Chap 7. Infinite series (무한급수)

7.1 Series and sequences

An infinite series is a special kind of (limit of) sequence.

$$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}, \quad s_n \to 2; \quad \text{geometric sum}$$

$$s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}, \quad s_n \to e(\text{later}); \quad \text{exponetial sum}$$

$$(s_n \ \text{ is obviously } \uparrow \ \& \ \ s_n \le 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3 \quad (\text{bdd above by } 3)$$

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \quad s_n \to \infty; \quad \text{harmonic sum}$$

⦿ A special property of the above sequences: $\boxed{s_{n+1} = s_n + \text{simple expression}}$

$$s_{n+1} = s_n + \frac{1}{2^{n+1}}, \quad s_{n+1} = s_n + \frac{1}{(n+1)!}, \quad s_{n+1} = s_n + \frac{1}{n+1}$$

Def. An **infinite series** is an expression of the form

$$a_0 + a_1 + a_2 + \cdots + a_n + \cdots \qquad (a_n \text{ is called the n-th term})$$

The sequence (s_n) defined by

$$s_n = a_0 + a_1 + a_2 + \cdots + a_n \quad (\text{or } \ s_0 = a_0; \ \ s_{n+1} = s_n + a_{n+1} \ \text{ for } n = 0,1,2,\cdots)$$

is called the **n-th partial sum**

If the seq (s_n) converges, with $\lim\limits_{n\to\infty} s_n = S$, we write symbolically

$$a_0 + a_1 + a_2 + \cdots + a_n + \cdots = S$$

and we say the series converges to the sum S ; If not, we say the series diverges

We write $a_0 + a_1 + a_2 + \cdots + a_n + \cdots$ as $\sum\limits_0^\infty a_n$, $\sum\limits_0 a_n$, or $\sum a_n$

ExaA.

$$\sum_0^\infty \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots = 2; \quad \text{geometric series}$$

$$\sum_0^\infty \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots \overset{\text{later}}{=} e; \quad \text{exponetial series}$$

$$\sum_1^\infty \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \text{diverges}; \quad \text{harmonic series}$$

ExaB. (geometric series)

$$\sum_0^\infty r^n = \begin{cases} \dfrac{1}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{otherwise} \end{cases}$$

ExaC. $s_0 = 0$; $s_{n+1} = s_n + (-1)^n \dfrac{1}{2^n}$ $(n \geq 0)$ $\lim\limits_{n\to\infty} s_n = ?$

(i.e., $(s_n)_0^\infty$: $0,\ 1,\ 1/2,\ 3/4,\ 5/8,\ 11/16,\ \cdots$; $\lim\limits_{n\to\infty} s_n = ?$)

Sol. s_n are the partial sums of the infinite series

$$0 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + (-1)^n \frac{1}{2^n} + \cdots$$

$$\therefore \quad \lim_{n\to\infty} s_n = \frac{1}{1-(-\frac{1}{2})} = \frac{2}{3}$$

⦿ Turning sequences (or limits of sequences) into infinite series

Goal: Given a sequence $s_n (n \geq 0)$, want to convert $\lim\limits_{n\to\infty} s_n$ into an infinite series

Idea: $s_n = s_0 + (s_1 - s_0) + (s_2 - s_1) + \cdots + (s_n - s_{n-1}) = s_0 + \sum\limits_1^n (s_k - s_{k-1})$

(RHS is called a telescoping sum)

$$\therefore \lim_{n\to\infty} s_n = s_0 + \lim_{n\to\infty} \sum_1^n (s_k - s_{k-1}) = s_0 + \sum_1^\infty (s_k - s_{k-1}) = \underbrace{s_0 + \sum_1^\infty (s_n - s_{n-1})}_{\text{telescoping series}}$$

$$\overset{\text{regard as}}{=} \quad a_0 + \sum_1^\infty a_n$$

Conclusion: Given a sequence $s_n (n \geq 0)$, we let

$$a_0 = s_0, \quad \& \quad a_n = s_n - s_{n-1} \quad \text{for } n \geq 1$$

$$\Rightarrow \quad \lim_{n\to\infty} s_n = \sum_0^\infty a_n$$

Remark. This converted form will be useful when $s_n - s_{n-1}$ has a simple expression in n

ExaD. Let $s_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} - \ln(n+1)$ for $n \geq 1$.

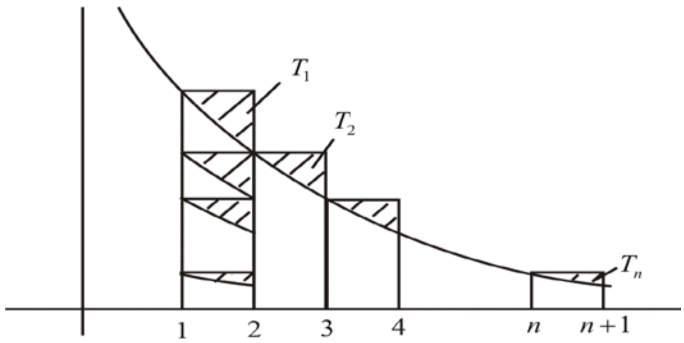
Convert the sequence into an infinite series.

Sol. Let $s_0 = 0$.

$$\begin{aligned} a_n &= s_n - s_{n-1} = \frac{1}{n} - \ln(n+1) + \ln n \\ &= \frac{1}{n} - \ln\frac{n+1}{n}, \quad \text{for } n \geq 1 \end{aligned}$$

$$\therefore \quad s_n \to \sum_1^\infty a_n = \sum_1^\infty \left(\frac{1}{n} - \ln\frac{n+1}{n}\right) = \gamma(\text{Euler's constant}) \ (\leftarrow \text{we know } \lim_{n\to\infty} s_n = \gamma)$$

Remark.



$$s_n = T_1 + T_2 + \cdots + T_n \qquad \therefore \quad a_n = s_n - s_{n-1} = T_n$$

$a_n =$ the area of the " triangle-like" region T_n

Ex. Convert $\displaystyle\sum_1^\infty \frac{1}{n(n+1)}$ & $\displaystyle\sum_1^\infty \ln\frac{n}{n+1}$ into **telescoping series**, respectively

Ans: $\displaystyle\sum_1^\infty \frac{1}{n(n+1)} = \sum_1^\infty\left(\frac{1}{n} - \frac{1}{n+1}\right)$ & $\displaystyle\sum_1^\infty \ln\frac{n}{n+1} = \sum_1^\infty(\ln n - \ln(n+1))$

7.2 Elementary convergence test

Theorem 7.2A The n-th term **test for divergence**

$$\sum a_n \text{ converges} \quad \Rightarrow \quad \lim_{n\to\infty} a_n = 0$$

Pf. Let s_n be the partial sums and $S = \sum a_n = \lim_{n\to\infty} s_n$

Then $a_n = s_n - s_{n-1}$ for $n \geq 1$

$$\therefore \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = S - S = 0$$

Remark (contrapositive statement(대우명제) of Theorem 7.2A):

$$\left[\begin{array}{l} \lim\limits_{n\to\infty} a_n \neq 0 \\ \text{or} \\ \lim\limits_{n\to\infty} a_n \text{ does not exist} \end{array}\right. \qquad \Rightarrow \qquad \sum a_n \text{ diverges}$$

Exa Are the series $\displaystyle\sum \frac{n}{n+1}$ & $\sum(-1)^n$ convergent?

