Chapter 6.   The Completeness Property

So far we learned essentially the **two** methods for finding (or proving) the limit of a sequence:

1st one: Squeeze Principle: can be applied to sequences whose good upper & lower sequences are expected

2nd one: Completeness Property: can be applied to sequences that are monotone (or monotone for $n \gg 1$)

Goal of this chapter is to give some new methods that can be used to construct or prove the existence of a limit.

More precisely, we will give "NIT (= Nested Intervals Theorem)" and "Cauchy criterion for convergence";

(each is equivalent to the completeness of $\mathbb{R}$; well-known to experts)

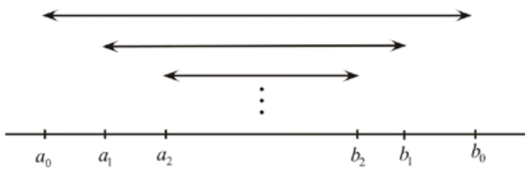
Main tools for describing those two methods: "LLT" & "the notion of convergence"

6.1 Nested intervals

Def. If a sequence $\left([a_n, b_n]\right)_{n=0}^{\infty}$ of closed intervals has the property that

$$[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n] \supset \cdots,$$

we say that the sequence $\left([a_n, b_n]\right)_{n=0}^{\infty}$ is nested.



Theorem (The Nested Intervals Theorem (for short NIT): 축소 (폐)구간열 정리)

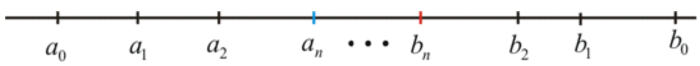
Suppose sequence $\left([a_n, b_n]\right)_{n=0}^{\infty}$ is a sequence of nested intervals & $\lim\limits_{n\to\infty}(b_n - a_n) = 0.$

Then $\bigcap\limits_{n=0}^{\infty}[a_n, b_n]$ consists of exactly one point.

Moreover, $\exists$ a real number L such that $\bigcap\limits_{n=0}^{\infty}[a_n, b_n] = \{L\}$ & $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} b_n$

Pf. $\left([a_n, b_n]\right)_{n=0}^{\infty}$ is nested: $[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n] \supset \cdots$

$\Rightarrow$ picture:



Hence it is clear that (a_n) is $\uparrow$ & bounded above by b_0.

By the Completeness Property, $\lim\limits_{n\to\infty} a_n$ $(\overset{\text{let}}{=} L)$ exists.

Since (a_n) is $\uparrow$, we get $a_n \leq L$ for all n ------ (*)

On the other hand, for any fixed n

$$a_k \leq b_k \leq b_n \quad \text{if} \quad k \geq n \quad (\leftarrow b_n \downarrow)$$

&

$$a_k \leq a_n \leq b_n \quad \text{if} \quad k \leq n \quad (\leftarrow a_n \uparrow)$$

Thus we have $a_k \leq b_n$ for all k.

So by LLT, $L = \lim_{k\to\infty} a_k \le b_n$ ------ (**)

(*) & (**) implies $a_n \le L \le b_n$ for all n

$$\therefore \quad \bigcap_{n=0}^{\infty}[a_n, b_n] \ni L$$

Claim: $\bigcap_{n=0}^{\infty}[a_n, b_n] = \{L\}$

Pf of Claim: If $M \in \bigcap_{n=0}^{\infty}[a_n, b_n]$, then $a_n \le M,\ L \le b_n$ for all n.

$$\Rightarrow \quad |L - M| \le (b_n - a_n) \quad \text{for all } n$$

$$\Rightarrow \quad |L - M| \le \lim_{n\to\infty}(b_n - a_n) = 0 \quad (\text{by LLT})$$

$$\therefore \quad M = L$$

Remains to show $\lim_{n\to\infty} b_n = L$; but it is obvious since

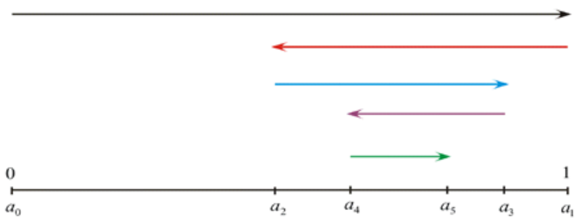
$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(b_n - a_n + a_n\right) = \lim_{n\to\infty}\left(b_n - a_n\right) + \lim_{n\to\infty} a_n = 0 + L = L$$

Exa. (An application of NIT)

Let $\begin{cases} a_n = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \cdots + (-1)^{n-1}\dfrac{1}{n} & \text{for } n \ge 1 \\ a_0 = 0 \end{cases}$. Show that (a_n) converges.

[We have already proved that (a_n) converges to $\ln 2$ by using error-term analysis. Only give an elementary pf of just its convergence]

Sol.



From the picture, we see that

$$[a_0, a_1], \quad [a_2, a_3], \quad [a_4, a_5], \ \cdots \ [a_{2k}, a_{2k+1}], \quad [a_{2k+2}, a_{2k+3}], \ \cdots$$

is a sequence of nested intervals.
It is also clear that

$$\left|a_{2k+1} - a_{2k}\right| = \frac{1}{2k+1} \quad \to \quad 0$$

Thus by NIT, $\exists$ a real number L such that

$$\lim_{k\to\infty} a_{2k} = L = \lim_{k\to\infty} a_{2k+1}$$

Moreover, $a_{2k} \le L \le a_{2k+1}$ for all k

It follows that

$$\left|a_n - L\right| = \begin{cases} L - a_{2k} & \text{if} \quad n = \text{even} = 2k \\ a_{2k+1} - L & \text{if} \quad n = \text{odd} = 2k+1 \end{cases}$$

$$\le a_{2k+1} - a_{2k} = \frac{1}{2k+1}$$

So, $\left|a_{2k} - L\right| \le \dfrac{1}{2k+1} < \dfrac{1}{2k}$ & $\left|a_{2k+1} - L\right| \le \dfrac{1}{2k+1}$

Consequently, $\left|a_n - L\right| \le \dfrac{1}{n}$ regardless of whether n is even or odd

$$\therefore \quad \lim_{n\to\infty} a_n = L$$

Ex. (A modification of NIT; it is also called the NIT)
Let $I_n = [a_n, b_n]$ for $n = 0, 1, 2, \cdots$.

