Example $y_1, \cdots, y_n \mid \mu, \sigma^2 \overset{iid}{\sim} N(\mu, \sigma^2)$, where both μ(real) and $\sigma(>0)$ are unknown.

Consider the noninformative prior $p(\mu, \sigma) \propto \sigma^{-1}$

- Then the joint posterior is

$$p(\mu, \sigma \mid y) \propto p(\mu, \sigma)\, p(y \mid \mu, \sigma)$$

$$\propto \sigma^{-1} \sigma^{-n} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2 \right]$$

$$= \sigma^{-(n+1)} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \bar{y} + \bar{y} - \mu)^2 \right]$$

$$= \sigma^{-(n+1)} \exp\left[-\frac{1}{2\sigma^2} \left\{ \sum_{i=1}^{n} (y_i - \bar{y})^2 + n(\bar{y} - \mu)^2 \right\} \right]$$

$$= \sigma^{-(n+1)} \exp\left[-\frac{1}{2\sigma^2} \left\{ (n-1)s^2 + n(\bar{y} - \mu)^2 \right\} \right] \quad \text{where } s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2$$

- The marginal posterior of μ is (integrating w.r.to σ)

$$p(\mu \mid y) = \int p(\mu, \sigma \mid y)\, d\sigma$$

$$\propto \int_0^\infty \sigma^{-(n+1)} \exp\left[-\frac{1}{2\sigma^2} \left\{ n(\bar{y} - \mu)^2 + (n-1)s^2 \right\} \right] d\sigma$$

$$\underset{\substack{\sigma^2 = z^{-1} \\ d\sigma = -\frac{1}{2} z^{-\frac{3}{2}} dz}}{=} \int_0^\infty z^{\frac{n+1}{2}} \exp\left[-\frac{z}{2} \left\{ n(\bar{y} - \mu)^2 + (n-1)s^2 \right\} \right] \frac{1}{2} z^{-\frac{3}{2}}\, dz$$

$$= \int_0^\infty \frac{1}{2} z^{\frac{n}{2}-1} \exp\left[-\frac{n(\bar{y}-\mu)^2 + (n-1)s^2}{2} z \right] dz$$

$$\text{Gamma}\left(\frac{n}{2}, \frac{n(\bar{y}-\mu)^2 + (n-1)s^2}{2} \right)$$

$$= \frac{1}{2} \frac{\Gamma(\frac{n}{2})}{\left(\frac{n(\bar{y}-\mu)^2 + (n-1)s^2}{2} \right)^{\frac{n}{2}}}$$

$$\propto \left(n(\bar{y}-\mu)^2 + (n-1)s^2 \right)^{-\frac{n}{2}}$$

$$\propto \left(1 + \frac{n(\mu - \bar{y})^2}{(n-1)s^2} \right)^{-\frac{n}{2}}$$

which is Student's t with location component $\bar{y}$, scale component $\frac{s}{\sqrt{n}}$ and df $= n-1$

student-t dist ($\theta \sim t_\nu(\mu, \sigma^2)$)

$$p(\theta) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\nu/2)\sqrt{\nu\pi}\,\sigma} \left(1 + \frac{1}{\nu} \left(\frac{\theta - \mu}{\sigma} \right)^2 \right)^{-(\nu+1)/2}$$

$E(\theta) = \mu$ for $\nu > 1$

$Var(\theta) = \frac{\nu}{\nu - 2} \sigma^2$ for $\nu > 2$.

• The posterior predictive distribution for a future observation $\tilde{y}$ is

$$P(\tilde{y}|y) = \int P(\tilde{y}|\theta, y)\, P(\theta|y)\, d\theta$$

$$= \int_0^\infty \int_{-\infty}^\infty P(\tilde{y}|\mu, \sigma, y)\, P(\mu, \sigma|y)\, d\mu\, d\sigma$$

$$\propto \int_0^\infty \int_{-\infty}^\infty \sigma^{-1} \exp\left[-\frac{1}{2\sigma^2}(\tilde{y}-\mu)^2\right] \sigma^{-(n+1)} \exp\left[-\frac{1}{2\sigma^2}\left\{ n(\mu-\bar{y})^2 + (n-1)s^2 \right\}\right] d\mu\, d\sigma$$

$$= \int_0^\infty \int_{-\infty}^\infty \sigma^{-(n+2)} \exp\left[-\frac{1}{2\sigma^2}\left\{ (\tilde{y}-\mu)^2 + n(\mu-\bar{y})^2 + (n-1)s^2 \right\}\right] d\mu\, d\sigma$$