Ans. $\displaystyle\frac{n}{n+1} \to 1 \neq 0 \quad \Rightarrow \quad \sum\frac{n}{n+1}$ diverges by the n-th term test

$\lim_{n\to\infty}(-1)^n$ does not exist $\qquad \therefore \sum(-1)^n$ diverges

Caution: The statement $\boxed{a_n \to 0 \quad \Rightarrow \quad \sum a_n \text{ converges}}$ is *false*

For example, $\frac{1}{n} \to 0$, but $\sum \frac{1}{n}$ diverges

Remark.

$$\sum_{n=0}^{\infty} a_n \text{ converges} \quad \Leftrightarrow \quad \text{given } \varepsilon > 0, \ \left|\sum_{n}^{m} a_k\right| < \varepsilon \ \text{ for } \ m,n \gg 1 \ \overset{\text{often}}{\Leftrightarrow} \ \sum_{n}^{m} a_k \to 0 \text{ as } m,n \to \infty$$

$$\sum_{n=0}^{\infty} a_n \text{ converges} \quad \Leftrightarrow \quad \text{given } \varepsilon > 0, \ \left|\sum_{n}^{\infty} a_k\right| < \varepsilon \ \text{ for } \ n \gg 1 \ \Leftrightarrow \ \sum_{n}^{\infty} a_k \to 0 \text{ as } n \to \infty$$

Pf. Let s_n be the partial sums $\left(\text{i.e., } s_n = \sum_{k=0}^{n} a_k\right)$. Then

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n \text{ converges} \quad &\Leftrightarrow \quad (s_n) \text{ is a Cauchy sequence} \\
&\Leftrightarrow \text{ given } \varepsilon > 0, \ \ |s_m - s_n| < \varepsilon \ \text{ for } \ m,n \gg 1 \\
&\Leftrightarrow \text{ given } \varepsilon > 0, \ \ \left|\sum_{n+1}^{m} a_k\right| < \varepsilon \ \text{ for } \ m,n \gg 1 \\
&\Leftrightarrow \text{ given } \varepsilon > 0, \ \ \left|\sum_{n}^{m} a_k\right| < \varepsilon \ \text{ for } \ m,n \gg 1 \\
&\Leftrightarrow \sum_{n}^{m} a_k \to 0 \ \left(\text{or, } \left|\sum_{n}^{m} a_k\right| \to 0\right) \text{ as } m,n \to \infty
\end{aligned}$$

Roughly,

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n \text{ converges} \quad &\Leftrightarrow \quad \sum_{k=0}^{n} a_k \to \sum_{n=0}^{\infty} a_n [\in \mathbb{R}] \quad \Leftrightarrow \quad \left|\sum_{n=0}^{\infty} a_n - \sum_{k=0}^{n} a_k\right| \to 0 \ (\text{as } n \to \infty) \\
&\Leftrightarrow \quad \left|\sum_{k=n+1}^{\infty} a_k\right| \to 0 \quad \Leftrightarrow \quad \sum_{k=n+1}^{\infty} a_k \to 0 \ (\text{as } n \to \infty)
\end{aligned}$$

Theorem 7.2B **Tail- convergence theorem**

$$\sum_{N_0}^{\infty} a_n \ \text{ converges for some } \ N_0 \ \overset{\text{(i)}}{\Rightarrow} \ \sum_{0}^{\infty} a_n \ \text{ converges } \ \overset{\text{(ii)}}{\Rightarrow} \ \sum_{N}^{\infty} a_n \ \text{ converges for every } \ N$$

Basic idea: $$\sum_{0}^{\infty} a_n = \underbrace{\sum_{0}^{N_0-1} a_n}_{\text{it is a fixed number}} + \sum_{N_0}^{\infty} a_n$$

Pf. (i) Let s'_k be the k-th partial sum of $\sum_{N_0}^{\infty} a_n$ (i.e., $s'_k = a_{N_0} + a_{N_0+1} + \cdots + a_{N_0+k}$), and

let s_k be the k-th partial sum of the series $\sum_{0}^{\infty} a_n$.

Then by hypo, $\lim_{k\to\infty} s'_k$ exists. Note that

$$s_{N_0+k} = (a_0 + a_1 + \cdots + a_{N_0-1}) + a_{N_0} + a_{N_0+1} + \cdots + a_{N_0+k} = s_{N_0-1} + s'_k$$

Hence $\lim_{k\to\infty} s_{N_0+k} = s_{N_0-1} + \lim_{k\to\infty} s'_k$ exists; and thus $\lim_{k\to\infty} s_{N_0+k} \left(\overset{\text{i.e.}}{=} \lim_{k\to\infty} s_k \right)$ exists

$$\therefore \ \sum_{0}^{\infty} a_n \ \text{ converges}$$

(ii) Let s_k' be the k-th partial sum of the series $\sum_{N}^{\infty} a_n$

(that is, $s_k' = a_N + a_{N+1} + \cdots + a_{N+k}$)

& let s_k be the k-th partial sum of the series $\sum_{0}^{\infty} a_n$

Then $\lim_{k\to\infty} s_k$ exists by hypothesis.

Since $s_k' = s_{N+k} - s_{N-1}$, $\lim_{k\to\infty} s_k' = \lim_{k\to\infty} s_{N+k} - s_{N-1} \left(\overset{\text{i.e.}}{=} \lim_{k\to\infty} s_k - s_{N-1} \right)$ exists.

$$\therefore \quad \sum_{N}^{\infty} a_n \quad \text{converges}$$

Since N is an arbitrary natural number, $\sum_{N}^{\infty} a_n$ converges for every N.

