Chap. 18 Integrability (정적분의 구체적인 값에는 관심이 없다)

A brief History

① Newton-Leibniz:

Suppose f is a real-valued function on an interval I. If there is a function F on I such that $F'(x) = f(x)$ on I, we say f has a primitive (function) F on I.

We say that a ft $f : I \to \mathbb{R}$ is Newton-integrable on I if it has a primitive ft on I.

All continuous functions on a compact interval are Newton-integrable --- will be proved later

It is traditional to write $F(x) = \int f(x)\, dx$ (and it is called an indefinite integral of f, or an anti-derivative, or a primitive) if $F'(x) = f(x)$. We can also define the **definite** Newton-integral of f on the compact interval $[a, b]$ as follows:

$$\boxed{\int_a^b f(x)dx \;\; \left(= (N)\int_a^b f(x)\, dx \;\; \right) = F(b) - F(a) \;\; \text{if } F'(x) = f(x) : \text{continuous on } [a, b]}$$

문제점: (i) Integration heavily depends on the differentiation

(ii) In some cases, we do not know whether such an anti-derivative $F(x)$ exists.

Cf: $\nexists\, F(x)$ s.t. $F'(x) = f(x)$ if $\underbrace{f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}}_{\text{has single discontinuity point at } 0}$ on $[-1, 1]$ $(by$ Darboux thm$)$

Moreover, $\exists f$ such that an explicit form of its anti-derivative $F(x)$ is **not** found, e.g., $f(x) = e^{x^2}$

Remark. Scientists and engineers regard

$$\int_a^b f(x)\, dx = \text{the (signed) area over } [a, b] \text{ and under the graph of } f(x)$$

문제점: "area" itself has not been defined

In fact, it is not easy to define the "area" of an arbitrary (planar) region.

② Cauchy: 미분개념을 사용하지 않고 극한을 사용하여 (최초로) 연속함수의 정적분을 정의함 (유한개의 불연속 점을 갖는 함수에 대해서도 정적분을 정의함)

③ Riemann: 임의의 유계인(bounded) 함수에 대하여 (정)적분가능의 개념을 정의하고 또한, 적분가능한 함수에 대하여 (정)적분의 개념을 도입함

결점: (리만)적분 불가능한 유계인 함수의 구체적인 예가 존재한다

예 (shortly later): $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ 0, & x \in \mathbb{Q}^c \cap [0,1] \end{cases}$ is not Riemann-integrable on $[0,1]$

④ Lebesgue(1902): 유계가 아닌 함수까지 정적분 개념을 확장함

Riemann : 정의구역을 분할 Lebesgue: 치역을 분할

예 (later): $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ 0, & x \in \mathbb{Q}^c \cap [0,1] \end{cases}$ is Lebesgue-integrable on $[0,1]$ & $\underbrace{(L)\int_0^1 f}_{\text{Lebesgue integral}} = 0$

18.1 Introduction. Partitions (분할)

Def A. A partition $\mathcal{P}$ of a compact interval $[a, b]$ is a strictly increasing **finite** sequence of numbers starting with a and ending with b : $\mathcal{P}: a = x_0 < x_1 < x_2 < \cdots < x_n = b$

Notation

A partition divides $[a, b]$ into smaller intervals $[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n]$

We use the notation: $[\Delta x_i] = [x_{i-1}, x_i], \quad \Delta x_i = x_i - x_{i-1} \quad (i = 1, 2, \cdots, n)$

Def B. The mesh $|\mathcal{P}|$ of a partition $\mathcal{P}$ is defined by $|\mathcal{P}| = \max\limits_{1 \le i \le n} \Delta x_i$

Thus $\mathcal{P}$ will be fine if its mesh $|\mathcal{P}|$ is small

Def C. • An **n**-partition is a partition containing n subintervals

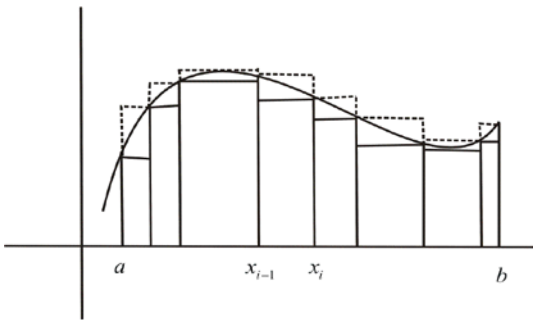
• The standard $n\text{-partition } \mathcal{P}^{(n)}$ is the one in which all subintervals have the same length.

