STA 3021: Stochastic Processes
Midterm 2 (6:15 PM - 7:30 PM on Nov 1, 2021)

Instructions:

- This test is a closed book exam, but you are allowed to use calculator. Clarity of your answer will also be a part of credit. When needed, use the notation $\Phi(z) = P(Z < z)$ for a standard normal distribution Z. Show your ALL work neatly.
- Your answer sheets must be written in English.
- Remind that you can submit your answer sheets over icampus in a **pdf** file format ONLY.
- By submitting your report online, it is assumed that you agree with the following pledge;

  Pledge: *I have neither given nor received any unauthorized aid during this exam.*
- Don't forget to write down your name and student ID on your answer sheet.

1. (10 points) Let X be exponential with mean $1/\lambda$ with density

$$f_X(x) = \lambda e^{-\lambda x}, x > 0.$$

Find $E(X|X > 1)$.

2. (10 points) Let Y be a Gamma random variable with parameters (s, α) with density

$$f_Y(y) = Ce^{-\alpha y}y^{s-1}, \quad y > 0,$$

where C is a constant does not depend on y. Suppose also the conditional distribution of X given $Y = y$ is Poisson with mean y. That is,

$$P(X = i|Y = y) = \frac{e^{-y}y^i}{i!}, \quad i \geq 0.$$

Find the conditional distribution of Y given $X = i$.

3. (10 points) For the transition probability matrix with state space $E = \{1, 2, 3, 4\}$, do a complete classification of states, that is, identify communicating classes, periodic/aperiodic, postive/null recurrent or transient.

$$P_1 = \begin{pmatrix} 0 & 0 & .4 & .6 & 0 \\ 0 & .2 & 0 & .5 & .3 \\ .5 & 0 & .5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ .3 & 0 & .5 & 0 & .2 \end{pmatrix}.$$

4. (15 points) Capa plays either one or two chess games everyday, with the number of games that she plays on successive days being a Markov Chain with transition probabilities

$$P_{1,1} = .2, \quad P_{1,2} = .8, \quad P_{2,1} = .4, \quad P_{2,2} = .6.$$

Capa wins each game with proabibility p. Suppose she plays two games on Monday.

(a) What is the probability that she wins all the games she plays on Tuesday?

(b) What is the expected number of games that she plays on Wednesday?

(c) In the long run, on what proportion of days does Capa win all her games?

5. (10 points) Sociologists often assume that the social classes of successive generations in a family can be regarded as a Markov chain. Thus, the occupation of a son is assumed to depend only on his father's occupation and not on his grandfather's. Suppose that such a model is appropriate with state space $E = \{Lower, Middle, Upper\}$ and that the transition probability matrix is given by

$$\begin{pmatrix} .4 & .4 & .2 \\ .05 & .7 & .25 \\ .05 & .5 & .45 \end{pmatrix}$$

For such a population, what fraction of people are middle class in the long run?

6. (20 points) A total of 4 white and 4 black balls are distributed among two urns, with each urn containing exactly 4 balls. At each stage, a ball is randomly selected from each urn and two selected balls are interchanged. Let X_n denote the number of black balls in run 1 after the nth interchange.

(a) Give the transition probabilities of the Markov Chain $X_n, n \geq 0$.

(b) Find the limiting probabilities.

7. (25 points) Let $\{X_n, n \geq 0\}$ be a DTMC with the state space $S = \{1, 2, 3, 4\}$ and following transition probability matrix

$$P = \begin{pmatrix} .4 & .3 & .2 & .1 \\ .5 & 0 & 0 & .5 \\ .5 & .0 & 0 & .5 \\ .1 & .1 & .4 & .4 \end{pmatrix}.$$

Suppose the initial distribution is given by $P(X_0 = 1) = 1$. Compute

(a) $P(X_2 = 3)$

(b) $P(X_1 = 2, X_2 = 4, X_3 = 1)$

(c) $P(X_1 = 2 | X_2 = 4, X_3 = 1)$

(d) $P(X_7 = 3 | X_5 = 4, X_3 = 2)$

(e) $E(X_3^2)$

1. $X \sim \exp\left(\frac{1}{\lambda}\right) \Rightarrow f_X(x) = \lambda e^{-\lambda x}, \quad x > 0.$

$\mathbb{E}(X \mid X > 1)$?

$$f_{X|X>1}(x) = \frac{P(X=x,\ X>1)}{P(X>1)}$$

$$= \frac{P(X=x)}{P(X>1)}, \quad x > 1$$

$$= \frac{\lambda e^{-\lambda x}}{\int_1^\infty \lambda e^{-\lambda x}\,dx}, \quad x > 1$$

$$= \frac{\lambda e^{-\lambda x}}{e^{-\lambda}} = \lambda e^{-\lambda(x-1)}, \quad x > 1$$

$$\mathbb{E}(X \mid X > 1) = \int_1^\infty x \cdot f_{X|X>1}(x)\,dx$$

$$= \int_1^\infty x \lambda e^{-\lambda(x-1)}\,dx$$

$$= 1 + \frac{1}{\lambda}$$

2. $Y \sim \text{Gamma}(s, \alpha)$

$$\Rightarrow f_Y(y) = Ce^{-\alpha y} y^{s-1}, \quad y > 0 \quad \text{where } C \text{ is constant.}$$

$$P(X = i \mid Y = y) = \frac{e^{-y} y^i}{i!}, \quad i \geq 0$$

$f(y|i)$?

