LINEAR FILTERS (REVIEW)

- Definition of a **linear filter**
  - A linear filter f is characterized by a sequence $(f_k)_k$ of real numbers
    - the f_k's are called the ***filter taps*** (and we write $f = (f_k)_k$
  - Filtering an input signal $x = (x_n)_n$ through filter f gives an output signal $y = (y_n)_n$:

$$y_n = \sum_k f_k x_{n-k} = \sum_k f_{n-k} x_k \text{ for all } n$$

- Mathematical notation: $y = f \otimes x$
  - That is called the ***convolution*** of f and x

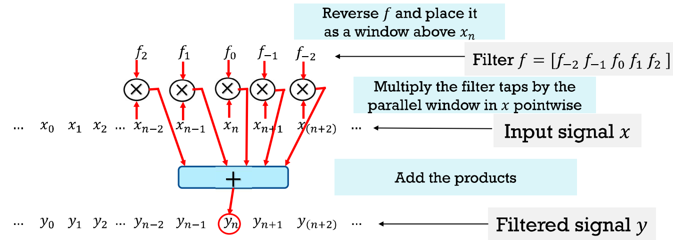
$$x \longrightarrow \boxed{f} \longrightarrow y = f \otimes x$$

- Notes about indexing notation:
  - Indices k can range from anywhere to anywhere
  - Any term where its index is "out of range" is by default = 0

If $x = [x_0, x_1, x_2, \ldots, x_{100}]$,
Then $x_{101} = 0, x_{-1} = 0, \ldots$

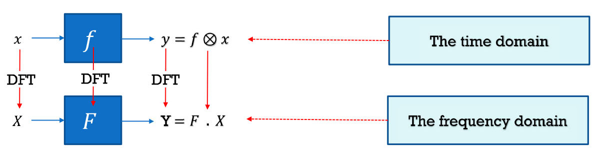
FILTERING AS A WEIGHTED "AVERAGE" (REVIEW)

-- THE FILTER TAPS ARE THE WEIGHTS --



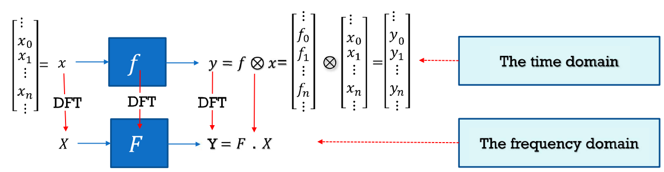
THE CONVOLUTION THEOREM (REVIEW) (1/2)

- **The convolution theorem:**
  - Let $x = (x_n)_n$ be a digital signal and $f = (f_k)_k$ be a filter, and let $y = (y_n)_n \stackrel{\text{def}}{=} f \otimes x$ be the output of filtering x with f.
  - Let X, Y and F denote the Fourier Transforms of x, y and f, respectively.
  - Then, $Y = F.X$ (pointwise multiplication).



THE CONVOLUTION THEOREM (REVIEW) (2/2)

- **The convolution theorem:**



$$\begin{bmatrix} \vdots \\ Y_0 \\ Y_1 \\ \vdots \\ Y_n \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ F_0 \\ F_1 \\ \vdots \\ F_n \\ \vdots \end{bmatrix} . \begin{bmatrix} \vdots \\ X_0 \\ X_1 \\ \vdots \\ X_n \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ F_0 . X_0 \\ F_1 . X_1 \\ \vdots \\ F_n . X_n \\ \vdots \end{bmatrix}$$

THE Z-TRANSFORM (REVIEW)

- Let $a = (a_k)_k$ be a sequence (like a discrete signal or a filter)
- The z-transform transforms a sequence $a = (a_k)_k$ into a complex function $A(z)$:

$$A(z) = \sum_k a_k z^k \quad \text{// a polynomial in z}$$

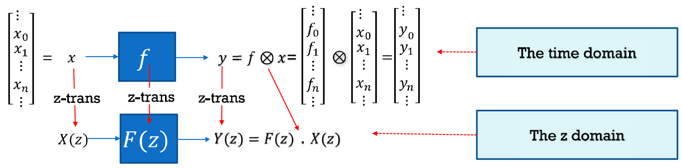
- We use the notation that the input sequence is denoted with a lower case letter, and its z-transform is denoted by the upper-case of the same letter:
  - $a = (a_k)_k \rightarrow A(z) = \sum_k a_k z^k$
  - $x = (x_k)_k \rightarrow X(z) = \sum_k x_k z^k$
  - $y = (y_k)_k \rightarrow Y(z) = \sum_k y_k z^k$
  - $f = (f_k)_k \rightarrow F(z) = \sum_k f_k z^k$

