# Ch4. Markov Chains

1. Define stochastic process / Examples
2. Characterize (i.e probability calculation)
3. Transient analysis $X_n$
4. First passage time $T = \min\{n \geq 0 | X_n \in \mathbb{B}\}$
   First visit time stopping time (the first time to hit set $\mathbb{B}$)
5. Limiting behavior $\lim_{n\to\infty} X_n$
6. Other applications

# Stochastic process (SP)

**Definition**

*A stochastic process* $\{X(t), t \in T\}$ *is a collection of random variables*

- Just a bunch of r.v's
- $t$: (usually) time or space
- $T$: index set for the process. If $T$ is countable (uncountable) then then it is called discrete process (continuous process).
- $X(t)$: the state of the process at time $t$
- The state space of a stochastic process is the set of all possible values $X(t)$ can take. Discrete state (continuous state) space means that state space is countable (uncountable).
- If $T = \mathbb{Z}$ (discrete), then we sometimes use notation $X_n$.

# Examples of SP

For example convenience start with time zero.

- $X(t)$: $\#$ of customers entered a supermarket by time $t$
- $X(t)$: $\#$ of customers in the supermarket
- $X(t)$: Total amount of sales up to time $t$
- $X_n$ : departure of $n^{th}$ customer
- $X_n$ : Dow-Jones stock index of the end of $n^{th}$ day.

Our ultimate goal is to describe the probabilistic model of a system evolving randomly.

# Outline of this course

We will learn the following stochastic processes

- (Discrete time/discrete state) Markov Chain (Ch. 4)
- (Continuous time/discrete state) Poisson process (Ch. 5)
- (Continuous time/continuous state) Brownian motion (Ch. 10)

If time permitted,

- (Continuous time/discrete state) Markov Chain (Ch. 6)
- (Continuous time/continuous state) Renewal process (Ch. 7)

# DTMC

**Definition: Discrete Time Markov Chain**

$\{X_n, n \geq 0\}$ is a DTMC on state space $\mathbb{S} = \{0, 1, , \cdots\}$ if

i) $X_n \in \mathbb{S} \quad \forall n \geq 0$

ii) Markov Property

$$P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0)$$

$$= P(X_{n+1} = j | X_n = i)$$

for all $i, j \in \mathbb{S}$, $i_n \in \mathbb{S}$

- Given the present, future is independent of the past.
- $X_{n+1}$ only depends on $X_n$

# DTMC

### Definition

*A DTMC is called time homogeneous if*

$$P(X_{n+1} = j | X_n = i) = P(X_1 = j | X_0 = i)$$

### Definition (Transition Probability)

$P(X_1 = j | X_0 = i) = p_{ij} =$ *(one-step) transition probablity*

$$\mathbb{P} = \begin{pmatrix} p_{00} & p_{01} & \cdots \\ p_{10} & p_{11} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$ *(one-step) transition matrix*

# Transition Matrix

- Properties of transition matrix

$$
\begin{aligned}
&i)\ p_{ij} \geq 0 \quad \forall i, j \\
&ii)\ \sum_j p_{ij} = P(X_1 \in \mathbb{S} | X_0 = i) = 1
\end{aligned}
$$

- Transition diagram
  - Graphical representation of MC
  - Node set $= \mathbb{S}$
  - Arc set $= \{(i, j), p_{ij} > 0\}$

For example $S = \{1, 2, 3\}$ and $P = \begin{pmatrix} .5 & 0 & .5 \\ 0 & 1 & 0 \\ 0 & .5 & .5 \end{pmatrix}$

# Examples of MC

1. Weather Forecasting. Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather conditions. Suppose also that if it rains today, then it will rain tomorrow with probability $\alpha$; and if it does not rain today, then it will rain tomorrow with probability $\beta$.

# On/Off process

2. A machine that is ON/OFF. Suppose that if the machine is ON on day $n$, it is ON on the $(n+1)$th day with probability $.95$ independent of past. On the other hand, if it is down, it stays down on the $(n+1)$th day with probability .02, also independent of the past. Let $X_n$ be the state of a machine

$$X_n = \begin{cases} 0 & \text{if OFF} \\ 1 & \text{if ON} \end{cases} \qquad n : 0, 1, 2, \cdots \text{(hours)}$$

Then, $X_n$ is a DTMC with state space $S = \{\mathrm{ON}, \mathrm{OFF}\}$ and transition probability

# Examples of MC

3. Communications system: Consider a communications system that transmits the digits 0 and 1. Each digit transmitted must pass through several stages, at each of which there is a probability $p$ that the digit entered will be unchanged when it leaves. Let $X_n$ denote the digit entering $n$th stage.

# More than 2 state

4. More than 2-state: On any given day Gary is either cheerful (C), so-so (S), or glum (G). If he is cheerful today, then he will be C, S, or G tomorrow with respective probabilities 0.5, 0.4, 0.1. If he is feeling so-so today, then he will be C, S, or G tomorrow with probabilities 0.3, 0.4, 0.3. If he is glum today, then he will be C, S, or G tomorrow with probabilities 0.2, 0.3, 0.5.

