Chap 16. Linearization and Convexity

16.1 Linearization

To study the local behavior of a function $f(x)$ near a point a, we usually approximate $f(x)$ by a polynomial function.

$$f(x) \approx \text{ a polynomial function near a point } a$$

The degree of the polynomial is called the order of the approximation

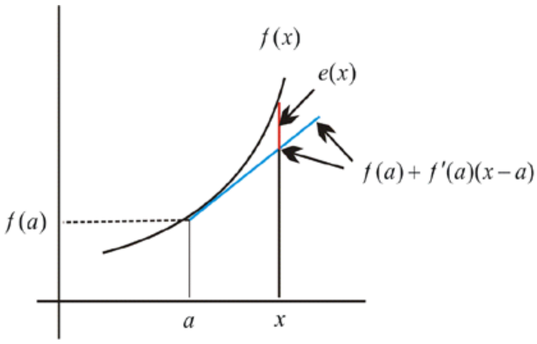
$$\begin{aligned} \text{first order approximation} &= \text{linear approximation} \\ \text{second order approximation} &= \text{quadratic approximation} \\ &\vdots \end{aligned}$$

Theorem-Definition A

$f(x)$: diff at a

$$\Rightarrow \quad f(x) = f(a) + f'(a)(x-a) + e(x), \quad \text{where } \lim_{x\to a} \frac{e(x)}{(x-a)} = 0$$

The linear polynomial $f(a) + f'(a)(x-a)$ is called the linearization of $f(x)$ at a.



Pf. $f(x)$: diff at a $\Rightarrow$ $\dfrac{f(x)-f(a)}{x-a} = f'(a) + e_1(x)$, where $\lim_{x\to a} e_1(x) = 0$

$$\therefore \quad f(x) - f(a) = f'(a)(x-a) + \underbrace{e_1(x)(x-a)}_{\equiv e(x)}$$

Hence $\displaystyle \lim_{x\to a} \frac{e(x)}{(x-a)} = \lim_{x\to a} e_1(x) = 0.$

Remark. (The converse of the above is also true)

$\exists$ a real number A s.t.

$$f(x) = \underbrace{f(a) + A(x-a)}_{\text{1차식}} + e(x), \text{ with } \lim_{x\to a} \frac{e(x)}{x-a} = 0$$

$$\Rightarrow \quad f \text{ is diff at } a \text{ and } A = f'(a),$$

Pf. Hypo says $\exists\ A(=\text{real})$ s.t. $\dfrac{f(x)-f(a)}{x-a} = A + \dfrac{e(x)}{x-a}$, *with* $\lim_{x\to a}\dfrac{e(x)}{x-a} = 0$

$$\Rightarrow \quad \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = A + \lim_{x\to a}\frac{e(x)}{x-a} = A$$

$\therefore$ $f'(a)$ exists and $f'(a) = A.$

Theorem B (Extended MVT or Linearization Error Theorem)

Suppose $f''(x)$ exists in some interval I with $I \ni a$. Then for each $x \in I$, $\exists$ a point c between a and x such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(c)}{2}(x-a)^2$$

Remark. If we write b in stead of x, the above can be written as

$(*)$ $f(b) = f(a) + f'(a)(b-a) + \dfrac{f''(c)}{2}(b-a)^2$, for some c between a and b.

For the pf, we need a lemma (a special case of $(*)$):

Lemma (Extended Rolle's theorem)

Suppose $f''(x)$ exists on $[a, b]$, and $f(a) = f'(a) = 0$, $f(b) = 0$.

$\Rightarrow$ $\exists\, c \in (a, b)$ s.t. $f''(c) = 0$

(This lemma is also valid for $[b, a]$, using $f(b) = 0$, $f(a) = f'(a) = 0$.)

Pf. Assume $a < b$. Then

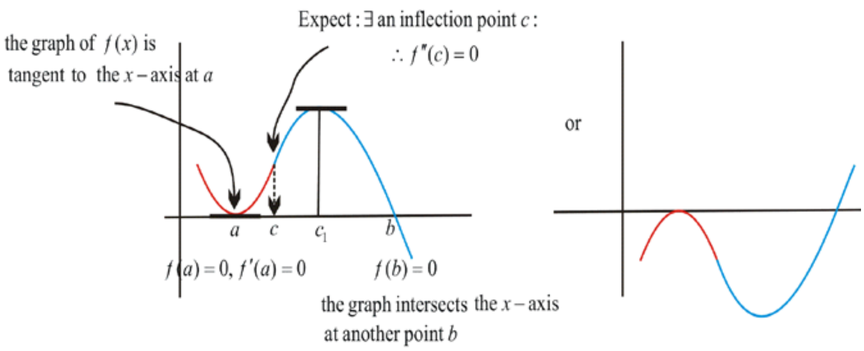
$$f(a) = f(b) = 0 \overset{\text{Rolle}}{\Rightarrow} f'(c_1) = 0, \quad \text{for some } c_1 \in (a, b)$$

$$f'(a) = f'(c_1) = 0 \overset{\text{Rolle}}{\Rightarrow} f''(c) = 0, \quad \text{for some } c \in (a, c_1)$$

Since $a < c < c_1 < b$, we get $c \in (a, b)$.