Z-TRANSFORM AND FILTERING (REVIEW)

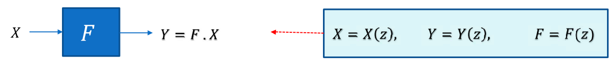
-- CONVOLUTION THEOREM IN TERMS OF Z-TRANSFORMS --

- **Convolution Theorem in terms of the z-transform**:
  - Let $x = (x_n)_n$ be a digital signal and $f = (f_k)_k$ be a filter, and let $y = (y_n)_n \stackrel{\text{def}}{=} f \otimes x$ be the output of filtering x with f.
  - Let $X(z), Y(z)$ and $F(z)$ denote the z-transforms of x, y and f, respectively.
  - Then, $Y(z) = F(z).X(z)$ (polynomial multiplication)

THE CONVOLUTION THEOREM (REVIEW)
-- AS A DIAGRAM--



Filtering in the z-domain as a simplified diagram:



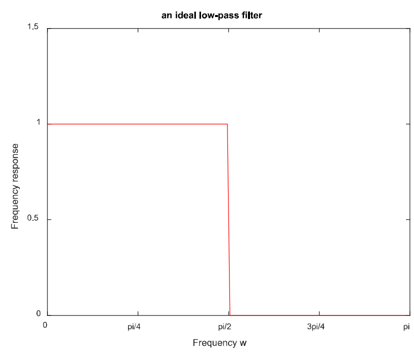
FREQUENCY RESPONSE OF A FILTER (REVIEW) (1)

- A filter $f = (f_k)_k$, as a spectrum-shaping tool, is best understood by its ***frequency response,*** $F(e^{-i\omega}) = \sum_k f_k e^{-ik\omega}$, where ω is a *continuous* frequency, and F is the z-transform of f

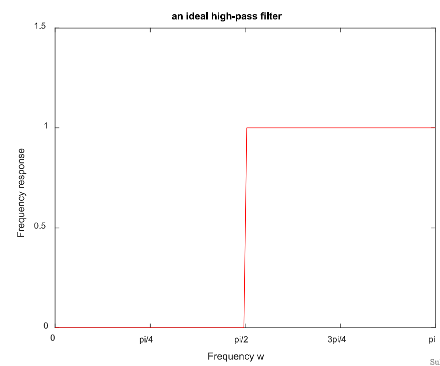
For convenience, people abuse the notation and write $F(\omega) = \sum_k f_k e^{-ik\omega}$

- The frequency response $F(\omega)$ is a complex function, periodic of period 2π
- Its magnitude $|F(\omega)|$ is periodic of period 2π and symmetric around the vertical axis, and thus it is enough to plot it in the $[0\ \pi]$ interval

LOW-PASS FILTERS (LPF) (REVIEW)
-- FREQUENCY RESPONSE --

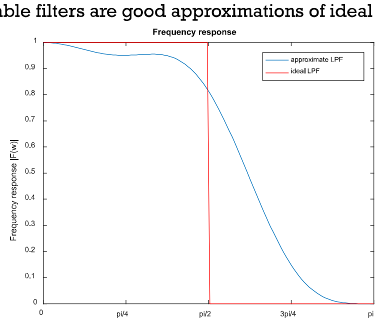


HIGH-PASS FILTERS (HPF) (REVIEW)
-- FREQUENCY RESPONSE --

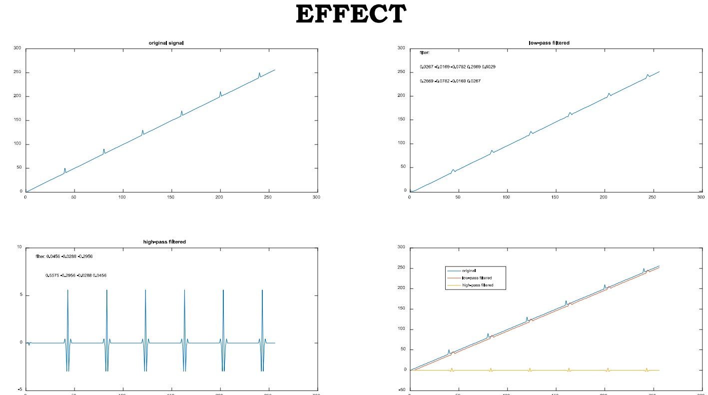


OBSERVATIONS ABOUT LPF'S AND HPF'S (REVIEW)

