Strict Stationarity - i) the mean is constant over time

ii) the covariance depends on the difference in time

Weak Stationarity - i) the second moment is finite

ii) the mean is constant over time

iii) the covariance does not depend on time t

$\hat{\gamma}(h) = \frac{1}{n} \sum_{j=1}^{n-h} (\bar{Y}_j - \bar{Y})(\bar{Y}_{j+h} - \bar{Y})$ => Estimation s.e. $\uparrow$ as $h \uparrow$ by the loss of information

=> Which is why we need parametric modeling

Moving Average of order q : $MA(q)$

$X_t = \theta(B) Z_t$

Auto-Regressive of order p : $AR(P)$

$X_t = \phi(B) Z_t + \phi^{p+1} X_{t-p-1}$

Conditions : - stationarity : the roots of $\phi(\cdot)$ are not on the unit circle

- causality : $\overset{AR(P)}{\phi(z)}$ has roots outside the unit circle

- invertibility : $\overset{MA(q)}{\theta(z)}$ has roots outside the unit circle

- identifiability : $\phi(z)$ and $\theta(z)$ has no common roots

$\frac{\bar{X} - \mu}{\sqrt{\nu / n}} \overset{d}{\sim} N(0,1)$, where $\nu = \sum_{h=-\infty}^{\infty} \gamma(h)$

=> 95% CI for μ : $\bar{X} \pm 1.96 \sqrt{\frac{\nu}{n}}$

ACVF of AR(P) : $\gamma(h) = Cov(X_t, X_{t+h}) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}$

Estimation of ACF :

- Asymptotic Normaltiy CLT

$$\begin{pmatrix} \hat{\rho}(1) \\ \hat{\rho}(2) \\ \vdots \\ \hat{\rho}(k) \end{pmatrix} \overset{d}{=} \mathcal{N}\left(\begin{pmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(k) \end{pmatrix}, \frac{1}{n} \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1k} \\ w_{21} & w_{22} & \dots & 2_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ w_{k1} & \dots & \dots & w_{kk} \end{pmatrix} \right)$$

$$w_{ij} = \sum_{k=1}^{\infty} \{\rho(k+i) + \rho(k-i) - 2\rho(i)\rho(k)\}$$
$$\times \{\rho(k+j) + \rho(k-j) - 2\rho(j)\rho(k)\}$$

is known as *Bartlett's* formula.

h-step ahead prediction : $\tilde{P}_t X_{t+h} = - \sum_{j=1}^{h-1} \pi_j X_{t+h-j} - \sum_{j=h}^{\infty} \pi_j X_{t+h-j}$

Mean-Square Prediction Error : $MSPE = E[(X_{t+h} - \tilde{P}_t X_{t+h})^2] = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2$

Prediction Interval : $\tilde{P}_n X_{n+h} \pm Z_{\alpha/2} \cdot \sqrt{MSPE}$

PACF : removing the effect of intermediate values

$$\alpha(0) = Corr(X_1, X_1) = 1$$

$$\alpha(1) = Corr(X_2, X_1) = \rho(1) = \frac{\gamma(h)}{\gamma(0)}$$

$$\vdots$$

$$\alpha(k) = Corr(X_{k+1} - P_k^* X_{k+1},\ X_1 - P_k^* X_1)$$

Yule - Walker Equation for $AR(P)$: given $AR(P)$ model, $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t$

$$E(X_{t-k} X_t) = E[X_{t-k}(\phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t)]$$

$$\gamma(k) = \phi_1 \gamma(k-1) + \phi_2 \gamma(k-2) + \cdots + \phi_p \gamma(k-P) + cov(Z_t, X_{t-k})$$

$$= \begin{cases} \phi_1 \gamma(k-1) + \phi_2 \gamma(k-2) + \cdots + \phi_p \gamma(k-P), & k \geq 1 \\ \phi_1 \gamma(-1) + \phi_2 \gamma(-2) + \cdots + \phi_p \gamma(-P) + \sigma^2, & k = 0 \end{cases}$$

$\Rightarrow$ then solve the system of linear equations for given $\phi_1, \phi_2, \ldots, \phi_p$

$$\underbrace{\begin{bmatrix} \gamma(1) \\ \gamma(2) \\ \vdots \\ \gamma(P-1) \\ \gamma(P) \end{bmatrix}}_{\gamma_p} = \underbrace{\begin{bmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(P-1) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(P-2) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(P-2) & & & \\ \gamma(P-1) & \gamma(P-2) & \cdots & \gamma(0) \end{bmatrix}}_{\Gamma_p} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{bmatrix}}_{\phi}$$

Yule - Walker Equation $\Rightarrow$ $\hat{\phi} = \hat{\Gamma}_p^{-1} \circ \hat{\gamma}_p$

$$\hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\phi}_1 \hat{\gamma}(1) - \cdots - \hat{\phi}_p \gamma(p)$$

$$\gamma(k) = \phi_1 \gamma(k-1) + \phi_2 \gamma(k-2) + \cdots + COV(Z_t,\ Z_{t-k} + \psi_1 Z_{t-k-1} + \psi_2 Z_{t-z-2} + \cdots)$$

$$\Rightarrow \sigma^2 = \gamma(0) - \phi_1 \gamma(1) - \phi_2 \gamma(2) - \cdots - \phi_{p-1} \gamma(p-1)$$

$$\vdots$$

95 % CI for ϕ_i's : since $\sqrt{n}(\hat{\phi} - \phi) \approx N(0, \hat{\sigma}^2 \hat{\Gamma}^{-1})$

$$\hat{\phi}_i \pm 1.96 \left(\frac{\hat{\sigma} \sqrt{\hat{\gamma}(0)}}{\sqrt{n}\, \hat{\Gamma}^{-1}} \right)$$

$$\psi(B)\phi(B) = \theta(B)$$