Remark. $\sum_{N_0}^{\infty} a_n$ diverges for some N_0 $\Rightarrow$ $\sum_{N}^{\infty} a_n$ diverges for every N

Theorem 7.2C Linearity theorem

Let p & q be real numbers. Then

$$\sum a_n \quad \& \quad \sum b_n : \text{conv} \quad \Rightarrow \quad \left\langle \begin{array}{l} \sum (pa_n + qb_n) \text{ converges, and} \\ \sum (pa_n + qb_n) = p\sum a_n + q\sum b_n \end{array} \right.$$

Pf. Let $s_k' = \sum_{0}^{k} a_n$ & $s_k'' = \sum_{0}^{k} b_n$. Then by hypo

$$\lim_{k\to\infty} s_k' \quad \& \quad \lim_{k\to\infty} s_k'' \ \text{ exist and } \ \lim_{k\to\infty} s_k' = \sum_{0}^{\infty} a_n \quad \& \quad \lim_{k\to\infty} s_k'' = \sum_{0}^{\infty} b_n$$

The sequence of partial sums of $\sum (pa_n + qb_n)$ is

$$s_k \equiv \sum_{0}^{k} (pa_n + qb_n) = p\sum_{0}^{k} a_n + q\sum_{0}^{k} b_n = ps_k' + qs_k''$$

$$\therefore \quad \underbrace{\lim_{k\to\infty} s_k}_{\substack{\| \\ \sum_{0}^{\infty}(pa_n + qb_n)}} = \lim_{k\to\infty} (ps_k' + qs_k'') = p\lim_{k\to\infty} s_k' + q\lim_{k\to\infty} s_k'' = p\sum_{0}^{\infty} a_n + q\sum_{0}^{\infty} b_n$$

Cor. $$\sum a_n \ \& \ \sum b_n : \text{conv} \Rightarrow \left\langle \begin{array}{lll} \sum (a_n \pm b_n) & \text{conv} & \& \ \sum (a_n \pm b_n) = \sum a_n \pm \sum b_n \\ \sum ca_n & \text{conv} & \& \ \sum ca_n = c\sum a_n \end{array} \right.$$

Note: $\sum a_n$ & $\sum b_n$: conv $\begin{cases} \not\Rightarrow & \sum a_n b_n \text{ : conv} \\ \not\Rightarrow & \sum \dfrac{a_n}{b_n} \text{ : conv} \end{cases}$

For example, take $a_n = b_n = \dfrac{(-1)^n}{\sqrt{n}} \quad \Rightarrow$

$$\sum a_n \ \ (\& \ \sum b_n) : \text{conv, but } \sum a_n b_n = \sum \frac{1}{n} : \text{div} \quad \& \quad \sum \frac{a_n}{b_n} = \sum 1 : \text{div}$$

Theorem 7.2D Comparison theorem for positive terms (: the most basic theorem)

Assume that $0 \le a_n \le a_n'$ for all n. Then

$$\sum a_n' \text{ converges } \Rightarrow \sum a_n \text{ converges, and } \sum a_n \le \sum a_n';$$

$$[\![(\text{대우}) \ \sum a_n \text{ diverges } \Rightarrow \sum a_n' \text{ diverges}]\!]$$

Pf. Let $s_k = \sum_0^k a_n$ & $s_k' = \sum_0^k a_n'$.

Since $a_n \ge 0$ and $a_n' \ge 0$ for all n, s_k & s_k' are increasing

By hypo, $\lim_{k\to\infty} s_k'$ exists, call this limit S'

Since s_k' is $\uparrow$ & $S' = \lim_{k\to\infty} s_k'$, we see $s_k' \le S'$ for all k (by Theorem $3.2B$)

Since $a_n \le a_n'$ for all n, it follows that $s_k \le s_k'$ for all k

$\therefore \ s_k \le S'$ for all k Thus (s_k) is $\uparrow$ & bounded above(by S')

By the Completeness Property, $\lim_{k\to\infty} s_k$ exists.

Now by LLT, $S \equiv \lim_{k\to\infty} s_k \le S'$

This shows that $\sum a_n$ converges, and that $\sum a_n \le \sum a_n'$

Caution. Non-negativity assumption $\boxed{0 \le a_n \le a_n' \ \forall n}$ is essential:

$$a_n := -1/n \le 1/n^2 =: a_n' \quad \Rightarrow \quad \sum a_n' : \text{ conv, but } \sum a_n : \text{diverges}$$

Exa. Is $\sum \dfrac{1}{\sqrt{n}}$ convergent?

Sol. $0 \le \dfrac{1}{n} \le \dfrac{1}{\sqrt{n}}$ for all $n \ge 1$

$$\sum \frac{1}{n} \text{ diverges} \qquad \therefore \ \sum \frac{1}{\sqrt{n}} \text{ diverges}$$

7.3 Convergence of **series with negative terms**

Def $\sum a_n$ is said to be <u>absolutely convergent</u> if $\sum | a_n |$ converges

$\sum a_n$ is called <u>conditionally convergent</u> if $\sum a_n$ converges, but $\sum | a_n |$ diverges

Cf (in some texts): $\sum a_n$ is called **<u>unconditionally</u>** <u>convergent</u> if every rearrangement of $\sum a_n$ converges (to the same limit); the notion of a rearrangement of $\sum a_n$ will be introduced in section 7.7

Exa
- $\sum \dfrac{(-1)^n}{2^n}$ & $\sum \dfrac{(-1)^n}{n!}$ are absolutely convergent
- $a_n \geq 0$ for all n & $\sum a_n$ conv $\Rightarrow$ $\sum a_n$: absolutely conv.
- $\sum \dfrac{(-1)^n}{n}$ is conditionally convergent

Theorem **Absolute convergence theorem**

$\sum | a_n |$ converges $\Rightarrow$ $\sum a_n$ converges

⦿ We will illustrate the idea on the series $\sum \dfrac{(-1)^n}{n!}$ (this series is clearly absolutely convergent).

To show it converges, write the series as

$$
\begin{aligned}
& 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots \\
\underset{\text{need to prove}}{\overset{\text{formally}}{=}} & \left[1 + 0 + \frac{1}{2!} + 0 + \frac{1}{4!} + \cdots \right] \\
& - \left[0 + \frac{1}{1!} + 0 + \frac{1}{3!} + 0 \cdots \right] \\
\overset{\text{write}}{=} & \sum b_n - \sum c_n
\end{aligned}
$$

Note that

$$
\begin{aligned}
& 0 \leq b_n \leq \frac{1}{n!} \qquad \& \qquad \sum \frac{1}{n!} \ \text{conv} \qquad \therefore \sum b_n \ \text{conv} \\
& 0 \leq c_n \leq \frac{1}{n!} \qquad \& \qquad \sum \frac{1}{n!} \ \text{conv} \qquad \therefore \sum c_n \ \text{conv}
\end{aligned}
$$

Thus $\sum (b_n - c_n)$ conv by Linearity theorem, and $\sum (b_n - c_n) = \sum b_n - \sum c_n$

Consequently, $\sum \dfrac{(-1)^n}{n!}$ converges.

