Chap8 Power series (거듭제곱급수, 멱급수)

8.1 Radius of convergence (수렴반지름, 수렴반경)

Def. A power series is an expression of the form

$$\sum_0^\infty a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots$$

where $a_n \in \mathbb{R}$ for $n = 0, 1, 2, \cdots$, $x_0 \in \mathbb{R}$ and x is an unspecified number.

⊙ The series $\sum_0^\infty a_n(x-x_0)^n$ is said to be a power series around (or centered at) $x = x_0$

$$\sum_0^\infty a_n(x-x_0)^n : \text{p.s. around } x = x_0 \quad \xrightarrow{\tilde{x}:=x-x_0} \quad \sum_0^\infty a_n(\tilde{x})^n : \text{p.s. around } \tilde{x} = 0$$

We shall treat only the case where $x_0 = 0$

Thus, whenever we refer to a power series, we shall mean a series of the form

$$\sum_0^\infty a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

● Power series are important because

(i) they are **used to represent functions**

(ii) the series are **useful in calculating values of the functions they represent**, since the first few terms of the power series give a good approximation to the function if x is small.

Eg1 ((i)의 관점)

$$\frac{1}{1-x} \overset{\text{represented as}}{=} \sum_0^\infty x^n \quad \text{when } |x|<1$$

$$\downarrow \leftarrow \text{integration}$$

$$-\ln(1-x) \overset{\text{represented as}}{=} \sum_1^\infty \frac{x^n}{n} \quad \text{when } |x|<1$$

Eg2 ((i)의 관점) $\quad f'(x) = f(x)$ and $f(0)=1 \Rightarrow f(x) = ?$

Sol. Method 1. $g(x) \overset{\text{let}}{=} e^{-x}f(x) \Rightarrow g'(x) = e^{-x}(f'(x)-f(x)) = 0$

$$\therefore \ g(x) = c(\text{constant}) \overset{f(0)=1}{\Rightarrow} g(0) = 1 = c$$

$$\therefore \ e^{-x}f(x) = 1 \qquad \text{i.e.,} \quad f(x) = e^x$$

Method 2. Assume $f(x) \overset{\text{represented as}}{=} \sum_0^\infty a_n x^n$. Then

$$f(0) = 1 \to \boxed{a_0 = 1} \quad \& \quad f'(x) = f(x) \to \boxed{na_n = a_{n-1} \quad \text{for } n \geq 1}$$

$$\therefore \ a_n = \frac{a_{n-1}}{n} = \frac{1}{n}\cdot\frac{a_{n-2}}{n-1} = \cdots = \frac{a_0}{n(n-1)(n-2)\cdots 1} = \frac{a_0}{n!} = \frac{1}{n!}$$

$$\therefore \quad f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{0}^{\infty} \frac{1}{n!} x^n \quad (\leftarrow \text{recall} \quad 0! = 1)$$

Remark. $\displaystyle\sum_{0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = e^x$

Ex. (i) Find an $f(x)$ such that

$$f''(x) - 2xf'(x) - 2f(x) = 0 \quad \text{with} \quad f(0) = 1, \ f'(0) = 0$$

(ii) Find an $f(x)$ such that

$$f''(x) - 2xf'(x) - 2f(x) = 0 \quad \text{with} \quad f(0) = 0, \ f'(0) = 1$$

Ans. (i) $\displaystyle f(x) = \sum_{0}^{\infty} \frac{x^{2n}}{n!} \quad (= e^{x^2})$ (ii) $\displaystyle f(x) = \sum_{0}^{\infty} \frac{2^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)} x^{2n+1}$

Eg3 ((ii)의 관점): Later it will be proved that

$$\sin x = \underbrace{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}_{\text{alternating series}} \quad \text{for every } x \in \mathbb{R}$$

Recall: If $a_n \downarrow 0$, then $\displaystyle\sum_{0}^{\infty} (-1)^n a_n$ converges. Moreover, we have seen that

$$\boxed{\sum_{0}^{\infty} (-1)^k a_k \overset{\text{write}}{=} S \quad \& \quad \sum_{0}^{n} (-1)^k a_k \equiv s_n \quad \Rightarrow \quad |\, s_n - S \,| \le a_{n+1}}$$

Easy to see that $x^n / n! \downarrow 0$ (as $n \to \infty$) for each $x \in (0,1]$. Hence by $\boxed{}$

$$\left| \sin x - (x - \frac{x^3}{3!}) \right| \le \frac{x^5}{5!} = \frac{|\,x\,|^5}{5!} \quad \text{for every } 0 \le x \le 1.$$ Accordingly, we also have

$$\left| \sin(-x) - \left((-x) - \frac{(-x)^3}{3!} \right) \right| \le \frac{(-x)^5}{5!} = \frac{|\,x\,|^5}{5!} \quad \text{for every } -1 \le x \le 0$$

$$\Vert$$

$$\left| -\sin(x) + \left(x - \frac{x^3}{3!} \right) \right| = \left| \sin(x) - \left(x - \frac{x^3}{3!} \right) \right|$$

Consequently, $\displaystyle\left| \sin x - (x - \frac{x^3}{3!}) \right| \le \frac{|\,x\,|^5}{5!} \quad \text{for every } -1 \le x \le 1.$

Thus if $|\,x\,| \ll 1$ (i.e., $|\,x\,|$ is small) $\displaystyle(\Rightarrow \frac{|\,x\,|^5}{5!} \text{ is very small})$, then

$$\sin x \approx x - \frac{x^3}{3!} \quad \text{for} \quad |\,x\,| \ll 1$$

$$\therefore \quad x - \frac{x^3}{3!} \text{ is a good approximation to } \sin x \text{ if } |\,x\,| \text{ is small}$$

⦿ In High School Math: $\displaystyle\lim_{x \to 0} \frac{\sin x}{x - \frac{x^3}{3!}} = 1$

◎ Note: If x is a fixed (real) number, then

$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$ is just a **series of numbers**.

Question: For which values of x, does the power series $\sum_0^\infty a_nx^n$ converge?

Eg. For which values of x, does the power series $\sum_1^\infty \frac{x^{2n}}{2^n n}$ converge?

