Chapter 1. Real numbers and monotone sequences

1.1 Real numbers

Question: What is a real number?

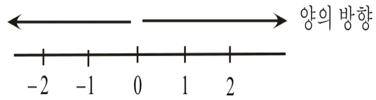
$\mathbb{N} = \{1, 2, 3, \cdots\}$: the set of natural numbers (or the *counting* numbers)

사용 예: 물건 또는 원소의 개수(1개, 2개,⋯), 시간의 흐름(하루, 이틀,⋯)

Kronecker: “God created the natural numbers, everything else is man’s handywork”

$\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, 3, \cdots\}$: the set of integers (introduced to solve problems such as $x + 5 = 2$)

즉, 방향개념도입



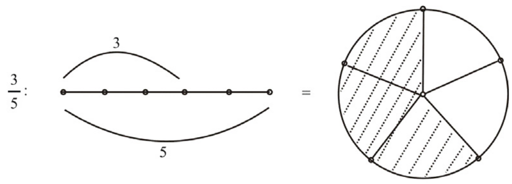
$\mathbb{Z}$ comes from the german word for number, zählen

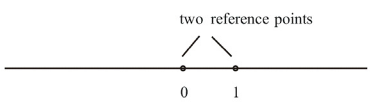
(The natural numbers are referred as the positive integers: $\mathbb{Z}^+ = \mathbb{N}$)

$\mathbb{Q}$ (stands for quotient) = $\left\{ \dfrac{m}{n} : m, n \in \mathbb{Z} \text{ with } n \neq 0 \right\}$:

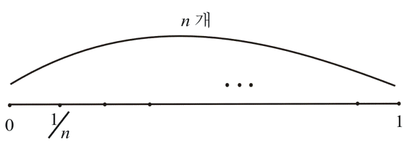
the set of rational numbers (= the ratio of two integers)

$\mathbb{Q}$ was introduced for “measuring” (for example, parts of a whole)





$\dfrac{1}{n}(n \in \mathbb{N})$: divide the original unit 1 into n equal parts



$\frac{m}{n}(m, n \in \mathbb{N}): \frac{1}{n}+\frac{1}{n}+\cdots+\frac{1}{n} \quad \left(\frac{1}{n}이\ m개\right)$

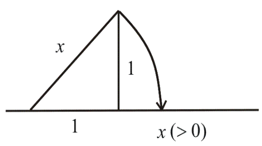
(또는 전체 n개 중에서 m개)

$-1/2:$ ($-$부호: 양의 반대방향)

Conclusion: The rational numbers include all of the **familiar** numbers that arise in everyday life.

Question: ***Is the set of rational numbers enough for the purpose of exact measurement?***

Ans: No (by Pythagoras)



By Pythagorean theorem, $\exists$ some length (= positive number) such that $x^2 = 2$

(We write $\sqrt{2}$ for the number $x(>0)$)

The ancient Greek mathematicians knew that this length could *not* be described by dividing a stick into equal parts. That is, $\sqrt{2}$ is *not* a rational number. Also, there are many other "lengths" that cannot be represented as the ratio of two integers. Therefore, the set of rational numbers is *not big enough* for the purpose of exact measurement.

At that time, there was no way of representing irrational numbers except as lengths (that is, as points on a line) --- such a representation is not well-suited to calculation ---

At any rate, the set of rational numbers had to be extended to the real numbers $\mathbb{R}$ (= "the real lengths" = the points on a line)

Many centuries later, Mathematicians discovered *another good way* to think of a real number (for computation):

A real number *can* be represented as an infinite decimal
(수렴성, 무한급수등을 공부해야 정확히 증명가능)

From now on, we assume that

$$\text{A real number} = \text{an integer} + \text{an infinite decimal}\left(\sum_{n=1}^{\infty}\frac{a_n}{10^n} = 0.a_1a_2\cdots a_n\cdots\right)$$

For example, $101.23000\cdots$, $-0.3333\cdots$, $3.141592\cdots$

(십진) 소수표현(infinite decimal representation)의 **단점:**

(단점-i) The (infinite) decimal representation of a real number need not be unique

예: $0.2000\cdots = 0.1999\cdots$

$(\because \quad x := 0.1999\cdots \quad \Rightarrow \quad 9x = 10x - x = 1.999\cdots - 0.1999\cdots = 1.8$

$$\therefore \quad x = \frac{1.8}{9} = 0.2)$$

(단점-ii) Not obvious how to calculate with infinite decimals (: details are discussed later)

장점: In terms of decimal expansions, there is a simple distinction between rational numbers and irrational numbers

Fact:

(a) A finite (or terminating) decimal represents a rational number: easy to prove

(b) An infinite (or non-terminating) decimal is a rational number if and only if it is a repeating decimal