If $\mathcal{P}^{(n)}$ is the standard $n\text{-partition}$ of $[a, b]$, then

$$\Delta x := \Delta x_i = \frac{b-a}{n} \quad (i = 1, 2, \cdots, n), \quad |\mathcal{P}^{(n)}| = \frac{b-a}{n}$$

18.2 Integrability (적분가능성)

Intuitive idea.



Given a partition $\mathcal{P}$ of $[a, b]$, we draw in the associated inscribed and circumscribed rectangles, and consider their total areas

The function $f(x)$ will be called (**Riemann-**) **integrable** if these two areas get arbitrarily close as the partition gets finer and finer.

The two areas then have a common limit, whose value will be called the **Riemann integral**,

$$\int_a^b f(x)\, dx \quad (\overset{\text{or}}{=} \ (R)\int_a^b f(x)\, dx \ )$$

Want to say these things analytically, **without referring to areas**

Def A. Let $f(x)$ be bounded on $[a,b]$, and $\mathcal{P}$ be a partition of $[a,b]$. Write

$$m_i = \inf_{[\Delta x_i]} f(x), \quad M_i = \sup_{[\Delta x_i]} f(x), \quad \Delta x_i = x_i - x_{i-1}$$

We define

$$L(\mathcal{P}) = L_f(\mathcal{P}) = \sum_{i=1}^{n} m_i \Delta x_i \quad \text{(the lower sum for } f(x) \text{ over } \mathcal{P})$$

$$U(\mathcal{P}) = U_f(\mathcal{P}) = \sum_{i=1}^{n} M_i \Delta x_i \quad \text{(the upper sum for } f(x) \text{ over } \mathcal{P})$$

Geometrically, if $f(x) > 0$, the upper (**lower**) sum represent the total area of the circumscribed (**inscribed**) rectangles, for the partition $\mathcal{P}$ and the function $f(x)$

Def B. A function f is called **integrable** (or Riemann-integrable) on $[a,b]$ (or $f \in \mathcal{R}[a,b]$ for short) if it is defined and **bounded** on $[a,b]$, and it satisfies

$$\boxed{\forall \varepsilon > 0, \quad \exists \text{ a partition } \mathcal{P} = \mathcal{P}(\varepsilon) \text{ of } [a,b] \text{ such that } U_f(\mathcal{P}) - L_f(\mathcal{P}) < \varepsilon}$$

This is known to be equivalent to the following formally stronger statement

$$\boxed{\begin{array}{c} \text{Given } \varepsilon > 0, \quad U_f(\mathcal{P}) \underset{\varepsilon}{\approx} L_f(\mathcal{P}) \quad \text{for all } \mathcal{P} \text{ such that } |\mathcal{P}| \approx 0 \\ \text{That is, } \forall \varepsilon > 0, \quad \exists \delta = \delta(\varepsilon) > 0 \text{ such that } U_f(\mathcal{P}) - L_f(\mathcal{P}) < \varepsilon \text{ for } \forall \mathcal{P} \text{ with } |\mathcal{P}| < \delta \end{array}}$$

In short (informally), $\boxed{\lim_{|\mathcal{P}| \to 0} (U_f(\mathcal{P}) - L_f(\mathcal{P})) = 0}$ or $\lim_{\delta \to 0} \{U_f(\mathcal{P}) - L_f(\mathcal{P}) : |\mathcal{P}| \le \delta\} = 0$

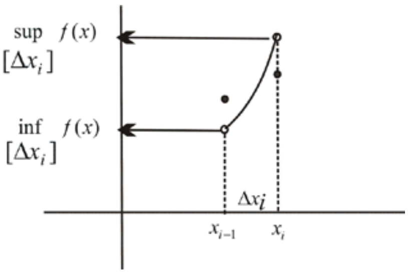
Note.

$$m_i = \inf_{[\Delta x_i]} f(x) \neq \min_{[\Delta x_i]} f(x) \quad \text{(in general)}$$

$$M_i = \sup_{[\Delta x_i]} f(x) \neq \max_{[\Delta x_i]} f(x) \quad \text{(in general)}$$

Indeed, bounded functions may not have a max or min on a compact interval;

$\inf f = \min f$ (& $\sup f = \max f$) is guaranteed only for continuous functions (최대-최소 정리)



$$\nexists \max_{[\Delta x_i]} f(x) \quad \& \quad \nexists \min_{[\Delta x_i]} f(x)$$

Ex. Show that x is integrable on any interval $[a, b]$

Pf. Given $\varepsilon > 0$, take any partition $\mathcal{P}$ of $[a, b]$ with $|\mathcal{P}| < \varepsilon$. Then

$$U(\mathcal{P}) - L(\mathcal{P}) = \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} x_i \Delta x_i - \sum_{i=1}^{n} x_{i-1} \Delta x_i$$

$$= \sum_{i=1}^{n} (x_i - x_{i-1}) \Delta x_i \le \varepsilon \sum_{i=1}^{n} \Delta x_i = \varepsilon(b-a)$$

Ex. $\quad f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number} \\ 0 & \text{otherwise} \end{cases}$

Prove that $f(x)$ is **not** integrable on $[0, 1]$

Pf. Let $\mathcal{P}$ be any partition of $[0, 1]$.

