Chap. 17 Taylor Approximation

17.1 Taylor polynomials

Def. Two fts f and g have **n-th order agreement** at a if they are n times diff at a and

$$\boxed{f(a) = g(a),\ \ f'(a) = g'(a),\ \ \cdots,\ f^{(n)}(a) = g^{(n)}(a)}$$

Exa. Show that $\sin x$ and $x - x^3$ have the **second-order** agreement at 0.

Pf. Follows since both functions satisfy $f(0) = 0,\ \ f'(0) = 1,\ \ f''(0) = 0$

However $\dfrac{d^3}{dx^3}(\sin x)\big|_{x=0} = -1, \qquad \dfrac{d^3}{dx^3}(x - x^3)\big|_{x=0} = -6$

$\therefore$ they do **not have third-order** agreement at 0.

Remark. $\sin x \approx x - \dfrac{x^3}{3!} \left(= x - \dfrac{x^3}{6}\right)$ for $x \approx 0$

Theorem-Definition 17.1

Suppose $f^{(n)}$ exists. Then the polynomial

$$\boxed{T_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

is the unique polynomial of degree n in powers of $x - a$ having n-th order agreement with $f(x)$ at a. $T_n(x)$ is called the **n-th order Taylor polynomial** for $f(x)$ at a.

Pf. Let $p(x)$ be a polynomial of degree n written in powers of $x - a$:

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n, \qquad c_i \in \mathbb{R}.$$

Differentiating $k\ (k \leq n)$ times gives

$$p^{(k)}(x) = k!\,c_k + \begin{cases} \text{terms having } (x-a) \text{ as a factor} & \text{if } \ k < n \\ 0 & \text{if } \ k = n \end{cases}$$

So $\quad p^{(k)}(a) = k!\,c_k$

Therefore,

$f(x)$ and $p(x)$ have n-th order agreement at a

$$\begin{aligned} &\Leftrightarrow \quad f^{(k)}(a) = k!\,c_k \qquad && k = 0, 1, 2, \cdots, n \\ &\Leftrightarrow \quad c_k = \frac{f^{(k)}(a)}{k!} && k = 0, 1, 2, \cdots, n \\ &\Leftrightarrow \quad p(x) \text{ is the n-th order Taylor polynomial of } f(x) \text{ at } a \end{aligned}$$

Ex. Taylor polynomials for the standard functions [**remember the result**]:

$$\frac{1}{1-x} \approx 1 + x + x^2 + \cdots + x^n \quad (T_n(x) \text{ at } x = 0)$$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \quad (T_n(x) \text{ at } x = 0)$$

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad (T_{2n+1}(x) \text{ or } T_{2n+2}(x) \text{ at } x = 0)$$

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} \quad (T_{2n}(x) \text{ or } T_{2n+1}(x) \text{ at } x = 0)$$

$$(1+x)^r \approx 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n, \text{ for } \forall r \in \mathbb{R} \quad (T_n(x) \text{ at } x = 0)$$

$$\left(\text{i.e., } (1+x)^r \approx 1 + \binom{r}{1}x + \binom{r}{2}x^2 + \cdots + \binom{r}{n}x^n, \text{ for } \forall r \in \mathbb{R} \quad (T_n(x) \text{ at } x = 0) \right)$$

$$\left[\begin{array}{l} \text{e.g., } \sqrt{1+x} \approx 1 + \frac{1}{2}x + \frac{1/2 \cdot (-1/2)}{2!}x^2 = 1 + \frac{1}{2}x - \frac{1}{8}x^2 \quad (T_2(x) \text{ at } x = 0) \\ \sqrt{1-x} = (1+(-x))^{1/2} \approx 1 + \frac{1}{2}(-x) + \frac{1/2 \cdot (-1/2)}{2!}(-x)^2 = 1 - \frac{1}{2}x - \frac{1}{8}x^2 \quad (T_2(x) \text{ at } x = 0) \end{array} \right]$$

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} \quad (T_n(x) \text{ at } x = 0)$$

Remark

① Let $T_n(x)$ be the Taylor polynomial for $\sin x$ at 0. Then

$$T_1(x) = T_2(x), \quad T_3(x) = T_4(x), \cdots$$
$$\left(\because \ (\sin x)^{(\text{even order})}\Big|_{x=0} = 0 \right)$$

② Let $T_n(x)$ be the Taylor polynomial for $\cos x$ at 0. Then

$$T_0(x) = T_1(x), \quad T_2(x) = T_3(x), \cdots$$
$$\left(\because \ (\cos x)^{(\text{odd order})}\Big|_{x=0} = 0 \right)$$

③ $\tan x \approx ?$ (not easy to expect the formula $T_n(x)$)

$$(\text{odd function}) \tan x \approx x + \frac{x^3}{3} + \frac{2x^5}{3 \cdot 5} + \frac{17x^7}{5 \cdot 7 \cdot 9} = T_7(x)$$

[the polynomial above gives **no** clue what $T_9(x)$ might be &

it is not easy to calculate $\frac{d^n}{dx^n} \tan x \Big|_{x=0}$]

Question: Assume the approximation $f(x) \approx T_n(x)$ for $x \approx a$.

