Chap 2. Estimations (추정, 어림) and Approximations (근사)

2.1 Inequality

Two simple tools for estimations:

inequalities (*for making comparisions*)

&

absolute values (*for measuring size and distance*)

Inequality laws (those are familiar):

[We will use $<$ in the statements; the laws using $\leq$ are analogous]

Addition

$$\begin{array}{rcl} a & < & b \\ +\, c & < & d \\ \hline a + c < & & b + d \end{array}$$

Subtraction (Please don't think of doing this)

Multiplication

$$a < b, \quad c < d \quad \Rightarrow \quad ac < bd \quad \text{if } \ a, b, c, d > 0$$

Sign-change law (changing signs reverses an inequality)

$$a < b \quad \Rightarrow \quad -a > -b$$

$$a < b \quad \Rightarrow \quad ka > kb \quad \text{if } k < 0$$

Reciprocal law

$$a < b \quad \Rightarrow \quad \frac{1}{a} > \frac{1}{b} \quad \text{if } \ a, b > 0$$

2.2 Estimations (추정, 어림)   Cf: estimate 추정하다(동), 추정값(명)

Def. If c is a number we are estimating, and $K < c < M$, we say that

K is a ***lower estimate*** (or lower bound) for c

&

M is an ***upper estimate*** (or upper bound) for c

If two sets of upper and lower estimates satisfy

$$K < K' < c < M' < M,$$

we say K', M' are stronger or sharper estimates for c, while K, M are weaker estimates

Ex A. Give upper & lower estimates for $\dfrac{1}{a^4+3a^2+1}$ $(a \in \mathbb{R})$

Sol. $0 \le a^2 < \infty \;\Rightarrow\; 1 \le a^4+3a^2+1 < \infty \qquad \therefore\; 0 < \dfrac{1}{a^4+3a^2+1} \le 1$

0이 될 수 없고 1은 될 수 있기 때문에

the upper estimate 1 is sharp(est) since equality $(=)$ is attained when $a = 0$

& the lower estimate 0 is also sharp since the fraction can be made arbitrarily close to 0 by taking a sufficiently large

Ex B. Give upper and lower estimates for $\dfrac{1+\sin^2 n}{1+\cos^2 n}$, for (integer) $n \ge 0$

$1+\sin^2 n$ 의 최소값

$1+\sin^2 n$ 의 최대값

Sol. $\dfrac{1}{2} \le \dfrac{1}{1+\cos^2 n} \le \dfrac{1+\sin^2 n}{1+\cos^2 n} \le 1+\sin^2 n \le 2$

$1+\cos^2 n$ 의 최대값

$1 \le 1+\cos^2 n$

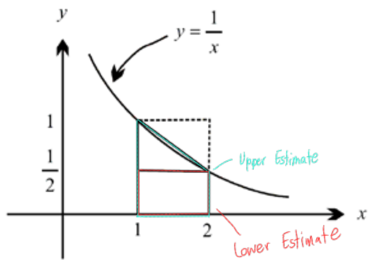
the upper estimate 2 is not sharp, but the lower estimate 1/2 is sharp (consider: $n = 0$)

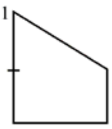
n은 자연수이기 때문에 $1+\sin^2 n = 2$ 는 불가능하다

$n=0$ 일 때

Ex C. Estimate $\ln 2 = \displaystyle\int_1^2 \frac{1}{x}\,dx$ (by interpreting the integral as the area under $1/x$ and over $[1, 2]$)

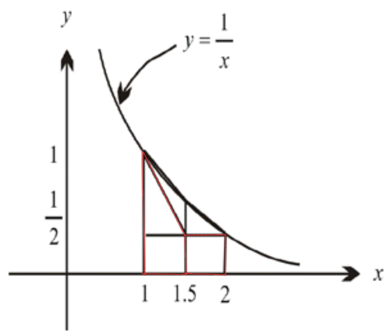
Sol.



