Chap11 Continuity and limits

11.1 Continuous functions

Suppose x and y are related by an equation $y = f(x)$. We say that y ***varies continuously with*** x if, roughly, small changes in x produce only small change in y.

$\boxed{\text{연속성: (원인}(x)\text{을 조금 변화시키면, 결과}(f(x))\text{도 조금 변한다)}}$

Exa1. Do the roots of $x^5 + ax + b = 0$ vary continuously with the coefficients a and b? That is, if we vary a and b a little, do the roots change by only a small amount?

Note that there is **no** explicit algebraic formula for the roots: we cannot find an explicit f representing $x = f(a,b)$: A concrete example will be given in section 2 of Chap12 [Exa B]

※Exa 2. Suppose we have an integral depending on a parameter, such as

$$y\,(= y(s)) = \int_0^\pi \frac{\sin st}{t}\,dt;$$

Does y vary continuously with s? Note that there is no elementary expression in s for the value of the integral. The answer will be given soon [Exa C]

Def. We say that $f(x)$ is continuous at x_0 if it is defined for $x \approx x_0$, and

$$\boxed{\text{given any } \varepsilon > 0, \quad f(x) \underset{\varepsilon}{\approx} f(x_0) \quad \text{for } x \approx x_0}$$

(The definition says roughly that

$f(x)$ should be arbitrarily close to $f(x_0)$, provided x stays sufficiently close to x_0.)

We say that $f(x)$ is continuous on the **open** interval I if it is continuous at every point of I.

Exa A. Show that x^2 is continuous on $I = (-a, a)$, $(a > 0)$

Pf. Fix any $x_0 \in I$. Then, given $\varepsilon > 0$, and any $x \in I$,

$$\begin{aligned} |\, x^2 - x_0{}^2 \,| &= |\, x - x_0 \,||\, x + x_0 \,| \\ &\le |\, x - x_0 \,|\,(|\, x \,| + |\, x_0 \,|) \\ &\le |\, x - x_0 \,| \cdot 2a < \varepsilon, \quad \text{if } x \underset{\varepsilon/2a}{\approx} x_0 \end{aligned}$$

Ex. Show that x^2 is continuous on $\mathbb{R} = (-\infty, \infty)$

Pf. Fix any $x_0 \in (-\infty, \infty)$ and let $\varepsilon > 0$ be given. Note that

$|\, x - x_0 \,| < \delta \quad \Rightarrow \quad |\, x \,| \le |\, x - x_0 \,| + |\, x_0 \,| < \delta + |\, x_0 \,| \le 1 + |\, x_0 \,|$, provided $\delta \le 1$

Thus if $|\, x - x_0 \,| < \delta$, we get

$$\begin{aligned}| x^2 - {x_0}^2 | &\le \big(| x | + | x_0 |\big) | x - x_0 | \\ &< \big(1 + 2 | x_0 |\big) | x - x_0 | \quad \text{if } \ \delta \le 1 \\ &< \big(1 + 2 | x_0 |\big) \delta < \varepsilon, \quad \text{if, in addition, } \ \delta \le \frac{\varepsilon}{1 + 2 | x_0 |}\end{aligned}$$

Therefore,

$$\text{given } \ \varepsilon > 0, \quad | x^2 - {x_0}^2 | < \varepsilon \ \text{ if } \ | x - x_0 | < \underbrace{\delta \overset{\text{let}}{=} \min\left\{1, \frac{\varepsilon}{1 + 2 | x_0 |}\right\}}_{\delta \, = \delta(\varepsilon, \, x_0) > 0}$$

This proves that x^2 is continuous at any point $x_0 \in (-\infty, \infty)$

Another proof: Fix any $x_0 \in (-\infty, \infty)$, and choose $a > 0$ such that $x_0 \in (-a, a)$

Then by the Previous Example, we know that

x^2 is continuous on $(-a, a)$. In particular, x^2 is continuous at x_0

Since x_0 was an arbitrary point in $(-\infty, \infty)$, this proves x^2 is continuous on $(-\infty, \infty)$

Ex. Show that $f(x) = 1 / x$ is continuous at $x = 2$

Pf. Note $f(2) = \frac{1}{2}$. Let $0 < \varepsilon < 1$. Then

$$| f(x) - f(2) | = \left| \frac{1}{x} - \frac{1}{2} \right| = \frac{| x - 2 |}{2 | x |} < \frac{| x - 2 |}{2} < \varepsilon, \ \text{if} \quad | x - 2 | < \varepsilon$$

Here we used the simple fact: $\quad | x - 2 | < \varepsilon\big(< 1\big) \quad \Rightarrow \quad x > 1$

This shows:

$$\text{given } \ 0 < \varepsilon < 1, \quad f(x) \underset{\varepsilon}{\approx} f(2), \ \text{ if } \ x \underset{\varepsilon}{\approx} 2$$

This proves the continuity of $f(x) = \frac{1}{x}$ at $x = 2$

Home Study. Show that $\frac{x}{1 + x}$ is continuous at $x = 1$

Continuity on the **closed intervals**:

We need to extend the definition of continuity to closed intervals I;

for example, $f(x) = \sqrt{1 - x^2}$: its natural domain $= [-1, 1]$

The problem is how to define ***continuity at the endpoints***.

Recall that “ for $x \approx a^+$ ” means “ for $x \approx a, \quad x \ge a$ ”

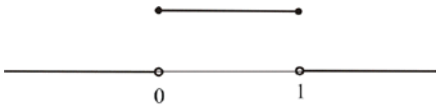
Def. Assuming $f(x)$ is defined for the relevant x -values, we say

$f(x)$ is **right-continuous** at x_0 if, given $\varepsilon > 0$, $f(x) \underset{\varepsilon}{\approx} f(x_0)$ for $x \approx x_0{}^+$;

$f(x)$ is **left-continuous** at x_0 if, given $\varepsilon > 0$, $f(x) \underset{\varepsilon}{\approx} f(x_0)$ for $x \approx x_0{}^-$.