# Examples of MC

5. Transforming to MC

   Suppose weather (rain) depends on last two days

$$
\begin{cases}
P(X_{n+1} = R | X_n = R, X_{n-1} = R) = .7 \\
P(X_{n+1} = R | X_n = R, X_{n-1} = N) = .5 \\
P(X_{n+1} = R | X_n = N, X_{n-1} = R) = .4 \\
P(X_{n+1} = R | X_n = N, X_{n-1} = N) = .2
\end{cases}
$$

   Can we embed this to MC? Yes, by defining state space as

$$\mathbb{S} = \{RR, NR, RN, NN\},$$

   where RR means it rained both today and yesterday, NR means it rained only today etc.

$$
\mathbb{P} = \begin{pmatrix}
.7 & 0 & .3 & 0 \\
.5 & 0 & .5 & 0 \\
0 & .4 & 0 & .6 \\
0 & .2 & 0 & .8
\end{pmatrix}
$$

# Example of MC - IID process

6. IID process (Independent trials process)
   $X_0, X_1, X_2, \cdots$ from $P(X_n = k) = a_k, \ k = 0, 1, 2, \cdots$

$$\mathbb{S} = \{0, 1, 2, \cdots\}$$

   Then, $\{X_n : n \geq 0\}$ is MC

- Indeed

$$P(X_{n+1} = i_{n+1} | X_n = i_n, X_{n-1} = i_{n-1}, \cdots X_0 = i_0)$$

$$= P(X_{n+1} = i_{n+1}) = a_{i_n+1} = P(X_{n+1} = i_{n+1} | X_n = i_n)$$

$$\mathbb{P} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & \cdots \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \end{array} & \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_0 & a_1 & a_2 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ & & & \end{pmatrix} \end{array}$$

# Example of MC - Random Walk

7. Sum of independent random variables (Random Walk)
   $\{X_n : n \geq 1\}$ be iid r.v taking values on integers with pmf $P(X_n = i) = p_i$. Define $Y_0 = 0$ (it starts at origin) and

   $$Y_n = X_1 + \cdots + X_n, \quad n \geq 1.$$

   Then, $\{Y_n\}$ is called a random walk. Our claim is that random walk is also a DTMC with $\mathbb{S} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

# Random Walk

- Indeed:

$$
\begin{aligned}
&P(Y_{n+1} = j | Y_n = i, Y_{n-1}, \cdots, Y_0) \\
&= P(Y_n + X_{n+1} = j | Y_n = i, Y_{n-1}, \cdots, Y_0) \\
&= P(X_{n+1} = j - i | Y_n = i, Y_{n-1}, \cdots, Y_0) \\
&= P(X_{n+1} = j - i | X_1 + \cdots + X_n = i, X_1 + \cdots + X_{n-1} = i_{n-1}, \cdots \\
&= P(X_{n+1} = j - i) = p_{j-i} = P(Y_{n+1} = j | Y_n = i)
\end{aligned}
$$

- Transition matrix

$$
\mathbb{P} = \begin{array}{cc}
 & \begin{array}{cccccc} -4 & -3 & -2 & -1 & 0 & 1 \end{array} \\
\begin{array}{c} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{array} &
\begin{pmatrix}
\cdots & p_{-1} & p_0 & p_1 & p_2 & \cdots & \\
\cdots & p_{-2} & p_{-1} & p_0 & p_1 & p_2 & \\
 & \cdots & p_{-2} & p_{-1} & p_0 & p_1 & p_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & & & & & &
\end{pmatrix}
\end{array}
$$

# Gambler's ruin model

8. Gambler's ruin model (Special case of random walk). Define

$$X_i = \begin{cases} +1 & \text{with } p \\ -1 & \text{with } q \end{cases}$$

The game will be stopped if the fortune after $n^{th}$ play is either 0 or N. Fortune after n-$th$ play is

$$Y_n = X_1 + \cdots + X_n$$

and $p_{00} = p_{NN} = 1$.

$$\mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ N \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & \cdots & & & & N \end{array} \\ \begin{pmatrix} 1 & & & & & & & \\ q & 0 & p & & & & & \\ & q & 0 & p & & & & \\ & & & & & q & 0 & p \\ & & & & & & & 1 \end{pmatrix} \end{array}$$

- State 0 $\&$ $N$ is called absorbing state (once entered never left that state)

# Total number of success in Bernoulli process

9. Let $\{X_n : n \geq 0\}$ be the number of success out of $n$ trials. Then, $\{X_n : n \geq 0\}$ is DTMC with $\mathbb{S} = \{0, 1, 2 \cdots\}$ and $P(X_0 = 0) = 1$.
   Indeed:

$$P(X_{n+1} = j | X_n = i) = \begin{cases} p & \text{if } j = i + 1 \\ q & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & \cdots \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \end{array} & \begin{pmatrix} q & p & 0 & \\ 0 & q & p & \cdots \\ & & & \\ 0 & & & \end{pmatrix} \end{array}$$

# Time to $n$th success in Bernoulli process

10. Let $\{X_n\}$ be the time of $n^{th}$ success in Bernoulli process. Then, $\{X_n, n \geq 0\}$ is DTMC with $\mathbb{S} = \{0, 1, 2, \cdots\}$ and $P(X_0 = 0) = 1$.
    Indeed:

$$P(X_{n+1} = j | X_n = i) = P(X_{n+1} - X_n = j - i)$$

$$= \begin{cases} pq^{j-i-1} & \text{if } j \geq i+1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & \cdots & \end{array} \\ \begin{pmatrix} 0 & p & qp & q^2p & \cdots \\ 0 & 0 & p & qp & \cdots \\ & & & & \\ & & & & \end{pmatrix} \end{array}$$

