varepsilon, L+\varepsilon)$ is given around L, then we can find a disk D_δ with center (a,b) and radius $\delta > 0$ such that f maps all the points in D_δ into the interval $(L-\varepsilon, L+\varepsilon)$
- If $\varepsilon > 0$ is given, we can find $\delta > 0$ such that if (x,y) is restricted to lie in the disk D_δ and $(x,y) \neq (a,b)$, then the corresponding part of S lies between the horizontal planes $z = L - \varepsilon$ and $z = L + \varepsilon$
- For functions of a single variable, when we let x approach a, there are only two possible directions of approach, from the left or from the right. For functions of two variables, we can let (x,y) approach (a,b) from an infinite number of directions in any manner as long as (x,y) stays within the domain of f
- If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2, where $L_1 = L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist

EXAMPLE 1 Show that $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

SOLUTION Let $f(x,y) = (x^2 - y^2)/(x^2 + y^2)$. First let's approach $(0,0)$ along the x-axis. Then $y = 0$ gives $f(x,0) = x^2/x^2 = 1$ for all $x \neq 0$, so

$$f(x,y) \to 1 \quad \text{as} \quad (x,y) \to (0,0) \text{ along the } x\text{-axis}$$

We now approach along the y-axis by putting $x = 0$. Then $f(0,y) = \frac{-y^2}{y^2} = -1$ for all $y \neq 0$, so

$$f(x,y) \to -1 \quad \text{as} \quad (x,y) \to (0,0) \text{ along the } y\text{-axis}$$

Continuity:

- A function f of two variables is called continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. We say f is continuous on D if f is continuous at every point (a,b) in D

☆ This means that a surface that is the graph of a continuous function has no hole or break

- Since any polynomial can be built up out of simple functions by multiplication and addition, it follows that all polynomials are continuous on $\mathbb{R}^2$. Likewise, any rational function is continuous on its domain because it is a quotient of continuous functions

- It can be shown that if f is a continuous function of two variables and g is a continuous function of a single variable that is defined on the range of f, then the composite function $h = g \circ f$ defined by $h(x,y) = g(f(x,y))$ is also a continuous function

Functions of Three or More Variables

- If (x, y, z) is in the domain of f and $0 < \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < \delta$ then $|f(x, y, z) - L| < \varepsilon$.

  The function f is continuous at (a, b, c) if $\lim_{(x,y,z)\to(a,b,c)} f(x, y, z) = f(a, b, c)$

- If f is defined on a subset D of R^n, then $\lim_{x\to a} f(x) = L$ means that for every number $\varepsilon > 0$, there is a corresponding number $\delta > 0$ such that,
  if $x \in D$ and $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$

14.3 Partial Derivatives

- In general, if f is a function of two variables x and y, suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant. Then we are really considering a function of a single variable x, $g(x) = f(x, b)$. If g has a derivative at a, then we call it the partial derivative of f with respect to x at (a, b) and denote it by $f_x(a, b)$

  $\Rightarrow f_x(a, b) = g'(a)$, where $g(x) = f(x, b)$

  $\Rightarrow f_x(a, b) = \lim_{h\to 0} \dfrac{f(a+h, b) - f(a, b)}{h}$

4 If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Interpretation of Partial Derivatives

- By fixing $y = b$, we are restricting our attention to the curve C_1 in which the vertical plane $y = b$ intersects S

☆ Both of the curve C_1 and C_2 pass through the point P

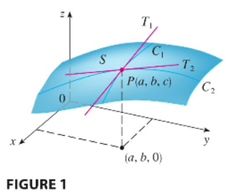


FIGURE 1
The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.