Sol. We use the **Ratio Test**. Set $a_n = \frac{x^{2n}}{2^n n}$. Then

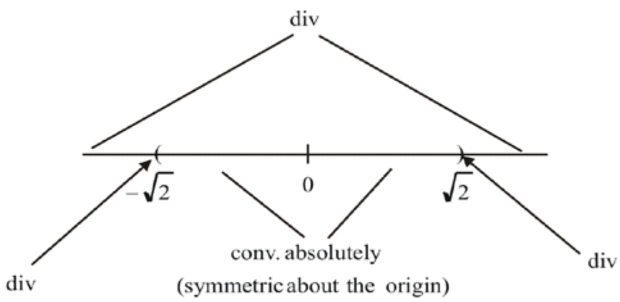
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^n n}{x^{2n}}\cdot\frac{x^{2(n+1)}}{2^{n+1}(n+1)}\right| = \lim_{n\to\infty}\frac{n\mid x^2\mid}{2(n+1)} = \frac{\mid x\mid^2}{2}$$

$$\therefore \quad \sum_1^\infty \frac{x^{2n}}{2^n n}\begin{cases}\text{conv. absolutely} & \text{for } \frac{\mid x\mid^2}{2} < 1 \quad (i.e., \text{ for } \mid x\mid < \sqrt{2}) \\ \text{div} & \text{for } \frac{\mid x\mid^2}{2} > 1 \quad (i.e., \text{ for } \mid x\mid > \sqrt{2})\end{cases}$$

Also, at the right endpoint $x = \sqrt{2}$, $\sum_1^\infty \frac{(\sqrt{2})^{2n}}{2^n n} = \sum_1^\infty \frac{1}{n}$: div

at the left endpoint $x = -\sqrt{2}$, $\sum_1^\infty \frac{(-\sqrt{2})^{2n}}{2^n n} = \sum_1^\infty \frac{1}{n}$: div

Therefore, $\sum_1^\infty \frac{x^{2n}}{2^n n}$ converges (absolutely) only for $\mid x\mid < \sqrt{2}$



Eg. For which values of x, does the power series $\sum_1^\infty \frac{x^n}{n}$ converge?

Sol. Set $a_n = \frac{x^n}{n}$. Then

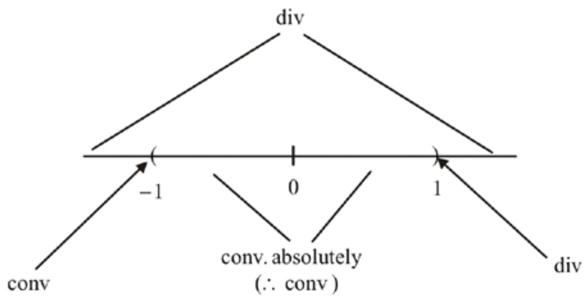
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n}{x^n}\cdot\frac{x^{n+1}}{(n+1)}\right| = \lim_{n\to\infty}\frac{n}{n+1}\mid x\mid = \mid x\mid$$

$$\therefore \quad \sum_1^\infty \frac{x^n}{n}: \begin{cases}\text{conv. absolutely} & \text{if } \mid x\mid < 1 \\ \text{div} & \text{if } \mid x\mid > 1\end{cases}$$

Also, at the right endpoint $x = 1$, $\sum_{1}^{\infty} \frac{1^n}{n} = \sum_{1}^{\infty} \frac{1}{n}$: div

at the left endpoint $x = -1$, $\sum_{1}^{\infty} \frac{(-1)^n}{n}$: conv (by Alternating series test)

Therefore, $\sum_{1}^{\infty} \frac{x^n}{n}$ converges for $-1 \le x < 1$



Eg. For which values of x, does the p.s. $\sum_{0}^{\infty} (n+1)^n x^n \left(= 1 + 2x + 3^2 x^2 + \cdots\right)$ converge?

Sol. At $x = 0$, $\sum_{0}^{\infty} (n+1)^n x^n = 1$ $\therefore$ conv

For any fixed $x \neq 0$, $\lim_{n\to\infty} (n+1)^n x^n \neq 0 \left[\leftarrow (n+1)^n \,|\, x \,|^n \geq (n \,|\, x \,|)^n \overset{if\ n \gg 1}{\geq} 2^n \to \infty\right]$

$\therefore$ $\sum_{0}^{\infty} (n+1)^n x^n$ diverges $\qquad$ $\therefore$ $\sum_{0}^{\infty} (n+1)^n x^n$ converges only at $x = 0$.

Theorem (Cauchy-Hadamard theorem)

For each p.s. $\sum_{0}^{\infty} a_n x^n$, there is a unique number $R \in [0, \infty]$ such that

$$\sum_{0}^{\infty} a_n x^n : \begin{cases} \text{conv. absolutely} & \text{for } |\, x \,| < R \\ \text{div} & \text{for } |\, x \,| > R \end{cases} \quad \left[\overset{\text{later}}{\Leftrightarrow} \sum_{0}^{\infty} a_n x^n : \begin{cases} \text{conv} & \text{for } |\, x \,| < R \\ \text{div} & \text{for } |\, x \,| > R \end{cases} \right]$$

(At $x = +R$ or $-R$, the series may converge or diverge)

Here, $R = \infty$ means that the series is absolutely convergent for every $x \in \mathbb{R}$;

$R = 0$ means that the series converges only for $x = 0$

The (extended) number R is called the radius of convergence of the power series;

Cf: $(-R, R)$ is called the "open interval of convergence" ($\neq$ interval of convergence, in general)

Note1: $R = \dfrac{1}{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|}$ if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$

Note2: $R \neq \dfrac{1}{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|}$ since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ may not exist

Note3. By Note2, we can not use Ratio test to prove the Cauchy-Hadamard theorem.

Proof of theorem.

1st step: $\underbrace{\boxed{\sum_0^\infty a_n x^n \text{ conv for } x = c, \text{ where } c \neq 0 \ \Rightarrow \ \sum_0^\infty | a_n x^n | \text{ conv for } | x | < | c |}}_{\text{key property of power series}}$

We prove this in two steps:

Case 1. $c = 1$. In this case, our hypothesis says: $\sum a_n$ converges

$$\begin{aligned}
\sum a_n \text{ converges} \quad &\Rightarrow \quad a_n \to 0 \quad \text{by the n-th term test} \\
&\Rightarrow \quad | a_n | \to 0 \ \text{(easy)} \\
&\Rightarrow \quad \lim_{n\to\infty} | a_n | < 1 \\
&\Rightarrow \quad | a_n | < 1 \quad \text{for } n \geq \text{(some)} N \quad \text{(by SLT)}
\end{aligned}$$

But

$$\sum_N^\infty | a_n x^n | = \sum_N^\infty | a_n || x |^n \quad \& \quad \begin{cases} 0 \leq | a_n || x |^n \leq | x |^n & \text{for } n \geq N \\ \sum_N^\infty | x |^n : \text{converges for } | x | < 1 \end{cases}$$

