Quiz 7

#1. Let $X_1, \dots, X_n$ be a random sample from a distribution with pdf $f(x;\theta) = \frac{1}{\theta^2} x e^{-\frac{x}{\theta}}$, $x > 0$.

In what statistic, the likelihood has mlr ?

$\rightarrow$ $L(\theta; \underline{x}) = \prod_{i=1}^{n} \frac{1}{\theta^2} x_i e^{-\frac{x_i}{\theta}} = \theta^{-2n} \prod_{i=1}^{n} x_i e^{-\frac{x_i}{\theta}}$

Let $\theta_1 < \theta_2$. Then the ratio of likelihood is

$$\frac{L(\theta_1; \underline{x})}{L(\theta_2; \underline{x})} = \frac{\theta_1^{-2n} \prod_{i=1}^{n} x_i e^{-\frac{x_i}{\theta_1}}}{\theta_2^{-2n} \prod_{i=1}^{n} x_i e^{-\frac{x_i}{\theta_2}}} = \left(\frac{\theta_2}{\theta_1}\right)^{2n} \exp\left(-\frac{\Sigma x_i}{\theta_1} + \frac{\Sigma x_i}{\theta_2}\right)$$

$$= \left(\frac{\theta_2}{\theta_1}\right)^{2n} \exp\Big(\underbrace{\Sigma x_i \left(\tfrac{1}{\theta_2} - \tfrac{1}{\theta_1}\right)}_{\text{monotone (MLR)}}\Big)$$

$\therefore \sum_{i=1}^{n} x_i$ is the mlr part.

#2. Let $X_1, \dots, X_n$ be a random sample from a distribution with pdf $f(x;\theta) = \sqrt{\frac{\theta}{2\pi}} e^{-\frac{\theta x^2}{2}}$, $-\infty < x < \infty$

For testing $H_0: \theta = 1$ against $H_1: \theta > 1$, find the UMP test.

$\rightarrow$ ① Let $\theta' > 1$. Then consider $H_0: \theta = 1$ vs $H_1: \theta = \theta'$

The ratio of likelihood is

$$\frac{L(\theta=1; \underline{x})}{L(\theta=\theta'; \underline{x})} = \frac{\prod_{i=1}^{n} \sqrt{\frac{1}{2\pi}} e^{-\frac{x_i^2}{2}}}{\prod_{i=1}^{n} \sqrt{\frac{\theta'}{2\pi}} e^{-\frac{\theta' x_i^2}{2}}} = \left(\frac{1}{\theta'}\right)^{\frac{n}{2}} \exp\Big(\underbrace{\frac{\Sigma x_i^2}{2}(-1+\theta')}_{\text{MLR : monotone increasing}}\Big) \le k$$

$$\Leftrightarrow \frac{\Sigma x_i^2}{2}(-1+\theta') \le k'$$

$$\Leftrightarrow \Sigma x_i^2 \le c$$

$$\therefore C^{MP} = \left\{ \sum_{i=1}^{n} x_i^2 \le c \right\}$$

Since it does not depend on θ', $C^{MP} = C^{UMP}$.

$$\therefore C^{UMP} = \left\{ \sum_{i=1}^{n} x_i^2 \le c \right\}$$

다른 풀이 ② Using a regular exponential family

$$f(x;\theta) = \exp\left(\frac{1}{2}\log\frac{\theta}{2\pi} - \frac{\theta x^2}{2}\right)$$

$\Rightarrow$ $\underbrace{P(\theta) = -\frac{\theta}{2}}_{\text{monotone decreasing}}$, $k(x) = x^2$ $\Rightarrow$ $C^{UMP} = \{ \Sigma k(x_i) \le c \}$

$$\therefore C^{UMP} = \left\{ \sum_{i=1}^{n} x_i^2 \le c \right\}$$