7.7 **Rearranging the terms of a series.**

Def. Given a series $\sum_{0}^{\infty} a_n$, we say that $\sum_{0}^{\infty} b_n$ is a rearrangement of $\sum_{0}^{\infty} a_n$ (or a rearranged series of $\sum_{0}^{\infty} a_n$) if $b_n = a_{\sigma(n)}$ for every n, where $\sigma : \mathbb{N}_0 \equiv \{0,1,2,\cdots\} \to \mathbb{N}_0$ is one-to-one & onto (i.e., σ is a permutation (자리바꿈) on $\mathbb{N}_0$).

[a rearrangement of a series = 급수의 자리바꿈(합) = 자리바꿈 급수 = a rearranged seies]

Note: *In general,* $\sum_{0}^{\infty} a_n \neq$ a rearrangement of $\sum_{0}^{\infty} a_n$. For example, we know

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2 \quad (\equiv L) \neq 0$$

Recall that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

$$\begin{aligned} \frac{L}{2} &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots \\ &= 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \cdots \quad --- (i) \end{aligned}$$

From this, we see that

$$L = 2 \times \frac{L}{2} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots \quad --- (ii)$$

$(i) + (ii) \Rightarrow$

$$\begin{aligned} \frac{3L}{2} &= 1 + 0 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + 0 + \frac{1}{7} - \frac{1}{4} + \cdots \\ &= \underbrace{1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots}_{\text{a rearrangement of the series } \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n}} \end{aligned}$$

Therefore,

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} \cdots \quad \neq \quad \underbrace{1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} +}_{\text{a rearrangement of the LHS}} \quad = \frac{3}{2} \ln 2$$

Theorem (Rearrangement Theorem) --- 결론을 기억할 것

① If $\sum a_n$ is **absolutely convergent**, and $\sum a_n = S$, then any rearrangement $\sum b_n$ of $\sum a_n$ is **still (absolutely) convergent** & $\sum b_n = S$.

(In other words, if $\sum a_n$ is absolutely convergent, then it is unconditionally convergent)

② Suppose $\sum a_n$ is **conditionally convergent**, and let c be either a real number or ∞ or $-\infty$. Then there is a rearrangement $\sum b_n$ of $\sum a_n$ such that $\sum b_n = c$.

Pf of ①: Let $\sum_{n=0}^{\infty} a_n = S$, and let $\sum_{n=0}^{\infty} b_n$ be a rearrangement of $\sum_{n=0}^{\infty} a_n$.

Let $\varepsilon > 0$, and choose N such that $\sum_{n=N+1}^{\infty} |a_n| < \varepsilon \left(\leftarrow \sum_{n=0}^{\infty} |a_n| \text{ is convergent} \right)$

Choose an $M \geq N$ such that all the terms $a_0, \cdots, a_N$ occur in the list $b_0, \cdots, b_M$.

If $n \geq M$, then in the sum $\sum_{k=0}^{n} b_k - \sum_{k=0}^{n} a_k$, all the terms $a_0, \cdots, a_N$ **cancel out**, and thus the remaining terms (in $\sum_{k=0}^{n} b_k - \sum_{k=0}^{n} a_k$) consist only of terms a_k with $k > N$.

$\therefore$ $\sum_{k=0}^{n} b_k - \sum_{k=0}^{n} a_k$ is a sum of some **non-repeating** terms in $\sum_{k=N+1}^{\infty} a_k$

$$\therefore \left| \sum_{k=0}^{n} b_k - \sum_{k=0}^{n} a_k \right| \leq 2 \sum_{k=N+1}^{\infty} |a_k| < 2\varepsilon$$

$$\therefore \left| \sum_{k=0}^{n} b_k - S \right| = \left| \sum_{k=0}^{n} b_k - \sum_{k=0}^{\infty} a_k \right| \leq \left| \sum_{k=0}^{n} b_k - \sum_{k=0}^{n} a_k \right| + \left| \sum_{k=n+1}^{\infty} a_k \right| < 2\varepsilon + \varepsilon = 3\varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\lim_{n \to \infty} \underbrace{\sum_{k=0}^{n} b_k}_{\text{partial sum of } \sum_{k=0}^{\infty} b_k} = S \qquad \text{i.e., } \sum_{k=0}^{\infty} b_k = S = \sum_{k=0}^{\infty} a_k$$

Remark. Another simple proof for $\boxed{\text{the case of all } a_n \geq 0}$:

Let s_n' be the n-th partial sum of the rearrangement $\sum b_n$.

