

$$1) \quad 1-1) \quad \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \leq \sum_{n=0}^{\infty} \frac{1}{(2n)^2} = \sum_{n=0}^{\infty} \frac{1}{4n^2}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{4n^2} = \frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{n^2} \quad \text{by Linearity Theorem}$$

$$\therefore \text{ Given } \sum_{n=0}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=0}^{\infty} \frac{1}{4n^2} = \frac{\pi^2}{24} \geq \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \quad (\text{Comparison Test})$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \text{ is convergent.}$$

1-2) Because $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent series with non-negative terms, they are both monotonically increasing and bounded above.

$$\text{Let } \sum_{n=0}^{\infty} a_n \leq L \quad \text{and} \quad \sum_{n=0}^{\infty} b_n \leq M, \text{ then,}$$

$$\sum_{n=0}^{\infty} a_n b_n \leq \sum_{n=0}^{\infty} L \cdot b_n \leq \sum_{n=0}^{\infty} LM, \text{ and } a_n b_n \text{ are non-negative terms.}$$

\therefore Because $\sum_{n=0}^{\infty} a_n b_n$ is monotonically increasing and bounded above by LM , $\sum_{n=0}^{\infty} a_n b_n$ converges by the Completeness Property.

$$2) \quad \sum_{k=0}^{\infty} b_k = b_1 + b_2 + \dots + b_k$$

$$= (a_0 + a_1) + (a_2 + a_3) + \dots + (a_{2k} + a_{2k+1})$$

$$= a_0 + a_1 + a_2 + a_3 + \dots + a_{2k} + a_{2k+1}, \text{ by the subsequence theorem,}$$

$$= \sum_{n=0}^{\infty} a_n = S$$

$$3) \quad 3-1) \quad \sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \frac{2 \sqrt[n]{n!}}{n} = 0, \text{ by } n^{\text{th}} \text{ root test}$$

$$3-2) \quad \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

$$\frac{a_{n+1}}{a_n} = \frac{n(\ln n)^p}{(n+1)(\ln(n+1))^p}, \quad \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1, \text{ ratio test fails}$$

$$\frac{1}{n(\ln n)^p} \leq \frac{1}{n \ln n}, \text{ because } n \geq 2 \text{ and } p > 0,$$

$$\frac{1}{n(\ln n)} \leq \frac{1}{n}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n} = 0 < 1$$

\therefore By comparison test, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges

4) 4-1) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n^2 n} = \dots + \frac{2}{\pi} - \frac{2}{\pi} + \frac{2}{\pi} - \frac{2}{\pi} + \dots$ for $n \gg 1$, so because it is alternating in signs, it is not convergent.

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n^2 n}$ is not conditionally convergent

4-2) $\sum_{n=2}^{\infty} (-1)^n \frac{n^5}{2^n}$ is absolutely convergent since,

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \frac{n^{\frac{5}{n}}}{2} = \frac{1}{2} < 1$$

$\therefore \sum_{n=2}^{\infty} (-1)^n \frac{n^5}{2^n}$ is not conditionally convergent

5) Let $\left| \frac{a_{n+1}}{a_n} \right| = x_n$ and $\left| \frac{b_{n+1}}{b_n} \right| = y_n$. By the Absolute Convergence Theorem, $\sum_{n=0}^{\infty} b_n$ is convergent. By the n^{th} term test of divergence, $\lim_{n \rightarrow \infty} b_n = 0$, which means $\lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = L < 1$.

Then it can be revised as, $\left| \frac{a_{n+1}}{a_n} \right| \leq L$, and L becomes the upper bound for $\left| \frac{a_{n+1}}{a_n} \right|$. Since $\left| \frac{a_{n+1}}{a_n} \right| \leq L < 1$, by Limit Location Theorem, $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \leq L < 1$, and this satisfies the ratio test, so $\sum_{n=0}^{\infty} a_n$ is absolutely convergent.

6) $\left| \frac{a_{n+1}}{a_n} \right| \leq r < 1$ for $n \gg 1$

pf) $\left| \frac{a_{n+1}}{a_n} \right| = \frac{|a_{n+1}|}{|a_n|}$

$$\Rightarrow |a_{n+1}| < |a_n| r \quad \text{for } n \gg 1$$

$$\Rightarrow |a_{n+2}| < |a_{n+1}| r < |a_n| r^2$$

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$$\Rightarrow |a_{n+k}| < |a_n| r^k \quad \text{for } k \geq 1.$$

$\Rightarrow \sum_{k=1}^{\infty} |a_n| r^k$ is convergent by the Linearity Theorem, and by the comparison test, $\sum_{k=1}^{\infty} |a_{n+k}|$ is also convergent. The Tail-Convergence Theorem can be then applied to verify $\sum_{n=0}^{\infty} |a_n|$ is convergent.