7.1 Integration by Parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx$$

7.2 Trigonometric Integrals

EX) $\int \cos^3 x\,dx$

$= \int \cos^2 x \cdot \cos x\,dx = \int (1-\sin^2 x)\cos x\,dx$ , $u = \sin x$

$\int \sin^5 x \cdot \cos^2 x\,dx$

$= \int (\sin^2 x)^2 \cos^2 x \cdot \sin x\,dx = \int (1-\cos^2 x)^2 \cos^2 x \sin x\,dx$ , $\cos x = u$

$\int_0^{\pi} \sin^2 x\,dx$ , using the half angle formula

$= \int_0^{\pi} \frac{1}{2}(1-\cos 2x)\,dx$

$\int \sin^4 x\,dx$ , apply the half angle formula

$= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$

Strategy for Evaluating $\int \sin^m x \cos^n x\,dx$

- if the power of cosine is odd, save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine
- if the power of sine is odd, save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine
- if the powers of both sine and cosine are even, use the half-angle identities

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$ , $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ , $\sin x \cos x = \frac{1}{2}\sin 2x$

Strategy for Evaluating $\int \tan^m x \cdot \sec^n x\,dx$

- if the power of secant is even, save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$
- if the power of tangent is odd, save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$

☆ $\int \tan x\,dx = \ln|\sec x| + C$ , $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

☆ if an even power of tangent appears with an odd power of secant, it is helpful to express the integrand in terms of $\sec x$

i) $\sin A \cos B = \frac{1}{2}\{\sin(A-B) + \sin(A+B)\}$

ii) $\sin A \sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}$

iii) $\cos A \cos B = \frac{1}{2}\{\cos(A-B) + \cos(A+B)\}$

## 7.3 Trigonometric Substitution

i) $\sqrt{a^2-x^2}$ , $x = a\sin\theta$ $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ , $1-\sin^2\theta = \cos^2\theta$

ii) $\sqrt{a^2+x^2}$ , $x = a\tan\theta$ $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ , $1+\tan^2\theta = \sec^2\theta$

iii) $\sqrt{x^2-a^2}$ , $x = a\sec\theta$ $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$ , $\sec^2\theta - 1 = \tan^2\theta$

## 7.4 Integration of Rational Functions by Partial Fractions

- method of integrating any rational function by expressing it as a sum of simpler fractions
- Suppose a rational function $f(x) = \frac{P(x)}{Q(x)}$
  - if $f$ is improper, where the degree of $P$ is greater than that of $Q$, we must first divide $Q$ into $P$ until a remainder is obtained
    $\Rightarrow f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$
- In the case of that the denominator $Q(x)$ is more complicated, the next step is to factor the denominator $Q(x)$ as far as possible
  - Case 1 : $Q(x)$ is a product of distinct linear factors
    - this means we can rephrase $Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$, where no factor is repeated. Then there exist constants $A_i$'s such that
      $\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}$
    - Solve for $A_i$'s
  - Case 2 : $Q(x)$ is a product of linear factors, some of which are repeated
    - Suppose the first linear factor $(a_1x+b_1)$ is repeated $r$ times ; $(a_1x+b_1)^r$ occurs in the factorization of $Q(x)$, $\frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_r}{(a_1x+b_1)^r}$
    - factor each of the repeated denominator $(a_ix+b_i)^{r_i}$ for $r_i$ times and solve for $A_i$'s
  - Case 3 : $Q(x)$ contains irreducible quadratic factors, none of which is repeated ($Q(x)$ has the factor $ax^2+bx+c$)
    - $f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$, and $\frac{R(x)}{Q(x)}$ will have a term of the form $\frac{Ax+B}{ax^2+bx+c}$
  - Case 4 : $Q(x)$ contains a repeated irreducible quadratic factor
    - if $Q(x)$ has the factor $(ax^2+bx+c)^r$, $\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$ occurs in the partial fraction decomposition of $\frac{R(x)}{Q(x)}$. ※ Similar process with Case 2

7.5 Strategy for Integration

- Below equations are not integrable

$e^{x^2}$, $\frac{e^x}{x}$, $\sin(x^2)$, $\cos(e^x)$, $\sqrt{x^3+1}$, $\frac{1}{\ln x}$, $\frac{\sin x}{x}$

7.6 Integration Using Tables and Computer Algebra Systems

7.7 Approximate Integration

2 situations in which it is impossible to find the exact value of a definite integral

- when $f(x)$ is not in integrable forms
- when the function is determined from a scientific experiment

Approximation Methods :

i) Riemann Sum

- Left-endpoint Approximation
- Right-endpoint Approximation
- Midpoint Rule

ii) Trapezoidal Rule

$$\int_a^b f(x)\,dx = \frac{1}{2}\left[\sum_{i=1}^{n} f(x_{i-1})\,\Delta x + \sum_{i=1}^{n} f(x_i)\,\Delta x\right] = \frac{\Delta x}{2}\left[\sum_{i=1}^{n} f(x_{i-1}) + \sum_{i=1}^{n} f(x_i)\right]$$

$$= \frac{\Delta x}{2}\left[(f(x_0)+f(x_1)) + (f(x_1)+f(x_2)) + \cdots + (f(x_{n-1})+f(x_n))\right]$$

$$= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

Error Bounds :

- Suppose $|f''(x)| \le K$ for $a \le x \le b$. If $E_T$ and $E_M$ are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \le \frac{K(b-a)^3}{24n^2}$$

Simpson's Rule :

$$\int_a^b f(x)\,dx = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 4(x_3) + 2(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

- Simpson's Rule gives us a much better approximation than does the Trapezoidal or Midpoint Rule.
- The approximation made using Simpson's Rule are weighted averages of those in the Trapezoidal and Midpoint Rule.

$\Rightarrow \quad S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n$

Error Bound for Simpson's Rule:

- Suppose that $|f^{(4)}(x)| \le K$ for $a \le x \le b$. If $E_s$ is the error involved in using Simpson's Rule, then $|E_s| \le \frac{K(b-a)^5}{180n^4}$

7.8 Improper Integrals

- a definite integral where the interval is infinite and also to the case where $f$ has an infinite discontinuity in $[a,b]$

Type 1 : Infinite Integrals

a) if $\int_a^t f(x)\,dx$ exists for every number $t \ge a$, then $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$

b) if $\int_t^b f(x)\,dx$ exists for every number $t \le b$, then $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_{-\infty}^b f(x)\,dx$

⇒ the improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called convergent if the corresponding limit exists and divergent if the limit does not exist

✱ $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \le 1$

Type 2 : Discontinuous Integrands

- Suppose $f$ is a positive continuous function defined on a finite interval $[a,b)$ but has a vertical asymptote at $b$.

a) if $f$ is continuous on $[a,b)$ and is discontinuous at $b$, then $\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$

b) if $f$ is continuous on $(a,b]$ and is discontinuous at $a$, then $\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$

⇒ the improper integral $\int_a^b f(x)\,dx$ is called convergent if the corresponding limit exists and divergent if the limit does not exist

Comparison Theorem:

- Sometimes it is impossible to find the exact value of an improper integral and yet it is important to know the convergence or divergence
- Suppose $f$ and $g$ are continuous functions with $f(x) \ge g(x) \ge 0$ for $x \ge a$

a) if $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

b) if $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.