Chap 21. Improper integrals

21.1 Basic definitions

Integral expressions such as

$$\int_1^\infty \frac{1}{x^2}\,dx \quad \text{or} \quad \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx \quad \text{are not (ordinary) Riemann integrals}$$

because the interval $[1, \infty)$ of the first is not finite & the integrand $\dfrac{1}{\sqrt{1-x^2}}$ of the second in not bounded on $(0, 1]$.

Def. • Improper integral of the first kind (제 1종 이상 (정)적분: 적분구간이 유한구간이 아님)

Assume that each $f(x)$ is bounded on the interval where the function is defined

$$\int_a^\infty f(x)\,dx \overset{\text{def}}{=} \lim_{R\to\infty}\int_a^R f(x)\,dx; \quad \int_{-\infty}^a f(x)\,dx \overset{\text{def}}{=} \lim_{R\to\infty}\int_{-R}^a f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx \overset{\text{def}}{=} \lim_{\substack{R\to\infty\\ S\to\infty}}\int_{-S}^R f(x)\,dx = \lim_{R\to\infty}\int_0^R f(x)\,dx + \lim_{S\to\infty}\int_{-S}^0 f(x)\,dx;$$

Caution: $\displaystyle\int_{-\infty}^\infty f(x)\,dx \neq \lim_{R\to\infty}\int_{-R}^R f(x)\,dx$

• Improper integral of the second kind (제 2종 이상 (정)적분: 피적분함수가 적분구간에서 유계가 아님)

If the integrand becomes infinite $(\infty$ or $-\infty)$ at the right endpoint b, we define

$$\int_a^{b^-} f(x)\,dx \overset{\text{def}}{=} \lim_{u\to b^-}\int_a^u f(x)\,dx$$

If the integrand becomes infinite $(\infty$ or $-\infty)$ at the left endpoint a, we define

$$\int_{a^+}^b f(x)\,dx \overset{\text{def}}{=} \lim_{u\to a^+}\int_u^b f(x)\,dx$$

(the $+$ or $-$ sign is often dropped)

In each case, we say that the improper integral converges (diverges) if the limit exists (does not exist)

Remark. If the integrand becomes infinite $(\infty$ or $-\infty)$ at some point $c \in (a, b)$, we define

$$\int_a^b f(x)\,dx \overset{\text{def}}{=} \int_a^{c^-} f(x)\,dx + \int_{c^+}^b f(x)\,dx \overset{\text{i.e.}}{=} \lim_{u\to c^-}\int_a^u f(x)\,dx + \lim_{v\to c^+}\int_v^b f(x)\,dx$$

We say that the improper integral converges if both the limits exist.

Exa A. (Remember the result: studied in Calculus)

$$(a)\ \int_1^\infty \frac{1}{x^p}\,dx \begin{cases}\text{converges} & \text{if } p > 1\\ \text{diverges} & \text{if } p \le 1\end{cases} \qquad (b)\ \int_{0^+}^1 \frac{1}{x^p}\,dx \begin{cases}\text{converges} & \text{if } p < 1\\ \text{diverges} & \text{if } p \ge 1\end{cases}$$

Remark.

$$\int_{a^+}^\infty f(x)\,dx \overset{\text{def}}{=} \int_{a^+}^b f(x)\,dx + \int_b^\infty f(x)\,dx \ \ (\forall b > a) \overset{\text{or}}{=} \lim_{R\to\infty,\ u\to a^+}\int_u^R f(x)\,dx$$

Exa B. Does $\int_{-\infty}^{\infty} \frac{t}{1+t^2}\,dt$ converge?

Sol. $\int_{-\infty}^{\infty} \frac{t}{1+t^2}\,dt = \lim_{S\to\infty,\,R\to\infty} \int_{-S}^{R} \frac{t}{1+t^2}\,dt \overset{\text{clear}}{=} \lim_{R\to\infty} \int_{0}^{R} \frac{t}{1+t^2}\,dt + \lim_{S\to\infty} \int_{-S}^{0} \frac{t}{1+t^2}\,dt$

$$= \underbrace{\lim_{R\to\infty} \frac{1}{2}\ln(1+R^2) - \lim_{S\to\infty} \frac{1}{2}\ln(1+S^2)}_{\text{each limit does not exist, so the integral diverges}}$$

Def (Cauchy's Principal value)

$$\mathrm{CPV}\int_{-\infty}^{\infty} \frac{t}{1+t^2}\,dt \overset{\text{def}}{=} \lim_{R\to\infty} \int_{-R}^{R} \frac{t}{1+t^2}\,dt = \lim_{R\to\infty} \frac{1}{2}\ln(1+t^2)\Big]_{-R}^{R} = 0$$

$$\mathrm{CPV}\int_{-1}^{1} \frac{1}{x}\,dx \overset{\text{def}}{=} \lim_{\varepsilon\to 0^+} \left(\int_{-1}^{-\varepsilon} \frac{1}{x}\,dx + \int_{\varepsilon}^{1} \frac{1}{x}\,dx \right) \overset{\frac{1}{x}\text{ is an odd function}}{=} 0$$

Note: $\int_{-1}^{1} \frac{1}{x}\,dx = \int_{-1}^{0^-} \frac{1}{x}\,dx + \int_{0^+}^{1} \frac{1}{x}\,dx$ diverges since each limit does not exist.

※ Ex

① Is $\int_{2^+}^{4} \frac{1}{\sqrt{x-2}}\,dx$ convergent?

