Analysis of BIBD (fixed blocks)

$$n_{ij} y_{ij} = n_{ij}(\mu + \tau_i + \beta_j + \varepsilon_{ij}) \qquad i = 1, \cdots, a$$
$$j = 1, \cdots, b$$

where $n_{ij} = \begin{cases} 1 & \text{if treatment } i \text{ appears in block } j \\ 0 & \text{o.w.} \end{cases}$

$$\sum_i \tau_i = \sum_j \beta_j = 0.$$

Then $n_{i\cdot} = r$ and $n_{\cdot j} = k$

Let $Q = \sum_i \sum_j n_{ij} \varepsilon_{ij}^2 = \sum_i \sum_j n_{ij}(y_{ij} - \mu - \tau_i - \beta_j)^2$

Then

i) $$\frac{\partial Q}{\partial \mu} = -2 \sum_i \sum_j n_{ij}(y_{ij} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j) = 0$$

$$\Rightarrow \underbrace{\sum_i \sum_j n_{ij} y_{ij}}_{y_{\cdot\cdot}} = N\hat{\mu} + r \underbrace{\sum_i \hat{\tau}_i}_{0} + k \underbrace{\sum_j \hat{\beta}_j}_{0}$$

$$\Rightarrow y_{\cdot\cdot} = N\hat{\mu}$$

$$\Rightarrow \boxed{\hat{\mu} = \frac{y_{\cdot\cdot}}{N}}$$

ii) $$\frac{\partial Q}{\partial \tau_i} = -2 \sum_j n_{ij}(y_{ij} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j) = 0 \quad \text{for } i = 1, \cdots, a$$

$$\Rightarrow y_{i\cdot} = r\hat{\mu} + r\hat{\tau}_i + \sum_j n_{ij} \hat{\beta}_j \quad -(A)$$

iii) $$\frac{\partial Q}{\partial \beta_j} = -2 \sum_i n_{ij}(y_{ij} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j) = 0 \quad \text{for } j = 1, \cdots, b$$

$$\Rightarrow y_{\cdot j} = k\hat{\mu} + \sum_i n_{ij} \hat{\tau}_i + k\hat{\beta}_j$$

$$\Rightarrow \hat{\beta}_j = \frac{1}{k} y_{\cdot j} - \frac{1}{k} k\hat{\mu} - \frac{1}{k} \sum_i n_{ij} \hat{\tau}_i \quad -(B)$$

Then we plug (B) in (A),

$$k y_{i\cdot} = kr\hat{\mu} + kr\hat{\tau}_i + \sum_j n_{ij}\left[y_{\cdot j} - k\hat{\mu} - \sum_{i'} n_{i'j} \hat{\tau}_{i'} \right] \quad \text{for } i = 1, \cdots, a$$

$-(C)$

Consider the last term in (C): $\sum_j n_{ij}\left[y_{\cdot j} - k\hat{\mu} - \sum_{i'} n_{i'j}\hat{\tau}_{i'}\right]$ —(*)

① $\sum_j n_{ij} y_{\cdot j} \equiv$ Sum of block totals for blocks containing Trt i

$\equiv B_i$

② $\sum_j n_{ij} k\hat{\mu} = k\hat{\mu} \underbrace{n_{i\cdot}}_{\substack{\shortparallel \\ r}} = kr\hat{\mu}$

③ $\sum_j n_{ij} \sum_{i'} n_{i'j}\hat{\tau}_{i'} = \sum_j \underbrace{n_{ij}^2}_{n_{ij}} \hat{\tau}_i + \sum_{i\neq i'}\sum_j n_{ij} n_{i'j} \hat{\tau}_{i'}$ —(D)

$n_{ij} n_{i'j} = 1$ if trts i and i' appear together in block j

Trts i and i' appear together in λ blocks.

$\Rightarrow \sum_{i\neq i'}\sum_j n_{ij} n_{i'j}\hat{\tau}_{i'} = \sum_{i\neq i'} \hat{\tau}_{i'} \underbrace{\sum_j n_{ij} n_{i'j}}_{\lambda} = \sum_{i\neq i'} \hat{\tau}_{i'}\lambda$

Thus, (D) is

$$\sum_j n_{ij}\sum_{i'} n_{i'j}\hat{\tau}_{i'} = r\hat{\tau}_i + \lambda\sum_{i\neq i'}\hat{\tau}_{i'}$$

$$\underset{\uparrow}{=} r\hat{\tau}_i - \lambda\hat{\tau}_i$$

$\sum_i \hat{\tau}_i = 0 \Rightarrow \hat{\tau}_i = -\sum_{i'\neq i}\hat{\tau}_{i'}$

Putting together ①, ②, and ③, (*) is

$$B_i - kr\hat{\mu} - (r-\lambda)\hat{\tau}_i \quad \text{—(E)}$$

Now consider (C), then

$$k y_{i\cdot} \underset{\substack{\uparrow \\ \text{by (E)}}}{=} \cancel{kr\hat{\mu}} + kr\hat{\tau}_i + B_i - \cancel{kr\hat{\mu}} - (r-\lambda)\hat{\tau}_i$$

$\Rightarrow k y_{i\cdot} - B_i = \hat{\tau}_i\left[kr - (r-\lambda)\right] = \hat{\tau}_i\left[\underbrace{r(k-1)}_{\substack{\shortparallel \\ \lambda(a-1)}} + \lambda\right]$ by the property.

$\Rightarrow k y_{i\cdot} - B_i = \hat{\tau}_i\left[\lambda(a-1) + \lambda\right] = \hat{\tau}_i \lambda a$

$$\Rightarrow \boxed{\hat{\tau}_i = \frac{k y_{i\cdot} - B_i}{\lambda a} = \frac{kQ_i}{\lambda a}}$$

where $Q_i = y_{i\cdot} - \frac{1}{k}B_i = y_{i\cdot} - \frac{1}{k}\sum_j n_{ij} y_{\cdot j}$

Similarly,

$$\boxed{\hat{\beta}_j = \frac{r Q_j'}{\lambda b}}$$

where $Q_j' = y_{\cdot j} - \frac{1}{r} \sum_i n_{ij} y_{i\cdot}$ ///

- Testing $H_0: \tau_1 = \cdots = \tau_a = 0$ vs H_a: not all τ_i are zero.

① Under Full model, obtain SSE_{F}

$$SSE_{F} = \sum_i \sum_j (Y_{ij} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j^{Full})^2$$

where $\hat{\beta}_j^{Full} = \bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot} - \frac{1}{k} \sum_i n_{ij} \hat{\tau}_i$.

② Under Reduced model (H_0), obtain SSE_R

$$SSE_R = \sum_i \sum_j (y_{ij} - \hat{\mu} - \hat{\beta}_j^{Reduced})^2$$

where $\hat{\beta}_j^{Reduced} = \bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot}$

③ Difference $\equiv SSE_R - SSE_{F} = SS$(Trts adjusted for blocks)

$$\Rightarrow \boxed{SS_{Trt}(\text{adjusted}) = \frac{k \sum_i Q_i^2}{\lambda a}}$$