15.1 Double Integrals over Rectangles

Volumes and Double Integrals

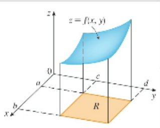


FIGURE 2

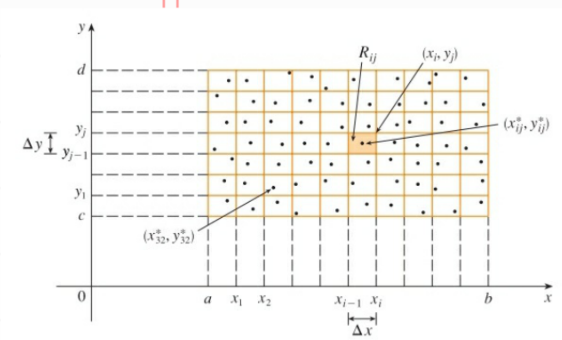
- Consider a function f of two variables defined on a closed rectangle $R = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d\}$.

  Let S be the solid that lies above R and under the graph of f; $S = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \le z \le f(x,y),\ (x,y) \in R\}$

- The first step is to divide the rectangle R into subrectangles. Divide the interval $[a,b]$ into m subintervals $[x_{i-1}, x]$ of equal width $\Delta x = \frac{(b-a)}{m}$ and divide $[c,d]$ into n subintervals $[y_{j-1}, y]$ of equal width $\Delta y = \frac{(d-c)}{n}$

  $$\Rightarrow \text{Volume} = \lim_{m,n\to\infty} \underbrace{\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta y\,\Delta x}_{\text{Double Riemann Sum}} = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

- The double integral of f over the rectangle R is $\iint f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$



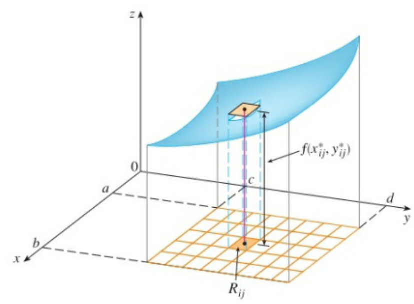


FIGURE 4

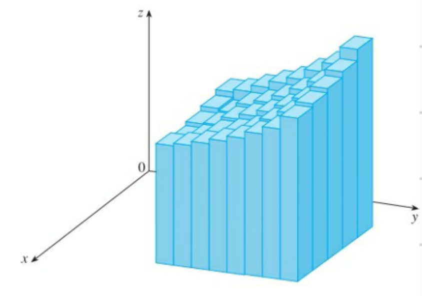


FIGURE 5

The Midpoint Rule

Midpoint Rule for Double Integrals:

$\iint f(x,y)\,dA = \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\,\Delta A$, where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x]$ and $\bar{y}_j$ is the midpoint of $[y_{i-1}, y_i]$

Iterated Integrals

Fubini's Theorem:

- If f is continuous on the rectangle $R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$, then

  $$\iint f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

  More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist

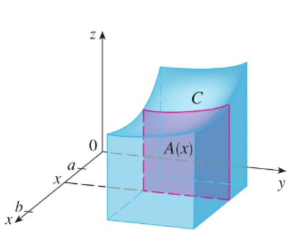


FIGURE 11

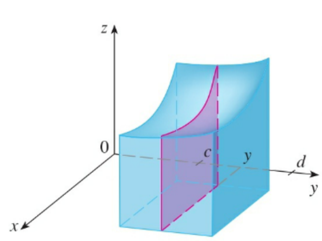


FIGURE 12

※ When we evaluate double integrals, it is wise to choose the order of integration that gives simpler integrals

$\Rightarrow \iint g(x)\,h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$, where $R = [a,b] \times [c,d]$

Average Value

- Recall that the average value of a function f of one variable defined on an interval $[a,b]$ is
  $f_{avg} = \frac{1}{b-a} \int_a^b f(x)\,dx$
- In a similar fashion, the average value of a function f of two variables defined on a rectangle R to be $f_{avg} = \frac{1}{A(R)} \iint f(x,y)\,dA$

15.2 Double Integrals over General Regions

Properties of Double Integrals

i) $\iint f(x,y) + g(x,y)\,dA = \iint f(x,y)\,dA + \iint g(x,y)\,dA$

ii) $\iint c f(x,y)\,dA = c \iint f(x,y)\,dA$

iii) If $f(x,y) \geq g(x,y)$ for all (x,y) in D, then $\iint f(x,y)\,dA \geq \iint g(x,y)\,dA$

iv) $\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$

15.4 Double Integrals in Polar Coordinates

- Recall that the polar coodinates (r,θ) of a point are related to the rectangular coordinates (x,y) by the equations
  i) $r^2 = x^2 + y^2$
  ii) $x = r\cos\theta$
  iii) $y = r\sin\theta$

