Chapter 5 Limit Theorems

5.1 Limits of *sums, products,* and *quotients*

Theorem. Assume that $a_n \to L$ and $b_n \to M$ as $n \to \infty$

(1) Linearity theorem $ra_n + sb_n \to rL + sM$ for every $r, s \in \mathbb{R}$

(2) Product theorem $a_n b_n \to LM$

(3) Quotient theorem $\dfrac{b_n}{a_n} \to \dfrac{M}{L}$ if $L \neq 0$ & $a_n \neq 0$ for all n

Pf. (1) $e_n \overset{\text{let}}{=} a_n - L$ $e_n' \overset{\text{let}}{=} b_n - M$

Then $e_n \to 0$ and $e_n' \to 0$ as $n \to \infty$

So, given $\varepsilon > 0$, $|e_n| < \varepsilon$ & $|e_n'| < \varepsilon$ for $n >> 1$.

$$|ra_n + sb_n - (rL + sM)| \le |r(a_n - L)| + |s(b_n - M)|$$
$$= |r||e_n| + |s||e_n'| < |r|\varepsilon + |s|\varepsilon = (|r| + |s|)\varepsilon \quad \text{for} \quad n >> 1$$

Thus by K-ε Principle, $ra_n + sb_n \to rL + sM$

(2) By hypo, given $\varepsilon > 0$, $|e_n| < \varepsilon$ & $|e_n'| < \varepsilon$ for $n >> 1$

Notice that

$|a_n b_n - LM| < \varepsilon$ holds when $\varepsilon < 1$ $\Rightarrow$ $|a_n b_n - LM| < \varepsilon'$ is (automatically) true for all $\varepsilon' \ge 1$

Thus we may and do assume $\varepsilon < 1$. Then

$$a_n b_n - LM = (e_n + L)(e_n' + M) - LM = e_n M + e_n' L + e_n e_n'$$
$$\therefore \quad |a_n b_n - LM| \le |M||e_n| + |L||e_n'| + |e_n||e_n'|$$
$$< \varepsilon|M| + \varepsilon|L| + \varepsilon \bullet \varepsilon < (|M| + |L| + 1)\varepsilon \equiv K\varepsilon$$

Thus by K-ε Principle, $a_n b_n \to LM$

(3) Enough to show $\dfrac{1}{a_n} \to \dfrac{1}{L} (L \neq 0)$ because (2) will then give

$$\frac{b_n}{a_n} = b_n \bullet \frac{1}{a_n} \quad \to \quad M \bullet \frac{1}{L} = \frac{M}{L}$$

Since $\left|\dfrac{1}{a_n} - \dfrac{1}{L}\right| = \dfrac{|a_n - L|}{|a_n||L|}$, to show the quotient on the right is *small*,

we must show the denominator is *not too small* (i.e., must show $|a_n|$ is not too small)

Given $\varepsilon > 0$, $a_n = L + e_n$ where $|e_n| < \varepsilon$ for $n >> 1$

$$|a_n| = |L + e_n| \ge |L| - |e_n| > |L| - \varepsilon \quad \text{for } n >> 1\text{, since } |e_n| < \varepsilon \text{ for } n >> 1$$
$$> \frac{|L|}{2} \quad \text{for } n >> 1\text{, since we can take } \varepsilon < \frac{|L|}{2}.$$

$$\therefore \quad \left|\frac{1}{a_n} - \frac{1}{L}\right| = \frac{|a_n - L|}{|a_n||L|} = \frac{|e_n|}{|a_n||L|} < \frac{\varepsilon}{\frac{|L|}{2} \bullet |L|} = \frac{2\varepsilon}{|L|^2} \quad \text{for} \quad n >> 1$$

Thus by K-ε Principle, $\dfrac{1}{a_n} \to \dfrac{1}{L}$ when $L \neq 0$

Remark to (3): $\boxed{\lim_{n\to\infty} a_n = L \quad \& \quad L \neq 0 \qquad \Rightarrow \qquad a_n \neq 0 \quad \text{for} \quad n >> 1}$

