22.3 Continuity and uniform convergence

Thm 22.3 (Continuity of uniform limits and sums)

(a) Let $f_n(x)$ be continuous on I for $n \geq 0$.
Suppose $f_n(x) \rightrightarrows f(x)$ on I.
Then $f(x)$ is continuous on I.

(b) Let $u_k(x)$ be continuous on I for $k \geq 0$, and suppose
$\sum u_k(x)$ converges uniformly on I.
Then the sum $f(x) = \sum u_k(x)$ is continuous on I.

Pf. (a) Suffices to show $f(x)$ is conti at an arbitrary point $x_0 \in I$.
Assume first x_0 is not an endpoint of I.
Basic idea for the pf: Fix $x_0 \neq$ end point of I. Then

$$|f(x) - f(x_0)| \leq \underbrace{|f(x) - f_N(x)|}_{\equiv I} + \underbrace{|f_N(x) - f_N(x_0)|}_{\equiv II} + \underbrace{|f_N(x_0) - f(x_0)|}_{\equiv III} \quad (N: \text{ a big natural number})$$

I & III are sufficiently small, since $f_n(x) \rightrightarrows f(x)$ on I
II is small for $x \approx x_0$, since $f_N(x)$ is continuous at x_0.

To give a rigorous pf of (a), let $\varepsilon > 0$ be given.
Since $f_n(x) \rightrightarrows f(x)$ on I, we can choose $N(\gg 1)$ so that

$$f_N(x) \underset{\varepsilon}{\approx} f(x) \quad \text{for all } x \in I \quad --- \text{(i)}$$

By the way, since $f_N(x)$ is continuous at x_0,

$$f_N(x) \underset{\varepsilon}{\approx} f_N(x_0) \quad \text{for } x \approx x_0 \quad --- \text{(ii)}$$

It follows that

$$f(x) \overset{\text{(i)}}{\underset{\varepsilon}{\approx}} f_N(x) \overset{\text{(ii)}}{\underset{\varepsilon}{\approx}} f_N(x_0) \overset{\text{(i)}}{\underset{\varepsilon}{\approx}} f(x_0) \quad \text{for } x \approx x_0$$

Therefore,

$$f(x) \underset{3\varepsilon}{\approx} f(x_0) \quad \text{for } x \approx x_0 \qquad \therefore \quad f(x) \text{ is continuous at } x_0$$

Alternative pf = essentially the above idea: Let $\varepsilon > 0$ be given. Then

$$\begin{aligned}
|f(x) - f(x_0)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\
&\leq \|f - f_N\|_I + |f_N(x) - f_N(x_0)| + \|f - f_N\|_I = 2\|f - f_N\|_I + |f_N(x) - f_N(x_0)| \\
&< 2\varepsilon + |f_N(x) - f_N(x_0)| \quad \text{when } N \gg 1, \text{ since } f_n \rightrightarrows f \text{ on } I \\
&< 2\varepsilon + \varepsilon = 3\varepsilon \quad \text{for } x \underset{\delta=\delta(\varepsilon, x_0)}{\approx} x_0, \text{ since } f_N \text{ is conti at } x_0
\end{aligned}$$

If x_0 is the left (or right) endpoint of I, the above pf can be easily modified by using $x \approx {x_0}^+$ (or $x \approx {x_0}^-$).

(b) $f_n(x) \overset{\text{let}}{=} \sum_0^n u_k(x) \Rightarrow f_n$ is conti on I for $n \geq 0$, since each u_k is conti on I.

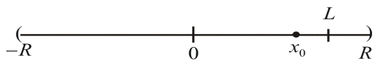
Hypo means: $\quad f_n(x) \rightrightarrows \sum_0^\infty u_k(x) \equiv f(x) \quad$ on I

$$\therefore \overset{\text{(a)}}{\Rightarrow} f(x) \text{ is conti on } I.$$

Cor. Any p.s. $\sum_0^\infty a_n x^n$ represents a continuous function inside its radius of convergence R.

That is, $f(x) \equiv \sum_0^\infty a_n x^n$ is continuous on $(-R, R)$.

Pf. Fix an arbitrary $x_0 \in (-R, R)$, and choose L so that $|\, x_0 \,| < L < R$.



Then $\overset{\text{by a property of } R \text{ (Thm 22.2-C)}}{\Rightarrow}$ $\sum_0^\infty a_n x^n$ converges uniformly on $[-L, L]$

i.e., $\underbrace{\sum_0^n a_k x^k}_{\text{conti on } (-\infty,\, \infty)} \rightrightarrows \sum_0^\infty a_k x^k$ on $[-L, L]$

$\therefore \quad f(x) = \sum_0^\infty a_n x^n$ is conti on $[-L, L]$ (by Thm 22.3-(b))

In particular, $f(x)$ is conti at x_0 since $x_0 \in [-L, L]$.

Since x_0 was an arbitrary point in $(-R, R)$, $f(x)$ is conti on $(-R, R)$.

22.4 Integration term-by-term (항별적분)

Thm A (Integration of a uniform limit)
Assume that, on a finite interval $[a, b]$, every $f_n(x)$ is continuous and $f_n \rightrightarrows f$.

$$\Rightarrow \qquad \int_a^b f_n(x)\,dx \quad \to \quad \int_a^b f(x)\,dx$$

i.e., $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx$ (i.e., limit and integral can be interchanged)

Pf. Every $f_n(x)$ is conti on $[a, b]$ and $f_n \rightrightarrows f$ on $[a, b]$

$\Rightarrow \quad f(x)$ is conti on $[a, b]$ (by Thm 22.3-(a))

$\Rightarrow \quad f(x)$ is integrable on $[a, b]$.