Pf of (the Absolute convergence) theorem. For every n, we let

$$a_n^+ = \begin{cases} a_n & \text{if } a_n > 0 \\ 0 & \text{if } a_n \le 0 \end{cases} \qquad a_n^- = \begin{cases} 0 & \text{if } a_n > 0 \\ -a_n & \text{if } a_n \le 0 \end{cases}$$

$$= \begin{cases} |\, a_n \,| & \text{if } a_n > 0 \\ 0 & \text{otherwise} \end{cases} \qquad = \begin{cases} |\, a_n \,| & \text{if } a_n < 0 \\ 0 & \text{otherwise} \end{cases}$$

Then for every n,

$$a_n^+ - a_n^- = \begin{cases} a_n & \text{if } a_n > 0 \\ a_n & \text{if } a_n \le 0 \end{cases} \qquad = a_n$$

$$\therefore \quad \sum a_n = \sum (a_n^+ - a_n^-)$$

And according to their definitions,

$$0 \le a_n^+ \le |\, a_n \,| \qquad \& \qquad 0 \le a_n^- \le |\, a_n \,| \quad \text{for all } n$$

Since by hypo $\sum |\, a_n \,|$ converges, the Comparison test shows that

$\sum a_n^+ \quad \& \quad \sum a_n^-$: are convergent

$\therefore \quad \sum (a_n^+ - a_n^-) = \sum a_n$ is convergent

Moreover, $\sum a_n = \sum (a_n^+ - a_n^-) = \sum a_n^+ - \sum a_n^-$

Another popular pf.

Suppose $\sum_0^{\infty} |\, a_n \,|$ converges. Let $s_n = \sum_{k=0}^{n} a_k$ and $\sigma_n = \sum_{k=0}^{n} |\, a_k \,|$. Then (σ_n) is convergent, and hence (σ_n) is a Cauchy sequence. Thus, for given $\varepsilon > 0$

$$|s_m - s_n| = \left| \sum_{k=n+1}^{m} a_k \right| \le \sum_{k=n+1}^{m} |\, a_k \,| = \sigma_m - \sigma_n = |\sigma_m - \sigma_n| < \varepsilon \quad \text{for } m > n \ge (\text{some})\, N$$

This shows the sequence (s_n) is also Cauchy; so (s_n) is convergent, and hence the series $\sum_0^{\infty} a_n$ is convergent.

Example. Show that $\sum_{n=0}^{\infty} \frac{\sin n}{2^n}$ is convergent

Sol. $\sum_{n=0}^{\infty} \left| \frac{\sin n}{2^n} \right| \le \sum_{n=0}^{\infty} \frac{1}{2^n} \quad \& \quad \sum_{n=0}^{\infty} \frac{1}{2^n}$ is convergent

So $\sum_{n=0}^{\infty} \frac{\sin n}{2^n}$ is absolutely convergent

$\therefore \quad \sum_{n=0}^{\infty} \frac{\sin n}{2^n}$ convergent by **Absolute convergence theorem**

7.4 Convergence tests: ratio and n-th root tests

Theorem A The **ratio test**

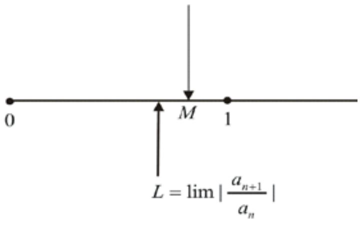
Suppose $a_n \neq 0$ for $n \gg 1$, and $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = L.$ Then

$$L < 1 \quad \Rightarrow \quad \sum a_n \ \text{ conv } \ (\text{absolutely})$$

$$L > 1 \quad \Rightarrow \quad \sum a_n \ \text{ diverges}$$

If $L = 1$ or there is no limit, the test fails and there is no conclusion.

Pf. Case1. $L < 1$



Choose a number M so that $L < M < 1.$ Then by SLT,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L \quad \Rightarrow \quad \left| \frac{a_{n+1}}{a_n} \right| < M \quad \text{ for } n \gg 1, \ \text{ say for } n \geq N$$

On the other hand,

$$\left| \frac{a_{n+1}}{a_n} \right| < M \quad \text{ for } n \geq N \quad \Rightarrow \quad | a_{n+1} | < | a_n | M \quad \text{ for } n \geq N$$

$$\therefore \quad | a_{N+1} | < | a_N | M$$

$$| a_{N+2} | < | a_{N+1} | M < | a_N | M^2$$

$$\vdots$$

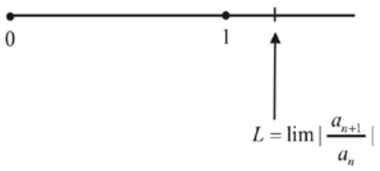
$$| a_{N+k} | < | a_N | M^k \quad \text{ for } k \geq 1$$

Since $M < 1,$ $\sum\limits_{k=1}^{\infty} M^k$ converges $\therefore$ $\sum\limits_{k=1}^{\infty} | a_N | M^k$ converges (by Linearity theorem)

Thus by the Comparison theorem, $\sum\limits_{k=1}^{\infty} | a_{N+k} | \ \left(= \sum\limits_{N+1}^{\infty} | a_n | \right)$ converges

Finally, by the Tail-convergence theorem, $\sum | a_n |$ converges

Case2. $L > 1$



By the SLT,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| (= L) > 1 \quad \Rightarrow \quad \left|\frac{a_{n+1}}{a_n}\right| > 1 \quad \text{ for } n \gg 1, \ \text{ say for } n \geq N$$

$$\Rightarrow \quad | a_{n+1} | > | a_n | \quad \text{ for } n \geq N$$

Since $a_n \neq 0$ for $n \gg 1$, we can assume that

$$| a_{n+1} | > | a_n | \quad \& \quad a_n \neq 0 \quad \text{for } n \geq N$$

$$\therefore \quad 0 < | a_N | < | a_{N+1} | < | a_{N+2} | \cdots$$

$$\therefore \quad | a_n | \text{ is (strictly)} \uparrow \quad \text{for} \quad n \geq N$$

$$\therefore \text{ either } \lim_{n\to\infty} | a_n | = \infty \text{ or } \underbrace{\lim_{n\to\infty} | a_n | \geq | a_N | > 0 \text{(by LLT) if the } \lim_{n\to\infty} | a_n | \text{ exists}}_{\substack{\Downarrow \\ \lim_{n\to\infty} a_n \neq 0}}$$

In any case, $\sum a_n$ diverges.