1 / 2 = area of  < ln2 < area of  = 3 / 4

Can you find a sharper estimate ?

A sharper estimate:



$$\left(\frac{15}{24} =\right)\frac{5}{8} = 0.625 < \ln 2 < \frac{17}{24} = 0.708\cdots < 0.71$$

Our textbook: $0.63 < \ln 2 < 0.71$ (why?) Compare with $\ln 2 \underset{\text{calculator}}{\approx} 0.69$

2.3 Proving boundedness

Our concern: How to show the boundedness or unboundedness of a sequence.

Often we want an estimate just in one direction

For example, we often assume that $a_n \geq 0$ for all n

(then it is trivial that a_n is bounded below (by 0))

1. To show (a_n) is bounded above, get one upper estimate B: $a_n \leq B \quad \forall n$

2. To show (a_n) is not bounded above, get a lower estimate for each term:

$$a_n \geq B_n \text{ such that } B_n \to \infty \text{ as } n \to \infty$$

Example

- $a_n = (1 + \frac{1}{n})^n$: we showed earlier that (a_n) is bdd above by the upper estimate;

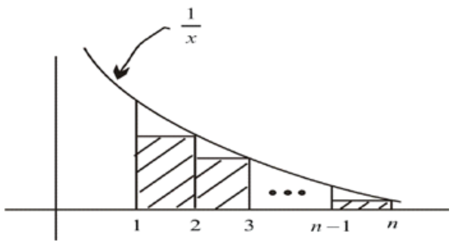
  $a_n < 3 \quad$ for all n

- $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$:

  We earlier showed $a_n > \ln(n+1) \quad (> \ln n \to \infty \text{ as } n \to \infty)$

Remark. (sometimes, trial & error is necessary for guessing boundedness or unboundedness of a given sequence)

Return to $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$:



From the picture, we see that $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \int_1^n \frac{1}{x}\,dx$

$$\therefore \quad a_n < 1 + \int_1^n \frac{1}{x}\,dx = 1 + \ln n \quad \text{(an upper estimate)} \text{ --- } \blacklozenge$$

$\sim \infty$

$1 + \ln n \to \infty$ (as $n \to \infty$); so the estimate $\blacklozenge$ is useless for showing the sequence (a_n) is **unbounded above** or for showing (a_n) is bounded

Question: $a_n \overset{\text{let}}{=} \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots + \frac{1}{p_n}$ (p_n denotes the n-th prime)

Is (a_n) bounded above or not ?

$a_n = \sum_{k=1}^{n} \frac{1}{p_k} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots \frac{1}{p_n}$

$\lim_{n\to\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{p_n}$, for $n \gg 1$, it is known that $\frac{1}{p_n} \approx \frac{1}{n \ln n} \approx \infty$, $\int_M^\infty \frac{1}{x \ln x}\,dx = \infty$

Ans. (a_n) is not bounded above (but the proof is very difficult)

For the pf, we need $\displaystyle\lim_{n\to\infty}\frac{n\ln n}{p_n}=1$ ($\overset{\text{tricky[Burton, pp358-359]}}{\Leftarrow}$ the Prime Number Theorem)
소수정의

Using this, we see that $\displaystyle\lim_{n\to\infty} a_n = \sum_{n=1}\frac{1}{p_n}\approx\sum\frac{1}{n\ln n}\overset{\text{integral test (studied later)}}{=}\infty$

Prime Number Theorem(proved independently by Hadamard and Poussin[1896]; 정수론 교재 참고):

$$\pi(x)\overset{\text{let}}{=}\sum_{p\le x}1 \quad (= \text{the number of primes that do not exceed } x)\ (\therefore\ \pi(p_n)=n\ )$$

