6.2 Paired Comparisons and a Repeated Measures Design

pdf read 295 - 296 for details

Paired Comparisons :

- Let $\bar{D}$ denotes the difference vector of means and δ is the population parameter of $\bar{D}$.

  Inferences about the vector of mean differences δ can be based on a T^2-statistic

  $$T^2 = n(\bar{D}-\delta)' S_d^{-1} (\bar{D}-\delta), \quad \text{where} \quad \bar{\mathbf{D}} = \frac{1}{n}\sum_{j=1}^{n} \mathbf{D}_j \text{ and } \mathbf{S}_d = \frac{1}{n-1}\sum_{j=1}^{n} (\mathbf{D}_j - \bar{\mathbf{D}})(\mathbf{D}_j - \bar{\mathbf{D}})'$$

  , and T^2 is distributed as an $\frac{(n-1)p}{(n-p)} F_{p,n-p}$.

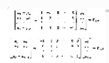
✩ pdf 298에 빨간색표시된 부분은 중요한 interpretation을 준다.

✩ If n and $n-p$ are both large, T^2 is approximately distributed as a χ^2 random variable, regardless of the form of the underlying population of differences.

✩✩ pdf 299-300에 빨간색 표시된 부분은 각 variable의 normality assumption을 확인하는 방법

A Repeated Measures Design for Comparing Treatments

- The name "repeated measures stems from the fact that all treatments are administered to each units.

 - comparing means within the samples ?

Test for Equality of Treatments in a Repeated Measures Design

— Consider an $N_q(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ population, and let $\mathbf{C}$ be a contrast matrix. An α-level test of H_0: $\mathbf{C}\boldsymbol{\mu} = \mathbf{0}$ (equal treatment means) versus H_1: $\mathbf{C}\boldsymbol{\mu} \neq \mathbf{0}$ is as follows: Reject H_0 if

$$T^2 = n(\mathbf{C}\bar{\mathbf{x}})'(\mathbf{CSC}')^{-1}\mathbf{C}\bar{\mathbf{x}} > \frac{(n-1)(q-1)}{(n-q+1)} F_{q-1,n-q+1}(\alpha) \qquad (6\text{-}16)$$

where $F_{q-1,n-q+1}(\alpha)$ is the upper (100α)th percentile of an F-distribution with $q-1$ and $n-q+1$ d.f. Here $\bar{\mathbf{x}}$ and $\mathbf{S}$ are the sample mean vector and covariance matrix defined, respectively, by

$$\bar{\mathbf{x}} = \frac{1}{n}\sum_{j=1}^{n} \mathbf{x}_j \quad \text{and} \quad \mathbf{S} = \frac{1}{n-1}\sum_{j=1}^{n} (\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})'$$

It can be shown that T^2 does not depend on the particular choice of $\mathbf{C}$.[1]

✩ T^2 does not depend on the particular choice of $\mathbf{C}$

└ This test is appropriate when the covariance matrix, $\mathrm{cov}(X) = \Sigma$ cannot be assumed to have any special structure.

A confidence region for contrasts $\mathbf{C}\boldsymbol{\mu}$, with $\boldsymbol{\mu}$ the mean of a normal population, is determined by the set of all $\mathbf{C}\boldsymbol{\mu}$ such that

$$n(\mathbf{C}\bar{\mathbf{x}} - \mathbf{C}\boldsymbol{\mu})'(\mathbf{CSC}')^{-1}(\mathbf{C}\bar{\mathbf{x}} - \mathbf{C}\boldsymbol{\mu}) \le \frac{(n-1)(q-1)}{(n-q+1)} F_{q-1,n-q+1}(\alpha) \qquad (6\text{-}17)$$

where $\bar{\mathbf{x}}$ and $\mathbf{S}$ are as defined in (6-16). Consequently, simultaneous $100(1-\alpha)\%$ confidence intervals for single contrasts $\mathbf{c}'\boldsymbol{\mu}$ for any contrast vectors of interest are given by (see Result 5A.1)

$$\mathbf{c}'\boldsymbol{\mu}: \quad \mathbf{c}'\bar{\mathbf{x}} \pm \sqrt{\frac{(n-1)(q-1)}{(n-q+1)} F_{q-1,n-q+1}(\alpha)} \sqrt{\frac{\mathbf{c}'\mathbf{S}\mathbf{c}}{n}} \qquad (6\text{-}18)$$

✩ T^2으로 전체적인 테스트를 해보고, T^2 통계량이 F분포 확률을 넘어가면, simultaneous confiden intervals를 통해 어느 요소끼리 유의미한 차이가 나는지 알아보는 건가?

→ 아님, ~~T^2 자체가 simultaneous~~ 한거임 ← 이게 아니었음 ↓ 이게 더 맞는거 같은데?

→ Contrast matrix를 활용한 T^2을 테스트해보고 유의미한 통계량을 보이면, simultaneous confidence intervals를 통해 어느 요소끼리 유의미한 차이가 나는지 확인.

