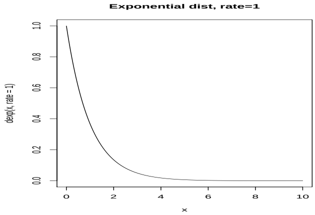
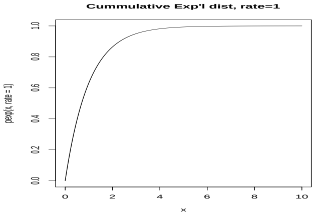
Ch5. Exponential distribution and Poisson process

1. Exponential distribution
2. Poisson process (PP)
3. Generalization of the PP: NPP and CPP

Exponential distribution

- We have argued in Chapter 2 that Exponential distribution is a continuous analogue of Geometric distribution. For rate λ, we denote
$$X \sim \mathrm{Exp}(\lambda)$$
- $f(x) = \lambda e^{-\lambda x} 1_{\{x \geq 0\}}$, $F(x) = 1 - e^{-\lambda x}, x \geq 0$





- Moments are
$$EX = \frac{1}{\lambda},\ EX^2 = \frac{2}{\lambda^2},\ \mathrm{Var}(X) = \frac{1}{\lambda^2}.$$

Properties of Exponential distribution

- $M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \dfrac{\lambda}{\lambda - t}, t < \lambda$
- $X_1 + \cdots + X_n \sim \mathrm{Gamma}(n, \lambda)$
- If $X_1 \sim \mathrm{Exp}(\lambda_1)$ and $X_2 \sim \mathrm{Exp}(\lambda_2)$ are independent,

$$\boxed{X = \min\{X_1, X_2\} \sim \mathrm{Exp}(\lambda_1 + \lambda_2)}$$

Indeed:

$$\begin{aligned} P(X > x) &= P(\min\{X_1, X_2\} > x) \\ &= P(X_1 > x \cap X_2 > x) \\ &= e^{-\lambda_1 x} e^{-\lambda_2 x} = e^{-(\lambda_1 + \lambda_2)x} \end{aligned}$$

Observe also that

$$EX = \frac{1}{\lambda_1 + \lambda_2} < \min\left(\frac{1}{\lambda_2}, \frac{1}{\lambda_1}\right)$$

Properties of Exponential distribution

- Ordering probability:

$$\begin{aligned} P(X_1 < X_2) &= \int_0^\infty P(X_1 < X_2 | X_1 = x_1) \lambda_1 e^{-\lambda_1 x_1} dx_1 \\ &= \int_0^\infty P(X_2 > x_1) \lambda_1 e^{-\lambda_1 x_1} dx_1 \\ &= \int_0^\infty e^{-\lambda_2 x_1} \lambda_1 e^{-\lambda_1 x_1} dx_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2} \end{aligned}$$

  Therefore,

$$P(X_1 = \min\{X_1, X_2\}) = P(X_1 < X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

- Similarly

$$P(X_2 = \min\{X_1, X_2\}) = P(X_2 < X_1) = \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

Minimum of Exponential distribution

- Define rank order (index of minimum) of X_1, X_2

$$N = \begin{cases} 1 & \text{if } X_1 < X_2 \\ 2 & \text{if } X_2 < X_1 \end{cases}$$

- Then, $X = \min\{X_1, X_2\}$ and N are independent
- In general, if $X_1, \cdots, X_n$ are independent Exponential random variables with rate $\lambda_1, \cdots, \lambda_n$. Then,

$$\min\{X_1, \cdots, X_n\} \sim \mathrm{Exp}(\lambda_1 + \ldots + \lambda_{\mathrm{n}})$$

$\min\{X_1, \cdots, X_n\}$ are independent with rank order N

Minimum of Exponential distribution

Indeed:

$$P(X > x, N = 1) = P(\min\{X_1, X_2\} > x, X_1 < X_2)$$

$$P(X_1 > x, X_2 > x, X_1 < X_2)$$

$$= \int_0^\infty P(X_1 > x, X_2 > x, X_1 < X_2 | X_1 = y)\lambda_1 e^{-\lambda_1 y} dy$$

$$= \int_0^\infty P(y > x, X_2 > x, y < X_2 | X_1 = y)\lambda_1 e^{-\lambda_1 y} dy$$

$$= \int_x^\infty P(X_2 > x, X_2 > y)\lambda_1 e^{-\lambda_1 y} dy$$

$$= \int_x^\infty e^{-\lambda_2 y}\lambda_1 e^{-\lambda_1 y} dy = \frac{\lambda_1}{\lambda_1 + \lambda_2}\int_x^\infty (\lambda_1 + \lambda_2) e^{-(\lambda_1 + \lambda_2)} dy$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_2)x} = P(N = 1)P(X > x)$$

Therefore, X and N are independent.