Then every subinterval of $\mathcal{P}$ contains a rational number and an irrational number.

$$\therefore \quad \sup_{[\Delta x_i]} f(x) = 1, \qquad \inf_{[\Delta x_i]} f(x) = 0$$

$$\therefore \; U_f(\mathcal{P}) = 1 \quad \text{and} \quad L_f(\mathcal{P}) = 0 \;\; \text{for any } \mathcal{P}$$

Therefore, $f(x)$ is not integrable on $[0, 1]$

18.3 Integrability of monotone and continuous functions

Question: What sort of functions are integrable?

Goal of this section is to prove that two kinds of functions (monotone functions & continuous functions) are always integrable.

Remark. Lebesgue succeeded in characterizing the integrable functions in terms of their discontinuities (the precise result will be given in Chap. 23)

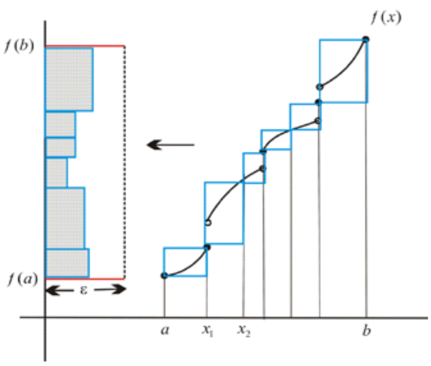
Recall (by K-ε Principle): f is integrable on $[a, b]$ if

$$\text{given } \varepsilon > 0, \; \exists \text{a partition } \mathcal{P} = \mathcal{P}(\varepsilon) \text{ of } [a, b] \;\; \text{s.t. } U_f(\mathcal{P}) \underset{K\varepsilon}{\approx} L_f(\mathcal{P})$$

Here K is a fixed constant depending only on $f(x)$ and not on the partition $\mathcal{P}$

Theorem A (Integrability of monotone functions)
If $f(x)$ is monotone on $[a, b]$, then $f \in \mathcal{R}[a, b]$

Geometric pf.
Suppose $f(x)$ is $\uparrow$ on $[a, b]$.



Given $\varepsilon > 0$, let $\mathcal{P}$ be any partition whose mesh $|\mathcal{P}| < \varepsilon$. Then

$$U_f(\mathcal{P}) - L_f(\mathcal{P}) = \text{shaded area} < (f(b) - f(a)) \cdot \varepsilon \equiv K\varepsilon$$

Analytic pf. Suppose $f(x)$ is $\uparrow$ on $[a, b]$.

Given $\varepsilon > 0$, let $\mathcal{P}$ be any partition whose mesh $|\mathcal{P}| < \varepsilon$.

Since $f(x)$ is $\uparrow$ on the sub-interval $[\Delta x_i] = [x_{i-1}, x_i]$, we have

$$M_i = \sup_{[\Delta x_i]} f(x) = f(x_i), \qquad m_i = \inf_{[\Delta x_i]} f(x) = f(x_{i-1})$$

$$U(\mathcal{P}) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} f(x_i)\Delta x_i, \qquad L(\mathcal{P}) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} f(x_{i-1})\Delta x_i$$

$$\begin{aligned}
\therefore \quad U(\mathcal{P}) - L(\mathcal{P}) &= \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big)\Delta x_i \le \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big) \,|\, \mathcal{P} \,| \\
&\big[\because \ f(x_i) - f(x_{i-1}) > 0 \ \text{ all } i, \ \text{ since we are assuming } f \text{ is } \uparrow\big] \\
&\le \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big)\varepsilon \quad (\leftarrow \ \big|\mathcal{P}\big| < \varepsilon) \\
&= \varepsilon(f(b) - f(a)) =: K\varepsilon
\end{aligned}$$

$\therefore$ $f(x)$ is integrable on $[a, b]$.