Thus, by Comparison Theorem, $\sum_N^\infty | a_n x^n |$ converges for $| x | < 1$

Now by Tail-Convergence Theorem, $\sum_0^\infty | a_n x^n |$ converges for $| x | < 1$

Case 2. $c \neq 0$

$$\begin{aligned}
\sum_0^\infty a_n x^n \text{ converges for } x = c \quad &\overset{x=cu}{\Rightarrow} \quad \sum_0^\infty a_n c^n u^n \text{ converges for } u = 1 \\
&\overset{\text{Case 1}}{\Rightarrow} \quad \sum_0^\infty | a_n c^n u^n | \text{ converges for } | u | < 1 \\
&\Rightarrow \quad \sum_0^\infty | a_n x^n | \text{ converges for } \left| \frac{x}{c} \right| < 1, \quad \text{or } | x | < | c |
\end{aligned}$$

Rk. Alternative combined proof :

$$\begin{aligned}
\sum_0^\infty a_n x^n \text{ converges for } x = c \ (c \neq 0) \quad &\Rightarrow \quad a_n c^n \to 0 \ \text{ by the n-th term test} \\
&\Rightarrow \quad | a_n c^n | \to 0 \\
&\Rightarrow \quad \lim_{n\to\infty} | a_n c^n | < 1 \\
&\Rightarrow \quad | a_n c^n | < 1 \ \text{ for } n \geq \text{(some)} N \ \text{(by SLT)}
\end{aligned}$$

Then

$$\sum_N^\infty |a_n x^n| = \sum_N^\infty |a_n c^n| \left|\frac{x}{c}\right|^n \le \sum_N^\infty \left|\frac{x}{c}\right|^n \text{ : converges for } \left|\frac{x}{c}\right| < 1 \text{ (i.e., } |x| < |c|)$$

Thus by Comparison Theorem, $\sum_N^\infty |a_n x^n|$ converges for $|x| < |c|$

Now by Tail-Convergence Theorem, $\sum_0^\infty |a_n x^n|$ converges for $|x| < |c|$

2nd step: Let $S = \{c \in [0,\infty) : \sum_0^\infty a_n c^n \text{ converges}\}$

Note that $S \neq \varnothing$ since $0 \in S$

If $S = [0,\infty)$, then $\sum_0^\infty a_n x^n$ converges for all $x \in \mathbb{R}$; so $R = \infty$

If $S \subsetneq [0,\infty)$, we can choose $b \in [0,\infty) \setminus S$; which says that

$$\sum_0^\infty a_n b^n \text{ diverges --- } \odot \quad \text{and } b \neq c \text{ for any } c \in S$$

Suppose $c > b$ for some $c \in S$. Then $\sum_0^\infty a_n b^n$ converges by 1st step; which violates $\odot$

So $c < b$ for every $c \in S$. So S is bounded above by b

Thus $\sup S$ exists. Write $\sup S =: R$

If $S = \{0\}$, then $\sum_0^\infty a_n x^n$ converges only for $x = 0$; so $R = 0$

Now we let $S \neq \{0\}$. Then $R[= \sup S] > 0$ [by the key property of power series]

Suffices to show that $\sum_0^\infty a_n x^n : \begin{cases} \text{conv. absolutely} & \text{for } |x| < R \\ \text{div} & \text{for } |x| > R \end{cases}$

$$\begin{aligned} |x| < R \quad & \overset{R \text{ is the least upper bound for } S}{\Rightarrow} \quad |x| (< R) \text{ is not an upper bound for } S \\ & \Rightarrow \quad |x| < d \le R \text{ for some } d \in S \\ & \Rightarrow \quad \sum_0^\infty a_n d^n \text{ converges} \\ & \Rightarrow \quad \sum_0^\infty |a_n x^n| \text{ conv} \quad (\Leftarrow \ |x| < d \ \ \& \ \text{1st step}) \end{aligned}$$

$$|x| > R \quad \Rightarrow \quad \sum_0^\infty a_n x^n : \text{ div}$$

[$\because$ Suppose, for a contradiction, that $\sum_0^\infty a_n x^n$ is convergent, for some $|x| > R$.

Choose any c such that $R < c < |x|$. Then $\sum_0^\infty a_n c^n$ (absolutely) converges (by 1st step)

$\therefore \quad c \in S$. This contradicts the fact that R is an upper bound for S]

Remark: $\sum_0^\infty a_n x^n$ & $\sum_N^\infty a_n x^n$ have the same radius of convergence (by Tail-Conv. Thm)

Eg. Find the radius of convergence for each of the following P.S.

$$\sum \frac{x^n}{3^{2n+1}} \quad (R=9); \qquad \sum n!\,x^n \quad (R=0)$$

$$\sum n^2 x^n \quad (R=1); \qquad \sum \frac{x^n}{n!} \quad (R=\infty)$$

※ Remember:

$$R \overset{\text{Ratio test}}{=} \frac{1}{\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|} \quad \text{if the limit } \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \text{ exists or } +\infty \text{ (easy to prove)}$$

$$\overset{\text{n-th root test}}{=} \frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{|\,a_n\,|}} \quad \text{if the limit } \lim_{n\to\infty}\sqrt[n]{|\,a_n\,|} \text{ exists or } +\infty \text{ (easy to prove)}$$

$$= \frac{1}{\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|\,a_n\,|}} \quad \text{(not easy to prove; will be given later)}$$

Note (← seen in the course of the proof of the Cauchy-Hadamard theorem)

Let $\sum_0^\infty a_n x^n$ be a power series.

