More criteria for (Riemann-) Integrability

⊙ **Another equivalent definition of (Riemann-) integrability (seen in most texts)**

Let $f(x)$ be a bounded function on $[a,b]$, and let $\mathcal{P}$: $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be a partition of $[a,b]$. Write

$$m_i = \inf_{[\Delta x_i]} f(x), \quad M_i = \sup_{[\Delta x_i]} f(x), \quad \Delta x_i = x_i - x_{i-1}$$

We define

$$L(\mathcal{P}) = L_f(\mathcal{P}) = \sum_{i=1}^{n} m_i \Delta x_i \quad \text{(the lower sum for } f(x) \text{ over } \mathcal{P})$$

$$U(\mathcal{P}) = U_f(\mathcal{P}) = \sum_{i=1}^{n} M_i \Delta x_i \quad \text{(the upper sum for } f(x) \text{ over } \mathcal{P})$$

The upper (**Darboux** or Riemann) integral of f on $[a,b]$ is defined by

$$\overline{\int_a^b} f(x)\,dx = \inf\{U_f(\mathcal{P}) : \mathcal{P} \text{ a partion of } [a,b]\} \overset{\text{simply}}{=} \inf_{\mathcal{P}} U_f(\mathcal{P}) \quad \text{(상적분)}$$

and the lower (**Darboux** or Riemann) integral of f on $[a,b]$ is defined by

$$\underline{\int_a^b} f(x)\,dx = \sup\{L_f(\mathcal{P}) : \mathcal{P} \text{ a partion of } [a,b]\} \overset{\text{simply}}{=} \sup_{\mathcal{P}} L_f(\mathcal{P}) \quad \text{(하적분)}$$

We say that f is integrable on $[a,b]$ (or $f \in \mathcal{R}[a,b]$) if $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$.

If $f \in \mathcal{R}[a,b]$, we define its (definite) integral $\int_a^b f(x)\,dx$ by

$$\int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx \overset{\text{or}}{=} \underline{\int_a^b} f(x)\,dx$$

Darboux's criterion for integrability. $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$ iff

for each $\varepsilon > 0$ there exists a partition $\mathcal{P} = \mathcal{P}_\varepsilon$ of $[a, b]$ such that $U_f(\mathcal{P}) - L_f(\mathcal{P}) < \varepsilon$.

Pf. Suppose that $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$, and let $\varepsilon > 0$ be given.

By the definition of $\overline{\int_a^b} f(x)\,dx$ & $\underline{\int_a^b} f(x)\,dx$, $\exists$ $\mathcal{P}_1 = \mathcal{P}_1(\varepsilon)$ and $\mathcal{P}_2 = \mathcal{P}_2(\varepsilon)$ such that

$$\underline{\int_a^b} f(x)\,dx - \varepsilon/2 < L_f(\mathcal{P}_1), \qquad U_f(\mathcal{P}_2) < \overline{\int_a^b} f(x)\,dx + \varepsilon/2$$

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then $\mathcal{P}$ is a common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$, and thus

$$L_f(\mathcal{P}_1) \le L_f(\mathcal{P}) \le U_f(\mathcal{P}) \le U_f(\mathcal{P}_2)$$

Hence $U_f(\mathcal{P}) - L_f(\mathcal{P}) \le U_f(\mathcal{P}_2) - L_f(\mathcal{P}_1) < \left(\overline{\int_a^b} f(x)\,dx + \varepsilon/2\right) - \left(\underline{\int_a^b} f(x)\,dx - \varepsilon/2\right) = \varepsilon$

For the converse, suppose that for any $\varepsilon > 0$, $\exists$ a partition $\mathcal{P} = \mathcal{P}_\varepsilon$ such that $U_f(\mathcal{P}) - L_f(\mathcal{P}) < \varepsilon$.