If $I_0 \supset I_1 \supset I_2 \supset \cdots$ (i.e., $\left(I_n\right)_1^\infty$ is nested), then

$$\bigcap_{n=0}^{\infty} I_n = [L,\ M], \text{ where } L = \lim_{n\to\infty} a_n \quad \& \quad M = \lim_{n\to\infty} b_n \quad (\&\ L \le M)$$

⊙ Archimedian Property (for short, AP)

$0 < \underbrace{a}_{\text{small}} < \underbrace{b}_{\text{big}} \quad \Rightarrow \quad \exists$ a natural number n_0 such that $n_0 a > b$

Pf. Suppose the conclusion were false; i.e., suppose $an \le b$ for every $n \in \mathbb{N}$.

Then the sequence

$a, \quad 2a, \quad 3a, \quad \cdots, \quad na, \quad \cdots$ is strictly $\uparrow$ & bounded above (by b)

$\overset{\text{Completeness Property}}{\Rightarrow} \quad \lim_{n\to\infty} na$ exists, call it L.

So, for given $\varepsilon > 0$, $\left|na - L\right| < \varepsilon$ for $n \gg 1$ (say, for $n \ge N$)

In particular, $\left|(N+1)a - L\right| < \varepsilon$ & $\left|Na - L\right| < \varepsilon$

$$\therefore \quad \left|(N+1)a - Na\right| < 2\varepsilon$$

i.e., $|\,a\,| < 2\varepsilon$ for any $\varepsilon > 0$ $\quad \therefore \quad a = 0;$ contradiction to $a > 0$

Key idea: $\lim_{n\to\infty} na = L$ [assume] $\Rightarrow \lim_{n\to\infty}(n+1)a = L$

$\Rightarrow \quad 0 = L - L = \lim_{n\to\infty}(n+1)a - \lim_{n\to\infty} na = \lim_{n\to\infty}\big((n+1)a - na\big) = \lim_{n\to\infty} a = a$: contrdicts $a > 0$

Note.

(1) Let $I_n = (0,\ 1/n)$ for $n = 1, 2, \cdots$. Then it is clear that

$I_1 \supset I_2 \supset I_3 \supset \cdots$ and $\ell(I_n) = 1/n \to 0$ as $n \to \infty$;

but $\bigcap_{n=1}^{\infty} I_n = \varnothing$

Pf. If $x > 0$, then by AP

$\exists\ n_0 \in \mathbb{N}$ such that $0 < 1/n_0 < x$

So $x \notin I_{n_0} = (0,\ 1/n_0)$, and thus $x \notin \bigcap_{n=1}^{\infty} I_n$

$$\therefore \quad \bigcap_{n=1}^{\infty} I_n = \varnothing$$

(2) Let $I_n = [n,\ \infty)$ for $n = 1, 2, \cdots$. Then it is clear that

$I_1 \supset I_2 \supset I_3 \supset \cdots$;

but $\bigcap_{n=1}^{\infty} I_n = \varnothing$

Pf. If $x > 0$, then by AP

$$\exists\ n_0 \in \mathbb{N} \text{ such that } n_0 > x$$

So $x \notin I_{n_0} = [n_0, \infty)$, and thus $x \notin \bigcap_{n=1}^{\infty} I_n$ $\quad \therefore \bigcap_{n=1}^{\infty} I_n = \varnothing$

(3) Let $I_n = (-1/n,\ 1/n)$ for $n = 1, 2, \cdots$. Then it is clear that

$$I_1 \supset I_2 \supset I_3 \supset \cdots \quad \text{and} \quad \ell(I_n) = 2/n \ \to\ 0 \text{ as } n \to \infty;$$

but $\bigcap_{n=1}^{\infty} I_n = \{0\}$

Pf.

$$I_n' \equiv [-1/2n,\ 1/2n] \subset I_n = (-1/n,\ 1/n) \subset I_n'' \equiv [-1/n,\ 1/n]$$

Each of $(I_n')_{n=1}^{\infty}$ & $(I_n'')_{n=1}^{\infty}$ is nested, and

$$\ell(I_n') = 1/n \to 0 \quad \& \quad \ell(I_n'') = 2/n \to 0$$

Thus by NIT

$\bigcap_{n=1}^{\infty} I_n'$ & $\bigcap_{n=1}^{\infty} I_n''$ consists of a single point, respectively

Since $0 \in \bigcap_{n=1}^{\infty} I_n'$ & $0 \in \bigcap_{n=1}^{\infty} I_n''$, we get

$$\bigcap_{n=1}^{\infty} I_n' = \{0\} = \bigcap_{n=1}^{\infty} I_n''$$

$$\therefore \bigcap_{n=1}^{\infty} I_n = \{0\}.$$

6.2 Cluster points of sequences

Def. K is called a cluster point (집적점) of the sequence (a_n) if

given $\varepsilon > 0$, $a_n \underset{\varepsilon}{\approx} K$ for infinitely many n.

Recall L is the limit of the sequence (a_n) if

given $\varepsilon > 0$, $a_n \underset{\varepsilon}{\approx} L$ for $n \gg 1$ (i.e., for all but finitely many n).

Trivial fact: If L is the limit of the seq (a_n), then

L is (automatically) a cluster point.

But there are cluster points which are not limits

Exa A.

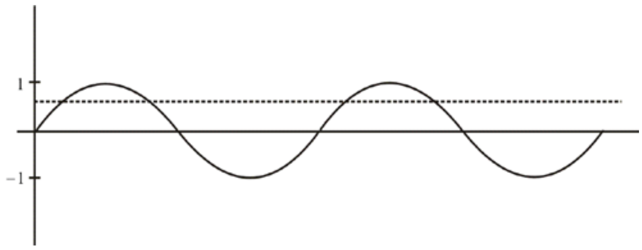
(a) $1,\ 1,\ 2,\ 1,\ 2,\ 3,\ 1,\ 2,\ 3,\ 4,\ 1,\ 2,\ 3,\ 4,\ 5, \cdots:$

Every positive integer is a cluster point, but the sequence has no limit.

(b) $1,\ \frac{1}{2},\ 1,\ \frac{1}{3},\ 1,\ \frac{1}{4}, \cdots:$

1 & 0 are cluster points, but the sequence has no limit.

(c) $\sin 1, \ \sin 2, \ \sin 3, \ \cdots\cdots, \sin n, \cdots$:



Every real number in $[-1, 1]$ is a cluster point of this sequence

(Its proof is very difficult)

But the sequence has no limit (already proved)

(d) $1, \quad 2, \quad 3, \quad 4, \ \cdots$ has no cluster point.

