Module 1: Preliminaries

1.1 Integers vs. reals

First, integers:

- We usually include both positive and negative integers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- Note: $\mathbb{N} = \{1, 2, 3, \ldots\}$ are the *natural* numbers.
- In many problems, we might only consider particular (finite or infinite) subsets of $\mathbb{Z}$
- Some famous problems:
  - (Solved) Fermat's Last Theorem: There are no integers x, y, z such that $x^n + y^n = z^n$ for some integer $n \geq 3$.
  - (Unsolved) Goldbach's conjecture (1742): Every even integer greater than 2 can be written as the sum of two primes.
  - (Unsolved - Millenium Prize) Riemann Hypothesis: The real part of any non-trivial zero of the Riemann zeta function is 1/2.
  - (Unsolved) Polynomial-time algorithm for factorizing an integer.

An example of computing with integers: Euclid's GCD algorithm:

- For positive integers m and n, $GCD(m, n) \triangleq$ the largest integer that divides both m and n.
- The symbol $\triangleq$ means "is defined as".
- Example:
  - 48 has divisors 1, 2, 3, 4, 6, 8, 12, 16, 24, 48.
  - 32 has divisors 1, 2, 4, 8, 16, 32.
  - Thus, GCD (32, 48) = 16.
- Algorithm 1:
  - Factorize m and n separately.
  - Search factors to find largest common factor.
- Analysis of Algorithm 1: factorization is hard.
  ⇒ No polynomial-time algorithm is known.
- Some observations:
  - Suppose $m > n$ and k divides both.
    ⇒ Then k divides $(m - n)$.
  - In fact, as long as these numbers are positive, k divides $(m - n), (m - 2n), (m - 3n) \ldots$
  - What is the smallest positive number in this list?
    ⇒ $m \mod n$
  - Thus, $GCD(m, n)$ divides $m \mod n$.
  - And because $GCD(m, n)$ already divides n, it's among the common divisors of both n and $m \mod n$.
  - Any of these common divisors of n and $m \mod n$ must also divide m, in particular, the largest one, $GCD(n, m \mod n)$.
  - Thus, $GCD(m, n) \quad = \quad GCD(n, m \mod n)$.
    ⇒ Divide and conquer!

Algorithm 2 (Euclid): (in Java)

```
Algorithm: gcd (m, n)
Input: integers m, n > 0

    // Check for bad input.
    if n > m
        return gcd (n,m)
    endif

    // The algorithm.
    if n = 0
        return m;
    else
        return gcd (n, m mod n)
    endif
```

- Analysis:
  - Note that $m \bmod n < n$.
  - If $n \leq \frac{m}{2}$, then $m \bmod n \ < \ n \ \leq \ \frac{m}{2}$.
  - If $n > \frac{m}{2}$, then $m \bmod n \ = \ m - n \ \leq \ \frac{m}{2}$.
  - Thus, the larger number is reduced by $\frac{1}{2}$ each time.
    ⇒ Thus, the running time is at most $O(\log m)$.

A few more observations:

- Euclid's algorithm is recursive.
- The size of the problem is small:
  ⇒ We need $O(\log m)$ bits to represent the numbers.
- The running time is *linear* in the size of the problem.

As it turns out, these characteristics (recursion, problem-size) are classically associated with discrete algorithms.
⇒ Algorithms for continuous structures rarely have these properties.

Now let's look at *real* numbers:

- Informally, we think of real numbers as "possibly having digits after the decimal point."
- Examples: 3.141, 2.718.

In-Class Exercise 1: Are integers also real numbers? What are rational numbers and how are they different from integers or reals?

- As it turns out, a careful definition is not trivial.
- We know that no integer exists between 5 and 6, but given any two real numbers, there is another real between them.

In-Class Exercise 2: What about rationals? Is there a rational number between every two rationals?

Countability:

- For finite sets, the size of the set is the number of elements.
- For infinite sets, it's a little more complicated:
  ⇒ Two infinite sets A and B have the same *cardinality* if there is a 1-1 mapping between their elements.

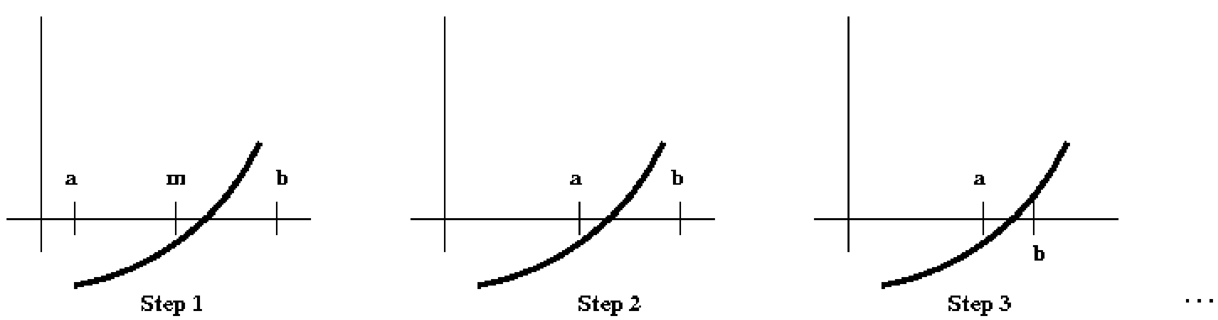
In-Class Exercise 3: How do the sizes of the naturals compare with the integers? What about the size of the rationals compared to integers?

- The size of the reals is much larger than that of integers: (Proof by contradiction)
  - Suppose that integers and reals had the same size.
    ⇒ Then, there is a 1-1 mapping
    ⇒ Then, one could list the reals according to this mapping

    1. r_1
    2. r_2
    3. r_3
    ....
  - Consider the real number s constructed as follows: the i-th digit of s differs from the i-th digit of r_i.
  - Then, s is not in the list.
    ⇒ A contradiction!

Consider a computational problem with real numbers: finding square roots.

- Suppose we want to find the square-root of a number, e.g., 3.
- Define the function $f(x) = x^2 - 3$.
- Then the solution of the equation $f(x) = 0$ is what we want.
- Notice: $f(x) = 0$ is where $f(x)$ cuts the x-axis.
  ⇒ It must be below 0 just before and above 0 just after (or vice-versa).
  ⇒ There is some interval $[a, b]$ that surrounds the solution where $f(a)$ and $f(a)$ are of opposite sign.
- For our problem, take $a = 0$ and $b = 2$.
- Key ideas in algorithm:



  - At each step, reduce the size of the interval by half.
  - Compute $f(m)$ where $m = \frac{(a+b)}{2}$.
  - Decide whether to "move" a or b based on whether m is on the same side of a or b.
    ⇒ i.e., whether $f(m)$ has the same sign as $f(a)$ or $f(b)$.
- Pseudocode:

```
Algorithm: Bisection (a, b, f)
Input: real numbers a,b where a < b, and function f
    m = (a + b) / 2
    while f(m) not close enough to zero
        if f(m) < 0
            a = m
        else
            b = m
        endif
        m = (a + b) / 2
    endwhile
    return m
```

- Sample Java code: Bisection.java.

In-Class Exercise 4: In the above example, we knew that $f(a) < 0$. Re-write the pseudocode to make it more general: replace the test $f(m) < 0$ with a test to see if $f(a)$ and $f(m)$ are of the same sign. Modify the code in Bisection.java accordingly.

