Homework II (2022)

1. Assume $y_1, \ldots, y_n \overset{iid}{\sim} N(\theta, 1/r)$ where θ and r are unknown. We consider the following priors for θ and r:

$$\theta|r \sim N\left(\mu, \frac{1}{\lambda r}\right),$$
$$r \sim Gamma\left(\frac{a}{2}, \frac{b}{2}\right),$$

where μ, λ, a and b are known. Show the following results:

$$\theta|r, y \sim N\left(\frac{n\bar{y} + \lambda\mu}{n + \lambda}, \frac{1}{r(n + \lambda)}\right),$$
$$r|y \sim Gamma\left(\frac{n + a}{2}, \frac{1}{2}\left\{\sum_i (y_i - \bar{y})^2 + \frac{n\lambda}{n + \lambda}(\bar{y} - \mu)^2 + b\right\}\right).$$

2. Suppose $y|\theta \sim Poisson(\theta)$. Find Jeffreys' prior density for θ, and then find α and β for which the Gamma(α, β) density is a close match to Jeffreys' density.

3. Assume our data $X_1, \cdots, X_n$ are i.i.d. Bernoulli(θ) with unknown θ, let $y = \sum_{i=1}^n x_i$. We derived in class the Jeffreys prior $p_J(\theta)$ for θ. Consider the transformation $\tau = \theta/(1 - \theta)$, so that τ is the *odds* of success.

   (a) Show that the joint density of the data (i.e., the likelihood) can be written in terms of τ as
   $$f(x|\tau) \propto \tau^y(\tau + 1)^{-n}.$$

   (b) Derive the Jeffreys prior $p_J(\tau)$ for τ.

   (c) Show that the Jeffreys prior for θ is invariant with respect to this particular transformation by showing that
   $$p_J(\theta) = p_J(\tau)\left|\frac{d\tau}{d\theta}\right|.$$

4. Suppose y has a binomial distribution for given n and unknown θ, where the prior distribution of θ is Beta(α, β).

   (a) Find the marginal distribution of y, $p(y)$, for $y = 0, \ldots, n$ (unconditional on θ). This discrete distribution is known as the *beta-binomial*, for obvious reasons.

   (b) Show that if the beta-binomial probability is constant in y, then the prior distribution has to have $\alpha = \beta = 1$.