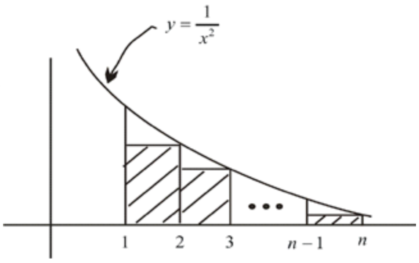
$$\Rightarrow\quad \lim_{x\to\infty}\frac{\pi(x)}{\dfrac{x}{\ln x}}=1\quad\left(\text{i.e., } \pi(x)\approx\frac{x}{\ln x}\ \text{ for } x\gg 1\right)$$

x가 충분히 크면 $\pi(x)$가 $\frac{x}{\ln x}$ 로 근접한다

Example $\quad a_n = 1+\dfrac{1}{4}+\dfrac{1}{9}+\cdots+\dfrac{1}{n^2}$

Is (a_n) bounded above or unbounded above ?

Sol 1.



total area of the shaded region = $\dfrac{1}{4}+\dfrac{1}{9}+\cdots+\dfrac{1}{n^2}$

$$\therefore\quad a_n < 1+\int_1^n\frac{1}{x^2}\,dx < 1+\int_1^\infty\frac{1}{x^2}\,dx = 1+1=2$$

Sol 2. (used in high-school math) $\displaystyle a_n=\sum_{k=1}^n\frac{1}{k^2}=1+\sum_{k=2}^n\frac{1}{k^2}<1+\sum_{k=2}^n\frac{1}{(k-1)k}\overset{\text{telescoping}}{=}1+\left(1-\frac{1}{n}\right)<2$

$= 1+\sum_{k=2}^{n}\frac{1}{k-1}-\frac{1}{k} = (1-\cancel{\tfrac{1}{2}})+(\cancel{\tfrac{1}{2}}-\cancel{\tfrac{1}{3}})+(\cancel{\tfrac{1}{3}}-\cancel{\tfrac{1}{4}})+\cdots(\cancel{\tfrac{1}{n-1}}-\tfrac{1}{n})$

$= 1+1-\frac{1}{n}<2$

2.4 Absolute values. Estimating size

Def $\quad |\,a\,| \overset{\text{def}}{=} \begin{cases} a & \text{if } a\ge 0\\ -a & \text{if } a<0\end{cases}$

Two good ways to think about absolute value:

- Absoulte value measures magnitude: $|\,a\,|$ is the size of a
- Absoulte value measures distance: $|\,a-b\,|$ is the distance between a & b

Easy fact

$$|\,a\,|\le M \quad\Leftrightarrow\quad -M\le a\le M$$
$$K\le a\le L\quad\Rightarrow\quad |\,a\,|\le M,\quad \text{where } M=\max\{|\,K\,|,|\,L\,|\}$$

Pf. $\quad -M\le -|\,K\,|\le K\le a\le L\le |\,L\,|\le M$

$\to \; |a-b| + |b| \geq |a|$

$|a| = |a-b+b| \leq |a-b| + |b|$, by triangular inequality

Absolute value laws

- multiplication law: $\mid ab \mid = \mid a \mid\mid b \mid$, $\quad \mid \dfrac{a}{b} \mid = \dfrac{\mid a \mid}{\mid b \mid}$ if $b \neq 0$
- triangle inequality: $\mid a + b \mid \leq \mid a \mid + \mid b \mid$
- extended triangle $\neq$: $\mid a_1 + a_2 + \cdots + a_n \mid \leq \mid a_1 \mid + \mid a_2 \mid + \cdots + \mid a_n \mid$
- difference form of triangle $\neq$: $\mid a - b \mid \geq \mid a \mid - \mid b \mid$, $\quad \mid a + b \mid \geq \mid a \mid - \mid b \mid$

  Or (by the symmetry of LHS): $|a-b| \geq \| a \mid - \mid b \|$, $\quad |a+b| \geq \| a \mid - \mid b \|$

$*$ To estimate the size, we have to use $\mid \quad \mid$:

$\to \; |a+b| \geq |a| - |b|$

$\Rightarrow \; |a-(-b)| \geq |a| - |b|$

to show is small in size, show $\mid a \mid < ($ a small number$)$

to show is large in size, show $\mid a \mid > ($ a large number$)$

Warning: to show a_n is small(in size), it does no good to show $a_n < \dfrac{1}{n}$

($\because$ a_n can be negatively large)

instead, have to show $\mid a_n \mid < \dfrac{1}{n}$

Ex. $\quad S_n = \dfrac{1}{2}\cos t + \dfrac{1}{2^2}\cos 2t + \cdots + \dfrac{1}{2^n}\cos nt$

Give an upper estimate for the size of S_n

Sol. By the extended triangle $\neq$,

$$\boxed{\begin{aligned} \mid S_n \mid &\leq \frac{1}{2} \mid \cos t \mid + \frac{1}{2^2} \mid \cos 2t \mid + \cdots + \frac{1}{2^n} \mid \cos nt \mid \\ &\leq \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} < 1 \quad (\text{for all } n) \end{aligned}}$$

Ex. $\quad \mid a \mid \geq 2 \quad \& \quad \mid b \mid \leq \dfrac{1}{2} \quad \Rightarrow \quad \mid a + b \mid \geq \boxed{\text{a good lower estimate ?}}$

Sol. $\quad \mid a + b \mid \geq \mid a \mid - \mid b \mid \geq 2 - \dfrac{1}{2} = \dfrac{3}{2}$

Warning (범하기 쉬운 추정과정의 실수)

- $\mid \sin n - \cos n \mid \geq \mid \sin n \mid - \mid \cos n \mid \geq 0 - 1 = -1$ (meaningless)

  절대값을 씌웠으니 당연히 음수보단 크거나 같으니까 ;;
- $\mid \pi - 3.14 \mid \leq \mid \pi \mid + 3.14 < 3.16 + 3.14 = 6.3$ (useless)
- $\mid a - b \mid \leq \mid a \mid - \mid b \mid$ (nonsense) (even for the case $\mid a \mid > \mid b \mid$)

Proposition

(a_n) is bounded $\Leftrightarrow$ $\exists\ B > 0$ such that $|\ a_n\ | \leq B$ for all n

Pf. $\Leftarrow$: trivial $\left(\because\ \ |\ a_n\ | \leq B \quad \Leftrightarrow \quad -B \leq a_n \leq B\right)$

$$\Rightarrow: \quad (a_n):\ \text{bdd} \quad \Rightarrow \quad \exists K \ \ \& \ \ L \quad \text{such that } K \leq a_n \leq L \quad \text{forall } n$$
$$\Rightarrow \quad |\ a_n\ | \leq \max(|\ K\ |, |\ L\ |) \equiv B \quad \text{for all } n$$

2.5 Approximations

In scientific work, one often write $a \approx b$ to mean that a & b are approximately equal.

The notation $a \approx b$ has no exact mathematical meaning

A slight modification of the notation:

$$a \ \ \& \ \ b \text{ are within } \varepsilon \text{ of each other} \overset{\text{the standard way of writing this}}{\Leftrightarrow} |a - b| < \varepsilon$$
$$\Leftrightarrow \quad \boxed{a \underset{\varepsilon}{\approx} b} \quad \text{(is often useful)}$$

where ε is some positive number (invariably thought of as small)

a와 b 가 근사적으로 가깝고, 차이는 ε 보다 작다.

Ex.

For $a > 0$,

$$a \underset{\text{decimal rep}}{=} a_0.a_1a_2a_3\cdots a_n\cdots$$

$$a^{(n)} \overset{def}{=} \underbrace{a_0.a_1a_2a_3\cdots a_n}_{n-\text{th truncation of } a} \text{ (it is a rational number)}$$

$\therefore$ $a^{(n)}$ is an approximation to a by a rational number

How close are they ?

