1. (20 points) If X_i, $i = 1, 2, 3$ are independent exponential random variables with rates λ_i, $i = 1, 2, 3$, find

   (a) $P(X_1 < X_2 < X_3)$

   (b) $P(X_1 < X_2 | \max(X_1, X_2, X_3) = X_3)$

   (c) $E(\min(X_1, X_2, X_3))$

   (d) $\mathrm{Var}(\min(X_1, X_2, X_3))$

a) $P(X_1 < X_2 < X_3)$

$$= P(X_2 < X_3 \mid X_1 = \min\{X_1, X_2, X_3\}) \cdot P(X_1 = \min\{X_1, X_2, X_3\})$$

$$= P(X_2 < X_3) \cdot P(X_1 = \min\{X_1, X_2, X_3\})$$

$$= \frac{\lambda_2}{\lambda_2 + \lambda_3} \cdot \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}$$

b) $P(X_1 < X_2 \mid \max(X_1, X_2, X_3) = X_3)$

$$= \frac{P(X_1 < X_2,\ \max(X_1, X_2, X_3) = X_3)}{P(\max(X_1, X_2, X_3) = X_3)}$$

$$= \frac{P(X_1 < X_2 < X_3)}{P(X_1 < X_2 < X_3) + P(X_2 < X_1 < X_3)}$$

$$= \frac{\frac{\lambda_2}{\lambda_2+\lambda_3} \cdot \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3}}{\frac{\lambda_2}{\lambda_2+\lambda_3} \cdot \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} + \frac{\lambda_1}{\lambda_1+\lambda_3} \cdot \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3}} \quad (\text{from a})$$

$$= \frac{\lambda_1 + \lambda_3}{\lambda_1 + \lambda_2 + 2\lambda_3}$$

c) $E(X) = \dfrac{1}{\lambda_1 + \lambda_2 + \lambda_3}$ $\quad (\because \min(X_1, X_2, X_3) \sim Exp(\lambda_1 + \lambda_2 + \lambda_3))$

d) $Var(X) = \dfrac{1}{(\lambda_1 + \lambda_2 + \lambda_3)^2}$

2. (10 points) A doctor has scheduled two appointments, one at 1:00 P.M. and the other at 1:30 P.M. The amounts of time that appointments last are independent exponential random variables with mean 30 minutes. Assuming that both patients are on time, find the expected amount of time that the 1:30 appointment spends at the doctor's office.

T_i : Amount of time that ith appointment lasts, $T_i \overset{iid}{\sim} Exp(\frac{1}{30})$

$E(T_2) = E(T_2 | T_1 > 30) + E(T_2 | T_1 < 30)$

$= 60e^{-1} + 30(1-e^{-1})$

$= 30 + 30e^{-1}$

3. (10 points) A businessman parks his car illegally in the streets for a period of exactly two hours. Parking surveillances occur according to a Poisson process with an average of λ passes per hour. What is the probability of the businessman getting a fine on a given day?

Parking surveillances : $N(t) \sim Poi(\lambda t)$

$P(N(2) \geq 1) = 1 - P(N(2) = 0)$

$= 1 - \frac{e^{-2\lambda}(2\lambda)^0}{0!}$

$= 1 - e^{-2\lambda}$

4. (15 points) Suppose that people arrive at a bus stop in accordance with a Poisson process with rate λ. The bus departs at time t. Let X denote the total amount of waiting time of all those who get on the bus at time t. Let $N(t)$ denote the number of arrivals by time t.

   (a) $E(X|N(t))$

   (b) $\mathrm{Var}(X|N(t))$

   (c) $\mathrm{Var}(X)$

$N(t) \sim Poi(\lambda t)$, $X = \sum_{i=1}^{N(t)} Y_i$ where Y_i : waiting time

a) $E(X|N(t)) = E\left(\sum_{i=1}^{N(t)} Y_i \,\middle|\, N(t)\right) = N(t)\, E(Y_1)$

b) $Var(X|N(t)) = Var\left(\sum_{i=1}^{N(t)} Y_i \,\middle|\, N(t)\right) = N(t)\, Var(Y_1)$

c) $Var(X) = Var[E(X|N(t))] + E[Var(X|N(t))]$

$= Var[N(t)E(Y_1)] + E[N(t)Var(Y_1)]$

$= \lambda t\, E(Y_1)^2 + \lambda t\, Var(Y_1)$

$= \lambda t\,(E(Y_1)^2 + Var(Y_1))$

5. (10 points) Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ that is independent of the nonnegative random variable T with mean μ and variance σ^2. Find