Since $L_f(\mathcal{P}) \le \underline{\int_a^b} f(x)\,dx$ and $\overline{\int_a^b} f(x)\,dx \le U_f(\mathcal{P})$, it follows that

$$\overline{\int_a^b} f(x)\,dx \le U_f(\mathcal{P}) < L_f(\mathcal{P}) + \varepsilon \le \underline{\int_a^b} f(x)\,dx + \varepsilon \qquad \therefore\ \overline{\int_a^b} f(x)\,dx < \underline{\int_a^b} f(x)\,dx + \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, we see that $\overline{\int_a^b} f(x)\,dx \le \underline{\int_a^b} f(x)\,dx$ $\left(\leftarrow \alpha + \varepsilon > \beta \text{ for } \forall \varepsilon > 0 \Rightarrow \alpha \ge \beta\right)$.

But $\overline{\int_a^b} f(x)\,dx \ge \underline{\int_a^b} f(x)\,dx$ is trivially true. Combining these two gives

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$$

Remember. Let $f(x)$ be bounded on $[a,b]$. Then

① $L_f(\mathcal{P}) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U_f(\mathcal{P})\left[\& \ U_f(\mathcal{Q})\right]$ $\forall$partition $\mathcal{P}\left[\& \ \mathcal{Q}\right]$ of $[a,b]$

② $\forall \varepsilon > 0$, $\exists$ a partition $\mathcal{P} = \mathcal{P}_\varepsilon$ of $[a,b]$ such that

$$\underline{\int_a^b} f(x)\,dx - \varepsilon < L_f(\mathcal{P}) \le U_f(\mathcal{P}) < \overline{\int_a^b} f(x)\,dx + \varepsilon$$

Remark: $f \in \mathcal{R}[a,b] \overset{②}{\Rightarrow} \forall \varepsilon > 0$, $\exists$ a partition $\mathcal{P} = \mathcal{P}_\varepsilon$ of $[a,b]$ such that

$$\int_a^b f(x)\,dx - \varepsilon < L_f(\mathcal{P}) \le U_f(\mathcal{P}) < \int_a^b f(x)\,dx + \varepsilon$$

Cor. (Limit criterion for integrability) Let f be bounded on $[a,b]$. Then

$f \in \mathcal{R}[a,b] \iff \exists$ a sequence of partitions $\mathcal{P}_n$ such that $\lim_{n\to\infty}\left(U_f(\mathcal{P}_n) - L_f(\mathcal{P}_n)\right) = 0$

Pf. Assume that $f \in \mathcal{R}[a,b]$. Then we see (from the Darboux's criterion for integrability) that

$\forall \varepsilon > 0$, $\exists$ a partition $\mathcal{P} = \mathcal{P}_\varepsilon$ such that $U_f(\mathcal{P}) - L_f(\mathcal{P}) < \varepsilon$

Take $\varepsilon = 1/n$ $(n = 1, 2, \cdots)$. Then there is a sequence of partitions $\mathcal{P}_n$ such that

$$U_f(\mathcal{P}_n) - L_f(\mathcal{P}_n) < 1/n$$

This clearly implies $\lim_{n\to\infty}\left(U_f(\mathcal{P}_n) - L_f(\mathcal{P}_n)\right) = 0$.

Conversely, assume that $\exists$ a seq of partitions $\mathcal{P}_n$ such that $\lim_{n\to\infty}\left(U_f(\mathcal{P}_n) - L_f(\mathcal{P}_n)\right) = 0$.

Then given any $\varepsilon > 0$, $U_f(\mathcal{P}_n) - L_f(\mathcal{P}_n) < \varepsilon$ for $\forall n \ge N = N(\varepsilon)$ (some N).

In particular, $U_f(\mathcal{P}_N) - L_f(\mathcal{P}_N) < \varepsilon$. Consequently,

given any $\varepsilon > 0$, $\exists$ a partition $\mathcal{P}_N = \mathcal{P}_N(\varepsilon)$ such that $U_f(\mathcal{P}_N) - L_f(\mathcal{P}_N) < \varepsilon$.

This gives $f \in \mathcal{R}[a,b]$.

Remark to Cor. (Common limit criterion for integrability): Let f be bounded on $[a,b]$.