$f(x)$ is continuous on $[a,b]$ if $f(x)$ is $\begin{cases} \text{continuous on } (a,b), \\ \text{right-continuous at } a, \\ \text{left-continuous at } b. \end{cases}$

Note. Even if $f(x)$ is defined on a bigger interval than $[a,b]$, for it to be continuous on $[a,b]$ we only ask it to be **one-sided continuous** at the endpoints. To see why, consider



$$h(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

The function $h(x)$ is not continuous at 0 or 1, yet we want to say it is continuous on $[0,1]$.

Def. We say $f(x)$ is continuous if its **domain is an interval** I of positive or infinite length, and it is continous on I.

- Why don't we just **say** $f(x)$ **is continuous** if **it is continuous on its domain**? Then we would have to say $\dfrac{1}{x}$ is continuous, which seems unreasonable.

(이와 같이 정의역이 연결되어 있지 않을 때(즉, 정의역이 구간이 아닐 때)는 함수 $f(x)$가 연속이라고 하는 것 보다, $f(x)$가 정의역에서 연속이라고 하는 것이 보다 더 자연스럽다)

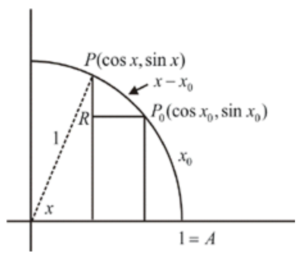
Remark 1. Continuity at x_0 is an aspect of the local behavior of $f(x)$ at x_0, since we verify it by looking at $f(x)$ in a nbd of x_0. (That is, the continuity of a function f at x_0 depends only on the behavior of f at points close to x_0)

Remark 2. Continuity on I is a local property of $f(x)$ since the definitions say we verify it by checking that $f(x)$ is continuous at each point of I.

Exa B. $\sin x$ is continuous.

Pf.

Represent x as the arc length $\overset{\frown}{AP}$, so the point P is $(\cos x, \sin x)$.



From the figure, we see that

$$|\ \overline{PR}\ | \le |\ \widehat{PP_0}\ | \qquad \text{i.e.,} \qquad |\ \sin x - \sin x_0\ | \le |\ x - x_0\ |$$

$$\therefore \quad \text{given } \varepsilon > 0, \quad \sin x \underset{\varepsilon}{\approx} \sin x_0 \quad \text{for } x \underset{\varepsilon}{\approx} x_0$$

Though the picture is drawn for x_0 in the first quadrant, the reasoning is valid regardless of the position of x_0. Since x_0 was arb, this shows $\sin x$ is continuous for every x, i.e., $\sin x$ is continuous.

※Exa C. Show $f(x) = \int_0^\pi \frac{\sin xt}{t}\,dt$ is continuous.

Note. $\int_0^\pi \frac{\sin xt}{t}\,dt \overset{\text{should be regarded as}}{=} \int_0^\pi h(x,t)\,dt$, where $h(x,t) = \begin{cases} \dfrac{\sin xt}{t} & t \ne 0 \\ x & t = 0 \end{cases}$

because $\lim_{t\to 0} \frac{\sin xt}{t} = \lim_{t\to 0} \frac{\sin xt}{xt} \cdot x = x$ for every $x \ne 0$ (and this also holds at $x = 0$)

Pf. Let x_0 be any fixed x-value. We then have

$$\begin{aligned} |\ f(x) - f(x_0)\ | &= \left| \int_0^\pi \frac{\sin xt}{t}\,dt - \int_0^\pi \frac{\sin x_0 t}{t}\,dt \right| \\ &= \left| \int_0^\pi \frac{\sin xt - \sin x_0 t}{t}\,dt \right| \\ &\le \int_0^\pi \frac{|\ \sin xt - \sin x_0 t\ |}{t}\,dt \quad \left(\leftarrow \text{Assume } \left| \int_a^b f(x)\,dx \right| \le \int_a^b |\ f(x)\ |\,dx \text{ if } a < b\right) \\ &\le \int_0^\pi \frac{|\ (x - x_0)t\ |}{t}\,dt = \pi\ |\ x - x_0\ | \end{aligned}$$

$$\therefore \quad \text{given } \varepsilon > 0, \quad f(x) \underset{\pi\varepsilon}{\approx} f(x_0) \quad \text{for } x \underset{\varepsilon}{\approx} x_0$$

Thus by the K-ε Principle, $f(x)$ is continuous at x_0.
Since x_0 was arbitrary, $f(x)$ is continuous (on $(-\infty, \infty)$)

- Discontinuities [= Isolated discontinuity points]

A point x_0, where f is not continuous, is called a point of discontinuity of f if it is **isolated** (i.e., it is continuous at other points near x_0), that is, if f is continuous for $x \underset{\ne}{\approx} x_0$

There are several (four) types of discontinuities, according to the geometric behavior of $f(x)$ at the point: See the text book (p. 154) for the pictures.

(i) removable discontinuity

$$f(x) = x \sin \frac{1}{x}$$

is undefined at $x = 0$. But since $\lim_{x\to 0} f(x) = \lim_{x\to 0} x \sin \frac{1}{x} = 0$, if we define $f(0) = 0$ then f will be continuous at $x = 0$.

(ii) jump //

(iii) infinite //

(iv) essential //

$$g(x) = \sin \frac{1}{x}$$

is undefined at $x = 0$. Since $\lim_{x\to 0} \sin \frac{1}{x}$ does not exist, there is no way one can define $g(0)$ so as to make $g(x)$ is continuous at $x = 0$.

