1) $\cancel{E(X|X>1) = \int_0^{\infty} x \cdot f(x|x>1)\, dx}$

$\cancel{f(x>1) = \lambda e^{-\lambda(x-1)}}$

$$E(X|X>1) = \int_1^{\infty} x\lambda e^{-\lambda x}\, dx$$

$$= -e^{-\lambda x} - \frac{1}{\lambda}e^{-\lambda x}\Big|_0^{\infty}$$

$$= 1 + \frac{1}{\lambda}$$

λx		$e^{-\lambda x}$
λ	$+$	$-\frac{1}{\lambda}e^{-\lambda x}$
0	$-$	$\frac{1}{\lambda^2}e^{-\lambda x}$

2) $C = \dfrac{1}{\Gamma(s)(\frac{1}{\alpha})^{s}}$ if $\int_0^\infty C e^{-\alpha y} y^{s-1}\, dy = 1$

$$f(x|y) = \frac{e^{-y} y^{x}}{x!}$$

$$f(x|y)\cdot f(y) = f(x,y) = \frac{e^{-y} y^{x}}{x!} \cdot \frac{y^{s-1} e^{-\alpha y}}{\Gamma(s)(\frac{1}{\alpha})^{s}}$$

$$= \frac{e^{-y(1+\alpha)}\, y^{x+s-1}}{x!\,\Gamma(s)\,(\frac{1}{\alpha})^{s}}$$

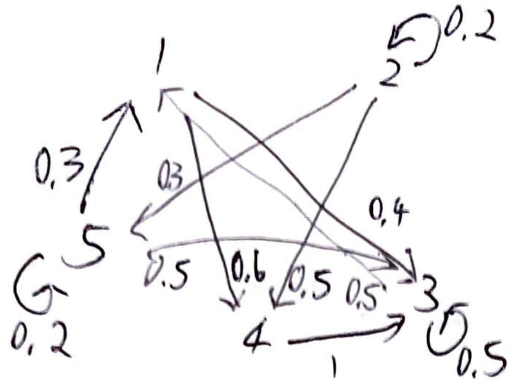
$$f(x) = \int_0^\infty \frac{y^{x+s-1} e^{-\frac{y}{\frac{1}{1+\alpha}}}}{x!\,\Gamma(s)\,(\frac{1}{\alpha})^{s}}\, dy$$

$$= \frac{\Gamma(x+s)(\frac{1}{1+\alpha})^{x+s}}{x!\,\Gamma(s)(\frac{1}{\alpha})^{s}} \underbrace{\int_0^\infty \frac{y^{x+s-1} e^{-\frac{y}{\frac{1}{1+\alpha}}}}{\Gamma(x+s)\,(\frac{1}{1+\alpha})^{x+s}}\, dy}_{=1}$$

$$= \frac{(x+s-1)!}{x!\,(s-1)!} \left(\frac{1}{1+\alpha}\right)^{x} \left(\frac{\alpha}{1+\alpha}\right)^{s}$$

$X \sim NegBin(x+s, \frac{1}{1+\alpha})$

3)



{1, 3} : closed, aperiodic, positive recurrent

{2} : open, aperiodic, transient

{4} : open, periodic, transient

{5} : open aperiodic, transient

4) $$P = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

a)

b)

c)

5) $\pi_L = 0.4\pi_L + 0.05\pi_m + 0.05\pi_u \Rightarrow 0.6\pi_L = 0.05\pi_m + 0.05\pi_u$

$\pi_M = 0.4\pi_L + 0.7\pi_m + 0.5\pi_u \Rightarrow 0.3\pi_m = 0.4\pi_L + 0.5\pi_u$

$\pi_u = 0.2\pi_L + 0.25\pi_m + 0.45\pi_u \Rightarrow 0.55\pi_u = 0.2\pi_L + 0.25\pi_m$

