

Chap 10. Local and Global behavior of functions.

10.1 Estimating functions

- ◎ We classify intervals according to their length and endpoints;

length: "finite" or "infinite" [most common terminology; bounded or unbounded]

endpoints: "closed" if the endpoints are included; "open" if they are not

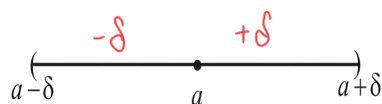
$$\left[\begin{array}{l} (a, b) = \{x : a < x < b\} : \text{finite (bounded) open interval} \\ [a, b] = \{x : a \leq x \leq b\} : \text{finite (bounded) closed interval} \\ (a, \infty) = \{x : x > a\} : \text{infinite (unbounded) open interval} \\ (-\infty, a] = \{x : x \leq a\} : \text{infinite (unbounded) closed interval} \end{array} \right.$$

Convention: We do not consider $(a, a) (= \emptyset)$ to be an interval. That is, any interval $I \neq \emptyset$

- ◎ An interval like $[a, b)$ is called half-open;

the interval $(-\infty, \infty)$, which has no endpoints, is considered as **both open and closed**.

- Def. The open interval $(a - \delta, a + \delta)$ is called the **δ -neighborhood of a** .



We call a its center and δ its radius.

- $x \in (a - \delta, a + \delta) \Leftrightarrow a - \delta < x < a + \delta \Leftrightarrow |x - a| < \delta \stackrel{\text{denote}}{\equiv} \frac{x \approx a}{\delta}$
 $\therefore x \underset{\delta}{\approx} a$ says that x is approximately equal to a if δ is small. $|x - a| < \delta$

- We write $f(I)$ for the set $\{f(x) : x \in I\}$, and we call $f(I)$ the range of $f(x)$ over I , or the image of I under the mapping f . For example if $f(x) = \sin x$, then

$$f((-\infty, \infty)) = [-1, 1], \quad f([0, \pi]) = [0, 1], \quad f((0, \pi)) = (0, 1]$$

Def. Given an interval I on which $f(x)$ is defined, we say

- b is an upper bound for $f(x)$ on $I \Leftrightarrow b$ is an upper bound for $f(I)$
 $\Leftrightarrow f(x) \leq b \quad \text{for } x \in I$

$f(x)$ is bounded above on $I \Leftrightarrow f$ has an upper bound on I

- a is a lower bound for $f(x)$ on $I \Leftrightarrow a$ is a lower bound for $f(I)$
 $\Leftrightarrow f(x) \geq a \quad \text{for } x \in I$

$f(x)$ is bounded below on $I \Leftrightarrow f$ has a lower bound on I

We say $f(x)$ is bounded on I if it is bounded above and bounded below on I .

$$\stackrel{\text{i.e.}}{\Leftrightarrow} \exists b_1 \text{ and } b_2 \text{ such that } b_1 \leq f(x) \leq b_2 \quad \forall x \in I$$

$$\Leftrightarrow \exists \text{ constant } K > 0 \text{ such that } |f(x)| \leq K \quad \forall x \in I$$

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Eg. Which of these functions is bounded below or above?

- (a) $3 \cos x$; $-3 \leq 3 \cos x \leq 3 \quad \therefore$ bounded
- (b) e^{-x} on $[0, \infty)$; $0 < e^{-x} \leq 1$ for $0 \leq x < \infty \quad \therefore$ bounded
- (c) $1 - x^2$; $1 - x^2 \leq 1$ for all $x \quad \therefore$ bounded above
- (d) $\tan x$; not bounded above or below

Def. Suppose $f(x)$ is defined on an interval I . We define

the supremum of $f(x)$ on $I = \sup f(I)$ (notation $\sup_I f(x)$)

the maximum of $f(x)$ on $I = \max f(I)$ (notation $\max_I f(x)$)

The infimum & minimum are defined analogously.

