

중간값 정리

Chap 12. Intermediate value theorem (IVT)

12.1 The existence of zeros

Continuity is a local property of functions. But we will prove it implies certain global properties, basic to analysis and its applications.

- an application: root-finding (i.e., want to solve equations $f(x) = 0$)

“Solving” means finding the real zeros of $f(x) = 0$ (i.e., those real c for which $f(c) = 0$)

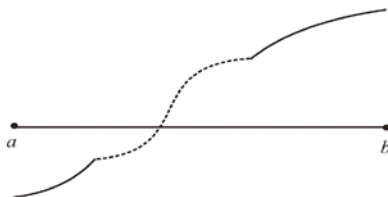
In detail, we can ask about

- existence: are there any zeros?
- number: are there finitely many? how many, or approximately how many?
- approximate location: find small intervals containing only one zero.
- calculation: determine the zero “exactly”, or to a given accuracy.

- **Bolzano’s theorem** (it is **about continuous functions which change sign**) is applicable to **all** of these questions

Def. We say $f(x)$ changes sign on $[a, b]$ if it is defined on this interval and has opposite signs at a and b :

$$f(a) < 0, f(b) > 0 \quad \text{or} \quad f(a) > 0, f(b) < 0 \quad (\text{equivalently, } f(a)f(b) < 0)$$

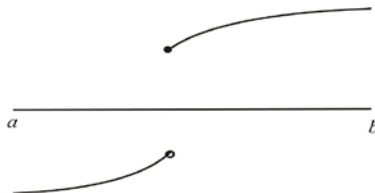


Theorem (**Bolzano’s theorem**)

Let $f(x)$ be continuous on $[a, b]$. Then

$$\text{If } f(x) \text{ changes sign on } [a, b] \text{ (i.e., } f(a)f(b) < 0) \Rightarrow f(x) \text{ has a zero on } [a, b] \text{ (} \therefore \text{ on } (a, b) \text{)}$$

Note. The property “ ” is not true in general when $f(x)$ is discontinuous on $[a, b]$; for example,



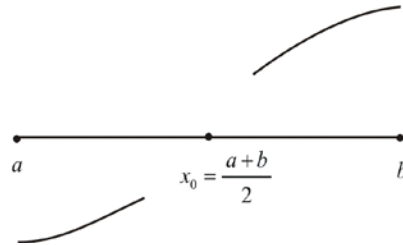
Remark1: “The conclusion is geometrically “obvious”

Remark2. The conclusion asserts that a certain real number exists, so we can expect that the proof will require the **“Completeness property”**. Actually, we will use “NIT” for the proof.

Proof of the Bolzano's theorem.

We consider the case where $f(x)$ changes from $-$ to $+$. We shall prove:

$$\boxed{f(a) < 0, \quad f(b) > 0 \quad \Rightarrow \quad f(c) = 0 \text{ for some } c \in [a, b] \quad (\text{actually, } c \in (a, b))}$$



Let $a_0 = a$ and $b_0 = b$. Divide $[a_0, b_0]$ into two by its midpoint x_0 .

- i) If $f(x_0) > 0$, let $[a_1, b_1] = [a, x_0]$.
- ii) If $f(x_0) < 0$, let $[a_1, b_1] = [x_0, b]$.
- iii) If $f(x_0) = 0$, we have found a zero ($= x_0$).

In each of the first two cases, we have

$$f(a_1) < 0, \quad f(b_1) > 0$$

& on the interval $[a_1, b_1]$, $f(x)$ still changes from $-$ to $+$.

We continue this process with $[a_1, b_1]$, bisecting it and choosing as $[a_2, b_2]$ the half on which $f(x)$ changes from $-$ to $+$.

If at any stage the midpoint is a zero of $f(x)$, we are done.

If not, we get an infinite sequence of nested intervals

$$[a, b] \supset [a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n] \supset \cdots \quad \text{NIT}$$

such that

$$(*) \quad f(a_n) < 0, \quad f(b_n) > 0, \quad \text{and} \quad b_n - a_n (= \frac{b-a}{2^n}) \rightarrow 0 \text{ as } n \rightarrow \infty$$

By NIT, $\exists c \in \bigcap_{n=0}^{\infty} [a_n, b_n]$ ($\because c \in [a, b]$) such that $\lim_{n \rightarrow \infty} a_n = c = \lim_{n \rightarrow \infty} b_n$.