Now

$$\left| \int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx \right| = \left| \int_a^b \Big(f(x) - f_n(x) \Big)\,dx \right| \le \int_a^b \Big| f(x) - f_n(x) \Big|\,dx$$

$$\le \sup_{x\in[a,b]} |\, f(x) - f_n(x) \,| \cdot (b-a) \overset{\text{i.e.}}{=} \| f - f_n \|_{[a,b]} \cdot (b-a) \to 0 \quad \text{since } f_n \rightrightarrows f \text{ on } [a, b]$$

Thm B (Term-by-term Integration of a series)
Assume
(i) for each k, $u_k(x)$ is conti on $[a, b]$

(ii) $\sum_0^\infty u_k(x) = f(x)$ uniformly on $[a, b]$

Then

$$\int_a^b \sum_0^\infty u_k(x)\,dx \left(\overset{\text{i.e.}}{=} \int_a^b f(x)\,dx \right) = \sum_0^\infty \int_a^b u_k(x)\,dx$$

Pf. Hypo says

$$\sum_0^n u_k(x) \text{ is conti on } [a,\, b] \text{ for every } n, \text{ and } \sum_0^n u_k(x) \rightrightarrows \sum_0^\infty u_k(x) \text{ on } [a,\, b]$$

$$\overset{\text{Thm 22.4-A}}{\Rightarrow} \quad \lim_{n\to\infty} \int_a^b \sum_0^n u_k(x)\, dx = \int_a^b \sum_0^\infty u_k(x)\, dx$$

$$\| \leftarrow \text{Linearity thm for integrals}$$

$$\lim_{n\to\infty} \sum_0^n \int_a^b u_k(x)\, dx = \sum_0^\infty \int_a^b u_k(x)\, dx$$

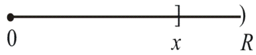
Cor. Any p.s. can be integrated term-by-term inside its open interval of convergence $(-R,\, R)$: that is,

$$\text{if } \sum_0^\infty a_n x^n = f(x) \text{ for } -R < x < R \text{ then}$$

$$\sum_0^\infty \frac{a_n}{n+1} x^{n+1} = \int_0^x f(t)\, dt \quad \text{for } -R < x < R$$

$$\text{(i.e., } \int_0^x \sum_0^\infty a_n t^n \, dt = \sum_0^\infty a_n \int_0^x t^n \, dt \quad \text{for } -R < x < R)$$

Pf. Let $0 < x < R$.



Then $\sum_0^\infty a_n t^n$ converges uniformly on $[0,\, x]$ (by Thm 22.2-C)

$$\overset{\text{Thm 22.4-B}}{\Rightarrow} \quad \int_0^x \sum_0^\infty a_n t^n \, dt = \sum_0^\infty a_n \int_0^x t^n \, dt = \sum_0^\infty \frac{a_n}{n+1} x^{n+1}$$

Same argument gives the result for $-R < x < 0$.

Question. Let $R(> 0)$ be the radius of convergence of the p.s. $\sum_0^\infty a_n x^n$.

What's the radius of convergence of the integrated series $\sum_0^\infty \frac{a_n}{n+1} x^{n+1}$?

Ans: It is R

A popular way 1: Let R' be the radius of convergence of $\sum_{n=0}^\infty \frac{a_n}{n+1} x^{n+1} = \sum_{n=1}^\infty \frac{a_{n-1}}{n} x^n$. Then

$$R' = \frac{1}{\overline{\lim_{n\to\infty}} \sqrt[n]{\frac{|a_{n-1}|}{n}}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} \cdot \overline{\lim_{n\to\infty}} \sqrt[n]{|a_{n-1}|}} = \frac{1}{\frac{1}{\lim_{n\to\infty} \sqrt[n]{n}} \cdot \overline{\lim_{n\to\infty}} \sqrt[n]{|a_{n-1}|}} = \frac{1}{\overline{\lim_{n\to\infty}} \left(|a_{n-1}|^{\frac{1}{n-1} \cdot \frac{n-1}{n}} \right)} = \frac{1}{\overline{\lim_{n\to\infty}} |a_{n-1}|^{\frac{1}{n-1}}} = R$$

Here we used: $a_n \geq 0 \;\&\; b_n \geq 0 (\forall n) \quad \Rightarrow \quad \overline{\lim_{n\to\infty}} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \overline{\lim_{n\to\infty}} b_n$ whenever $\lim_{n\to\infty} a_n$ exists

A popular way 2: Note that $\sum_{n=0}^\infty \frac{a_n}{n+1} x^{n+1} = x \sum_{n=0}^\infty \frac{a_n}{n+1} x^n$. Since multiplication by x does not change the set on which the series $\sum_{n=0}^\infty \frac{a_n}{n+1} x^n$ converges, we have

$$\frac{1}{R'} = \overline{\lim_{n\to\infty}} \sqrt[n]{\frac{|a_n|}{n+1}} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n+1}} \cdot \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{n+1}} \cdot \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = \frac{1}{R} \qquad \therefore\; R' = R$$

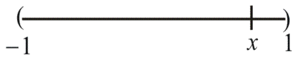
Another way will be given later: we need some theorems about term-by-term differentiation.

