10.1 Curves Defined by Parametric Equations

- there are situations when it is impossible to describe by an equation of the form $y = f(x)$
- suppose that x and y are both given as functions of a third variable t, named "parameter", by the equations $x = f(t)$ and $y = g(t)$, named "parametric equations."
- sometimes we restrict t to lie in a finite interval $a \le t \le b$ $\Rightarrow$ initial point $(f(a), g(a))$, terminal point $(f(b), g(b))$
- different sets of parametric equations can represent the same curve
- if we need to graph an equation of the form $x = g(y)$, we can use the parametric equations $x = g(t)$, $y = t$.
- if we need to graph an equation of the form $y = f(x)$, we can use the parametric equations $x = t$, $y = f(t)$

☆ One of the most important uses of parametric curves is in computer-aided design CAD

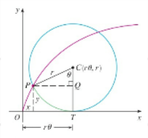


FIGURE 14

- Suppose the circle has rotated through θ radians, the parameter. The distance it has rolled from the origin is $|OT| = \text{arc } PT = r\theta$. Then the center of the circle is $C(r\theta, r)$. Let the coordinates of P be (x, y),

$\Rightarrow$ $x = |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta)$

$y = |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta)$, therefore parametric equations of the cycloid are

$x = r(\theta - \sin\theta)$ & $y = r(1 - \cos\theta)$

10.2 Calculus with Parametric Curves

Tangents:

- Suppose f and g are differentiable functions and we want to find the tangent line at a point of the parametric curve $x = f(x)$, $y = g(t)$

$\Rightarrow$ $\frac{dy}{dx} = \left(\frac{dy}{dt}\right) / \left(\frac{dx}{dt}\right)$, given $\frac{dx}{dt} \neq 0$

☆ a curve has a horizontal tangent when $\frac{dy}{dt} = 0$ and it has a vertical tangent when $\frac{dx}{dt} = 0$

☆ $\frac{d^2y}{dx^2} \neq \left(\frac{d^2y}{dt^2}\right) / \left(\frac{d^2x}{dt^2}\right)$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \left[\frac{d}{dt}\left(\frac{dy}{dx}\right)\right] / \left(\frac{dx}{dt}\right)$

Area:

- if the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \le t \le \beta$ then we can calculate an area formula by using the Substitution Rule

$\Rightarrow$ $A = \int_a^b y\, dx = \int_\alpha^\beta g(t) f'(t)\, dt$

Arc Length :

- Remember the formula of arc length, $L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$
- Suppose that C can also be described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \le t \le \beta$ where $\frac{dx}{dt} = f'(t) > 0$. ($f(\alpha) = a$, $f(\beta) = b$)

$$\Rightarrow L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_\alpha^\beta \sqrt{1+\left(\frac{dy/dt}{dx/dt}\right)^2}\,\frac{dx}{dt}\,dt = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Surface Area

$$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

10.3 Polar Coordinates

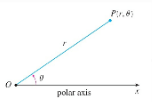


FIGURE 1

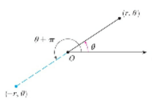


FIGURE 2

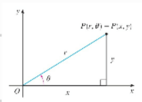


FIGURE 5

- If P is any other point in the plane, let r be the distance from O to P and let θ be the angle between the polar axis and the line OP
- In the Cartesian coordinate system, every point has only one representation, but in the polar coordinate system each point has many representations.
- the connection between polar and Cartesian coordinates can be seen, $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$

$\Rightarrow x = r\cos\theta$, $y = r\sin\theta$, and to find r and θ when x and y are known, we use

$\Rightarrow r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$

Tangents to Polar Curve :

- to find a tangent line to a polar curve $r = f(\theta)$,

$\Rightarrow x = r\cos\theta = f(\theta)\cos\theta$, $y = r\sin\theta = f(\theta)\sin\theta$

$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{d\theta}\right) / \left(\frac{dx}{d\theta}\right) = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$, we locate horizontal tangents by finding the points where $\frac{dy}{d\theta} = 0$. Likewise, we locate vertical tangents at the points where $\frac{dx}{d\theta} = 0$

☆ notice that if we are looking for tangent lines at the pole, then $r = 0$ and the equation will be simplified to $\frac{dy}{dx} = \tan\theta$ if $\frac{dr}{d\theta} \ne 0$

10.4 Areas and Lengths in Polar Coordinates

Formula for the Area of a sector of a circle :

$A = \frac{1}{2} r^2 \theta$, r is the radius and θ is the radian measure of the central angle

