Standard Canonical Form for Linear Operators over $\mathbb{C}((t))$

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Abstract

In this paper we look at linear operators over the field $\mathbb{C}((\mathrm{t}))$ of formal power series and we provide a canonical form (which we call *standard canonical form*) for such operators based on the theorem of *Newton-Puiseux*. To do this we introduce the *standard matrix* of an irreducible polynomial and use it to show how the standard canonical form is related to the characteristic polynomial.

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1 Introduction

Let K be a field and let V be a finite-dimensional vector space over K. Then every linear operator $T : V \to V$ over K can be written in a specific form, namely its **rational canonical form** (also called the Frobenius Canonical Form). This canonical form is obtained from companion matrices that are derived from the characteristic polynomial of T. However, in the case of an algebraically closed field such as $\mathbb{C}$, one can provide a simpler form that is as close to a diagonal matrix as possible, called the **Jordan canonical form**. There is a basis $\beta \subset V$ such that $[T]_\beta$ is a block sum of matrices of the form

$$A = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix} \tag{1.1}$$

where λ is an eigenvalue of T. This is due to the fact that in this case, the characteristic polynomial splits into linear factors. See Chapter 12 of [1]. It should be emphasized that two linear operators S and T, over $\mathbb{C}$, are similar if and only if they have the same Jordan canonical form.

When we consider the field of real numbers $\mathbb{R}$, we are immediately faced with the fact that it is not algebraically closed. However, it does support a canonical form similar to the Jordan canonical form, called the **real canonical form**. In this case there is a basis $\beta \subset V$ such that $[T]_\beta$ is a block sum of matrices of the form

$$A = \begin{pmatrix} B & I & 0 & \dots & 0 \\ 0 & B & I & \dots & 0 \\ 0 & 0 & B & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & B \end{pmatrix} \tag{1.2}$$

where B is either as above and I is the 1 by 1 identity matrix, or $B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $b \neq 0$, and I is the 2 by 2 identity matrix.

This bifurcation for $\mathbb{R}$ results directly from the following well-known fact about the real numbers. Given an irreducible monic polynomial p over $\mathbb{R}$, the degree of p is either one or two. If p has degree one, then any block matrix associated with p is of the form (1.1). If p instead has degree two, then its roots are in complex conjugate pairs. Indeed, every monic, irreducible quadratic over $\mathbb{R}$ can be written as $p(x) = x^2 - 2ax + a^2 + b^2 = (x - (a + ib))(x - (a - ib))$ where $a, b \in \mathbb{R}$ and $b \neq 0$. In this case any block matrix associated with p is of the form (1.2). See Section 3.4 and Theorem 3.4.5 of [3] for more information about the real canonical form.

A decisive property of the real numbers $\mathbb{R}$ is that it has only one field extension of finite degree other than itself, namely the complex numbers $\mathbb{C}$. The field $\mathbb{C}$ is another familiar field with a decisive property (algebraically closed). In both cases, we have fields that are "special" in the sense that we can describe explicitly all the finite field extensions. This special property leads to a dramatic simplification of the discussion of canonical form for linear operators.

We now consider **the field of formal power series** over the complex numbers.

$$\mathbb{C}((t)) = \{\Sigma_{i \geq k} a_i t^i \mid a_i \in \mathbb{C}\}.$$

These include any power series over $\mathbb{C}$ where there is a lower bound on the exponents of the non-zero terms. So $\Sigma_{i \geq -5} t^i$ is allowed, but $\Sigma_{i \leq 0} t^i$ is not allowed. By the **Newton-Puiseux Theorem**, $\mathbb{C}((t))$ has exactly one field extension of degree n for any $n > 0$, namely $\mathbb{C}((\theta))$ where $\theta^n = t$. See Corollary 13.15 of [2]. In particular, the algebraic closure of $\mathbb{C}((t))$ is obtained by adjoining all n-th roots $\{t^{1/n} \mid n > 0\}$ to $\mathbb{C}((t))$.

Since $\mathbb{C}((t))$ has this special property concerning finite field extensions, we might wonder whether we can find a simplified canonical form for linear operators in this case. So, with an abundance of hindsight, let us pose the following questions.

> *What can we do for linear operators over the field $\mathbb{C}((t))$? Is there a specialized canonical form for such operators? And how does it incorporate the theorem of Newton-Puiseux?*

It turns out that we have found some very satisfying answers for these questions.