# Successive runs

11. Free throw shooting with independent Bernoulli trials. Let $X_n$ be the number of successive goals. Then, $\{X_n\}$ is DTMC with transition probability

$$\mathbb{P} =$$

# Characterization of MC

- For given MC $\{X_n\}$ with state space $\mathbb{S}$ and transition matrix $\mathbb{P} = (p_{ij})$, We want to characterize MC by doing all probability calculation.
- For example, we want to calculate

$$\begin{aligned} \{X_0, X_1\} \Rightarrow P(X_0 = i, X_1 = j) &\\ = P(X_1 = j | X_0 = i) P(X_0 = i) &\\ = p_{ij} P(X_0 = i) & \end{aligned}$$

$$\begin{aligned} \{X_0, X_1, X_2\} \Rightarrow & P(X_2 = k, X_1 = j, X_0 = i) \\ &= P(X_2 = k | X_1 = j, X_0 = i) P(X_1 = j, X_0 = i) \\ &= p_{jk} p_{ij} P(X_0 = i) \end{aligned}$$

- To fully characterize MC, we need to know transition probability $\mathbb{P}$ and initial distribution

$$\boxed{P(X_0 = i) = a_i, \quad \forall i \in \mathbb{S}}$$

# Characterization of MC

**Characterization of MC**:
DTMC is completely characterized by

i) initial distribution
ii) transition prob matrix $\mathbb{P}$

- With initial distribution and transition probability $\mathbb{P}$, observe

$$\boxed{P(X_n = j) = \sum_{i \in \mathbb{S}} P(X_n = j | X_0 = i) P(X_0 = i)}$$

implies that we need to calculate $n$-step transition probability matrix

$$p_{ij}^{(n)} = P(X_n = j | X_0 = i)$$
$$\mathbb{P}^{(n)} = (p_{ij}^{(n)})$$

to calculate current state probability.

# Transient analysis Chapman - Kolmogorov equation

- How to calculate this?

**Chapman - Kolmogorov equation**

$$p_{ij}^{(n+m)} = \sum_{k \in \mathbb{S}} p_{ik}^{(n)} p_{kj}^{(m)}$$

(Summing over all intermediate states $k$ after $n^{th}$ transition, then move $m^{th}$ step to arrive $j$ state.)

# Transient analysis

Indeed:

$$
\begin{aligned}
p_{ij}^{(n+m)} &= P(X_{n+m} = j | X_0 = i) \\
&= \sum_{k=0}^{\infty} P(X_{n+m} = j, X_n = k | X_0 = i) \\
&= \sum_{k} P(X_{n+m} = j | X_n = k, X_0 = i) P(X_n = k | X_0 = i) \\
&= \sum_{k} P(X_{n+m} = j | X_n = k) P(X_n = k | X_0 = i) \\
&= \sum_{k} p_{kj}^{(m)} p_{ik}^{(n)}
\end{aligned}
$$

In a matrix notation

$$\boxed{\mathbb{P}^{(n+m)} = \mathbb{P}^{(n)} \mathbb{P}^{(m)}}$$

# Transient analysis

Even further, $\mathbb{P}^{(n)} = \mathbb{P}^n$ (matrix power).

Indeed:

$$\mathbb{P}_{ij}^{(0)} = P(X_0 = j | X_0 = i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{o.w} \end{cases}$$

$$\therefore \mathbb{P}_{ij}^{(0)} = I$$

$$\mathbb{P}_{ij}^{(1)} = P(X_1 = j | X_0 = i) = p_{ij} \quad \therefore P^{(1)} = \mathbb{P}$$

Therefore, by induction

$$\mathbb{P}^{(n+1)} = \mathbb{P}^{(1)} \cdot \mathbb{P}^{(n)} = \mathbb{P}\mathbb{P}^n = \mathbb{P}^{n+1}$$

# Example

- Weather forecasting

$$\mathbb{P} = \begin{pmatrix} .7 & .3 \\ .4 & .6 \end{pmatrix}$$

$$P(\text{It will rain 4 days after} \mid \text{ it rains today}) = \mathbb{P}^{(4)}_{11}$$

$$\mathbb{P}^{(4)} = \mathbb{P}^4 = \begin{pmatrix} .7 & .3 \\ .4 & .6 \end{pmatrix} \begin{pmatrix} .7 & .3 \\ .4 & .6 \end{pmatrix} \cdots \begin{pmatrix} .7 & .3 \\ .4 & .6 \end{pmatrix}$$

$$= \begin{pmatrix} .5749 & .4251 \\ .5668 & .4332 \end{pmatrix}$$

# Example

Consider a DTMC $\{X_n, n \geq 0\}$ on $\mathbb{S} = \{1, 2, 3, 4\}$ with $P(X_0 = 1) = 1$, and the transition probability matrix

$$P = \begin{pmatrix} .4 & .3 & .2 & .1 \\ .5 & 0 & 0 & .5 \\ .5 & 0 & 0 & .5 \\ .1 & .2 & .3 & .4 \end{pmatrix}.$$