Suffices to show: $f(c) = 0$.

Since f is conti on $[a, b]$, the **Sequential Continuity Theorem** implies that

$$\lim_{n \rightarrow \infty} f(a_n) = f(c) = \lim_{n \rightarrow \infty} f(b_n) \quad \text{그런데!}$$

According to $(*)$, we have

$$f(a_n) < 0 \text{ and } f(b_n) > 0 \text{ for all } n$$

Thus by LLT (for sequences), we have

$$\lim_{n \rightarrow \infty} f(a_n) \leq 0 \text{ and } \lim_{n \rightarrow \infty} f(b_n) \geq 0. \quad \therefore f(c) = 0$$

If $f(x)$ changes from $+$ to $-$, then $-f(x)$ changes from $-$ to $+$.

$$\Rightarrow \exists c \in [a, b] \text{ such that } (-f)(c) = 0$$

$$\Rightarrow f(c) = 0.$$

Corollary. **Intermediate Value Theorem (IVT)**

Assume $f(x)$ is continuous on $[a, b]$, and $f(a) \leq f(b)$ ($f(a) \geq f(b)$ resp.).

Then for $k \in \mathbb{R}$,

$$f(a) \leq k \leq f(b) \quad (f(a) \geq k \geq f(b) \text{ resp.}) \Rightarrow \exists c \in [a, b] \text{ such that } f(c) = k.$$

i.e., if f is conti on $[a, b]$, it takes on all values between $f(a)$ and $f(b)$ as x varies over $[a, b]$.

More common statement: Assume $f(x)$ is continuous on $[a, b]$. Then

$$\text{whenever } f(a) < \underline{k} < f(b) \text{ or } f(a) > \underline{k} > f(b) \Rightarrow \exists c \in (a, b) \text{ such that } f(c) = k$$

Pf of Corollary. If $k = f(a)$ or $k = f(b)$, we are done

If not (i.e., $f(a) < k < f(b)$), we consider

$$\underline{f(x) - k} : \text{ it is conti on } [a, b] \text{ and } \underline{f(a) - k < 0 \text{ and } f(b) - k > 0}$$

Bolzano's thm $\Rightarrow f(x) - k$ has a zero c on $[a, b]$ (Actually, has a zero c on (a, b))

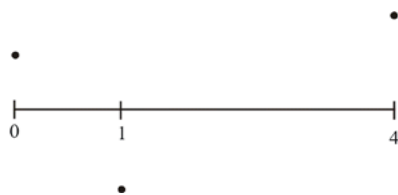
$$\text{i.e., } f(c) = k \text{ for some } c \in [a, b] \quad (\text{actually, } c \in (a, b)).$$

Consequently, in any case, $\exists c \in [a, b]$ such that $f(c) = k$

Exa. $e^x - 3x$ has at least two positive zeros.

Sol. $f(x) = e^x - 3x$: conti on $(-\infty, \infty)$

$$f(0) = \underline{1 > 0}, \quad f(1) = \underline{e - 3 < 0}, \quad f(4) = \underline{e^4 - 12 > 0}$$



Thus, by Bolzano's theorem, $f(x)$ has at least two positive zeros.

고정점

HS1 [Fixed-Point Theorem] If $f : [a, b] \rightarrow [a, b]$ is continuous, then prove that f has at least one fixed point; that is, $\exists x_0 \in [a, b]$ such that $f(x_0) = x_0$

HS2. Prove that a continuous function **whose values are always rational numbers** is a **constant function**

12.2 Applications of Bolzano's theorem

Exa A (existence of a zero): A poly of odd degree has at least one (real) zero.

