Homework I (2022)

1. Consider the $n \times n$ compound symmetric covariance matrix Σ with main diagonal elements σ^2 and off-diagonal elements $\rho\sigma^2$. Show that

$$\epsilon = \frac{n^2(\bar{\sigma}_{ii} - \bar{\sigma}_{..})^2}{(n-1)(S - 2n\sum \bar{\sigma}_{i.}^2 + n^2\bar{\sigma}_{..}^2)} \tag{1}$$

is equal to 1, where $\bar{\sigma}_{ii}$ is the mean of the entries on the main diagonal of Σ, $\bar{\sigma}_{..}$ is the mean of all elements of Σ, $\bar{\sigma}_{i.}$ is the mean of the entries in row i of Σ, and S is the sum of the squares of the elements of Σ. (Note that the compound symmetry is a special case of a more general situation, sphericity. (1) is an alternative expression of the sphericity condition)

2. Suppose that repeated measurements are obtained at time points $1, \ldots, n$ for each of n subjects. Consider the mixed model

$$y_i = X\beta + Z\gamma_i + \epsilon_i$$

for $i = 1, \ldots, m$, where y_i is the $n \times 1$ vector of responses for subject i, X is the $n \times 2$ design matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & n \end{pmatrix},$$

$\beta^T = (\beta_0, \beta_1)$, $Z = (1, \ldots, 1)^T$, γ_i are independent $N(0, \sigma^2)$, the $n \times 1$ vectors ϵ_i are independent $N(0, \sigma_e^2 I_n)$, where I_n is the $n \times n$ identity matrix, and γ_i and ϵ_i are independent. Derive the variance-covariance matrix of y_i.

3. Consider the balanced one-way random model:

$$\begin{aligned} &y_{ij}|a_i \sim indep.N(\mu + a_i, \sigma^2), \quad i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n, \\ &a_i \sim iidN(0, \sigma_a^2). \end{aligned}$$

Find a $100(1-\alpha)\%$ confidence interval for the intraclass correlation coefficient.

4. Suppose that Y is a random vector with mean vector μ, where

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{pmatrix}.$$

(a) Suppose that we are interested in the null hypothesis $H_0 : \mu_1 - \mu_2 = 0,\ \mu_1 - \mu_3 = 0,\ \mu_1 - \mu_4 = 0,\ \mu_1 - \mu_5 = 0$, which may be written alternatively as

$$H_0 : \begin{pmatrix} \mu_1 - \mu_2 \\ \mu_1 - \mu_3 \\ \mu_1 - \mu_4 \\ \mu_1 - \mu_5 \end{pmatrix} = 0,$$

where 0 is a (4×1) vector of zeros. Give an appropriate matrix L so that H_0 may be written in the form $H_0 : L\mu = 0$.

(b) Now find an appropriate matrix L corresponding to the null hypothesis $H_0 : \mu_1 - \mu_2 = 0,\ \mu_2 - \mu_3 = 0,\ \mu_3 - \mu_4 = 0,\ \mu_4 - \mu_5 = 0$. Do the hypotheses in (a) and (b) address the same issue or different issues? Explain.

(c) Find the matrix U such that you can express the hypothesis in (b) in the form $H_0 : \mu^T U = 0^T$. Note that now 0^T is a (1×4) vector, so H_0 is being expressed as a row vector instead of a column vector.

(d) Now suppose we are interested in the null hypothesis $H_0 : \mu_1 - (\mu_2 + \mu_3 + \mu_4 + \mu_5)/4 = 0,\ \mu_2 - (\mu_3 + \mu_4 + \mu_5)/3 = 0,\ \mu_3 - (\mu_4 + \mu_5)/2 = 0,\ \mu_4 - \mu_5 = 0$. Reexpress H_0 in the form $H_0 : \mu^T U = 0^T$ by finding the appropriate matrix U.