STA 3021: Stochastic Processes
Quiz 3 (May 30, 2018)

Student ID: ______________________ **Name**: ______________________

1. (10 points) A total of 3 white and 3 black balls are distributed among two urns, with each urn containing exactly 3 balls. At each stage, a ball is randomly selected from each urn and two selected balls are interchanged. Let X_n denote the number of black balls in run 1 after the nth interchange.

   (a) Give the transition probabilities of the Markov Chain $X_n, n \geq 0$.

$E = \{0, 1, 2, 3\}$

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & \frac{1}{9} & \frac{4}{9} & \frac{4}{9} & 0 \\ 2 & 0 & \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ 3 & 0 & 0 & 1 & 0 \end{array}$$

$P_{11} = P(\text{black from urn 1 \& black from urn 2})$

$+ P(\text{white from urn 1 \& white ''})$

$= \frac{1}{3} \cdot \frac{2}{3} + \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{9}$

   (b) Find the limiting probabilities.

irreducible, aperiodic chain.

$$\begin{cases} \pi = \pi P \\ \sum \pi_i = 1 \end{cases} \Rightarrow \pi = \left(\frac{1}{20}, \frac{9}{20}, \frac{9}{20}, \frac{1}{20}\right)$$

2. (5 points) Write down two equivalent definitions of Poisson processes.

① $\{N(t), t \geq 0\}$ is a $PP(\lambda)$

$\Leftrightarrow$ $\{T_i\}$'s are IID $Exp(\lambda)$ random variables.

(where, T_n is the sequence of inter-arrivals)

② (i) $N(0) = 0$

(ii) $\{N(t)\}$ has independent increments

(iii) $P(N(t+h) - N(t) = 1) = \lambda \cdot h + o(h)$ for all $t, h > 0$

(iv) $P(N(t+h) - N(t) \geq 2) = o(h)$ for all $t, h > 0$

3. (5 points) Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. Let S_n denote the time of the nth event. Find

(a) $E(S_4)$

Let T_n be the sequence of inter-arrivals. Then $S_n = T_1 + T_2 + \cdots + T_n$

$T_i \overset{iid}{\sim} Exp(\lambda)$

$\therefore E(S_4) = E(T_1 + T_2 + T_3 + T_4) = \dfrac{4}{\lambda}$

(b) $E(S_3 | N(1) = 1)$.

$= E(T_1 + T_2 + T_3 \mid N(1) = 1)$

$= E(T_1 + (1 - T_1) + (T_1 + T_2 - 1) + T_3 \mid N(1) = 1)$

$= 1 + E((T_1 + T_2 - 1) + T_3 \mid N(1) = 1)$

$= 1 + E(T_1' + T_2')$

$= 1 + \dfrac{2}{\lambda}$,

End of quiz.