Chap 22. Sequences and series of functions

22.1 Pointwise and uniform convergence

Basic questions for two popular series

for power series:

$$\text{Let} \quad \sum_0^\infty a_n x^n = f(x) \quad \text{or} \quad \text{assume} \quad f(x) \overset{\text{can be expressed as}}{=} \sum_0^\infty a_n x^n \quad \text{on} \quad (-R, R)$$

Are the following statements true?

$$f'(x) = \sum_1^\infty n a_n x^{n-1} \quad \text{on} \quad (-R, R)$$

$$\int_0^x f(t)\,dt = \sum_{n=0}^\infty \int_0^x a_n t^n\,dt \quad \text{on} \quad |x| < R$$

for "Fourier series"(= trigonometric series):

$$\text{Let} \quad f(x) = a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx) \quad \text{on} \quad [-\pi, \pi]$$

(periodic functions are usually represented not by power series but by trigonometric series)

$$\text{On } [-\pi, \pi], \quad f'(x) = ?$$

$$\int_{-\pi}^{\pi} f(x)\,dx = ?$$

Generally we should study the series of the form $\sum_0^\infty u_n(x)$, where each $u_n(x)$ is a function of some unspecified type.

⦿ Three standard questions about $\sum_0^\infty u_n(x)$.

1. If every $u_n(x)$ is conti on an interval I, on I is $\sum_0^\infty u_n(x)$ conti?

2. If every $u_n(x)$ is diff on an interval I, on I is $\sum_0^\infty u_n(x)$ diff?

   If so, does $\left(\sum_0^\infty u_n(x)\right)' = \sum_0^\infty u_n{}'(x)$ on I?

3. If every $u_n(x)$ is integrable on a compact interval $[a, b]$, on $[a, b]$ is $\sum_0^\infty u_n(x)$ integrable?

   If so, does $\int_a^b \sum_0^\infty u_n(x)\,dx = \sum_0^\infty \int_a^b u_n(x)\,dx$?

(Equivalent) reformulations for these problems are as follows:

Let $f_n(x) = \sum_0^n u_k(x)$. Then we may write $\sum_0^\infty u_k(x) = \lim_{n\to\infty} f_n(x)$.

So,

Question 1 is equivalent to:

Is $\lim_{n\to\infty} f_n(x)$ conti on I whenever each $f_n(x)$ is conti on I?

Question 2 is equivalent to:

Is $\lim_{n\to\infty} f_n(x)$ diff on I whenever each $f_n(x)$ is diff on I?

Moreover, if so, does

$$\left(\lim_{n\to\infty} f_n(x)\right)' = \lim_{n\to\infty} f_n{}'(x) \quad \text{for} \quad x \in I\,?$$

Question 3 is equivalent to:

Is $\lim_{n\to\infty} f_n(x)$ integrable on $[a,b]$ whenever each $f_n(x)$ is integrable on $[a,b]$?

Moreover, if so, does

$$\int_a^b \lim_{n\to\infty} f_n(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx \quad \text{for} \quad x \in I\,?$$

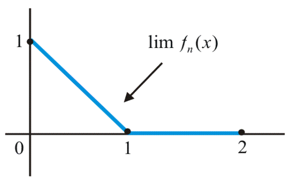
※ Each answer of these questions is no in general.

1. Let $f_n(x) = x^n$. Then each $f_n(x)$ is conti on $[0,1]$. However,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \underbrace{\begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}}_{\text{disconti at } x=1}$$

2. Let $f_n(x) = \dfrac{1-x}{1+x^n}$. Then each $f_n(x)$ is diff on $[0,2]$. However,

$$\lim_{n\to\infty} f_n(x) = \underbrace{\begin{cases} 1-x, & 0 \le x \le 1 \\ 0, & 1 < x \le 2 \end{cases}}_{\text{not diff at } x=1}$$



Another example: each $f_n(x) := |x|^{1+1/n}$ $(n \ge 1)$ is easily seen to be diff on $[-1,1]$. (Draw)

But $f(x) := \lim_{n\to\infty} f_n(x) = |x|$ is clearly not diff at $x = 0$.

More sophisticated example: $f_n(x) \equiv \dfrac{\sin(nx)}{n} (n \ge 1)$ $\rightarrow$ each $f_n(x)$ is diff on $(-\infty, \infty)$

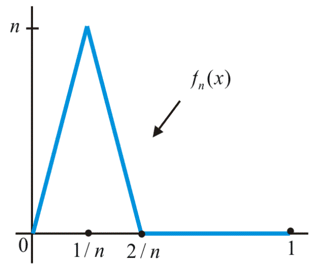
Moreover, $f_n'(x) = \cos nx$ $\quad\therefore\quad f_n'(0) = 1 \quad \forall n,$ so $\lim\limits_{n\to\infty} f_n'(0) = 1$

But clearly, $\lim\limits_{n\to\infty} f_n(x) = 0 \quad \forall x \in (-\infty, \infty) \quad \therefore \quad \left(\lim\limits_{n\to\infty} f_n(x)\right)' = 0 \quad \forall x \in (-\infty, \infty)$

$$\text{Thus} \quad \lim_{n\to\infty} f_n'(0) = 1 \neq 0 = \left(\lim_{n\to\infty} f_n(x)\right)'\Big|_{x=0}$$

3. Let $f_n(x)(n \geq 2)$ be defined by

$$f_n(x) = \begin{cases} n^2 x, & 0 \leq x \leq 1/n \\ 2n - n^2 x, & 1/n \leq x \leq 2/n \\ 0, & 2/n \leq x \leq 1 \end{cases}$$



Then each $f_n(x)$ is conti on $[0, 1]$. So each $f_n(x)$ is integrable on $[0, 1]$.