Pf. Let $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$, where n is odd. 홀수 차수 다항식

We may assume that $a_0 > 0$ (otherwise use $-f$)

$$f(x) = x^n(a_0 + a_1 \frac{1}{x} + \dots + a_n \frac{1}{x^n}) \equiv x^n g(x)$$

$$\begin{aligned} |x| \rightarrow \infty &\Rightarrow \frac{1}{x}, \dots, \frac{1}{x^n} \rightarrow 0 \\ &\Rightarrow g(x) \rightarrow a_0 > 0 \end{aligned}$$

Since $a_0 > 0$, by **FLT** we get $g(x) > 0$ for $|x| \gg 1$

$\therefore x^n g(x) < 0$ for $x \ll -1$ & $x^n g(x) > 0$ for $x \gg 1$ since n is odd.

i.e., $f(x) < 0$ for $x \ll -1$ & $f(x) > 0$ for $x \gg 1$

•

•

Since $f(x)$ is continuous on $(-\infty, \infty)$, $f(x)$ has a zero (by Bolzano's theorem).

Remark. In computer searches for zeros of a polynomial, one looks for intervals on which $f(x)$ changes sign, and then uses the **bisection process** or **Newton's method** to find a zero inside. (Newton's method is **faster**; the **bisection method** is more reliable and **doesn't require calculation of derivatives**)

⊙ The isolation of the zeros (want to have each interval on which $f(x)$ changes sign so small there is **only one zero of $f(x)$ inside it**)

Exa B. **Approximate location** of a zero.

Let $f(x) = x^3 + hx - 1$, $h \approx 0^+$

Think of $f(x)$ as a small perturbation of $x^3 - 1$.

Since $x^3 - 1$ has a zero at 1, $f(x)$ should have a zero close to 1: call it $z(h)$.

Give the approximate value of $z(h)$, and prove it is **right-continuous** at 0.

Sol. Write $z = z(h) = 1 + \varepsilon$, where $\varepsilon = \varepsilon(h) \approx 0$.

$$f(z) = 0 \Rightarrow (1 + \varepsilon)^3 + h(1 + \varepsilon) - 1 = 0$$

$$\therefore 1 + 3\varepsilon + 3\varepsilon^2 + \varepsilon^3 + h + h\varepsilon - 1 = 0$$

Since $h(> 0)$ is small (i.e., $h \approx 0^+$ by hypo) & $\varepsilon \approx 0$, we get $1 + 3\varepsilon + h - 1 \approx 0$

$$\therefore \varepsilon \approx -\frac{h}{3} \quad \therefore z \approx 1 - \frac{h}{3} \text{ (for } h \approx 0^+)$$

To prove $z(h)$ is right-continuous at 0 (i.e., $\lim_{h \rightarrow 0^+} z(h) = 1$; which seems plausible by $z(h) \approx 1 - \frac{h}{3}$),

it suffices to prove

$$z = z(h) \in [1 - h, 1] \text{ for small } h > 0. \quad \text{i.e., } 1 - h \leq z = z(h) \leq 1 \text{ for small } h > 0$$

Note that

$$f(1 - h) = (1 - h)^3 + h(1 - h) - 1 = -2h + 2h^2 - h^3 < 0 \text{ for } h \approx 0^+ \quad \text{]} \rightarrow 1 - h$$

$$\& f(1) = h > 0$$

Thus by Bolzano's theorem, $1 - h \leq z \leq 1$ for small $h > 0$.

© In trying to establish the existence of a zero, it may be better to rewrite the equation in the form

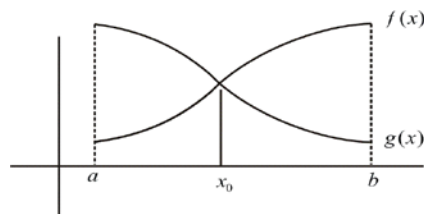
$f(x) = g(x)$ since the graph of $f(x)$ and $g(x)$ may be easier to plot or visualize than the graph of $f(x) - g(x)$.

$$F(x) = f(x) - g(x) \quad \text{복잡할 땐 분해를 해보자}$$

Intersection Principle

교점

- (a) The roots of $f(x) = g(x)$ are the x -coordinates of the points where the graph of $f(x)$ and $g(x)$ intersect.



(it is evident from the picture)

- (b) If $f(x)$ and $g(x)$ are continuous on $[a, b]$, and

$$[f(a) < g(a) \text{ and } f(b) > g(b)] \quad \text{or} \quad [f(a) > g(a) \text{ and } f(b) < g(b)],$$

then the two graphs intersect over some point $c \in [a, b]$.

