Chap 15. Differentiation: Global Properties

15.1 The mean-value theorem

Theorem **MVT**

$$f(x):\begin{cases}\text{conti on } [a,\, b] \\ \text{diff on } (a,\, b)\end{cases} \quad\Rightarrow\quad \exists\, c \in (a,\, b) \ \text{ s.t. } \ f(b)-f(a)=f'(c)(b-a)$$

• Everybody knows its geometric meaning.

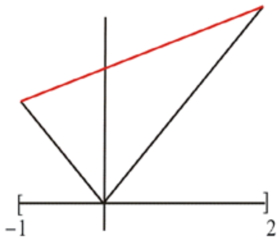
⦿ Note. The hypothesis of MVT can be stated as:

$$f(x):\begin{cases}\text{diff on } (a,\, b) & \left(\overset{\text{known}}{\Rightarrow} \quad \text{conti on } (a,\, b)\ \right) \\ \underbrace{\text{conti at the endpoints } \ a \ \ \&\ b}_{\text{one-sided continuity}} & \end{cases}$$

Ex. Discuss the applicability of MVT to

(a) $f(x)=|\, x\,|$ on $[-1,\, 2]$; (b) $f(x)=\sqrt{x}$ on $[0,\, a]$

Sol. (a)



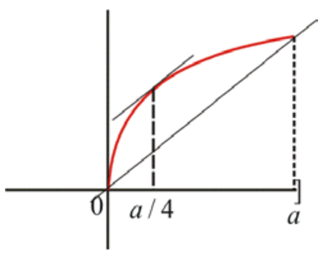
$f(x)=|\, x\,|$ is not diff at 0, so the MVT does not apply.

In fact, the conclusion does not hold; because

$$\frac{f(2)-f(-1)}{2-(-1)}=\frac{2-1}{3}=\frac{1}{3},$$

while $f'(x)$ can only be ± 1

(b)



$f(x)=\sqrt{x}$ is diff on $(0,\, a]$ but not at 0.

However, since $f(x)$ is right conti at 0, it satisfies the Hypo of MVT. So the conclusion also holds. For example, we can see:

$$\frac{f(a)-f(0)}{a-0}=\frac{1}{\sqrt{a}}; \qquad f'(c)=\frac{1}{\sqrt{a}},\ \text{ if } c=\frac{a}{4} \quad \text{since } \ f'(x)=\frac{1}{2\sqrt{x}}$$

Lemma **Rolle's theorem** (The special case of MVT where $f(x)$ is zero at both ends)

$$f(x):\begin{cases}\text{conti on } [a,\, b] \\ \text{diff on } (a,\, b)\end{cases}, \quad \text{and } \ f(b)=f(a)=0 \quad\Rightarrow\quad \exists\, c\in(a,\, b) \ \text{ s.t. } \ f'(c)=0$$

Pf $\quad f \in C[a,b] \overset{\text{MmT}}{\Rightarrow} \exists x_{\max}, x_{\min} \in [a,b]$ s.t. $f(x_{\min}) \le f(x) \le f(x_{\max})\ \forall x \in [a,b]$

Suppose $x_{\max} = c$ for some $c \in (a,b)$; so that c is a local extremum pt for $f(x)$. Then

$f'(c) = 0$ (by Theorem 14.3 B --- Fermat's Critical Point Theorem)

Suppose $x_{\min} = c$ for some $c \in (a,b)$; so that c is a local extremum pt for $f(x)$. Then

$f'(c) = 0$ (again by Fermat's Critical Point Theorem)

If both $x_{\max}$ & $x_{\min}$ are end pts of $[a,b]$ $\left(\text{i.e., } \{x_{\max}, x_{\min}\} = \{a,b\}\right)$, then we have

$f(x) \equiv 0$ on $[a,b]\left(\Leftarrow f(a) = f(b) = 0\right)$; and thus (trivially) $f'(c) = 0\ \ \forall c \in (a,b)$.

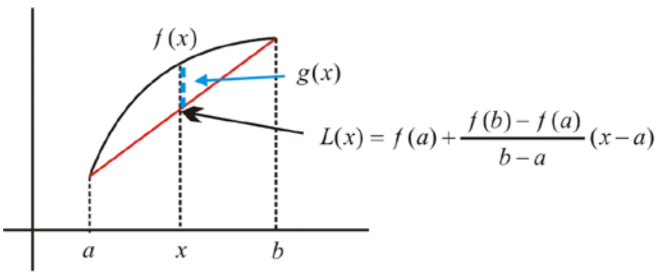
Remark. A version of Rolle's theorem (often called just Rolle's theorem)

[A special case of MVT where $f(x)$ has the same value at both ends]

$$f(x) : \begin{cases} \text{conti on } [a,b] \\ \text{diff on } (a,b) \end{cases}, \text{ and } f(b) = f(a) \text{ (it is not nec zero)} \Rightarrow \exists\, c \in (a,b) \text{ s.t. } f'(c) = 0$$

Pf. Apply the usual Rolle's theorem to $g(x) = f(x) - f(a)$.

Pf of the MVT



Set $g(x) = f(x) - L(x)$. Then

$$g(x) : \begin{cases} \text{conti on } [a,b] \\ \text{diff on } (a,b) \end{cases}, \quad \text{and} \quad g(b) = g(a) = 0 \quad (\leftarrow L(a) = f(a),\ \ L(b) = f(b))$$

$$\overset{\text{Rolle's theorem}}{\Rightarrow} \quad \exists\, c \in (a,b) \text{ s.t. } g'(c) = 0, \text{ that is, } L'(c) = f'(c) \text{ for some } c \in (a,b)$$

$$\Leftrightarrow \quad \exists\, c \in (a,b) \text{ s.t. } \frac{f(b) - f(a)}{b - a} = f'(c)$$

15.2 Applications of the MVT

Theorem

Let $f(x)$ be diff on the interval I. Then on I

$$f'(x) > 0 \quad \Rightarrow \quad f(x) \text{ is strictly inc}$$

$$f'(x) < 0 \quad \Rightarrow \quad f(x) \text{ is strictly dec}$$

$$f'(x) \ge 0 \quad \Rightarrow \quad f(x) \text{ is inc} \qquad \surd$$

$$f'(x) \le 0 \quad \Rightarrow \quad f(x) \text{ is dec}$$

$$f'(x) = 0 \quad \Rightarrow \quad f(x) \text{ is constant}$$

Pf of $\sqrt{}$. Suppose $x_1 < x_2$, where $x_1, x_2 \in I$.

