13.1 Vector Functions and Space Curves

- a function whose domain is a set of real numbers and whose range is a set of vectors

$$\Rightarrow r(t) = \langle \underbrace{f(t),\ g(t),\ h(t)}_{\text{component functions}} \rangle = f(t)i + g(t)j + h(t)k$$

✻ the domain of $r(t)$ is defined on the $D(f) \cap D(g) \cap D(h)$

Limits and Continuity :

Theorem :

- If $r(t) = \langle f(t), g(t), h(t) \rangle$, then $\lim_{t \to a} r(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$, provided the limits of the component functions exist. A vector function r is continuous at a if $\lim_{t \to a} r(t) = r(a)$

Space Curves :

- The set C of all points (x, y, z) in space, where $\underbrace{f(t) = x,\ g(t) = y,\ h(t) = z}_{\text{parametric equations of } C}$, and $\underbrace{t}_{\text{parameter}}$ varies throughout the interval I

13.2 Derivatives and Integrals of Vector Functions

Theorem :

- If $r(t) = \langle f(t), g(t), h(t) \rangle = f(t)i + g(t)j + h(t)k$, where f, g, and h are differentiable functions, then $r'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)i + g'(t)j + h'(t)k$

Differentiation Rules :

3 Theorem Suppose **u** and **v** are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
5. $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

✻ Geometrically, if a curve lies on a sphere with center the origin, then the tangent vector $r'(t)$ is always perpendicular to the position vector $r(t)$

Integrals :

$$\int_a^b r(t)\,dt = \left(\int_a^b f(t)\,dt\right)i + \left(\int_a^b g(t)\,dt\right)j + \left(\int_a^b h(t)\,dt\right)k$$

13.3 Arc Length and Curvature

Length of a curve:

- Suppose that the curve has the vector equation $r(t) = \langle f(t), g(t), h(t) \rangle$, $a < t < b$, where f', g', and h' are continuous. If the curve is traversed exactly once as t increases from a to b, then it can be shown that its length is $L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt = \int_a^b |r'(t)|\, dt$

The Arc Length Function:

$$s(t) = \int_a^t |r'(u)|\, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

- If a curve $r(t)$ is already given in terms of a parameter t and $s(t)$ is the arc length function, then we may be able to solve for t as a function of s: $t = t(s)$. Then the curve can be reparametrized in terms of s by substituting for t: $r = r(t(s))$.

Curvature:

- A parametrization $r(t)$ is called smooth on an interval I if r' is continuous and $r'(t) \neq 0$ on I
- If C is a smooth curve defined by the vector function $r(t)$, recall that the unit tangent vector $T(t)$ is given by $T(t) = \frac{r'(t)}{|r'(t)|}$
- The curvature of C at a given point is a measure of how quickly the curve changes direction at that point. (the magnitute of the rate of change of the unit tangent vector with respect to arc length)

$\Rightarrow \kappa = \left|\frac{dT}{ds}\right| = \left|\frac{dT/dt}{ds/dt}\right| = \left|\frac{T'(t)}{r'(t)}\right|$, where T is the unit tangent vector

☆ Small circles have large curvature and large circles have small curvature, and the curvature of a straight line is always 0

$\Rightarrow \kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$

For the special case of a plane curve with equation $y = f(x)$, we choose x as the parameter and write $\mathbf{r}(x) = x\,\mathbf{i} + f(x)\,\mathbf{j}$. Then $\mathbf{r}'(x) = \mathbf{i} + f'(x)\,\mathbf{j}$ and $\mathbf{r}''(x) = f''(x)\,\mathbf{j}$. Since $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{j} = \mathbf{0}$, it follows that $\mathbf{r}'(x) \times \mathbf{r}''(x) = f''(x)\,\mathbf{k}$. We also have $|\mathbf{r}'(x)| = \sqrt{1 + [f'(x)]^2}$ and so, by Theorem 10,

11 $$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

The Normal and Binormal Vectors:

- $T'(t)$ itself is not a unit vector but at any point where $\kappa \neq 0$ we can define the principal unit normal vector $N(t)$, $N(t) = \frac{T'(t)}{|T'(t)|}$

☆ think of the unit normal vector as indicating the direction in which the curve is turning at each point

- The vector $B(t) = T(t) \times N(t)$ is called the binormal vector; it is perpendicular to both T and N and is also a unit vector

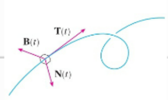

FIGURE 6

- The plane determined by the normal and binormal vectors N and B at a point P on a curve C is called the normal plane at C at P; it consists of all lines that are orthogonal to the tangent vector T
- The plane determined by the vectors T and N is called the osculating plane; it is the plane that comes closest to containing the part of the curve near P.

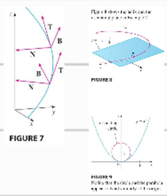
FIGURE 7

- The circle that lies in the osculating plane of C at P, has the same tanget as C at P, lies on the concave side of C, and has radius $\rho = \frac{1}{\kappa}$, the reciprocal of the curvature, is called osculating circle of C at P

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

13.4 Motion in Space: Velocity and Acceleration

- Suppose a particle moves through space so that its position vector at time t is $r(t)$. For small values of h, the vector $\frac{r(t+h)-r(t)}{h}$ approximates the direction of the particle moving along the curve $r(t)$. Its magnitude measures the size of the displacement vector per unit time.
  $\Rightarrow$ Velocity vector, $v(t) = \lim_{h \to 0} \frac{r(t+h)-r(t)}{h} = r'(t)$
  $\Rightarrow$ Speed of the particle at time t is the magnitude of the velocity vector, $|v(t)|$

- In general, vector integrals allow us to recover velocity when acceleration is known and position when velocity is known:
$$v(t) = v(t_0) + \int_{t_0}^{t} a(u)\,du$$
$$r(t) = r(t_0) + \int_{t_0}^{t} v(u)\,du$$

Projectile Motion:

Tangential and Normal Components of Acceleration:

- When we study the motion of a particle, it is often useful to resolve the acceleration into two components, one in the direction of the tangent and the other in the direction of the normal

normal. If we write $v = |\mathbf{v}|$ for the speed of the particle, then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\mathbf{v}}{v}$$

and so $$\mathbf{v} = v\mathbf{T}$$

If we differentiate both sides of this equation with respect to t, we get

5 $$\mathbf{a} = \mathbf{v}' = v'\mathbf{T} + v\mathbf{T}'$$

If we use the expression for the curvature given by Equation 13.3.9, then we have

6 $$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{T}'|}{v} \quad \text{so} \quad |\mathbf{T}'| = \kappa v$$

no matter how an object moves through space, its acceleration always lies in the plane of T and N

The unit normal vector was defined in the preceding section as $\mathbf{N} = \mathbf{T}'/|\mathbf{T}'|$, so (6) gives

$$\mathbf{T}' = |\mathbf{T}'|\mathbf{N} = \kappa v\mathbf{N}$$

and Equation 5 becomes

7 $$\boxed{\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}}$$

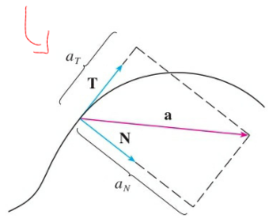

FIGURE 7

Writing a_T and a_N for the tangential and normal components of acceleration, we have

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$$

where

8 $$a_T = v' \quad \text{and} \quad a_N = \kappa v^2$$

This resolution is illustrated in Figure 7.

$\Rightarrow$ Let $V = |v|T = vT$,

$$V \cdot a = vT \cdot (v'T + \kappa v^2 N)$$
$$= vv'T \cdot T + \kappa v^3 T \cdot N$$
$$= vv', \text{ and therefore } a_T = v' = \frac{V \cdot a}{v} = \frac{r'(t) \cdot r''(t)}{|r'(t)|}$$

Kepler's Laws of Planetary Motion :

Kepler's Laws

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.
2. The line joining the sun to a planet sweeps out equal areas in equal times.
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.