Chap 10. Local and Global behavior of functions.

10.1 Estimating functions

◎ We classify intervals according to their length and endpoints;

length: finite or infinite [most common terminology; bounded or unbounded]

endpoints: closed if the endpoints are included; open if they are not

$(a,b)=\{x : a<x<b\}$: finite (bounded) open interval

$[a,b]=\{x : a\le x\le b\}$: finite (bounded) closed interval

$(a,\infty)=\{x : x>a\}$: infinite (unbounded) open interval

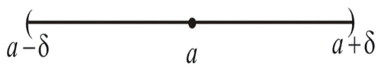
$(-\infty,a]=\{x : x\le a\}$: infinite (unbounded) closed interval

Convention: We do not consider $(a,a)\ (=\varnothing)$ to be an interval. That is, any interval $I\neq\varnothing$

◎ An interval like $[a,b)$ is called half-open;

the interval $(-\infty,\infty)$, which has no endpoints, is considered as both open and closed.

• Def. The open interval $(a-\delta,\,a+\delta)$ is called the δ − neighborhood of a.



We call a its center and δ its radius.

• $x\in(a-\delta,\,a+\delta)\ \Leftrightarrow\ a-\delta<x<a+\delta\ \Leftrightarrow\ |\,x-a\,|<\delta\ \underset{\text{denote}}{\equiv}\ x\underset{\delta}{\approx}a$

$\therefore$ $x\underset{\delta}{\approx}a$ says that x is approximately equal to a if δ is small.

• We write $f(\mathrm{I})$ for the set $\{f(x) : x\in\mathrm{I}\}$, and we call $f(\mathrm{I})$ the range of $f(x)$ over I, or the image of I under the mapping f. For example if $f(x)=\sin x$, then

$$f((-\infty,\infty))=[-1,1],\quad f([0,\pi])=[0,1],\quad f((0,\pi))=(0,1]$$

Def. Given an interval I on which $f(x)$ is defined, we say

- b is an upper bound for $f(x)$ on I $\Leftrightarrow$ b is an upper bound for $f(\mathrm{I})$
  $\Leftrightarrow$ $f(x)\le b$ for $x\in\mathrm{I}$

$f(x)$ is bounded above on I $\Leftrightarrow$ f has an upper bound on I

- a is a lower bound for $f(x)$ on I $\Leftrightarrow$ a is a lower bound for $f(\mathrm{I})$
  $\Leftrightarrow$ $f(x)\ge a$ for $x\in\mathrm{I}$

$f(x)$ is bounded below on I $\Leftrightarrow$ f has a lower bound on I

We say $f(x)$ is bounded on I if it is bounded above and bounded below on I.

$\overset{\text{i.e.}}{\Leftrightarrow}$ $\exists$ b_1 and b_2 such that $b_1\le f(x)\le b_2\quad\forall x\in\mathrm{I}$

$\Leftrightarrow$ $\exists$ constant $K>0$ such that $|\,f(x)\,|\le K\quad\forall x\in\mathrm{I}$

Eg. Which of these functions is bounded below or above?

(a) $3\cos x$; $\quad -3 \le 3\cos x \le 3 \quad \therefore$ bounded

(b) e^{-x} on $[0,\infty)$; $\quad 0 < e^{-x} \le 1$ for $0 \le x < \infty \quad \therefore$ bounded

(c) $1-x^2$; $\quad 1-x^2 \le 1$ for all $x \quad \therefore$ bounded above

(d) $\tan x$; not bounded above or below

Def. Suppose $f(x)$ is defined on an interval I. We define

the supremum of $f(x)$ on I $= \sup f(\mathrm{I})$ $(\text{notation } \sup\limits_{\mathrm{I}} f(x))$

the maximum of $f(x)$ on I $= \max f(\mathrm{I})$ $(\text{notation } \max\limits_{\mathrm{I}} f(x))$

The infimum & minimum are defined analogously.

