1) $\sum a_n$ conditionally converges $\Rightarrow \sum a_n^+ = \infty$ & $\sum a_n^- = \infty$

Proof by contrapositive : i) If $\sum a_n^+ = L$ & $\sum a_n^- = \infty$, then $\sum a_n = \sum a_n^+ - \sum a_n^- = -\infty$ diverges

ii) If $\sum a_n^+ = \infty$ & $\sum a_n^- = L'$, then $\sum a_n = \sum a_n^+ - \sum a_n^- = \infty$ diverges

iii) If $\sum a_n^+ = L$ & $\sum a_n^- = L'$, then $\sum a_n = L - L'$ is absolutely convergent

2) $\sum (-1)^n a_n = \sum_{n=2}^{\infty} (-1)^n (\sqrt[n]{n} - 1) = \sum a_n^+ + \sum a_n = \sum_{k=2}^{\infty} \sqrt[2k]{2k} - 1 - \left\{ \sum_{k=3}^{\infty} \sqrt[2k+1]{2k+1} - 1 \right\}$

i) By the Cauchy's test for alternating series, $\sum_{n=2}^{\infty} (-1)^n (\sqrt[n]{n} - 1)$ converges

ii) $\left| \sum (-1)^n a_n \right| = \sum_{n=2}^{\infty} \sqrt[n]{n} - 1 \sim \sum_{n=2}^{\infty} \sqrt[n]{n}$ diverges (Note that $\lim_{n\to\infty} \frac{|\sqrt[n]{n} - 1|}{|\sqrt[n]{n}|} = 1$)

$\therefore$ By the Asymptotic Comparison Test, $\sum_{n=2}^{\infty} \sqrt[n]{n} - 1$ diverges.

Since (i) converges but (ii) diverges, $\sum_{n=2}^{\infty} (-1)^n (\sqrt[n]{n} - 1)$ is conditionally convergent

3) i) $\sum_{n=1}^{\infty} \frac{\cos n}{n}$

pf) $\sum_{n=1}^{\infty} \frac{\cos n}{n} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)(\cos n)$ so we may apply Dirichlet Test.

(1) $a_n = \frac{1}{n}$ is a monotonically decreasing function and $a_n > 0$ for all n.

(2) We need to prove $\sum_{n=1}^{\infty} \cos n$ is bounded

First, note Euler's formulas, $\cos(x) = (e^{ix} + e^{-ix})/2$ and $\sin(x) = (e^{ix} - e^{-ix})/2i$

$$\Rightarrow \sum_{x=1}^{n} \cos(x) = \frac{e^{i} + e^{-i}}{2} + \frac{e^{2i} + e^{-2i}}{2} + \cdots + \frac{e^{ni} + e^{-ni}}{2}$$

$$= \frac{1}{2} \Big\{ \underbrace{e^{-ni} + e^{-(n-1)i} + \cdots + e^{-i} + 1 + e^{i} + e^{2i} + \cdots + e^{(n-1)i} + e^{ni}}_{= \sum_{x=-n}^{n} e^{xi} = \frac{e^{-ni}(e^{(2n+1)i} - 1)}{e^{i} - 1}}) - 1 \Big\}$$

$$= \frac{1}{2} \left\{ \frac{e^{-ni}(e^{(2n+1)i} - 1)}{e^{i} - 1} - 1 \right\}$$

$$= \frac{1}{2} \left\{ \frac{e^{(n+1)i} - e^{-ni} - e^{i} + 1}{e^{i} - 1} \right\} \times \left(\frac{2i\, e^{-\frac{1}{2}i}}{2i\, e^{-\frac{1}{2}i}} \right)$$

$$= \frac{1}{2} \left\{ \frac{e^{(n+\frac{1}{2})i} - e^{-(n+\frac{1}{2})i}}{2i} - \frac{e^{\frac{1}{2}i} - e^{-\frac{1}{2}i}}{2i} \right\} \Big/ \left\{ \frac{e^{\frac{1}{2}i} - e^{-\frac{1}{2}i}}{2i} \right\}, \quad \text{where } \sin(x) = (e^{ix} - e^{-ix})/2i$$

$$= \frac{1}{2} \left\{ \frac{\sin(n + \frac{1}{2}) - \sin(\frac{1}{2})}{\sin(\frac{1}{2})} \right\}$$

$$= \frac{\cos(\frac{n+1}{2}) \sin(\frac{n}{2})}{\sin(\frac{1}{2})} \leq \frac{1}{\sin(\frac{1}{2})} = M$$

$\therefore$ The two conditions for Dirichlet Test are satisfied, thus $\sum_{n=1}^{\infty} \frac{\cos n}{n}$ converges

ii) $\sum_{n=1}^{\infty} \frac{\sin n}{n}$

pf) $\sum_{n=1}^{\infty} \frac{\sin n}{n} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)(\sin n)$ so we may apply Dirichlet Test.

(1) $a_n = \frac{1}{n}$ is a monotonically decreasing function and $a_n > 0$ for all n.

(2) We need to prove $\sum_{n=1}^{\infty} \sin n$ is bounded

First, note that $2\sin(\alpha)\sin(\beta) = -\cos(\alpha+\beta) + \cos(\alpha-\beta)$

$$\sum_{n=1}^{\infty} \sin n = \frac{1}{2\sin(1)} \sum_{n=1}^{\infty} 2\sin(1) \cdot \sin n$$

$$= \frac{1}{2\sin(1)} \sum_{n=1}^{\infty} \cos(n-1) - \cos(n+1)$$

$$= \frac{1}{2\sin(1)}\left\{[\cos(0)-\cos(2)]+[\cos(1)-\cos(3)]+\cdots[\cos(n-2)-\cos(n)]+[\cos(n-1)-\cos(n+1)]\right\}$$
$$= \frac{1}{2\sin(1)}\left\{\cos(0)+\cos(1)-\cos(n)-\cos(n+1)\right\}$$

Using the triangular inequalities,

$$\left|\sum_{n=1}^{\infty}\sin n\right| = \frac{1}{2\sin(1)}\left\{\left|\cos(0)+\cos(1)-\cos(n)-\cos(n+1)\right|\right\}$$
$$\leq \frac{1}{2\sin(1)}\left\{|\cos(0)|+|\cos(1)|+|-\cos(n)|+|-\cos(n+1)|\right\}$$
$$\leq \frac{1}{2\sin(1)}\{4\}$$
$$= \frac{2}{\sin(1)} = M$$

$\therefore$ The two conditions for Dirichlet Test are satisfied, thus $\sum_{n=1}^{\infty}\frac{\sin n}{n}$ converges

4) $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} a_n \frac{n^{2021}}{e^n}$

$\Rightarrow \lim_{n\to\infty}\frac{b_{n+1}}{b_n} = \lim_{n\to\infty}\frac{e^n(n+1)^{2021}}{e^{n+1}\,n^{2021}} = e^{-1}\left\{\lim_{n\to\infty}\frac{(n+1)^{2021}}{n^{2021}}\right\} = e^{-1} < 1 \quad \therefore \sum_{n=1}^{\infty}\frac{n^{2021}}{e^n}$ converges thus bounded

$\therefore$ Since both a_n is convergent and $\sum_{n=1}^{\infty}\frac{n^{2021}}{e^n}$ is bounded, by Dirichlet Test,
$\sum_{n=1}^{\infty} a_n \frac{n^{2021}}{e^n}$ converges