Value at Risk (VaR)

- Definition: VaR is a quantile of the loss function.
- VaR by RiskMetrics
- VaR by Extreme Value Theory
- VaR by Peak Over Threshold
- Ch 7. of Tsay (2010), 'Analysis of Financial Time Series'

Value at Risk (VaR)

- Risks in financial markets
  - Credit risk
  - Operational risk (legal issues, political risk, moral hazard, etc.)
  - Market risk
- VaR is measuring market risk

$$P(L > \mathrm{VaR}) = p$$

  where L is a random variable measuring the risk for the next l periods.
- VaR is a quantile of the loss function.
- VaR is a loss associated with a rare event under normal market
- Naive estimator; sample p-th quantile

$$x_p = \inf\{x \mid F_l(x) \geq p\}$$

VaR

- $p = .01$ is used for risk management, and $p = .001$ in stress testing
- Time horizon to measure risk, 1-10 days for market risk, 1-5 years for credit risk
- RiskMetrics: J.P. Morgan use this to estimate VaR

$$r_t \mid F_{t-1} \sim N(0, \sigma_t^2)$$
$$\begin{cases} r_t = \sigma_t \epsilon_t \\ \sigma_t^2 = \alpha \sigma_{t-1}^2 + (1-\alpha) r_{t-1}^2 \end{cases}$$

- $\alpha \in (0,1)$, and typically $alpha$ is large (.94).
- This is a $GARCH(1,1)$ model with constraint $\alpha + \beta = 1$, i.e. $IGARCH$ model.

VaR - RiskMetrics

- Since $r_t = \sigma_t \epsilon_t$,

  $$\sigma_t^2 = \alpha\sigma_{t-1}^2 + (1-\alpha)r_{t-1}^2 = \sigma_{t-1}^2 + (1-\alpha)\sigma_{t-1}^2(\epsilon_{t-1}^2 - 1)$$

  and $\mathbb{E}(\epsilon_{t+i}^2 - 1 \mid \mathcal{F}_t) = 0$ for $i \geq 1$ implies that

  $$\mathbb{E}(\sigma_{t+i}^2 | F_t) = \sigma_{t+i-1}^2, \quad i \geq 2,$$

  equivalently

  $$\mathrm{Var}(r_{t+i}|F_t) = \sigma_{t+1}^2, \quad i \geq 1.$$

  Therefore, a k-horizon return

  $$r_t[k] = r_{t+1} + \ldots + r_{t+k} \sim \mathcal{N}(0, k\sigma_{t+1}^2)$$

  conditionally on F_t. Therefore, $\mathrm{VaR}(p)$ is given by

  $$\xi_p = z_{1-p}\sqrt{k}\hat{\sigma}_{t+1}$$

  e.g. $p = .05$ gives $z_{.05} = 1.65$, $p = .01$ gives $z_{.01} = 2.326$.

VaR - RiskMetrics

- This is *square root of time rule* in VaR under RiskMetrics.
- Example. Daily IBM stock log-return giving

$$\sigma_t^2 = .94\sigma_{t-1}^2 + (1 - .94)r_{t-1}^2$$

and last trading day $r_{9190} = -.0128, \hat{\sigma}_{9190} = .0003472$
Find 1-day 5% VaR if you invest $10 million.

$$\mathrm{VaR} = \$10\text{million} \times \left(1.65 \times \sqrt{1} \times \sqrt{.94 \cdot .0003472 + .06 \times .0128^2}\right)$$

$$= \$302,500$$

It means that with 95% probabiliity, the loss will be less than or equal to $\$302,500$ for a long position of IBM stock when you invest 10 million dollars.

RiskMetrics

+ It is simple, easy to understand and apply
- Stylized facts say returns are heavy-tailed, non-Gaussian.
- Even stationary assumption fails due to non-constant mean ($\mu \neq 0$)
- May use other volatility models such as ARMA-GARCH(p, q).
- Here, we focus on VaR calculation using
  - extreme value theory (EVT)
  - Daily return data $\{r_1, \ldots, r_n\}$ assuming i.i.d.
  - $r_{(n)} = \max\{r_1, \ldots, r_n\}$
  - Be careful that we are goint to use "loss", negative log-returns, instead of log-returns, to calculate VaR.

