# Stochastic Processes (STA3021)
# HW7 Solution

1. Chapter 5 #4
   The problem is asking
   $$P(A > B + C, A > B)$$
   under the following service time assumptions.

   (a) For $A = B = C = 10$,
   $$P\left(A > B + C, A > B\right) = 0$$
   since it is impossible.

   (b) Desired event can be expressed as
   $$P\left(A > B + C, A > B\right) = P\left(A > 1 + C, A > 1|B = 1\right)P\left(B = 1\right)$$
   $$+P\left(A > 2 + C, A > 1|B = 2\right)P\left(B = 2\right)+P\left(A > 3 + C, A > 3|B = 3\right)P\left(B = 3\right)$$
   $$= P\left(A = 3, C = 1\right)P\left(B = 1\right) + 0 + 0 = \frac{1}{27}.$$

   (c)
   $$P\left(A > B + C, A > B\right) = \int_0^\infty P\left(A > b + C, A > b|B = b\right)f_B(b)db$$
   $$= \int_0^\infty \int_0^\infty P\left(A > b + c, A > b|B = b, C = c\right)f_C(d)dcf_B(b)db$$
   $$= \int_0^\infty \int_0^\infty P\left(A > b + c|B = b, C = c\right)f_C(d)dcf_B(b)db$$
   $$= \int_0^\infty \int_0^\infty e^{-\mu(b+c)}\mu^2 e^{-(b+c)\mu}dbdc = \mu^2 \int_0^\infty e^{-2b\mu}db \int_0^\infty e^{-2c\mu}dc = \frac{1}{4}.$$

2. Chapter 5 #18

(a)(b)
$$P\left(X_{(1)} \le x\right) = 1 - P\left(X_{(1)} > x\right) = 1 - P\left(X_1 > x, X_2 > x\right)$$
$$= 1 - P\left(X_1 > x\right)P\left(X_2 > x\right) = 1 - e^{-2\mu x}.$$

Thus, $X_{(1)}$ follows Exponential distribution with rate $2\mu$. It gives that
$$EX_{(1)} = \frac{1}{2\mu}, \quad \mathrm{Var}X_{(1)} = \frac{1}{4\mu^2}.$$

(c)(d) For $X_{(2)}$, which is the maximum of $(X_1, X_2)$, note that

$$P\left(X_{(2)} \le x\right) = P\left(X_1 < x, X_2 < x\right) = (1 - e^{-\mu x})^2.$$

Thus, the pdf is given by

$$f(x) = \frac{d}{dx}(1 - e^{-\mu x})^2 = 2\mu e^{-\mu x}(1 - e^{-\mu x}).$$

$$\begin{aligned} EX_{(2)} &= \int_0^\infty x 2\mu e^{-\mu x}(1 - e^{-\mu x})dx = 2\mu \left( \int_0^\infty x e^{-\mu x} dx - \int_0^\infty x e^{-2\mu x} \right) dx \\ &= 2\mu \left( \frac{1}{\mu^2} - \frac{1}{(2\mu)^2} \right) = \frac{3}{2\mu}, \end{aligned}$$

where we use the property of Exponential distribution

$$\int_0^\infty x\mu e^{-\mu x} dx = \frac{1}{\mu}.$$

Similarly,

$$\begin{aligned} \mathrm{Var} X_{(2)} &= \int_0^\infty x^2 2\mu e^{-\mu x}(1 - e^{-\mu x})dx - (EX_{(2)})^2 \\ &= 2\mu \left\{ \int_0^\infty x^2 e^{-\mu x} dx - \int_0^\infty x^2 e^{-2\mu x} dx \right\} - \left( \frac{3}{2\mu} \right)^2 \\ &= 2\mu \left\{ \frac{\Gamma(3)}{\mu^3} - \frac{\Gamma(3)}{(2\mu)^3} \right\} - \left( \frac{3}{2\mu} \right)^2 = \frac{5}{4\mu^2}, \end{aligned}$$

where the Gamma distribution integrates to 1 implies that

$$\int_0^\infty x^{\alpha - 1} e^{-\mu x} dx = \frac{\Gamma(\alpha)}{\mu^\alpha}.$$