Case3. $L = 1$

$$\sum \frac{1}{n^2} \text{ conv with } \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1; \quad \text{whereas} \quad \sum \frac{1}{n} \text{ div with } \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$$

Theorem B The **n-th root test**

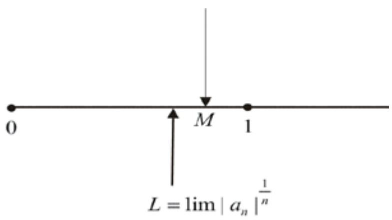
Suppose $\lim_{n\to\infty} \sqrt[n]{| a_n |} = L$. Then

$$L < 1 \quad \Rightarrow \quad \sum a_n \text{ conv (absolutely)}$$

$$L > 1 \quad \Rightarrow \quad \sum a_n \text{ diverges}$$

If $L = 1$ or there is no limit, the test fails and there is no conclusion

Pf. Case1. $L < 1$



Choose a number M so that $L < M < 1$. Then by SLT,

$$\lim_{n\to\infty} \sqrt[n]{| a_n |} = L \quad \Rightarrow \quad \sqrt[n]{| a_n |} < M \quad \text{for } n \gg 1, \text{ say for } n \geq N$$

$$\text{i.e.,} \quad | a_n | < M^n \quad \text{for } n \geq N$$

$\sum_N^\infty M^n$ converges since $M < 1$ $\quad \therefore \sum_N^\infty | a_n |$ conv (by the Comparison thm)

Finally, by the Tail-convergence theorem, $\sum | a_n |$ converges

Case2. $L > 1$: Exercise. Case3. $L = 1$: Give examples

Exa. Test for convergence: $(a)\ \sum \frac{(-1)^n n}{2^n}$ $(b)\ \sum \frac{1}{n^2}$

Sol. (a) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n} \to \frac{1}{2} = L < 1$

Thus by Ratio test, the series conv absolutely $\therefore$ conv

(b) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n^2}{(n+1)^2} \to 1 = L$ $\therefore$ the Ratio test fails.

$\sqrt[n]{|\, a_n\,|} = n^{-2/n} = (n^{\frac{1}{n}})^{-2} \to 1^{-2} = 1 = L$ $\therefore$ the n-th root test also fails.

However, $$\sum_1^\infty \frac{1}{n^2} = 1 + \sum_2^\infty \frac{1}{n^2} \le 1 + \overbrace{\sum_2^\infty \frac{1}{(n-1)n}}^{(*)}$$

(*) converges since its partial sums $\sum_2^N \frac{1}{(n-1)n} = \sum_2^N \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1 - \frac{1}{N} \to 1$

Thus, by Comparison test, $\sum_2^\infty \frac{1}{n^2}$ converges $\therefore \sum_1^\infty \frac{1}{n^2}$ converges.

7.5 The **integral and asymptotic comparison tests** (: very useful)

These tests are shown to be useful for series like $\sum \frac{1}{n^2}$

(Seen that the Ratio and the n-th root test fail for the series)

Theorem A **The integral test**

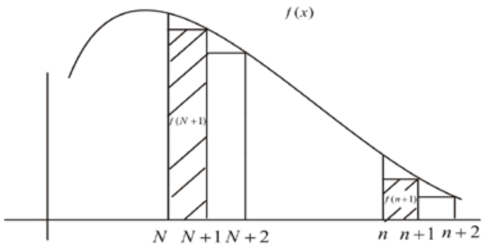
Suppose $f(x) \ge 0$ and decreasing, for $x \ge$ some positive integer N. Then

$\sum f(n)$ **converges if** the area under $f(x)$ and over $[N, \infty)$ is **finite,** i.e., $\int_N^\infty f(x)\,dx < \infty$

&

$\sum f(n)$ **diverges if** the area under $f(x)$ and over $[N, \infty)$ is **infinite,** i.e., $\int_N^\infty f(x)\,dx = \infty$

Pf.



Case1. the area is finite

From the picture, we see that

$$0 \le \underbrace{f(n+1)}_{\text{area of shaded rectangle}} \le A_n \equiv \text{the area under } f(x) \ \& \ \text{over } [n, n+1] \quad \text{for } n \ge N$$

Hypo $\Rightarrow \quad \sum_{N}^{\infty} A_n$ converges

$$(\because) \qquad s_k = \sum_{N}^{N+k} A_n \ (= \text{the seq of partial sums of } \sum_{N}^{\infty} A_n\ )$$

$$= \text{area under } f(x) \ \ \& \ \ \text{over } [N, N+k+1]$$

$$\therefore \quad \lim_{k\to\infty} s_k = \text{the area over } \ [N, \infty) < \infty \ \ \text{by assumption}$$

Thus by Comparison theorem

$$\sum_{N}^{\infty} f(n+1) \quad \text{converges} \qquad\qquad i.e., \ \sum_{N+1}^{\infty} f(n) \ \ \text{converges}$$

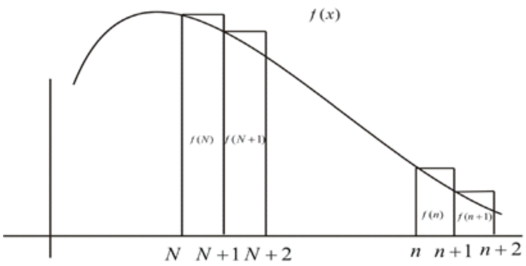
$\therefore \quad \sum f(n)$ converges as well (by Tail-convergence theorem)

Short pf : Figure shows $\displaystyle\sum_{N+1}^{n} f(k) \le \int_N^n f(x)\,dx \le \int_N^\infty f(x)\,dx \overset{\text{Hypo}}{<} \infty$

$$\therefore \quad \sum_{N+1}^{n} f(k) \ \ \uparrow \ \ \& \text{ bounded above (by } \int_N^\infty f(x)\,dx) \quad \therefore \ \sum_{N}^{\infty} f(k) : \ \text{converges}$$

$\therefore \quad \sum f(n)$ converges (by Tail-convergence theorem)

Case2. the area is infinite



From the picture, we see that

$$0 \le \underbrace{A_n}_{\text{area under } f(x)\ \&\ \text{over } [n,n+1]} \le \underbrace{f(n)}_{\text{area of rectangle}} \qquad \text{for } n \ge N$$