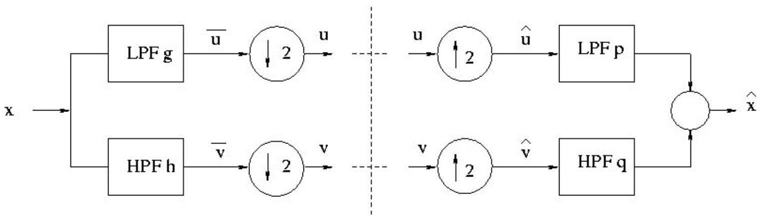
- Ideal LPF's and HPF's are not realizable in practice, but
  - many realizable filters are good approximations of ideal filters



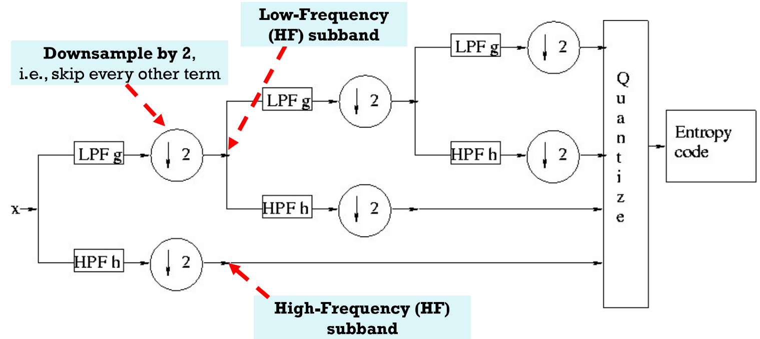
EXAMPLES OF LPF'S AND HPF'S AND THEIR EFFECT



THE MAIN SCHEME OF SUBBAND CODING/DECODING (REVIEW)



HOW SUBBAND CODING IS GENERALLY APPLIED
-- A TREE-LIKE STRUCTURE: THE ENCODER --



HOW SUBBAND CODING IS GENERALLY APPLIED
-- A TREE-LIKE STRUCTURE: THE DECODER --

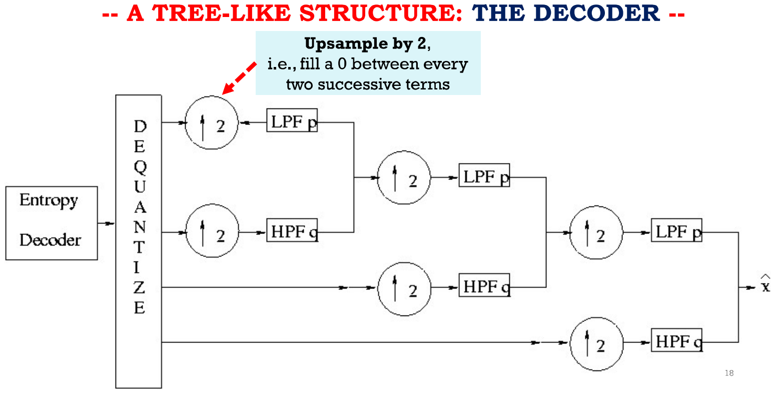


ILLUSTRATION OF SUBBAND CODING ON 1D SIGNALS

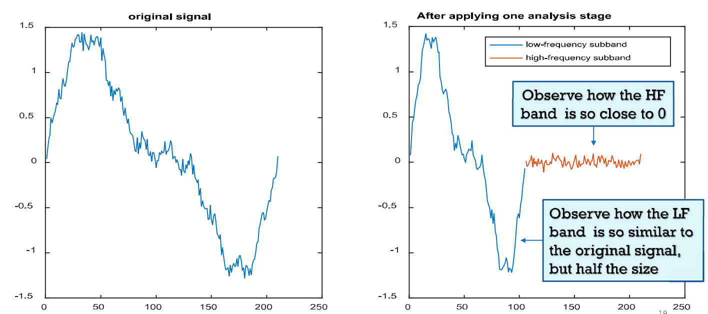


ILLUSTRATION OF SUBBAND CODING ON 2D SIGNALS

Original Lena

A one-level transform of Lena (Lena 1)