(a) If $f\in\mathcal{R}[a,b]$, then $\exists$ a seq. of partitions $\mathcal{P}_n$ such that $\int_a^b f(x)\,dx=\lim_{n\to\infty}U_f(\mathcal{P}_n)=\lim_{n\to\infty}L_f(\mathcal{P}_n)$

(b) If $\exists$ a seq. of partitions $\mathcal{P}_n$ such that $\lim_{n\to\infty}U_f(\mathcal{P}_n)=\lim_{n\to\infty}L_f(\mathcal{P}_n)$ (i.e., both exist & are equal), then

$f\in\mathcal{R}[a,b]$ and in this case, $\int_a^b f(x)\,dx=\lim_{n\to\infty}U_f(\mathcal{P}_n)=\lim_{n\to\infty}L_f(\mathcal{P}_n)$

Pf. (a) $f\in\mathcal{R}[a,b]$ $\Rightarrow$ $\exists$ a seq. of partitions $\mathcal{P}_n$ such that $\lim_{n\to\infty}\left(U_f(\mathcal{P}_n)-L_f(\mathcal{P}_n)\right)=0$ (by Cor)

Note that $L_f(\mathcal{P}_n)\le\underline{\int_a^b} f(x)\,dx=\int_a^b f(x)\,dx=\overline{\int_a^b} f(x)\,dx\le U_f(\mathcal{P}_n)$ $(\Leftarrow f\in\mathcal{R}[a,b])$

$\therefore\ 0\le\int_a^b f(x)\,dx-L_f(\mathcal{P}_n)\le U_f(\mathcal{P}_n)-L_f(\mathcal{P}_n)$ $\quad\therefore\ \int_a^b f(x)\,dx=\lim_{n\to\infty}L_f(\mathcal{P}_n)$ (by letting $n\to\infty$)

Also, $U_f(\mathcal{P}_n)=L_f(\mathcal{P}_n)+\left(U_f(\mathcal{P}_n)-L_f(\mathcal{P}_n)\right)\to\int_a^b f(x)\,dx$ as $n\to\infty$

(b) Hypothesis clearly implies $\lim_{n\to\infty}\left(U_f(\mathcal{P}_n)-L_f(\mathcal{P}_n)\right)=0$. Thus $f\in\mathcal{R}[a,b]$ (by Corollary)

Note that $L_f(\mathcal{P}_n)\le\underline{\int_a^b} f(x)\,dx\le\overline{\int_a^b} f(x)\,dx\le U_f(\mathcal{P}_n)$ for $\forall n\in\mathbb{N}$.

Taking limits $\overset{\text{Hypo + Sandwich thm}}{\Rightarrow}$ $\lim_{n\to\infty}U_f(\mathcal{P}_n)=\lim_{n\to\infty}L_f(\mathcal{P}_n)=\underline{\int_a^b} f=\overline{\int_a^b} f=\int_a^b f(x)\,dx\ [\Rightarrow f\in\mathcal{R}[a,b]]$

Theorem (Another criterion for integrability) Let f be bounded on $[a,b]$. Then

$\forall\varepsilon>0$, $\exists\,\mathcal{P}=\mathcal{P}_\varepsilon$ of $[a,b]$ such that $U_f(\mathcal{P})-L_f(\mathcal{P})<\varepsilon$ (i.e., $f\in\mathcal{R}[a,b]$)

$\Leftrightarrow$ $\forall\varepsilon>0$, $\exists\delta=\delta(\varepsilon)>0$ such that $U_f(\mathcal{P})-L_f(\mathcal{P})<\varepsilon$ for $\forall\mathcal{P}$ with $|\mathcal{P}|<\delta$

(i.e., $U_f(\mathcal{P})-L_f(\mathcal{P})\to0$ as $|\mathcal{P}|\to0$, which is used as the definition of integrability of f in our text)

---This is already proved in the last paragraph of Chapter 19 ---

Exa1. $f(x)=\begin{cases}1 & \text{if } x \text{ is a rational number}\\ 0 & \text{otherwise}\end{cases}$ Prove that $f\notin\mathcal{R}[0,1]$

Pf. Let $\mathcal{P}$ be any partition of $[0,1]$.