Remark. Actually, we proved

$$\boxed{f \text{ is } \uparrow \text{ on } [a, b] \quad \Rightarrow \quad U(\mathcal{P}) - L(\mathcal{P}) \le |\mathcal{P}|\big(f(b) - f(a)\big) \ \ \forall\, \mathcal{P}}$$

Theorem B (Integrability of continuous functions)

$$f(x) \text{ is conti on } [a, b] \quad \Rightarrow \quad f \in \mathcal{R}[a, b]$$

Pf. Let $\mathcal{P}$ be a partition of $[a, b]$. Then

$$\begin{aligned}
U_f(\mathcal{P}) - L_f(\mathcal{P}) &= \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \\
&= \sum_{i=1}^{n} (\sup_{[\Delta x_i]} f(x) - \inf_{[\Delta x_i]} f(x))\Delta x_i \\
&\overset{\text{Max-min theorem for conti fts}}{=} \sum_{i=1}^{n} (f(x_i') - f(x_i''))\Delta x_i, \quad \text{for some } x_i' \ \& \ x_i'' \in [\Delta x_i] \ \ (i = 1, 2, \cdots, n)
\end{aligned}$$

Recall $f \in C[a, b] \quad \Rightarrow \quad f \in UC[a, b]$

Thus, given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $\big|f(x') - f(x'')\big| < \varepsilon$ if $\big|x' - x''\big| < \delta$

Now we take $\mathcal{P}$ so that $\big|\mathcal{P}\big| < \delta$ $[\Rightarrow \mathcal{P} = \mathcal{P}(\varepsilon)]$. Then

$$\big|x_i' - x_i''\big| \le \Delta x_i < \delta \quad \text{for all } i \quad (\text{since } \ |\mathcal{P}| = \max_i \Delta x_i < \delta)$$

$$f(x_i') - f(x_i'') = \big|f(x_i') - f(x_i'')\big| < \varepsilon \quad \text{for all } i$$

$$\therefore \quad U(\mathcal{P}) - L(\mathcal{P}) \le \sum_{i=1}^{n} \varepsilon \Delta x_i = \varepsilon \sum_{i=1}^{n} \Delta x_i = \varepsilon(b - a)$$

$\therefore$ $f(x)$ is integrable on $[a, b]$.

18.4 Basic properties of integrable functions

Review: Let $A,\ B \subset \mathbb{R}$ and $c \in \mathbb{R}$.

① $c > 0 \quad \Rightarrow \quad \sup cA = c \sup A, \quad \inf cA = c \inf A, \quad \text{where } cA = \{ca \mid a \in A\}$

② $\sup(-A) = -\inf A, \qquad \inf(-A) = -\sup A$

③ $\sup(A+B) \le \sup A + \sup B, \qquad \inf(A+B) \ge \inf A + \inf B$

Pf. ③: $\quad A + B = \{a + b : a \in A,\ b \in B\} \qquad a \in A \ \ \&\ \ b \in B \ \Rightarrow$

$a \le \sup A \ \ \&\ \ b \le \sup B \ \Rightarrow\ a + b \le \sup A + \sup B \ \Rightarrow\ \sup(A+B) \le \sup A + \sup B$

Theorem A (Linearity property of integrability)

Let $c_1, c_2 \in \mathbb{R}$. Then

$f(x)$ & $g(x)$ are integrable on $[a,b]$ $\Rightarrow$ $c_1 f(x) + c_2 g(x)$ is integrable on $[a,b]$

Pf. It suffices to prove:

(i) $f(x)$ integrable $\Rightarrow$ $-f(x)$ integrable

(ii) $f(x)$ integrable $\Rightarrow$ $cf(x)$ $(c : \text{real})$ integrable

(iii) $f(x)$ & $g(x)$ integrable $\Rightarrow$ $f(x) + g(x)$ integrable

(i): Hypo implies:

given $\varepsilon > 0,\ \exists \mathcal{P} = \mathcal{P}(\varepsilon)$ s.t. $U_f(\mathcal{P}) \underset{\varepsilon}{\approx} L_f(\mathcal{P})$

$$\sup_{[\Delta x_i]}(-f(x)) \overset{②}{=} -\inf_{[\Delta x_i]} f(x) \quad \Rightarrow \quad U_{-f}(\mathcal{P}) = -L_f(\mathcal{P})$$

$$\inf_{[\Delta x_i]}(-f(x)) \overset{②}{=} -\sup_{[\Delta x_i]} f(x) \quad \Rightarrow \quad L_{-f}(\mathcal{P}) = -U_f(\mathcal{P})$$

$$\therefore \quad U_{-f}(\mathcal{P}) - L_{-f}(\mathcal{P}) = -L_f(\mathcal{P}) + U_f(\mathcal{P}) = U_f(\mathcal{P}) - L_f(\mathcal{P})$$

$$\therefore \quad U_{-f}(\mathcal{P}) \underset{\varepsilon}{\approx} L_{-f}(\mathcal{P}) \quad \text{since } U_f(\mathcal{P}) \underset{\varepsilon}{\approx} L_f(\mathcal{P}) \ \text{ for our } \mathcal{P}$$

