Time Series Analysis (STA 5015)
Chapter 2 Solution

1. Problem 2.3

   a. Note that

$$\mathrm{Cov}(X_{t+h}, X_t) = \mathrm{Cov}(Z_{t+h} + .3Z_{t+h-1} - .4Z_{t+h-2}, Z_t + .3Z_{t-1} - .4Z_{t-2})$$

$$= \gamma_Z(h) + .3\gamma_Z(h-1) - .4\gamma_Z(h-2) + .3\gamma_Z(h+1) + .09\gamma_Z(h) - .12\gamma_Z(h-1)$$
$$-.4\gamma_Z(h+2) - .12\gamma_Z(h+1) + .16\gamma_Z(h)$$
$$= 1.25\gamma_Z(h) + .18\gamma_Z(h-1) - .4\gamma_Z(h-2) + .18\gamma_Z(h+1) - .4\gamma_Z(h+2),$$

   where $\gamma_Z(h)$ is ACVF of WN(0,1), that is $\gamma_Z(h) = 1$ if $h = 1$ and 0 otherwise. Therefore, ACVF of X is given by

$$\gamma_X(h) = \begin{cases} 1.25, & h = 0 \\ .18, & h = \pm 1 \\ -.4, & h = \pm 2 \\ 0 & \text{otherwise.} \end{cases}$$

   Or, you can plug-in general formula introduced in Proposition 2.2.1.

   b. Similar calculation with $\gamma_{\tilde{Z}}(h) = .25$ if $h = 1$ and 0 otherwise gives that

$$\gamma_Y(h) = \begin{cases} 1.25, & h = 0 \\ .18, & h = \pm 1 \\ -.4, & h = \pm 2 \\ 0 & \text{otherwise.} \end{cases}$$

   Therefore, it has the **same** ACVF with that of part a. This shows that ACVF does not uniquely determine stationary process.

2. (a) Let $\gamma_X(h)$ be the ACVF of $\{X_t\}$ Then,

$$\mathrm{Cov}(Y_{t+h}, Y_t) = \begin{cases} \mathrm{Cov}(X_{t+h}, X_t) & \text{if } t+h \text{ odd, } t \text{ odd,} \\ \mathrm{Cov}(X_{t+h}, X_t + 3) & \text{if } t+h \text{ odd, } t \text{ even,} \\ \mathrm{Cov}(X_{t+h} + 3, X_t) & \text{if } t+h \text{ even, } t \text{ odd,} \\ \mathrm{Cov}(X_{t+h} + 3, X_t + 3) & \text{if } t+h \text{ even, } t \text{ even,} \end{cases}$$

$$= \begin{cases} \gamma_X(h) & \text{if } t+h \text{ odd, } t \text{ odd,} \\ \gamma_X(h) & \text{if } t+h \text{ odd, } t \text{ even,} \\ \gamma_X(h) & \text{if } t+h \text{ even, } t \text{ odd,} \\ \gamma_X(h) & \text{if } t+h \text{ even, } t \text{ even,} \end{cases}$$

   Therefore, ACVF of $\{Y_t\}$ does not depends of t.

(b) However, $\{Y_t\}$ is NOT a stationary process since

$$EY_t = \begin{cases} EX_t = 0 & \text{if } t \text{ is odd,} \\ EX_t + 3 = 3 & \text{if } t \text{ is even.} \end{cases}$$

implies that EY_t depends on t.

3. First note that

$$EX_t = E\sin(2\pi Ut) = \int_0^1 \sin(2\pi ut)du = \frac{-\cos(2\pi ut)}{2\pi t}\Big|_0^1 = \frac{-\cos(2\pi t) + \cos 0}{2\pi t} = 0,$$

since t is an integer value. For the covariance calculation, observe that

$$\mathrm{Cov}(X_{t+h}, X_t) = E(X_{t+h}X_t) = E(\sin(2\pi U(t+h))\sin(2\pi Ut)).$$

Therefore, if $h = 0$, then

$$\mathrm{Var}(X_t) = E(\sin^2(2\pi Ut)) = \int_0^1 \sin^2(2\pi ut)du = \int_0^1 \frac{1-\cos(4\pi ut)}{2}du$$
$$= \frac{1}{2}\left(u - \frac{1}{4\pi t}\sin(4\pi ut)\Big|_0^1\right) = \frac{1}{2}$$

If $h \neq 0$, then

$$\mathrm{Cov}(X_{t+h}, X_t) = \int_0^1 \sin(2\pi u(t+h))\sin(2\pi ut)du = \int_0^1 \frac{1}{2}\left\{\cos(2\pi uh) - \cos(2\pi u(2t+h))\right\}du$$
$$= \frac{1}{2}\left(\frac{1}{2\pi h}\sin(2\pi uh) - \frac{1}{2\pi(2t+h)}\sin(2\pi u(2t+h))\Big|_0^1\right)$$
$$= \frac{1}{2}\left(\frac{\sin(2\pi h) - \sin 0}{2\pi h} - \frac{\sin(2\pi(2t+h)) - \sin 0}{2\pi(2t+h)}\right) = 0.$$

Hence, $\{X_t\}$ is a weakly stationary time series.

