8.1 Introduction. Radius of Convergence

Power Series : $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$

Theorem : Radius of Convergence

- For each power series $\sum a_n x^n$, there is a unique $R \geq 0$ such that $\sum a_n x^n$ converges absolutely for $|x| < R$, diverges for $|x| > R$

✲ the number $R$ is called the radius of convergence. By convention, we say $R = \infty$ if the series is absolutely convergent for all $x$

✲ $R = \sup A$

8.2 Convergence at the Endpoints. Abel Summation

- One must replace the real variable $x$ by the complex variable $z$ ; the function on the right is not defined when $z = \pm i$, so the complex series can only converge for $|z| < 1$, which means in turn that the real series can only converge when $|x| < 1$

Abel Summation :

- Suppose $\sum a_n x^n = f(x)$, for $|x| < 1$, where $f(x)$ is defined and continuous at $x = 1$, but the series diverges at $1$. Then we say the series $\sum a_n$ is Abel-summable to $f(1)$, and write $\sum a_n = f(1)$

8.3 Operations on Power Series : addition

- Power series can be formally manipulated by the operations of algebra or calculus – added, multiplied, differentiated, or integrated

Theorem : Linearity Theorem for Power Series

- If $\sum a_n x^n = f(x)$ and $\sum b_n x^n = g(x)$, for $|x| < K$, then for any constants $p$ and $q$,

$\sum (p a_n + q b_n) x^n = p f(x) + q g(x)$, for $|x| < K$

8.4 Multiplication of Power Series

Theorem : Multiplication of Power Series

$\sum a_n x^n = f(x)$ and $\sum b_n x^n = g(x) \Rightarrow \sum c_n x^n = f(x) g(x)$, where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots a_n b_0 = \sum a_i b_j$

Theorem : Multiplication Theorem for Series

- Suppose $\sum a_n$ and $\sum b_n$ converge absolutely, to the sums $A$ and $B$ respectively. Then if we put

$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum_{i+j=n} a_i b_j$, the series $\sum c_n$ converges absolutely to the sum $A \cdot B$