Chap14. Differentiation: local properties

14.1 The derivative

Def. Let $f(x)$ be defined for $x \approx a$. We write

$$(*): \qquad \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = f'(a)$$

if the limit exists; its value $f'(a)$ is called the derivative of $f(x)$ at a, and we say that f is differentiable at a, or f has a derivative at a.

Alternative ways of writing $(*)$:

▷ Let $x - a = \Delta x \quad \Rightarrow \quad (*) \quad \Leftrightarrow \quad \lim_{\Delta x\to 0}\frac{f(a+\Delta x)-f(a)}{\Delta x} = f'(a)$

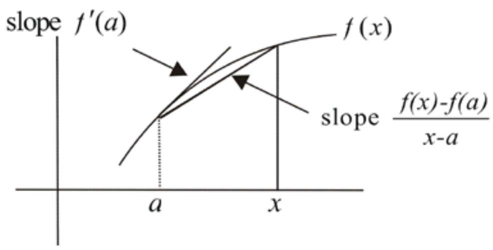
▷ $y = f(x), \quad \Delta y := y - f(a) = f(x) - f(a) = f(a+\Delta x) - f(a)$

$$(*) \quad \Leftrightarrow \quad \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \left.\frac{dy}{dx}\right|_{x=a}$$

Interpretations of the ***derivative*** and the ***difference quotient***

▷ $f(x)$ is defined by its graph:

$$f'(a) = \begin{array}{l}\text{slope of the tangent line}\\ \text{to the graph of } f(x) \text{ at } \big(a, f(a)\big)\end{array} \qquad \frac{f(x)-f(a)}{x-a} = \text{slope of the secant}$$



▷ $f(x)$ is a relation between variables:

$$\left.\frac{dy}{dx}\right|_{x=a} = \begin{array}{l}\text{rate of change of } y\\ \text{w.r.t. } x \text{ when } x = a\end{array} \qquad \frac{\Delta y}{\Delta x} = \begin{array}{c}\text{average rate of change}\\ \text{over } [a, a+\Delta x]\end{array}$$

Def. We say that $f(x)$ is diff on the open interval I if it is diff at every point of I; when that is so, its derivative on I is defined to be the function $f'(x)$ given by the rule: $x_0 \mapsto f'(x_0), \ x_0 \in I$

⊙ Differentiability at the endpoints: one-sided differentiability (See below)

Def. Assuming the limits exist, we define

$$f'({x_0}^+) = \lim_{x \to {x_0}^+} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{(right hand derivative)} \quad \text{assume } f(x) \text{ is defined for } x \approx {x_0}^+$$

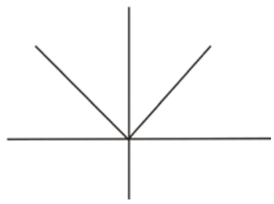
$$f'({x_0}^-) = \lim_{x \to {x_0}^-} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{(left hand derivative)} \quad \text{assume } f(x) \text{ is defined for } x \approx {x_0}^-$$

Fact (trivial): Let $f(x)$ be defined for $x \approx a$. Then

$$f'(a) \ \text{exists} \qquad \Leftrightarrow \qquad f'(a^+) = f'(a^-)$$

Exa A.

$f(x) = |x|$



Find $f'(x)$ on (a) $I = (-\infty, \infty)$; (b) $I = [0, \infty)$; (c) $I = (-\infty, 0]$

Sol. $$f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \quad = \quad \operatorname{sgn} x$$

At $x = 0$, we have

$$f'(0^+) = \lim_{x \to 0^+} \frac{|\,x\,| - 0}{x - 0} = 1, \qquad f'(0^-) = \lim_{x \to 0^-} \frac{|\,x\,| - 0}{x - 0} = \lim_{x \to 0^-} \frac{-x}{x} = -1$$

$$\therefore \quad f'(0) \ \text{does not exist.}$$

However, $f'(x)$ is differentiable on $[0, \infty)$ and $f'(x) = 1$ on $[0, \infty)$, since in this case

$$f'(0) \overset{\text{means}}{=} f'(0^+) = 1.$$

Exa B. $f(x) = \sqrt{1 - x^2}$; $\qquad g(x) = \left(\sqrt{1 - x^2}\right)^3 \quad \Rightarrow \quad f', \ g' \ ?$