By MVT, $\exists\, c \in I$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Thus

$$\begin{aligned} f'(x) \geq 0 \text{ on } I \quad &\Rightarrow \quad f'(c) \geq 0 \\ &\Rightarrow \quad f(x_2) \geq f(x_1) \\ &\Rightarrow \quad f(x) \text{ is inc} \end{aligned}$$

Exa. Using MVT, show that

(a) $\sin x < x$, for all $x > 0$

(b) given $\varepsilon > 0$, $\ln x_2 - \ln x_1 < \varepsilon(x_2 - x_1)$ if $x_2 > x_1 \gg 1$

Pf. (a) $x > 1 \Rightarrow \sin x \leq 1 < x \Rightarrow \sin x < x$

$$\begin{aligned} 0 < x \leq 1 \quad &\Rightarrow \quad \sin x - \sin 0 \overset{\text{MVT}}{=} (\cos c)\cdot(x - 0) \text{ for some } c \text{ s.t. } 0 < c < x \leq 1 \\ &\Rightarrow \quad \sin x < x \quad (\because\ 0 < c < 1 (< \pi/2) \Rightarrow 0 < \cos c < 1) \end{aligned}$$

(b) Let $\varepsilon > 0$. Then, for $x_2 > x_1$,

$$\begin{aligned} \ln x_2 - \ln x_1 &\overset{\text{MVT}}{=} \frac{1}{c}(x_2 - x_1) \quad \text{for some } c \text{ s.t. } x_1 < c < x_2 \\ &< \varepsilon(x_2 - x_1) \quad \text{since } \frac{1}{c} < \frac{1}{x_1} < \varepsilon \text{ if } x_1 > \frac{1}{\varepsilon} \end{aligned}$$

Remark. MVT remains true even if $b < a$:

$$\begin{aligned} &\text{If } b < a, \text{ then } f(a) - f(b) \overset{\text{MVT}}{=} f'(c)(a - b), \ (b < (\text{some})c < a) \\ &\overset{\times(-1)}{\Rightarrow} \quad f(b) - f(a) = f'(c)(b - a), \ (b < c < a) \\ &\overset{\text{can write}}{\Rightarrow} \quad \underbrace{f(b) = f(a) + f'(c)(b - a),}_{\text{exactly the same as the standard form of MVT}(a<b)} \quad \text{for some } c \text{ between } a \text{ and } b \end{aligned}$$

So MVT has the same form, regardless of $a < b$ or $b < a$.

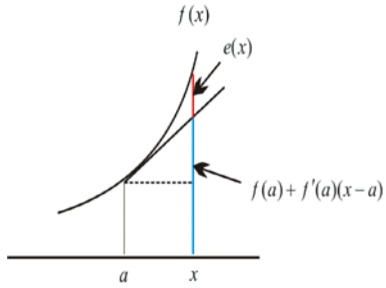
Therefore, if we replace b by x, we can write the MVT in the following useful form.

Approximation form of the MVT
$f(x)$: diff on an interval I & $a \in I$
$\Rightarrow$ $\forall x \in I$, $\exists\, c(= c_{a,x})$ between a and x such that $f(x) = f(a) + f'(c)(x - a)$
($Note$: The form is simple, but we do not know where the point c is exactly)
(Caution: $f(a) + f'(c)(x - a)$는 x 에 관한 1차식이 아님 (why?)

$$\boxed{\begin{array}{l} \qquad\qquad\qquad \text{Linear approximation} \\ \qquad f(x) \approx \underbrace{f(a) + f'(a)(x-a)}_{x\text{ 에관한 1차식}} \quad \text{for } x \approx a, \\ \text{with} \quad \lim_{x \to a} \frac{e(x)}{x-a} = 0; \quad e(x) = f(x) - \{f(a) + f'(a)(x-a)\} \end{array}}$$

즉, 한 점 a 에서의 $f(a)$ 와 $f'(a)$ 의 값을 알면, a 근방에서의 $f(x)$ 의 근사값을 알 수 있다.

Pf.



$$\lim_{x \to a} \frac{e(x)}{x-a} = \lim_{x \to a} \frac{f(x) - f(a) - f'(a)(x-a)}{x-a}$$
$$= \lim_{x \to a} \frac{f(x) - f(a)}{x-a} - f'(a) \quad = \quad f'(a) - f'(a) = 0. \qquad //$$

Remark. Consider $\sin x$, for $x > 0$

MVT says: $\sin x < x$, for $x > 0$ (seen earlier)

Linear Approx says: $\sin x \approx x$, for $x \approx 0$ $(\because \; f(x) = \sin x \Rightarrow f(0) = 0, \; f'(0) = 1)$

15.3 Extension of the MVT

Theorem Cauchy's MVT (a parametric form of the ordinary MVT)

Hypo: $\boxed{f(t) \;\&\; g(t): \text{ conti on } [a,b] \;\&\; \text{diff on } (a,b)}$

$\Rightarrow \quad \exists\, c \in (a,b) \quad \text{s.t.} \quad f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)]$

Accordingly, $\exists\, c \in (a,b)$ s.t. $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{g'(c)}$ if, in addition, $g'(t) \neq 0$ on (a,b)

Note1. Hypo $g(t) : \left\{ \begin{array}{l} \text{conti on } [a,b] \\ \text{diff on } (a,b) \end{array} \right.$ *plus* $g'(t) \neq 0$ on (a,b) $\Rightarrow$ $g(b) - g(a) \neq 0$

$(\because \; g(b) - g(a) = 0 \quad \Rightarrow \quad 0 = g(b) - g(a) \underset{\text{MVT}}{\overset{\exists\, c \in (a,b)}{=}} \underbrace{g'(c)}_{\neq 0} \cdot (b-a) \neq 0; \; \text{contradiction})$

Note 2. This does not follow directly from the ordinary MVT

$$(\because \; \frac{f(b) - f(a)}{g(b) - g(a)} \overset{\text{ordinary MVT only says}}{=} \frac{f'(c_1)}{g'(c_2)}, \text{ with different } c_1 \;\&\; c_2)$$

First pf. (It is simple but it is geometrically “less” appealing)

$$h(t) := f(t) - \left[f(a) + \frac{f(b)-f(a)}{g(b)-g(a)}(g(t)-g(a)) \right]$$

Seen that

$$g: \text{conti on } [a,b],\ \text{diff on } (a,b)\ \ \&\ \ g'(t) \neq 0 \ \text{ on } \ (a,b) \ \overset{\text{MVT}}{\Rightarrow} \ g(b) \neq g(a)$$

$\therefore \quad \dfrac{f(b)-f(a)}{g(b)-g(a)}$ is well defined, so is $h(t)$.