Note that every term of $\sum b_n$ is among the terms of the original series $\sum_{n=0}^{\infty} a_n$, and hence

$$s_n' \leq S \left(= \sum_{n=0}^{\infty} a_n \right) \quad \text{for every } n \quad (\text{i.e., } \{s_n'\} \text{ is bounded above by } S)$$

But s_n' is $\uparrow (\leftarrow a_n \geq 0 \ \forall n)$. Thus $\lim_{n \to \infty} s_n'$ exists. Write $S' = \lim_{n \to \infty} s_n'$.

Then we have $\lim_{n \to \infty} s_n' \leq S$ (by LLT) That is, $S' \leq S$

That is, the rearrangement $\sum b_n$ converges, & to a sum $S' \leq S$.

By symmetry, since $\sum a_n$ can be regarded as a rearrangement of $\sum b_n$, we must have $S \leq S'$.

Consequently, $S = S'$.

"Pf" of ②: (optional) We will not prove this statement; Instead we shall show that there is a rearrangement $\sum b_n$ of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ such that $\sum b_n = \pi$.

⟦ A slight modification of the line of the argument below will show the statement in ② is true. ⟧

Recall that $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ is conditionally convergent & $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=\ln 2.$

Note first that the series of **positive terms** and the series of **negative terms** both diverge;

That is,

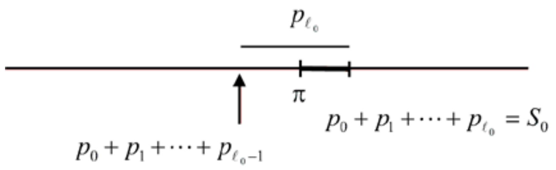
$$1+\frac{1}{3}+\frac{1}{5}+\cdots \overset{\text{diverges to}}{\rightarrow} \infty \quad (\text{we write } \sum_{0}^{\infty} p_n = \infty)$$

$$-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}+\cdots \overset{\text{diverges to}}{\rightarrow} -\infty \quad (\text{we write } \sum_{0}^{\infty} q_n = -\infty)$$

Let ℓ_0 be the first integer such that

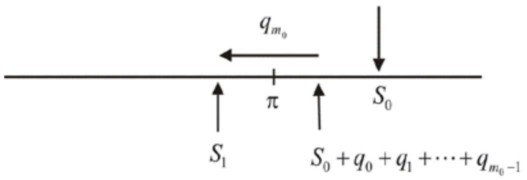
$$S_0 \equiv p_0 + p_1 + \cdots + p_{\ell_0} > \pi$$

That is, $p_0 + p_1 + \cdots + p_{\ell_0 - 1} < \pi < p_0 + p_1 + \cdots + p_{\ell_0} = S_0$ & $|\, S_0 - \pi \,| < p_{\ell_0}.$



Let m_0 be the first integer such that

$$S_1 \equiv S_0 + q_0 + q_1 + \cdots + q_{m_0} < \pi$$



Note that $|\, S_1 - \pi \,| < |\, q_{m_0} \,|.$

Let ℓ_1 be the first integer such that

$$S_2 \equiv S_1 + p_{\ell_0+1} + p_{\ell_0+2} + \cdots + p_{\ell_1} > \pi$$

So, $|\, S_2 - \pi \,| < p_{\ell_1}.$

Let m_1 be the first integer such that

$$S_3 \equiv S_2 + q_{m_0+1} + q_{m_0+2} + \cdots + q_{m_1} < \pi$$

So, $|\, S_3 - \pi \,| < |\, q_{m_1} \,|.$

$$\vdots$$

$$\vdots$$

Let $\sum b_n$ be this rearranged series. That is,

$$\sum b_n = \underbrace{p_0 + p_1 + p_2 + \cdots + p_{\ell_0}} + \overbrace{q_0 + q_1 + \cdots + q_{m_0}} + \underbrace{p_{\ell_0+1} + \cdots + p_{\ell_1}}$$