In-Class Exercise 5: Examine the code in Bisection.java:

- Use a calculator to compute $\sqrt{3}$ and compare with the output of the program. How would you increase the accuracy of the program?
- Can the `bisection()` method be written recursively? What would that do to efficiency?

1.2 Some interesting tidbits about real numbers

In no particular order, here are some interesting facts about real numbers:

- What is an irrational number? A real number that's not rational.
  - A *rational* can be expressed as a ratio of integers: $r = \frac{p}{q}$, where p, q are integers.
  - How do we know *irrational* numbers exist? Example: $\sqrt{2}$.
- Proof that $\sqrt{2}$ is irrational:
  - Assume that $\sqrt{2}$ is rational.
  - Express $\sqrt{2}$ as $r = \frac{p}{q}$ where the fraction $\frac{p}{q}$ cannot be reduced.
  - Then, $\frac{p^2}{q^2} = 2$

$$\begin{array}{lrcl} \Rightarrow & p^2 & = & 2q^2 \\ \Rightarrow & & p^2 \text{ is even} & \\ \Rightarrow & & p \text{ is even} & \end{array}$$

  - By a similar argument (can you explain?), q is even.
    $\Rightarrow$ Contradicts the assumption that the fraction is reduced.
- We know that rationals and integers are of the same cardinality.
  $\Rightarrow$ The irrationals are uncountable.
- Most famous real number constants are irrational, e.g. π, e. (Proofs are not straightforward)

- Numbers that are solutions of some polynomial with integer coefficients are called *algebraic*, e.g., $\sqrt{2}$ is the solution of $x^2 - 2 = 0$.
- Another example: Φ, the golden ratio is the solution to $x^2 - x - 1 = 0$.
- Real numbers that are not algebraic are called *transcendental* e.g, π, e
- The real number system is unique:
  - One can describe properties that define the real number system.
    ⇒ For example: closure with respect to arithmetic, existence of a least-upper-bound.
  - One can show: any number system with these properties must be equivalent to the real number system.
- The continuum hypothesis: there is no set whose countability lies between that of the naturals and the reals. What is known: cannot be proved or disproved using standard set theory.

Ways of writing proofs:

- Even when the key idea in a proof makes sense, actually *writing* the proof may not be trivial.
- For example: should there be a figure? should the proof written free-flowing, like English text?
- Look at **these variations** of proofs of the irrationality of $\sqrt{2}$.

1.3 Sequences

Consider Zeno's paradox (circa 430 BC):

- To move from A to B, you must first reach the half-way point.
- But to reach the half-way point, you must first reach a quarter of the distance.
- ... there are an infinite number of such moves.
  ⇒ Therefore, it's not possible to move to B.

Let's take a closer look:

- Clearly, Zeno is suggesting that the infinite sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ *diverges* to infinity.
- Define S_n the sum of the first n terms
  $S_n \triangleq \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n}$

In-Class Exercise 6: Add code to Zeno.java to compute S_n for any value of n. Try $n = 5, 10, 100$. What do you observe?

- Consider the general sum $S'_n \triangleq 1 + x + x^2 + x^3 + \ldots + x^n$
  - We've changed Zeno's sum slightly by adding 1 at the start.
  - Observe that
    $S'_n \;=\; S'_{n-1} + x^n$
  - Also, by factoring,
    $S'_n \;=\; 1 + xS'_{n-1}$
  - Equating the two gives us
    $S'_{n-1} + x^n \;=\; 1 + xS'_{n-1}$

- Solving for S'_{n-1}
  $S'_{n-1} \quad = \quad \frac{x^n - 1}{x-1}$

In-Class Exercise 7: Add the formula for S_n to Zeno.java and compare the for-loop sum (with 1 added) and the formula result. Plot S_n vs. n.

Now, let's examine a different sum:

- Define $H_n \quad = \quad 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$
- This is sometimes called the Harmonic sum, and H_n is called the n-th Harmonic number.

In-Class Exercise 8: Add code to Harmonic.java and compare the output from two computations: using `double` vs. using `float`. Try different value of n. What do you think the sum is converging to? Plot H_n vs. n.

In-Class Exercise 9: Write code to compute $A_n = (1 + \frac{1}{n})^n$ for various values of n. Write your code in the `computeA()` method of SequenceExample.java. Plot A_n vs. n.

Let's make a few observations:

- These are the first few A_n values: $A_1 = 2, \ A_2 = 2.25, \ A_3 = 2.370370370, \ A_4 = 2.44140625$
- There are infinitely many A_n's because there are infinitely many values of n.
  $\Rightarrow A_1, A_2, A_3, \ldots$ is called a *sequence* of numbers.
- Where there's no confusion, we'll use the terminology "the sequence A_n" to mean the collection of numbers (in order) $A_1, A_2, A_3, \ldots$
- Notice that H_n and S_n in the earlier examples are also sequences.

What comes at the "end" of a sequence?

- Consider the simple sequence $B_n = \frac{1}{n}$.
- For large n, B_n is approximately 0.
- Now, there is no value of n for which $B_n = 0$.
  $\Rightarrow B_n$ gets closer and closer to 0 but never actually "hits" 0.
- Similarly, the sequence $S_n = \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n}$ gets closer and closer to 1.0 but never actually "hits" 1.0.
- In both cases, we say that the sequence has a *limit*.
  - The *limit* of the sequence B_n is 0.
  - The *limit* of the sequence S_n is 1.
- We write this as:

$$\lim_{n \to \infty} B_n \quad = \quad 0$$
$$\lim_{n \to \infty} S_n \quad = \quad 1$$

In-Class Exercise 10: Write code in SequenceExample2.java to print the first few terms of the sequence $C_n = \frac{\sin(n)}{n}$. How is this sequence different from the ones we've seen so far?

Some strangeness with limits:

- Because sequences can behave strangely, we need a more careful definition of a limit.
- Wrong definition #1:
  Each successive term is closer to the limit.
- Wrong definition #2:
  Sequence X_n has limit L if the distance $X_n - L$ keeps decreasing.
- Wrong definition #3:
  Given some arbitrarily small number ϵ, the distance $|X_n - L|$ is less than ϵ.
- Correct definition #1:
  Given some arbitrarily small number ϵ, the distance $|X_n - L|$ is greater than ϵ only finitely many times.
- Correct definition #2:
  For every small number ϵ, $|X_n - L| \ < \ \epsilon$ for all large enough n.
- Correct definition #3 (most common):
  For every small number ϵ, there exists an integer N such that $|X_n - L| \ < \ \epsilon$ for all $n > N$.

Informal version of last definition:

- For any "close-to-the-limit" distance ϵ, you can go far enough down the sequence (large enough N so that the rest of the sequence from there is no further than ϵ from the limit L.

1.4 Random sequences

Consider the following code: (source file)

```
public class RandomSequence {

    public static void main (String[] argv)
    {
        for (int n=1; n<=10; n++) {
            System.out.println ("Un, n=" + n + ": " + computeU(n));
        }
    }

    static double computeU (int n)
    {
        return RandTool.uniform ();
    }

}
```

Note:

- This prints out a sequence of numbers, each of which is randomly drawn from the range *[0, 1]*.
- Let U_n denote this sequence.
- Let $V_n \triangleq \frac{1}{n}(U_1 + U_2 + \ldots + U_n)$.