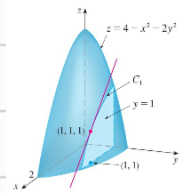


FIGURE 2

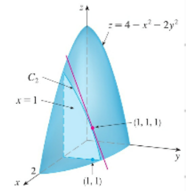


FIGURE 3

Functions of More Than Two Variables

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$
$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$
$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$
$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Higher Derivatives

- If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $f(x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the second partial derivatives of f

✱ $f_{xy} = \frac{d^2 f}{dy\,dx}$ means that we first differentiate with respect to x and then with respect to y

Clairaut's Theorem :

- Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$

✱ The theorem can also be applied for higher order derivatives

Partial Differential Equations

Laplace's Equation :

$\Rightarrow \frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} = 0$, and solutions of the form of equations are called Harmonic Functions.

The Cobb - Douglass Production Function

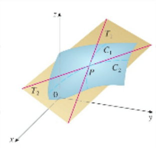
Assumptions made by Cobb and Douglass :

(i) If either labor or capital vanishes, then so will production.

(ii) The marginal productivity of labor is proportional to the amount of production per unit of labor.

(iii) The marginal productivity of capital is proportional to the amount of production per unit of capital.

14.4 Tangent Planes and Linear Approximations



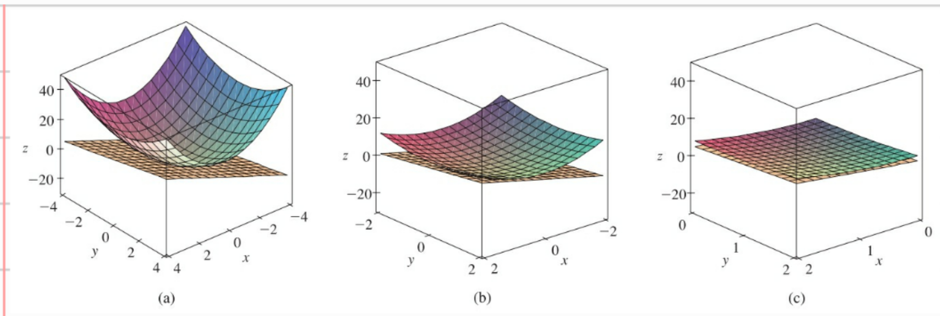
Tangent Planes

- the tangent plane to the surface at the point P is defined to be the plane that contains both tangent lines T_1 and T_2

✱ think of the tangent plane to S at P as consisting of all possible tangent lines at P to curves that lie on S and pass through P.

- Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

- The tangent plane at P is the plane that most closely approximates the surface S near the point P.



↑ As the scale of y decreases from (a) - (c), the tangent plane becomes similar to the suface

Linear Approximations

- An equation of the tangent plane to the graph of a function f of two variables at the point $(a, b, f(a,b))$ is

$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$, and $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$ is called the linearization of f at (a,b), and the approximation $f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$ is called the linear approximation or the tangent plane approximation of f at (a,b).

- If $z = f(x,y)$, then f is differentiable at (a,b) if Δz can be expressed in the form

$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$, where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$

✲ $\Delta y = f(a+\Delta x) - f(a) = f'(a)\Delta x + \varepsilon \Delta x$

Theorem :

- If the partial derivatives f_x and f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b)

Differentials

- In case of $f(x) = y$, while $\Delta x = dx$, Δy represents the change in height of the curve $y = f(x)$ and dy represents the change in height of the tangent line when x changes by an amount $dx = dx$,

Total Differential :

$dz = f_x(x,y)\,dx + f_y(x,y)\,dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \approx f_x(a,b)(x-a) + f_y(a,b)(y-b)$

✲ $\Delta z = f(x,y) - f(a,b)$

Functions of Three or More Variables

- Linear approximations, differentiability, and differentials can be defined in a similar manner for functions of more than two variables

$\Rightarrow f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$, and if $w = f(x,y,z)$,

$\Delta w = f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z)$ & $dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$

14.5 The Chain Rule

- If $y=f(x)$ and $x=g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable functions, then y is indirectly a differentiable function of t and $\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt}$

The Chain Rule (Case 1):

- Suppose that $z=f(x,y)$ is a differentiable function of x and y, where $x=g(t)$ and $y=h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$$