The pf is similar if $b < a$.

Geometric meaning:



Pf of Theorem B

Want to find a point c between a and x such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(c)}{2}(x-a)^2$$

Let $P(x) = f(a) + f'(a)(x-a) + C(x-a)^2$, ($C$: constant which is determined later)

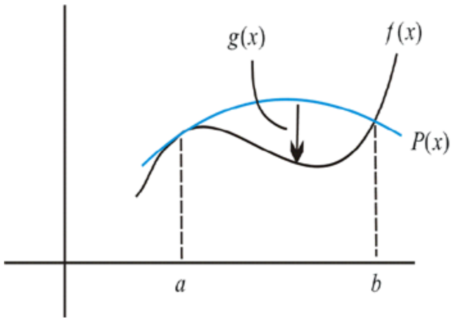
Note that $P(x)$ is a quadratic polynomial s.t.

$$P(a) = f(a)$$

$$P'(a) = f'(a)$$

We choose C so that $P(b) = f(b)$. In other words, C is taken to satisfy

$$f(b) = f(a) + f'(a)(b-a) + C(b-a)^2$$



Now we let $g(x) = f(x) - P(x)$. Then

$g(x)$ satisfies the Hypo of the Extended Rolle's theorem.

$$\therefore \quad c \in (a, b) \text{ s.t. } g''(c) = 0 \qquad i.e., \ f''(c) = P''(c)$$

$$P''(c) = 2C \quad \Rightarrow \quad C = \frac{P''(c)}{2} = \frac{f''(c)}{2}.$$

Exa. Using the Linearization Error Thm, show that $\cos x > 1 - x^2/2$, for $x \underset{\neq}{\approx} 0$

Pf.

$$f(x) = \cos x \quad \Rightarrow \quad f'(x) = -\sin x, \ f''(x) = -\cos x$$

$$\overset{\text{Linear E-T}}{\Rightarrow} \quad f(x) = f(0) + f'(0)x + \frac{f''(c)}{2}x^2, \quad \text{for } c \text{ between } 0 \text{ and } x$$

$$\Rightarrow \quad \cos x = 1 - \frac{\cos c}{2}x^2, \quad \text{for } c \text{ between } 0 \text{ and } x$$

Since $0 < \cos c < 1$ in the interval $0 < |x| < \pi/2$, we conclude that

$$\cos x > 1 - x^2/2, \ \text{for } 0 < |x| < \pi/2 \quad (\therefore \text{ for } x \underset{\neq}{\approx} 0)$$

16.2 Applications (of the Linearization Error Theorem) to convexity

Def. We say $f(x)$ has a strict local maximum (minimum) at a if

$$f(x) < f(a) \quad (f(x) > f(a)) \quad \text{for } x \underset{\neq}{\approx} a$$

Theorem A Second derivative test for local extrema

Suppose $f''(x)$ is continuous at $x = a$ and $f'(a) = 0$. Then

(i) $\quad f''(a) > 0 \quad \Rightarrow \quad f(x)$ has a strict local minimum at a

(ii) $\quad f''(a) < 0 \quad \Rightarrow \quad f(x)$ has a strict local maximum at a

(iii) $\quad f''(a) = 0 \quad \Rightarrow$ give no conclusion (Need another information)

Pf. (i) $\quad f''(x)$: conti at $x = a \quad \Rightarrow \quad f''(x)$ exists for $x \approx a$

$$\underset{\text{plus } f'(a)=0}{\overset{\text{Linear E-T}}{\Rightarrow}} \quad \exists\, c \text{ between } a \text{ and } x \text{ s.t.}$$

$$(*) : \quad f(x) - f(a) = \frac{f''(c)}{2}(x-a)^2 \quad \text{for } x \approx a$$

$f''(a) > 0 \quad \Rightarrow \quad f''(x) > 0 \quad$ for $x \approx a$, since $f''(x)$ is conti at $x = a$

$\Rightarrow \quad f''(c) > 0$, since c lies between a and x

$\Rightarrow \quad f(x) > f(a) \quad$ for $x \underset{\neq}{\approx} a \quad$ (by $(*)$)

$\therefore \quad x = a$ is a strict local minimum point for $f(x)$.