✩ check Example 6.2 in pdf 302

6.3 Comparing Mean Vectors from Two Populations

Assumptions Concerning the Structure of the Data

1. The sample $\mathbf{X}_{11}, \mathbf{X}_{12}, \ldots, \mathbf{X}_{1n_1}$, is a random sample of size n_1 from a p-variate population with mean vector $\boldsymbol{\mu}_1$ and covariance matrix $\boldsymbol{\Sigma}_1$.
2. The sample $\mathbf{X}_{21}, \mathbf{X}_{22}, \ldots, \mathbf{X}_{2n_2}$, is a random sample of size n_2 from a p-variate population with mean vector $\boldsymbol{\mu}_2$ and covariance matrix $\boldsymbol{\Sigma}_2$.
3. Also, $\mathbf{X}_{11}, \mathbf{X}_{12}, \ldots, \mathbf{X}_{1n_1}$, are independent of $\mathbf{X}_{21}, \mathbf{X}_{22}, \ldots, \mathbf{X}_{2n_2}$. (6-19)

Further Assumptions When n_1 and n_2 Are Small

1. Both populations are multivariate normal.
2. Also, $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$ (same covariance matrix). (6-20)

- Having the same covariance matrix is very important, and with Σ_1 and Σ_2, we are able to compute the common covariance Σ, using the below equation.

$$\mathbf{S}_{\text{pooled}} = \frac{\sum_{j=1}^{n_1} (\mathbf{x}_{1j} - \bar{\mathbf{x}}_1)(\mathbf{x}_{1j} - \bar{\mathbf{x}}_1)' + \sum_{j=1}^{n_2} (\mathbf{x}_{2j} - \bar{\mathbf{x}}_2)(\mathbf{x}_{2j} - \bar{\mathbf{x}}_2)'}{n_1 + n_2 - 2}$$

$$= \frac{n_1 - 1}{n_1 + n_2 - 2}\mathbf{S}_1 + \frac{n_2 - 1}{n_1 + n_2 - 2}\mathbf{S}_2 \qquad (6\text{-}21)$$

Since the independence assumption in (6-19) implies that $\overline{\mathbf{X}}_1$ and $\overline{\mathbf{X}}_2$ are independent and thus $\text{Cov}(\overline{\mathbf{X}}_1, \overline{\mathbf{X}}_2) = \mathbf{0}$ (see Result 4.5), by (3-9), it follows that

$$\text{Cov}(\overline{\mathbf{X}}_1 - \overline{\mathbf{X}}_2) = \text{Cov}(\overline{\mathbf{X}}_1) + \text{Cov}(\overline{\mathbf{X}}_2) = \frac{1}{n_1}\boldsymbol{\Sigma} + \frac{1}{n_2}\boldsymbol{\Sigma} = \left(\frac{1}{n_1} + \frac{1}{n_2}\right)\boldsymbol{\Sigma} \qquad (6\text{-}22)$$

Because $\mathbf{S}_{\text{pooled}}$ estimates $\boldsymbol{\Sigma}$, we see that

$$\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\mathbf{S}_{\text{pooled}}$$

is an estimator of $\text{Cov}(\overline{\mathbf{X}}_1 - \overline{\mathbf{X}}_2)$.

The likelihood ratio test of

$$H_0: \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 = \boldsymbol{\delta}_0$$

is based on the square of the statistical distance, T^2, and is given by (see [1]). Reject H_0 if

$$T^2 = (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2 - \boldsymbol{\delta}_0)'\left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\mathbf{S}_{\text{pooled}}\right]^{-1}(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2 - \boldsymbol{\delta}_0) > c^2 \qquad (6\text{-}23)$$

Result 6.2. If $\mathbf{X}_{11}, \mathbf{X}_{12}, \ldots, \mathbf{X}_{1n_1}$ is a random sample of size n_1 from $N_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma})$ and $\mathbf{X}_{21}, \mathbf{X}_{22}, \ldots, \mathbf{X}_{2n_2}$ is an independent random sample of size n_2 from $N_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma})$, then

$$T^2 = [\overline{\mathbf{X}}_1 - \overline{\mathbf{X}}_2 - (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)]'\left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\mathbf{S}_{\text{pooled}}\right]^{-1}[\overline{\mathbf{X}}_1 - \overline{\mathbf{X}}_2 - (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)]$$

is distributed as

$$\frac{(n_1 + n_2 - 2)p}{(n_1 + n_2 - p - 1)}F_{p,n_1+n_2-p-1}$$

Consequently,

$$P\left[(\overline{\mathbf{X}}_1 - \overline{\mathbf{X}}_2 - (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2))'\left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\mathbf{S}_{\text{pooled}}\right]^{-1}(\overline{\mathbf{X}}_1 - \bar{\mathbf{y}}\ \sqrt{\lambda_i}\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)c^2} = \sqrt{\lambda_i}\sqrt{.25}\right] \qquad (6\text{-}24)$$

where

$$c^2 = \frac{(n_1 + n_2 - 2)p}{(n_1 + n_2 - p - 1)}F_{p,n_1+n_2-p-1}(\alpha)$$

the lengths of the major/minor axes of an ellipse

$$\sqrt{\lambda_i}\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)c^2}$$

Simultaneous Confidence Intervals

Result 6.3. Let $c^2 = [(n_1 + n_2 - 2)p/(n_1 + n_2 - p - 1)]F_{p,n_1+n_2-p-1}(\alpha)$. With probability $1 - \alpha$.

$$\mathbf{a}'(\overline{\mathbf{X}}_1 - \overline{\mathbf{X}}_2) \pm c\sqrt{\mathbf{a}'\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\mathbf{S}_{\text{pooled}}\mathbf{a}}$$

will cover $\mathbf{a}'(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)$ for all $\mathbf{a}$. In particular $\mu_{1i} - \mu_{2i}$ will be covered by

$$(\overline{X}_{1i} - \overline{X}_{2i}) \pm c\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)s_{ii,\text{pooled}}} \qquad \text{for } i = 1, 2, \ldots, p$$

The *Bonferroni* $100(1 - \alpha)\%$ *simultaneous confidence intervals* for the p population mean differences are

$$\mu_{1i} - \mu_{2i}: \quad (\bar{x}_{1i} - \bar{x}_{2i}) \pm t_{n_1+n_2-2}\left(\frac{\alpha}{2p}\right)\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)s_{ii,\text{pooled}}}$$

where $t_{n_1+n_2-2}(\alpha/2p)$ is the upper $100(\alpha/2p)$th percentile of a t-distribution with $n_1 + n_2 - 2$ d.f.

