Chapter 4. Error term analysis

Goal: Convergence & its speed (= rate of convergence); attack by one shot

4.1 The error term

It is an important practical (and often theoretical) matter to know not just that a sequence (a_n) converges to a limit L, but also to have some idea of **how rapidly** it converges to L.

For example, it can be proved that

$$a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} \quad \to \quad \ln 2 \quad (\text{very slowly})$$

$$b_n = \frac{2}{1\cdot 3} + \frac{2}{3\cdot 3^3} + \frac{2}{5\cdot 3^5} + \cdots + \frac{2}{(2n-1)\cdot 3^{2n-1}} \quad \to \quad \ln 2 \quad (\text{rapidly})$$

"Expect" for the limit: ln2

(i) $\quad \ln(1+x) \xleftarrow{\text{integrate}} \dfrac{1}{1+x}$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \quad \text{for } |x| < 1$$

Take $\int_0^x$: where $0 < x < 1 \Rightarrow$

$$\ln(1+x) \overset{\text{expect}}{=} x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for } 0 < x < 1 \quad \text{(true: will be proved in section 4.2)}$$

$$\therefore \lim_{x\to 1^-} \ln(1+x) = \ln 2 = \lim_{x\to 1^-}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \right]$$

$$\overset{\text{expect}}{=} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} + \cdots \text{(true; there are 'two' ways to verify this)}$$

(An elementary proof will be given soon)

(ii) $$f(x) \overset{\text{let}}{=} \sum_{n=1}^{\infty} \frac{2}{2n-1}\left(\frac{x}{3}\right)^{2n-1} \quad \left(\text{assume } \left|\frac{x}{3}\right| < 1\right)$$

$$f'(x) \overset{\text{expect}}{=} \frac{2}{3}\sum_{n=1}^{\infty}\left(\frac{x}{3}\right)^{2n-2} \quad \text{(true for } |x|<3\text{: will be proved in Chap22)}$$

$$= \frac{2}{3}\left[1 + \left(\frac{x}{3}\right)^2 + \left(\frac{x}{3}\right)^4 + \cdots\right] = \frac{2}{3}\frac{1}{1-\left(\frac{x}{3}\right)^2} \quad \text{for } |x| < 3$$

$$= \frac{6}{9-x^2} = \frac{1}{3-x} + \frac{1}{3+x} \quad \text{for } |x| < 3$$

Thus for $|x|<3$,

$$\underbrace{\int_0^x f'(x)\,dx}_{\substack{\| \leftarrow f(0)=0 \\ f(x)}} = \int_0^x \left[\frac{1}{3-x}+\frac{1}{3+x}\right]dx = \ln\left(\frac{3+x}{3-x}\right)$$

$$\therefore \quad \sum_{n=1}^{\infty}\frac{2}{2n-1}\left(\frac{x}{3}\right)^{2n-1} = \ln\left(\frac{3+x}{3-x}\right) \qquad \text{for } |x|<3$$

Take $x=1 \Rightarrow \quad \ln 2 = \sum_{n=1}^{\infty}\frac{2}{(2n-1)}\left(\frac{1}{3}\right)^{2n-1} = \frac{2}{1\cdot 3}+\frac{2}{3\cdot 3^3}+\frac{2}{5\cdot 3^5}+\cdots$

The first sequence ($=a_n$) is useless for computing ln2, because it converges too slowly

(Since $a_{100} = a_{99} - \frac{1}{100}$, at the 100-th term of the sequence, the second decimal place is still changing)

By contrast, (b_n) converges rapidly; the term b_3 gives ln2 to three decimal places.