Ans. $$a \underset{\varepsilon}{\approx} a^{(n)}, \quad \text{where } \varepsilon = \frac{1}{10^n}$$

For example, since $\pi = 3.14159\cdots$,

$$\pi \underset{0.1}{\approx} 3.1, \quad \pi \underset{0.01}{\approx} 3.14 \quad (\text{Actually } \pi \underset{0.05}{\approx} 3.1, \quad \pi \underset{0.002}{\approx} 3.14)$$

- Well-known fact

유리수집합 $\mathbb{Q}$: 사칙연산에 관해 닫혀있다 (단, 0으로 나누는 것은 제외)

$\sqrt{2}$: an irrational number

※ Archimedian Property

Let $\varepsilon > 0$ (small). Then for any **(large)** $a > 0$, $\exists\, N \in \mathbb{N}$ such that $N\varepsilon > a$

어떠한 작은 수 ε 라도 어떠한 자연수 N번 더하면, 임의의 큰 수 a보다 커진다.

Or

Let $\varepsilon > 0$ (small). Then for any **(large)** $a > 0$, $\exists\, n \in \mathbb{N}$ such that $10^n\varepsilon > a$

Theorem 2.5 Suppose $a, b \in \mathbb{R}$ with $a < b$. Then

(i) $\exists\, r \in \mathbb{Q}$ such that $a < r < b$ 서로 다른 숫자 a, b가 있을 때, a, b 사이에 적어도 하나의 유리수가 존재하며,

(ii) $\exists\, s \in \mathbb{Q}^c$ such that $a < s < b$ 서로 다른 숫자 a, b가 있을 때, a, b 사이에 적어도 하나의 무리수가 존재한다.

Pf. (i) $\boxed{\text{First assume } b > 0}$

If b is rational, we can choose n so large that $a + \dfrac{1}{10^n} < b$. This is possible

($\because$ may assume $(0) < b - a < 1$ (otherwise, the assertion is trivial)

$$\therefore \quad b - a = 0.0\cdots0 * \bullet\cdots \qquad (*: \text{ the first nonzero digit})$$
$$\geq 0.0\cdots0 *$$

, $*$는 1~9 중에 하나인 숫자이고, 이 중 가장 작은건 1 이니까

$$\geq 0.\underbrace{0\cdots01}_{m\text{개의 digits}} = \frac{1}{10^m} \quad (\text{some } m) > \frac{1}{10^n} \quad \text{if } n > m \quad)$$

Thus $a < b - \dfrac{1}{10^n} < b \qquad \therefore\ b - \dfrac{1}{10^n} \overset{\text{let}}{=} r$: rational ($\therefore$ OK)

If b is not rational, we can choose n so large that

$$\frac{1}{10^n} < b - a \qquad ------(\#1) \Rightarrow$$

$-n\ln 10 < \ln(b-a)$
$n\ln 10 > -\ln(b-a)$
$n > \dfrac{-\ln(b-a)}{\ln 10}$

(possible by taking a natural number n such that $n > -\dfrac{\ln(b-a)}{\ln 10}$)

Note that $$|\, b - b^{(n)}\,| \overset{\overset{b>0}{\downarrow}}{=} b - b^{(n)} < \frac{1}{10^n} \qquad ------(\#2)$$

$\underbrace{a < b - \tfrac{1}{10^n}}_{\#1}$, $\underbrace{b - \tfrac{1}{10^n} < b^{(n)}}_{\#2} < b$

$(\#1)$ & $(\#2)$ imply $\quad a < \underbrace{b^{(n)}}_{\|\text{take as } r} < b \quad (\therefore \text{ OK})$

If $b \leq 0$, then $\exists$ an integer N such that $b + N > 0$

$$\overset{\text{previous case}}{\Rightarrow} \quad \exists\, r \in \mathbb{Q} \text{ such that } a + N < r < b + N$$
$$\therefore \quad a < r - N (= \text{rational number}) < b$$

무리수

(ii) From (i) we have $a < r < b$, where $r \in \mathbb{Q}$.