Pf. By hypo, given $\varepsilon > 0$, $L - \varepsilon < a_n < L + \varepsilon$ for $n >> 1$

If $L > 0$, take $\varepsilon = \frac{L}{2} (> 0)$, then $0 < \frac{L}{2} < a_n$ for $n >> 1$

$$\therefore \quad a_n \neq 0 \quad \text{for} \quad n >> 1$$

If $L < 0$, take $\varepsilon = -\frac{L}{2} (> 0)$, then $a_n < \frac{L}{2} < 0$ for $n >> 1$

$$\therefore \quad a_n \neq 0 \quad \text{for} \quad n >> 1$$

Consequently, in each case, we have $a_n \neq 0$ for $n >> 1$

In particular, $\frac{1}{a_n}$ is defined for $n >> 1$

Alternative (short) pf.

By hypo, given $\varepsilon > 0$, $a_n \underset{\varepsilon}{\approx} L$ for $n >> 1$

Take $\varepsilon = \frac{|L|}{2} (> 0)$. Then

$$|a_n - L| < \frac{|L|}{2} \quad \text{for} \quad n >> 1$$

Thus, for $n >> 1$,

$$|a_n| = |a_n - L + L| = |L - (L - a_n)| \geq |L| - |L - a_n| > |L| - |L|/2 = |L|/2 (> 0)$$

$$\therefore \quad a_n \neq 0 \quad \text{for} \quad n >> 1$$

Exa A. $\lim_{n\to\infty} \frac{3n^2 - 2n - 1}{n^2 + 1} = ?$

Sol. $\frac{3n^2 - 2n - 1}{n^2 + 1} = \frac{3 - \frac{2}{n} - \frac{1}{n^2}}{1 + \frac{1}{n^2}} \to \frac{3}{1} = 3 \quad (\because \quad \frac{1}{n} \to 0, \quad \frac{1}{n^2} \to 0)$

Theorem (Algebraic operations for infinite limits)

(1) $a_n \to \infty$ and $b_n \to \infty$ { or $b_n \to L(\text{finite})$, or $b_n \geq (\text{some number}) C$ for all n }

$$\Rightarrow \quad a_n + b_n \to \infty$$

(2) $a_n \to \infty$ and $b_n \to \infty$ { or $b_n \to L(> 0)$, or $b_n \geq (\text{some number}) K > 0$ for all n }

$$\Rightarrow \quad a_n \bullet b_n \to \infty$$

(3) $a_n \to \infty \quad \Rightarrow \quad \dfrac{1}{a_n} \to 0$ (but the converse is *false* in general)

(4) $a_n \to 0 \quad \& \quad a_n > 0 \quad \text{for all } n \quad \Rightarrow \quad \dfrac{1}{a_n} \to \infty$

Pf (1) Assume $a_n \to \infty$ and $b_n \geq C$ for all n. Then

given $M > 0$, $a_n > M + |C|$ for $n >> 1$

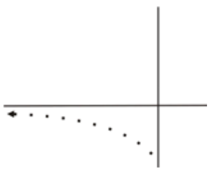
Thus $a_n + b_n > M + |C| + C \geq M$ for $n >> 1$

That is, $a_n + b_n > M$ for $n >> 1$

$$\therefore \quad \lim_{n\to\infty}(a_n + b_n) = \infty$$

(2), (3), (4): Exercise

Note: The converse of (3) is false:



$$a_n = -1/n \quad \to 0 \quad \text{but } 1/a_n = -n \quad \to -\infty$$

Exa B Find $\lim_{n\to\infty} n(a + \cos n\pi)$, for different values of a

Sol. $\cos n\pi = (-1)^n$ for all n

If $a > 1$, then $a + \cos n\pi = a + (-1)^n \geq a - 1 > 0$ for all n

$$n(a + \cos n\pi) \geq n(a-1) \to \infty \qquad \therefore \quad n(a + \cos n\pi) \to \infty$$