The mathematical description of different types of discontinuity is most easily given using the idea of "limit for a function''

11.2 Limits of functions

The essential difference between *continuity* and *limit* ;

"to be conti at x_0, the ft $f(x)$ must be defined at x_0, but

to have a limit as $x \to x_0$, $f(x)$ need not be defined at x_0 ''

For example, let $f(x) = x^2 / x \overset{\text{means}}{=} \begin{cases} x^2 / x & \text{if } x \neq 0 \\ \text{undefined} & \text{if } x = 0 \end{cases}$

$\Rightarrow$ $f(0)$ does not exist, so $f(x)$ cannot be continuous at $x = 0$, but we can say that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0$$

Def A. (The limit of a function)

Let $f(x)$ be defined for $x \approx x_0$, but not necessarily at x_0

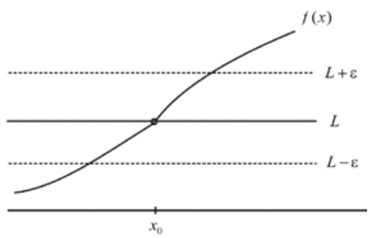
(that is, $f(x)$ is defined for $x \underset{\neq}{\approx} x_0$)

We say $f(x)$ has the limit L as $x \to x_0$ if ($\exists$ a number L such that)

$$\text{given } \varepsilon > 0, \quad f(x) \underset{\varepsilon}{\approx} L \quad \text{for } x \underset{\neq}{\approx} x_0$$

If this is so, we write

$$\lim_{x\to x_0} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to x_0$$



Def B Assume $f(x)$ is defined for $x \underset{\neq}{\approx} {x_0}^+$ or $x \underset{\neq}{\approx} {x_0}^-$, respectively.

(right hand limit) $\lim\limits_{x \to {x_0}^+} f(x) = L$: given $\varepsilon > 0$, $f(x) \underset{\varepsilon}{\approx} L$ for $x \underset{\neq}{\approx} {x_0}^+$

(left hand limit) $\lim\limits_{x \to {x_0}^-} f(x) = L$: given $\varepsilon > 0$, $f(x) \underset{\varepsilon}{\approx} L$ for $x \underset{\neq}{\approx} {x_0}^-$

Theorem. $\lim\limits_{x \to x_0} f(x) = L \quad \Leftrightarrow \quad \lim\limits_{x \to {x_0}^+} f(x) = L$ and $\lim\limits_{x \to {x_0}^-} f(x) = L$

Pf. $(\Rightarrow)$ Obvious

$(\Leftarrow)$ By hypo,

$$\text{given } \varepsilon > 0, \quad f(x) \underset{\varepsilon}{\approx} L \quad \text{for } x \underset{\substack{\neq \\ \delta_1 > 0}}{\approx} {x_0}^+$$

$$f(x) \underset{\varepsilon}{\approx} L \quad \text{for } x \underset{\substack{\neq \\ \delta_2 > 0}}{\approx} {x_0}^-$$

Thus, given $\varepsilon > 0$, $f(x) \underset{\varepsilon}{\approx} L$ for $x \underset{\substack{\neq \\ \delta}}{\approx} x_0$ (where $\delta = \min\{\delta_1, \delta_2\} > 0$)

$$\therefore \quad \lim_{x \to x_0} f(x) = L$$

Exa A. Show directly from the definition that

(a) $\lim\limits_{x \to 0} x \sin \dfrac{1}{x} = 0$ (b) $\lim\limits_{x \to 1^-} \sqrt{1 - x^2} = 0$

Pf. (a) Given $\varepsilon > 0$,

$$|\, x \sin \frac{1}{x} \,| = |\, x \,||\, \sin \frac{1}{x} \,| \le |\, x \,| < \varepsilon, \quad \text{for } |x\,| < \varepsilon, \ x \neq 0 \ \ (i.e., \ \text{for } x \underset{\substack{\neq \\ \varepsilon}}{\approx} 0)$$

(b) Note that the function $\sqrt{1 - x^2}$ is not defined for $x > 1$.
Given $\varepsilon > 0$,

$$\sqrt{1 - x^2} = \sqrt{1 + x}\sqrt{1 - x} < \sqrt{2}\sqrt{1 - x} \quad \text{for } x < 1$$

$$< \varepsilon \quad \text{if } 1 - x < \frac{\varepsilon^2}{2} \quad (i.e., \ \text{for } x \underset{\substack{\neq \\ \varepsilon^2/2}}{\approx} 1^-)$$

Exa B. $f(x) = \dfrac{|\, x^2 - 4 \,|}{x + 2}$ Find $\lim\limits_{x \to -2} f(x)$

Sol.
$$f(x) = \frac{|\, x + 2 \,||\, x - 2 \,|}{x + 2} = \begin{cases} |\, x - 2 \,| & \text{if } x > -2 \\ -\,|\, x - 2 \,| & \text{if } x < -2 \end{cases}$$

$$\therefore \quad \lim_{x \to -2^+} f(x) = 4, \qquad \lim_{x \to -2^-} f(x) = -4$$

$\therefore \quad \lim\limits_{x \to -2} f(x)$ does not exist.

Def C. Limits at infinity

$$\lim_{x\to\infty} f(x) = L \overset{\text{def}}{\Leftrightarrow} \text{given } \varepsilon > 0, \quad f(x) \underset{\varepsilon}{\approx} L \quad \text{for } x \gg 1$$

$$\lim_{x\to-\infty} f(x) = L \overset{\text{def}}{\Leftrightarrow} \text{given } \varepsilon > 0, \quad f(x) \underset{\varepsilon}{\approx} L \quad \text{for } x \ll -1$$

Exa C. Show directly from the definition that

(a) $\displaystyle\lim_{x\to\infty} \frac{1}{1+x^2} = 0$ (b) $\displaystyle\lim_{x\to\infty} \frac{2x}{1+x} = 2$

Pf. (a) Given $\varepsilon > 0$, $\displaystyle\frac{1}{1+x^2} < \varepsilon$ if $1 + x^2 > \displaystyle\frac{1}{\varepsilon}$, for example if $x > \displaystyle\frac{1}{\sqrt{\varepsilon}}$

(b) Left as an exercise.