Memoryless property

Memoryless property

A continuous r.v is said to have memoryless property if

$$P(X > s + t | X > t) = P(X > s) \text{ for all } \; s, t \geq 0$$

$$\text{i.e. } P(X > s + t) = P(X > s) P(X > t)$$

- Assume life time of an instrument has memoryless property. If the instrument survived at time t, its remaining life time is the same as the original life time.

Exponential distribution is memoryless

- For $\mathrm{Exp}(\lambda)$, note that

  $$P(X > t + s) = e^{-\lambda(s+t)} = e^{-\lambda s} e^{-\lambda t} = P(X > s) P(X > t)$$

  Therefore, Exponential distribution is memoryless (and in fact the only continuous random variable with memoryless property).
- Memoryless property simplifies calculations remarkably. For example, let X be the number of miles (in thousands) that car runs before battery is dead. Suppose it follows Exponential distribution with rate λ and interested in the probability that car complete 5 (thousands) miles without replacing battery.

  $$P(X > t + 5 | X > t) = P(X > 5) = e^{-\lambda \cdot 5}$$

Example 5.2

Suppose that the amount of time one spends in a bank is exponentially distributed with mean ten minutes, that is, $\lambda = 1/10$. What is the probability that a customer will spend more than fifteen minutes in the bank? What is the probability that a customer will spend more than fifteen minutes in the bank given that she is still in the bank after ten minutes?

Example 5.8

Suppose you arrive at a post office having two clerks at a moment when both are busy but there is no one else waiting in line. You will enter service when either clerk becomes free. If service times for clerk i are exponential with rate λ_i, $i = 1, 2$, find $E(T)$, where T is the amount of time that you spend in the post office.

Counting Process

Counting Process
A stochastic process $\{N(t), t \geq 0\}$ is said to be a counting process if $N(t)$ represents the total number of events occur up to (and including) time t.

If we draw sample path of $N(t)$, then

Counting Process

Some examples include:

1. $\#$ of persons entering a store at time t
2. $\#$ of people were born by time t
3. $\#$ of goals a given soccer player scored by time t

Observations for $N(t)$:

(i) $N(t) \geq 0$

(ii) $N(t)$ is integer-valued

(iii) If $s < t$, then $N(s) \leq N(t)$

(iv) For $s < t$, $N(t) - N(s) = \#$ of events in $(s, t]$

Counting Process

Some remarks in order:

- For fixed time t, $N(t)$ is a r.v
- If we are interested in the whole time interval $(0, t]$, we call it a process.
- $N(t) - N(s)$ is called increments.
- A counting process is said to have independent increments if the numbers of events that occur in disjoint time intervals are independent. That is,

$$N(t+s) - N(s) \text{ are indepenent with } \{N_u : u \leq s\}$$

Independent increments

- Independent increments make calculation easy because

$$N(t) = \sum_{i=0}^{k} \Big(N(t_{i+1}) - N(t_i) \Big)$$

  for $0 = t_0 < t_1 < t_2 \ldots < t_k < t_{k+1} = t$.
- Therefore, $N(t)$ can be represented as the sum of independent r.v's.
- Poisson process is a particular example of counting process with independent increments. We will define Poisson process in a moment, but need technical notation called $o(h)$.

little $o(h)$

- Measuring convergence speed of a function $f(\cdot)$ around zero.

The function f is said to be $o(h)$ (say little oh of h) if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0$$

- It means that $f(h)$ converges to zero even faster than h (linear function). For example, function $f(x) = x^2$ is $o(h)$ since

$$\lim_{h \to 0} \frac{h^2}{h} = 0$$

- However, function $f(x) = x$ is NOT $o(h)$ since

$$\lim_{h \to 0} \frac{h}{h} = 1 \neq 0$$

little $o(h)$

Useful facts on $o(h)$:

1. $f = o(h), g = o(h) \Rightarrow f + g = o(h)$. Indeed:

$$\lim_{h \to 0} \frac{(f+g)(h)}{h} = \lim_{h \to 0} \frac{f(h)}{h} + \lim_{h \to 0} \frac{g(h)}{h} = 0$$

2. If $f = o(h)$, then $cf = o(h)$.
3. Thus, any finite linear combination of $o(h)$ functions is again $o(h)$.

In practice, you can regard $o(h)$ as the remainder (error) vanishes faster than straight line. Little $o(h)$ is used to represent error terms disappear as h getting smaller.