Ex. Apply term-by-term integration if possible to

(a) $\dfrac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^n x^n + \cdots, \quad |\,x\,| < 1 \quad (R = 1)$

(b) $x \underset{\text{assume}}{=} \underbrace{\dfrac{\pi}{2} - \dfrac{4}{\pi}\left(\cos x + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^2} + \cdots\right)}_{\text{trigonometric series}}, \quad 0 \le x \le \pi$

Sol. (a) Let $|\,x\,| < 1$.



$$\overset{\text{Thm 22.4 -B}}{\Rightarrow} \quad \underset{\underset{\ln(1+x)}{\|}}{\int_0^x \frac{1}{1+t}\,dt} = \int_0^x 1\,dt - \int_0^x t\,dt + \int_0^x t^2\,dt - \cdots = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

(b) (Note that the RHS of (b) is not a p.s.)

$$\left|\frac{\cos(2n+1)x}{(2n+1)^2}\right| \le \frac{1}{(2n+1)^2} \quad 0 \le \forall x \le \pi \qquad \& \qquad \sum_0^\infty \frac{1}{(2n+1)^2} \text{ converges}$$

$$\overset{\text{Weierstrass M-test}}{\Rightarrow} \quad \sum_0^\infty \frac{\cos(2n+1)x}{(2n+1)^2} \text{ converges uniformly on } [0, \pi]$$

$$\therefore \quad \frac{\pi}{2} - \frac{4}{\pi}\sum_0^\infty \frac{\cos(2n+1)x}{(2n+1)^2} \text{ converges uniformly on } [0, \pi]$$

By Thm 22.4-B, the series can be integrated term-by-term:

$$\int_0^x t\,dt = \int_0^x \frac{\pi}{2}\,dt - \frac{4}{\pi}\left[\int_0^x \cos t\,dt + \int_0^x \frac{\cos 3t}{3^2}\,dt + \cdots\right], \quad 0 \le \forall x \le \pi$$

i.e., $\dfrac{x^2}{2} = \dfrac{\pi x}{2} - \dfrac{4}{\pi}\left(\sin x + \dfrac{\sin 3x}{3^3} + \dfrac{\sin 5x}{5^3} + \cdots\right), \quad 0 \le \forall x \le \pi$

22.5 Differentiation term-by-term (항별미분)

Theorem A (Differentiation of a limit of functions)

Suppose that for every n

$f_n'(x)$ is conti on an interval I,

and

$f_n(x) \longrightarrow f(x)$ and $f_n'(x) \rightrightarrows g(x)$ on I

Then f is diff, and $f'(x) = g(x)$ on I (Therefore, $f_n'(x) \rightrightarrows f'(x)$ on I)

Pf. Fix an arbitrary point $a \in I$.

Since $f_n'(x) \rightrightarrows g(x)$ on I, we have for any $x \in I$

$$\int_a^x f_n'(t)\,dt \quad \longrightarrow \quad \int_a^x g(t)\,dt \quad \text{(by Thm 22.4-A)}$$

$$\|\!\leftarrow \text{1st FTC}$$

$$f_n(x) - f_n(a) \quad \longrightarrow \quad f(x) - f(a) \quad \text{since } f_n \longrightarrow f \text{ pointwise on } I.$$

$$\therefore \quad f(x) - f(a) = \int_a^x g(t)\,dt \quad \text{(by the uniqueness of limit)}$$

Note that g is conti on I because $f_n'(x)$ is conti on I & $f_n'(x) \rightrightarrows g(x)$ on I

2nd FTC implies that $\displaystyle\int_a^x g(t)\,dt$ is diff and $\dfrac{d}{dx}\displaystyle\int_a^x g(t)\,dt = g(x)$ on I

$\therefore \quad f(x)$ is diff and $f'(x) = g(x)$ on I

Theorem B (Term-by-term differentiation of series)

Assume that on an interval I,

(a) for each k, $u_k'(x)$ is conti

(b) $\sum_0^\infty u_k(x)$ converges (pointwise)

(c) $\sum_0^\infty u_k'(x)$ converges uniformly

Then on I,

the sum $f(x) \equiv \sum u_k(x)$ is diff and $f'(x) = \sum u_k'(x)$ (i.e., $\left(\sum u_k(x)\right)' = \sum u_k'(x)$)

Pf. Let $S_n(x) = \sum_0^n u_k(x)$ and $g(x) = \sum_0^\infty u_k'(x)$. Then for every n,

$S_n'(x)$ is conti on I and

$$S_n(x) \to f(x) \quad \text{and} \quad S_n'(x) \rightrightarrows g(x) \quad \text{on } I$$

Thus by Thm 22.5-A,

$f(x)$ is diff and $f'(x) = g(x)$ on I $\therefore\ S_n'(x) \rightrightarrows f'(x)$ on I

In particular, $S_n'(x) \to f'(x)$ on I

$$\therefore \quad f'(x) = \lim_{n\to\infty} S_n'(x) = \lim_{n\to\infty}\left(\sum_0^n u_k(x)\right)' = \lim_{n\to\infty}\sum_0^n u_k'(x) = \sum_0^\infty u_k'(x) \quad \text{on } I$$

Exa. Let $f(x) = \sum_1^\infty \frac{\cos nx}{n^3}$.

Prove that $f(x)$ converges and $f'(x)$ exists $\forall x \in (-\infty, \infty)$, and find $f'(x)$.

