Experimental Design
Note 14-1
Split-plot Design

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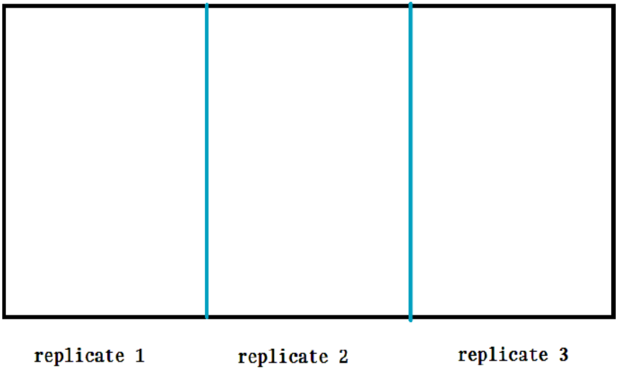
Example 1 I

Example: Study six corn varieties and four fertilizers and yield is the response. Three replicates are needed.

- Method 1: Completely randomized full factorial design, 24 level combinations of variety and fertilizer are applied to $24 \times 3 = 72$ pieces of land (each to three).
- Method 2: Select three fields of large area. Each field is divided into four areas (four whole-plots), four fertilizers are randomly assigned to the four whole-plots. Each area is further divided into six subareas (sub-plots), and the six varieties are randomly planted in these sub-plots.

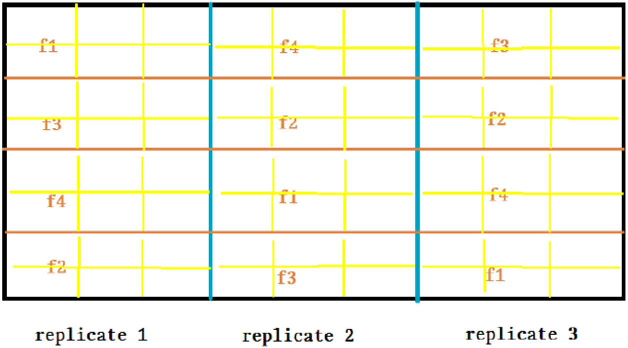
Example 1 II



Example 1 III

replicate 1	replicate 2	replicate 3
f1	f4	f3
f3	f2	f2
f4	f1	f4
f2	f3	f1

Example 1 IV



f1
f4
f3
f3
f2
f2
f4
f1
f4
f2
f3
f1
replicate 1
replicate 2
replicate 3

Example 1 V

This leads to a **split-plot design**:

- whole-plot (treatment) factor: fertilizer
- sub-plot (treatment) factor: corn variety

Example 2 I

- Example 2: A paper manufacturer is investigating three different pulp preparation methods an four different cooking temperatures for the pulp and study their effect on the tensile strength of the paper. Three replicates are needed.
- Because the pilot plant is only capable of making 12 runs per day, so the experimenter decides to run one replicate on each of the three days and to consider the days as blocks.

Example 2 II

On any day, a batch of pulp is produced by one of the three methods (a whole-plot). Then the batch is divided into four samples (four sub-plots), and each sample is cooked at one of the four temperatures. Then a second batch of pulp is made up using another of the three methods. This second batch is also divided into four samples that are tested at the four temperatures. The process is then repeated for the third method.

- **The split-plot is a multifactor experiment where it is not possible to completely randomize the order of the runs:**
  - In replicate 1, select a pulp preparation method, prepare a batch
  - Divide the batch into four sections or samples, and assign one of the temperature levels to each
  - Repeat for each pulp preparation method

Example 2 III

    - Conduct replicates 2 and 3 similarly
- Each replicate (sometimes called **blocks**) has been divided into three parts, called the **whole plots**
- Pulp preparation methods is the **whole plot treatment**
- Each whole plot has been divided into four **subplots** or **split-plots**
- Temperature is the **subplot treatment**
- Generally, the hard-to-change factor is assigned to the whole plots
- This design requires only 9 batches of pulp (assuming three replicates)

The data is given below.

Example 2 IV

	Day 1			Day 2			Day 3		
Method	1	2	3	1	2	3	1	2	3
Temp									
200	30	34	29	28	31	31	31	35	32
225	35	41	26	32	36	30	27	40	34
250	27	38	33	40	42	32	41	39	39
275	36	42	36	41	40	40	40	44	45

Split-Plot Structure

- factors are crossed (different frome nested)
- randomization restriction (different from completely randomized)
- information on factor effects from two levels (or strata)
- split-plot can be considered as two superimposed blocked designs:
  - A: whole-plot factor (a); B: sub-plot factor (b); r replicates
  - $RCBD_A$: number of treatment: a, number of blocks: r.
  - $RCBD_B$: number of treatment: b, number of blocks: ra. for whole-plots, subdivision to smaller sub-plots are ignored. For sub-plots, whole-plots considered blocks.
- More power for main subplot effect and interaction
- Should use design only for practical reasons
- Randomized factorial design more powerful if feasible

A Typical Data Layout for Split-Plot Design

	Block 1			Block 2			Block 3		
WP-Factor A	1	2	3	1	2	3	1	2	3
SP-Factor B									
1	y_{111}	y_{121}	.	.	.	.	.	.	y_{331}
2	y_{112}	y_{122}	.	.	.	.	.	.	y_{332}
3	y_{113}	y_{123}	.	.	.	.	.	.	y_{333}
4	y_{114}	y_{124}	.	.	.	.	.	.	y_{334}

In general:
y_{ijk} where i denotes Block i, j denotes the jth level of the whole-plot factor A, and k denotes the kth level of the sub-plot factor B.

