① 1-1) $w \in \overline{\cap A_i} \iff w \notin \cap A_i$

$\iff w \notin \{A_i\}$ for all i

$\iff w \in \{\overline{A_i}\}$ for at least one i

$\iff w \in \cup \overline{A_i}$

$\therefore \overline{\cap A_i} = \cup \overline{A_i}$

1-2) i) $n=1$, $P(A_1|B) = P(A_1|B)$

ii) $n=2$, $P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B) - P(A_1 \cap A_2|B)$

$\leq P(A_1|B) + P(A_2|B)$

iii) $n-1$일 때 성립한다면, $P(\bigcup^{n-1} A_i|B) \leq \sum^{n-1} P(A_i|B)$

n일 때, $P(\bigcup^{n} A_i|B) = P(\bigcup^{n-1} A_i \cup A_n|B) \leq \sum_{i=1}^{n-1} P(A_i|B) + P(A_n|B)$

$= \sum_{i=1}^{n} P(A_i|B)$

1-3) i) $n=1$, $P(A_1) = P(A_1)$

ii) $n=2$, $P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2)$

$\geq P(A_1) + P(A_2) - 1$

iii) $n-1$일 때 성립한다면, $P(\bigcap^{n-1} A_i) \geq \sum^{n-1} P(A_i) - (n-2)$

n일 때, $P(\bigcap^{n} A_i) = P(\bigcap^{n-1} A_i \cap A_n) \geq P(\bigcap^{n-1} A_i) + P(A_n) - 1$

$\geq \sum^{n-1} P(A_i) - (n-2) + P(A_n) - 1$

$= \sum^{n} P(A_i) - (n-1)$

1-4)

i) $n=1$, $P(A_1)=P(A_1)$

ii) $n=2$, $P(A_1 \cup A_2) = P(A_1)+P(A_2)-P(A_1 \cap A_2)$

iii) $n-1$일 때, 성립한다고 가정하면,

$$P\left(\bigcup_{i=1}^{n-1} A_i\right) = \sum^{n-1} P(A_i) - \sum_{i<j}\sum P(A_i \cap A_j) + \cdots + (-1)^n P(A_1 \cap \cdots \cap A_{n-1})$$

n일 때, $P\left(\bigcup_{i=1}^{n} A_i\right) = P\left(\bigcup_{i=1}^{n-1} A_i \cup A_n\right)$, let $C = \bigcup_{i=1}^{n-1} A_i$

$= P(C \cup A_n) = P(C) + P(A_n) - P(C \cap A_n)$, 분배법칙을 이용해 전개하면

$= \sum P(A_i) - \sum_{i<j}\sum P(A_i \cap A_j) + \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cdots \cap A_n)$

② 2-1) $A_2 = (A_2 \cap A_1) \cup (A_2 \cap \bar{A}_1)$, $A_1 \subset A_2$ 이므로

$$= A_1 \cup (A_2 \cap \bar{A}_1)$$

$\Rightarrow$ 공리 3에 의해 $P(A_2|B) = P(A_1|B) + P(A_2 \cap \bar{A}_1|B)$ 이고,

공리 1에 의해 $P(A_2 \cap \bar{A}_1|B) \geq 0$ 이므로

$$\therefore P(A_2|B) \geq P(A_1|B)$$

or

$$P(A_1|B) \leq P(A_2|B)$$

2-2) $\{A_2|B\} = \{A_1|B \cap A_2|B\} \cup \{\bar{A}_1|B \cap A_2|B\}$

$$= \{A_1|B\} \cup \{\bar{A}_1 \cap A_2|B\}$$

$$P(A_2|B) = P(A_1|B) + P(\bar{A}_1 \cap A_2|B) - \underbrace{P(A_1 \cap \bar{A}_1 \cap A_2|B)}_{=0}$$

$$= P(A_1|B) + P(A_2 - A_1|B)$$

$$\therefore P(A_2|B) - P(A_1|B) = P(A_2 - A_1|B)$$

3

$A_i \supset A_{i+1}$, $\lim\limits_{n\to\infty} A_n = \bigcap\limits_{i=1}^{\infty} A_i$

$$\Rightarrow 1 - P(\lim_{n\to\infty} A_n) = 1 - P\left(\bigcap_{i=1}^{\infty} A_i\right)$$

$$= P\left(\overline{\bigcap_{i=1}^{\infty} A_i}\right)$$

$$= P\left(\bigcup_{i=1}^{\infty} \bar{A}_i\right)$$, $\bar{A}_i$는 증가사상 ··· ①

$\bar{A}_i \subset \bar{A}_{i+1}$, $\lim\limits_{n\to\infty} \bar{A}_n = \bigcup\limits_{i=1}^{\infty} \bar{A}_i = \bigcup\limits_{i=1}^{\infty} B_i$, 이때 $B_i = \bar{A}_i - \bar{A}_{i-1} = \bar{A}_i \cap \left(\bigcup\limits_{j=1}^{i-1} \bar{A}_j\right)$

$$P(\lim_{n\to\infty} \bar{A}_n) = P\left(\bigcup_{i=1}^{\infty} \bar{A}_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} P(B_i) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} B_i\right)$$

$$= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} \bar{A}_i\right) = \lim_{n\to\infty} P(\bar{A}_n) = \lim_{n\to\infty} 1 - P(A_n) = 1 - \lim_{n\to\infty} P(A_n)$$ ···· ②