The Two-Sample Situation When $\Sigma_1 \neq \Sigma_2$

- When $\Sigma_1 \neq \Sigma_2$, we are unable to find a "distance" measure like T^2
- The conclusions can be seriously misleading when the populations are nonnormal. Nonnormality and unequal covariances cannot be separated with Barlett's test.

Result 6.4. Let the sample sizes be such that $n_1 - p$ and $n_2 - p$ are large. Then, an approximate $100(1 - \alpha)\%$ confidence ellipsoid for $\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2$ is given by all $\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2$ satisfying

$$[\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2 - (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)]'\left[\frac{1}{n_1}\mathbf{S}_1 + \frac{1}{n_2}\mathbf{S}_2\right]^{-1}[\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2 - (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)] \le \chi_p^2(\alpha)$$

where $\chi_p^2(\alpha)$ is the upper (100α)th percentile of a chi-square distribution with p d.f. Also, $100(1 - \alpha)\%$ simultaneous confidence intervals for all linear combinations $\mathbf{a}'(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)$ are provided by

$$\mathbf{a}'(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) \text{ belongs to } \mathbf{a}'(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) \pm \sqrt{\chi_p^2(\alpha)}\sqrt{\mathbf{a}'\left(\frac{1}{n_1}\mathbf{S}_1 + \frac{1}{n_2}\mathbf{S}_2\right)\mathbf{a}}$$

* the effect of unequal variances is least when $n_1 = n_2$ and greatest when n_1 is much less than n_2 or vice-versa.

Remark. If $n_1 = n_2 = n$, then $(n - 1)/(n + n - 2) = 1/2$, so

$$\frac{1}{n_1}\mathbf{S}_1 + \frac{1}{n_2}\mathbf{S}_2 = \frac{1}{n}(\mathbf{S}_1 + \mathbf{S}_2) = \frac{(n-1)\mathbf{S}_1 + (n-1)\mathbf{S}_2}{n + n - 2}\left(\frac{1}{n} + \frac{1}{n}\right)$$
$$= \mathbf{S}_{\text{pooled}}\left(\frac{1}{n} + \frac{1}{n}\right)$$

With equal sample sizes, the large sample procedure is essentially the same as the procedure based on the pooled covariance matrix. (See Result 6.2.) In one dimension, it is well known that the effect of unequal variances is least when $n_1 = n_2$ and greatest when n_1 is much less than n_2 or vice versa.

An Approximation to the Distribution of T^2 for Normal Populations When Sample Sizes Are Not Large

- One can test $H_0: \mu_1 - \mu_2 = 0$ even when the population covariance matrices are unequal and the two sample sizes are not large. The result requires that both sample sizes n_1 and n_2 are greater than p.

$T^2 = (\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2))' \left[\frac{1}{n_1}S_1 + \frac{1}{n_2}S_2\right]^{-1} (\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2))$, and compare this with

$T^2 = \frac{vp}{v-p+1} F_{p,v-p+1}$, where the degrees of freedom v are estimated from the sample covariance matrices using the relation.

$$v = \frac{p + p^2}{\sum_{i=1}^{2} \frac{1}{n_i} \left\{ \text{tr}\left[\left(\frac{1}{n_i}\mathbf{S}_i\left(\frac{1}{n_1}\mathbf{S}_1 + \frac{1}{n_2}\mathbf{S}_2\right)^{-1}\right)^2\right] + \left(\text{tr}\left[\frac{1}{n_i}\mathbf{S}_i\left(\frac{1}{n_1}\mathbf{S}_1 + \frac{1}{n_2}\mathbf{S}_2\right)^{-1}\right]\right)^2 \right\}},$$

where $\min(n_1, n_2) \le n_1 + n_2$

6.4 Comparing Several Multivariate Population Means (One-Way MANOVA)

- MANOVA is used to investigate whether the mean vectors of more than two population are the same and, if not, which mean components differ significantly.

Assumptions about the Structure of the Data for One-Way MANOVA

1. Let $X_{\ell 1}, X_{\ell 2}, \ldots, X_{\ell n_\ell}$ is a random sample of size n_ℓ from a population with mean μ_ℓ, $\ell = 1, 2, 3, \ldots, g$. The random samples from different populations are independent.
2. All populations have a common covariance matrix Σ.
3. Each population is multivariate normal.

A Summary of Univariate ANOVA

- The *reparameterization*

$$\underset{\left(\begin{array}{c}\ell\text{th population}\\ \text{mean}\end{array}\right)}{\mu_\ell} = \underset{\left(\begin{array}{c}\text{overall}\\ \text{mean}\end{array}\right)}{\mu} + \underset{\left(\begin{array}{c}\ell\text{th population}\\ \text{(treatment) effect}\end{array}\right)}{\tau_\ell} \qquad (6\text{-}32)$$

τ = treatment

leads to a restatement of the hypothesis of equality of means. The null hypothesis becomes

treatment effect 가 모두 없다...?

$$H_0: \tau_1 = \tau_2 = \cdots = \tau_g = 0$$

The response $X_{\ell j}$, distributed as $N(\mu + \tau_\ell, \sigma^2)$, can be expressed in the suggestive form

$$\underset{}{X_{\ell j}} = \underset{\text{(overall mean)}}{\mu} + \underset{\left(\begin{array}{c}\text{treatment}\\ \text{effect}\end{array}\right)}{\tau_\ell} + \underset{\left(\begin{array}{c}\text{random}\\ \text{error}\end{array}\right)}{e_{\ell j}} \qquad (6\text{-}33)$$

where the $e_{\ell j}$ are independent $N(0, \sigma^2)$ random variables. To define uniquely the model parameters and their least squares estimates, it is customary to impose the constraint $\sum_{\ell=1}^{g} n_\ell \tau_\ell = 0$.

