STA 3021: Stochastic Processes
Midterm 1 (6:15 PM - 7:30 PM on Sep 27, 2021)

Instructions:

- This test is a closed book exam, but you are allowed to use calculator. Clarity of your answer will also be a part of credit. When needed, use the notation $\Phi(z) = P(Z < z)$ for a standard normal distribution Z. Show your ALL work neatly.
- Your answer sheets must be written in English.
- Remind that you can submit your answer sheets over icampus in a **pdf** file format ONLY.
- By submitting your report online, it is assumed that you agree with the following pledge;

  Pledge: *I have neither given nor received any unauthorized aid during this exam.*
- Don't forget to write down your name and student ID on your answer sheet.

1. (10 points) State the following theorems/definitions as precisely as you can.
   (a) Axioms of Probability.
   (b) Let $X_1, \ldots, X_n$ be a sequence of IID random variables with mean μ and variance σ^2. State the central limit theorem.

2. (10 points) A fair die is tossed until a 2 is obtained. If X is the number of trials required to obtain the first 2, what is the smallest value of x such that $P(X \leq x) \geq .5$?

3. (10 points) For a random variable Z with cdf

$$F(z) = \begin{cases} 0, & z < 1, \\ \frac{z^2-2z+2}{2}, & 1 \leq z < 2, \\ 1, & z \geq 2 \end{cases}$$

   Sketch the cdf on a graph and find $\mathrm{E}(Z^2)$.

4. (10 points) In a class there are four freshman boys, six freshman girls, and six sophomore boys. How many sophomore girls must be present if sex and class are to be independent when a student is selected at random?

5. (15 points) Consider n people and suppose that each of them has a birthday that is equally likely to be any of the 365 days of the year. Furthermore, assume that their birthdays are independent, and let A be the event that no two of them share the same birthday. Employ the Poisson paradigm to approximate $P(A)$.

6. (15 points) Let (X, Y) be a bivariate random variable with pdf

$$f(x, y) = \begin{cases} 8xy, & 0 \leq x \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

   Find the marginal pdf of X and Y.

7. (15 points) Let $X \sim \text{Gamma}(r, \lambda)$, $r > 0, \lambda > 0$ with pdf

$$\frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x} 1_{\{x>0\}}.$$

Find the MGF of $M_X(t)$.

8. (15 points) Show that

$$P\Big(\cup_{i=1}^n E_i \Big) \leq \sum_{i=1}^n P(E_i).$$

1 - (a) Axioms of Probability

A probability measure P (on a σ - field of subsets $\mathcal{F}$ of a set S) is a real - valued set function satisfying.

i) $P(S) = 1$ (add up to 1)

ii) $P(A) \geq 0$ for all $A \in \mathcal{F}$ (non-negative)

iii) If $A_n \in \mathcal{F}$, $n = 1, 2, \cdots$ are mutually disjoint sets, that is

$A_i \cap A_j = \phi$ if $i \neq j$, then

$$P\left(\bigcup_{i=1}^{\infty} A_n\right) = \sum_{i=1}^{\infty} P(A_n) \quad \text{(countably additive)}$$

(b) Definition of Central Limit Theorem

$$Z_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1),$$

where $\xrightarrow{d}$ represents convergence in distribution.

2.

X : number of trials

p : probability to obtain a 2 $\left(=\frac{1}{6}\right)$

$X \sim GEO\left(\frac{1}{6}\right)$, $f_X(x) = \left(\frac{5}{6}\right)^{x-1}\left(\frac{1}{6}\right)$

$$P(X \le x) = \sum_{i=1}^{x} P(X=i) = \sum_{i=1}^{x}\left(\frac{5}{6}\right)^{i-1}\left(\frac{1}{6}\right)$$

$$= \frac{\frac{1}{6}\cdot\left(1-\left(\frac{5}{6}\right)^{x}\right)}{1-\frac{5}{6}} = 1-\left(\frac{5}{6}\right)^{x} \ge 0.5$$

$$\Rightarrow \left(\frac{5}{6}\right)^{x} \le 0.5$$

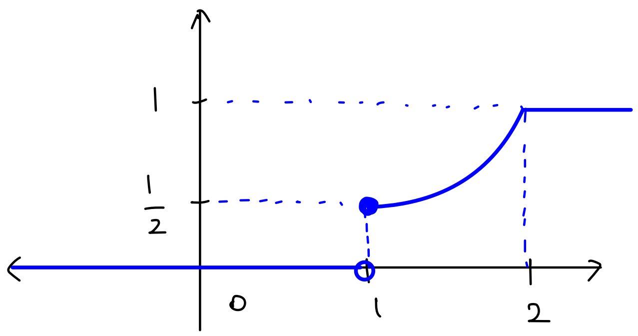
$\left(\frac{5}{6}\right)^{3} \fallingdotseq 0.58$, $\left(\frac{5}{6}\right)^{4} \fallingdotseq 0.48$