(iii)

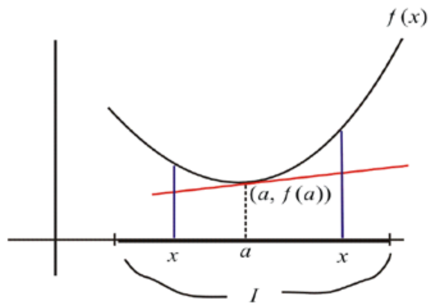
$f(x) = x^4 \quad \Rightarrow \quad (f'(0) = 0) \;\; f''(0) = 0; \quad$ it has a local min at $x = 0$

$f(x) = -x^4 \quad \Rightarrow \quad (f'(0) = 0) \;\; f''(0) = 0; \quad$ it has a local max at $x = 0$

⦿ Graphing Technique

Def A. Let $f(x)$ be diff on I. We call $f(x)$ convex on I if over all of I its graph lies above the tangent line at $(a, f(a))$, for all $a \in I$, i.e.,

$$f(x) \geq f(a) + f'(a)(x-a), \quad \text{for all } a, x \in I.$$



We call $f(x)$ is strictly convex on I if

$$f(x) > f(a) + f'(a)(x-a), \quad \text{for all } a, x \in I \text{ with } x \neq a.$$

Similarly, $f(x)$ is concave on I (strictly concave if $<$ holds) if

$$f(x) \leq f(a) + f'(a)(x-a), \quad \text{for all } a, x \in I.$$

Theorem B (Second derivative test for convexity)

Assume $f''(x)$ exists on the open interval I. Then

$f''(x) \geq 0$ on $I \quad \Rightarrow \quad f(x)$ is convex on I

Pf. For any $a, x \in I$, we have

$$f(x) \overset{\text{Linear E-T}}{=} f(a) + f'(a)(x-a) + \frac{f''(c)}{2}(x-a)^2, \quad \text{where } a < c < x \text{ or } x < c < a$$

$$\overset{\text{obviously } c\in I \ \& \ f''(c)(x-a)^2 \geq 0}{\Rightarrow} \quad f(x) \geq f(a) + f'(a)(x-a) \quad \text{for any } a, x \in I.$$

Remark. $f''(x) > 0$ on $I \Rightarrow f(x)$ is strictly convex on I

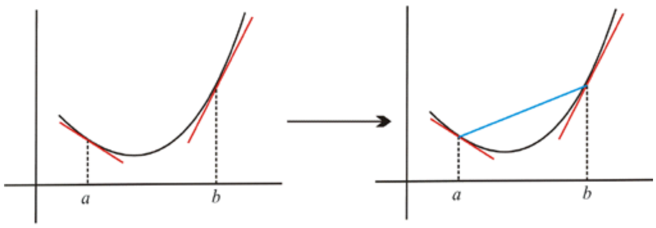
Theorem C (First derivative test for convexity)
If $f(x)$ is diff on I, then

$$f(x) \text{ is convex on } I \Leftrightarrow f'(x) \text{ is inc on } I$$

Pf. $(\Rightarrow)$ Let $a < b$ be two points on I. Then by Hypo

① $f(b) \geq f(a) + f'(a)(b-a)$

② $f(a) \geq f(b) + f'(b)(a-b)$



$\therefore f'(a) \leq \frac{f(b)-f(a)}{b-a} \leq f'(b)$

① $\Rightarrow \frac{f(b)-f(a)}{b-a} \geq f'(a)$

② $\underset{a-b<0}{\Rightarrow} \frac{f(a)-f(b)}{a-b} \leq f'(b)$ i.e., $\frac{f(b)-f(a)}{b-a} \leq f'(b)$

So $f'(a) \leq f'(b)$ $\therefore f'(x)$ is inc on I.

$(\Leftarrow)$ We shall prove: $f(x)$ is not convex on $I \Rightarrow f'(x)$ is not inc on I.
If $f(x)$ is not convex on I, then for some $a, b \in I$

$$f(b) < f(a) + f'(a)(b-a)$$

Case1 $a < b$

$\Rightarrow \frac{f(b)-f(a)}{b-a} < f'(a)$

$\| \leftarrow$ MVT

$f'(c)$ for some $a < c < b$

$\therefore c > a$, but $f'(c) < f'(a)$

$\therefore f'(x)$ is not inc on I

Case2 $a > b$

$\Rightarrow \frac{f(b)-f(a)}{b-a} > f'(a)$

$\| \leftarrow$ MVT

$f'(c)$ for some $b < c < a$

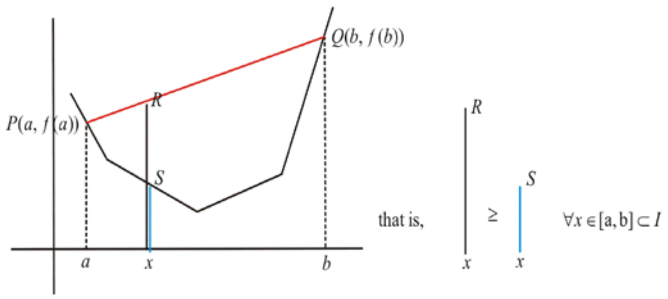
$\therefore a > c$, but $f'(a) < f'(c)$

$\therefore f'(x)$ is not inc on I

Note. The notion of convexity is often used for continuous functions which are not differentiable. How can we define convexity without assuming differentiability?