From this, we conclude that

a real number is a rational number iff its decimal expansion has a repeating pattern

« repeating pattern: $0.a_1a_2\cdots a_n b_1b_2\cdots b_m b_1b_2\cdots b_m b_1b_2\cdots b_m\cdots$

If all $b_i (i = 1, 2, \cdots, m)$ of the above expansions are zero, it becomes a finite decimal »

Examples:

$$\begin{aligned} 1 &= 1.000\cdots \\ 3/2 &= 1.5000\cdots \\ 22/7 &= 3.142857142857\cdots \\ 33/8 &= 4.125000\cdots \\ \pi &= 3.141592653589\cdots \quad \text{(nonrepeating) by Lambert(1766)} \end{aligned}$$

Idea for the pf of the above conclusion:

(i) Repeating decimal is a rational number:

For example, if $x = 0.143143143\cdots$, then

$$1000x - x = 143.143.143\cdots - 0.143143143\cdots = 143$$

$$\therefore \quad x = \frac{143}{999} \quad \text{(rational number)}$$

Key idea: $\boxed{x \ = \ \text{repeating decimal} \quad \Rightarrow \quad x \ \text{ is a geometric series}}$

(ii) (converse of (i)) A rational number is a repeating decimal

Case1: $\frac{33}{8} = ? = 4.125$ (zero remainder: After some steps, division process stops)

Case2: $\frac{2}{7} = ? = 0.285714285714\cdots$ (has no zero remainder: division process repeats)

When dividing, for example, by 7, the only possible remainders are $0,1,2,3,4,5,6$

If the zero remainder occurs, the division process stops (so get a *finite decimal*)

If the zero remainder can never occur in the division process (= 7번의 나누는 과정), one of them (i.e., one of $1,2,3,4,5,6$) should be appear again (so get a *repeating decimal*)

Ex: $0.101001000100001\cdots = \text{rational number?}$

Ans: No, because the above is not a repeating pattern

※ An application of decimal representations of real numbers

Question: What is the major part of the real numbers?

(i.e., 유리수와 무리수중 어느 쪽이 (“개수”가) 많을까?)

Attack by **Probabilistic idea**

The above question can be represented in the alternative form as follows:

면이 10개이고 각면에 숫자 0 ~ 9 가 각각 1개씩 적힌 연필을 생각하자

(각 면이 나올 확률이 동일하도록 제작한다: 정 10면체는 존재하지 않는다는 것을 기억)

연필을 계속해서 굴리면서

첫 번째 나온면의 숫자	$\rightarrow$	a_1(에 대응시킴)
두 번째 나온면의 숫자	$\rightarrow$	a_2(에 대응시킴)
	$\vdots$	
그리고 숫자	$0.a_1a_2a_3\cdots$	를 생각하자

위 문제의 변형: 이와 같이 만들어진 숫자 (a decimal representation of a real number)가 유리수(repeating pattern)와 비슷한가 아니면 무리수(non-repeating pattern)와 비슷한가?

Ans: Surely (in probabilistic sense), it will have a non-repeating pattern (따라서, 무리수가 훨씬 많다)

Two more well known facts:

① The set $\mathbb{Q}$ is countable [countable: 자연수 집합과 일대일 대응]

② The set $\mathbb{R}$ is uncountable

Return to (단점-ii): Not obvious how to calculate with infinite decimals

How to add or multiply two decimals

◎ Usual approach

$$\begin{array}{|r|}\hline 2.389 \\ +\ 2.389 \\ \hline \cdots 78 \\ \hline \end{array} \qquad\qquad \begin{array}{|r|}\hline 2.849 \\ \times.\quad 09 \\ \hline \cdots 41 \\ \hline \end{array}$$

This approach has no problem for finite decimals, but, for infinite decimals, serious problem can occur since an infinite decimal has no right end.

⦿ Another (useful) approach

To get around this, we use its *truncations* to finite decimals, viewing these as approximations to the infinite decimal

Ex. Use the idea of truncations to calculate $\pi + \sqrt[3]{2}$

$$\pi \text{ is the "limit" of } \quad 3,\quad 3.1,\quad 3.14,\quad 3.141,\quad 3.1415,\quad 3.14159,\quad \cdots \nearrow$$

$$\sqrt[3]{2} \text{ is the "limit" of } \quad 1,\quad 1.2,\quad 1.25,\quad 1.259,\quad 1.2599,\quad 1.25992,\quad \cdots \nearrow$$

$$\therefore\quad \pi + \sqrt[3]{2} \text{ is the "limit" of } \quad 4,\quad 4.3,\quad 4.39,\quad 4.400,\quad 4.4014,\quad 4.40151,\quad \cdots \nearrow$$

Comment: 이와 같이 증가하는 소수열(decimal representations)로 접근하는 방법이 일견 초기 몇단계에서 앞자리수들에 변화가 일어나므로 좋은 방법이 아닌 것처럼 생각할 수 있다. 그러나 조금만 더 계속하면 더 이상 변화하지 않는 자리수들이 나타나게 되어 근사값을 구하는데 문제가 되지 않는다.