Sol. $$\sum_1^\infty \left|\frac{\cos nx}{n^3}\right| \le \sum_1^\infty \frac{1}{n^3} \text{ conv}$$

$$\overset{\text{Comparison thm}}{\Rightarrow} \quad \sum_1^\infty \frac{\cos nx}{n^3} \text{ conv absolutely for } \forall x \in (-\infty, \infty)$$

$$\therefore \quad \sum_1^\infty \frac{\cos nx}{n^3} \text{ converges for } \forall x \in (-\infty, \infty).$$

On the other hand, for every $n \ge 1$

$$\left(\frac{\cos nx}{n^3}\right)' = -\frac{\sin nx}{n^2} \quad \text{is conti } \forall x \in (-\infty, \infty)$$

&

$$\sum_1^\infty \left(\frac{\cos nx}{n^3}\right)' \text{ converges uniformly on } (-\infty, \infty).$$

$$(\because \quad \sum_1^\infty \left(\frac{\cos nx}{n^3}\right)' = \sum_1^\infty \frac{-\sin nx}{n^2} \qquad \& \qquad \sum_1^\infty \left|\frac{-\sin nx}{n^2}\right| \le \sum_1^\infty \frac{1}{n^2} \text{ conv}$$

$$\overset{\text{M-test}}{\Rightarrow} \quad \sum_1^\infty \frac{-\sin nx}{n^2} \text{ conv uniformly on } (-\infty, \infty) \quad)$$

Therefore, by Thm 22.5-B,

$$\left(\sum_1^\infty \frac{\cos nx}{n^3}\right)' = \sum_1^\infty \left(\frac{\cos nx}{n^3}\right)' = -\sum_1^\infty \frac{\sin nx}{n^2} \qquad \forall x \in (-\infty, \infty)$$

Most general version of the previous Theorem A and Theorem B [on Differentiation]

Theorem A [General version] (Differentiation of a limit of functions)
Suppose that for each n

$f_n(x)$ is diff on the interval $[a,b]$,

and

$f_n(x_0) \to f(x_0)$ at some $x_0 \in [a,b]$ and $f_n'(x) \rightrightarrows$ (a fct) $g(x)$ on $[a,b]$

Then $f_n \rightrightarrows \left[\text{a fct} =: \lim_{n\to\infty} f_n =: f\right]$ on $[a,b]$ and f is diff on $[a,b]$.

Moreover, $f'(x) = g(x)$ on $[a,b]$ $\left[\Rightarrow f_n'(x) \rightrightarrows \left(\lim_{n\to\infty} f_n(x)\right)' \text{ for each } x \in [a,b]\right]$

Namely, $f'(x)\left[=\left(\lim_{n\to\infty} f_n(x)\right)'\right] = \lim_{n\to\infty} f_n'(x)$ for each $x \in [a,b]$

Its proof is not easy [A proof can be found in Rudin-PMA]

Theorem B [General version] (Differentiation term-by-term)
Assume that

(a) for each k, $u_k(x)$ is diff on a bounded interval $[a,b]$

(b) $\sum_{k=0}^{\infty} u_k(x_0)$ converges at some point $x_0 \in [a,b]$

(c) $\sum_{k=0}^{\infty} u_k'(x)$ converges uniformly on $[a,b]$

Then $\left(\sum_{k=0}^{\infty} u_k(x)\right)' = \sum_{k=0}^{\infty} u_k'(x)$ for each $x \in [a,b]$

Its proof comes from the above Theorem A [General version]

Weierstrass Theorem [**remember the statement**]:
Let $f \in C[a,b]$[with complex-valued] . Then $\exists$ a sequence of polynomials P_n such that

$P_n \rightrightarrows f$ on $[a,b]$ i.e., $\lim_{n\to\infty} \sup_{x\in[a,b]} |P_n(x) - f(x)| = 0$

If f is real, the P_n may be taken real.

Application: Prove

$f \in C[0,1]$ and $\int_0^1 f(x)x^n dx = 0 \ \forall n = 0,1,2,\cdots \Rightarrow f(x) = 0$ on $[0,1]$

Hint: Use the Weierstrass theorem to show that $\int_0^1 f^2(x)dx = 0$

Return to sophisticated but important examples:

Exa. Show that (the Weierstrass nowhere-diff function) $f(x)=\sum_{n=0}^{\infty}\frac{1}{2^n}\sin(5^n\pi x)$ is continuous on $\mathbb{R}$

Pf. $$\sum_{n=0}^{\infty}\left|\frac{1}{2^n}\sin(5^n\pi x)\right|\le\sum_{n=0}^{\infty}\frac{1}{2^n}\ :\ \text{conv}$$

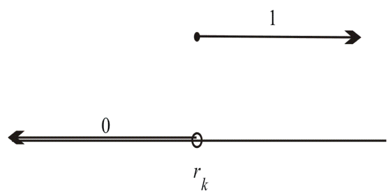
Thus

$$\underbrace{\sum_{n=0}^{\ell}\frac{1}{2^n}\sin(5^n\pi x)}_{\text{conti on }\mathbb{R}}\quad\rightrightarrows\quad\sum_{n=0}^{\infty}\frac{1}{2^n}\sin(5^n\pi x)\ \ \text{(by M-test)}$$

By Uniform Convergence Theorem, $f(x)=\sum_{n=0}^{\infty}\frac{1}{2^n}\sin(5^n\pi x)$ is continuous on $\mathbb{R}$

Exa. Let $r_1,r_2,\cdots,r_n,\cdots$ be an enumeration of the rational numbers & let

$$f_k(x)=\begin{cases}1 & \text{if } x\ge r_k\\ 0 & \text{if } x<r_k\end{cases}\qquad (k=1,2,\cdots)$$



It is clear that each $f_k(x)$ is conti at every point except r_k

Now we define $f(x)=\sum_{k=1}^{\infty}2^{-k}f_k(x)$

Prove:

① $f(x)$ is continuous at every irrational number

② $f(x)$ is discontinuous at every rational number

③ $0\le f(x)\le 1$ for every $x\in\mathbb{R}$

④ $f(x)$ is $\uparrow$ (increasing) on $\mathbb{R}$

Pf. ③ $$0\le f(x)=\sum_{k=1}^{\infty}2^{-k}f_k(x)=\sum_{k=1}^{\infty}2^{-k}\,|\,f_k(x)\,|\le\sum_{k=1}^{\infty}2^{-k}=1$$