Obviously,

$$\int_0^1 f_n(x)\,dx = \text{area of the corresponding triangle} = 1 \quad \text{for each } n.$$

$$\therefore \quad \lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1$$

Claim: $\lim\limits_{n\to\infty} f_n(x) = 0$ for each $x \in [0, 1]$

Pf of claim: Clearly, $f_n(0) = 0$ for every n $\quad\therefore \lim\limits_{n\to\infty} f_n(0) = 0.$

For each fixed $x \in (0, 1]$, $\exists$ a natural number N such that $\dfrac{2}{N} < x$.

Hence $f_n(x) = 0$ for all $n > N\left(> \dfrac{2}{x}\right)$

$$\therefore \quad \lim_{n\to\infty} f_n(x) = 0$$

Consequently, $\lim\limits_{n\to\infty} f_n(x) = 0$ for each $x \in [0, 1]$.

Therefore,

$$\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 0\,dx = 0 \neq 1 = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx.$$

More sophisticated example:

Let $\{r_n\}_1^\infty$ be an enumeration of the rational numbers in $[0, 1]$.

Define $f_n(x)\ (n = 1, 2, \cdots)$ on $[0, 1]$ by

$$f_n(x) = \begin{cases} 0 & \text{if } x \neq r_1, r_2, \cdots, r_n \\ 1 & \text{if } x = r_1, r_2, \cdots, r_n \end{cases}$$

It is clear that $f_n(x) \to f(x) := \begin{cases} 0 & \text{if } x \text{ is any irrational number in } [0,1] \\ 1 & \text{if } x \text{ is any rational number in } [0,1] \end{cases}$

Note that each $f_n(x)$ is integrable on $[0,1]$, since it has finitely many discontinuity points on $[0,1]$.

But we have already seen that $f(x)$ is not integrable on $[0,1]$.

Def A. (Pointwise convergence of a sequence of functions: **점별수렴**)

Let $f_n(x)$ $(n = 0, 1, 2, \cdots)$ be a sequence of functions, defined on an interval I. We say that $\{f_n(x)\}$ converges pointwise to $f(x)$ on I (as $n \to \infty$) provided that

for each (fixed) $x \in I$, $f_n(x) \to f(x)$ as $n \to \infty$.

(특징: 구간 I 에서 임의로 점 x 를 택해 고정한 후 수렴성 조사)

Remark. We say that $\{f_n(x)\}$ converges pointwise on I if $\exists$ a function $f : I \to \mathbb{R}$ such that $f_n \to f$ pointwise on I.

Notation. $\{f_n(x)\}$ converges pointwise to $f(x)$ on I:

$$\boxed{\begin{array}{c} f_n(x) \to f(x) \text{ on } I \quad \text{or} \quad \lim_{n\to\infty} f_n(x) = f(x) \text{ on } I \quad \text{or} \quad \lim_{n\to\infty} f_n(x) = f(x),\ \forall\,(\text{fixed})\ x \in I \\ \text{or equivalently, } \left|f_n(x) - f(x)\right| =: \varepsilon_n(x) \to 0 \quad \forall\,(\text{fixed})\ x \in I \end{array}}$$

Ex A. Let $f_n(x) = x^n$. $\lim_{n\to\infty} f_n(x) = ?$ on $[0,1]$

Sol. For each $x \in [0,1]$,

$$f_n(x) = x^n \xrightarrow{n\to\infty} \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases} \equiv f(x)$$

$$\therefore \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Ex B. Let $f_n(x) = \dfrac{n}{1+nx}$. $\lim_{n\to\infty} f_n(x) = ?$ on $(0,\infty)$

Sol. For each $x > 0$,

$$\frac{n}{1+nx} = \frac{1}{1/n + x} \xrightarrow{n\to\infty} \frac{1}{x}$$

$$\therefore \lim_{n\to\infty} \frac{n}{1+nx} = \frac{1}{x} \quad \text{on } (0,\infty)$$

Remark. $\lim_{n\to\infty} f_n(x) = f(x)$ on I (i.e., $\boxed{f(x_0) = \lim_{n\to\infty} f_n(x_0), \quad \forall\,(\text{fixed})\ x_0 \in I}$)

$$\overset{\text{i.e.}}{\Leftrightarrow} \text{ given } \varepsilon > 0, \quad f_n(x_0) \underset{\varepsilon}{\approx} f(x_0) \quad \text{for } n \ge N = N(\varepsilon, x_0) \quad \underbrace{\text{whenever } x_0 \in I}_{x_0\text{를 먼저 고정}}.$$

$$\overset{\text{i.e.}}{\Leftrightarrow} \text{ for each (fixed) } x_0 \in I, \text{ and for every } \varepsilon > 0,\ \exists\ N = N(\varepsilon, x_0) \text{ s.t.}$$
$$n \ge N \quad \Rightarrow \quad \left|f_n(x_0) - f(x_0)\right| < \varepsilon$$

※ Def B (Uniform convergence: 고른 수렴 = 균등수렴 = 균일수렴 = 평등수렴)

Notation (standard): For a function $g(x)$ defined on an interval I, we write

$$\|g\|_I \overset{\text{write}}{=} \sup_{x\in I} |g(x)|$$

Let $f_n(x)$ $(n = 0, 1, 2, \cdots)$ be a sequence of functions, defined on an interval I. We say that $\{f_n(x)\}$ converges "uniformly" on I to $f(x)$ if

$$\boxed{\lim_{n\to\infty} \|f_n - f\|_I = 0 \quad \text{i.e.,} \quad \lim_{n\to\infty} \underbrace{\sup_{x\in I} |f_n(x) - f(x)|}_{\text{It is a function of } n \text{ alone}} = 0}$$

Notation. $\{f_n(x)\}$ converges uniformly on I to $f(x)$:

$$\boxed{f_n(x) \rightrightarrows f(x) \text{ on } I \quad \text{or} \quad f_n \rightrightarrows f \text{ on } I}$$