Notice that

$$f(x) \text{ has a maximum on } \mathrm{I} \Leftrightarrow \sup_{\mathrm{I}} f(x) = f(\overline{m}), \text{ for some } \overline{m} \in \mathrm{I}$$

$$f(x) \text{ has a minimum on } \mathrm{I} \Leftrightarrow \inf_{\mathrm{I}} f(x) = f(\underline{m}), \text{ for some } \underline{m} \in \mathrm{I}$$

Eg. Find the sup, inf, max, and min of $f(x)$ over I and J:

$$f(x) = \sin x, \quad \mathrm{I} = (-\infty, \infty), \quad \mathrm{J} = (0, \pi/2)$$

Sol. Over I, $\sup f(x) = 1 = \max f(x)$, $\quad \inf f(x) = -1 = \min f(x)$

Over J,
$$\sup f(x) = 1; \quad \max f(x) \text{ does not exist}$$
$$\inf f(x) = 0; \quad \min f(x) \text{ does not exist}$$

Note that $f(\mathrm{I}) \neq \varnothing$ since any interval I is assumed to be non-empty.

Theorem (Completeness Property for functions)

Suppose $f(x)$ is defined on an interval I.

If $f(x)$ is bounded above on I, then $\sup\limits_{\mathrm{I}} f(x)$ exists;

If $f(x)$ is bounded below on I, then $\inf\limits_{\mathrm{I}} f(x)$ exists

⦿ Estimating functions: inequalities and absolute values

Recall that $f(x)$ is a real number for each $x \in D_f$. Therefore,

$$| f(x)g(x) | = | f(x) || g(x) |, \qquad | f(x) + g(x) | \le | f(x) | + | g(x) |$$

&

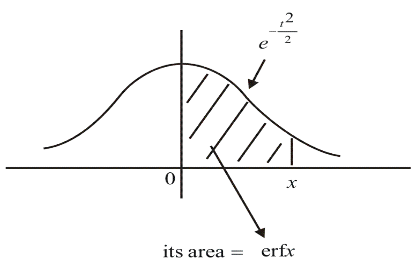
$$f(x) \text{ is bounded on } \mathrm{I} \iff \exists \text{ constant } K > 0 \text{ such that } | f(x) | \le K \quad \forall x \in \mathrm{I}$$

Eg. $f(x)$ and $g(x)$ are bounded on I $\Rightarrow$ $f(x)g(x)$ is bounded on I

Pf. $\exists$ constants K and L such that $| f(x) | \le K$ and $| g(x) | \le L$

$$\therefore \quad | f(x)g(x) | = | f(x) || g(x) | \le KL (= \text{constant})$$

Eg. $\operatorname{erf} x = \int_0^x e^{-t^2/2} dt$ (the error function)



Show that $\operatorname{erf} x$ is bounded above on $[0, \infty)$

Pf. Since $e^{-t^2/2} > 0$, $\int_0^x e^{-t^2/2} dt$ is $\uparrow$ on $[0, \infty)$.

Thus, it suffices to show that

$$\int_0^x e^{-t^2/2} dt \quad \text{is bounded above for } x \ge 1$$

Since $\operatorname{erf} x = \underbrace{\int_0^1 e^{-t^2/2} dt}_{\substack{= \text{ a fixed finite value} \\ \text{or, clearly } \le 1}} + \int_1^x e^{-t^2/2} dt$ for $x \ge 1$,

it suffices to show the last integral is bounded above for $x \ge 1$.

Note that

$$\begin{aligned} & t^2 \ge t \quad \text{for } t \ge 1 \\ \Rightarrow\ & e^{t^2/2} \ge e^{t/2} \quad \text{for } t \ge 1 \text{ since } e^x \text{ is } \uparrow \\ \Rightarrow\ & e^{-t^2/2} \le e^{-t/2} \quad \text{for } t \ge 1 \end{aligned}$$

Thus the above implies that, for $x \ge 1$,

$$\int_1^x e^{-t^2/2} dt \le \int_1^x e^{-t/2} dt = -2e^{-t/2}\Big|_1^x \le 2e^{-1/2}$$

$$\therefore \quad \int_1^x e^{-t^2/2} dt \quad \text{is bounded above for } x \ge 1.$$

10.2 Approximating functions

Notation:

$$|\ f(x)-g(x)\ |<\varepsilon \quad \text{for} \quad x\in \mathrm{I} \quad \Leftrightarrow \quad f(x)\underset{\varepsilon}{\approx} g(x) \quad \text{for} \quad x\in \mathrm{I}$$

$$\overset{\text{inequality form}}{\Leftrightarrow} \quad g(x)-\varepsilon < f(x) < g(x)+\varepsilon \quad \text{for} \quad x\in \mathrm{I}$$

$$\overset{\text{the error form}}{\Leftrightarrow} \quad f(x) = g(x)+e(x), \quad \text{where} \quad |\ e(x)\ |<\varepsilon \quad \text{for} \quad x\in \mathrm{I}$$