④ $x\ge y$

$$\begin{aligned}&\Rightarrow\ f_k(x)\ge f_k(y)\quad(\because\ f_k\ \text{is}\ \uparrow)\\ &\Rightarrow\ 2^{-k}f_k(x)\ge 2^{-k}f_k(y)\\ &\Rightarrow\ \sum_{k=1}^{\infty}2^{-k}f_k(x)\ge\sum_{k=1}^{\infty}2^{-k}f_k(y)\\ &\therefore\ f(x)\ge f(y)\end{aligned}$$

$\therefore$ $f(x)$ is $\uparrow$ (increasing)

① Choose an arbitrary irrational number x_0 and fix it.

We will show that $f(x)$ is continuous at x_0

Note that every $2^{-k} f_k(x)$ is continuous at x_0.

We have seen that

$$| 2^{-k} f_k(x) | \le 2^{-k} \qquad \forall x \in \mathbb{R}$$

$$\& \qquad \sum_{k=1}^{\infty} 2^{-k} \ : \text{converges} \quad \left(\text{in fact, } \sum_{k=1}^{\infty} 2^{-k} = 1 < \infty\right)$$

Thus by **M-test**, $\sum_{k=1}^{\infty} 2^{-k} f_k(x)$ converges uniformly on $\mathbb{R}$.

Since $\underbrace{\sum_{k=1}^{n} 2^{-k} f_k(x)}_{\text{conti at } x_0} \rightrightarrows \sum_{k=1}^{\infty} 2^{-k} f_k(x)$ on $\mathbb{R}$,

the limit function $\sum_{k=1}^{\infty} 2^{-k} f_k(x)$ is conti at x_0 (by Uniform Convergence Theorem)

② Choose any rational number x_0 and fix it. Then $x_0 = r_m$ for some m

We will show that $f(x)$ is not continuous at r_m

Write $f(x) = \frac{1}{2^m} f_m(x) + \sum_{k \neq m}^{\infty} \frac{1}{2^k} f_k(x)$

Recall that each $\frac{1}{2^k} f_k(x)$ is conti at every point x *if* $x \neq r_k$

& disconti at $x = r_k$

Thus if $k \neq m$, then $\frac{1}{2^k} f_k(x)$ is conti at r_m

So $\sum_{k \neq m}^{\infty} \frac{1}{2^k} f_k(x)$ is conti at the point r_m (by **M-test + Uniform Convergence Theorem**)

& clearly $\frac{1}{2^m} f_m(x)$ is disconti at r_m

If $f(x)$ is conti at r_m, then $f(x) - \sum_{k \neq m}^{\infty} \frac{1}{2^k} f_k(x)$ should be conti at r_m.

Then $\frac{1}{2^m} f_m(x)$ is conti at r_m. This is a contradiction.

Therefore, $f(x)$ is not continuous at r_m

HS. Prove that $f(x) = \sum_{n=1}^{\infty} \frac{\cos(2^n x)}{3^n}$ is differentiable everywhere on $\mathbb{R}$, and that

$$f'(x) = -\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \sin(2^n x) \quad \forall x \in \mathbb{R}$$

22.6 Power series and analytic functions

Theorem (Differentiation of P.S.)

Any p.s. $\sum_0^\infty a_n x^n$ can be differentiated term-by-term within its radius R of convergence. That is, if

$$f(x) = \sum_0^\infty a_n x^n, \quad |\, x \,| < R$$

then $f(x)$ is differentiable and

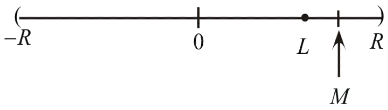
$$f'(x) = \sum_1^\infty n a_n x^{n-1}, \quad |\, x \,| < R$$

Pf. Claim: $\boxed{\text{Let } R' \text{ be the radius of convergence of the p.s. } \sum_1^\infty n a_n x^{n-1}. \text{ Then } R' \geq R}$

To verify the claim, it suffices to show that

$$\boxed{0 < L < R \quad \Rightarrow \quad \sum_1^\infty n a_n x^{n-1} \text{ converges at } x = L}$$

Choose any M such that $0 < L < M < R.$



Note that $\boxed{(*) : \quad 0 < r < 1 \quad \Rightarrow \quad \lim_{n\to\infty} nr^n = 0}$ because

$$\lim_{n\to\infty} nr^n = \lim_{x\to\infty} xr^x = \lim_{x\to\infty} xe^{x\ln r} \underset{\ln r < 0}{=} \lim_{x\to\infty} xe^{-kx} \ (k > 0) = \lim_{x\to\infty} \frac{x}{e^{kx}} \overset{\text{L'Hospital}}{=} 0\,.$$

Since $0 < \dfrac{L}{M} < 1,$ it follows from $(*)$ that $\lim_{n\to\infty} n\left(\dfrac{L}{M}\right)^n = 0$

$\therefore \quad n\left(\dfrac{L}{M}\right)^n < L \quad$ for $n \gg 1,$ say for $n \geq N \qquad$ i.e., $nL^{n-1} < M^n \quad$ for $n \geq N$

$\therefore \quad \sum_N^\infty n \,|\, a_n \,|\, L^{n-1} < \underbrace{\sum_N^\infty |\, a_n \,|\, M^n}$: converges since $\sum_0^\infty a_n x^n$ converges absolutely inside its radius of convergence R and $M < R$

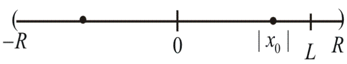
$\therefore \quad \sum_N^\infty n \,|\, a_n \,|\, L^{n-1}$ converges by Comparison theorem. $\quad \therefore \quad \sum_N^\infty n a_n L^{n-1}$ converges.