Sol. The behavior of $\frac{1}{\sqrt{x-2}}$ for $x \approx 2^+$ is the same as that of $\frac{1}{\sqrt{x}}$ for $x \approx 0^+$

Since $\int_{0^+}^{1} \frac{1}{\sqrt{x}}\,dx$ converges, $\int_{2^+}^{3} \frac{1}{\sqrt{x-2}}\,dx$ is also convergent.

$$\therefore \int_{2^+}^{4} \frac{1}{\sqrt{x-2}}\,dx = \int_{2^+}^{3} \frac{1}{\sqrt{x-2}}\,dx + \underbrace{\int_{3}^{4} \frac{1}{\sqrt{x-2}}\,dx}_{\text{integrable since } \frac{1}{\sqrt{x-2}} \text{ is conti on } [3,\,4]} \quad \text{is convergent}$$

Remark.

$$\int_{2^+}^{4} \frac{1}{\sqrt{x-2}}\,dx = \underbrace{\int_{2^+}^{2+\varepsilon} \frac{1}{\sqrt{x-2}}\,dx}_{\| \atop \int_{0+}^{\varepsilon} \frac{1}{\sqrt{x}}\,dx\text{:converges}} + \underbrace{\int_{2+\varepsilon}^{4} \frac{1}{\sqrt{x-2}}\,dx}_{\text{integrable since } \frac{1}{\sqrt{x-2}} \text{ is conti on } [2+\varepsilon,\,4]}$$

$\therefore \int_{2^+}^{4} \frac{1}{\sqrt{x-2}}\,dx$ is convergent.

② Is $\int_{0}^{1^-} \frac{1}{\sqrt{1-x^2}}\,dx$ convergent?

Sol. The behavior of $\frac{1}{\sqrt{1-x^2}}$ for $x \approx 1^-$ is the same as that of $\frac{1}{\sqrt{2(1-x)}}$ for $x \approx 1^-$.

It is also the same as that of $\dfrac{1}{\sqrt{2x}}$ for $x \approx 0^+$

Since $\displaystyle\int_{0^+}^{1} \frac{1}{\sqrt{2x}}\,dx$ converges, $\displaystyle\int_0^{1^-} \frac{1}{\sqrt{1-x^2}}\,dx$ is also convergent.

Remark.

$$\int_0^{1^-} \frac{1}{\sqrt{1-x^2}}\,dx = \underbrace{\int_0^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}}\,dx}_{\text{integrable since } \frac{1}{\sqrt{1-x^2}} \text{ is conti on } [0,1-\varepsilon]} + \underbrace{\int_{1-\varepsilon}^{1^-} \frac{1}{\sqrt{1-x^2}}\,dx}_{(*)}$$

$$(*) \approx \int_{1-\varepsilon}^{1^-} \frac{1}{\sqrt{2(1-x)}}\,dx \overset{1-x=t}{=} \int_{0^+}^{\varepsilon} \frac{1}{\sqrt{2t}}\,dt : \ \text{converges}$$

$\therefore\ \displaystyle\int_0^{1^-} \frac{1}{\sqrt{1-x^2}}\,dx$ is convergent.

③ Is $\displaystyle\int_0^{\infty} \frac{x^2}{1+x^3}\,dx$ convergent?

Sol.

$$\int_0^{\infty} \frac{x^2}{1+x^3}\,dx = \underbrace{\int_0^{R} \frac{x^2}{1+x^3}\,dx}_{\text{integrable since the integrand is conti on [0, R]}} + \int_R^{\infty} \frac{x^2}{1+x^3}\,dx \ (R \gg 1)$$

The behavior of $\dfrac{x^2}{1+x^3}$ for $x \gg 1$ is the same as that of $\dfrac{1}{x}$ for $x \gg 1$ because of

$$\frac{\dfrac{x^2}{1+x^3}}{\dfrac{1}{x}} = \frac{x^3}{1+x^3} \to 1 \ \text{ as } \ x \to \infty$$

But $\displaystyle\int_R^{\infty} \frac{1}{x}\,dx$ $(R \gg 1)$ diverges, and thus $\displaystyle\int_0^{\infty} \frac{x^2}{1+x^3}\,dx$ is also divergent.

Ex. Is $\displaystyle\int_0^{1} \frac{1}{\sqrt{x(1-x^2)}}\,dx$ convergent?

Sol. Note that the integral is improper at $x = 0$ and $x = 1$.

Thus we split the integral as

$$\int_0^{1} \frac{1}{\sqrt{x(1-x^2)}}\,dx$$
$$= \underbrace{\int_0^{\varepsilon_1} \underbrace{\frac{1}{\sqrt{x(1-x^2)}}}_{\approx \frac{1}{\sqrt{x}}}\,dx}_{\text{conv}} + \underbrace{\int_{\varepsilon_1}^{1-\varepsilon_2} \frac{1}{\sqrt{x(1-x^2)}}\,dx}_{\text{integrable}} + \underbrace{\int_{1-\varepsilon_2}^{1} \underbrace{\frac{1}{\sqrt{x(1-x^2)}}}_{\approx \frac{1}{\sqrt{2(1-x)}}}\,dx}_{\text{conv as before}}$$

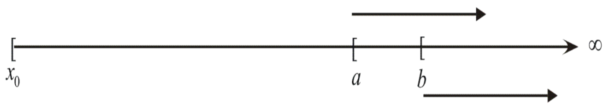
21.2 Comparison theorems

The improper integrability (convergence) of $\int_0^\infty f(x)\,dx$ is similar to the convergence of the infinite series $\sum_0^\infty a_n$ [or $\sum_0^\infty a(n)$]