$$+ \overbrace{q_{m_0+1} + \cdots + q_{m_1}} + \underbrace{p_{\ell_1+1} + \cdots + p_{\ell_2}} + \overbrace{q_{m_1+1} + q_{m_1+2} + \cdots + q_{m_2}} + \cdots$$

$$\overset{\text{rename}}{=} b_0 + b_1 + b_2 + \cdots + \underline{b_{n_0}} + b_{n_0+1} + \cdots + \underline{b_{n_1}} + b_{n_1+1} + \cdots + \underline{b_{n_2}} + \cdots$$

(where $b_{n_0} = p_{\ell_0}$, $b_{n_1} = q_{m_0}$, $b_{n_2} = p_{\ell_1}$, $b_{n_3} = q_{m_1}$, $\cdots$)

Then the sequence s_n of partial sums of $\sum b_n$ has S_i as a subsequence. That is,

$$\begin{aligned}
S_0 &= s_{n_0} \overset{\text{i.e.}}{=} b_0 + \cdots + b_{n_0} \\
S_1 &= s_{n_1} \overset{\text{i.e.}}{=} b_0 + \cdots + b_{n_1} \\
S_2 &= s_{n_2} \overset{\text{i.e.}}{=} b_0 + \cdots + b_{n_2} \\
S_3 &= s_{n_3} \overset{\text{i.e.}}{=} b_0 + \cdots + b_{n_3} \\
&\vdots \\
S_i &= s_{n_i} \overset{\text{i.e.}}{=} b_0 + \cdots + b_{n_i} \\
&\vdots
\end{aligned}$$

The construction shows that

$$| S_i - \pi | < | b_{n_i} | \quad \text{for every } i.$$

Since $\lim_{n\to\infty} p_n = 0$ & $\lim_{n\to\infty} q_n = 0$, we have $\lim_{i\to\infty} b_{n_i} = 0$

$$\therefore \quad \lim_{i\to\infty} S_i = \pi$$

On the other hand, for any fixed n, s_n lies between S_i and S_{i+1} for some i.
And it is clear that $n \to \infty \iff i \to \infty$.

$$\therefore \quad \lim_{n\to\infty} s_n = \pi \quad \text{by Squeeze Principle.}$$

Remark. An idea for the proof of the statement in ②:

(i) $\sum a_n$: conditionally converges $\Rightarrow$ $\sum a_n^+ = \infty$ & $\sum a_n^- = \infty$ (easy Ex)

(ii) Apply the above argument to $\sum a_n^+$ & $\sum(-a_n^-)$ $\left(\text{instead of } \sum_0^\infty p_n \ \& \ \sum_0^\infty q_n \right)$

Ex. (optional)

Show that there is a rearrangement $\sum b_n$ of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ such that $\sum b_n = \infty$.

Pf. Recall that we are using the following notations:

$$\sum_0^{\infty} p_n = 1 + \frac{1}{3} + \frac{1}{5} + \cdots = \infty, \qquad \sum_0^{\infty} q_n = -\frac{1}{2} - \frac{1}{4} - \frac{1}{6} + \cdots = -\infty.$$

Let ℓ_0 be the first integer such that

$$p_0 + p_1 + \cdots + p_{\ell_0} > 1 - q_0$$

and set $S_0 = p_0 + p_1 + \cdots + p_{\ell_0} + q_0$. Then $S_0 > 1$.

Let ℓ_1 be the first integer such that

$$S_0 + p_{\ell_0+1} + p_{\ell_0+2} + \cdots + p_{\ell_1} > 2 - q_1$$

and set $S_1 = S_0 + p_{\ell_0+1} + p_{\ell_0+2} + \cdots + p_{\ell_1} + q_1$. Then $S_1 > 2$.