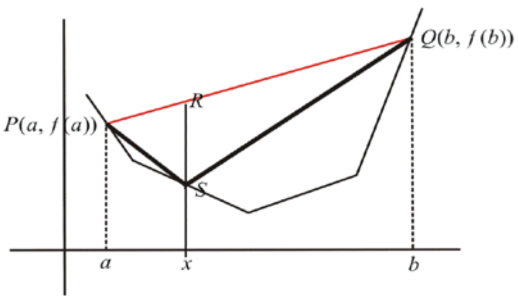
Def B (Geometric convexity; In many texts, Geo-convexity is just used for convexity)
Let $f(x)$ be defined on any type of interval I. For any subinterval $[a,b] \subset I$, we let $P:(a,f(a))$ and $Q:(b,f(b))$ be the two points of the graph lying over the endpoints of the subinterval. We say that $f(x)$ is geometrically convex on I if

$$\boxed{(*):\quad \text{the graph of } f(x) \text{ lies on or below the chord } PQ, \quad \text{for all } [a,b] \subset I}$$



Remark. $f(x)$ is geo-convex on I iff

$$\boxed{\frac{f(x)-f(a)}{x-a} \le \frac{f(b)-f(a)}{b-a}, \quad \text{for any triple } a<x<b \text{ in } I}$$



Pf 1 (Geometric view) For any $x \in (a,b)$, let $S=(x,f(x))$. Then

$$f \text{ is geo-convex on } I \quad \Leftrightarrow \quad S \text{ lies on or below } R$$

$$\overset{\text{obvious}}{\Leftrightarrow} \quad \text{slope } PS \le \text{slope } PQ \quad \Leftrightarrow \quad \frac{f(x)-f(a)}{x-a} \le \frac{f(b)-f(a)}{b-a}$$

Pf 2 (Analytic view) Using the equation for PQ,

$$y-\text{coordinate of } R = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

$$(\ \&\ \ y-\text{coordinate of } S = f(x)\ )$$

Thus

$$(*) \quad \Leftrightarrow \quad f(x) \le f(a) + \frac{f(b)-f(a)}{b-a}(x-a), \quad a<x<b$$

$$\Leftrightarrow \quad \frac{f(x)-f(a)}{x-a} \le \frac{f(b)-f(a)}{b-a}, \qquad a<x<b$$

Proposition (Ex 16.2-#2 + Pb 16-2)

Assume $f(x)$ is diff on I. Then

$$f \text{ is convex on } I \quad \Leftrightarrow \quad f \text{ is geo-convex on } I$$

Pf. $(\Rightarrow)$

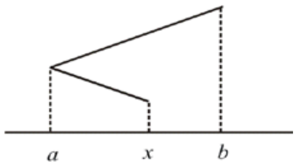
$$f \text{ is convex on } I \overset{\text{seen}}{\Leftrightarrow} f'(x) \text{ is } \uparrow \text{ on } I$$

$$\Updownarrow \text{ def}$$

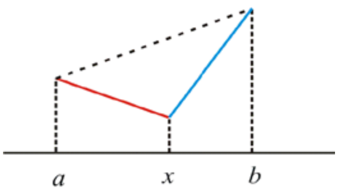
$$f(x) \geq f(a) + f'(a)(x-a), \quad \text{for all } a, x \in I$$

Let $a < x < b$.

Goal: $$\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a}$$



We first show: $$\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(x)}{b-x}$$ (remember!!)



$[\because$ LHS $\overset{\text{MVT}}{=} f'(c_1)$ with $a < c_1 < x$; RHS $\overset{\text{MVT}}{=} f'(c_2)$ with $x < c_2 < b$

But $f'(x)$ is $\uparrow$ on I so $f'(c_1) \leq f'(c_2)$ $\therefore$ LHS $\leq$ RHS$]$

Now we can easily verify (see Proposition below, or see figure above) that

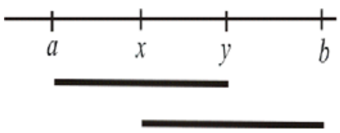
Mediant Property
$\frac{b}{a} \leq \frac{d}{c} \Rightarrow \frac{b}{a} \leq \frac{b+d}{a+c} \leq \frac{d}{c}$, whenever $a, c > 0$ [easy if $a, b, c, d > 0$]

Accordingly, $$\frac{f(x)-f(a)}{x-a} \leq \frac{f(x)-f(a)+f(b)-f(x)}{(x-a)+(b-x)} = \frac{f(b)-f(a)}{b-a}$$

$(\Leftarrow)$ Let f be geo-convex on I. Suffices to show: $f'(x)$ is $\uparrow$ on I.