Fisher-Tippett theorem (three types theorem) I

- We are interested in the distribution of maximum:

$$P(r_{(n)} \leq x) = P(r_1 \leq x, \ldots, r_n \leq x)$$

$$= P(r_1 \leq x) \cdots P(r_n \leq x) = \{F(x)\}^n$$

- Interested in the limiting distribution as $n \to \infty$ when F is unknown.

Theorem (Fisher-Tippet)

If there exist sequence of constants $\alpha_n > 0$, β_n such that, as $n \to \infty$

$$P(r^* \leq x) := P\left(\frac{r_{(n)} - \beta_n}{\alpha_n} \leq x\right) \longrightarrow F_*(x)$$

for some non-degenerate distribution F_*, then F_* has the same type as one of the following distribution.

Fisher-Tippett theorem (three types theorem) II

(i) Type I: $\xi = 0$, the Gumbell family.

$$F_*(x) = \exp(-\exp(-x)), \quad -\infty < x < \infty$$

(ii) Type II: $\xi > 0$, the Frechet family.

$$F_*(x) = \begin{cases} \exp(-(1+\xi x)^{-1/\xi}), & \text{if } x > \frac{1}{\xi} \\ 1, & \text{o.w.} \end{cases}$$

(iii) Type III: the Weibull family.

$$F_*(x) = \begin{cases} \exp(-(1+\xi x)^{-1/\xi}), & \text{if } x < -\frac{1}{\xi} \\ 1, & \text{o.w.} \end{cases}$$

EVT I

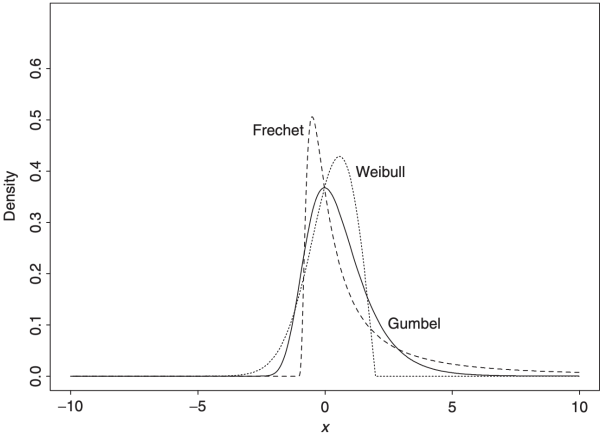

Figure 7.2 Probability density functions of extreme value distributions for maximum. Solid line is for Gumbel distribution, dotted line is for Weibull distribution with $\xi = -0.5$, and dashed line is for Fréchet distribution with $\xi = 0.9$.

EVT II

- The distribution is called the generalized extreme value (GEV) distribution.
- Gumbel distribution can be considered as the limit $\xi \to 0$.
- In finance, we're more interested in Frechet-type since Weibull distribution has finite right end point.
- ξ is referred to as the shape parameter. In fact, tail behavior of $F(x)$ determines EVT type
  - $\xi = 0$: light-tail such as normal, log-normal distribution
  - $\xi > 0$: heavy-tailed including student-t, stable distribution.
  - $\xi < 0$: short-tailed, for example uniform.
- $\alpha = 1/\xi$ is called the tail-index.

GEV - estimation

- MLE can be used. See p.346 of Tsay (2015)
- ξ (or $\alpha = 1/\xi$) determines limiting distribution type, so the estimation of ξ is very important in EVT.
- Hill estimator is the most popular non-parametric estimator.

$$\hat{\xi}^H = \frac{1}{q} \sum_{i=1}^{q} \left\{ \log(r_{(T-i+1)}) - \log(r_{T-q}) \right\}$$

$$\sqrt{q}(\hat{\xi}^H - \xi) \to N(0, \xi^2)$$

- The choice of q is critical in applying Hill estimator.