What is its remainder (or error)? How can we estimate the error?

17.2 Taylor's theorem with Lagrange remainder

Theorem 17.2 (**Taylor Theorem** with Lagrange remainder)

Suppose $f(x)$ is $(n+1)$ times diff in an open interval $I \ni a, x$.

Then $\exists\, c$ between a and x such that

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n\,!}(x-a)^n + R_n(x);$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)\,!}(x-a)^{n+1} \qquad \text{where } a < c < x \text{ or } x < c < a$$

$R_n(x)$ is called the remainder term or the error term

[The remainder of the above form is usually called the ***Lagrange* remainder**]

Remark. If we write b instead of x, the above result can be stated as

$$f(b) = f(a) + f'(a)(b-a) + \cdots + \frac{f^{(n)}(a)}{n\,!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)\,!}(b-a)^{n+1},$$

for some c between a and b.

Remark.

$n = 0$: Taylor theorem is just the MVT

$n = 1$: Taylor theorem is just the Extended MVT (or the Linearization Error Theorem)

Lemma (**Extended Rolle's theorem**)

If $f^{(n+1)}(x)$ exists on $[a,b]$ (with $a < b$), and

$$f(a) = f'(a) = \cdots = f^{(n)}(a) = 0 = f(b),$$

then $\exists\, c$ between a and b such that $f^{(n+1)}(c) = 0$.

(Lemma is also valid on $[b,a]$, under the same hypothesis)

Pf. Assume $a < b$.

$$f(a) = 0 = f(b) \quad \overset{\text{Rolle's Theorem}}{=} \quad f'(c_1) = 0 \quad \text{for some } c_1,\ a < c_1 < b$$

$$f'(a) = 0 = f'(c_1) \quad \overset{\text{Rolle's Theorem}}{=} \quad f''(c_2) = 0 \quad \text{for some } c_2,\ a < c_2 < c_1$$

$$\vdots$$

$$f^{(n)}(a) = 0 = f^{(n)}(c_n) \quad \overset{\text{Rolle's Theorem}}{=} \quad f^{(n+1)}(c) = 0 \quad \text{for some } c,\ a < c < c_n$$

$$a < c < c_n < \cdots < c_1 < b \quad \Rightarrow \quad c \in (a,b)$$

Proof of the Taylor theorem

$$P(x) \overset{\text{let}}{=} f(a) + f'(a)(x-a) + \frac{f''(a)}{2\,!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n\,!}(x-a)^n + C(x-a)^{n+1},$$

where C is a constant chosen to satisfy $P(b) = f(b)$. That is, C is a number such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + C(b-a)^{n+1}.$$

Note that $P(a) = f(a), \quad P'(a) = f'(a), \quad \cdots \quad , \quad P^{(n)}(a) = f^{(n)}(a) \;\;\&\;\; P(b) = f(b)$

Now we let $g(x) = f(x) - P(x)$

$$\Rightarrow \quad g(a) = g'(a) = \cdots = g^{(n)}(a) = 0 = g(b)$$

$$\overset{\text{Lemma}}{\Rightarrow} \quad \exists c \in (a,b) \text{ such that } g^{(n+1)}(c) = 0 \;\; (\textit{i.e. } f^{(n+1)}(c) = P^{(n+1)}(c) = C(n+1)!)$$

$$\therefore \; C = \frac{f^{(n+1)}(c)}{(n+1)!} \text{ for some } c \in (a,b)$$

Remark: **"**Taylor Theorem with Lagrange remainder" can be stated as follows:

Suppose $f(x)$ is $(n+1)$ times diff in an open interval I.

Then for any $a, a+h \in I\left[\Rightarrow [a, a+h] \subset I \text{ if } h > 0; \;\; [a+h, a] \subset I \text{ if } h < 0\right]$, $\exists 0 < \theta < 1$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(a+\theta h)$$

That is, $\exists 0 < \theta < 1$ s.t. $f(a+h) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a) h^k + \frac{1}{(n+1)!} f^{(n+1)}(a+\theta h) h^{n+1}$

17.3 Estimating error in Taylor approximation

Recall: The expression for the error (or remainder) term in $f(x) \approx T_n(x)$ at a is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}, \quad \text{for some } c \text{ between } a \text{ and } x$$

Note: Since we don't know exactly where c is, we can not find the exact error.

