1) $$\varepsilon = \frac{n^2(\bar{\sigma}_{ii} - \bar{\sigma}_{..})^2}{(n-1)(S - 2n\sum \bar{\sigma}_{i.}^2 + n^2 \bar{\sigma}_{..}^2)}$$

$$\overset{n\times n}{\Sigma} = \begin{bmatrix} \sigma^2 & \rho\sigma^2 & \rho\sigma^2 & \cdots & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 & \rho\sigma^2 & \cdots & \rho\sigma^2 \\ \vdots & \rho\sigma^2 & \sigma^2 & \cdots & \rho\sigma^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho\sigma^2 & \rho\sigma^2 & \rho\sigma^2 & \cdots & \sigma^2 \end{bmatrix}$$

$$\Rightarrow \sigma_{ii} = \sigma^2, \quad \sigma_{ij} = \rho\sigma^2 \quad \forall i \neq j$$

$$\Rightarrow \bar{\sigma}_{ii} = \frac{1}{n}\sum_{i=1}^{n} \sigma_{ii} = \sigma^2, \quad \bar{\sigma}_{..} = \frac{1}{n^2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij}\right) = \frac{1}{n^2}\left(n\sigma^2 + n(n-1)\rho\sigma^2\right) = \frac{1}{n}\sigma^2 + \frac{n-1}{n}\rho\sigma^2 = \frac{1}{n}\sigma^2\left(1+(n-1)\rho\right)$$

$$S = \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij}^2 = n(\sigma^2)^2 + n(n-1)(\rho\sigma^2)^2, \quad \bar{\sigma}_{i.} = \frac{1}{n}\sum_{j=1}^{n} \sigma_{ij} = \frac{1}{n}\left(\sigma^2 + (n-1)\rho\sigma^2\right)$$

$$\Rightarrow n^2(\bar{\sigma}_{ii} - \bar{\sigma}_{..})^2 = \left(n(\bar{\sigma}_{ii} - \bar{\sigma}_{..})\right)^2$$

$$= \left[n\left(\sigma^2 - \frac{1}{n}\sigma^2\left(1+(n-1)\rho\right)\right)\right]^2$$

$$= \left[n\sigma^2 - \sigma^2\left(1+(n-1)\rho\right)\right]^2$$

$$= (n-1)^2(1-\rho)^2(\sigma^2)^2$$

$$\Rightarrow \varepsilon = 1 \quad \text{iff} \quad (n-1)(1-\rho)^2(\sigma^2)^2 = S - 2n\sum_{i=1}^{n} \bar{\sigma}_{i.}^2 + n^2\bar{\sigma}_{..}^2$$

$$\Rightarrow S - 2n\sum_{i=1}^{n} \bar{\sigma}_{i.}^2 + n^2\bar{\sigma}_{..}^2 = n(\sigma^2)^2 + n(n-1)\rho^2(\sigma^2)^2 - 2(\sigma^2)^2 - 4(n-1)\rho(\sigma^2)^2 - 2(n-1)^2\rho^2(\sigma^2)^2 + (\sigma^2)^2 + 2(n-1)\rho(\sigma^2)^2 + (n-1)^2\rho^2(\sigma^2)^2$$

$$= n(\sigma^2)^2 + n(n-1)\rho^2(\sigma^2)^2 - (\sigma^2)^2 - 2(n-1)\rho(\sigma^2)^2 - (n-1)^2\rho^2(\sigma^2)^2$$

$$= (n-1)(\sigma^2)^2 - 2(n-1)\rho(\sigma^2)^2 + (n-1)\rho^2(\sigma^2)^2$$

$$= (n-1)(\sigma^2)^2(1 - 2\rho + \rho^2)$$

$$= (n-1)(\sigma^2)^2(1-\rho)^2 \qquad \therefore \quad \varepsilon = \frac{n^2(\bar{\sigma}_{ii} - \bar{\sigma}_{..})^2}{(n-1)(S - 2n\sum \bar{\sigma}_{i.}^2 + n^2\bar{\sigma}_{..}^2)} = 1$$

