1) a) $y_{ij} | a_i \overset{IID}{\sim} N(\mu + a_i + \beta_j, \sigma^2)$, $a_i \overset{IID}{\sim} N(0, \sigma_a^2)$

$$y|a \in \mathbb{R}^{m\times n}, \quad y|a = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{bmatrix} = \begin{bmatrix} \mu + a_1 + \beta_1 + \varepsilon_{11} & \mu + a_1 + \beta_2 + \varepsilon_{12} & \cdots & \mu + a_1 + \beta_n + \varepsilon_{1n} \\ \mu + a_2 + \beta_1 + \varepsilon_{21} & \mu + a_2 + \beta_2 + \varepsilon_{22} & \cdots & \mu + a_2 + \beta_n + \varepsilon_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mu + a_m + \beta_1 + \varepsilon_{m1} & \mu + a_m + \beta_2 + \varepsilon_{m2} & \cdots & \mu + a_m + \beta_n + \varepsilon_{mn} \end{bmatrix}$$

Marginal Mean

$\Rightarrow E(y_{ij}) = E[E(y_{ij}|a_i)]$, using the double expectation theorem

$= E[\mu + a_i + \beta_j]$, using the assumption $a_i \sim N(0, \sigma_a^2)$

$= \mu + \beta_j$

Marginal Variance

$\Rightarrow Var(y_{ij}) = E[Var(y_{ij}|a_i)] + Var[E(y_{ij}|a_i)]$, using the double expectation theorem

$= E[\sigma^2] + Var(\mu + a_i + \beta_j)$

$= \sigma^2 + \sigma_a^2$

$\therefore E(y) = [\mu + \beta_1 \quad \mu + \beta_2 \quad \cdots \quad \mu + \beta_n]$

$Var(y) = (\sigma^2 + \sigma_a^2) I_n$

b) Marginal Mean

$\Rightarrow E(y_{ij}) = E[E(y_{ij}|a_i, b_j)]$, using the double expectation theorem

$= E[\mu + a_i + b_j]$, using $a_i \overset{IID}{\sim} N(0, \sigma_a^2)$, $b_j \overset{IID}{\sim} N(0, \sigma_b^2)$

$= \mu$

Marginal Variance

$\Rightarrow Var(y_{ij}) = E[Var(y_{ij}|a_i, b_j)] + Var[E(y_{ij}|a_i, b_j)]$, using the double expectation theorem

$= E[\sigma^2] + Var(\mu + a_i + \beta_j)$

$= \sigma^2 + \sigma_a^2 + \sigma_b^2$

c) Marginal Mean

$$\Rightarrow E(y_{ij}) = E\left[E(y_{ij} \mid a_i, b_j, g_{ij})\right]$$, using the double expectation theorem

$$= E\left[\mu + a_i + b_j + g_{ij}\right]$$, using $a_i \overset{IID}{\sim} N(0, \sigma_a^2)$, $b_j \overset{IID}{\sim} N(0, \sigma_b^2)$, $g_{ij} \overset{IID}{\sim} N(0, \sigma_g^2)$

$$= \mu$$

Marginal Variance

$$\Rightarrow Var(y_{ij}) = E\left[Var(y_{ij} \mid a_i, b_j, g_{ij})\right] + Var\left[E(y_{ij} \mid a_i, b_j, g_{ij})\right]$$, using the double expectation theorem

$$= E\left[\sigma^2\right] + Var(\mu + a_i + \beta_j + g_{ij})$$

$$= \sigma^2 + \sigma_a^2 + \sigma_b^2 + \sigma_g^2$$

2) a) $$Var(Y_{ij}) = Var(\beta_{0i} + \beta_{1i} t_{ij} + \varepsilon_{ij})$$

$$= Var(\beta_{0i}) + Var(\beta_{1i} t_{ij}) + Var(\varepsilon_{ij}) + 2 \cdot Cov(\beta_{0i}, \beta_{1i} t_{ij})$$