- The question of equality of means is answered by assessing whether the contribution of the treatment array is large relative to the residuals.

$$\underset{\text{(observation)}}{x_{\ell j}} = \underset{\left(\begin{array}{c}\text{overall}\\ \text{sample mean}\end{array}\right)}{\bar{x}} + \underset{\left(\begin{array}{c}\text{estimated}\\ \text{treatment effect}\end{array}\right)}{(\bar{x}_\ell - \bar{x})} + \underset{\text{(residual)}}{(x_{\ell j} - \bar{x}_\ell)} \qquad (6\text{-}34)$$

- The size of an array is quantified by stringing the rows of the array out into a vector and calculating its squared length. (L_2 norm) == Sum of Squares

$$SS_{obs} = SS_{mean} + SS_{tr} + SS_{res}$$

If H_0 is true, variances computed from SS_{tr} and SS_{res} should be approximately equal.

The sum of squares decomposition illustrated numerically in Example 6.7 is so basic that the algebraic equivalent will now be developed.

Subtracting $\bar{x}$ from both sides of (6-34) and squaring gives

$$(x_{\ell j} - \bar{x})^2 = (\bar{x}_\ell - \bar{x})^2 + (x_{\ell j} - \bar{x}_\ell)^2 + 2(\bar{x}_\ell - \bar{x})(x_{\ell j} - \bar{x}_\ell)$$

We can sum both sides over j, note that $\sum_{j=1}^{n_\ell} (x_{\ell j} - \bar{x}_\ell) = 0$, and obtain

$$\sum_{j=1}^{n_\ell} (x_{\ell j} - \bar{x})^2 = n_\ell(\bar{x}_\ell - \bar{x})^2 + \sum_{j=1}^{n_\ell} (x_{\ell j} - \bar{x}_\ell)^2$$

Next, summing both sides over ℓ we get

$$\sum_{\ell=1}^{g} \sum_{j=1}^{n_\ell} (x_{\ell j} - \bar{x})^2 = \sum_{\ell=1}^{g} n_\ell(\bar{x}_\ell - \bar{x})^2 + \sum_{\ell=1}^{g} \sum_{j=1}^{n_\ell} (x_{\ell j} - \bar{x}_\ell)^2 \tag{6-35}$$

$$\begin{pmatrix} SS_{cor} \\ \text{total (corrected) SS} \end{pmatrix} = \begin{pmatrix} SS_{tr} \\ \text{between (samples) SS} \end{pmatrix} + \begin{pmatrix} SS_{res} \\ \text{within (samples) SS} \end{pmatrix}$$

or

$$\begin{array}{ccccccccc} \sum_{\ell=1}^{g} \sum_{j=1}^{n_\ell} x_{\ell j}^2 & = & (n_1 + n_2 + \cdots + n_g)\bar{x}^2 & + & \sum_{\ell=1}^{g} n_\ell(\bar{x}_\ell - \bar{x})^2 & + & \sum_{\ell=1}^{g} \sum_{j=1}^{n_\ell} (x_{\ell j} - \bar{x}_\ell)^2 & \\ (SS_{obs}) & = & (SS_{mean}) & + & (SS_{tr}) & + & (SS_{res}) & (6\text{-}36) \end{array}$$

d.f. of $SS_{mean} = 1$, d.f. of $SS_{tr} = g-1$, d.f. of $SS_{reg} = n-g$

ANOVA Table for Comparing Univariate Population Means

Source of variation	Sum of squares (SS)	Degrees of freedom (d.f.)
Treatments	$SS_{tr} = \sum_{\ell=1}^{g} n_\ell(\bar{x}_\ell - \bar{x})^2$	$g - 1$
Residual (error)	$SS_{res} = \sum_{\ell=1}^{g} \sum_{j=1}^{n_\ell} (x_{\ell j} - \bar{x}_\ell)^2$	$\sum_{\ell=1}^{g} n_\ell - g$
Total (corrected for the mean)	$SS_{cor} = \sum_{\ell=1}^{g} \sum_{j=1}^{n_\ell} (x_{\ell j} - \bar{x})^2$	$\sum_{\ell=1}^{g} n_\ell - 1$

The degrees of freedom above are known to be for the χ^2 distributions associated with the corresponding sums of squares.

The usual F-test rejects $H_0: \tau_1 = \tau_2 = \cdots = \tau_g = 0$ at level α if

$$F = \frac{SS_{tr}/(g-1)}{SS_{res}\Big/\left(\sum_{\ell=1}^{g} n_\ell - g\right)} > F_{g-1, \Sigma n_\ell - g}(\alpha)$$

where $F_{g-1, \Sigma n_\ell - g}(\alpha)$ is the upper (100α)th percentile of the F-distribution with $g - 1$ and $\Sigma n_\ell - g$ degrees of freedom. This is equivalent to rejecting H_0 for large values of SS_{tr}/SS_{res} or for large values of $1 + SS_{tr}/SS_{res}$. The statistic appropriate for a multivariate generalization rejects H_0 for *small* values of the reciprocal

$$\frac{1}{1 + SS_{tr}/SS_{res}} = \frac{SS_{res}}{SS_{res} + SS_{tr}} \tag{6-37}$$

Multivariate Analysis of Variance

MANOVA Model For Comparing a Population Mean Vectors

$$\sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell}(\mathbf{x}_{\ell j}-\bar{\mathbf{x}})(\mathbf{x}_{\ell j}-\bar{\mathbf{x}})' = \sum_{\ell=1}^{g} n_\ell(\bar{\mathbf{x}}_\ell-\bar{\mathbf{x}})(\bar{\mathbf{x}}_\ell-\bar{\mathbf{x}})' + \sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell}(\mathbf{x}_{\ell j}-\bar{\mathbf{x}}_\ell)(\mathbf{x}_{\ell j}-\bar{\mathbf{x}}_\ell)' \tag{6-40}$$