Thm A (Tail-convergence theorem)

If $f(x)$ is "integrable" (= "locally integrable" in most of texts) on $I=[x_0,\infty)$ ($\overset{\text{means}}{\Leftrightarrow}$ $f(x)$ is integrable on every compact subinterval of I), and $a, b \in I$, then

$$\int_a^\infty f(x)\,dx \ \text{ converges } \ \Leftrightarrow \ \int_b^\infty f(x)\,dx \ \text{ converges}$$



(There are similar statements for the other kinds of improper integrals)

Pf. For R large enough,

$$\int_a^R f(x)\,dx = \underbrace{\int_a^b f(x)\,dx}_{(*)} + \int_b^R f(x)\,dx \quad \text{(by the Interval addition theorem)}$$

$(*)$ is a fixed finite value since $f(x)$ is integrable on every compact suninterval of I

Thus

$$\lim_{R\to\infty}\int_a^R f(x)\,dx = \lim_{R\to\infty}\left(\int_a^b f(x)\,dx + \int_b^R f(x)\,dx\right) = \int_a^b f(x)\,dx + \lim_{R\to\infty}\int_b^R f(x)\,dx$$

Therefore,

$$\lim_{R\to\infty}\int_a^R f(x)\,dx \ \text{ exists } \ \Leftrightarrow \ \lim_{R\to\infty}\int_b^R f(x)\,dx \ \text{ exists}$$

In other words, $\int_a^\infty f(x)\,dx$ converges $\Leftrightarrow$ $\int_b^\infty f(x)\,dx$ converges

Proposition (A version of FLT & a version of LLT for functions)

(a) $f(x)$ is inc for $x \gg 1$, & $\lim_{x\to\infty} f(x) = L \quad \Rightarrow \quad f(x) \le L$ for $x \gg 1$

(b) $f(x)$ is inc and $f(x) \le B$ for $x \gg 1 \quad \Rightarrow \quad \lim_{x\to\infty} f(x)$ exists, and $\lim_{x\to\infty} f(x) \le B$

(There are similar statements for $\lim_{x\to a} f(x)$ etc.)

Pf. (a) Suppose the conclusion were false.

Then $\exists\ x_0 \gg 1$ s.t. $f(x_0) > L$.

Then $f(x) \ge f(x_0) > L$ for $x \ge x_0$ since f is $\uparrow$

Taking $\lim_{x\to\infty}$ $\Rightarrow$

$\lim_{x\to\infty} f(x) \ge f(x_0) > L$ by LLT for functions. This violates the given hypo.

(b) By hypo, $f(n)$ is $\uparrow$ and $f(n) \leq B$ for natural numbers $n \gg 1$.
Then by "Completeness property" for sequences, $\lim\limits_{n\to\infty} f(n)$ exists; call it L.
Then by LLT for sequences,

$$\lim_{n\to\infty} f(n) \leq B.$$

It remains to prove: $L = \lim\limits_{x\to\infty} f(x)$.
Since $\lim\limits_{n\to\infty} f(n) = L$ and $f(n)$ is $\uparrow$, we have

given $\varepsilon > 0$, $L - \varepsilon < f(n) \leq L$ for $n \geq$ some N.

If $x > N$, let n' be an integer with $n' > x$; then

$$L - \varepsilon < \underbrace{f(N) \leq f(x) \leq f(n')}_{(\because\ f \text{ is } \uparrow \text{ for } x \gg 1)} \leq L$$

Therefore, $f(x) \underset{\varepsilon}{\approx} L$ for $x > N$ i.e., $\lim\limits_{x\to\infty} f(x) = L$.

Thm B (Comparison theorems for improper integrals)

(i) Assume $f(x)$ and $g(x)$ are locally integrable and
$0 \leq f(x) \leq g(x)$, for $x \geq a$.
Then

$$\int_a^\infty g(x)\,dx \text{ converges } \Rightarrow \int_a^\infty f(x)\,dx \text{ converges}$$

and $$\int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx$$

(ii) Assume $f(x)$ and $g(x)$ are locally integrable and
$0 \leq f(x) \leq g(x)$, on $[a, b)$.
Then

$$\int_a^{b^-} g(x)\,dx \text{ converges } \Rightarrow \int_a^{b^-} f(x)\,dx \text{ converges}$$

and $$\int_a^{b^-} f(x)\,dx \leq \int_a^{b^-} g(x)\,dx$$

Pf. We prove only (i).
Since $0 \leq f(x) \leq g(x)$ for $x \geq a$,

$$\int_a^R f(x)\,dx \leq \underbrace{\int_a^R g(x)\,dx}_{(*)}$$

$$\underset{\text{Proposition-(a)}}{\overset{(*) \text{ is } \uparrow \text{ as a ft of } R \text{ since } g(x)\geq 0 \text{ for } x\geq a}{\leq}} \quad \lim_{R\to\infty} \int_a^R g(x)\,dx = \int_a^\infty g(x)\,dx$$

$$\therefore \int_a^R f(x)\,dx \text{ is } \uparrow \quad \& \quad \int_a^R f(x)\,dx \leq \underbrace{\int_a^\infty g(x)\,dx}_{\substack{\| \\ B}}$$

Thus by Proposition-(b),

$$\lim_{R\to\infty} \int_a^R f(x)\,dx \text{ exists } \& \quad \lim_{R\to\infty} \int_a^R f(x)\,dx \leq \int_a^\infty g(x)\,dx$$

$$\text{i.e., } \int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx$$

A short way:

$$\forall R > a, \quad \int_a^R f(x)\,dx \le \int_a^R g(x)\,dx \le \underbrace{\int_a^\infty g(x)\,dx}_{\text{indep of } R}$$

Taking $R \to \infty \quad \Rightarrow \quad \lim_{R\to\infty} \int_a^R f(x)\,dx \le \int_a^\infty g(x)\,dx$ [LLT for fcts]

$$\text{i.e., } \int_a^\infty f(x)\,dx \le \int_a^\infty g(x)\,dx$$

Exa A. Does $\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx$ converge?