So what we get is rather a way of estimating the error.

For example, we consider $f(x) = e^x$

$$e^x \overset{\text{Taylor's formula at } a=0}{=} 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^c}{(n+1)!}x^{n+1}, \quad 0 < c < x \text{ or } x < c < 0$$

In using the formula, there are three things to handle with:

- the size of the error
- the size of the interval
- n (the order of approximation)

We can fix two of these, and ask how the third is affected.

Exa A. Is $e^x \underset{0.01}{\approx} T_3(x)$ at 0?, whenever $|x| \leq 0.5$

That is, $|R_3(x)| < 0.01$?, whenever $|x| \leq 0.5$

Sol. $R_3(x) = \dfrac{e^c}{4!}x^4, \quad \textit{where } 0 < |\, c \,| \le 0.5 \;\; (\leftarrow 0 < c < x \text{ or } x < c < 0)$

$$e^c \underset{e<3,\, |c|\le 0.5}{\le} 3^{0.5} = \sqrt{3} < 1.75$$

$$\therefore \quad |\, R_3(x) \,| < \frac{1.75}{4!}(0.5)^4 \underset{\text{easy}}{<} 0.005 \quad \text{for } |\, x \,| \le 0.5$$

$$\therefore \quad |\, R_3(x) \,| < 0.01 \quad \text{is true} \quad \text{whenever } |\, x \,| \le 0.5$$

Exa B. Calculate e to two decimal places.

Sol. Taking $x = 1$ in Taylor's formula of e^x at 0 gives

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{e^c}{(n+1)!}, \quad 0 < c < 1$$

We know that $0 < e < 3$, so that

$$1 < e^c < 3 \quad (\Leftarrow \quad 0 < c < 1)$$

$$\therefore \quad \frac{1}{(n+1)!} < \frac{e^c}{(n+1)!} = R_n(1) < \frac{3}{(n+1)!}$$

We want to find (smallest) n such that

$$\frac{3}{(n+1)!} < 0.01 = \frac{1}{10^2} \quad (\textit{i.e.} \;\; (n+1)! > 300)$$

$$n = 4 : \quad 5! = 120 < 300$$
$$n = 5 : \quad 6! = 720 > 300 \quad (\therefore \; n = 5 \text{ is the desired one})$$

In fact, $R_5(1) = \dfrac{e^c}{6!} < \dfrac{3}{6!} = \dfrac{1}{240} < \dfrac{1}{200} = 0.005$

So, taking $n = 5$ and calculating give

$$e \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = 2.716 **$$

Since the missing later terms are all positive,

$$e = 2.716 + E, \quad 0 < E < 0.005 \text{ --- } \blacklozenge$$

Moreover, since $\dfrac{1}{6!} = \dfrac{1}{720} > \dfrac{1}{1000} = 0.001$, we see that $0.001 < E < 0.005$

$$\Rightarrow \; e = 2.72 + (E - 0.004), \; -0.003 < \hat{E} := E - 0.004 < 0.001; \quad |\hat{E}| < 0.003 < 0.01$$

$$\therefore \quad e \underset{0.01}{\approx} 2.72$$

Remark: From ♦ we see that $e < 2.8$. Hence

$$R_5(1) = \frac{e^c}{6!} < \frac{e}{6!} < \frac{2.8}{720} = 0.00388\cdots < 0.0039 \text{; so } e = 2.716 + \tilde{E}, \quad 0.001 < \tilde{E} < 0.0039$$

$$\Rightarrow e = 2.71 + \left(\tilde{E} + 0.006\right), \quad 0.007 < \breve{E} := \tilde{E} + 0.006 < 0.0099; \qquad |\breve{E}| < 0.0099 < 0.01$$

Thus $e \underset{0.01}{\approx} 2.71$ is also true. However

$$e \underset{0.01}{\approx} 2.72 \text{ is better than } e \underset{0.01}{\approx} 2.71, \text{ since } |\hat{E}| < 0.003 \text{ but } \breve{E} > 0.007$$

(This is also clear if we remember $e = 2.7182\cdots$)

Caution: In doing this sort of estimation, one should take advantage of ***missing terms*** in the Taylor polynomial.