Statistical Model I I

$$y_{ijk} = \mu + r_i + \alpha_j + (r\alpha)_{ij} + \beta_k + (r\beta)_{ik} + (\alpha\beta)_{jk} + (r\alpha\beta)_{ijk} + \epsilon_{ijk}$$

for $i = 1, \cdots, r;\ j = 1, \cdots, a;\ k = 1, \cdots, b$

- r_i: block effects (random)$\sim N(0, \sigma_r^2)$
- α_j: whole-plot factor (A) main effects (fixed)
- $(r\alpha)_{ij}$: whole-plot error (random)$\sim$ normal with $\sigma^2_{r\alpha}$
- β_k: sub-plot factor (B) main effects (fixed)
- $(r\beta)_{ik}$: block-B interaction (random) $\sim$ normal with $\sigma^2_{r\beta}$
- $(\alpha\beta)_{jk}$: interaction between A and B (fixed)
- $(r\alpha\beta)_{ijk}$: sub-plot error (random) $\sim$ normal with $\sigma^2_{r\alpha\beta}$
- ϵ_{ijk}: random error $\sim N(0, \sigma^2)$

Sum of Squares

- $SS_r = ab\sum_i (\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot})^2$, $df = r-1$
- $SS_A = rb\sum_j (\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdot\cdot\cdot})^2$, $df = a-1$
- $SS_{rA} = b\sum_i \sum_j (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdot\cdot\cdot})^2$, $df = (r-1)(a-1)$
- $SS_B = ar\sum_k (\bar{y}_{\cdot\cdot k} - \bar{y}_{\cdot\cdot\cdot})^2$, $df = b-1$
- $SS_{rB} = a\sum_i \sum_k (\bar{y}_{i\cdot k} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot k} + \bar{y}_{\cdot\cdot\cdot})^2$, $df = (r-1)(b-1)$
- $SS_{AB} = r\sum_j \sum_k (\bar{y}_{\cdot jk} - \bar{y}_{\cdot j\cdot} - \bar{y}_{\cdot\cdot k} + \bar{y}_{\cdot\cdot\cdot})^2$, $df = (a-1)(b-1)$
- $SS_{rAB} = \sum_i \sum_j \sum_k (y_{ijk} - \bar{y}_{ij\cdot} - \bar{y}_{i\cdot k} - \bar{y}_{\cdot jk} + \bar{y}_{i\cdot\cdot} + \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdot\cdot k} - \bar{y}_{\cdot\cdot\cdot})^2$, $df = (r-1)(a-1)(b-1)$.
- $SSE =$

Statistical Model I III

Expected mean squares (restricted)

		r	a	b	1	
		R	F	F	R	
	term	i	j	k	h	$E(MS)$
	r_i	1	a	b	1	$\sigma^2 + ab\sigma_r^2$
whole plot	α_j	r	0	b	1	$\sigma^2 + b\sigma_{r\alpha}^2 + \frac{rb\Sigma\alpha_j^2}{a-1}$
	$(r\alpha)_{ij}$	1	0	b	1	$\sigma^2 + b\sigma_{r\alpha}^2$
	β_k	r	a	0	1	$\sigma^2 + a\sigma_{r\beta}^2 + \frac{ra\Sigma\beta_k^2}{b-1}$
	$(r\beta)_{ik}$	1	a	0	1	$\sigma^2 + a\sigma_{r\beta}^2$
subplot	$(\alpha\beta)_{jk}$	r	0	0	1	$\sigma^2 + \sigma_{r\alpha\beta}^2 + \frac{r\Sigma\Sigma(\alpha\beta)_{jk}^2}{(a-1)(b-1)}$
	$(r\alpha\beta)_{ijk}$	1	0	0	1	$\sigma^2 + \sigma_{r\alpha\beta}^2$
	ϵ_{ijk}	1	1	1	1	σ^2 (not estimable)

There are two error structures; the whole-plot error and the subplot error

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Statistical Model I IV

Estimates and tests of fixed effects

- $\hat{\alpha}_j = \bar{y}_{.j.} - \bar{y}_{...}$ for $j = 1, \cdots, a$
- $\hat{\beta}_k = \bar{y}_{..k} - \bar{y}_{...}$ for $k = 1, \cdots, b$
- $(\hat{\alpha\beta})_{jk} = \bar{y}_{.jk} - \bar{y}_{.j.} - \bar{y}_{..k} + \hat{y}_{...}$
- Test $\alpha_j = 0$, $F_0 = MS_A / MS_{rA}$
- Test $\beta_k = 0$, $F_0 = MS_B / MS_{rB}$
- Test $(\alpha\beta)_{jk} = 0$, $F_0 = MS_{AB} / MS_{rAB}$

Statistical Model I V

Example See split_plot.SAS.