Suppose $a < b$ $(a, b \in I)$, and will show $f'(a) \leq f'(b)$.

Choose x, y so that $a < x < y < b$



f is geo-convex on I $\Rightarrow$

$$\frac{f(x)-f(a)}{x-a} \le \frac{f(y)-f(a)}{y-a} \quad \left(\overset{\text{Subtraction form below}}{\Rightarrow} \quad \frac{f(x)-f(a)}{x-a} \le \frac{f(y)-f(x)}{y-x} \right)$$

$$\frac{f(y)-f(x)}{y-x} \le \frac{f(b)-f(x)}{b-x} \quad \left(\overset{\text{Subtraction form below again}}{\Rightarrow} \quad \frac{f(y)-f(x)}{y-x} \le \frac{f(b)-f(y)}{b-y} \right)$$

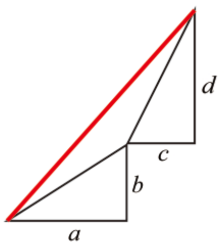
$$\therefore \quad \frac{f(x)-f(a)}{x-a} \le \frac{f(y)-f(x)}{y-x} \le \frac{f(b)-f(y)}{b-y} = \frac{f(y)-f(b)}{y-b}$$

Ignore the middle term & let $x \to a^+$; $y \to b^-$

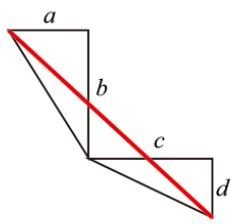
$$\Rightarrow \quad f'_+(a) \le f'_-(b) \quad \overset{f\text{: diff on } I}{\Rightarrow} \quad f'(a) \le f'(b)$$

Proposition [Mediant Property]: $a, c > 0$ & $\frac{b}{a} \le \frac{d}{c} \Rightarrow \frac{b}{a} \le \frac{b+d}{a+c} \le \frac{d}{c}$

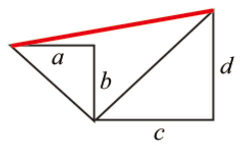
Pf.



$a,c,b,d>0$



$a,c>0$ & $b,d<0$



$a,c>0$ & $b<0,\ d>0$

Subtraction form of Mediant Property: $0<a<c$ & $\frac{b}{a} \le \frac{d}{c} \Rightarrow \frac{b}{a} \le \frac{d-b}{c-a}$

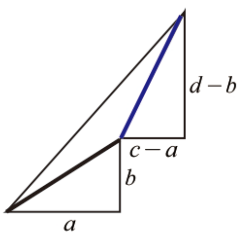
Pf1. $0<a<c$ & $\frac{b}{a} \le \frac{d}{c} \Rightarrow bc \le ad \Rightarrow bc-ad \le 0$

$\therefore\ b(c-a)-a(d-b) = bc-ab-(ad-ab) = bc-ad \le 0$

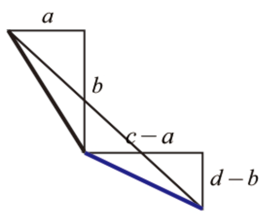
$\therefore\ b(c-a) \le a(d-b)$

$\times \frac{1}{a(c-a)}$ [note $a>0$ & $c-a>0$] $\Rightarrow \frac{b}{a} \le \frac{d-b}{c-a}$

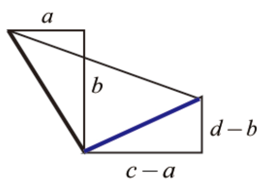
Pf2. Note: $0<a<c$ & $\frac{b}{a} \le \frac{d}{c} \Rightarrow bc \le ad \Rightarrow d \ge b\frac{c}{a} > b$ if $b>0$



$d>b>0$



$b<0$ & $d<b$

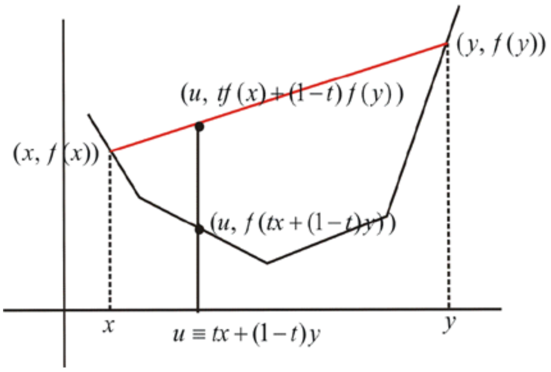


$b<0$ & $d>b$

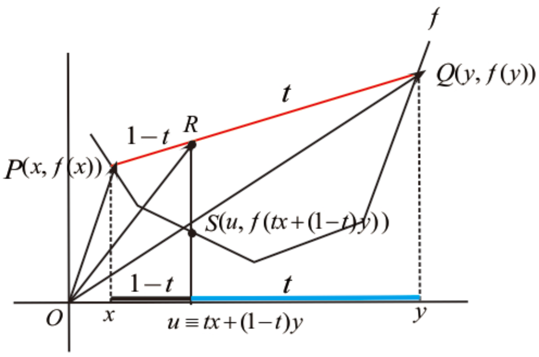
※ Claim: f is geo-convex on an interval I