Note that $h:$ conti on $[a,b]$, diff on (a,b) & $h(a) = h(b) = 0$. Thus

$$\overset{\text{Rolle's thm}}{\Rightarrow} \quad \exists\, c \in (a,b) \ \ s.t. \ \ h'(c) = 0$$

$$\Rightarrow \ f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) = 0; \quad \text{so} \quad \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}, \quad c \in (a,b).$$

A variant of first proof: Let $h(t) := f(t) - \left(\dfrac{f(b)-f(a)}{g(b)-g(a)} \right) g(t)$. Then

$h:$ conti on $[a,b]$, diff on (a,b) & $h(a) = \dfrac{f(a)g(b)-g(a)f(b)}{g(b)-g(a)} = h(b)$. Thus

$$\overset{\text{Rolle's thm}}{\Rightarrow} \quad \exists\, c \in (a,b) \ \ s.t. \ \ h'(c) = 0$$

$$\Rightarrow \ f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) = 0; \quad \text{so} \quad \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}, \quad c \in (a,b).$$

Second pf. [similar to the first pf] (most common and simple pf)

$$F(t) : \overset{\text{let}}{=} \ f(t)[g(b)-g(a)] - g(t)[f(b)-f(a)]$$

$$\Rightarrow \quad F(t): \ \begin{array}{l} \text{conti on } [a,b] \\ \text{diff on } (a,b) \end{array} \quad \& \quad F(a) = f(a)g(b) - g(a)f(b) = F(b) \ \ (\text{easy})$$

$$\overset{\text{Rolle's thm}}{\Rightarrow} \quad \exists\, c \in (a,b) \quad \text{s.t.} \quad F'(c) = 0$$

$$\therefore \quad 0 = f'(c)[g(b)-g(a)] - g'(c)[f(b)-f(a)]$$

Third pf. [similar to the second pf]

$$F(t) : \overset{\text{let}}{=} \ [f(t)-f(a)][g(b)-g(a)] - [g(t)-g(a)][f(b)-f(a)]$$

$$\Rightarrow \quad F(t): \ \begin{array}{l} \text{conti on } [a,b] \\ \text{diff on } (a,b) \end{array} \quad \& \quad F(a) = F(b) = 0$$

$$\overset{\text{Rolle's thm}}{\Rightarrow} \ \exists\, c \in (a,b) \quad \text{s.t.} \ \ 0 = F'(c) = f'(c)[g(b)-g(a)] - g'(c)[f(b)-f(a)]$$

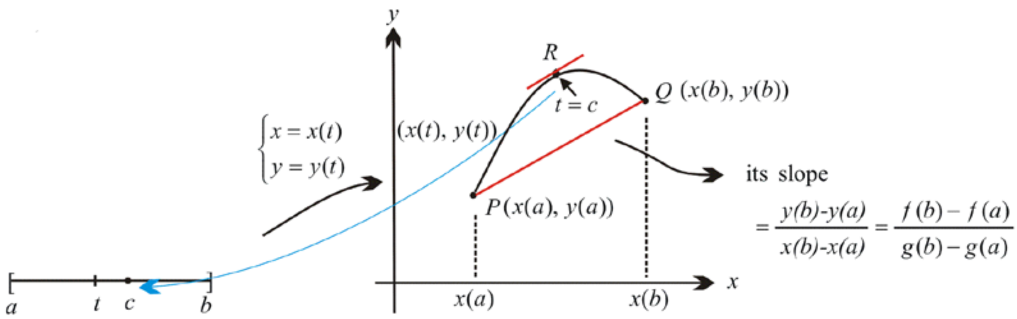
Remark: Each of the above three proofs is elementary (; not beyond high-school math)

※ A geometric interpretation of the Cauchy’s MVT:

Set $x = g(t)$, $y = f(t)$. Then the map

$$(x,y) : [a,b] \ni t \mapsto (x(t), y(t)) \ \ (\overset{\text{i.e.}}{=} \ (g(t), f(t)) \in \mathbb{R}^2$$

gives a parametric representation of a plane curve that begins at $P(x(a), y(a))$ and ends at $Q(x(b), y(b))$



Claim: Suppose $\begin{cases} f: \text{ conti on } [a,b] \text{ and diff on } (a,b) \\ \& \\ g' \in C[a,b] \quad \& \quad g'(t) \neq 0 \text{ on } (a,b) \end{cases}$

--- slightly stronger hypo than the usual statement of Cauchy's MVT ---

$\Rightarrow$ The (parametrically defined) curve is the graph of a diff -function $y = y(x)$ on the open interval $(x(a), (x(b))$ (or on $(x(b), x(a))$)

Pf of claim: Assume $g' \in C[a, b]$ & $g'(t) \neq 0$ on (a, b)

$\overset{\text{Bolzano's thm}}{\Rightarrow}$ $g'(t)$ does not change sign on (a, b)

[Fact (proved later): Assume g : diff on (a,b) & $g'(t) \neq 0$ on (a, b)

$\Rightarrow$ $g'(t)$ does not change sign on (a, b)]

That is, either $g'(t) > 0$ or $g'(t) < 0$ on (a, b)

$\swarrow \qquad \searrow$

g is strictly $\uparrow$ $\qquad$ g is strictly $\downarrow$

$\searrow \qquad \swarrow$

$x = g(t)$ has an inverse, write $t = g^{-1}(x)$ on $(x(a), x(b))$ (or on $(x(b), x(a))$

Assume, wlog, $g'(t) > 0$ on (a, b). Then by the inverse function theorem for diff,

$g^{-1}(x)$ is diff on $(x(a), x(b))$

$\therefore \quad y = f(t) = f(g^{-1}(x)) \equiv y(x)$ is diff on $(x(a), x(b))$. ///

Thus by the chain rule, say, on the interval $(x(a), x(b))$,

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy\,/\,dt}{dx\,/\,dt} = \frac{f'(t)}{g'(t)} \quad ---(\#)$$

By the interpretation of ordinary MVT ,

the slope of the secant $PQ =$ the slope of the curve at some point R

Thus if we let $t = c \in (a, b)$ be the corresponding value to the point R, then we have

$$\underbrace{\frac{f(b)-f(a)}{g(b)-g(a)}}_{\text{the slope of } PQ} = \left.\frac{dy}{dx}\right|_{t=c} \underset{(\#)}{\overset{\uparrow}{=}} \left.\frac{f'(t)}{g'(t)}\right|_{t=c} = \frac{f'(c)}{g'(c)}$$

Consequently, we have

$$\underbrace{\frac{f(b)-f(a)}{g(b)-g(a)}}_{\text{the slope of } PQ} = \underbrace{\frac{f'(c)}{g'(c)}}_{\substack{\text{slope of the tangent line to the curve } (x(t),\, y(t))(a\le t\le b) \\ \text{at some point } c\in(a,\, b)}}$$

Applications of Cauchy's MVT:

Exa. Prove $1-\dfrac{x^2}{2} < \cos x$ for $x \neq 0$

Pf. Since $1-\dfrac{x^2}{2}$ & $\cos x$ are even functions, it suffices to show that

$$1-\frac{x^2}{2} < \cos x \quad \text{for } x>0 \quad \text{or} \quad 1-\cos x < \frac{x^2}{2} \quad \text{for } x>0$$

By Cauchy's MVT applied to the pair of functions $f(x) = 1-\cos x$ & $g(x) = \dfrac{x^2}{2}$ where $x > 0$ $\Rightarrow$

$$\frac{1-\cos x}{\frac{x^2}{2}} = \frac{f(x)-f(0)}{g(x)-g(0)} = \frac{f'(c)}{g'(c)} = \frac{\sin c}{c} < 1 [\leftarrow \sin x < x \text{ for } x>0] \text{ for some } 0< c < x$$