Def D. Infinite limits

Let $f(x)$ be defined for $x \underset{\neq}{\approx} x_0$, etc

$$\lim_{x\to x_0} f(x) = \infty \overset{\text{def}}{\Leftrightarrow} \text{given any } b > 0, \quad f(x) > b \quad \text{for } x \underset{\neq}{\approx} x_0, \quad \text{etc}$$

Exa D. (a) $\displaystyle\lim_{x\to 0^+} \frac{1}{x} = \infty$, $\displaystyle\lim_{x\to 0^-} \frac{1}{x} = -\infty$, $\displaystyle\lim_{x\to 0} \frac{1}{x^2} = \infty$

(b) $\displaystyle\lim_{x\to\infty} x^2(k + \cos x) = \infty \quad \Leftrightarrow \quad k > 1$

Sol. (b) $(\Leftarrow)$ Assume $k > 1$.

Since $k + \cos x \geq k - 1$ for all x, we have, given $b > 0$,

$$x^2(k + \cos x) \geq x^2(k-1) > b \quad \text{for} \quad x > \sqrt{\frac{b}{k-1}}$$

$(\Rightarrow)$ If $k \leq 1$, then $x^2(k + \cos x) \leq 0$ when $x = \pi, 3\pi, 5\pi, \cdots$.

Thus, it is not true that

$$x^2(k + \cos x) > b > 0, \quad \text{for} \quad x \gg 1.$$

11.3 Limit theorems for functions

Principle A. Error form for limit

Write $f(x) = L + e(x)$. Then

$$f(x) \to L \quad \Leftrightarrow \quad e(x) \to 0, \quad \text{as} \quad x \to x_0, \ \text{etc}$$

Principle B. The K-ε Principle for limits of functions

If one can prove, for some K not depending on x and ε, that

$$\text{given } \varepsilon > 0, \quad f(x) \underset{K\varepsilon}{\approx} L \quad \text{for } x \underset{\neq}{\approx} x_0, \ \text{etc},$$

then $f(x) \to L$ as $x \to x_0$.

Theorem A. Algebraic limit theorems

If a, b are constants, and $f(x) \to L$, $g(x) \to M$ as $x \to x_0$, etc.,

(i) Linearity theorem $af(x) + bg(x) \to aL + bM$ as $x \to x_0$

(ii) Product theorem $f(x) \cdot g(x) \to L \cdot M$ as $x \to x_0$

(iii) Quotient theorem $f(x) / g(x) \to L / M$ as $x \to x_0$

$$(\text{when } g(x) \neq 0 \ \text{ for } \ x \underset{\neq}{\approx} x_0, \ \text{ and } \ M \neq 0)$$

Pf. Exercise (use Principle A and Principle B).

Theorem A_∞ Infinite limit theorems

In the statements below, the limits are taken as $x \to x_0$, etc., while the properties are assumed to hold for $x \underset{\neq}{\approx} x_0$, etc.

(i) $$f(x) \to \infty \quad \& \quad \begin{cases} g(x) \to \infty, \ \text{or} \\ g(x) \ \text{bounded below} \end{cases} \quad \Rightarrow \quad f(x) + g(x) \to \infty$$

(ii) $$f(x) \to \infty \quad \& \quad \begin{cases} g(x) \to L, \ L > 0 \quad \text{or} \\ g(x) > k > 0 \ \text{ for some } k \end{cases} \quad \Rightarrow \quad f(x) \cdot g(x) \to \infty$$

(iii) $$f(x) \to \infty \quad \underset{\text{if } f(x)>0}{\overset{\Rightarrow}{\Leftarrow}} \quad \frac{1}{f(x)} \to 0$$

Pf. Ex

Theorem B. Squeeze theorem

Suppose $f(x) \le g(x) \le h(x)$ for $x \underset{\neq}{\approx} x_0$, etc. Then

$$f(x) \to L \ \text{ and } \ h(x) \to L \ \text{ as } \ x \to x_0 \quad \Rightarrow \quad g(x) \to L \ \text{ as } \ x \to x_0$$

Theorem B_∞. Squeeze theorem for infinite limits

Suppose $f(x) \geq g(x)$ for $x \underset{\neq}{\approx} x_0$, etc. Then

$$\lim_{x\to x_0} g(x) = \infty \quad \Rightarrow \quad \lim_{x\to x_0} f(x) = \infty, \text{ etc.}$$

Pf. Ex

Exa A. Show that $\sqrt[n]{x} \longrightarrow 1$ as $x \longrightarrow 1$.

Sol.

$$\begin{array}{lll} 1 < \sqrt[n]{x} < x & \text{if } x > 1 & \\ \downarrow \qquad\quad \downarrow & & \text{Thus by Squeeze theorem, } \lim_{x\to 1^+} \sqrt[n]{x} = 1 \quad -(i) \\ 1 \qquad\quad\; 1 & \text{as } x \longrightarrow 1^+ & \end{array}$$

$$\begin{array}{lll} x < \sqrt[n]{x} < 1 & \text{if } 0 < x < 1 & \\ \downarrow \qquad\quad \downarrow & & \text{Thus by Squeeze theorem, } \lim_{x\to 1^-} \sqrt[n]{x} = 1 \quad -(ii) \\ 1 \qquad\quad\; 1 & \text{as } x \longrightarrow 1^- & \end{array}$$

$$(i) \text{ and } (ii) \quad \Rightarrow \quad \lim_{x\to 1} \sqrt[n]{x} = 1$$

Exa B. Let $f(x) = \displaystyle\int_1^x \frac{\sqrt{1+t}}{t}\, dt$. Show $f(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Pf. $\dfrac{\sqrt{1+t}}{t} \geq \dfrac{\sqrt{t}}{t} = \dfrac{1}{\sqrt{t}}$ if $t > 0$

$$\therefore \quad \int_1^x \frac{\sqrt{1+t}}{t} dt \geq \int_1^x \frac{1}{\sqrt{t}} dt = 2\sqrt{x} - 2 \text{ if } x \geq 1 \quad (\therefore \text{ for } x \gg 1)$$

$$\& \quad \lim_{x\to\infty} (2\sqrt{x} - 2) = \infty$$

$$\therefore \quad \lim_{x\to\infty} f(x) = \infty$$

Theorem C. **LLT** (for functions)