For example,

$$\begin{aligned} f(x) := \sin x &= x - \frac{x^3}{3!} + R_3(x) \\ &\overset{\text{or}}{=} x - \frac{x^3}{3!} + \underset{\substack{\uparrow \\ \text{4th-order term}}}{0} + R_4(x) \end{aligned}$$

$(R_4(x)$ is much smaller than $R_3(x)$ for $|x| \approx 0)$

$$\ulcorner \because \quad R_3(x) = \frac{f^{(4)}(c)}{4!}x^4 = \frac{\sin c}{4!}x^4$$

$$R_4(x) = \frac{f^{(5)}(c)}{5!}x^5 = \frac{\cos c}{5!}x^5 : \text{much smaller than } R_3(x) \text{ for } |x| \approx 0 \ \lrcorner$$

Consequently, in the approximation $\sin x \approx x - \frac{x^3}{3!}$, the polynomial on the right should be viewed as $T_4(x)$, *not* $T_3(x)$

Ex1. Use Taylor's theorem to prove that

$$1 + \frac{x}{2} - \frac{x^2}{8} < \sqrt{1+x} < 1 + \frac{x}{2}, \text{ for all } x > 0$$

Sol. Taylor's theorem (applied to $f(x) = \sqrt{1+x}$ $(x > 0)$) gives

$$\sqrt{1+x} = 1 + \frac{x}{2} + R_1(x); \quad \sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + R_2(x),$$

where $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for some $c \in (0, x)$.

Since $f'(x) = \frac{1}{2\sqrt{1+x}}$, $f^{(2)}(x) = -\frac{1}{4(1+x)^{3/2}}$, $f^{(3)}(x) = \frac{3}{8(1+x)^{5/2}}$, we obtain

$$R_1(x) = \frac{f^{(2)}(c)}{2!}x^2 < 0 \quad \& \quad R_2(x) = \frac{f^{(3)}(c)}{3!}x^3 > 0 \quad \text{whenever} \quad x > 0$$

Therefore, $\quad 1 + \frac{x}{2} - \frac{x^2}{8} < \sqrt{1+x} < 1 + \frac{x}{2}, \quad \text{for all } x > 0.$

Ex2. Prove that $|\ln(1+x) - x| < \frac{x^2}{2}$ for $x > 0$

Pf. $\quad f(x) := \ln x \ \ (x > 0) \quad \Rightarrow \quad f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}$

By Taylor's theorem,

$$f(1+x) = f(1) + xf'(1) + \frac{x^2}{2}f''(1+\theta x), \quad \text{for some } 0 < \theta < 1$$

$$\therefore \quad \ln(1+x) = x - \frac{1}{2(1+\theta x)^2}x^2 \quad (x > 0) \quad \text{for some } 0 < \theta < 1$$

$$\therefore \quad |\ln(1+x) - x| = \frac{1}{2(1+\theta x)^2}x^2 < \frac{1}{2}x^2 \quad (x > 0) \quad \left[\leftarrow 1 + \theta x > 1 \ \Rightarrow \ \frac{1}{(1+\theta x)^2} < 1 \right]$$

Ex3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is nonnegative, twice differentiable, and $f''(x) \le 1/2$ for all $x \in \mathbb{R}$. Use Extended Mean Value Theorem (or Taylor's theorem) to prove that

$$|f'(x)| \le \sqrt{f(x)} \qquad\qquad \text{Hint: Consider } f(x+h).$$

Pf. By Extended MVT (or Taylor's theorem),

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(\xi) \ \text{ for some } \xi \text{ between } x \text{ and } x+h.$$

Since $f(x+h) \ge 0 \ (\forall h \in \mathbb{R})$ and $f''(\xi) \le 1/2$, we get

$$0 \le \underbrace{f(x) + hf'(x) + \frac{h^2}{4}}_{\text{quadratic polynomial in } h} \quad \forall h \in \mathbb{R}$$

This implies $D \le 0$, and so $f'(x)^2 - f(x) \le 0$. Equivalently, $|f'(x)| \le \sqrt{f(x)}$.

17.4 Taylor series

Recall (Taylor theorem)

If $f(x)$ is $(n+1)$ times diff in some open interval $I \ni 0$, then for each $x \in I$,

$\exists c$ between 0 and x such that

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

Question: What about if $n \to \infty$?

Suppose $f(x)$ is infinitely diff in some open interval $I \ni 0$.

