STA: 3021 Stochastic Processes

Midterm (NOON - 1:15 PM on Oct 26, 2017)

Student ID & Full Name: ______________________________

Instructions: This test is a closed book exam, but you are allowed to use calculator. Clarity of your answer will also be a part of credit. When needed, use the notation $\Phi(z) = P(Z < z)$ for a standard normal distribution Z. Show your ALL work neatly.

1. (10 points) A deck of 52 playing cards, containing all 4 aces, is randomly divided into 4 piles of 13 cards each. Find the probability that each pile has an ace.

E_i = i-th pile has an ace.

$$P(E_1E_2E_3E_4) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)P(E_4|E_1E_2E_3)$$

$$= \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}} \cdot \frac{\binom{3}{1}\binom{36}{12}}{\binom{39}{13}} \cdot \frac{\binom{2}{1}\binom{24}{12}}{\binom{26}{13}} \cdot \frac{\binom{1}{1}\binom{12}{12}}{\binom{13}{13}}$$

$$\approx 0.105$$

2. (10 points) There are three coins in a barrel. Theses coins, when flipped, will come up heads with respective probabilities .3, .5, .7. A coin is randomly selected from among these three and is then flipped ten times. Let N be the number of heads obtained on the ten flips. Find $P(N = n), n = 0, 1, \ldots, 10$.

C_i: the event that i-th coin is selected

$N | C_1 \sim B(10, 0.3)$

$N | C_2 \sim B(10, 0.5)$

$N | C_3 \sim B(10, 0.7)$

$$P(N=n) = \sum_{i=1}^{3} P(N=n | C_i) P(C_i)$$

$$= \frac{1}{3}\left[\binom{10}{n}(0.3)^n(0.7)^{10-n} + \binom{10}{n}(0.5)^n(0.5)^{10-n} + \binom{10}{n}(0.7)^n(0.3)^{10-n}\right]$$

3. (10 points) Approximately 80,000 marriages took place in the state of New York last year. Use Poisson approximations to estimate the probability that for at least three of these couples both partners celebrated their birthday on the first day of the same (but any) month.

X : # of the couples that both partners celebrated ~.

$$X \sim B\left(80000, \binom{12}{1}\left(\frac{1}{365}\right)^2\right)$$

$$\approx \text{Poisson}\left(80000\binom{12}{1}\left(\frac{1}{365}\right)^2 \approx 7.2\right)$$

$$\therefore P(X \geq 3) = 1 - P(X \leq 2)$$

$$\approx 1 - \sum_{x=0}^{2} \frac{e^{-7.2}(7.2)^x}{x!}$$

$$\approx 0.97$$

4. (10 points) Find $E(X|Y = y)$ for the joint density function of X and Y given by

$$f(x, y) = \frac{e^{-x/y}e^{-y}}{y}, \quad 0 < x < \infty, \quad 0 < y < \infty.$$

$$f_{X|Y} = \frac{f(x,y)}{\int_0^\infty \frac{e^{-x/y}e^{-y}}{y}\,dx} = \frac{1}{y}e^{-x/y}$$

$$\therefore E(X|Y=y) = \int_0^\infty x \cdot \frac{1}{y} e^{-x/y}\,dx$$

$$\overset{\frac{x}{y}=t}{=} \int_0^\infty t e^{-t} y\,dt$$

$$= y\int_0^\infty t e^{-t}\,dt$$

$$= y$$

5. (15 points) Show for continuous random variable with joint density $f_{X,Y}(x,y)$ that

$$E(E(X|Y)) = E(X).$$

$$E(E(X|Y)) = \int_{-\infty}^{\infty} E(X|Y)\, f(y)\, dy$$

$$= \int_{-\infty}^{\infty} f(y) \left(\int_{-\infty}^{\infty} x \cdot f(x|y)\, dx \right) dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{\infty} f(x,y)\, dy \right) dx$$

$$= \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= E(X)$$

6. (15 points) Let $X \sim Poisson(\lambda_1)$ and $Y \sim Poisson(\lambda_2)$ independent of X. Find the conditional distribution of X given $X + Y = n$.