If $a < -1$, then $a + \cos n\pi = a + (-1)^n \leq a + 1 < 0$ for all n

$$n(a + \cos n\pi) \leq n(a+1) \to -\infty \qquad \therefore \quad n(a + \cos n\pi) \to -\infty$$

If $a = 1$, then $n(a + \cos n\pi) = n(a + (-1)^n) = n(1 + (-1)^n) = \begin{cases} 2n, & n = \text{even} \\ 0, & n = \text{odd} \end{cases}$

If $a = -1$, then $n(a + \cos n\pi) = n(a + (-1)^n) = n(-1 + (-1)^n) = \begin{cases} 0, & n = \text{even} \\ -2n, & n = \text{odd} \end{cases}$

$$\therefore \quad \lim_{n\to\infty} n(a + \cos n\pi) \text{ does not exist if } a = 1 \text{ or } a = -1$$

If $|a| < 1$, then $n(a + \cos n\pi) = n(a + (-1)^n) = \begin{cases} n(a+1) \quad (\to \infty), & n = \text{even} \\ n(a-1) \quad (\to -\infty), & n = \text{odd} \end{cases}$

$$\therefore \quad \lim_{n\to\infty} n(a + \cos n\pi) \text{ does not exist if } |a| < 1$$

5.2 Comparison theorems

Theorem (*Squeeze Theorem* or *Sandwich Theorem*)

Suppose that there are three sequences (a_n), (b_n), and (c_n) satisfying

$$a_n \le b_n \le c_n \quad \text{for} \quad n >> 1$$

If $a_n \to L$ & $c_n \to L$, then $b_n \to L$ also.

Pf. By hypo, given $\varepsilon > 0$, $a_n \underset{\varepsilon}{\approx} L$ & $c_n \underset{\varepsilon}{\approx} L$ for $n >> 1$

That is, given $\varepsilon > 0$, $L - \varepsilon < a_n < L + \varepsilon$ & $L - \varepsilon < c_n < L + \varepsilon$ for $n >> 1$

So, $L - \varepsilon < a_n \le b_n \le c_n < L + \varepsilon$ for $n >> 1$

This shows: given $\varepsilon > 0$, $b_n \underset{\varepsilon}{\approx} L$ for $n >> 1$

Exa A. Show that $\sqrt[n]{2 + \cos na} \to 1$, for any fixed number a

Pf. Recall easy facts:

- $a, b > 0$ & $a \le b \Rightarrow \sqrt[n]{a} \le \sqrt[n]{b}$ (대우로 증명가능)
- $a > 0 \Rightarrow \lim_{n\to\infty} \sqrt[n]{a} = 1$

Pf of the second fact:

Case1. $a > 1 \Rightarrow \sqrt[n]{a} > 1$

$$\sqrt[n]{a} \overset{\text{let}}{=} 1 + h_n, \quad h_n > 0 \qquad \text{Have to show } h_n \to 0$$

$$a = (1+h_n)^n = 1 + nh_n + \frac{n(n-1)}{2!}h_n^{\,2} + \frac{n(n-1)(n-2)}{3!}h_n^{\,3} + \cdots + h_n^{\,n}$$
$$\ge 1 + nh_n > nh_n$$

$$\therefore \quad 0 \quad < \quad h_n \quad < \frac{a}{n}$$
$$\downarrow \qquad\qquad \downarrow \quad \text{as} \quad n \to \infty$$
$$0 \quad \le \qquad \le \quad a \bullet 0 = 0$$

$$\therefore \quad \lim_{n\to\infty} h_n = 0$$

Case2. $0 < a < 1 \Rightarrow 1/a > 1$

$$\overset{\text{Case1}}{\Rightarrow} \lim_{n\to\infty} \sqrt[n]{1/a} = 1 \quad (\text{ i.e., } \lim_{n\to\infty} \frac{1}{\sqrt[n]{a}} = 1 \ )$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{a} = 1 \quad (\because \quad \lim_{n\to\infty} a_n = L(\ne 0) \Rightarrow \lim_{n\to\infty} 1/a_n = 1/L)$$