ILLUSTRATION OF SUBBAND CODING ON 2D SIGNALS

A one-level transform of Lena (Lena 1)

A two-level transform of Lena (Lena 2)

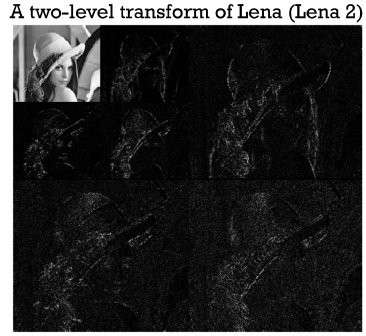


ILLUSTRATION OF SUBBAND CODING ON 2D SIGNALS

Original Lena

Lena reconstructed from just the low-frequency subband of Lena 2 (CR=16)

SUBBAND CODING ISSUES

- **Filter design**
- Quantization Method
- Shape of the tree
- Same or different filter sets per image or class of images?

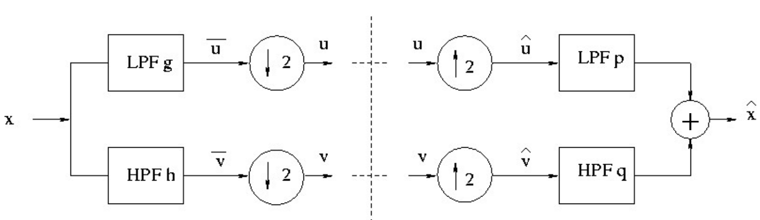
FILTER DESIGN

- Classical filter design techniques for LPF's and HPF's
  - Least Mean Square technique
  - Butterworth technique
  - Chebychev technique
- Those techniques are for designing **single** filters, rather than a bank of four filters working together
- The four filters (g, h, p, q) for a subband coding system must have the **perfection reconstruction** property (to be seen later)
  - the output signal is identical to the input signal if no quantization takes place

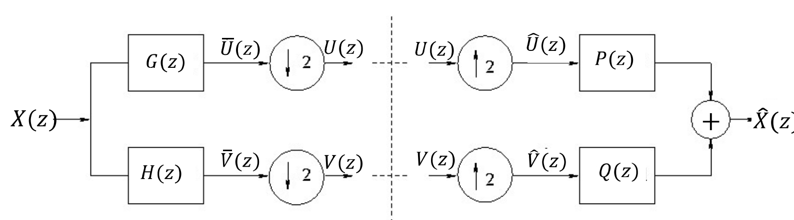
HOW SHOULD THE FOUR FILTERS RELATE TO ONE ANOTHER

- The four filters $g, h, p,$ and q should be designed so that if there is no quantization, then the reconstructed signal $\hat{x}$ is identical to the original signal x
- That is, $\hat{x} = x$
- That is referred to as the perfection reconstruction (PR) condition
- We will see next how that the PR condition translates into conditions on the four filters
- To do so, we will recall the subband coding/decoding scheme, and make use of the z-transform and the convolution theorem

THE MAIN SCHEME OF SUBBAND CODING IN THE TIME DOMAIN



THE MAIN SCHEME OF SUBBAND CODING IN THE Z-DOMAIN



$X(z), G(z), H(z), \bar{U}(z), \bar{V}(z), U(z), V(z), \hat{U}(z), \hat{V}(z), P(z), Q(z),$ and $\hat{X}(z)$ are the z-transorms of $x, g, h, \bar{u}, \bar{v}, u, v, \hat{u}, \hat{v}, p, q, and\ \hat{x}$.