To think about questions of this kind, we slightly change our point of view about limits;
Instead of looking at the approximation itself, $a_n \underset{\varepsilon}{\approx} L$, we focus our attention on the error term $e_n = a_n - L$

Theorem (Error-form Principle)
Let $a_n = L + e_n$. Then $a_n \to L \Leftrightarrow e_n \to 0$

4.2 The error in the geometric series

Proposition (geometric sum limit)

$$a_n \overset{\text{let}}{=} 1+a+a^2+\cdots+a^n \Rightarrow \lim_{n\to\infty} a_n = \frac{1}{1-a} \quad \text{if } |a|<1$$

Pf. $\quad 1+a+a^2+\cdots+a^n = \frac{1-a^{n+1}}{1-a} = \frac{1}{1-a}-\frac{a^{n+1}}{1-a} \quad \text{if } a\neq 1$

i.e., $\quad e_n = -\frac{a^{n+1}}{1-a} \quad \text{if } a\neq 1$

Since $|a|<1$, we have $a^n \to 0$ as $n\to\infty \quad \therefore \; e_n \to 0$

Ex. Let $a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n}$

Show $\lim_{n\to\infty} a_n = \ln 2$

Pf. Idea: $\left. x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} \right|_{x=1} = a_n$

$$\uparrow \quad \leftarrow \quad \left. \int_0^x (*) \right|_{x=1}$$

$$(*): \quad 1 - x + x^2 - x^3 + \cdots + (-1)^{n-1} x^{n-1}$$

Based on this, we consider

$$1 - x + x^2 - x^3 + \cdots + (-1)^{n-1} x^{n-1} = \frac{1}{1+x} - (-1)^n \frac{x^n}{1+x} : \quad x \neq -1$$

Take $\int_0^1 \quad \Rightarrow$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} = \ln 2 \pm \int_0^1 \frac{x^n}{1+x} dx$$

Suffices to show: $e_n = \pm \int_0^1 \frac{x^n}{1+x} dx \to 0$

Clearly, $|e_n| = \int_0^1 \frac{x^n}{1+x} dx \underset{\substack{\uparrow \\ (1/2 \le)\frac{1}{1+x} \le 1 \text{ for } 0 \le x \le 1}}{\le} \int_0^1 x^n \, dx = \frac{1}{n+1} \to 0$

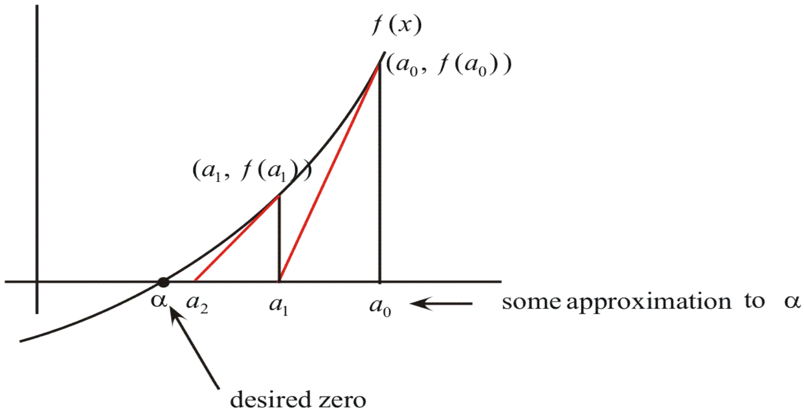
$$\therefore \lim_{n\to\infty} a_n = \ln 2$$

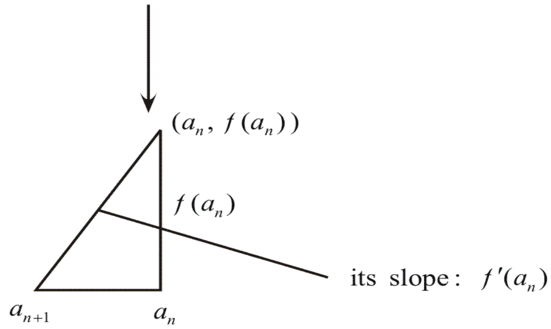
4.3 A sequence converging to $\sqrt{2}$: Newton's method