Case3. $a = 1$: Trivial

Remark (later): $a > 0 \Rightarrow \lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} a^{\frac{1}{n}} \overset{a^x \text{ is continuous on } \mathbb{R}}{=} a^{\lim_{n\to\infty}\frac{1}{n}} = a^0 = 1$

We turn now to the problem:

$$1=\sqrt[n]{1} \le \sqrt[n]{2+\cos na} \le \sqrt[n]{3}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$1 \qquad\qquad\qquad\qquad 1 \quad \text{as} \quad n\to\infty$$

Hence by Squeeze Theorem, $\sqrt[n]{2+\cos na} \to 1$

Theorem (Squeeze Theorem for infinite limits)

$$b_n \ge a_n \quad \& \quad a_n \to \infty \qquad \Rightarrow \qquad b_n \to \infty$$

Pf. (Easy) Given $M>0$, $\quad a_n > M \quad \text{for} \quad n>>1$

$$\downarrow \quad \Leftarrow \quad b_n \ge a_n$$

$$b_n > M \quad \text{for} \quad n>>1$$

Review

$$1+1/2+1/3+\cdots+1/n \quad > \quad \ln(n+1) > \ln n \quad \to \infty$$

$$\therefore \quad \lim_{n\to\infty}(1+1/2+1/3+\cdots+1/n)=\infty$$

Exa B $\quad a>1 \quad \Rightarrow \quad a^n \to \infty$

Pf. $\quad a>1 \quad \Rightarrow \quad a=1+k, \quad k>0$

$$\Rightarrow \quad a^n=(1+k)^n=1+nk+\text{positive terms}$$

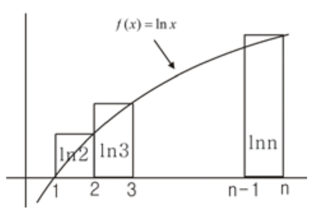
$$> \quad 1+nk \quad \to \quad \infty \quad \text{as} \quad n\to\infty$$

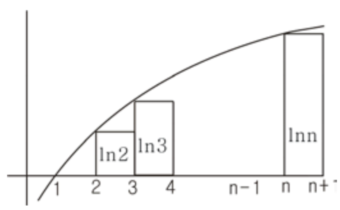
$$\therefore \quad a^n \to \infty$$

Exa C $\quad$ ($n!$ ~ ? when n is large enough)

Show that $\displaystyle \lim_{n\to\infty}\frac{\ln n!}{n\ln n}=1$ (In symbols, $\ln n! \sim n\ln n$)

Sol. $\quad \ln n! = \ln 1+\ln 2+\cdots+\ln n$





the total area of the above rectangles

$$= \quad \ln 1+\ln 2+\cdots+\ln n$$

$$\int_1^n \ln x\,dx \le \ln 1+\ln 2+\cdots\ln n \le n\ln n \quad \left(\text{or } \int_2^{n+1} \ln x\,dx\right)$$

Integral in the LHS = $\left[x\ln x - x\right]_1^n = n\ln n - n + 1$

$$\times\frac{1}{n\ln n} \;\Rightarrow\; \underbrace{1-\frac{1}{\ln n}+\frac{1}{n\ln n}}_{\downarrow \atop 1} \;\le\; \frac{\ln n!}{n\ln n} \;\le\; \underbrace{1}_{\downarrow \atop 1}$$

$$\therefore \quad \lim_{n\to\infty}\frac{\ln n!}{n\ln n} = 1$$

5.3 Location Theorems

Theorem A (*Limit Location Theorem* : **LLT** for short)

If (a_n) is convergent, then

(a) $a_n \le M$ for $n >> 1$ $\Rightarrow$ $\lim_{n\to\infty} a_n \le M$

(b) $a_n \ge M$ for $n >> 1$ $\Rightarrow$ $\lim_{n\to\infty} a_n \ge M$

[결론: $\le$ (또는 $\ge$) 의 양변에 limit를 택한 결과가 성립한다; if the limit is known to exist]

For example, $a_n \ge 0$ for $n >> 1$ $\Rightarrow$ $\lim_{n\to\infty} a_n \ge 0$ if (a_n) is convergent.