Compute

- $P(X_2 = 4)$
- $P(X_1 = 2 | X_2 = 4, X_3 = 1)$

# Other probability calculation depending on the sample path

Suppose now that we're interested in the sample path of MC

$$P(X_k \in \mathbb{B} \text{ for some } k = 1, \cdots, m | X_0 = i)$$

$$= P(X_k \text{ in } \mathbb{B} \text{ within } m \text{ steps } | X_0 = i)$$

Graphically,

# Example

- Consider new MC

$$W_n = \begin{cases} X_n & \text{if } n < N \\ C & \text{if } n \geq N \end{cases}$$

Define first passage time as

$$N = \min\{n : X_n \in \mathbb{B}\}$$

Then, $\{W_n\}$ is a MC with state space

$$S_1 = \{S \backslash \{\mathbb{B}\}, C\}$$

$$Q_{ij} = P_{ij} \quad \text{if } ij \neq \mathbb{B}$$

$$Q_{iC} = \sum_{j \in \mathbb{B}} P_{ij} \quad \text{if } i \neq \mathbb{B}$$

$$Q_{CC} = 1 \ \ \text{(absorbing state)}$$

Therefore, $P(X_k \in \mathbb{B} \text{ for some } k = 1, \cdots, m | X_0 = i)$
$= P(W_m = C | X_0 = i) = P(W_m = C | W_0 = i) = Q^m_{i,C}$

# Example: Successive runs

Let $N$ be the number of flips required to observe three successive heads.

(a) $P(N \leq 8)$

For example, we have following realizations:

H T T H H T T H H H

Then, $N = 10$. First define MC as the number of successive heads $\{X_n\}$ with $\mathbb{S} = \{0, 1, 2, 3, 4 \cdots\}$ Then, it is a MC with

$$\mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \ldots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \ldots \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \ldots \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & \ldots \end{pmatrix} \end{array}$$

# Example

- Now, define "new MC"

$$W_n = \begin{cases} X_n & \text{if } n < N \\ 3 & \text{if } n \geq N \end{cases}$$

with $N = \min\{n : X_n = 3\}$, Then, $\{W_n\}$ is MC with

$$\mathbb{Q} = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \multicolumn{4}{c}{} \\ 1 & & & & \\ 2 & & & & \\ 3 & & & & \end{array}$$

$$\mathbb{Q} = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\therefore P(N \leq 8) = P(W_8 = 3 | W_0 = 0) P(W_0 = 0) = Q_{0,3}^8 = 107/256$$

(b) $P(N = 8)$

$$P(N = 8) = P(N \leq 8) - P(N \leq 7) = Q_{0,3}^8 - Q_{0,3}^7$$

## 4.3 Classification of states

- Now, we are interested in

$$\lim_{n\to\infty} \mathbb{P}^n = P(X_\infty = j | X_0 = i)$$
$$= P(\text{starts at } i \text{ and end at } j^{th} \text{ state})$$

  In what cases such limit exists and how to calculate?

- First we need the classification of states to say about existence. Two main concepts

$$i) \text{ Irreducibility}$$
$$ii) \text{ Aperiodicity}$$

- Formal definitions

**Definition**

*A state j is accessible from state $i (i \to j)$ if there exists $n \geq 0$ s.t*

$$p_{ij}^{(n)} > 0$$

*(starting from i, it eventually enter state j)*

# Classification of states

## Definition

*Two states $i$ and $j$ are said to communicate if*

$$i \to j \text{ and } j \to i$$

## Remark

$i \leftrightarrow j$ *is an "equivalent" relation*

$$i) \text{ reflexive } i \leftrightarrow i$$
$$ii) \text{ symmetric } i \leftrightarrow j \iff j \leftrightarrow i$$
$$iii) \text{ transitivity } i \leftrightarrow j, j \leftrightarrow k \Longrightarrow i \leftrightarrow k$$
$$\left( \because \; P_{ik}^{n+m} = \sum_{r=0}^{\infty} P_{ir}^{n} \cdot P_{rk}^{m} \geq P_{ij}^{n} \cdot P_{jk}^{m} > 0 \right)$$

# Classification of states

**Definition**

*A subset $C \subset \mathbb{S}$ is said to be communicating class if*

$$i)\ i, j \in C \Rightarrow i \leftrightarrow j$$
$$ii)\ i \in C,\ i \leftrightarrow j \Rightarrow j \in C$$

*(the largest subset all communicating each other)*

**Definition**

*A communicating calss $C$ is closed if $i \in C$ and $j \notin C \Rightarrow i \nrightarrow j$*
*(entry is allowed but not exist)*

**Irreducible MC**

A DTMC is called irreducible if all state communicate with each other. That is $\mathbb{S}$ is a single closed communicating class.
(MC can be reached from any state to any state.)

# Example

- $$\mathbb{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{pmatrix}$$

  Irreducible MC

- $$\mathbb{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  Since $3 \nrightarrow 2$, it is NOT irreducible (reducible). If MC is reducible, then we can partition the state space as communicating classes:

  $$C_1 = \{0, 1\} \quad C_2 = \{2\} \quad C_3 = \{3\}$$

  $$S = C_1 \cup C_2 \cup C_3$$

# Periodicity

## Definition (Periodicity)

*A state $i$ is said to be periodic with period $d(i)$ if $d(i)$ is the greatest common divisor (gcd) of n s.t*

$$p_{ii}^{(n)} > 0$$

- If $d(i) = 1$, then it is called aperiodic
- If $n$ does not exist, then $d(i) := \infty$

Example:

## Theorem

*Periodicity is a class property. That is, if $i \leftrightarrow j$, then $d(i) = d(j)$*

# Recurrent/Transient

**Recurrence and Transience**

State $i$ is called recurrent if the process starting in state $i$ will revisit state $i$ for sure. More formally, define

$$N_i = \min\{n > 0 | X_n = i\}$$

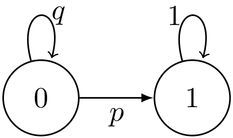
known as first passage time, first visit time, stopping time, then

$$f_{ii} := P(N_i < \infty | X_0 = i) = 1$$

It is called that the state $i$ is transient if $f_{ii} < 1$.