Hypo $\Rightarrow \quad \sum_{N}^{\infty} A_n$ diverges

$$(\because) \qquad s_k = \sum_{N}^{N+k} A_n \ = \text{area under } f(x) \ \ \& \ \ \text{over } [N, N+k+1]$$

$$\therefore \quad \lim_{k\to\infty} s_k = \text{the area over } \ [N, \infty) = \infty \ \ \text{by assumption}$$

Thus by Comparison theorem, $\displaystyle\sum_{N}^{\infty} f(n)$ diverges

$\therefore \quad \sum f(n)$ diverges as well (by Tail-convergence theorem)

Short pf : Figure $\Rightarrow$ $\displaystyle\sum_{N}^{\infty} f(k) \left(= \lim_{n\to\infty} \sum_{N}^{n} f(k) \ge \lim_{n\to\infty} \int_N^{n+1} f(x)\,dx \right) = \int_N^\infty f(x)\,dx \overset{\text{Hypo}}{=} \infty$

$$\therefore \quad \sum_{N}^{\infty} f(k) \ \ \text{diverges} \qquad\qquad \therefore \quad \sum f(n) \ \ \text{diverges (by Tail-convergence theorem)}$$

Summary of the key idea of the integral test:

Suppose $f(x) \geq 0$ and decreasing, on the interval $[N, \infty)$ $(N = \text{some positive integer})$

$$\Rightarrow \qquad \sum_{n=N+1}^{\ell} f(n) \leq \int_N^{\ell} f(x)dx \leq \sum_{n=N}^{\ell-1} f(n) \quad \text{(draw the picture)}$$

Important: If $f(x) \geq 0$ and decreasing, on the interval $[1, \infty)$ & if $\int_1^{\infty} f(x)\,dx < \infty$, then

$$\sum_{n=2}^{\ell} f(n) \leq \int_1^{\ell} f(x)dx \leq \sum_{n=1}^{\ell-1} f(n)$$

By letting $\ell \to \infty$, we obtain

$$\sum_{n=2}^{\infty} f(n) \leq \int_1^{\infty} f(x)dx \leq \sum_{n=1}^{\infty} f(n)$$

$$\therefore \quad \int_1^{\infty} f(x)dx \leq \sum_{n=1}^{\infty} f(n) \leq f(1) + \int_1^{\infty} f(x)dx$$

This gives $\left| \sum_{n=1}^{\infty} f(n) - \int_1^{\infty} f(x)dx \right| \leq f(1)$ ($\int_1^{\infty} f(x)dx$ is an approximation of $\sum_{n=1}^{\infty} f(n)$)

Ex. By the same way, we have

$$\int_k^{\infty} f(x)dx \leq \sum_{n=k}^{\infty} f(n) \leq f(k) + \int_k^{\infty} f(x)dx \qquad \forall k \geq 1$$

So $\sum_{n=1}^{k-1} f(n) + \int_k^{\infty} f(x)dx \leq \sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{k-1} f(n) + \sum_{n=k}^{\infty} f(n) \leq \sum_{n=1}^{k-1} f(n) + f(k) + \int_k^{\infty} f(x)dx$

This gives $\left| \sum_{n=1}^{\infty} f(n) - \underbrace{\left(\sum_{n=1}^{k-1} f(n) + \int_k^{\infty} f(x)dx \right)}_{\text{a better approximation}} \right| \leq f(k)$

Application: $\sum_{1}^{\infty} n^{-4} \underset{f(10)}{\approx} 1^{-4} + 2^{-4} + \cdots + 9^{-4} + \underbrace{\int_{10}^{\infty} x^{-4}\,dx}_{=\frac{1}{3}10^{-3}}$,

where $f(x) = x^{-4}$, so $f(10) = 10^{-4} = 0.0001$

Home study: Use integral test to show that

(i) $\sum \frac{1}{n^p}$ & $\sum \frac{1}{n(\ln n)^p}$: $\begin{cases} \text{conv} & \text{if } p > 1 \\ \text{div} & \text{if } p \leq 1 \end{cases}$ (ii) $\sum_{n=2}^{\infty} \frac{\ln n}{n^p}$ converges if $p > 1$

※ Theorem B **Asymptotic (or limit) comparison test**

If $|\, a_n \,| \sim |\, b_n \,|$ $\left(\text{meaning} : \lim_{n\to\infty} \frac{|\, a_n \,|}{|\, b_n \,|} = 1\right)$, then

$$\sum |\, a_n \,| \text{ converges} \quad \Leftrightarrow \quad \sum |\, b_n \,| \text{ converges}$$

Pf. By the hypo & SLT,

$$\frac{1}{2} < \left| \frac{a_n}{b_n} \right| < \frac{3}{2} \quad \text{for } n \gg 1, \quad \text{say for } n \geq N$$

$$\therefore \quad \frac{1}{2} |\, b_n \,| < |a_n| < \frac{3}{2} |\, b_n \,| \quad \text{for } n \geq N \quad --- (*)$$

$$\sum |a_n| \text{ conv} \quad \Rightarrow \quad \sum_N^\infty |a_n| \text{ conv} \quad \text{(by Tail-conv Thm)}$$
$$\overset{(*)}{\Rightarrow} \quad \sum_N^\infty \frac{1}{2}|b_n| \text{ conv} \quad \text{(by Comparison theorem)}$$
$$\Rightarrow \quad \sum_N^\infty |b_n| \text{ conv} \quad \text{(by Linearity theorem)}$$
$$\Rightarrow \quad \sum |b_n| \text{ conv} \quad \text{(by Tail-conv theorem)}$$

On the other hand,

$$\sum |b_n| \text{ conv} \quad \Rightarrow \quad \sum_N^\infty |b_n| \text{ conv} \quad \text{(by Tail-conv theorem)}$$
$$\overset{(*)}{\Rightarrow} \quad \sum_N^\infty \frac{3}{2}|b_n| \text{ conv} \quad \text{(by Linearity theorem)}$$
$$\Rightarrow \quad \sum_N^\infty |a_n| \text{ conv} \quad \text{(by Comparison theorem)}$$
$$\Rightarrow \quad \sum |a_n| \text{ conv} \quad \text{(by Tail-conv theorem)}$$

Exa. Do these converge or diverge ?

$$(a) \quad \sum_2^\infty \frac{1}{n^3 - 2n + 1} \qquad\qquad (b) \quad \sum \sqrt{\frac{4n}{n^2+1}}$$

Sol. (a)
$$\frac{1}{n^3 - 2n + 1} \sim \frac{1}{n^3} \quad \& \quad \sum_2^\infty \frac{1}{n^3} \text{ conv} \quad \Rightarrow \quad \sum_2^\infty \frac{1}{n^3 - 2n + 1} \text{ conv}$$

(b)
$$\sqrt{\frac{4n}{n^2+1}} \sim \sqrt{\frac{4n}{n^2}} = \frac{2}{\sqrt{n}} \quad \& \quad \sum \frac{2}{\sqrt{n}} \text{ div} \quad \Rightarrow \quad \sum \sqrt{\frac{4n}{n^2+1}} \text{ div}$$

Ex. Is $\displaystyle\sum_1^\infty \frac{1}{n^{1+\frac{1}{n}}}$ convergent ?

