Experimental Design
Note 6
Introduction to factorial design

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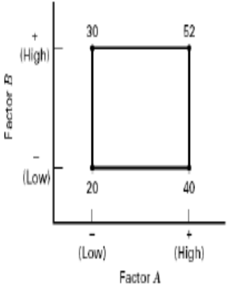
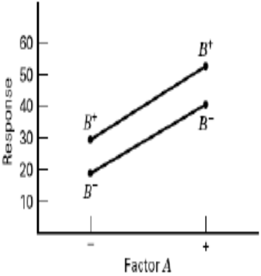
Introduction to Factorial Design of Engineering Experiments

- Many experiments involve the study of the effects of two or more factors
  - Factorial design are most efficient
  - All possible combinations of the levels of the factors are investigated
    - crossed
  - Effect of a factor: change in response produced by a change in the level of the factor
    - ie. main effect (primary factor)

Some Basic Definitions I



■ **FIGURE 5.1** A two-factor factorial experiment, with the response (y) shown at the corners



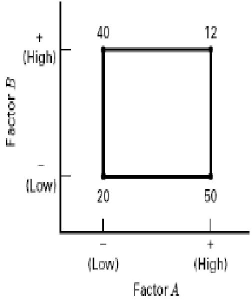
■ **FIGURE 5.3** A factorial experiment without interaction

Some Basic Definitions II

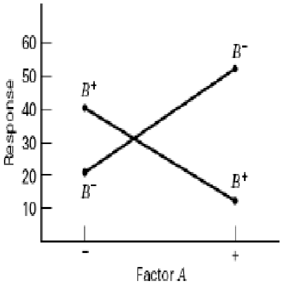
Definition of a factor effect: The change in the mean response when the factor is changed from low to high

$$A = \bar{y}_{A^+} - \bar{y}_{A^-} = \frac{40+52}{2} - \frac{20+30}{2} = 21$$
$$B = \bar{y}_{B^+} - \bar{y}_{B^-} = \frac{30+52}{2} - \frac{20+40}{2} = 11$$
$$AB = \frac{52+20}{2} - \frac{30+40}{2} = -1$$

The Case of Interaction I



■ FIGURE 5.2 A two-factor factorial experiment with interaction



■ FIGURE 5.4 A factorial experiment with interaction

The Case of Interaction II

$$A = \bar{y}_{A^+} - \bar{y}_{A^-} = \frac{50+12}{2} - \frac{20+40}{2} = 1$$

$$B = \bar{y}_{B^+} - \bar{y}_{B^-} = \frac{40+12}{2} - \frac{20+50}{2} = -9$$

$$AB = \frac{12+20}{2} - \frac{40+50}{2} = -29$$

Another way for interaction: Regression Model & Associated Response Surface I

Assume design factors are quantitative (e.g. temperature, time)

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \epsilon$$

The least squares fit is

$$\hat{y} = 35.5 + 10.5 x_1 + 5.5 x_2 + 0.5 x_1 x_2$$

We ignore the interaction. Then $\hat{y} = 35.5 + 10.5 x_1 + 5.5 x_2$.

Another way for interaction: Regression Model & Associated Response Surface II

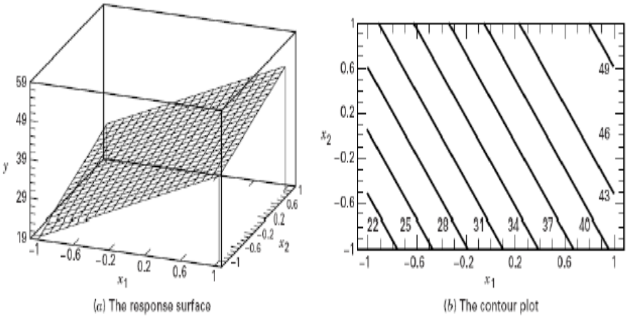


FIGURE 5.5 Response surface and contour plot for the model $\hat{y} = 35.5 + 10.5x_1 + 5.5x_2$