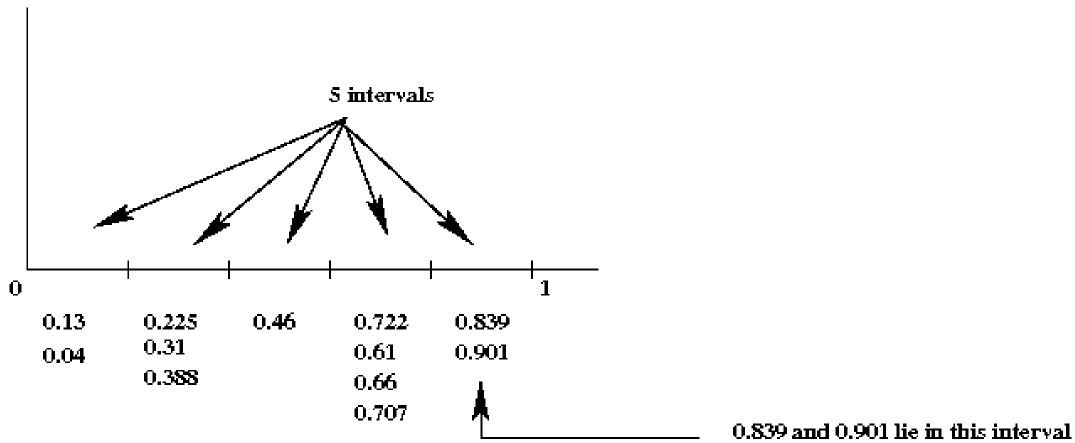
In-Class Exercise 11: Write code in RandomSequence2.java to print the first 10 terms of the sequence V_n defined above. You will also need RandTool.java.

- Does the sequence U_n have a limit?
- Does the sequence V_n have a limit?

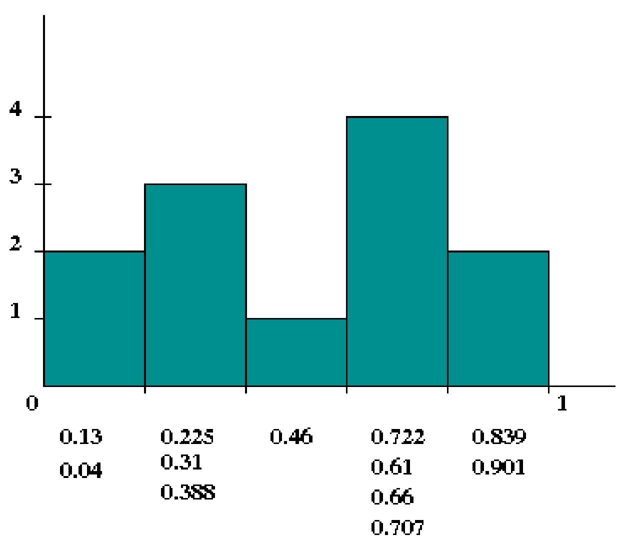
How are these sequences different from the ones we've seen before? *After* writing your code to compute V_n above, examine the code in RandomSequence3.java. What is the difference in the two approaches?

Next, we'll compute a few histograms:

- What is a histogram? Suppose we construct a 5-bin histogram on the interval *[0,1]*.
  - Consider this data set:
    0.901, 0.13, 0.225, 0.46, 0.722, 0.61, 0.04, 0.3, 0.388, 0.839, 0.66, 0.707
  - First, divide the range *[0,1]* into 5 intervals:



  - Then, count the number of data items in each interval:



- We will write some code to compute a histogram for values in the sequence U_n. (source file)

```
        // Create space: one counter per interval.
        int[] bins = new int [size];

        // Interval size.
        double interval = 1.0 / size;

        // Generate the data set, and identify the intervals.
        for (int k=0; k<numSamples; k++) {
            double u = computeU (n);                    // Get the n-th in sequence.
            int b = (int) Math.floor (u / interval);    // Which bin or interval?
            bins[b] ++;                                 // Increment count accordingly.
        }
```

In-Class Exercise 12: First, examine the code in RandomSequence4.java and verify that it prints out possible random values, or *samples*, of U_5. Modify the code to print values of U_{493}. Next, download Histogram.java and print a histogram for U_5 with different numbers of samples: 10, 100 and 10000. What do you notice? Is it intuitive? How do the histograms for U_5 differ from the equivalent histograms for U_{493}?

We need to point out a sublety:

- Consider the sequence U_n
  - Each U_n is drawn randomly in the interval $[0, 1]$.
  - Thus, for example, U_7 does not depend on the value of U_3.
- However, is this property is true for the sequence V_n?

In-Class Exercise 13: Modify Histogram.java and to compute and print a histogram for V_n with different numbers of samples: 10, 100 and 10000.

- How does the histogram for V_5 differ from the histogram for V_{493}?
- Print the histogram for V_{10000} and explain the result.

A scaled sequence:

- Define the sequence $W_n = \sqrt{n}(V_n - 0.5)$.

In-Class Exercise 14:

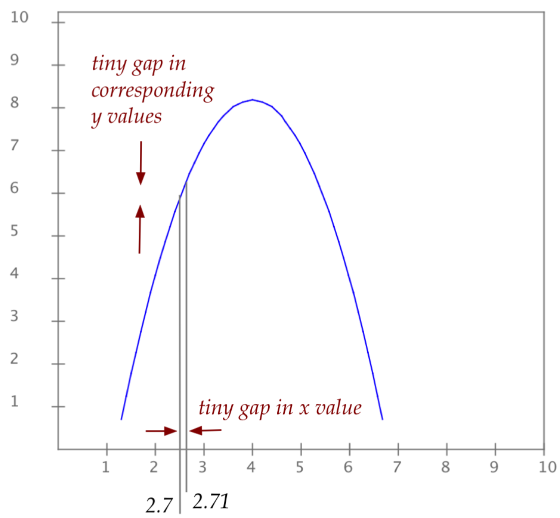
- First, let's get a feel for this by printing out example values: add code in RandomSequence5.java. Does this appear to converge to a number for large values of n?
- Next, let's examine the histograms for various values of n. Modify Histogram.java and to compute and print a histogram for W_n with different numbers of samples: 10, 100 and 10000.
  - How does the histogram for W_5 differ from the histogram for W_{493}?
  - Print and then plot the histogram for W_{10000}.
- We'll now change the type of random generation. In Histogram2.java, create a means of generating U'_n, a random variable that's different from U_n.
  - Then, define $V'_n = \frac{1}{n}(U'_1 + \ldots U'_n)$.
  - Then, run your program to estimate what V'_n is converging to. Let's call this number μ.
  - Now define $W'_n = \sqrt{n}(V'_n - \mu)$.
  - Plot the histograms for W'_5 and W'_{493}.

The histograms of W_n and W'_n highlight one of the most important results in the history of science.