GEV I

- Illustration with IBM stock from July 3, 1962 - Dec 31, 1998.
- Remember historical crash on Oct 9, 1987 (black monday)

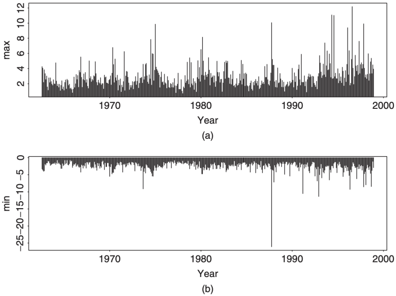

Figure 7.3 Maximum and minimum daily log returns of IBM stock when subperiod is 21 trading days. Data span is from July 3, 1962, to December 31, 1998: (a) positive returns and (b) negative returns.

TABLE 7.1 Results of Hill Estimator for Daily Log Returns of IBM Stock from July 3, 1962, to December 31, 1998[a]

q	190	200	210
r_t	0.300(0.022)	0.299(0.021)	0.305(0.021)
$-r_t$	0.290(0.021)	0.292(0.021)	0.289(0.020)

[a] Standard errors are in parentheses.

GEV II

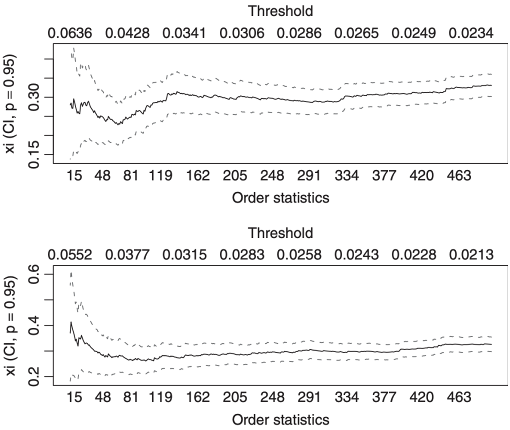

Figure 7.4 Scatterplots of Hill estimator for daily log returns of IBM stock. Sample period is from July 3, 1962, to December 31, 1998: upper plot is for positive returns and lower one for negative returns.

Threshold is selected by finding stable region

GEV III

TABLE 7.2 Maximum-Likelihood Estimates of Extreme Value Distribution for Daily Log Returns of IBM Stock from July 3, 1962 to December 31, 1998[a]

Length of Subperiod	Scale α_n	Location β_n	Shape Par. ξ_n
	Minimal Returns		
1 mon. ($n = 21$, $g = 437$)	0.823(0.035)	1.902(0.044)	0.197(0.036)
1 qur ($n = 63$, $g = 145$)	0.945(0.077)	2.583(0.090)	0.335(0.076)
6 mon. ($n = 126$, $g = 72$)	1.147(0.131)	3.141(0.153)	0.330(0.101)
1 year ($n = 252$, $g = 36$)	1.542(0.242)	3.761(0.285)	0.322(0.127)
	Maximal Returns		
1 mon. ($n = 21$, $g = 437$)	0.931(0.039)	2.184(0.050)	0.168(0.036)
1 qur ($n = 63$, $g = 145$)	1.157(0.087)	3.012(0.108)	0.217(0.066)
6 mon. ($n = 126$, $g = 72$)	1.292(0.158)	3.471(0.181)	0.349(0.130)
1 year ($n = 252$, $g = 36$)	1.624(0.271)	4.475(0.325)	0.264(0.186)

[a] Standard errors are in parentheses.

Block of size ≥ 63 is suggested for negative return

VaR with GEV I

- VaR is $(1-p^*)$th percentile of F, i.e.

$$P(r_t \leq r_n^*) = 1 - p^*$$

Since $GEV \approx P(r_t \leq r_n^*)^n$,

$$\begin{aligned} \therefore P(r_t \leq r_n^*) &\approx (GEV)^{\frac{1}{n}} \\ &= \exp\left(\frac{1}{n}\log GEV\right) \end{aligned}$$