The Effect of Interaction on the Response Surface I

Suppose that we add an interaction term to the model:

$$\hat{y} = 35.5 + 10.5x_1 + 5.5x_2 + 8x_1x_2$$

Interaction is actually a form of curvature

The Effect of Interaction on the Response Surface II

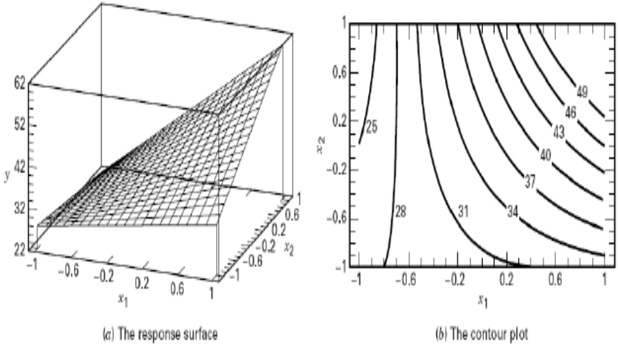


FIGURE 5.6 Response surface and contour plot for the model $\hat{y} = 35.5 + 10.5x_1 + 5.5x_2 + 8x_1x_2$

The Effect of Interaction on the Response Surface III

One-factor-a-time design as the opposite of factorial design.
Advantages of factorial over one-factor-a-time

- more efficient (runsize and estimation precision)
- able to accommodate interactions
- results are valid over a wider range of experimental conditions

Example I: Battery life experiment I

An engineer is studying the effective life of a certain type of battery. Two factors, plate material and temperature, are involved. There are three types of plate materials (1, 2, 3) and three temperature levels (15, 70, 125). Four batteries are tested at each combination of plate material and temperature, and all 36 tests are run in random order. The experiment and the resulting observed battery life data are given as following:

Example I: Battery life experiment II

TABLE 5.1

Life (in hours) Data for the Battery Design Example

Material Type	Temperature (°F) 15		70		125	
1	130	155	34	40	20	70
	74	180	80	75	82	58
2	150	188	136	122	25	70
	159	126	106	115	58	45
3	138	110	174	120	96	104
	168	160	150	139	82	60

A = Material type; B = Temperature (a quantitative variable)

- What effects do material type & temperature have on life?
- Is there a choice of material that would give long life regardless of temperature (a robust product)?

Example II: Bottling Experiment

A soft drink bottler is interested in obtaining more uniform fill heights in the bottles produced by his manufacturing process. An experiment is conducted to study three factors of the process, which are

the percent carbonation (A):	10, 12, 14 percent
the operating pressure (B):	25, 30 psi
the line speed (C):	200, 250 bpm

The response is the deviation from the target fill height. Each combination of the three factors has two replicates and all 24 runs are performed in a random order. The experiment and data are shown below.

	Pressure (B)			
	25 psi		30 psi	
	Line Speed (C)		Line Speed (C)	
Carbonation (A)	200	250	200	250
10	-3, -1	-1, 0	-1, 0	1, 1
12	0, 1	2, 1	2, 3	6, 5
14	5, 4	7, 6	7, 9	10,11

Factorial Design

- a number of factors: $F_1, F_2, \cdots, F_r$.
- each with a number of levels: $l_1, l_2, \cdots, l_r$
- number of all possible level combinations (treatments): $l_1 \times l_2 \times \cdots \times l_r$
- interested in (main) effects, 2-factor interactions (2 fi), 3-factor interactions (3 fi), etc

The General Two-Factor Factorial Experiment

TABLE 5.2

General Arrangement for a Two-Factor Factorial Design

		Factor B			
		1	2	...	b
Factor A	1	$y_{111}, y_{112}, \ldots, y_{11n}$	$y_{121}, y_{122}, \ldots, y_{12n}$		$y_{1b1}, y_{1b2}, \ldots, y_{1bn}$
	2	$y_{211}, y_{212}, \ldots, y_{21n}$	$y_{221}, y_{222}, \ldots, y_{22n}$		$y_{2b1}, y_{2b2}, \ldots, y_{2bn}$
	$\vdots$				
	a	$y_{a11}, y_{a12}, \ldots, y_{a1n}$	$y_{a21}, y_{a22}, \ldots, y_{a2n}$		$y_{ab1}, y_{ab2}, \ldots, y_{abn}$

a levels of factor A; b levels of factor B; n replicates

This is a **completely randomized design**.