Thus, by Tail-convergence theorem, $\sum_1^\infty n a_n L^{n-1}$ converges.

i.e., $\sum_1^\infty n a_n x^{n-1}$ converges at $x = L$

Now, let $|\, x_0 \,| < R$ and will show that f is diff at x_0.

Choose L so that $0 < |\, x_0 \,| < L < R.$



Then by Theorem 22.2-C,

$\sum_1^\infty n a_n x^{n-1}$ converges uniformly on $[-L, L]$ since the $R' \geq R > L$

Thus by Theorem 22.5-B,

$$f(x)\ (=\sum_0^\infty a_n x^n)\ \text{ is diff and }\ f'(x)=\sum_1^\infty n a_n x^{n-1}\quad \text{on } [-L, L].$$

In particular, f is diff at x_0 and $f'(x_0)=\sum_1^\infty n a_n {x_0}^{n-1}$

Since x_0 was an arbitrary point satisfying $|x_0|<R$,

$$f \ \text{ is diff and } \ f'(x)=\sum_1^\infty n a_n x^{n-1} \quad \text{on } \ |x|<R.$$

Remark. Indeed, $R'=R$

Pf. Just proved $R'\geq R$. We now show $R'\leq R$. By the definition of R',

$$g(x):=f'(x)=\sum_1^\infty n a_n x^{n-1}=a_1+2a_2x+3a_3x^2+\cdots \ \text{ converges for } \ |x|<R'$$

$$\overset{\text{Cor 22.4}}{\Rightarrow}\quad \underbrace{\int_0^x g(t)\,dt}_{=f(x)-f(0)}=a_1x+a_2x^2+a_3x^3+\cdots=\sum_1^\infty a_n x^n \quad \text{converges for } \ |x|<R'$$

$$\therefore\quad f(x)=f(0)+\sum_1^\infty a_n x^n=a_0+\sum_1^\infty a_n x^n=\sum_0^\infty a_n x^n \quad \text{for } \ |x|<R'$$

$$\therefore\quad R\geq R'$$

Another popular way:

Let R & R' be the resp. radius of convergence of $\sum_{n=0}^\infty a_n x^n$ & $\sum_{n=1}^\infty n a_n x^{n-1} (=\sum_{n=0}^\infty (n+1)a_{n+1}x^n)$. Then

$$R'=\frac{1}{\overline{\lim\limits_{n\to\infty}}\sqrt[n]{(n+1)\,|\,a_{n+1}\,|}}=\frac{1}{\overline{\lim\limits_{n\to\infty}}\left(\{(n+1)\,|\,a_{n+1}\,|\}^{\frac{1}{n+1}\cdot\frac{n+1}{n}}\right)}=\frac{1}{\overline{\lim\limits_{n\to\infty}}\left(\{(n+1)\,|\,a_{n+1}\,|\}^{\frac{1}{n+1}}\right)}=\frac{1}{\overline{\lim\limits_{n\to\infty}}\,|\,a_{n+1}\,|^{\frac{1}{n+1}}}=R$$

Another easy way: Note that $x\sum_{n=1}^\infty n a_n x^{n-1}=\sum_{n=1}^\infty n a_n x^n$. Since multiplication by x does not change the set on which the series $\sum_{n=1}^\infty n a_n x^{n-1}$ converges , we have

$$\frac{1}{R'}=\overline{\lim_{n\to\infty}}\sqrt[n]{n|a_n|}=\lim_{n\to\infty}\sqrt[n]{n}\cdot\overline{\lim_{n\to\infty}}\sqrt[n]{|\,a_n\,|}=\overline{\lim_{n\to\infty}}\sqrt[n]{|\,a_n\,|}=\frac{1}{R}\qquad \therefore\ R'=R$$

Exa A. Evaluate $f(x)=\dfrac{x^2}{1\cdot 2}+\dfrac{x^3}{2\cdot 3}+\cdots+\dfrac{x^n}{(n-1)n}+\cdots=\left(=\sum_2^\infty \dfrac{x^n}{(n-1)n}\right)$

Sol. The radius of convergence for the series is 1 because

$$a_n=\frac{1}{(n-1)n}\quad\Rightarrow\quad R=\frac{1}{\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|}=\lim_{n\to\infty}\frac{n(n+1)}{(n-1)n}=1$$

Thus by term-by-term differentiation (twice)

$$f''(x)=1+x+x^2+x^3+\cdots\quad\text{for } \ |x|<1$$

$$\text{RHS (of the above)}=\frac{1}{1-x}\quad\text{for } \ |x|<1$$

$$\therefore\quad f''(x)=\frac{1}{1-x}\quad\text{for } \ |x|<1$$

Integrate both sides $\Rightarrow$

$$\begin{array}{ccc} \displaystyle\int_0^x f''(t)\,dt & = & \displaystyle\int_0^x \frac{1}{1-t}\,dt, \quad |\,x\,|<1 \\ \| & & \| \\ f'(x)-f'(0) & & -\ln(1-x) \end{array}$$

$$\overset{f'(0)=0}{\Rightarrow} \quad f'(x) = -\ln(1-x), \quad |\,x\,|<1$$

Integrate once again $\Rightarrow$

$$\begin{array}{ccc} \displaystyle\int_0^x f'(t)\,dt & = \displaystyle\int_0^x -\ln(1-t)\,dt = \int_0^x (1-t)'\ln(1-t)\,dt, & |\,x\,|<1 \\ \| & \|\ \text{integration by parts} & \\ f(x)-f(0) \overset{f(0)=0}{=} f(x) & x+(1-x)\ln(1-x), & |\,x\,|<1 \end{array}$$

Cor A (Taylor series theorem) [Every power series (with $R>0$) is a Taylor series]

Let $\sum_0^\infty a_n x^n$ have the radius of convergence $R>0$. Then the function

$f(x) = \sum_0^\infty a_n x^n$ is infinitely diff in $(-R,\, R)$, and

$\sum_0^\infty a_n x^n$ is its Taylor series around $x=0$, that is, $a_n = \dfrac{f^{(n)}(0)}{n\,!}$ for every $n \geq 0$.