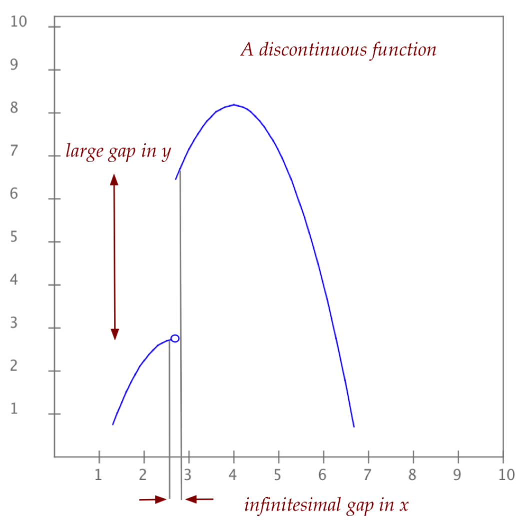
1.5 Two key ideas in the continous world: limits and continuity

Two important concepts pervade continuous mathematics:

- The idea of a *limit* (which we have just encountered).
- The idea of *continuity*
  ⇒ We will need to understand *functions* first.
- The general idea in continuity: consider $f(x) = 8 - (x-4)^2$
  - If you pick *any* two values of x very close to each other, e.g., $x = 2.7$ and $x = 2.71$
    ⇒ Then, $f(2.7)$ and $f(2.71)$ are "close" for continuous functions.



- In contrast, a discontinuous function will have discontinuities somewhere, as in



About limits:

- Many key results in continuous mathematics are really limit results.

- Generally, limits *simplify* expressions, and consequently, the mathematics.
- Generally, limits result in more powerful analytic tools.
- Although we never really reach "infinity", limits are good approximations for large enough n
  ⇒ Sometimes, they're all we have.
- Theorems about limits aren't always easy to prove
  ⇒ But that shouldn't make the theorem statement hard to understand
- Always test a result with practical data or values before using it.

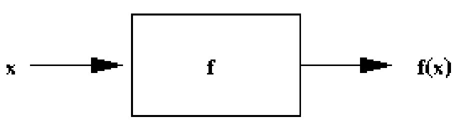
About continuity:

- Continuity is usually a theoretical requirement to make proofs work.
- Many real-world functions are actually "well-behaved" (continuous) and, therefore, easy to work with.
- However, many are not - in these cases, we have to struggle with each individual case.
  ⇒ It's almost always messy.
- Continuity can be explained in terms of limits:
  - Suppose x_n is a sequence with limit L, and $f(x)$ is a function.
  - Then, consider the sequence of *function values* $f_n = f(x_n)$.
    ⇒ f_n is a sequence of real numbers.
  - For a continuous function, the limit of f_n is $f(L)$.

1.6 Functions

What is a *function*?

- Engineers often think of a function as something that takes an input value and produces an output:



- Mathematicians define it this way: (e.g., Thomas and Finney. *Calculus and Analytic Geometry*)

  A function from a set D to a set R is a rule that assigns an element of R to each element in D.

- The set D is often called the *domain* of the function, meaning, the set of *allowable* input values.
  ⇒ We don't want inputs that are "bad"
- The set R is often called the *range*, the set of *possible* outputs.
- For mathematical purity, we'd have to worry about whether infinity is an allowed value or in the range
  ⇒ For most applications, we won't have to worry about such odd cases.

Exercise 15: What types of "odd" cases do we have to worry about in real life? Can you think of a function such that the number 2 is a "bad" input value (i.e., shouldn't be allowed)?

Exercise 16: Get out a piece of paper and draw (by hand) the function $f(x) = 3x^2 + 5$. What are the domain and range of this function? How much of the domain and range did you sketch out?

Now let's write a simple program to compute $f(x) = 3x^2 + 5$: (source file)

```
import java.util.*;

public class FunctionExample1 {

    public static void main (String[] argv)
    {
        // This is what reads from the keyboard:
        Scanner scanner = new Scanner (System.in);

        // Put out a prompt:
        System.out.print ("Enter x: ");

        // Read in a "double" (real) value:
        double x = scanner.nextDouble ();

        // Compute function:
        double f = 3*x*x + 5;

        // Print result (output):
        System.out.println (f);
    }

}
```

Exercise 17: Download, compile and execute the above program, FunctionExample1.java to confirm your results from the previous exercise.

Exercise 18: Modify the above program to compute the function $f(x) = \frac{1}{(x-2)}$. Use the program to generate some values and draw a graph of the function. What happens when $x = 2$?

Next, let's write out several values using a loop: (source file)

```
import java.util.*;

public class FunctionExample2 {

    public static void main (String[] argv)
    {
        System.out.println (" x   f(x)");          // Table header
        System.out.println ("---------");
        for (double x=1; x<=10; x=x+1) {
            double f = 3*x*x + 5;                  // Compute function.
            System.out.println (x + "  " + f);     // Print result.
        }
    }

}
```

Exercise 19: Modify the above program to print out function values for x in the range $[-10, -1]$.

We'll next graphically depict a function using a few points: (source file)

```
import java.util.*;

public class FunctionExample3 {

    public static void main (String[] argv)
    {
        // Make a Function object and give it a name.
        Function F = new Function ("Silly function");

        for (double x=1; x<=10; x=x+1) {
            double f = 3*x*x + 5;
            // Feed the x,f(x) combinations into the object.
            F.add (x, f);
        }

        // Write to screen.
        System.out.println (F);

        // Display.
        F.show ();
    }

}
```

Exercise 20: Modify the above program to show the shape of the function $f(x) = \frac{1}{(x-2)}$ in the range $[0, 5]$. You will need to download Function.java and SimplePlotPanel.java.

Exercise 21: For each of the functions below, generate 100 values in the range $[0, 10]$ and feed them into a `Function` object. Then, display the result.

- $f(x) = 3x + 5$
- $f(x) = x^2 - 2$
- $f(x) = \frac{5}{x^2}$
- $f(x) = e^{-2x}$

Use `Function.show(F,G,...)` to plot multiple `Function`'s together.

1.7 Distance between functions

"Comparison" vs. "distance":

- To see if $f(x)$ "bigger" than $g(x)$, we can draw both and ask: in which intervals is $f(x)$ bigger?
- This lets us compare functions just as we can compare *numbers*.
- Is there a function analog to the *distance* between two numbers?

- If there were, we could say function $f(x)$ is "closer" to function $h(x)$ than $g(x)$ is.
  ⇒ For example, is $f(x) = 3x + 5$ closer to $h(x) = \frac{4}{x^2}$ or is $g(x) = 4e^{-2x}$ closer?

First, let's start examining "distance" by displaying two functions together:

- Let's draw $f(x) = 3x + 5$ and $g(x) = 4x + 5$.
- To do this, we'll generate 100 points in $[0, 10]$ for each function and display the results.

