

1d.1 Three - Dimensional Coordinate Systems

2D Coordinate \Rightarrow Quadrants

3D Coordinate \Rightarrow Octants

3D Space :

Surfaces :

Distance and Spheres :

Distance Formula in Three Dimensions The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Equation of a Sphere An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

In particular, if the center is the origin O , then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$

1d.2 Vectors

Combining Vectors :

Components :

Properties of Vectors If \mathbf{a} , \mathbf{b} , and \mathbf{c} are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$	2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$	4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$	6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$	8. $1\mathbf{a} = \mathbf{a}$

1d.3 The Dot Product

1 Definition If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of \mathbf{a} and \mathbf{b} is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

2 Properties of the Dot Product If \mathbf{a} , \mathbf{b} , and \mathbf{c} are vectors in V_3 and c is a scalar, then

- | | |
|---|---|
| 1. $\mathbf{a} \cdot \mathbf{a} = \mathbf{a} ^2$ | 2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ |
| 3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ | 4. $(ca) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (cb)$ |
| 5. $\mathbf{0} \cdot \mathbf{a} = 0$ | |

3 Theorem If θ is the angle between the vectors \mathbf{a} and \mathbf{b} , then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

6 Corollary If θ is the angle between the nonzero vectors \mathbf{a} and \mathbf{b} , then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$$

7 Two vectors \mathbf{a} and \mathbf{b} are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Scalar projection of \mathbf{b} onto \mathbf{a} : $\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

Vector projection of \mathbf{b} onto \mathbf{a} : $\text{proj}_{\mathbf{a}} \mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \right) \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a}$

12.4 The Cross Product

4 Definition If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of \mathbf{a} and \mathbf{b} is the vector

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

* If you get a cross product of \mathbf{a} and \mathbf{b} , you are getting another vector \mathbf{c} perpendicular to \mathbf{a} and \mathbf{b} .

8 Theorem The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both \mathbf{a} and \mathbf{b} .

9 Theorem If θ is the angle between \mathbf{a} and \mathbf{b} (so $0 \leq \theta \leq \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$$

10 Corollary Two nonzero vectors \mathbf{a} and \mathbf{b} are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by \mathbf{a} and \mathbf{b} .

11 Properties of the Cross Product If \mathbf{a} , \mathbf{b} , and \mathbf{c} are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ scalar triple product $\simeq \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

↳ vector triple product

Triple Products

14 The volume of the parallelepiped determined by the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

* If we use the above formula and discover that the volume of the parallelepiped determined by \mathbf{a} , \mathbf{b} , and \mathbf{c} is 0, then the vectors must lie in the same plane; that is, they are coplanar.

12.5 Equations of Lines and Planes

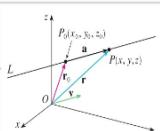


FIGURE 1

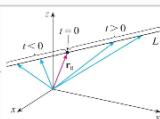


FIGURE 2

← If \mathbf{a} is the vector representation $\overrightarrow{P_0 P}$, then the Triangle Law for vector addition gives $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$, but since \mathbf{a} and \mathbf{v} are parallel vectors, there is a scalar t such that $\mathbf{a} = t\mathbf{v}$, thus $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$. As t varies, the line is traced out by the tip of the vector.

2 Parametric equations for a line through the point (x_0, y_0, z_0) and parallel to the direction vector $\langle a, b, c \rangle$ are

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

- The vector equation and parametric equations of a line are not unique. If we change the point or the parameter or choose a different parallel vector, then the equations change.

Direction Numbers of L :

- the components a, b, c of a vector $\mathbf{v} = \langle a, b, c \rangle$ used to describe the direction of a line L

* Any 3 numbers proportional to a, b , and c could also be used as a set of direction numbers for L

Symmetric Equations of L :

- a way of describing a line L is to eliminate the parameter t from parametric equations

$$\Rightarrow t = \frac{x - x_0}{a}, \quad t = \frac{y - y_0}{b}, \quad t = \frac{z - z_0}{c}$$

$\leadsto \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$, these equations are "symmetric equations of L ", and notice that

the numbers a, b , and c are direction numbers of L

* If one of a, b, c is 0, $x = x_0$, $\frac{y - y_0}{b} = \frac{z - z_0}{c}$

- In general, direction numbers of the line L through the points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ are

$x_1 - x_0, y_1 - y_0, z_1 - z_0$ and so symmetric equations of L are

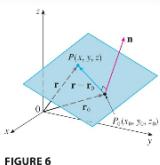
$$\Rightarrow \frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

- In general, the vector equation of a line through the vector r_0 in the direction of a vector v

is $r = r_0 + tv$. If the line also passes through r_1 , then we can take $v = r_1 - r_0$ and so its vector equation is $r = r_0 + t(r_1 - r_0) = (1-t)r_0 + tr_1$, $0 \leq t \leq 1$

Planes:

- A single vector parallel to a plane is not enough to convey the "direction" of the plane, but a vector perpendicular to the plane does completely specify its direction. Thus a plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector n that is orthogonal to the plane.



- Let $P(x, y, z)$ be an arbitrary point in the plane, and let r_0 and r be the position vectors of P_0 and P . Then the vector $r - r_0$ is represented by $\vec{P_0P}$. The normal vector n is orthogonal to every vector in the given plane.

$$\Rightarrow n \cdot (r - r_0) = 0$$

$$\Rightarrow n \cdot r = n \cdot r_0$$

- Scalar Equation of the plane through point $P_0(x_0, y_0, z_0)$ with normal vector $n = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

- Two planes are parallel if their normal vectors are parallel. If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors.

Vector equation
of the plane

NOTE Since a linear equation in x , y , and z represents a plane and two nonparallel planes intersect in a line, it follows that two linear equations can represent a line. The points (x, y, z) that satisfy both $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ lie on both of these planes, and so the pair of linear equations represents the line of intersection of the planes (if they are not parallel). For instance, in Example 7 the line L was

Distances :

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

12.6 Cylinders and Quadric Surfaces

Traces : (or Cross-Sections)

- the curves of intersection of the surface with planes parallel to the coordinate planes

Cylinders :

- a surface that consists of all lines (called rulings) that are parallel to a given line and pass through a given plane curve

* If one of the variables x , y , or z is missing from the equation of a surface, then the surface is a cylinder.

* When you are dealing with surfaces, it is important to recognize that an equation like $x^2 + y^2 = 1$ represents a cylinder, not a circle. The trace of the cylinder $x^2 + y^2 = 1$ in the xy -plane is the circle with equations $x^2 + y^2 = 1$, $z = 0$.

Quadric Surfaces :

- the graph of a second-degree equation in three variables x , y , and z .