해석학 2 과제 1

1) $f(x) = e^{-x}$

$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

$= e^{-a} - e^{-a}(x-a) + \frac{1}{2}e^{-a}(x-a)^2$, where $a=0$

$= 1 - x + \frac{x^2}{2}$

2) $f(x) = \frac{1}{\sqrt{1+x}}$, $f'(x) = -\frac{1}{2}(1+x)^{-\frac{3}{2}}$, $f^{(2)}(x) = \frac{3}{4}(1+x)^{-\frac{5}{2}}$, $f^{(3)}(x) = -\frac{15}{8}(1+x)^{-\frac{7}{2}}$

$T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{6}(x-a)^3$

$= \frac{1}{\sqrt{1+a}} - \frac{1}{2}(1+a)^{-\frac{3}{2}}(x-a) + \frac{3}{8}(1+a)^{-\frac{5}{2}}(x-a)^2 - \frac{5}{16}(1+a)^{-\frac{7}{2}}(x-a)^3$, $a=0$

$= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3$

3) a) $f(x) = \sin x$

$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, where $c \in (a, x)$

$= \sin(a) + \cos(a)(x-a) - \frac{\sin(a)}{2}(x-a)^2 - \frac{\cos(a)}{3!}(x-a)^3 + \frac{\sin(a)}{4!}(x-a)^4 + \cdots$, $a=0$

$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \frac{f^{(2n+3)}(c)}{(2n+3)!}x^{2n+3}$, note that x can be defined for all x.

For any fixed x in $x \in (-\infty, \infty)$, $\lim_{n\to\infty} \frac{f^{(2n+3)}(c)}{(2n+3)!}x^{2n+3} = 0$, which is the Lagrangian Remainder.

$\therefore$ The Taylor Series at 0 of $f(x) = \sin x$ converges to the function for all x.

b) $f(x) = (1-x)^{-1}$, $f'(x) = (1-x)^{-2}$, $f''(x) = 2(1-x)^{-3}$, ..., $f^{(n)}(x) = n!(1-x)^{-n-1}$

$f(x) = (1-a)^{-1} + (1-a)^{-2}(x-a) + (1-a)^{-3}(x-a)^2 + (1-a)^{-4}(x-a)^3 + \cdots + (1-a)^{-n-1}(x-a)^n + (1-c)^{-n-2}(x-a)^{n+1}$, where $c \in (a, x)$ and let $a=0$.

$= 1 + x + x^2 + x^3 + \cdots + x^n + c^{n+1}$

$\lim_{n\to\infty} c^{n+1} = 0$ iff $x \in (-1, 1)$, and $(-1, 0] \subset (-1, 1)$

$\therefore$ the Taylor Series at 0 of $f(x) = \frac{1}{1-x}$ converges to the function for $x \in (-1, 0]$.

c) $f(x) = \ln(1+x)$, $f'(x) = (1+x)^{-1}$, $f''(x) = -(1+x)^{-2}$, $f^{(3)}(x) = 2(1+x)^{-3}$, $f^{(4)}(x) = -6(1+x)^{-4}$ ..., $f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$

$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

$= \ln(1+a) + (1+a)(x-a) - \frac{1}{2}(1+a)^{-2}(x-a)^2 + \cdots + (-1)^{n-1}\frac{1}{n}(1+a)^{-n}(x-a)^n + (-1)^n\frac{1}{n+1}(1+c)^{-n-1}(x-a)^{n+1}$, $a=0$

$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + (-1)^{n-1}\frac{1}{n}x^n + (-1)^n\frac{1}{n+1}(1+c)^{-n-1}x^{n+1}$

$\lim_{n\to\infty} (-1)^n \frac{1}{n+1}(1+c)^{-n-1}x^{n+1} = 0$ iff $x \in (-1, 1)$, and $[0, 1) \subset (-1, 1)$

$\therefore$ the Taylor Series at 0 of $f(x) = \ln(1+x)$ converges to the function for $x \in [0, 1)$.

4) a) $p(x) = (x-a)^k Q(x)$

$p'(x) = k(x-a)^{k-1}Q(x) + (x-a)^k Q'(x)$

$p^{(2)}(x) = k(k-1)(x-a)^{k-2}Q(x) + 2k(x-a)^{k-1}Q'(x) + (x-a)^k Q^{(2)}(x)$

$\vdots$

$p^{(d)}(x) = \sum_{j=1}^{d-1} d \cdot \frac{k!}{(k-j)!}(x-a)^{k-j}Q^{(k-j)}(x) + \frac{k!}{(k-d)!}(x-a)^{k-d}Q(x) + (x-a)^k Q^{(d)}(x)$

Note that $p^{(d)}(a) = 0$ for all $d = 0, 1, 2, \ldots, k-1$

$p^{(k)}(x) = \sum_{j=1}^{k-1} k \cdot \frac{k!}{(k-j)!}(x-a)^{k-j}Q^{(k-j)}(x) + k!Q(x) + (x-a)^k Q^{(k)}(x)$

$p^{(k)}(a) = k!Q(a) \neq 0$ since it is given that $Q(a) \neq 0$

b) It is given that $f(x) = 2x^3 - bx^2 + 1$ and that it is double zero at some point, which gives $f(a) = f'(a) = 0$, $f''(a) \neq 0$ for some a.

Using the Taylor's Expansion, $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$, where $f'(x) = 6x^2 - 2bx$ and $f''(x) = 12x - 2b$.

$f(x) = (x-a)^2 Q(x)$, where $Q(x)$ is a polynomial, $Q(a) \neq 0$.

$\Rightarrow$ $f(x) = (x-a)^2 Q(x)$

$f'(x) = 2(x-a)Q(x) + (x-a)^2 Q'(x)$

$f''(x) = 2Q(x) + 2(x-a)Q'(x) + 2(x-a)Q'(x) + (x-a)^2 Q''(x)$

$= 2Q(x) + 4(x-a)Q'(x) + (x-a)^2 Q''(x)$

$\Rightarrow$ $f''(x) = 12x - 2b = 2Q(x) + 4(x-a)Q'(x) + (x-a)^2 Q''(x)$

$f''(a) = 12a - 2b = 2Q(a) \neq 0$

$\therefore$ $b \neq 6x$