2 Simple Operators

Unless otherwise stated, $F = \mathbb{C}((t))$ and *V* is a vector space over *F*of dimension *n*. The main purpose in this section is to discuss a special case of the canonical form problem for linear operators over F.

We start with the following definition.

Definition 2.1. Let $T : V \to V$ be a linear operator. We say that *T* is *simple* if the only *T*-variant subspaces of *V* are *V* and $\{0\}$.

If $T : V \to V$ is any linear operator we can consider the F-subalgebra $F[T] \subseteq End_F(V)$. The commutative algebra $F[T]$ consists of all operators of the form

$$a_0 I + a_1 T + a_2 + ... + a_m T^m$$

where $a_i \in \mathbb{C}((t))$, and T^m denotes T composed with itself m times.

2.1 Newton-Puiseux Theorem for Linear Operators

The following proposition can be considered as *"The Newton-Puiseux Theorem for Linear Operators"*.

Proposition 2.2. *Let $T : V \to V$ be a simple linear operator and let $I \in End_F(V)$ be the identity operator. Then $F[T]$ is a field and $F[T] = F[\theta]$ where $\theta \in F[T]$ and $\theta^n = tI$. Furthermore there is a basis β of V such that*

$$[\theta]_\beta = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ t & 0 & 0 & \dots & 0 \end{pmatrix} \tag{2.1}$$

Proof. By Corollary 13.15 of [2], there is an element $\theta \in F[T]$ such that $\theta^n = tI$. Thus, the minimal polynomial min(θ) of θ, divides the polynomial $x^n - t \in F[x]$. Note that $D = \mathbb{C}[[\mathrm{t}]]$ is a principal ideal domain and tD is the maximal ideal of *D*. It follows from Eisenstein's Criterion that $x^n - t$ is irreducible over $D[x]$. It then follows from Gauss's Lemma, that $x^n - t$ is irreducible over $F[x]$ since *F* is the field of fractions of *D*. Since min$(\theta) \mid x^n - t$, an irreducible polynomial, we have min(θ)= $x^n - t$. Since $x^n - t$ has degree *n*, the characteristic polynomial of θ must be equal to $x^n - t$. Hence, $F[\theta] = F[T]$ by degree count. Finally, the rational canonical form of θ is the companion matrix of $x^n - t$, as claimed in (2.1). □

Proposition 2.2 will help us identify all simple linear operators $T \in End_F(V)$.

Definition 2.3. The $n \times n$ matrix A is in *standard form* over $\mathbb{C}((t))$ if it is in the form

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ a_{n-1}t & a_0 & a_1 & \dots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_3t & a_4t & a_5t & \dots & a_1 & a_2 \\ a_2t & a_3t & a_4t & \dots & a_0 & a_1 \\ a_1t & a_2t & a_3t & \dots & a_{n-1}t & a_0 \end{pmatrix} \tag{2.2}$$

where $a_i \in \mathbb{C}((t))$, for $i = 1, ..., n-1$. This is the same as saying that $A \in F[\theta]$ where θ is as in 2.1. Indeed, $A = \sum_{i=0}^{n-1} a_i \theta^i$.

It follows that such a matrix is semisimple. We determine below in Theorem 2.5 when a matrix in standard form is simple, and in Theorem 3.1 when two simple matrices in standard form are similar.

2.2 The Standard Matrix of an Irreducible Polynomial

Let $p(x) \in F[x]$ be an irreducible polynomial of degree $n > 0$. By The Newton Puiseux Theorem,

$$K = F[x]/(p(x)) = F[s].$$

for some $s \in K$ with $s^n = t$. But $y = \overline{x} \in K$ and $p(y) = 0$. Thus we can write

$$y = \sum_{i=0}^{n-1} a_i s^i, \quad a_i \in F.$$

We define *the standard canonical matrix of p(x)* to be

$$S(p) = \sum_{i=0}^{n-1} a_i \theta^i$$

where

$$\theta = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ t & 0 & 0 & \dots & 0 \end{pmatrix} \tag{2.3}$$

Notice that $S(p)$ is a matrix in standard form, in the sense of Definition 2.3, and the characteristic polynomial of $S(p)$ is $p(x)$. There is an easy test to identify which matrices in standard form come from *irreducible* polynomials. See Theorem 2.5 below.