$X+Y \sim \text{Poisson}(\lambda_1+\lambda_2)$

$$\therefore f(x \mid X+Y=n) = \frac{P(X=x,\ Y=n-x)}{P(X+Y=n)}$$

$$= \frac{P(X=x)\cdot P(Y=n-x)}{P(X+Y=n)}$$

$$= \frac{\dfrac{e^{-\lambda_1}\lambda_1^{x}}{x!}\cdot\dfrac{e^{-\lambda_2}\lambda_2^{n-x}}{(n-x)!}}{\dfrac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}}$$

$$= \binom{n}{x}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{x}\left(1-\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n-x}$$

$$\therefore X \mid X+Y=n \sim B\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

7. The first patient is given either drug 1 or drug 2 at random. Suppose that the drug 1 is effective with probability p_1 and drug 2 is effective with probability p_2. If the nth patient is given drug $i (i = 1, 2)$ and it is observed to be effective for that patient, then the same drug is given to patient $n+1$. If not, that is it is observed to be ineffective, then the $(n+1)$th patient is given the other drug. Find the probability that the third patient receive drug1 by formulating this by DTMC. (You are required to define X_n, state space S and its transition probability also.)

$X_n =$ type of drug given to nth patient

$\{X_n, n \geq 1\}$ is DTMC with $S = \{1, 2\}$

$$\mathbb{P} = \begin{array}{c|cc} & 1 & 2 \\ 1 & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{pmatrix} p_1 & 1-p_1 \\ 1-p_2 & p_2 \end{pmatrix}$}} \\ 2 & & \end{array}$$

Problem is asking $P(X_3 = 1)$

$$P(X_3 = 1) = P(X_3 = 1 \mid X_1 = 1) P(X_1 = 1) + P(X_3 = 1 \mid X_1 = 2) P(X_1 = 2)$$

$$= \frac{1}{2} \left(P(X_3 = 1 \mid X_1 = 1) + P(X_3 = 1 \mid X_1 = 2) \right)$$

$$\mathbb{P}^2 = \begin{pmatrix} p_1 & 1-p_1 \\ 1-p_2 & p_2 \end{pmatrix} \begin{pmatrix} p_1 & 1-p_1 \\ 1-p_2 & p_2 \end{pmatrix}$$

$$= \begin{pmatrix} p_1^2 + (1-p_1)(1-p_2) & p_1(1-p_1) + p_2(1-p_1) \\ p_1(1-p_2) + p_2(1-p_2) & (1-p_1)(1-p_2) + p_2^2 \end{pmatrix}$$

$$= \frac{1}{2} \left\{ p_1^2 + (1-p_1)(1-p_2) + p_1(1-p_2) + p_2(1-p_1) \right)$$

8. (15 points) Let $\{X_n, n \geq 0\}$ be a DTMC with the state space $S = \{1, 2, 3, 4\}$ and following transition probability matrix

$$P = \begin{pmatrix} .4 & .3 & .2 & .1 \\ .5 & 0 & 0 & .5 \\ .5 & 0 & 0 & .5 \\ .1 & .2 & .3 & .4 \end{pmatrix}.$$

Suppose the initial distribution is given by $P(X_0 = 1) = 1$. Compute

(a) $P(X_2 = 4)$

$P(X_2 = 4 | X_0 = 1) P(X_0 = 1) = P_{14}^{(2)} = 1/3$

$$P^2 = \begin{pmatrix} .42 & .14 & .11 & .33 \\ .25 & .25 & .25 & .25 \\ .25 & .25 & .25 & .25 \\ .33 & .11 & .14 & .42 \end{pmatrix}$$

(b) $E(X_3^2)$

$$P^3 = \begin{pmatrix} .3260 & .1920 & .1830 & .2990 \end{pmatrix}$$

$P(X_3 = 1) = P(X_3 = 1 | X_0 = 1) P(X_0 = 1) = P_{11}^3$

$\therefore EX_3^2 = 1 \times P_{11}^3 + 2^2 \times P_{12}^3 + \cdots + 4^2 P_{14}^3$

$= 7.525$

(c) $P(X_1 = 2 | X_2 = 4; X_3 = 1) = \dfrac{P(X_3 = 1, X_2 = 4, X_1 = 2)}{P(X_3 = 1, X_2 = 4)}$

$= \dfrac{P(X_3 = 1 | X_2 = 4, X_1 = 2) P(X_2 = 4, X_1 = 2)}{P(X_3 = 1 | X_2 = 4) P(X_2 = 4)}$

Markov

$= \dfrac{P(X_3 = 1 | X_2 = 4) P(X_2 = 4 | X_1 = 2) P(X_1 = 2 | X_0 = 1) P(X_0 = 1)}{P(X_3 = 1 | X_2 = 4) P(X_2 = 4)}$

$= \dfrac{.1 \times .5 \times .3}{.1 \times 1/3} = .45$