Ch 5. Consistency and Limiting distributions

Convergence in probability

Definition

Let $\{X_n\}$ be a sequence of random variables and let X be a random variable. X_n converges to X in probability if and only if, for all $\epsilon > 0$, $\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0$

- In this case, we will write $X_n \xrightarrow{p} X$

- Some useful tools to show the convergence in probability

- Markov inequality (p. 93): For $u(X) > 0$,

$$P(u(X) \geq c) \leq \frac{E(u(X))}{c}$$

  Chebyshev's inequality (p. 93)

$$P((X-\mu)^2 \geq c) \leq \frac{\sigma^2}{c} \quad \text{or} \quad P(|X-\mu| \geq a) \leq \frac{\sigma^2}{a^2}$$

- Weak Law of Large Numbers (WLLN):
  For iid random sample $X_1, \ldots, X_n$, $\bar{X}_n \xrightarrow{p} \mu$ if $\mu = E(X_1)$ and $Var(X_1) < \infty$.

- Continuous Mapping Theorem (CMT): If $\bar{X}_n \xrightarrow{p} X$ and $\bar{Y}_n \xrightarrow{p} Y$, then $g(X_n, Y_n) \xrightarrow{p} g(X, Y)$.

- Example 5.1

  - $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

    $\rightarrow \frac{\sum X_i}{n-2} \xrightarrow{p} \mu$ and $S_n^2 = \frac{\sum (X_i - \bar{X})^2}{n-1} \xrightarrow{p} \sigma^2$

- $X_1, X_2, \ldots \overset{iid}{\sim} Gamma(3, \beta)$

  $\rightarrow \frac{\bar{X}_n}{3} \xrightarrow{p} \beta$

- Example 5.2

- $X_1, \ldots, X_n \stackrel{iid}{\sim} U[0, \theta]$

  $\rightarrow Y_n = \max(X_1, \ldots, X_n) \xrightarrow{p} \theta$

Convergence in distribution

Definition

For a sequence of random variables $X_1, X_2, \ldots$, if $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ for all x, then we say that X_n converges to X in distribution, and write $X_n \xrightarrow{d} X$.

- Two useful theorems for $\xrightarrow{d}$.

1. Continuous mapping theorem:
   $X_n \xrightarrow{d} X$ implies that $g(X_n) \xrightarrow{d} g(X)$ for any continuous function g.
2. Slutsky's theorem:
   If $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{p} c_1$, $Z_n \xrightarrow{p} c_2$, where c_1 and c_2 are constants, then

$$Y_n X_n + Z_n \xrightarrow{d} c_1 X + c_2$$

- Relationship between $\xrightarrow{d}$ and $\xrightarrow{p}$

1. $\xrightarrow{p}$ implies $\xrightarrow{d}$
2. $\xrightarrow{d}$ may not impliy $\xrightarrow{p}$

- Example 5.3: $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

Let $T_n = \frac{\bar{X}_n - \mu}{s/\sqrt{n}}$ where $s = \sqrt{\frac{\sum (X_i - \bar{X}_n)^2}{n-1}}$, then

$$T_n \xrightarrow{d} Z, \text{ where } Z \sim N(0,1)$$

Central Limit Theorem (CLT)

Theorem

Suppose that $X_1, X_2, \ldots$ *is a random sample from a distribution having mean zero and unit variance. Then* $Y_n = \sqrt{n}\bar{X}_n \xrightarrow{d} N(0,1)$

Proof.

Neet to show $P(Y_n \leq y) \to \Phi(y)$ for all y, where $\Phi(y)$ is the cdf of $N(0,1)$. This is equivalent to showing $M_{Y_n}(t) \to M_Z(t) = e^{t^2/2}$. First we assume that $M_X(t)$ exists for $-h < t < h$, $h > 0$. From Taylor expansion, we have

$$M_X(t) = M_X(0) + M_X'(0) + M_X''(\xi)t^2/2 = 1 + M_X''(\xi)t^2/2$$

for some $0 < \xi < t$. □

Now,

$$M_{Y_n}(t) = E(e^{tY_n}) = E\left(\exp\left(\frac{tX_1}{\sqrt{n}} + \ldots + \frac{tX_n}{\sqrt{n}}\right)\right)$$
$$= \left[E\left(\exp\left(\frac{tX_1}{\sqrt{n}}\right)\right)\right]^n = \left[M_X\left(\frac{t}{\sqrt{n}}\right)\right]^n$$

for some $-h < t/\sqrt{n} < h$. This means that

$$M_{Y_n}(t) \approx \left(1 + \frac{t^2/2}{n}\right)^n \to e^{t^2/2}$$

You can easily generalize this theorem as follows:
Suppose that $X_1, X_2, \ldots$ is a random sample from a distribution having mean μ and variance σ^2. Then $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2)$

- Example 5.4

1. $X_1, \ldots, X_n \overset{iid}{\sim} b(1,p) \Rightarrow \sqrt{n}(\bar{X}_n - p) \xrightarrow{d} N(0, p(1-p))$

2. $X_1, \ldots, X_n \overset{iid}{\sim} \chi^2_1 \Rightarrow \sqrt{n}(\bar{X}_n - 1) \xrightarrow{d} N(0, 2)$

Theorem (Delta method)

If $\sqrt{n}(X_n - \theta) \xrightarrow{d} N(0, \sigma^2)$*, then*

$$\sqrt{n}(g(X_n) - g(\theta)) \xrightarrow{d} N(0, \sigma^2(g'(\theta))^2),$$

for any twice differential function $g(\cdot)$ *at* θ *and* $g'(\theta) \neq 0$*.*

Proof.

$g(X_n) = g(\theta) + g'(\theta)(X_n - \theta) + \frac{g''(\xi)}{2}(X_n - \theta)^2$ for some $0 < \xi < X_n$

$\square$

- Example 5.5: $X_1, \ldots, X_n \overset{iid}{\sim} Gamma(\alpha, \beta)$

$$(1)\sqrt{n}\left(\frac{1}{\bar{X}_n} - \ ? \ \right) \xrightarrow{d} N(0, \ ? \ )$$

$$(2)\sqrt{n}\left(\log(\bar{X}_n) - \ ? \ \right) \xrightarrow{d} N(0, \ ? \ )$$

Exercises: 5.1.2, 5.1.3, 5.1.7, 5.2.2, 5.2.3, 5.2.12, 5.3.9, 5.3.11