Poisson Process (PP)

Poisson Process Axioms

The counting process $N(t)$ is said to be a Poisson process with rate $\lambda > 0$, denote as $PP(\lambda)$, if

(i) $N(0) = 0$ (starts at 0)

(ii) $\{N(t)\}$ has independent increments

(iii) $P(N(t+h) - N(t) = 1) = \lambda h + o(h)$ for all $t, h > 0$

(iv) $P(N(t+h) - N(t) \geq 2) = o(h)$ for all $t, h > 0$

- $(iii) - (iv)$ implies that for small time interval h, event can happen at most once. It also says that the rate λ remains the same for all time interval.

Properties of PP

Independent increments implies Markov property!

For $t_0 < t_1 < t_2$

$$
\begin{aligned}
&P(N(t_2) = x_2 | N(t_1) = x_1, N(t_0) = x_0) \\
&= P(N(t_2) - N(t_1) = x_2 - x_1 | N(t_1) = x_1, N(t_0) = x_0) \\
&= P(N(t_2) - N(t_1) = x_2 - x_1) \\
&= P(N(t_2) - N(t_1) = x_2 - x_1 | N(t_1) = x_1) \\
&= P(N(t_2) = x_2 | N(t_1) = x_1)
\end{aligned}
$$

Bernoulli process and PP

- It is also very closely related to Bernoulli process. Recall

$$Bin(k,p) \approx Poisson(\lambda) \text{ as } k \uparrow \infty \quad kp \approx \lambda$$

- Consider $N(t)$ with finer intervals $h = t/k$.
- At this small interval h, condition $(iii) - (iv)$ implies that it is only possible to observe one event with probability $p = \lambda h$ or not.

- Then, we can represent

$$N(t) \approx \sum_{j=1}^{k} Y_j, \quad Y_j \sim \text{Bernoulli}(\lambda h)$$

Bernoulli process and PP

- Note that

$$P(N(t) = m) = P(A) + P(B) \text{ with}$$

$$A = m \text{ out of } k \text{ interval contains exactly 1 event}$$

$$B = \text{total } m \text{ events but there is at least one interval} \geq 2 \text{ events}$$

- $P(B) \leq P(\text{there is at least one interval} \geq 2)$

$$= P\left(\cup_{i=1}^{k}(i^{th} \text{ interval } \geq 2)\right) \leq \sum_{i=1}^{k} P(i^{th}\text{interval } \geq 2)$$

$$= ko(h) \to 0 \text{ as } h \to 0$$

- $P(A) = P(Bin(k, \lambda h) = m) \approx P(\mathrm{Poisson}(k\lambda h) = m) = P(\mathrm{Poisson}(\lambda t) = m)$
- Therefore, $N(t)$ is the continuous version of Bernoulli process with

$$N(t) \approx \mathrm{Poisson}(\lambda t)$$

Properties of the Poisson Process(PP)

Theorem 5.1

$$N(t) \stackrel{d}{=} N(t+s) - N(s) \sim \mathrm{Poisson}(\lambda t)$$

Proof: Recall MGF of $\mathrm{Poisson}(\lambda)$ is given by

$$M_X(u) = E(e^{uX}) = \exp(\lambda(e^u - 1))$$

Thus, Laplace transform is

$$M_X(-u) = \exp(\lambda(e^{-u} - 1))$$

It is enough to show that

$$g(t) = E(e^{-uN(t)}) = \exp(\lambda t(e^{-u} - 1))$$

$N(t) \sim \text{Poisson}(\lambda t)$

Note that

$$\begin{aligned} g(t+h) &= E(e^{-uN(t+h)}) \\ &= E(e^{-u\{N(t+h)-N(t)+N(t)\}}) \\ &= E(e^{-u\{N(t+h)-N(t)\}})E(e^{-uN(t)}) \\ &= g(t)E(e^{-u\{N(t+h)-N(t)\}}) \end{aligned}$$

The later term becomes

$$E(e^{-u\{N(t+h)-N(t))\}})=e^{-u0}(1-\lambda h+o(h))+e^{-u}(\lambda h+o(h))+\textstyle\sum_{k=2}^{\infty} e^{-ku}P_k$$

$$= 1 - \lambda h + e^{-u}\lambda h + o(h)$$

since $|e^{-u}| \leq 1$ implies that

$$\left|\sum_{k=2}^{\infty} e^{-ku}P_k\right| \leq \left|\sum_{k=2}^{\infty} P_k\right| = o(h)$$