Sol. The integral is improper at both ends. So both must be studied separately.

Write $\int_0^\infty = \int_0^1 + \int_1^\infty$

$$x > 1 \Rightarrow \frac{e^{-x^2}}{\sqrt{x}} < e^{-x} \quad \overset{\text{Comparison Thm B}}{\Rightarrow} \quad \int_1^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx \le \underbrace{\int_1^\infty e^{-x}\,dx}_{\text{convergent}}$$

$$0 < x \le 1 \Rightarrow \frac{e^{-x^2}}{\sqrt{x}} < \frac{1}{\sqrt{x}} \quad \overset{\text{Comparison Thm B}}{\Rightarrow} \quad \int_0^1 \frac{e^{-x^2}}{\sqrt{x}}\,dx \le \underbrace{\int_0^1 \frac{1}{\sqrt{x}}\,dx}_{\text{convergent}}$$

$\therefore \quad \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx$ is convergent.

Exa B. Does $\int_0^1 \frac{\ln^3 x}{\sqrt{x}}\,dx$ converge?

Sol.

$$\int_0^1 \frac{\ln^3 x}{\sqrt{x}}\,dx \underset{\text{i.e., } x=e^u}{\overset{\ln x = u}{=}} \int_{-\infty}^0 u^3 e^{u/2}\,du = -\int_0^\infty t^3 e^{-t/2}\,dt$$

$$\lim_{t\to\infty} \frac{t^3}{e^{t/4}} \overset{\text{L'Hospital}}{=} 0 \quad \Rightarrow \quad \frac{t^3}{e^{t/4}} < 1 \ \text{ for } \ t \gg 1 \quad \Rightarrow \quad t^3 e^{-t/2} < e^{-t/4} \ \text{ for } \ t \gg 1$$

$$\therefore \quad a \gg 1 \quad \Rightarrow \quad \int_a^\infty \underbrace{t^3 e^{-t/2}}_{\ge 0}\,dt < \underbrace{\int_a^\infty \underbrace{e^{-t/4}}_{\ge 0}\,dt}_{\text{converges}}$$

$\therefore \quad \int_0^\infty t^3 e^{-t/2}\,dt$ is convergent by Tail-convergence theorem.

※ Thm C (Asymptotic comparison test for improper integrals); very useful

(i) Assume $f(x)$ and $g(x)$ are conti & ≥ 0 for $x \ge a$,

and $f(x) \sim g(x)$ as $x \to \infty$.

Then

$$\int_a^\infty f(x)\,dx \text{ converges} \quad \Leftrightarrow \quad \int_a^\infty g(x)\,dx \text{ converges}$$

(ii) Assume $f(x)$ and $g(x)$ are conti & ≥ 0 on $(a, b]$.

Suppose also that

$f(x)$ and $g(x) \to \infty$ as $x \to a^+$, and

$f(x) \sim g(x)$ at a^+ (i.e., $\frac{f(x)}{g(x)} \to 1$ as $x \to a^+$)

Then

$$\int_{a^+}^{b} f(x)\,dx \ \text{ converges } \quad \Leftrightarrow \quad \int_{a^+}^{b} g(x)\,dx \ \text{ converges}$$

Pf. Exercise

Exa C. Does $\displaystyle\int_0^\infty \frac{1}{\sqrt{x(1+x)}}\,dx$ converge?

Sol. Both ends are improper.

At 0, $\quad \displaystyle\frac{1}{\sqrt{x(1+x)}} \sim \frac{1}{\sqrt{x}}$ and $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges

At ∞, $\quad \displaystyle\frac{1}{\sqrt{x(1+x)}} \sim \frac{1}{x}$ and $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges

Thus by Asymptotic comparison test,

$$\int_0^\infty \frac{1}{\sqrt{x(1+x)}}\,dx \ \text{ is divergent}$$

21.3 The Gamma function(; the generalized factorial function)

Motivation: $\quad \displaystyle\int_0^\infty t^n e^{-t}\,dt = n! \quad \text{for} \quad n = 0,1,2,\cdots \quad (0! = 1)$

A natural idea for deriving the above formula:

$$a > 1 \quad \Rightarrow \quad \int_0^\infty a^{-t}\,dt = -\frac{a^{-t}}{\ln a}\Big|_{t=0}^{t=\infty} = \frac{1}{\ln a}$$

(In particular(taking $a = e$), $\displaystyle\int_0^\infty e^{-t}\,dt = 1$)