Note: Other names for cluster point are "**accumulation point**" or "**limit point**"

Caution: **limit point** $\neq$ **limit**

※Theorem (Cluster point theorem)

K is a cluster point of the sequence (a_n)

$\Leftrightarrow$ K is the limit of some subsequence (a_{n_i})

Pf. $\Leftarrow$ $(\text{easy}):$ given $\varepsilon > 0$, $a_{n_i} \underset{\varepsilon}{\approx} K$ for $i \gg 1$, say, for all $i \geq I$

So, $a_{n_I}, \quad a_{n_{I+1}}, \quad a_{n_{I+2}}, \ \cdots \underset{\varepsilon}{\approx} K$

$\therefore \quad a_n \underset{\varepsilon}{\approx} K$ for infinitely many n

$\therefore \quad K$ is a cluster point of (a_n)

$\Rightarrow$: By hypo, we can choose a_{n_1} so that

$$a_{n_1} \underset{1}{\approx} K$$

By hypo again, we can choose a_{n_2} so that

$$a_{n_2} \underset{\frac{1}{2}}{\approx} K \quad \text{and} \quad n_2 > n_1$$

By the same way, we can choose a_{n_3} so that

$$a_{n_3} \underset{\frac{1}{3}}{\approx} K \quad \text{and} \quad n_3 > n_2$$

$$\vdots$$

$$a_{n_i} \underset{1/i}{\approx} K \quad (\forall i) \quad \text{and} \quad n_i > n_{i-1}$$

Since $n_i > n_{i-1}$, (a_{n_i}) forms a subsequence of (a_n).

Moreover,

$$\text{given } \varepsilon > 0, \quad a_{n_i} \underset{\varepsilon}{\approx} K \quad \text{for } \frac{1}{i} < \varepsilon, \quad \text{i.e., for } i > \frac{1}{\varepsilon}$$

$\therefore \quad K = \lim_{i\to\infty} a_{n_i}$ i.e., K is the limit of the subsequence (a_{n_i})

Exa B. Let $a_n = \dfrac{1}{n} + (-1)^n$. Show (a_n) has -1 & 1 as cluster points, but no limit

Pf.

$$a_{2k+1} = \frac{1}{2k+1} - 1 \quad \to \quad -1 \quad \text{as } k \to \infty$$

$$a_{2k} = \frac{1}{2k} + 1 \quad \to \quad 1 \quad \text{as } k \to \infty$$

$\therefore$ -1 & 1 are cluster points (by the Cluster point theorem)

Since $\lim\limits_{k\to\infty} a_{2k+1} \neq \lim\limits_{k\to\infty} a_{2k}$, $\lim\limits_{n\to\infty} a_n$ does not exist.

Exa1. Find the cluster points of $\left(\sin \dfrac{n\pi}{2}\right)_0^\infty$.

Sol. $\left(\sin \dfrac{n\pi}{2}\right)_0^\infty$: $0,\ 1,\ 0,\ -1;\ 0,\ 1,\ 0,\ -1;\ \cdots$

$\therefore$ the cluster points are 0, 1, -1

Exa2. Prove that if a sequence is convergent, it has only one cluster point.

Pf. Say $a_n \to L$. Then L is a cluster point.

If K is also a cluster point, then by the **Cluster point theorem**, $\exists$ a subsequence (a_{n_i}) such that

$\lim\limits_{i\to\infty} a_{n_i} = K$.

But since $\lim\limits_{n\to\infty} a_n = L$, we have $\lim\limits_{i\to\infty} a_{n_i} = L$ by the **Subsequence Theorem**.

Hence $K = L$

Exa3. Find a sequence that having only one cluster point, yet not convergent.

Sol. $1,\ 2,\ 1,\ 3,\ 1,\ 4,\ \cdots$

Its cluster point is 1, but clearly it has no limit.

6.3 The Bolzano-Weierstrass theorem

Sequences in general do not converge, but they often have subsequences which converge.

Question: What kind of sequence has a convergent subsequence?

※※ Theorem (Bolzano-Weierstrass Theorem: **BWT** for short)

If (x_n) is a bounded sequence, then it has a convergent subsequence.

Pf. **key idea**: the Method of Bisection plus NIT

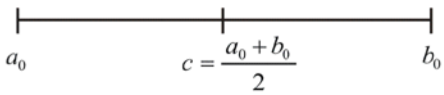
By the Cluster point theorem, it suffices to show that the bounded sequence (x_n) has a cluster point.

Since (x_n) is bounded, there are points a_0 and b_0 such that

$$a_0 \le x_n \le b_0 \quad \text{for all } n$$

We set $\text{length}[a_0,\ b_0] = d$

We can assume $d > 0$, otherwise, (x_n) is constant ($\Rightarrow$ OK)



At least one of the half-intervals $[a_0,\ c]$ & $[c,\ b_0]$ contains infinitely many x_n .

Call this half-interval $[a_1,\ b_1]$ (; if both do, use the left-hand one)

We then have

$$[a_0,\ b_0] \supset [a_1,\ b_1], \qquad \text{length}[a_1,\ b_1] = \frac{d}{2}$$

$$[a_1,\ b_1] \ \text{contains infinitely many } x_n$$

Similiarly, by dividing $[a_1,\ b_1]$ in half, we get an $[a_2,\ b_2]$ such that

$$[a_1,\ b_1] \supset [a_2,\ b_2], \qquad \text{length}[a_2,\ b_2] = \frac{d}{2^2}$$

$$[a_2,\ b_2] \ \text{contains infinitely many } x_n$$

Continuing, we get a sequence of nested intervals such that

$$[a_0,\ b_0] \supset [a_1,\ b_1] \supset [a_2,\ b_2] \supset \cdots \supset [a_n,\ b_n] \supset \cdots$$

$$\text{length}[a_n,\ b_n] = \frac{d}{2^n}$$

$$[a_n,\ b_n] \ \text{contains infinitely many } x_n$$

Since $\dfrac{d}{2^n} \to 0$ as $n \to \infty$, by NIT

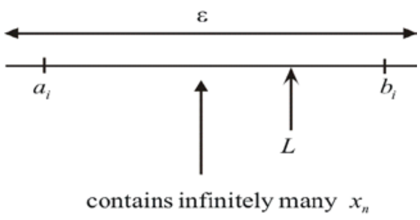
$\exists$ a unique point L such that $\bigcap_{n=1}^{\infty} [a_n,\ b_n] = \{L\}$

Claim: L is a cluster point of (x_n)

Pf of claim: Given $\varepsilon > 0$, choose an i so that $\text{length}[a_i, \ b_i] = \dfrac{d}{2^i} < \varepsilon$

Since $[a_i, \ b_i]$ contains an infinitely many x_n & $a_i \le L \le b_i$, we get

$$x_n \underset{\varepsilon}{\approx} L \quad \text{for infinitely many } x_n$$



$\therefore$ L is a cluster point of (x_n)

Summary of the pf (without using Cluster point theorem):

(x_n) is bounded $\Rightarrow$ all $x_n \in [a_0, b_0] =: I_0$

Bisect I_0 into two equal halves.