같은 방법으로 곱셈 $\pi \times \sqrt[3]{2}$ 의 근사값(approximation) 또는 극한값("limit")도 구할 수 있다.

$$3,\quad 3.72,\quad 3.9259,\quad 3.924519,\quad \cdots \nearrow$$

결론: 실수들의 덧셈·곱셈 이라는 간단해보이는 연산을 위해서도 수열 및 극한의 개념을 이해하는 것이 필요하다. 특히, **증가 (또는 감소)하는 수열(또는 수열의 극한)을 실수를 파악하기 위한 도구로 사용하려는 시도는 비교적 자연스럽다.**

1.2 Increasing sequences

Def. An infinite list $a_0,\ a_1, a_2,\ \cdots,\ a_n,\ \cdots$ of (real) numbers is called a *sequence* of (real) numbers. We call a_n the **n-th term** of the sequence.

Notation (for sequence):

$$a_0,\ a_1, a_2,\ \cdots, a_n,\ \cdots \quad \text{or} \quad \{a_n\}_{n=0}^{\infty} \quad \text{or} \quad \{a_n\}, n \geq 0 \quad \text{or} \quad \{a_n\}$$

Sometimes: $(a_n)_{n=0}^{\infty}$ or $(a_n)_{n\geq 0}$ or (a_n) or a_n

Some examples of sequences:

$$1,\quad 1/2,\quad 1/3,\quad 1/4,\quad \cdots:\quad \left(\frac{1}{n}\right)_{n=1}^{\infty},\qquad \left\{\frac{1}{n}\right\}, n \geq 1$$

$$1,\quad -1,\quad 1,\quad -1,\quad \cdots:\quad \left((-1)^n\right)_{n=0}^{\infty},\qquad \left\{(-1)^n\right\}, n \geq 0$$

$$1,\quad 4,\quad 9,\quad 16,\quad \cdots:\quad \left(n^2\right)_{n=1}^{\infty},\qquad \left\{n^2\right\}, n \geq 1$$

$$3,\quad 3.1,\quad 3.14,\quad 3.141,\quad 3.1415,\quad \cdots$$

Def. We say that the sequence $\{a_n\}$ is increasing if $a_n \leq a_{n+1}$ for all n

(strictly increasing if $a_n < a_{n+1}$ for all n)

decreasing if $a_n \geq a_{n+1}$ for all n

(strictly decreasing if $a_n > a_{n+1}$ for all n)

We often use the terminology *non-decreasing* for increasing

1.3 The limit of an increasing sequence

In this section, we will give a provisional definition for *the limit of an increasing sequence*

Note: In the provisional definition below,

★−1 we assume that none of **the sequence** (a_n) ends with all 9's

(i.e., they are written as terminating decimal, if possible)

Recall that each member of (a_n) is an infinite decimal

★−2 **The limit** L however might appear in either form

(i.e., it can be terminating or non-terminating form)

(we will call the form a *suitable decimal representation* for L)

★−1의 예) (어떤 자리수 이후 9가 반복되는 **수열**의 경우: 2가지 표현이 가능)

(a) $0.5,\quad 0.59,\quad 0.599,\quad \underbrace{0.5999\cdots9\cdots}_{\substack{\downarrow\text{should change}\\ 0.6}},\quad 0.59999,\quad 0.59999,\quad \cdots$

(b) $0.1,\quad \underbrace{0.09999\cdots}_{\substack{\downarrow\text{should change}\\ 0.1}},\quad 0.1,\quad \underbrace{0.09999\cdots}_{\substack{\downarrow\text{should change}\\ 0.1}},\quad 0.1,\quad 0.1,\quad 0.1,\quad \cdots$

(c) $0.3,\quad 0.33,\quad \underbrace{0.3333\cdots}_{\substack{\downarrow\text{impossible}\\ \text{terminating decimal}}},\quad 0.34,\quad 0.341,\quad 0.3411,\quad \cdots$

★−2의 예) (usual possible form for L) (*suitable form* for L will be defined soon)

$(a)\quad 0.9,\quad 0.99,\quad 0.999,\quad \cdots \to\quad 1(1.000\cdots) = 0.9999\cdots$

$(b)\quad 1/2(=0.5),\quad 2/3(=0.666\cdots),\quad 3/4(=0.75),\quad 4/5(=0.8),\quad \cdots \to\quad 1 = 0.9999\cdots$

