Experimental Design
Note 14
Nested Design

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Nested and Split-Plot Designs

- Text reference, Chapter 14
- These are **multifactor** experiments that have some important industrial/ agricultural applications
- Nested and split-plot designs frequently involve one or more **random** factors, so the methodology of Chapter 13 (expected mean squares, variance components) is important
- There are many variations of these designs- we consider only some basic situations

Crossed vs Nested Factors I

- Factors A (a levels) and B (b levels) are considered **crossed** if every combinations of A and B (ab of them) occurs.
  An example:

	Factor A			
Factor B	1	2	3	4
1	xx	xx	xx	xx
2	xx	xx	xx	xx
3	xx	xx	xx	xx

A	1			2			3			4		
B	1	2	3	1	2	3	1	2	3	1	2	3
	x	x	x	x	x	x	x	x	x	x	x	x
	x	x	x	x	x	x	x	x	x	x	x	x

Crossed vs Nested Factors II

- Factor B is considered nested under A (a levels) if
  1. under each fixed level (i) of A, B has b_i levels.
  2. the levels of B under the sample level of A are comparable.
  3. under a level of A, the levels of B can be arbitrarily numbered.

A	1			2			3			4		
B	1	2	3	4	5	6	7	8	9	10	11	12
	X	X	X	X	X	X	X	X	X	X	X	X
	X	X	X	X	X	X	X	X	X	X	X	X

Material Purity Experiment I

Consider a company that buys raw material in batches from three different suppliers. The purity of this raw material varies considerably, which causes problems in manufacturing the finished product. We wish to determine if the variability in purity is attributable to difference between the suppliers. Four batches of raw material are selected at random from each supplier, three determinations of purity are made on each batch. The data, after coding by subtracting 93 are given below.

Material Purity Experiment II

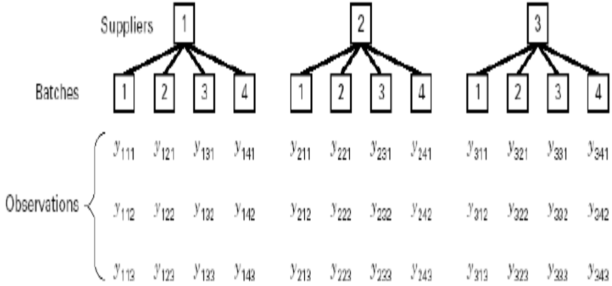


Figure 14-1 A two-stage nested design.

Material Purity Experiment III

	Supplier 1				Supplier 2				Supplier 3			
Batches	1	2	3	4	1	2	3	4	1	2	3	4
	1	-2	-2	1	1	0	-1	0	2	-2	1	3
	-1	-3	0	4	-2	4	0	3	4	0	-1	2
	0	-4	1	0	-3	2	-2	2	0	2	2	1
$y_{ij.}$	0	-9	-1	5	-4	6	-3	5	6	0	2	6
$y_{i..}$	-5				4				14			

Other Examples for Nested Factors

1. Drug company interested in stability of product
   - Two manufacturing sites
   - Three batches from each site
   - Ten tablets from each batch
2. Stratified random sampling procedure
   - Randomly sample five states
   - Randomly select three counties
   - Randomly select two towns
   - Randomly select five households

Statistical Model

- Two factor nested model

$$y_{ijk} = \mu + \tau_i + \beta_{j(i)} + \epsilon_{k(ij)}$$

for $i = 1, \cdots, a$; $j = 1, \cdots, b$; $k = 1, \cdots, n$.

