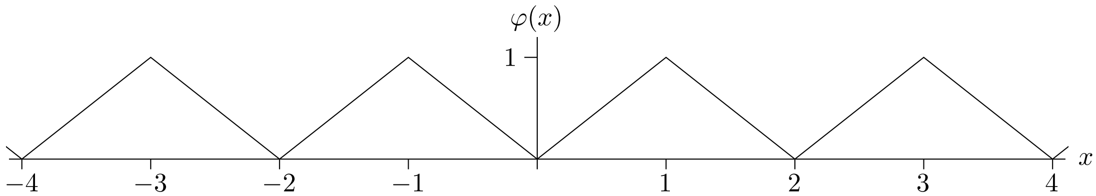
A Nowhere Differentiable Continuous Function

These notes contain a standard[1] example of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous everywhere but differentiable nowhere. Define the function $\varphi : \mathbb{R} \to \mathbb{R}$ by the requirements that $\varphi(x) = |x|$ for $x \in [-1, 1]$ and that $\varphi(x+2) = \varphi(x)$ for all real x. So φ is periodic of period 2.



Now define

$$f(x) = \sum_{n=0}^{\infty} \left(\tfrac{3}{4}\right)^n \varphi(4^n x)$$

As $|\varphi(x)| \le 1$, the series converges uniformly by the Weiersrtrass M–test with $M_n = \left(\frac{3}{4}\right)^n$. As φ is a continuous function, $f(x)$ is a uniform limit of continuous functions and hence is continuous.

We now fix any $x \in \mathbb{R}$ and prove that f is *not* differentiable at x by exhibiting a sequence $\{h_m\}_{m\in\mathbb{N}}$ of real numbers converging to 0 such that $\frac{1}{h_m}\big[f(x+h_m) - f(x)\big]$ diverges as $m \to \infty$. In fact $h_m = \pm\frac{1}{2}4^{-m}$ with the sign chosen[2] so that there is no integer strictly between $4^m x$ and $4^m(x+h_m)$. We next compute the magnitude of the n^{th} term in $\frac{1}{h_m}\big[f(x+h_m) - f(x)\big]$. That is, we compute $|\gamma_{m,n}|$ where

$$\gamma_{m,n} = \tfrac{1}{h_m}\left(\tfrac{3}{4}\right)^n \big[\varphi(4^n x + 4^n h_m) - \varphi(4^n x)\big] = \pm 2(3^n)4^{m-n}\big[\varphi(4^n x \pm \tfrac{1}{2}4^{n-m}) - \varphi(4^n x)\big]$$

Case $n > m$: In this case $\frac{1}{2}4^{n-m}$ is an even integer. So $\gamma_{m,n} = 0$ because $\varphi(4^n x \pm \frac{1}{2}4^{n-m}) = \varphi(4^n x)$ because φ has period 2.

Case $n = m$: Recall that the sign of h_m was chosen so that so that there is no integer strictly between $4^m x$ and $4^m(x+h_m)$. So $\big(4^m x, \varphi(4^m x)\big)$ and $\big(4^m(x+h_m), \varphi(4^m x + 4^m h_m)\big)$ lie on the same ramp (i.e. straight line segment) in the graph of φ, above. Each of those ramps has slope -1 or $+1$. So $\big|\varphi(4^m x + 4^m h_m) - \varphi(4^m x)\big| = 4^m |h_m| = \frac{1}{2}$ and $\big|\gamma_{m,n}\big| = 2(3^m)4^{m-m}\frac{1}{2} = 3^m$.

Case $n < m$: Since $\big|\varphi(y) - \varphi(x)\big| \le |y - x|$ for all $x, y \in \mathbb{R}$, we always have that

$$\big|\gamma_{m,n}\big| \le 2(3^n)4^{m-n}\tfrac{1}{2}4^{n-m} = 3^n$$

Putting these bounds together

$$\begin{aligned}\Big|\tfrac{1}{h_m}\big[f(x+h_m) - f(x)\big]\Big| &= \Big|\sum_{n=0}^{\infty}\gamma_{m,n}\Big| = \Big|\sum_{n=0}^{m}\gamma_{m,n}\Big| \ge |\gamma_{m,m}| - \sum_{n=0}^{m-1}|\gamma_{m,n}| \ge 3^m - \sum_{n=0}^{m-1}3^n = 3^m - \tfrac{1-3^m}{1-3} \\ &= \tfrac{1}{2}(3^m + 1)\end{aligned}$$

Sure enough, this diverges as $m \to \infty$. So f is not differentiable at x.

---

[1] This particular example is due to John McCarthy and appeared in the American Mathematical Monthly, Vol. LX, No. 10, December 1953. In 1872, Weierstrass gave the example $f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x)$ for $b < 1$ and $ab > 1 + \frac{3}{2}\pi$. It is discussed in *A Course in Mathematical Analysis* by E. Goursat (translated by E. R. Hedrick).

[2] To see that the sign may be chosen in this way, observe that $4^m[x+\frac{1}{2}4^{-m}] - 4^m[x-\frac{1}{2}4^{-m}] = 1$. Either $4^m[x+\frac{1}{2}4^{-m}]$ and $4^m[x-\frac{1}{2}4^{-m}]$ are both integers, in which case there are no integers in the open interval $(4^m[x-\frac{1}{2}4^{-m}], 4^m[x+\frac{1}{2}4^{-m}])$ and we may choose either sign for h_m. Or there is exactly one integer in the open interval $(4^m[x-\frac{1}{2}4^{-m}], 4^m[x+\frac{1}{2}4^{-m}])$. This one integer is either $4^m x$, in which case we may choose either sign for h_m, or is in $(4^m[x-\frac{1}{2}4^{-m}], 4^m x)$, in which case we choose $h_m = +\frac{1}{2}4^{-m}$, or is in $(4^m x, 4^m[x+\frac{1}{2}4^{-m}])$ in which case we choose $h_m = -\frac{1}{2}4^{-m}$.