If the limits exist,

$$f(x) \leq M \text{ for } x \underset{\neq}{\approx} x_0 \quad \Rightarrow \quad \lim_{x\to x_0} f(x) \leq M$$

$$f(x) \leq g(x) \text{ for } x \underset{\neq}{\approx} x_0 \quad \Rightarrow \quad \lim_{x\to x_0} f(x) \leq \lim_{x\to x_0} g(x), \text{ etc}$$

Theorem D. Function Location Theorem (**FLT**)

If the limit exists,

$$\lim_{x\to x_0} f(x) < M \quad \Rightarrow \quad f(x) < M \quad \text{for} \quad x \underset{\neq}{\approx} x_0$$

Exa C. Let $f(x) = \int_0^x \frac{dt}{\sqrt{1-t^4}}$. Estimate $\lim_{x\to 1^-} f(x)$ from above

Sol. For $0 \le t < 1$, we have

$$\begin{aligned} & t^4 \le t^2 \\ \Rightarrow & \sqrt{1-t^4} \ge \sqrt{1-t^2} \\ \Rightarrow & \int_0^x \frac{dt}{\sqrt{1-t^4}} < \int_0^x \frac{dt}{\sqrt{1-t^2}}, \quad \text{for} \quad 0 < x < 1 \\ & \qquad\qquad\qquad = \sin^{-1} x \le \frac{\pi}{2} \end{aligned}$$

Thus $\lim_{x\to 1^-} f(x) \le \frac{\pi}{2}$ by **LLT** (for functions)

Exa D. Let $f(x) = \frac{x^3+9}{1-x^2-x^3}$. Show $f(x) < -0.9$ for $x \gg 1$.

Sol. $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1+\frac{9}{x^3}}{\frac{1}{x^3} - \frac{1}{x} - 1} = -1 < -0.9$$

$\therefore \quad f(x) < -0.9 \quad \text{for} \quad x \gg 1$ by **FLT**

11.4 Limits and continuous functions

Theorem A. Limit form of continuity [the most popular definition of continuity]

Let $f(x)$ be defined for $x \approx x_0$. Then

$$f(x) \text{ is continuous at } x_0 \iff \lim_{x\to x_0} f(x) = f(x_0)$$

Pf. What we must show is: given $\varepsilon > 0$,

$$f(x) \underset{\varepsilon}{\approx} f(x_0) \quad \text{for} \quad x \approx x_0 \quad \Leftrightarrow \quad f(x) \underset{\varepsilon}{\approx} f(x_0) \quad \text{for} \quad x \underset{\neq}{\approx} x_0$$

$\Rightarrow$ is trivial

$\Leftarrow$ is also true since $f(x) = f(x_0)$ if $x = x_0$

※**Theorem B** (Sign preserving property of continuous functions)

$$f(x) \text{ is continuous at } x_0 \text{ and } f(x_0) > 0 \quad \Rightarrow \quad f(x) > 0 \quad \text{for} \quad x \approx x_0.$$

First pf. The hypo says $\lim_{x\to x_0} f(x)(= f(x_0)) > 0$ (by Theorem A)

But according to the FLT,

$$\lim_{x\to x_0} f(x) > 0 \quad \Rightarrow \quad f(x) > 0 \ \text{ for } \ x \underset{\neq}{\approx} x_0;$$

This holds for $x \approx x_0$ as well, since by hypo $f(x_0) > 0$

Second pf. Choose an ε so that $f(x_0) > \varepsilon > 0$.

Since $f(x)$ is conti at x_0, $f(x) \underset{\varepsilon}{\approx} f(x_0)$ for $x \approx x_0$.

These imply that

$$0 < f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon \quad \text{for} \ \ x \approx x_0$$

$$\therefore \quad f(x) > 0 \ \text{ for } \ x \approx x_0.$$

Remark. $f(x)$ is continuous at x_0 and $f(x_0) < 0 \quad \Rightarrow \quad f(x) < 0$ for $x \approx x_0$.

Theorem C. Algebraic operations on continuous fts

Suppose f & g are conti at x_0, and a, b are constants. Then

$$\left.\begin{array}{ll} \text{(i)} & af + bg \\ \text{(ii)} & fg \\ \text{(iii)} & f\,/\,g \ \ (\text{if } g(x_0) \neq 0) \end{array}\right\} \ \text{are conti at } x_0$$

Note. To show (iii), we must first verify that $g(x) \neq 0$ for $x \approx x_0$

This can be verified as follows;

$$g(x_0) \neq 0 \quad \Rightarrow \quad \text{either} \ \ g(x_0) > 0 \quad \text{or} \quad g(x_0) < 0$$

$$\overset{g:\ \text{conti at } x_0}{\Rightarrow} \quad \text{either} \ \ g(x) > 0 \quad \text{or} \quad g(x) < 0 \ \text{ for } \ x \approx x_0 \quad (\text{by Thm B})$$

$$\Rightarrow \quad g(x) \neq 0 \ \text{ for } \ x \approx x_0$$

Exa A. (a) Any polynomial $a_0x^n + a_1x^{n-1} + \cdots + a_n$ is conti for all x

(b) All rational functions are conti, except at the points where the denominator is 0.