# Example

- Consider two-state MC




- $N_0 = \min\{n > 0, X_n = 0\}$.

$$P(N_0 = n | X_0 = 0) = \begin{cases} q & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases}$$

$$f_{00} = P(N_0 < \infty | X_0 = 0) = \sum_{n=1}^{\infty} P(N_0 = n | X_0 = 0) = q$$

- Therefore, if $q = 1$, state 0 is recurrent. While $q < 1$ implies that state 0 is transient.
- If state $i$ is recurrent, then starting in state $i$, the process eventually revisit state $i, \cdots$, hence MC visits state $i$ infinitely often times.

# Recurrence and Transience

- Alternative definition for recurrence/transience by counting the number of times MC visits state $i$.
- Define indicator function

$$I_n = \begin{cases} 1 & \text{if } X_n = i \\ 0 & \text{if } X_n \neq i \end{cases}$$

  Then,

$$R_i = \sum_{n=0}^{\infty} I_n$$

  counts the number of times MC visits state $i$.
- If $f_{ii} < 1$, then $P(R_i = k | X_0 = i) = f_{ii}^{k-1}(1 - f_{ii})$, $k = 1, 2 \cdots$. Hence $R_i \sim \text{Geo}(1 - f_{ii})$. (define success as not coming back to state $i$)
- $E(R_i | X_o = i) = \frac{1}{1-f_{ii}}$.

# Recurrence and Transience

- Also observe that

$$E(R_i|X_0 = i) = E(\sum_{n=0}^{\infty} I_n|X_0 = i) = \sum_{n=0}^{\infty} E(I_n|X_0 = i)$$
$$= \sum_{n=0}^{\infty} P(X_n = i|X_0 = i) = \sum_{n=0}^{\infty} p_{ii}^n$$

$$\boxed{E(R_i|X_0 = i) = \frac{1}{1 - f_{ii}} = \sum_{n=1}^{\infty} p_{ii}^n}$$

**Proposition 4.1**
State $i$ is recurrent iff $\sum_{n=0}^{\infty} p_{ii}^n = \infty$ and transient iff $\sum_{n=0}^{\infty} p_{ii}^n < \infty$

# Recurrence and Transience

## Remark

1. *Transient = it will only be visited for a finite number of times*
2. *Recurrent = it will be visited infinitely many times*
3. *For a finite state MC, all states cannot be transient!*
   *That is, at least one state should be recurrent!*

**Theorem**
Transient/ Recurrence are class property. That is if $i$ is recurrent(transient) and $i \leftrightarrow j$, then $j$ is also recurrent (transient)

# Recurrence and Transience

Indeed:

$$\sum_{r=1}^{\infty} P_{jj}^{r} \geq \sum_{r=1}^{\infty} P_{jj}^{m+n+r} \geq \sum_{r} \sum_{i} P_{ji}^{m} P_{ii}^{r} P_{ij}^{n}$$

$$\geq \sum_{r} P_{ji}^{m} P_{ii}^{r} P_{ij}^{n} \quad \text{(summing less terms)}$$

$$= P_{ji}^{m} P_{ij}^{n} \sum_{r} P_{ii}^{r} = \infty$$

b/c $i \leftrightarrow j$

Practical conclusion for irreducible MC = they are all recurrent!

# Example

1. $$\mathbb{P} = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

   All commute each other, hence irreducible MC.
   All states are recurrent

2. $$\mathbb{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

   Three communicating classes $\{0, 1\}$ $\{2, 3\}$ $\{4\}$

# Positive/Null recurrent

- We will distinguish recurrent state into two as in the below:

**Positive/Null recurrence**
State $i$ is said to be positive recurrent if

$$m_i = E(N_i | X_0 = i) < \infty$$

and null recurrent if $m_i = \infty$

- Starting in state $i$, $m_i$ is the average number of steps needed to come back to state $i$.
- If state $i$ is positive recurrent, then starting in $i$, the expected time to return to state $i$ should be finite.

# Positive/Null recurrent

- Why introducing positive recurrence? Because it is possible to have $m_i = \infty$ even if $f_{ii} = 1$ (recurrent).
- For example,

$$P(N_i = n | X_0 = i) = \frac{1}{n(n+1)}, n = 1, 2, \cdots$$

$$f_{ii} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

However,

$$m_i = \sum_{n=1}^{\infty} n \cdot P(N_i = n | X_0 = i) = \sum_{n=1}^{\infty} n \cdot \frac{1}{n(n+1)} = \infty$$

- Later on, in studying limit of MC, two cases give different limit!
- Positive recurrence is also class property.