$$= D_{11} + t_{ij}^2 D_{22} + \sigma^2 + 2 t_{ij} D_{12}$$

$$COV(Y_{ij}, Y_{ik}) = COV(\beta_{0i} + \beta_{1i} t_{ij} + \varepsilon_{ij}, \beta_{0i} + \beta_{1i} t_{ik} + \varepsilon_{ik})$$

$$= Var(\beta_{0i}) + COV(\beta_{0i}, \beta_{1i} t_{ik}) + COV(\beta_{1i} t_{ij}, \beta_{0i}) + COV(\beta_{1i} t_{ij}, \beta_{1i} t_{ik})$$

$$= D_{11} + t_{ik} D_{12} + t_{ij} D_{12} + t_{ij} t_{ik} D_{22}$$

b) $$COV(Y_{ij}, Y_{ik}) = D_{11} + t_{ik} D_{12} + t_{ij} D_{12} + t_{ij} t_{ik} D_{22}$$, let $D_{12} = 0$

$$= D_{11} + t_{ij} t_{ik} D_{22}$$

$\therefore$ Y_{ij} and Y_{ik} are correlated since within-subject variation still exists.

c) $$Var(Y_{ij}) = COV(\beta_{0i} + \beta_{1i} t_{ij} + \varepsilon_{ij}, \beta_{0i} + \beta_{1i} t_{ij} + \varepsilon_{ij})$$

$$= Var(\beta_{0i}) + t_{ij}^2 Var(\beta_{1i}) + 2 COV(\beta_{0i}, \beta_{1i} t_{ij}) + Var(\varepsilon_{ij})$$

$$= D_{11} + t_{ij}^2 D_{22} + \sigma_1^2 + \sigma_2^2$$

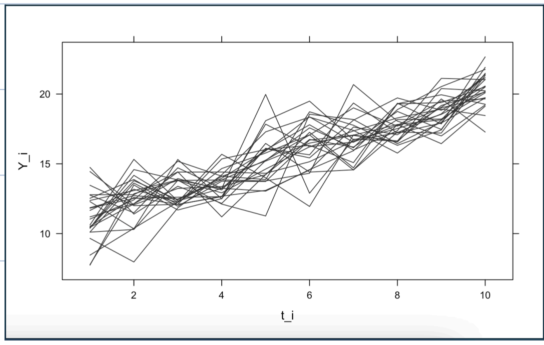
$$COV(Y_{ij}, Y_{ik}) = COV(\beta_{0i} + \beta_{1i} t_{ij} + \varepsilon_{ij}, \beta_{0i} + \beta_{1i} t_{ik} + \varepsilon_{ik})$$

$$= Var(b_{0,i}) + COV(b_{1,i} t_{ij}, b_{1,i} t_{ik}) + (t_{ij} + t_{ik}) COV(b_{0,i}, b_{1,i}) + COV(\varepsilon_{ij}, \varepsilon_{ik})$$

$$= D_{11} + t_{ij} t_{ik} D_{22} + (t_{ij} + t_{ik}) D_{12} + \rho^{|j-k|} \sigma_1^2$$

3) default conditions : (i) $n_i = 10$, $t_{ij} = j$, $E(Y_{ij}|X_i) = \beta_0 + \beta_1 t_{ij}$, $m = 25$ subjects , $\beta = [10 \quad 1]$