Also remark that the general pdf formula for $r$-th order statistics from the IID random sample $X_1, \ldots, X_n$ is given by

$$\frac{n!}{(r-1)!1!(n-r)!} F(x)^{r-1} f(x) (1 - F(x))^{n-r}.$$

3. Chapter 5 #30

Let $T_d$ and $T_c$ denote the life time of dog and cat which follows Exponential distribution with $\lambda_d$ and $\lambda_c$ respectively. Denote $A$ be the additional life time, then

$$EA = E(A|T_d > T_c)P\left(T_d > T_c\right) + E(A|T_d < T_c)P\left(T_d < T_c\right).$$

As done in class,

$$P\left(T_d > T_c\right) = \frac{\lambda_c}{\lambda_c + \lambda_d}, \quad P\left(T_d < T_c\right) = \frac{\lambda_d}{\lambda_c + \lambda_d}$$

Hence,

$$EA = \frac{1}{\lambda_d} \frac{\lambda_c}{\lambda_c + \lambda_d} + \frac{1}{\lambda_c} \frac{\lambda_d}{\lambda_c + \lambda_d}.$$

4. Chapter 5 #31

   Let $T_i$ denote the amount of time that $i$-th appointment takes at the doctor's office. Then, the expected time can be expressed as

$$ET_2 = E(T_2|T_1 < 30)P(T_1 < 30) + E(T_2|T_1 > 30)P(T_1)$$

   Since, the first patient see a doctor with Exponential distribution with mean 30,

$$P(T_1 < 30) = 1 - e^{-1}, \quad P(T_1 > 30) = e^{-1}.$$

   When the second patient came to doctor's office and found that the first patient was still seeing a doctor (which is the latter case), then the second patient should wait till the first patient finish the appointment and her own amount of time seeing a doctor. This is on average 60 min due to memoryless property of Exponential distribution. All is all,

$$ET_2 = 30(1 - e^{-1}) + 60(e^{-1}) = 30 + 30e^{-1}.$$

5. Chapter 5 #42

   (a) Straightforward calculation gives

$$ES_4 = E\left(\sum_{i=1}^{4} T_i\right) = \frac{4}{\lambda}.$$

   (b) Note that since $N(1) = 2$, the arrivals are given by

$$S_1 < S_2 \leq 1 < S_3 < S_4.$$

   Now consider the counting process start at time 1, that is,

$$N'(t) = \sum_{i=3}^{\infty} 1_{\{S_i \leq t\}},$$

   then $N'(t)$ is a Poisson process with rate $\lambda$. Denote the inter-arrivals of $N'(t)$ as $T_i'$, then

$$\begin{aligned} E(S_4|N(1) = 2) &= E(T_1 + T_2 + T_3 + T_4|N(1) = 2) \\ &= E(T_1 + T_2 + (1 - T_1 - T_2) + (T_1 + T_2 + T_3 - 1) + T_4|N(1) = 2) \\ &= 1 + E((T_1 + T_2 + T_3 - 1) + T_4|N(1) = 2) = 1 + E(T_1' + T_2') = 1 + \frac{2}{\lambda}. \end{aligned}$$

   (c) Since the PP has independent increments

$$E(N(4) - N(2)|N(1) = 3) = E(N(4) - N(2)) = 2\lambda.$$

6. Chapter 5 #50

   Let $T$ and $X$ denote the number of hours between successive train arrivals and the number of people who get on the next train respectively. Since the train just leaved, we should wait $T$ amount of time to get on the next train. During the $T$ time, the passengers arrive with Poisson process process with rate 7, it is deduced that $X|T$ follows poisson distribution with rate $7T$.

   (a) By conditioning on $T$, it follows that

   $$EX = E(E(X|T)) = E(7T) = \frac{7}{2}.$$

   (b) Similarly, by conditioning on $T$,

   $$\mathrm{Var}X = E(\mathrm{Var}(X|T)) + \mathrm{Var}(E(X|T)) = E(7T) + \mathrm{Var}(7T) = \frac{7}{2} + \frac{49}{12} = \frac{91}{12}.$$