Let ℓ_2 be the first integer such that

$$S_1 + p_{\ell_1+1} + p_{\ell_1+2} + \cdots + p_{\ell_2} > 3 - q_2$$

and set $S_2 = S_1 + p_{\ell_1+1} + p_{\ell_1+2} + \cdots + p_{\ell_2} + q_2$. Then $S_2 > 3$.

$$\vdots$$

Let $\sum b_n$ be this rearranged series. That is,

$$\sum b_n = \underbrace{p_0 + p_1 + p_2 + \cdots + p_{\ell_0} + q_0} + \overbrace{p_{\ell_0+1} + \cdots + p_{\ell_1} + q_1} +$$

$$+ \underbrace{p_{\ell_1+1} + \cdots + p_{\ell_2} + q_2} + \cdots$$

$$\overset{\text{rename}}{=} b_0 + b_1 + b_2 + \cdots + \underline{b_{n_0}} + b_{n_0+1} + \cdots + \underline{b_{n_1}} + b_{n_1+1} + \cdots + \underline{b_{n_2}} + \cdots$$

$$(\text{with } b_{n_0} = q_0, \quad b_{n_1} = q_1, \quad b_{n_2} = q_2, \quad \cdots)$$

Then the sequence s_n of partial sums of $\sum b_n$ has S_i as a subsequence:

$$\begin{aligned} S_0 &= s_{n_0} \\ S_1 &= s_{n_1} \\ &\vdots \\ S_i &= s_{n_i} \\ &\vdots \end{aligned}$$

The construction shows that $S_i > i + 1$ for every i

So $\lim_{i\to\infty} S_i = \infty$.

On the other hand, for any fixed n, s_n lies between S_i and S_{i+1} for some i.

This implies $\lim_{n\to\infty} s_n = \infty$ by Squeeze Principle.

● Another three tests. [Cauchy's 2^n test: well-known; Raabe's test, Dirichlet test: advanced]

Cauchy's 2^n test (or **Cauchy's condensation test**) [기억할 것]

If $a_n \downarrow 0$, then $\sum_1^\infty a_n$ converges $\Leftrightarrow$ $\sum_0^\infty 2^n a_{2^n}$ converges

Pf. Let s_n, t_n, respectively, denote the n-th partial sums of $\sum_1^\infty a_n$ & $\sum_0^\infty 2^n a_{2^n}$.

Given n, there is a k satisfying $n < 2^k$, and hence

$$
\begin{aligned}
s_n &= a_1 + a_2 + \cdots + a_n \\
&\le a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + (a_{2^k} + a_{2^k+1} + \cdots + a_{2^{k+1}-1}) \\
(a_n \downarrow) \Rightarrow \quad &\le a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k} = t_k
\end{aligned}
$$

Thus if $\sum_0^\infty 2^n a_{2^n}$ converges, then (t_k) is bounded. Consequently, (s_n) is bounded above and hence $\sum_1^\infty a_n$ converges since (s_n) is monotonically increasing.

Conversely,

$$
\begin{aligned}
s_{2^n} &= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{n-1}+1} + a_{2^{n-1}+2} + \cdots + a_{2^n}) \\
&\ge \frac{1}{2}a_1 + a_2 + 2a_4 + 2^2 a_8 + \cdots + 2^{n-1}a_{2^n} \qquad (\Leftarrow a_n \downarrow) \\
&= \frac{1}{2}(a_1 + 2a_2 + 2^2 a_4 + 2^3 a_8 + \cdots + 2^n a_{2^n}) = \frac{1}{2}t_n
\end{aligned}
$$

If $\sum_1^\infty a_n$ converges, then in particular (s_{2^n}) is bounded. So $(\frac{1}{2}t_n)$ is bounded above and hence (t_n) is bounded above. Since (t_n) is also increasing, it is convergent. This means $\sum_0^\infty 2^n a_{2^n}$ converges.