Note that $\boxed{\sup_{x\in I} |f_n(x) - f(x)| \le \varepsilon \quad \Leftrightarrow \quad |f_n(x) - f(x)| \le \varepsilon, \text{ for all } x \in I}$

Hence

$f_n(x) \rightrightarrows f(x)$ on I

$$\Leftrightarrow \boxed{\begin{array}{l} \text{given } \varepsilon > 0, \quad \exists\ N = N(\varepsilon) \text{ (depends on } \varepsilon, \text{ but not on } x\text{) such that} \\ n \ge N \quad \Rightarrow \quad |f_n(x) - f(x)| < \varepsilon \text{ (or } \le \varepsilon) \quad \text{for all } x \in I \end{array}}$$

$$\Leftrightarrow \boxed{\begin{array}{l} \text{given } \varepsilon > 0, \quad \exists\ N = N(\varepsilon) \text{ such that} \\ n \ge N \quad \Rightarrow \quad \sup_{x\in I} |f_n(x) - f(x)| \le \varepsilon \text{ (or } < \varepsilon) \end{array}}$$

Remark (obvious)

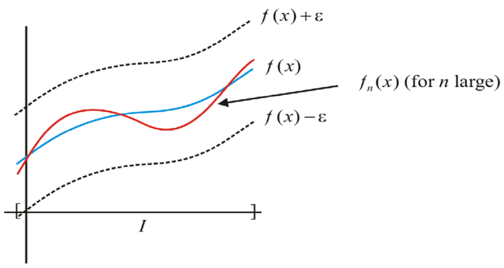
$$f_n(x) \rightrightarrows f(x) \text{ on } I \quad \Rightarrow \quad f_n \to f \text{ on } I$$

Pf. Follows from $|f_n(x) - f(x)| \le \|f_n - f\|_I \quad \forall x \in I$

"Remember"

$$\boxed{f_n(x) \rightrightarrows f(x) \text{ on } I \quad \Leftrightarrow \quad \underbrace{\underbrace{\sup_{x\in I} \left|f_n(x) - f(x)\right|}_{\text{maximum error on } I}}_{\text{it is (just) a ft of } n \text{ alone}} \left(= \| f_n - f \|_I\right) \to 0 \text{ as } n \to \infty}$$

This means "in geometric sense" that



i.e., all $f_n(x)$ $(n = 0, 1, 2, \cdots)$ lie in the (curved) band $(f(x)-\varepsilon,\ f(x)+\varepsilon)$ on I, for $n \gg 1$

(the band is a neighborhood of f in some sense)

Summary:

(i) pointwise convergence: "**vertical**" test (i.e., x-test) on I

(ii) uniform convergence: "**band**" test (i.e., y-test) on I

Ex C. Show, as $n \to \infty$,

$$f_n(x) = \frac{nx}{1+n^2x^2} \to 0 \text{ on } [0,1] \quad \text{but} \quad \not\rightrightarrows \quad 0 \text{ on } [0,1]$$

Pf. $x = 0$: $f_n(0) = 0$ for every n $\therefore$ $f_n(0) \to 0$ as $n \to \infty$

$$0 < \underbrace{x}_{\text{fix}} \le 1 : \quad 0 \le f_n(x) = \frac{nx}{1+n^2x^2} < \frac{nx}{n^2x^2} = \frac{1}{nx} \to 0 \cdot \frac{1}{x} = 0$$

$\therefore$ $f_n(x) \to 0$ (pointwise) on $[0,1]$

But $$\sup_{x\in[0,1]} |f_n(x) - 0| \ge |f_n(1/n)| = \frac{n \cdot 1/n}{1+n^2(1/n)^2} = 1/2 \not\to 0$$

$\therefore$ $f_n(x) \not\rightrightarrows 0$ on $[0,1]$

Note: $$f_n'(x) = \frac{n(1+n^2x^2) - nx(2n^2x)}{(1+n^2x^2)^2} = \frac{n(1-n^2x^2)}{(1+n^2x^2)^2} = 0 \Leftrightarrow x = \frac{1}{n}$$

f_n :		$\nearrow$	max	$\searrow$	
f_n' :		$+$	0	$-$	
x :	0		$\frac{1}{n}$		1

$$\therefore \quad \sup_{x\in[0,1]} |f_n(x) - 0| \underset{f_n \ge 0}{=} \sup_{x\in[0,1]} f_n(x) \underset{f_n \in C[0,1]}{=} \max_{x\in[0,1]} f_n(x) = f_n(1/n)$$

Ex D. Does $\dfrac{n}{1+nx} \rightrightarrows \dfrac{1}{x}$ on $(0,\infty)$?

(Seen, in Ex B, that $\dfrac{n}{1+nx} \to \dfrac{1}{x}$ pointwise on $(0,\infty)$)

Sol. We need to estimate

$$\sup_{x\in(0,\infty)} \left| \frac{n}{1+nx} - \frac{1}{x} \right| = \sup_{x\in(0,\infty)} \frac{1}{(1+nx)x}$$

$$g_n(x) \overset{\text{let}}{=} \frac{1}{(1+nx)x} \quad \to \quad g_n'(x) = \frac{-(1+2nx)}{(1+nx)^2 x^2} < 0 \ \text{ on } (0, \infty) \quad \therefore\ g_n(x) \text{ is strictly } \downarrow \ \text{ for } x > 0$$

Hence we must investigate the behavior of $g_n(x)$ at $x \approx 0^+$

$$\sup_{x\in(0,\infty)} \frac{1}{(1+nx)x} \underset{\substack{\uparrow \\ \text{take } x=1/n}}{\geq} \frac{n}{2} \to \infty$$

$$\therefore \quad \frac{n}{1+nx} \not\rightrightarrows \frac{1}{x} \quad \text{on } (0, \infty)$$

Remark. Does $\dfrac{n}{1+nx} \rightrightarrows \dfrac{1}{x}$ on $[1, \infty)$?