Statistical Model (Two Factors: A and B) I

- Statistical model is

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk}$$

for $i = 1, 2, \cdots, a$; $j = 1, 2, \cdots, b$; $k = 1, 2, \cdots, n$, where

$\mu =$ grand mean

$\tau_i = i$th level effect of factor A (controls B) (main effects of A)

$\beta_j = j$th level effect of factor B (controls A) (main effects of B)

$(\tau\beta)_{ij} =$ interaction effect of combination ij

(Explain variation not explained by main effects)

$\epsilon_{ijk} \sim N(0, \sigma^2)$

Statistical Model (Two Factors: A and B) II

- Over-parameterized model: must include certain parameter constraints. Typically

$$\sum_i \tau_i = 0, \quad \sum_j \beta_j = 0, \quad \sum_i (\tau\beta)_{ij} = 0, \quad \sum_j (\tau\beta)_{ij} = 0$$

Estimates

- Rewrite observation as:

$$\begin{aligned} y_{ijk} &= \bar{y}_{...} + (\bar{y}_{i..} - \bar{y}_{...}) + (\bar{y}_{.j.} - \bar{y}_{...}) \\ &+ (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}) + (y_{ijk} - \bar{y}_{ij.}) \end{aligned}$$

- Results in estimates

$$\hat{\mu} = \bar{y}_{...}, \quad \hat{\tau}_i = \bar{y}_{i..} - \bar{y}_{...}, \hat{\beta}_j = \bar{y}_{.j.} - \bar{y}_{...},$$
$$(\hat{\tau\beta})_{ij} = \bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}$$

- Predictive value at level combination ij is

$$\hat{y}_{ijk} = \bar{y}_{ij.}$$

- Residuals are

$$\hat{\epsilon}_{ijk} = y_{ijk} - \bar{y}_{ij.}$$

Effects Estimation (Battery Experiment)

0. $\hat{\mu} = \bar{y}_{...} = 105.5278$
1. Treatment mean response, or cell mean, or predicted value,

$$\hat{y}_{ij} = \hat{\mu}_{ij} = \bar{y}_{ij.} = \hat{\mu} + \hat{\tau}_i + \hat{\beta}_j + (\hat{\tau\beta})_{ij}$$

	temperature		
material	1	2	3
1	134.75	57.25	57.50
2	155.75	119.75	49.50
3	144.00	145.75	85.50

Battery Example II

2. Factor level means
   row means $\bar{y}_{i..}$ for A; column means $\bar{y}_{.j.}$ for B
   material: $\bar{y}_{1..} = 83.166$, $\bar{y}_{2..} = 108.3333$, $\bar{y}_{3..} = 125.0833$;
   temperature: $\bar{y}_{.1.} = 144.8333$, $\bar{y}_{.2.} = 107.5833$,
   $\bar{y}_{.3.} = 64.1666$
3. Main effects estimates

$$\hat{\tau}_1 = -22.3612, \quad \hat{\tau}_2 = 2.8055, \quad \hat{\tau}_3 = 19.555$$
$$\hat{\beta}_1 = 39.3055, \quad \hat{\beta}_2 = 2.0555, \quad \hat{\beta}_3 = -41.3611$$

4. Interactions $((\hat{\tau\beta})_{ij})$

	temperature		
material	1	2	3
1	12.2779	-27.9721	15.6946
2	8.1112	9.3612	-17.4722
3	-20.3888	18.6112	1.7779

Battery Example III

Partitioning the Sum of Squares

- Based on

$$\begin{aligned} y_{ijk} &= \bar{y}_{\cdots} + (\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots}) + (\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdots}) \\ &\quad + (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdots}) + (y_{ijk} - \bar{y}_{ij\cdot}) \end{aligned}$$

- Calculate

$$SS_T = \sum_i \sum_j \sum_k (y_{ijk} - \bar{y}_{\cdots})^2$$

Battery Example IV

- Right hand side simplifies to

$$
\begin{aligned}
SS_T &= bn\sum_i(\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot})^2 + an\sum_j(\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdot\cdot\cdot})^2 \\
&+ n\sum_i\sum_j(\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdot\cdot\cdot})^2 + \sum_i\sum_j\sum_k(y_{ijk} - \bar{y}_{ij\cdot})^2 \\
&= SS_A + SS_B + SS_{AB} + SSE \\
df &= (a-1) + (b-1) + (a-1)(b-1) + ab(n-1)
\end{aligned}
$$

- Using SS/df leads to MS_A, MS_B, MS_{AB}, and MSE.