2.3 The Characterization of Simple Operators

Because of Definition 2.3 and the construction in §2.2 we have good insight about the structure of simple operators. In this section (Theorem 2.5 below) we complete the characterization. All that remains here is to determine which operators in standard canonical form are simple.

Let θ be as in Proposition 2.2. Without loss of generality, $V = F^n$ and $\beta = \{e_1, ..., e_n\}$, where $e_1 = (0, 0, \ldots, 0, 1)$, $e_2 =$ (0,0,...,1,0), and $e_n = (1, 0, \ldots, 0, 0)$, an ordered basis of V. Then we see that $\theta(e_1) = e_2$, $\theta(e_2) = e_3, \ldots, \theta(e_{n-1}) = e_n,$ and $\theta(e_n) = te_1.$

Lemma 2.4. *Let θ be as in Proposition 2.2, and let $s, l \in \mathbb{Z}_+$ be such that $sl = n.$ Then $V = \bigoplus_{i=1}^{s} V_i$ where $V_i = Span\left\{e_i, e_{s+i}, \ldots, e_{(l-1)s+i}\right\}$, where e_k is defined as above for all k. Furthermore, for each $i \le s$, $\theta^s(V_i) \subseteq V_i$. Also the matrix of $\theta^s|_{V_i}$ with respect to the ordered basis $\left\{e_i, e_{s+i}, \ldots, e_{(l-1)s+i}\right\}$ of V_i is the $l \times l$-matrix as depicted in Proposition 2.2.*

Proof. This is a straightforward, explicit calculation. □

Theorem 2.5. *Let $x \in F[\theta] \subseteq End_F(V)$, where θ is as above in Proposition 2.2. Write*

$$x = \sum_{i=0}^{n-1} f_i \theta^i$$

where $f_i \in F[\theta]$, and let $k = gcd\left(\{i \mid f_i \neq 0\}\right)$. Then the following are equivalent.

(1) $(k, n) = 1$.

(2) $x \in End_F(V)$ makes V into a simple $F[x]$-module.

Proof. Suppose that $s = (k, n) > 1$. Then $s|k$ so the only non-zero terms in the expression for *x* are $f_i\theta^i$ where $s|i$. Thus $x \in F\left[\theta^s\right]$. By Lemma 2.4, for each $i = 1, 2, \ldots, s$, $\theta^s(V_i) \subseteq V_i$ where V_i is defined as in Lemma 2.4. Since $x \in F\left[\theta^s\right]$, we have $x(V_i) \subseteq V_i$ for all *i*. Thus for each *i*, V_i is a non-zero proper *x*-invariant subspace of *V* so *V* is not a simple $F[x]$-module.

Now suppose that $(k,n)=1$. Let $s \in \mathbb{Z}_{>1}$ be such that $s|n$. Suppose $x \in F\left[\theta^s\right]$. Then $s|k$, so that $s|(k,n)=1$. But this is impossible, since $(k,n)=1$. Thus $x \notin F\left[\theta^s\right]$. Now it follows from the Fundamental Theorem of Galois Theory (Theorem 11.1 of [4]) that

$$\{F\left[\theta^s\right] \mid 1 \leq s \leq n,\, s|n\}$$

is a complete list of the F-subfields of $F[\theta]$. But $F[x]$ is a subfield of $F[\theta]$ containing F. Hence, $F[x]=F[\theta]$. Thus V is a simple $F[x]$-module since it is a simple $F[\theta]$-module. ☐

3 Characteristic Polynomial

If $x, y \in End_F(V)$, we write $x \sim y$ if x and y are similar. We denote by $char(x)$ the characteristic polynomial of x.

Theorem 3.1. *Let*

$$x=\sum_{i=0}^{n-1} f_i\theta^i,\; y=\sum_{i=0}^{n-1} g_i\theta^i \in F[\theta]$$

be simple, where $\theta \in End_F(V)$ is as defined in Proposition 2.2. Then the following are equivalent.

(1) $x \sim y$

(2) $char(x)=char(y)$

(3) There exists $\omega \in \mathbb{C}^*$ such that $\omega^n=1$ and, for each i, $g_i=\omega^i f_i$.

Proof. That (1) implies (2) is obvious.

We now show that (2) implies (1). Suppose (2) is true. Since x and y are both simple, then $x \sim Comp(char(x))=Comp(char(y)) \sim y$, where *Comp* is the companion matrix.