Choose I_1 to be one of the two equal halves of I_0 containing infinitely many number of terms of x_n;

and take $x_{n_1} \in I_1$.

Choose I_2 to be one of the two equal halves of I_1 containing infinitely many number of terms of x_n;

and take $x_{n_2} \in I_2$ with $n_2 > n_1$.

Continuing in this fashion, we obtain, for every index $i \in \mathbb{N}$,

an interval $I_i =: [a_i, b_i]$ & a point $x_{n_i} \in I_i$ with $n_i > n_{i-1}$ and $I_{i-1} \supset I_i$

Note that $\text{length}\,(I_i) = b_i - a_i = \dfrac{\ell(I_0)}{2^i} = \dfrac{b_0 - a_0}{2^i} \to 0$ as $i \to \infty$.

Thus by NIT, $\exists$ a unique point L such that $\bigcap_{i=1}^{\infty} I_i = \{L\}$

Notice that L & $x_{n_i} \in I_i$. It follows that

$$\left| x_{n_i} - L \right| \le b_i - a_i \to 0 \quad \text{as } i \to \infty$$

$\therefore$ $\lim\limits_{i \to \infty} x_{n_i} = L$ (i.e., (x_{n_i}) is a convergent subsequence of (x_n))

Exa. Let $(a_n)_{n=0}^{\infty} = \{\sin(n^2 + n + 1)\}_{n=0}^{\infty}$; $\quad (b_n)_0^{\infty} = \{e^{\sin n}\}_{n=0}^{\infty}$.

Then it is clear that (a_n) & (b_n) are bounded sequences.

$\therefore \overset{\text{BWT}}{\Longrightarrow}$ each has a convergent subsequence.

6.4 Cauchy sequences

Def. We say that the sequence (a_n) is a Cauchy sequence if,

$$\text{given } \varepsilon > 0, \quad a_m \underset{\varepsilon \text{ or } K\varepsilon}{\approx} a_n \quad \text{for } m,\ n \gg 1 \quad (\text{or for } m > n \gg 1)$$

i.e., given $\varepsilon > 0$, $\exists$ a number $N(= N(\varepsilon))$ such that

$$a_m \underset{\varepsilon \text{ or } K\varepsilon}{\approx} a_n \quad \text{for all } m,\ n \ge N \quad (\text{or for all } m > n \ge N)$$

Exa. $a_n = \dfrac{3n+1}{n+2}$ Show (a_n) is Cauchy.

Pf. Let $\varepsilon > 0$ be given. Then

$$\begin{aligned}|a_m - a_n| &= \left|\frac{3m+1}{m+2} - \frac{3n+1}{n+2}\right| = \left|\frac{5(m-n)}{(m+2)(n+2)}\right| \\ &\le \frac{5m}{(m+2)(n+2)} + \frac{5n}{(m+2)(n+2)} \\ &\le \frac{5}{n+2} + \frac{5}{m+2} < \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \text{if} \quad n+2 > \frac{10}{\varepsilon} \ \& \ m+2 > \frac{10}{\varepsilon}\end{aligned}$$

Thus

$$|a_m - a_n| < \varepsilon \quad \text{if} \quad m, n > \frac{10}{\varepsilon} - 2$$

So $a_m \underset{\varepsilon}{\approx} a_n$ for $m,\ n \gg 1$ $\therefore$ (a_n) is Cauchy

⊙ Fact (easy): (a_n) is convergent $\Rightarrow$ (a_n) is a Cauchy sequence

Pf. Suppose $\lim_{n\to\infty} a_n = L$. Then

$$\text{given } \varepsilon > 0, \quad a_n \underset{\varepsilon}{\approx} L \quad \text{for } n \gg 1$$

So, $a_m \underset{\varepsilon}{\approx} L$ & $a_n \underset{\varepsilon}{\approx} L$ for $m,\ n \gg 1$

Thus

$$a_m \underset{2\varepsilon}{\approx} a_n \quad \text{for } m,\ n \gg 1$$

$\therefore$ (a_n) is a Cauchy sequence

Question: What about the converse?

Ans is yes (Next theorem)

Theorem (The Cauchy criterion for convergence)

If (a_n) is a Cauchy sequence, then (a_n) converges.

Pf. Let (a_n) be a Cauchy sequence.

1st step: (a_n) is bounded

To prove this, take $\varepsilon = 1$. Then by the def of Cauchy sequence $\exists$ an N such that

$$a_n \underset{1}{\approx} a_m \quad \text{for all } n,\ m \geq N$$

In particular,

$$a_n \underset{1}{\approx} a_N \quad \text{for all } n \geq N \qquad \text{i.e.,} \quad a_N - 1 < a_n < a_N + 1 \quad \text{for all } n \geq N$$

This says (a_n) is bounded for $n \gg 1$

This gives (a_n) is bounded for all n

2nd step: (a_n) has a convergent subsequence (a_{n_i})

$\llbracket \because$ proved (a_n) is bounded $(\leftarrow$ 1st step)

$\overset{\text{BWT}}{\Rightarrow}$ (a_n) has a convergent subsequence; call it (a_{n_i}) $\rrbracket$

3rd step: Claim: Write $L = \lim\limits_{i\to\infty} a_{n_i}$ Then $\lim\limits_{n\to\infty} a_n = L$

To prove the Claim, let $\varepsilon > 0$ be given.

Since (a_n) is Cauchy, $\exists\, N \in \mathbb{N}$ such that

$$n, m \geq N \quad \Rightarrow \quad |a_n - a_m| < \varepsilon$$

Since $L = \lim\limits_{i\to\infty} a_{n_i}$, $\exists I \in \mathbb{N}$ such that

$$i \geq I \quad \Rightarrow \quad |a_{n_i} - L| < \varepsilon .$$

Now take an integer $i_0 \geq I$ so that $n_{i_0} \geq N$. Then

$$n \geq N \quad \Rightarrow \quad |a_n - L| < |a_n - a_{n_{i_0}}| + |a_{n_{i_0}} - L| < \varepsilon + \varepsilon = 2\varepsilon$$

$$\therefore \quad \lim_{n\to\infty} a_n = L \text{ by } K\text{-}\varepsilon \text{ principle}$$

Question: If a sequence (a_n) satisfies;

$$\text{given } \varepsilon > 0, \quad a_{n+1} \underset{\varepsilon}{\approx} a_n \quad \text{for } n \gg 1,$$

is (a_n) convergent?