Short proof: $a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + a_8 \cdots \le a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$

$$\underbrace{\frac{a_1}{2} + a_2 + 2a_4 + 4a_8 + \cdots}_{=\frac{1}{2}(a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots)} \le a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8)\cdots$$

Applications of Cauchy's 2^n test

Eg1. (p – series) $\sum_1^\infty \frac{1}{n^p}$ $(p > 0)$

Sol. $\frac{1}{n^p} \downarrow 0$ as $n \to \infty$. $\sum_1^\infty 2^n \cdot \frac{1}{(2^n)^p} = \sum_1^\infty (2^{1-p})^n = \begin{cases} \text{conv} & \text{if } 2^{1-p} < 1 \quad (\Leftrightarrow p > 1) \\ \text{div} & \text{if } 2^{1-p} \ge 1 \quad (\Leftrightarrow p \le 1) \end{cases}$

Eg2. $\sum_2^\infty \frac{1}{n(\ln n)^p}$ $(p > 0)$

Sol. $\frac{1}{n(\ln n)^p} \downarrow 0$ as $n \to \infty$ (& for $n \gg 1$)

$$\sum_{N_0}^{\infty} \cancel{2^n} \cdot \frac{1}{\cancel{2^n}(\ln 2^n)^p} = \sum_{N_0}^{\infty} \frac{1}{(\ln 2^n)^p} = \sum_{N_0}^{\infty} \frac{1}{(n \ln 2)^p} = \frac{1}{(\ln 2)^p} \sum_{N_0}^{\infty} \frac{1}{n^p} = \begin{cases} \text{conv} & \text{if } p > 1 \\ \text{div} & \text{if } p \le 1 \end{cases}$$

Eg3. $\displaystyle\sum_{1000}^{\infty} \frac{1}{n \ln n (\ln \ln n)^p} \quad (p > 0)$

Sol. $\displaystyle\frac{1}{n \ln n (\ln \ln n)^p} \downarrow 0 \quad \text{as } n \to \infty \quad (\& \text{ for } n \gg 1)$

$$\sum_{N_0}^{\infty} \cancel{2^n} \cdot \frac{1}{\cancel{2^n} \ln 2^n (\ln \ln 2^n)^p} = \frac{1}{\ln 2} \sum_{N_0}^{\infty} \frac{1}{n(\ln n + \ln \ln 2)^p} = \begin{cases} \text{conv} & \text{if } p > 1 \\ \text{div} & \text{if } p \le 1 \end{cases} \quad \text{(by Eg2)}$$

because of $\displaystyle\frac{1}{n(\ln n + \ln \ln 2)^p} \sim \frac{1}{n(\ln n)^p}$.

Eg4. Let $a_n \downarrow 0$. Then $\displaystyle\sum_{1}^{\infty} a_n$ converges $\Rightarrow$ $n a_n \to 0$ as $n \to \infty$.

Pf. $\displaystyle\sum_{1}^{\infty} a_n : \text{conv} \overset{2^n \text{ test}}{\Rightarrow} \sum_{1}^{\infty} 2^n a_{2^n} : \text{conv} \Rightarrow \lim_{n\to\infty} 2^n a_{2^n} = 0$

Given any k, we can choose an integer n such that $2^n \le k \le 2^{n+1}$. Then

$$a_k \le a_{2^n} \quad (\because \ a_n \downarrow)$$

$$\therefore \quad k a_k \le 2^{n+1} a_{2^n} = 2 \cdot (2^n a_{2^n}) \to 0 \quad \text{as } n \to \infty$$

Note that $n \to \infty \Leftrightarrow k \to \infty$. Therefore, $\displaystyle\lim_{k\to\infty} k a_k = 0$.

- **Raabe's test** (often useful in the case that $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \ (a_n > 0)$ or $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$; Ratio test fails)

Lemma: $p > 1$ & $x \in (0,1) \Rightarrow (1-x)^p > 1 - px$

Pf. M1. Follows from "Bernoulli inequality": $p > 1 \Rightarrow (1+x)^p > 1 + px$ for $\forall x > -1$ (Ex)

M2. (A direct pf) Let $f(x) = (1-x)^p - 1 + px \quad (p > 1)$

$f'(x) = -p(1-x)^{p-1} + p = p\left(1 - (1-x)^{p-1}\right) > 0$ for $\forall x \in (0,1)$, since $0 < 1 - x < 1$ & $p - 1 > 0$

$\therefore$ $f(x)$ is strictly $\uparrow$ on $(0,1)$ & $f(0) = 0$; and so $f(x) > 0$ for $\forall x \in (0,1)$ --- done