$N(t) \sim \text{Poisson}(\lambda t)$

Therefore,

$$g(t+h) = g(t)(1 + \lambda h(e^{-u} - 1) + o(h))$$

$$\frac{g(t+h) - g(t)}{h} = g(t)\lambda(e^{-u} - 1) + \frac{o(h)}{h}$$

By lettting $h \to 0$, it leads to

$$g'(t) = g(t)\lambda(e^{-u} - 1)$$

$$\iff \frac{g'(t)}{g(t)} = \lambda(e^{-u} - 1)$$

$$\iff \{\log g(t)\}' = \lambda(e^{-u} - 1)$$

$$\iff \log g(t) = \int_0^t \lambda(e^{-u} - 1)dx + C$$

Hence,

$$\log g(t) = \lambda t(e^{-u} - 1) + C$$

$N(t) \sim \text{Poisson}(\lambda t)$

Now, use initial condition $N(0) = 0$

$$g(0) = E(e^{-uN(0)}) = 1$$

$$\therefore C = 0$$

Therefore,

$$g(t) = \exp(\lambda t(e^{-u} - 1)),$$

which is the Laplace transformation of Poisson(λt).

$$\therefore N(t) \sim \text{Poisson}(\lambda t)$$

Since we can think $N(t+s) - N(s)$ be the new Poisson process starting at time s, and (iii)-(iv) implies time homogeneousness of Poisson distribution,

$$N(t+s) - N(s) \overset{d}{=} N(t) \sim \text{Poisson}(\lambda t)$$

$N(t) \sim \text{Poisson}(\lambda t)$

- For fixed t, we have

$$\boxed{\begin{gathered} E(N(t)) = \lambda t, \quad \text{Var}(N(t)) = \lambda t \\ \text{Cov}(N(t), N(t+s)) = \lambda t \end{gathered}}$$

Because,

$$\begin{aligned} EN(t)N(t+s) &= E\Big(N(t)\{N(t+s) - N(t) + N(t)\}\Big) \\ &= E(N(t))E(N(t+s) - N(t)) + EN(t)^2 \\ &= \lambda t \ \lambda s + \lambda t + (\lambda t)^2 \end{aligned}$$

Hence,

$$\text{Cov}(N(t), N(t+s)) = E(N(t)N(t+s)) - E(N(t))E(N(t+s))$$

$$= \lambda t \ \lambda s + \lambda t + (\lambda t)^2 - \lambda t \ \lambda(t+s) = \lambda t$$

$N(t) \sim \text{Poisson}(\lambda t)$

- Even, we can do exact probability calculation. Consider $0 < t_1 < t_2$ and $k_1 \leq k_2$. Note that

$$P(N(t_1) = k_1, N(t_2) = k_2)$$

$$= P(N(t_1) = k_1, N(t_2) - N(t_1) = k_2 - k_1)$$

$$= P(N(t_1) = k_1)P(N(t_2) - N(t_1) = k_2 - k_1)$$

$$= \frac{e^{-\lambda t_1}(\lambda t_1)^{k_1}}{k_1!} \frac{e^{-\lambda(t_2 - t_1)}(\lambda(t_2 - t_1))^{k_2 - k_1}}{(k_2 - k_1)!}$$

- Example: Let $N(t) \sim PP(8)$

$$P(N(2.5) = 17, N(3.7) = 22, N(4.3) = 36)$$

$$= P(N(2.5) = 17)P(N(3.7) - N(2.5) = 5)P(N(4.3) - N(3.7) = 14)$$

$$= \frac{e^{-20}20^{17}}{17!} \frac{e^{-9.6}(9.6)^5}{5!} \frac{e^{-4.8}(4.8)^{14}}{14!}$$

PP with inter-arrivals and jumps

- We can understand PP with inter-arrivals and jump of size 1.
- Let T_n be the sequence of inter- arrivals. Then, the occurrence of n-th event can be represented as

$$S_0 = 0$$
$$S_1 = T_1$$
$$S_n = T_1 + \cdots + T_n$$

- Now $N(t)$ can be represented as

$$N(t) = \sum_{i=1}^{\infty} \mathbf{1}_{\{s_i \leq t\}}$$

- Therefore, once we can know the distribution of $\{T_i\}$, PP is completely characterized.