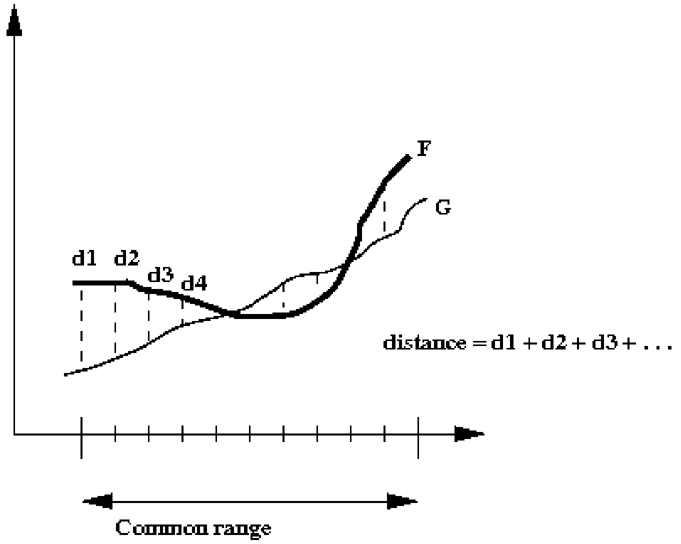
Here's the program: (source file)

```
public class FunctionComparison {

    public static void main (String[] argv)
    {
        // Make two objects, one for each function.
        Function F = new Function ("3x+5");
        Function G = new Function ("4x+5");

        // Generate 100 values in the range [0,10]
        for (double x=0; x<=10; x+=0.1) {
            F.add (x, 3*x+5);
            G.add (x, 4*x+5);
        }

        // Display both together.
        Function.show (F, G);
    }

}
```

Exercise 22 : Modify the above code to draw $f(x) = 3x + 5$ and $g(x) = 3x + 10$ in the range $[0, 10]$. Use only 50 points.

Next, we'll consider *quantifying* distance:

- We can quantify the distance between two *numbers*:
  ⇒ The distance between numbers 12.33 and 10.01 is $12.33 - 10.01 = 2.32$.
- Can we apply the "subtraction" idea to functions?
- One way to do this:



Let's write a small program to compute the "distance" between $f(x) = 3x + 5$ and $g(x) = 3x + 10$.

- We'll identify 50 x-values on the x-axis and take the difference between the two functions at these x-values.
- We'll sum up these differences.

Here's the program: (source file)

```
public class FunctionComparison2 {

    public static void main (String[] argv)
    {
        // Initialize sum.
        double distance = 0;

        // Generate 50 values in the range [0,10]
        for (double x=0; x<=10; x+=0.2) {
            double f = 3*x + 5;
            double g = 3*x + 10;
            double diff = g - f;
            distance = distance + diff;
        }

        System.out.println ("Distance: " + distance);
    }

}
```

Exercise 23: Did it matter whether we used $g - f$ or $f - g$ above? What is the distance between the functions $f(x) = 3x + 5$ and $h(x) = 20$? What happens when you use $f - h$ vs. $h - f$?

Exercise 24: Coding exercise: modify the above example so that the loop has only one line of code (and is therefore more compact).

Exercise 25: Modify the example to use 100 points (in the same range [0,10]) instead of 50. What do you notice? What does this tell you about the method we're using to compute distance?

Consider the following modification: (source file)

```
public class FunctionComparison3 {

    public static void main (String[] argv)
    {
        // Initialize sum.
        double sum= 0;

        // Generate 50 values in the range [0,10]
        for (double x=0; x<=10; x+=0.2) {
            double f = 3*x + 5;
            double g = 3*x + 10;
            sum += Math.abs (f - g);
        }

        // Compute the average distance:
        double distance = sum / 50;

        System.out.println ("Distance f to g: " + distance);
    }

}
```

Note:

- We have now used an average of the distances instead of the sum.

Exercise 26: Modify the above example as follows:

- First, use 100 points instead of 50. Then use 1000 points. What do you observe?
- Next, change the range from $[0, 10]$ to $[0, 5]$. What is the distance between f and g using this range?
- Repeat the above for f and $h(x) = 20$. Does the distance measure for functions make sense?

Finally, consider this variation of measuring functional distance: (source file)

```
public class FunctionComparison4 {

    public static void main (String[] argv)
    {
        // Initialize sum.
        double sum= 0;

        // Generate 50 values in the range [0,10]
        for (double x=0; x<=10; x+=0.2) {
            double f = 3*x + 5;
            double g = 3*x + 10;
            sum += Math.abs (f - g);
        }

        // Compute the average distance, multiplied by length of interval:
        double distance = (10-0) * sum / 50;

        System.out.println ("Distance f to g: " + distance);
    }

}
```

Note:

- Here, we have multiplied the distance by the size of the range.

Exercise 27: Now modify the above example to use $[0, 5]$ as the range:

- What is the distance between f and g using this range?
- Next, use 100 points instead of 50. Then use 1000 points. Is our modified measure severely affected by the number of points?

Let's re-write the above code slightly: (source file)

```
public class FunctionComparison5 {

    public static void main (String[] argv)
    {
        // Initialize sum.
        double sum= 0;

        // Compute interval = range/number-of-points:
        double interval = (10.0 - 0.0) / 50.0;

        // Generate 50 values in the range [0,10]
        for (double x=0; x<=10; x+=0.2) {
            double f = 3*x + 5;
            double g = 3*x + 10;
            sum += interval * Math.abs (f - g);
        }

        System.out.println ("Distance f to g: " + sum);
    }

}
```

Exercise 28: Why does this produce exactly the same result?

Exercise 29: Coding exercise: Change the second line to

```
      double interval = (10 - 0) / 50;
```

Then compile and execute - what do you observe? Explain.

We can use our distance measure to compute the distance of a function from the x-axis:
⇒ The x-axis is merely the function $z(x) = 0$.

Here's an example: (source file)

```
public class FunctionComparison6 {

    public static void main (String[] argv)
    {
        // Initialize sum.
        double sum= 0;

        // Compute interval = range/number-of-points:
        double interval = (10.0 - 0.0) / 50.0;

        // Generate 50 values in the range [0,10]
```

```
        for (double x=0; x<=10; x+=interval) {
            double f = 3*x + 5;
            double z = 0;
            sum += interval * Math.abs (f - z);
        }

        System.out.println ("Distance f to x-axis: " + sum);
    }

}
```

Exercise 30: Consider the function $h(x) = 20$ in the range $[0, 10]$. Draw this on paper - you should have a rectangle of width 10 and height 20. Next, use 4 equally-spaced x-values and hand-execute the above program with $h(x)$ instead of $f(x)$. Can you relate the calculations to the area of the rectangle?

Exercise 31: Now consider the function $f(x) = 3x + 5$ in the range $[0, 10]$. Draw this on paper and repeat the above steps (using 4 intervals) to compute the distance to the x-axis. How does this relate to the area? What happens when we use more intervals (e.g., 50 intervals)? Try 50 intervals, then try 1000 intervals. Can you calculate the area exactly by hand?

Exercise 32: Draw the two functions $f(x) = 3x + 5$ and $g(x) = 20$ on paper. What is the relationship between our distance measure and area?

