STA 3021: Stochastic Processes
Quiz 1 (Sep 26, 2017)

Student ID: ______________________ **Name:** ______________________

1. (10 points) For a random variable Z with cdf

$$F(z) = \begin{cases} 0, & z < 0, \\ .5, & z = 0, \\ .5 + .5z^2, & 0 < z < 1, \\ 1, & z \geq 1 \end{cases}$$

Find Var(Z).

$$f(z) = \begin{cases} 0.5 & , z=0 \\ z & , 0<z<1 \\ 0 & , \text{o.w} \end{cases}$$

$$E(Z) = 0 \times 0.5 + \int_0^1 z^2 \, dz = \frac{1}{3}$$

$$E(Z^2) = 0^2 \times 0.5 + \int_0^1 z^3 \, dz = \frac{1}{4}$$

$$\therefore \text{Var}(Z) = E(Z^2) - \{E(Z)\}^2 = \frac{1}{4} - \frac{1}{9} = \frac{5}{36}$$

2. (10 points) The joint density of X and Y is given by

$$f(x,y) = \frac{e^{-x/y}e^{-y}}{y}, \quad 0 < x < \infty, 0 < y < \infty$$

Find $E(X|Y = y)$.

$$f_{X|Y} = \frac{f(x,y)}{\int_0^\infty \frac{e^{-x/y}e^{-y}}{y}\,dx} = \frac{1}{y}e^{-x/y}$$

$$\therefore E(X|Y=y) = \int_0^\infty x \cdot \frac{1}{y}e^{-x/y}\,dx$$

$$\overset{\frac{x}{y}=t}{=} \int_0^\infty te^{-t} \cdot y\,dt$$

$$= y\int_0^\infty te^{-t}\,dt$$

$$= y$$

3. (10 points) Consider 23 people and suppose that each of them has a birthday that is equally likely to be any of the 365 days of the year. Furthermore, assume that their birthdays are independent, and let A be the event that no two of them share the same birthday. Employ the Poisson paradigm to approximate $P(A)$.

let X be # of the pairs of people sharring the same birthday among 23 people.

Then, $X \sim B\left(\binom{23}{2}, \frac{1}{365}\right)$

$\approx Poisson\left(\binom{23}{2}\frac{1}{365}\right)$

$\therefore P(A) \approx P(X=0) = \frac{e^{-\binom{23}{2}\frac{1}{365}}}{0!} = 0.4999$

4. (10 points) For a compound random variable $S = \sum_{i=1}^{N} X_i$ find $\mathrm{Cov}(N, S)$. Assume that N and $X_i's$ are independent and X_i's are IID random variables.

$$\mathrm{Cov}(N,S) = E\left(N\sum_{i=1}^{N}X_i\right) - E(N)E\left(\sum_{i=1}^{N}X_i\right)$$

$$\begin{aligned} E\left(N\sum_{i=1}^{N}X_i\right) &= E_N\left(E\left(N\sum_{i=1}^{N}X_i \mid N\right)\right) \\ &= E\left(N\sum_{i=1}^{N}E(X_i)\right) \\ &= E\left(N^2E(X_1)\right) \\ &= E(X_1)E(N^2) \end{aligned}$$

$$\begin{aligned} E\left(\sum_{i=1}^{N}X_i\right) &= E\left(E\left(\sum_{i=1}^{N}X_i \mid N\right)\right) \\ &= E\left(\sum_{i=1}^{N}E(X_i)\right) \\ &= E\left(N E(X_1)\right) \\ &= E(X_1)E(N) \end{aligned}$$

$$\begin{aligned} \therefore \mathrm{Cov}(N,S) &= E(X_1)E(N^2) - E(X_1)\{E(N)\}^2 \\ &= E(X_1)\mathrm{Var}(N) \end{aligned}$$

End of quiz.