In the rest of this chapter (sections 4.3 & 4.4), we illustrate the use of the error form on sequences whose general term a_n is not given explicitly in terms of n, but instead is given recursively by a formula involving a_{n-1} and previous terms as well

(Such sequences are the normal thing one encounters in numerical analysis and computation)

Newton's method (a numerical method for locating a zero α of a given function $f(x)$ to any accuracy desired)





$$\therefore \quad f'(a_n) = \frac{f(a_n)}{a_n - a_{n+1}}$$

This gives the formula $\boxed{a_{n+1} = a_n - \dfrac{f(a_n)}{f'(a_n)}}$

$$\lim_{n\to\infty} a_n \overset{\text{expect}}{=} \alpha$$

That is, start with $a_0 \to a_1 \to a_2 \to \cdots \overset{\text{hope}}{\to} \alpha$

Ex. Find a sequence (a_n) s.t. $a_n \to \sqrt{2}$, by using Newton's method

(& investigate its rate of convergence)

Sol. $\sqrt{2}$: the positive zero of $f(x) = x^2 - 2$

Recall

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)} = a_n - \frac{a_n^{\ 2} - 2}{2a_n}$$

$$= \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \quad --- \quad (*)$$

Expect: any starting value " a_0 close enough to $\sqrt{2}$ " will generate a sequence converging to $\sqrt{2}$

$$(a_0 \underset{\text{should determine}}{\approx} \sqrt{2})$$

$$e_n \overset{\text{let}}{=} a_n - \sqrt{2} \quad \& \quad \text{show } e_n \to 0$$

(Notice that we have no explicit formula for a_n)

Key idea: $\boxed{\text{Use } (*) \text{ to relate } e_{n+1} \text{ to } e_n}$

$$e_{n+1} = a_{n+1} - \sqrt{2} \quad = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) - \sqrt{2}$$

$$\therefore \quad e_{n+1} = \frac{1}{2}\left(\left(\sqrt{2}+e_n\right) + \frac{2}{\sqrt{2}+e_n}\right) - \sqrt{2} \quad (\leftarrow a_n = \sqrt{2}+e_n)$$

$$= \frac{e_n^{\ 2}}{2\left(\sqrt{2}+e_n\right)}$$

To show e_{n+1} gets small, must show the denominator is *not* small

$$\left|\sqrt{2}+e_n\right| \geq \sqrt{2} - |e_n| > 1.4 - 0.9 = 0.5 \quad \text{provided } |e_n| < 0.9$$

So if $|e_n| < 0.9$, then $|e_{n+1}| < e_n^{\ 2}$

Thus if we choose a starting value a_0 satisfying $|e_0| < 0.9$, we see that

$$|e_1| < e_0^{\ 2}, \quad |e_2| < e_1^{\ 2} < e_0^{\ 4}, \quad \cdots, \quad |e_n| < e_0^{\ 2^n} \to 0 \ \text{ very rapidly as } n \to \infty$$

$$\therefore \quad e_n \to 0 \ \text{ very rapidly}$$

Remark:

If we take a_0 such that $|e_0| < 0.1$, then

$$|e_1| < 0.01 = 10^{-2}$$
$$|e_2| < 0.0001 = 10^{-4}$$
$$\vdots$$

Home Study: Let $a > 0$.

(i) Find a sequence (a_n) converging to $\sqrt{a}$

(ii) Find a sequence (a_n) converging to $\sqrt[3]{a}$

Remark: $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$ with $a_0 \approx \sqrt{2}$ (or $a_0 > 0$)

$\Rightarrow$ (a_n) is convergent

Pf. $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \overset{AG}{\geq} \sqrt{a_n \cdot \frac{2}{a_n}} = \sqrt{2} \quad \forall n \geq 0$

$\therefore$ $(a_n)_1^\infty$ is bounded below by $\sqrt{2}$ (even if $0 < a_0 < \sqrt{2}$)