Is it possible that for any $x \in I$:

$$f(x) \overset{??\text{(can be represented as)}}{=} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

Ans: Not true in general

$$f(x) \overset{\text{def}}{=} \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Claim: $f(x)$ is infinitely diff at each $x \in \mathbb{R}$. [It is clear that $f(x)$ is infinitely diff at each $x \neq 0$]

Moreover, $f^{(n)}(0) = 0 \quad \forall n = 1, 2, 3, \cdots$ (i.e., f is infinitely flat at $x = 0$)

Pf of claim:

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{e^{-1/x^2}}{x}$$
$$\underset{\frac{1}{x}=t}{=} \lim_{t\to\pm\infty} te^{-t^2} = \lim_{t\to\pm\infty} \frac{t}{e^{t^2}} \left(\frac{\pm\infty}{\infty} - \text{form} \right) \overset{\text{L'Hospital}}{=} \lim_{t\to\pm\infty} \frac{1}{2te^{t^2}} = 0$$

$$f''(0) = \lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0} \frac{2\frac{1}{x^3} e^{-1/x^2}}{x}$$
$$= 2 \lim_{x\to 0} \frac{e^{-1/x^2}}{x^4} \underset{\frac{1}{x^2}=t}{=} 2 \lim_{t\to\infty} \frac{t^2}{e^t} \overset{\text{L'Hospital}}{=} 0$$

From these, one may expect that $f^{(n)}(0) = 0 \quad \forall n \in \mathbb{N}$

Indeed, we can prove this by using **Mathematical Induction**:

Assume $f^{(k)}(0) = 0$. Then for $x \neq 0$, we can verify (by Math. Induction) that

$f^{(k)}(x) = R(1/x)e^{-1/x^2}$, where $R(1/x)$ is a polynomial in $1/x$ --- **Check**

So is $\dfrac{R(1/x)}{x}$. Thus $\dfrac{R(1/x)}{x} = a_m \left(\dfrac{1}{x}\right)^m + a_{m-1} \left(\dfrac{1}{x}\right)^{m-1} + \cdots + a_1 \left(\dfrac{1}{x}\right)$ (for some m)

Note that for every $n \in \mathbb{N}$, $\displaystyle \lim_{x\to 0} x^{-n} e^{-1/x^2} \underset{\frac{1}{x}=t}{=} \lim_{t\to\pm\infty} \frac{t^n}{e^{t^2}} \overset{\text{L}}{=} 0$.

Hence $\displaystyle \lim_{x\to 0} \frac{R(1/x)e^{-1/x^2}}{x} = 0$. Therefore,

$$f^{(k+1)}(0) = \lim_{x\to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x} = \lim_{x\to 0} \frac{R(1/x)e^{-1/x^2}}{x} = 0$$

Now suppose that $\displaystyle f(x) = \sum \frac{f^{(n)}(0)}{n!} x^n$ for $x \neq 0$.

Notice that $\text{RHS} = 0$ since $f^{(n)}(0) = 0 \quad \forall n = 0, 1, 2, \cdots$

Then we must have $f(x) = 0$ for $x \neq 0$, which is **absurd** since $f(x) > 0 \ \ \forall x \neq 0$.

Def. Let f be infinitely diff at 0. The power series

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \left(= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n\right)$$

is called the **Taylor series** of f at 0 (or the **MacLaurin series** of f)

Def. If $\exists R > 0$ such that $f(x)$ can be expressed as

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n \qquad \forall x \text{ with } |x| < R,$$

we say that f is (real) **analytic** at 0.

Remark. Let f be infinitely diff on $(-R, R)$. Then we know that for each $n \in \mathbb{N}$

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \\ &\qquad\qquad \left(0 < c < x < R, \text{ or } -R < x < c < 0\right) \\ &= T_n(x) \quad + \quad R_n(x) \end{aligned}$$

Thus if $R_n(x) \to 0$ $\forall x$ with $|x| < R$ (i.e. $T_n(x) \to f(x)$ $\forall x$ with $|x| < R$) as $n \to \infty$,

(which means $\displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(x) \quad \forall x$ with $|x| < R$)

then f is (real) analytic at $x = 0$.

Consequently, we have

$$\begin{aligned} f \text{ is analytic at } 0 &\overset{\text{def}}{\Leftrightarrow} f(x) \overset{\text{can be expressed}}{=} \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n \qquad \forall x \text{ with } |x| < (\text{some})R \\ &\Leftrightarrow \quad \exists R > 0 \quad \text{s.t. } T_n(x) \to f(x) \quad \forall\, |x| < R \text{ as } n \to \infty \\ &\Leftrightarrow \quad \exists R > 0 \quad \text{s.t. } R_n(x) \to 0 \quad \forall\, |x| < R \text{ as } n \to \infty \end{aligned}$$

Later (Chapter 22: Cor 22.6A)