1.8 Sine's and signs

Let us revisit some elementary trigonometry:

- Consider a right-angled triangle with hypotenuse of length r and sides x and y as shown in Figure A below:

Figure A

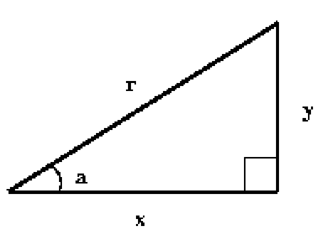
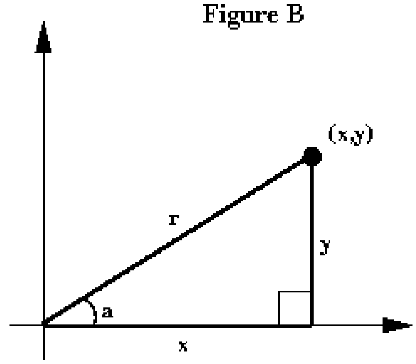
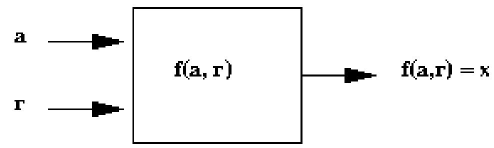


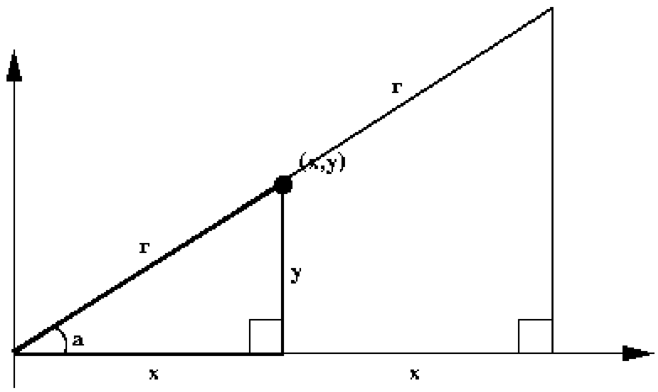
Figure B



- Alternatively, consider a point with coordinates (x, y) that happens to be a distance r from the origin (Figure B above).
- We'd like to know: for a point at distance r from the origin, what are the coordinates (x, y)?
- Clearly, it depends on the angle a.
  ⇒ The smaller a is, the smaller y is (and the larger x is).
- Thus, we ask: what function gives us x, when given r and a?



- First, notice what happens if we keep a fixed and double r:



  ⇒ Both x and y are also doubled.

  Exercise 33: Why is this true?

- Thus, $2x = f(a, 2r)$. Since this is true for any such multiple, r must really be a multiplier.
  ⇒ We can write the function as $x = rg(a)$.
  ⇒ $g(a) = \frac{x}{r}$.
  ⇒ $g(a)$ = ratio of side x to hypotenuse r when the angle is a

- There is a name for this function: *cosine*
  ⇒ $\cos(a) = \frac{x}{r}$.

- Similarly for y, we give that function a special name: $\sin(a) = \frac{y}{r}$.

- To summarize: we can get x and y in terms of r and a as follows:
  $x = r\cos(a)$
  $y = r\sin(a)$

Let's write a program to print out some values of these functions: (source file)

```
public class SinCos {

    public static void main (String[] argv)
    {
        // First, some sin examples;
        double s = Math.sin (0);
        System.out.println ("sin(0)=" + s);
        s = Math.sin (Math.PI/2);
        System.out.println ("sin(pi/2)=" + s);
        s = Math.sin (Math.PI/4);
        System.out.println ("sin(pi/4)=" + s);

        // Now some cos examples:
        double c = Math.cos (0);
        System.out.println ("cos(0)=" + c);
        c = Math.cos (Math.PI/2);
        System.out.println ("cos(pi/2)=" + c);
        c = Math.cos (Math.PI/4);
```

```
        System.out.println ("cos(pi/4)=" + c);
    }

}
```

Exercise 34: Modify the above program to print out $\sin(\frac{4\pi}{3})$ and answer the following:

- Why is the result negative? What does it have to do with actual angles and our earlier definition of the sin function?
- Convert the angle from radians to degrees (perhaps by modifying the program). Draw the result on paper. See if this helps answering the above question.
- Modify the code to print out $\sin(2\pi + \frac{4\pi}{3})$. Can you explain the result?

We'll now write code to draw the functions in the range $[0, 8\pi]$: (source file)

```
public class SinCos2 {

    public static void main (String[] argv)
    {
        // Create two Function objects.
        Function sinFunc = new Function ("sin");
        Function cosFunc = new Function ("cos");

        // Put values into them.
        for (double x=0; x<=8*Math.PI; x+=0.1) {
            sinFunc.add (x, Math.sin(x));
            cosFunc.add (x, Math.cos(x));
        }

        // Display.
        Function.show (sinFunc, cosFunc);
    }

}
```

Exercise 35: Explain why both functions are *periodic* (that is, the structure repeats). What is the *period* (the size of the interval after it repeats)?

We will now try to find the maximum value of the sin function in the range $[0, 2\pi]$:

- The graph shows that the maximum is 1.
- If we didn't know, we could write a small program to find the maximum: (source file)

```
public class SinMaximum {

    public static void main (String[] argv)
    {
        // Initial guess: max occurs at 0, and has value 0.
        double bestX = 0;
        double max = Math.sin (bestX);

        // Now try values in the range [0,2*pi]
        for (double x=0; x<=2*Math.PI; x+=0.1) {
            double f = Math.sin (x);
            if (f > max) {
                max = f;
                bestX = x;
```

```
            }
        }

        // Print result.
        System.out.println ("Max value " + max + " occurred at x=" + bestX);
    }

}
```

Exercise 36: Use the above approach to find the minimum value of $\cos(x)$ for x in the range $[0, 2\pi]$. What could we do to make the result more accurate?

1.9 Guessing Functions

Often we have data and want a *rule* that explains the data:

- We want to decipher or extract the *function* that relates the data.
- Example: suppose we have this data:

x	*f(x)*
1	1
2	4
3	9
4	16
5	25

  ⇒ Then we could easily determine $f(x) = x^2$.

- Is it so obvious if the data were:

x	*f(x)*
1.5	2.25
2.33	5.4289
2.71	7.3441
4.6	21.16
5.3	28.09

Exercise 37: What good are such "rules"? Why are they useful? Why not just use the data directly?

Exercise 38: Can you guess the function that produced this data:

x	*f(x)*
1.0	5.86
2.0	9.00

3.0	12.14
4.0	15.28
5.0	18.43
6.0	21.57

Techniques for guessing functions:

- A first step is to see if the data can be reasonably approximated by a *linear* function such as $f(x) = 3x + 5$.
- If so, we try to find values a and b so that $f(x) = ax + b$ best explains the data.
- For example, consider the data

x	***f(x)***
1.0	8.0
2.0	11.00
3.0	14.00
4.0	17.00
5.0	20.00

- We will first display the data to get a "feel" for the shape of the function: (source file)

```
public class DataAnalysis {

    public static void main (String[] argv)
    {
        // Make a Function object.
        Function F = new Function ("mystery");

        // Put the data in.
        F.add (1, 8);
        F.add (2, 11);
        F.add (3, 14);
        F.add (4, 17);
        F.add (5, 20);

        // Display it.
        F.show ();
    }

}
```

- Now that we know that it's *linear*, i.e., of the form $f(x) = ax + b$ how do we discover a and b? This is where a little math will be useful.
    - Consider two points x_1 and x_2 in the data.
    - For example, $x_1 = 1$ and $x_2 = 2$ (the first two x-values).
    - For $x_1 = 1$, we know that $f(x_1) = ax_1 + b$ must equal 8.
      $\Rightarrow ax_1 + b = 8$
      $\Rightarrow a + b = 8$
    - Similarly, for the second point
      $\Rightarrow ax_2 + b = 11$
      $\Rightarrow 2a + b = 11$
    - Subtracting the first equation from second gives us $a = 3$, which means $b = 5$.