(ii) If $c > 0$,

$$\sup_{[\Delta x_i]}(cf(x)) \overset{①}{=} c \sup_{[\Delta x_i]} f(x) \quad \Rightarrow \quad U_{cf}(\mathcal{P}) = cU_f(\mathcal{P})$$

$$\inf_{[\Delta x_i]}(cf(x)) \overset{①}{=} c \inf_{[\Delta x_i]} f(x) \quad \Rightarrow \quad L_{cf}(\mathcal{P}) = cL_f(\mathcal{P})$$

$$\therefore \quad U_{cf}(\mathcal{P}) - L_{cf}(\mathcal{P}) = c(U_f(\mathcal{P}) - L_f(\mathcal{P}))$$

$$\therefore \quad U_{cf}(\mathcal{P}) \underset{c\varepsilon}{\approx} L_{cf}(\mathcal{P}) \quad \text{since } U_f(\mathcal{P}) \underset{\varepsilon}{\approx} L_f(\mathcal{P}) \ \text{ for our } \mathcal{P}$$

$\therefore \quad cf\ (c > 0)$ is integrable

If $c < 0$, then $(-c)f$ is integrable

$\therefore \quad cf$ is integrable, by (i)

(iii) On any interval I, we have by ③

$$\sup_I (f+g) \le \sup_I f + \sup_I g \quad \& \quad \inf_I (f+g) \ge \inf_I f + \inf_I g$$

$$\therefore\ U_{f+g}(\mathcal{P}) \le U_f(\mathcal{P}) + U_g(\mathcal{P}) \quad \& \quad L_{f+g}(\mathcal{P}) \ge L_f(\mathcal{P}) + L_g(\mathcal{P})$$

$$\therefore \quad U_{f+g}(\mathcal{P}) - L_{f+g}(\mathcal{P}) \le \big(U_f(\mathcal{P}) - L_f(\mathcal{P})\big) + \big(U_g(\mathcal{P}) - L_g(\mathcal{P})\big)$$

$$\therefore\ U_{f+g}(\mathcal{P}) \underset{2\varepsilon}{\approx} L_{f+g}(\mathcal{P}) \ \text{ since } \ U_f(\mathcal{P}) \underset{\varepsilon}{\approx} L_f(\mathcal{P}) \ \& \ U_g(\mathcal{P}) \underset{\varepsilon}{\approx} L_g(\mathcal{P}) \ \text{ for our } \mathcal{P}$$

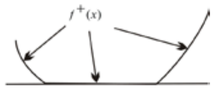
$\therefore \quad f + g$ is integrable

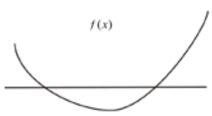
Theorem B (Absolute value property of integrability)

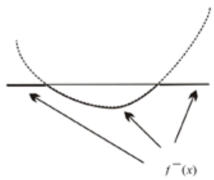
$$f(x) \text{ is integrable on } [a, b] \ \Rightarrow \ |\, f(x)\,| \text{ is integrable on } [a, b]$$

$$\not\Leftarrow$$

Pf.







Define

$$f^+(x) = \begin{cases} f(x) & \text{for } \{x : f(x) \ge 0\} \\ 0 & \text{otherwise} \end{cases} \qquad \text{(positive part of } f(x)\text{)}$$

$$f^-(x) = \begin{cases} f(x) & \text{for } \{x : f(x) \le 0\} \\ 0 & \text{otherwise} \end{cases} \qquad \text{(negative part of } f(x)\text{)}$$

Note that $|\, f(x)\,| = f^+(x) - f^-(x) \quad ---(*)$

We shall show:

$$f(x) \text{ is integrable on } [a, b] \quad \Rightarrow \quad f^+(x) \text{ is integrable on } [a, b]$$

If this is proved, then since $f^-(x) = -(-f(x))^+$, $f^-(x)$ is also integrable on $[a, b]$

Then, by $(*)$, $|\, f(x)\,|$ is integrable on $[a, b]$

For any bounded function $f(x)$ on an interval I,

$$f(x) \geq 0 \text{ on } I \underset{f^+(x)=f(x) \text{ on } I}{\Rightarrow} \sup_I f^+(x) = \sup_I f(x), \quad \inf_I f^+(x) = \inf_I f(x)$$

