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1) a) i) $\forall A \in S$, $P(A) \geq 0$ (non-negativity)

ii) $P(S) = 1$ (add up to 1)

iii) If A_i's are disjoints for all i's, $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$

b) Central Limit Theorem : $\dfrac{\sum_{i=1}^{n} X_i - np}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$

2)

$$P(X \leq x) = 1-(1-p)^x \geq \frac{1}{2}$$

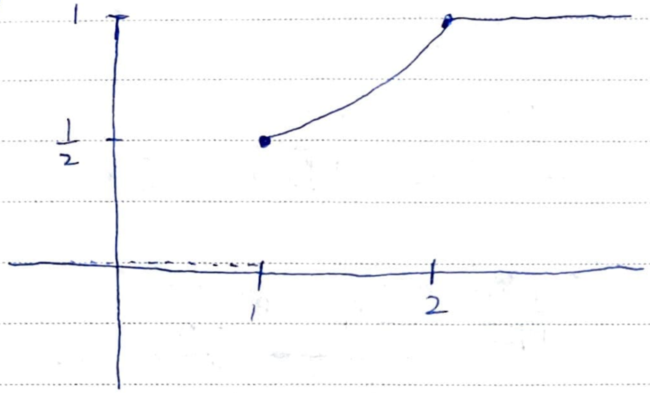
$$\ln(1-p)^x \geq \ln\frac{1}{2}$$

$$x \geq \frac{\ln(\frac{1}{2})}{\ln(\frac{5}{6})} \simeq 3.8$$

$\therefore$ $x = 4$

answer for #2

3)



~~$f(z) = z - 1$~~

$E(z^2) = \int_1^2 z^3 - z^2 \, dz + \frac{1}{2}$

$= \frac{1}{4}z^4 - \frac{1}{3}z^3 \Big|_1^2 + \frac{1}{2}$

$= 4 - \frac{8}{3} - \frac{1}{4} + \frac{1}{3} = \underline{1.4167}$

answer for #3

4) $P(FB) = \frac{4}{14+n}$, $P(FG) = \frac{6}{14+n}$

$P(B) \cdot P(F) = P(FB)$

$\frac{10}{14+n} \cdot \frac{10}{14+n} = \frac{4}{14+n}$

$\frac{100}{14+n} = 4$

$25 = 14 + n$

$\underline{11 = n}$ $\therefore$ 11 sophomore girls are needed.

answer for #4

5) $A \sim Pois\left(\binom{n}{2}\frac{1}{365}\right)$

$P(A=0) = e^{\binom{n}{2}\frac{1}{365}}$

Let $X \sim B\left(\binom{n}{2}, \frac{1}{365}\right)$. Then using the Poisson paradigm,

$$P(A) = \frac{\lambda^x e^{-\lambda}}{x!}, \text{ where } x=0 \text{ and } \lambda = \binom{n}{2}\frac{1}{365}$$

$$\therefore = e^{-\binom{n}{2}\frac{1}{365}}$$

answer for #5

6) $f_X(x) = \int_x^1 8xy\,dy = 4xy^2\Big|_x^1 = 4x(1-x^2) = 4x - 4x^3, \quad 0<x<1$

$f_Y(y) = \int_0^y 8xy\,dx = 4x^2y\Big|_0^y = 4y(y^2) = 4y^3, \quad 0<y<1$

answers for #6

7)

$$M_X(t) = \int_0^\infty e^{tx} \cdot \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha}\,dx = \int_0^\infty \frac{x^{\alpha-1}e^{-\left(\frac{\beta}{1-\beta t}\right)^{-1}x}}{\Gamma(\alpha)\beta^\alpha}\,dx$$

$$= \int_0^\infty \frac{x^{\alpha-1}e^{-\left(\frac{\beta}{1-\beta t}\right)^{-1}x}}{\Gamma(\alpha)\beta^\alpha} \cdot \frac{\left(\frac{\beta}{1-\beta t}\right)^\alpha}{\left(\frac{\beta}{1-\beta t}\right)^\alpha}\,dx$$

$$= \left(\frac{\beta}{1-\beta t}\right)^\alpha \cdot \frac{1}{\beta^\alpha} \underbrace{\int_0^\infty \frac{x^{\alpha-1}e^{-\left(\frac{\beta}{1-\beta t}\right)^{-1}x}}{\Gamma(\alpha)\cdot\left(\frac{\beta}{1-\beta t}\right)^\alpha}}_{=1}$$

$$= \left(\frac{1}{1-\beta t}\right)^\alpha, \quad \text{let } \alpha = r, \ \lambda = \frac{1}{\beta}$$

$$= \left(\frac{1}{1-\frac{t}{\lambda}}\right)^r = \left(\frac{\lambda}{\lambda - t}\right)^r, \quad r>0, \ \lambda>0, \ \lambda>t$$

answer for #7

8) $p(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} p(E_i) - \sum_{i_1<i_2}^{n-1} p(E_{i_1} \cap E_{i_2}) + \sum_{i_1<i_2<i_3}^{n-2} p(E_{i_1} \cap E_{i_2} \cap E_{i_3}) + \cdots$

using Inclusion-Exclusion ~~the~~ Formula,

By the axioms of Probability, we can assume,

$\Rightarrow p(\bigcap_{i=1}^{n} E_i) \leq p(\bigcap_{i=1}^{m} E_i)$ for any positive $n>m$

Therefore, the sum of the terms of the expansion after $\sum_{i=1}^{n} p(E_i)$ is non-positive.

$$\therefore p(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} p(E_i) \underbrace{- \sum_{i_1<i_2}^{n-1} p(E_{i_1} \cap E_{i_2}) + \cdots + (-1)^{r+1} \sum_{i_1<\cdots<i_r} p(E_{i_1} \cap \cdots \cap E_{i_r}) + \cdots}_{\leq 0}$$

$$\leq \sum_{i=1}^{n} p(E_i)$$