$\pi_L = 0.0833\pi_m + 0.0833\pi_u$

$0.3\pi_m = 0.4(0.0833\pi_m + 0.0833\pi_u) + 0.5\pi_u$

$= 0.03332\pi_m + 0.03332\pi_u + 0.5\pi_u$

$0.26668\pi_m = 0.53332\pi_u$

$\pi_m = 1.9999\pi_u$

$0.55\pi_u = 0.2(0.0833\pi_m + 0.0833\pi_u) + 0.25(1.9999\pi_u)$

$= 0.2(0.1666\pi_u + 0.0833\pi_u) + 0.49998\pi_u$

$\Rightarrow \pi_m = 0.6154$

6) X_n : the number of black balls after the n^{th} interchange

$\Rightarrow S = \{0, 1, 2, 3, 4\}$

a)
$$\mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{16} & \frac{6}{16} & \frac{9}{16} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{9}{16} & \frac{6}{16} & \frac{1}{16} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

0B 4W

1B 3W   3B 1W

2B 2W   2B 2W

3B 1W   1B 3W

b) $\pi_0 = \frac{1}{16}\pi_1$ Given: $\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$

$\pi_1 = \frac{6}{16}\pi_1 + \frac{1}{4}\pi_2 + \pi_0 \Rightarrow \pi_0 = \frac{5}{8}\pi_1 - \frac{1}{4}\pi_2$

$\pi_2 = \frac{9}{16}\pi_1 + \frac{1}{2}\pi_2 + \frac{9}{16}\pi_3 \Rightarrow \pi_2 = \frac{9}{8}\pi_1 + \frac{9}{8}\pi_3$

$\pi_3 = \frac{1}{4}\pi_2 + \frac{6}{16}\pi_3 + \pi_4$

$\pi_4 = \frac{1}{16}\pi_3$

$\Rightarrow \frac{9}{16}\pi_3 = \frac{1}{4}\pi_2$

$\pi_3 = \frac{4}{9}\pi_2$

$\pi_4 =$

$\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$

$\left(\frac{5}{8}\pi_1 - \frac{1}{4}\pi_2\right) + \pi_1 + \pi_2 + \frac{4}{9}\pi_2 + \pi_4 = 1$

$\therefore$ $\pi_0 = 0.01429$, $\pi_1 = 0.2294$, $\pi_2 = 0.5143$,

~~$\pi_1 = 0.2285$~~ $\pi_3 = 0.22779$, $\pi_4 = 0.01419$

7) a) $\cancel{P(X_2=3) = P(X_2=3 \mid X_0=1)P(X_0=1) + P(X_2=3 \mid X_0=2) + P(X_2=3 \mid X_0=3)}$
$\cancel{+P(X}$

$P(X_2=3) = P(X_2=3 \mid X_0=1)\,P(X_0=1)$, since $P(X_0=1)=1$ is given,

$= P_{13}^{(2)} = 0.12$

b) $$P(X_3=1, X_2=4, X_1=2) = \frac{P(X_3=1, X_2=4, X_1=2)}{P(X_2=4, X_1=2)} \cdot \frac{P(X_2=4, X_1=2)}{P(X_1=2)} \cdot P(X_1=2)$$

$$= P(X_3=1 \mid X_2=4) \cdot P(X_2=4 \mid X_1=2) \cdot P(X_1=2)$$

$$= 0.1 \cdot 0.5 \cdot 0.13$$

$$= 0.0065$$

c) $\cancel{P(X_1=2)}$ $$P(X_1=2 \mid X_2=4, X_3=1) = \frac{P(X_3=1, X_2=4, X_1=2)}{P(X_2=4, X_3=1)} \cdot \frac{P(X_2=4, X_1=2)}{P(X_2=4, X_1=2)}$$

$$= P(X_3=1 \mid X_2=4) \cdot \frac{P(X_2=4, X_1=2)}{P(X_2=4, X_3=1)} \cdot \frac{P(X_1=2)}{P(X_1=2)} \cdot \frac{P(X_2=4)}{P(X_2=4)}$$

$$= P(X_2=4 \mid X_1=2) \cdot \frac{P(X_1=2)}{P(X_2=4)}$$

$$= 0.5 \cdot \frac{0.3}{0.33}$$

$$= 0{,}45$$

d) $P(X_7=3 \mid X_5=4, X_3=2) = P(X_7=3 \mid X_5=4)$, by Markov Property

$= 0.18$

e) $$E(X_3^2) = 1^2 \cdot P(X_3=1) + 2^2 \cdot P(X_3=2) + 3^2 \cdot P(X_3=3) + 4^2 P(X_3=4)$$

$$= 0.326 + 4 \cdot 0.159 + 9 \cdot 0.216 + 16 \cdot 0.299$$

$$= 7.69$$