Pf. (b) $h(x) \stackrel{\text{let}}{=} f(x) - g(x) \stackrel{\text{easy}}{\Rightarrow} \text{conclusion}$

Exa C. Counting zeros

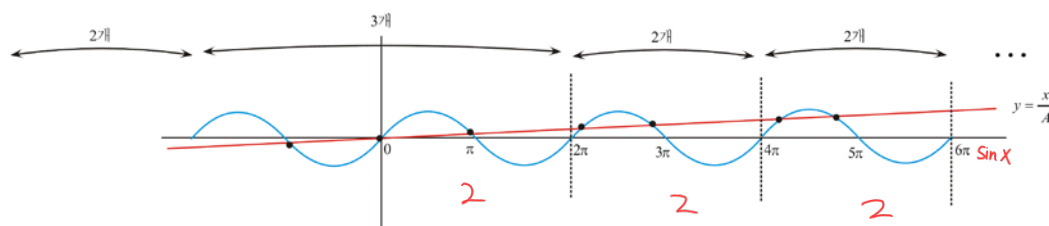
Approximately how many zeros has $x - A \sin x$, if A is large?

Sol. Note that $x - A \sin x = 0 \Leftrightarrow \sin x = \frac{x}{A}$.

Hence the number of zeros of $x - A \sin x$ is the same as that of the roots of $\sin x = \frac{x}{A}$.

Since $|\sin x| \leq 1$, the roots of $\sin x = \frac{x}{A}$ satisfy $\left|\frac{x}{A}\right| \leq 1$ (i.e., $|x| \leq A$).

That is, all the roots lie in $[-A, A]$.



From the figure, we see that

“over each interval $[2n\pi, (2n+2)\pi]$ lying inside $[-A, A]$, there will be two intersections”
 (this **isn't** quite right at the ends of $[-A, A]$ and 0 (= origin) is in two intervals, but we don't need to worry about it since we are assuming that A is large)
 Since there are **about** $2A/2\pi$ numbers of these intervals of length 2π inside $[-A, A]$, each with two intersections, there are in all approximately $2A/\pi$.

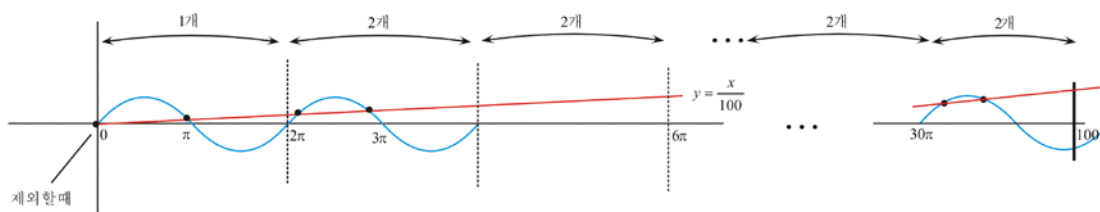
Conclusion: Approximately $x - A \sin x$ has $\frac{2A}{\pi}$ zeros.

Ex. (Exactly) How many zeros of $\sin x = \frac{x}{100}$?

Sol. Note that, since $\sin x$ and $\frac{x}{100}$ are odd functions, if x_0 is a root of the equation, then $-x_0$ is also a root. Thus “the number of negative roots = the number of positive roots”.
 Also, 0 is clearly a root of the equation.

It suffices to find how many positive roots there are.

If $\sin x = \frac{x}{100}$, then $\left|\frac{x}{100}\right| \leq 1$ since $|\sin x| \leq 1$. i.e., $|x| \leq 100$ (i.e., |every root| ≤ 100)



$$2\pi = 6.28 \dots \Rightarrow \frac{100}{6.28 \dots} = 15.88 \dots \Rightarrow 15 < \frac{100}{2\pi} < 16 \Rightarrow \left\lfloor \frac{100}{2\pi} \right\rfloor = 15$$

$$\left\lfloor \frac{100}{2\pi} \right\rfloor \cdot 2\pi = 30\pi \doteq 94.2 \Rightarrow \pi < 5 < 100 - \left\lfloor \frac{100}{2\pi} \right\rfloor \cdot 2\pi < 2\pi$$

94.2

Homesta!