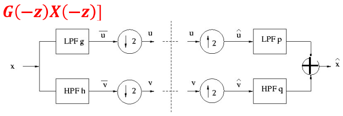
By the Conv. Thm: $\bar{U}(z) = G(z)X(z), \bar{V}(z) = H(z)X(z), \hat{X}(z) = P(z)\hat{U}(z) + Q(z)\hat{V}(z)$

DERIVATION OF THE PR CONDITION FOR THE FOUR FILTERS (1/3)

1. $\bar{u} = [\bar{u}_0, \bar{u}_1, \bar{u}_2, \bar{u}_3, \ldots, \bar{u}_{N-2}, \bar{u}_{N-1}] \rightarrow \bar{U}(z) = \bar{u}_0 + \bar{u}_1 z + \bar{u}_2 z^2 + \bar{u}_3 z^3 + \cdots + \bar{u}_{N-2} z^{N-2} + \bar{u}_{N-1} z^{N-1}$
2. $u = [\bar{u}_0, \bar{u}_2, \bar{u}_4 \ldots, \bar{u}_{N-2}] \rightarrow \hat{u} = [\bar{u}_0, 0, \bar{u}_2, 0, \ldots, \bar{u}_{N-2}, 0] \rightarrow \hat{U}(z) = \bar{u}_0 + \bar{u}_2 z^2 + \bar{u}_4 z^4 + \cdots + \bar{u}_{N-2} z^{N-2}$
3. Observe that
   - $\bar{U}(z) = \bar{u}_0 + \bar{u}_1 z + \bar{u}_2 z^2 + \bar{u}_3 z^3 + \cdots + \bar{u}_{N-2} z^{N-2} + \bar{u}_{N-1} z^{N-1}$ (From (1))
   - $\bar{U}(-z) = \bar{u}_0 - \bar{u}_1 z + \bar{u}_2 z^2 - \bar{u}_3 z^3 + \cdots + \bar{u}_{N-2} z^{N-2} - \bar{u}_{N-1} z^{N-1}$ (Replace z by $-z$ in the line above)
4. Therefore, $\bar{U}(z) + \bar{U}(-z) = 2(\bar{u}_0 + \bar{u}_2 z^2 + \bar{u}_4 z^4 + \cdots + \bar{u}_{N-2} z^{N-2}) = 2\hat{U}(z)$
5. Therefore, $\hat{U}(z) = \frac{1}{2}[\bar{U}(z) + \bar{U}(-z)]$

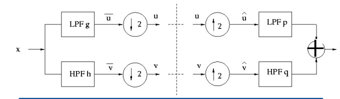
$\bar{U}(z) = G(z)X(z), \bar{V}(z) = H(z)X(z),$
$\hat{X}(z) = P(z)\hat{U}(z) + Q(z)\hat{V}(z)$

6. By the convolution theorem, $\bar{U}(z) = G(z)X(z)$ and thus $\bar{U}(-z) = G(-z)X(-z)$
7. From (5) and (6) above, we have $\hat{U}(z) = \frac{1}{2}[G(z)X(z) + G(-z)X(-z)]$
8. Similarly, $\hat{V}(z) = \frac{1}{2}[H(z)X(z) + H(-z)X(-z)]$



DERIVATION OF THE PR CONDITION (2/3)

7. $\hat{U}(z) = \frac{1}{2}[G(z)X(z) + G(-z)X(-z)]$
8. $\hat{V}(z) = \frac{1}{2}[H(z)X(z) + H(-z)X(-z)]$
9. Recall that $\hat{X}(z) = P(z)\hat{U}(z) + Q(z)\hat{V}(z)$
10. Using 7-9, we get:
    - $\hat{X}(z) = P(z)\hat{U}(z) + Q(z)\hat{V}(z) = \frac{1}{2}P(z)[G(z)X(z) + G(-z)X(-z)] + \frac{1}{2}Q(z)[H(z)X(z) + H(-z)X(-z)]$
    - $\hat{X}(z) = \frac{1}{2}[P(z)G(z) + Q(z)H(z)]X(z) + \frac{1}{2}[P(z)G(-z) + Q(z)H(-z)]X(-z)$



$\bar{U}(z) = G(z)X(z), \bar{V}(z) = H(z)X(z),$
$\hat{X}(z) = P(z)\hat{U}(z) + Q(z)\hat{V}(z)$

11. Therefore: $\hat{X}(z) = \boxed{\frac{1}{2}[P(z)G(z) + Q(z)H(z)]}X(z) + \boxed{\frac{1}{2}[P(z)G(-z) + Q(z)H(-z)]}X(-z)$
12. For perfect reconstruction $\hat{x} = x$, so we must have $\hat{X}(z) = X(z)$
13. Therefore, $\boxed{\frac{1}{2}[P(z)G(z) + Q(z)H(z)]=1}$ & $\boxed{\frac{1}{2}[P(z)G(-z) + Q(z)H(-z)]=0}$

