Ch3. Conditional probability and Expectation

Conditional distribution of X given $Y = y$

The conditional distribution of X given $Y = y$ defined as

- Discrete : $P_{X|Y}(x|y) = \frac{P(X=x, Y=y)}{P(Y=y)}$
- Continuous : $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$

$$\text{Interpretation} \quad \approx \quad \frac{P(X \in (x, x+dx), Y \in (y, y+dy))}{P(Y \in (y, y+dy))}$$

Thus, conditional Expectation of X given $Y = y$ is calculated from

$$E(X|Y = y) = \begin{cases} \sum xP(X = x|Y = y) \\ \int_{-\infty}^{\infty} xf_{X|Y}(x|y)dx \end{cases}$$

Remark that $E(X|Y = y)$ is a function of y since it is integrated w.r.t. x.

Example:

Suppose the joint density of X and Y is given by

$$f(x,y) = \begin{cases} 6xy(2-x-y) & \text{if } 0 < x < 1, 0 < y < 1 \\ 0 & \text{o.w} \end{cases}$$

Compute the conditional expectation of X give that $Y = y$, where $0 < y < 1$.
Sol:

Computing Expectations by conditioning

Recall that conditional expectation of X given $Y = y$ is a function of y, say

$$h(y) := E(X|Y = y).$$

Thus we can define

$$E(X|Y) := h(Y)$$

as the function of random variable Y whose value at $Y = y$ is $h(y)$. Indeed $E(X|Y)$ is a r.v, and computing expectation gives

$$E(E(X|Y)) = \int_{-\infty}^{\infty} h(y) f_Y(y) dy = \int_{-\infty}^{\infty} E(X|Y = y) f_Y(y) dy$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx \right\} f_Y(y) dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot \frac{f_{X,Y}(x,y)}{f_Y(y)} f_Y(y) dy dx = \int_{-\infty}^{\infty} x \cdot f_X(x) dx = EX$$

Therefore, we have identity

$$\boxed{EX = E_Y(E(X|Y))}$$

Example

Sam will read either one chapter of his probability book or one chapter of his history book. If the number of misprints in a chapter of his probability book is Poisson distributed with mean 2 and if the number of misprints in his history chapter is Poisson distributed with mean 5, then assuming Sam is equally likely to choose either book, what is the expected number of misprints that Sam will come across?
Sol:

Computing Expectations by conditioning

- Similarly we can compute conditional Variance of X given $Y = y$

$$\mathrm{Var}(X|Y=y) = E(X^2|Y=y) - \Big\{E(X|Y=y)\Big\}^2$$

$$= E((X - E(X|Y=y))^2|Y=y)$$

- Conditional variance formula

$$\boxed{\mathrm{Var}(X) = E_Y(\mathrm{Var}(X|Y)) + \mathrm{Var}_Y(E(X|Y))}$$

Compound random variable

Let $X_1, \ldots, X_n$ be IID random variable with distribution F.

- Sum of (fixed) n IID random variables

$$\sum_{i=1}^{n} X_i$$

- Suppose that N is a non-negative integer valued random variales independent with $X_1, \ldots,$. Define the random variable

$$\sum_{i=1}^{N} X_i, \quad X_i \text{ ind. } N$$

  is called a compound random variable (sum of a random number of rvs).

(e.g)
1. Total claim for a single policy for a fixed time
2. # of accident (or # of casualty) for a given day

Expectation of a Compound Random Variable

Using conditional expectation, first observe that

$$E(\sum_{i=1}^{N} X_i | N = n) = E(\sum_{i=1}^{n} X_i | N = n) = \sum_{i=1}^{n} E(X_i) = nE(X_1)$$

since N and X_i are independent. Hence,

$$E\left(\sum_{i=1}^{N} X_i\right) = E\left(NE(X_1)\right) = E(N)E(X_1)$$

The last equation is called the Wald's identity

$$\boxed{E\left(\sum_{i=1}^{N} X_i\right) = E(N)E(X_1)}$$

Variance of a Compound Random Variable

Let $S = \sum_{i=1}^{N} X_i$. Then the variance of S is calculated from

$$\mathrm{Var}(S) = E(\mathrm{Var}(S|N)) + \mathrm{Var}(E(S|N)).$$

Note that

$$\mathrm{Var}(S|N=n) = \mathrm{Var}\left(\sum_{i=1}^{n} X_i | N = n\right) = \mathrm{Var}(\sum_{i=1}^{n} X_i) = n\sigma^2$$

$$E(S|N=n) = E\left(\sum_{i=1}^{N} X_i | N = n\right) = nEX_1 = n\mu$$

Therefore,

$$\mathrm{Var}(S) = E(N\sigma^2) + \mathrm{Var}(N\mu) = \sigma^2 E(N) + \mu^2 \mathrm{Var}(N)$$

MGF of a Compound Random Variable

Observe that

$$\begin{aligned}
M_s(t) = E(e^{St}) &= E_N\Big(E(e^{St}|N)\Big) \\
&= E_N\Big(E(e^{(X_1+\cdots+X_N)t}|N)\Big) \\
&= E_N\Big(E(e^{X_1 t})\cdots E(e^{X_N t})\Big) \\
&= E_N\Big(\{E(e^{X_1 t}\}^N\Big) \\
&= E_N\Big(M_X(t)^N\Big) \\
&= E_N\Big(\exp\{N \ \log M_X(t)\}\Big) \\
&= M_N\Big(\log M_X(t)\Big)
\end{aligned}$$

Example

Suppose that $N \sim$ Poisson(λ). Find the MGF of the compound Poisson random variable.
Sol: Recall that the MGF of a Poisson(λ) is given by

$$M_N(t) = \exp\{\lambda(e^t - 1)\}.$$

Thus, the MGF of a compound Poisson r.v is given by

$$M_s(t) = \exp\left\{\lambda(e^{\log M_X(t)} - 1)\right\}$$

$$= \exp\left\{\lambda(M_X(t) - 1)\right\}.$$

We can also derive from MGF that

$$E(S) = \lambda E(X_1)$$

$$\mathrm{Var}(S) = \lambda\sigma^2 + \mu^2\lambda = \lambda(\sigma^2 + \mu^2) = \lambda E X_1^2$$