Ans: No

For example, the sequence $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ satisfies;

$$\text{given } \varepsilon > 0, \quad |a_{n+1} - a_n| = \frac{1}{n+1} < \frac{1}{n} < \varepsilon \quad \text{if } n > \frac{1}{\varepsilon}$$

So, given $\varepsilon > 0$, $a_{n+1} \underset{\varepsilon}{\approx} a_n$ for $n \gg 1$.

However, we have already seen that (a_n) is divergent.

⊙ Typical examples of Cauchy sequences.

Type1.

$$|a_m - a_n| \le c_n \quad \text{for every } m, n \in \mathbb{N} \quad (\text{or } m, n \gg 1) \quad \& \quad \lim_{n\to\infty} c_n = 0$$

$\Rightarrow \quad (a_n)$ is Cauchy

Pf. $\lim_{n\to\infty} c_n = 0 \quad \Rightarrow \quad$ given $\varepsilon > 0$, $|c_n| < \varepsilon$ for $n \gg 1$

So $|a_m - a_n| \le |c_n| < \varepsilon$ for $m, n \gg 1$

i.e., given $\varepsilon > 0$, $a_m \underset{\varepsilon}{\approx} a_n$ for $m, n \gg 1$.

Exa. If $|a_m - a_n| \le \dfrac{1}{m+n}$, then show that (a_n) is Cauchy

Pf. For every $m, n \in \mathbb{N}$,

$$|a_m - a_n| \le \frac{1}{m+n} < \frac{1}{n} \qquad \& \qquad \frac{1}{n} \to 0$$

※Type2. Let (a_n) be a sequence.

If $\exists$ constants $C > 0$ and K, with $0 < K < 1$, such that

$$|a_{n+1} - a_n| \le CK^n \quad \text{for every } n \quad (\text{or } n \gg 1),$$

then (a_n) is a Cauchy sequence

Pf. Let $m > n$. Then

$$\begin{aligned}|a_m - a_n| &\le |a_n - a_{n+1}| + |a_{n+1} - a_{n+2}| + \cdots + |a_{m-1} - a_m| \\ &\le CK^n + CK^{n+1} + \cdots + CK^{m-1} \\ &< CK^n\left(1 + K + K^2 + \cdots\right) = \frac{CK^n}{1-K} \equiv c_n; \qquad c_n \to 0 \text{ since } 0 < K < 1\end{aligned}$$

Thus (a_n) is a sequence of Type1

$\therefore \quad (a_n)$ is Cauchy

Exa. If (a_n) satisfies $|a_{n+1} - a_n| \le (1/2)^n$ for every n (or $n \gg 1$), then (a_n) is Cauchy.

Def. A sequence (a_n) is said to be contractive if $\exists$ a constant K with $0 < K < 1$, such that

$$|a_{n+2} - a_{n+1}| \le K|a_{n+1} - a_n| \quad \text{for all } n$$

Type3. If (a_n) is a contractive sequence, then (a_n) is Cauchy.

Pf. By hypo, we have for every n

$$|a_{n+2} - a_{n+1}| \le K|a_{n+1} - a_n| \le K^2|a_n - a_{n-1}| \le \cdots \le K^n|a_2 - a_1|$$

$\therefore \quad (a_n)$ is a sequence of Type2 $\qquad \therefore \quad (a_n)$ is Cauchy

Exa. Recall the sequence of Fibonacci fractions is defined by

$$a_1 = 1, \qquad a_{n+1} = \frac{1}{a_n + 1} \quad \text{for } n \ge 1.$$

Using Cauchy criterion for convergence, prove that (a_n) converges, and determine its limit.

Remark. We already proved that (a_n) converges, by using an error-term analysis.

Pf. To prove (a_n) is convergent, it suffices to show (a_n) is a Cauchy sequence.

$$|a_{n+2}-a_{n+1}| = \left|\frac{1}{a_{n+1}+1}-\frac{1}{a_n+1}\right| = \frac{|a_{n+1}-a_n|}{(a_{n+1}+1)(a_n+1)}$$

$$\overset{?}{\le} \frac{1}{(1/2+1)(1/2+1)}|a_{n+1}-a_n| = \frac{2}{3}\cdot\frac{2}{3}|a_{n+1}-a_n| = \frac{4}{9}|a_{n+1}-a_n|$$

⟦ $\because$

$$a_1 = 1 \quad \Rightarrow \quad a_2 = \frac{1}{1+a_1} = \frac{1}{2} \quad \Rightarrow \quad 1 \le 1+a_2 \le 3/2$$

$$\Rightarrow \quad 2/3 \le a_3 = \frac{1}{1+a_2} \le 1$$

$$\Rightarrow \quad 1/2 \le a_3 \le 1$$

Expect: $1/2 \le a_n \le 1$ for all n

Suppose $1/2 \le a_n \le 1$ for all n. Then

$$3/2 \le a_n + 1 \le 2$$

$$\Rightarrow \quad \frac{1}{2} \le \frac{1}{a_n+1} \le \frac{2}{3}$$

$$\Rightarrow \quad \frac{1}{2} \le \frac{1}{a_n+1} = a_{n+1} \le 1$$

Thus by Math. Induction, $1/2 \le a_n \le 1$ for all n. ⟧

Alternative easy way: It is clear that $a_n \le 1$ for $\forall n \ge 1$;

thus we see also that $a_n \ge 1/2$ for $\forall n \ge 1$ because $a_{n+1} = \dfrac{1}{a_n+1}$ for $n \ge 1$

$\therefore$ (a_n) is contractive So (a_n) is Cauchy $\therefore$ (a_n) is convergent

Writing $\lim\limits_{n\to\infty} a_n = L$, and taking limits on the relation $a_{n+1} = \dfrac{1}{a_n+1}$ give

$$L = \frac{1}{L+1} \qquad \text{i.e.,} \quad L^2 + L - 1 = 0 \qquad \therefore \quad L = \frac{-1+\sqrt{5}}{2} \quad (\because L > 0)$$

Remark. Another way of expecting that $1/2 \le a_n \le 1$ for all n:

Draw the graph $y = \dfrac{1}{x+1}$

Exa. Assume $x_0 = a$, $x_1 = b$ with $0 < a < b$ &

$$x_{n+1} = \frac{x_n + 3x_{n-1}}{4} \quad \text{for } n \ge 1$$

Show that (x_n) converges, and determine its limit.

