7.1 ## Series and Sequences

Definition :

Infinite Series $= S_n = a_0 + a_1 + \cdots a_n$

Geometric Series :

- $\sum_0^\infty r^n = \frac{1}{1-r}$, if $|r| < 1$

Telescoping Series :

$\sum_0^\infty a_n = S_0 + (S_1 - S_0) + (S_2 - S_1) + \cdots (S_n - S_{n-1}) + \cdots$

7.2 ## Elementary Convergence Tests

Theorem : The n-th Term Test for Divergence

$\sum a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$

* the contrast is false EX) $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum \frac{1}{n}$ diverges

- The contrapositive of the above is " n-th term divergence test "

$\Rightarrow$ $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum a_n$ diverges

Theorem : Tail-Convergence Theorem

$\sum_{N_0}^\infty a_n$ converges for some N_0 $\Rightarrow$ $\sum_0^\infty a_n$ converges $\Rightarrow$ $\sum_N^\infty a_n$ converges for all N

* the partial series starting with the N-th term is often called a tail of the original series

* The contrapositive of the Tail-convergence Theorem is the same statement about divergence : if one tail diverges , then the series diverges and all its tails diverges.

Theorem : Linearity Theorem

- Let p and q be real numbers , then $\sum a_n$ and $\sum b_n$ converge $\Rightarrow$ $\sum p a_n + q b_n$ converges , and $\sum p a_n + q b_n = p \sum a_n + q \sum b_n$

- If convergent series are added , subtracted , or multiplied by a constant factor , the resulting series converge and to the corresponding sums :

$\sum a_n \pm b_n = \sum a_n \pm \sum b_n$, $\sum c \cdot a_n = c \sum a_n$

* By contrast , if two convergent series are multiplied or divided term-by-term , the resulting series will not necessarily converge , and even if it does , it will certainly not converge to the product or quotient of the two sums

Theorem : Comparison Theorem for Positive Series

- Assume that $0 \le a_n \le a'_n$ for all n, then

  $\sum a'_n$ converges $\Rightarrow$ $\sum a_n$ converges, and $\sum a_n \le \sum a'_n$

  $\sum a_n$ diverges $\Rightarrow$ $\sum a'_n$ diverges

☆ The comparison test applies only to series whose terms are positive

7.3 The Convergence of Series with Negative Terms

Definition :

- $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges
- $\sum a_n$ is conditionally convergent if $\sum a_n$ converges, but $\sum |a_n|$ diverges

☆ A conditionally convergent series converges only because the negative terms partly cancel the positive ones, and so keep the total sum down. An absolutely convergent series is one whose terms are so small in size that even if you make them all positive, the resulting series converges.

Theorem : Absolute Convergence Theorem

$$\sum |a_n| \text{ converges} \Rightarrow \sum a_n \text{ converges}$$

7.4 Convergence Tests : Ratio and n-th Root Tests.

Theorem : The ratio test

- Suppose $a_n \neq 0$ for $n \gg 1$, and $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, then $L < 1 \Rightarrow \sum a_n$ converges absolutely

  $L > 1 \Rightarrow \sum a_n$ diverges

Theorem : The n-th root test

- Suppose $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L$, then $L < 1 \Rightarrow \sum a_n$ converges absolutely

  $L > 1 \Rightarrow \sum a_n$ diverges

  ☆ $L = 1$ or no limit $\Rightarrow$ test fails, no conclusion

7.5 The Integral and Asymptotic Comparison Tests

Theorem : The Integral Test

- Suppose $f(x) \ge 0$ and decreasing, for $x \ge$ some positive integer N. Then $\sum f(n)$ converges if the area under $f(x)$ and over $[N, \infty)$ is finite, and diverges if the area is infinite

If $p > 1$, then $\lim_{r\to\infty} r^{1-p} = 0$, so the area is finite; this proves the convergence statement.

For the divergence statement, it's conceptually simplest to compare the series when $p < 1$ with the known-to-be-divergent harmonic series ($p = 1$). However, to get practice with the integral test, let's continue with that.

If $p = 1$, the integral evaluates to $\lim_{r\to\infty} \ln r$, so the area is infinite, and the series diverges.

If $0 \le p < 1$, $\lim_{r\to\infty} r^{1-p} = \infty$, so the area is again infinite, and the series diverges.

If $p < 0$, the function is no longer decreasing, but the series is divergent since its individual terms tend to ∞. □

Theorem : Asymptotic Comparison Test

- If $|a_n| \sim |b_n|$, then $\sum |a_n|$ converges $\Leftrightarrow$ $\sum |b_n|$ converges

7.6 Series with Alternating Signs : Cauchy's Test

Theorem : Cauchy's Test for Alternating Series

- If $\{a_n\}$ is positive and strictly decreasing, and $\lim_{n \to \infty} a_n = 0$, then $\sum (-1)^n a_n$ converges

7.7 Rearranging the Terms of a Series

- If the terms of an absolutely convergent series are rearranged, the new series is still absolutely convergent, and has the same sum as the old one.
- If the series is conditionally convergent, by rearranging its terms one can get a new series which will converge to any prescribed real number, or if one wishes, diverge to ∞ or $-\infty$.