Home Study. Show that $x - \dfrac{x^3}{3!} < \sin x$ for $x > 0$

Exa. Show that if $r > 0$ & $x > 1$, then $\ln x < \dfrac{x^r - 1}{r}$

In particular, $\ln x < \dfrac{x^3-1}{3}$ & $\ln x < 3\left(x^{1/3}-1\right)$ for every $x > 1$

Pf. Apply Cauchy MVT to the pair of functions $f(x) = x^r$ & $g(x) = \ln x$ $\Rightarrow$

$$\frac{x^r - 1}{\ln x} = \frac{f(x)-f(1)}{g(x)-g(1)} = \frac{f'(c)}{g'(c)} (1 < c < x) = \frac{rc^{r-1}}{1/c} = rc^r > r$$

This proves that $\ln x < \dfrac{x^r-1}{r}$ for $x > 1$

⊙ **Another interpretation** of Cauchy's MVT:

If $F=(f,g):[a,b]\to\mathbb{R}^2$ is conti on $[a,b]$ & diff on (a,b), then

$\exists c\in(a,b)$ such that $F'(c)$ and $F(b)-F(a)$ [as two-dimensional vectors] are **parallel**

i.e., $\exists c\in(a,b)$ such that $\big(f'(c),g'(c)\big)//\big(f(b)-f(a),g(b)-g(a)\big)$

In particular, $\exists c\in(a,b)$ s.t. $\dfrac{f(b)-f(a)}{g(b)-g(a)}=\dfrac{f'(c)}{g'(c)}$ provided that $g'(t)\neq 0\ \forall t\in(a,b)$

♦ Parametrically defined curves in $\mathbb{R}^2$

- A (differentiable) plane curve can be parametrized by one parameter as $\boldsymbol{r}=\boldsymbol{r}(t):[a,b]\to\mathbb{R}^2$

  $\boldsymbol{r}(t)=(x(t),y(t))\big[\mathbf{r}=(x,y)=x\mathbf{i}+y\mathbf{j}\,\big]$ $\qquad\boldsymbol{r}'(t)=\big(x'(t),y'(t)\big)$ (see below)

Def. We say that $\boldsymbol{r}(t)=(x(t),y(t)):[a,b]\to\mathbb{R}^2$ is differentiable at $t_0\in[a,b]$ if

$$\boldsymbol{r}'(t_0)=\lim_{h\to 0}\frac{\boldsymbol{r}(t_0+h)-\boldsymbol{r}(t_0)}{h}\ \text{ exists.}$$

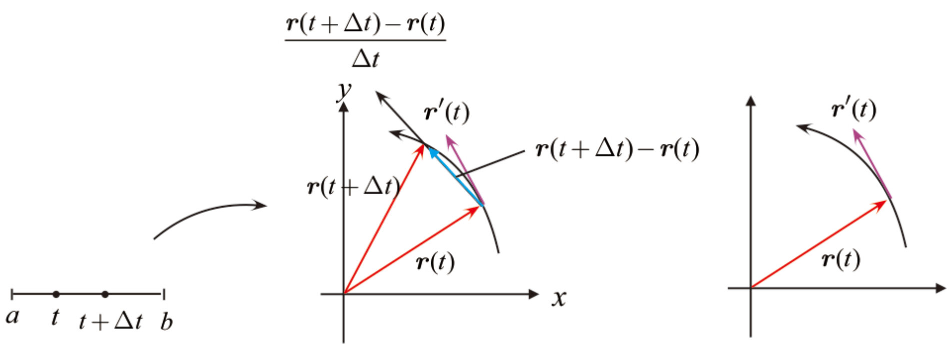
Note that $\dfrac{\boldsymbol{r}(t_0+h)-\boldsymbol{r}(t_0)}{h}=\left(\dfrac{x(t_0+h)-x(t_0)}{h},\dfrac{y(t_0+h)-y(t_0)}{h}\right)$. Hence

$\lim\limits_{h\to 0}\dfrac{\boldsymbol{r}(t_0+h)-\boldsymbol{r}(t_0)}{h}$ exists iff both $\lim\limits_{h\to 0}\dfrac{x(t_0+h)-x(t_0)}{h}$ & $\lim\limits_{h\to 0}\dfrac{y(t_0+h)-y(t_0)}{h}$ exist.

That is, $\boldsymbol{r}(t)=(x(t),y(t))$ is diff at t_0 $\Leftrightarrow$ both $x(t)$ and $y(t)$ are diff at t_0.

Thus if $\boldsymbol{r}(t)=(x(t),y(t))$ is diff at t_0, then $\boldsymbol{r}'(t_0)=\big(x'(t_0),y'(t_0)\big)$

A useful geometric meaning of $\boldsymbol{r}'(t)$:



From the above figure, we see that $\boldsymbol{r}'(t)$ is a **tangent vector** to the curve at $\boldsymbol{r}(t)$

15.3 **L'Hospital's rule** for indeterminate forms ($\leftarrow$ actually, due to Bernoulli)
(an application of Cauchy's MVT)

Q: How can we evaluate $\lim_{x\to a} \dfrac{f(x)}{g(x)}$?

An easy case:

$$\lim_{x\to a} \frac{f(x)}{g(x)} \underset{\text{if } \lim_{x\to a} f(x) \text{ exists } \& \lim_{x\to a} g(x)(\text{exists})\neq 0}{\overset{\uparrow}{=}} \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \underset{\text{if } f \,\&\, g \text{ are conti, and } g(a)\neq 0}{=} \frac{f(a)}{g(a)}$$

- How about if $\lim_{x\to a} g(x) = 0$ (or $g(a) = 0$) ?

  (i) If, in addition, $\lim_{x\to a} f(x) \neq 0$ (or $f(a) \neq 0$), then

  $$\lim_{x\to a} \frac{f(x)}{g(x)} = \infty \quad \text{or} \quad -\infty \quad (\text{Ex})$$

  (ii) If, in addition, $\lim_{x\to a} f(x) = 0$ (or $f(a) = 0$), then

  $$\lim_{x\to a} \frac{f(x)}{g(x)} \quad \text{is said to be "indeterminate"}$$

Theorem A L'Hospital's rule **(elementary case)**

If $f(a) = g(a) = 0$, and the right side below is defined, then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{f'(a)}{g'(a)}$

Pf (easy).

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)} \ (\text{if the right exists})$$

Exa.

(a) $$\lim_{x\to 0} \frac{x^3-2x}{x^3+x} \overset{\text{L}}{=} \left.\frac{3x^2-2}{3x^2+1}\right|_{x=0} = -2$$

※ (b) $$\lim_{x\to 0} \frac{\sin x}{x} \overset{\text{L}}{=} \frac{\sin'(0)}{1} = \frac{\cos 0}{1} = 1 \quad (\text{Is the argument right?})$$

Ans. It is a cheat. Why?

In fact, $\lim_{x\to 0} \dfrac{\sin x}{x} \overset{\text{L}}{=} \dfrac{\sin'(0)}{1}$ can only happen if $\sin'(0)$ exists.