■ **TABLE 14.16**

Analysis of Variance for the Split-Plot Design Using the Tensile Strength Data from Table 14.14

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Replicates (or blocks)	77.55	2	38.78		
Preparation method (A)	128.39	2	64.20	7.08	0.05
Whole plot error (replicates (or blocks) $\times$ A)	36.28	4	9.07		
Temperature (B)	434.08	3	144.69	41.94	<0.01
Replicates (or blocks) $\times$ B	20.67	6	3.45		
AB	75.17	6	12.53	2.96	0.05
Subplot error (replicates (or blocks) $\times$ AB)	50.83	12	4.24		
Total	822.97	35			

Statistical Model II I

$$y_{ijk} = \mu + r_i + \alpha_j + (r\alpha)_{ij} + \beta_k + (\alpha\beta)_{jk} + \epsilon_{ijk}$$

for $i = 1, \cdots, r$; $j = 1, \cdots, a$; $k = 1, \cdots, b$

- r_i: block effects (random)$\sim N(0, \sigma_r^2)$
- α_j: whole-plot factor (A) main effects (fixed)
- $(r\alpha)_{ij}$: whole plot error$\sim$ normal with $\sigma_{r\alpha}^2$
- β_k: sub-plot factor (B) main effects (fixed)
- $(\alpha\beta)_{jk}$: A and B interaction (fixed)
- ϵ_{ijk}: sub-plot error $N(0, \sigma_\epsilon^2)$

Statistical Model II II

Expected mean square

Term	$E(MS)$
r_i	$\sigma_\epsilon^2 + ab\sigma_r^2$
$\alpha_j(A)$	$\sigma_\epsilon^2 + b\sigma_{r\alpha}^2 + \frac{rb\sum_j \alpha_j^2}{a-1}$
$(r\alpha)_{ij}$	$\sigma_\epsilon^2 + b\sigma_{r\alpha}^2$ (whole plot error)
β_k (B)	$\sigma_\epsilon^2 + \frac{ra\sum_j \alpha_j^2}{b-1}$
$(\alpha\beta)_{jk}$ (AB)	$\sigma_\epsilon^2 + \frac{r\sum_j \sum_k (\alpha\beta)_{jk}^2}{(a-1)(b-1)}$
ϵ_{ijk}	σ_ϵ^2 (subplot error)

General Split-Plot Designs I

- Can have $>$ one whole-plot factor and $>$ one subplot factor with various blocking schemes
- split-plot design consists of two superimposed block design
  Whole plot
  - CRD, RCBD, Factorial D, BIBD, etc.

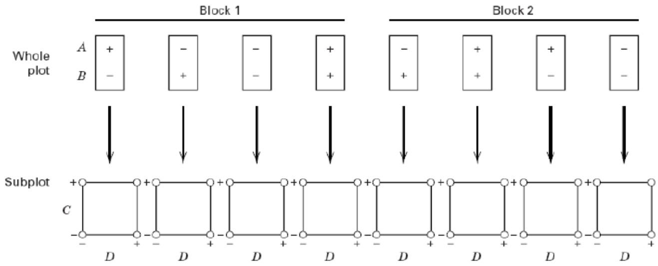
  Subplot
  - RCBD,BIBD, Factorial Design, etc.
- Analysis of Covariance

Variations of the Basic Split-Plot Design

More than two factors - see page 627

A & B (gas flow & temperature) are hard to change;

C & D (time & wafer position) are easy to change.



■ **FIGURE 14.7** A split-plot design with four design factors, two in the whole plot and two in the subplot

Other Variations I

- Split-split-plot design
  1. randomization restriction can occur at any number of levels within the experiment
  2. two-level: split-split-plot design

  Example: Absorption times of antibiotic capsule
  - 3 technicians
  - 4 dosage strengths
  - 4 capsule wall thickness
  - 4 replicates/days
  - A split-split-plot design
  - Two randomization restrictions present within each replicate

Other Variations II

- Strip-split-plot design (or Criss cross design, or Split-block design)
  Example: we want to compare the yield of a certain crop under different systems of soil preparation ($A : a_1, a_2, a_3, a_4$) and different density of seeding ($B : b_1, b_2, b_3, b_4, b_5$). Both operations (tilling and seeding) are done mechanically and it is impossible to perform both on small pieces of land. The arrangement shown below (strip-split-plot design) is then replicated r times, each time using different randomizations for A and B.

Other Variations III

	b1	*b4*	*b2*	*b3*	*b5*
a4	a_4b_1	a_4b_4	a_4b_2	a_4b_3	a_4b_5
a1	a_1b_1	a_1b_4	a_1b_2	a_1b_3	a_1b_5
a2	a_2b_1	a_2b_4	a_2b_2	a_2b_3	a_2b_5
a3	a_3b_1	a_3b_4	a_3b_2	a_3b_3	a_3b_5

Strip plots $\downarrow$

Whole plots $\leftarrow$