$$f(y|i) = \frac{f(y,i)}{f(i)}$$

$$f(y,i) = f(i|y)\, f(y) = Ce^{-\alpha y} y^{s-1} \times \frac{e^{-y} y^i}{i!}$$

$$f(i) = \int_0^\infty f(y,i)\, dy$$

$$= \int_0^\infty Ce^{-\alpha y} y^{s-1} \frac{e^{-y} y^i}{i!}\, dy \qquad C = \frac{\alpha^s}{\Gamma(s)}$$

$$= \frac{C}{i!} \int_0^\infty e^{-(\alpha+1)y}\, y^{(s+i)-1}\, dy$$

$$= \frac{C}{i!} \cdot \frac{\Gamma(s+i)}{(1+\alpha)^{s+i}}$$

$$f(y|i) = \frac{\cancel{C} e^{\alpha y} y^{s-1} \cdot \frac{e^{-y} y^i}{\cancel{i!}}}{\frac{\cancel{C} \cdot \Gamma(s+i)}{\cancel{i!}\,(1+\alpha)^{s+i}}} = \frac{(1+\alpha)^{s+i}}{\Gamma(s+i)} e^{-(1+\alpha)y}\, y^{(s+i-1)}$$

$$\therefore \quad Y \mid X = i \sim \text{Gamma}(s+i, \alpha+1)$$

3. state space $E = \{1, 2, 3, 4, 5\}$

$$\mathbb{P}_1 = \begin{pmatrix} 0 & 0 & 0.4 & 0.6 & 0 \\ 0 & 0.2 & 0 & 0.5 & 0.3 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0.3 & 0 & 0.5 & 0 & 0.2 \end{pmatrix}$$

$C_1 = \{1, 3, 4\}$: closed / aperiodic / positive recurrent

$C_2 = \{2\}$: open / aperiodic / transient

$C_3 = \{5\}$: open / aperiodic / transient

4. Transition probability $\mathbb{P} = \begin{bmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{bmatrix}$

State space $E = \{1, 2\}$: # of games

Let {월, 화, 수, 목, 금} be $\{X_{MO}, X_{TU}, X_{WE}, X_{TH}, X_{FR}\}$

(a) $P(X_{TU}=1 \mid X_{MO}=2) \times p + P(X_{TU}=2 \mid X_{MO}=2) \times p^2$

$= 0.4p + 0.6p^2$

(b) $\mathbb{P}^2 = \begin{bmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{bmatrix} \begin{bmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.36 & 0.64 \\ 0.32 & 0.68 \end{bmatrix}$

$P(X_{WE}=1 \mid X_{MO}=2) = p_{21}^{(2)} = 0.32$

$P(X_{WE}=2 \mid X_{MO}=2) = p_{22}^{(2)} = 0.68$

$\mathbb{E}(X_{WE}=2) = 1 \times 0.32 + 2 \times 0.68 = 1.68$

(c) $\pi = \pi \mathbb{P}$, where $\pi = \{\pi_1, \pi_2\}$

$(\pi_1 \ \pi_2) = (\pi_1 \ \pi_2)\begin{pmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{pmatrix} = (0.2\pi_1 + 0.4\pi_2, \ 0.8\pi_1 + 0.6\pi_2)$

$$\left.\begin{aligned} \pi_1 &= 0.2\pi_1 + 0.4\pi_2 \\ \pi_2 &= 0.8\pi_1 + 0.6\pi_2 \\ \pi_1 + \pi_2 &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} &(\pi_1, \pi_2) = \left(\tfrac{1}{3}, \tfrac{2}{3}\right) \\ &\therefore \tfrac{1}{3}p + \tfrac{2}{3}p^2 \end{aligned}$$