Pf.
$$f(x) = \sum_0^\infty a_n x^n, \qquad |\,x\,|<R$$

$$\overset{\text{term-by-term differentiation}}{\Rightarrow} \quad f'(x) = \sum_1^\infty n a_n x^{n-1}, \qquad |\,x\,|<R$$

$$\overset{//\text{ - again}}{\Rightarrow} \quad f''(x) = \sum_2^\infty n(n-1)a_n x^{n-2}, \qquad |\,x\,|<R$$

$$\vdots$$

$$\therefore \quad f \in C^\infty(-R,\, R)$$

To calculate the a_n, we observe that

$$f^{(n)}(x) = n\,!\,a_n + c_1 x + c_2 x^2 + \cdots, \quad |\,x\,|<R \ \text{ for some constants } c_1, c_2, \cdots$$

Putting $x=0 \quad \Rightarrow \quad a_n = \dfrac{f^{(n)}(0)}{n\,!}$

Cor B (Zero theorem for p.s.)

Let $R>0$. If $\sum_0^\infty a_n x^n = 0$ for $|\,x\,|<R$, then $a_n = 0$ for every $n \geq 0$

Pf. $f(x) \overset{\text{let}}{=} \sum_0^\infty a_n x^n \overset{\text{Hypo}}{=} 0$ for $\forall x$ with $|\,x\,|<R$

$\Rightarrow \quad \forall n \geq 0, \quad f^{(n)}(x) = 0$ for $|\,x\,|<R$. In particular, $f^{(n)}(0) = 0 \quad \forall n \geq 0$

$$\therefore \quad a_n \overset{\text{Taylor series thm (CorA)}}{=} \frac{f^{(n)}(0)}{n\,!} = 0 \quad \forall n \geq 0$$

Cor C (Uniqueness of p.s.)

Let $R > 0$. If $\sum_0^\infty a_n x^n = \sum_0^\infty b_n x^n$ for $|x| < R$, then $a_n = b_n$ for every $n \geq 0$

Pf. $\sum_0^\infty a_n x^n \overset{\text{let}}{=} f(x) = \sum_0^\infty b_n x^n$ for $|x| < R$

$$\Rightarrow \quad a_n = \frac{f^{(n)}(0)}{n!} = b_n \quad \text{for every } n \geq 0$$

Exa B. Find a power series solution $y(x)$ to the differential equation

$$\begin{cases} y' + xy = 0 \\ y(0) = 1 \end{cases}$$

Sol. Assume that for some $R > 0$, $y(x)$ has a p.s. representation

$$y = \sum_0^\infty a_n x^n, \quad \text{for} \quad |x| < R$$

Then by using term-by-term differentiation

$$\begin{aligned} y' + xy &= \sum_1^\infty n a_n x^{n-1} + \sum_0^\infty a_n x^{n+1}, \quad \text{for} \quad |x| < R \\ &= \sum_0^\infty (n+1) a_{n+1} x^n + \sum_0^\infty a_{n-1} x^n, \quad \text{where } a_{-1} \overset{\text{def}}{=} 0 \end{aligned}$$

Since $y' + xy = 0$, we get

$$0 = \sum_0^\infty [(n+1)a_{n+1} + a_{n-1}] x^n \quad \text{for} \quad |x| < R$$

$\Rightarrow \quad (n+1)a_{n+1} + a_{n-1} = 0 \quad$ for $n \geq 0$ (by the Zero theorem for p.s.)

Replacing n by $n+1$ $\Rightarrow$

$$(n+2)a_{n+2} + a_n = 0 \quad \text{for } n \geq -1$$

i.e., $a_{n+2} = -\dfrac{a_n}{(n+2)}$ for $n \geq -1$

$a_{-1} = 0$ (by def) $\rightarrow$ $a_1 = 0$ $\rightarrow$ $a_3 = 0$ $\rightarrow$ $\cdots$ $a_{2n-1} = 0$

$a_0 = 1 (\leftarrow y(0) = 1)$ $\rightarrow$ $a_2 = -\dfrac{a_0}{2} = -\dfrac{1}{2}$ $\rightarrow$ $a_4 = -\dfrac{a_2}{4} = \left(-\dfrac{1}{4}\right)\left(-\dfrac{1}{2}\right)$

$\rightarrow \cdots \quad a_{2n} = \left(-\dfrac{1}{2n}\right)\left(-\dfrac{1}{(2n-2)}\right)\cdots\left(-\dfrac{1}{4}\right)\left(-\dfrac{1}{2}\right) = \dfrac{(-1)^n}{2n(2n-2)\cdots 4\cdot 2} = \dfrac{(-1)^n}{2^n n!}$

$$\therefore \quad y = \sum_0^\infty a_{2n} x^{2n} = \sum_0^\infty \frac{(-1)^n}{2^n n!} x^{2n} = \sum_0^\infty \frac{1}{n!}\left(-\frac{x^2}{2}\right)^n = e^{-\frac{x^2}{2}}$$