- Bracket notation represents nested factor
- Cannot include interaction
- Factors may be random or fixed
- Can use EMS algorithm to derive tests

Sum of Squares Decomposition

$$y_{ijk} = \bar{y}_{...} + (\bar{y}_{i..} - \bar{y}_{...}) + (\bar{y}_{ij.} - \bar{y}_{i..}) + (y_{ijk} - \bar{y}_{ij.})$$

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk} - \bar{y}_{...})^2 = bn\sum_{i=1}^{a}(\bar{y}_{i..} - \bar{y}_{...})^2 + n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{y}_{ij.} - \bar{y}_{i..})^2$$

$$+ \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk} - \bar{y}_{ij.})^2$$

$$SS_T = SS_A + SS_{B(A)} + SSE$$

Analysis of Variance Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
A	SS_A	$a-1$	MS_A	
B(A)	$SS_{B(A)}$	$a(b-1)$	$MS_{B(A)}$	
Error	SSe	$ab(n-1)$	MSE	
Total	SS_T	$abn-1$		

$$SS_T = \sum_i \sum_j \sum_k y_{ijk}^2 - y_{...}^2/abn$$

$$SS_A = \frac{1}{bn} \sum_i y_{i..}^2 - y_{...}^2/abn$$

$$SS_{B(A)} = \frac{1}{n} \sum_i \sum_j y_{ij.}^2 - \frac{1}{bn} \sum_i y_{i..}^2$$

$$SSE = \sum_i \sum_j \sum_k y_{ijk}^2 - \frac{1}{n} \sum_i \sum_j y_{ij.}^2$$

Use EMS to define proper tests

Two-Factor Nested Model with Fixed Effects

$$y_{ijk} = \mu + \tau_i + \beta_{j(i)} + \epsilon_{k(ij)}$$

where $\sum_{i=1}^{a} \tau_i = 0$ and $\sum_{j=1}^{b} \beta_{j(i)} = 0$ for each i.

	F	F	R	
	a	b	n	
Term	i	j	k	EMS
τ_i	0	b	n	$\sigma^2 + \frac{bn\sum_i \tau_i^2}{a-1}$
$\beta_{j(i)}$	1	0	n	$\sigma^2 + \frac{n\sum_i \sum_j \beta_{j(i)}^2}{a(b-1)}$
$\epsilon_{k(ij)}$	1	1	1	σ^2

- Estimates: $\hat{\tau}_i = \bar{y}_{i..} - \bar{y}_{...}$; $\hat{\beta}_{j(i)} = \bar{y}_{ij.} - \bar{y}_{i..}$.
- Tests: MS_A/MSE for $H_0 : \tau_i = 0$; $MS_{B(A)}/MSE$ for $H_0 : \beta_{j(i)} = 0$.

Two-Factor Nested Model with Random Effects

$$y_{ijk} = \mu + \tau_i + \beta_{j(i)} + \epsilon_{k(ij)}$$

where $\tau_i \sim N(0, \sigma_\tau^2)$ and $\beta_{j(i)} \sim N(0, \sigma_\beta^2)$.

	R	R	R	
	a	b	n	
Term	i	j	k	EMS
τ_i	1	b	n	$\sigma^2 + n\sigma_\beta^2 + bn\sigma_\tau^2$
$\beta_{j(i)}$	1	1	n	$\sigma^2 + n\sigma_\beta^2$
$\epsilon_{k(ij)}$	1	1	1	σ^2

- Estimates: $\hat{\sigma}_\tau^2 = (MS_A - MS_{B(A)})/nb$; $\hat{\sigma}_\beta^2 = (MS_{B(A)} - MSE)/n$.
- Tests: $MS_A/MS_{B(A)}$ for $H_0 : \sigma_\tau^2 = 0$; $MS_{B(A)}/MSE$ for $H_0 : \sigma_\beta^2 = 0$.

Two-Factor Nested Model with Mixed Effects

$$y_{ijk} = \mu + \tau_i + \beta_{j(i)} + \epsilon_{k(ij)}$$

where $\sum_{i=1}^{a} \tau_i = 0$ and $\beta_{j(i)} \sim N(0, \sigma_\beta^2)$.