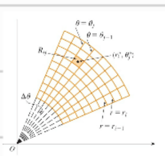


FIGURE 4 Dividing R into polar subrectangles

- In order to compute the double integral $\iint f(x,y)\,dA$, where R is a polar rectangle, we divide the interval $[a,b]$ into m subintervals $[r_{i-1}, r_i]$ of equal width $\Delta r = \frac{(b-a)}{m}$ and we divide the interval $[\alpha,\beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$ of equal width $\Delta\theta = \frac{(\beta-\alpha)}{n}$. Then the circles $r = r_i$ and the rays $\theta = \theta_j$ divide the polar rectangle R into the small polar rectangles R_{ij}

Change to Polar Coordinates in a Double Integral:

- If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then $\iint f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) \cdot r\,dr\,d\theta$

- If f is continuous on a polar region of the form $D = \{(r,\theta) \mid \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$, then
  $\iint f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$

15.4 Application of Double Integrals

Density and Mass

- To find the total mass m of the lamina, we divide a rectangle R containing D into subrectangles R_{ij} of the same size and consider $\rho(x,y)$ to be 0 outside D.

$\Rightarrow m = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint \rho(x,y)\, dA$

Moments and Centers of Mass

Moment about the x-axis :

$M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint y\rho(x,y)\, dA$

Moment about the y-axis :

$M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint x\rho(x,y)\, dA$

- The coordinates $(\bar{x},\bar{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x,y)$ are $\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint x\cdot\rho(x,y)\, dA$, $\bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint y\rho(x,y)\, dA$, where the mass m is given by $m = \iint \rho(x,y)\, dA$

Moment of Inertia

- The moment of inertia (also called the second moment) of a particle of mass m about an axis is defined to be mr^2, where r is the distance from the particle to the axis.

Moment of Inertia about the x-axis :

$I_x = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint y^2\rho(x,y)\, dA$

Moment of Inertia about the y-axis :

$I_y = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint x^2\rho(x,y)\, dA$

Moment of Inertia about the origin :

$I_0 = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \left[(x_{ij}^*)^2+(y_{ij}^*)^2\right] \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint (x^2+y^2)\rho(x,y)\, dA$

* $I_x + I_y = I_0$

⋮

Probability

Expected Values

15.5 Surface Area

- We divide D into small rectangles R_{ij} with area $\Delta A = \Delta x \Delta y$. If (x_i, y_j) is the corner of R_{ij} closest to the origin, let $P(x_i, y_j, f(x_i, y_j))$ be the point on S directly above it. The tangent plane to S at P_{ij} is an approximation to S near P_{ij}. So the area ΔT_{ij} of the part of this tangent plane that lies directly above R_{ij} is an approximation to the area ΔS_{ij} of S that lies directly above R_{ij}

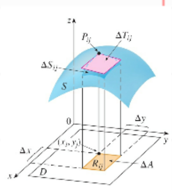


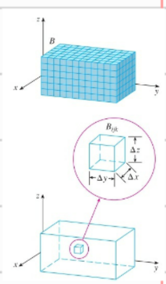
FIGURE 1

$\Rightarrow$ $A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta T_{ij}$, where $\Delta T_{ij} = \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\ \Delta A$

- The area of the surface with equation $z = f(x, y)$, $(x, y) \in D$, where f_x and f_y are continuous, is

$\Rightarrow$ $A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\ \Delta A$

15.6 Triple Integrals



- The first step is to divide B into sub-boxes. We do this by dividing the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal width Δx, dividing $[c, d]$ into m intervals of width Δy, and dividing $[r, s]$ into n subintervals of width Δz.

$\Rightarrow$ $\Delta V = \Delta x \cdot \Delta y \cdot \Delta z$

Triple Riemann Sum :

$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V$

- The triple integral of f over the box B is $\iiint f(x, y, z)\, dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, k_{ijk}^*)\Delta V$

Fubini's Theorem for Triple Integrals

- If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$\iiint f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\ dx\, dy\, dz$

$\Rightarrow$ for $E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\}$,

$\iiint f(x, y, z)\, dV = \iint \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\right] dA$

$\Rightarrow$ for $E = \{(x, y, z) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x, y) \le z \le u_2(x, y)\}$

$\iiint f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\ dz\, dy\, dx$

* The most difficult step in evaluating a triple integral is setting up an expression for

the region of integration

Applications of Triple

- The interpretation of a triple integral is not very useful because it would be the "hypervolume" of a 4-dimensional object and that is difficult to visualize. Nonetheless, the triple integral can be interpreted in different ways in different physical situations depending on the physical interpretations of x, y, z and $f(x,y,z)$)