Testing Hypotheses I

1. Main effects of A:
   $H_0 : \tau_1 = \cdots = \tau_a = 0$ vs H_1 : at least one $\tau \neq 0$.
2. Main effects of B:
   $H_0 : \beta_1 = \cdots = \beta_b = 0$ vs H_1 : at least one $\beta_j \neq 0$.
3. Interaction effects of AB:
   $H_0 : (\tau\beta)_{ij} = 0$ for all i, j vs
   H_1 : at least one $(\tau\beta)_{ij} \neq 0$.

Use F-statistic for testing the hypotheses above:
1: $F_0 = \frac{SS_A/(a-1)}{SSE/(ab(n-1))}$ 2: $F_0 = \frac{SS_B/(b-1)}{SSE/(ab(n-1))}$ 3: $F_0 = \frac{SS_{AB}/(a-1)(b-1)}{SSE/(ab(n-1))}$.

Testing Hypotheses II

Note that

$$
\begin{aligned}
E(MSE) &= \sigma^2 \\
E(MS_A) &= \sigma^2 + bn \sum_i \tau_i^2 / (a-1) \\
E(MS_B) &= \sigma^2 + an \sum_j \beta_j^2 / (b-1) \\
E(MS_{AB}) &= \sigma^2 + n \sum_i \sum_j (\tau\beta)_{ij}^2 / (a-1)(b-1)
\end{aligned}
$$

Analysis of Variance Table I

■ **TABLE 5.3**

The Analysis of Variance Table for the Two-Factor Factorial, Fixed Effects Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
A treatments	SS_A	$a-1$	$MS_A = \frac{SS_A}{a-1}$	$F_0 = \frac{MS_A}{MS_E}$
B treatments	SS_B	$b-1$	$MS_B = \frac{SS_B}{b-1}$	$F_0 = \frac{MS_B}{MS_E}$
Interaction	SS_{AB}	$(a-1)(b-1)$	$MS_{AB} = \frac{SS_{AB}}{(a-1)(b-1)}$	$F_0 = \frac{MS_{AB}}{MS_E}$
Error	SS_E	$ab(n-1)$	$MS_E = \frac{SS_E}{ab(n-1)}$	
Total	SS_T	$abn-1$		

$$\text{SS}_\text{T} = \sum\sum y_{ijk}^2 - y_{...}^2/abn; \ \ \text{SS}_\text{A} = \frac{1}{bn}\sum y_{i..}^2 - y_{...}^2/abn$$

$$\text{SS}_\text{B} = \frac{1}{an}\sum y_{.j.}^2 - y_{...}^2/abn; \ \ \text{SS}_\text{subtotal} = \frac{1}{n}\sum\sum y_{ij.}^2 - y_{...}^2/abn$$

$$\text{SS}_\text{AB} = \text{SS}_\text{subtotal} - \text{SS}_\text{A} - \text{SS}_\text{B}; \ \ \text{SS}_\text{E}\text{=Subtraction}$$

Analysis of Variance Table II

$df_E > 0$ only if $n > 1$. When $n = 1$, no SSE is available.
So we cannot test the effects.

- If we can assume that the interactions are negligible $((\tau\beta)_{ij} = 0)$, MS_{AB} becomes a good estimate of σ and it can be used as MSE
- Caution: if the assumption is wrong, then error and interaction are confounded and testing results can go wrong.

See Battery-Life.SAS.