(a) $\mathrm{Cov}(T, N(T))$

(b) $\mathrm{Var}(N(T))$

$N(T) | T \sim Poi(\lambda T)$ where $E(T) = \mu$, $var(T) = \delta^2$

a) $COV(T, N(T)) = E(TN(T)) - E(T)E(N(T))$

$$E(TN(T)) = E[E(TN(T)|T)]$$
$$= E[TE(N(T)|T)]$$
$$= E(\lambda T^2)$$
$$= \lambda(\mu^2 + \delta^2)$$

$$E(N(T)) = E[E(N(T)|T)]$$
$$= E(\lambda T)$$
$$= \lambda\mu$$

$\therefore$ $COV(T, N(T)) = E(TN(T)) - E(T)E(N(T)) = \lambda(\mu^2 + \delta^2) - \lambda\mu^2 = \lambda\delta^2$

b) $Var(N(T)) = Var[E(N(T)|T)] + E[Var(N(T)|T)]$

$$= Var(\lambda T) + E(\lambda T)$$
$$= \lambda^2\delta^2 + \lambda\mu$$

6. (10 points) For a standard Brownian motion $\{B(t), t \geq 0\}$, and $0 \leq s \leq t$, find

(a) $E(B(t)|B(s) = y)$

(b) Variance of $B(t) - tB(1)$, $t \in [0, 1]$.

a) $E(B(t)|B(s) = y) = E(B(t) - B(s) + B(s) | B(s) = y)$

$$= E(B(t) - B(s)) + y$$
$$= E(B(t-s)) + y$$
$$= 0 + y$$
$$= y$$

b) $Var(B(t) - B(s)) = Var(B(t) - B(t^2))$

$$= Var(B(t - t^2))$$
$$= t - t^2$$
$$= t(1-t)$$

7. (10 points) Consider a random walk

$$X_t = \sum_{k=1}^{t} Z_k, \quad X_0 = 0, \quad t = 1, 2, \dots,$$

and $\{Z_i\}$'s are i.i.d. with $P(Z_k = 1) = p$, $P(Z_k = -1) = 1 - p$, $p \in (0, 1)$. Find

(a) $P(X_4 = 0)$

(b) $P(Z_2 = 1 | X_3 = 1)$

a) $P(X_4 = 0) = P(Z_1 + Z_2 + Z_3 + Z_4 = 0)$

$= {}_4C_2 \, p^2 (1-p)^2$

$= 6p^2(1-p)^2$

b) $P(Z_2 = 1 | X_3 = 1) = P(Z_2 = 1 | \sum_{k=1}^{3} Z_k = 1)$

$$= \frac{P(Z_2 = 1, \ Z_1 + Z_3 = 0)}{P(Z_1 + Z_2 + Z_3 = 1)}$$

$$= \frac{p \cdot {}_2C_1 \cdot p(1-p)}{{}_3C_2 \, p^2 (1-p)}$$

$$= \frac{2}{3}$$

8. (15 points) Consider a random walk

$$X_t = \sum_{k=1}^{t} Z_k, \quad X_0 = 0, \quad t = 1, 2, \dots,$$

and $\{Z_i\}$'s are i.i.d. and **symmetric** random variables. Show that

$$P\left(\max_{0 \le i \le t} |X_i| \ge a\right) \le 2P(|X_t| > a).$$

Let $T_a := \min_t (X_t \ge a)$,

$P(X_t > a) = P(X_t > a | T_a \le t) P(T_a \le t) + P(X_t > a | T_a > t) P(T_a > t)$

$= P(X_t > a | T_a \le t) P(T_a \le t)$

$= \frac{1}{2} P(T_a \le t)$

$\therefore P(X_t > a) = \frac{1}{2} P(T_a \le t)$ ⋯①

$P(\max |X_i| \ge a) \le P(\max X_i \ge a) + P(\max -X_i \ge a)$, $P(|X_t| > a) = 2P(X_t > a)$ $(\because$ Symm$)$ ⋯③

$\le 2P(\max X_i \ge a)$ $(\because$ Symm$)$ ⋯②

$P(\max |X_i| \ge a) \overset{②}{\le} 2 \cdot P(\max X_i \ge a) = 2 \cdot P(T_a \le t) \overset{①}{=} 4 \cdot P(X_t > a) \overset{③}{=} 2P(|X_t| > a)$

$\therefore P(\max |X_i| \ge a) \le 2P(|X_t| > a)$