Sol. $$x_{n+1} = \frac{4x_n - 3x_n + 3x_{n-1}}{4}$$

$$\therefore \quad x_{n+1} - x_n = -\frac{3}{4}(x_n - x_{n-1})$$

$$\therefore \quad |x_{n+1} - x_n| = \frac{3}{4}|x_n - x_{n-1}|$$

$\therefore \quad (x_n)$ is a contractive sequence. So (x_n) is convergent.

Writing $\lim\limits_{n\to\infty} x_n = L$, and taking limits on both sides of the given relation $x_{n+1} = \dfrac{x_n + 3x_{n-1}}{4}$ give

$$L = \frac{L+3L}{4} \qquad \text{i.e., } L = L \quad \text{(give no conclusion)}$$

Thus we need new idea.

Back to the relation: $x_{n+1} - x_n = -\dfrac{3}{4}(x_n - x_{n-1})$

From this, we get

$$x_2 - x_1 = -\frac{3}{4}(x_1 - x_0) = -\frac{3}{4}(b-a)$$

$$x_3 - x_2 = -\frac{3}{4}(x_2 - x_1) = \left(-\frac{3}{4}\right)^2 (x_1 - x_0) = \left(-\frac{3}{4}\right)^2 (b-a)$$

$$\vdots$$

$$x_n - x_{n-1} = \left(-\frac{3}{4}\right)^{n-1} (b-a)$$

$$\therefore \quad x_n = x_1 + (b-a)\left[-\frac{3}{4} + \left(-\frac{3}{4}\right)^2 + \cdots + \left(-\frac{3}{4}\right)^{n-1}\right]$$

$$= b + (b-a)\frac{-\frac{3}{4}\left(1 - \left(-\frac{3}{4}\right)^{n-1}\right)}{1+\frac{3}{4}} \quad \to \quad b + (b-a)\frac{-\frac{3}{4}}{\frac{7}{4}} = \frac{3}{7}a + \frac{4}{7}b$$

Def. A function $f : \overset{\text{an interval}}{\overbrace{I}} \to \mathbb{R}$ is said to be contractive if $\exists$ a constant $K > 0$, with $0 < K < 1$, such that

$$|f(x) - f(y)| \le K|x-y| \quad \text{for all } x, y \in I$$

Exa. Suppose $f : I \to \mathbb{R}$ is a contractive function on I, and define

$$a_{n+1} = f(a_n) \quad \text{for } n \ge 1.$$

Then show that (a_n) is a Cauchy sequence.

Pf. $|a_{n+2} - a_{n+1}| = |f(a_{n+1}) - f(a_n)| \le K|a_{n+1} - a_n|$ (for some $0<K<1$) $\forall n \ge 1$

$\therefore \quad (a_n)$ is contractive So (a_n) is a Cauchy sequence.

Ex. Let $f : I \to \mathbb{R}$ be differentiable on I.

If $\exists$ a constant $K > 0$, with $0 < K < 1$, such that

$$|f'(x)| \le K \quad \text{for all } x \in I,$$

then show that f is contractive

Pf. $x, y \in I \Rightarrow f(x) - f(y) \overset{\text{MVT}}{=} f'(c)(x-y)$ for some $c \in (x,y)$ or $c \in (y,x)$

$$\therefore \quad |f(x) - f(y)| \overset{\text{MVT}}{=} |f'(c)|\,|x-y| \le K|x-y| \quad \text{for all } x, y \in I$$

6.5 The Completeness Property *for sets*

So far we discussed the **"Completeness Property for *sequences of numbers*"**.

We now discuss the **"Completeness Property for *a set of numbers*"**

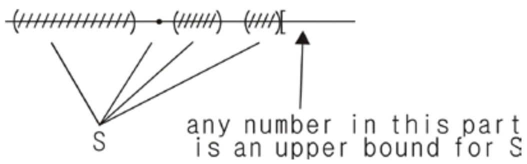
* *a sequence of numbers* : (a countable set) $\therefore$ the numbers are ordered **in a list**

* *a set of numbers* (for example, the set of irrational numbers): an unordered collection

Def. Let S be a set of real numbers (i.e., $S \subset \mathbb{R}$).

If a number b has the property that $x \le b$ for all $x \in S$, then b is called an *upper bound* for S.

A set S is said to be *bounded above* if S has an upper bound.



A number m is called the maximum of S if

(i) m is an upper bound for S, and (ii) $m \in S$

(i.e., $x \le m$ for every $x \in S$, *and* $m \in S$)

Notation: $m = \max S$

Ex. $S = [0,1]$: bounded above (by 1) & $\max S = 1$

Ex. $S = (0,1)$: bounded above (by 1), but it has no maximum

($\because$ if m is an upper bound for S, then $m \ge 1$. But such $m \notin S$)

Def. Let $S \subset \mathbb{R}$. The supremum of S is a number $\overline{m}$ satisfying:

sup-1: $\overline{m}$ is an upper bound for S (i.e., $x \le \overline{m}$ for all $x \in S$)

sup-2: $\overline{m} \le$ any upper bound of S (i.e., $\overline{m}$ is the least upper bound for S)

i.e., $x \le b$ for all $x \in S$ $\Rightarrow$ $\overline{m} \le b$

(In other words, b is any upper bound for S $\Rightarrow$ $\overline{m} \le b$)

Equivalently(대우), if $b < \overline{m}$, then b is not an upper bound for S

That is, $\boxed{b < \overline{m} \;\Rightarrow\; \exists x \in S \text{ such that } b < x}$

Or, $\boxed{\text{for any } \varepsilon > 0, \; \exists x \in S \text{ such that } \overline{m} - \varepsilon < x \le \overline{m}}$

Notation: $\overline{m} = \sup S$ ($\leftarrow$ supremum : Latin language) = $\operatorname{lub} S$ ($\leftarrow$ least upper bound)

Caution: $\sup S \in S$ is false in general [즉, 일반적으로 $\sup S \in S$ 라는 보장은 없음]

◎ Simple fact: $\sup S$ is unique, if it exists

Pf. Let $\overline{m_1} = \sup S$ and $\overline{m_2} = \sup S$

Since $\overline{m_1}$ is an upper bound for S & $\overline{m_2}$ is a least upper bound for S,

$$\overline{m_2} \le \overline{m_1}$$

Interchanging the role of m_1 and m_2, we have

$$\overline{m_1} \le \overline{m_2}$$

$$\therefore \quad \overline{m_1} = \overline{m_2}$$

Exa. $S = \left\{ 1 - \dfrac{1}{n} \;:\; n = 1, 2, 3, \cdots \right\}$ Find $\sup S$ and $\max S$

Sol. Any $b \ge 1$ is an upper bound.