⦿ We return to describe the types of discontinuity (talked about 11-1) by using the limit

1. Removable discontinuity. $\lim_{x\to x_0} f(x) = L$, but $f(x_0)$ is undefined or $L \neq f(x_0)$

   (a) $f(x) = \dfrac{x^2 - 1}{x - 1}$ has a removable discontinuity at $x = 1$

   Since $\lim_{x\to 1} f(x) = 2$, we can remove it by defining $f(1) = 2$

(b) $f(x) = \dfrac{\sin x}{x}$ has a removable discontinuity at $x = 0$

(can remove it by defining $f(0) = 1$)

(c) $f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1} & x \neq 1 \\ 3 & x = 1 \end{cases}$ has a removable discontinuity at $x = 1$

2. Jump discontinuity.

$$\lim_{x \to x_0^+} f(x) \neq \lim_{x \to x_0^-} f(x), \text{ but both limits exist}$$

(a) $\operatorname{sgn} x = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \end{cases}$ has a jump discontinuity at 0,

since $\lim_{x \to 0^+} \operatorname{sgn} x = 1 \neq -1 = \lim_{x \to 0^-} \operatorname{sgn} x$

(b) $f(x) = \dfrac{|x^2 - 1|}{x - 1} = \dfrac{|x - 1||x + 1|}{x - 1}$ has a jump discontinuity at 1

3. Infinite discontinuity

$$\lim_{x \to x_0^+} f(x) \ (\text{or } \lim_{x \to x_0^-} f(x)\ ) = \infty \text{ or } -\infty$$

(a) $\dfrac{1}{x^2}$ at 0 since $\lim_{x \to 0} \dfrac{1}{x^2} = \infty$

(b) $\dfrac{1}{x}$ at 0, $\tan x$ at $\pi / 2$

4. Essential discontinuity

Any discontinuity not of the preceding three types; for example,

$\sin \dfrac{1}{x}$ at 0

Pf. (we use the 'sequential continuity theorem' that will be proved in 11-5)

(i) $\lim_{x \to 0} \sin \dfrac{1}{x}$ does not exist

$\because$ $x_n = \dfrac{1}{n\pi} \to 0$ & $y_n = \dfrac{1}{2n\pi + \pi / 2} \to 0$

but $\lim_{n \to \infty} \sin(1/x_n) = \lim_{n \to \infty} \sin n\pi = 0 \neq 1 = \lim_{n \to \infty} \sin(2n\pi + \pi/2) = \lim_{n \to \infty} \sin(1/y_n)$

$\therefore$ 0 is not a removable discontinuity

(ii) $\lim_{x\to 0^+} \sin\frac{1}{x}$ does not exist (; this can be verified as above).

$\therefore$ 0 is not a jump discontinuity

(iii) $|\sin\frac{1}{x}| \le 1 \quad \forall x \ne 0$

$\therefore$ $\lim_{x\to 0^+} \sin\frac{1}{x} \ne \infty$ or $-\infty$ & $\lim_{x\to 0^-} \sin\frac{1}{x} \ne \infty$ or $-\infty$

$\therefore$ 0 is not an infinite discontinuity

Consequently, 0 is an essential discontinuity

- How to understand the continuity of the function $f(x) = 1/x$?

Answer 1: $f(x) = 1/x$ is continuous on the natural domain $\{x : x \ne 0\}$

Remark: The natural domain $\{x : x \ne 0\} = \mathbb{R} \setminus \{0\}$ is **not** an interval

Answer 2 [Most natural answer to high-school students]:

$f(x) = \frac{1}{x}$ is not continuous on the extended domain $\mathbb{R}$; this means that

$$f(x) \overset{\text{extended def}}{:=} \begin{cases} \frac{1}{x}, & x \ne 0 \\ \text{any (finite) value}, & x = 0 \end{cases} \quad \text{is discontinuous at } x = 0$$

More precisely,

$$f(x) \overset{\text{extended def}}{:=} \begin{cases} \frac{1}{x}, & x \ne 0 \\ \text{any (finite) value}, & x = 0 \end{cases} \quad \text{is} \quad \begin{cases} \text{continuous if } x \ne 0 \\ \text{discontinuous at } x = 0 \end{cases}$$

A related exercise: How to answer the continuity of the rational function $\frac{x+3}{x(x-1)(x+2)}$?

A natural answer: $\frac{x+3}{x(x-1)(x+2)}$ is $\begin{cases} \text{continuous if } x \ne 0, 1, -2 \\ \text{discontinuous at the points } x = 0, 1, -2 \end{cases}$

Exa. What can we say about the continuity of $f(x) = \frac{\sin x}{x}$?

A natural answer: $f(x) = \frac{\sin x}{x}$ is continuous at 0 [by defining $f(0) = 1 = \lim_{x\to 0} \frac{\sin x}{x}$]

Indeed, $f(x) = \frac{\sin x}{x}$ is continuous at any point non-zero x, and

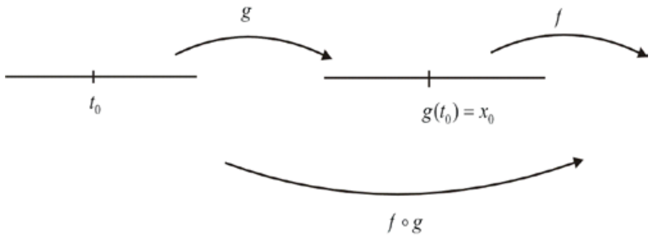
0 is a removable discontinuity point of $f(x)$

Therefore, $f(x) = \frac{\sin x}{x}$ is continuous on $\mathbb{R}$

Theorem D. Composition theorem

Let $x = g(t), \quad x_0 = g(t_0)$

$$\left.\begin{array}{l} g(t) \text{ is conti at } t_0 \\ f(x) \text{ is conti at } x_0 \end{array}\right\} \quad \Rightarrow \quad f(g(t)) \text{ is conti at } t_0$$