$$\boxed{\Leftrightarrow \quad \forall x, y \in I \ \ \& \ \ 0 \le \forall t \le 1, \quad f(tx + (1-t)y) \ \le \ tf(x) + (1-t)f(y)}$$

(In most texts, $\boxed{}$ is used for the definition of convexity)



Pf1.



From the figure above

$$PR : RQ = (1-t) : t \quad [\text{here } \ 0 \le t \le 1]$$

Hence

$$\overrightarrow{OR} = t\overrightarrow{OP} + (1-t)\overrightarrow{OQ} = t(x, f(x)) + (1-t)(y, f(y))$$
$$= (tx + (1-t)y, tf(x) + (1-t)f(y)) = (u, tf(x) + (1-t)f(y))$$
i.e., R [as a point in xy-plane] $= (u, tf(x) + (1-t)f(y))$

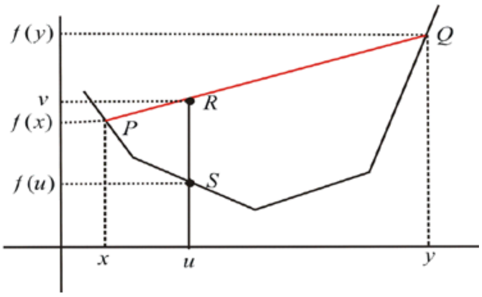
But it is clear that $\quad S = (u, f(tx + (1-t)y))$

f is geometrically convex means that

$$y-\text{coordinate of } \ S \le y-\text{coordinate of } \ R$$

$$\therefore \quad f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \ \text{ whenever } \ 0 \le t \le 1 \ \& \ x, y \in I$$

Pf2.



Recall f is geo-convex on I $\overset{\text{means}}{\Leftrightarrow}$ S lies on or below R $\Leftrightarrow$ $f(u) \leq v$

Let $t = \dfrac{y-u}{y-x}$ $\Rightarrow$ $0 \leq t \leq 1$ & $u = tx + (1-t)y$

On the other hand, since $y-x : y-u = \overline{QP} : \overline{QR}$,

$$t = \frac{\overline{QR}}{\overline{QP}} = \frac{f(y)-v}{f(y)-f(x)} \qquad \text{so} \quad v = tf(x) + (1-t)f(y)$$

$$\therefore \quad f(u) \leq v \quad \Leftrightarrow \quad f(tx+(1-t)y) \leq tf(x) + (1-t)f(y)$$

Ex 1. Suppose f is convex on $[0, \infty)$ and $f(0) = 0$. Show that

$$f(x) + f(y) \le f(x+y), \quad \text{for all } x, y > 0$$

Pf. For $x > 0$, $f(x) = f\left(\frac{x}{x+y} \cdot (x+y) + \frac{y}{x+y} \cdot 0\right)$

$$\le \frac{x}{x+y} f(x+y) + \frac{y}{x+y} f(0) \qquad (\leftarrow\ f \text{ is convex })$$

$$= \frac{x}{x+y} f(x+y) \ \ (\leftarrow f(0) = 0)$$

For $y > 0$, $f(y) = f\left(\frac{x}{x+y} \cdot 0 + \frac{y}{x+y} \cdot (x+y)\right)$

$$\le \frac{x}{x+y} f(0) + \frac{y}{x+y} f(x+y) \qquad (\leftarrow\ f \text{ is convex })$$

$$= \frac{y}{x+y} f(x+y) \ \ (\leftarrow f(0) = 0)$$

Summing these, we obtain

$$f(x) + f(y) \le \frac{x}{x+y} f(x+y) + \frac{y}{x+y} f(x+y) = f(x+y), \quad \forall x, y > 0$$

Another pf. By symmetry, we may assume that $0 < x < y$. Then $0 < x < y < x+y$

f is convex on $[0, \infty)$ $\Rightarrow$

$$\frac{f(x) - f(0)}{x - 0} \overset{0<x<y}{\le} \frac{f(y) - f(x)}{y - x} \overset{x<y<x+y}{\le} \frac{f(x+y) - f(y)}{(x+y) - y} = \frac{f(x+y) - f(y)}{x} \quad (\text{by } \textbf{remember})$$