Then every subinterval of $\mathcal{P}$ contains a rational number and an irrational number.

$$\therefore\ \sup_{[\Delta x_i]} f(x)=1,\qquad \inf_{[\Delta x_i]} f(x)=0$$

$$\therefore\ U_f(\mathcal{P})=1\quad\text{and}\quad L_f(\mathcal{P})=0\quad\text{for any }\mathcal{P}$$

Hence $\overline{\int_0^1} f(x)\,dx=\inf_{\mathcal{P}}U_f(\mathcal{P})=1$ & $\underline{\int_0^1} f(x)\,dx=\sup_{\mathcal{P}}L_f(\mathcal{P})=0$

Therefore, $\overline{\int_0^1} f(x)\,dx\ \neq\ \underline{\int_0^1} f(x)\,dx$ $\quad\therefore\ f\notin\mathcal{R}[0,1]$

Exa2. Let $f(x)=\begin{cases}x & \text{if } x\in\mathbb{Q}\cap[0,1]\\ 0 & \text{if } x\in\mathbb{Q}^c\cap[0,1]\end{cases}$

(i) Prove that $f\notin\mathcal{R}[0,1]$. (ii) Prove also that $\underline{\int_0^1} f(x)\,dx=0$ and $\overline{\int_0^1} f(x)\,dx=\dfrac{1}{2}$

Pf. (i) Let $\mathcal{P}: 0 = x_0 < x_1 < \cdots < x_n = 1$ be a partition of $[0,1]$. Then

$m_i = 0$ $(i = 1, 2, \cdots, n)$ (since each $[\Delta x_i]$ contains an irrational number) $\therefore L_f(\mathcal{P}) = 0$

Next, since $x_{i-1} < x_i$, $\exists$ a rational number r_i such that $\frac{1}{2}(x_{i-1} + x_i) < r_i < x_i$

$$\therefore \quad M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \geq f(r_i) = r_i$$

$$\therefore \quad U_f(\mathcal{P}) = \sum_{i=1}^{n} M_i \Delta x_i \geq \sum_{i=1}^{n} r_i \Delta x_i \geq \sum_{i=1}^{n} \frac{1}{2}(x_{i-1} + x_i) \Delta x_i = \frac{1}{2} \sum_{i=1}^{n} (x_i + x_{i-1})(x_i - x_{i-1})$$

$$= \frac{1}{2} \sum_{i=1}^{n} (x_i^2 - x_{i-1}^2) = \frac{1}{2}(x_n^2 - x_0^2) = \frac{1}{2}$$

$$\therefore \quad U_f(\mathcal{P}) \geq 1/2$$

Since $\mathcal{P}$ is arbitrary, $0 = L_f(\mathcal{P}) < 1/2 \leq U_f(\mathcal{P})$ $\qquad \therefore \quad f \notin \mathcal{R}[0,1]$

(ii) Already seen that $L_f(\mathcal{P}) = 0$ $\forall \mathcal{P}$ of $[0, 1]$ $\qquad \therefore \underline{\int_0^1} f(x)\,dx = 0$

Also already proved that $U_f(\mathcal{P}) \geq 1/2$ $\forall \mathcal{P}$ of $[0,1]$ $\qquad \therefore \overline{\int_0^1} f(x)\,dx \geq \frac{1}{2}$

To prove $\overline{\int_0^1} f(x)\,dx \leq \frac{1}{2}$, let $\mathcal{P}^{(n)}$ be the standard n-partition of $[0,1]$. Then

$$\Delta x_i = \frac{1}{n} \quad \text{and} \quad x_i = \frac{i}{n}, \quad \text{for } \forall i$$