2) $$Var(y_i) = Var(X\beta + Z\gamma_i + \varepsilon_i)$$

$$= Var(Z\gamma_i + \varepsilon_i)$$

$$= Var(Z\gamma_i) + Var(\varepsilon_i)$$

$$= Z\,Var(\gamma_i)Z^T + Var(\varepsilon_i)$$

$$= \sigma^2 ZZ^T + \sigma_e^2 I_n$$

$$= \begin{bmatrix} \sigma^2+\sigma_e^2 & \sigma^2 & \sigma^2 & \cdots & \sigma^2 \\ \sigma^2 & \sigma^2+\sigma_e^2 & \sigma^2 & \cdots & \sigma^2 \\ \sigma^2 & \sigma^2 & \sigma^2+\sigma_e^2 & \cdots & \sigma^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma^2 & \sigma^2 & \sigma^2 & \cdots & \sigma^2+\sigma_e^2 \end{bmatrix}$$

3) Let $y_{ij} = \mu + a_i + \varepsilon_i$, where $a_i \sim N(0, \sigma_a^2)$, $\varepsilon_i \sim N(0, \sigma_e^2)$

Then, $Var(y_{ij}) = Var(\mu + a_i + \varepsilon_i)$

$$= Var(a_i + \varepsilon_i)$$

$$= Var(a_i) + Var(\varepsilon_i)$$

$$= \sigma_a^2 + \sigma_e^2$$

$$= \sigma^2$$

$$\Rightarrow \quad SST = \sum_{i=1}^{m}\sum_{j=1}^{n}(y_{ij} - \bar{y})^2$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}(y_{ij} - \hat{y}_i + \hat{y}_i - \bar{y})^2$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}(\hat{y}_i - \bar{y})^2 + \sum_{i=1}^{m}\sum_{j=1}^{n}(y_{ij} - \hat{y}_i)^2$$

$$= SS_{treatment} + SSE$$

$$\Rightarrow \frac{SS_{treatment}}{\sigma^2} \sim \chi^2(m-1) \quad , \quad \frac{SSE}{\sigma^2} \sim \chi^2(n-m)$$

$$\Rightarrow \frac{\left(\frac{SS_{treatment}}{\sigma^2}\right)^2/(m-1)}{\left(\frac{SSE}{\sigma^2}\right)^2/(n-m)} \sim F(m-1, n-m)$$

$\therefore$ reject if $F_0 = \dfrac{\left(\frac{SS_{treatment}}{\sigma^2}\right)^2/(m-1)}{\left(\frac{SSE}{\sigma^2}\right)^2/(n-m)} > F_\alpha(m-1, n-m)$

4) a) $\begin{bmatrix} \mu_1-\mu_2 \\ \mu_1-\mu_3 \\ \mu_1-\mu_4 \\ \mu_1-\mu_5 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} = L\mu$

$\therefore \quad L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}$

b) $\begin{bmatrix} \mu_1-\mu_2 \\ \mu_2-\mu_3 \\ \mu_3-\mu_4 \\ \mu_4-\mu_5 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} = L\mu$

$\therefore \quad L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

$\Rightarrow$ Let $L_1 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}$ and $L_2 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$.

$$L_1 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} = L_2$$

$\therefore$ Since we can obtain L_2 from L_1 throug elementary row operations, (a) and (b) address the same issue.

c) $\begin{bmatrix} \mu_1-\mu_2 & \mu_2-\mu_3 & \mu_3-\mu_4 & \mu_4-\mu_5 \end{bmatrix} = \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \mu^T U$

$\therefore \quad U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

d) $\begin{bmatrix} \mu_1 - \frac{1}{4}(\mu_2+\mu_3+\mu_4+\mu_5) \\ \mu_2 - \frac{1}{3}(\mu_3+\mu_4+\mu_6) \\ \mu_3 - \frac{1}{2}(\mu_4+\mu_5) \\ \mu_4 - \mu_5 \end{bmatrix}^T = \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{3} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{3} & -\frac{1}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{3} & -\frac{1}{2} & -1 \end{bmatrix} = \mu^T U$

$\therefore \quad U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{3} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{3} & -\frac{1}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{3} & -\frac{1}{2} & -1 \end{bmatrix}$