Midterm 2022

1. $P(X_1=x_1, X_2=x_2 \mid X_1+X_2=y)$

$\longrightarrow$ does not depend on y if $y=0,1,3,4$

But, when $y=2$, $P(X_1+X_2=2) = P(X_1=1, X_2=1) + 2P(X_1=0, X_2=2)$

$= 2(1-e^{-\theta}-\theta e^{-\theta})e^{-\theta} + \theta^2 e^{-2\theta}$

$$\frac{P(X_1=x_1, X_2=x_2, X_1+X_2=2)}{P(X_1+X_2=2)}$$

$$= \frac{P(X_1=x_1, X_2=2-x_1)}{2(1-e^{-\theta}-\theta e^{-\theta})e^{-\theta}+\theta^2 e^{-2\theta}}$$

$$= \frac{P(X_1=x_1)P(X_2=2-x_1)}{2(1-e^{-\theta}-\theta e^{-\theta})e^{-\theta}+\theta^2 e^{-2\theta}}$$

(i) $x_1=0$, $x_2=2$

$$P(X_1=x_1, X_2=x_2 \mid X_1+X_2=y) = \frac{e^{-\theta}(1-e^{-\theta}-\theta e^{-\theta})}{2(1-e^{-\theta}-\theta e^{-\theta})e^{-\theta}+\theta^2 e^{-2\theta}} \Rightarrow \text{depends on } \theta$$

(ii) $x_1=2$, $x_2=2$ $\Rightarrow$ Similar to (i), $P(X_1=x_1, X_2=x_2 \mid X_1+X_2=y)$ depends on θ

(iii) $x_1=1$, $x_2=1$

$$P(X_1=x_1, X_2=x_2 \mid X_1+X_2=y) = \frac{\theta^2 e^{-2\theta}}{2(1-e^{-\theta}-\theta e^{-\theta})e^{-\theta}+\theta^2 e^{-2\theta}} \Rightarrow \text{depends on } \theta$$

$\therefore$ X_1+X_2 is NOT SS for θ.

2.

(a) $L(\lambda) = 9\lambda^6 x_1^2 x_2^2 e^{-\lambda^3 x_1^3 - \lambda^3 x_2^3}$

$\Rightarrow X_1^3 + X_2^3$: s.s. by factorization Theorem

(b) $\ell(\lambda) \propto 6\log\lambda - \lambda^3(x_1^3 + x_2^3)$

$\ell'(\lambda) = \frac{6}{\lambda} - 3\lambda^2(x_1^3 + x_2^3) = 0$

$\Rightarrow \frac{2}{\lambda^3} = x_1^3 + x_2^3 \Rightarrow \lambda^3 = \frac{2}{x_1^3 + x_2^3}$

$\hat{\lambda} = \left(\frac{2}{x_1^3 + x_2^3}\right)^{\frac{1}{3}} = \left(\frac{1}{14}\right)^{\frac{1}{3}}$

3.

(a) $L(\theta) = \theta^n 2^{n\theta} \left(\prod_{i=1}^{n} X_i\right)^{-(\theta+1)}$, $\ell(\theta) = n\log\theta + n\theta\log 2 - (\theta+1)\Sigma\log X_i$

$\ell'(\theta) = \frac{n}{\theta} + n\log 2 - \Sigma\log X_i = 0 \Rightarrow \frac{n}{\theta} = \Sigma\log\left(\frac{X_i}{2}\right)$

$\therefore \hat{\theta}^{MLE} = \frac{n}{\Sigma\left(\log\frac{X_i}{2}\right)}$

(b) $\log f(x;\theta) = \log\theta + \theta\log 2 - (\theta+1)\log X$

$\frac{d}{d\theta}\log f(x;\theta) = \frac{1}{\theta} - \log X$

$\frac{d^2}{d\theta^2}\log f(x;\theta) = -\frac{1}{\theta^2} \Rightarrow I(\theta) = \frac{1}{\theta^2}$

(c) $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \theta^2)$

4.