EgA. Use $\boxed{\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots \qquad \text{for all } x\in\mathbb{R}}$ to find a δ – neighborhood of 0 over which $\sin x \underset{\varepsilon}{\approx} x$ with $\varepsilon = 0.001$

Sol. Since $\sin x$ and x are odd functions, we can do the work for $x \geq 0$.

$$\boxed{\sin x = \underbrace{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots}_{\text{alternating series}} \qquad \text{for any fixed } x\in[0,1]}$$

By the Alternating series test, we get

$$|\ \sin x - x\ | \leq \frac{x^3}{3!} = \frac{x^3}{6}, \quad \text{for } 0<x<1$$

$$\therefore \quad < 0.001 \quad \text{if} \quad x^3 < 0.006 \quad (\text{i.e., } x < \sqrt[3]{0.006} \doteq 0.18171)$$

Thus if we take $\delta = 0.18$, then $\sin x \underset{0.001}{\approx} x$

◎ Elementary inequality for definite integral (will be proved later):

$$\boxed{\left.\begin{array}{l} f(x) < g(x) \quad \text{for } x\in \mathrm{I} \quad \& \\ \int_a^b f(x)\,dx \ \ \& \int_a^b g(x)\,dx \text{ exists for } a,b\in \mathrm{I} \text{ with } a<b \end{array}\right\rangle \quad \Rightarrow \quad \int_a^b f(x)\,dx < \int_a^b g(x)\,dx}$$

⊙ Assume $a<b$. Then

$$(*) \qquad \boxed{f(x)\underset{\varepsilon}{\approx} g(x) \text{ on } [a,b] \quad \Rightarrow \quad \int_a^b f(x)\,dx \underset{\varepsilon(b-a)}{\approx} \int_a^b g(x)\,dx}$$

Pf. $\qquad f(x)\underset{\varepsilon}{\approx} g(x)$ for $x\in[a,b]$

$$\Rightarrow g(x)-\varepsilon < f(x) < g(x)+\varepsilon \text{ for } x\in[a,b]$$

$$\Rightarrow \int_a^b (g(x)-\varepsilon)\,dx < \int_a^b f(x)\,dx < \int_a^b (g(x)+\varepsilon)\,dx$$

$$\text{i.e., } \int_a^b g(x)\,dx - \varepsilon(b-a) < \int_a^b f(x)\,dx < \int_a^b g(x)\,dx + \varepsilon(b-a)$$

$$\Rightarrow \int_a^b f(x)\,dx \underset{\varepsilon(b-a)}{\approx} \int_a^b g(x)\,dx$$

EgB. Estimate the error in $\cos x \underset{\varepsilon}{\approx} 1 - \dfrac{x^2}{2}$ for $|x| < 0.1$

Sol. From the result in EgA, we know

$$\sin x \underset{0.001}{\approx} x \quad \text{for} \quad |x| < 0.18, \quad \text{so for} \quad |x| < 0.1$$

$$\overset{(*)}{\Rightarrow} \quad \int_0^x \sin t \, dt \underset{0.001x}{\approx} \int_0^x t \, dt \quad \text{for} \quad 0 < x < 0.1$$

$$\therefore \quad 1 - \cos x \underset{0.0001}{\approx} \frac{x^2}{2} \quad \text{for } 0 < x < 0.1$$

Since $1 - \cos x$ & $\dfrac{x^2}{2}$ are even, it is also true that

$$1 - \cos x \underset{0.0001}{\approx} \frac{x^2}{2} \quad \text{for} \quad -0.1 < x < 0$$

$$\therefore \quad \cos x \underset{0.0001}{\approx} 1 - \frac{x^2}{2} \quad \text{for} \quad |x| < 0.1$$

10.3 Local behavior

To study the **continuity** or **differentiability** of $f(x)$ at x_0, we need to its local behavior near x_0. (i.e., its behavior in some $\delta -$nbd of x_0)

For that purpose, we use the notation

$$\text{for} \quad x \approx x_0 \qquad \text{or} \qquad \text{for } x \text{ near } x_0$$

which means

$$\text{for} \quad x \text{ in some } \delta - \text{nbd of } x_0 \qquad (i.e., \quad x \underset{\delta}{\approx} x_0 \quad \text{for some } \delta > 0)$$

EgA. T or F ?