① (key property) If $\sum_0^\infty a_n x^n$ conv at $x = c (\neq 0)$, then $\sum_0^\infty a_n x^n$ conv abso for $|\,x\,| < |\,c\,|$

② If $\sum_0^\infty a_n x^n$ conv abso at $x = c$, then $\sum_0^\infty a_n x^n$ conv abso for $|\,x\,| \leq |\,c\,|$

That is, if $\sum_0^\infty a_n x^n$ conv abso at $x = c$, then $\sum_0^\infty a_n x^n$ conv abso at $x = -c$

Pf of ②: Follows from $\sum_0^\infty |\,a_n x^n\,| \underset{(\because |x| \leq |c|)}{\leq} \sum_0^\infty |\,a_n c^n\,|$ & **Comparison Theorem**

※③ If $\sum_0^\infty a_n x^n$ conv conditionally at $x = c$, then R(the radius of conv) $= |\,c\,|$

Pf of ③: $\sum_0^\infty a_n x^n$ conv (conditionally) at $x = c$ $\Rightarrow$ $\sum_0^\infty a_n x^n$ conv abso for $|\,x\,| < |\,c\,|$ (by ①)

$\Downarrow$

$\sum_0^\infty a_n x^n$ is not absolutely convergent at $x = c$

$\Downarrow$

$\sum_0^\infty a_n x^n$ diverges for $|\,x\,| > |\,c\,|$

($\because$ if $\sum_0^\infty a_n x^n$ converges at some point x with $|\,x\,| > |\,c\,|$, then (by the key property of p.s.) the series converges absolutely at c; contradiction)

Thus we have $\sum_0^\infty a_n x^n : \begin{cases} \text{conv. absolutely} & \text{for } |\,x\,| < |\,c\,| \\ \text{div} & \text{for } |\,x\,| > |\,c\,| \end{cases}$. Therefore, $R = |\,c\,|$

※④ $\sum_0^\infty a_n x^n$ conv for $|x| < |c| \Rightarrow \sum_0^\infty a_n x^n$ conv abso for $|x| < |c|$

(the converse "$\Leftarrow$" is trivial)

Pf. Choose any x such that $|x| < |c|$.

Need only show that $\sum_0^\infty a_n x^n$ converges absolutely at x

We can choose x_0 such that $|c| > |x_0| > |x|$

$$\Rightarrow \sum_0^\infty a_n x_0^n \text{ conv (by hypo)} \overset{①(\text{key property})}{\Rightarrow} \sum_0^\infty a_n x^n \text{ conv abso for } |x| < |x_0|$$

$$\Rightarrow \sum_0^\infty a_n x^n \text{ conv abso at } x$$

⑤ [proved later; need Weierstrass M-test]

If $\sum_0^\infty a_n x^n$ is convergent for $x = R$, then for every r such that $0 \le r < R$,

$\sum_0^\infty a_n x^n$ is absolutely and **uniformly** convergent in $[-r, r]$

Remark: **convergence property is a pointwise property**

- **Alternative way of understanding the radius of convergence of a given power series:**

Proposition. ($\limsup_{n\to\infty}$ - version of SLT)

Let $\{a_n\}$ be a bounded sequence. Then

$$\limsup_{n\to\infty} a_n = M \Rightarrow \forall \varepsilon > 0,\quad a_n > M - \varepsilon \text{ for infinitely many } n$$

Equivalently, $\limsup_{n\to\infty} a_n > M' \Rightarrow a_n > M'$ for infinitely many n

Pf. This was proved in Chapter6---Appendix

Theorem. (Generalized n-th root test; often called **n-th root test**)

Suppose $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = M$. Then

$$M < 1 \Rightarrow \sum a_n \text{ conv (absolutely)}$$
$$M > 1 \Rightarrow \sum a_n \text{ diverges}$$

If $M = 1$, the test fails and there is no conclusion

Pf. Case1. $M < 1$

Choose a number M' so that $M < M' < 1$. Then

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} \left(= \lim_{n\to\infty} \sup\left\{ \sqrt[n]{|a_n|}, \sqrt[n+1]{|a_{n+1}|}, \cdots \right\} \right) = M < M'$$

$$\overset{\text{SLT}}{\Rightarrow} \sup\left\{ \sqrt[n]{|a_n|}, \sqrt[n+1]{|a_{n+1}|}, \cdots \right\} < M' \quad \text{for } n \gg 1, \text{ say for } n \ge N$$

$$\Rightarrow |a_n| < (M')^n \quad \text{for } n \ge N$$

$\sum_{N}^{\infty}(M')^n$ converges since $M' < 1$ $\quad \therefore \sum_{N}^{\infty} | a_n |$ converges (by the Comparison thm)

$\therefore \sum_{0}^{\infty} | a_n |$ converges (by Tail-convergence Thm)

Case2. $M > 1$

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = M > 1 \overset{\text{Proposition}}{\Rightarrow} \sqrt[n]{|a_n|} > 1 \text{ for infinitely many } n$$

$$\Rightarrow |a_n| > 1 \text{ for infinitely many } n$$

$$\Rightarrow |a_n| \not\to 0 \text{ as } n \to \infty \text{ (i.e., } \{a_n\} \text{ does not conv to } 0\text{)}$$

$$\Rightarrow \sum a_n \text{ diverges}$$

Theorem (Cauchy-Hadamard theorem: a consequence of the Generalized n-th root test)

Let $\sum_{0}^{\infty} a_n x^n$ be a given power series, and let $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = M$ $(0 \le M \le \infty$ is possible$)$. Then

$$\sum_{0}^{\infty} a_n x^n \begin{cases} \text{conv (absolutely)} & \text{if } |x| < \dfrac{1}{M} \\ \text{div} & \text{if } |x| > \dfrac{1}{M} \end{cases}$$

As a consequence,

$$R \text{ (= the radius of convergence of } \sum_{0}^{\infty} a_n x^n) = \frac{1}{M} = \frac{1}{\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}}$$

Pf Assume $0 < M < \infty$ (The case $M = 0$ or ∞: Home Study)

Since $\sqrt[n]{|a_n x^n|} = |x| \sqrt[n]{|a_n|}$, we have $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n x^n|} = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} \cdot |x| = M|x|$

Applying the Generalized n-th root test to $\sum_{0}^{\infty} a_n x^n$ gives

$$\sum_{0}^{\infty} a_n x^n \begin{cases} \text{conv (absolutely)} & \text{if } M|x| < 1 \\ \text{div} & \text{if } M|x| > 1 \end{cases}$$

Ex1. Let $\sum_{0}^{\infty} a_n x^n = 1 + x + (2x)^2 + (2x)^4 + (2x)^8 + \cdots$. Show $R = 1/2$

Ex2. $\sum_{n=0}^{\infty} \left(1 + \sin\frac{n\pi}{2}\right)^n \frac{x^n}{2^n}$ Show $R = 1$

8.2 Convergence at the endpoints. Abel summation

Let R be the radius of convergence of the P.S. $\sum_{0}^{\infty} a_n x^n$. Then we know that

$$\sum_{0}^{\infty} a_n x^n \begin{cases} \text{conv absolutely for } |x| < R \\ \text{div} \qquad\qquad \text{for } |x| > R \end{cases}$$

Question: What about convergence at two endpoints $x = R$ and $x = -R$?