Definition
$\{N(t), t \geq 0\}$ is a $PP(\lambda)$ iff $\{T_i\}$'s are IID $\mathrm{Exp}(\lambda)$ random variables.

Inter-arrivals of PP(λ) are IID $\mathrm{Exp}(\lambda)$

- We can show that if T_i's are IID Exponentially distributed, $N(t)$ follows Poisson distribution with mean λt.
- Note that

$$P(N(t) = k) = P(N(t) \geq k) - P(N(t) \geq k+1)$$

One key relationship between number of events and arrival time is

$$\boxed{\{N(t) \geq k\} = \{S_k \leq t\}}$$

Also note that $S_k = T_1 + \ldots + T_k \sim \mathrm{Gamma}(k, \lambda)$ since S_k is the sum of k IID $\mathrm{Exp}(\lambda)$. Hence, it equals to

$$P(S_k \leq t) - P(S_{k+1} \leq t)$$

$$= \left(1 - e^{-\lambda t} \sum_{r=0}^{k-1} \frac{(\lambda t)^r}{r!}\right) - \left(1 - e^{-\lambda t} \sum_{r=0}^{k} \frac{(\lambda t)^r}{r!}\right)$$

$$= e^{-\lambda t} \frac{(\lambda t)^k}{k!} \sim \mathrm{Poisson}(\lambda t).$$

Inter-arrivals of PP(λ) are IID $\mathrm{Exp}(\lambda)$

- Independent increment comes from $\{T_i\}$ are IID Exponential dist^n with memoryless property.
- Consider new $PP(\lambda)$ starting at time s,

$$N_s(t) := N(t+s) - N(s)$$

It is again PP(λ). Graphically,

Inter-arrivals of PP(λ) are IID $\mathrm{Exp}(\lambda)$

- Conversely, we can also show from the first Definition using axioms. That is, two definitions are equivalent.
- For example, for T_1

$$P(T_1 \leq t) = P(N(t) \geq 1) = 1 - e^{-\lambda t} \sim \mathrm{Exp}(\lambda)$$

- For T_2:

$$P(T_2 \leq s | T_1 = t) = P(N(t+s) - N(t) \geq 1 | T_1 = t)$$

$$= P(N(t+s) - N(t) \geq 1 | N(t) = 1, N(u) = 0, u < t)$$

$$= P(N(t+s) - N(t) \geq 1) = P(N(s) \geq 1)$$

$$= 1 - e^{-\lambda s} \sim \mathrm{Exp}(\lambda)$$

implies that T_2 is independent from T_1 and follows $\mathrm{Exp}(\lambda)$.

Example 5.13

Suppose that people immigrate into a territory at a Poisson rate $\lambda = 1$ per day.

(a) What is the expected time until the tenth immigrant arrives?

(b) What is the probability that the elapsed time between the tenth and the eleventh arrival exceeds two days?

- $E(S_{10}) = E(T_1 + \cdots T_{10}) = 10 \cdot \frac{1}{\lambda} =$ 10 days
- $P(T_{11} > 2) = e^{-2\lambda} = e^{-2} = .133$

Exercise 57

Events occur according to a Poisson process with rate $\lambda = 2$ per hour.

(a) What is the probability that no event occurs between 8 P.M. and 9 P.M.?

(b) Starting at noon, what is the expected time at which the fourth event occurs?

(c) What is the probability that two or more events occur between 6 P.M. and 8 P.M.?

Conditional distribution of arrival time $\{S_n\}$

- Recall, $\{N(t), t \geq 0\} \sim PP(\lambda)$ and n-th event time is given by $S_n = T_1 + \ldots + T_n$. We are interested in the conditional distribution of event times under the total number of events happened by time t is known.

Theorem 5.2 (Campell's theorem)
Given $N(t) = n$,

$$(S_1, \ldots, S_n) \stackrel{d}{=} (U_{(1)}, \cdots, U_{(n)}),$$

where $U_{(1)} \leq \ldots \leq U_{(n)}$ are order statistics from Uniform$(0, t)$ distribution.

Note, it does not depend on λ at all!