Ans: Easy to check that

$$f'(x) = \frac{-x}{\sqrt{1 - x^2}} \quad \text{on} \ (-1, 1)$$

$$g'(x) = -3x\sqrt{1 - x^2} \quad \text{on} \ [-1, 1]$$

$\odot$ Different notations for the derivative:

* No indep variable is explicitly named: f' or Df

* An indep variable x is given:

$$f'(x), \qquad Df(x), \qquad D_x f(x), \qquad \frac{df}{dx}, \qquad \frac{d}{dx} f(x)$$

At specific points;

$$f'(a); \quad f'(x_0); \quad \left.\frac{df}{dx}\right|_{x_0}, \text{ etc}$$

Theorem

$$f(x): \text{ diff at } a \quad \Rightarrow \quad f(x): \text{ conti at } a$$
$$f(x): \text{ diff on } I \quad \Rightarrow \quad f(x): \text{ conti at } I$$

Pf. $$\lim_{x\to a}\big(f(x)-f(a)\big) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}(x-a)$$
$$= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}(x-a) = f'(a)\cdot 0 = 0$$

$\therefore \quad \lim_{x\to a} f(x) = f(a),$ which shows $f(x)$ is conti at a

Remark1. If f is diff on the right (or left) at a, it is right (or left)-conti at a.

Pf: $$\lim_{x\to a^+}(f(x)-f(a)) = \lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a^+}(x-a) = f'(a^+)\cdot 0 = 0$$

Remark2. $f(x)$ is diff on I

$\Rightarrow$ $f(x)$ is diff at every point of I

$\Rightarrow$ $f(x)$ is conti at every point of I

$\Rightarrow$ $f(x)$ is conti on I

$\odot$ Differentiability on I is a local property.

So we can say that the preceding theorem is of the type : ***local*** $\Rightarrow$ ***local*** (it is easy)

Note: The converse of the above theorem is not true

For example, $f(x) = |x|$ is conti at 0, but not diff at 0.

$\odot$ A curious result (due to Weierstrass):

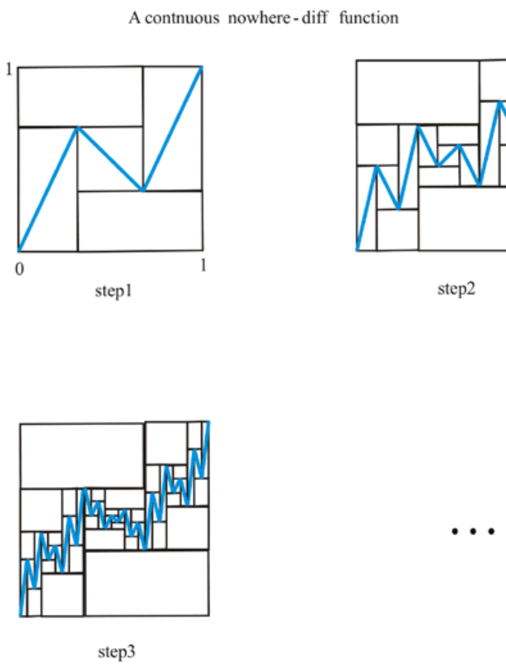
There exists a continuous function which is nowhere differentiable on $(-\infty, \infty)$:

$$f(x) = \sum_{n=0}^{\infty}\frac{1}{2^n}\cos(3^n x)$$ (its graph is a typical fractal curve)

To expect the result , draw $\sum_{n=0}^{1} \frac{1}{2^n} \cos(3^n x), \quad \sum_{n=0}^{2} \frac{1}{2^n} \cos(3^n x), \quad \sum_{n=0}^{3} \frac{1}{2^n} \cos(3^n x), \quad \cdots$

Its rigorous proof is not so easy.

Another simple construction (due to H. Katsuura): Its proof is also not easy.