	F	R	R	
	a	b	n	
Term	i	j	k	EMS
τ_i	0	b	n	$\sigma^2 + n\sigma_\beta^2 + \frac{bn\sum_i \tau_i^2}{a-1}$
$\beta_{j(i)}$	1	1	n	$\sigma^2 + n\sigma_\beta^2$
$\epsilon_{k(ij)}$	1	1	1	σ^2

- Estimates: $\hat{\tau}_i = \bar{y}_{i..} - \bar{y}_{...}$; $\hat{\sigma}_\beta^2 = (MS_{B(A)} - MSE)/n$.
- Tests: $MS_A/MS_{B(A)}$ for $H_0 : \tau_i = 0$; $MS_{B(A)}/MSE$ for $H_0 : \sigma_\beta^2 = 0$.

Expected Mean Squares in the Two-Stage Nested Design

TABLE 14.1

Expected Mean Squares in the Two-Stage Nested Design

$E(MS)$	A Fixed B Fixed	A Fixed B Random	A Random B Random
$E(MS_A)$	$\sigma^2 + \frac{bn\sum \tau_i^2}{a-1}$	$\sigma^2 + n\sigma_\beta^2 + \frac{bn\sum \tau_i^2}{a-1}$	$\sigma^2 + n\sigma_\beta^2 + bn\sigma_\tau^2$
$E(MS_{B(A)})$	$\sigma^2 + \frac{n\sum\sum \beta_{j(i)}^2}{a(b-1)}$	$\sigma^2 + n\sigma_\beta^2$	$\sigma^2 + n\sigma_\beta^2$
$E(MS_E)$	σ^2	σ^2	σ^2

Purity Experiment

See nested.SAS.

Other Scenarios for Nested Factors I

- Staggered Nested Designs
- General *m*-Stage Nested Designs

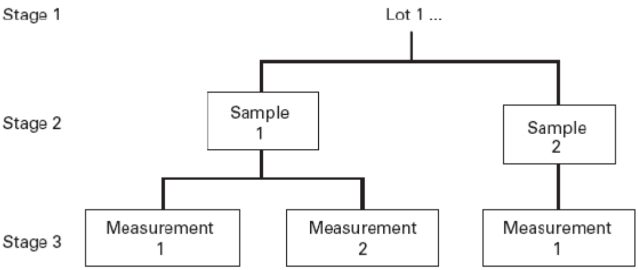
$$y_{ijkl} = \mu + \tau_i + \beta_{j(i)} + \gamma_{k(ij)} + \epsilon_{l(ijk)}$$

- Designs with Both Nested and Factorial Factors

$$y_{ijkl} = \mu + \tau_i + \beta_j + \gamma_{k(j)} + (\tau\beta)_{ij} + (\tau\gamma)_{ik(j)} + \epsilon_{l(ijk)}$$

- Sections 14.2, 14.3 in Textbook.

