Cluster Algebra Cryptography

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1 Introduction

Asymmetric encryption algorithms are based on the idea of a "one-way function", which is a function that is easy to compute, but difficult to reverse. For example, it is easy to multiply two numbers together but hard to factor a given number. This project looks to create such a one-way function based on cluster algebras and see how useful it is as a cryptographic algorithm.

2 Background

2.1 Cluster Algebras

Let $n \in \mathbb{Z}_{\geq 1}$ and $\mathcal{F}$ be the field of rational functions over $\mathbb{Q}$ in n variables. A **seed** of $\mathcal{F}$ is a pair $\Sigma = (\mathbf{x}, Q)$ where $\mathbf{x} = \{x_1, ..., x_n\}$ generates $\mathcal{F}$ freely and Q is a finite quiver with n vertices. We assume Q has no loops and no 2-cycles. Instead of Q, we will often use a skew-symmetric matrix $B = (b_{ij})$ to display the same information as Q where

$$b_{ij} = \#\{\text{arrows } i \to j\} - \#\{\text{arrows } j \to i\}.$$

The quiver Q and matrix B are called the **exchange quiver/matrix**.

For $1 \leq k \leq n$, we define a new seed $\mu_k(\Sigma) = (\mu_k(\Sigma), \mu_k(Q))$ called the **mutation of Σ in direction k**. We have $\mu_k(\mathbf{x}) = (x_1, ..., x_k^*, ..., x_n)$ where

$$x_k^* = \frac{\prod_{i \to k} x_i + \prod_{k \to i} x_i}{x_k} = \frac{\prod_{b_{ik}>0} x_i + \prod_{b_{ik}<0} x_i}{x_k}.$$

These mutation equations are called **exchange relations**. The quiver $\mu_k(Q)$ is constructed from Q by

1. adding a new arrow $i \to j$ for each pair of arrows $i \to k$ and $k \to j$;
2. reversing the orientation of every arrow with target or source k;
3. and erasing every pair of opposite arrows.

In terms of matrices, we obtain a new matrix $\mu_k(B) = (b'_{ij})$ where

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \operatorname{sgn}(b_{ik}) \max\{0, b_{ik}b_{kj}\} & \text{otherwise.} \end{cases}$$

Note that $\mu_k(\mu_k(\Sigma)) = \Sigma$.

If we have a seed $\Sigma = (\mathbf{x}, Q)$, then $\mathbf{x} = \{x_1, ..., x_n\}$ is called a **cluster** and its elements are called **cluster variables**. The **cluster monomials** are the elements that are a product of cluster variables in a fixed cluster. The **cluster algebra** is defined as the subalgebra generated by all cluster variables.

It is proved in [FZ03] that a cluster algebra has finitely many distinct cluster variables if and only if there exists a seed whose quiver is of Dynkin type. Furthermore, the classification of cluster algebras with finitely many exchange matrices is shown in [FST12].

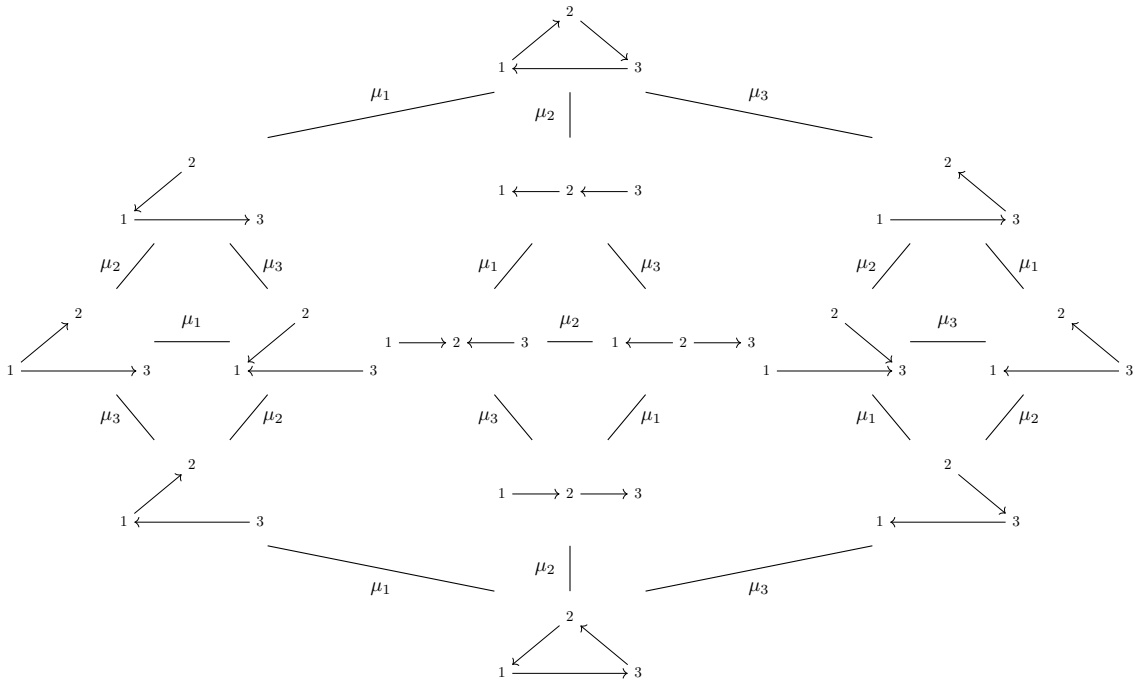
Example 2.1. Suppose Q is an A_3 quiver with the following orientation:

$$Q = 1 \to 2 \to 3.$$

The associated matrix B is

$$B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

There are a total of 14 different seeds. We present them as well as what they mutate to below:



From the example, there are a number of things to note. We see that mutation is not a commutative operation in general, that is, $\mu_i\mu_j\Sigma \neq \mu_j\mu_i\Sigma$. In addition, the number of edges does not remain constant so it is not an invariant to the mutation class of a seed. There is a clear duality or symmetry we see among the exchange quivers. We see that for an exchange quiver Q in the example, its dual quiver Q^*, which is derived from Q by reversing all edges, is also an exchange quiver for the same cluster algebra. Another symmetry is if we "placed a mirror" along the 2 vertex of Q and reflected all edges across the mirror, we again obtain an exchange quiver. The question is whether these duality observations are true in general as they could undermine the security of a cryptoalgorithm based on cluster algebras.

We see that mutation of a seed is quite simple, either graphically using a quiver or algebraically with matrices, but given two seeds, it is in general difficult to see if they are mutation equivalent. This will be the basis of how we will create a cryptographic algorithm.

2.2 Asymmetric Encryption

A lot of popular asymmetric encryption algorithms such as RSA and ECC are based on the idea of a "one-way function", but these use commutative properties, such as the product of numbers in RSA and an abelian group in ECC. However, the mutation of seeds is not commutative, that is in general, $\mu_i\mu_j\Sigma \neq \mu_j\mu_i\Sigma$, even up to reordering.

The following is a key exchange protocol using a non-abelian group G created by Anshel-Anshel-Goldfeld [AAG99] (for $a, b \in G$, we denote $b^a := a^{-1}ba$):

1. Select and publish $a_1, ..., a_k, b_1, ..., b_m \in G$.

2. A picks a secret word x in $a_1, ..., a_k$. We view $x = x(a_1, ..., a_k)$ as a function in the a_i's.

3. A sends $b_1^x, ..., b_m^x$ to B.

4. B picks a secret word y in $b_1, ..., b_m$. We view $y = y(b_1, ..., b_m)$ as a function in the b_i's.

5. B sends $a_1^y, a_2^y, ..., a_k^y$ to A.

6. A computes $x(a_1^y, ..., a_k^y) = y^{-1}xy$, and then $x^{-1}y^{-1}xy$.

7. B computes $y(b_1^x, ..., b_m^x) = x^{-1}yx$, and then $y^{-1}x^{-1}yx = (x^{-1}y^{-1}xy)^{-1}$.

8. A and B use the shared key $K = x^{-1}y^{-1}xy$.

Fix a seed Σ of n variables. We can view mutation operations on Σ as the free group G on the set $\{\mu_i : 1 \leq i \leq n\} \cup \{\mu_0\}$ where μ_0 is the identity element representing no mutation. We will denote $\mu_i\mu_j$ to be mutation Σ first in direction j and then in direction i. Since $\mu_i\mu_i\Sigma = \Sigma$, then $\mu_i^{-1} = \mu_i$ so G is a group. Instead of using the entire seed Σ, we can just use the mutation quiver or matrix. If the mutation matrix is chosen so that it is not mutation equivalent to one of the matrices in [FST12], then we are guaranteed G to be infinite.

References

[AAG99] Iris Anshel, Michael Anshel, and Dorian Goldfeld. An algebraic method for public-key cryptography. Math. Res. Lett., 6:287–291, 1999. 2

[FST12] Anna Felikson, Michael Shapiro, and Pavel Tumarkin. Skew-symmetric cluster algebras of finite mutation type. J. Eur. Math. Soc., 14:1135–1180, 2012. 1, 3

[FZ03] Sergey Fomin and Andrei Zelevinsky. Cluster algebras II: Finite type classification. Invent. Math., 154:63–121, 2003. 1