Clearly, $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \leq \sup_{x \in [x_{i-1}, x_i]} x = x_i$

$$\therefore \quad U_f(\mathcal{P}^{(n)}) = \sum_{i=1}^{n} M_i \Delta x_i \leq \sum_{i=1}^{n} x_i \Delta x_i = \frac{1}{n} \sum_{i=1}^{n} x_i = \frac{1}{n} \sum_{i=1}^{n} \frac{i}{n} = \frac{1}{n^2} \sum_{i=1}^{n} i = \frac{n(n+1)}{2n^2}$$

On the other hand, it is obvious that $\overline{\int_0^1} f(x)\,dx \leq U_f(\mathcal{P})$ $\forall \mathcal{P}$ of $[0,1]$

In particular, $\overline{\int_0^1} f(x)\,dx \leq U_f(\mathcal{P}^{(n)}) \leq \frac{n(n+1)}{2n^2}$ $\qquad \therefore \overline{\int_0^1} f(x)\,dx \leq \lim_{n \to \infty} \frac{n(n+1)}{2n^2} = \frac{1}{2}$ (by LLT)

Another short pf. Let $\mathcal{P}: 0 = x_0 < x_1 < \cdots < x_n = 1$ be a partition of $[0,1]$.

Claim: $\quad M_i = \sup_{x \in [\Delta x_i]} f(x) = x_i$

Pf of claim. Clearly, $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \leq \sup_{x \in [x_{i-1}, x_i]} x = x_i$

$\therefore$ x_i is an upper bound for the set $\{f(x) : x \in [x_{i-1}, x_i]\}$ --- ①

On the other hand, we know that $0 < \forall \varepsilon \ll 1$, $\exists\, r_i \in \mathbb{Q} \cap [x_{i-1}, x_i]$ such that $x_i - \varepsilon < r_i < x_i$.

$\therefore$ $x_i - \varepsilon < r_i = f(r_i) < x_i$ (for some $r_i \in [x_{i-1}, x_i]$),

which shows $x_i - \varepsilon$ is not an upper bound for the set $\{f(x) : x \in [x_{i-1}, x_i]\}$ --- ②

$$\text{① \& ②} \quad \Rightarrow \quad M_i = \sup_{x \in [\Delta x_i]} f(x) = x_i$$

Note that $m_i = 0$ $(i = 1, 2, \cdots, n)$ $\qquad \therefore 0 = L_f(\mathcal{P})$ $\forall \mathcal{P}$ of $[0,1]$

$$\therefore \quad U_f(\mathcal{P}) - L_f(\mathcal{P}) = U_f(\mathcal{P}) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} x_i \Delta x_i \quad \to \quad \int_0^1 x\,dx = \frac{1}{2} \quad \text{as } |\mathcal{P}| \to 0$$

Thus $f \notin \mathcal{R}[0,1]$. Here we used: $f \in \mathcal{R}[0,1]$ iff $U_f(\mathcal{P}) - L_f(\mathcal{P}) \to 0$ as $|\mathcal{P}| \to 0$

HS. (i) Let $f(x)=\begin{cases} x & \text{if } x\in\mathbb{Q}\cap[0,1] \\ -x & \text{if } x\in\mathbb{Q}^c\cap[0,1] \end{cases}$

Prove that $f\notin\mathcal{R}[0,1]$ and, moreover, that $\overline{\int_0^1} f(x)\,dx=1/2$ and $\underline{\int_0^1} f(x)\,dx=-1/2$

(ii) Let $f(x)=\begin{cases} x^2 & \text{if } x\in\mathbb{Q}\cap[0,1] \\ -x^2 & \text{if } x\in\mathbb{Q}^c\cap[0,1] \end{cases}$

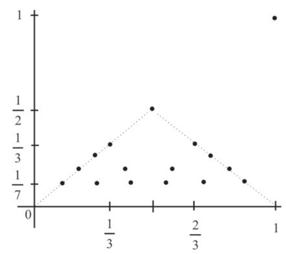
Prove that $f\notin\mathcal{R}[0,1]$ and, moreover, that $\overline{\int_0^1} f(x)\,dx=1/3$ and $\underline{\int_0^1} f(x)\,dx=-1/3$