This gives $f(x) - f(0) \le f(x+y) - f(y)$, which (by $f(0) = 0$) says $f(x) + f(y) \le f(x+y)$

Ex 2. If f is convex on I, show that

$$(*): \quad f(a - b + c) \le f(a) - f(b) + f(c), \quad \text{whenever } a < b < c \text{ with } a, b, c \in I.$$

Pf. $a < b < c \quad \Rightarrow \quad b = ta + (1-t)c$ for some $t \in (0, 1)$ (indeed, $t = \frac{c - b}{c - a}$)

Since f is convex,

$$f(b) \le t f(a) + (1-t) f(c) \qquad --- ①$$

Note that $a - b + c = a - (ta + (1-t)c) + c = (1-t)a + tc$. Hence

$$f(a - b + c) \le (1-t) f(a) + t f(c) \qquad --- ②$$

$$① + ② \qquad \Rightarrow \quad f(b) + f(a - b + c) \le f(a) + f(c)$$

$$\therefore \qquad f(a - b + c) \le f(a) - f(b) + f(c)$$

An application of Ex 2.

$f(x) = x^3$ is easily seen to be convex on $(0, \infty)$. Thus by $(*)$ of the above

$$(a - b + c)^3 \le a^3 - b^3 + c^3, \quad \text{whenever } 0 < a < b < c.$$

Ex. Give another proof of the above inequality.

HS (very useful, but not easy to prove)

If $f(x)$ is continuous on I, and $f\left(\frac{x+y}{2}\right) \le \frac{f(x) + f(y)}{2}$ $\forall x, y \in I$, then show that $f(x)$ is convex on I.

Recall: f is geo-convex (or convex) on an interval I

$$\boxed{\Leftrightarrow \quad \forall x, y \in I \ \ \& \ \ 0 \le \forall t \le 1, \quad f(tx + (1-t)y) \ \le \ tf(x) + (1-t)f(y)}$$

In particular, if f is geo-convex (or convex) on an interval I, then

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} \qquad \forall x, y \in I$$

Ex. If f is geo-convex (or convex) on an interval I, then show that

$$f\left(\frac{x+y+z}{3}\right) \le \frac{f(x)+f(y)+f(z)}{3} \qquad \forall x, y, z \in I$$

Pf. Using the convexity of f twice, we obtain, whenever $x, y, z \in I$,

$$f\left(\frac{x+y+z}{3}\right) = f\left(\frac{2}{3}\left(\frac{x+y}{2}\right) + \frac{1}{3}z\right) \le \frac{2}{3}f\left(\frac{x+y}{2}\right) + \frac{1}{3}f(z)$$

$$\le \frac{2}{3}\left(\frac{f(x)+f(y)}{2}\right) + \frac{1}{3}f(z) = \frac{f(x)+f(y)+f(z)}{3}$$

Application1: Let $n \in \mathbb{N}$. Show that $\dfrac{a^n+b^n+c^n}{3} \ge \left(\dfrac{a+b+c}{3}\right)^n$ for all $a, b, c > 0$

Pf. $n=1$: trivial

Assume $n \ge 2$ (an integer):

Consider $f(x) := x^n$ on $(0, \infty)$

$$f''(x) = n(n-1)x^{n-2} > 0 \quad \text{for } x > 0 \quad \Rightarrow \quad f(x) \text{ is (strictly) convex on } (0, \infty)$$

Thus $\dfrac{f(a)+f(b)+f(c)}{3} \ge f\left(\dfrac{a+b+c}{3}\right)$ if $a, b, c > 0$

That is, $\dfrac{a^n+b^n+c^n}{3} \ge \left(\dfrac{a+b+c}{3}\right)^n$ for all $a, b, c > 0$

Application2: If $x, y, z > 0$, show that

$$\frac{x}{2x+y+z} + \frac{y}{x+2y+z} + \frac{z}{x+y+2z} \le \frac{3}{4}$$

Sol. Let $s = x+y+z \ (>0)$. Then the above expression is written as.

$$\frac{x}{s+x} + \frac{y}{s+y} + \frac{z}{s+z} \le \frac{3}{4} \quad \text{(which is the one we are going to prove)}$$

To see why, let $f(t) = \dfrac{t}{s+t}$ $(t > 0)$. Then

$$f''(t) = -\frac{2s}{(s+t)^3} < 0 \quad \text{for all } t > 0$$

Thus, $f(t)$ concave on $(0, \infty)$.

$$\therefore \quad \frac{f(x)+f(y)+f(z)}{3} \le f\left(\frac{x+y+z}{3}\right) = f\left(\frac{s}{3}\right) = \frac{\frac{s}{3}}{s+\frac{s}{3}} = \frac{1}{4}$$

i.e., $f(x)+f(y)+f(z) \le \dfrac{3}{4}$ $\qquad \therefore \quad \dfrac{x}{s+x} + \dfrac{y}{s+y} + \dfrac{z}{s+z} \le \dfrac{3}{4}$