Notice that

$f(x)$ has a maximum on $I \Leftrightarrow \sup_I f(x) = f(\bar{m})$, for some $\bar{m} \in I$

$f(x)$ has a minimum on $I \Leftrightarrow \inf_I f(x) = f(\underline{m})$, for some $\underline{m} \in I$

Eg. Find the sup, inf, max, and min of $f(x)$ over I and J :

$$f(x) = \sin x, \quad I = (-\infty, \infty), \quad J = (0, \pi/2)$$

Sol. Over I , $\sup f(x) = 1 = \max f(x)$, $\inf f(x) = -1 = \min f(x)$

Over J , $\sup f(x) = 1$; $\max f(x)$ does not exist
 $\inf f(x) = 0$; $\min f(x)$ does not exist

Note that $f(I) \neq \emptyset$ since any interval I is assumed to be non-empty.

Theorem (Completeness Property for functions)

Suppose $f(x)$ is defined on an interval I .

If $f(x)$ is bounded above on I , then $\sup_I f(x)$ exists;

If $f(x)$ is bounded below on I , then $\inf_I f(x)$ exists

● Estimating functions: inequalities and absolute values

Recall that $f(x)$ is a real number for each $x \in D_f$. Therefore,

$$|f(x)g(x)| = |f(x)| |g(x)|, \quad |f(x) + g(x)| \leq |f(x)| + |g(x)|$$

&

$$f(x) \text{ is bounded on } I \Leftrightarrow \exists \text{ constant } K > 0 \text{ such that } |f(x)| \leq K \quad \forall x \in I$$

Eg. $f(x)$ and $g(x)$ are bounded on $I \Rightarrow f(x)g(x)$ is bounded on I

Pf. \exists constants K and L such that $|f(x)| \leq K$ and $|g(x)| \leq L$

$$\therefore |f(x)g(x)| = |f(x)| |g(x)| \leq KL (= \text{constant})$$

Eg. $\operatorname{erf} x = \int_0^x e^{-t^2/2} dt$ (the error function)

Show that $\operatorname{erf} x$ is bounded above on $[0, \infty)$

Pf. Since $e^{-t^2/2} > 0$, $\int_0^x e^{-t^2/2} dt$ is \uparrow on $[0, \infty)$.

Thus, it suffices to show that

$$\int_0^x e^{-t^2/2} dt \text{ is bounded above for } x \geq 1$$

Since $\operatorname{erf} x = \underbrace{\int_0^1 e^{-t^2/2} dt}_{\text{= a fixed finite value or, clearly } \leq 1} + \int_1^x e^{-t^2/2} dt$ for $x \geq 1$,

it suffices to show the last integral is bounded above for $x \geq 1$.

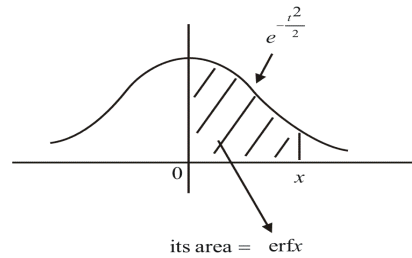
Note that

$$\begin{aligned} t^2 &\geq t \text{ for } t \geq 1 \\ \Rightarrow e^{t^2/2} &\geq e^{t/2} \text{ for } t \geq 1 \text{ since } e^x \text{ is } \uparrow \\ \Rightarrow e^{-t^2/2} &\leq e^{-t/2} \text{ for } t \geq 1 \end{aligned}$$

Thus the above implies that, for $x \geq 1$,

$$\int_1^x e^{-t^2/2} dt \leq \int_1^x e^{-t/2} dt = -2e^{-t/2} \Big|_1^x \leq 2e^{-1/2}$$

$$\therefore \int_1^x e^{-t^2/2} dt \text{ is bounded above for } x \geq 1.$$



10.2 Approximating functions

Notation:

$$\begin{aligned}
 |f(x) - g(x)| < \varepsilon \text{ for } x \in I &\Leftrightarrow f(x) \underset{\varepsilon}{\approx} g(x) \text{ for } x \in I \\
 &\overset{\text{inequality form}}{\Leftrightarrow} g(x) - \varepsilon < f(x) < g(x) + \varepsilon \text{ for } x \in I \\
 &\overset{\text{the error form}}{\Leftrightarrow} f(x) = g(x) + e(x), \text{ where } |e(x)| < \varepsilon \text{ for } x \in I
 \end{aligned}$$