Remark. Alternative way for finding the solution of the diff equation $\begin{cases} y' + xy = 0 \\ y(0) = 1 \end{cases}$

Sol. $\times\ e^{x^2/2} \quad \Rightarrow \quad \underbrace{e^{x^2/2} y' + x e^{x^2/2} y}_{\left(e^{x^2/2} y\right)'} = 0 \qquad \therefore\ e^{x^2/2} y = c$ (constant)

$$y(0) = 1 \quad \Rightarrow \quad c = 1 \quad \Rightarrow \quad y = e^{-x^2/2}$$

Def. (Real) Analytic functions (= 급수 전개가능한 함수)

A function $f(x)$ which can be represented as a p.s. around the origin with a positive radius of convergence is said to be (real) analytic at the origin. That is, if $\exists\, R > 0$ such that

$$f(x) \underset{\text{can be represented as}}{=} \sum_0^\infty a_n x^n, \quad |\, x \,| < R$$

then we say that f is (real) analytic at the origin.

Fact

- f is (real) analytic at the origin $\Leftrightarrow$ The Taylor series for $f(x)$ around the origin converges to $f(x)$
- f is (real) analytic at the origin $\Rightarrow$ f has a unique p.s. representation

  i.e., if f is (real) analytic at the origin , then $\exists\, R > 0$ such that

$$f(x) = \sum_0^\infty \frac{f^{(n)}(0)}{n!} x^n \quad \forall\, |\, x \,| < R$$

  Accordingly, if f is real analytic at the origin $\overset{\text{Cor 22.6 A}}{\Rightarrow}$ $f \in C^\infty(-R, R)$

Question: $f \in C^\infty(-R, R) \overset{?}{\Rightarrow} f$ is real analytic at the origin

Ans is NO:

$$f(x) \equiv \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$\overset{\text{not hard}}{\Rightarrow} \quad f \in C^\infty(-\infty, \infty) \quad \& \quad \underbrace{f^{(n)}(0) = 0 \ \text{ for } n \geq 0}_{\text{seen earlier}}$$

Suppose f is real analytic at the origin. Then

$$\exists\, R > 0 \ \text{ such that } \ f(x) = \sum_0^\infty \frac{f^{(n)}(0)}{n!} x^n \quad \forall\, |\, x \,| < R$$

$$\overset{f^{(n)}(0)=0 \ \ \forall n \geq 0}{\Rightarrow} \quad f(x) = 0 \quad \forall\, |\, x \,| < R \qquad \text{--- contradiction}$$

Ex. Find an $f \in C^\infty(-\infty, \infty)$, such that

its Taylor series at the origin $\left(= \sum_0^\infty \frac{f^{(n)}(0)}{n!} x^n\right)$ is $\sum_0^\infty \frac{x^n}{n!}$,

yet $f(x) \neq e^x$ for $x \approx 0$

Sol.

$$f(x) \equiv \begin{cases} e^x + e^{-1/x^2} & x \neq 0 \\ e^x & x = 0 \end{cases}$$

$$\Rightarrow \quad f \in C^\infty(-\infty, \infty) \quad \& \quad \sum_0^\infty \frac{f^{(n)}(0)}{n!} x^n = \sum_0^\infty \frac{x^n}{n!} \quad \text{(see the previous example)}$$

But, obviously

$$f(x) \neq e^x \quad \text{for } x \neq 0 \qquad \therefore\ f(x) \neq e^x \quad \text{for } x \approx 0$$

Proposition. Suppose $f \in C^{\infty}(-R,R)$, $R>0$.

If $\exists M>0$ such that $|f^{(n)}(x)| \leq M$ (or M^n) for all $x \in (-R,R)$ & $n \in \mathbb{N}$, then

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k \quad \text{for all } x \in (-R,R)$$

Pf. Fix any $x \in (-R,R)$. Recall $R_{n-1}(x)=\dfrac{f^{(n)}(c)}{n!}x^n$

$$|R_{n-1}(x)|=\frac{|f^{(n)}(c)|}{n!}|x|^n \leq \frac{M}{n!}|x|^n \left[\text{or, } \frac{M^n}{n!}|x|^n\right] \quad \forall n \in \mathbb{N}$$

Notice that $\dfrac{M|x|^n}{n!}\left[\text{or, } \dfrac{M^n|x|^n}{n!}\right] \to 0$ as $n \to \infty$ $[\leftarrow \text{Ratio test}]$

$$\therefore \quad R_{n-1}(x) \to 0 \quad \text{for every } x \in (-R,R)$$

Applications:

(i) $\sin x$ & $\cos x$ are real analytic at $x=0$, and

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots, \qquad \cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$$

(ii) e^x is real analytic at $x=0$, and $e^x=1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\cdots$

Pf. (i) $f(x):=\sin x \Rightarrow |f^{(n)}(x)| \leq 1$ for all $x \in \mathbb{R}$

$\therefore$ f is real analytic at $x=0$, and $f(x)=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\cdots$ holds for all $x \in \mathbb{R}$

(ii) $f(x):=e^x \Rightarrow |f^{(n)}(x)|=|e^x| \leq e^R =: M$ for all $x \in [-R,R]$ & all $n \in \mathbb{N}$

$\therefore$ f is real analytic at $x=0$, and $f(x)=1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\cdots$ on $[-R,R]$

Since $R>0$ was arbitrary, we conclude that

$$f(x)=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots \text{ holds for all } x \in \mathbb{R}$$