4. (a) Since $X_2 - \rho(\sigma_2/\sigma_1)X_1$ is the linear combination of X_1 and X_2, it follows normal distribution (recall HW#1) with mean

$$E(X_2 - \rho(\sigma_2/\sigma_1)X_1) = E(X_2) - \rho(\sigma_2/\sigma_1)E(X_1) = \mu_2 - \rho(\sigma_2/\sigma_1)\mu_1.$$

$$\mathrm{Var}(X_2 - \rho(\sigma_2/\sigma_1)X_1) = \mathrm{Cov}(X_2 - \rho(\sigma_2/\sigma_1)X_1, X_2 - \rho(\sigma_2/\sigma_1)X_1)$$
$$= \mathrm{Var}(X_2) - 2\rho(\sigma_2/\sigma_1)\mathrm{Cov}(X_1, X_2) + (\rho(\sigma_2/))^2\mathrm{Var}(X_1)$$
$$= \sigma_2^2 - 2\rho^2\sigma_2^2 + \rho^2\sigma_2^2 = (1-\rho^2)\sigma_2^2.$$

All in all,

$$X_2 - \rho\frac{\sigma_2}{\sigma_2}X_1 \sim \mathcal{N}(\mu_2 - \rho(\sigma_2/\sigma_1)\mu_1, (1-\rho^2)\sigma^2).$$

(b) For multivariate normal distribution, zero correlation implies independence. Note that

$$\begin{aligned}\mathrm{Cov}(X_2 - \rho\sigma_2/\sigma_1 X_1, X_1) &= \mathrm{Cov}(X_2, X_1) - \rho(\sigma_2/\sigma_1)\mathrm{Cov}(X_1, X_1)\\ &= \rho\sigma_1\sigma_2 - \rho(\sigma_2/\sigma_1)\sigma_1^2 = 0,\end{aligned}$$

in turn $X_2 - \rho\sigma_2/\sigma_1 X_1$ and X_1 are independent.

5. In a compact form, we have

$$(1-.5B)X_t = (1+.5B)Z_t \Rightarrow \psi(B) = (1+.5B)(1-.5B)^{-1}, \quad \pi(B) = (1-.5B)(1+.5B)^{-1}.$$

Using geometric sum, we have

$$\psi(B) = (1+.5B)(1+.5B+.5^2B^2+\ldots),$$

thus

$$\psi_1 = 1, \psi_2 = 2(.5)^2, \psi_3 = 2(.5)^3, \psi_4 = 2(.5)^4, \psi_5 = 2(.5)^5.$$

Alternatively (which I prefer) is to use identity

$$(1+\psi_1 B+\psi_2 B^2+\psi_3 B^3+\ldots)(1-.5B) = 1+.5B$$

Similar calculation for $\pi(B)$ gives that

$$\pi_1 = -1, \pi_2 = 2(-.5)^2, \pi_3 = 2(-.5)^3, \pi_4 = 2(-.5)^4, \pi_5 = 2(-.5)^5.$$

6. DIY

7. Problem 2.12
For MA(1) process $X_t = Z_t - .6Z_{t-1}$, note that

$$\gamma_X(h) = \begin{cases} (1+\theta^2)\sigma^2 & , h=0 \\ \theta\sigma^2 & , h=\pm 1 \\ 0 & , \text{ o.w} \end{cases} = \begin{cases} 1.36 & , h=0 \\ -.6 & , h=\pm 1 \\ 0 & , \text{ o.w} \end{cases}$$

Thus, long-run variance is given by

$$\nu = \sum_{h=-\infty}^{\infty} \gamma_X(h) = 1.36 + 2(-.6) = .16.$$

The 95% confidence interval is therefore given by

$$\overline{X} \pm 1.96\sqrt{\frac{\nu}{n}} = (.0786, .2354).$$

Since 0 is not include in the confidence interval we reject the null hypotheses of $\mathrm{H}_0 : \mu = 0$ under 5% significance level.

8. Problem 2.13

(a) Recall that AR(1) model $X_t = \rho X_{t-1} + Z_t$ has ACVF and ACF by

$$\gamma_X(h) = \frac{\phi^{|h|}}{1-\phi^2}\sigma^2, \quad \rho_X(h) = \phi^{|h|}.$$

Thus, plug-into Bartlett's formula gives

$$\begin{aligned} w_{ii} &= \sum_{k=1}^{i} \phi^{2i}(\phi^k - \phi^{-k})^2 + \sum_{k=i+1}^{\infty} \phi^{2k}(\phi^i + \phi^{-i})^2 \\ &= (1-\phi^{2i})(1+\phi^2)(1-\phi^2)^{-1} - 2i\phi^{2i}. \end{aligned}$$