Sol. $g_n'(x) < 0$ for $x \geq 1$ $\quad \therefore\ g_n(x)$ is strictly $\downarrow$ for $x \geq 1$

$\therefore\quad g_n(x)$ has its max at $x = 1$

$$\therefore \quad \sup_{x\in[1,\infty)} \frac{1}{(1+nx)x} \underset{\substack{\uparrow \\ x=1}}{=} \frac{1}{1+n} \to 0$$

$$\therefore \quad \frac{n}{1+nx} \rightrightarrows \frac{1}{x} \quad \text{on } [1, \infty)$$

Ex E. Show that $f_n(x) := x^n \to 0$ pointwise on $[0, 1)$ (obvious) but $x^n \not\rightrightarrows 0$ on $[0, 1)$

Pf. For any $n \geq 1$, $x^n \to 1$ as $x \to 1^-$

$\therefore\quad x^n > 1/2$ for $x \approx 1^-$

$\therefore\quad |x^n - 0| > 1/2$ for $x \approx 1^-$

$\therefore\quad x^n \not\rightrightarrows 0$ on $[0, 1)$

Alternative easy pf.

$$\sup_{x\in[0,1)} |f_n(x) - f(x)| = \sup_{x\in[0,1)} |x^n - 0| = \sup_{x\in[0,1)} x^n \underset{\substack{\uparrow \\ \text{take } x=x_n=1/\sqrt[n]{2}}}{\geq} (1/\sqrt[n]{2})^n = 1/2$$

Or, $$\sup_{x\in[0,1)} |x^n - 0| = \sup_{x\in[0,1)} x^n \underset{\substack{\uparrow \\ \text{take } x=x_n=(1-1/n)}}{\geq} (1-1/n)^n \to 1/e \neq 0$$

$$\therefore \quad x^n \not\rightrightarrows 0 \quad \text{on } [0, 1)$$

Remark. Indeed, we can see that $\sup_{x\in[0,1)} x^n = 1$

($\because$ Clearly 1 is an upper bound for the set $\{x^n : 0 \leq x < 1\}$

Let $0 < \varepsilon < 1$. Then $\exists\, y$ (depend on n) such that $\sqrt[n]{1-\varepsilon} < y < 1$ $(\leftarrow \sqrt[n]{1-\varepsilon} < 1)$

i.e., $\forall \varepsilon \in (0,1), \exists y \in (0,1)$ such that $1 - \varepsilon < y^n < 1$ for some n

$\therefore\quad 1 - \varepsilon$ is not an upper bound for the set $\{x^n : 0 \leq x < 1\}$

Therefore, 1 is the least upper bound for the set $\{x^n : 0 \leq x < 1\}$)

⊙ Basic Theorem for uniform convergence **[an equivalent characterization for uniform convergence]**:

$$f_n(x) \rightrightarrows f(x) \ \text{ on } I \iff \boxed{\begin{array}{l} \exists \text{ a (nonnegative) real sequence } (\varepsilon_n) \text{ such that} \\ \text{(i)} \quad |f_n(x) - f(x)| \leq \varepsilon_n \quad \text{for all } x \in I \ \ (\therefore\ \varepsilon_n \text{ is indep of } x \in I) \\ \text{(ii)} \quad \varepsilon_n \to 0 \ \text{ as } n \to \infty \end{array}}$$

Pf. Already seen that $f_n(x) \rightrightarrows f(x)$ on I $\Leftrightarrow$ $\lim\limits_{n\to\infty} \sup\limits_{x\in I} |f_n(x) - f(x)| = 0$

$\Rightarrow$: Letting $\varepsilon_n = \sup\limits_{x\in I} |f_n(x) - f(x)|$ $\Rightarrow$ (i) & (ii) are clearly satisfied.

$\Leftarrow$: $$0 \le \sup_{x\in I} |f_n(x) - f(x)| \overset{\text{(i)}}{\le} \underbrace{\varepsilon_n \to 0}_{\text{(ii)}} \quad \text{as } n \to \infty$$

$$\therefore \quad \lim_{n\to\infty} \sup_{x\in I} |f_n(x) - f(x)| = 0$$

Ex F. Show $\dfrac{x+n}{1+nx} \rightrightarrows \dfrac{1}{x}$ on $[1, a]$ $(a > 1)$

Pf. For $x \ge 1$,

$$\left|\frac{x+n}{1+nx} - \frac{1}{x}\right| = \frac{x^2-1}{x(1+nx)} \le \frac{a^2-1}{1+n} \equiv \underbrace{\varepsilon_n}_{\text{indep of } x} \to 0$$

Ex G. Show $e^{\frac{x}{n}} \rightrightarrows 1$ on $[0, 1]$

Pf. $$\left|e^{\frac{x}{n}} - 1\right| = \left|e^{\frac{x}{n}} - e^0\right| \overset{\text{MVT}}{=} e^c \cdot \frac{x}{n}, \quad \text{where } 0 < c < \frac{x}{n} (< 1)$$

$$< \underbrace{\frac{e}{n}}_{\text{indep of } x\in[0,1]} \to 0$$

Def. (Pointwise and Uniform convergence of series)

Let $u_k(x)$ $(k = 0,1,2,\cdots)$ be defined on I, and let

$$S_n(x) = u_0(x) + u_1(x) + \cdots + u_n(x) \quad \text{(the nth partial sum of the series)}$$

We say that $\sum\limits_0^\infty u_k(x)$ converges pointwise (uniformly) on I if the sequence $\{S_n(x)\}_0^\infty$ converges pointwise (uniformly) on I. If the series converges, its sum(= its limit) is the function $f(x)$ defined by

$$f(x) = \lim_{n\to\infty} S_n(x) = \sum_0^\infty u_k(x), \quad x \in I$$

⦿ $f(x) = \sum\limits_0^\infty u_k(x)$ on I $\overset{\text{means}}{\Leftrightarrow}$ $f(x) = \sum\limits_0^\infty u_k(x)$ converges (pointwise) for every $x \in I$

※ Ex A.