- All the applications of double integrals can be extended to triple integrals

$\Rightarrow$ $m = \iiint \rho(x,y,z)\,dV$) mass

$\Rightarrow$ $M_{zy} = \iiint x\rho(x,y,z)\,dV$

$M_{xz} = \iiint y\rho(x,y,z)\,dV$

$M_{xy} = \iiint z\rho(x,y,z)\,dV$

moments

$\Rightarrow$ $[\bar{x}, \bar{y}, \bar{z}] = \left[\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right]$) center of mass

$\Rightarrow$ $I_x = \iiint (y^2+z^2)\,\rho(x,y,z)\,dV$

$I_y = \iiint (x^2+z^2)\,\rho(x,y,z)\,dV$

$I_z = \iiint (x^2+y^2)\,\rho(x,y,z)\,dV$

moments of inertia

15.7 Triple Integrals in Cylindrical Coordinates

- Cylindrical coordinates is the extension of polar coordinates to 3 dimensions

Cylindrical Coordinates

- In the cylindrical coordinate system, a point P in 3-dimensional space is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P.

$\Rightarrow$ $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$\Rightarrow$ $r^2 = x^2+y^2$, $\tan\theta = \frac{y}{x}$, $z = z$

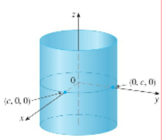


FIGURE 4
$r = c$, a cylinder

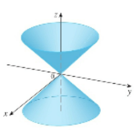


FIGURE 5
$z = r$, a cone

* Cylindrical coordinates are useful in problems that involve symmetry about an axis, and the z-axis is chosen to coincide with this axis of symmetry

Evaluating Triple Integrals with Cylindrical Coordinates

- for $E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$,

$$\iiint f(x,y,z)\,dV = \iint \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$

this can be converted to cylindrical coordinates as,

$$\Rightarrow \iiint f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\cdot r\,dz\,dr\,d\theta$$

15.8 Triple Integrals in Spherical Coordinates

Spherical Coordinates

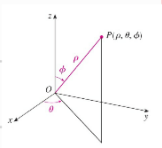


FIGURE 1 The spherical coordinates of a point

- From the point (ρ, θ, ϕ), $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment OP. Note that $\rho \ge 0$, $0 \le \phi \le \pi$

✱ the spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point.

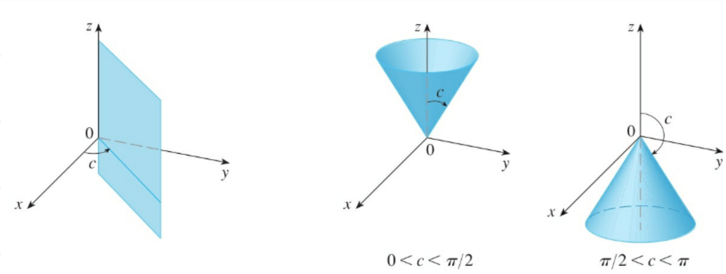


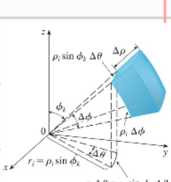
FIGURE 3 $\theta = c$, a half-plane FIGURE 4 $\phi = c$, a half-cone

$\Rightarrow z = \rho\cos\phi$, $r = \rho\sin\phi$

$\Rightarrow x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$

$\Rightarrow \rho^2 = x^2 + y^2 + z^2$

Evaluating Triple Integrals with Spherical Coordinates



- In the spherical coordinate system, the counterpart of a rectangular box is a spherical wedge,
  $E = \{(\rho,\theta,\phi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\}$, where $a \ge 0$, $\beta - \alpha \le 2\pi$, $d - c \le \pi$
- We divide E into smaller spherical wedges E_{ijk} by means of equally spaced spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$, and half-cones $\phi = \phi_k$

$$\Rightarrow \Delta V_{ijk} = (\Delta\rho)(\rho_i\Delta\phi)(\rho_i \sin\phi_k\,\Delta\theta) = \rho_i^2 \sin\phi_k\,\Delta\rho\,\Delta\theta\,\Delta\phi$$

$$\Rightarrow \iiint f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V_{ijk}$$

triple integration in spherical coordinates $= \displaystyle\lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\tilde\rho_i \sin\tilde\phi_k\cos\tilde\theta_j,\ \tilde\rho_i\sin\tilde\phi_k\sin\tilde\theta_j,\ \tilde\rho_i\cos\tilde\phi_k)\,\tilde\rho_i^2\sin\tilde\phi_k\,\Delta\rho\,\Delta\theta\,\Delta\phi$

15.9 Change of Variables in Multiple Integrals

- Consider a change of variables that is given by a transformation T from the uv-plane to the xy-plane, $T(u,v) = (x,y)$, where x and y are related to u and v by the equations $x = x(u,v)$, $y = y(u,v)$, and we usually assume that T is a C^1 transformation

⋮

Jacobian