$a < r < r + \frac{\sqrt{2}}{n} < b$

$\hookrightarrow b - r > \frac{\sqrt{2}}{n}$

$n > \frac{\sqrt{2}}{b-r}$

Choose n so large that $n > \dfrac{\sqrt{2}}{b-r}$ $(b - r > 0)$. Then

$$a < r < r + \frac{\sqrt{2}}{n} \left(\stackrel{\text{let}}{=} s : \text{irrational} \right) < b \quad (\therefore \text{ OK})$$

Alternative popular way of proving Theorem 2.5 (by means of Archimedian Property)

(i) $\exists\, r \in \mathbb{Q}$ such that $a < r < b$

Pf. Case1. $b > 0$ ($\&$ $a < b$)

By AP (Archimedian Property),

$$\exists n \in \mathbb{N} \text{ such that } \underset{n\varepsilon > 1,\ \varepsilon > \frac{1}{n}}{b - a > \frac{1}{n}} \left(\text{i.e., } a - b < -\frac{1}{n} \right) \quad \text{---}(\odot)$$

Again by AP applied to the positive number $\dfrac{1}{b}(\backsim \varepsilon)$, we see that $\exists(\text{big})\, m \in \mathbb{N}$ such that $\dfrac{1}{b} m \geq n$

$$\text{i.e., } b \leq \frac{m}{n} \quad \text{for some big } m \in \mathbb{N}$$

Let m be the **smallest** positive integer such that $b \leq \dfrac{m}{n}$. Then we have

$$\frac{m-1}{n} < b \qquad \text{--- ①}$$

$b \leq \frac{m}{n}$

$a - b < -\frac{1}{n}$

$a - b + b < \frac{m}{n} - \frac{1}{n}$

and by $(\odot)$

$$a = b + (a - b) < \frac{m}{n} - \frac{1}{n} = \frac{m-1}{n} \qquad \text{--- ②}$$

$$\text{② \& ①} \quad \Rightarrow \quad a < \underbrace{\frac{m-1}{n}}_{=\text{rational number}} < b$$

Case2. $b \leq 0$ ($\&$ $a < b$)

By AP (taking $\varepsilon = 1$), we can choose a positive integer n such that $n \cdot 1 > \underbrace{-b}_{\geq 0}$

$$\text{i.e., } b + n > 0 \quad \text{for some } n \in \mathbb{N}$$

$$\stackrel{\text{Case1}}{\Rightarrow} \quad \exists r' \in \mathbb{Q} \text{ such that } a + n < r' < b + n$$

$$\therefore \quad a < \underbrace{r' - n}_{\in \mathbb{Q}} < b$$

In any case, we proved (i)

(ii) $\exists\, s \in \mathbb{Q}^c$ such that $a < s < b$

Pf. $a < b \quad \Rightarrow \quad a - \sqrt{2} < b - \sqrt{2}$

$$\stackrel{(i)}{\Rightarrow} \quad \exists s' \in \mathbb{Q} \text{ such that } a - \sqrt{2} < s' < b - \sqrt{2}$$

$$\text{i.e., } a < s' + \sqrt{2} =: s < b \quad (\text{with } s \in \mathbb{Q}^c)$$

◊ Laws for calculating with approximations

① transitive law: $a \underset{\varepsilon}{\approx} b \quad \& \quad b \underset{\varepsilon'}{\approx} c \quad \Rightarrow \quad a \underset{\varepsilon+\varepsilon'}{\approx} c$

② addition law: $a \underset{\varepsilon}{\approx} a' \quad \& \quad b \underset{\varepsilon'}{\approx} b' \quad \Rightarrow \quad a+b \underset{\varepsilon+\varepsilon'}{\approx} a'+b'$

