Quiz 4

#1. Suppose that $X_1, \dots, X_n$ is a random sample from $N(0, \frac{\theta}{2})$.

Find the MVUE of θ^2.

$$\rightarrow f(x;\theta) = \frac{1}{\sqrt{2\pi \cdot \frac{\theta}{2}}} \exp\left(-\frac{x^2}{2 \cdot \frac{\theta}{2}}\right)$$

$$= \exp\Big(\underbrace{-\frac{x^2}{\theta}}_{p(\theta)k(x)\ :\ \text{S.S.}} - \log\sqrt{\pi} - \log\sqrt{\theta}\Big)$$

By property of exponential family, Σx_i^2 is CSS.

Since $\frac{X_i}{\sqrt{\theta/2}} \sim N(0,1)$, $\left(\frac{X_i}{\sqrt{\theta/2}}\right)^2$ follows $\chi^2(1)$.

$\Rightarrow \sum_{i=1}^{n} \left(\frac{X_i}{\sqrt{\theta/2}}\right)^2 \sim \chi^2(n)$

Let $Y = \Sigma X_i^2$. Then $Y \sim \frac{\theta}{2}\chi^2(n)$.

$$E(Y^2) = Var(Y) + (E(Y))^2 = \left(\frac{\theta}{2}\right)^2 \cdot 2n + \left(\frac{\theta}{2} \cdot n\right)^2$$

$$= \frac{n}{2}\theta^2 + \frac{n^2}{4}\theta^2 = \left(\frac{n}{2} + \frac{n^2}{4}\right)\theta^2 = \frac{n(n+2)}{4}\theta^2$$

$\therefore \frac{4Y^2}{n(n+2)} = \frac{4(\Sigma X_i^2)^2}{n(n+2)}$ is unbiased and a function of CSS.

$\frac{4(\Sigma X_i^2)^2}{n(n+2)}$ is the MVUE of θ^2 by Lehmann – Scheffe.

#2. $X_1, \dots, X_n$ be a random sample from a distribution with pdf $f(x;\theta) = \theta^2 x e^{-\theta x}$, $x>0$, $\theta>0$.

Find the MVUE of θ.

$\rightarrow$ $f(x;\theta) = \theta^2 x e^{-\theta x} = \exp(\underbrace{-\theta x}_{p(\theta)k(x)\ :\ \text{s.s.}} + \log x + 2\log\theta)$

By property of exponential family, Σx_i is CSS.

Let $Y = \sum_{i=1}^{n} X_i$

Since $X_1, \dots, X_n \overset{iid}{\sim} \text{Gamma}(2, \frac{1}{\theta})$, $Y = \Sigma X_i$ follows $\text{Gamma}(2n, \frac{1}{\theta})$.

Therefore $E(Y) = \frac{2n}{\theta}$.

$$E\left(\frac{1}{Y}\right) = \int_0^\infty \frac{1}{y} f(y)\,dy$$

$$= \int_0^\infty \frac{1}{y} \cdot \frac{\theta^{2n}}{\Gamma(2n)} y^{2n-1} e^{-\theta y}\,dy$$

$$= \frac{\theta^{2n}}{\Gamma(2n)} \underbrace{\int_0^\infty y^{(2n-1)-1} e^{-\theta y}\,dy}_{\Gamma(2n-1)\left(\frac{1}{\theta}\right)^{2n-1}} \quad \text{by Gamma integration}$$

$$= \frac{\theta^{2n}}{\Gamma(2n)} \cdot \Gamma(2n-1) \cdot \left(\frac{1}{\theta}\right)^{2n-1}$$

$$= \frac{\theta}{2n-1}$$

$\therefore$ $\frac{2n-1}{Y} = \frac{2n-1}{\Sigma X_i}$ is unbiased and a function of CSS.

$\frac{2n-1}{\Sigma X_i}$ is the MVUE of θ by Lehmann-Scheffe.