EgA. Use $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ for all $x \in \mathbb{R}$ to find a δ -neighborhood of 0 over

which $\sin x \underset{\varepsilon}{\approx} x$ with $\varepsilon = 0.001$

Sol. Since $\sin x$ and x are odd functions, we can do the work for $x \geq 0$.

$$\sin x = x - \underbrace{\frac{x^3}{3!} + \frac{x^5}{5!} - \dots}_{\text{alternating series}} \text{ for any fixed } x \in [0, 1]$$

By the Alternating series test, we get

$$\begin{aligned}
 |\sin x - x| &\leq \frac{x^3}{3!} = \frac{x^3}{6}, \text{ for } 0 < x < 1 \\
 \therefore &< 0.001 \text{ if } x^3 < 0.006 \text{ (i.e., } x < \sqrt[3]{0.006} \doteq 0.18171)
 \end{aligned}$$

Thus if we take $\delta = 0.18$, then $\sin x \underset{0.001}{\approx} x$

© Elementary inequality for definite integral (will be proved later):

$$\left. \begin{aligned} &f(x) < g(x) \text{ for } x \in I \quad \& \\ &\int_a^b f(x) dx \text{ \& } \int_a^b g(x) dx \text{ exists for } a, b \in I \text{ with } a < b \end{aligned} \right\} \Rightarrow \int_a^b f(x) dx < \int_a^b g(x) dx$$

• Assume $a < b$. Then

$$(*) \quad f(x) \underset{\varepsilon}{\approx} g(x) \text{ on } [a, b] \Rightarrow \int_a^b f(x) dx \underset{\varepsilon(b-a)}{\approx} \int_a^b g(x) dx$$

Pf. $f(x) \underset{\varepsilon}{\approx} g(x)$ for $x \in [a, b]$

$$\Rightarrow g(x) - \varepsilon < f(x) < g(x) + \varepsilon \text{ for } x \in [a, b]$$

$$\Rightarrow \int_a^b (g(x) - \varepsilon) dx < \int_a^b f(x) dx < \int_a^b (g(x) + \varepsilon) dx$$

$$\text{i.e., } \int_a^b g(x) dx - \varepsilon(b-a) < \int_a^b f(x) dx < \int_a^b g(x) dx + \varepsilon(b-a)$$

$$\Rightarrow \int_a^b f(x) dx \underset{\varepsilon(b-a)}{\approx} \int_a^b g(x) dx$$

EgB. Estimate the error in $\cos x \approx 1 - \frac{x^2}{2}$ for $|x| < 0.1$

Sol. From the result in EgA, we know

$$\sin x \underset{0.001}{\approx} x \text{ for } |x| < 0.18, \text{ so for } |x| < 0.1$$

$$\stackrel{(*)}{\Rightarrow} \int_0^x \sin t \, dt \underset{0.001x}{\approx} \int_0^x t \, dt \text{ for } 0 < x < 0.1$$

$$\therefore 1 - \cos x \underset{0.0001}{\approx} \frac{x^2}{2} \text{ for } 0 < x < 0.1$$

Since $1 - \cos x$ & $\frac{x^2}{2}$ are even, it is also true that

$$1 - \cos x \underset{0.0001}{\approx} \frac{x^2}{2} \text{ for } -0.1 < x < 0$$

$$\therefore \cos x \underset{0.0001}{\approx} 1 - \frac{x^2}{2} \text{ for } |x| < 0.1$$

10.3 Local behavior

To study the **continuity** or **differentiability** of $f(x)$ at x_0 , we need to its local behavior near x_0 .
(i.e., its behavior in some δ -nbd of x_0)

For that purpose, we use the notation

$$\text{for } x \approx x_0 \quad \text{or} \quad \text{for } x \text{ near } x_0$$

which means

$$\text{for } x \text{ in some } \delta\text{-nbd of } x_0 \quad (\text{i.e., } x \underset{\delta}{\approx} x_0 \text{ for some } \delta > 0)$$