How can we know whether $\sin'(0)$ exists?

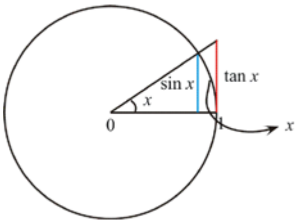
Recall that $\sin'(0) = \lim_{x\to 0} \dfrac{\sin x - \sin 0}{x-0} = \lim_{x\to 0} \dfrac{\sin x}{x}$ (if the limit exists).

So, to seek the value $\sin'(0)$, we first need to prove that $\lim_{x\to 0} \dfrac{\sin x}{x}$ exists.

Therefore, we can not apply L'Hospital's rule to calculate $\lim_{x\to 0} \dfrac{\sin x}{x}$.

Ex. (Seen in High-School Math) Show that $\sin x < x < \tan x \ (= \frac{\sin x}{\cos x})$, for $x \underset{\neq}{\approx} 0^+$

Pf of Ex.



Comparing areas of small triangle, circular sector , and big triangle, we see that

$$\sin x < x < \tan x \quad \text{for } 0 < x < \pi/2 \quad (\therefore \ \text{for } x \underset{\neq}{\approx} 0^+)$$

From this, we easily get

$$\cos x < \frac{\sin x}{x} < 1, \ \text{ for } x \underset{\neq}{\approx} 0^+, \quad \text{which clearly implies} \quad \lim_{x \to 0^+} \frac{\sin x}{x} = 1.$$

We also have $\lim_{x \to 0^-} \frac{\sin x}{x} = \lim_{t \to 0^+} \frac{\sin(-t)}{-t} = \lim_{t \to 0^+} \frac{\sin t}{t} = 1.$ (or since $\frac{\sin x}{x}$ is even)

Accordingly, $\lim_{x \to 0} \frac{\sin x}{x} = 1.$

Comment [seen]: $0 < x < \pi/2 \ \Rightarrow \ x\cos x < \sin x < x$

$$\text{(c)} \quad \lim_{x \to 0} \frac{1 - \cos x}{\sin x} \overset{\mathrm{L}}{=} \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0.$$

$$\text{(d)} \quad \lim_{x \to 0} \frac{1 - \cos x}{x \sin x} \overset{\mathrm{L}}{=} \left. \frac{\sin x}{\sin x + x\cos x} \right|_{x=0} \ (\text{still } \tfrac{0}{0})$$

$$\overset{\mathrm{L}}{=} \left. \frac{(\sin x)'}{(\sin x + x\cos x)'} \right|_{x=0} \quad ??$$

To answer the last question (??), we need a variant of ordinary L'Hospital's rule.

Theorem B [L'Hospital's rule for $0/0$ as $x \to a$]

Suppose that

$$f(x) \text{ and } g(x) \text{ are of class } C^1 \text{ for } x \approx a, \text{ and}$$

$$f(a) = g(a) = 0, \ \text{ but } g'(x) \neq 0 \ \text{ for } x \underset{\neq}{\approx} a$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \quad \text{if the limit on the right exists.}$$

Pf. For $x \approx a$, we have

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} \overset{\text{Cauchy's MVT}}{=} \frac{f'(c_x)}{g'(c_x)} \quad \text{for some } c_x \text{ between } a \text{ and } x.$$

Letting $x \to a$ gives

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(c_x)}{g'(c_x)} \underset{\substack{\uparrow \\ x \to a \,\Rightarrow\, c_x \to a}}{=} \lim_{x \to a} \frac{f'(x)}{g'(x)}, \quad \text{since the last limit exists by hypo.}$$

Exa. (a) $$\lim_{x\to 0}\frac{1-\cos x}{x\sin x} \overset{\text{L}}{=} \lim_{x\to 0}\frac{\sin x}{\sin x + x\cos x} \overset{\text{L}}{=} \lim_{x\to 0}\frac{\cos x}{2\cos x - x\sin x} = \frac{1}{2}$$

Reasoning : the 3rd limit exists(easy) $\Rightarrow$ the 2nd limit exists $\Rightarrow$ the 1st limit exists

(b) $$\lim_{x\to 0}\frac{e^x-1-x}{x^2} \overset{\text{L}}{=} \lim_{x\to 0}\frac{e^x-1}{2x} \overset{\text{L}}{=} \lim_{x\to 0}\frac{e^x}{2} = \frac{1}{2}$$

Remark.
L'Hospital's rule also works for one-sided limits as $x \to a^+$ or $x \to a^-$. (Theorem B' below)

It also holds for limits taken as $x \to \infty$, since the change of variable $x = 1/t$ reduces it to the case $t \to 0^+$. (Theorem B'' below)

Theorem B' [L'Hospital's rule for $0/0$ as $x \to a^+$]
Suppose that

$f(x)$ and $g(x)$ are of class C^1 for $x \approx a^+$, and

$f(a) = g(a) = 0$, but $g'(x) \neq 0$ for $x \underset{\neq}{\approx} a^+$

Then
$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = \lim_{x\to a^+}\frac{f'(x)}{g'(x)} \quad \text{if the limit on the right exists.}$$

Pf. For $x \underset{\neq}{\approx} a^+$, we have
$$\frac{f(x)}{g(x)} = \frac{f(x)-f(a)}{g(x)-g(a)} \overset{\text{Cauchy's MVT}}{=} \frac{f'(c_x)}{g'(c_x)} \quad \text{for some } c_x \in (a, x).$$

Letting $x \to a^+$ gives
$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = \lim_{x\to a^+}\frac{f'(c_x)}{g'(c_x)} \underset{\substack{\uparrow \\ x\to a^+ \Rightarrow c_x \to a^+}}{=} \lim_{x\to a^+}\frac{f'(x)}{g'(x)}, \quad \text{since the last limit exists by hypo.}$$

Exa. $$\lim_{x\to 0^+}\frac{\sin x}{\sqrt{x}}\left(\frac{0}{0}\right) \overset{\text{L}}{=} \lim_{x\to 0^+}\frac{\cos x}{\dfrac{1}{2\sqrt{x}}} = \lim_{x\to 0^+} 2\sqrt{x}\cos x = 0$$

Theorem B'' [L'Hospital's rule for $0/0$ as $x \to \infty$]
Suppose that

$f(x)$ and $g(x)$ are of class C^1 for $x \gg 1$, and

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$, but $g(x)$ & $g'(x) \neq 0$ for $x \gg 1$

Then
$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)} \quad \text{if the limit on the right exists.}$$

Pf. Note that $x \overset{\text{let}}{=} 1/t \quad \Rightarrow \quad x \to \infty \Leftrightarrow t \to 0^+$, and apply Theorem B'

Alternative direct pf (for High-School Math Teachers). Let $y > x \gg 1$. Then

$$\frac{f(x)-f(y)}{g(x)-g(y)} \overset{\text{Cauchy's MVT}}{=} \frac{f'(c_{x,y})}{g'(c_{x,y})}, \quad x < c_{x,y} < y$$