Thus the number of positive roots:

$$\underbrace{\frac{1}{\text{in } (0, 2\pi)} + \frac{2}{\text{in } [2\pi, 4\pi)} + \frac{2}{\text{in } [4\pi, 6\pi)} + \cdots + \frac{2}{\text{in } [28\pi, 30\pi)} + \frac{2}{\text{in } [30\pi, 100]}}_{14 \text{ intervals}} = 1 + 2 \cdot 14 + 2 = 31(7\|)$$

Therefore, the number of all roots: $\frac{1}{x=0} + 31 \cdot 2 = 63(7\|)$

12.3 Graphical continuity

IVP (Intermediate Value Property): A function $f(x)$ defined on $[a, b]$ **is said to have the IVP** on $[a, b]$ if for each k between $f(a)$ and $f(b)$, $\exists c \in [a, b]$ such that $f(c) = k$.
 [or, if whenever $f(a) < k < f(b)$ or $f(a) > k > f(b)$, then $\exists c \in (a, b)$ such that $f(c) = k$]

Recall: IVT (Intermediate Value Theorem) says “Any continuous function on $[a, b]$ has the IVP on $[a, b]$ ”

We now **prove the converse** of this, for strictly monotone functions.

Theorem (Continuity theorem for monotone functions)

If $f(x)$ is strictly monotone and has the IVP on $[a, b]$, then it is continuous on $[a, b]$.

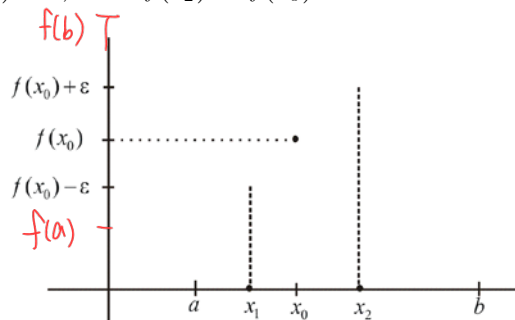
Pf. Suppose $f(x)$ is strictly inc on $[a, b]$ ($\therefore f(a) < f(b)$). Let x_0 be a point in (a, b)

(i.e., assume x_0 is not an endpoint of $[a, b]$). Then $f(a) < f(x_0) < f(b)$.

We show f is conti at x_0 .

Let $\varepsilon > 0$ be small. Then by IVP on $[a, b]$, $\exists x_1, x_2 \in [a, b]$ such that

$$f(x_1) = f(x_0) - \varepsilon, \quad f(x_2) = f(x_0) + \varepsilon$$



Indeed, they are unique and $x_1 < x_2$ ($\therefore x_1 < x_0 < x_2$) since $f(x)$ is strictly inc.

$$\begin{aligned} x_1 < x < x_2 &\stackrel{f \text{ is strictly inc}}{\Rightarrow} f(x_1) < f(x) < f(x_2) \\ \therefore f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon &\text{ for } x \in (x_1, x_2) \\ \therefore f(x) \underset{\varepsilon}{\approx} f(x_0) &\text{ for } x \underset{\varepsilon}{\approx} x_0 \end{aligned}$$

If x_0 is an endpoint, for example say $x_0 = a$, then by the preceding argument

$x_0 = a$ 로 가정했으니깐

$$(f(a) - \varepsilon <) f(a) < f(x) < f(a) + \varepsilon \text{ for } a < x < x_2$$

$$\therefore f(x) \underset{\varepsilon}{\approx} f(a) \text{ for } x \approx a^+$$

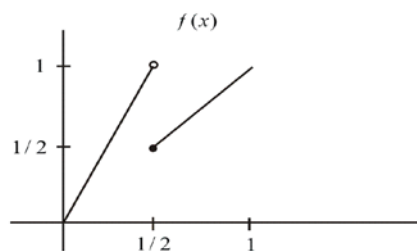
This implies that $f(x) \rightarrow f(a)$ as $x \rightarrow a^+$ which means $f(x)$ is right-conti at $x = a$.

Conclusion: Suppose $f(x)$ is **strictly monotone** on $[a, b]$. Then

$$f \text{ has the IVP on } [a, b] \Leftrightarrow f \text{ is continuous on } [a, b].$$

Ex. Give an example of a ft $f(x)$ having the IVP on $[a, b]$ but which is not continuous on $[a, b]$.