(total (corrected) sum of squares and cross products) (treatment (Between) sum of squares and cross products) (residual (Within) sum of squares and cross products)

The *within* sum of squares and cross products matrix can be expressed as

$$\begin{aligned}\mathbf{W} &= \sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell}(\mathbf{x}_{\ell j}-\bar{\mathbf{x}}_\ell)(\mathbf{x}_{\ell j}-\bar{\mathbf{x}}_\ell)' \\ &= (n_1-1)\mathbf{S}_1+(n_2-1)\mathbf{S}_2+\cdots+(n_g-1)\mathbf{S}_g\end{aligned} \tag{6-41}$$

generalization of $(n_1+n_2-2)\,S_{pooled}$

MANOVA Table for Comparing Population Mean Vectors

Source of variation	Matrix of sum of squares and cross products (SSP)	Degrees of freedom (d.f.)
Treatment	$\mathbf{B}=\sum_{\ell=1}^{g} n_\ell(\bar{\mathbf{x}}_\ell-\bar{\mathbf{x}})(\bar{\mathbf{x}}_\ell-\bar{\mathbf{x}})'$	$g-1$
Residual (Error)	$\mathbf{W}=\sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell}(\mathbf{x}_{\ell j}-\bar{\mathbf{x}}_\ell)(\mathbf{x}_{\ell j}-\bar{\mathbf{x}}_\ell)'$	$\sum_{\ell=1}^{g} n_\ell - g$
Total (corrected for the mean)	$\mathbf{B}+\mathbf{W}=\sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell}(\mathbf{x}_{\ell j}-\bar{\mathbf{x}})(\mathbf{x}_{\ell j}-\bar{\mathbf{x}})'$	$\sum_{\ell=1}^{g} n_\ell - 1$

Testing Parameter

Wishart densities. (See [1].)

One test of $H_0: \boldsymbol{\tau}_1=\boldsymbol{\tau}_2=\cdots=\boldsymbol{\tau}_g=\mathbf{0}$ involves generalized variances. We reject H_0 if the ratio of generalized variances

$$\Lambda^* = \frac{|\mathbf{W}|}{|\mathbf{B}+\mathbf{W}|} = \frac{\left|\sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell}(\mathbf{x}_{\ell j}-\bar{\mathbf{x}}_\ell)(\mathbf{x}_{\ell j}-\bar{\mathbf{x}}_\ell)'\right|}{\left|\sum_{\ell=1}^{g}\sum_{j=1}^{n_\ell}(\mathbf{x}_{\ell j}-\bar{\mathbf{x}})(\mathbf{x}_{\ell j}-\bar{\mathbf{x}})'\right|} \tag{6-42}$$

is too small. The quantity $\Lambda^* = |\mathbf{W}|/|\mathbf{B}+\mathbf{W}|$, proposed originally by Wilks

Bartlett (see [4]) has shown that if H_0 is true and $\Sigma n_\ell = n$ is large,

$$-\left(n-1-\frac{(p+g)}{2}\right)\ln\Lambda^* = -\left(n-1-\frac{(p+g)}{2}\right)\ln\left(\frac{|\mathbf{W}|}{|\mathbf{B}+\mathbf{W}|}\right) \tag{6-43}$$

has approximately a chi-square distribution with $p(g-1)$ d.f. Consequently, for $\Sigma n_\ell = n$ large, we reject H_0 at significance level α if

$$-\left(n-1-\frac{(p+g)}{2}\right)\ln\left(\frac{|\mathbf{W}|}{|\mathbf{B}+\mathbf{W}|}\right) > \chi^2_{p(g-1)}(\alpha) \tag{6-44}$$

where $\chi^2_{p(g-1)}(\alpha)$ is the upper (100α)th percentile of a chi-square distribution with $p(g-1)$ d.f.

Table 6.3 Distribution of Wilks' Lambda, $\Lambda^* = |\mathbf{W}|/|\mathbf{B}+\mathbf{W}|$

No. of variables	No. of groups	Sampling distribution for multivariate normal data
$p=1$	$g\ge 2$	$\left(\frac{\Sigma n_\ell - g}{g-1}\right)\left(\frac{1-\Lambda^*}{\Lambda^*}\right) \sim F_{g-1,\Sigma n_\ell - g}$
$p=2$	$g\ge 2$	$\left(\frac{\Sigma n_\ell - g - 1}{g-1}\right)\left(\frac{1-\sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right) \sim F_{2(g-1),2(\Sigma n_\ell - g-1)}$
$p\ge 1$	$g=2$	$\left(\frac{\Sigma n_\ell - p - 1}{p}\right)\left(\frac{1-\Lambda^*}{\Lambda^*}\right) \sim F_{p,\Sigma n_\ell - p-1}$
$p\ge 1$	$g=3$	$\left(\frac{\Sigma n_\ell - p - 2}{p}\right)\left(\frac{1-\sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right) \sim F_{2p,2(\Sigma n_\ell - p-2)}$

or $\left(\frac{\Sigma n_\ell - p - 2}{p}\right)\left(\frac{1-\sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right) \le F_{2(p),2(N)}(0.01)$

pdf 325 - 329, check Example 6.9 & 6.10

6.5 Simultaneous Confidence Intervals for Treatment Effects

- When the hypothesis of equal treatment effects is rejected, those effects that led to the rejection of the hypothesis are of interest. (Use Bonferroni approach!)
- Since the treatment effect is $\hat{\tau}_k = \bar{x}_k - \bar{x}$, so the difference between the treatments will be $\hat{\tau}_k - \hat{\tau}_\ell = \bar{x}_k - \bar{x} - (\bar{x}_\ell - \bar{x}) = \bar{x}_k - \bar{x}_\ell$, and the two-sample t-based confidence interval is valid with an appropriately modified α.