Caution: It is *not* true that "$a_n > 0$ for $n >> 1$ $\Rightarrow$ $\lim_{n\to\infty} a_n > 0$" even if (a_n) is convergent.

For example, $\frac{1}{n} > 0$ for all n ($\therefore$ $\frac{1}{n} > 0$ for $n >> 1$) but $\lim_{n\to\infty}\frac{1}{n} = 0$

Pf of (a). The statement (a) can be written as

$$a_n \le M \text{ for } n >> 1,\quad a_n \to L \;\Rightarrow\; L \le M$$

$a_n \to L$ $\Rightarrow$ given $\varepsilon > 0$, $a_n \underset{\varepsilon}{\approx} L$ for $n >> 1$

That is, $L-\varepsilon < a_n < L+\varepsilon$ for $n >> 1$

Since $a_n \le M$ for $n >> 1$, we have

$$L-\varepsilon < M, \quad \text{for any } \varepsilon > 0 \qquad \text{----} \quad (*)$$

This implies $L \le M$.

($\because$ If $L > M$, choose $\varepsilon = L - M (> 0)$, then

$L-\varepsilon = L-(L-M) = M$; contradiction to $(*)$)

(b) can be proved in a similar way.

Note: Only the following conclusion can be guaranteed:

(i) $a_n < M$ for $n >> 1$ $\Rightarrow$ $\lim_{n\to\infty} a_n \le M$ if (a_n) is convergent

(ii) $a_n > M$ for $n >> 1$ $\Rightarrow$ $\lim_{n\to\infty} a_n \ge M$ if (a_n) is convergent

⊙ A variant of the Limit Location Theorem

(a_n) & (b_n) : convergent, $a_n \le b_n$ for $n >> 1$ $\Rightarrow$ $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$

Pf. $a_n - b_n \le 0$ for $n >> 1$ $\overset{(a)}{\Rightarrow}$ $\lim_{n\to\infty}(a_n - b_n) \le 0$ (since $\lim_{n\to\infty}(a_n - b_n)$ exists; why?)

$$\Rightarrow \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n \le 0$$

$$\Rightarrow \lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$$

Theorem B (*Sequence Location Theorem*: **SLT** for short)

Assuming (a_n) converges,

(a) $\lim_{n\to\infty} a_n < M \Rightarrow a_n < M$ for $n >> 1$

(b) $\lim_{n\to\infty} a_n > M \Rightarrow a_n > M$ for $n >> 1$

Pf. (a) Let $L = \lim_{n\to\infty} a_n$. Then

given $\varepsilon > 0$, $L - \varepsilon < a_n < L + \varepsilon$ for $n >> 1$

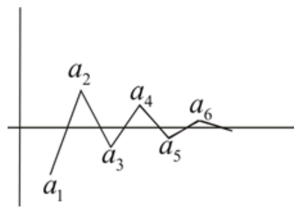
Hypo means $L < M$.

Take $\varepsilon = M - L (> 0)$. Then

$$a_n < L + \varepsilon = L + (M - L) = M \quad \text{for} \quad n >> 1$$

(b) can be proved in a similar way.