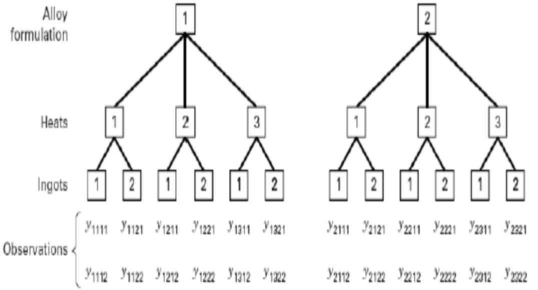
Other Scenarios for Nested Factors II



■ FIGURE 14.4 A three-stage staggered nested design

General m-Stage Nested Designs I

■ FIGURE 14.5 A three-stage nested design



$$y_{ijkl} = \mu + \tau_i + \beta_{j(i)} + \gamma_{k(ij)} + \epsilon_{l(ijk)}$$

General m-Stage Nested Designs II

TABLE 14.7

Analysis of Variance for the Three-Stage Nested Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square
A	$bcn \sum_i (\bar{y}_{i\ldots} - \bar{y}_{\ldots.})^2$	$a - 1$	MS_A
B (within A)	$cn \sum_i \sum_j (\bar{y}_{ij..} - \bar{y}_{\ldots.})^2$	$a(b - 1)$	$MS_{B(A)}$
C (within B)	$n \sum_i \sum_j \sum_k (\bar{y}_{ijk.} - \bar{y}_{\ldots.})^2$	$ab(c - 1)$	$MS_{C(B)}$
Error	$\sum_i \sum_j \sum_k \sum_l (y_{ijkl} - \bar{y}_{ijk.})^2$	$abc(n - 1)$	MS_E
Total	$\sum_i \sum_j \sum_k \sum_l (y_{ijkl} - \bar{y}_{\ldots.})^2$	$abcn - 1$	

General m-Stage Nested Designs III

■ **TABLE 14.8**

Expected Mean Squares for a Three-Stage Nested Design with A and B Fixed and C Random

Model Term	Expected Mean Square
τ_i	$\sigma^2 + n\sigma_\gamma^2 + \frac{bcn \sum \tau_i^2}{a-1}$
$\beta_{j(i)}$	$\sigma^2 + n\sigma_\gamma^2 + \frac{cn \sum \sum \beta_{j(i)}^2}{a(b-1)}$
$\gamma_{k(ij)}$	$\sigma^2 + n\sigma_\gamma^2$
$\epsilon_{l(ijk)}$	σ^2

Example 14.2 Nested and Factorial Factors I

TABLE 14.9
Assembly Time Data for Example 14.2

	Layout 1				Layout 2				
Operator	1	2	3	4	1	2	3	4	$y_{i..}$
Fixture 1	22	23	28	25	26	27	28	24	404
	24	24	29	23	28	25	25	23	
Fixture 2	30	29	30	27	29	30	24	28	447
	27	28	32	25	28	27	23	30	
Fixture 3	25	24	27	26	27	26	24	28	401
	21	22	25	23	25	24	27	27	
Operator totals, $y_{.jk.}$	149	150	171	149	163	159	151	160	
Layout totals, $y_{.j..}$	619				633				$1252 = y_{....}$

Example 14.2 Nested and Factorial Factors II

$$y_{ijkl} = \mu + \tau_i + \beta_j + \gamma_{k(j)} + (\tau\beta)_{ij} + (\tau\gamma)_{ik(j)} + \epsilon_{(ijk)l}$$

for $i = 1, 2, 3$ (Fixtures); $j = 1, 2$ (Layouts); $k = 1, 2, 3, 4$ (Operators); $l = 1, 2$ (replicates).
Assume that fixtures and layouts are fixed, operators are random - gives a mixed model (use restricted form)

Example 14.2 Nested and Factorial Factors III

TABLE 14.10
Expected Mean Squares for Example 14.2

Model Term	Expected Mean Square
τ_i	$\sigma^2 + 2\sigma^2_{\tau\gamma} + 8\sum \tau_i^2$
β_j	$\sigma^2 + 6\sigma^2_{\gamma} + 24\sum \beta_j^2$
$\gamma_{k(j)}$	$\sigma^2 + 6\sigma^2_{\gamma}$
$(\tau\beta)_{ij}$	$\sigma^2 + 2\sigma^2_{\tau\gamma} + 4\sum\sum (\tau\beta)^2_{ij}$
$(\tau\gamma)_{ik(j)}$	$\sigma^2 + 2\sigma^2_{\tau\gamma}$
$\epsilon_{(ijk)l}$	σ^2

Example 14.2 Nested and Factorial Factors IV

TABLE 14.11

Analysis of Variance for Example 14.2

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
Fixtures (F)	82.80	2	41.40	7.54	0.01
Layouts (L)	4.08	1	4.09	0.34	0.58
Operators (within layouts), $O(L)$	71.91	6	11.99	5.15	<0.01
FL	19.04	2	9.52	1.73	0.22
$FO(L)$	65.84	12	5.49	2.36	0.04
Error	56.00	24	2.33		
Total	299.67	47			