(a) $x^4 \leq x^2$ for $x \approx 0$ (b) $x^3 \leq x$ for $x \approx 0$ (c) $x^3 \leq x$ for $x \approx 0, x \geq 0$

Sol. (a) is True because $x^4 \leq x^2$ for $|x| < 1$ ($\therefore$ $x^4 \leq x^2$ for $x \underset{1}{\approx} 0$)

(b) is **False** because $x^3 > x$ for $-1 < x < 0$

(c) is True because $x^3 \leq x$ for $0 \leq x < 1$

EgB. If $f(x)$ and $g(x)$ are bounded for $x \approx x_0$, so is $f(x) + g(x)$

Pf. By hypo,

$$|f(x)| < K \quad \text{for } x \approx x_0, \quad \text{say for} \quad |x - x_0| < \delta'$$

$$\& \quad |g(x)| < L \quad \text{for } x \approx x_0, \quad \text{say for} \quad |x - x_0| < \delta''$$

Thus, $|f(x) + g(x)| \leq |f(x)| + |g(x)| < K + L$ for $|x - x_0| < \min\{\delta', \delta''\} \equiv \delta$

Therefore, $f(x) + g(x)$ is bounded for $x \approx x_0$

⦿ Behavior at infinity

Sometimes, one wants to know the behavior of $f(x)$ on some interval like (a,∞) or $(-\infty,a)$.
For that purpose, we use the notation

$$\text{for } x \gg 1, \qquad \text{for } x \text{ large}, \qquad \text{for } x \text{ in some interval } (a,\infty)$$
$$\text{for } x \ll -1, \qquad \text{for negatively large } x\ , \qquad \text{for } x \text{ in some interval } (-\infty,a)$$
$$\text{for } |x| \gg 1, \qquad \text{for large } |x|\ , \qquad \text{for } |x| > \text{ some positive number } a$$

where in each case, an appropriate value of a exists, but is unspecified.
We can also use the terminology like

$$\text{local behavior of } f(x) \text{ at } \infty \quad (= \text{ for } x \gg 1)$$
$$\text{local behavior of } f(x) \text{ at } -\infty \quad (= \text{ for } x \ll -1)$$
$$\text{local behavior of } f(x) \text{ at } \pm\infty \quad (= \text{ for } |x| \gg 1)$$

EgC. Let $f(x)$ be a polynomial with positive leading coefficient. That is,

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n, \quad a_0 > 0\,.$$

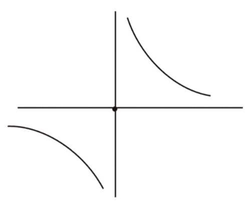
Then
(a) $f(x) > 0$ at ∞ (i.e., $f(x) > 0$ for $x \gg 1$)
(b) if n is even, then $f(x) > 0$ at $-\infty$ (i.e., $f(x) > 0$ for $x \ll -1$);
if n is odd, then $f(x) < 0$ at $-\infty$
(c) $\dfrac{1}{f(x)}$ is bounded at $\pm\infty$ $\quad$ (*i.e.*, for $|x| \gg 1$)

⦿ Local properties at a point

$f(x)$ is *locally increasing* at x_0 means $f(x)$ is inc for $x \approx x_0$
$f(x)$ is *locally bounded* at x_0 means $f(x)$ is bounded for $x \approx x_0$
$f(x)$ is *locally positive* at x_0 means $f(x)$ is positive for $x \approx x_0$

EgD. (Easy to prove)
(a) $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ (with const term $a_n > 0$) is locally positive at 0.
(b) $\sin x$ is locally inc at every $x_0 \in (-\pi/2, \pi/2)$, but not at $\pm\pi/2$.

(c) The function $f(x) = \begin{cases} 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$ is locally bounded at any $x_0 \neq 0$, but not at 0.