Differentiate both sides w.r.t. $a \quad \Rightarrow$

$$\frac{d}{da}\int_0^\infty a^{-t}\,dt \overset{\text{expect}}{=} \int_0^\infty \frac{\partial}{\partial a}(a^{-t})\,dt = \int_0^\infty -ta^{-t-1}\,dt = -\frac{1/a}{(\ln a)^2}$$

$$\therefore \quad \int_0^\infty ta^{-t}\,dt = \frac{1}{(\ln a)^2}$$

Differentiate both sides w.r.t. a again $\quad \Rightarrow$

$$\int_0^\infty -t^2a^{-t-1}\,dt = -2\frac{1/a}{(\ln a)^3}$$

$$\therefore \quad \int_0^\infty t^2a^{-t}\,dt = 2\frac{1}{(\ln a)^3}$$

Differentiate both sides w.r.t. a again $\quad \Rightarrow$

$$\int_0^\infty -t^3a^{-t-1}\,dt = -3!\frac{1/a}{(\ln a)^4}$$

$$\therefore \quad \int_0^\infty t^3a^{-t}\,dt = 3!\frac{1}{(\ln a)^4}$$

Repeat this process to obtain

$$\int_0^\infty t^n a^{-t}\,dt = n!\frac{1}{(\ln a)^{n+1}} \quad \text{for } n = 0,1,2,\cdots$$

Takeing $a = e$ gives $\boxed{(*):\quad \int_0^\infty t^n e^{-t}\,dt = n! \quad \text{for } n = 0,1,2,\cdots}$

A rigorous proof of $(*)$:

$$n = 0: \quad \int_0^\infty e^{-t}\,dt = \lim_{R\to\infty}\int_0^R e^{-t}\,dt = 1(\text{easy})$$

$n \geq 1$:

$$\int_0^R \underbrace{t^n}_{\text{그}}\underbrace{e^{-t}}_{\text{적}}\,dt \overset{\text{부분적분}}{=} -t^n e^{-t}\Big]_0^R + n\int_0^R t^{n-1}e^{-t}\,dt = -R^n e^{-R} + n\int_0^R t^{n-1}e^{-t}\,dt$$

$R \to \infty \quad \Rightarrow$

$$\lim_{R\to\infty}\int_0^R t^n e^{-n}\,dt = -\underbrace{\lim_{R\to\infty} R^n e^{-R}}_{=0 \text{ by L'Hospital}} + n\lim_{R\to\infty}\int_0^R t^{n-1}e^{-t}\,dt$$

$$\begin{aligned}\therefore \quad \int_0^\infty t^n e^{-t}\,dt &= n\int_0^\infty t^{n-1}e^{-t}\,dt \\ &\overset{\text{same argument}}{=} n(n-1)\int_0^\infty t^{n-2}e^{-t}\,dt \\ &= \cdots \\ &= n(n-1)(n-2)\cdots 1\underbrace{\int_0^\infty e^{-t}\,dt}_{=1} = n!\end{aligned}$$

Def. For $x > 0$, $\Gamma(x) \overset{\text{def}}{=} \int_0^\infty t^{x-1}e^{-t}\,dt$

$\Gamma(x)$ is called the Gamma function (or the generalized factorial function)

⦿ Some properties of the Gamma function

G1 $\Gamma(n+1) = n!$ for all integers $n \geq 0$; already seen

G2 $\Gamma(x)$ is convergent for $x > 0$

Pf. Note that the integral is improper at both ends

$$\Gamma(x) = \int_{0^+}^1 t^{x-1}e^{-t}\,dt + \int_1^\infty t^{x-1}e^{-t}\,dt$$

$$t \gg 1 \ (\text{say } t \geq M) \quad \Rightarrow \quad t^{x-1} < e^{t/2} \quad \forall x \in \mathbb{R}$$

(because $\lim_{t\to\infty}\frac{t^{x-1}}{e^{t/2}}\begin{cases} = 0 & \forall x > 1 \ (\text{by L'Hospital}) \\ = 0 & \forall x \leq 1 \ (\text{trivial})\end{cases}$ Or, for any $x \in \mathbb{R}$, we have

$$\lim_{t\to\infty}\frac{(x-1)\ln t}{t} \overset{\text{L'Hospital}}{=} (x-1)\lim_{t\to\infty}\frac{1/t}{1} = 0 \quad \therefore \frac{(x-1)\ln t}{t} < 1/2 \quad \left(\text{i.e., } t^{x-1} < e^{t/2}\right) \quad \text{for } t \gg 1\,)$$

$$\therefore \quad \int_M^\infty t^{x-1}e^{-t}\,dt \le \int_M^\infty e^{-t/2}\,dt \ \text{ is convergent}$$

$$\therefore \quad \int_1^\infty t^{x-1}e^{-t}\,dt = \underbrace{\int_1^M \underbrace{t^{x-1}e^{-t}}_{\substack{\text{conti on [1, M]} \\ \therefore\ \text{integrable}}}\,dt + \underbrace{\int_M^\infty t^{x-1}e^{-t}\,dt}_{\text{convergent}}}_{\therefore\ \text{convergent}}$$

$$0 < t < 1 \quad \Rightarrow \quad t^{x-1}e^{-t} \le t^{x-1}$$

$$\Rightarrow \quad \int_{0^+}^1 t^{x-1}e^{-t}\,dt \le \int_{0^+}^1 t^{x-1}\,dt \ \text{ is convergent if } x-1>-1 \ \text{ i.e., if } x>0$$

By Comparison theorem,

$$\int_{0^+}^1 t^{x-1}e^{-t}\,dt \ \text{ is convergent for } \ x>0.$$

G3 $\quad \Gamma(x+1) = x\Gamma(x)$ for $x > 0$

Pf. $\Gamma(x+1) = \lim_{R\to\infty}\int_0^R t^x e^{-t}\,dt$ [cf: not improper at $t=0$, since $x>0$]