Exercise 39: Identify the linear function in the previous exercise.

Exercise 40: Draw the graph for this data:

d	***v***
8.33	1666.67
22.22	3666.67
23.61	4833.33
30.55	5000
36.81	5166.67
47.22	8000
69.44	11333.33
105.56	19666.67

Is the curve linear or non-linear? The data is from a historic 1929 paper.

What if the data is obviously "noisy" (has errors)?

- There are well-established techniques for fitting linear functions to noisy data.
  ⇒ Example: *least-squares regression*
- Key idea: find the line so that the "distance" from data points to the line is minimized.
  ⇒ The "distance" measure is very similar to the one we used earlier.

Exercise 41: Draw the graph for this data:

x	***f(x)***
1.0	1
2.0	13
3.0	33
4.0	61
5.0	97

Is the curve linear or non-linear?

What if the data is obviously non-linear?

- Non-linear functions are hard to fit to data.
- If the underlying function has a simple form, such as $f(x) = ax^2$, it is easy.
- However, non-linear functions can be quite complicated.
  ⇒ This is a sub-area called *interpolation* or *curve-fitting*.

1.10 Dealing with Nonlinearity

One way to get a grip around nonlinearity is to *transform* the data or your guessed function:

- For example, consider the following data:

x	$f(x)$
0.5	4.97
1.5	4.77
2.5	4.33
3.5	3.57
4.5	2.18

  - This function is obviously non-linear, as a plot will show.

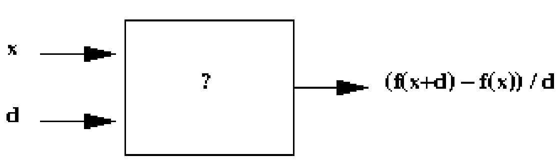
    Exercise 42: Write a small program to compute $x^2 + (f(x))^2$.

- This is an example of transforming the data to see a pattern (and infer an underlying function).

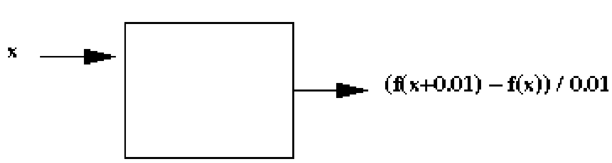
One way of transforming a function is to study its *derivative*:

- Consider some function $f(x)$ and some particular value of x such as $x = 5$.
- Suppose we pick some small value such as 0.01 and compute the following:
  - Compute $f(5 + 0.01)$
  - Compute $f(5)$
  - Then compute $\frac{f(5.01) - f(5)}{(5.01 - 5)}$.

  This value answers the question: if you move along the x-axis from 5 to 5.01 (a small step), how much does f change?
- Notice: this defines "change" starting at 5.
  $\Rightarrow$ At the x-value of 6, the change would be $\frac{f(6.01) - f(6)}{6.01 - 6}$.
- In more general terms, the change at some value x is: $\frac{f(x+0.01) - f(x)}{0.01}$.
- What is so special about 0.01?
  Nothing. We should really define "change" at x as: $\frac{f(x+d) - f(x)}{d}$.
- We could use $d = 0.01$ or $d = 0.0000001$ or whatever's appropriate, depending on our needs.
- Notice that the result is a *number*.
  - There is one such *number* for every x and d.
  - Thus, the output is a *function* of x and d:

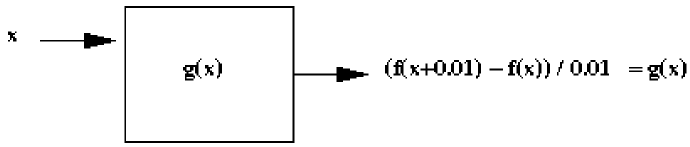


  - For a fixed value of $d = 0.01$ (for example), each input x results in a output number $\frac{f(x+0.01) - f(x)}{0.01}$.



  - Thus, the computation is a *function*.

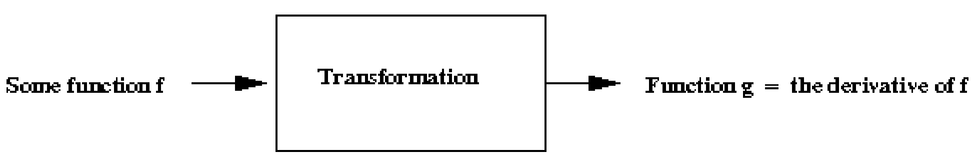
- Let's give this function a name: $g(x) = \frac{f(x+0.01)\ -\ f(x)}{0.01}$.



- Let's look an example:
  - Consider the function $f(x) = 3x^2$.
  - We'll write a program to compute $g(x) = \frac{f(x+0.01)\ -\ f(x)}{0.01}$: (source file)

```
public class Diff {

    public static void main (String[] argv)
    {
        // The value of d is fixed throughout.
        double d = 0.01;

        // Compute for x-values in [0,10]
        for (double x=0; x<=10; x+=1) {
            // Compute f.
            double f = 3*x*x;

            // Compute (f(x+d) - f(x)) / d
            double f_xd = 3 * (x+d)*(x+d);
            double g = (f_xd - f) / d;

            // Print.
            System.out.println ("x=" + x + "  g(x)=" + g);
        }
    }

}
```

- This type of transformation of a function is called a *derivative*



- There is a more precise mathematical definition, which we will defer to later.

Exercise 43: Execute the above program, Diff.java. Does the function $g(x)$ look familiar? Modify the above code to fill in the f and g values into `Function` objects and display them.

Exercise 44:

- Use $d = 0.01$ and plot the derivative function $g(x)$ along with $f(x)$ when $f(x) = 3x^2 + 5$.
- Compare this derivative function with that of $f(x) = 3x^2$. Explain what you observe.
- Use two points on the derivative function and write down the function as a formula.
- Explore what happens with different d values, for example try $d = 1, d = 0.1, d = 0.0001, d = 0.00001$.

Reconstructing the original function:

- Suppose I know the derivative function $g(x)$, can I reconstruct the original $f(x)$ such that $g(x) = \frac{f(x+d) - f(x)}{d}$?
- We can re-arrange the above to see that $f(x+d) = f(x) + dg(x)$
- But how do we know the $f(x)$ on the right side?
- As a first step, suppose we happen to know $f(0)$.
  - Then we can compute $f(0+d) = f(0) + dg(0)$.
    $\Rightarrow$ We know $g(x)$, remember?
  - Now that we know $f(d)$, we can compute $f(d+d) = f(d) + dg(d)$.
  - Now that we know $f(2d)$, we can compute $f(3d) = f(2d) + dg(2d)$.
  - ... and so on until we have a whole bunch of values of f.
- Let's see how this works in an example:
  - Suppose we know the derivative function is $g(x) = 6x$ and that $f(0) = 0$.
  - We want to ask: what function $f(x)$ is such that its derivative is $g(x) = 6x$?
  - Let's write a program to successively compute $f(d), f(2d), f(3d), f(4d), \ldots$: (source file)

```
public class Integration {

    public static void main (String[] argv)
    {
        // The value of d is fixed throughout.
        double d = 0.01;

        // Initial value is assumed known:
        double f = 0;

        // Compute for x-values in [0,10]
        for (double x=0.01; x<=2; x+=0.01) {
            // We are given g, so we can compute it.
            double g = 6 * x;

            // Find the new value at f(x+d) that becomes the new f.
            f = f + d * g;

            // Print.
            System.out.println ("x=" + x + "   f(x)=" + f);
        }
    }

}
```