Order Statistics

- If the Y_i, $i = 1, \ldots, n$, are independent identically distributed continuous random variables with probability density f , then the joint density of the order statistics $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ is given by

$$f(y_1, \ldots, y_n) = n! \Pi_{i=1}^{n} f(y_i), \quad y_1 \leq \ldots \leq y_n$$

- If the Y_i,$i = 1, \ldots, n$, are uniformly distributed over $(0, t)$, the joint density of the order statistics $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ is

$$f(y_1, \ldots, y_n) = \frac{n!}{t^n}, \quad 0 < y_1 \leq \ldots \leq y_n < t$$

Campell's theorem

If $N(t) = 1$, for $s \le t$,

$$
\begin{aligned}
P(S_1 < s | N(t) = 1) = P(T_1 < s | N(t) = 1) &= \frac{P(T_1 < s, N(t) = 1)}{P(N(t) = 1)} \\
&= \frac{P(\text{one event in } [0, s),\ \text{no event in } [s, t])}{\lambda t e^{-\lambda t}} \\
&= \frac{\lambda s e^{-\lambda s} e^{-\lambda (t-s)}}{\lambda t e^{-\lambda t}} = \frac{s}{t}
\end{aligned}
$$

Hence,

$$
f(s | N(t) = 1) = \frac{\partial}{\partial s} \frac{s}{t} = \frac{1}{t}.
$$

Conditional distribution of arrival time $\{S_n\}$

If $N(t) = 2$ and $S_1 < S_2$ (also $s_1 < s_2$)

$$
\begin{aligned}
&P(S_1 \le s_1, S_2 \le s_2 | N(t) = 2) \\
&= \frac{P(S_1 \le s_1, S_2 \le s_2, N(t) = 2)}{P(N(t) = 2)} \\
&= \frac{P(1 \text{ event in } [0, s_1])P(1 \text{ event in } (s_1, s_2])P(0 \text{ event in } (s_2, t])}{P(N(t) = 2)} \\
&= \frac{\lambda s_1 e^{-\lambda s_1} \lambda(s_2 - s_1) e^{-\lambda(s_2 - s_1)} e^{-\lambda(t - s_2)}}{(\lambda t)^2 e^{-\lambda t}/2!} \\
&= \frac{2! s_1 (s_2 - s_1)}{t^2}
\end{aligned}
$$

Hence,

$$
f(s_1, s_2 | N(t) = 2) = \frac{\partial^2}{\partial s_1 \partial s_2} \frac{2! s_1 (s_2 - s_1)}{t^2} = \frac{2!}{t^2}.
$$

Campell's theorem

- Campell's theorem tells that given the number of arrivals, event happen randomly on $[0, t]$. For example if $N(t) = 3$, our intuition says
$$E(S_3|N(t) = 3) = \frac{3}{4}t$$
- Exact calculation shows that
$$P(S_3 \leq x|N(t) = 3) = P(U_{(3)} \leq x)$$
$$= P(\max\{U_1, U_2, U_3\} \leq x|N(t) = 3) = \left(\frac{x}{t}\right)^3$$
Therefore,
$$E(S_3|N(t) = 3) = \int_0^t P(S_3 > x|N(t) = 3)dx$$
$$= \int_0^t \left(1 - \frac{x^3}{t^3}\right) dx = t - \frac{1}{t^3}\frac{1}{4}t^4 = t - \frac{1}{4}t = \frac{3}{4}t$$

Exercise 60

Customers arrive at a bank at a Poisson rate λ per minute. Suppose two customers arrived during the first hour. What is the probability that

(a) both arrived during the first 20 minutes?

(b) at least one arrived during the first 20 minutes?

Superposition of two PPs

Superposition of two PPs

Suppose that $\{N_1(t), t \geq 0\} \sim PP(\lambda_1)$ and $\{N_2(t), t \geq 0\} \sim PP(\lambda_2)$ are independent. Superposition of two PPs

$$N(t) = N_1(t) + N_2(t) \sim PP(\lambda_1 + \lambda_2)$$

Recall $\min(X_1, X_2) \sim \mathrm{Exp}(\lambda_1 + \lambda_2)$

Splitting of two PPs

Reverse procedure is called splitting. Under Bernoulli splitting (i.e randomly split into two) produces two independent PPs.

Proposition 5.2
Consider a Poisson process $\{N(t), t \geq\}$ having rate λ, and suppose that each time an event occurs it is classified as either a type I with probability p or a type II event. $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are independent PP with rate $p\lambda$ and $(1-p)\lambda$.