Checking Assumptions I

- Errors are normally distributed
  Histogram or QQ plot of residuals
- Constant variance
  Residuals vs $\hat{y}_{ij}$ plot, Residuals vs factor A plot, and Residuals vs factor B plot

Checking Assumptions II

- If $n = 1$, no interaction.
  Tukey's Test of Nonadditivity Assumption $(\tau\beta)_{ij} = \gamma\tau_i\beta_j$.
  $H_0 : \gamma = 0$

$$SS_N = \frac{\left[\sum_i \sum_j y_{ij} y_{i.} y_{.j} - y_{..}(SS_A + SS_B + y_{..}^2/ab)\right]^2}{abSS_A SS_B}$$

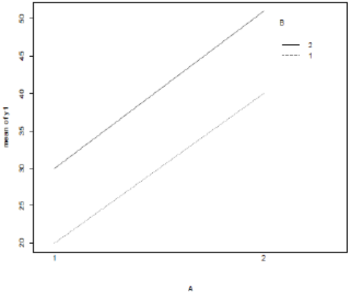
$$F_0 = \frac{SS_N/1}{(SSE - SS_N)/((a-1)(b-1)-1)} \sim F_{1,(a-1)(b-1)-1}$$

the convenient procedure used for RCBD can be employed.

See Battery-Life.SAS.

Interaction I

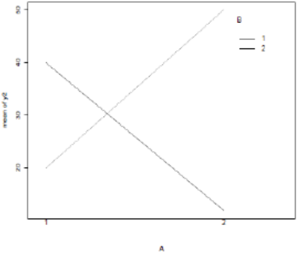
Interaction plot for A and B (No Interaction)



Difference between level means of B (with A fixed at a level) does not depend on the level of A; demonstrated by two parallel lines.

Interaction II

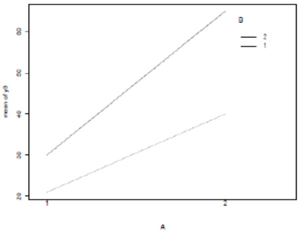
Interaction Plot for A and B (Antagonistic Interaction from B to A)



Difference between level means of B (with A fixed at a level) depends on the level of A. If the trend of mean response over A reverses itself when B changes from one level to another, the interaction is said to be antagonistic from B to A. Demonstrated by two lines with slopes of opposite signs.

Interaction III

Interaction Plot for A and B (Synergistic Interaction from B to A)



Difference between level means of B (with A fixed at a level) depends on the level of A. If the trend of mean response over A do not change when B changes from one level to another, the interaction is said to be synergistic; demonstrated by two unparalleled lines with slopes of the same sign.

See Battery-Life.SAS.

Multiple comparison when factors do not interact I

When factors do not interact, i.e., the F test for interaction is not significant in the ANOVA, factor level means can be compared to draw conclusions regarding their effects on response.

- $var(\bar{y}_{i\cdot\cdot}) = \frac{\sigma^2}{nb}$, $var(\bar{y}_{\cdot j\cdot}) = \frac{\sigma^2}{na}$
- For A or rows: $var(\bar{y}_{i\cdot\cdot} - \bar{y}_{i'\cdot\cdot}) = \frac{2\sigma^2}{nb}$;
  For B or columns: $var(\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdot j'\cdot}) = \frac{2\sigma^2}{na}$;
- Tukey's method
  For rows: $CD = \frac{q_\alpha(a, ab(n-1))}{\sqrt{2}}\sqrt{MSE\frac{2}{nb}}$
  For columns: $CD = \frac{q_\alpha(b, ab(n-1))}{\sqrt{2}}\sqrt{MSE\frac{2}{na}}$
- Bonferroni method: $CD = t_{\alpha/2m, ab(n-1)}SE$ where SE depends on whether for rows or columns.

Multiple comparison when factors do not interact II

However, when interactions are present, be careful interpreting factor level means (row or column) comparisons because it can be misleading. Usually we will directly compare treatment means (or cell means) instead.