Caution: It is *not* true that $\lim_{n\to\infty} a_n \le M \Rightarrow a_n \le M$ for $n >> 1$



$\lim_{n\to\infty} a_n = 0$ ($\therefore$ $\lim_{n\to\infty} a_n \le 0$) but $a_n > 0$ only for every even $n (> 0)$

5.4 Subsequences (Commonly used for proving "non-existence of limits")

Def. A subsequence of (a_n) is a sequence consisting of terms (a_n) and having the form

$$a_{n_1},\ a_{n_2},\ a_{n_3},\cdots,\ a_{n_i},\ \cdots,\quad \text{where}\ \ n_1 < n_2 < n_3 < \cdots < n_i < \cdots.$$

(Remember: n_i's are nonnegative integers & "strictly" increasing)

Exa. Let $(a_n):\ 1,\ 2,\ 1,\ 3,\ 1,\ 4,\ 1,\ 5,\ \cdots$

Each list in the left is a subseq of (a_n):

$1,\ 1,\ 1,\ 1,\ \cdots$

$1,\ 2,\ 3,\ 4,\ \cdots$

$3,\ 4,\ 5,\ 6,\ \cdots$

Each list in the right is **not** a subseq of (a_n):

$2,\ 2,\ 2,\ 2,\ \cdots$

$3,\ 2,\ 5,\ 4,\ \cdots$

Theorem (**Subsequence Theorem**)

If (a_n) converges, every subsequence also converges, and to the same limit.

In other words,

$$\lim_{n\to\infty} a_n = L \quad \Rightarrow \quad \lim_{i\to\infty} a_{n_i} = L \ \text{ for every subsequence } (a_{n_i})$$

Pf. $\displaystyle\lim_{n\to\infty} a_n = L \quad \Rightarrow \quad \text{given}\ \ \varepsilon > 0,\quad a_n \underset{\varepsilon}{\approx} L \quad \text{for}\ \ n \gg 1$

That is, $\exists$ a number N (depending only on ε) such that

$$a_n \underset{\varepsilon}{\approx} L \quad \text{for}\ \ n > N \quad ---(*)$$

Remind the indices $n_1,\ n_2,\ \cdots, n_i,\ \cdots$ are strictly increasing & nonnegative integers.

(Thus n_i is strictly $\uparrow$ & $\displaystyle\lim_{i\to\infty} n_i = \infty$)

$$\therefore\quad n_i > N \quad \text{for}\ \ i \gg 1 \quad ---(**)$$

$$(*) \quad \& \quad (**) \quad \Rightarrow \quad a_{n_i} \underset{\varepsilon}{\approx} L \quad \text{for } i \gg 1 \qquad\qquad \therefore\ \lim_{i\to\infty} a_{n_i} = L$$

Exa A. It is true that $\displaystyle\lim_{n\to\infty} a_n^{\,2} = 0 \quad \Rightarrow \quad \lim_{n\to\infty} a_n = 0$

What's wrong in the following argument?

Wrong pf: By contraposition, we shall show that $\displaystyle\lim_{n\to\infty} a_n \neq 0 \quad \Rightarrow \quad \lim_{n\to\infty} a_n^{\,2} \neq 0$

Suppose therefore that $\displaystyle\lim_{n\to\infty} a_n = L,\ \text{ where } L \neq 0.$

Then $\displaystyle\lim_{n\to\infty} a_n^{\,2} = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} a_n = L^2 \ \neq\ 0$

Note: $\overset{\text{negation}}{\sim}(\lim\limits_{n\to\infty} a_n = 0)$ is ***not*** equivalent to $\lim\limits_{n\to\infty} a_n \neq 0$

In fact,

$$\sim(\lim_{n\to\infty} a_n = 0) \quad \Leftrightarrow \quad \begin{cases} \text{either } \lim\limits_{n\to\infty} a_n \text{ does not exist, or} \\ \lim\limits_{n\to\infty} a_n \text{ exists } \ \& \ \lim\limits_{n\to\infty} a_n \neq 0 \end{cases}$$

Right pf:

$$\lim_{n\to\infty} {a_n}^2 = 0 \quad \Rightarrow \quad \text{given } \ \varepsilon > 0, \quad {a_n}^2 \underset{\varepsilon^2}{\approx} 0 \quad \text{for } \ n \gg 1$$

$$\overset{\text{clearly}}{\Rightarrow} \quad \text{given } \ \varepsilon > 0, \quad a_n \underset{\varepsilon}{\approx} 0 \quad \text{for } \ n \gg 1$$

$$\therefore \quad \lim_{n\to\infty} a_n = 0$$

Exa B. Prove $\lim\limits_{n\to\infty} \sin\dfrac{n\pi}{2}$ does not exist.