$|ab - a'b'| = |ab - ab' + ab' - a'b'| \leq |ab - ab'| + |ab' - a'b'|$

$= |a|\cdot|b-b'| + |b'|\cdot|a-a'| < |a|\varepsilon + |b'|\varepsilon$

③ multiplication law: $a \underset{\varepsilon}{\approx} a' \quad \& \quad b \underset{\varepsilon'}{\approx} b' \quad \Rightarrow \quad ab \underset{?}{\approx} a'b'$

$|a^2 - b^2| = |a+b||a-b| < \varepsilon|a+b|$

④ power law: $a \underset{\varepsilon}{\approx} b$ with $a+b \neq 0 \quad \Rightarrow \quad a^2 \underset{|a+b|\varepsilon}{\approx} b^2$

⑤ reciprocal law: $a \underset{\varepsilon}{\approx} b$ with $a \;\&\; b \neq 0 \quad \Rightarrow \quad \dfrac{1}{a} \underset{\frac{\varepsilon}{|ab|}}{\approx} \dfrac{1}{b}$ $\quad \left|\dfrac{1}{a} - \dfrac{1}{b}\right| = \left|\dfrac{b-a}{ab}\right| < \dfrac{\varepsilon}{|ab|}$

Pf. Easy exercise

2.6 The terminology "for n large"

In estimating or approximating the terms of a seq (a_n), sometimes the estimate is **not valid for all terms of the seq.**

Eg A. Let $a_n = \dfrac{5n}{n^2-2}$, $n \geq 1$. For what n is $|a_n| < 1$?

Sol. For $n = 1$, the estimate is not valid

If $n > 1$, then $a_n > 0$.

$$\therefore \quad |a_n| = a_n = \frac{5n}{n^2-2} < 1 \text{ holds} \quad \Leftrightarrow \quad 5n < n^2 - 2 \quad \Leftrightarrow \quad 5 < n - \frac{2}{n}$$

It is clear $5 < n - \dfrac{2}{n}$ holds for all $n \geq 6$. Therefore, $|a_n| < 1$ for all $n \geq 6$.

Eg B. Let $a_n = \dfrac{n^2+2n}{n^2-2}$. For what n is $a_n \underset{0.1}{\approx} 1$?

$$\text{Sol.} \quad |a_n - 1| = \left|\frac{n^2+2n}{n^2-2} - 1\right| \underset{n\geq 2}{=} \frac{2n+2}{n^2-2} < 0.1 = \frac{1}{10}$$

$$\Updownarrow$$

$$n^2 - 2 > 20n + 20 \quad \Leftrightarrow \quad n(n-20) > 22$$

By inspection, the last inequality holds for $n \geq 22$

※ Def. The sequence (a_n) has the property P **for n large** if
$\exists$ a number N such that a_n has the property P for all $n \geq N$

One can say instead **for large** n, **for** n **sufficiently large**, etc

We will use the symbolic notation $\boxed{\text{for } n \gg 1}$

Note: In the definition of the above, N need not be an integer.

Return to Eg A & Eg B:

If $a_n = \dfrac{5n}{n^2 - 2}$, then $|\, a_n \,| < 1$ for $n \gg 1$

If $a_n = \dfrac{n^2 + 2n}{n^2 - 2}$, then $a_n \underset{0.1}{\approx} 1$ for $n \gg 1$

Remark. $a \gg b$, with $a, b > 0$, have the meaning that

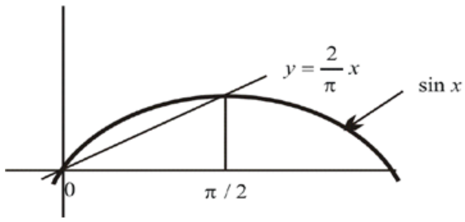
"a is relatively large compared with b", that is, $a\,/\,b$ is large

Thus we do not write "for $n \gg 0$";

Intuitively, every positive integer is relatively large compared with 0

Eg C. Show that the sequence $\left(\sin \dfrac{10}{n}\right)$ is (strictly) decreasing for large n

Pf.