Pf. Given $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$f(x) \underset{\varepsilon}{\approx} f(x_0) \quad \text{for} \quad x \underset{\delta}{\approx} x_0$$ (by the continuity of f at x_0)

Also,

$$g(t) \underset{\delta}{\approx} g(t_0) \quad \text{for} \quad t \approx t_0$$ (by the continuity of g at t_0)

This means that $x \underset{\delta}{\approx} x_0$ for $t \approx t_0$ (since $g(t) = x$)

Therefore, we get

$$\text{given } \varepsilon > 0, \quad f(g(t)) \underset{\varepsilon}{\approx} f(g(t_0)) \quad \text{for} \quad t \approx t_0$$

Theorem D'

Let $x = g(t)$, and I and J be intervals. Then

$$\left.\begin{array}{l} g(t) \text{ is conti on } I \\ g(I) \subset J \\ f(x) \text{ is conti on } J \end{array}\right\} \quad \Rightarrow \quad f(g(t)) \text{ is conti on } I$$

Pf. Comes from "Continuity of f on I $\overset{def}{\Leftrightarrow}$ Continuity of f at each point $x_0 \in I$"

Exa B.

(a) (We have seen that) $\sin x$ is conti

$\therefore$ $\cos x = \sin(x + \dfrac{\pi}{2})$ is conti by Theorem D'

$\tan x = \dfrac{\sin x}{\cos x}$, $\sec x = \dfrac{1}{\cos x}$, $\csc x = \dfrac{1}{\sin x}$ are conti whenever they are defined

(b) $\sin(x^2 + 1)$ is conti on $\mathbb{R}$

$\cos^3\left(\dfrac{1}{x}\right)$ is conti on $\mathbb{R} \setminus \{0\}$

Exa. Show that $f(x)$ is conti, then $|\, f(x) \,|$ is also conti

Pf. This follows from:

$$|\, f(x) \,| = |\quad| \circ f(x) \quad \& \quad |\quad| \text{ is conti (on } \mathbb{R})$$

11.5 Continuity and sequences

- Is there any good way to prove the followings?

$$\sin\frac{1}{n} \to 0; \qquad e^{\frac{1}{n}} \to 1;$$

$$a_n \geq 0 \text{ and } a_n \to L \quad \Rightarrow \quad \sqrt{a_n} \to \sqrt{L}$$

These naturally lead to the **question**: $\quad x_n \to a \;+\; \boxed{f: \;?} \quad \Rightarrow \quad f(x_n) \to f(a)$

Theorem. Sequential Continuity Theorem [very useful]

$$x_n \to a \quad \text{and} \quad \boxed{f(x) \text{ is continuous at } a} \quad \Rightarrow \quad f(x_n) \to f(a)$$

Pf. Given $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$f(x) \underset{\varepsilon}{\approx} f(a) \quad \text{for} \quad x \underset{\delta}{\approx} a, \quad \text{since} \quad f(x) \quad \text{is continuous at} \quad a.$$

Also, we see that $\quad x_n \underset{\delta}{\approx} a \quad$ for $\quad n \gg 1$, since $\quad x_n \to a.$

$$\therefore \quad f(x_n) \underset{\varepsilon}{\approx} f(a) \quad \text{for} \quad n \gg 1, \quad \text{which shows} \quad f(x_n) \to f(a)$$

Remark. (one-sided continuity)

$$\underbrace{x_n \to a, \;\; x_n \geq a}_{\text{i.e., } x_n \to a^+ \text{ (for short)}} \quad \text{and} \quad \boxed{f(x) \text{ is right-conti at } a} \quad \Rightarrow \quad f(x_n) \to f(a)$$

Cor. If $\exists$ a seq $\{x_n\}$ such that $x_n \to a$, but $\lim\limits_{n\to\infty} f(x_n) \neq f(a)$, then f is not conti at a.

Or, if $\exists$ two seqs $\{x_n\}$ and $\{x_n'\}$ s.t. $x_n \to a$ & $x_n' \to a$, but $\lim\limits_{n\to\infty} f(x_n) \neq \lim\limits_{n\to\infty} f(x_n')$, then f is not conti at a.

Exa. Show that $f(x) = \cos\dfrac{1}{x}$ has an essential discontinuity at 0.

Pf. $\lim\limits_{x\to 0^+} \cos\dfrac{1}{x} \;\; (= \lim\limits_{x\to 0^-} \cos\dfrac{1}{x}\,) \neq \pm\infty$, since $|\cos\dfrac{1}{x}| \leq 1$ for all $x \neq 0$.

Thus, it suffices to show $\lim\limits_{x\to 0^+} \cos\dfrac{1}{x}$ does not exist.

Suppose $\lim\limits_{x\to 0^+} f(x) = L$. Define $f(0) = L$; then $f(x)$ becomes right-continuous at 0.

Consider the two sequences

$$x_n = \frac{1}{2n\pi} \quad (f(x_n) = 1 \;\text{ for all } n\,), \qquad x_n' = \frac{1}{(2n+1)\pi} \quad (f(x_n') = -1 \;\text{ for all } n\,)$$

Since $x_n \to 0^+$ and $x_n' \to 0^+$,

$$f(x_n) \to f(0) \quad \text{and} \quad f(x_n') \to f(0) \qquad \text{by the Sequential Continuity Theorem}$$

Hence $f(0) = 1$ and $f(0) = -1$. This is absurd.

$$\therefore \quad \lim_{x\to 0^+} f(x) \quad \text{does not exist}$$

Theorem A. Limit form of sequential continuity

Let $f(x)$ be defined for $x \underset{\neq}{\approx} a$, and assume $\lim\limits_{x\to a} f(x) = L$. Then

$$x_n \longrightarrow a, \;\; x_n \neq a \qquad \Rightarrow \quad f(x_n) \to L$$

Pf. Let $\varepsilon > 0$. Then $\exists\, \delta > 0$ such that $f(x) \underset{\varepsilon}{\approx} L$ for $x \overset{\neq}{\underset{\delta}{\approx}} a$

Since $x_n \longrightarrow a, \;\; x_n \neq a$, we also find that $x_n \overset{\neq}{\underset{\delta}{\approx}} a$ for $n \gg 1$,

$\therefore \quad f(x_n) \underset{\varepsilon}{\approx} L$ for $n \gg 1$, which shows $f(x_n) \to L$

Exa. Show that $\lim\limits_{x\to\infty} \sin x$ does not exist

Pf. Suppose that $\lim\limits_{x\to\infty} \sin x = L$. Then by Theorem A,

$\lim\limits_{n\to\infty} \sin(n\pi) = L$ since $n\pi \to \infty$ $\quad \therefore\; L = 0$

& : a contradiction.