The Chain Rule (Case 2):

- Suppose that $z=f(x,y)$ is a differentiable function of x and y, where $x=g(s,t)$ and $y=h(s,t)$ are differentiable functions of s and t. Then,

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \quad \frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

☆ s and t are independent variables, x and y are intermediate variables, and z is dependent variable

The Chain Rule (General Version):

- Suppose that u is a differentiable function of the n variables $x_1, x_2, \dots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \dots, t_m$. Then u is a function of $t_1, t_2, \dots, t_m$ and $\frac{\partial u}{\partial t_i}=\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i}+\frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i}+\cdots+\frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$ for each $i=1,2,3,\dots,m$.

Implicit Differentiation

- Suppose that an equation of the form $F(x,y)=0$ defines y implicitly as a differentiable function of x, that is, $y=f(x)$, where $F(x,f(x))=0$ for all x in the domain of f

$$\frac{dy}{dx}\frac{dx}{dx}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0, \quad \frac{dx}{dx}=1, \quad \frac{\partial F}{\partial y}\neq 0$$

14.6 Directional Derivatives and the Gradient Vector

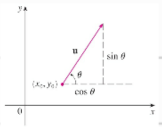
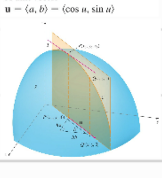


FIGURE 2
A unit vector
$\mathbf{u}=\langle a, b\rangle=\langle\cos\theta, \sin\theta\rangle$



- Suppose that we now wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $u=[a,b]$. To do this, we consider the surface S with the equation $z=f(x,y)$ and we let $z_0=f(x_0,y_0)$.
- Then the point $P(x_0,y_0,z_0)$ lies on S. The vertical plane that passes through P in the direction of u intersects S in a curve C. The slope of the tangent line T to C at the point P

is the rate of change of z in the direction of u

- If $Q(x, y, z)$ is another point on C and P', Q' are the projections of P, Q onto the xy-plane, then the vector $\overrightarrow{P'Q'}$ is parallel to u and so $\overrightarrow{P'Q'} = hu = [ha, hb]$ for some scalar h.

$\Rightarrow \quad x - x_0 = ha \quad , \quad y - y_0 = hb$

$\Rightarrow \quad \dfrac{\Delta z}{h} = \dfrac{z - z_0}{h} = \dfrac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$

- The directional derivative of f at (x_0, y_0) in the direction of a unit vector $u = [a, b]$ is

$D_u f(x_0, y_0) = \lim_{h \to 0} \dfrac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$, if the limit exists

Theorem :

- If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $u = [a, b]$ and $D_u f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$

The Gradient Vector

- If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by $\nabla f(x, y) = [f_x(x, y), f_y(x, y)] = \dfrac{\partial f}{\partial x} i + \dfrac{\partial f}{\partial y} j$

* $D_u f(x, y) = \nabla f(x, y) \cdot u$, this expresses the directional derivative in the direction of a unit vector u as the scalar projection of the gradient vector onto u

Functions of Three Variables

- The directional derivative of f at (x_0, y_0, z_0) in the direction of a unit vector $u = [a, b, c]$ is $D_u f(x_0, y_0, z_0) = \lim_{h \to 0} \dfrac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$, if the limit exists

$\Rightarrow \quad \nabla f(x, y, z) = [f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)] = \dfrac{\partial f}{\partial x} i + \dfrac{\partial f}{\partial y} j + \dfrac{\partial f}{\partial z} k$

$\Rightarrow \quad D_u f(x, y, z) = \nabla f(x, y, z) \cdot u$

Maximizing the Directional Derivative

Theorem :

- Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_u f(x)$ is $|\nabla f(x)|$ and it occurs when u has the same direction as the gradient vector $\nabla f(x)$

Tangent Planes to Level Surfaces

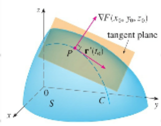


FIGURE 9

- The gradient vector at P, $\nabla F(x_0, y_0, z_0)$ is perpendicular to the tangent vector $r'(t)$ to any curve C on S that passes through P. If $\nabla F(x_0, y_0, z_0) \neq 0$, it is therefore natural to define the tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$.