Let τ_{ki} be the ith component of $\boldsymbol{\tau}_k$. Since $\boldsymbol{\tau}_k$ is estimated by $\hat{\boldsymbol{\tau}}_k = \bar{\mathbf{x}}_k - \bar{\mathbf{x}}$

$$\hat{\tau}_{ki} = \bar{x}_{ki} - \bar{x}_i \tag{6-45}$$

and $\hat{\tau}_{ki} - \hat{\tau}_{\ell i} = \bar{x}_{ki} - \bar{x}_{\ell i}$ is the difference between two independent sample means. The two-sample t-based confidence interval is valid with an appropriately modified α. Notice that

$$\text{Var}(\hat{\tau}_{ki} - \hat{\tau}_{\ell i}) = \text{Var}(\bar{X}_{ki} - \bar{X}_{\ell i}) = \left(\frac{1}{n_k} + \frac{1}{n_\ell}\right)\sigma_{ii}$$

where σ_{ii} is the ith diagonal element of $\boldsymbol{\Sigma}$. As suggested by (6-41), $\text{Var}(\bar{X}_{ki} - \bar{X}_{\ell i})$ is estimated by dividing the corresponding element of $\mathbf{W}$ by its degrees of freedom. That is,

$$\widehat{\text{Var}}(\bar{X}_{ki} - \bar{X}_{\ell i}) = \left(\frac{1}{n_k} + \frac{1}{n_\ell}\right)\frac{w_{ii}}{n-g}$$

where w_{ii} is the ith diagonal element of $\mathbf{W}$ and $n = n_1 + \cdots + n_g$.

It remains to apportion the error rate over the numerous confidence statements. Relation (5-28) still applies. There are p variables and $g(g-1)/2$ pairwise differences, so each two-sample t-interval will employ the critical value $t_{n-g}(\alpha/2m)$, where

$$m = pg(g-1)/2 \tag{6-46}$$

is the number of simultaneous confidence statements.

Result 6.5. Let $n = \sum_{k=1}^{g} n_k$. For the model in (6-38), with confidence at least $(1-\alpha)$,

$$\tau_{ki} - \tau_{\ell i} \quad \text{belongs to} \quad \bar{x}_{ki} - \bar{x}_{\ell i} \pm t_{n-g}\left(\frac{\alpha}{pg(g-1)}\right)\sqrt{\frac{w_{ii}}{n-g}\left(\frac{1}{n_k} + \frac{1}{n_\ell}\right)}$$

for all components $i = 1, \ldots, p$ and all differences $\ell < k = 1, \ldots, g$. Here w_{ii} is the ith diagonal element of $\mathbf{W}$.

6.6 Testing for Equality of Covariance Matrices

- One of the assumptions made when comparing two or more multivariate mean vectors is that the covariance matrices of the potentially different populations are the same.

Box's M-test:

- testing for equal covariance

With g populations, the null hypothesis is

$$H_0: \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = \cdots = \boldsymbol{\Sigma}_g = \boldsymbol{\Sigma} \tag{6-47}$$

where $\boldsymbol{\Sigma}_\ell$ is the covariance matrix for the ℓth population, $\ell = 1, 2, \ldots, g$, and $\boldsymbol{\Sigma}$ is the presumed common covariance matrix. The alternative hypothesis is that at least two of the covariance matrices are not equal.

- Likelihood Ratio Statistic for testing equal variance

$$\Lambda = \prod_\ell \left(\frac{|S_\ell|}{|S_{pooled}|} \right)^{\frac{n_\ell - 1}{2}}, \quad S_{pooled} = \frac{1}{\sum(n_\ell - 1)} \{ (n_1-1)S_1 + (n_2-1)S_2 + \cdots + (n_g-1)S_g \}$$

Box's test is based on his χ^2 approximation to the sampling distribution of $-2 \ln \Lambda$ (see Result 5.2). Setting $-2 \ln \Lambda = M$ (Box's M statistic) gives

$$M = \left[\sum_\ell (n_\ell - 1) \right] \ln |\mathbf{S}_{\text{pooled}}| - \sum_\ell [(n_\ell - 1) \ln |\mathbf{S}_\ell|] \qquad (6\text{-}50)$$

- note that the determinant of the pooled covariance matrix $|S_{pooled}|$ will lie somewhere near the "middle" of the determinants $|S_\ell|$'s.

Box's Test for Equality of Covariance Matrices

Set

$$u = \left[\sum_\ell \frac{1}{(n_\ell - 1)} - \frac{1}{\sum_\ell (n_\ell - 1)} \right] \left[\frac{2p^2 + 3p - 1}{6(p + 1)(g - 1)} \right] \qquad (6\text{-}51)$$

where p is the number of variables and g is the number of groups. Then

$$C = (1 - u)M = (1 - u) \left\{ \left[\sum_\ell (n_\ell - 1) \right] \ln |\mathbf{S}_{\text{pooled}}| - \sum_\ell [(n_\ell - 1) \ln |\mathbf{S}_\ell|] \right\} \qquad (6\text{-}52)$$

has an approximate χ^2 distribution with

$$v = g \frac{1}{2} p(p + 1) - \frac{1}{2} p(p + 1) = \frac{1}{2} p(p + 1)(g - 1) \qquad (6\text{-}53)$$

degrees of freedom. At significance level α, reject H_0 if $C > \chi^2_{p(p+1)(g-1)/2}(\alpha)$.

Box's χ^2 approximation works well if each n_ℓ exceeds 20 and if p and g do not exceed 5. In situations where these conditions do not hold, Box ([7], [8]) has provided a more precise F approximation to the sampling distribution of M.