$\lim\limits_{n\to\infty} \sin(\frac{\pi}{2} + 2n\pi) = L$ since $\frac{\pi}{2} + 2n\pi \to \infty$ $\quad \therefore\; L = 1$

Theorem B (the converse of Theorem A)

Let $f(x)$ be defined for $x \underset{\neq}{\approx} a$, and suppose that $f(x_n) \to L$ for all $\{x_n\}$ s.t. $x_n \longrightarrow a$ with $x_n \neq a$. Then $\lim\limits_{x\to a} f(x) = L$.

Pf. Suppose that the conclusion does not hold. That is,

$$\sim\Big(\forall\varepsilon > 0, \;\; \exists\delta{>}0 \;\text{ such that } \forall x \text{ with } 0 < |\, x - a \,| < \delta \;\; \Rightarrow \;\; |f(x) - L\,| < \varepsilon\Big)$$

This is equivalent to:

$$\Big(\exists\varepsilon_0 > 0 \text{ such that } \forall\delta{>}0, \;\; \exists x \text{ with } 0{<}\,|\, x - a \,| < \delta, \;\; \text{but } \;|f(x) - L\,| \geq \varepsilon_0\Big)$$

Taking $\delta = 1/n$ for $n \in \mathbb{N}$, we see that $\exists x$ with $0 <| \, x - a \,| < \frac{1}{n}(=\delta)$, but $|\, f(x) - L \,| \geq \varepsilon_0 (> 0)$

This means precisely that for every $n \in \mathbb{N}$, $\exists x_n \in D_f$ such that

$$0 <|\, x_n - a \,| < \frac{1}{n}, \text{ but } \;|\, f(x_n) - L \,| \geq \varepsilon_0$$

Note that $0 <|\, x_n - a \,| < \frac{1}{n}$ clearly implies that $x_n \to a$ with $x_n \neq a$.

Accordingly, we have a sequence $\{x_n\}$ such that

$$x_n \to a \;\text{ with } x_n \neq a \text{, while } \; f(x_n) \not\to L \;\text{ since } |\, f(x_n) - L \,| \geq \varepsilon_0$$

This contradicts our assumption, so we conclude that $\lim\limits_{x\to a} f(x) = L$.

TheoremA + TheoremB:

Let $f(x)$ be defined for $x \underset{\neq}{\approx} a$. Then

$\lim\limits_{x\to a} f(x) = L \Leftrightarrow f(x_n) \to L$ for every sequence $x_n \in D_f$, $x_n \neq a$ such that $x_n \to a$

Remark (Sequential Continuity Theorem, revisited)
Let f be defined on an interval I and $a \in I$. Then

$$\boxed{f(x) \text{ is continuous at } a}$$

iff

$f(x_n) \to f(a)$ for every sequence $x_n \in I$ with $x_n \to a$

Exa. Let $f(x) = \begin{cases} 1, & \text{if } x \text{ is a rational number} \\ 0, & \text{if } x \text{ is an irrational number} \end{cases}$

Show that f is discontinuous at every $c \in \mathbb{R}$.

Pf. If c is a rational, then $x_n := c + \dfrac{\sqrt{2}}{n}$ is a sequence of irrational numbers such that $x_n \to c$.

Hence $f(x_n) = 0$ for every $n \in \mathbb{N}$, while $f(c) = 1$.

$\therefore\ x_n \to c$ but $f(x_n) \not\to f(c)$; so f is discontinuous at every rational c

If c is an irrational, then

$x_n := c^{(n)} [= \text{ the n-th truncation of } c]$ is a sequence of rational numbers such that $x_n \to c$.

Hence $f(x_n) = 1$ for every $n \in \mathbb{N}$, while $f(c) = 0$.

$\therefore\ x_n \to c$ but $f(x_n) \not\to f(c)$; so f is discontinuous at every irrational c

Revisit to Composition theorem: Let $x = g(t)$, $x_0 = g(t_0)$

$$\left.\begin{array}{l} g(t) \text{ is conti at } t_0 \\ f(x) \text{ is conti at } x_0 \end{array}\right\} \Rightarrow f(g(t)) \text{ is conti at } t_0$$

An alternative proof by **using Sequential Continuity Theorem**:

Let $t_n \to t_0$

$\Rightarrow\ g(t_n) \to g(t_0) = x_0$ $[\leftarrow g$ is continuous at $t_0]$

$\Rightarrow\ f(g(t_n)) \to f(x_0) = f(g(t_0))$ $[\leftarrow f$ is continuous at $x_0]$

HS1. Let $f(x) = \begin{cases} x, & \text{if } x \text{ is a rational number} \\ 1-x, & \text{if } x \text{ is an irrational number} \end{cases}$

Show that the function $f(x)$ is continuous only at $x = 1/2$.

HS2. Let $f(x) = \begin{cases} x, & x \text{ is a rational number} \\ x^2, & x \text{ is an irrational number} \end{cases}$

Show that the function $f(x)$ is continuous only at $x = 0$ and $x = 1$.