5. $$\mathbb{P} = \begin{pmatrix} 0.4 & 0.4 & 0.2 \\ 0.05 & 0.7 & 0.25 \\ 0.05 & 0.5 & 0.45 \end{pmatrix}$$

$$\pi = (\pi_L \ \ \pi_M \ \ \pi_u)$$

$\pi \mathbb{P} = \pi$ $\quad \because$ positive recurrent & aperiodic

$$\Rightarrow \left.\begin{aligned} \pi_L &= 0.4\pi_L + 0.05\pi_M + 0.05\pi_u \\ \pi_M &= 0.4\pi_L + 0.7\pi_M + 0.5\pi_u \\ \pi_u &= 0.2\pi_L + 0.25\pi_M + 0.45\pi_u \\ \pi_L &+ \pi_M + \pi_u = 1 \end{aligned}\right\}$$

$$\Rightarrow \pi_L = \frac{1}{13}, \quad \pi_M = \frac{8}{13}, \quad \pi_u = \frac{4}{13}$$

$$\therefore \pi_M = \frac{8}{13}$$

6. State space $E = \{0, 1, 2, 3, 4\}$: state of # of black

(a)

$$
\mathbb{P} = \begin{array}{c c}
 & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} &
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
\frac{1}{16} & \frac{6}{16} & \frac{9}{16} & 0 & 0 \\
0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\
0 & 0 & \frac{9}{16} & \frac{6}{16} & \frac{1}{16} \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\end{array}
$$

(b)

$$
\begin{tikzcd}
0 \arrow[r, shift left] & 1 \arrow[l, shift left] \arrow[r, shift left] \arrow[loop above] & 2 \arrow[l, shift left] \arrow[r, shift left] \arrow[loop above] & 3 \arrow[l, shift left] \arrow[r, shift left] \arrow[loop above] & 4 \arrow[l, shift left]
\end{tikzcd}
$$

: positive recurrent, aperiodic

$\Rightarrow \pi = \pi \mathbb{P}$, $\Sigma \pi_j = 1$ where $\pi = \{\pi_0, \pi_1, \pi_2, \pi_3, \pi_4\}$.

$$
\left.
\begin{aligned}
\pi_0 &= \frac{1}{16}\pi_1 \\
\pi_1 &= \pi_0 + \frac{6}{16}\pi_1 + \frac{1}{4}\pi_2 \\
\pi_2 &= \frac{9}{16}\pi_1 + \frac{1}{2}\pi_2 + \frac{9}{16}\pi_3 \\
\pi_3 &= \frac{1}{4}\pi_2 + \frac{6}{16}\pi_3 + \pi_4 \\
\pi_4 &= \frac{6}{16}\pi_3 \\
\pi_0 &+ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1
\end{aligned}
\right\}
\Rightarrow
\begin{aligned}
\pi_0 &= \frac{1}{70} \\
\pi_1 &= \frac{8}{35} \\
\pi_2 &= \frac{18}{35} \\
\pi_3 &= \frac{8}{35} \\
\pi_4 &= \frac{1}{70}
\end{aligned}
$$

7.

$$\mathbb{P} = \begin{pmatrix} 0.4 & 0.3 & 0.2 & 0.1 \\ 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 \\ 0.1 & 0.1 & 0.4 & 0.4 \end{pmatrix}, \quad P(X_0=1)=1$$

(a) $P(X_2 = 3) = P(X_2=3 \mid X_0=1)\, P(X_0=1)$

$$= P_{13}^{(2)} \times 1 = 0.12$$

$$\mathbb{P}^2 = \begin{bmatrix} 0.42 & 0.13 & 0.12 & 0.33 \\ 0.25 & 0.2 & 0.3 & 0.25 \\ 0.25 & 0.2 & 0.3 & 0.25 \\ 0.33 & 0.07 & 0.18 & 0.42 \end{bmatrix}$$

(b) $P(X_1 = 2, X_2 = 4, X_3 = 1)$

$$= P(X_3=1 \mid X_2=4)\, P(X_2=4 \mid X_1=2)\, P(X_1=2 \mid X_0=1)$$

$$= P_{41} \times P_{24} \times P_{12} = 0.1 \times 0.5 \times 0.3 = 0.015$$

(c) $P(X_1 = 2 \mid X_2 = 4, X_3 = 1)$

$$= \frac{P(X_1=2, X_2=4, X_3=1)}{P(X_2=4, X_3=1)} = \frac{15}{33} = \frac{5}{11}$$

$\because P(X_2=4, X_3=1) = P(X_3=1 \mid X_2=4)\, P(X_2=4 \mid X_0=1)$

$$= P_{41} \times P_{14}^{(2)} = 0.1 \times 0.33 = 0.033$$

(d) $P(X_7 = 3 \mid X_5 = 4, X_3 = 2)$

$$= \frac{P(X_7=3, X_5=4, X_3=2)}{P(X_5=4, X_3=2)}$$

$$= \frac{P(X_7=3 \mid X_5=4)\,P(X_5=4 \mid X_3=2)\,P(X_3=2 \mid X_0=1)}{P(X_5=4 \mid X_3=2)\,P(X_3=2 \mid X_0=1)}$$

$$= P(X_7=3 \mid X_5=4) = p_{43}^{(2)} = 0.18$$

(e) $\mathbb{E}(X_3^2) = 1^2 \times 0.326 + 2^2 \times 0.159 + 3^2 \times 0.216 + 4^2 \times 0.299$

$= 7.69$

$$\mathbb{P}^3 = \begin{pmatrix} 0.326 & 0.159 & 0.216 & 0.299 \\ & \vdots & & \\ & & & \end{pmatrix}$$