$(c)\quad 0.3,\quad 0.33,\quad 0.333,\quad \cdots \to\quad 0.3333\cdots$

$(d)\quad 1/3(=0.333\cdots),\quad 1/3(=0.333\cdots),\quad 1/3(=0.333\cdots),\quad \cdots \to 1/3(=0.333\cdots)$

$(e)\quad 1(1.0000\cdots),\quad 1(1.0000\cdots),\quad 1(1.0000\cdots),\quad \cdots \to\quad 1(1.0000\cdots) = 0.9999\cdots$

※ (Provisional) Def. Let (a_n) be an **increasing** sequence. We say that a number L, in a suitable decimal representation, is the limit of (a_n) if,

given any positive integer k, all the a_n after some place $= L$ to k decimal places

(i.e., ∃ a natural number $N = N(k)$ such that if $n \geq N$ then $a_n = L$ to k decimal places).

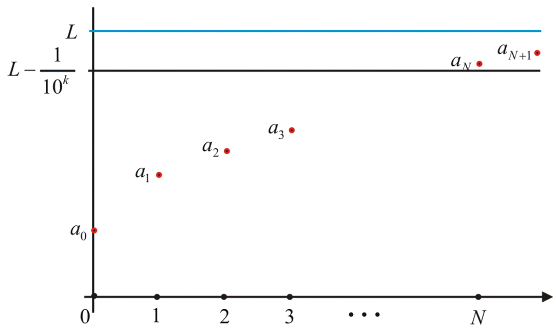
(번호 N 이후부터 나타나는 (수열의) 항은 L 과 소수점 이하 k 자리까지 같다)

In other words, given any positive integer k, ∃ a natural number $N = N(k)$ such that

$$L - a_n (= |a_n - L|) < \frac{1}{10^k} \quad \text{for every } n \geq N$$

In this case, we write $\lim\limits_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$

⊙ 번호 N 이후부터 나타나는 항들이 L 과 소수점 이하 k 자리까지 같다는 의미의 기하적 해석



주의: N 은 k 가 변하면 일반적으로 변한다. 실제로 k 가 커지면 일반적으로 N 도 커진다.

예) (a) $0.9,\quad 0.99,\quad 0.999,\quad \cdots \to \quad L = ?$ (in a suitable representation)

Ans: $L = 0.9999\cdots$이면 위의 정의를 만족한다.

$L = 1.0000\cdots$이면 위의 정의를 만족하지 않는다.

(물론 결과적으로는 $L = 1$과 같다)

(b) $1/2(=0.5),\quad 2/3(=0.666\cdots),\quad 3/4(=0.75),\quad 4/5(=0.8),\quad \cdots \to \quad L = ?$ (//)

Ans: $L = 0.9999\cdots \quad (L \neq 1.0000\cdots)$

(c) $1(1.0000\cdots),\quad 1(1.0000\cdots),\quad 1(1.0000\cdots),\quad \cdots \to \quad L = ?$ (//)

Ans: $L = 1(=1.0000\cdots)$

Remark1. (Remember that we are assuming (a_n) is an increasing sequence)

If $\lim_{n\to\infty} a_n$ exists, then it must be unique

Pf. Suppose that there are real numbers L and L' such that

$$\lim_{n\to\infty} a_n = L \quad \text{and} \quad \lim_{n\to\infty} a_n = L'$$

Let k be any positive integer. Then

$\lim_{n\to\infty} a_n = L \quad \Rightarrow \quad \exists$ a natural number $N_1 = N_1(k)$ such that

$a_n = L$ to k decimal places for $n \geq N_1$

$\lim_{n\to\infty} a_n = L' \quad \Rightarrow \quad \exists$ a natural number $N_2 = N_2(k)$ such that

$a_n = L'$ to k decimal places for $n \geq N_2$

Thus for $n \geq \max\{N_1, N_2\}$, we have

$L = a_n = L'$ to k decimal places

Note that L and L' are independent of k. Therefore $L = L'$

Remark2. $\lim_{n\to\infty} a_n$ need not exist.

For example, $1,\quad 2,\quad 3,\cdots$ has no limit

Def. A sequence (a_n) is said to be ***bounded above*** if $\exists$ a number B such that

$$a_n \leq B \quad \text{for all } n$$

(Any such B is called **an upper bound** for the sequence (a_n))

예) $1,\quad 1/2,\quad 1/3,\quad 1/4,\quad \cdots$: bounded above (by any number ≥ 1)

$1,\quad -1,\quad 1,\quad -1,\quad \cdots$: //

$1,\quad 4,\quad 9,\quad 16,\quad \cdots$: not bounded above

※ Theorem 1.3 [**Completeness Property** (or axiom) of the real numbers]

A *positive* sequence (a_n) is $\uparrow$ and bounded above $\Rightarrow$ $\lim_{n\to\infty} a_n$ exists

주의: 유리수 집합에서는 위정리에 대응되는 결과가 성립하지 않는다.