Need to solve

$$\exp\left(\frac{1}{n}\log GEV\right) = 1 - p^*$$

VaR with GEV II

(e.g) If $\xi_n = 0$

$$\exp\left(-\frac{1}{n}\exp\left(-\left(\frac{r_n^* - \beta_n}{\alpha_n}\right)\right)\right) = 1 - p^*$$

$$-\frac{1}{n}\exp\left(\exp\left(-\left(\frac{r_n^* - \beta_n}{\alpha_n}\right)\right)\right) = \log(1 - p^*)$$

$$r_n^* = \beta_n - \alpha_n \log\left\{-n\log(1 - p^*)\right\}$$

Similarly if $\xi_n \neq 0$

$$r_n^* = \beta_n - \frac{\alpha_n}{\xi_n}\left\{1 - \left(-n\log(1 - p^*)\right)^{-\xi_n}\right\}$$

VaR with GEV III

1. Select the length of subperiod n to calculate max/min.
2. Find MLE $\hat{\beta}_n$, $\hat{\alpha}_n$, $\hat{\xi}_n$
3. Model checking
4. Plug-into formula to calculate VaR

- Example: IBM stock

$$\hat{\alpha}_n = .945,\ \hat{\beta}_n = 2.583,\ \hat{\xi}_n = .335,\ n = 63, p = .01$$

$$\begin{aligned}\mathrm{VaR} &= 2.583 - \frac{.945}{.335}\left\{1 - (-63\log(.99))^{-.335}\right\} \\ &= 3.049\%\end{aligned}$$

For \$10 million investment, estimated VaR becomes

$$\$10\text{million} \times 0.03049 = \$304,969$$

VaR & GEV

- The choice of block size n is critical.
- Cannot consider effects of other variables such as macroeconomic variables.
- In fact, returns are "strongly" correlated, but assumed iid.
- To overcome these difficulties, we will consider conditional approaches, called POT (peak over threshold) suggested by Smith (1989)

$$\text{Exceedance} = r_t - \eta$$

η is a threshold

POT I

More precisely, we're interested in the conditional distribution

$$P\left(r > x + \eta \mid r > \eta\right) = \frac{P(r > x + \eta)}{P(r > \eta)} = \frac{1 - F(x + \eta)}{1 - F(\eta)},\ x > 0$$

Since

$$\{P(r < x)\}^n \approx \exp\left\{-\left(1 + \xi\left(\frac{x - \beta}{\alpha}\right)\right)^{1/\xi}\right\}$$

$$n \log\{P(r < x)\} \approx -\left(1 + \xi\left(\frac{x - \beta}{\alpha}\right)\right)^{1/\xi}$$

Now approximate $\log F(x) \approx -(1 - F(x))$ from $e^{-y} \approx 1 - y$

$$\therefore 1 - P(r < x) \approx \frac{1}{n}\left[1 + \xi\left(\frac{x - \beta}{\alpha}\right)\right]^{-1/\xi}$$

POT II

Finally

$$\frac{1 - F(x+\xi)}{1 - F(\xi)} \approx \left[1 + \frac{\xi x}{\alpha + \xi(\eta - \beta)}\right]_{+}^{-1/\xi},\ x > 0$$

As $\xi \to 0$, it becomes $\exp\left(-x/\alpha\right)$

POT III

Theorem (Peak Over Threshold)

Under the condition that Fisher-Tippett theorem holds. For large n, the distribution of $r = x + \eta$ given $r > \eta$ is given by the generalized Pareto distribution with cdf

$$G(x) = \begin{cases} 1 - \left(1 + \frac{\xi x}{\alpha + \xi(\eta - \beta)}\right)_+^{-1/\xi} & \textit{if } \xi \neq 0 \\ 1 - \exp\left(-x/\alpha\right) & \textit{if } \xi = 0 \end{cases}$$

- POT assumes that the exceedances happen according to (non-homogeneous) Poisson process. If we assume time-homogeneous Poisson process on the arrival of exceedance, it is equivalent to i.i.d. arriaval of exceedances.
- Use MLE to estimate parameters.