(a) $\sum\limits_0^n \dfrac{x^k}{k!} \left(= 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!}\right) \rightrightarrows e^x$ on any interval $[-R, R]$, where $R > 0$

(b) (i) $\sum\limits_0^n \dfrac{x^k}{k!} \to e^x$ on $(-\infty, \infty)$ i.e., $\sum\limits_0^\infty \dfrac{x^k}{k!} = e^x$ (pointwise) on $(-\infty, \infty)$ ($\Leftarrow$ (a))

(ii) $\sum\limits_0^n \dfrac{x^k}{k!} \not\rightrightarrows e^x$ on $(-\infty, \infty)$ i.e., $\sum\limits_0^\infty \dfrac{x^k}{k!} \ne e^x$ uniformly on $(-\infty, \infty)$

Pf. (a) Given any $x \in [-R, R]$,

$$e^x \overset{\text{Taylor' theorem}}{=} 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^c x^{n+1}}{(n+1)!}, \quad 0 < c < x \quad \text{or} \quad x < c < 0$$

$$\therefore \quad \left|e^x - \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right)\right| = \frac{e^c\, |x|^{n+1}}{(n+1)!} \le \frac{e^R R^{n+1}}{(n+1)!}$$

Remains to show: $(*)$: $\displaystyle\lim_{n\to\infty}\frac{e^R R^{n+1}}{(n+1)!}=0$ (note that R is fixed)

To prove $(*)$, choose N so large that $R<\dfrac{N+1}{2}$ ---(▲)

Thus if $n>N$, then

$$\begin{aligned}\frac{e^R R^{n+1}}{(n+1)!}&=e^R\cdot\frac{R^{N+1}}{(N+1)!}\cdot\frac{R}{(N+2)}\cdots\frac{R}{(n+1)}\\&<e^R\cdot\frac{R^{N+1}}{(N+1)!}\cdot\frac{1}{2}\cdots\frac{1}{2}\quad(\text{by } (▲))\\&=\underbrace{e^R\cdot\frac{R^{N+1}}{(N+1)!}}_{\text{fixed number}}\left(\frac{1}{2}\right)^{n-N}\quad\to\quad 0\ \text{ as }\ n\to\infty\end{aligned}$$

$$\therefore\quad\sum_0^n\frac{x^k}{k!}\ \rightrightarrows\ e^x\ \text{ on any interval } [-R,R]$$

Another way of showing $(*)$: Set $a_n=\dfrac{e^R R^{n+1}}{(n+1)!}$ (R is fixed).

Then $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{R}{n+2}=0$, and thus $\sum a_n$ is convergent; so $a_n\to 0$ as $n\to\infty$

(b) (i) Fix any $x_0\in(-\infty,\infty)$. Then $\exists\, R>0$ s.t. $x_0\in[-R,R]$.

From the result (a), $\displaystyle\sum_0^n\frac{x^k}{k!}\ \rightrightarrows\ e^x$ on $[-R,R]$

$$\overset{\text{obviously}}{\Rightarrow}\quad\sum_0^n\frac{x^k}{k!}\ \to\ e^x\ \text{ on }\ [-R,R]$$

In particular, we see that

$$\sum_0^n\frac{{x_0}^k}{k!}\ \to\ e^{x_0}\quad\text{since } x_0\in[-R,R]$$

Since $x_0\in(-\infty,\infty)$ was an arbitrary point,

$$\sum_0^n\frac{{x_0}^k}{k!}\ \to\ e^{x_0}\quad\forall x_0\in(-\infty,\infty)\qquad\text{i.e., }\sum_0^n\frac{x^k}{k!}\ \to\ e^x\quad\text{on } (-\infty,\infty)$$

Alternative way of showing (b)-(i):

Given any fixed $x\in(-\infty,\infty)$

$$e^x\overset{\text{Taylor' theorem}}{=}1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\frac{e^c x^{n+1}}{(n+1)!},\quad 0<c<x\ \text{ or }\ x<c<0$$

$$\therefore\quad\left|e^x-\left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\right)\right|=\frac{e^c\,|\,x\,|^{n+1}}{(n+1)!}\le\frac{e^{|x|}\,|\,x\,|^{n+1}}{(n+1)!}$$

Remains to show:

$$(♦):\quad\lim_{n\to\infty}\frac{e^{|x|}\,|\,x\,|^{n+1}}{(n+1)!}=0\quad(\text{note that } x \text{ is fixed } \& \ e^{|x|} \text{ is indep of } n)$$

To prove $(♦)$, we may assume $x>0$ and we let $A_n=\dfrac{1}{n!}x^n$. Then

$$\frac{A_{n+1}}{A_n} = \frac{x}{n+1} < 1/2 \quad \text{if} \quad n > 2x - 1$$

Choose N so that $N > 2x - 1$. Then

$$A_{N+1} < \frac{1}{2} A_N$$

$$A_{N+2} < \frac{1}{2} A_{N+1} < \frac{1}{2^2} A_N$$

$$\vdots$$

$$A_{N+p} < \frac{1}{2^p} A_N$$

Thus $\lim_{n\to\infty} A_n = 0$ $\quad \therefore \quad \lim_{n\to\infty} \frac{e^{|x|} \mid x \mid^{n+1}}{(n+1)!} = 0$

$$\therefore \quad \sum_0^n \frac{x^k}{k!} \quad \to \quad e^x \qquad \text{i.e.,} \quad \sum_0^\infty \frac{x^k}{k!} \quad = \quad e^x \quad \text{on } (-\infty, \infty)$$

Another way of showing $(\blacklozenge)$: Set $a_n = \dfrac{e^{|x|} \mid x \mid^{n+1}}{(n+1)!}$ (x is fixed).