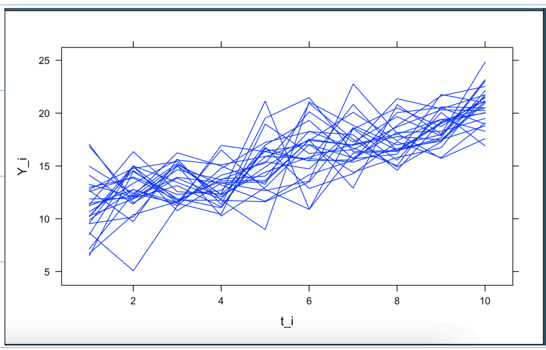
a) $\sigma = 1$, $\tau = 1$



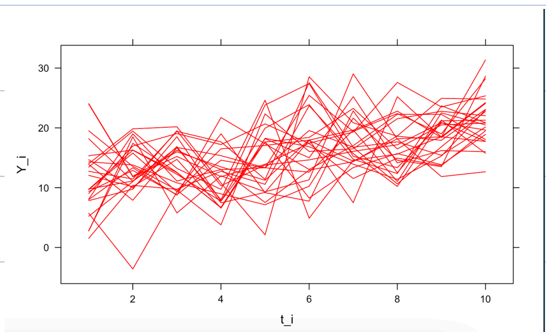
$$\Rightarrow \ cov(Y_{ij}, Y_{ik}) = cov(b_{0,i}, b_{0,i}) + cov(\varepsilon_{ij}, \varepsilon_{jk})$$

$$= \begin{cases} \tau^2 & , \forall i \neq k \\ \sigma^2 + \tau^2 & , \forall i = k \end{cases}$$

$\sigma = 1$, $\tau = 2$



$\sigma = 1$, $\tau = 5$



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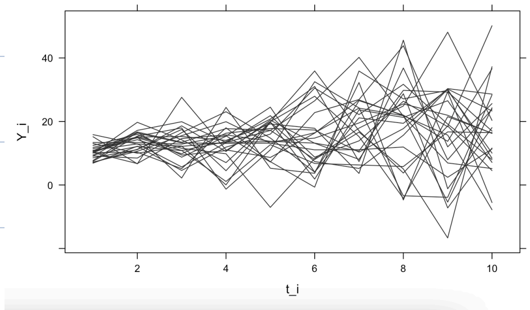
b) $\sigma = 1 \,,\quad D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

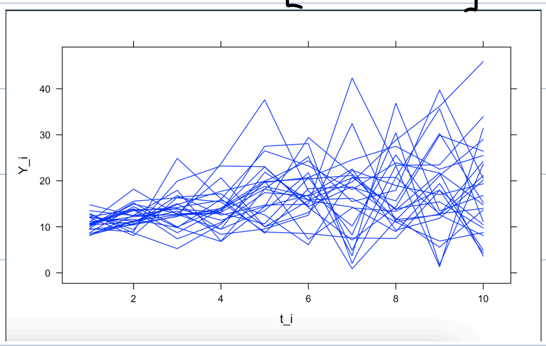


$$\mathrm{COV}(Y_{ij}, Y_{ik}) = \mathrm{COV}(\beta_0 + \beta_1 t_{ij} + b_{0,i} + b_{1,i} t_{ij} + \varepsilon_{ij} \,,\; \beta_0 + \beta_1 t_{ik} + b_{0,i} + b_{1,i} t_{ik} + \varepsilon_{ik})$$

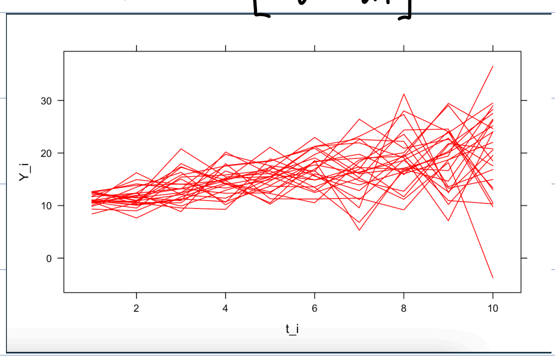
$$= Var(\beta_0) + cov(b_{1,i} t_{ij}, b_{1,i} t_{ik}) + cov(b_{0,i}, b_{1,i} t_{ij}) + cov(b_{0,i}, b_{1,i} t_{ik}) + cov(\varepsilon_{ij}, \varepsilon_{ik})$$

$$= \begin{cases} D_{11} + t_{ij} t_{ik} D_{22} + (t_{ij} + t_{ik}) D_{12} \,, & \forall j \neq k \\ D_{11} + t_{ij}^2 D_{22} + 2 t_{ij} D_{12} + \sigma^2 \,, & \forall j = k \end{cases}$$