This is often hard to determine.
However, it is not hard to determine the conv at $x = \pm R$ for the power series of the form

$\sum_{0}^{\infty} a_n x^n$ **with** $a_n \ge 0$ **for all** n (or $a_n \le 0$ for all n)

Eg. Determine the convergence at the endpoints $x = \pm R$ for the power series:

$$(a)\ \sum x^n \qquad (b)\ \sum \frac{x^n}{n} \qquad (c)\ \sum \frac{x^n}{n^2}$$

(These series all have $R = 1$)

Sol.

(a) $(\pm 1)^n \not\to 0$ $\therefore$ diverges by n-th term test

(b) $\sum \frac{1}{n}$ diverges, but $\sum \frac{(-1)^n}{n}$ converges by Alternating series test

(c) $\sum \frac{1}{n^2}$ conv , and so $\sum \frac{(-1)^n}{n^2}$ is also conv by Absolute convergence theorem

Question: Is there any **way of predicting** the radius of convergence **in advance**?

(without using the Ratio Test or n-th root test)

Sometimes this is possible if we can **calculate the sum** of the power series **explicitly**

Note: First predict R and then next should verify it !!!

Eg. We know: $\sum_0^\infty x^n = \frac{1}{1-x}$ for $|x| < 1$. Use this to show $R(\text{of left p.s.}) = 1$.

Pf. Remind that R is the unique number s.t. $\sum_0^\infty x^n \begin{cases} \text{conv absolutely for } |x| < R \\ \text{div} \qquad\qquad\qquad \text{for } |x| > R \end{cases}$

Predict: We know $\sum_0^\infty x^n$ conv absolutely for $|x| < 1$. $\therefore$ $R \geq 1$

Since the RHS becomes undefined when $x = 1$, it is reasonable to **expect that** $R = 1$

Now we will **verify** $R = 1$

It is clear that $\sum_0^\infty x^n$ diverges at $x = 1$.

Thus, $\sum_0^\infty x^n$ diverges for $|x| > 1$ by the property of power series.

Consequently, we know that

$$\sum_0^\infty x^n \quad \text{converges absolutely for } |x| < 1$$

$$\& \quad \text{diverges} \quad \text{for } |x| > 1$$

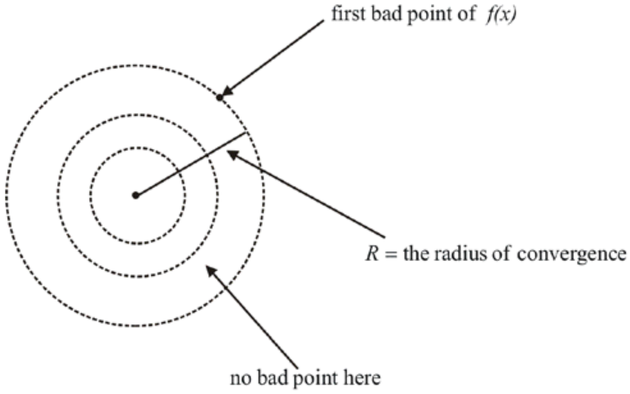
Therefore, $\sum_0^\infty x^n$ has $R = 1$

● Advanced result (optional): will be proved in **complex analysis** (3학년)

급수 $\sum_{0}^{\infty} a_n x^n$ 의 합($\equiv f(x)$) 의 "구체적 표현"을 알 때 수렴반경을 구하는 방법:

정리: 원점이 중심인 원들을 반지름을 증가시키며 그려나갈 때, **원점에서** 처음 나타나는 $f(z)$ 의 bad point (= **bad complex number**)**까지의 거리**가 $\sum_{0}^{\infty} a_n x^n$ 의 수렴반경이다.

(만일, bad point가 없으면 $R = \infty$ 이다)



Remark. $f(x) = \dfrac{1}{1-x}$ $\left(\to f(z) = \dfrac{1}{1-z}\right)$ $\to$ the first bad point (from the origin) is

$$z = 1 \ \ (\text{i.e.}, x = 1)$$

Eg (optional: caution!!) We know that

$$\sum_{0}^{\infty}(-1)^n x^{2n} \ \ \left(= \sum_{0}^{\infty}(-x^2)^n\right) = \frac{1}{1+x^2} \quad \text{for } |\, x \,| < 1 \ \ (\leftarrow \text{for } |\, -x^2 \,| < 1)$$

Use this to find its radius R of convergence.

Sol. Note that $\text{RHS} = \dfrac{1}{1+x^2}$ is defined for all $x \in \mathbb{R}$

Thus it has no **real** bad point at all. Is it true that $R = \infty$?

Ans is NO!!! We should **find the first complex bad point** if exists.

So we should consider the complex function $\dfrac{1}{1+z^2}$ instead of $\dfrac{1}{1+x^2}$

The complex function is not defined at $z = \pm i$

Thus the first bad points (from the origin) are $\pm i$. Therefore, $R = 1$

Another popular way: we know that $\sum_{0}^{\infty} y^n : \left\langle \begin{array}{ll} \text{conv absolutely for} & |\, y \,| < 1 \\ \text{div} \quad\quad\quad\quad \text{for} & |\, y \,| > 1 \end{array} \right.$

$$\text{Thus } \sum_{0}^{\infty}(-1)^n x^{2n} \ \ \left(= \sum_{0}^{\infty}(-x^2)^n\right): \left\langle \begin{array}{ll} \text{conv abso for} & |\, -x^2 \,| < 1 \\ \text{div} \quad\quad\quad \text{for} & |\, -x^2 \,| > 1 \end{array} \right. \Leftrightarrow \begin{cases} |\, x \,| < 1 \\ |\, x \,| > 1 \end{cases} \quad \therefore\ R = 1$$

◎ Abel introduced another sum ($\neq$ usual sum) for the power series which diverges at a point a, yet which has the explicit sum that is defined at a.

Def. (Abel summation) [생략해도 무방]

Suppose

$$\sum_0^\infty a_n x^n = f(x), \quad \text{for } |x| < 1,$$

where $f(x)$ is defined & **continuous** at $x = 1$, but the series diverges at $x = 1$.

Then we say that

$$\boxed{\sum_0^\infty a_n \text{ is Abel-summable to } f(1)} \quad \text{and write}$$

$$\sum_0^\infty a_n = f(1) \quad (\text{Abel summation}) \quad \left(\text{ or } \sum_0^\infty a_n \overset{\text{Abel summation}}{=} f(1) \right)$$

Warning: $\sum_0^\infty a_n \overset{\text{Abel summation}}{=} f(1)$ does not mean $\sum_0^\infty a_n = f(1)$ (usual sum)

Eg. Find the Abel sum of $1 - 1 + 1 - 1 + \cdots + (-1)^n + \cdots$

Sol. The corresponding power series & its sum are

$$1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x}, \quad \text{for } |x| < 1$$

Note that the series diverges at $x = 1$ since $(-1)^n \not\to 0$.