For example,

$$w_{11} = (1-\phi^2)(1+\phi^2)(1-\phi^2)^{-1} - 2\phi^2 = 1-\phi^2 = .36$$

$$w_{22} = (1-\phi^4)(1+\phi^2)(1-\phi^2)^{-1} - 4\phi^4 = (1+\phi^2)^2 - 4\phi^4 = 1.0512$$

Therefore, 95% confidence interval for $\rho(i)$ is calculated as

$$\widehat{\rho}(1) \pm \frac{1.96}{\sqrt{n}}\sqrt{w_{11}} = (.3204, .5556)$$

$$\widehat{\rho}(2) \pm \frac{1.96}{\sqrt{n}}\sqrt{w_{22}} = (-.056, .346)$$

Since CI do not include the true values of $\rho(1) = .8$ and $\rho(2) = .64$, the data are not consistent with an AR(1) model with $\phi = .8$.

(b) For MA(1) model $X_t = Z_t + \theta Z_{t-1}$,

$$\rho_X(h) = \begin{cases} 1 & , h = 0 \\ \frac{\theta}{1+\theta^2} & , h = \pm 1 \\ 0 & , \text{o.w} \end{cases}$$

Thus,

$$w_{11} = 1 - 3\rho(1)^2 + 4\rho(1)^4 = .5676, \quad w_{22} = 1 + 2\rho(1)^2 = 1.3893.$$

95% confidence interval is

$$\widehat{\rho}(1) \pm 1.96\frac{1.96}{\sqrt{n}}\sqrt{w_{11}} = (.290, .585)$$

$$\widehat{\rho}(2) \pm \frac{1.96}{\sqrt{n}}\sqrt{w_{22}} = (-.086, .376)$$

Now, confidence intervals contain true values of $\rho(1) = .441$ and $\rho(2) = 0$, thus data is consistent with an MA(1) model with $\theta = .6$.

9. (a) We will use trigonometric identity

$$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta)+\cos(\alpha-\beta)\right)$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right)$$

Note that

$$(2\cos\omega)X_{n-1} - X_{n-2}$$
$$= (2\cos\omega)\left(A\cos(\omega(n-1)) + B\sin(\omega(n-1))\right) - A\cos(\omega(n-2)) - B\sin(\omega(n-2))$$
$$= 2A\cos\omega\cos(\omega(n-1)) + 2B\cos\omega\sin(\omega(n-1)) - A\cos(\omega(n-2)) - B\sin(\omega(n-2))$$
$$= A\left(\cos(\omega n) + \cos(\omega(n-2))\right) + B\left(\sin(\omega n) + \sin(\omega(n-2))\right) - A\cos(\omega(n-2)) - B\sin(\omega(n-2))$$
$$= A\cos(\omega n) + A\cos(\omega(n-2)) + B\sin(\omega n) + B\sin(\omega(n-2)) - A\cos(\omega(n-2)) - B\sin(\omega(n-2))$$
$$= A\cos(\omega n) + B\sin(\omega n) = X_n.$$

Thus,

$$X_n = (2\cos\omega)X_{n-1} - X_{n-2}.$$

(b) We can understand this model as AR(2) with zero error, so

$$\widetilde{P}_n X_{n+1} = (2\cos\omega)X_n - X_{n-1}.$$

10. For MA(2) $X_t = Z_t + 2Z_{t-1} - 2Z_{t-2}$, observe that $P_1X_2 = a_0 + a_1X_1$. Thus, we want to find a_0 and a_1 minimizing MSPE $= E(X_2 - P_1X_2)^2 = E(X_2 - a_0 - a_1X_1)^2$. Solving "derivative =0" gives

$$\frac{\partial \text{MSPE}}{\partial a_0} = E(X_2 - a_0 - a_1X_1)(-2) = 0$$

$$\frac{\partial \text{MSPE}}{\partial a_1} = E(X_2 - a_0 - a_1X_1)(-X_1) = 0$$

Therefore, we have

$$a_0 = 0, \quad a_1 = \frac{EX_1X_2}{EX_1^2} = \frac{\gamma(1)}{\gamma(0)} = -\frac{2}{9}.$$

since

$$\gamma(0) = E(Z_t + 2Z_{t-1} - 2Z_{t-2}, Z_t + 2Z_{t-1} - 2Z_{t-2}) = 1 + 4 + 4 = 9$$
$$\gamma(1) = E(Z_{t+1} + 2Z_t - 2Z_{t-1}, Z_t + 2Z_{t-1} - 2Z_{t-2}) = 2 - 4 = -2.$$

Also, MSPE is calculated as

$$\text{MSPE} = E\left(X_2 + \frac{2}{9}X_1\right)^2 = E(X_2^2) + \frac{4}{9}E(X_1X_2) + \frac{4}{81}EX_1^2$$

$$= \frac{85}{81}\gamma(0) + \frac{4}{9}\gamma(1) = \frac{85}{81}9 + \frac{4}{9}(-2) = \frac{77}{9}$$