$\Rightarrow$ Approximation to the total area A of R :

$$A \approx \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta$$

$$\approx \int_a^b \frac{1}{2}[f(\theta_i^*)]^2\,d\theta$$

- let R be a region that is bounded by curves with polar equations $r = f(\theta)$, $r = g(\theta)$, $a \le \theta \le b$, where $f(\theta) \ge g(\theta) \ge 0$, $0 < b - a \le 2\pi$. Then the area A of R is found by subtracting the area inside $r = g(\theta)$ from the area inside $r = f(\theta)$

$\Rightarrow$ $A = \frac{1}{2}\int_a^b [f(\theta)]^2 - [g(\theta)]^2 \, d\theta$

☆ the fact that a single point has many representations in polar coordinates sometimes makes it difficult to find all the points of intersection of two polar curves since points don't collide at the origin because they reach the origin at different times, but the curves intersect there nonetheless

☆ some of intersection points may be found by $r \Rightarrow -r$

Arc Length :

- Using the equation of conversing polar coordinates to Cartesian Coordinates,

$x = r\cos\theta = f(\theta)\cos\theta$, $y = r\sin\theta = f(\theta)\sin\theta$, and if we differentiate, we get

$\Rightarrow \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$, $\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$, these give rise to,

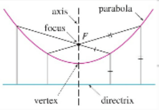
$$\Rightarrow \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta + \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta$$

$= \left(\frac{dr}{d\theta}\right)^2 + r^2$, and assuming f' is continuous,

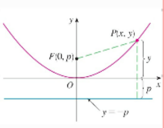
$$\Rightarrow L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta$$

$$= \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

10.5 Conic Sections



- Vertex is in the middle of the focus and directrix



- Suppose we place its vertex at the origin O and its directrix parallel to the x-axis. The distance from any point on the parabola $P(x, y)$ to the focus $F(0, p)$ is $|PF| = \sqrt{x^2 + (y-p)^2}$, and the distance from P to the directrix $|y + p|$

$\Rightarrow$ By the characteristic of directrix,

$$\sqrt{x^2 + (y-p)^2} = |y + p|$$

$$x^2 + (y-p)^2 = (y+p)^2$$

$x^2 = 4py$, and this is an equation of the parabola with focus $(0, p)$ and directrix $y = -p$

- if we write $a = \frac{1}{4p}$, then the standard equation of a parabola becomes $y = ax^2$, It opens upward if $p > 0$ and downward if $p < 0$.

Ellipses :

- assuming the foci are on the x-axis at the points $(-c,0)$ and $(c,0)$ so that the origin is half way between the foci. let the sum of distances from a point on the ellipse to the foci be $2a>0$. Then $P(x,y)$ is a point on the ellipse when $|PF_1| + |PF_2| = 2a$

$\Rightarrow \sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a$

$\vdots$

$\Rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a \geq b > 0$, has foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$ and vertices $(\pm a, 0)$

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a \geq b > 0$, has foci $(0, \pm c)$, where $c^2 = a^2 - b^2$ and vertices $(0, \pm a)$

Hyperbolas :

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ has foci $(\pm c, 0)$, where $c^2 = a^2 + b^2$, vertices $(\pm a, 0)$, and asymptotes $y = \pm \left(\dfrac{b}{a}\right)x$

10.6 Conic Sections in Polar Coordinates

- Let F be a fixed point called the focus and l be a fixed line called the directrix in a plane. Let e be a fixed positive number called the eccentricity. The set of all points P in the plane such that $\dfrac{|PF|}{|Pl|} = e$, the ratio of the distance from F to the distance from l is the constant e, is a conic section.

The conic is, a) an ellipse if $e < 1$

b) a parabola if $e = 1$

c) a hyperbola if $e > 1$

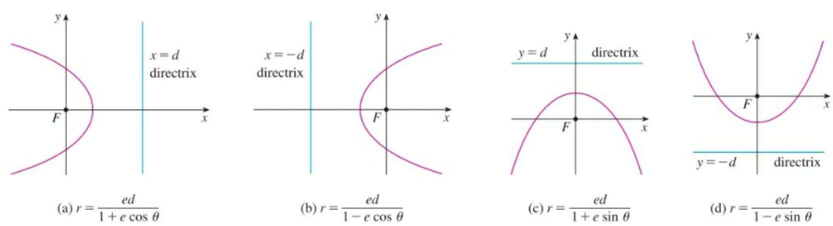


FIGURE 2
Polar equations of conics

6 Theorem A polar equation of the form

$$r = \frac{ed}{1 \pm e\cos\theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e\sin\theta}$$

represents a conic section with eccentricity e. The conic is an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola if $e > 1$.