POT IV

$\{r_t\}$ is used to estiamte GPD parameters

TABLE 7.3 Estimation Results of a Two-Dimensional Homogeneous Poisson Model for Daily Negative Log Returns of IBM Stock from July 3, 1962 to December 31, 1998[a]

Thr.	Exc.	Shape Parameter ξ	Log(Scale) $\ln(\alpha)$	Location β
		Original Log Returns		
3.0%	175	0.30697(0.09015)	0.30699(0.12380)	4.69204(0.19058)
2.5%	310	0.26418(0.06501)	0.31529(0.11277)	4.74062(0.18041)
2.0%	554	0.18751(0.04394)	0.27655(0.09867)	4.81003(0.17209)
		Removing the Sample Mean		
3.0%	184	0.30516(0.08824)	0.30807(0.12395)	4.73804(0.19151)
2.5%	334	0.28179(0.06737)	0.31968(0.12065)	4.76808(0.18533)
2.0%	590	0.19260(0.04357)	0.27917(0.09913)	4.84859(0.17255)

[a]The baseline time interval is 252 (i.e., 1 year). The numbers in parentheses are standard errors, where Thr. and Exc. stand for threshold and the number of exceedings.

VaR using POT

- Note that GEV and GPD shares the same parameters. Hence, replacing parameter estimates will give you a VaR.
- For a given upper probability p, the VaR is given by

$$\mathrm{VaR} = \begin{cases} \beta - \frac{\alpha}{\xi}\left[1 - \{-D\log(1-p)\}^{1/\xi}\right], & \text{if } \xi \neq 0 \\ \beta - \alpha \log\left(-D\log(1-p)\right), & \text{if } \xi = 0 \end{cases}$$

D is the baseline time interval. For daily stock return, typically use $D = 252$, the number of trading days.

Example

Example 7.8. Consider again the case of holding a long position of IBM stock valued at \$10 million. We use the estimation results of Table 7.3 to calculate 1-day horizon VaR for the tail probabilities of 0.05 and 0.01.

- Case I: Use the original daily log returns. The three choices of threshold η result in the following VaR values:
  1. $\eta = 3.0\%$: VaR(5%) = \$228,239, VaR(1%) = \$359.303.
  2. $\eta = 2.5\%$: VaR(5%) = \$219,106, VaR(1%) = \$361,119.
  3. $\eta = 2.0\%$: VaR(5%) = \$212,981, VaR(1%) = \$368.552.
- Case II: The sample mean of the daily log returns is removed. The three choices of threshold η result in the following VaR values:
  1. $\eta = 3.0\%$: VaR(5%) = \$232,094, VaR(1%) = \$363,697.
  2. $\eta = 2.5\%$: VaR(5%) = \$225,782, VaR(1%) = \$364,254.
  3. $\eta = 2.0\%$: VaR(5%) = \$217,740, VaR(1%) = \$372,372.

- POT gives more stable estimates than GEV.
- But, still depends on threshold. Threshold should be large enough to satisfy asymptotics and small enough to have enough exceedances to be used in estimation.

Diagnostics I

- GPD distribution implies that for $\xi > 0$,

$$\bar{F}(\xi) := P\left(r > x + \eta \mid r > \eta\right) = \left(1 + \frac{\xi x}{\beta + \xi(\eta - \alpha)}\right)^{-1/\xi}$$

$$\approx c x^{-1/\xi} \quad \text{as } x \to \infty$$

Hence,

$$\log \bar{F}(\eta) \approx \log c - \frac{1}{\xi} \log x$$

Since $\bar{F}(X_{(n-i+1)}) \approx \frac{i}{n}$, it suggests that

$$\log \left(\frac{i}{n}\right) \approx \log c - \frac{1}{\xi} \log X_{(n-i+1)}$$

Therefore, $\log$-$\log$ scale plot on $\bar{F}$ and order-statistics should be approximately linear. This is called tail probability plot.