  $\Rightarrow F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$

Significance of the Gradient Vector

- On the one hand, the gradient vector $\nabla f(x_0, y_0, z_0)$ gives the direction of fastest increase of f. On the other hand, $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface S of f through P.

14.7 Maximum and Minimum Values

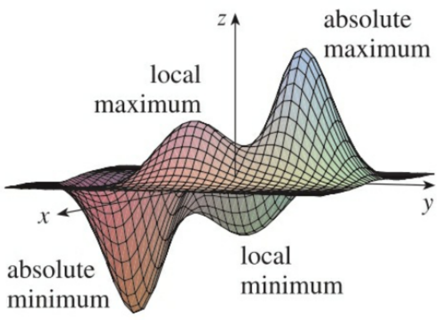


FIGURE 1

1 Definition A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leqslant f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leqslant f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geqslant f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

2 Theorem If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

- A point (a,b) is called a critical point of f if $f_x(a,b) = 0$ and $f_y(a,b) = 0$, or if one of these partial derivatives does not exist. At a critical point, a function could have a local maximum or minimum or neither.
- The point where it becomes a local maximum on the direction of the one axis and the local minimum on the other axis is called the saddle point.

Second Derivative Test :

- Suppose the second partial derivatives of f are continuous on a disk with center (a,b), and suppose that $f_x(a,b) = 0$ and $f_y(a,b) = 0$. Let $D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$,
  a) If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum
  b) If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum
  c) If $D < 0$, then $f(a,b)$ is neither. (Saddle point)

☆ $D = 0$ gives no information.

$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$

Absolute Maximum and Minimum Values

Extreme Value Theorem for Functions of Two Variables :

- If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D

  ⇒ If f has an extreme value at (x_1, y_1), then (x_1, y_1) is either a critical point of f or a boundary point of D

- To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:
  1) Find the values of f at the critical points of f in D
  2) Find the extreme values of f on the boundary of D

3) The largest of the values from steps 1 and 2 is the absolute maximum value ; the smallest of these is the absolute minimum value

14.8 Lagrange Multipliers

- Lagrange's method for maximizing or minimizing a general function $f(x,y,z)$ subject to a constraint of the form $g(x,y,z) = k$
- Start by tring to find the extreme values of $f(x,y)$ subject to a constraint of the form $g(x,y) = k$
- To maximize $f(x,y)$ subject to $g(x,y) = k$ is to find the largest value of c such that the level curve $f(x,y) = c$ intersects $g(x,y) = k$
  => This means the gradient vectors are parellel ; that is, $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some scalar λ

- As for $f(x,y,z)$, the point (x,y,z) is restricted to lie on the level surface S with equation $g(x,y,z) = k$; we consider the level surfaces $f(x,y,z) = c$ and argue that if the maximum value of f is $f(x_0, y_0, z_0) = c$, then the level surface $f(x,y,z) = c$ is tangent to the level surface $g(x,y,z) = k$ and so the corresponding gradient vectors are parellel
  => Lagrange Multiplier : λ from the $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$

Method of Lagrange Multipliers :

- To find the maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$ [assuming that these extreme values exist and $\nabla g \neq 0$ on the surface $g(x,y,z) = k$]:
  a) Find all values of x, y, z and λ such that $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ and $g(x,y,z) = k$
  b) Evaluate f at all the points (x,y,z) that result from step (a). The largest of these values is the maximum value of f ; the smallest is the minimum value of f.

Two Constraints

- We are looking for the extreme values of f when (x,y,z) is restricted to lie on the curve of intersection C of the level surfaces $g(x,y,z) = k$ and $h(x,y,z) = c$
  => $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$