Pf. Note that

$$\sin\frac{n\pi}{2} = 0 \quad \text{if} \quad \frac{n\pi}{2} = k\pi$$

$$\& \qquad\qquad\qquad\qquad\qquad (\text{where } k \in \mathbb{N})$$

$$\sin\frac{n\pi}{2} = 1 \quad \text{if} \quad \frac{n\pi}{2} = (2k + \frac{1}{2})\pi$$

Thus $(\sin\dfrac{2k\pi}{2})_{k=1}^{\infty}$ & $(\sin\dfrac{(4k+1)\pi}{2})_{k=1}^{\infty}$ are two subsequences of $(\sin\dfrac{n\pi}{2})_1^{\infty}$ such that

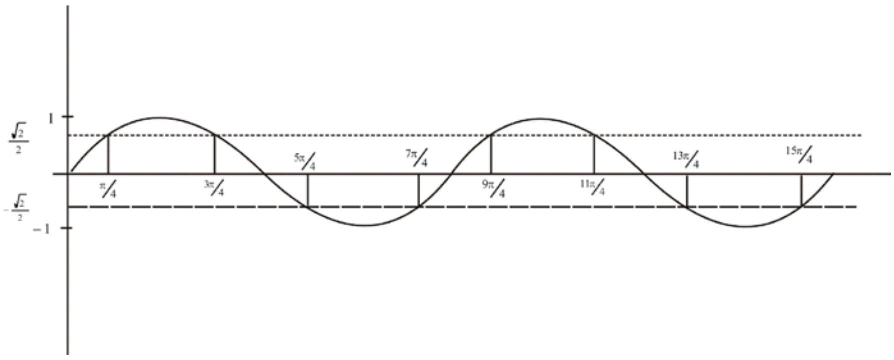
$$\sin\frac{2k\pi}{2} = 0 \to 0 \quad \neq \quad 1 \leftarrow 1 = \sin\frac{(4k+1)\pi}{2}$$

Therefore, $\lim\limits_{n\to\infty} \sin\dfrac{n\pi}{2}$ does not exist, by Subsequence Theorem,

※Exa C. Prove that $\lim\limits_{n\to\infty} \sin n$ does not exist.

(This is harder than the preceding example, since we don't know the exact values of $\sin n$)

Pf.



From the graph, we see that there are infinitely many intervals of length $\frac{\pi}{2}$ on which $\sin x \geq \frac{\sqrt{2}}{2}$.

Since each of these intervals has length > 1, in each of them we can choose an integer;

let k_i be the integer chosen from the i-th interval.

This gives a subsequence $(\sin k_i)$ such that

$$\sin k_i \geq \frac{\sqrt{2}}{2} \quad --- (\circledast)$$

Similarly, we can choose an integer m_i from each of the successive intervals of length $\frac{\pi}{2}$

on which $\sin x \leq -\frac{\sqrt{2}}{2}$, giving a subsequence $(\sin m_i)$ such that

$$\sin m_i \leq -\frac{\sqrt{2}}{2} \quad --- (\circledast\circledast)$$

Suppose now that $\lim_{n\to\infty} \sin n \overset{\text{let}}{=} L$ exists. Then by Subsequence Theorem,

$$\lim_{i\to\infty} \sin k_i = L \quad \& \quad \lim_{i\to\infty} \sin m_i = L$$

But $(\circledast)$ & $(\circledast\circledast)$ & LLT imply that

$$\lim_{i\to\infty} \sin k_i \geq \frac{\sqrt{2}}{2} \quad \& \quad \lim_{i\to\infty} \sin m_i \leq -\frac{\sqrt{2}}{2}$$

$$\text{i.e.,} \quad L \geq \frac{\sqrt{2}}{2} \quad \& \quad L \leq -\frac{\sqrt{2}}{2} \qquad : \text{contradiction}$$

$\therefore$ $\lim_{n\to\infty} \sin n$ does not exist.