6.7 Two-Way Multivariate Analysis of Variance

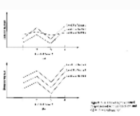
Univariate Two-Way Fixed-Effects Model with Interaction

- Suppose there are g levels of factor 1 and b levels of factor 2, and n independent observations can be observed at each of the gb combinations of levels.

$$\underbrace{X_{\ell k r}}_{\text{response}} = \underbrace{\mu}_{\text{overall level}} + \underbrace{\tau_\ell}_{\text{effect of factor 1}} + \underbrace{\beta_k}_{\text{effect of factor 2}} + \underbrace{\gamma_{\ell k}}_{\text{factor 1 \& 2 interaction}} + e_{\ell k r}, \quad \ell = 1, 2, \ldots, g; \; k = 1, 2, \ldots, b; \; r = 1, 2, \ldots, n$$

$$* \sum_{\ell=1}^{g} \tau_\ell = \sum_{k=1}^{b} \beta_k = \sum_{\ell=1}^{g} \gamma_{\ell k} = \sum_{k=1}^{b} \gamma_{\ell k} = 0$$

$$\rightarrow X_{\ell k r} = \bar{x} + (\bar{x}_{\ell \cdot} - \bar{x}) + (\bar{x}_{\cdot k} - \bar{x}) + (\underbrace{\bar{x}_{\ell k}}_{\text{average for the } \ell^{th} \text{ level of factor 1 and the } k^{th} \text{ level of factor 2}} - \bar{x}_{\ell} - \bar{x}_{\cdot k} + \bar{x}) + (X_{\ell k r} - \bar{x}_{\ell k})$$



factor 1 *and* the kth level of factor 2. Squaring and summing the deviations $(x_{\ell kr} - \bar{x})$ gives

$$\sum_{\ell=1}^{g}\sum_{k=1}^{b}\sum_{r=1}^{n}(x_{\ell kr} - \bar{x})^2 = \sum_{\ell=1}^{g} bn(\bar{x}_{\ell\cdot} - \bar{x})^2 + \sum_{k=1}^{b} gn(\bar{x}_{\cdot k} - \bar{x})^2 + \sum_{\ell=1}^{g}\sum_{k=1}^{b} n(\bar{x}_{\ell k} - \bar{x}_{\ell\cdot} - \bar{x}_{\cdot k} + \bar{x})^2 + \sum_{\ell=1}^{g}\sum_{k=1}^{b}\sum_{r=1}^{n}(x_{\ell kr} - \bar{x}_{\ell k})^2 \qquad (6\text{-}57)$$

or

$$SS_{cor} = SS_{fac1} + SS_{fac2} + SS_{int} + SS_{res}$$

The corresponding degrees of freedom associated with the sums of squares in the breakup in (6-57) are

$$gbn - 1 = (g - 1) + (b - 1) + (g - 1)(b - 1) + gb(n - 1) \qquad (6\text{-}58)$$

ANOVA Table for Comparing Effects of Two Factors and Their Interaction

Source of variation	Sum of squares (SS)	Degrees of freedom (d.f.)
Factor 1	$SS_{fac1} = \sum_{\ell=1}^{g} bn(\bar{x}_{\ell\cdot} - \bar{x})^2$	$g - 1$
Factor 2	$SS_{fac2} = \sum_{k=1}^{b} gn(\bar{x}_{\cdot k} - \bar{x})^2$	$b - 1$
Interaction	$SS_{int} = \sum_{\ell=1}^{g}\sum_{k=1}^{b} n(\bar{x}_{\ell k} - \bar{x}_{\ell\cdot} - \bar{x}_{\cdot k} + \bar{x})^2$	$(g - 1)(b - 1)$
Residual (Error)	$SS_{res} = \sum_{\ell=1}^{g}\sum_{k=1}^{b}\sum_{r=1}^{n}(x_{\ell kr} - \bar{x}_{\ell k})^2$	$gb(n - 1)$
Total (corrected)	$SS_{cor} = \sum_{\ell=1}^{g}\sum_{k=1}^{b}\sum_{r=1}^{n}(x_{\ell kr} - \bar{x})^2$	$gbn - 1$

The F-ratios of the mean squares, $SS_{fac1}/(g-1)$, $SS_{fac2}/(b-1)$, and $SS_{int}/(g-1)(b-1)$ to the mean square, $SS_{res}/(gb(n-1))$ can be used to test for the effects of factor 1, factor 2, and factor 1–factor 2 interaction, respectively. (See

Multivariate Two-Way Fixed-Effects Model with Interaction

Again, the generalization from the univariate to the multivariate analysis consists simply of replacing a scalar such as $(\bar{x}_{\ell\cdot} - \bar{x})^2$ with the corresponding matrix $(\bar{\mathbf{x}}_{\ell\cdot} - \bar{\mathbf{x}})(\bar{\mathbf{x}}_{\ell\cdot} - \bar{\mathbf{x}})'$.

The MANOVA table is the following:

MANOVA Table for Comparing Factors and Their Interaction

Source of variation	Matrix of sum of squares and cross products (SSP)	Degrees of freedom (d.f.)
Factor 1	$SSP_{fac1} = \sum_{\ell=1}^{g} bn(\bar{\mathbf{x}}_{\ell\cdot} - \bar{\mathbf{x}})(\bar{\mathbf{x}}_{\ell\cdot} - \bar{\mathbf{x}})'$	$g - 1$
Factor 2	$SSP_{fac2} = \sum_{k=1}^{b} gn(\bar{\mathbf{x}}_{\cdot k} - \bar{\mathbf{x}})(\bar{\mathbf{x}}_{\cdot k} - \bar{\mathbf{x}})'$	$b - 1$
Interaction	$SSP_{int} = \sum_{\ell=1}^{g}\sum_{k=1}^{b} n(\bar{\mathbf{x}}_{\ell k} - \bar{\mathbf{x}}_{\ell\cdot} - \bar{\mathbf{x}}_{\cdot k} + \bar{\mathbf{x}})(\bar{\mathbf{x}}_{\ell k} - \bar{\mathbf{x}}_{\ell\cdot} - \bar{\mathbf{x}}_{\cdot k} + \bar{\mathbf{x}})'$	$(g - 1)(b - 1)$
Residual (Error)	$SSP_{res} = \sum_{\ell=1}^{g}\sum_{k=1}^{b}\sum_{r=1}^{n}(\mathbf{x}_{\ell kr} - \bar{\mathbf{x}}_{\ell k})(\mathbf{x}_{\ell kr} - \bar{\mathbf{x}}_{\ell k})'$	$gb(n - 1)$
Total (corrected)	$SSP_{cor} = \sum_{\ell=1}^{g}\sum_{k=1}^{b}\sum_{r=1}^{n}(\mathbf{x}_{\ell kr} - \bar{\mathbf{x}})(\mathbf{x}_{\ell kr} - \bar{\mathbf{x}})'$	$gbn - 1$

