Stochastic Processes (STA3021)
HW1 Solution

1. Since $E \cup F \subset S$, and $E \cup F = E \cup (F \cap E^c)$

$$P(S) = 1 \geq P(E \cup F) = P(E) + P(F \cap E^c) = P(E) + P(F) - P(E \cap F).$$

2. First, construct a disjoint collection of sets $E_1^*, E_2^*, E_3^*, \ldots, E_n^*$ with the property that

$$\bigcup_{i=1}^{n} E_i^* = \bigcup_{i=1}^{n} E_i.$$

We can do so by taking E_i^* by $E_1^* = E_1,\ E_i^* = E_i \setminus \left(\bigcup_{j=1}^{i-1} E_j\right)$. Now we have

$$P\left(\bigcup_{i=1}^{n} E_i\right) = P\left(\bigcup_{i=1}^{n} E_i^*\right) = \sum_{i=1}^{n} P(E_i^*),$$

where the last equality follows from the third axiom of probability since E_i^* are disjoint. Also note from the above construction of E_i^*'s, $E_i^* \subseteq E_i$ implies that $P(E_i^*) \leq P(E_i)$. Therefore, we have that

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i^*) \leq \sum_{i=1}^{n} P(E_i).$$

3. Our experiment is to distribute 4 piles of 13 cards each to four players. Let $E_i = \{i^{th}$ player has exactly 1 ace $\}$, i=1, 2, 3, and 4. Then,

$$P(E_1) = \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}} \cdot \frac{\binom{39}{13\ 13\ 13}}{\binom{39}{13\ 13\ 13}} = \frac{39 \cdot 38 \cdot 37}{51 \cdot 50 \cdot 49}.$$

Now, consider second pile has exactly 1 ace after the event E_1 is given. Then,

$$P(E_2 \mid E_1) = \frac{\binom{3}{1}\binom{36}{12}}{\binom{39}{13}} \cdot \frac{\binom{26}{13\ 13}}{\binom{26}{13\ 13}} = \frac{26 \cdot 25}{38 \cdot 37}$$

since only 3 Aces are left. Similar to above reasonings,

$$P(E_3 \mid E_1E_2) = \frac{\binom{2}{1}\binom{24}{12}}{\binom{26}{13}} = \frac{13}{25},$$

$$P(E_4 \mid E_1E_2E_3) = 1$$

Thus, $P(E_1E_2E_3E_4) = P(E_1) \cdot P(E_2 \mid E_1) \cdot P(E_3 \mid E_1E_2) \cdot P(E_4 \mid E_1E_2E_3) = \frac{39 \cdot 26 \cdot 13}{51 \cdot 50 \cdot 49} = .105$ by conditional property.

Without using conditional probability, we can directly calculate this using multinomial coefficient by

$$P(E_1E_2E_3E_4) = \frac{4!\binom{48}{12\ 12\ 12\ 12}}{\binom{52}{13\ 13\ 13\ 13}} = .105,$$

where 4! comes from the possible ways to distribute Aces to 4 players.

4. Let E_i to be the event that person i selects own hat with $n \geq 2$ (If $n = 1$, then obviously the probability is 1). Note that

$$P\left(\text{no one selects own hat}\right) = 1 - P\left(E_1 \cup E_2 \cup \cdots \cup E_n\right)$$

$$= 1 - [\sum_{i_1} P\left(E_{i_1}\right) - \sum_{i_1<i_2} P\left(E_{i_1}E_{i_2}\right) + \cdots + (-1)^{n+1} P\left(E_{i_1}E_{i_2}\cdots E_{i_n}\right)] \tag{1}$$

by inclusion-exclusion identity. Now let $k \in 1, 2, \ldots, n$, then we have

$$P\left(E_{i_1}E_{i_2}\cdots E_{i_k}\right) = \frac{\text{number of ways k specific men can select own hats}}{\text{total number of ways hats can be arranged}} = \frac{(n-k)!}{n!},$$

$$\sum_{i_1<i_2\cdots<i_k} = \text{number of ways to choose k variables out of n variables} = \binom{n}{k}$$

Thus,

$$\sum_{i_1<i_2\cdots<i_k} P\left(E_{i_1}E_{i_2}\cdots E_{i_k}\right) = \binom{n}{k}\frac{(n-k)!}{n!} = \frac{1}{k!}$$

Therefore, by plug-into (1),

$$P\left(\text{no one selects own hat}\right) = 1 - \frac{1}{1!} + \frac{1}{2!} + \cdots + (-1)^n \frac{1}{n!}.$$

5. Let $(S, \mathcal{F}, P(\cdot))$ is a given probability model, that is, $P(\cdot)$ satisfies three axioms to be a probability measure. Then, for the conditional probability $P(\cdot|B)$ with $P(B) > 0$, observe that

Axiom 1 Since $P(A) \geq 0$ for all $A \in \mathcal{F}$,

$$P(A|B) = \frac{P(AB)}{P(B)} \geq 0.$$

Axiom 2

$$P(S \mid B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1.$$

Axiom 3 If A_i's are mutually exclusive, then

$$P\left(\cup_{i=1}^{\infty} A_i | B\right) = \frac{P\left(\cup_{i=1}^{\infty} A_i \cap B\right)}{P\left(B\right)} = \frac{P\left(\cup_{i=1}^{\infty} (A_i \cap B)\right)}{P\left(B\right)}.$$

Note that sets $(A_i \cap B)$, $i = 1, \ldots, n$ are disjoint because

$$(A_i \cap B) \cap (A_j \cap B) = A_i \cap A_j \cap B = \emptyset$$

if $i \neq j$ since A_i's are all disjoint. Therefore, we have that

$$P\left(\cup_{i=1}^{\infty} A_i | B\right) = \frac{\sum_{i=1}^{\infty} P\left(A_i \cap B\right)}{P\left(B\right)} = \sum_{i=1}^{\infty} \frac{P\left(A_i \cap B\right)}{P\left(B\right)} = \sum_{i=1}^{\infty} P\left(A_i | B\right)$$

as required.