Homework Chapter 2 (STA 5015)

Remark. This assignment will not be collected nor graded, but will be considered in the midterm.

Reading: Chapter 2

1. Problem 2.3

2. Let $\{X_t\}$ be a stationary process zero mean. Define

$$Y_t = \begin{cases} X_t & \text{if } t \text{ is odd,} \\ X_t + 3 & \text{if } t \text{ is even.} \end{cases}$$

   (a) Find ACVF of $\{Y_t\}$.

   (b) Is it a stationary process?

3. Show that $X_t = \sin(2\pi U t)$, $t = 1, 2, \ldots$, where U has a uniform distribution on the interval (0, 1) is a weakly stationary process. Hint, use formula

$$\sin\alpha \sin\beta = \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{2}$$

4. Let $\mathbf{X} = (X_1, X_2)'$ follow multivariate normal distribution with parameters

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

   (a) Find the distribution of

$$X_2 - \rho\frac{\sigma_2}{\sigma_1}X_1.$$

   (b) Show that $X_2 - \rho\sigma_2/\sigma_1 X_1$ and X_1 are independent. Recall that for MVN, zero-covariance implies independence.

5. For ARMA(1,1) process given by

$$X_t - .5X_{t-1} = Z_t + .5Z_{t-1}, \quad Z_t \sim WN(0, \sigma^2)$$

   calculate ψ_j and π_j, $j = 1, \ldots, 5$.

6. Recall the key results on the convergence of series. Review your calculus book.

   - The series $\sum a_n$ is called **convergent** if its partial sum $s_n = \sum_{i=1}^{n} a_i = a_1 + \ldots + a_n$ is convergent and $\lim_{n\to\infty} s_n = s$. Otherwise, we say that the series **divergent**.
   - A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent

- If a series $\sum a_n$ is **absolutely convergent**, then it is convergent. (That is, absolute convergence always guarantee convergence of series!)
- A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.
- (The alternating series test) If the alternating series

  $$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \ldots (b_n > 0)$$

  satisfies i) $b_{n+1} \leq b_n$ for all n, ii) $\lim_{n\to\infty} b_n = 0$, then the series is convergent

(a) Show that $\displaystyle\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$ converges absolutely.

(b) Show that $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges, but not absolutely.

7. Problem 2.12

8. Problem 2.13

9. For the stationary process

   $$X_t = A\cos(\omega t) + B\sin(\omega t), \quad t = 0, \pm 1, \pm 2, \ldots,$$

   where A and B are uncorrelated random variables with mean 0 and variance 1 and ω is a fixed frequency in $[0, \pi)$.

   (a) Show that

   $$X_n = (2\cos\omega)X_{n-1} - X_{n-2}.$$

   (b) Find $\widetilde{P}_n X_{n+1}$ (note that it is based on the infinite past).

10. Consider the MA(2) time series $X_t = Z_t + 2Z_{t-1} - 2Z_{t-2}$ with $\{Z_t\} \sim \mathrm{WN}(0, 1)$. Find $P_1 X_2$ and its mean squared error. First note that it is based on the finite past. To find $P_1 X_2$, rather than using matrix formula introduced in class, applying "partial derivatives = 0" equations directly gives simpler calculation. For MSPE calculation, plug-in formula.