예): $1,\quad 1.4,\quad 1.41,\quad 1.414,\quad \cdots \quad \to \quad \sqrt{2}\ (\notin \mathbb{Q})$

Sketch of the idea for the proof (한가지 수열을 예로 들어 그 이유를 알아보자)

Write out the decimal expansions of the numbers a_n and arrange them as follows:

$$\begin{cases} a_0 = 15.34576\cdots \\ a_1 = 16.26745\cdots \\ a_2 = 16.33654\cdots \\ a_3 = 16.34722\cdots \\ a_4 = 16.34745\cdots \\ a_5 = 16.34747\cdots \\ a_6 = 16.34748\cdots \\ \quad \vdots \end{cases} \qquad \Big((a_n) \text{ is } > 0 \;\; \& \;\; \uparrow \text{ and bdd above} \Big)$$

Look down $(\downarrow)$ the list of numbers

Choose any positive integer k, and fix it.

Claim: After some index, *the integer part and (first)* k *decimal places* of the numbers on the list *no longer change*

To see this, look first at the integer part of the numbers.

They are $\uparrow$ (넓은 의미로), but they *cannot* strictly $\uparrow$ infinitely often.

$[\![\because$ (a_n) is bdd above $\Rightarrow$ "the integer part of a_n" is also bdd above $]\!]$

So after some index $n = n_0$, the integer part never changes

Starting from this term a_{n_0}, continue down the list, looking now just at the first decimal places.

It is $\uparrow$ (넓은 의미로), but it is ≤ 9 $---(*)$

$[\![\because$ otherwise, it turned into 0, the integral part would have to change $\Rightarrow$ $\otimes$ since we are assuming that the integral part never changes $]\!]$

So $(*)$ $\Rightarrow$ after some index $n_1 (\ge n_0)$, the first decimal place will stay constant.

Continue down from the term a_{n_1}. Then after some index $n_2 (\ge n_1)$, the second decimal place will stay constant.

$[\![\because$ otherwise, it would get beyond 9 and the first decimal place would have to change $\Rightarrow$ $\otimes$ $]\!]$

Continuing in this way, we see that after some index (depending on k), the integer part and the first k decimal places remain constant.

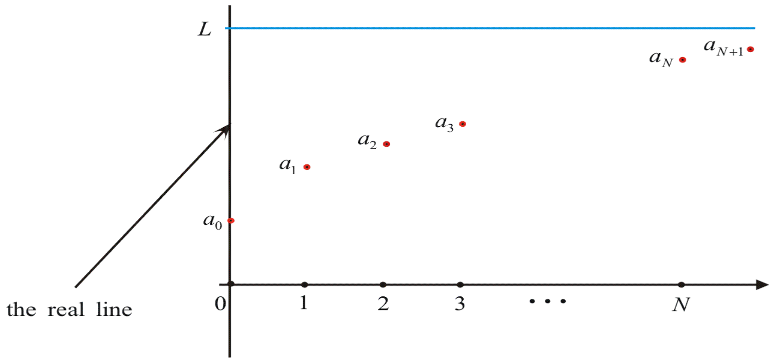
Define these unchanging values as the integer part and the first k decimal places of the limit L.

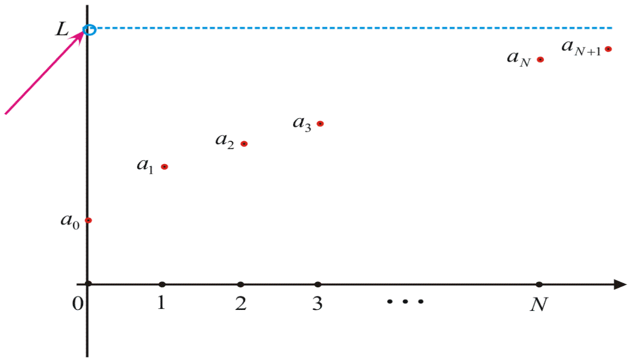
Since k was arbitrary positive integer, the integer part and every decimal place is determined.

Therefore we have defined L (i.e., (a_n) has a limit).

Remember: L 을 안다 $\Leftrightarrow$ (L 을 십진소수로 전개할 때) L 의 각 자리수를 안다

☻ (실수집합의) 완비성(Completeness)의 직관적 해석





만일 real line 상의 L 의 위치에 hole이 있다면, 이 값으로 단조증가하는 수열의 극한값은 존재할 수 없다.