Then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\mid x \mid}{n+2} = 0$, and thus $\sum a_n$ is convergent; so $a_n \to 0$ as $n \to \infty$

(ii) **An indirect pf (in our text)**

Suppose $\sum_0^n \frac{x^k}{k!} \quad \rightrightarrows \quad e^x$ on $(-\infty, \infty)$. Then

given $\varepsilon > 0$, $e^x \underset{\varepsilon}{\approx} 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$, $\quad \forall x \in \mathbb{R}$, for $n \gg 1$

i.e., given $\varepsilon > 0$, $\left| e^x - \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \right) \right| < \varepsilon$, $\quad \forall x \in \mathbb{R}$, for $n \gg 1$

In particular, $\left| \frac{e^x}{x^n} - \frac{(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!})}{x^n} \right| < \frac{\varepsilon}{\mid x \mid^n} < \varepsilon$, $\quad \forall x > 1$, for $n \gg 1$

i.e., $(\bigstar)$: $\frac{e^x}{x^n} \underset{\varepsilon}{\approx} \frac{1}{x^n} + \frac{1}{x^{n-1}} + \frac{1}{2!\,x^{n-2}} + \cdots + \frac{1}{n!}$ $\quad \forall x > 1$, for $n \gg 1$

For any fixed such n (with $n \gg 1$), let $x \to \infty$ $\quad \Rightarrow$

LHS of $(\bigstar)$ $\to \infty$ by L'Hospital's rule, but RHS of $(\bigstar)$ $\to \frac{1}{n!}$

Contradiction!!

$$\therefore \quad \sum_0^n \frac{x^k}{k!} \quad \not\rightrightarrows \quad e^x \quad \text{on } (-\infty, \infty)$$

Another direct proof of showing (b)-(ii):

$$\sup_{x\in(-\infty,\,\infty)}\left|e^x-\left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\right)\right|$$

$$\geq \sup_{x\in(0,\,\infty)}\left|e^x-\left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\right)\right|$$

Note: $h(x)\overset{\text{let}}{=}e^x-\left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\right)=\frac{e^c x^{n+1}}{(n+1)!}>0$ for $x>0$ (by Taylor theorem)

$$h'(x)=e^x-\left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^{n-1}}{(n-1)!}\right)\overset{\text{Taylor theorem again}}{=}\underbrace{\frac{e^c x^n}{n!}}_{>0},\quad 0<c<x$$

$\therefore\ h(x)$ is strictly $\uparrow$ on $(0,\infty)$ ---$(*)$

$$\overset{\text{take } x=n\ (\leftarrow(*))}{\geq}\ e^n-\left(1+n+\frac{n^2}{2!}+\cdots+\frac{n^n}{n!}\right)\underset{\text{Taylor theorem}}{=}\frac{e^c n^{n+1}}{(n+1)!},\quad 0<c<n$$

$$>\frac{n^{n+1}}{(n+1)!}=\frac{n}{n+1}\cdot\frac{n}{n}\cdot\frac{n}{n-1}\cdot\cdots\cdot\frac{n}{1}>\frac{1}{2}\cdot 1\cdot\frac{n}{n-1}\cdots\frac{n}{1}>\frac{n}{2}\to\infty$$

$$\therefore\quad \sup_{x\in(-\infty,\,\infty)}\left|e^x-\left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\right)\right|\not\to 0$$

$$\therefore\quad \sum_0^n\frac{x^k}{k!}\ \not\rightrightarrows\ e^x\ \text{ on } (-\infty,\infty)$$

22.2 Criteria for uniform convergence

Recall the notation: $\|u_n\|_I\overset{\text{write}}{=}\sup_{x\in I}|u_n(x)|$

Thm A (A necessary condition for uniform convergence)

Suppose $\sum_0^\infty u_k(x)$ converges uniformly on I. Then

$$\|u_n\|_I\to 0\ \text{ as }\ n\to\infty$$

Pf. Let $S_n(x)=\sum_0^n u_k(x)$ and $S(x)=\sum_0^\infty u_k(x)$. Then hypo says

$S_n(x)\rightrightarrows S(x)$ on I. i.e., $\|S_n-S\|_I\to 0$ as $n\to\infty$ ---($\odot$)

$$u_n(x)=S_n(x)-S_{n-1}(x)\quad(n\geq 1)$$

$$=S_n(x)-S(x)+S(x)-S_{n-1}(x)$$

$$|u_n(x)|\leq|S_n(x)-S(x)|+|S(x)-S_{n-1}(x)|\quad\forall x\in I$$

$$\leq\|S_n-S\|_I+\|S_{n-1}-S\|_I$$

$$\therefore\quad \|u_n\|_I\leq\|S_n-S\|_I+\|S_{n-1}-S\|_I\overset{\text{as } n\to\infty}{\to}0+0=0\quad\text{by }\odot$$

$$\therefore\quad \|u_n\|_I\to 0\ \text{ as }\ n\to\infty.$$

Ex. An easy way of proving: $\sum_0^\infty\frac{x^k}{k!}\ \not\rightrightarrows\ e^x$ uniformly on $(-\infty,\infty)$

Sol. $\quad u_n(x) = \dfrac{x^n}{n!}$

$$\|u_n\|_{(-\infty,\,\infty)} = \sup_{x\in(-\infty,\,\infty)} \frac{|\,x\,|^n}{n!} \underset{\text{take } x=n}{\geq} \frac{n^n}{n!} = \frac{n}{n}\cdot\frac{n}{n-1}\cdots\frac{n}{1} > n \to \infty$$

$$\therefore \quad \|u_n\|_{(-\infty,\,\infty)} \not\to 0 \quad \text{as} \quad n\to\infty$$

$$\therefore \quad \sum_0^\infty \frac{x^k}{k!} \not\rightrightarrows e^x \quad \text{uniformly on } (-\infty,\,\infty) \quad \text{by Thm A}$$

※ Thm B (Weierstrass M-test) [Here M means **majorant**]

Suppose that for $k \geq 0$, $|u_k(x)| \leq M_k$ on I, and $\sum_0^\infty M_k$ converges.