If a power series $\displaystyle\sum_{0}^{\infty} a_n x^n$ has radius of convergence $R > 0$, it will be proved that

$$f(x) := \sum_{0}^{\infty} a_n x^n \in C^{\infty}(-R\ R) \ \ \& \ \ \sum_{0}^{\infty} a_n x^n = \sum_{o}^{\infty}\frac{f^{(n)}(0)}{n!}x^n \ (= \text{T.S. of } f(x)) \text{ on } (-R, R)$$

$$\therefore \ \boxed{f \text{ is analytic at } x = 0 \ \Leftrightarrow \ \exists R > 0 \ \text{ s.t. } f(x) \overset{\text{expressed as}}{=} \sum_{0}^{\infty} a_n x^n \ \text{ on } (-R, R)}$$

Ex. Show that

(i) $\displaystyle\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \quad |x| < 1$

(ii) $\displaystyle e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots, \quad \text{all } x \in \mathbb{R}$

Pf (i) Exercise

(ii) Have to show: Given any $x \in \mathbb{R}$, $R_{n-1}(x) \, (= \frac{e^c}{n!} x^n) \to 0 \; (0 < c < x \text{ or } x < c < 0)$.

Take an arb $x \in \mathbb{R}$ and fix it. Choose N so that $|x| < N/2$.

If $0 < c < x$, then $0 < e^c < e^x$, let $A = e^x$

If $x < c < 0$, then $e^x < e^c < 1$, let $A = 1$

For $n > N$, we then have

$$\frac{|e^c x^n|}{n!} \le \frac{A|x|^n}{n!} = \frac{A|x|^N \cdot |x|^{n-N}}{N! \cdot (N+1)\cdots n}$$
$$= \frac{A|x|^N}{N!} \cdot \frac{|x|}{N+1} \cdot \frac{|x|}{N+2} \cdots \frac{|x|}{n}$$
$$\le \underbrace{\frac{A|x|^N}{N!}}_{\text{indep of } n} \cdot \underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2}}_{n-N\text{개}} \le \frac{K}{2^{n-N}} \; (K : \text{indep of } n) \to 0 \text{ as } n \to \infty$$

$$\therefore \quad \lim_{n\to\infty} \frac{e^c}{n!} x^n = 0, \quad \text{for each } x \in \mathbb{R}.$$

Alternative way:
Given any fixed $x \in (-\infty, \infty)$, we have

$$e^x \overset{\text{Taylor's theorem}}{=} 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^c x^{n+1}}{(n+1)!}, \quad 0 < c < x \text{ or } x < c < 0$$

$$\therefore \quad \left| e^x - \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right) \right| = \frac{e^c |x|^{n+1}}{(n+1)!} \le \frac{e^{|x|} |x|^{n+1}}{(n+1)!}$$

Remains to show:

$$(*) : \lim_{n\to\infty} \frac{e^{|x|} |x|^{n+1}}{(n+1)!} = 0 \quad (\text{note: } x \text{ is fixed } \& \; e^{|x|} \text{ is indep of } n)$$

Actually, it suffices to show

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} \left[=: \lim_{n\to\infty} a_n(x) \right] = 0 \quad \text{---}\blacklozenge$$

$\blacklozenge$ can be proved by applying Ratio test as follows:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x)}{a_n(x)} \right| = \lim_{n\to\infty} \frac{(n+1)!}{|x|^{n+1}} \times \frac{|x|^{n+2}}{(n+2)!} = \lim_{n\to\infty} \frac{|x|}{n+2} = 0 < 1$$

Ex. Use Taylor's theorem to prove

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots \left(= \sum_{n=1}^{\infty} (-1)^{n-1}/n \right) = \ln 2$$

Pf. **Idea**: $1-\frac{1}{2}+\frac{1}{3}-\cdots = x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\Big|_{x=1}$

$$f(x) := x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots \quad\Rightarrow\quad f'(x) = 1-x+x^2-\cdots = \frac{1}{1+x} \quad \text{for } |x|<1$$

Based on this observation, we let $f(x)=\ln(1+x)$.

Then $f'(x)=\frac{1}{1+x}$, $f^{(2)}(x)=\frac{-1}{(1+x)^2}$, $f^{(3)}(x)=\frac{(-1)^2\,2!}{(1+x)^3}$, $\cdots$

$$f^{(n)}(x)=\frac{(-1)^{n-1}(n-1)!}{(1+x)^n} \quad \text{for every } n\ge 1$$

Taylor's theorem applied to $f(x)=\ln(1+x)$ $(x>0)$ gives

$$\ln(1+x) = f(0)+f'(0)x+\cdots+\frac{f^{(n)}(0)}{n!}x^n + R_n(x)$$

$$= x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+\frac{(-1)^{n-1}}{n}x^n + R_n(x), \quad R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \ (0<c<x)$$