THE PR CONDITION (3/3)

13. Therefore, $\frac{1}{2}[P(z)G(z) + Q(z)H(z)]=1$ & $\frac{1}{2}[P(z)G(-z) + Q(z)H(-z)]=0$
14. Hence, the perfect reconstruction (PR) condition becomes:

$$\boxed{\begin{array}{l} P(z)G(z) + Q(z)H(z)=2 \\ P(z)G(-z) + Q(z)H(-z)=0 \end{array}}$$

15. Consequently, to get a subband filter bank (of four filters), one has to solve the two equations above, subject to the constraints that
    - $H(1) = Q(1) = 0, H(-1) \neq 0, Q(-1) \neq 0$ (because h and q are high-pass filters)
    - $G(-1) = P(-1) = 0, G(1) \neq 0, P(1) \neq 0$ (because g and p are low-pass filters)

EXERCISES

- Let $f = (f_k)_k$ be a filter, and let $F(z) = \sum_k f_k z^k$ be its z-transform. Prove that:
  1. If f is a LPF, then $F(z=1) \neq 0$ and $F(z=-1) = 0$
  2. If f is a LPF, then $\sum_k f_k \neq 0$ and $\sum_k f_k(-1)^k = 0$
  3. If f is a HPF, then $F(z=1) = 0$ and $F(z=-1) \neq 0$
  4. If f is a HPF, then $\sum_k f_k = 0$ and $\sum_k f_k(-1)^k \neq 0$
- Let $g = (g_k)_k$ where $g_k = (-1)^k f_k$. Prove that
  1. If f is a LPF, then g is a HPF
  2. If f is a HPF, then g is a LPF

EXISTENCE OF FILTER BANKS THAT SATISFY THE PR CONDITION

- Are there filter banks that satisfy the PR condition?
  - Yes, there are many!!!!!!!
- Are they all good for compression?
- In other terms, is the PR condition sufficient?
- Answer:
  - No, the PR condition alone is not sufficient
  - Not all filter banks that satisfy the PR condition perform well in lossy compression
  - When quantization is applied (and loss incurred), the reconstructed signal can be of very low quality and exhibit serious distortions and artifacts

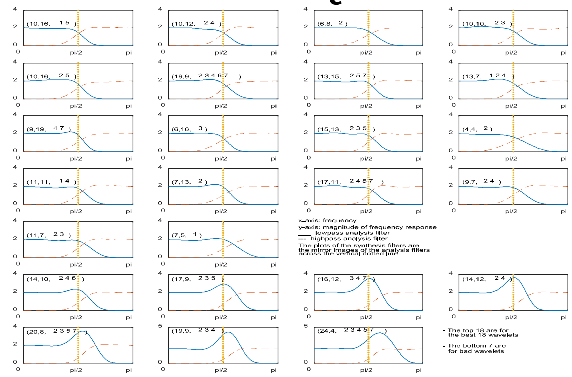
EXISTENCE OF GOOD FILTER BANKS (1/2)

- Are there <u>good</u> filter banks that satisfy the PR condition?
- Answer:
  - Yes, there are!!!!!!!
  - Most filter banks that satisfy the PR condition are bad for compression, but a few are good
- All we need is just one good filter bank
- Luckily, there are a few families of filter banks that are quite good for compression, and satisfy the PR condition

EXISTENCE OF GOOD FILTER BANKS (2/2)

- I have generated all filter quartets (i.e., filter banks) that
  - Satisfy the perfect reconstruction condition, and
  - Have a combined length (of the two analysis filters g and h) of at most 56 taps
- There were more than 4000 such quartets
- I measured their goodness for subband coding
- Findings:
  - The overwhelming majority of the quartets are bad or not good enough
  - Only about 18 quartets were very good
- The good quartets are illustrated next (by their frequency responses)
- Also illustrated are 7 bad quartets

GOOD FILTER QUARTETS AND BAD FILTER QUARTETS



SUBBAND CODING ISSUES

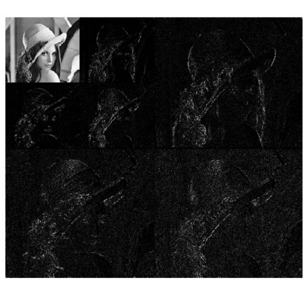
- Filter design
- **Quantization Method**
- Shape of the tree
- Same or different filter sets per image or class of images?