Ans:



Another example: $f(x) = \begin{cases} \sin(1/x), & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}$

Cf: More interesting example --- see the Corollary below: [needs Darboux's theorem]

미분에 관한 사이값 정리

Darboux's theorem [Intermediate Value Theorem for derivative] (will be proved in chap 15)

If f is diff on $[a, b]$, and if k between $f'(a)$ and $f'(b)$, then $\exists c \in [a, b]$ such that $f'(c) = k$.

[If f is diff on $[a, b]$, and if $f'(a) < k < f'(b)$ or $f'(a) > k > f'(b)$, then $\exists c \in (a, b)$ s.t. $f'(c) = k$]

Remark. In the theorem above, the continuity of f' is not necessary.

Cor (of Darboux): There is a **discontinuous** function $g(x)$ having IVP on $[a, b]$.

Pf. Take an f that is diff on $[a, b]$ but whose derivative f' is not conti at some point $x_0 \in (a, b)$.

(e.g., $f(x) := \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$: diff on $[-1, 1]$ but $f'(x)$ is not conti at $x = 0 \in (-1, 1)$)

Set $g = f'$. Then g has the IVP on $[a, b]$ (by Darboux's theorem), but g is not conti at $x_0 \in (a, b)$.

12.4 Inverse functions

Theorem (Inverse function theorem for continuity).

If $y = f(x)$ is **continuous & strictly inc** on $[a, b]$, then it has an inverse function $x = g(y)$ on $[f(a), f(b)]$ which is **continuous & strictly inc**.

(Remark. The theorem is also true for strictly dec functions; in that case $[f(a), f(b)]$ should be replaced by $[f(b), f(a)]$)

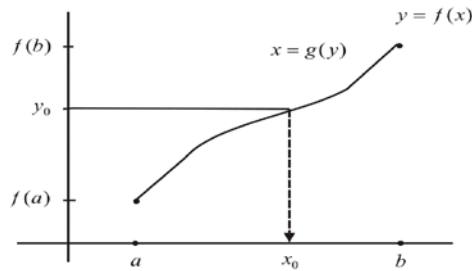
다음학기 내용

Pf. There are three things to prove.

(A) The inverse function is defined on $[f(a), f(b)]$

For, if $y_0 \in [f(a), f(b)]$, then by IVT

$\exists x_0 \in [a, b]$ such that $f(x_0) = y_0$; it is unique because $f(x)$ is strictly inc.



Therefore one can define g at y_0 by $g(y_0) = x_0$.

(B) The function $g(y)$ is strictly inc (i.e., $y_0 < y_1 \Rightarrow g(y_0) < g(y_1)$)

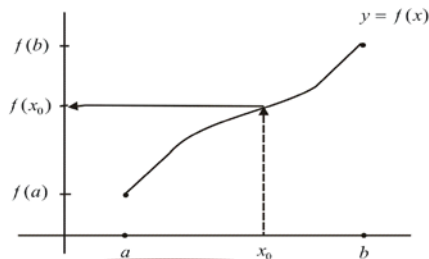
To prove this, set $x_0 = g(y_0)$ and $x_1 = g(y_1)$ where $y_0 < y_1$. Then

결론부정 $x_0 \geq x_1 \xRightarrow{f \text{ is strictly } \uparrow} f(x_0) \geq f(x_1) \quad \text{i.e., } f(g(y_0)) \geq f(g(y_1)) \quad \text{i.e., } y_0 \geq y_1$
contradiction

Therefore (B) holds.

(C) The function $g(y)$ is continuous.

By (B), $g(y)$ is strictly inc on $[f(a), f(b)]$ & $g(y)$ has the IVP on this interval, since if $a \leq x_0 \leq b$, then $g(f(x_0)) = x_0$ where $f(x_0) \in [f(a), f(b)]$.



Continuity thm for monotone fts
 $\Rightarrow g$ is conti on $[f(a), f(b)]$.

An application.

$f(x) = x^n$ ($n \in \mathbb{N}$): conti & strictly inc on $[0, \infty)$

$\therefore \xRightarrow{\text{Inverse ft thm}} \sqrt[n]{x} (= x^{1/n})$ is also conti & strictly inc on $[0, \infty)$.