(iii) Let $f(x)=\begin{cases} \sin x & \text{if } x\in\mathbb{Q}\cap[0,\frac{\pi}{4}] \\ \cos x & \text{if } x\in\mathbb{Q}^c\cap[0,\frac{\pi}{4}] \end{cases}$

Prove that $f\notin\mathcal{R}[0,\frac{\pi}{4}]$ and find the values: $\overline{\int_0^{\pi/4}} f(x)\,dx$ and $\underline{\int_0^{\pi/4}} f(x)\,dx$

More examples (Riemann integrable functions with infinitely many discontinuities):

Exa3. (Thomae's function or Tree function or Popcorn function)



Let $f:[0,1]\to\mathbb{R}$ be a function defined by

$$f(x)=\begin{cases} 0 & \text{if } x \text{ is irrational or } x=0 \\ \dfrac{1}{q} & \text{if } x=\dfrac{p}{q} \text{ with } \gcd(p,q)=1 \text{ and } p,q\in\mathbb{N} \end{cases}$$

Prove that

$$f\in\mathcal{R}[0,1] \quad\text{and}\quad \int_0^1 f(x)\,dx=0\,.$$

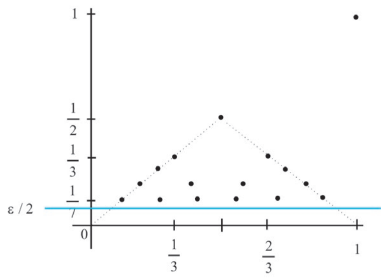
Pf. [An elementary way of showing the integrability of f]:

First note that if I is any open interval that intersects $[0,1]$, then

$$f(x)=0 \text{ for some } x\in[0,1]\cap I \qquad \therefore\quad L(\mathcal{P})=0 \text{ for } \forall\text{partition } \mathcal{P} \text{ of } [0,1]$$

Let $\varepsilon>0$ be arbitrary, and let $A:=\{x\in[0,1]:|f(x)|\ge\varepsilon/2\}=\{x\in[0,1]:f(x)\ge\varepsilon/2\}$

Key fact 1: A is a **finite set**, since it consists of only those rational numbers $x=p/q$, where $1/q\ge\varepsilon/2$ and $0<p\le q$ (or see the figure of f below)



Write $A = \{x \in [0,1] : f(x) \geq \varepsilon/2\} = \{z_1, z_2, \cdots, z_n := 1\}$ with $z_k < z_{k+1}$ for each k

Choose a finite number of (positive) points $x_k, y_k \ (1 \leq k \leq n)$ such that

(i) $x_k < z_k < y_k$, with $y_k < x_{k+1}$ for $\forall k = 1, 2, \cdots, n-1$, and $x_n < z_n = y_n = 1$

(ii) $y_k - x_k < \dfrac{\varepsilon}{2n}$ for every $1 \leq k \leq n$.

That is, $(0 <) x_1 < z_1 < y_1 < x_2 < z_2 < y_2 < x_3 < \cdots, x_{n-1} < z_{n-1} < y_{n-1} < x_n < z_n = y_n = 1$,

where $y_k - x_k < \dfrac{\varepsilon}{2n}$ for every $1 \leq k \leq n$

Consider the partition $\mathcal{P}$ of $[0,1]$ given by

$$\mathcal{P} = \{0 = y_0 < x_1 < y_1 < x_2 < y_2 < x_3 < \cdots < x_{n-1} < y_{n-1} < x_n < y_n = 1\}$$

[Note that any point of the set A (i.e., z_k's) are not contained in the partition $\mathcal{P}$]

Let $M_k = \sup\{f(x) : x_k \leq x \leq y_k\}$, for $1 \leq k \leq n$, and

$\widetilde{M_k} = \sup\{f(x) : y_{k-1} \leq x \leq x_k\}$, for $1 \leq k \leq n$