Multiple comparison when factors interact: treatment (cell) mean comparison I

When factors interact, multiple comparison is usually directly applied to treatment means

$$\mu_{ij} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} \text{ vs } \mu_{i'j'} = \mu + \tau_{i'} + \beta_{j'} + (\tau\beta)_{i'j'}$$

- $\hat{\mu}_{ij} = \bar{y}_{ij\cdot}$ and $\hat{\mu}_{i'j'} = \bar{y}_{i'j'\cdot}$.
- $var(\bar{y}_{ij\cdot} - \bar{y}_{i'j'\cdot}) = \frac{2\sigma^2}{n}$
- there are ab treatment means and $m_0 = \frac{ab(ab-1)}{2}$ pairs.
- Tukey's method: $CD = \frac{q_\alpha(ab, ab(n-1))}{\sqrt{2}} \sqrt{MSE \frac{2}{n}}$

Multiple comparison when factors interact: treatment (cell) mean comparison II

- Bonferroni's method: $CD = t_{\alpha/2m, ab(n-1)} \sqrt{MSE \frac{2}{n}}$

See Battery-Life.SAS.

Quantitative and Qualitative Factors

- The basic ANOVA procedure treats every factor as if it were qualitative
- Sometimes an experiment will involve both quantitative and qualitative factors, such as in Example 5.1
- This can be accounted for in the analysis to produce regression models for the quantitative factors at each level (or combination of levels) of the qualitative factors
- These response curves and/or response surfaces are often a considerable aid in practical interpretation of the results

Fitting Response Curves or Surfaces I

Goal: Model the functional relationship between lifetime and temperature at every material level. In Battery Experiment:

- Material is qualitative while temperature is quantitative.
- Want to fit the response using effects of material, temperature, and their interactions.
- Temperature has quadratic effect. Here we will simply t and t^2.
- Levels of material need to be converted to indicator variables denoted by x_1 and x_2 as follows

material	x_1	x_2
1	1	0
2	0	1
3	-1	-1

Fitting Response Curves or Surfaces II

- For convenience, convert temperature to -1, 0, and 1 using $t = \frac{\text{temperature}-70}{55}$
- The following model is used:

$$\begin{aligned} y_{ijk} &= \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 t + \beta_4 x_1 t + \beta_5 x_2 t + \beta_6 t^2 \\ &\quad + \beta_7 x_1 t^2 + \beta_8 x_2 t^2 + \epsilon_{ijk} \end{aligned}$$

Want to estimate the coefficients: β_0, β_1, $\cdots$, β_8 using regression.

Fitting Response Curves or Surfaces III

See Battery-Life-Quantity.SAS.
Results: From the SAS output, the fitted model is

$$\begin{aligned}\hat{y} &= 107.58 - 50.33x_1 - 12.17x_2 - 40.33t + 1.71x_1t - 12.79x_2t \\ &\quad -3.08t^2 + 41.96x_1t^2 - 14.04x_2t^2\end{aligned}$$

Note that terms with insignificant coefficients are still kept in the fitted model here. In practice, model selection may be employed to remove unimportant terms and choose the best fitted model. But we will not pursue it in this course.
The model above are in terms of both x_1, x_2 and t. We can specify the level of material, that is, the values of dummy variable x_1 and x_2, to derive fitted response curves for material at different levels.
Three response curves:

Fitting Response Curves or Surfaces IV

- Material at level 1 ($x_1 = 1$, $x_2 = 0$)

$$\hat{y}_{1t} = 57.25 - 38.62t + 38.88t^2$$

- Material at level 2 ($x_1 = 0$, $x_2 = 1$)

$$\hat{y}_{2t} = 119.75 - 53.12t - 17.12t^2$$

- Material at level 3 ($x_1 = -1$, $x_2 = -1$)

$$\hat{y}_{3t} = 145.74 - 29.25t - 31t^2$$

These curves can be used to predict lifetime of battery at any temperature between 15 and 125 degree. But one needs to be careful about extrapolation. For example, the fitted curve at Material level 1 suggests that lifetime of a battery can be infinity when temperature goes to infinity, which is clearly false.