EgA. T or F ?

$$(a) \quad x^4 \leq x^2 \text{ for } x \approx 0 \quad (b) \quad x^3 \leq x \text{ for } x \approx 0 \quad (c) \quad x^3 \leq x \text{ for } x \approx 0, x \geq 0$$

Sol. (a) is True because $x^4 \leq x^2$ for $|x| < 1$ ($\therefore x^4 \leq x^2$ for $x \underset{1}{\approx} 0$)

(b) is **False** because $x^3 > x$ for $-1 < x < 0$

(c) is True because $x^3 \leq x$ for $0 \leq x < 1$

EgB. If $f(x)$ and $g(x)$ are bounded for $x \approx x_0$, so is $f(x) + g(x)$

Pf. By hypo,

$$|f(x)| < K \text{ for } x \approx x_0, \quad \text{say for } |x - x_0| < \delta'$$

$$\& \quad |g(x)| < L \text{ for } x \approx x_0, \quad \text{say for } |x - x_0| < \delta''$$

Thus, $|f(x) + g(x)| \leq |f(x)| + |g(x)| < K + L$ for $|x - x_0| < \min\{\delta', \delta''\} \equiv \delta$

Therefore, $f(x) + g(x)$ is bounded for $x \approx x_0$

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● Behavior at infinity

Sometimes, one wants to know the behavior of $f(x)$ on some interval like (a, ∞) or $(-\infty, a)$.

For that purpose, we use the notation

for $x \gg 1$, for x large, for x in some interval (a, ∞)
 for $x \ll -1$, for negatively large x , for x in some interval $(-\infty, a)$
 for $|x| \gg 1$, for large $|x|$, for $|x| >$ some positive number a

where in each case, an appropriate value of a exists, but is unspecified.

We can also use the terminology like

local behavior of $f(x)$ at ∞ (= for $x \gg 1$)
 local behavior of $f(x)$ at $-\infty$ (= for $x \ll -1$)
 local behavior of $f(x)$ at $\pm\infty$ (= for $|x| \gg 1$)

EgC. Let $f(x)$ be a polynomial with positive leading coefficient. That is,

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n, \quad a_0 > 0.$$

Then

- (a) $f(x) > 0$ at ∞ (i.e., $f(x) > 0$ for $x \gg 1$)
- (b) if n is even, then $f(x) > 0$ at $-\infty$ (i.e., $f(x) > 0$ for $x \ll -1$);
 if n is odd, then $f(x) < 0$ at $-\infty$
- (c) $\frac{1}{f(x)}$ is bounded at $\pm\infty$ (i.e., for $|x| \gg 1$)

● Local properties at a point

$f(x)$ is *locally increasing* at x_0 means $f(x)$ is inc for $x \approx x_0$

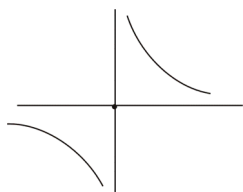
$f(x)$ is *locally bounded* at x_0 means $f(x)$ is bounded for $x \approx x_0$

$f(x)$ is *locally positive* at x_0 means $f(x)$ is positive for $x \approx x_0$

EgD. (Easy to prove)

- (a) $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ (with const term $a_n > 0$) is locally positive at 0.
- (b) $\sin x$ is locally inc at every $x_0 \in (-\pi/2, \pi/2)$, but not at $\pm\pi/2$.

- (c) The function $f(x) = \begin{cases} 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$ is locally bounded at any $x_0 \neq 0$, but not at 0.



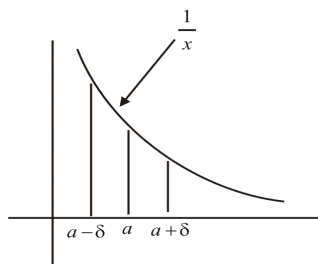
10.4 Local and global properties of functions

Def. We say that $f(x)$ is locally bounded on the open interval I if it is locally bounded at every point of I :

for all $x_0 \in I$, $f(x)$ is bounded for $x \approx x_0$

EgA. Show that $\frac{1}{x}$ is locally bounded on $(0, \infty)$

Sol.