$f_X(x) = \theta(1-x)^{\theta-1} \Rightarrow E(X) = \frac{1}{1+\theta}, \quad V(X) = \frac{\theta}{(1+\theta)^2(2+\theta)}$

Let $Y = -\log(1-X) \Rightarrow f_Y(y) = \theta e^{-\theta y}$: $Y \sim Gamma(1, \frac{1}{\theta})$

(a) $\hat{\theta}_1 = \frac{1}{\frac{1}{n}\sum -\log(1-X_i)} = \frac{1}{\bar{Y}}$

From CLT, $\sqrt{n}(\bar{Y} - \frac{1}{\theta}) \xrightarrow{d} N(0, \frac{1}{\theta^2})$

Let $g(x) = \frac{1}{x} \Rightarrow g'(x) = -\frac{1}{x^2}$

From Delta method, $\underbrace{\sqrt{n}(\frac{1}{\bar{Y}} - \theta)}_{= \sqrt{n}(\hat{\theta}_1 - \theta)} \xrightarrow{d} N(0, \theta^2)$

$\hat{\theta}_2 = \frac{1}{1-\bar{X}}$

From CLT, $\sqrt{n}(\bar{X} - \frac{1}{1+\theta}) \xrightarrow{d} N(0, \frac{\theta}{(1+\theta)^2(2+\theta)})$

Let $g(x) = \frac{1}{1-x} \Rightarrow g'(x) = \frac{1}{(1-x)^2}$

From Delta method, $\sqrt{n}(\frac{1}{1-\bar{X}} - \frac{\theta+1}{\theta}) \xrightarrow{d} N(0, \frac{(1+\theta)^4}{\theta^4}\frac{\theta}{(1+\theta)^2(2+\theta)})$

$$\sqrt{n}(\hat{\theta}_2 - \frac{\theta+1}{\theta}) \longrightarrow N(0, \frac{(1+\theta)^2}{\theta^3(2+\theta)})$$

(b) There is a mistake in my problem (a).

$\hat{\theta}_2$ should be $\frac{1-\bar{X}}{\bar{X}}$! So computing ARE is not relavant..

I gave you full points for (b) regardless of your answer.

Sorry about that.

5. (a)

$$L(\theta) = \frac{2^n}{\theta^{2n}} \prod_{i=1}^{n} x_i \, I(0 < x_i \leq \theta) = \frac{2^n}{\theta^{2n}} I(0 < \max_{1 \leq i \leq n} x_i \leq \theta)$$

$$\Longrightarrow \hat{\theta} = Y = \max_{1 \leq i \leq n} X_i$$

$$\because F(x;\theta) = \frac{x^2}{\theta^2} = \frac{1}{2} \Longrightarrow \text{median} = \frac{\theta}{\sqrt{2}}$$

MLE for median $= \dfrac{Y}{\sqrt{2}}$

(b) $f_Y(y) = \dfrac{2n y^{2n-1}}{\theta^{2n}}, \quad 0 < y \leq \theta, \quad F_Y(y) = \left(\dfrac{y}{\theta}\right)^{2n}$

$$E(Y) = \int_0^\theta \frac{2n y^{2n}}{\theta^{2n}} \, dy = \frac{2n}{2n+1}\theta$$

$$\rightarrow E\left(\frac{Y}{\sqrt{2}}\right) = \frac{2n\theta}{(2n+1)\sqrt{2}} \Longrightarrow \frac{(2n+1)Y}{2n\sqrt{2}}$$ is unbiased estimator for the median

(c)

$$\lim_{n\to\infty} P\left(\left|\frac{Y}{\sqrt{2}} - \frac{\theta}{\sqrt{2}}\right| > \varepsilon\right)$$

$$= \lim_{n\to\infty} P(|Y-\theta| > \sqrt{2}\varepsilon)$$

$$= \lim_{n\to\infty} \left[P(Y > \theta + \sqrt{2}\varepsilon) + P(Y < \theta - \sqrt{2}\varepsilon)\right]$$

$$= \lim_{n\to\infty} P(Y < \theta - \sqrt{2}\varepsilon) = \lim_{n\to\infty} \left(\frac{\theta - \sqrt{2}\varepsilon}{\theta}\right)^{2n} = \lim_{n\to\infty} \left(1 - \frac{\sqrt{2}\varepsilon}{\theta}\right)^{2n} = 0$$