Raabe's test. ($\leftarrow$ comparison with $p-$series) Assume $a_n > 0$ $(\forall n)$ and

$$\lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) = L \quad \left(\overset{\text{easy}}{\Rightarrow} \frac{a_{n+1}}{a_n} \to 1\right)$$

If $L > 1 \Rightarrow \sum a_n$: converges (main interest)
If $L < 1 \Rightarrow \sum a_n$: diverges
If $L = 1 \Rightarrow$ no conclusion

Pf. We prove only the case $L>1$; the case $L<1$: Ex

Choose p such that $L>p>1$. Then by SLT

$$n\left(1-\frac{a_{n+1}}{a_n}\right)>p \quad \text{for } n\gg 1 \qquad \therefore\ \frac{a_{n+1}}{a_n}<1-\frac{p}{n} \quad \text{for } n\gg 1$$

Applying $x=\dfrac{1}{n}$ $(n\gg 1)$ to $\boxed{\text{Lemma: } p>1 \ \& \ x\in(0,1) \ \Rightarrow \ (1-x)^p>1-px}$

$$\Rightarrow \quad \frac{a_{n+1}}{a_n}<\underbrace{1-\frac{p}{n}<\left(1-\frac{1}{n}\right)^p}_{\text{key idea}}<\left(1-\frac{1}{n+1}\right)^p=\frac{n^p}{(n+1)^p} \quad \text{for } n\gg 1$$

$$(n+1)^p a_{n+1}<n^p a_n \quad \text{for } n\gg 1 \qquad \text{i.e., } n^p a_n \text{ is strictly} \downarrow \text{ for } n\geq N$$

$$\Rightarrow n^p a_n<N^p a_N \quad \text{for } n\geq N \qquad \Rightarrow \quad a_n<\left(N^p a_N\right)n^{-p} \text{ for } n\geq N$$

$$\therefore \sum_N^\infty a_n<\left(N^p a_N\right)\sum_N^\infty n^{-p}: \text{ converges since } p>1 \qquad \therefore \sum a_n: \text{ converges} \quad \text{(by Tail Conv Thm)}$$

Eg1. Test the convergence of $\displaystyle\sum\frac{(2n)!}{4^n(n!)^2}$

Sol. $a_n:=\dfrac{(2n)!}{4^n(n!)^2}$ (>0) $\qquad \dfrac{a_{n+1}}{a_n}=\dfrac{1}{2}\dfrac{2n+1}{n+1}\to 1 \qquad \therefore$ ratio test fails

But $\displaystyle\lim_{n\to\infty} n\left(1-\frac{a_{n+1}}{a_n}\right)=\lim_{n\to\infty} n\left(1-\frac{1}{2}\frac{2n+1}{n+1}\right)=\frac{1}{2}<1 \qquad \therefore$ div

Eg2. Test the convergence of $\displaystyle\sum\frac{1\cdot 4\cdot 7\cdots(3n+1)}{n^2 3^n n!}$

Sol. $a_n:=\dfrac{1\cdot 4\cdot 7\cdots(3n+1)}{n^2 3^n n!} \qquad \dfrac{a_{n+1}}{a_n}=\dfrac{(3n+4)n^2}{3(n+1)^3}\to 1 \qquad \therefore$ ratio test fails

But $\displaystyle n\left(1-\frac{a_{n+1}}{a_n}\right)=n\left(1-\frac{(3n+4)n^2}{3(n+1)^3}\right)=\frac{5n^3+9n^2+3}{3(n+1)^3} \ \to \ \frac{5}{3}>1 \qquad \therefore$ conv

- **Dirichlet test**

⊙ **Summation by parts formula**:

$$\boxed{\begin{aligned}\sum_{k=1}^n a_k b_k &= a_n B_n+\sum_{k=1}^{n-1}(a_k-a_{k+1})B_k, \quad \text{where } B_k=\sum_{\ell=1}^k b_\ell \\ &= a_n B_n-\sum_{k=1}^{n-1}(a_{k+1}-a_k)B_k=\underbrace{a_n}_{\text{그}}\underbrace{B_n}_{\text{적}}-\sum_{k=1}^{n-1}\underbrace{(\Delta a_k)}_{\text{미}}\underbrace{B_k}_{\text{적}}, \quad \text{where } \Delta a_k=a_{k+1}-a_k\end{aligned}}$$