But the the function $f(x) = \dfrac{1}{1+x}$ is defined at $x = 1$ & continuous at $x = 1$.

Thus, $\sum_0^\infty (-1)^n = \dfrac{1}{2}$ (Abel summation).

8.3 Operations on power series; addition

Theorem (Linearity theorem for p.s.)

If $\sum a_n x^n = f(x)$ and $\sum b_n x^n = g(x)$ (conv) for $|x|<K$, then for any constants p and q,

$$\sum (pa_n + qb_n)x^n = pf(x)+qg(x) \overset{\text{i.e.}}{=} p\sum a_n x^n + q\sum b_n x^n \quad \text{for } |x|<K$$

Pf. For each x with $|x|<K$,

$$\sum (pa_n + qb_n)x^n = \sum (pa_n x^n + qb_n x^n) \overset{\text{Linearity thm for infinite series}}{=} p\sum a_n x^n + q\sum b_n x^n$$

Remark.

$$\sum a_n x^n = f(x) \text{ for } |x|<K_1 \quad \& \quad \sum b_n x^n = g(x) \text{ for } |x|<K_2$$

$$\Rightarrow \quad \sum (pa_n + qb_n)x^n = pf(x)+qg(x) \quad \text{for } |x|<K \text{, where } K=\min\{K_1, K_2\}.$$

Eg. $$1+x+x^2+x^3+x^4+\cdots = \frac{1}{1-x} \quad \text{for } |x|<1$$

& $$1-x+x^2-x^3+x^4+\cdots = \frac{1}{1+x} \quad \text{for } |x|<1$$

Addiding these (a special case of Linearity thm) gives

$$2(1+x^2+x^4+\cdots) = \frac{1}{1-x}+\frac{1}{1+x} = \frac{2}{1-x^2} \quad \text{for } |x|<1$$

8.4 Multiplication of p.s.

$$(a_0+a_1x+a_2x^2+\cdots)(b_0+b_1x+b_2x^2+\cdots) \overset{\text{def}}{=} ?$$

A natural product (for two power series) is defined as follows:

$$(a_0+a_1x+a_2x^2+\cdots)(b_0+b_1x+b_2x^2+\cdots)$$

$$\overset{\text{def}}{=} a_0b_0 + (a_0b_1+a_1b_0)x + (a_0b_2+a_1b_1+a_2b_0)x^2+\cdots$$

This is called the Cauchy product of $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$

Eg. $$(1+x+x^2+x^3+\cdots)(1-x+x^2-x^3+\cdots)=?$$

Sol.

$$1+x+x^2+x^3+\cdots = \frac{1}{1-x} \quad \text{for } |x|<1$$

$$1-x+x^2-x^3+\cdots = \frac{1}{1+x} \quad \text{for } |x|<1$$

Cauchy Product (of left sides):

$$(1+x+x^2+x^3+\cdots)(1-x+x^2-x^3+\cdots)$$

$$=1+(-1+1)x+(1-1+1)x^2+\cdots=1+x^2+x^4+\cdots=\frac{1}{1-x^2} \quad \text{for } |x|<1$$

Usual product of right sides: $\frac{1}{1-x}\cdot\frac{1}{1+x}=\frac{1}{1-x^2}$ for $|x|<1$

Is the result "$1+x^2+x^4+\cdots=\frac{1}{1-x^2}$ for $|x|<1$" natural ? Yes:

Recall that $\quad 1+x+x^2+x^3+\cdots \;=\; \dfrac{1}{1-x} \quad$ for $\quad |x|<1$

Substituting x^2 for x gives

$$1+x^2+x^4+\cdots \;=\; \frac{1}{1-x^2} \quad \text{for} \quad |x^2|<1 \quad (\text{i.e., for} \quad |x|<1)$$

Remark: Given two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, the new series $\sum_{n=0}^{\infty} c_n$ defined by

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum_{i+j=n} a_i\, b_j = \sum_{k=0}^{n} a_k b_{n-k}$$

is called the Cauchy product ($\overset{\text{for short}}{=}$ CP) of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$.

⊙ The Cauchy product $\sum_{n=0}^{\infty} c_n$ of the given two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is geometrically visualized as follows:

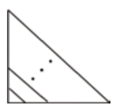
$$\begin{array}{lllll}
c_0 & c_1 & c_2 & & c_n \\
\cancel{a_0 b_0} + & \cancel{a_0 b_1} + & \cancel{a_0 b_2} + \cdots\cdots & & + \cancel{a_0 b_n} + \cdots \\
\cancel{a_1 b_0} + & \cancel{a_1 b_1} + & a_1 b_2 + \cdots & + \cancel{a_1 b_{n-1}} & + a_1 b_n + \cdots \\
\cancel{a_2 b_0} + & a_2 b_1 + & a_2 b_2 + & & \\
 & & & & \\
\cancel{a_n b_0} + & a_n b_1 + & a_n b_2 + \cdots\cdots & & + a_n b_n + \cdots
\end{array}$$

(the Cauchy product is the summation by **triangles**)

Or (in our text book)

$$\begin{array}{llll}
\cancel{a_n b_0} & a_n b_1 & & a_n b_n \\
\vdots & & & \\
\cancel{a_2 b_0} & & & \\
\cancel{a_1 b_0} & \cancel{a_1 b_1} & \cdots & a_1 b_n \\
\cancel{a_0 b_0} & \cancel{a_0 b_1} & \cancel{a_0 b_2} & \cancel{a_0 b_n} \\
c_0 & c_1 & c_2 & c_n
\end{array}$$

$c_0 + c_1 + \cdots + c_n =: C_n \quad$ (= the n-th partial sum of the Cauchy product)

= the total sum of lower triangle 

Theorem A (Multiplication of p.s.)

$$\sum_{n=0}^{\infty} a_n x^n \quad = \quad f(x) \quad \text{(converges) for } |x| < K$$

$$\& \quad \sum_{n=0}^{\infty} b_n x^n \quad = \quad g(x) \quad \text{(converges) for } |x| < K$$

$$\Rightarrow \quad \text{(the Cauchy product) } \sum_{n=0}^{\infty} c_n x^n \quad = \quad f(x)g(x) \quad \text{(converges) for } |x| < K$$

(Here $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum_{i+j=n} a_i b_j = \sum_{k=0}^{n} a_k b_{n-k}$)

Theorem B (Multiplication theorem for series)