General Factorial Design and Model I

- Factorial Design - including all possible level combinations
- a levels of Factor A, b levels of Factor B, $\cdots$
- Straightforward ANOVA if all fixed effects
- In 3 factor mdoel $\rightarrow$ $nabc$ observations
- Need $n > 1$ to test for all possible interactions
- Statistical Model (3 factors)

$$y_{ijkl} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\beta\gamma)_{jk} + (\tau\gamma)_{ik} + (\tau\beta\gamma)_{ijk} + \epsilon_{ijkl}$$

where $i = 1, 2, \cdots, a$; $j = 1, 2, \cdots, b$; $k = 1, 2, \cdots, c$; $l = 1, 2, \cdots, n$.

General Factorial Design and Model II

- ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F
Factor A	SS_A	$a-1$	MS_A	MS_A/MSE
Factor B	SS_B	$b-1$	MS_B	MS_B/MSE
Factor C	SS_C	$c-1$	MS_C	MS_C/MSE
AB	SS_{AB}	$(a-1)(b-1)$	MS_{AB}	MS_{AB}/MSE
AC	SS_{AC}	$(a-1)(c-1)$	MS_{AC}	MS_{AC}/MSE
BC	SS_{BC}	$(b-1)(c-1)$	MS_{BC}	MS_{BC}/MSE
ABC	SS_{ABC}	$(a-1)(b-1)(c-1)$	MS_{ABC}	MS_{ABC}/MSE
Error	SSE	$abc(n-1)$	MSE	
Total	SS_T	$abcn-1$		

- See Bottling.SAS.

General Factorial Model-2

- Usual assumptions and diagnostics.
- Multiple comparisons: simple extensions of the two-factor case.
- Often higher order interactions are negligible.
- Beyond three-way interactions difficult to picture
- Pooled together with error (increase df_E).

Blocked Factorial Experiment I

Revisit Battery Life Experiment

If we assume further that four operators (1,2,3,4) were hired to conduct the experiment. It is known that different operators can cause systematic difference in battery lifetime. Hence operators should be treated as blocks.
For each treatment (treatment combination), the observations were in the order of the operators 1, 2, 3, and 4.
This is a **blocked factorial design**.

Blocked Factorial Experiment II

Statistical Model:

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \delta_k + \epsilon_{ijk}$$

for $i = 1, 2, \cdots, a$; $j = 1, 2, \cdots, b$; and $k = 1, 2, \cdots, n$; δ_k is the effect of the $k$the block.

- randomization restriction is imposed (complete block factorial design).
- interactions between blocks and treatment effects are assumed to be negligible.

Blocked Factorial Experiment III

- The previous ANOVA table for the experiment should be modified as follows:
  Add: Block Sum of Square

$$SS_{Blocks} = \frac{1}{ab}\sum_k y_{..k}^2 - \frac{y_{...}^2}{abn}, \quad df = n - 1$$

  Modify: Error Sum of Squares:

$$(new)SSE = (old)SSE - SS_{Blocks}, \quad df = (ab-1)(n-1)$$

- other inferences should be modified accordingly.

Blocked Factorial Experiment IV

ANOVA Table

TABLE 5.20

Analysis of Variance for a Two-Factor Factorial in a Randomized Complete Block

Source of Variation	Sum of Squares	Degrees of Freedom	Expected Mean Square	F_0
Blocks	$\frac{1}{ab}\sum_k y_{..k}^2 - \frac{y_{...}^2}{abn}$	$n-1$	$\sigma^2 + ab\sigma_\delta^2$	
A	$\frac{1}{bn}\sum_i y_{i..}^2 - \frac{y_{...}^2}{abn}$	$a-1$	$\sigma^2 + \frac{bn\sum \tau_i^2}{a-1}$	$\frac{MS_A}{MS_E}$
B	$\frac{1}{an}\sum_j y_{.j.}^2 - \frac{y_{...}^2}{abn}$	$b-1$	$\sigma^2 + \frac{an\sum \beta_j^2}{b-1}$	$\frac{MS_B}{MS_E}$
AB	$\frac{1}{n}\sum_i\sum_j y_{ij.}^2 - \frac{y_{...}^2}{abn} - SS_A - SS_B$	$(a-1)(b-1)$	$\sigma^2 + \frac{n\sum\sum(\tau\beta)_{ij}^2}{(a-1)(b-1)}$	$\frac{MS_{AB}}{MS_E}$
Error	Subtraction	$(ab-1)(n-1)$	σ^2	
Total	$\sum_i\sum_j\sum_k y_{ijk}^2 - \frac{y_{...}^2}{abn}$	$abn-1$		

Blocked Factorial Experiment V

Revisit Battery Example See Battery-Life-Blocked-Factorial.SAS.

