2.1 Introduction. Inequalities

Inequality Laws:

- Addition: you can only add inequalities in the same direction
- Subtraction
- Multiplication: This is legal provided that the inequalities are in the same direction and the numbers are positive
- Sign-Change Law: Changing signs reverses an inequality
- Reciprocal law: The inequality reverses, if both numbers are positive or negative

2.2 Estimations

- If c is a number we are estimating, and $K < c < M$, we say K is a lower estimate or lower bound for c; M is an upper estimate or upper estimate for c
- If two sets of upper and lower estimates satisfy the inequalities $K < K' < c < M' < M$, we say K', M' are stronger or sharper estimates for c, while K, M are weaker estimates

2.3 Proving Boundedness

- To show $\{a_n\}$ is bounded above, get one upper estimate: $a_n \leq B$ for all n
- To show $\{a_n\}$ is not bounded above, get a lower estimate for each term: $a_n \geq B_n$, such that B_n tends to ∞ as $n \to \infty$

2.4 Absolute Values. Estimating Size.

Absolute Value:

- Define $|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$

☆ absolute value measures magnitude

- $|a|$ is guaranteed to be non-negative
- The absolute value is also an efficient way to give symmetric bounds: $|a| \leq M \Rightarrow -M \leq a \leq M$

☆ $K \leq a \leq L \Rightarrow |a| \leq M$, where $M = \max(|K|, |L|)$

Absolute Value Laws

i) Multiplication law:

- $|ab| = |a||b|$; $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, if $b \neq 0$

ii) Triangle Inequality:

- $|a+b| \leq |a| + |b|$

iii) Extended Triangle Inequality

- $|a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|$

☆ Difference Forms of the Triangle Inequality : $|a-b| \geq |a|-|b|$, $|a+b| \geq |a|-|b|$

$\Rightarrow$ In either case, $|a|$ should be larger than $|b|$

- To estimate size, you have to use $|\ |$, and the estimations have to be in the right direction :
  - to show a is small in size, show $|a| <$ a small number
  - to show a is large in size, show $|a| >$ a large number

- You can't use the triangle inequality mechanically.
- If a and b are close, you won't see the triangle inequality.

2.5 Approximations

- The standard way of writing the distance between a and b is less than ϵ : $|a-b| < \epsilon$ or $a \underset{\epsilon}{\approx} b$

Theorem :

- For any two real numbers $a < b$, there is
  i) a rational number $r \in \mathbb{Q}$ between $a < r < b$
  ii) an irrational number $s \notin \mathbb{Q}$ between $a < r < b$

Transitive Law :

$a \underset{\epsilon}{\approx} b$ and $b \underset{\epsilon'}{\approx} c \Rightarrow a \underset{\epsilon+\epsilon'}{\approx} c$

Additive Law :

$a \underset{\epsilon}{\approx} a'$ and $b \underset{\epsilon'}{\approx} b' \Rightarrow a+b \underset{\epsilon+\epsilon'}{\approx} a'+b'$

2.6 The Terminology "for n large"

- In estimating or approximating the terms of a sequence $\{a_n\}$, sometimes the estimate is not valid for all terms of the sequence; for example, it might fail for the first few terms, but be valid for the later terms. In such a case, one has to specify the values of n for which the estimate holds.
- Sometimes a property of a sequence a_n is not true for the first few terms, but only starts to hold after a certain place in the sequence
- The sequence $\{a_n\}$ has property P for n large if there is a number N such that a_n has property P for all $n \geq N$ $\quad (n \gg 1)$

☆ N need not be an integer

- We have to make the N explicit whenever we have to prove that some property holds for large n, We, in general, don't want to have to specify exactly what N is

More General Form of the Completeness Property

- A sequence which is bounded and monotone for $n \gg 1$ has a limit