Here the hypothesis '$g'(x) \neq 0$ for $x \gg 1$' is used

Fix x, and let $y \to \infty$ $\Rightarrow$

$$\text{LHS} \to \frac{f(x)}{g(x)}\left[g(x) \neq 0 \text{ for } x \gg 1 \text{ is used}\right] \quad \text{RHS} \to \frac{f'(c_x)}{g'(c_x)} \quad (x < c_x)$$

$$\therefore \quad \frac{f(x)}{g(x)} = \frac{f'(c_x)}{g'(c_x)}, \quad c_x \in (x, \infty)$$

Letting $x \to \infty$ gives

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(c_x)}{g'(c_x)} \underset{\substack{\uparrow \\ x\to\infty \Rightarrow c_x \to\infty}}{=} \lim_{x\to\infty} \frac{f'(x)}{g'(x)}, \text{ since the last limit exists by hypo}$$

Theorem C [L'Hospital's rule for ∞/∞ as $x \to \infty$]

Suppose that

$f(x)$ and $g(x)$ are diff, $g'(x) \neq 0$ for $x \gg 1$, and

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty.$

Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} \quad \text{if the limit on the right exists.}$$

Pf (for High-School Math Teachers). Let $y > x \gg 1$. Then

$$\frac{f(y)-f(x)}{g(y)-g(x)} \overset{\text{Cauchy's MVT}}{=} \frac{f'(c_{x,y})}{g'(c_{x,y})}, \quad x < c_{x,y} < y$$

Since $f(y)$ & $g(y) \neq 0 (>0)$, we get

$$\text{LHS} = \frac{f(y)\left[1 - \dfrac{f(x)}{f(y)}\right]}{g(y)\left[1 - \dfrac{g(x)}{g(y)}\right]} = \frac{f'(c_{x,y})}{g'(c_{x,y})}$$

Thus if we fix x and letting $y \to \infty$ $\left(\Rightarrow \frac{f(x)}{f(y)} \,\&\, \frac{g(x)}{g(y)} \to 0\right)$ $\Rightarrow$

$$\lim_{y\to\infty} \frac{f(y)}{g(y)} = \frac{f'(c_x)}{g'(c_x)}, \quad \text{where } x < c_x$$

Since LHS is independent of x, letting $x \to \infty$ $(\Rightarrow c_x \to \infty)$ gives

$$\lim_{y\to\infty} \frac{f(y)}{g(y)} = \lim_{x\to\infty} \frac{f'(c_x)}{g'(c_x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

※ Remark. Theorem C above also holds, under somewhat weaker hypothesis (below):

Theorem C' [L'Hospital's rule for $\dfrac{\text{anything}}{\infty}$ as $x \to \infty$]

($i.e.$, don't need to know the behavior of $f(x)$ as $x \to \infty$)

Suppose that

$f(x)$ and $g(x)$ are diff, $g'(x) \neq 0$ for $x \gg 1$, and

$\lim\limits_{x\to\infty} g(x) = \infty$.

Then $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \dfrac{f'(x)}{g'(x)}$ if the limit on the right exists.

Pf. Let $L = \lim\limits_{x\to\infty} \dfrac{f'(x)}{g'(x)}$. Then

$$\forall\ \varepsilon > 0, \quad \frac{f'(x)}{g'(x)} \underset{\varepsilon}{\approx} L \quad \text{for } x \gg 1, \quad \text{say for } x > a \qquad ---(\#)$$

For that a,

$$\frac{f(x)-f(a)}{g(x)-g(a)} \ \underset{\substack{\uparrow\\\uparrow\\\uparrow\\\uparrow}}{=}\ \frac{\dfrac{f(x)}{g(x)} - \dfrac{f(a)}{g(x)}}{1 - \dfrac{g(a)}{g(x)}} \quad \underset{\substack{\varepsilon\\ \uparrow \\ \frac{f(a)}{g(x)} \,\&\, \frac{g(a)}{g(x)} \to 0 \text{ by hypo}}}{\approx} \quad \frac{f(x)}{g(x)} \quad \text{for } x \gg 1 \ \ (\text{say for } x > b)$$

$$(\ g(x) \neq 0 (>0) \ \text{ for } x \gg 1 \ \text{ by hypo }\ )$$

On the other hand,

$$\frac{f(x)-f(a)}{g(x)-g(a)} \overset{\text{Cauchy's MVT}}{=} \frac{f'(c_x)}{g'(c_x)}, \quad a < c_x < x$$

$$\underset{\varepsilon}{\approx} \ L \quad (\text{by } (\#)) \ \text{ since } c_x > a$$

So $\dfrac{f(x)}{g(x)} \underset{2\varepsilon}{\approx} L$ for $x \gg 1$ (say for $x > \max\{a,b\}$) $\quad \therefore\ \lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = L.$

Alternative pf (for High-School Math Teachers). Let $y > x \gg 1$. Then

$$\frac{f(y)-f(x)}{g(y)-g(x)} \overset{\text{Cauchy's MVT}}{=} \frac{f'(c_{x,y})}{g'(c_{x,y})}, \quad x < c_{x,y} < y$$

Since $g(y) \neq 0 (>0)$, we get

$$\text{LHS} = \frac{\dfrac{f(y)}{g(y)} - \dfrac{f(x)}{g(y)}}{1 - \dfrac{g(x)}{g(y)}} = \frac{f'(c_{x,y})}{g'(c_{x,y})}$$

Thus if we fix x and letting $y \to \infty$ $\left(\Rightarrow \dfrac{f(x)}{g(y)} \ \& \ \dfrac{g(x)}{g(y)} \to 0\right) \Rightarrow$

$$\lim_{y\to\infty} \frac{f(y)}{g(y)} = \frac{f'(c_x)}{g'(c_x)}, \quad \text{where } \ x < c_x$$

Since LHS is independent of x, letting $x \to \infty$ $(\Rightarrow c_x \to \infty)$ gives

$$\lim_{y\to\infty} \frac{f(y)}{g(y)} = \lim_{x\to\infty} \frac{f'(c_x)}{g'(c_x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

Exa. $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} \left(\frac{\infty}{\infty}\right) \overset{L}{=} \lim_{x\to\infty} \frac{1/x}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$

Theorem C'' [L'Hospital's rule for $\frac{\text{anything}}{\infty}$ as $x \to a^+$]

($i.e.$, don't need to know the behavior of $f(x)$ as $x \to a^+$)

Suppose that

$f(x)$ and $g(x)$ are diff, $g'(x) \neq 0$ for $x \underset{\neq}{\approx} a^+$, and

$\lim_{x\to a^+} g(x) = \infty$.