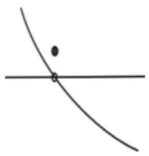
$$\therefore \sup_I f^+(x) - \inf_I f^+(x) = \sup_I f(x) - \inf_I f(x)$$

$$f(x) \leq 0 \text{ on } I \Rightarrow \sup_I f^+(x) = 0, \quad \inf_I f^+(x) = 0$$

$$\therefore \sup_I f^+(x) - \inf_I f^+(x) \ (= 0) \leq \underbrace{\sup_I f(x) - \inf_I f(x)}_{\geq 0}$$

$f(x)$ has both positive & negative values on I $\Rightarrow$

$$\sup_I f^+(x) = \sup_I f(x), \quad \inf_I f^+(x) \ (= 0) > \inf_I f(x) \ (= \text{negative})$$



$$\therefore \sup_I f^+(x) - \inf_I f^+(x) \leq \sup_I f(x) - \inf_I f(x)$$

Consequently, in any case

$$\sup_I f^+(x) - \inf_I f^+(x) \leq \sup_I f(x) - \inf_I f(x)$$

$$\therefore U_{f^+}(\mathcal{P}) - L_{f^+}(\mathcal{P}) \leq U_f(\mathcal{P}) - L_f(\mathcal{P}) \ \forall \text{partition } \mathcal{P}$$

$\therefore$ $f^+(x)$ is integrable if $f(x)$ is integrable on $[a, b]$

A direct way (without using the decomposition $|f| = f^+ - f^-$) of showing

$$f(x) \text{ is integrable on } [a, b] \Rightarrow |f(x)| \text{ is integrable on } [a, b]$$

Notice that

$$f(x) \geq 0 \text{ on } I \Rightarrow |f| = f \text{ on } I$$

$$\Rightarrow \sup_I |f| = \sup_I f, \quad \inf_I |f| = \inf_I f$$

$$\therefore \sup_I |f| - \inf_I |f| = \sup_I f - \inf_I f$$

$$f(x) \leq 0 \text{ on } I \Rightarrow |f| = -f \text{ on } I$$

$$\Rightarrow \sup_I |f| = \sup_I(-f) = -\inf_I f, \quad \inf_I |f| = -\sup_I f$$

$$\therefore \sup_I |f| - \inf_I |f| = -\inf_I f + \sup_I f = \sup_I f - \inf_I f$$

$f(x)$ has both positive & negative values on I $\Rightarrow$

$$\sup_I |f| = \text{the larger part of } \sup_I f \text{ and } -\inf_I f$$

$$\& \quad \inf_I |f| \geq 0 \quad \textit{and} \quad \inf_I f < 0$$

$$\therefore \quad \sup_I |f| - \underbrace{\inf_I |f|}_{\geq 0} \leq \sup_I |f| < \sup_I f - \inf_I f$$

In any case, we thus have

$$\sup_I |f| - \inf_I |f| \;\leq\; \sup_I f - \inf_I f \quad \text{---}(\blacktriangle)$$

$\therefore$ $|f(x)|$ is integrable if $f(x)$ is integrable on $[a, b]$

Example: $|f(x)|$ is integrable on $[a, b]$ $\not\Rightarrow$ $f(x)$ is integrable on $[a, b]$

$$f(x) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{otherwise} \end{cases}$$

Then f is not integrable on $[0, 1]$ $\ulcorner \because$ $U_f(\mathcal{P}) = 1, \quad L_f(\mathcal{P}) = -1$ for any $\mathcal{P}$ $\lrcorner$

However, $|f(x)| = 1 \quad \forall x \in \mathbb{R}$ $\qquad \therefore$ $|f|$ is integrable on $[0, 1]$.

Ex

1. $f(x)$ is integrable on $[a, b]$ $\Rightarrow$ $(f(x))^2$ is integrable on $[a, b]$

Pf. $\qquad \forall x \in I, \quad |f(x)| \leq \sup_I |f(x)| \quad \therefore \quad |f(x)|^2 \leq \left(\sup_I |f(x)|\right)^2$

$$\Rightarrow \sup_I \left(f(x)^2\right) = \sup_I \left(|f(x)|^2\right) \leq \left(\sup_I |f(x)|\right)^2$$