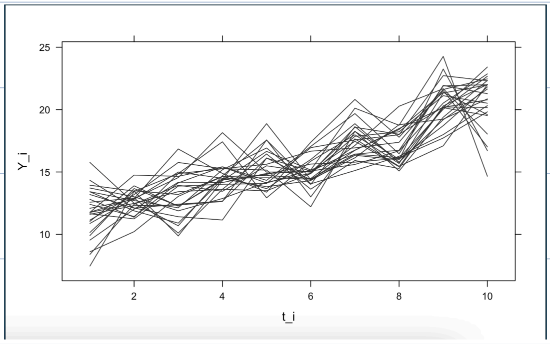
$\sigma = 1 \,,\quad D = \begin{bmatrix} 2 & -0.2 \\ -0.2 & 2 \end{bmatrix}$



$\sigma = 1 \,,\quad D = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix}$



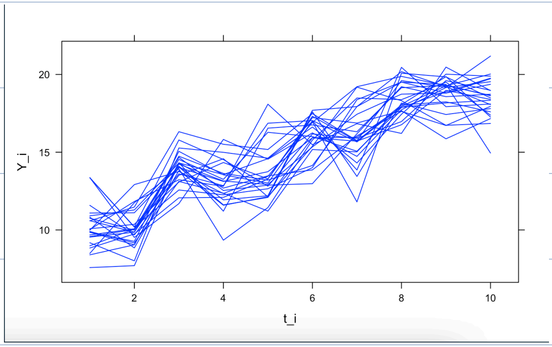
c) $\sigma = 1$, $\tau = 2$



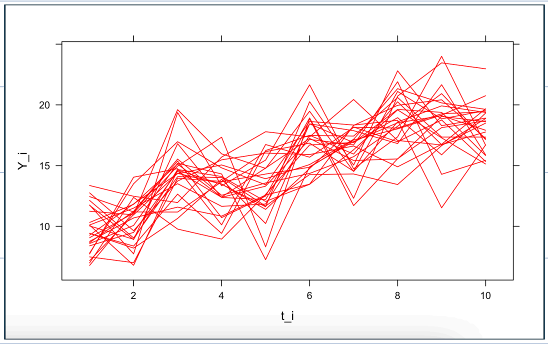
$$COV(Y_{ij}, Y_{ik}) = COV(\beta_0 + \beta_1 t_{ij} + W_i(t_{ij}) + \varepsilon_{ij}, \beta_0 + \beta_1 t_{ik} + W_i(t_{ik}) + \varepsilon_{ik})$$

$$= COV(W_i(t_{ij}), W_i(t_{ik})) + COV(\varepsilon_{ij}, \varepsilon_{ik}) = \begin{cases} \tau^2 \rho^{|j-k|}, & \forall j \neq k \\ \tau^2 + \sigma^2, & \forall j = k \end{cases}$$

$\sigma = 1$, $\tau = 2$



$\sigma = 2$, $\tau = 2$



4) 120 Children : greater than or equal to 15 $\mu g/dL$

↳ 40 $\mu g/dL$ housing

↳ randomized
- 40 children to placebo
- 40 to low dose of succimer
- 40 to higher dose of succimer

a) R로 풀어서 답안 적음

b) (i) lead level과 상관없는 error가 포함되어 있다 → 결국 근데 에러가 일괄적으로 포함이 되었고, 에러가 lead level과 독립적이라는 말 아닐까

(ii) 0, 2, 4, 6, 8 주마다 각 어린이들에게서 lead level 검사를 해왔는데, 이 검사간격 텀이 길어서 within-subject variation이 반영되지 못했다. linear mixed model - serial correlation

(iii) 모든 treatment group에 관해서 within-Child variation이 동일한가

(iv) 그냥 모든 어린이들에 대해서 동일한 모델을 사용할 수 있느냐 인가

c)

d)