For any $x_0 = a > 0$, take $\delta = \frac{a}{2}$. Then

$$a - \delta < x < a + \delta \quad (\text{i.e., } \frac{a}{2} < x < \frac{3a}{2}) \quad \Rightarrow \quad \frac{2}{3a} < \frac{1}{x} < \frac{2}{a}$$

$\therefore f(x)$ is bounded for $x \approx_\delta a$ where $\delta = \frac{a}{2}$.

Remark. $\frac{1}{x}$ is not bounded on $(0, \infty)$ since $\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$

Def. We say $f(x)$ is locally inc on an open interval I if it is locally inc at every point $x_0 \in I$

EgB. $f(x) = \tan x$ is locally inc on every interval of its domain: $(-\frac{\pi}{2}, \frac{\pi}{2}), (\frac{\pi}{2}, \frac{3\pi}{2}), \dots$

However, it is not increasing ($\because f(\pi/4) = 1 > 0 = f(\pi)$)

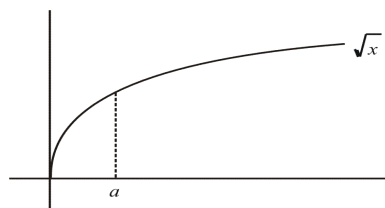
Remark(on local behavior at endpoints). In the preceding definitions, if $I = [a, b]$,

replace “for $x \approx a$ ” by “for $x \approx a, x \geq a$ ” (notation: for $x \approx a^+$)

“for $x \approx b$ ” by “for $x \approx b, x \leq b$ ” (notation: for $x \approx b^-$)

Thus we say f is locally inc at the left endpoint of $[a, b]$ if $f(x)$ is inc for $x \approx a^+$

EgC. Show that \sqrt{x} is locally bounded on $[0, \infty)$



Sol. If $a > 0$, take $\delta = a$, say; then $0 \leq \sqrt{x} \leq \sqrt{2a}$ for $x \approx_{\delta} a$

If $a = 0$, take $\delta = 1$, say; then $0 \leq \sqrt{x} \leq 1$ for $x \approx_{\delta} 0^+$

Note that \sqrt{x} is not bounded on $[0, \infty)$ since $\lim_{x \rightarrow \infty} \sqrt{x} = \infty$

More precisely, if \sqrt{x} is bounded on $[0, \infty)$, then

$\exists M > 0$ such that

$$|\sqrt{x}| = \sqrt{x} \leq M \quad \forall x \in [0, \infty)$$

Let $x \rightarrow \infty \Rightarrow LHS \rightarrow \infty$, but $RHS (= M)$ is a fixed positive number; a contradiction

Local vs Global

- local property:

A property is **local** if to verify that it holds on an interval, it is enough to check that it holds in a **nbhd of each point** on this interval; (for ex, locally inc or locally bdd on I)

- global property:

A property is **global** if to see if it holds on an interval I , one must *look at* the function on the interval I **as a whole**; (for example, f is bounded on I or f is periodic on $(-\infty, \infty)$)

© In general, if a property P is global, then P is also local.

For example,

$$f(x) \text{ is bounded on } I \Rightarrow f(x) \text{ is locally bounded on } I$$

$\not\Leftarrow$

$$\left(f(x) = \frac{1}{x} \text{ on } (0, \infty) \quad \text{or} \quad \sqrt{x} \text{ on } [0, \infty) \right)$$

Theorem (will be proved later)

Let $f(x)$ be a function defined on a **closed and bounded** interval $I = [a, b]$. Then

$$f(x) \text{ is locally bounded on } I \Rightarrow f(x) \text{ is (globally) bounded on } I$$

- pointwise property:

This is a property of $f(x)$ on an interval I which can be verified **point-by-point** in I ;

for example, *positivity* is a pointwise property

$$(\because f(x) \text{ is positive on } I \Leftrightarrow f(a) > 0 \text{ for each } a \in I)$$