# Recurrence and Transience

### Definition

*An irreducible DTMC is called transient/ recurrent/ null recurrent/ positive recurrent if every state in it is transient/ recurrent/ null recurrent / positive recurrent respectively.*

Below is the key theorem in classifying states of a finite state MC.

**For finite state MC**

A finite closed communicating class is positive recurrent

A finite open communicating class is transient

No null recurrent states for a finite state MC

# Example

$$\mathbb{P} = \begin{pmatrix} .5 & .5 & 0 \\ 0 & .5 & .5 \\ 0 & 0 & 1 \end{pmatrix}$$

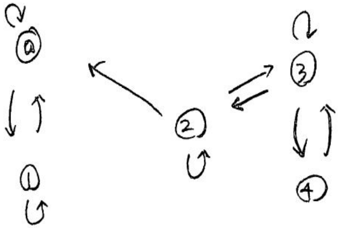
Reducible chain

$C_1 = \{0\} \quad \rightarrow$ open/ aperiodic/ transient

$C_2 = \{1\} \quad \rightarrow$ open/ aperiodic/ transient

$C_3 = \{2\} \quad \rightarrow$ closed/ aperiodic/ positive recurrent

# Example



## 4.4 Long-run proportions and limiting probabilities

- We study

$$\boxed{\lim_{n\to\infty} \mathbb{P}^n =?}$$

- That is, we are interested in

$$\boxed{\pi_j = P(X_\infty = j) = \lim_{n\to\infty} P(X_n = j)}$$

  which is the long-run proportion of time (limiting probability) MC in state $j$.

- For brevity, we only consider irreducible MC. For reducible MC, see graduate level textbooks.
- Depending on the classification of states, consider
  - Transient
  - Null recurrent
  - Positive recurrent and aperiodic
  - Positive recurrent but periodic

# Limiting Behavior - Transient

- First note that irreducible transient MC only possible for infinite state MC. For example, random walk with $p = .5$
- Since it is transient, it implies that

$$\sum_{n=0}^{\infty} P_{jj}^n < \infty$$

Since sum converges, we have

$$\lim_{n\to\infty} P_{jj}^n = 0$$

We can similarly argue that $\lim_{n\to\infty} P_{ij}^n = 0$. Hence, red regardless of initial state

$$\boxed{\pi_j = 0}$$

# Limiting Behavior - Null recurrent

- Also, irreducible null recurrent chain only possible for infinite state MC.
- Short answer: If MC is null recurrent, then

$$\boxed{\pi_j = 0}$$

- Before starting our discussion, we needed the following fact. Define

$$f_{ij} = P(X_n = j \text{ for some } n > 0 | X_0 = i)$$

be the probability of the MC starting $i$ will ever return to state $j$. Then,

**Proposition (Proposition 4.3)**

*If $i$ is recurrent ($f_{ii} = 1$) and $i$ communicate with $j$, then $f_{ij} = 1$.*

# Limiting Behavior - Null recurrent

- Consider irreducible and recurrent MC starting in state $i$, and define

$$T_1 : \text{first time the chain revisit state } j$$
$$T_2 : \text{second time the chain revisit state } j$$

- Observe from Markov properties

$$T_1 < \infty$$

since Proposition 4.3 implies that $f_{ij} = 1$, hence it returns to state $j$ for sure in finite time.

- Also,

$$T_2, T_3, \ldots \sim \text{IID Geometric distribution}$$

with mean

$$m_j = E(N_j | X_0 = j)$$

# Limiting Behavior - Null recurrent

- Law of large numbers implies that

$$\lim_{n\to\infty} \frac{T_1 + \cdots + T_n}{n} = \lim_{n\to\infty} \frac{T_1}{n} + \lim_{n\to\infty} \frac{T_2 + \cdots + T_n}{n}$$
$$= 0 + E(T_2) = m_j.$$

- Therefore,

$$\pi_j = P(X_\infty = j) = \lim_{n\to\infty} \frac{n}{T_1 + \cdots + T_n} = \frac{1}{m_j}$$

(out of $T_1 + \ldots + T_n$ trials, MC visits state $j$ $n$ times)

- Remark that $T_1/n \to 0$ implies that $\pi_j$ do not depend on initial state.

# Limiting Behavior - Null recurrent

**Proposition 4.4**
For irreducible and recurrent MC, for any initial state

$$\pi_j = \frac{1}{m_j} = P(X_\infty = j)$$

- In also means that $\lim_{n\to\infty} P_{ij}^n = 1/m_j$ regardless of initial state $i$.
- If MC is furthermore NULL recurrent, that is $m_j = \infty$, we have

$$\boxed{\pi_j = 0}$$

# Limiting Behavior - Positive recurrent and aperiodic

- Suppose there is a limit

$$\pi_j = P(X_\infty = j) = \lim_{n\to\infty} P(X_n = j).$$

- Then, it should satisfy

$$\begin{aligned} P(X_{n+1} = j) &= \sum_{i\in S} P(X_{n+1} = j | X_n = i) P(X_n = i) \\ &= \sum_{i\in S} P(X_n = i) P_{ij} \end{aligned}$$

  Thus, by taking the limit on both sides, we have

$$\pi_j = \sum_{i\in S} \pi_i P_{ij}$$

- Also, $\pi_j$ should add up to one

$$\sum_{j\in S} \pi_j = 1$$

# Limiting Behavior - Positive recurrent and aperiodic

- In a matrix form,

$$\boxed{\boldsymbol{\pi} = \boldsymbol{\pi}\mathbb{P}, \quad \sum_j \pi_j = 1} \tag{1}$$

where

$$\boldsymbol{\pi} = (\pi_0, \pi_1, \cdots), \quad \mathbf{1} = (1, \cdots, 1)$$

(Be careful that it is written as row vector)

**Theorem**

If irreducible MC is positive recurrent and aperiodic, then (1) has the unique solution and

$$\lim_{n\to\infty} P(X_n = j | X_0 = i) = \pi_j = \frac{1}{m_j}$$

for any initial state $i$.