따라서 (직관적으로 말하면)

임의의 단조증가하는 수열이 항상 극한값을 갖는다

⇔ real line에 hole (gap)이 없다

(실수집합의 이 성질을 **완비성** (또는 연속성)이라고 한다: i.e., $\mathbb{R}$ is complete)

1.4 Example: The number e

Review

$\odot$ Binomial theorem: $(1+x)^k = 1 + kx + \binom{k}{2}x^2 + \cdots + \binom{k}{i}x^i + \cdots + x^k,$

$$\text{where} \quad \binom{k}{i} = \frac{k!}{i!(k-i)!} \quad (0! = 1)$$

$\odot$ $\quad 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} < 2$

$\odot$ $\quad 1 + r + r^2 + \cdots + r^n = \frac{1-r^{n+1}}{1-r} \quad (\text{if } r \neq 1)$

Motivation (Compound interest formula: 복리법)

P : 원금(principal)

r : 연이율(annual interest rate) ($r=1$; 100% annual interest)

1년에 n회로 균등하게 나누어서 복리로 지급

1년후 총액 $A_n = P(1+\frac{r}{n})^n$

For example, if $r = 1$ & $P = 1$ $\Rightarrow$

$A_1 = 1 + 1 = 2$ simple interest

$A_2 = (1+\frac{1}{2})^2 = 2.25$ compound semiannually

$A_4 = (1+\frac{1}{4})^4 \approx 2.44$ compound quarterly

(Expect: this sequence will $\uparrow$ strictly)

Proposition. $\quad \lim_{n\to\infty} (1+\frac{1}{2^n})^{2^n}$ exists

Def. $\quad e \overset{\text{denote}}{=} \lim_{n\to\infty} (1+\frac{1}{2^n})^{2^n}$ (Euler named the limit e)

$e = 2.718281\cdots$

e : an irrational number by Lambert (임용고시에 출제됨)

e : a transcendental number by Hermite

Pf of Proposition. Let $a_n = (1+\frac{1}{2^n})^{2^n}$.

Suffices to show: $\{a_n\}$ is $\uparrow$ & bdd above

First we will prove that $\{a_n\}$ is $\uparrow$:

To prove this, observe that if $b > 0$ then $(1+b)^2 > 1 + 2b$ holds.

Taking 2^n power $\Rightarrow$ $(1+b)^{2^{n+1}} = (1+b)^{2\cdot 2^n} > (1+2b)^{2^n}$

Thus if we take $b = \dfrac{1}{2^{n+1}}$ then $(1+\dfrac{1}{2^{n+1}})^{2^{n+1}} > (1+\dfrac{1}{2^n})^{2^n}$

$$i.e., \quad a_{n+1} > a_n \qquad \therefore \ \{a_n\} \text{ is (strictly) } \uparrow$$

Next, we will show that $\{a_n\}$ is bounded above. Moreover, we can show that

$$(1+\frac{1}{n})^n \text{ is bounded above}$$

$$(1+\frac{1}{n})^n = 1 + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \cdots + \binom{n}{k}\frac{1}{n^k} + \cdots + \binom{n}{n}\frac{1}{n^n}$$

$$= 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3}\cdots + \frac{n(n-1)(n-2)\cdots(n-(k-1))}{k!}\frac{1}{n^k}$$

$$+ \cdots\cdots + \frac{1}{n^n}$$

$$\leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!} + \cdots + \frac{1}{n!}$$

$$(\because n(n-1)(n-2)\cdots(n-(k-1)) \leq n^k \quad (\text{for } k = 1, 2, \cdots, n))$$

$$\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{k-1}} + \cdots + \frac{1}{2^{n-1}}$$

$$(\because \frac{1}{k!} = \frac{1}{k(k-1)\cdots 2} \leq \frac{1}{2\cdot 2\cdot \cdots 2} = (\frac{1}{2})^{k-1} \ (\text{for } k = 2, \cdots, n))$$

$$< 1 + 2 = 3$$

$$\therefore \quad \{a_n\} \text{ is bounded above (by 3)}$$

Ex. Prove that $(1+\dfrac{1}{n})^n$ is (strictly) increasing (Already seen that $(1+\dfrac{1}{n})^n$ is bounded above)

(If this is proved, we conclude that $\lim\limits_{n\to\infty}(1+\dfrac{1}{n})^n$ exists $(=e)$)

Pf. (short) Recall that for $x_1, x_2, \cdots, x_n > 0$,

$$x_1 x_2 \cdots x_n \leq (\frac{x_1 + x_2 + \cdots + x_n}{n})^n \quad (AG-\text{mean inequality})$$

Take $x_1 = x_2 = \cdots = x_{n-1} = 1 + \dfrac{1}{n-1}$ & $x_n = 1$. Then

$$(1+\frac{1}{n-1})^{n-1} \leq (1+\frac{1}{n})^n \quad (\text{by AG})$$

$$i.e., \quad a_{n-1} \leq a_n \qquad \therefore \ \{a_n\} \text{ is } \uparrow$$