- A test is conducted by, $H_0: \gamma_{11} = \gamma_{12} = \cdots = \gamma_{gb} = 0$ (No interaction effects), and it is (the likelihood ratio test) rejected if Wilk's lambda, $\Lambda^* = \frac{|SSP_{res}|}{|SSP_{int} + SSP_{res}|}$, applied to

$$-\left[gb(n-1) - \frac{p+1-(g-1)(b-1)}{2}\right] \ln \Lambda^* > \chi^2_{(g-1)(b-1)p}(\alpha)$$

, is greater than χ^2 statistic

- Ordinarily, the test for interaction is carried out before the tests for main factor effects. If interaction effects exist, p univariate two-way analyses of variance are often conducted to see whether the interaction appears in some responses but not others, and those responses without interaction may be interpreted in terms of additive factor 1 and 2 effects.

Testing for factor 1 and factor 2

In the multivariate model, we test for factor 1 and factor 2 main effects as follows. First, consider the hypotheses $H_0: \boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 = \cdots = \boldsymbol{\tau}_g = \mathbf{0}$ and H_1: at least one $\boldsymbol{\tau}_\ell \neq \mathbf{0}$. These hypotheses specify *no* factor 1 effects and *some* factor 1 effects, respectively. Let

$$\Lambda^* = \frac{|\mathrm{SSP}_{\mathrm{res}}|}{|\mathrm{SSP}_{\mathrm{fac1}} + \mathrm{SSP}_{\mathrm{res}}|} \tag{6-66}$$

so that small values of Λ^* are consistent with H_1. Using Bartlett's correction, the likelihood ratio test is as follows:

Reject $H_0: \boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 = \cdots = \boldsymbol{\tau}_g = \mathbf{0}$ (no factor 1 effects) at level α if

$$-\left[gb(n-1) - \frac{p+1-(g-1)}{2}\right] \ln \Lambda^* > \chi^2_{(g-1)p}(\alpha) \tag{6-67}$$

where Λ^* is given by (6-66) and $\chi^2_{(g-1)p}(\alpha)$ is the upper (100α)th percentile of a chi-square distribution with $(g-1)p$ d.f.

In a similar manner, factor 2 effects are tested by considering $H_0: \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \cdots = \boldsymbol{\beta}_b = \mathbf{0}$ and H_1: at least one $\boldsymbol{\beta}_k \neq \mathbf{0}$. Small values of

$$\Lambda^* = \frac{|\mathrm{SSP}_{\mathrm{res}}|}{|\mathrm{SSP}_{\mathrm{fac2}} + \mathrm{SSP}_{\mathrm{res}}|} \tag{6-68}$$

are consistent with H_1. Once again, for large samples and using Bartlett's correction: Reject $H_0: \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \cdots = \boldsymbol{\beta}_b = \mathbf{0}$ (no factor 2 effects) at level α if

$$-\left[gb(n-1) - \frac{p+1-(b-1)}{2}\right] \ln \Lambda^* > \chi^2_{(b-1)p}(\alpha) \tag{6-69}$$

where Λ^* is given by (6-68) and $\chi^2_{(b-1)p}(\alpha)$ is the upper (100α)th percentile of a chi-square distribution with $(b-1)p$ degrees of freedom.

Simultaneous Confidence Intervals for $\tau_{\ell i} - \tau_{mi}$

The $100(1-\alpha)\%$ simultaneous confidence intervals for $\tau_{\ell i} - \tau_{mi}$ are

$$\tau_{\ell i} - \tau_{mi} \quad \text{belongs to} \quad (\bar{x}_{\ell \cdot i} - \bar{x}_{m \cdot i}) \pm t_\nu\left(\frac{\alpha}{pg(g-1)}\right)\sqrt{\frac{E_{ii}}{\nu}\frac{2}{bn}} \tag{6-70}$$

where $\nu = gb(n-1)$, E_{ii} is the ith diagonal element of $\mathbf{E} = \mathrm{SSP}_{\mathrm{res}}$, and $\bar{x}_{\ell \cdot i} - \bar{x}_{m \cdot i}$ is the ith component of $\bar{\mathbf{x}}_{\ell \cdot} - \bar{\mathbf{x}}_{m \cdot}$.

Similarly, the $100(1 - \alpha)$ percent simultaneous confidence intervals for $\beta_{ki} - \beta_{qi}$ are

$$\beta_{ki} - \beta_{qi} \quad \text{belongs to} \quad (\bar{x}_{\cdot ki} - \bar{x}_{\cdot qi}) \pm t_\nu\left(\frac{\alpha}{pb(b-1)}\right)\sqrt{\frac{E_{ii}}{\nu}\frac{2}{gn}} \tag{6-71}$$

where ν and E_{ii} are as just defined and $\bar{x}_{\cdot ki} - \bar{x}_{\cdot qi}$ is the ith component of $\bar{\mathbf{x}}_{\cdot k} - \bar{\mathbf{x}}_{\cdot q}$.