Taking $x=1$ gives

$$\ln 2 = 1-\frac{1}{2}+\frac{1}{3}-\cdots+\frac{(-1)^{n-1}}{n}+R_n(1), \quad R_n(1)=\frac{f^{(n+1)}(c)}{(n+1)!} \ (0<c<1)$$

Notice that

$$|R_n(1)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}\right| = \frac{1}{(n+1)!}\left|\frac{(-1)^n n!}{(1+c)^{n+1}}\right| \le \frac{1}{n+1} \to 0 \quad \text{as} \quad n\to\infty$$

$$\therefore \quad \sum_{n=1}^{\infty}(-1)^{n-1}/n = \ln 2$$

Def. Let $a\in\mathbb{R}$. If $\exists R>0$ such that $f(x)$ can be expressed as

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n \qquad \forall x \ \text{ with } \ |x-a|<R,$$

we say that f is (real) **analytic** at a.

Or, equivalently, f is (real) **analytic** at a if $\exists R>0$ such that

$$R_n(x) := \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \to 0 \quad \text{as } n\to\infty, \ \text{for any } x \text{ with } |x-a|<R$$

(equivalently, $\lim_{n\to\infty}\frac{f^{(n)}(c)}{n!}(x-a)^n = 0 \quad \forall x$ with $|x-a|<R$)

Later, it will be proved that

$$\boxed{f \text{ is analytic at } x=a \iff \exists R>0 \text{ s.t. } f(x) \overset{\text{is expressed as}}{=} \sum_{0}^{\infty} a_n(x-a)^n \ \text{ on } \ (a-R, a+R)}$$

More on Taylor's theorem (보충)

⊙ **Taylor's theorem with integral remainder**

Version1. Let I be an interval in $\mathbb{R}$

① Suppose $f \in C^1(I)$ (i.e., f' is continuous on I) and $a \in I$. Then

$$f(a+h) - f(a) = h\int_0^1 f'(a+th)\,dt$$

Pf. $h\int_0^1 f'(a+th)\,dt \overset{a+th=:u}{=} \int_a^{a+h} f'(u)\,du = f(a+h) - f(a)$

② Suppose $f \in C^2(I)$ (i.e., f'' is continuous on I) and $a \in I$. Then

$$f(a+h) = f(a) + hf'(a) + h^2\int_0^1 (1-t)f''(a+th)\,dt$$

Pf. By ①, $f(a+h) - f(a) = h\int_0^1 f'(a+th)\,dt$

$$h\int_0^1 f'(a+th)\,dt = h\int_0^1 -(1-t)' f'(a+th)\,dt$$

$$\overset{\text{integration by parts}}{=} \left[-h(1-t)f'(a+th)\right]_{t=0}^{t=1} + h\int_0^1 (1-t)f''(a+th)h\,dt$$

$$= hf'(a) + h^2\int_0^1 (1-t)f''(a+th)\,dt$$

Therefore, the result follows.

③ Suppose $f \in C^3(I)$ (i.e., $f^{(3)}$ is continuous on I) and $a \in I$. Then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2}f''(a) + \frac{h^3}{2!}\int_0^1 (1-t)^2 f^{(3)}(a+th)\,dt$$

Pf. By ②, $f(a+h) = f(a) + hf'(a) + h^2\int_0^1 (1-t)f''(a+th)\,dt$

$$h^2\int_0^1 (1-t)f''(a+th)\,dt = h^2\int_0^1 \left(-\frac{(1-t)^2}{2}\right)' f''(a+th)\,dt$$

$$\overset{\text{integration by parts}}{=} \left[h^2\frac{-(1-t)^2}{2}f''(a+th)\right]_{t=0}^{t=1} + h^2\int_0^1 \frac{(1-t)^2}{2}f^{(3)}(a+th)h\,dt$$

$$= \frac{h^2}{2}f''(a) + \frac{h^3}{2}\int_0^1 (1-t)^2 f^{(3)}(a+th)\,dt$$

Therefore, the result follows.

④ Suppose $f \in C^{(n+1)}(I)$ and $a \in I$. Then

$$f(a+h) = \sum_{k=0}^{n}\frac{h^k}{k!}f^{(k)}(a) + \frac{h^{n+1}}{n!}\int_0^1 (1-t)^n f^{(n+1)}(a+th)\,dt$$

Ex. Another equivalent form ($\Leftarrow$ apply **integration by parts**, or substitute $x = a + h$ in version 1)

① $f \in C^1(I)$ and $a \in I$ $\Rightarrow$ for every $x \in I$, $f(x) = f(a) + \int_a^x f'(t)\,dt$ (FTC)

② $f \in C^2(I)$ and $a \in I$ $\Rightarrow$ for every $x \in I$, $f(x) = f(a) + f'(a)(x-a) + \int_a^x (x-t)f^{(2)}(t)\,dt$