- Bernoulli splitting is also called thinning.
- Hence, inter-arrivals of $N_1(t) \sim \mathrm{Exp(p\lambda)}$ and $N_2(t) \sim \mathrm{Exp((1-p)\lambda)}$

Example

Suppose cars arrive at three-way-intersection according to

$$N(t) \sim PP(\lambda = 30/\text{min})$$

With probability .6 it turns left and dete N_1 be the number of cars turning left. Similarly denote N_2 be the number of cars turning right. Then,

$$N_1 \sim PP(.6 \times 30) = PP(18)$$

$$N_2 \sim PP(.4 \times 30) = PP(12)$$

and N_1 and N_2 are independent.

Example 5.14

If immigrants to area A arrive at a Poisson rate of ten per week, and if each immigrant is of English descent with probability $1/12$, then what is the probability that no people of English descent will emigrate to area A during the month of February?

Examples

Vehicles stopping at a restaurant in accordance with a Poisson process with rate 20 per hour. On each vehicle arriving, it has following number of persons with probability

# of person	1	2	3	4	5
probability	.3	.3	.2	.1	.1

Find the expected number of persons arriving at the restaurant in 1 hour.

Nonhomogeneous PP

- Nonhomogeneous PP handles case where rate λ depends on time.
- Example 5.24: Siegbert runs a hot dog stand that opens at 8 A.M. From 8 until 11 A.M. customers seem to arrive, on the average, at a steadily increasing rate that starts with an initial rate of 5 customers per hour at 8 A.M. and reaches a maximum of 20 customers per hour at 11 A.M. From 11 A.M. until 1 P.M. the (average) rate seems to remain constant at 20 customers per hour. However, the (average) arrival rate then drops steadily from 1 P.M. until closing time at 5 P.M. at which time it has the value of 12 customers per hour.

Nonhomogeneous PP

Then, it is possible to describe rate as a function of time t.

$$\lambda(t) = \begin{cases} 0 & 0 \leq t \leq 8 \\ 5 + 5(t-8) & 8 \leq t \leq 11 \\ 20 & 11 \leq t \leq 13 \\ 20 - 2(t-13) & 13 \leq t \leq 17 \\ 0 & 0 < t \leq 24 \end{cases}$$

Hence defining $\lambda(t) = \lambda(t-24)$ if $t > 24$ gives rate for all days.

Nonhomogeneous PP

Nonhomegeneous PP Axioms

Counting process $\{N(t), t \geq 0\}$ is said to be nonhomegeneous PP (NPP) with intensity function $\lambda(t), t \geq 0$ if

$$i)\ N(0) = 0$$

$$ii)\ \{N(t), t \geq 0\} \text{ has independent increments}$$

$$iii)\ P(N(t+h) - N(t) \geq 2) = o(h)$$

$$iv)\ P(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$$

Again remark that the difference between PP and NPP is that the arrival rate at time t to be a function of t.

NPP $N(t) \sim \text{Poisson}(m(t))$

Theorem 5.3 If $\{N(t), t \geq 0\}$ is a NPP with intensity function $\lambda(t), t \geq 0$, then

$$N(t+s) - N(s) \sim \text{Poisson}(m(t+s) - m(s)),$$

where $m(t)$ is called the mean function of NPP defined by

$$m(t) = \int_0^t \lambda(y)dy.$$

$$N(t+s) - N(s) \sim \text{Poisson}\left(\int_s^{t+s} \lambda(y)dy\right)$$

NPP and Bernoulli process

- We can also related NPP to Bernoulli process as in the case of PP.
- However, Bernoulli process $\{X_1, X_2, \ldots\}$ indicating whether event happened or not on the k-th subinterval of $[0, t]$ are INDEPENDENT but NOT IDENTICALLY distributed.
- Probability of observing event is given by

$$p(t) = \lambda(t)h$$

- This suggest how to get NPP using splitting scheme from homogeneous PP.

NPP sampling

- Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. When an event occurred at time t, split this process into two subprocess according to probability $p(t)$ independently of what has occurred prior to t
- Then,

$$\{N_1(t), t \geq 0\} \sim \mathrm{NPP}\left(\lambda p(t)\right)$$

$$\{N_2(t), t \geq 0\} \sim \mathrm{NPP}\left(\lambda(1 - p(t))\right)$$

- See proof in the textbook on page 324. They verify that $N_t(t)$ satisfies NPP axioms.

Example 5.24 Continued

If we assume that the numbers of customers arriving at Siegberts stand during disjoint time periods are independent, then what is a good probability model for the preceding? What is the probability that no customers arrive between 8:30 A.M. and 9:30 A.M. on Monday morning? What is the expected number of arrivals in this period?