Then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)} \quad \text{if the limit on the right exists.}$$

Pf. Let $a < x < y < a + \delta$. Then

$$\frac{f(x) - f(y)}{g(x) - g(y)} \overset{\text{Cauchy's MVT}}{=} \frac{f'(c_{x,y})}{g'(c_{x,y})}, \quad x < c_{x,y} < y$$

Since $g(x) \neq 0 (> 0)$, we get

$$\text{LHS} = \frac{\frac{f(x)}{g(x)} - \frac{f(y)}{g(x)}}{1 - \frac{g(y)}{g(x)}} = \frac{f'(c_{x,y})}{g'(c_{x,y})}$$

Thus if we fix y and letting $x \to a^+$ $\left(\Rightarrow \frac{f(y)}{g(x)} \ \& \ \frac{g(y)}{g(x)} \to 0\right) \Rightarrow$

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \frac{f'(c_y)}{g'(c_y)}, \quad \text{where } a < c_y < y$$

Since LHS is independent of y, letting $y \to a^+$ $(\Rightarrow c_y \to a^+)$ gives

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{y\to a^+} \frac{f'(c_y)}{g'(c_y)} = \lim_{y\to a^+} \frac{f'(y)}{g'(y)}$$

Ex. $\lim_{x\to 0^+} x \ln x = ??$

Sol. $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} \left(\frac{\text{anything}}{\infty} \overset{\text{in fact}}{=} \frac{-\infty}{\infty}\right) \overset{L}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$

HS. If f is twice differentiable on some open interval containing a, prove that

$$\lim_{x\to a} \frac{f(x) - f(a) - f'(a)(x-a)}{\frac{1}{2}(x-a)^2} = f''(a)$$

Ex. (Technical)

Let $f(x)$ be diff for $x \gg 1$, and suppose that

$$\lim_{x\to\infty}\big(f(x)+f'(x)\big)=2010.$$

Prove that

$$\lim_{x\to\infty} f(x)=2010 \quad \text{and} \quad \lim_{x\to\infty} f'(x)=0.$$

Pf. $$\frac{\big(e^x f(x)\big)'}{(e^x)'}=\frac{e^x\big(f(x)+f'(x)\big)}{e^x}=f(x)+f'(x) \underset{\text{Hypo}}{\longrightarrow} 2010 \quad \text{as } x\to\infty$$

and $e^x \to \infty$ as $x\to\infty$.

Thus by L'Hospital's rule [$\dfrac{\text{anything}}{\infty}$ as $x\to\infty$: Theorem C'],

$$\frac{e^x f(x)}{e^x}=f(x) \underset{\text{Hypo}}{\longrightarrow} 2010 \quad \text{as } x\to\infty$$

$$\therefore \quad \lim_{x\to\infty} f(x)=2010 \quad \text{and} \quad \lim_{x\to\infty} f'(x)=0.$$

Ex (easy) Using L'Hospital's rule or using the definition of derivative, show that

(i) If $a, b>0$, then show that $\displaystyle\lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n=\lim_{x\to\infty}\left(\frac{\sqrt[x]{a}+\sqrt[x]{b}}{2}\right)^x=\sqrt{ab}$

(ii) If f is twice diff on an interval containing a, show that

$$\lim_{h\to 0}\frac{f(a+2h)-2f(a+h)+f(a)}{h^2}=f''(a)$$

(iii) If f is three times diff on an interval containing a, show that

$$\lim_{h\to 0}\frac{f(a+3h)-3f(a+2h)+3f(a+h)-f(a)}{h^3}=f^{(3)}(a)$$

Suggestion for (ii) & (iii): Regard the respective numerator as the function of h

Home Study.

1. Summarize several types of L'Hospital's rule
2. Apply L'Hospital's rule to solve (related) problems in High-School Math.

Ex (**Darboux's IVT for derivative**) [Any derivative has IVP (Intermediate Value Property)]
Let f be diff on $[a, b]$. If k is a number such that $f'(a) < k < f'(b)$ or $f'(a) > k > f'(b)$. Then show that

$$\exists\, c \in (a, b) \text{ such that } f'(c) = k.$$

Note: Here we do not assume the continuity of f' on $[a, b]$.

Lemma. Let $f : I(=\text{open interval}) \to \mathbb{R},\ c \in I,$ and assume that $f'(c)$ exists. Then

(a) $\quad f'(c) > 0 \quad \Rightarrow \quad \exists \delta > 0 \text{ s.t.} \begin{cases} f(x) > f(c) & \text{for all } x \in I \text{ with } c<x<c+\delta \\ f(x) < f(c) & \text{for all } x \in I \text{ with } c-\delta<x<c \end{cases}$

In particular, f has no (local) minimum at c $\quad[\leftarrow$ consider the interval $c-\delta<x<c]$

(b) $\quad f'(c) < 0 \quad \Rightarrow \quad \exists \delta > 0 \text{ s.t.} \begin{cases} f(x) < f(c) & \text{for all } x \in I \text{ with } c<x<c+\delta \\ f(x) > f(c) & \text{for all } x \in I \text{ with } c-\delta<x<c \end{cases}$

In particular, f has no (local) minimum at c $\quad[\leftarrow$ consider the interval $c<x<c+\delta]$

Pf [already seen in Chap14]. We (re)prove only (a): Hypo says $\displaystyle\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = f'(c) > 0$

$$\overset{\text{FLT}}{\Rightarrow} \quad \frac{f(x)-f(c)}{x-c} > 0 \text{ for } x \underset{\delta}{\approx} c \quad (\text{i.e., for } x \in I \text{ with } x \in (c-\delta, c+\delta))$$

$\Rightarrow \quad f(x) - f(c) > 0 \quad$ for all $x \in I$ with $c < x < c+\delta$

$\quad [\&\ \ f(x) - f(c) < 0 \quad$ for all $x \in I$ with $c-\delta < x < c]$

$\Rightarrow \quad f(x) > f(c) \quad$ for all $x \in I$ with $c < x < c+\delta$

$\quad [\&\ \ f(x) < f(c) \quad$ for all $x \in I$ with $c-\delta < x < c]$

Pf of Darboux. WLOG, we may assume that $f'(a) < k < f'(b)$.
Define $\varphi : [a, b] \to \mathbb{R}$ by $\varphi(x) = f(x) - kx$. Then obviously φ is diff on $[a, b]$, and

$$\varphi'(a) = f'(a) - k < 0 \quad --(i) \qquad \text{and} \qquad \varphi'(b) = f'(b) - k > 0 \quad --(ii)$$

Note that
(i) implies that the minimum of φ can not occur at $x = a$ ---① [by Lemma-(b)]
Similarly,
(ii) implies that the minimum of φ can not occur at $x = b$ ---② [by Lemma-(a)]
However, since $\varphi \in C[a, b]$, it attains a minimum value on $[a, b]$ ---③ (by MmT).
Combining ①,② &③ yields that φ must have its minimum at some point c in (a, b).
Thus $c \in (a, b)$ is a local minimum(extremum) point of the differentiable function φ
Therefore, we conclude that $0 = \varphi'(c) = f'(c) - k$
(by **Theorem B**, in Section 14.3 [= **Fermat's Critical Point Theorem**])
Hence $f'(c) = k,$ for some $c \in (a, b)$.