- Suppose we wanted to compute $f(x)$ at some particular value of x, for example, $x = 3$.
  - We would still need to start at $d = 0$, and compute $f(d), f(2d), f(3d), \ldots,$, until we "hit" $f(3)$.
- Let's re-arrange the code to compute $f(x)$ at any desired value: (source file)

```
public class Integration2 {

    public static void main (String[] argv)
    {
        // We'll compute f(x) at x=1, 2, ..., 10 and put that in a graph.
        Function F = new Function ("f");

        // Notice: we are computing f(1), f(2), ..., and NOT f(d), f(2d), ...
        for (double x=1; x<=10; x+=1) {
            // Our method compute_f() does all the dirty work.
            double f = compute_f (x);
            F.add (x, f);
            System.out.println ("x=" + x + "  f(x)=" + f);
        }
```

```
        // Display.
        F.show ();
    }


    static double compute_f (double x)
    {
        // The value of d is fixed throughout.
        double d = 0.01;

        // Initial value is assumed known:
        double f = 0;

        // Compute for x-values in [0,10]. We are now using a variable called z.
        // Notice: goes over the range 0.01 up through x (which is input to the method).
        for (double z=0.01; z<=x; z+=0.01) {
            // We are given g, so we can compute it.
            double g = 6 * z;

            // Find the new value at f(z+d) that becomes the new f.
            f = f + d * g;

            // No need to print intermediate values.
        }

        // Return f(x)
        return f;
    }

}
```

Exercise 45: Suppose the derivative of $f(x)$ is $g(x) = -sin(x)$ and that we know $f(0) = 1$. Modify the above code to compute $f(x)$ in the interval $[0, 2\pi]$ and display the result.

Working with a simulation:

- Suppose we were able to simulate some physical system of interest.
- Let's say that we are interested in the relationship between two variables x and y.
  ⇒ We want to figure out the function f such that $y = f(x)$.
- If the relationship is nonlinear, we can try to use derivatives to see if that helps.
- For example, suppose our simulation was a piece of software called `Simulator`: (source file)

```
public class UnknownFunctionDerivative {

    public static void main (String[] argv)
    {
        // Make an instance of the object.
        Simulator S = new Simulator ();

        double d = 0.01;

        for (double x=0; x<=10; x+=1) {
            // Compute derivative at x.
            double f = S.getValue (x);
            double fd = S.getValue (x+d);
```

```
            double g = (fd - f) / d;
            System.out.println ("x=" + x + "  g(x)=" + g);
        }

    }

}
```

Exercise 46: Download Simulator.class and UnknownFunctionDerivative.java, and execute. What is the relationship between x and $f(x)$?

1.11 Computing and storing functions

First, consider how functions are *specified*:

- If the domain is discrete and finite, we could simply list function values:
  - For example

d	*v*
8.33	1666.67
22.22	3666.67
23.61	4833.33
30.55	5000
36.81	5166.67
47.22	8000
69.44	11333.33
105.56	19666.67

  - Here, *v* is a function of *d*, but there are only finitely many values
    ⇒ One can list all of them.

- For an uncountable domain, this is impossible
  ⇒ Must have use some other means
  ⇒ function is specified by construction, e.g., $f(x) = 3x^2 + 5$.

There are several different ways of writing code to compute functions:

1. If the functional form is known, simply implement the code to compute, e.g., for $f(x) = 3x^2 + 5$

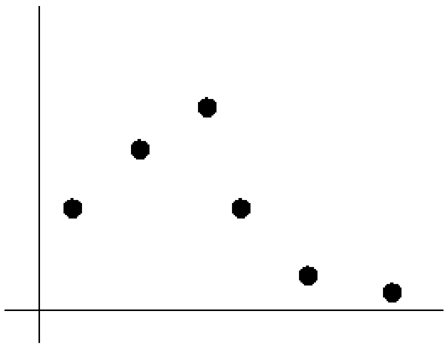
```
double computef (double x)
{
    return 3*x*x + 5;
}
```

2. If an approximate functional form is known, write code to implement the approximation:
   - For example, $sin(x)$ is approximately $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$
   - Then, sample code for this calculation:

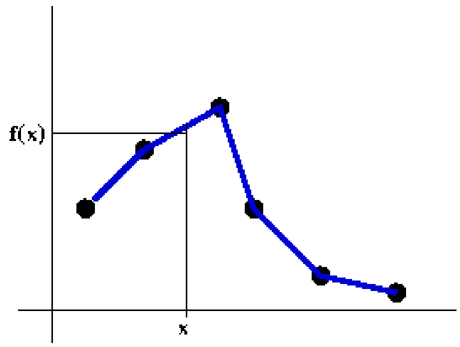
```
static double sin (double x)
{
    double s = x - (x*x*x)/(1*2*3) + (x*x*x*x*x)/(1*2*3*4*5) - (x*x*x*x*x*x*x)/(1*2*3*4*5*6*7);
    return s;
}
```

Exercise 47: How can you simplify and speed up the code above? What does the Java library use to compute sin(x)? Compare the accuracy of the above function with Java's sin function.

3. If no functional form is known, we might have some sample points:



Collection of sample points



Piecewise linear approximation

In this case, the approximate function might suffice.

4. Sometimes we have sample points, but can construct a *smoother curve*
   ⇒ Use curved interpolating segments (e.g., Bezier curves)

Exercise 48: Download AccelCar.java, Function.java and SimplePlotPanel.java. This is a simple model of an accelerating vehicle - the goal is to start at the left and reach the right in the least time possible but also such that the velocity at the end is as close to zero as possible.

- Compile and execute `AccelCar`, and try to achieve as low a score as possible.
- Then try to determine a good acceleration "function" by hand by reasoning.
- Download MyController.java, compile and execute. Examine the code - you should see a sample acceleration function.
- Implement a better acceleration function in `MyController`.
- How would you systematically search for the optimal acceleration function?

1.12 Limits of functions

A sequence of *functions*:

- We've seen examples of sequences of real numbers
  ⇒ e.g., $C_n = \frac{\sin(n)}{n}$.
- Consider the function $f_n(x) = \frac{x^2}{n}$ in the interval $[0, 1]$.
- The functions
  $f_1(x) = x^2$

$f_2(x) = \frac{x^2}{2}$
$f_3(x) = \frac{x^2}{3}$
...
form a sequence of functions.

- The limit of a sequence of functions is itself a *function*.
- Some of the most important results in mathematics arise from limits of functions.
  ⇒ Often the limiting function is more useful and practical
- Example: the Central Limit Theorem.
- The derivative function is an example of a limiting function:
  - For a function $f(x)$, define
    $g_d(x) = \frac{f(x+d) \ - \ f(x)}{d}$.
  - Then, as $d \rightarrow 0$, we get a sequence of functions.
  - The derivative is defined as the limit of this sequence:
    $f'(x) = \lim_{d \rightarrow 0} g_d(x) = \lim_{d \rightarrow 0} \frac{f(x+d) \ - \ f(x)}{d}$.

Exercise 49: Construct a sequence of distinct functions so that the limit is $f(x) = x$.
