1.1 Introduction. Real numbers.

- 2 difficulties with decimal representation

1) 2 different infinite decimals can represent the same real number

2) it is not immediately obvious how to calculate with them because an infinite decimal has no right-hand end

1.2 Increasing Sequences

Sequence :

- an infinite list of numbers, written in a definite order

EX) $a_0, a_1, \ldots, a_n, \ldots$ or $\{a_n\}, n \geq 0$

Definition :

- We say the sequence $\{a_n\}$ is increasing if $a_n \leq a_{n+1}$ for all n.
- We say the sequence $\{a_n\}$ is strictly increasing if $a_n < a_{n+1}$ for all n.
- We say the sequence $\{a_n\}$ is decreasing if $a_n \geq a_{n+1}$ for all n.
- We say the sequence $\{a_n\}$ is strictly decreasing if $a_n > a_{n+1}$ for all n.

1.3 The limit of an increasing sequence

Limit :

- A number L, in a suitable decimal representation, is the limit of the increasing sequence $\{a_n\}$ if, given any integer $k > 0$, all the a_n after some place in the sequence agree with L to k decimal places.

notations : $\lim_{n \to \infty} \{a_n\} = L$ or $\{a_n\} \to L$ as $n \to \infty$

- If such an L exists, it must be unique. On the other hand, such an L need not exist.

- A sequence $\{a_n\}$ is said to be bounded above if there is a number B such that $a_n \leq B$ for all n

Theorem :

- A positive increasing sequence $\{a_n\}$ that is bounded above has a limit

1.4 Example : the number e

Binomial Theorem :

$(1+x)^k = 1 + kx + \cdots + \binom{k}{i} x^i + \cdots + x^n$

Geometric Sum :

$1 + r + r^2 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$

Compound Interest Formula :

$A_n = p\left(1 + \frac{r}{n}\right)^n$

- investing p dollars at the annual interest rate r, with the interest compounded at equal time intervals n times a year

1.5 Example : the harmonic sum and Euler's number

Harmonic Sums :

- an increasing sequence that does not have a limit

Proposition 1.5A *Let* $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\,, \quad n \geq 1\,.$
The sequence $\{a_n\}$ *is strictly increasing, but not bounded above.*

Proposition 1.5B *Let* $b_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n+1)\,, \quad n \geq 1.$
Then $\{b_n\}$ *has a limit (denoted by* γ *and called "Euler's number").*

1.6 Decreasing sequences, The Completeness Property

- A sequence $\{a_n\}$ is said to be bounded below if there is a number C such that $a_n \geq C$ for all n

Theorem :

- A positive decreasing sequence has a limit

3 cases :

i) The sequence also contains a positive term a_N. In this case, all the terms after a_N will be positive.

ii) All the terms are negative. In this case, just change the sign of all the terms : the sequence $\{-a_n\}$ will be a positive decreasing sequence, so it will have a limit

L; then $-L$ is the limit of $\{a_n\}$. For, since the decimal places of L agree with those of the $\{-a_n\}$, the places of $-L$ agree with those of the $\{a_n\}$

iii) Neither of the above.

- Sequence $\{a_n\}$ is bounded if it is bounded above and below; there are constants B and C such that $C \leq a_n \leq B$ for all n

Completeness Property:

- A bounded monotone sequence has a limit