Sol.
$$\frac{1}{n^{1+1/n}} \sim \frac{1}{n} \quad \text{because} \quad \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n^{1+1/n}}} = \lim_{n\to\infty} n^{\frac{1}{n}} \underset{\text{already seen}}{=} 1$$

Since $\displaystyle\sum_1^\infty \frac{1}{n}$ diverges, $\displaystyle\sum_1^\infty \frac{1}{n^{1+1/n}}$ is also divergent.

Another way: $\forall n \geq 1, \quad n < 2^n \qquad \therefore \quad n^{1/n} < 2 \quad \forall n \geq 1;$ so $\displaystyle\frac{1}{n^{1+1/n}} > \frac{1}{2n}$

Since $\displaystyle\sum_1^\infty \frac{1}{2n}$ diverges, $\displaystyle\sum_1^\infty \frac{1}{n^{1+1/n}}$ is also divergent.

Ex. Assume $a_n \to 0$ as $n \to \infty$. Show that $\displaystyle\sum_1^\infty \sin|a_n|$ converges $\Leftrightarrow$ $\displaystyle\sum_1^\infty |a_n|$ converges

Pf.
$$\sin|a_n| \sim |a_n| \quad \text{since} \quad \lim_{n\to\infty} \frac{\sin|a_n|}{|a_n|} \overset{a_n \to 0}{=} \lim_{x\to 0^+} \frac{\sin x}{x} = 1.$$

7.6 Series with alternating signs: Cauchy's test

Theorem. **Cauchy's test for alternating series** (or Alternating series test)

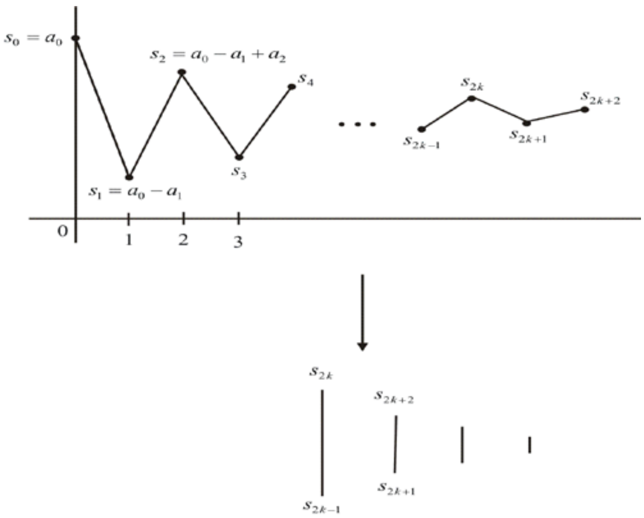
$$a_n > 0 \quad \text{for all } n \quad \& \quad a_n \downarrow \text{ strictly, and } \lim_{n\to\infty} a_n = 0.$$

(for short, $a_n \downarrow 0$ strictly)

$$\Rightarrow \sum_0^{\infty} (-1)^n a_n \quad \text{converges}$$

Moreover, if we let $\sum_{n=0}^{\infty} (-1)^n a_n = S$, then $\left| \sum_{k=0}^{n} (-1)^k a_k - S \right| \left(= \left| \sum_{n+1}^{\infty} (-1)^k a_k \right| \right) < a_{n+1}$

Pf.



From the figure (←hypothsis), we see that

$$s_{2k-1} < s_{2k+1} < \cdots < s_{2k+2} < s_{2k} \quad \text{for } k \geq 1 \quad \&$$

$$s_{2k} - s_{2k-1} = a_{2k} \quad and \quad s_{2k+1} - s_{2k} = -a_{2k+1}$$

So $\{[s_{2k-1}, s_{2k}]\}_{k\geq 1}$ is nested & $\text{length}[s_{2k-1}, s_{2k}] \overset{\text{hypothsis}}{\longrightarrow} 0$ as $k \to \infty$

Thus by NIT, $\exists$ a unique real number S such that $\bigcap_{k=1}^{\infty} [s_{2k-1}, s_{2k}] = \{S\}$

In fact, $\lim_{k\to\infty} s_{2k-1} = S = \lim_{k\to\infty} s_{2k}$

$$\therefore \quad s_{2k-1} < s_{2k+1} < \cdots < S < \cdots < s_{2k+2} < s_{2k} \quad \text{for } k \geq 1$$

From the figure again, we see that

$$|\, s_{2k} - S \,| = s_{2k} - S < s_{2k} - s_{2k+1} = a_{2k+1}$$

&

$$|\, s_{2k-1} - S \,| = S - s_{2k-1} < s_{2k} - s_{2k-1} = a_{2k}$$

Consequently, $|\, s_n - S \,| < a_{n+1}$ for any n. This implies $\lim_{n\to\infty} s_n = S$ since $\lim_{n\to\infty} a_n = 0$

Alternative way of showing

$$a_n \downarrow 0 \quad \Rightarrow \quad \sum_{n=0}^{\infty}(-1)^n a_n \left[= a_0 - a_1 + a_2 - \cdots + (-1)^n a_n + \cdots\right] \text{ converges}$$

Set $s_n = \sum_{k=0}^{n}(-1)^k a_k = a_0 - a_1 + a_2 - \cdots + (-1)^n a_n$

Key observation:

$$s_{2n-1}(n \geq 1) = (a_0 - a_1) + (a_2 - a_3) + \cdots + (a_{2n-2} - a_{2n-1}) \leq s_{2n+1} \quad [\leftarrow \text{each } () \geq 0]$$

$$\begin{aligned} s_{2n-1}(n \geq 1) &= (a_0 - a_1) + (a_2 - a_3) + \cdots + (a_{2n-2} - a_{2n-1}) \\ &= a_0 - \underbrace{(a_1 - a_2)}_{\geq 0} - \underbrace{(a_3 - a_4)}_{\geq 0} - \cdots - \underbrace{(a_{2n-3} - a_{2n-2})}_{\geq 0} - \underbrace{a_{2n-1}}_{\geq 0} \\ &\leq a_0 \end{aligned}$$

$\therefore \quad s_{2n-1} \uparrow$ and bounded above by a_0; so $s_{2n-1} \uparrow (\text{some}) S (\leq a_0 < \infty)$

Also,

$$s_{2n} = s_{2n-1} + a_{2n} \to S + 0 = S \quad [\leftarrow a_{2n} \to 0]$$

Consequently, $s_{2n-1} \to S$ & $s_{2n} \to S$

$\therefore \quad s_n \to S$ $[\leftarrow$ Claim below$]$

Claim: Let $\{a_n\}$ be a sequence of real numbers.