Similarly,

$$\forall x \in I, \quad |f(x)| \geq \inf_I |f(x)| \quad \therefore \quad |f(x)|^2 \geq \left(\inf_I |f(x)|\right)^2$$

$$\Rightarrow \inf_I \left(f(x)^2\right) = \inf_I \left(|f(x)|^2\right) \geq \left(\inf_I |f(x)|\right)^2$$

$$\begin{aligned} \therefore \quad \sup_I \left(f(x)^2\right) - \inf_I \left(f(x)^2\right) &\leq \left(\sup_I |f(x)|\right)^2 - \left(\inf_I |f(x)|\right)^2 \\ &\leq \left(\sup_I |f| + \inf_I |f|\right)\left(\sup_I |f| - \inf_I |f|\right) \\ &\leq 2M\left(\sup_I |f| - \inf_I |f|\right) \\ &\leq 2M\left(\sup_I f - \inf_I f\right) \;\; (\text{by } (\blacktriangle)), \quad \text{where } M = \sup_{x \in [a,b]} |f(x)| \end{aligned}$$

$$\therefore \quad U_{f^2}(\mathcal{P}) - L_{f^2}(\mathcal{P}) \leq 2M\left(U_f(\mathcal{P}) - L_f(\mathcal{P})\right)$$

2. $(\Leftarrow 1)$ $f(x)$ & $g(x)$ are integrable on $[a, b]$ $\Rightarrow$ $f(x)g(x)$ is integrable on $[a, b]$

Pf. $\qquad fg = \dfrac{1}{2}\left[(f+g)^2 - f^2 - g^2\right]$

⊙ **Alternative unifying approach for testing integrability**

Def. Let f be a bounded function on $[a, b]$, and let J be any subinterval of $[a, b]$

$$\sup_{x\in J} f(x) - \inf_{x\in J} f(x) \overset{\text{denote}}{=} \operatorname{Osc}(f, J)$$

$\operatorname{Osc}(f, J)$ is called the oscillation of f over J

Fact (easy but useful): Let f and J be as above. Then

$$\operatorname{Osc}(f, J) = \sup_{x,\, y\in J} |\, f(x) - f(y)\, |$$

i.e., $\displaystyle \sup_{x\in J} f(x) - \inf_{x\in J} f(x) = \sup_{x,\, y\in J} |\, f(x) - f(y)\, |$

Pf. First we prove: $\displaystyle \sup_{x,y\in J}\left|f(x)-f(y)\right| \le \sup_{x\in J} f(x) - \inf_{y\in J} f(y)$

Notice that

$$f(x) \le \sup_{x\in J} f(x), \quad \text{for all } x\in J \qquad \text{and} \qquad f(y) \ge \inf_{y\in J} f(y), \quad \text{for all } y\in J$$

i.e., $\displaystyle f(x) \le \sup_{x\in J} f(x), \quad \text{for all } x\in J$ and $\displaystyle -f(y) \le -\inf_{y\in J} f(y), \quad \text{for all } y\in J$

Adding these inequalities gives

$$f(x) - f(y) \le \sup_{x\in J} f(x) - \inf_{y\in J} f(y), \quad \text{for all } x, y\in J$$

$$\therefore \quad \left|f(x)-f(y)\right| \le \sup_{x\in J} f(x) - \inf_{y\in J} f(y), \quad \text{for all } x, y\in J \quad \text{(by symmetry of RHS)}$$

Taking $\displaystyle \sup_{x,y\in J} \Rightarrow \sup_{x,y\in J}\left|f(x)-f(y)\right| \le \sup_{x\in J} f(x) - \inf_{y\in J} f(y)$

Next we prove: $\displaystyle \sup_{x\in J} f(x) - \inf_{y\in J} f(y) \le \sup_{x,y\in J}\left|f(x)-f(y)\right|$.

Obviously, $\displaystyle f(x) - f(y) \le \sup_{x,y\in J} |\, f(x) - f(y)\, |, \quad \text{for all } x, y\in J$

$$\text{i.e.,} \quad f(x) \le \sup_{x,y\in J} |\, f(x) - f(y)\, | + f(y), \quad \text{for all } x, y\in J$$

Fix any $x\in J$ and take $\displaystyle \inf_{y\in J} \Rightarrow$

$$f(x) \le \sup_{x,y\in J} |\, f(x) - f(y)\, | + \inf_{y\in J} f(y), \quad \text{for any } x\in J$$

Take $\displaystyle \sup_{x\in J} \Rightarrow$

$$\sup_{x\in J} f(x) \le \sup_{x,y\in J} |\, f(x) - f(y)\, | + \inf_{y\in J} f(y)$$

$$\therefore \quad \sup_{x\in J} f(x) - \inf_{y\in J} f(y) \le \sup_{x,y\in J} |\, f(x) - f(y)\, |$$

Theorem Let f & g be bounded fts on $[a, b]$, and let J be any subinterval of $[a, b]$. Then