Application3: Show that

$$a^a b^b c^c \ge (abc)^{\frac{a+b+c}{3}} \quad \text{for all } a,b,c>0$$

Pf. It suffices to show:

$$a\ln a + b\ln b + c\ln c \ge \frac{a+b+c}{3}\left(\ln a + \ln b + \ln c\right) \quad \text{for all } a,b,c>0$$

Or,

$$\frac{a\ln a + b\ln b + c\ln c}{3} \ge \frac{a+b+c}{3}\cdot\frac{1}{3}\ln(abc) \quad \text{for all } a,b,c>0$$

Take $f(x) = x\ln x\,(x>0) \Rightarrow$

$$f'(x) = \ln x + 1 \quad \to \quad f''(x) = \frac{1}{x} > 0 \text{ for } x>0$$

$\therefore$ $f(x)$ is (strictly) convex for $x>0$

$$\therefore \quad f\left(\frac{a+b+c}{3}\right) \le \frac{f(a)+f(b)+f(c)}{3} \quad \text{for all } a,b,c>0$$

Hence

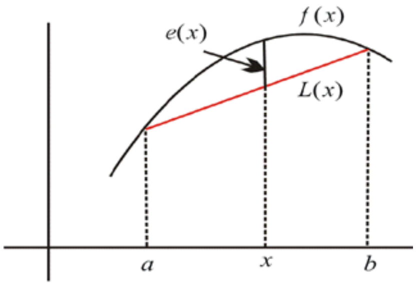
$$\begin{aligned}
\frac{a\ln a + b\ln b + c\ln c}{3} &\ge \frac{a+b+c}{3}\cdot\ln\left(\frac{a+b+c}{3}\right) \quad \text{for all } a,b,c>0 \\
&\ge \frac{a+b+c}{3}\ln\left(\sqrt[3]{abc}\right) \quad \text{for all } a,b,c>0 \quad [\leftarrow \text{AG} \ne] \\
&= \frac{a+b+c}{3}\cdot\frac{1}{3}\ln(abc) \quad \text{for all } a,b,c>0
\end{aligned}$$

16.3 The error in linear interpolation

Suppose $f''(x)$ exists on $[a, b]$, and let $L(x)$ be the linear function agreeing with $f(x)$ at the endpoints (i.e., $f(a) = L(a)$ & $f(b) = L(b)$)

Then for $x \in [a, b]$

$e(x) = f(x) - L(x)$ measures the error in the approximation $f(x) \approx L(x)$



What is the maximum value of $|\, e(x)\,|$ on $[a, b]$?

Ans. If $|\, f''(x)\,| \le M$ on $[a, b]$, then

$$|\, e(x)\,| \le \frac{M}{8}(b-a)^2, \quad a \le x \le b.$$

Pf. Let $x_0 \in [a, b]$ be a maximum point for $|\, e(x)\,|$. We may assume x_0 is not an endpoint of $[a, b]$ ($\because$ $e(a) = 0$ & $e(b) = 0$).

Thus we can assume x_0 is a local max pt or a local min pt for $e(x)$. So $e'(x_0) = 0$.

Note that x_0 lies in either $(a, \frac{a+b}{2}]$ or $[\frac{a+b}{2}, b)$.

WLOG, we may assume $x_0 \in [\frac{a+b}{2}, b)$. Then by Extended MVT

$$0 = e(b) = e(x_0) + \underbrace{e'(x_0)}_{=0}(b - x_0) + \frac{e''(c)}{2}(b-x_0)^2, \quad \text{where } x_0 < c < b$$

$$\therefore \quad e(x_0) = -\frac{e''(c)}{2}(b-x_0)^2$$

$$\therefore \quad |\, e(x_0)\,| = \frac{|\, e''(c)\,|}{2}(b-x_0)^2 \underset{\because\ b-x_0 \le (b-a)/2}{\leqq} \frac{|\, e''(c)\,|}{2}\frac{(b-a)^2}{4} = \frac{|\, e''(c)\,|}{8}(b-a)^2$$

Note that, since $e(x) = f(x) - L(x)$, we get $e''(x) = f''(x)$

$$\therefore\ |\, f''(x)\,| \le M \quad \Rightarrow \quad |\, e''(x)\,| \le M$$

$$\therefore \quad |\, e(x)\,| \underset{x_0 \text{ is a max pt for } |e(x)|}{\leqq} |\, e(x_0)\,| \le \frac{|\, e''(c)\,|}{8}(b-a)^2 \le \frac{M}{8}(b-a)^2, \quad a \le \forall x \le b$$