One more construction (due to **J. McCarthy**): see the attached File (made by **J. Feldman**)

14.2 Differentiation formulas

Theorem A (Algebraic differentiation rules)

Suppose u & v are diff on an interval I. Then

(i) $au + bv$ is diff on I, and $(au+bv)' = au' + bv'$ $(a, b : \text{constants})$

(ii) uv is diff on I, and $(uv)' = u'v + uv'$

(iii) $\frac{u}{v}$ is diff on I on the set $v \neq 0$, and $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

Pf (ii) :

$$\frac{u(x)v(x) - u(a)v(a)}{x-a} = \frac{u(x)-u(a)}{x-a} v(a) + u(x) \frac{v(x)-v(a)}{x-a} \quad \text{for } x \approx a$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$u'(a)v(a) \quad + \quad u(a)v'(a) \qquad \text{as } x \to a$$

Theorem B (Chain Rule: 합성함수 미분법)

If f and g are diff, then over any interval where $f(g(x))$ is defined,

$$f \circ g \text{ is diff} \quad \text{and} \quad (f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In other words, if $y = f(x)$ and $x = g(t)$, then $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt}$

Pf. See the textbook.

Theorem C (Differentiation of inverse functions: 역함수 미분법)

Let $y = f(x)$ be strictly monotone on an interval I, and $x = g(y)$ be the inverse function defined on the interval $J = f(I)$. If f is diff on I and $f'(x) \neq 0$ on I, then $g(y)$ is differentiable on J and $\dfrac{dx}{dy} = \dfrac{1}{dy\,/\,dx}$.

More precisely, if we set $y_0 = f(x_0)$, then whenever $f'(x_0) \neq 0$,

$$g(y) \text{ is diff at } y_0, \text{ and } \boxed{g'(y_0) = \frac{1}{f'(x_0)}}$$

Pf. Recall $y = f(x)$ is diff on I $\Rightarrow$ conti on I

So $f(x)$ is strictly monotone & conti on I

Thus by **Inverse function theorem for continuity**, we have

$x = g(y)$ is conti and strictly monotone on $J = f(I)$.

Let $f'(x_0) \neq 0$ and set $y_0 = f(x_0)$.

For $y \underset{\neq}{\approx} y_0$ ($\Rightarrow$ $x \underset{\neq}{\approx} x_0$ since x is strictly monotone), we have

$$\frac{g(y) - g(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{\dfrac{f(x) - f(x_0)}{x - x_0}}$$

$$y \to y_0 \underset{x=g(y) \text{ is conti}}{\Longrightarrow} x \to x_0 \qquad \therefore \quad \downarrow$$

$$\frac{1}{f'(x_0)} \quad (\text{since } f'(x_0) \neq 0)$$

$$\therefore \quad g'(y_0) = \frac{1}{f'(x_0)}$$

Remark: In the case that $f(x)$ is **not strictly monotone on the whole interval** I, **but** it is **strictly monotone on some subinterval** $\tilde{I}$, then we can *apply the theorem to the subinterval* $\tilde{I}$.

14.3 Derivatives and local properties

Theorem A. Suppose $f(x)$ is diff on an open interval I. Then

(i) $f(x)$ is locally inc on $I \quad \Rightarrow \quad f'(x) \geq 0$ on I

(ii) $f(x)$ is locally dec on $I \quad \Rightarrow \quad f'(x) \leq 0$ on I

Pf. We shall show (i) holds for any point $a \in I$

$$\begin{aligned} f(x) \text{ is locally inc at } a \quad &\Rightarrow \quad f(x) \geq f(a) \quad \text{for } x \approx a^+ \\ &\Rightarrow \quad \frac{f(x)-f(a)}{x-a} \geq 0 \quad \text{for } x \underset{\neq}{\approx} a^+ \\ &\Rightarrow \quad \lim_{x \to a^+} \frac{f(x)-f(a)}{x-a} \geq 0 \quad \text{(by LLT)} \end{aligned}$$

$$\therefore \; f'(a^+) \geq 0$$

Since by hypothesis $f'(a)$ exists, we have $f'(a) = f'(a^+) \geq 0$.

