1.

$$\Sigma = \begin{pmatrix} \sigma^2 & \rho\sigma^2 & \rho\sigma^2 & \cdots & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 & \rho\sigma^2 & \cdots & \rho\sigma^2 \\ \vdots & \vdots & \vdots & \ddots & \\ \rho\sigma^2 & \rho\sigma^2 & \rho\sigma^2 & \cdots & \sigma^2 \end{pmatrix}_{n\times n}.$$

$$= \sigma^2(1-\rho)I + \rho\sigma^2 J.$$

Then $\bar{\sigma}_{\cdot\cdot} = \frac{1}{n^2}\left(n\sigma^2 + 2\times\frac{n(n-1)}{2}\rho\sigma^2\right)$

$$= \frac{1}{n^2}\{n\sigma^2(1+(n-1)\rho)\}$$

$$= \frac{\sigma^2}{n}(1+(n-1)\rho).$$

$$\bar{\sigma}_{ii} = \frac{1}{n} n\sigma^2 = \sigma^2$$

$$\bar{\sigma}_{i\cdot} = \frac{1}{n}(\sigma^2 + (n-1)\rho\sigma^2) = \frac{\sigma^2}{n}(1+(n-1)\rho) = \bar{\sigma}_{\cdot\cdot}$$

and $S = n\sigma^4 + n(n-1)\rho^2\sigma^4 = n\sigma^4(1+(n-1)\rho^2)$

Therefore, the numerator of ε is

$$n^2(\bar{\sigma}_{ii} - \bar{\sigma}_{\cdot\cdot})^2 = n^2\left(\sigma^2 - \frac{\sigma^2}{n}(1+(n-1)\rho)\right)^2$$

$$= \sigma^4(n-1)^2(1-\rho)^2$$

and the denominator of ε is

$$(n-1)\left(S - 2n\sum_{i=1}^{n}\bar{\sigma}_{i\cdot}^2 + n^2\bar{\sigma}_{\cdot\cdot}^2\right) = (n-1)\left(S - 2n^2\bar{\sigma}_{\cdot\cdot}^2 + n^2\bar{\sigma}_{\cdot\cdot}^2\right)$$

$$= (n-1)(S - n^2\bar{\sigma}_{\cdot\cdot}^2) = (n-1)\{n\sigma^4(1+(n-1)\rho^2) - \sigma^4(1+(n-1)\rho)^2\}$$

$$= (n-1)\sigma^4\{n + n(n-1)\rho^2 - (1+(n-1)\rho)^2\}.$$

$$= (n-1)\sigma^4 \{ n + n^2\rho^2 - n\rho^2 - (1 + 2(n-1)\rho + (n-1)^2\rho^2) \}$$

$$= (n-1)\sigma^4 \{ n-1 - 2n\rho + 2\rho + n\rho^2 - \rho^2 \}$$

$$= (n-1)^2\sigma^4 (1 - 2\rho + \rho^2)$$

$$= (n-1)^2\sigma^4 (1-\rho)^2$$

Thus $\varepsilon = \dfrac{\sigma^4 (n-1)^2 (1-\rho)^2}{(n-1)^2 \sigma^4 (1-\rho)^2} = 1.$

2. $$\underset{\sim}{y_i} = X\underset{\sim}{\beta} + \underset{\sim}{Z}\gamma_i + \underset{\sim}{\varepsilon_i}$$

where $$\underset{\sim}{Z} = \underset{\sim}{1}_n, \quad \gamma_i \overset{ind}{\sim} N(0, \sigma^2), \quad \underset{\sim}{\varepsilon_i} \overset{iid}{\sim} N(0, \sigma_e^2 I_n)$$

$$Var(\underset{\sim}{y_i}) = \underset{\sim}{Z}\, Var(\gamma_i)\, \underset{\sim}{Z}^T + Var(\underset{\sim}{\varepsilon_i}) = \underset{\sim}{1}_n \sigma^2 \underset{\sim}{1}_n^T + \sigma_e^2 I_n$$

$$= \sigma^2 J_n + \sigma_e^2 I_n$$

$$= \begin{pmatrix} \sigma^2+\sigma_e^2 & \sigma^2 & \sigma^2 & \cdots & \sigma^2 \\ & \sigma^2+\sigma_e^2 & \sigma^2 & \cdots & \sigma^2 \\ & & & \ddots & \vdots \\ & \text{sym} & & & \sigma^2+\sigma_e^2 \end{pmatrix}$$

3.

$$y_{ij} \mid a_i \overset{\text{indep}}{\sim} N(\mu + a_i, \sigma^2), \quad i=1,\cdots,m; \ j=1,\cdots,n$$

$$a_i \overset{\text{iid}}{\sim} N(0, \sigma_a^2)$$

$$\text{Intraclass correlation coefficient } (\rho) = \frac{\sigma_a^2}{\sigma_a^2 + \sigma^2}$$