Warning:
$\varphi'(a) < 0$ does **not** implies that φ is locally strictly decreasing at $x = a$
Similarly
$\varphi'(b) > 0$ does **not** implies that φ is locally strictly increasing at $x = b$
--- see the next question ---

Question: $f'(c) > 0 \overset{??}{\Rightarrow} f$ is $\nearrow$ in some nbd $(c-\delta, c+\delta)$ of the point c

Answer is unexpectedly no.

Set $f(x) = \begin{cases} \dfrac{x}{2} + x^2 \sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Then $f'(0) = \lim\limits_{x\to 0}\left(\dfrac{1}{2} + x\sin\dfrac{1}{x}\right) = \dfrac{1}{2} > 0$

Note that $f'(x) = \begin{cases} \dfrac{1}{2} - \cos\dfrac{1}{x} + 2x\sin\dfrac{1}{x} & \text{if } x \neq 0 \\ \dfrac{1}{2} & \text{if } x = 0 \end{cases}$: conti for $x \neq 0$; but not conti at $x = 0$

Choose two sequences $x_n = \dfrac{1}{2n\pi}$ and $y_n = \dfrac{1}{2n\pi + \dfrac{\pi}{2}}$ $(n \gg 1)$; x_n & $y_n \approx 0$ if $n \gg 1$.

Obviously $x_n > y_n$, but

$$f(x_n) - f(y_n) = \frac{1}{4n\pi} - \left[\frac{1}{4n\pi + \pi} + \frac{1}{(2n\pi + \pi/2)^2}\right]$$
$$= \frac{1}{4n\pi} - \left[\frac{1}{4n\pi + \pi} + \frac{4}{(4n\pi + \pi)^2}\right] = \frac{1 + 4n(1 - 4/\pi)}{4\pi n(4n+1)^2} < 0 \text{ for } n \gg 1 \left[\leftarrow 1 - 4/\pi < 0\right]$$

This shows that f is not increasing in any neighborhood of 0

HS. Prove that if f is diff on a nbd of c with $f'(c) > 0$ and f' is continuous at c

$\Rightarrow$ f is strictly $\nearrow$ on some nbd of c

Application of Darboux.

Ex. Let $g : [-1, 1] \to \mathbb{R}$ be the function defined by $g(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$

Show that there is no differentiable function f such that $f'(x) = g(x)$ for all $x \in [-1, 1]$.

In other words, g is not the derivative on $[-1, 1]$ of any function on this interval.

Pf. Suppose that such a function f were exist.

That is, suppose that $\exists$ a diff fct f with $f'(x) = g(x)$ on $[-1,1]$

Then $g = f'$ clearly fails to satisfy the IVP on the interval $[-1, 1]$.

This contradicts the Darboux's theorem. Therefore, such a function f cannot exist.

Comment: We know that

$g' \in C[a, b]$ & $g'(t) \neq 0$ on (a, b)

$\overset{\text{Bolzano's thm}}{\Rightarrow}$ $g'(t)$ does not change sign on (a, b)

The hypothesis can be slightly weakened as:
Ex (A reformulation of Darboux's IVT for derivative) [or Bolzano's theorem for derivative]

Let g be diff on (a, b) & $g'(t) \neq 0$ on (a, b)

$\Rightarrow$ $g'(t)$ does not change sign on (a, b)

Pf 1. (Use Darboux' theorem) By contraposition, it suffices to prove:

if $g'(c) > 0$ and $g'(d) < 0$ (with $c, d \in (a,b)$), then $\exists \xi$ between c and d such that $g'(\xi) = 0$.

But this is obviously true by Darboux's IVT for derivative

Pf 2. (a direct pf) By contraposition, it suffices to prove:

if $g'(c) > 0$ and $g'(d) < 0$ (with $c, d \in (a,b)$), then $\exists \xi$ between c and d such that $g'(\xi) = 0$.

Suppose $c < d$ and let $\xi \in [c, d]$ be a point such that $g(\xi) = \max\limits_{x \in [c,d]} g(x)$

Shall show: $\xi \neq c$ and $\xi \neq d$.

$$g'(c) > 0 \overset{\text{Lemma-(a)}}{\Rightarrow} g(x) > g(c) \text{ for } x \underset{\neq}{\approx} c^+ \Rightarrow \xi \neq c$$

$$g'(d) < 0 \overset{\text{Lemma-(b)}}{\Rightarrow} g(x) > g(d) \text{ for } x \underset{\neq}{\approx} d^- \Rightarrow \xi \neq d$$

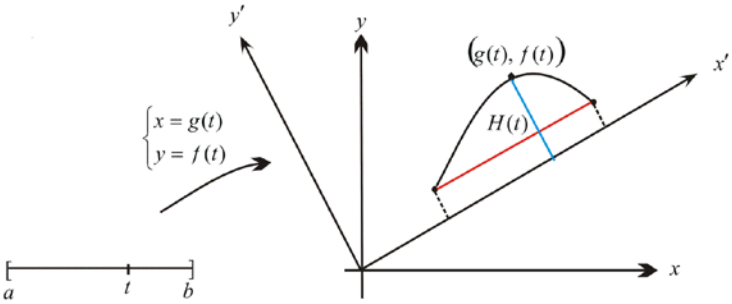
Thus we showed that $c < \exists \xi < d$ & $g(\xi) = \max\limits_{x \in [c,d]} g(x)$.

This says that, on the open interval (c, d), $g(x)$ has a local maximum at ξ; which implies $g'(\xi) = 0$.

Another pf of Cauchy's MVT (using more natural auxiliary function) --- **optional**

Suppose $f(t)$ & $g(t)$: conti on $[a, b]$, diff on (a, b), and $g'(t) \neq 0$ on (a, b).

Then $\exists\, c \in (a, b)$ such that $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{g'(c)}$



Observe that the equation of the x' − axis is $y = mx = \dfrac{f(b) - f(a)}{g(b) - g(a)} x$;

[A parametric form of the line $y = mx$ is given by $L(t) = (t, mt)$]

$\therefore$ $H(t) = \dfrac{f(t) - mg(t)}{\sqrt{m^2 + 1}}$ (= the directed distance from the line $y = mx$ to a pt $(g(t), f(t))$)

Note that hypo implies

$$H(t) = \frac{f(t) - mg(t)}{\sqrt{m^2 + 1}} : \left\{ \begin{array}{l} \text{conti on } [a, b] \\ \text{diff on } (a, b) \end{array} \right. , \quad \text{and} \quad H(a) = H(b)$$

$$\overset{\text{Rolle's theorem}}{\Rightarrow} \quad \exists\, c \in (a, b) \text{ s.t. } H'(c) = 0, \text{ that is, } \frac{f'(c) - mg'(c)}{\sqrt{m^2 + 1}} = 0 \text{ for some } c \in (a, b)$$

$$\Leftrightarrow \quad \exists\, c \in (a, b) \text{ s.t. } f'(c) - mg'(c) = 0 \left(\text{i.e., } \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \right)$$