Ch10. Brownian motion

1. Brownian motion
2. BM with drift and Geometric Brownian Motion
3. Levy process

Construction of Brownian Motion

- Continuous version of random walk.

$$X_i = \begin{cases} 1 & \text{with } \frac{1}{2} \\ -1 & \text{with } \frac{1}{2} \end{cases}$$

Then we will argue that the scaled random walk

$$B_n(t) = \frac{1}{\sqrt{n}} S_{[nt]} \xrightarrow{d} B(t)$$

Observation

1. Since $EX_i = 0$, $\operatorname{Var} X_i = 1$, CLT implies that

$$\sqrt{n}\left(\frac{X_1 + \cdots + X_n}{n} - 0\right) \xrightarrow{d} \mathcal{N}(0,1)$$

$$\therefore \frac{S_n}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1)$$

   Observe that

$$B_n(t) = \frac{S[nt]}{\sqrt{n}} = \frac{S[nt]}{\sqrt{[nt]}}\sqrt{\frac{[nt]}{n}} \approx \sqrt{t}\mathcal{N}(0,1) = \mathcal{N}(0,t)$$

   Thus, for fixed t,

$$\boxed{B_n(t) \sim \mathcal{N}(0,t)}$$

   See, Bean machine example for illustration.

2. $B(t)$ has stationary and independent increments since $X_i's$ are IID.

Brownian Motion

Definition

A stochastic process $\{B(t), t \geq 0\}$ *is said to be a (standard) Brownian Motion (BM) if*

$i)$ $B(0) = 0$

$ii)$ Stationary$\&$ independent increments

– $B(t_2) - B(t_1),\ B(t_3) - B(t_2),\ \cdots, B(t_n) - B(t_{n-1})$ are independent for all $0 \leq t_1 < t_2 \cdots < t_n$

– Distribution of $B(t+s) - B(t)$ does not depend on t

$iii)$ For every $t > 0,\ \ B(t) \sim N(0, t)$

Therefore, we have from $ii)$ an $iii)$ that

$$\boxed{B(t) - B(s) \overset{d}{=} B(t-s) \sim \mathcal{N}(0, t-s)}$$

Properties

1. Sample path is continuous

$$B(t+h) - B(t) = B(h) \sim N(0,h)$$

   Thus, $h \downarrow 0$, variance becomes zero, therefore

$$P(B(t+h) - B(t) \to 0) = 1 \quad \text{as } h \to 0.$$

   That is, $\lim_{h\to 0} B(t+h) = B(t)$.

2. However, sample path of $B(t)$ is nowhere differentiable.

$$\lim_{h\downarrow 0} \frac{B(t+h) - B(t)}{h} \approx \lim_{h\downarrow 0} \mathcal{N}(0, 1/h) = \mathcal{N}(0, \infty).$$

   It means that such limit does not exist.

Properties of BM

3. Note that

$$B(ct) = B(ct) - B(0) \sim \mathcal{N}(0, ct)$$

$$B(t) = B(t) - B(0) \sim \mathcal{N}(0, t)$$

$$B(ct) \stackrel{d}{=} \sqrt{c}B(t)$$

4. $EB(t)B(s) = \frac{1}{2}(|t| + |s| - |t - s|) = \min(t, s)$.

5. We can define BM with $\sigma^2 \neq 1$ by taking $\sigma B(t)$. That is, consider the limit of random walk with $X_i \sim IID(0, \sigma^2)$

Example of BM calculation

Let $Y(t)$ be the amount time racer 1 is ahead than racer 2, when 100t$\%$ of race is done. Suppose $Y(t) \sim BM$ with variance σ^2. If racer 1 is leading by σ seconds at the midpoint of the race, find the probability that racer1 is winner.
Solution:

$$\begin{aligned}
&P(Y(1) > 0|Y(1/2) = \sigma) \\
&= P(Y(1) - Y(1/2) > -\sigma \mid Y(1/2) = \sigma) \\
&= P(Y(1) - Y(1/2) > -\sigma) \\
&= P(Y(1/2) > -\sigma) = P(\mathcal{N}(0, \sigma^2/2) > -\sigma) \\
&= P(\frac{\sigma}{\sqrt{2}} Z > -\sigma) = P(Z > -\sqrt{2}) \approx .9213
\end{aligned}$$

Properties of BM

6. For $s \leq t$,

$$\begin{pmatrix} B(s) \\ B(t) \end{pmatrix} \sim MVN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} s & s \\ s & t \end{pmatrix} \right)$$

   Thus, the conditional distribution

$$B(t)|B(s) = B(t) - B(s) + B(s)|B(s) \sim \mathcal{N}(B(s), t - s)$$

7. Define the first time BM hits a by $T_a := \min_t\{B(t) \geq a\}$. Then,

$$P(T_a \leq t) = P\left(\max_{0 \leq s \leq t} B(s) \geq a \right) = 2P(B(t) \geq a)$$

Properties of BM

- Indeed: First observe that

$$P(B(t) \geq a) = P(B(t) \geq a | T_a \leq t) P(T_a \leq t)$$

$$+P(B(t) \geq a | T_a > t) P(T_a > t) = P(B(t) \geq a | T_a \leq t) P(T_a \leq t)$$

since $B(t) \geq a$ cannot happen when $T_a > t$. Also note that

$$P(B(t) \geq a | T_a \leq t) = \frac{1}{2}$$

due to symmetry of BM, so

$$P(T_a \leq t) = 2P(B(t) \geq a) = 2 \int_a^{\infty} \mathcal{N}(0, t) dx.$$

Other variations of BM - BM with drift

BM with drift:

$$X(t) = \mu t + \sigma B(t)$$

- $X(0) = 0$
- $X(t)$ is a stationary, independent increment process.
- $X(t) \sim \mathcal{N}(\mu t, \sigma^2 t)$

BM with drift mu is related to asymmetric random walk $p \neq \frac{1}{2}$

$$X_i = \begin{cases} 1 & \text{with } p \\ -1 & \text{with } 1 - p \end{cases}$$

Other variations of BM - Geometric BM

- Geometric BM is an exponential of BM.
- Recall $X \sim LN(\mu, \sigma^2)$ if $\log X = \mathcal{N}(\mu, \sigma^2)$. That is,

$$X \stackrel{d}{=} \exp\{\mu + \sigma Z\}, \quad Z \sim \mathcal{N}(0, 1).$$

- Similarly for BM define

$$X(t) = \exp\{\mu t + \sigma B(t)\} = e^{Y(t)}$$

  then $X(t)$ is called the Geometric BM (GBM) where $Y(t)$ is BM with drift μ.
- GBM is widely used in finance, for example in Black-Scholes option pricing formula.

Lévy process

Definition (Lévy process)

A RCLL(right continuous left limit) stochastic process $Y = \{Y(t), t \geq 0\}$ *with* $Y(0) = 0$ *is a Lévy process iff it has*

$i)$ Independence of increments
(if time intervals are not overlapping, they are independent)
$ii)$ Stationary increments
$(Y(t+s) - Y(s) \stackrel{d}{=} Y(t) - Y(0) \quad \forall s.t)$

Examples of Levy process

1. Poisson process $PP(\lambda)$

$$N(t+s) - N(t) = N(s) \sim Poisson(\lambda s)$$

$$\therefore \text{ increments are Poisson}$$

2. Brownian motion $B(t)$

$$B(t+s) - B(t) = B(s) \sim N(0, s)$$

$$\therefore \text{ increments are Normal}$$

3. Compound poisson process
4. Renewal process - Counting process with iid inter-arrivals