Chap 9. Functions of one variable (studied in set theory course or mostly well-known: 생략해도 무방)

9.1 Functions

⦿ A real-valued function of one (real) variable (for short, **function**):

Roughly, it is *a rule assigning*, to each real number a in its domain, a corresponding (real) number b.

Def. (idea: identify a function with its graph)

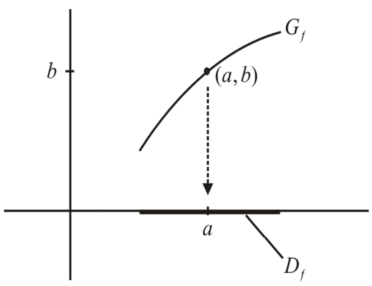
A **function** f is a set G_f of ordered pairs (a,b) of numbers, such that no two pairs have the same first entry:

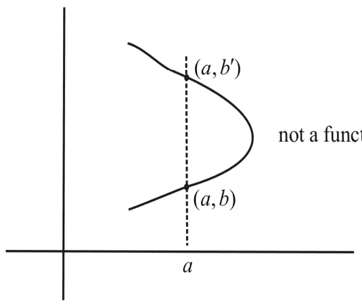
$$(a,b) \ \ and \ \ (a,b') \in G_f \quad \Rightarrow \quad b = b' \quad ---(*)$$

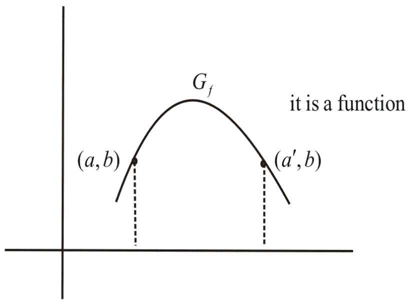
If $(a,b) \in G_f$, we say that f is defined at a, and write $b = f(a)$.

The domain of f is the set of numbers for which f is defined. That is,

$$D_f = \{a : (a,b) \in G_f \quad \text{for some } b\}$$







When the ordered pairs (a,b) are visualized as points in the xy −plane, the set G_f is called the **graph of the function**; **it is essentially the same as the function**.

Two functions f and g are called *equal* if $G_f = G_g$.

<u>There are three view points to understand functions</u>:

1. Geometric viewpoint of function (기하적 관점): 한 평면(xy −plane)위에 정의역과 치역을 모두 나타냄

   By $(*)$, a subset G of the plane is the graph of a function if and only if each vertical line $x = a$ contains at most one point of the graph

2. Analytic viewpoint of function (해석적 관점): 수식으로 표현

Thinks of a function as a rule $a \overset{\text{assigning}}{\longrightarrow} f(a), \quad a \in D_f$.

Note: In this rule, a must determine $f(a)$ uniquely. That is, the function must be single-valued.

This rule is usually given by an expression in an independent variable, like

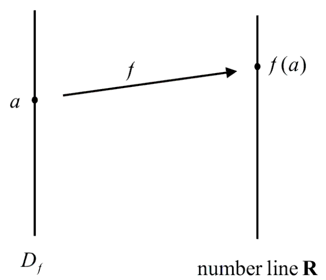
$$\sqrt{4x+1}, \quad |\, e^u \tan^{-1}(1+u)\,|, \quad erf(erf\,t), \quad \int_0^x \sin(J_0(t))^2 dt$$

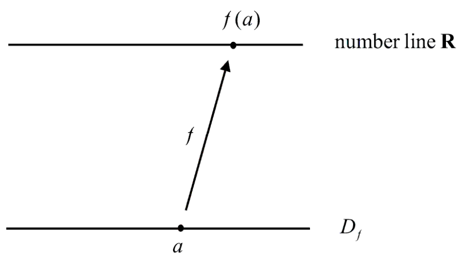
Here $erf\,x = \int_0^x e^{-t^2/2}dt$: error fct, $\quad J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)d\theta$: Bessel fct

This view point is not general enough: ∃ many graphs that having no analytic representation in terms of known functions

3. Mapping viewpoint (대응으로서의 관점): (일반적으로) 정의역과 공변역 (또는 치역)을 따로 나타냄

Regards the function as a mapping (or map) $f : D_f \to \mathbb{R}$, which associates with each point $a \in D_f$ the corresponding point $b = f(a)$ on the number line





(the mapping viewpoint is most useful when dealing with functions of several variables)

● The three different viewpoints are necessary, because they suggest different sorts of questions to ask about functions:

- From the geometric view, one might ask if f is **increasing or decreasing**, is **convex**, or has **maxima and minima**
- The analytic view leads to the operations of **algebra and calculus**, with the resulting equations and inequalities
- Thinking of f as a mapping leads one to ask whether it is **injective**, whether it has an **inverse** and what it does to different types of sets: Does it take intervals into intervals? Are there points which are mapped to themselves?