Someone often regards the integral $\int_0^R t^x e^{-t}\,dt$ as $\lim_{u\to 0^+}\int_u^R t^x e^{-t}\,dt$, even when $x>0$

$$\begin{aligned}
\int_0^R \underbrace{t^x}_{\text{그}}\underbrace{e^{-t}}_{\text{적}}\,dt &\overset{\text{부분적분}}{=} -t^x e^{-t}\Big]_0^R + x\int_0^R t^{x-1}e^{-t}\,dt \\
&= \left(-R^x e^{-R} + \lim_{t\to 0^+}\frac{t^x}{e^t}\right) + x\int_0^R t^{x-1}e^{-t}\,dt \\
&\xrightarrow{R\to\infty} x\int_0^\infty t^{x-1}e^{-t}\,dt
\end{aligned}$$

Here we used:

$$\lim_{t\to 0^+}\frac{t^x}{e^t} = 0 \qquad \text{and} \qquad \lim_{R\to\infty} R^x e^{-R} = \lim_{R\to\infty}\frac{R^x}{e^R} \overset{\text{L'Hospital}}{=} 0$$

$$\therefore \quad \Gamma(x+1) = \lim_{R\to\infty}\int_0^R t^x e^{-t}\,dt = x\int_0^\infty t^{x-1}e^{-t}\,dt = x\Gamma(x)$$

G4 $\quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

$$\text{Pf.} \quad \Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\,dt = \lim_{R\to\infty,u\to 0^+}\int_u^R \frac{e^{-t}}{\sqrt{t}}\,dt \overset{t=s^2}{=} \lim_{R\to\infty,u\to 0^+}\int_{\sqrt{u}}^{\sqrt{R}} \frac{e^{-s^2}}{s}2sds$$

$$= 2\int_0^\infty e^{-s^2}\,ds \overset{\text{use double integral}}{=} 2\cdot\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

G5 $\lim_{x\to 0^+}\Gamma(x)=\infty$ $\quad(\therefore\ \Gamma(x)$ is not convergent at $x=0)$

Pf. For $x>0$,
$$\begin{aligned}\Gamma(x)&=\int_{0^+}^{\infty} t^{x-1}e^{-t}\,dt\\ &>\int_{0^+}^{1} t^{x-1}e^{-t}\,dt \quad \text{since the integrand is positive}\\ &\geq \frac{1}{e}\int_{0^+}^{1} t^{x-1}\,dt \quad \left(\leftarrow\quad e^{-t}\geq\frac{1}{e}\ \text{ on } (0,1]\right)\\ &\overset{\text{since } x>0}{=} \frac{1}{e}\cdot\frac{1}{x}\quad (\to\infty\quad\text{as}\quad x\to 0^+)\end{aligned}$$
$$\therefore\quad \lim_{x\to 0^+}\Gamma(x)=\infty$$

G6 $\Gamma(x)$ is continuous, for all $x>0$

Pf. Exercise

⊙ An extension of the definition of $\Gamma(x)$: Optional

$$\Gamma(x)=\int_0^{\infty} t^{x-1}e^{-t}\,dt \quad \text{is not convergent for}\quad x\leq 0$$

However, the definition of $\Gamma(x)$ is extended to non-integer $x\leq 0$ as follows:

Using G3, we get

$$\text{for}\quad x>0,\qquad \Gamma(x+1)=x\Gamma(x)$$

If $0<x+1$, then $\Gamma(x+1)$ is defined (convergent). Thus if $0<x+1$ and $x\neq 0$, we can define $\Gamma(x)\equiv\dfrac{\Gamma(x+1)}{x}$

In particular, for $-1<x<0$, $\Gamma(x)=\dfrac{\Gamma(x+1)}{x}$

We know $\Gamma(x+2)=(x+1)\Gamma(x+1)=(x+1)x\Gamma(x)$ for $x>0$

Thus if $0<x+2$ and $x\neq 0,\ -1$, we can define $\Gamma(x)\equiv\dfrac{\Gamma(x+2)}{(x+1)x}$

In particular, for $-2<x<0\ \&\ x\neq -1$, $\Gamma(x)=\dfrac{\Gamma(x+2)}{(x+1)x}$

In general, for integer $n\geq 0$,

$$\Gamma(x+n)=(x+n-1)\cdots(x+1)x\Gamma(x)\quad \text{for } x>0$$

Thus if $0<x+n$ and $x\neq 0,\ -1,\ \cdots,\ -(n-1)$, we define

$$\Gamma(x)=\frac{\Gamma(x+n)}{(x+n-1)\cdots(x+1)x}$$

In other words, for $\forall x\in\mathbb{R}$ with $x\neq 0,-1,-2,\cdots,$ we define

$$\Gamma(x)=\frac{\Gamma(x+n)}{(x+n-1)\cdots(x+1)x}\qquad (x+n>0\ \text{ for some }\ n)$$

Applications.