Key fact 2: For each k, we have $M_k \leq 1$ and $\widetilde{M_k} \leq \varepsilon/2$. Hence,

$$\begin{aligned} U(\mathcal{P}) - L(\mathcal{P}) = U(\mathcal{P}) &= \sum_{k=1}^{n} M_k (y_k - x_k) + \sum_{k=1}^{n} \widetilde{M_k}(x_k - y_{k-1}) \\ &\leq \sum_{k=1}^{n} 1 \cdot \frac{\varepsilon}{2n} + \varepsilon/2 \underbrace{\sum_{k=1}^{n}(x_k - y_{k-1})}_{<1} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

Therefore, $f \in \mathcal{R}[0,1]$ and $\int_0^1 f(x)\,dx = \sup_{\mathcal{P}} L(\mathcal{P}) = 0$.

Exa4 [**advanced**]. Let C be the Cantor set in $[0, 1]$, and let

$f : [0, 1] \to \mathbb{R}$ be a function defined by $f(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases}$

Prove that $f \in \mathcal{R}[0,1]$ and $\int_0^1 f(x)\,dx = 0$.

Cf: Later (in Chap 23) we shall prove that $f \in \mathcal{R}[0,1]$ and $\int_0^1 f(x)\,dx = 0$ by a different method

Remark. (Definition of the Cantor set and its basic properties: will be studied in Chapter23)

1. (the Cantor set) $C = \bigcap_{n=1}^{\infty} C_n$, where

   $C_1 = [0, 1/3] \cup [2/3, 1]$,

   $C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$

   $\vdots$

2. It is easy to check that the "length" of the Cantor set is zero
3. It is known that the Cantor set C is an uncountable set
4. f is discontinuous at all points in C, so it has uncountably many discontinuities (proved later)

Pf of the Final Example. Recall that

$$C_1 = [0, 1/3] \cup [2/3, 1] =: I_{1,1} \cup I_{1,2} = \bigcup_{k=1}^{2} I_{1,k},$$

$$C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1] =: \bigcup_{k=1}^{4} I_{2,k} = \bigcup_{k=1}^{2^2} I_{2,k}$$

$$\vdots$$

$$C_n = \bigcup_{k=1}^{2^n} I_{n,k} \quad (\text{disjoit union of } 2^n - \text{intervals of length } \frac{1}{3^n})$$

$$\vdots$$

Note that $0 \le f = \chi_C \le \chi_{C_n}$ for every n. Hence

$$\overline{\int_0^1} \chi_C(x)\,dx \le \overline{\int_0^1} \chi_{C_n}(x)\,dx \overset{\chi_{C_n} \in \mathcal{R}[0,1]}{=} \int_0^1 \chi_{C_n}(x)\,dx = \int_0^1 \chi_{\bigcup_{k=1}^{2^n} I_{n,k}}(x)\,dx$$

$$= \int_0^1 \sum_{k=1}^{2^n} \chi_{I_{n,k}}(x)\,dx \quad \left[\Leftarrow \bigcup_{k=1}^{2^n} I_{n,k} \text{ is disjoint union} \right]$$

$$\overset{\text{by linearity}}{=} \sum_{k=1}^{2^n} \int_0^1 \chi_{I_{n,k}}(x)\,dx = \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n \quad \to \quad 0 \quad \text{as } n \to \infty$$

$$\therefore \quad \overline{\int_0^1} \chi_C(x)\,dx = 0, \quad \text{and so } \underline{\int_0^1} \chi_C(x)\,dx = 0$$

$$\therefore \quad \overline{\int_0^1} \chi_C(x)\,dx = \underline{\int_0^1} \chi_C(x)\,dx = 0$$

$$\therefore \quad \chi_C \in \mathcal{R}[0,1] \text{ and } \int_0^1 \chi_C(x)\,dx = 0.$$

Summary of the "Equivalent conditions for Riemann integrability"

Let f be a bounded function on $[a, b]$ & let $\mathcal{P}$ be a partition of $[a, b]$. Recall that

$$\operatorname{Osc}(f, J) \overset{\text{denote}}{=} \sup_{x\in J} f(x) - \inf_{x\in J} f(x) = \sup_{x,\, y\in J} |\, f(x) - f(y)\,| \quad (J : \text{a subinterval of } [a, b])$$