Pf.
$$\begin{aligned}\sum_{k=1}^n a_k b_k &= a_1 b_1+a_2 b_2+\cdots+a_n b_n=a_1 B_1+a_2(B_2-B_1)+\cdots+a_n(B_n-B_{n-1}) \\ &= (a_1-a_2)B_1+(a_2-a_3)B_2+\cdots+(a_{n-1}-a_n)B_{n-1}+a_n B_n \\ &= \sum_{k=1}^{n-1}(a_k-a_{k+1})B_k+a_n B_n\end{aligned}$$

※ **Dirichlet Test**

Suppose (i) a_n is $\downarrow 0$ (i.e., $a_1 \geq a_2 \geq a_3 \geq \cdots \downarrow 0$) &

(ii) $\left|\sum_{k=1}^{n} b_k\right| \leq \underbrace{M}_{\text{indep of } n}$ $(n = 1, 2, \cdots)$ (i.e., the sequence of partial sums of (b_n) is bounded).

Then $\sum_{1}^{\infty} a_n b_n$ is convergent.

Remark: Dirichlet test is a generalization of Alternating series test (why?)

Pf of the Dirichlet test:

$$\sum_{k=1}^{n} a_k b_k = \underbrace{a_n B_n} + \underbrace{\sum_{k=1}^{n-1} (a_k - a_{k+1}) B_k} \quad \text{---}(*)\ (\textbf{Summation by parts formula})$$

Enough to show that $\lim_{n\to\infty} a_n B_n$ exists and the series $\sum_{k=1}^{\infty} (a_k - a_{k+1}) B_k$ converges.

(1) Clearly $|a_n B_n| \leq M a_n \to 0$ as $n \to \infty$

(2) Will show $\sum_{k=1}^{\infty} (a_k - a_{k+1}) B_k$ converges absolutely

Pf of (2): $\sum_{k=1}^{n-1} |(a_k - a_{k+1}) B_k| \overset{\text{"}a_n \text{ is } \downarrow\text{" is used}}{\leq} M\left(\sum_{k=1}^{n-1} (a_k - a_{k+1})\right) = M(a_1 - a_n)$

& $\sum_{k=1}^{\infty} (a_k - a_{k+1}) = \lim_{n\to\infty} \sum_{k=1}^{n-1} (a_k - a_{k+1}) = a_1 - \lim_{n\to\infty} a_n = a_1$; converges

$\therefore$ $\sum_{k=1}^{\infty} (a_k - a_{k+1}) B_k$ converges absolutely $\therefore$ it converges.

(3) (optional) $\sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k b_k \overset{(*)+(1)+(2)}{=} \underbrace{\sum_{k=1}^{\infty} (a_k - a_{k+1}) B_k}_{\text{converges by (2)}}$

Eg. Show that the series $1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \cdots$ is convergent.

Pf. Note that

$$1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \cdots = 1 \cdot 1 + \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot (-2) + \frac{1}{4} \cdot 1 + \frac{1}{5} \cdot 1 + \frac{1}{6}(-2) + \cdots =: \sum_{n=1}^{\infty} a_n b_n$$

That is, $a_n = 1/n$ & $\{b_n\}_1^{\infty} = (1, 1, -2, 1, 1, -2, \cdots)$

Clearly $a_n \downarrow 0$ as $n \to \infty$

Let $B_n = \sum_{k=1}^{n} b_k$. Then

$\{B_n\}_1^{\infty} = (1, 2, 0, 1, 2, 0, \cdots)$ $\therefore$ $|B_n| \leq 2$ for every $n \geq 1$

Thus by Dirichlet test, the series $1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \cdots$ is convergent.

HS. Prove that $\sum_{n=1}^{\infty} \frac{\cos n}{n}$ & $\sum_{n=1}^{\infty} \frac{\sin n}{n}$ are both convergent