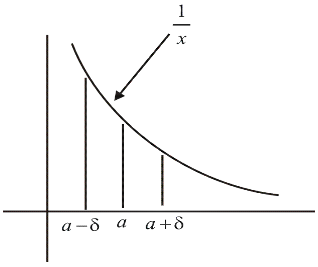
10.4 Local and global properties of functions

Def. We say that $f(x)$ is locally bounded on the open interval I if it is locally bounded at every point of I:

for all $x_0 \in I, \quad f(x)$ is bounded for $x \approx x_0$

EgA. Show that $\frac{1}{x}$ is locally bounded on $(0,\infty)$

Sol.



For any $x_0 = a > 0$, take $\delta = \frac{a}{2}$. Then

$$a-\delta < x < a+\delta \ \text{ (i.e., } \frac{a}{2} < x < \frac{3a}{2}) \quad \Rightarrow \quad \frac{2}{3a} < \frac{1}{x} < \frac{2}{a}$$

$\therefore \quad f(x)$ is bounded for $x \underset{\delta}{\approx} a$ where $\delta = \frac{a}{2}$.

Remark. $\frac{1}{x}$ is not bounded on $(0,\infty)$ since $\lim_{x\to 0^+} \frac{1}{x} = \infty$

Def. We say $f(x)$ is locally inc on an open interval I if it is locally inc at every point $x_0 \in I$

EgB. $f(x) = \tan x$ is locally inc on every interval of its domain: $(-\frac{\pi}{2},\frac{\pi}{2}),\ (\frac{\pi}{2},\frac{3\pi}{2}),\cdots$

However, it is not increasing ($\because \quad f(\pi/4) = 1 > 0 = f(\pi)$)

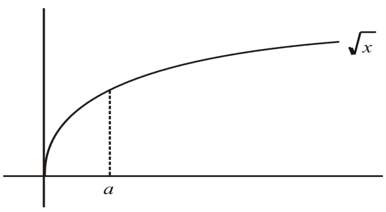
Remark(on local behavior at endpoints). In the preceding definitions, if $I = [a,b]$,

replace "for $x \approx a$" by "for $x \approx a,\ x \ge a$" (notation: for $x \approx a^+$)

"for $x \approx b$" by "for $x \approx b,\ x \le b$" (notation: for $x \approx b^-$)

Thus we say f is locally inc at the left endpoint of $[a,b]$ if $f(x)$ is inc for $x \approx a^+$

EgC. Show that $\sqrt{x}$ is locally bounded on $[0,\infty)$



Sol. If $a > 0$, take $\delta = a$, say; then $0 \le \sqrt{x} \le \sqrt{2a}$ for $x \underset{\delta}{\approx} a$

If $a = 0$, take $\delta = 1$, say; then $0 \le \sqrt{x} \le 1$ for $x \underset{\delta}{\approx} 0^+$

Note that $\sqrt{x}$ is not bounded on $[0,\infty)$ since $\lim\limits_{x\to\infty} \sqrt{x} = \infty$

More precisely, if $\sqrt{x}$ is bounded on $[0,\infty)$, then
$\exists\ M > 0$ such that

$$|\sqrt{x}| = \sqrt{x} \le M \qquad \forall x \in [0,\infty)$$

Let $x \to \infty \Rightarrow LHS \to \infty$, but $RHS(= M)$ is a fixed positive number; a contradiction

Local vs **Global**

- local property:

  A property is **local** if to verify that it holds on an interval, it is enough to check that it holds in **a nbd of each point** on this interval; (for ex, locally inc or locally bdd on I)

- global property:

  A property is **global** if to see if it holds on an interval I, one must *look at* the function on the interval I **as a whole**; (for example, f is bounded on I or f is periodic on $(-\infty,\infty)$)

◎ In general, if a property P is global, then P is also local.

For example,

$f(x)$ is bounded on I $\Rightarrow$ $f(x)$ is locally bounded on I

$\not\Leftarrow$

($f(x) = \dfrac{1}{x}$ on $(0,\infty)$ or $\sqrt{x}$ on $[0,\infty)$)

Theorem (will be proved later)

Let $f(x)$ be a function defined on a <u>**closed and bounded**</u> interval $I = [a,b]$. Then
$f(x)$ is locally bounded on I $\Rightarrow$ $f(x)$ is (globally) bounded on I

- pointwise property:

This is a property of $f(x)$ on an interval I which can be verified **point-by-point** in I;
for example, *positivity* is a pointwise property

($\because$ $f(x)$ is positive on I $\Leftrightarrow$ $f(a) > 0$ for each $a \in I$)