If $b < 1$, then $1 - b > 0$

Thus by AP, $\exists$ a natural number n_0 such that $\dfrac{1}{n_0} < 1 - b$

That is, $b < 1 - \dfrac{1}{n_0}$ (& RHS $\in S$)

So b is not an upper bound for S

$$\therefore \quad \sup S = 1$$

Since any upper bound of S *can not* belong to S, $\max S$ does not exist.

Exa. $S = \left\{ 1 + \dfrac{1}{n} \;:\; n = 1, 2, 3, \cdots \right\}$ Find $\sup S$ and $\max S$

Sol. Any $b \ge 2$ is an upper bound.

If $b < 2$, b is not an upper bound for S

$$\therefore \quad \sup S = 2$$

Since $2 \in S$, $\max S = 2$

Proposition

If $\max S$ exists, then $\sup S$ exists and $\sup S = \max S$

Pf. Let $m = \max S$. Then by the def of maximum

$$m \in S \quad \text{and} \quad x \le m \quad \text{for all } x \in S$$

So m is an upper bound for S --- (i)

On the other hand,

if $x \le b$ for all $x \in S$, then $m \le b$ (since $m \in S$) --- (ii)

From (i) and (ii), we conclude that $m = \sup S$ (that is, $\max S = \sup S$)

※ **Theorem (Completeness Property for sets)**

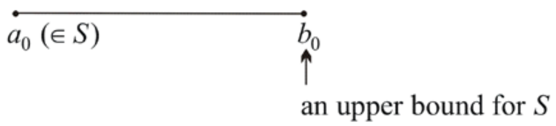
If $S(\subset \mathbb{R}) \neq \varnothing$ and bounded above, then $\sup S$ exists.

(that is, **if a nonempty set in $\mathbb{R}$ has an upper bound, it has a least upper bound**)

Pf. Let b_0 be an upper bound for S

We can choose $a_0 \in S$ since $S \neq \varnothing$

$\therefore \quad a_0 \le b_0$



Bisect the interval $[a_0, b_0]$ with its midpoint $c\left(= \dfrac{a_0 + b_0}{2}\right)$.

Choose the half-interval $[a_0, c]$ if c is an upper bound for S.

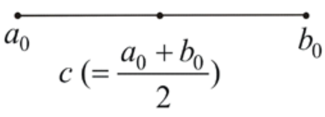
Otherwise, choose $[c, b_0]$. Call this half-interval $[a_1, b_1]$. Then

b_1 (= the right endpoint) is an upper bound for S

&

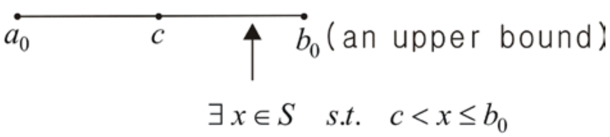
$[a_1, b_1]$ contains a point of S.

$\ulcorner\because$ If $c\left(= \dfrac{a_0 + b_0}{2}\right)$ is an upper bound for S, then



$[a_0, c]$ contains a_0 & $a_0 \in S$.

Otherwise (i.e., if c is not an upper bound for S), we have



$\therefore \quad [c, b_0]$ contains x & $x \in S$ $\lrcorner$

Repeat this halving process with $[a_1, b_1]$ and continue. Then we can get a sequence of nested intervals

$$[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n] \supset \cdots$$

such that

b_n is an upper bound for S & $[a_n, b_n]$ contains a point of S, and also

$$\text{length}\,[a_n, b_n] \to 0$$

By NIT, $\exists\ \bar{m} \in \bigcap_{n=1}^{\infty} [a_n, b_n]$ with $\lim_{n\to\infty} a_n = \bar{m}$ & $\lim_{n\to\infty} b_n = \bar{m}$ (notice that $b_n \downarrow \bar{m}$)

We now show that $\bar{m} = \sup S$ (it is expected from the fact that $b_n \downarrow \bar{m}$)

(i) **sup-1**: $\bar{m}$ is an upper bound for S

$\ulcorner\because$ $x \in S \Rightarrow x \le b_n$ for all n, since each b_n is an upper bound for S

$$\Rightarrow\quad x \le \lim_{n\to\infty} b_n = \bar{m}, \quad \text{by LLT}$$

$\therefore$ $\bar{m}$ is an upper bound for S $\lrcorner$

(ii) **sup-2**: $\bar{m} \le$ any upper bound of S

$\ulcorner\because$ Let $x \le b$ for all $x \in S$ (i.e., let b be an upper bound for S)

$\Rightarrow$ $a_n \le b$ for all n, since each $[a_n, b_n]$ contains a point of S

(that is, $a_n \le x_n$ for some $x_n \in S$)

$$\Rightarrow\quad \lim_{n\to\infty} a_n (= \bar{m}) \le b, \quad \text{by LLT} \quad \lrcorner$$

Exa. Let $S = \{r : r \text{ is an irrational number s.t. } r < 1\}$. $\sup S = ?$

Sol. Any $b \ge 1$ is an upper bound.

If $b < 1$, then (by the density of rational numbers)

$$\exists \text{ a rational number } \frac{m}{n} \text{ such that } b < \frac{m}{n} < 1$$

We can choose a sufficiently small $\varepsilon > 0$ so that

$$b < \frac{m}{n} < \underbrace{\varepsilon\sqrt{2} + \frac{m}{n}}_{\text{irrational number}} < 1;$$

which shows any b $(b < 1)$ is not an upper bound for the set S.

Therefore, $\sup S = 1$.

Ex. Let $S = \{r : r \text{ is a rational number s.t. } r < 1\}$. Determine $\sup S$.