Write $\operatorname{Osc}(f : \mathcal{P}) = \sum_{i=1}^{n} \operatorname{Osc}(f : [\Delta x_i])\Delta x_i = \sum_{i=1}^{n} \left(\sup_{[\Delta x_i]} f(x) - \inf_{[\Delta x_i]} f(x) \right) \Delta x_i = U_f(\mathcal{P}) - L_f(\mathcal{P})$

Theorem A. [Integrability] Let f be bounded on $[a, b]$. Then TFAE

① $\forall \varepsilon > 0$, $\exists$ a partition $\mathcal{P} = \mathcal{P}_\varepsilon$ such that $U_f(\mathcal{P}) - L_f(\mathcal{P}) \left(= \operatorname{Osc}(f : \mathcal{P})\right) < \varepsilon$

② $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$ i.e., $\inf_{\mathcal{P}} U_f(\mathcal{P}) = \sup_{\mathcal{P}} L_f(\mathcal{P})$

③ $f \in \mathcal{R}[a, b]$ ($\lim_{|\mathcal{P}|\to 0} \left(U_f(\mathcal{P}) - L_f(\mathcal{P}) \right) = 0$) i.e., $\operatorname{Osc}(f : \mathcal{P}) \to 0$ as $|\mathcal{P}| \to 0$. More precisely,

$\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ such that $\operatorname{Osc}(f : \mathcal{P}) < \varepsilon$ for all $\mathcal{P}$ with $|\mathcal{P}| < \delta$

④ $\exists$ a seq. of partitions $\mathcal{P}_n$ such that $U_f(\mathcal{P}_n) - L_f(\mathcal{P}_n) \left(= \operatorname{Osc}(f : \mathcal{P}_n)\right) \to 0$ as $n \to \infty$

⑤ $\exists$ a seq. of partitions $\mathcal{P}_n$ such that $\lim_{n\to\infty} U_f(\mathcal{P}_n) = \lim_{n\to\infty} L_f(\mathcal{P}_n) =: I$ (i.e., both exist & are equal)

[i.e., $\exists$ a seq. of partitions $\mathcal{P}_n$ and a real number I such that

$U_f(\mathcal{P}_n) \to I$ and $L_f(\mathcal{P}_n) \to I$ as $n \to \infty$]

In this case, $I = \int_a^b f(x)\, dx$

⑥ **[Riemann's definition of integrability]**

$I = \lim_{|\mathcal{P}|\to 0} \underbrace{\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})}_{\text{Riemann sums}}$ (uniquely) exists, for all choices of $t_i \in [x_{i-1}, x_i]$

Here, $|\mathcal{P}| = \max_{1\le i\le n}(x_i - x_{i-1})$. In this case, $I = \int_a^b f(x)\, dx$

Theorem B. [Useful result for actual computation of Riemann integral] (cf: Theorem A-①)

Let $f \in \mathcal{R}[a, b]$. Then the following results hold

① for any seq $\mathcal{P}_n$ of partitions of $[a, b]$ such that $|\mathcal{P}_n| \to 0$,

$$\lim_{n\to\infty} L(\mathcal{P}_n) = \int_a^b f(x)\, dx \quad \& \quad \lim_{n\to\infty} U(\mathcal{P}_n) = \int_a^b f(x)\, dx.$$

② for any seq $\mathcal{P}_n$ of partitions of $[a, b]$ such that $|\mathcal{P}_n| \to 0$,

$\lim_{n\to\infty} S_f(\mathcal{P}_n) = \int_a^b f(x)\, dx$, where $S_f(\mathcal{P}_k)$ be a Riemann sum for $f(x)$ over $\mathcal{P}_n$.

HS. Suppose $f, g \in \mathcal{R}[a,b]$ and $g(x) = f(x)$ for all $x \in [a,b] \setminus S$, where the set $S \subset [a,b]$ has a finite number of points. Then show that $\int_a^b f(x)dx = \int_a^b g(x)dx$