5.5 Two common mistakes

Exa A. Prove: $a_n \to 0$ & b_n is bounded $\Rightarrow$ $a_n b_n \to 0$

What's wrong in the following argument?

Wrong pf. Since (b_n) is bounded, $\exists$ two real numbers L & M such that

$$\begin{array}{ccccc} & L \leq & b_n & \leq M & \\ & & \Downarrow & & \\ a_n L & \leq & a_n b_n & \leq & a_n M \\ \downarrow & & & & \downarrow \quad \text{as } n \to \infty \\ 0 & & & & 0 \end{array}$$

$$\therefore \quad a_n b_n \to 0$$

Note: In the above, $\Downarrow$ is *not* true (; $\Downarrow$ is true only if $a_n \geq 0$)

◎ A modification of the above argument: Start with "$L \leq b_n \leq M$" ($\Leftarrow$ (b_n) is bounded)

Case1 $a_n \geq 0 \Rightarrow$

$$\begin{array}{ccccc} a_n L & \leq & a_n b_n & \leq & a_n M \\ \downarrow & & & & \downarrow \quad \text{as } n \to \infty \\ 0 & & & & 0 \end{array}$$

$$\therefore \quad a_n b_n \to 0$$

Case2 $a_n \leq 0 \Rightarrow$

$$\begin{array}{ccccc} a_n L & \geq & a_n b_n & \geq & a_n M \\ \downarrow & & & & \downarrow \quad \text{as } n \to \infty \\ 0 & & & & 0 \end{array}$$

$$\therefore \quad a_n b_n \to 0$$

This is also wrong, since a_n might alternate between positive & negative.

Right argument (use absolute values)

Since (b_n) is bounded, $\exists$ a number $K > 0$ such that

$$|b_n| \leq K \quad \text{for all } n$$

Then $0 \leq |a_n b_n| = |a_n||b_n| \leq |a_n| \cdot K \to 0 \cdot K = 0$ as $n \to \infty$

By Squeeze Theorem, $\lim_{n\to\infty} |a_n b_n| = 0$. This clearly implies $\lim_{n\to\infty} a_n b_n = 0$.

Exa B. Prove: $a_n \to L,\ L \neq 0 \quad \Rightarrow \quad \dfrac{1}{a_n} \to \dfrac{1}{L}$

Sol. (a reproof of the result) Hypo says: given $\varepsilon > 0$, $|a_n - L| < \varepsilon$ for $n \gg 1$

Assume first that $L > 0$. Then $\lim_{n\to\infty} a_n > \dfrac{L}{2} (> 0)$.

Thus by SLT (or by taking $\varepsilon = \dfrac{L}{2}$), we have $a_n > \dfrac{L}{2}$ for $n \gg 1$

Then $\left|\dfrac{1}{a_n} - \dfrac{1}{L}\right| = \dfrac{|a_n - L|}{a_n \cdot L} < \dfrac{\varepsilon}{\frac{L}{2} \cdot L} = \dfrac{2\varepsilon}{L^2}$ for $n \gg 1$

$$\therefore \quad \lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L} \quad \text{by } K\text{-}\varepsilon \text{ Principle.}$$

If $L < 0$, then

$$a_n \to L \underset{\text{know}}{\Rightarrow} -a_n \to -L \overset{-L>0}{\underset{\text{prev case}}{\Rightarrow}} \frac{1}{-a_n} \to -\frac{1}{L} \underset{\text{know}}{\Rightarrow} \frac{1}{a_n} \to \frac{1}{L}$$