Empirical Bayes

$\underset{\sim}{y} \mid \underset{\sim}{\theta} \sim N(\underset{\sim}{\theta}, \sigma^2 I)$, $\quad \underset{\sim}{\theta} \sim N(\mu \underset{\sim}{1}, \tau^2 I)$.

$$p(\underset{\sim}{\theta} \mid \underset{\sim}{y}) \propto p(y \mid \theta)\, p(\theta)$$

$$\propto \exp\left\{-\frac{1}{2}(\underset{\sim}{y}-\theta)^T \frac{1}{\sigma^2} I \,(\underset{\sim}{y}-\underset{\sim}{\theta})\right\} \cdot \exp\left\{-\frac{1}{2}(\underset{\sim}{\theta}-\mu \underset{\sim}{1})^T \frac{1}{\tau^2} I\,(\underset{\sim}{\theta}-\mu \underset{\sim}{1})\right\}$$

$$= \exp\left\{-\frac{1}{2\sigma^2}\left(\underset{\sim}{\theta}^T \underset{\sim}{\theta} - 2\underset{\sim}{y}^T \underset{\sim}{\theta} + \underset{\sim}{y}^T \underset{\sim}{y}\right) - \frac{1}{2\tau^2}\left(\underset{\sim}{\theta}^T \underset{\sim}{\theta} - 2\mu \underset{\sim}{1}^T \underset{\sim}{\theta} + \mu^2 \underset{\sim}{1}^T \underset{\sim}{1}\right)\right\}$$

$$\propto \exp\left[-\frac{1}{2}\left\{\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)\underset{\sim}{\theta}^T \underset{\sim}{\theta} - 2\left(\frac{1}{\sigma^2}\underset{\sim}{y} + \frac{1}{\tau^2}\mu \underset{\sim}{1}\right)^T \underset{\sim}{\theta}\right\}\right]$$

$$\propto \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)\left(\underset{\sim}{\theta} - \left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)^{-1}\left(\frac{1}{\sigma^2}\underset{\sim}{y} + \frac{1}{\tau^2}\mu \underset{\sim}{1}\right)\right)^T\right.$$

$$\left.\left(\underset{\sim}{\theta} - \left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)^{-1}\left(\frac{1}{\sigma^2}\underset{\sim}{y} + \frac{1}{\tau^2}\mu \underset{\sim}{1}\right)\right)^T\right\}$$

kernal of multivariate normal dist. with

mean $\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)^{-1}\left(\frac{1}{\sigma^2}\underset{\sim}{y} + \frac{1}{\tau^2}\mu \underset{\sim}{1}\right)$

and variance $\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)^{-1} I_n$

Since

$$\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right)^{-1} I_n = \frac{\sigma^2\tau^2}{\sigma^2+\tau^2} I_n = \sigma^2(1-B)\, I_n,$$

$$E(\underset{\sim}{\theta} \mid \underset{\sim}{y}) = \sigma^2(1-B)\, I_n \left(\frac{1}{\sigma^2}\underset{\sim}{y} + \frac{1}{\tau^2}\mu \underset{\sim}{1}_n\right)$$

$$= (1-B)\underset{\sim}{y} + \frac{\sigma^2}{\tau^2}\underbrace{(1-B)}_{\frac{\tau^2}{\sigma^2+\tau^2}}\mu \underset{\sim}{1}_n$$

$$= (1-B)\underset{\sim}{y} + B\mu \underset{\sim}{1}_n$$

$$Var(\underset{\sim}{\theta} \mid \underset{\sim}{y}) = \sigma^2(1-B)\, I_n.$$