We now show that (3) implies (1). Let $P=diag(1,\omega,\omega^2,...,\omega^{n-1})$. Then $P^{-1}=diag(1,\omega^{n-1},\omega^{n-2},...,\omega)$. Thus $P\theta P^{-1}=\omega\theta$ so that, for each i, $P\theta^i P^{-1}=\omega^i\theta^i$. Thus $PxP^{-1}=y$ so $x \sim y$.

Finally we show that (1) implies (3). Let $P \in End_F(V)$ be such that $PxP^{-1}=y$. Thus $PF[\theta]P^{-1} \subseteq F[\theta]$ since both x and y generate $F[\theta]$. Then $PF[\theta]P^{-1}=F[\theta]$ since this operation is an automorphism of $Aut(F[\theta]/F)$. Thus $P\theta P^{-1}=\omega\theta$ for some $\omega \in \mathbb{C}^*$ with $\omega^n=1$. Then $P\theta^i P^{-1}=\omega^i\theta^i$ for each *i* so $g_i=\omega^i f_i$ for each *i*. ☐

3.1 Finding the Eigenvalues

Let θ be as in Proposition 2.2. Let $s^n = t$ be fixed. Then the set of solutions to the equation $x^n = t$ is $\{s, \omega s, \omega^2 s, \dots, \omega^{n-1} s\}$ where $\omega \in \mathbb{C}^*$and $\omega^n = 1$. Then the set of eigenvectors in $F[s]$ of θ is $\left\{\left(1, \omega^j s, \omega^{2j} s^2, \dots, \omega^{(n-1)j} s^{n-1}\right)^T\right\}_{j=0}^{n-1}$ (Here $(\cdot)^T$ means "transpose"). Furthermore, $\left(1, \omega^j s, \omega^{2j} s^2, \dots, \omega^{(n-1)j} s^{n-1}\right)^T$ has eigenvalue $\omega^j s$ for all j. Thus θ is conjugate to $diag\left(s, \omega s, \omega^2 s, \dots, \omega^{n-1} s\right)$. Then θ^i is conjugate to

$$diag\left(s^i, \omega^i s^i, \omega^{2i} s^i, \dots, \omega^{(n-1)i} s^i\right).$$

Let $x = \sum_{i=0}^{n-1} f_i \theta^i \in \mathbb{C}[[t]]$. Then x is conjugate to the diagonal matrix whose $(j+1, j+1)$-entry is

$$P(\omega^j s) = \sum_{i=0}^{n-1} f_i [\omega^j s]^i$$

where $P(Z) = \sum_{i=0}^{n-1} f_i Z^i$. Notice also that the coefficients of the characteristic polynomial of x are the elementary symmetric functions of $\{P\left(\omega^j s\right)\}_{j=0}^{n-1}$. Therefore x is conjugate over $F[s]$ to the diagonal matrix

$$diag\left(P\left(s\right), P\left(\omega s\right), P\left(\omega^2 s\right), \dots, P\left(\omega^{n-1} s\right)\right).$$

From the above discussion, the next theorem follows naturally.

Theorem 3.2. *The characteristic polynomial of* $x = P(\theta) = \sum_{i=0}^{n-1} f_i \theta^i \in \mathbb{C}[[\theta]]$ *is given by*

$$char(x) = (-1)^n \prod_{i=0}^{n-1} \left(\lambda - P\left(\omega^i s\right)\right)$$

where $s^n = t$ *is fixed and* $\omega \in \mathbb{C}^*$*is a primitive n-th root of unity.*

Example 3.3. Let $n = 2$ and let $x = \begin{pmatrix} a & b \\ bt & a \end{pmatrix} = P(\theta) = a + b\theta \in \mathbb{C}[\theta]$, where $\theta = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$. We compute $char(x)$ directly. Indeed,

$$char(x) = det \begin{pmatrix} a - \lambda & b \\ bt & a - \lambda \end{pmatrix} = (\lambda - P(s))(\lambda - P(-s)),$$

where $s = \sqrt{t}$.

4 More General Operators

But what about the general case? We now have the necessary building blocks to determine the standard canonical form for any linear operator over $\mathbb{C}((t))$.

