Quiz 2

\# 1. Suppose that Y_n is a χ^2 random variable with degrees of freedom n.
In this case, $\frac{Y_n}{n}$ converges to a in probability.
What is a?

$\Rightarrow$ $Y_n \overset{d}{=} X_1 + X_2 + \cdots + X_n$ where X_i's independently follow χ_1^2

where χ_1^2 means χ^2 with degree of freedom 1.

$$\frac{Y_n}{n} = \frac{\Sigma X_i}{n} = \overline{X}_n \xrightarrow{P} 1 \quad \text{by WLLN} \quad (\because E(X_i) = 1)$$

$$\therefore \frac{Y_n}{n} \xrightarrow{P} 1 \qquad \Rightarrow \quad a = 1.$$

\# 2. Suppose that $X_1, X_2, \ldots$ is a sequence of IID random variable having pdf $f(x) = 1$, $0 < x < 1$.
In this case, $\sqrt{n}(\log \overline{X}_n - a) \xrightarrow{d} N(0, b)$.
Find a and b.

$$\Rightarrow \quad \sqrt{n}\left(\overline{X}_n - \tfrac{1}{2}\right) \xrightarrow{d} N\left(0, \tfrac{1}{12}\right)$$

$$\left(\begin{array}{l} \because \ E(X) = \int_0^1 x\,dx = \frac{1}{2}. \ = \mu \\ \mathrm{Var}(X) = E(X^2) - [E(X)]^2 = \int_0^1 x^2 dx - \left[\int_0^1 x\,dx\right]^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} = \sigma^2 \end{array} \right)$$

Let $g(x) = \log x$, and by delta Method,

$$\sqrt{n}(g(\overline{X}_n) - g(\mu)) \xrightarrow{d} N(0, \sigma^2 (g'(\mu))^2)$$

$$\Rightarrow \quad \sqrt{n}(\log \overline{X}_n - \log \mu) \xrightarrow{d} N\left(0, \sigma^2 \cdot \left(\tfrac{1}{\mu}\right)^2\right)$$

$$\therefore \sqrt{n}\left(\log \overline{X}_n - \log \tfrac{1}{2}\right) \xrightarrow{d} N\left(0, \tfrac{1}{12} \cdot 2^2\right)$$

$$= N\left(0, \tfrac{1}{3}\right)$$

$$\therefore \quad a = \log \tfrac{1}{2} = -\log 2, \qquad b = \tfrac{1}{3}$$