Diagnostics II

- For $\xi = 0$, it is simply Exponential distribution, hence QQ plot against exponential distribution can be used to infer whether $\xi = 0$ or not.
- Mean excess plot (mean residual life plot)
  Let $\psi(\eta_0) = \alpha + \xi(\eta_0 - \beta)$.
  (i) $\mathbb{E}(r - \eta_0 \mid \eta_0) = \frac{\psi(\eta_0)}{1-\xi}$
  (ii) If GPD holds for η_0, then it also holds for any threshold above η_0. The excess distribution for an arbitrary threshold $\eta > \eta_0$ is GPD with the same ξ but with $\psi(\eta) = \psi(\eta_0) + \xi(\eta - \eta_0)$
  (iii) For any $\eta > \eta_0$, we have

$$\mathbb{E}(r - \eta \mid r > \eta) = \frac{\psi(\eta_0) + \xi(\eta - \eta_0)}{1 - \xi}$$

Diagnostics III

- Therefore, empirical mean excess function

$$\frac{1}{N_\eta}\sum_{i=1}^{N_\eta}(r_i - \eta), \quad N_\eta = \#\text{ of returns} \geq \eta$$

and threshold η should be linear. If $\xi > 0$, it is also increasing linear trend. This can be also used to determine threshold value η.

- Expected shortfall (ES) = Expected loss given that VaR is exceeded

$$ES_q = E(r \mid r > \mathrm{VaR}_q) = \mathrm{VaR}_q + \mathbb{E}(r - \mathrm{VaR}_q \mid r > \mathrm{VaR}_q)$$

$$= \frac{\mathrm{VaR}_q}{1-\xi} + \frac{\psi(\eta) - \xi\eta}{1-\xi}$$

Diagnostics IV

Example: IBM stock (negative) return data

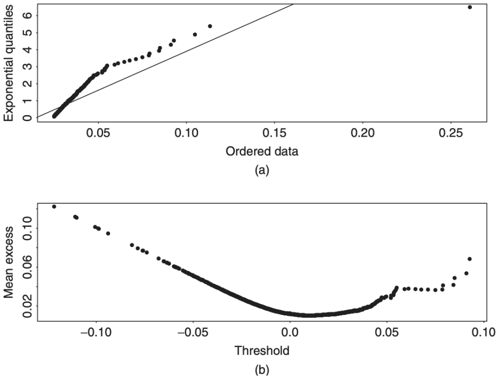

Figure 7.6 Plots for daily negative IBM log returns from July 3, 1962, to December 31, 1998. (a) QQ plot of excess returns over threshold 2.5% and (b) mean excess plot.

$\xi \neq 0$ from (a) and mean excess plot suggest threshold around .025-.05

Diagnostics V

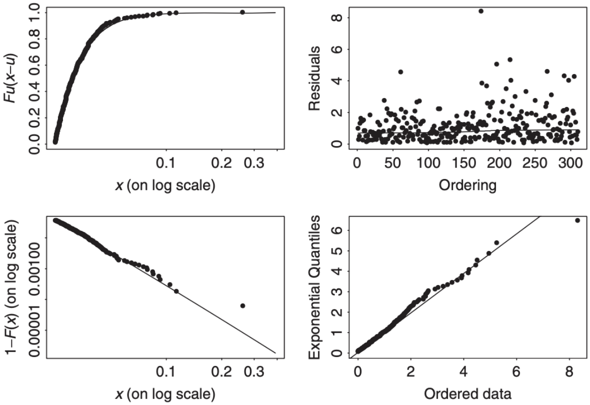

Figure 7.7 Diagnostic plots for GPD fit to daily negative log returns of IBM stock from July 3, 1962, to December 31, 1998.

When $\eta = .025$, tail-probability plot is linear. $\hat{\xi} = .264$. May do better by increasing threshold.

VaR for a stationary time series

- If data $\{x_t\}$ is weakly correlated, then three types theorem still holds but with different parameterizations.

$$P\left(\max\{x_1, \ldots, x_n\} \leq u_n\right) \approx \{F(x)\}^{n\theta}$$

  where $\theta \in (0, 1)$ is an extremal index.
- θ can be estimated by blocks method or runs method.
- VaR is modified by multiplying n by $n\hat{\theta}$. For example,

$$\mathrm{VaR} = \beta_n - \frac{\alpha_n}{\xi_n}\left[1 - \left\{-n\hat{\theta}\log(1-p)\right\}^{-1/\xi_n}\right],\ \xi \neq 0$$