Factorial Experiment with Two blocking factors I

- Use Latin square as blocking scheme
  Suppose the experimental factors are F_1 and F_2. F_1 has three levels (1,2,3) and F_2 has 2 levels. There are $3 \times 2 = 6$ treatment combinations. These treatments can be represented by Latin letters

F_1	F_2	Treatment
1	1	A
1	2	B
2	1	C
2	2	D
3	1	E
3	2	F

Factorial Experiment with Two blocking factors II

- Two blocking factors are Block 1 and Block 2, each with 6 blocks
  A 6×6 Latin square can be used as the blocking scheme:

	Block 1					
Block 2	1	2	3	4	5	6
1	A	B	C	D	E	F
2	B	C	D	E	F	A
3	C	D	E	F	A	B
4	D	E	F	A	B	C
5	E	F	A	B	C	D
6	F	A	B	C	D	E

Factorial Experiment with Two blocking factors III

- Statistical Model

$$y_{ijkl} = \mu + \alpha_i + \tau_j + \beta_k + (\tau\beta)_{jk} + \theta_l + \epsilon_{ijkl}$$

where α_i and θ_l are blocking effects, τ_j, β_k, and $(\tau\beta)_{jk}$ are the treatment main effects and interactions.

- Radar Detection Experiment Run in a 6×6 Latin Square

Radar Detection Experiment Run in a 6 X 6 Latin Square

	Operator					
Day	**1**	**2**	**3**	**4**	**5**	**6**
1	$A(f_1g_1 = 90)$	$B(f_1g_2 = 106)$	$C(f_1g_3 = 108)$	$D(f_2g_1 = 81)$	$F(f_2g_3 = 90)$	$E(f_2g_2 = 88)$
2	$C(f_1g_3 = 114)$	$A(f_1g_1 = 96)$	$B(f_1g_2 = 105)$	$F(f_2g_3 = 83)$	$E(f_2g_2 = 86)$	$D(f_2g_1 = 84)$
3	$B(f_1g_2 = 102)$	$E(f_2g_2 = 90)$	$G(f_2g_3 = 95)$	$A(f_1g_1 = 92)$	$D(f_2g_1 = 85)$	$C(f_1g_3 = 104)$
4	$E(f_2g_2 = 87)$	$D(f_2g_1 = 84)$	$A(f_1g_1 = 100)$	$B(f_1g_2 = 96)$	$C(f_1g_3 = 110)$	$F(f_2g_3 = 91)$
5	$F(f_2g_3 = 93)$	$C(f_1g_3 = 112)$	$D(f_2g_1 = 92)$	$E(f_2g_2 = 80)$	$A(f_1g_1 = 90)$	$B(f_1g_2 = 98)$
6	$D(f_2g_1 = 86)$	$F(f_2g_3 = 91)$	$E(f_2g_2 = 97)$	$C(f_1g_3 = 98)$	$B(f_1g_2 = 100)$	$A(f_1g_1 = 92)$

Factorial Experiment with Two blocking factors V

- Analysis of Variance for Radar Detection Experiment Run as a 3×2

Source of Variation	Sum of Squares	Degrees of Freedom	General Formula for Degrees of Freedom	Mean Square	F_0	P-Value
Ground clutter, G	571.50	2	$a - 1$	285.75	28.86	<0.0001
Filter type, F	1469.44	1	$b - 1$	1469.44	148.43	<0.0001
GF	126.73	2	$(a - 1)(b - 1)$	63.37	6.40	0.0071
Days (rows)	4.33	5	$ab - 1$	0.87		
Operators (columns)	428.00	5	$ab - 1$	85.60		
Error	198.00	20	$(ab - 1)(ab - 2)$	9.90		
Total	2798.00	35	$(ab)^2 - 1$			