12.1 The Existence of Zeros

Questions about the Existence of zeros :

i) Existence : are there any zeros ?

ii) Number : are there infinitely many ? how many , or about how many ?

iii) Approximate Location : Find small intervals containing only one zero

iv) Calculation : Determine the zero "exactly", or to a given accuracy

- We say $f(x)$ changes sign on the closed finite interval $[a, b]$ if it is defined on this interval and has opposite signs at a and b : $f(a) \cdot f(b) < 0$

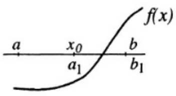
Bolzano's Theorem :

- Let $f(x)$ be continuous on $[a, b]$, then $f(x)$ changes sign on $[a,b]$ $\Rightarrow$ $f(x)$ has a zero on $[a,b]$

The starting interval $[a_0, b_0]$ is just $[a, b]$ itself. To get the next interval in the sequence, divide $[a_0, b_0]$ in two by its midpoint x_0; then choose as $[a_1, b_1]$ the half-interval on which $f(x)$ goes from $-$ to $+$:

if $f(x_0) > 0$, let $[a_1, b_1] = [a, x_0]$;

if $f(x_0) < 0$, let $[a_1, b_1] = [x_0, b]$.



In either case, we have $f(a_1) < 0$. $f(b_1) > 0$. This gives a new interval $[a_1 . b_1]$ of half the length, on which $f(x)$ still changes sign from $-$ to $+$.

(If at the midpoint we find that $f(x_0) = 0$, the above doesn't apply, but in that case we can stop and pack up: we've found a zero.)

We continue this process with $[a_1, b_1]$, bisecting it and choosing as $[a_2, b_2]$ the half on which $f(x)$ goes from $-$ to $+$. If at any stage the midpoint is a zero of $f(x)$, we are done; if not, we get an infinite sequence of nested intervals

$$[a, b] \supset [a_1, b_1] \supset [a_2, b_2] \supset \ldots \supset [a_n, b_n] \supset \ldots$$

such that

(2) $$f(a_n) < 0, \quad f(b_n) > 0, \quad \text{and} \quad b_n - a_n \to 0 .$$

By the Nested Interval Theorem 6.1, there is a unique c inside all these intervals, and

$$\lim a_n = c , \qquad \lim b_n = c.$$

To finish, we show that $f(c) = 0$. Since $f(x)$ is continuous on $[a, b]$, the Sequential Continuity Theorem 11.5 implies that

$$\lim f(a_n) = f(c) , \qquad \lim f(b_n) = f(c) .$$

According to (2), we have $f(a_n) < 0$ and $f(b_n) > 0$ for all n; it follows by the Limit Location Theorem for sequences 5.3A that

$$\lim f(a_n) \le 0 , \qquad \lim f(b_n) \ge 0 , \quad \text{i.e.,}$$

$$f(c) \le 0 , \qquad f(c) \ge 0 ,$$

which implies that $f(c) = 0$, proving (1). $\square$

Intermediate Value Theorem :

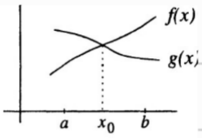
- Assume $f(x)$ is continuous on $[a, b]$ $f(a) \le f(b)$. Then for $k \in \mathbb{R}$,

$f(a) \le k \le f(b)$ $\Rightarrow$ $k = f(c)$ for some $c \in [a , b]$

✳ Since Bolzano's Theorem is essentially the special case of the Intermediate Value Theorem when $k = 0$, the two theorems are equivalent

12.2 Applications of Bolzano's Theorem

- A polynomial of odd degree has at least one zero



Intersection Principle 12.2

(a) *The roots of $f(x) = g(x)$ are the x-coordinates of the points where the graphs of $f(x)$ and $g(x)$ intersect.*

(b) *If $f(x)$ and $g(x)$ are continuous on $[a, b]$, and on this interval their graphs change their "above-below" position, i.e., the graph that is below at a becomes the one above at b:*

$$f(a) < g(a) \quad \text{and} \quad f(b) > g(b), \qquad \text{or} \qquad f(a) > g(a) \quad \text{and} \quad f(b) < g(b)\,,$$

then the two graphs intersect over some point $c \in [a, b]$.

12.3 Graphical Continuity

Continuity Theorem for Monotone Functions

- If the function $f(x)$ is strictly monotone and has the Intermediate Value Property on $[a, b]$, then it is continuous on $[a, b]$

12.4 Inverse Functions

Inverse Function Theorem:

- If $y = f(x)$ is continuous and strictly increasing on $[a, b]$, it has an inverse function $x = g(x)$ on $[f(a), f(b)]$ which is continuous and strictly increasing

☆ The theorem is also true for strictly decreasing functions; in that case $[f(a), f(b)]$ which is continuous and strictly increasing