◎ A word about functional notation:

The analytic viewpoint suggests the notation $f(x)$, or introducing a **dependent variable** y and writing $y = f(x)$ or $y = y(x)$

The geometric and mapping viewpoints which often do not use variables, suggest using just f as the notation ("**sin**" and "**exp**" are all right but those are awkward, so $\sin x$, $\exp x$ will be used).

Domain D_f

The natural domain: all values of x for which the expression makes sense

If the domain is for some reason taken to be smaller than the natural domain, we get the restricted function (with its restricted domain)

$$\begin{array}{lll} \sin x & \text{periodic} & \\ \sin x, \quad 0 \le x \le \pi & \ge 0 & : \text{ all are different} \\ \sin x, \quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2} & \uparrow & \end{array}$$

$$\underbrace{\frac{x^2}{x}}_{\text{its natural domain is } \mathbb{R}\setminus\{0\}} \neq \underbrace{x}_{\text{its natural domain is } \mathbb{R}}$$

Remark: the sequence $\{a_n\}_0^\infty$ is actually a very special type of function;

$$\{a_n\}_0^\infty \quad \leftrightarrow \quad a:\{0,1,2,\cdots\} \to \mathbb{R} \quad (\text{identify } \ a(n) = a_n)$$

9.2 Algebraic operations on functions

$$f(x)+g(x), \quad f(x)g(x), \quad cf(x), \quad f(x)/\,g(x)$$

The natural domain of $f(x)/\,g(x) = \{x : f(x) \text{ and } g(x) \text{ are defined \& } g(x) \neq 0\}$

Composition: the best way to think of it;

$$w = f(y), \quad y = g(x) \quad \Rightarrow \quad w = f(g(x)) = f \circ g(x)$$

The formal def of composition is given in terms of mapping: $\underbrace{a \xrightarrow{g} g(a) \xrightarrow{f}}_{f\circ g}$

Def. Given two functions $g: D_g \to \mathbb{R}$ and $f: D_f \to \mathbb{R}$, we define their composition

$$f \circ g : D_{f\circ g} \to \mathbb{R} \ \text{ by } \ f\circ g(a) = f(g(a)), \quad a \in D_{f\circ g};$$

$$D_{f\circ g} = \{a \in \mathbb{R} : g \text{ is defined at } a \ \text{ and } \ f \text{ is defined at } g(a)\}$$

Eg. $\sin x \circ \sqrt{x} = \sin\sqrt{x}$; $\quad$ domain $= \{x : x \ge 0\}$

$\sqrt{x} \circ \sin x = \sqrt{\sin x}$; $\quad$ domain $= \{x : \sin x \ge 0\} = \{[2k\pi, (2k+1)\pi]\}(k = 0, \pm 1, \pm 2, \cdots)$

Two special compositions:

- **translation:** if $a > 0$

  the graph $f(x+a)$ is the graph G_f moved to the left a units

  $f(x-a)$ // moved to the right a units

  the first follows from: $g(x) \equiv f(x+a) \quad \Rightarrow \quad g(0) = f(a), \quad g(-a) = f(0)$

- **change of scale:** if $a > 1$

  the graph $f(x/a)$ is the graph expanded horizontally by the factor a

  the graph $f(ax)$ is the graph compressed horizontally by the factor $1/a$

$$\sin\frac{x}{2} \quad\quad\quad\quad \sin x \quad\quad\quad\quad \sin 2x$$

$\xleftarrow[\text{(expanded)}]{\text{주기가 늘어남}}$ $\quad\quad$ $\xrightarrow[\text{(compressed)}]{\text{주기가 줄어듬}}$

9.3 Some properties of functions

Def. Let $f(x)$ be a function with domain D. We say f is

increasing	if $f(a) \le f(b)$ for all pairs $a < b$ in D
strictly increasing	if $f(a) < f(b)$ for all pairs $a < b$ in D
decreasing	if $f(a) \ge f(b)$ for all pairs $a < b$ in D
strictly decreasing	if $f(a) > f(b)$ for all pairs $a < b$ in D
monotone	if f is either increasing in D or decreasing in D
strictly monotone	if f is either strictly inc in D or strictly dec in D

Eg. On their natural domains,

(a) e^x, x^3, and $\ln x$ are strictly inc $\quad$ (b) e^{-x} is strictly dec

(c) $\operatorname{sgn} x \left(= \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} \right)$ is inc