# Example 4.21: Mood of an individual

- Recall Example 4.21 with transition probability

$$\mathbb{P} = \begin{pmatrix} .5 & .4 & .1 \\ .3 & .4 & .3 \\ .2 & .3 & .5 \end{pmatrix}$$

- Then, it is irreducible, aperiodic and positive recurrent MC. Hence limiting behavior is completely determined by solving

$$(\pi_1, \pi_2, \pi_3) = (\pi_1, \pi_2, \pi_3)\mathbb{P}, \quad \pi_1 + \pi_2 + \pi_3 = 1$$

# Example 4.20: Weather forecasting

- Weather forecasting with $\alpha > 0, \beta > 0$

$$\mathbb{P} = \begin{array}{c c} & \begin{array}{cc} r & n \end{array} \\ \begin{array}{c} r \\ n \end{array} & \begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix} \end{array}$$

Find the limiting distribution.

# Example 4.27: Bonus Malus (Good-Bad) system for car insurance

- A policy holder's state chages depending on the number of claims made by that policyholder. Next state if

| state | premium | 0 claims | 1 claims | 2 claims | $\geq$3 claims |
|---|---|---|---|---|---|
| 1 | 200 | 1 | 2 | 3 | 4 |
| 2 | 250 | 1 | 3 | 4 | 4 |
| 3 | 400 | 2 | 4 | 4 | 4 |
| 4 | 600 | 3 | 4 | 4 | 4 |

- Let $X_n$ be the state of policy holder at $n$-th year and $a_k =$ P(Policyholder makes $k$ claims).
- Then, the transition probability is given by

$$\mathbb{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} a_0 & a_1 & a_2 & 1-a_0-a_1-a_2 \\ a_0 & 0 & a_1 & 1-a_0-a_1 \\ 0 & a_0 & 0 & 1-a_0 \\ 0 & 0 & a_0 & 1-a_0 \end{pmatrix} \end{array}$$

## Bonus Malus -continued

- Assume that $a_k \sim \text{Poisson}(1/2)$ (so that $a_0 = .6065$, $a_1 = .3033$, $a_2 = .0758$).
- Interested in the average annual premium paid in the long-run. Let $r$ be the annual premium function, then it is given by

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} r(X_n)}{N}$$

**Proposition 4.6**

For irreducible MC with stationary probabilities $\pi_j$,

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} r(X_n)}{N} = \sum_{j\in S} r(j)\pi_j$$

for any initial state $i$.

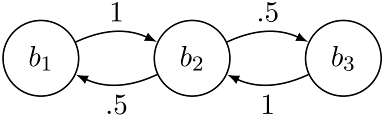
# Bonus Malus -continued

- Hence, the average annual premium paid in the long-run is

$$200\pi_1 + 250\pi_2 + 400\pi_3 + 600\pi_4$$

# Limiting Behavior - Positive recurrent but periodic

- Consider brand switching between $\{b_1, b_2, b_3\}$.




- The transition probability is given by

$$\mathbb{P} = \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \end{array} \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ .5 & 0 & .5 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

- Then, it is irreducible, positive recurrent but periodic with $d = 2$.

# Limiting Behavior - Positive recurrent but periodic

- Then, one can observe that the limit is alternating in the sense that

$$\lim_{n\to\infty} \mathbb{P}^{2n} = \begin{pmatrix} .5 & 0 & .5 \\ 0 & 1 & 0 \\ .5 & 0 & .5 \end{pmatrix}, \quad \lim_{n\to\infty} \mathbb{P}^{2n+1} = \begin{pmatrix} 0 & 1 & 0 \\ .5 & 0 & .5 \\ 0 & 1 & 0 \end{pmatrix}$$

- Limit do not exists for positive recurrent but periodic MC.
- However, observe that the solution of $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbb{P}$ and $\sum \pi_j = 1$ exists uniquely.

# Limiting Behavior - Positive recurrent but periodic

- $$\mathbb{P}^* = \begin{pmatrix} .25 & .5 & .25 \\ .25 & .5 & .25 \\ .25 & .5 & .25 \end{pmatrix}$$
- We can interpret the solution $\{\pi_j\}$ as the average of limits along the subperiods;

$$\lim_{n\to\infty}\left(\frac{P^{2n}+P^{2n+1}}{2}\right)$$

# Limiting behavior - summary

All in all, for irreducible MC the limiting behavior is

- Transient: Limit exits but $\pi_j = 0$.
- Null recurrent: Limit exits but $\pi_j = 0$.
- Positive recurrent and aperiodic: Limit exits and give by the solution of $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbb{P}$, $\sum \pi_j = 1$.
- Positive recurrent but periodic: Limit only exits along the subperiod. (Hence, $\lim_{n\to\infty} \mathbb{P}^n$ do not exists). However, the solution of $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbb{P}$, $\sum \pi_j = 1$ exists and can be interpreted as the average of limits on subperiods.