③ $f \in C^3(I)$ and $a \in I \Rightarrow$ for every $x \in I$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{1}{2!}\int_a^x (x-t)^2 f^{(3)}(t)\,dt$$

④ $f \in C^{(n+1)}(I)$ and $a \in I \Rightarrow$ for every $x \in I$,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$$

Pf. ②:

$$f(x) = f(a) + \int_a^x f'(t)\,dt = f(a) + \int_a^x [-(x-t)']f'(t)\,dt$$

$$= f(a) - (x-t)f'(t)\Big|_{t=a}^{t=x} - \int_a^x [-(x-t)]f''(t)\,dt \quad [\leftarrow \text{integration by parts}]$$

$$= f(a) + (x-a)f'(a) + \int_a^x (x-t)f''(t)\,dt$$

$$= f(a) + (x-a)f'(a) + \int_a^x -\left(\frac{(x-t)^2}{2}\right)' f''(t)\,dt$$

$$= f(a) + (x-a)f'(a) - \frac{(x-t)^2}{2} f''(t)\Bigg|_{t=a}^{t=x} + \frac{1}{2}\int_a^x (x-t)^2 f^{(3)}(t)\,dt$$

[$\leftarrow$ integration by parts again]

$$= f(a) + (x-a)f'(a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{1}{2!}\int_a^x (x-t)^2 f^{(3)}(t)\,dt$$

Version2. Let I be an interval in $\mathbb{R}$

① Suppose $f \in C^1(I)$ (i.e., f' is continuous on I) and $a \in I$. Then

$$f(a+h) - f(a) - hf'(a) = h\int_0^1 \left[f'(a+th) - f'(a)\right]dt$$

Pf. This comes from subtracting $hf'(a)$ on both sides of version1-①)

② Suppose $f \in C^2(I)$ (i.e., f'' is continuous on I) and $a \in I$. Then

$$f(a+h) - f(a) - hf'(a) - \frac{h^2}{2} f''(a) = \frac{h^2}{1!}\int_0^1 (1-t)\left[f''(a+th) - f''(a)\right]dt$$

Pf. By version1-②, $f(a+h) - f(a) - hf'(a) = h^2\int_0^1 (1-t)f''(a+th)\,dt$

Subtracting $f''(a)\dfrac{h^2}{2}$ on both sides of the above gives

$$f(a+h) - f(a) - hf'(a) - f''(a)\frac{h^2}{2} = h^2\int_0^1 (1-t)f''(a+th)\,dt - f''(a)\frac{h^2}{2}$$

$$= h^2\int_0^1 (1-t)\left[f''(a+th) - f''(a)\right]dt \quad \left(\leftarrow \frac{h^2}{2} = h^2\int_0^1 (1-t)\,dt\right)$$

③ Suppose $f \in C^3(I)$ (i.e., $f^{(3)}$ is continuous on I) and $a \in I$. Then

$$f(a+h) - f(a) - hf'(a) - \frac{h^2}{2} f''(a) - \frac{h^3}{3!} f^{(3)}(a) = \frac{h^3}{2!}\int_0^1 (1-t)^2\left[f^{(3)}(a+th) - f^{(3)}(a)\right]dt$$

Pf. Use $\dfrac{h^k}{k!} = \dfrac{h^k}{(k-1)!}\int_0^1 (1-t)^{k-1}\,dt$

Cor. Suppose $f \in C^k(I)$ (i.e., $f^{(k)}$ is continuous on I) and $a \in I$. Then

$$\frac{R_{a,k}(h)}{h^k} \to 0 \quad \text{as } h \to 0,$$

where $R_{a,k}(h) := \dfrac{h^k}{(k-1)!}\int_0^1 (1-t)^{k-1}\left[f^{(k)}(a+th) - f^{(k)}(a)\right]dt$

Pf.

$$\begin{aligned}
\left|\frac{R_{a,k}(h)}{h^k}\right| &\le \frac{1}{(k-1)!}\int_0^1 (1-t)^{k-1}\left| f^{(k)}(a+th) - f^{(k)}(a)\right| dt \\
&\le \sup_{0\le t\le 1}\left| f^{(k)}(a+th) - f^{(k)}(a)\right| \cdot \frac{1}{(k-1)!}\int_0^1 (1-t)^{k-1}\,dt \\
&= \frac{1}{k!}\cdot \sup_{0\le t\le 1}\left| f^{(k)}(a+th) - f^{(k)}(a)\right| \ \to\ 0 \ \text{ as } h \to 0, \text{ since } f^{(k)} \text{ is conti at } a
\end{aligned}$$