Ex1. $\Gamma\left(\frac{5}{2}\right) = ?$

Sol. $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$

Ex2. $\Gamma\left(-\frac{1}{2}\right) = ?$

Sol. $\Gamma\left(-\frac{1}{2}+1\right) = -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right) \qquad \therefore \quad \Gamma\left(-\frac{1}{2}\right) = -2\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$

Ex3. $\Gamma(x) = 2\int_0^\infty t^{2x-1}e^{-t^2}dt \ (\text{for } x > 0) \ \overset{\text{or}}{=} \ \int_0^1 (-\ln t)^{x-1}\, dt \ (\text{for } x > 0)$

In particular, $\int_0^1 \sqrt{-\ln t}\, dt = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$ and $\int_0^1 \frac{dt}{\sqrt{-\ln t}} = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Pf. $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\, dt :$ $\quad t = u^2 \ \rightarrow$ first; $\quad t = -\ln u \ \rightarrow$ second

21.4 Absolute and conditional convergence

Def. $\int_a^\infty f(x)\, dx$ converges absolutely if $\int_a^\infty |f(x)|\, dx$ converges

$\int_a^\infty f(x)\, dx$ converges conditionally if $\int_a^\infty f(x)\, dx$ converges, but $\int_a^\infty |f(x)|\, dx$ diverges.

Theorem (Absolute convergence theorem for improper integrals): very useful

If $f(x)$ is locally integrable on $[a, \infty)$, and $\int_a^\infty f(x)\, dx$ is absolutely convergent, then it is convergent.

Pf.
$$f^+(x) \overset{\text{def}}{=} \max\{f(x), 0\} = \frac{|f(x)| + f(x)}{2}$$
$$f^-(x) \overset{\text{def}}{=} \max\{-f(x), 0\} = \frac{|f(x)| - f(x)}{2}$$
(Draw each picture)

Then $f(x) = f^+(x) - f^-(x)$ (note that $f^+(x),\ f^-(x) \geq 0$)

It is clear that
$$0 \leq f^+(x) \leq |f(x)|, \qquad 0 \leq f^-(x) \leq |f(x)| \quad ---(*)$$

If $f(x)$ is locally integrable on $[a, \infty)$, then we see that $|f(x)|$ is locally integrable on $[a, \infty)$,

and so by $(*)$, $f^+(x)$ and $f^-(x)$ are also locally integrable on $[a, \infty)$.

Since $\int_a^{\infty} |f(x)|\,dx$ is convergent (by hypo), $(*)$ and Comparison theorem imply that

$$\int_a^{\infty} f^+(x)\,dx \quad \text{and} \quad \int_a^{\infty} f^-(x)\,dx \quad \text{are also convergent.}$$

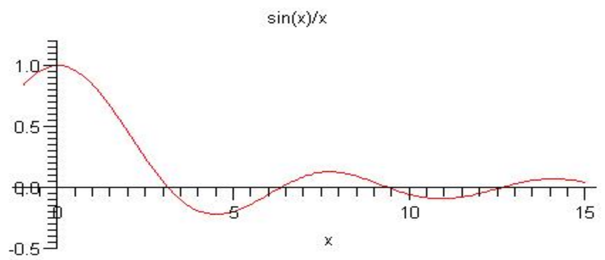
Therefore,

$$\int_a^{\infty} f(x)\,dx \underset{f=f^+-f^-}{=} \int_a^{\infty} f^+(x)\,dx - \int_a^{\infty} f^-(x)\,dx \quad \text{is also convergent}$$

Ex. Prove that $\int_0^{\infty} \frac{\sin x}{x}\,dx$ converges conditionally.

Sol. Recall that $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$. Thus if we (re)define $\left.\frac{\sin x}{x}\right|_{x=0} \overset{\text{def}}{=} 1$, then $\frac{\sin x}{x}$ becomes a continuous function on $[0, \infty)$. In particular, $\frac{\sin x}{x}$ is integrable on $[0, \pi]$.

$$\therefore \quad \int_0^{\infty} \frac{\sin x}{x}\,dx \quad \text{is not improper at } 0.$$



In fact, according to the Endpoint Lemma, $\int_0^{\infty} \frac{\sin x}{x}\,dx$ would not be improper at 0 even if we define $\left.\frac{\sin x}{x}\right|_{x=0} \overset{\text{def}}{=} \text{any real number}$. In other words, $\frac{\sin x}{x}$ is integrable on $[0, \pi]$ for any choice of the value of $\frac{\sin x}{x}$ at 0. Therefore,

$$\int_0^{\infty} \frac{\sin x}{x}\,dx \quad \text{converges if and only if} \quad \int_{\pi}^{\infty} \frac{\sin x}{x}\,dx \quad \text{converges}$$

We first show: $\int_0^{\infty} \left|\frac{\sin x}{x}\right| dx$ is not convergent.

To prove this, it suffices to show

$$\int_{\pi}^{\infty} \left|\frac{\sin x}{x}\right| dx \quad \text{is not convergent, since} \quad \left|\frac{\sin x}{x}\right| \quad \text{is integrable on } [0, \pi].$$

Let $A_n = \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x} dx$. It is clear that

$$\int_{\pi}^{R} \frac{|\sin x|}{x} dx > \sum_{n=1}^{N} \underbrace{\int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x} dx}_{=|A_n| \text{ (why ?)}} \quad \text{if } R > (N+1)\pi$$

For every $n \geq 1$, $\quad |A_n| = \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x} dx \geq \frac{1}{(n+1)\pi} \int_{n\pi}^{(n+1)\pi} |\sin x| dx$

$$= \frac{1}{(n+1)\pi} \int_{0}^{\pi} |\sin x| dx = \frac{1}{(n+1)\pi} \int_{0}^{\pi} \sin x \, dx$$

$$= \frac{2}{(n+1)\pi} > \frac{2}{(n+1)4} = \frac{1}{2n+2}$$

$$\therefore \quad \int_{\pi}^{R} \frac{|\sin x|}{x} dx > \sum_{n=1}^{N} \frac{1}{2n+2} \quad \text{if } R > (N+1)\pi$$

$$\geq \sum_{n=1}^{N} \frac{1}{4n} = \frac{1}{4} \sum_{n=1}^{N} \frac{1}{n} \to \infty \quad \text{as } N \to \infty \ (\Rightarrow R \to \infty)$$

$\therefore \lim_{R\to\infty} \int_{\pi}^{R} \frac{|\sin x|}{x} dx = \infty$ i.e., $\int_{\pi}^{\infty} \left|\frac{\sin x}{x}\right| dx$ is not convergent.