(ii) If $f(x)$ is locally dec on I, then $-f(x)$ is locally inc on I. So we have by (i)

$$-f'(x) \geq 0 \quad \text{on} \quad I. \qquad \text{i.e. } f'(x) \leq 0 \quad \text{on } I.$$

Remark: The theorem is also true for a closed interval I, interpreting the derivative at an endpoint as the left (or right) hand derivative, and "locally inc at an endpoint" as only applying to that side of the endpoint lying inside I

Note: $f(x)$ is locally strictly inc on $I \quad \overset{?}{\Rightarrow} \quad f'(x) > 0$ on I

Answer is **No**: Think about $f(x) = x^3$ at $x = 0$

LLT would **only say**: $\dfrac{f(x)-f(a)}{x-a} > 0 \quad \Rightarrow \quad \lim_{x \to a} \dfrac{f(x)-f(a)}{x-a} \geq 0$

Def A. Let $f(x)$ be defined on an "open interval I"

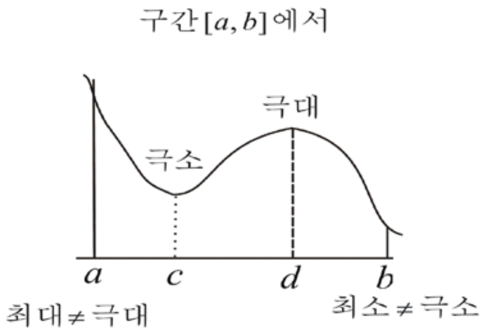
$c \in I$ is called a local (or relative) maximum pt if $f(c) \geq f(x)$ for $x \approx c$

$c \in I$ is called a local (or relative) minimum pt if $f(c) \leq f(x)$ for $x \approx c$

The terminology " local extremum" covers both local maximum and local minimum.

Note: 위의 정의(Def A)에 의하면 극대점 또는 극소점은 구간 I 의 경계가 아닌 내부에 있는 점이 되어야 한다.

이렇게 정의하는 이유: 함수 f 가 미분가능일 때, 극대점과 극소점에서 도함수가 영이 되는 것이 자연스럽다고 생각하기 때문이다. (아래 그림 참조)



Theorem B [**Fermat's Critical Point Theorem:** standard form] --- **enough for most applications**

Suppose $f(x)$ is differentiable on an open interval I.

$a \in I$ is a local extremum point $\Rightarrow$ $f'(a) = 0$

Pf. Suppose for example that a is a local maximum point. Then

$$\frac{f(x)-f(a)}{x-a} \le 0 \ \ for\ x \underset{\neq}{\approx} a^+; \qquad \frac{f(x)-f(a)}{x-a} \ge 0 \ \ for\ x \underset{\neq}{\approx} a^-$$

Taking limits as $x \to a^+$ and $x \to a^-$ respectively $\Rightarrow$

$$f'(a^+) \le 0 \quad \text{and} \quad f'(a^-) \ge 0 \quad (\text{by LLT})$$

Since $f'(a)$ exists, $0 \le f'(a^-) = f'(a) = f'(a^+) \le 0 \qquad \therefore\ f'(a) = 0$

Def B. A point where $f'(x) = 0$ is called a *critical point* for $f(x)$.

Note: A critical point need not be a local extreme point.

For example, $x = 0$ is a critical pt, but not a local extremum pt of $f(x) = x^3$

Theorem C (Isolation Principle): See the textbook

-- A "simple" way to decide if a critical point is actually an extremum point --

생략해도 무방: 미분을 조금 더 공부하면 더 쉽게 알 수 있다 ($\Leftarrow$ 도함수의 부호조사)

Example. Let $f(x) = xe^{-x}$. Find and classify its extremum points.

Sol. $f(x)$ is clearly diff on $(-\infty, \infty)$

$$f'(x) = e^{-x} + x(-e^{-x}) = e^{-x}(1-x)$$

$$\therefore\ f'(x) = 0 \quad \Leftrightarrow \quad x = 1$$

Thus there is a unique critical point, at $x = 1$

Question: Is it an extremum point?

$$f'(x) = e^{-x}(1-x)$$

$$\Downarrow$$

f	$\nearrow$		$\searrow$
f'	$+$	0	$-$
x		1	

$\therefore$ f has a local maximum at (the critical point) $x = 1$

H-S problems:

Pb1.