Then $\sum_0^\infty u_k(x)$ converges uniformly on I.

Pf. For each $x_0 \in I$,

$\sum_0^\infty u_k(x_0)$ is absolutely convergent by Comparison test. $\therefore$ $\sum_0^\infty u_k(x_0)$ converges.

Thus, we can write: $\sum_0^\infty u_k(x) = f(x), \quad x \in I.$

i.e., $\sum_0^\infty u_k(x)$ converges pointwise to its sum $f(x)$ on I.

Let $S_n(x) = \sum_0^n u_k(x)$. Then

$$|f(x) - S_n(x)| = \left|\sum_{n+1}^\infty u_k(x)\right| \underset{(*)}{\leq} \sum_{n+1}^\infty |u_k(x)| \leq \sum_{n+1}^\infty M_k \equiv \underbrace{\varepsilon_n}_{\text{indep of } x\in I}$$

$(*)$ follows from: Ex. $\sum a_n$: (abso.) conv $\Rightarrow$ $\left|\sum a_n\right| \leq \sum |a_n|$

We will show $\varepsilon_n \to 0$.

$$\varepsilon_n = \sum_{n+1}^\infty M_k = \sum_0^\infty M_k - \sum_0^n M_k \xrightarrow{\sum_0^\infty M_k\text{: converges}} \sum_0^\infty M_k - \sum_0^\infty M_k = 0$$

Therefore,

$\sum_0^\infty u_k(x)$ converges uniformly on I (by **Basic Theorem for uniform convergence**)

Equivalent form of M-test: $\sum_0^\infty u_k(x)$ converges uniformly on I if $\sum_0^\infty \|u_k\|_I$ converges.

Ex. Show that $\sum_1^\infty \dfrac{\cos nx}{n^2}$ converges uniformly on $(-\infty,\,\infty)$.

Pf. $\quad \left|\dfrac{\cos nx}{n^2}\right| \leq \dfrac{1}{n^2} \quad \forall x \in (-\infty,\,\infty)$ and $\sum_1^\infty \dfrac{1}{n^2}$ converges

$$\overset{\text{Weierstrass M-test}}{\Rightarrow} \quad \sum_1^\infty \frac{\cos nx}{n^2} \text{ converges uniformly on } (-\infty,\,\infty).$$

Ex. Show that $f(x)=\sum_{n=1}^{\infty}\frac{x}{n(1+nx^2)}$ converges uniformly on $\mathbb{R}$

Pf. Note that $1+nx^2 \geq 2|x|\sqrt{n}$. Hence for $x \neq 0$

$$\sum_{n=1}^{\infty}\left|\frac{x}{n(1+nx^2)}\right| \leq \sum_{n=1}^{\infty}\frac{|x|}{n\cdot 2|x|\sqrt{n}} \leq \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}} : \text{ conv}$$ (This also holds for $x=0$)

By M-test, $f(x)=\sum_{n=1}^{\infty}\frac{x}{n(1+nx^2)}$ converges uniformly on $\mathbb{R}$

Ex . We know that $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ for $0 \leq x < 1$

(i) Show that $\sum_{n=0}^{\infty}x^n$ converges uniformly on $[0,t]$ when $0<t<1$.

(ii) Show that $\sum_{n=0}^{\infty}x^n$ does not converge uniformly on $[0,1)$.

Pf. (i) $|x^n| = x^n \leq t^n \quad \forall x \in [0,t]$ & $\sum_{n=0}^{\infty}t^n$: converges since $t<1$

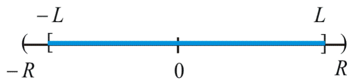
Then by Weierstrass M-test, $\sum_{n=0}^{\infty}x^n$ converges uniformly on $[0,t]$

(ii) $u_n(x)=x^n \quad \Rightarrow \quad \|u_n\|_{[0,1)}=1 \not\to 0$

Thus $\sum_{n=0}^{\infty}x^n$ is not uniformly convergent on $[0,1)$

Thm C (Uniform convergence of power series)

If $\sum_{0}^{\infty}a_n x^n$ has the radius of convergence $R>0$, then the series converges uniformly on every interval $[-L, L]$, where $0 \leq L < R$.



Pf. Let $x \in [-L, L]$. Then $|a_n x^n| \leq |a_n| L^n$.

By the definition of radius of convergence of P.S. , $\sum_{0}^{\infty}a_n x^n$ converges absolutely for $\forall\, |x| < R$.

In particular, $\sum_{0}^{\infty}|a_n| L^n$ converges (since $0 \leq L < R$).

$\overset{\text{Weierstrass M-test}}{\Rightarrow} \sum_{0}^{\infty}a_n x^n$ converges uniformly on $[-L, L]$.

Two additional theorems on uniform convergence:

Theorem D (Cauchy criterion for uniform convergence)

Let $\{F_n\}$ be a sequence of functions defined on an interval I. Then

$\{F_n\}$ is uniformly convergent on I

$\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\|F_m - F_n\|_I < \varepsilon$ for all $m > n > N$

Pf. $(\Rightarrow$: easy part$)$

Suppose $\{F_n\}$ is uniformly convergent to F on I. Then

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\|F_n - F\|_I < \varepsilon / 2$ for all $n > N$.