Note that "$=$" holds in AG $\Leftrightarrow$ $x_1 = x_2 = \cdots = x_n$

Since $x_1 (= x_2 = x_3 = \cdots = x_{n-1}) \neq x_n$, "$=$" does not hold.

$$\therefore \quad a_{n-1} < a_n \qquad \therefore \quad \{a_n\} \text{ is strictly } \uparrow$$

1.5 Example: the harmonic sum and *Euler's number*

Experimental calculation으로는 수렴하지 않는다는 것을 판정하기 쉽지않거나
거의 불가능한 수열의 예 한가지를 생각해보자 (이런 경우에는 수학적 증명이 필요하다)

Proposition. Let $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ $(n \geq 1)$ (call it the harmonic sums)

Show that (a_n) is strictly inc, but not bounded above ($\therefore \lim_{n\to\infty} a_n$ does not exist)

Remark. When (a_n) is (strictly)$\uparrow$ & is not bounded above, we often write

$$\lim_{n\to\infty} a_n = \infty$$

Pf1 (of the Proposition) Suffices to show that $a_1, a_2, a_4, a_8, a_{16}, \cdots$ is not bounded above

$$(\text{i.e., } a_{2^k} \to \infty \quad as \quad k \to \infty)$$

$$a_{2^k} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{2\text{개}} + \underbrace{\frac{1}{5} + \cdots + \frac{1}{8}}_{4\text{개}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{8\text{개}} + \cdots + \underbrace{\cdots + \frac{1}{2^k}}_{2^{k-1}\text{개}}$$

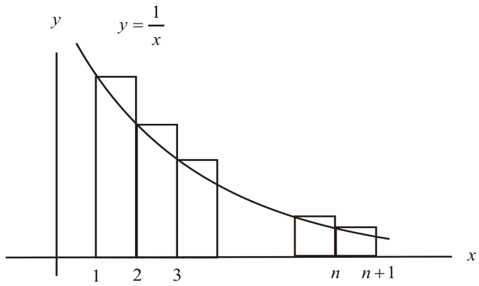
$$> 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{8} + \cdots + \frac{1}{8}} + \underbrace{\frac{1}{16} + \cdots + \frac{1}{16}} + \cdots + \underbrace{(\frac{1}{2^k} \cdots + \frac{1}{2^k})}$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}$$

$$= 1 + \frac{k}{2} \quad \to \infty \quad \text{as} \quad k \to \infty$$

$$\therefore \quad a_{2^k} \to \infty \quad as \quad k \to \infty$$

Pf2 (better than Pf1; geometric pf)



Total area of the above rectangles $= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = a_n$

$$a_n > \text{the area under the curve } y = 1/x \quad \& \quad \text{over } [1, n+1]$$

$$\|$$

$$\int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1)$$

i.e., $a_n > \ln(n+1) \to \infty$ $\qquad \therefore (a_n)$ is unbounded above

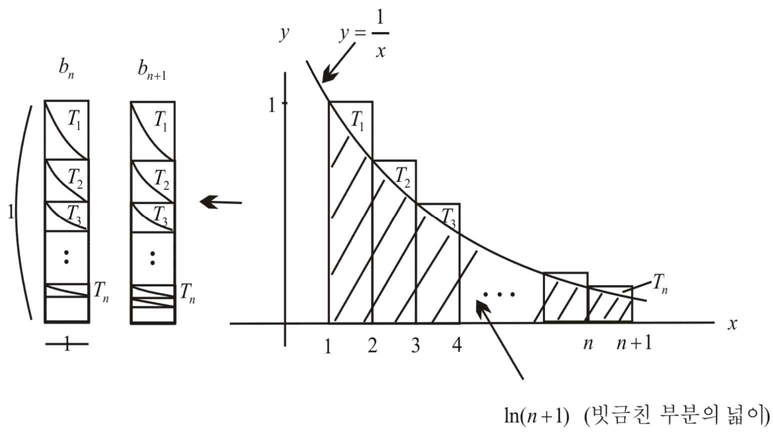
Proposition. $b_n \overset{\text{let}}{=} 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+1) \quad (n \geq 1)$

Then $\lim_{n\to\infty} b_n$ exists (usually, one write $\lim_{n\to\infty} b_n = \gamma$: called the Euler's constant)

Note: $c_n \overset{\text{let}}{=} 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n$

$$\Rightarrow \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n \quad (\because \lim_{n\to\infty} (\ln(n+1) - \ln n) = \lim_{n\to\infty} \ln(1+\frac{1}{n}) = \ln 1 = 0)$$

Pf. Suffices to prove that (b_n) is $\uparrow$ & bdd above



$$\begin{aligned} b_n &= \text{area of rectangles} - \text{area under the curve} \\ &= T_1 + T_2 + \cdots + T_n \end{aligned}$$

