5.1 Areas and Distances

- the area of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

☆ it can be proved that the limit above always exists as long as f is continuous. It can also be shown that we get the same value if we use left endpoints

☆ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

5.2 The Definite Integral

Definite Integral :

- if f is a function defined for $a \le x \le b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0 (= a), x_1, \ldots, x_n (= b)$ be the endpoints of these subintervals and we let $x_1^*, \ldots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the i^{th} subinterval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i^*)\Delta x}_{\text{Riemann Sum}}$$

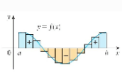

FIGURE 3
$\sum f(x_i^*)\,\Delta x$ is an approximation to the net area.

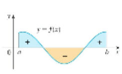

FIGURE 4
$\int_a^b f(x)\,dx$ is the net area.

☆ if f takes on both positive and negative values, then the Riemann Sum is the sum of the areas of the rectangles that lie above the x-axis and the negatives of the areas of the rectangles that lie below the x-axis

== the areas of the blue rectangles minus the areas of the gold rectangles

Theorem :

- if f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

- if f is integrable on $[a, b]$, then $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x$, where $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$

☆
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

- We often choose the sample point x_i^* to be the right endpoint of the i^{th} subinterval because it is convenient for computing the limit. But if the purpose is to find an approximation to an integral, it is usually better to choose x_i^* to be the midpoint of the interval.

Midpoint Rule :

$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x = \Delta x[f(\bar{x}_1)+f(\bar{x}_2)+\cdots+f(\bar{x}_n)]$, $\Delta x = \frac{b-a}{n}$, $\bar{x}_i = \frac{1}{2}(x_{i-1}+x_i)$

Properties of the Definite Integral

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

Properties of the Integral

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant
2. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant
4. $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

5. $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

Comparison Properties of the Integral

6. If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant 0$.
7. If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$.
8. If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a)$$

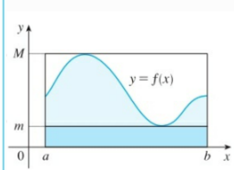

FIGURE 16

5.3 The Fundamental Theorem of Calculus

Part 1

- if f is continuous on $[a,b]$, then the function g defined by $g(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$

Part 2

- if f is continuous on $[a,b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, $F' = f$.

5.4 Indefinite Integrals and the Net Change Theorem

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C$$

5.5 The Substitution Rule

- if $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

The Substitution Rule for Definite Integrals

- if g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Integrals of Symmetric Functions

- Suppose f is continuous on $[-a, a]$

i) if f is even, $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

ii) if f is add, $\int_{-a}^{a} f(x)\,dx = 0$

Problem Plus pdf 425