Note) $(\tilde{y}-\mu)^2 + n(\mu-\bar{y})^2 = (n+1)\mu^2 - 2(\tilde{y}+n\bar{y})\mu + \tilde{y}^2 + n\bar{y}^2$

$$= (n+1)\left(\mu^2 - 2\frac{\tilde{y}+n\bar{y}}{n+1}\mu\right) + \tilde{y}^2 + n\bar{y}^2$$

$$= (n+1)\left(\mu - \frac{\tilde{y}+n\bar{y}}{n+1}\right)^2 - \frac{(\tilde{y}+n\bar{y})^2}{n+1} + \tilde{y}^2 + n\bar{y}^2$$

$$= (n+1)\left(\mu - \frac{\tilde{y}+n\bar{y}}{n+1}\right)^2 + \frac{n}{n+1}(\tilde{y}-\bar{y})^2$$

$$P(\tilde{y}|y) = \int_0^\infty \sigma^{-(n+2)} \exp\left[-\frac{1}{2\sigma^2}\left\{\frac{n}{n+1}(\tilde{y}-\bar{y})^2 + (n-1)s^2\right\}\right] \int_{-\infty}^\infty \exp\left[-\frac{n+1}{2\sigma^2}\left(\mu - \frac{\tilde{y}+n\bar{y}}{n+1}\right)^2\right] d\mu\, d\sigma$$

Normal$\left(\frac{\tilde{y}+n\bar{y}}{n+1}, \frac{\sigma^2}{n+1}\right)$

$$= \int_0^\infty \sigma^{-(n+2)} \exp\left[-\frac{1}{2\sigma^2}\left\{\frac{n}{n+1}(\tilde{y}-\bar{y})^2 + (n-1)s^2\right\}\right] \sqrt{2\pi}\sqrt{\frac{\sigma^2}{n+1}}\, d\sigma$$

$$\propto \int_0^\infty \sigma^{-(n+1)} \exp\left[-\frac{1}{2\sigma^2}\left\{\frac{n}{n+1}(\tilde{y}-\bar{y})^2 + (n-1)s^2\right\}\right] d\sigma$$

$$\underset{\sigma^2 = z^{-1}}{=} \int_0^\infty z^{\frac{n+1}{2}} \exp\left[-\frac{z}{2}\left\{\frac{n}{n+1}(\tilde{y}-\bar{y})^2 + (n-1)s^2\right\}\right] \frac{1}{2} z^{-\frac{3}{2}}\, dz$$

$$= \int_0^\infty \frac{1}{2} z^{\frac{n}{2}-1} \exp\left[-\frac{\frac{n}{n+1}(\tilde{y}-\bar{y})^2 + (n-1)s^2}{2} z\right] dz$$

Gamma$\left(\frac{n}{2}, \frac{\frac{n}{n+1}(\tilde{y}-\bar{y})^2 + (n-1)s^2}{2}\right)$

$$= \frac{1}{2} \frac{\Gamma(\frac{n}{2})}{\left(\frac{\frac{n}{n+1}(\tilde{y}-\bar{y})^2 + (n-1)s^2}{2}\right)^{\frac{n}{2}}}$$

$$\propto \left(\frac{n}{n+1}(\tilde{y}-\bar{y})^2 + (n-1)s^2\right)^{-\frac{n}{2}}$$

$$\propto \left(1 + \frac{(\tilde{y}-\bar{y})^2}{(1+\frac{1}{n})(n-1)s^2}\right)^{-\frac{n}{2}}$$

which is Student's t with location $\bar{y}$, Scale $\sqrt{1+\frac{1}{n}}\, s$ and df $n-1$.

- The marginal posterior of σ is (integrating w.r.to μ)

$$P(\sigma|y) = \int p(\mu,\sigma|y)\, d\mu$$

$$\propto \int_{-\infty}^{\infty} \sigma^{-(n+1)} \exp\left[-\frac{1}{2\sigma^2}\left\{(n-1)s^2 + n(\bar{y}-\mu)^2\right\}\right] d\mu$$

$$= \sigma^{-(n+1)} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right) \int_{-\infty}^{\infty} \exp\left(-\frac{n}{2\sigma^2}(\mu-\bar{y})^2\right) d\mu$$

Normal $(\bar{y}, \frac{\sigma^2}{n})$

$$= \sigma^{-(n+1)} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right) \sqrt{2\pi} \sqrt{\frac{\sigma^2}{n}}$$

$$\propto \sigma^{-n} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right)$$

Hence, writing $z = \sigma^{-2}$,

$$P(z|y) \propto z^{\frac{n-1}{2}-1} \exp\left(-\frac{(n-1)s^2}{2} z\right)$$

which is Gamma with shape parameter $\frac{n-1}{2}$ and failure rate $\frac{1}{2}\sum_{i=1}^{n}(y_i-\bar{y})^2$

- Now $\mu|\sigma, y \sim N(\bar{y}, \frac{\sigma^2}{n})$

Given μ and y, σ has conditional pdf

$$P(\sigma|\mu,y) = \frac{P(\sigma|y)\,P(\mu|\sigma,y)}{P(\mu|y)}$$

$$\propto \sigma^{-n} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\bar{y})^2\right) \sigma^{-1} \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right)$$

$$= \sigma^{-(n+1)} \exp\left[-\frac{1}{2\sigma^2}\left\{\sum_{i=1}^{n}(y_i-\bar{y})^2 + n(\bar{y}-\mu)^2\right\}\right]$$

$$= \sigma^{-(n+1)} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2\right)$$

Hence, writing $z = \sigma^{-2}$,

$$P(z|\mu,y) \propto z^{\frac{n}{2}-1} \exp\left(-\frac{\sum_{i=1}^{n}(y_i-\mu)^2}{2} z\right)$$

which is Gamma with shape parameter $\frac{n}{2}$ and failure rate $\frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)^2$