Goal: $(a_n)_1^\infty$ is $\downarrow$

$$a_n - a_{n+1} = a_n - \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) = \frac{1}{2}\left(a_n - \frac{2}{a_n}\right)$$

$$= \frac{1}{2}\frac{a_n^2 - 2}{a_n} \geq \frac{1}{2} \cdot \frac{0}{a_n} = 0 \quad \forall n \geq 1 \text{ (note } a_n \geq \sqrt{2} > 0 \text{ for } n \geq 1)$$

$\therefore$ $(a_n)_1^\infty$ is $\downarrow$

Thus (a_n) is convergent (by the Completeness Property of $\mathbb{R}$)

Now, we let $\lim_{n\to\infty} a_n = \alpha$ $(\Rightarrow \alpha \geq \sqrt{2})$

Since $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$, we have (by taking limits)

$$\alpha = \frac{1}{2}\left(\alpha + \frac{2}{\alpha}\right) \qquad \therefore \quad \alpha^2 = 2 \qquad \therefore \quad \alpha = \sqrt{2}$$

4.4 The sequence of Fibonacci fractions

The Fibonacci sequence is given by

$1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad \cdots$: often written as $F_0, F_1, F_2, \cdots$

Let $a_n = \frac{F_n}{F_{n+1}} (n \geq 0)$, the ratios of successive terms of the Fibonacci sequence:

$$\frac{1}{1}(=1) \quad \frac{1}{2}(=0.5) \quad \frac{2}{3}(\doteq 0.667) \quad \frac{3}{5}(=0.6) \quad \frac{5}{8}(=0.625) \quad \frac{8}{13} \quad \frac{13}{21} \quad \cdots$$

Question: $a_n \rightarrow ?$

Note that if $a_{n+1} = \frac{F_{n+1}}{F_{n+2}} = \frac{F_{n+1}}{F_n + F_{n+1}} = \frac{1}{\frac{F_n}{F_{n+1}} + 1} = \frac{1}{a_n + 1} (n \geq 0)$ and $a_0 = 1$

$$\therefore \quad a_{n+1} = \frac{1}{a_n + 1} \quad (a_0 = 1, \quad a_1 = 0.5, \quad a_2 \doteq 0.667)$$

If we **assume** $\lim_{n\to\infty} a_n \equiv M$ exists, it is easy to find M

Indeed, if $\lim_{n\to\infty} a_n \equiv M$ exists, then

$$\lim_{n\to\infty} a_{n+1} = \frac{1}{\lim_{n\to\infty} a_n + 1} \qquad \text{i.e.,} \quad M = \frac{1}{M+1}$$

$$\text{i.e.,} \quad M^2 + M - 1 = 0 \qquad \therefore \quad M = \frac{\sqrt{5}-1}{2} \quad (\because \quad M > 0)$$

Target: $\lim_{n\to\infty} a_n = \dfrac{\sqrt{5}-1}{2} \overset{\text{let}}{\equiv} M$

Must examine $e_n = a_n - M$ & try to show $e_n \to 0$

$$e_{n+1} = a_{n+1} - M = \frac{1}{a_n+1} - M$$

$$= \frac{1}{e_n + M + 1} - M = \frac{1 - M - M^2 - Me_n}{e_n + M + 1}$$

$$= -\frac{M}{e_n + M + 1} e_n \quad (\leftarrow \quad M^2 + M - 1 = 0)$$

$$= -\frac{\sqrt{5}-1}{2e_n + \sqrt{5} + 1} e_n \quad (\leftarrow \quad M = \frac{\sqrt{5}-1}{2}) \quad (\text{note that } \sqrt{5} - 1 < 2.3 - 1 = 1.3)$$

$$\left|\sqrt{5} + 1 + 2e_n\right| \geq 2.2 + 1 - 2|e_n|$$

$$\geq 2.2 + 1 - 2(0.2) = 2.8 \quad \text{if } |e_n| \leq 0.2$$

Using $\sqrt{5} < 2.3$ $(\to \sqrt{5} - 1 < 1.3)$, we get

$$(*): \quad |e_{n+1}| < \frac{1.3}{2.8}|e_n| < \frac{1}{2}|e_n| \quad \text{if } |e_n| \leq 0.2$$