(d) $\dfrac{1}{x}$ is not dec $\quad (\leftarrow \ f(-1) < f(1) \ )$

- $f(x)$ is (strictly) inc $\Leftrightarrow$ $-f(x)$ is (strictly) dec

Def. $f(x)$ is **even** if $f(-x) = f(x)$ for all $x \in D_f$

$f(x)$ is **odd** if $f(-x) = -f(x)$ for all $x \in D_f$

For both definitions the domain D_f must be symmetric about the point 0 (i.e., $x \in D_f \Leftrightarrow -x \in D_f$)

Geometrically,

"f is even" means that G_f is symmetric about the $y-$axis

"f is odd" means that G_f is symmetric about the origin

Eg (a) A polynomial with only odd powers of x is an odd function

A polynomial with only even powers of x is an even function

The function 0 is both even and odd

(b) $f \cdot g$ and f/g are even if f and g are both even or both odd

odd if one function is even and the other is odd

(c) $\cos x$ is even; $\sin x$, $\tan x$ are odd

Proposition. Suppose D_f is symmetric around 0. Then f has a unique representation as the sum of an even and an odd function:

$$f(x) = E(x) + O(x), \quad E(x) \text{ is even}, \ \mathrm{O}(x) \text{ is odd}$$

Pf. $$f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} = E(x) + O(x)$$

Uniqueness: easy exercise

Def. We say that $f(x)$ is periodic if $\exists\, c > 0$ such that $f(x+c) = f(x)$ for all $x \in D_f$

The number c is called **a period** of f; the smallest such c (if it exists) is called the minimal period of f, or simply **the period** of f.

- $\sin x$ & $\cos x$ have period 2π

  $\tan x$ has period π

- If c is a period, so is $2c, 3c,$ and so on.

  A constant function is periodic, but it has no minimal period.

Eg. If a function f is even and monotone, it is constant

Pf. f is even $\Rightarrow$ D_f is symmetric about 0

Let $a, b \in D_f$ and suppose $0 \le a < b$; then $-b, -a \in D_f$, and $-b < -a$

So if, say f is inc, we have

$$f(a) \le f(b) \quad \text{and} \quad \underbrace{f(-b)}_{\substack{\| \leftarrow f \text{ is even} \\ f(b)}} \le \underbrace{f(-a)}_{\substack{\to \| \\ f(a)}}$$

Thus $f(a) = f(b)$.

Since a and b were arbitrary, f is constant on the right half of D_f, and thus on all of D_f, since it is an even function.

If f is dec, $-f$ is inc, therefore constant by the preceding: thus f is also constant.

Eg. Show that $\sin x$ is not a polynomial function

Pf. A polynomial ft has at most a finite number of zeros equal to its degree, whereas $\sin x$ has infinitely many zeros.

9.4 Inverse functions

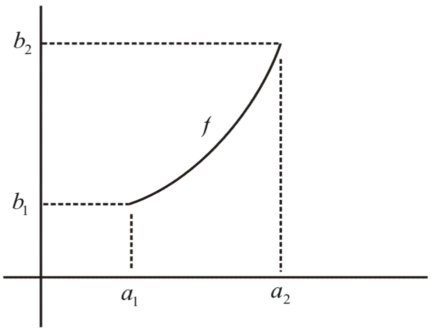
- the mapping viewpoint, to define inverse functions;
- the geometric viewpoint, to understand them intuitively;
- the analytic viewpoint, to calculate with them.

• **The mapping viewpoint**. Let f be a function defined on the interval $[a_1,\ a_2]$, and assume that on this interval f has these two properties

Inv-1 f is strictly inc

Inv-2 f takes on every value between $b_1 = f(a_1)$ and $b_2 = f(a_2)$,

i.e., if $b \in [b_1,\ b_2]$, there is an $a \in [a_1,\ a_2]$ such that $f(a) = b$.



Then f is a mapping of intervals:

$$f : [a_1,\ a_2] \ \longrightarrow\ [b_1,\ b_2];$$

for each $b \in [b_1,\ b_2]$, there is an $a \in [a_1,\ a_2]$ such that $f(a) = b$, and there is only one such a,

since the function is strictly inc.

It allows us to define the "backwards", or inverse, mapping $f^{-1} : [b_1,\ b_2] \ \longrightarrow\ [a_1,\ a_2]$

by the rule $f^{-1}(b) = a \quad \Leftrightarrow \quad f(a) = b$.