4.1 Standard Canonical Form in General

Let $T : V \to V$ be a linear operator. Assume that the characteristic polynomial of T factors as

$$char(T) = \phi_1^{n_1}\phi_2^{n_2}\cdots\phi_k^{n_k}$$

where each ϕ_i is irreducible. Then there is a basis $\beta = \sqcup_i \beta_i$ of V so that the matrix of $A = [T]_\beta$ can be written as follows.

$$A = \begin{pmatrix} C_1 & 0 & 0 & \dots & 0 \\ 0 & C_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \dots & C_{k-1} & 0 \\ 0 & 0 & \dots & 0 & C_k \end{pmatrix}$$

where each C_i is a cyclic transformation on some T - invariant subspace $V_i \subseteq V$ with characteristic polynomial $\phi_{m_i}^{l_i}$ for some $l_i \leq n_i$. In any case, $V = \oplus V_i$. The theory of rational canonical forms says that we can arrange it so that each C_i is the companion matrix of the polynomial $\phi_{m_i}^{l_i}$. **However**, we can also arrange it so that C_i is a $k_i \times k_i$ matrix of the form

$$C_i = \begin{pmatrix} B_i & I & 0 & \dots & 0 \\ 0 & B_i & I & \dots & 0 \\ 0 & 0 & B_i & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & B_i \end{pmatrix} \tag{4.1}$$

where B_i has (irreducible) characteristic polynomial ϕ_{m_i} of degree a_i, I is an $a_i \times a_i$ identity matrix, and $k_i = l_i a_i$. Indeed, this is just another way of representing an indecomposable, cyclic linear transformation as a matrix.

It then follows from Proposition 2.2 that there exists an invertible matrix $\gamma_i \in Gl_{a_i}(F)$ such that

$$\gamma_i B_i \gamma_i^{-1} = D_i$$

is in standard canonical form, with irreducible characteristic polynomial (so $D_i \in F[\theta]$ where θ is as in Proposition 2.2 and D_i satisfies the conditions of Theorem 2.5). See also Definition 2.3. Let

$$P_i = \begin{pmatrix} \gamma_i & 0 & 0 & \dots & 0 & 0 \\ 0 & \gamma_i & 0 & \dots & 0 & 0 \\ 0 & 0 & \gamma_i & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \gamma_i & 0 \\ 0 & 0 & 0 & \dots & 0 & \gamma_i \end{pmatrix}$$

an invertible matrix. Then

$$P_i C_i P_i^{-1} = S_i = \begin{pmatrix} D_i & I & 0 & \dots & 0 & 0 \\ 0 & D_i & I & \dots & 0 & 0 \\ 0 & 0 & D_i & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & 0 \\ 0 & 0 & 0 & \dots & D_i & I \\ 0 & 0 & 0 & \dots & 0 & D_i \end{pmatrix} \tag{4.2}$$

Recall that for any operator $T : V \to V$ is similar to a block sum of operators of the form C_i, with C_i as above in (4.1). This brings us to our main result.

Theorem 4.1. *Let $T : V \to V$ be a linear operator over $\mathbb{C}((t))$. Then there exists a basis $\beta = \sqcup_{i=1}^{s} \beta_i \subset V$ such that $[T]_\beta$ is of the form*

$$[T]_\beta = \begin{pmatrix} S_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & S_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & S_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & S_{s-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & S_s \end{pmatrix}$$

where V_i is the span of β_i and each $S_i = [T|_{V_i}]_{\beta_i}$ is as above in (4.2). In particular, D_i is simple, and it is in standard form in the sense of Definition 2.3.

We refer to the matrix in Theorem 4.1 as the **standard canonical form** for the linear operator T. Recall from Theorem 2.5 that it is easy to determine by inspection if a given matrix, in standard form, has irreducible characteristic polynomial. Notice also that Theorem 3.1 yields a simple criterion to decide when two matrices in standard canonical form are similar.

5 Conclusions and Questions

The authors view the results of this paper as a successful departure from the companion matrix approach to finding canonical forms for linear operators over $\mathbb{C}((t))$. One natural problem suggested by these results is to identify other fields F for which there is a notion of "standard canonical form". Does this depend on some kind of Newton-Puiseux Theorem? Or is it enough to start with a field F that comes equipped with a discrete valuation $R \subset F$? Said differently, what is the natural generality for this approach, and what assumptions about F are required to define a matrix in "standard form"?

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