1. $y_1, \cdots, y_n \overset{iid}{\sim} N(\theta, \sigma^2)$ ← known

$\theta \sim N(\mu_0, \tau_0^2)$

- The posterior pdf is

$$p(\theta|y_1,\cdots,y_n) \propto \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\theta)^2\right\} \times \exp\left\{-\frac{1}{2\tau_0^2}(\theta-\mu_0)^2\right\}$$

$$= \exp\left\{-\frac{1}{2\sigma^2}\left[n(\bar{y}-\theta)^2 + \sum_{i=1}^n (y_i-\bar{y})^2\right] - \frac{1}{2\tau_0^2}(\theta-\mu_0)^2\right\}$$

$\uparrow$

$y_i - \theta = \bar{y} - \theta + y_i - \bar{y}$

$$\propto \exp\left\{-\frac{1}{2\sigma^2}\left[n\theta^2 - 2n\bar{y}\theta\right] - \frac{1}{2\tau_0^2}\left[\theta^2 - 2\mu_0\theta\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left[\left(\frac{n}{\sigma^2}+\frac{1}{\tau_0^2}\right)\theta^2 - 2\left(\frac{n}{\sigma^2}\bar{y} + \frac{\mu_0}{\tau_0^2}\right)\theta\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left(\frac{n}{\sigma^2}+\frac{1}{\tau_0^2}\right)\left[\theta^2 - 2\frac{\frac{n}{\sigma^2}\bar{y}+\frac{\mu_0}{\tau_0^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau_0}}\theta + \left(\frac{\frac{n}{\sigma^2}\bar{y}+\frac{\mu_0}{\tau_0^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau_0^2}}\right)^2 - \left(\frac{\frac{n}{\sigma^2}\bar{y}+\frac{\mu_0}{\tau_0^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau_0^2}}\right)^2\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left(\frac{n}{\sigma^2}+\frac{1}{\tau_0^2}\right)\left(\theta - \frac{\frac{n\bar{y}}{\sigma^2}+\frac{\mu_0}{\tau_0^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau_0^2}}\right)^2\right\}$$

This is the kernel of $N(\mu_1, \tau_1^2)$

where $\mu_1 = \dfrac{\frac{n\bar{y}}{\sigma^2}+\frac{\mu_0}{\tau_0^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau_0^2}}$ and $\tau_1^2 = \left(\frac{n}{\sigma^2}+\frac{1}{\tau_0^2}\right)^{-1}$

- $\tilde{y}$ is a future observation

$$p(\tilde{y}|y_1,\cdots,y_n) = \int_{-\infty}^{\infty} p(\tilde{y}|\theta)\,p(\theta|y_1,\cdots,y_n)\,d\theta$$

$$\propto \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}(\tilde{y}-\theta)^2\right\} \exp\left\{-\frac{1}{2\tau_1^2}(\theta-\mu_1)^2\right\} d\theta$$

$$= \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}(\theta^2 - 2\tilde{y}\theta + \tilde{y}^2) - \frac{1}{2\tau_1^2}(\theta^2 - 2\mu_1\theta + \mu_1^2)\right\} d\theta$$

$$= \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left[\left(\frac{1}{\sigma^2}+\frac{1}{\tau_1^2}\right)\theta^2 - 2\left(\frac{\tilde{y}}{\sigma^2}+\frac{\mu_1}{\tau_1^2}\right)\theta + \left(\frac{\tilde{y}^2}{\sigma^2}+\frac{\mu_1^2}{\tau_1^2}\right)\right]\right\} d\theta$$

$$= \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma^2}+\frac{1}{\tau_1^2}\right)\left[\theta^2 - 2\,\frac{\frac{\tilde{y}}{\sigma^2}+\frac{\mu_1}{\tau_1^2}}{\frac{1}{\sigma^2}+\frac{1}{\tau_1^2}}\,\theta + \left(\frac{\frac{\tilde{y}}{\sigma^2}+\frac{\mu_1}{\tau_1^2}}{\frac{1}{\sigma^2}+\frac{1}{\tau_1^2}}\right)^2 - \left(\frac{\frac{\tilde{y}}{\sigma^2}+\frac{\mu_1}{\tau_1^2}}{\frac{1}{\sigma^2}+\frac{1}{\tau_1^2}}\right)^2\right]\right\}$$

$$\exp\left\{-\frac{1}{2}\left(\frac{\tilde{y}^2}{\sigma^2}+\frac{\mu_1^2}{\tau_1^2}\right)\right\}\, d\theta$$

$$= \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma^2}+\frac{1}{\tau_1^2}\right)\left(\theta - \frac{\frac{\tilde{y}}{\sigma^2}+\frac{\mu_1}{\tau_1^2}}{\frac{1}{\sigma^2}+\frac{1}{\tau_1^2}}\right)^2\right\} d\theta$$

$\rightarrow$ kernel of $N\left(\frac{\frac{\tilde{y}}{\sigma^2}+\frac{\mu_1}{\tau_1^2}}{\frac{1}{\sigma^2}+\frac{1}{\tau_1^2}},\ \left(\frac{1}{\sigma^2}+\frac{1}{\tau_1^2}\right)^{-1}\right)$

$$\exp\left\{\frac{1}{2}\,\frac{\left(\frac{\tilde{y}}{\sigma^2}+\frac{\mu_1}{\tau_1^2}\right)^2}{\frac{1}{\sigma^2}+\frac{1}{\tau_1^2}} - \frac{1}{2}\left(\frac{\tilde{y}^2}{\sigma^2}+\frac{\mu_1}{\tau_1^2}\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left(\frac{\tilde{y}^2}{\sigma^2} - \frac{\frac{\tilde{y}^2}{(\sigma^2)^2} + 2\,\frac{\mu_1}{\sigma^2\tau_1^2}\,\tilde{y}}{\frac{1}{\sigma^2}+\frac{1}{\tau_1^2}}\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\underbrace{\left(\frac{1}{\sigma^2} - \frac{\frac{1}{(\sigma^2)^2}}{\frac{1}{\sigma^2}+\frac{1}{\tau_1^2}}\right)}_{= \frac{1}{\sigma^2+\tau_1^2}}\tilde{y}^2 - 2\,\underbrace{\frac{\frac{\mu_1}{\sigma^2\tau_1^2}}{\frac{1}{\sigma^2}+\frac{1}{\tau_1^2}}}_{= \frac{\mu_1}{\sigma^2+\tau_1^2}}\,\tilde{y}\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\,\frac{1}{(\sigma^2+\tau_1^2)}\left(\tilde{y}^2 - 2\mu_1\tilde{y}\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\,\frac{1}{(\sigma^2+\tau_1^2)}\left(y - 2\mu_1\right)^2\right\}$$

$\therefore$ This is the kernel of $N(\mu_1,\ \sigma^2+\tau_1^2)$.