해석학2　　중간고사 (타전공생용)　　2021. 10. 19

배점: 각 문항 10점

주의사항: 문제의 증명 또는 풀이과정을 상세히 기술하시오

1. Let $f(x)=\dfrac{\sin^3(\pi x)-\sin^2(\pi x)+6x^3}{e^{x^2-1}+x^2+1}$.

   Show that $f'(x)=2$ somewhere on $[0,1]$, without solving any equations.

2. Let $P(x)$ be a polynomial and let a,b,c,d (with $a<b<c$) be real roots of the equation $P(x)=0$. Prove that $\exists\, x\in(a,c)$ such that $P''(x)-6P'(x)+9P(x)=0$ ..

3. Determine $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin^2 x}-\dfrac{1}{x^2}\right)$

4. Suppose that f is such that $f'(x)$ and $f''(x)$ are conti on $(-\delta,\delta)$ for some $\delta>0$, and that $f(0)=0$. Determine the value $\lim\limits_{x\to 0}\dfrac{d}{dx}\left(\dfrac{f(x)}{x}\right)$

5. Let f be twice differentiable for $x\gg 1$, and suppose that

   $\lim\limits_{x\to\infty}f(x)=2021$ and $\lim\limits_{x\to\infty}f''(x)=0$. Determine the value $\lim\limits_{x\to\infty}f'(x)$.

   Hint: Use Taylor's theorem

6. Prove that $(a-b+c)^5\le a^5-b^5+c^5$ holds whenever $0<a<b<c$

7. Prove that $|\cos x-1|\le\dfrac{x^2}{2}$ for all $x\in\mathbb{R}$

8. Evaluate $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{2n}\dfrac{k}{n^2+k^2}$

9. Suppose that $|f(x)-f(y)|\le|x-y|^{1/3}$ for all $x,y\in[a,b]$. Then show that

   $U_f(\mathcal{P})-L_f(\mathcal{P})\le|\mathcal{P}|^{1/3}(b-a)\to 0$ for every partition $\mathcal{P}$ of $[a,b]$

   Here $|\mathcal{P}|$ is the mesh of the partition $\mathcal{P}$: $a=x_0<x_1<\cdots<x_n=b$ of $[a,b]$,

   i.e., $|\mathcal{P}|=\max\limits_{1\le i\le n}(x_i-x_{i-1})$