$\therefore$ smallest $x = 4$

3.

$$F(z) = \begin{cases} 0 & , \quad z < 1 \\ \dfrac{z^2 - 2z + 2}{2} & , \quad 1 \le z < 2 \\ 1 & , \quad z \ge 2 \end{cases}$$

i) Graph of cdf :



ii) Find $\mathbb{E}(Z^2)$

$$f(z) = \begin{cases} \frac{1}{2} & , \quad z = 1 \\ z-1 & , \quad 1 < z < 2 \\ 0 & , \quad \text{otherwise} \end{cases}$$

$$\mathbb{E}(Z^2) = 1 \times \frac{1}{2} + \int_1^2 z^2 (z-1)\, dz$$

$$= \frac{1}{2} + \left[\frac{1}{4} z^4 - \frac{1}{3} z^3 \right]_1^2 = \frac{23}{12}$$

4.

Let

F : freshman , S : sophomore , B : boys , G : girls

and x : number of sophomore girls ,

then $P(S \cap G) = P(S) \cdot P(G)$. ($\because$ sex and class are indep.) .

$$\Rightarrow P(S \cap G) = \frac{x}{16+x} = \frac{6+x}{16+x} \cdot \frac{6+x}{16+x} = P(S) \cdot P(G)$$

$$\Rightarrow (16+x)x = (6+x)^2 \qquad \therefore \quad x = 9$$

5.

A: event of 'no pair' sharing same birthday

X: number of pairs sharing birthday

$$X \sim Bin\left(\binom{n}{2}, \frac{1}{365}\right) \approx Poi\left(\binom{n}{2}\cdot\frac{1}{365}\right)$$ by Poisson Pardigm

$$f(x) = \frac{e^{-\binom{n}{2}\cdot\frac{1}{365}} \cdot \left(\binom{n}{2}\cdot\frac{1}{365}\right)^{x}}{x!}$$

$$\Rightarrow P(A) = P(X=0) = e^{-\binom{n}{2}\cdot\frac{1}{365}}$$

6.

$$f(x,y) = \begin{cases} 8xy, & 0 \le x \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

i) marginal pdf of X

$$f(x) = \int_x^1 8xy\,dy = 8x\left[\frac{1}{2}y^2\right]_x^1 = 4x(1-x^2), \quad 0 \le x \le 1$$

ii) marginal pdf of Y

$$f(y) = \int_0^y 8xy\,dx = 8y\left[\frac{1}{2}x^2\right]_0^y = 4y^3, \quad 0 \le y \le 1$$

7.

$X \sim Gamma(r, \lambda)$, $r > 0$, $\lambda > 0$

$$\frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x} \mathbb{1}_{\{x>0\}}$$

$$M_X(t) = \mathbb{E}(e^{tX})$$

$$= \int_0^\infty e^{tx} \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x} dx$$

$$= \int_0^\infty \frac{1}{\Gamma(r)} (\lambda - t)^r \cdot \frac{\lambda^r}{(\lambda-t)^r} x^{r-1} e^{-(\lambda-t)x} dx$$

$$= \left(\frac{\lambda}{\lambda-t}\right)^r \int_0^\infty \underbrace{\frac{1}{\Gamma(r)} (\lambda-t)^r x^{r-1} e^{-(\lambda-t)x}}_{= \text{pdf of Gamma}(r,\, \lambda-t)} dx$$

$$= \left(\frac{\lambda}{\lambda-t}\right)^r, \quad \lambda > t$$

8.

Define $F_1 = E_1$, $F_2 = E_2 - E_1$, $\cdots$, $F_n = E_n - \bigcup_{j=1}^{n-1} E_j$.

Then,

$$P\left(\bigcup_{i=1}^{n} E_i\right) = P\left(\bigcup_{i=1}^{n} F_i\right) \quad \because \bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{n} F_i$$

$$= \sum_{i=1}^{n} P(F_i) \quad \because F_i\text{'s are disjoint subsets, by Axiom of probability}$$

$$\leq \sum_{i=1}^{n} P(E_i) \quad \because F_i \subseteq E_i \Rightarrow P(F_i) \leq P(E_i)$$

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