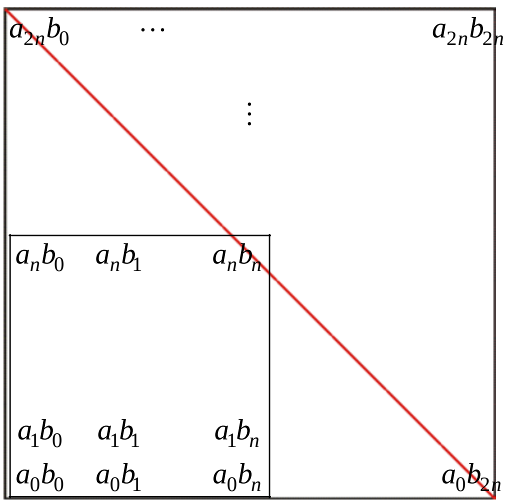
Suppose $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converges absolutely, and set $\sum_{n=0}^{\infty} a_n = A$, $\sum_{n=0}^{\infty} b_n = B$

Then the Cauchy product $\sum_{n=0}^{\infty} c_n$ converges absolutely, and $\sum_{n=0}^{\infty} c_n = A \cdot B$

Pf. Case1: all a_n and b_n are ≥ 0

(Have to show: $\sum_{n=0}^{\infty} c_n$ (= CP) $= \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right)$ whenever $\sum_{n=0}^{\infty} a_n$ & $\sum_{n=0}^{\infty} b_n$ converge)

Note that all the possible products $a_i b_j$ occur in the following matrix array.



If we write A_n, B_n, C_n for n- th partial sums of $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$, $\sum_{n=0}^{\infty} c_n$ respectively, then

the small square $=$ all $a_i b_j$ occurring in $A_n B_n$

the lower triangle $=$ all $a_i b_j$ occurring in C_{2n}

the big square $=$ all $a_i b_j$ occurring in $A_{2n} B_{2n}$

Hence

small square $\subseteq$ lower triangle $\subseteq$ big square

$$\therefore \quad \underbrace{A_n B_n}_{\substack{\downarrow \\ A\cdot B}} \quad \le \quad C_{2n} \quad \le \underbrace{A_{2n} B_{2n}}_{\substack{\downarrow \\ A\cdot B}} \quad (\because \ \text{all} \ \ a_i b_j \ge 0)$$

$$(\because \quad A_{2n} \,\&\, B_{2n} \ \text{ are subsequences of } \ A_n \ \&\, B_n, \ \text{respectively})$$

Thus by Squeeze Principle

$$\underbrace{C_{2n}}_{\text{a subseq of } C_n} \quad \to \quad A\cdot B$$

Note that $\quad C_{2n} \uparrow A\cdot B$, so $\quad C_n \le C_{2n} \le A\cdot B$, and hence C_n is bounded above.

But clearly, C_n is $\uparrow$. Thus $\lim\limits_{n\to\infty} C_n$ exists.

$$\Downarrow \quad \text{Use } \lim_{n\to\infty} C_{2n} = A\cdot B, \quad \text{together with Subsequence thm}$$

$$\lim_{n\to\infty} C_n = A\cdot B$$

In other words, we proved $\displaystyle\sum_{n=0}^{\infty} c_n = A\cdot B$

Case2: all a_n and b_n are ≤ 0

$-a_n$ and $-b_n \ge 0$ for all n

Denote the partial sums of $\displaystyle\sum_{n=0}^{\infty}(-a_n)$, $\displaystyle\sum_{n=0}^{\infty}(-b_n)$ by A'_n, B'_n, respectively;

$$\text{i.e.,} \quad A'_n = \sum_{k=0}^{n}(-a_k), \qquad B'_n = \sum_{k=0}^{n}(-b_k)$$

and let C'_n be the n-th partial sum of the Cauchy product $\displaystyle\sum_{n=0}^{\infty} c'_n$ of $\displaystyle\sum_{n=0}^{\infty}(-a_n)$ & $\displaystyle\sum_{n=0}^{\infty}(-b_n)$.

Note that

$$\boxed{\begin{array}{lll}
\cancel{(-a_n)(-b_0)} & & \\
\vdots & & \\
\vdots & & \\
\vdots & & \\
\vdots & & \\
\cancel{(-a_1)(-b_0)} & & \\
\cancel{(-a_0)(-b_0)} & \cancel{(-a_0)(-b_1)} & \cancel{(-a_0)(-b_n)} \\
c'_0 & c'_1 & c'_n
\end{array}}$$

$$\begin{aligned}
c'_0 &= (-a_0)(-b_0) = c_0 \\
c'_1 &= (-a_0)(-b_1) + (-a_1)(-b_0) = c_1 \\
&\vdots \\
&\vdots \\
c'_n &= (-a_0)(-b_n) + \cdots + (-a_n)(-b_0) = c_n
\end{aligned}$$

Hence $C'_n = c'_0 + c'_1 + \cdots + c'_n = C_n$. Thus by the result of Case1,

$$\underbrace{A'_n B'_n}_{\substack{\| \\ A_n\cdot B_n}} \quad \le \quad \underbrace{C'_{2n}}_{\substack{\| \\ C_{2n}}} \quad \le \underbrace{A'_{2n} B'_{2n}}_{\substack{\| \\ A_{2n}\cdot B_{2n}}}$$

Then by the same argument seen in Case1,

$$\lim_{n\to\infty} C_n = A\cdot B \qquad \text{i.e.,} \quad \sum_{n=0}^{\infty} c_n = A\cdot B$$

Case3 (optional): the series contains both positive and negative terms

Write $a_n = a_n^+ - a_n^-$, $b_n = b_n^+ - b_n^-$.