Thus, for all $m > n > N$, we have

$$\|F_m - F_n\|_I = \|F_m - F + F - F_n\|_I \le \|F_m - F\|_I + \|F - F_n\|_I < \varepsilon$$

$(\Leftarrow)$ Let $\varepsilon > 0$ be given. Then by hypo, we can choose $N \in \mathbb{N}$ such that

$$\|F_m - F_n\|_I < \varepsilon / 2 \quad \text{for all} \quad m > n > N \quad \text{---} \ (\odot)$$

For any fixed $n > N$, we let $a_m = \|F_m - F_n\|_I \ (m > n)$. Then this implies that

$$a_m < \varepsilon / 2 \ \text{ for all } \ m > n, \quad \text{and so} \ \lim_{m\to\infty} a_m \le \varepsilon / 2 \ \text{(by LLT)}$$

Now fix any $x \in I$. Then the number sequence $\{F_n(x)\}$ has the property that

$$|F_m(x) - F_n(x)| < \varepsilon / 2 < \varepsilon \ \text{ for all } \ m > n > N \ \ (\text{by } (\odot))$$

This says the sequence $\{F_n(x)\}$ is a Cauchy sequence. Hence $\{F_n(x)\}$ is convergent

Thus we can let $\lim_{n\to\infty} F_n(x) = F(x)$, $x \in I$.

Then for all $n > N$ and all $x \in I$

$$|F(x) - F_n(x)| = \lim_{m\to\infty} |F_m(x) - F_n(x)| \le \lim_{m\to\infty} \|F_m - F_n\|_I \le \varepsilon / 2$$

$$\therefore \quad \sup_{x\in I} |F(x) - F_n(x)| \le \varepsilon / 2 \ \text{ for all } \ n > N$$

i.e., $\|F - F_n\|_I \le \varepsilon / 2 < \varepsilon$ for all $n > N$ $\quad \therefore \quad F_n \rightrightarrows F$ on I

Corollary.

Let $\{u_n\}_0^\infty$ be a sequence of functions on an interval I. Then

$\sum_0^\infty u_k(x)$ converges uniformly on I

$\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\left\| \sum_{k=n+1}^{m} u_k \right\|_I < \varepsilon$ for all $m > n > N$

[Shortly, $\left\| \sum_{k=n+1}^{m} u_k \right\|_I \to 0$ as $m, n \to \infty$]

Pf. Have only to notice that $\left\| \sum_{k=n+1}^{m} u_k \right\|_I = \left\| \sum_{k=0}^{m} u_k - \sum_{k=0}^{n} u_k \right\|_I$

Theorem E (**Tail convergence test** for uniform convergence of series of functions)

Suppose $\sum_{0}^{\infty} u_k(x)$ converges pointwise on an interval I. Then

$$\sum_{0}^{\infty} u_k(x) \text{ converges uniformly on } I \iff \lim_{n\to\infty}\left(\sup_{x\in I}\left|\sum_{k=n}^{\infty} u_k(x)\right|\right) = 0 \text{ i.e., } \sup_{x\in I}\left|\sum_{k=n}^{\infty} u_k(x)\right| \to 0$$

Pf. Follows from the simple fact that

$$\sum_{0}^{\infty} u_k(x) \text{ converges uniformly on } I \iff \sup_{x\in I}\left|\sum_{0}^{\infty} u_k(x) - \sum_{0}^{n} u_k(x)\right| = \sup_{x\in I}\left|\sum_{n+1}^{\infty} u_k(x)\right| \to 0$$

Ex [Advanced]. Let $S(x) = \sum_{n=1}^{\infty} (-1)^{n+1}\dfrac{1}{n+x^2}$. Prove that

(i) $S(x)$ converges uniformly on $\mathbb{R} = (-\infty, \infty)$.

(ii) $S(x)$ converges absolutely at no point of $(-\infty, \infty)$ --- easy

Pf of (i). M1 [Use M-test]

Let $u_n(x) = (-1)^{n+1}\dfrac{1}{n+x^2} \quad \Rightarrow \quad \sup_{x\in(-\infty,\infty)} |u_n(x)| = \dfrac{1}{n} := M_n$ and $\sum_{n=1}^{\infty} M_n = \infty$

So Weierstrass M-test does not work.

M2 [Use n-th term test]

$\|u_n\|_{\mathbb{R}} = \dfrac{1}{n} \to 0$ as $n \to \infty$; So n-th term test also does not work.

M3 [Use **Theorem E** plus "Cauchy's alternating series test" (since it is alternating)]

Since $\dfrac{1}{n+x^2}$ is $\downarrow 0$ for each (fixed) $x \in \mathbb{R}$, we see that

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n+x^2} \text{ converges (pointwise) by Alternating series test,}$$

Now let $S_n(x) = \sum_{k=1}^{n} (-1)^{k+1}\dfrac{1}{k+x^2} =: \sum_{k=1}^{n} u_k(x)$. Then

$$\left|S_n(x) - S(x)\right| \left[= \left|\sum_{n+1}^{\infty} u_k(x)\right| = |\text{ Tail }|\right] \le |u_{n+1}(x)| = \frac{1}{n+1+x^2} \le \frac{1}{n+1} \quad \forall x \in \mathbb{R}$$

So $\|S_n - S\|_{\mathbb{R}} \left[= \sup_{x\in\mathbb{R}}\left|\sum_{n+1}^{\infty} u_k(x)\right|\right] \le \dfrac{1}{n+1} \to 0$ as $n \to \infty$; $\therefore$ $S_n(x) \rightrightarrows S(x)$ on $\mathbb{R}$

i.e., $\sum_{n=1}^{\infty} (-1)^{n+1}\dfrac{1}{n+x^2}$ converges uniformly on $\mathbb{R}$.

Remark (cf: see the previous Ex) Assume that on an interval I,

(i) $u_n(x)$ is nonnegative and $\downarrow$ &

(ii) $\|u_n\|_I \to 0$ (i.e., $u_n \rightrightarrows 0$ on I)

Then $\sum_{n=0}^{\infty} (-1)^n u_n(x)$ is uniformly convergent on I.

Pf. Exercise

Homework: ① Does $\sum_{n=1}^{\infty} \dfrac{x^n}{n}$ converge uniformly on $[-1,0]$?

② Does $\sum_{n=1}^{\infty} \dfrac{x^n}{n}$ converge uniformly on $[0,1)$?