Since $|e_2| = a_2 - \dfrac{\sqrt{5}-1}{2} \doteq 0.667 - 0.618 \approx 0.05,$ we have

$$|e_n| < 0.2 \quad \text{for all } n \geq 2 \quad \text{by } (*)$$

Therefore

$$|e_3| < \frac{1}{2}|e_2|, \quad |e_4| < \frac{1}{2}|e_3| < \left(\frac{1}{2}\right)^2 |e_2|, \quad \cdots, \quad |e_n| < \underbrace{\left(\frac{1}{2}\right)^{n-2} |e_2|}_{\to 0 \text{ as } n\to\infty}$$

$$\therefore \quad a_n \to M$$

Home Study: Let $a_{n+1} = \dfrac{1}{a_n + 1}$ with $a_0 = A$ & $A \neq -1$

For what values of A, is (a_n) convergent

(Hint: Draw a graph suggested by the recursive formula)

Return to a rigorous but elementary pf of: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad -1 < x \le 1$

Idea: $\frac{d}{dx}\ln(1+x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ for $|x| < 1$

Integrating ($\int_0^x \ dt$) term by term $\Rightarrow$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad -1 < x < 1$$

(Later, we will prove that every power series can be integrated term by term "within the (open) interval of convergence"; the radius of convergence R of the RHS = 1) --- not studied in high school math

Remember that we already proved (by only using high school math) that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2$$

Thus, it suffices to verify the following:

Claim: Using only "High School-Math" (the same idea as seen in Example of section 4.2), prove that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad -1 < x < 1$$

Pf. Start with $1 - x + x^2 - x^3 + \cdots + (-1)^{n-1}x^{n-1} = \frac{1}{1+x} - (-1)^n \frac{x^n}{1+x}: \quad x \ne -1$

Case 1: $0 \le x < 1$ Take $\int_0^x () \ dt \Rightarrow$

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} = \ln(1+x) \pm \int_0^x \frac{t^n}{1+t} dt$$

Suffices to show: $e_n(x) := \pm \int_0^x \frac{t^n}{1+t} dt \to 0$ as $n \to \infty$

Clearly, $|e_n(x)| = \int_0^x \frac{t^n}{1+t} dt \underset{\substack{\uparrow \\ (1/2 \le) \frac{1}{1+t} \le 1 \text{ for } 0 \le t \le x < 1}}{\le} \int_0^x t^n \ dt = \frac{x^{n+1}}{n+1} \le \frac{1}{n+1} \to 0$

Case 2: $-1 < x < 0$ Want: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad -1 < x < 0$

By letting $x = -y \ (0 < y < 1)$, we need to show

$$-\ln(1-y) = y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \cdots \quad 0 < y < 1$$

Start with $1 + y + y^2 + y^3 + \cdots + y^{n-1} = \frac{1}{1-y} - \frac{y^n}{1-y}: \quad y \ne 1$

Take $\int_0^y () \ dt \ (0 < y < 1) \Rightarrow$

$$y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \cdots + \frac{y^n}{n} = -\ln(1-y) - \int_0^y \frac{t^n}{1-t} dt$$

Suffices to show: $\int_0^y \frac{t^n}{1-t} dt \ (0 < y < 1)$ as $n \to \infty$

$$\int_0^y \frac{t^n}{1-t} dt \le \int_0^y \frac{t^n}{1-y} dt = \frac{1}{1-y}\int_0^y t^n \ dt = \frac{1}{1-y}\frac{y^{n+1}}{n+1} \le \frac{1}{1-y}\frac{1}{n+1} \to 0 \text{ as } n \to \infty$$