The ANOVA table for this design is

Source	S.S	df	MS	E(MS)
Group	SS_{Group}	$m-1$	$SS_{Group}/(m-1)$	$n\sigma_a^2 + \sigma^2$
Error	SSE	$nm-m$	$SSE/(nm-m)$	σ^2
Total	SS_T	$mn-1$		

where
$$SS_T = \sum_i \sum_j (y_{ij} - \bar{y}_{..})^2$$
$$SS_{Group} = \sum_i \sum_j (\bar{y}_{i.} - \bar{y}_{..})^2$$
$$SSE = SS_T - SS_{Group}.$$

It can be shown that

$$\frac{SSE}{\sigma^2} \sim \chi^2_{(nm-m)}$$

$$\frac{SS_{Group}}{\sigma^2 + n\sigma_a^2} \sim \chi^2_{(m-1)}$$

and SSE and SS_{Group} are indep.

Therefore, $$MS_{Group} \sim (\sigma^2 + n\sigma_a^2)\frac{\chi^2_{(m-1)}}{m-1}$$

$$MS_E \sim \frac{\sigma^2 \chi^2_{(mn-m)}}{mn-m}$$

and MS_{Group}, MS_E are indep.

$$\Rightarrow \frac{MS_{Group}}{MS_E} \sim \underbrace{\frac{\sigma^2 + n\sigma_a^2}{\sigma^2}}_{\substack{\shortparallel \\ 1+n\theta}} F_{(m-1,\, nm-m)}$$

where $\theta = \frac{\sigma_a^2}{\sigma^2} = \frac{\rho}{1-\rho}$

$$\Rightarrow \frac{MS_{Group}/MSE}{1+n\theta} \sim F_{(m-1,\, nm-m)}$$

$$\Rightarrow P\left(F_{\alpha/2}(m-1, nm-m) \le \frac{MS_{Group}/MSE}{1+n\theta} \le F_{1-\frac{\alpha}{2}}(m-1, nm-m)\right) = 1-\alpha$$

$$\Rightarrow P\left(F_{\alpha/2}(m-1, nm-m)^{-1}\frac{MS_{Group}}{MS_E} \ge 1+n\theta \ge F_{1-\frac{\alpha}{2}}(m-1, nm-m)^{-1}\frac{MS_{Group}}{MS_E}\right) = 1-\alpha$$

$\Rightarrow P(U \ge \theta \ge L) = 1-\alpha$ where $U = \frac{1}{n}\left[F_{\frac{\alpha}{2}}(m-1, nm-m)^{-1}\frac{MS_{Group}}{MS_E} - 1\right]$

and $L = \frac{1}{n}\left[F_{1-\frac{\alpha}{2}}(m-1, nm-m)^{-1}\frac{MS_{Group}}{MS_E} - 1\right]$

Since $\rho = \frac{\theta}{1+\theta}$, this 95% C.I. for θ can be transformed to a 95% C.I. for ρ as follows

$$\left[\frac{L}{1+L}, \frac{U}{1+U}\right]$$

4. (a) We need L to be such that "sweeping" each row of L down the column vector $\underset{\sim}{\mu}$ gives the corresponding element of H_0. Note that $\mu_1 - \mu_2 = 1\cdot\mu_1 + (-1)\mu_2 + 0\cdot\mu_3 + 0\cdot\mu_4 + 0\cdot\mu_5$, so the first row of L should be $(1, -1, 0, 0, 0)$. The other rows are similar, leading to

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

(b) Using the same reasoning as in (a), the first row of this L should be the same as above. The second row should be $\mu_2 - \mu_3 = 0\cdot\mu_1 + 1\cdot\mu_2 + (-1)\mu_3 + 0\cdot\mu_4 + 0\cdot\mu_5$. The entire matrix looks like

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

It is clear that both this hypothesis and that in (a) are addressing the same issue. Both correspond to asking whether $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5$, that is, they both ask whether all five means are equal to the same value.
Note: Neither hypothesis corresponds to $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0$.

(c) We can immediately find U as the transpose of the matrix L in (b); that is

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Alternatively, we could have started from first principles and gone through the same reasoning as in (b) but with columns instead of rows.

(d) Note that we want to write H_0 as a (1×4) vector; i.e.,

$$H_0: \quad \mu_1 - \frac{1}{4}\mu_2 - \frac{1}{4}\mu_3 - \frac{1}{4}\mu_4 - \frac{1}{4}\mu_5 = 0$$

$$\mu_2 - \frac{1}{3}\mu_3 - \frac{1}{3}\mu_4 - \frac{1}{3}\mu_5 = 0$$

$$\mu_3 - \frac{1}{2}\mu_4 - \frac{1}{2}\mu_5 = 0$$

$$\mu_4 - \mu_5 = 0 .$$

Now $\underset{\sim}{\mu}^T = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$, so we want to find U such that "sweeping" the row $\underset{\sim}{\mu}^T$ down each column of U gives these elements (which are columns of a row vector). We get

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{3} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{3} & -\frac{1}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{3} & -\frac{1}{2} & -1 \end{pmatrix}$$