1. $\mathrm{Osc}(cf, J) = |\, c \,|\, \mathrm{Osc}(f, J)$ for any real number c
2. $\mathrm{Osc}(f+g, J) \le \mathrm{Osc}(f, J) + \mathrm{Osc}(g, J)$
3. $\mathrm{Osc}(|\, f\,|, J) \le \mathrm{Osc}(f, J)$
4. $\mathrm{Osc}(f^2, J) \le 2M \cdot \mathrm{Osc}(f, J), \quad \text{where } M = \sup\limits_{x\in[a,\,b]} |\, f(x) \,|$
5. $fg = \dfrac{1}{4}\{(f+g)^2 - (f-g)^2\} \overset{\text{or}}{=} \dfrac{1}{2}\{(f+g)^2 - f^2 - g^2\}$: **obvious**
6. $\mathrm{Osc}(\dfrac{1}{f}, J) \le \dfrac{1}{m^2}\mathrm{Osc}(f, J)$ if $|\, f(x) \,| \ge m > 0 \quad \forall x \in [a, b]$

Pf. 1. $$\mathrm{Osc}(cf, J) = \sup_{x\in J} cf(x) - \inf_{x\in J} cf(x) \overset{\text{Fact}}{=} \sup_{x,y\in J} |\, cf(x) - cf(y) \,|$$
$$\underset{|cf(x)-cf(y)| = |c|\cdot|f(x)-f(y)|}{=} |\, c \,| \cdot \sup_{x,y\in J} |\, f(x) - f(y) \,|$$
$$= |\, c \,|\, \mathrm{Osc}(f, J)$$

2. $$\mathrm{Osc}(f+g, J) = \sup_{x,y\in J} |\, (f+g)(x) - (f+g)(y) \,|$$
$$\underset{|(f+g)(x)-(f+g)(y)| \le |f(x)-f(y)|+|g(x)-g(y)|}{\le} \sup_{x,y\in J} |\, f(x) - f(y) \,| + \sup_{x,y\in J} |\, g(x) - g(y) \,|$$
$$= \mathrm{Osc}(f, J) + \mathrm{Osc}(g, J)$$

3. $$\mathrm{Osc}(|\, f\,|, J) = \sup_{x,y\in J} \big\|\, f(x) \,| - |\, f(y) \,\big\| \underset{\|f(x)|-|f(y)\| \le |f(x)-f(y)|}{\le} \sup_{x,y\in J} \big|f(x) - f(y)\big| = \mathrm{Osc}(f, J)$$

4. $$\mathrm{Osc}(f^2, J) = \sup_{x,y\in J} \big|f(x)^2 - f(y)^2\big|$$
$$= \sup_{x,y\in J} \big|f(x) + f(y)\big| \cdot \big|f(x) - f(y)\big|$$
$$\le 2M \cdot \sup_{x,y\in J} \big|f(x) - f(y)\big| = 2M \cdot \mathrm{Osc}(f, J), \text{ where } M = \sup_{x\in[a,\,b]} |\, f(x) \,|$$

6. $$\left|\frac{1}{f(x)} - \frac{1}{f(y)}\right| = \frac{\big|f(x) - f(y)\big|}{\big|f(x)\big|\big|f(y)\big|} \le \frac{1}{m^2}\big|f(x) - f(y)\big| \quad \text{for any } x, y \in J,$$
$$\text{if } |\, f(x) \,| \ge m > 0 \quad \forall x \in [a, b]$$
$$\therefore \quad \mathrm{Osc}(\frac{1}{f}, J) \le \frac{1}{m^2}\mathrm{Osc}(f, J)$$

Cor. Let $f,\, g \in \mathcal{R}[a, b]$ (note that f and g are bounded on $[a, b]$). Then

$$cf (c : \text{real}),\ \ f+g,\ \ |\, f\,|,\ \ f^2,\ \ fg,\ \ \frac{1}{f}(\text{if } |\, f \,| \ge m > 0 \text{ on } [a, b]) \quad \in \mathcal{R}[a, b]$$

Pf. We only prove that $cf \in \mathcal{R}[a, b]$. Let $\mathcal{P}$ be a partition of $[a, b]$. Then

$$U_{cf}(\mathcal{P}) - L_{cf}(\mathcal{P}) = \sum_{i=1}^{n} (\sup_{[\Delta x_i]} cf(x) - \inf_{[\Delta x_i]} cf(x))\Delta x_i = \sum_{i=1}^{n} \mathrm{Osc}(cf, [\Delta x_i])\Delta x_i$$
$$\overset{\text{prev. Theorem - 1}}{=} |\, c \,| \sum_{i=1}^{n} \mathrm{Osc}(f, [\Delta x_i])\Delta x_i = |\, c \,| \Big(U_f(\mathcal{P}) - L_f(\mathcal{P})\Big)$$