Inv-1 guarantees that the map f is injective ($a \neq a' \Rightarrow f(a) \neq f(a')$)

Inv-2 says that f is surjective (for each b, $\exists$ an a such that $b = f(a)$)

Therefore, f is bijective; $\left(\overset{\text{well- known}}{\Leftrightarrow} \quad \exists \text{ an inverse map } f^{-1} \right)$

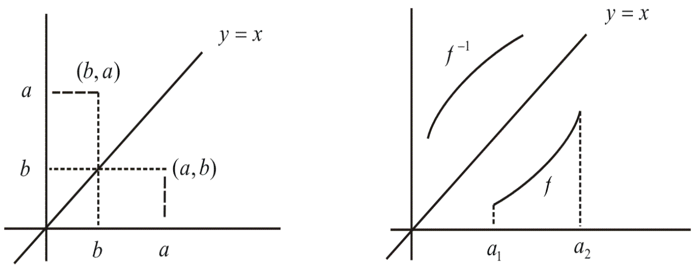
- **The geometric viewpoint**

  We consider the graphs of f and f^{-1}.

  The defining property $\boxed{f^{-1}(b) = a \Leftrightarrow f(a) = b}$ of the inverse map translates into

$$\boxed{(b,a) \in G_{f^{-1}} \quad \Leftrightarrow \quad (a,b) \in G_f}$$

  From this we get the intuitive geometric picture of f^{-1} :
  flipping the plane around $y = x$ carries G_f into $G_{f^{-1}}$



- **The analytic viewpoint**

Expressed in terms of variables, the defining property $\boxed{f^{-1}(b) = a \Leftrightarrow f(a) = b}$ of inverse functions becomes $\quad y = f^{-1}(x) \quad \Leftrightarrow \quad x = f(y)$

($\therefore$ get $f^{-1}(x)$ by solving $x = f(y)$ for y in terms of x)

Composing f and f^{-1} gives us also the useful relations

$$f(f^{-1}(x)) = x \qquad \text{for} \quad b_1 \le x \le b_2$$

$$f^{-1}(f(x)) = x \qquad \text{for} \quad a_1 \le x \le a_2$$

(Warning: $f \circ f^{-1} \ne f^{-1} \circ f$)

Remark. All of the preceding is also valid, making the appropriate changes, for strictly decreasing functions.

Eg. Find f^{-1} if $f(x) = x^2 + 1$

Sol. To satisfy **Inv-1** and **Inv-2,** we restrict the domain to the set $x \ge 0,$ on which $f(x)$ is strictly inc. Interchange the two variables: the restriction $x \ge 0$ turns into $y \ge 0$. Thus

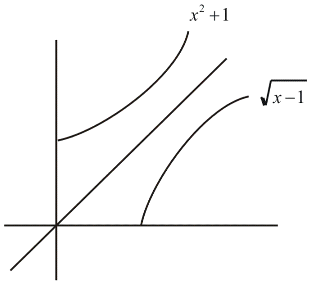
$$x = y^2 + 1, \quad y \ge 0$$

$$\Updownarrow$$

$$y = \sqrt{x - 1}, \quad x \ge 1 \quad (\text{we use the positive square root since } y \ge 0)$$

The domain of $f^{-1}(x)$ is the range of $f(x)$ $\quad$ (i.e.,$x \ge 1$)

Therefore, $f^{-1}(x) = \sqrt{x-1}, \quad x \geq 1$



9.5 The elementary functions

(a) the rational functions: fts in the form $p(x)/q(x)$ where $p(x)$ and $q(x)$ are polynomials

(b) the basic trigonometric functions: $\cos x$, $\sin x$, $\tan x$, $\sec x$, $\csc x$, $\cot x$ and

the six inverses $\cos^{-1} x$, $\sin^{-1} x, \cdots$

(c) e^x, $\ln x$

(d) the **algebraic functions**: those fts $y = y(x)$ which satisfy an equation of the form

$$y^n + a_1(x)y^{n-1} + \cdots + a_{n-1}(x)y + a_n(x) = 0,$$

where coefficients $a_k(x)$ are rational functions.

(For example, any expression involving some combination of **n-th** roots ($y = \sqrt[n]{x} \leftarrow y^n = x$), non-negative integer powers of x, and arithmetic operations is an algebraic function, but there are many other algebraic functions.)

The elementary functions are all functions that we can get from the **four classes above** by $+, -, \times, \div$, and **composition** of functions. Thus it includes combinations such as

$$\sin^3(\sqrt{x-2}) \cdot 10^{x^2}, \quad \ln\left(\tan^{-1}(e^{\sqrt{x}} - x^3)\right)\sec(22x)$$

$$y = x^\alpha (x > 0) \ (\alpha : \text{real}) \quad (\leftarrow x^\alpha = e^{\alpha \ln x} = e^x \circ \alpha \ln x)$$

Remark. transcendental functions (초월 함수): those functions that are not algebraic