Show that $a_{2n} \to L$ & $a_{2n+1} \to L \quad \Rightarrow \quad \lim_{n\to\infty} a_n$ exists & $\lim_{n\to\infty} a_n = L$

Pf. Let $\varepsilon > 0$. Then

$\exists N_1$ such that $|a_{2n} - L| < \varepsilon$ for all $n \geq N_1$(i.e., $2n \geq 2N_1$) $\left[\leftarrow \lim_{n\to\infty} a_{2n} = L\right]$ &

$\exists N_2$ such that $|a_{2n+1} - L| < \varepsilon$ for all $n \geq N_2$(i.e., $2n+1 \geq 2N_2+1$) $\left[\leftarrow \lim_{n\to\infty} a_{2n+1} = L\right]$

Now we take $N = \max\{2N_1, 2N_2+1\}$ & let $k \geq N$. Then

$|a_k - L| < \varepsilon$, regardless of whether k is even or odd

$\therefore |a_k - L| < \varepsilon$ for all $k \geq N$ i.e., $\lim_{k\to\infty} a_k = L$

Comment: Let $a_n \downarrow 0$. Then

$$\sum_{k=0}^{\infty}(-1)^k a_k =: S \quad [\Rightarrow s_{2n-1} \uparrow S \;\&\; s_{2n} \downarrow S] \Rightarrow \begin{cases} 0 \leq S - s_{2n-1} \leq a_{2n} & [\leftarrow s_{2n} = s_{2n-1} + a_{2n}] \\ 0 \leq s_{2n} - S \leq a_{2n+1} \end{cases}$$

$\therefore \quad |s_m - S| \leq a_{m+1}$ for every $m \geq 0$

Exa. $\displaystyle\sum_{2}^{\infty}\frac{(-1)^n}{\sqrt{n}}$: converges by Alternating series test since $\dfrac{1}{\sqrt{n}} \downarrow 0$ strictly.

Remark. The alternating series test is **still true** if

① $a_n \downarrow 0$ (without strictly decreasing assumption)

or

② $a_n \downarrow 0$ (or strictly $\downarrow 0$) for $n \gg 1$

Recall (i) $\displaystyle\sum_{0}^{\infty} a_n$: conv $\Rightarrow$ $\displaystyle\lim_{n\to\infty} a_n = 0$

(ii) $a_n \downarrow 0 \Rightarrow \displaystyle\sum_{0}^{\infty}(-1)^n a_n$: conv & $\left|\displaystyle\sum_{n+1}^{\infty}(-1)^k a_k\right| \leq a_{n+1}$

Question1. Suppose $a_n \geq 0$ & $a_n \to 0$ $\overset{?}{\Rightarrow}$ $\displaystyle\sum_{0}^{\infty}(-1)^n a_n$: conv

Ans. No; for example,

$$0-1+0-\frac{1}{3}+0-\frac{1}{5}+\cdots \quad (\text{i.e., } a_{2n}=0,\ a_{2n-1}=\frac{1}{2n-1}) : \text{div}$$

$$2-1+\frac{1}{2^2}-\frac{1}{3}+\frac{1}{4^2}-\frac{1}{5}+\cdots : \text{div} \quad (\text{easy to check})$$

Question2. $a_n \geq 0$ & $a_n \downarrow$ $\overset{?}{\Rightarrow}$ $\displaystyle\sum_{0}^{\infty}(-1)^n a_n$: conv

Ans. No; for example, $\displaystyle\sum_{0}^{\infty}(-1)^n\frac{n+2}{n+1}$ is not convergent.

($\because$ If the series were convergent,

$$\overset{\text{n-th term test}}{\Rightarrow} \quad \lim_{n\to\infty}(-1)^n\frac{n+2}{n+1}=0 \quad\Rightarrow\quad \lim_{n\to\infty}\frac{n+2}{n+1}=0; \quad \text{absurd}\ )$$

Return to Claim: $e = 1 + 1! + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}$

Pf. We first prove $e \geq \sum_{n=0}^{\infty} \frac{1}{n!}$. Recall that $e \left(= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \right)$

Moreover, we can prove the next result

Ex. Show that $e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$ for every $x \geq 0$.

In particular, $e \geq 1 + 1! + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}$ --- ($\odot$)

Pf. We use the trivial fact: $e^x \geq 1$ if $x \geq 0$

Take $\int_0^x dt \quad \Rightarrow \quad e^x - 1 \geq x \quad$ i.e., $e^x \geq 1 + x$

Take $\int_0^x dt$ again $\quad \Rightarrow \quad e^x - 1 \geq x + \frac{x^2}{2} \quad$ i.e., $e^x \geq 1 + x + \frac{x^2}{2}$

Take $\int_0^x dt$ again $\quad \Rightarrow \quad e^x - 1 \geq x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3} \quad$ i.e., $e^x \geq 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

Continue this process to get $e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$ for every $x \geq 0$

Next we prove $e \leq \sum_{n=0}^{\infty} \frac{1}{n!}$.

We note that

$$\left(1 + \frac{1}{n}\right)^n \overset{\text{Binomial theorem}}{=} 1 + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \cdots + \binom{n}{k}\frac{1}{n^k} + \cdots + \binom{n}{n}\frac{1}{n^n}$$

$$= 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} \cdots + \frac{n(n-1)(n-2)\cdots(n-(k-1))}{k!}\frac{1}{n^k}$$
$$+ \cdots\cdots + \frac{1}{n^n}$$

$$\leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!} + \cdots + \frac{1}{n!}$$

$$\leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!} + \cdots + \frac{1}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Letting $n \to \infty$ shows

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \leq \sum_{n=0}^{\infty} \frac{1}{n!} \quad [\leftarrow \text{LLT}] \quad \text{--- } (\oplus)$$

Combining ($\odot$) & ($\oplus$) shows that $e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}$

Remark. Seen that $e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$ for every $x \geq 0$.

Later (Chapter 22), we shall prove that

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \quad \text{for every } x \in \mathbb{R}.$$