Since $\sum_{n=0}^{\infty} a_n$ & $\sum_{n=0}^{\infty} b_n$ are both absolutely convergent, we know that

$\sum_{n=0}^{\infty} a_n^+$, $\sum_{n=0}^{\infty} a_n^-$; $\sum_{n=0}^{\infty} b_n^+$, $\sum_{n=0}^{\infty} b_n^-$ are all convergent, and

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_n^+ - \sum_{n=0}^{\infty} a_n^- \quad \& \quad \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} b_n^+ - \sum_{n=0}^{\infty} b_n^-$$

Now

$$c_n = \sum_{i+j=n} a_i b_j = \sum_{i+j=n} \left(a_i^+ - a_i^-\right)\left(b_j^+ - b_j^-\right)$$

$$\overset{\text{OK since it is the sum of finite \# of terms}}{=} \sum_{i+j=n} \left(a_i^+ b_j^+ + a_i^- b_j^-\right) - \sum_{i+j=n} \left(a_i^- b_j^+ + a_i^+ b_j^-\right)$$

$$\overset{\text{write}}{=} c_n^+ - c_n^- \quad \text{(respectively)}$$

Let $\sum_{n=0}^{\infty} a_n^+ = A^+$, $\sum_{n=0}^{\infty} a_n^- = A^-$; $\sum_{n=0}^{\infty} b_n^+ = B^+$, $\sum_{n=0}^{\infty} b_n^- = B^-$. Then

$$\sum_{n=0}^{\infty} c_n^+ = \sum_{n=0}^{\infty} \sum_{i+j=n} \left(a_i^+ b_j^+ + a_i^- b_j^-\right)$$

$$\overset{?}{=} \sum_{n=0}^{\infty} \sum_{i+j=n} a_i^+ b_j^+ + \sum_{n=0}^{\infty} \sum_{i+j=n} a_i^- b_j^- \quad \text{(i.e., Is each convergent ?)}$$

Since a_i^+, b_j^+, a_i^-, b_j^- are all ≥ 0 & $\sum_{n=0}^{\infty} a_n^+$, $\sum_{n=0}^{\infty} a_n^-$; $\sum_{n=0}^{\infty} b_n^+$, $\sum_{n=0}^{\infty} b_n^-$ are all convergent,

$$\sum_{n=0}^{\infty} \sum_{i+j=n} a_i^+ b_j^+ + \sum_{n=0}^{\infty} \sum_{i+j=n} a_i^- b_j^- \overset{\text{Case1}}{=} \sum_{n=0}^{\infty} a_n^+ \cdot \sum_{n=0}^{\infty} b_n^+ + \sum_{n=0}^{\infty} a_n^- \cdot \sum_{n=0}^{\infty} b_n^- = A^+B^+ + A^-B^-$$

This shows each of $\sum_{n=0}^{\infty} \sum_{i+j=n} a_i^+ b_j^+$ & $\sum_{n=0}^{\infty} \sum_{i+j=n} a_i^- b_j^-$ is convergent, and hence ? is OK

Similarly,

$$\sum_{n=0}^{\infty} c_n^- = \sum_{n=0}^{\infty} \sum_{i+j=n} \left(a_i^- b_j^+ + a_i^+ b_j^-\right) = A^-B^+ + A^+B^-$$

Therefore,

$$\sum c_n = \sum \left(c_n^+ - c_n^-\right) \overset{\sum c_n^+ \ \& \ \sum c_n^- :\ \text{convergent (proved)}}{=} \sum c_n^+ - \sum c_n^-$$

$$= \left(A^+B^+ + A^-B^-\right) - \left(A^-B^+ + A^+B^-\right) = \left(A^+ - A^-\right)\left(B^+ - B^-\right) = AB$$

Pf of Theorem A

Let $A_n = a_n x^n$, $B_n = b_n x^n$ (for every $n \geq 0$). Then we see that

$\sum_{0}^{\infty} A_n$ & $\sum_{0}^{\infty} B_n$ are absolutely convergent for $|x| < R$

Then

$$\left(\sum_0^\infty A_n\right)\left(\sum_0^\infty B_n\right) \overset{\text{Theorem B}}{=} \sum_0^\infty C_n,$$

where $C_n = \sum_{k=0}^n A_k B_{n-k} = \sum_{k=0}^n \left(a_k x^k\right)\cdot\left(b_{n-k}x^{n-k}\right) = x^n \sum_{k=0}^n a_k b_{n-k} = c_n x^n$.

This means

$$\left(\sum_0^\infty a_n x^n\right)\left(\sum_0^\infty b_n x^n\right) = \sum_0^\infty c_n x^n \quad \text{for } |x|<R$$

Caution: In general, $\boxed{\sum a_n \ \ \& \ \ \sum b_n : \text{ both conv (but not absolutely)} \ \not\Rightarrow \ \underbrace{\sum c_n}_{=\text{Cauchy product}} : \text{conv}}$

For example,

$$\sum a_n = \sum b_n \overset{\text{take}}{=} \sum_{n=0}^\infty \frac{(-1)^n}{\sqrt{n+1}} \quad \left(=1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots\right): \text{conditionally converges}$$

Then

$$c_n = \sum_{k=0}^n a_k b_{n-k} = \sum_{k=0}^n \frac{(-1)^k}{\sqrt{k+1}}\frac{(-1)^{n-k}}{\sqrt{n-k+1}} = (-1)^n \sum_{k=0}^n \frac{1}{\sqrt{(k+1)(n-k+1)}}$$

Note that

$$(k+1)(n-k+1) = -k^2+nk+n+1 = -\left(k-\frac{n}{2}\right)^2+\left(\frac{n}{2}+1\right)^2 \le \left(\frac{n}{2}+1\right)^2$$

$$\therefore \quad |c_n| = \sum_{k=0}^n \frac{1}{\sqrt{(k+1)(n-k+1)}} \ge \sum_{k=0}^n \frac{1}{n/2+1} = \sum_{k=0}^n \frac{2}{n+2} = \frac{2(n+1)}{n+2} \to 2$$

This shows $\lim_{n\to\infty} c_n \ne 0$, so the Cauchy product $\sum_{n=0}^\infty c_n$ diverges.

Eg. Find the p.s. for the function $\frac{1}{(1-x)^2}$.

Sol. $\sum_0^\infty x^n = 1+x+x^2+\cdots+x^n+\cdots = \frac{1}{1-x}$ for $|x|<1$

By Theorem A,

the Cauchy product $\sum_{n=0}^\infty c_n x^n$ of $\sum_0^\infty x^n$ and $\sum_0^\infty x^n$ converges to $\frac{1}{(1-x)^2}$ for $|x|<1$

Note that $c_n = \sum_{k=0}^n a_k b_{n-k} = \sum_{k=0}^n 1\cdot 1 = n+1$. Hence

$$\sum_{n=0}^\infty c_n x^n = \sum_{n=0}^\infty (n+1)x^n = 1+2x+3x^2+\cdots \quad \text{for } |x|<1.$$

Cf: $1+x+x^2+\cdots+x^n+\cdots = \frac{1}{1-x}$ for $|x|<1$

$\Downarrow \leftarrow$ formally differentiate term-by-term (within the (open) interval of convergence)

$1+2x+3x^2+\cdots+nx^{n-1}+\cdots = \frac{1}{(1-x)^2}$ for $|x|<1$ (This is true: will be proved later)