Ex. Let $S = \left\{r : r \text{ is a rational number s.t. } r < \sqrt{2}\right\}$. Determine $\sup S$

Theorem (easy to expect) [Remember the conclusion]

(i) (a_n) is $\uparrow$ & bounded above $\Rightarrow$ $\lim\limits_{n\to\infty} a_n$ exists & $\lim\limits_{n\to\infty} a_n = \sup\{a_n : n \in N\}$

(ii) (a_n) is $\downarrow$ & bounded below $\Rightarrow$ $\lim\limits_{n\to\infty} a_n$ exists & $\lim\limits_{n\to\infty} a_n = \inf\{a_n : n \in N\}$

Pf. (i) $S \overset{\text{let}}{=} \{a_n : n \in N\}$

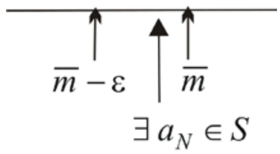
$\Rightarrow$ $S \neq \varnothing$ and bdd above

$\overset{\text{Completeness Property}}{\Rightarrow}$ $\sup S \ (\overset{\text{let}}{=} \overline{m})$ exists

Suffices to show: $\overline{m} = \lim\limits_{n\to\infty} a_n$

For this, let $\varepsilon > 0$ be given. Then

$\overset{\sup-(2)}{\Rightarrow}$ $\exists\ a_N \in S$ s.t. $\overline{m} - \varepsilon < a_N \leq \overline{m}$



Since (a_n) is $\uparrow$, it follows that

$$\overline{m} - \varepsilon < a_N \leq a_n \leq \overline{m} \quad \text{for every} \quad n \geq N$$

$$\Downarrow \text{(clearly)}$$

$$\overline{m} - \varepsilon < a_n < \overline{m} + \varepsilon \quad \text{for every} \quad n \geq N$$

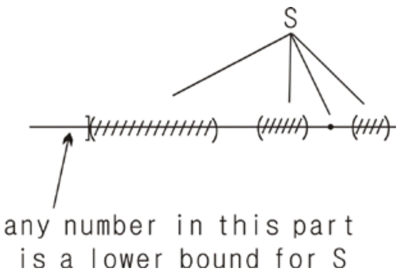
i.e., $a_n \underset{\varepsilon}{\approx} \overline{m}$ for every $n \geq N$

$\therefore$ $\overline{m} = \lim\limits_{n\to\infty} a_n$

(ii) The proof is similar to that of (i)

Def. Let $S \subset \mathbb{R}$. If a number b has the property that $x \geq b$ for all $x \in S$, then b is called a *lower bound* for S.

A set S is said to be *bounded below* if S has a lower bound.



A number m is called the minimum of S if

(i) m is a lower bound for S, and (ii) $m \in S$

(i.e., $x \ge m$ for every $x \in S$, *and* $m \in S$)

Notation: $m = \min S$

Def. Let $S \subset \mathbb{R}$. The infimum of S is a number $\underline{m}$ satisfying:

inf-1: $\underline{m}$ is a lower bound for S (i.e., $x \ge \underline{m}$ for all $x \in S$)

inf-2: $\underline{m} \ge$ any lower bound of S (i.e., $\underline{m}$ is the greatest lower bound for S)

i.e., $x \ge b$ for all $x \in S \quad \Rightarrow \quad \underline{m} \ge b$

(In other words, b is any lower bound for $S \quad \Rightarrow \quad \underline{m} \ge b$)

Equivalently(대우), if $b > \underline{m}$, then b is not a lower bound for S

That is, $\boxed{b > \underline{m} \quad \Rightarrow \quad \exists x \in S \text{ such that } x < b}$

Or, $\boxed{\text{for any } \varepsilon > 0, \quad \exists x \in S \text{ such that } \underline{m} \le x < \underline{m} + \varepsilon}$

Notation: $\underline{m} = \inf S$ ($\leftarrow$ infimum) $=$ glb$S$ ($\leftarrow$ greatest lower bound)

Ex. $S = \left\{ 1 - \dfrac{1}{n} : n = 1, 2, 3, \cdots \right\} \quad \Rightarrow \quad \inf S = 0$ and $\min S = 0$

Ex. $S = \left\{ 1 + \dfrac{1}{n} : n = 1, 2, 3, \cdots \right\} \quad \Rightarrow \quad \inf S = 1$ and $\min S$ does not exist

Proposition

If $\min S$ exists, then $\inf S$ exists and $\inf S = \min S$

Pf. Exercise

Theorem

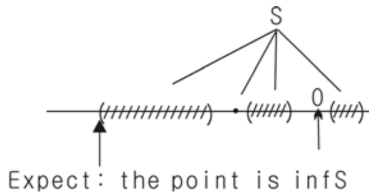
If $S(\subset \mathbb{R}) \ne \varnothing$ and bounded below, then $\inf S$ exists.

Pf.

M1. This can be proved by using the same argument seen before

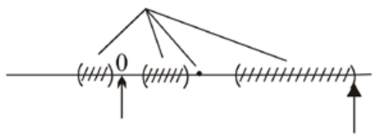
M2. Let $S\ (\neq \emptyset)$ be bounded below. Then

$\exists$ a number b s.t. $x \geq b \quad \forall x \in S$ --- (*)



How can we prove the existence of $\inf S$

$-S := \{-x : x \in S\}$



$\sup(-S)$
$= -\inf S$ (by inspection)

(*) $\Leftrightarrow$ $\exists$ a number b s.t. $-x \leq -b \quad \forall x \in S$

$\therefore$ $-S \neq \emptyset$ and it is bounded above by $-b$

$\overset{\text{Completeness Property}}{\Rightarrow}$ $\sup(-S)$ exists

We shall show: $\sup(-S) = -\inf S$

(If this is proved, $\inf S = -\sup(-S)$, so that $\inf S$ exists)

$\because$ $\sup(-S) \overset{\text{let}}{=} \alpha$. Then

(i) α is an upper bound for $-S$

(i.e., $-x \leq \alpha$ for any $x \in S$)

(ii) if b is an upper bound for $-S$, then $\alpha \leq b$

(i.e., $-x \leq b$ for any $x \in S$ $\Rightarrow$ $\alpha \leq b$)

Note that

(i) $\Leftrightarrow$ $x \geq -\alpha$ for any $x \in S$

(i.e., $-\alpha$ is a lower bound for S)

(ii) $\Leftrightarrow$ if $x \geq -b$ for any $x \in S$, then $-\alpha \geq -b$

(i.e., $-\alpha \geq$ any lower bound of S)

Therefore

$-\alpha = \inf S$

i.e., $\left(\sup(-S) =\right) \alpha = -\inf S$ $\lrcorner$

Ex. Let $S = \left\{ r : r \text{ is a rational number s.t. } r > \sqrt{2} \right\}$. Determine $\inf S$