Example

- Suppose that customers buy iPhone in SKKU follows NPP with

$$\lambda(t) = 5625te^{-3t}, \quad t \geq 0$$

Then,

$$N(t) \sim \text{Poisson}\left(\int_0^t \lambda(u)du\right)$$

$$= \text{Poisson}(625(1 - e^{-3t} - 3te^{-3t}))$$

As $t \to \infty$, it converges to Poisson(625), so on average 625 iPhones sold in this model.

Compound Poisson process

Compound Poisson Process A stochastic process $\{X(t), t \geq 0\}$ is said to be a compound Poisson process if it can be represented as

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0,$$

where $\{N(t), t \geq 0\}$ is a Poisson process, and $\{Y_i, i \geq 1\}$ is a family of independent and identically distributed random variables that is also independent of $\{N(t), t \geq 0\}$.

That is, CPP extends PP by having general jump size determined by $\{Y_i, i \geq 1\}$.

Examples

1. If $Y_i = 1$, then $X(t) \sim PP(\lambda)$
2. Buses arrive according to $PP(\lambda)$ and $\#$ of persons on each bus is Y_i. $X(t)$ is the total number of person in the bus and follows CPP.
3. Suppose customers leave a supermarket in accordance with a Poisson process. If Y_i is the amount spent by the ith customer. Then $X(t)$ is CPP representing total amount of money spent by time t.
4. Computer fails by $PP(\lambda)$. When you fix it, it associated a cost of repair Y_i. Then $X(t)$ is CPP with total money spent to fix your computer.

Properties

- $E(X(t)) = \lambda t E(Y_1)$.
- $\mathrm{Var}(X(t)) = \lambda t E(Y_1^2)$.

Indeed:

$$E(X(t)) = E_N(E(X(t)|N(t))) = E(Y_1)E(N(t)) = \lambda t E(Y_1)$$

Similarly, using conditional expectation we have

$$\mathrm{Var}(X(t)) = E(N)\,\mathrm{Var}(Y_1) + \mathrm{Var}(N)E(Y_1)^2$$

$$= \lambda t \left(\mathrm{Var}(Y_1) + E(Y_1)^2\right) = \lambda t E(Y_1^2)$$

Example 5.26

Suppose that families migrate to an area at a Poisson rate $\lambda = 2$ per week. If the number of people in each family is independent and takes on the values 1, 2, 3, 4 with respective probabilities $1/6$, $1/3$, $1/3$, $1/6$ then what is the expected value and variance of the number of individuals migrating to this area during a fixed five-week period?

Laplace transformation of CPP

- Laplace transform is given by

$$E(e^{-sX(t)}) = E(e^{-s\sum_{i=1}^{N(t)} Y_i})$$

$$= \sum_{k=0}^{\infty} E(e^{-s\sum_{i=1}^{N(t)} Y_i} | N(t) = k) P(N(t) = k)$$

$$= \sum_{k=0}^{\infty} E(e^{-s\sum_{i=1}^{k} Y_i}) \frac{e^{-\lambda t}(\lambda t)^k}{k!} = \sum_{k=0}^{\infty} \left\{E(e^{-sY_1})\right\}^k \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

$$= e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(E(e^{-sY_1})\lambda t)^k}{k!}$$

$$= e^{-\lambda t} \exp\{E(e^{-sY_1})(\lambda t)\} = \exp\{-\lambda t(1 - E(e^{-sY_1}))\}$$

- Note that PP(λ) has Laplace transformation as $\exp\{-\lambda t(1 - e^{-s})\}$

Superposition of two CPPs

- Let
$$X(t) \sim CPP(\lambda_1, F_1)$$
$$Y(t) \sim CPP(\lambda_2, F_2),$$
where F_1 and F_2 represent the distribution of magnitude of jumps. Then, superimposing two CPPs give
$$X(t) + Y(t) \sim CPP\left(\lambda_1 + \lambda_2; \quad \frac{\lambda_1}{\lambda_1 + \lambda_2} F_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} F_2\right)$$
- Combined process events will occur according to a Poisson process with rate $\lambda_1 + \lambda_2$, and each event independently will be from the first CPP with probability $\lambda_1/(\lambda_1 + \lambda_2)$.

Normal Approximation

- Because this is a sum of (random number of) random variable, CLT still holds. We can approximate it by Normal distribution when t is large.

$$X(t) \approx N(\lambda t\ E(Y_1), \lambda t\ E(Y_1^2))$$

- Example 5.26 continued: find the approximate probability that at least 240 people migrate to the area within the next 50 weeks. $P(X(50) \geq 240)$.