Also observe that limiting behavior free from initial state!

# 4.7 Branching Process

- Arising from Francis Galton's statistical investigation of the extinction of family names.
- Let $X_n$ be the size of a population at $n$-th generation.
- $X_0$ is the size of the zeroth generation and assume to be 1.
- Each individual produce offspring of the same kind, and by the end of lifetime, has produces $j$ new offspring with probability $P_j, j \geq 0$.
- Graphically, it looks like

# Branching process

- Mathematically, it can be represented as

$$\boxed{X_{n+1} = \sum_{i=1}^{X_n} Z_{i,n}, \quad X_0 = 1,}$$

  where $\{Z_{i,n}\}$ are IID rvs (over all $i$ and $n$) with pmf $P_j, j \geq 0$.
- Denote $E(Z_{i,n}) = \mu$ and $\mathrm{Var}(Z_{i,n}) = \sigma^2$.
- We are interested in the long-run behavior of population size. WIll it die out or remain stable or infinity? In particular, probability of extinction?
- Can apply MC thoery.

# Branching process

Branching process $\{X_n\}$ is a MC with $\mathbb{S} = \{0, 1, 2, \cdots\}$.

Indeed:

$$P(X_{n+1} = j | X_n = k, X_{n-1}, \cdots X_0 = 1)$$

$$= P(\sum_{i=1}^{X_n} Z_{i,n} = j | X_n = k, \cdots, X_0 = 1)$$

$$= P(\sum_{i=1}^{k} Z_{i,n} = j) = P(X_{n+1} = j | X_n = k)$$

# Branching Process - Population size

Consider the size of population at $n$-th generation.

- Let $\mu_n = EX_n$ and $\operatorname{Var} X_n = \sigma_n^2$. Then,

$$\mu_0 = 1, \quad \sigma_0^2 = 0.$$

- Observe

$$\mu_{n+1} = E\left(\sum_{i=1}^{X_n} Z_{i,n}\right) = E\left(E\left(\sum_{i=1}^{X_n} Z_{i,n} | X_n\right)\right)$$

$$= E(X_n \cdot E(Z_{i,n})) = EX_n \cdot E(Z_{i,n}) = \mu_n \cdot \mu$$

Since $\mu_0 = 1$, we have

$$\boxed{\mu_n = \mu^n}$$

# Branching Process

- Similarly, for variance, we have

$$\sigma_{n+1}^2 = \operatorname{Var}(X_{n+1}) = \operatorname{Var}\left(\sum_{i=1}^{X_n} Z_{i,n}\right)$$

$$= E(X_n)\operatorname{Var}(Z_{i,n}) + \operatorname{Var}(X_n)E(Z_{i,n})^2$$

Hence,

$$\sigma_{n+1}^2 = \mu^n \sigma^2 + \mu^2 \sigma_n^2$$

Since $\sigma_0^2 = 0$, we have

$$\sigma_1^2 = \sigma^2$$

$$\vdots$$

$$\sigma_n^2 = \sigma^2(\mu^{n-1} + \mu^n + \ldots + \mu^{2n-2})$$

$$\boxed{\sigma_n^2 = \begin{cases} \sigma^2 \mu^{n-1} \frac{\mu^n - 1}{\mu - 1} & \text{if } \mu \neq 1 \\ n\sigma^2 & \text{if } \mu = 1 \end{cases}}$$

# Branching Process

Therefore, according for value of $\mu$, the average size of population becomes graphically,

# Branching Process - extinction probability

- Furthermore, using limiting behavior of MC we can show

$$\lim_{n\to\infty} P(X_n = 0|X_0 = 1) = \begin{cases} 1 & \text{if } \mu \le 1 \\ \pi_0 & \text{if } \mu > 1 \end{cases}$$

- For example, if $\mu < 1$,

$$\begin{aligned} 0 = \mu^n = E(X_n) &= \sum_{j=1}^{\infty} jP(X_n = j) \\ &\geq \sum_{j=1}^{\infty} P(X_n = j) = P(X_n \geq 1). \end{aligned}$$

  Since probability is always non-negative, we have

$$P(X_n = 0) = 1 - P(X_n \geq 1) \to 1 \text{ as } n \to \infty.$$

- It is also interesting to see that a population dies out even when $\mu = 1$.

# Branching process - extinction probability

- For $\pi_0$ calculation, observe that

$$\pi_0 = P(\text{pop}^n \text{ dies out}) = \sum_{j=0}^{\infty} P(\text{pop}^n \text{ dies out}|X_1 = j)P(X_1 = j)$$

$$\sum_{j=0}^{\infty} P(\text{pop}^n \text{ dies out}|X_1 = j)P_j = \sum_{j=0}^{\infty} \pi_0^j P_j$$

- Therefore $\pi_0$ should satisfy

$$\pi_0 = \sum_{j=0}^{\infty} \pi_0^j P_j$$

and in fact $\pi_0$ is the smallest positive solution.

# Branching Process - extinction probability

Example 4.3. If $P_0 = \frac{1}{4}, P_1 = \frac{1}{4}, P_2 = \frac{1}{2}$, then determine $\pi_0$.