From the picture, we see

$$b_n < b_{n+1} \quad (\text{in fact, } b_{n+1} = b_n + T_{n+1}) \quad \& \quad b_n < 1$$

$$\therefore \lim_{n\to\infty} b_n \text{ exists}$$

It easy easy to see that $\frac{1}{2} < \gamma = \lim_{n\to\infty} b_n = T_1 + T_2 + \cdots + T_n + \cdots < 1$

It is known that $\gamma = 0.577\cdots$

Application: Estimate the size $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{999}$

Ans: $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{999} \approx \ln 1000 = 3 \cdot \ln 10 \approx 3 \times 2.3 = 6.9$

Remark. $0 <$ the error of the estimation <1 $(\because b_n < 1 \ \forall n)$

Open (for over 200 years):

It is not known whether γ is an algebraic number (an irrational number)

Algebraic number: a zero of some polynomial with integer coefficients

1.6 Decreasing sequences

Def. Let (a_n) be a decreasing sequence.

A number L, in a suitable decimal representation, is called the limit of (a_n) if

given any integer $k > 0$, $\exists$ an index N such that

for all $n \geq N$, $a_n = L$ to k decimal places.

That is, for each $k \in \mathbb{N}$, $\exists$ a nonnegative integer N such that

$$(a_n - L \;=)\; |\, a_n - L \,| < \frac{1}{10^k} \quad \text{for all } n \geq N$$

Def. A sequence (a_n) is said to be bounded below if $\exists$ a number C s.t. $a_n \geq C$ for all n.

(Any such C is called a lower bound for (a_n))

Theorem 1.6 If a ***positive*** seq (a_n) is $\downarrow$(dec) [$\Rightarrow$ bounded below], then $\lim\limits_{n\to\infty} a_n$ exists.

Pf. Repeat the argument in Theorem 1.3

Def. A sequence (a_n) is said to be bounded if it is bdd above & bdd below.

i.e., $\exists$ constants B & C such that $C \leq a_n \leq B$ for all n

Def. A sequence is called monotone if it is $\uparrow$(increasing) or $\downarrow$(decreasing) for all n

Remark.

ⓐ (extension of Thm 1.3) Any increasing sequence (a_n) which is bounded above has a limit

(i) all terms in (a_n) are > 0 $\Rightarrow$ already done (Theorem 1.3)

(ii) (a_n) has a term ≤ 0.

Case 1. (a_n) also contains a positive term a_N

Then $a_N, a_{N+1}, a_{N+2}, \cdots\cdots$: are all positive

$$\overset{\text{apply the argument for Theorem 1.3}}{\Rightarrow} \quad \text{OK}$$

Case 2. All terms in (a_n) are negative

Then $(-a_n)$ is positive & dec

$$\overset{\text{Thm1.6}}{\Rightarrow} \quad \lim_{n\to\infty} (-a_n) \text{ exists, } \quad \text{call it } L$$

$\therefore$ given any positive integer k, $\exists N = N(k)$ such that

for all $n \geq N$, $-a_n = L$ to k decimal places

i.e., for all $n \geq N$, $a_n = -L$ to k decimal places

$\therefore \lim\limits_{n\to\infty} a_n = -L$

Case 3. (a_n) contains no positive terms, but not all the terms are negative

Then the seq contains the term 0, but no positive terms.

Since (a_n) is $\uparrow$, all terms after 0 must be 0.

$$\therefore \lim_{n\to\infty} a_n = 0$$

ⓑ (extension of Thm 1.6) Any dec sequence (a_n) which is bounded below has a limit

(i) all terms in (a_n) are > 0 $\Rightarrow$ done (Theorem 1.6)

(ii) (a_n) contains a non-positive term can be handled similarly.

Thm 1.3 (& its extension) plus Thm 1.6 (& its extension) $\Rightarrow$

$$\boxed{\begin{array}{c}\text{Completeness Property of } \mathbb{R} : \\ \text{A bounded monotone sequence has a limit}\end{array}}$$

Completeness means that $\mathbb{R}$ has no holes (or gaps). For example,

$$\sqrt{2} \in \mathbb{R}$$

$$\underbrace{1, \quad 1.4, \quad 1.41, \quad 1.414, \quad \cdots}_{\text{rational numbers}} \quad \to \sqrt{2}$$

($\sqrt{2}$ can be regarded as the increasing limit of a sequence of **rational** numbers)

- 지금까지 실수를 이해하기 위한 접근방법: 다소 직관적

문제점: ① **단조 수열**인 경우에만 극한개념을 정의함

② 실수의 **decimal representation에 의존**함

해석학에서 가장 중요한 개념인 **estimation** ($\Delta \leq$ goal $\leq \square$) 과 **approximation** ($\approx$) 을 이용하여 위의 2가지 한계를 극복하게 될 것임