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1)

1-1 $a_n = \dfrac{1\cdot 3\cdot 5\cdot \cdots (2n+1)}{2\cdot 4\cdot 6\cdots 2n}$ for $n \geq 1$

pf)

i) To show the sequence is increasing,

$a_1 = \frac{3}{2} > 1$

$a_2 = \frac{3}{2}\left(\frac{5}{4}\right) > 1$

$a_3 = \frac{3}{2}\left(\frac{5}{4}\right)\left(\frac{7}{6}\right) > 1$

$\vdots$

It is shown that $\dfrac{a_n}{a_{n-1}} = \dfrac{2n+1}{2n} > 1$, which indicates that the sequence is strictly increasing.

ii) From above, we can notice $\frac{2n+1}{2n}$, which is always greater than 1, is multiplied. Therefore, by applying geometric series test, we can assume the series has no upper limit.

$\therefore$ a_n is strictly increasing and not bounded above.

1-2 $\lim_{n\to\infty} \frac{1}{n} \left\{ \frac{2\cdot 4\cdot \cdots \cdot 2n}{1\cdot 3\cdot \cdots (2n-1)} \right\}^2$

$\lim_{n\to\infty} \frac{1}{n} \left\{ \prod_{i=1}^{n} \frac{2i}{2i-1} \right\}^2 > \lim_{n\to\infty} \frac{1}{n} \left\{ \prod_{i=1}^{n} \frac{2i}{2i} \right\}^2 = \lim_{n\to\infty} \frac{1}{n} = 0$

2)

2-a Suppose $\{a_n\}$ is a non-decreasing sequence for $n \gg 1$. Then $a_n \leq a_{n+1}$ for all $n \geq 1$.

$\Rightarrow$ $\dfrac{a_{n+1}}{a_n} \geq 1$, which contradicts $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = L < 1$. Therefore, $\{a_n\}$ is decreasing for $n \gg 1$

2-b Indirect proof) It is given $a_n > 0$, and it is proved that $\{a_n\}$ is a decreasing sequence.

So not until reaching to the lower bound, 0, $\{a_n\}$ keeps decreasing.

Direct proof) $|a_n - 0| < \varepsilon$

$a_n < \varepsilon$

$\dfrac{1}{a_n} > \dfrac{1}{\varepsilon}$, $\therefore$ there is $n > N$ such that $a_n \underset{\varepsilon}{=} 0$

3) Let $A = 1+\frac{1}{4}+\frac{1}{7}+\cdots+\frac{1}{3n-2} = \sum_{i=1}^{n}\frac{1}{3i-2} > \sum_{i=1}^{n}\frac{1}{3i} = \frac{1}{3}\sum_{i=1}^{n}\frac{1}{i}$, and we know $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{i} = \infty$. By the comparison test, $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{3i-2}$ also goes to ∞, and it is known that $\lim_{n\to\infty}\ln n = \infty$.

Accordingly, we have the form $\frac{\infty}{\infty}$, which we can apply L'hospital's rule.

=> setting $n \approx x$,

i) $\frac{d}{dx}\left(\sum_{i=1}^{x}\frac{1}{3i-2}\right)$

$= \sum_{i=1}^{x}\frac{d}{dx}\left(\frac{1}{3i-2}\right)$

$= \sum_{i=1}^{x}\frac{-3}{(3i-2)^2}$

ii) $\frac{d}{dx}(\ln x) = \frac{1}{x}$

=> $\lim_{n\to\infty}\frac{1+\frac{1}{4}+\frac{1}{7}+\cdots+\frac{1}{3n-2}}{\ln n}$

$= \lim_{n\to\infty}\frac{-3n}{1+16+49+\cdots+(3n-2)^2}$

$< \lim_{n\to\infty}\frac{-3n}{9+36+81+\cdots(3n)^2} = 0$, we can verify the limit is 0; therefore, by the comparison test, the given equation $\lim_{n\to\infty}\frac{1+\frac{1}{4}+\frac{1}{7}+\cdots+\frac{1}{3n-2}}{\ln n}$ also converges

4 $\lim_{n\to\infty}\cos 3n \overset{?}{\underset{\varepsilon}{=}} L$

Since $-1 \le \cos 3n \le 1$ for any $n \ge 0 \in \mathbb{N}$, we may find infinitely many intervals of length $\frac{\pi}{2}$ on which $\cos(3x) \ge \frac{\sqrt{2}}{2} > 1$, which means that there is at least one integer k_i within each interval. (For instance, an integer k_1 could fit between $x = [\frac{7}{4}\pi, \frac{9}{4}\pi]$). This gives a subsequence $\sin k_i$ such that $\cos(3k_i) \ge \frac{\sqrt{2}}{2}$.

Similarly, we can choose an integer m_i from each of the successive intervals of length $\frac{\pi}{2}$ on which $\cos(3x) \le -\frac{\sqrt{2}}{2}$, giving a subsequence $\cos(3k_i)$ such that $\sin 3k_i \le -\frac{\sqrt{2}}{2}$.

We now suppose the limit of $\cos 3n$ exists. Then by Subsequence Theorem, $\lim_{i\to\infty}\cos 3k_i = L = \lim_{i\to\infty}\cos 3m_i$. However, referring to the above conclusions, $\lim_{i\to\infty}\cos 3k_i \ge \frac{\sqrt{2}}{2}$ and $\lim_{i\to\infty}\cos 3m_i \le -\frac{\sqrt{2}}{2}$.

$\therefore$ Because of the restrictions stated, $\lim_{n\to\infty}\cos 3n$ does not exist.