$\sin x$ is strictly increasing on the interval $0 < x < \pi / 2$

$$\therefore \quad 0 < a < b < \pi / 2 \quad \Rightarrow \quad \sin a < \sin b$$

$$\therefore \quad \sin \frac{10}{n+1} < \sin \frac{10}{n} \quad \text{if} \quad \frac{10}{n} < \frac{\pi}{2}$$

$$i.e., \quad \text{if} \quad n > \frac{20}{\pi} = 6.***$$

Take $N = \dfrac{20}{\pi}$ or take $N = 7(\text{the first integer after } \dfrac{20}{\pi})$

Eg D If (a_n) is bounded above for $n \gg 1$, it is bounded above

Pf. By hypo, $\exists$ B & N (we may take N to be an integer) such that

$$a_n \le B \qquad \text{for } n \ge N$$

Let B' be a number greater than $a_0, a_1, \cdots, a_N$ & B. Then

$$\begin{cases} a_n < B' \quad \text{for} \quad n = 0,1,2,\cdots,N \\ \& \\ a_n \le B < B' \quad \text{for} \quad n \ge N \end{cases} \qquad \therefore \quad a_n < B' \quad \text{for all} \quad n \ge 0$$

Eg E. Let (a_n) & (b_n) be $\uparrow$ for $n \gg 1$. Prove that $(a_n + b_n)$ is $\uparrow$ for $n \gg 1$

Pf. By hypo, $\exists$ numbers N_1 $\&$ N_2 such that

$$a_n \le a_{n+1} \quad \text{for } n \ge N_1 \qquad \& \qquad b_n \le b_{n+1} \quad \text{for } n \ge N_2$$

Choose any $N \ge N_1, N_2$ (for example, $N = \max(N_1, N_2)$).

Then $a_n \le a_{n+1}$ & $b_n \le b_{n+1}$ for $n \ge N$

$\therefore$ $a_n + b_n \le a_{n+1} + b_{n+1}$ for $n \ge N$

i.e., $(a_n + b_n)$ is $\uparrow$ for $n \gg 1$

Remark: Most of the time, we don't want to have to specify exactly what N is.

※ We can use the terminology " for n large" to weaken the hypothesis of

the **completeness Property of $\mathbb{R}$**

Suppose (a_n) is $\uparrow$ & bdd above.

In finding its limit, we see that

how the sequence behaves near its beginning is not important.

the early terms would be changed, but the limit would stay the same

Indeed, if the sequence is bounded (above), but it is increasing only after some term a_N

i.e., $a_n \le a_{n+1}$ for $n \ge N$,

it still has a limit, & exactly the same procedure we used before will produce it.

The same observation applies to $\downarrow$ & bounded (below) sequences.

Consequently, we are lead to a slightly **general form** of the **completeness Property of $\mathbb{R}$:**

$\boxed{\text{A sequence which is bounded \& monotone for } n \gg 1 \text{ have a limit}}$

※ Proposition

Suppose that a $\&$ b are two numbers . Then

$$\forall \varepsilon > 0 \quad (i.e., \text{for every } \varepsilon > 0), \quad a \underset{\varepsilon}{\approx} b \qquad \Rightarrow \quad a = b$$

Equivalently,

$$\forall n \in \mathbb{N}, \quad a \underset{1/n}{\approx} b \;\; [\text{i.e., } |\, a - b \,| < 1/n] \qquad \Rightarrow \quad a = b$$

Pf. Suppose $a \ne b$. Then $|\, a - b \,| > 0$.

Choose $\varepsilon = \dfrac{|\, a - b \,|}{2}$. Then by hypo $|\, a - b \,| < \dfrac{|\, a - b \,|}{2}$: a contradiction