Using **integration by parts**, we next show: $\int_{\pi}^{\infty} \frac{\sin x}{x} dx$ is convergent.

$$\int_{\pi}^{R} \frac{\sin x}{x} dx \overset{\text{integration by parts}}{=} -\frac{\cos x}{x}\Big]_{\pi}^{R} - \int_{\pi}^{R} \frac{\cos x}{x^2} dx$$

$$-\frac{\cos x}{x}\Big]_{\pi}^{R} = -\frac{\cos R}{R} - \frac{1}{\pi} \to -\frac{1}{\pi} \quad \text{as } R \to \infty$$

But $\int_{\pi}^{R} \left|\frac{\cos x}{x^2}\right| dx \leq \int_{\pi}^{R} \frac{1}{x^2} dx \leq \int_{\pi}^{\infty} \frac{1}{x^2} dx$ is convergent

$\therefore \int_{\pi}^{\infty} \left|\frac{\cos x}{x^2}\right| dx$ is convergent (by the Comparison theorem for improper integrals)

$\therefore \int_{\pi}^{\infty} \frac{\cos x}{x^2} dx$ is convergent (by the Absolute convergence theorem for improper integrals)

Therefore, $\int_{\pi}^{\infty} \frac{\sin x}{x} dx$ is convergent.

Ex. Give an another proof of the fact that $\int_{\pi}^{\infty} \frac{\sin x}{x} dx$ is convergent .

Hint: Use Alternating series test

Remark. It is known that $\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$ (but its calculus proof is not easy)

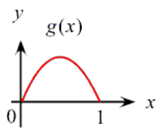
The most popular proof of $\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$ will be studied in complex analysis

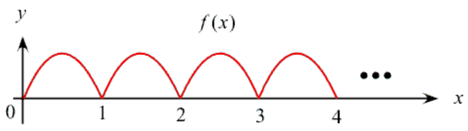
Additional exercises:

Ex1. Give an example of a continuous function f on $[0, \infty)$ with the property that $\sum_{1}^{\infty} f(n)$ converges, but $\int_0^\infty f(x)dx$ diverges

Sol. Let $g(x) = x(1-x)$, for $0 \le x \le 1$. Define, for $x \ge 0$,

$$f(x) = g(x)\chi_{[0,1]}(x) + g(x-1)\chi_{[1,2]}(x) + \cdots + g(x-n)\chi_{[n,n+1]}(x) + \cdots = \sum_{n=0}^{\infty} g(x-n)\chi_{[n,n+1]}(x)$$





Note that $f(n) = 0$ for $n = 0, 1, 2, \cdots$.

Thus $\sum_{1}^{\infty} f(n) = 0$ is trivially convergent, but for $N \in \mathbb{N}$, we have

$$\int_0^N f(x)dx = N\int_0^1 x(1-x)dx = \frac{N}{6} \to \infty; \quad \text{so} \quad \int_0^\infty f(x)dx \quad \text{is divergent.}$$

Ex2. We have seen that "$\sum a_n$ converges $\Rightarrow$ $a_n \to 0$ as $n \to \infty$"

Its integral analogue would say: $\int_a^\infty f(x)\,dx$ converges $\Rightarrow$ $\lim_{x\to\infty} f(x) = 0$.

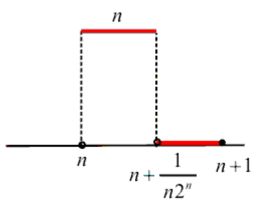
Give a proof or counterexample.

(i) Assume $f(x)$ is nonnegative

(ii) Assume $f(x)$ is nonnegative and continuous

Sol. (i) We cannot say that $\lim_{x\to\infty} f(x) = 0$ even if $\int_a^\infty f(x)\,dx$ converges:

Define $f(x) = \sum_{n=1}^{\infty} n\chi_{[n, n+\frac{1}{n2^n}]}(x)$.



Then $f \ge 0$ on $[1, \infty)$

$$\int_1^\infty f(x)\,dx = \sum_{n=1}^{\infty} n \cdot \frac{1}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

Thus $\int_1^\infty f(x)\,dx$ converges, but clearly f is not bounded, so $\lim_{x\to\infty} f(x)$ does not exist.

(ii) Exercise

Cf: It can be shown that $\int_0^\infty \sin(x^2)\,dx \left(\overset{x^2=t}{=} \int_0^\infty \frac{\sin t}{2\sqrt{t}}dt \right) \left(\sin(x^2) \text{ need not be } \ge 0 \text{ on } [0, \infty) \right)$

is convergent, but clearly $\lim_{x\to\infty} \sin(x^2)$ does not exist.

It is more safe to start with $\int_0^\infty \sin(x^2)\,dx = \int_0^{\sqrt{\pi}} \sin(x^2)\,dx + \int_{\sqrt{\pi}}^\infty \sin(x^2)\,dx$