(a) Suppose that f is defined for $x \approx 0$, and that $f(0) = f'(0) = 0$. Find $\lim_{x\to 0} \frac{f(x)}{x}$

(b) Suppose $|f(x)| \le x^2$ for $x \approx 0$. Prove that $f'(0) = 0$

Pb2.

(a) Let $f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

Prove that f is continuous at 0, but not differentiable at 0

(b) Let $f(x) = \begin{cases} x^2\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

Prove that f is diff for all x, but $f'(x)$ is not conti at 0

Pb3. Find a function f defined for all x which is **twice differentiable** at every x, but $f''(x)$ is not continuous at 0

Pb4. Give an example of a function defined for all x which is differentiable at 0, but not even continuous at any other point.

Critical Point Theorem [Most General version] --- will be treated once more in Chap15

Let f be defined on $[a,b]$. If $f(x)$ has a local maximum or a local minimum at an **interior** point $c\in(a,b)$, and if $f'(c)$ exists, then $f'(c)=0$

Alternative statement:

Suppose f is defined on an open interval I, f is diff at a point $c\in I$, and c is a local extremum point. Then $f'(c)=0$

This can be proved by the same way as in the proof of Theorem B[= Standard form of Critical Point Theorem] --- Check

We give a slightly different proof below:

Lemma. Let $f: I(=\text{open interval})\to\mathbb{R}$, $c\in I$, and assume that $f'(c)$ exists. Then

$$f'(c)>0 \quad\Rightarrow\quad \exists\delta>0 \text{ s.t.} \begin{cases} f(x)>f(c) & \text{for all } x\in I \text{ with } c<x<c+\delta \\ f(x)<f(c) & \text{for all } x\in I \text{ with } c-\delta<x<c \end{cases}$$

In particular, f has no local maximum at c $[\leftarrow$ consider the interval $c<x<c+\delta]$

Similarly,

$$f'(c)<0 \quad\Rightarrow\quad \exists\delta>0 \text{ s.t.} \begin{cases} f(x)<f(c) & \text{for all } x\in I \text{ with } c<x<c+\delta \\ f(x)>f(c) & \text{for all } x\in I \text{ with } c-\delta<x<c \end{cases}$$

In particular, f has no (local) maximum at c $[\leftarrow$ consider the interval $c-\delta<x<c]$

Pf. We prove only the first statement.

Hypo says $\displaystyle\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=f'(c)>0$

$$\overset{\text{FLT}}{\Rightarrow}\ \frac{f(x)-f(c)}{x-c}>0 \text{ for } x\underset{\delta}{\approx}c \ \ (\text{i.e., for } x\in I \text{ with } x\in(c-\delta,c+\delta))$$

$$\Rightarrow\ f(x)-f(c)>0 \quad \text{for all } x\in I \text{ with } c<x<c+\delta$$
$$\left[\&\ f(x)-f(c)<0 \quad \text{for all } x\in I \text{ with } c-\delta<x<c\right]$$

$$\Rightarrow\ f(x)>f(c) \quad \text{for all } x\in I \text{ with } c<x<c+\delta$$
$$\left[\&\ f(x)<f(c) \quad \text{for all } x\in I \text{ with } c-\delta<x<c\right]$$

Pf of the Critical Point Theorem [Most General version]:

WLOG, we assume that $f(x)$ has a local **maximum** at an **interior** point $c\in(a,b)$

If $f'(c)>0$, then $f(x)>f(c)$ for $x\approx c$ with $x>c$ $[\leftarrow$ Lemma]. So c is not a local max pt.

Similarly, if $f'(c)<0$, then $f(x)>f(c)$ for $x\approx c$ with $x<c$ $[\leftarrow$ Lemma]. So c is again not a local maximum point.

Therefore, $f'(c)=0$ [since $f'(c)$ can be neither *positive* nor *negative*], if it exists