By ①, $P\left(\overline{\bigcap\limits_{i=1}^{\infty} A_i}\right) = 1 - P(\lim\limits_{n\to\infty} A_n)$

By ②, $P\left(\bigcup\limits_{i=1}^{\infty} \bar{A}_i\right) = 1 - \lim\limits_{n\to\infty} P(A_n)$

$\therefore$ $P(\lim\limits_{n\to\infty} A_n) = \lim\limits_{n\to\infty} P(A_n)$

④

4-1) $P\left(\bigcup_{i=1}^{n} A_i\right) = P\left(\bigcup_{i=1}^{n}\left(\overline{\bigcup^{i-1} A_j} \cap A_i\right)\right) = \sum_{i=1}^{n} P\left(\overline{\bigcup^{i-1} A_j} \cap A_i\right) = \sum_{i=1}^{n} P(A_i) P\left(\overline{\bigcup^{i-1} A_j}\right)$

여기서 $P\left(\overline{\bigcup^{i-1} A_j}\right) = P\left(\bigcap^{i-1} \bar{A}_j\right) = \prod_{j=1}^{i-1} P(\bar{A}_j) = \prod_{j=1}^{i-1}[1-P(A_j)]$

$\therefore P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) \prod_{j=1}^{i-1}[1-P(A_j)]$

4-2) $P(A_1 \cap A_2 \cap \bar{A}_3) = P(\bar{A}_3) - P((\bar{A}_1 \cup \bar{A}_2) \cap \bar{A}_3)$

$= P(\bar{A}_3) - P(\bar{A}_3) P(\bar{A}_1 \cup \bar{A}_2)$

$= P(\bar{A}_3)(1 - P(\bar{A}_1 \cup \bar{A}_2))$

$= P(\bar{A}_3) P(A_1 \cap A_2)$

$= P(\bar{A}_3) P(A_1) P(A_2)$

4-3) $P(A_1 \cap \bar{A}_2 \cap \bar{A}_3) = P(A_1) - P((A_2 \cup A_3) \cap A_1)$

$= P(A_1) - P(A_1) P(A_2 \cup A_3)$

$= P(A_1)(1 - P(A_2 \cup A_3))$

$= P(A_1) P(\bar{A}_2 \cap \bar{A}_3)$

$= P(A_1) P(\bar{A}_2) P(\bar{A}_3)$

4-4) $P(\bar{A}_1 \cap \bar{A}_2) P(\bar{A}_3)$

$= (1 - P(A_1 \cup A_2))(1 - P(A_3))$

$= 1 - P(A_3) - P(A_1 \cup A_2) + P(A_3) P(A_1 \cup A_2)$

$= 1 \underbrace{- P(A_3) - P(A_1 \cup A_2) + P((A_1 \cup A_2) \cap A_3)}_{= -P(A_1 \cup A_2 \cup A_3)}$

$= 1 - P(A_1 \cup A_2 \cup A_3)$

$= P(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3)$

$\therefore P(\bar{A}_1 \cap \bar{A}_2) P(\bar{A}_3) = P(\bar{A}_1) P(\bar{A}_2) P(\bar{A}_3) = P(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3)$

⑤ 5-1) 거짓 ; $\Omega = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 2, 3\}$, $B = \{3, 5, 6\}$ 이라면

$\bar{A} \neq B$ 이지만 $P(A) = P(\bar{B})$ 로 동일하다.

5-2) 참 ; $P(A|B) = P(B|A) \iff \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A \cap B)}{P(A)}$

$\iff P(A) = P(B)$

5-3) 거짓 ; A = 주사위 소수 , B = 주사위 2이하 , C = 주사위 4의 약수 라고

둔다면 $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, $P(C) = \frac{1}{2}$ 이지만

$P(A|C) = \frac{1}{3} < P(B|C) = \frac{2}{3}$

5-4) 거짓 ; $\Omega = \{1 \sim 9\}$, $A = \{8, 9\}$, $B = \{3, 6, 9\}$ 라고 하면

$P(A|B) = \frac{1}{3} = P(B)$ 이지만

$P(A \cap B) = \frac{1}{9} \neq P(A)P(B) = \frac{2}{9} \times \frac{1}{3} = \frac{2}{27}$

5-5) 참 ; $P(B|\bar{A}) = \dfrac{P(\bar{A} \cap B)}{P(\bar{A})} = \dfrac{P(A \cap B)}{P(A)} = P(B|A)$

$\iff P(A)P(\bar{A} \cap B) = P(\bar{A})P(A \cap B)$

$\iff P(A)(P(B) - P(A \cap B)) = P(A \cap B)(1 - P(A))$

$\iff P(A)P(B) - P(A)P(A \cap B) = P(A \cap B) - P(A)P(A \cap B)$

$\iff P(A \cap B) = P(A)P(B)$

$\therefore$ A와 B는 독립

⑤ 5-6) 거짓 ; A = 주사위 소수 , B = 주사위 5 이상, C = 주사위 6의 약수 라고

할 때, $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, $P(C) = \frac{2}{3}$ 로

$\Rightarrow P(AB|C) = \frac{1}{4} \neq \frac{1}{2} \times \frac{1}{4} = P(A|C)\,P(B|C)$