QUANTIZATION QUESTIONS

- As we saw in previous lectures, we have scalar quantizers, and vector quantizers
- Scalar quantizer work best when the data is decorrelated
  - And different scalar quantizers can be used: uniform, semi-uniform, and optimal Max-Lloyd quantizers
- Vector quantizers are preferable if the data is (still) correlated
- So, for subband coding, which quantizers to use?
- Same type of quantizer for all subbands?
- If not, why not and which type for which subband?
- If same type of quantizer, which type?
- If scalar quantizers, which subtype, and what quantizer parameters to use?

QUANTIZATION OF SUBBANDS

- If we choose the filters really well, and subbands are split deep enough, then the high-frequency (HF) subbands should be largely decorrelated
  - Therefore, scalar quantization are quite suitable for high-frequency subbands
- The LL subband can still have a lot of correlation. Therefore,
  - Either use a vector quantizer, or
  - Apply DCT on it then use a scalar quantizer like in JPEG, or
  - Use a (uniform) scalar quantizer of fine granularity (i.e., large number of small intervals)



QUANTIZATION OF HIGH-FREQUENCY SUBBANDS (1/8)

- We said that for HF subbands, use scalar quantizers
- Uniform or non-uniform?
  - Uniform might be sufficient, especially if HF has been split several times
  - But as we'll see, a non-uniform (optimal) Max-Lloyd quantizer could be justified
- Non-uniform quantizers have a higher overhead to represent, increasing the bitrate
- But, as will be seen, it turns out that HF subbands can be modeled statistically, requiring only a couple of parameters
- This implies that optimal non-uniform quantizers can be specified with a small number of data values, thus keeping the overhead (and the bitrate) low

QUANTIZATION OF HIGH-FREQUENCY SUBBANDS (2/8)
-- STATISTICAL MODEL OF HF SUBBANDS --

- Recall that if we have the probability distribution $p(x)$ of the data x to be quantized, then the Max-Lloyd algorithm can be executed nicely to compute the decision levels $d_1, d_2, \dots, d_{n-1}$ and reconstruction values $r_0, r_1, \dots, r_{n-1}$:

$$r_i = \frac{\int_{d_i}^{d_{i+1}} xp(x)dx}{\int_{d_i}^{d_{i+1}} p(x)dx},\ d_i = \frac{r_{i-1}+r_i}{2} \text{ for all } i$$

- Therefore, if we know $p(x)$, we can solve those equations using the iterative Max-Lloyd algorithm
- Do we know $p(x)$, when the data x is the pixels in a HF subband?
- Answer: Yes, that has been computed by researchers (see next)

QUANTIZATION OF HIGH-FREQUENCY SUBBANDS (3/8)
-- STATISTICAL MODEL OF HF SUBBANDS --

- Probability distribution $p(x)$ of the pixel values in HF subbands: The generalized Gaussian distribution

$$p(x) = ae^{-|bx|^r}$$

where

$$b = \frac{1}{\sigma}\left(\frac{\Gamma(\frac{3}{r})}{\Gamma(\frac{1}{r})}\right)^{\frac{1}{2}}$$

and

$$a = \frac{br}{2\Gamma(\frac{1}{r})}$$

and σ = the standard deviation of the data in the HF subband

Γ is a well-known math function:
$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$$
and is implemented and available in Matlab

r is a parameter that was studied and estimated by scientists to be: $r = 0.7$

QUANTIZATION OF HIGH-FREQUENCY SUBBANDS (4/8)
-- STATISTICAL MODEL OF HF SUBBANDS --

- Taking $r = 0.7$, we get
  - $\Gamma\left(\frac{1}{r}\right) = \Gamma\left(\frac{1}{0.7}\right) = 0.8861$, $\Gamma\left(\frac{3}{r}\right) = \Gamma\left(\frac{3}{0.7}\right) = 8.6879$
  - $b = \frac{1}{\sigma}\left(\frac{\Gamma(\frac{3}{r})}{\Gamma(\frac{1}{r})}\right)^{\frac{1}{2}} = \frac{3.1313}{\sigma}$, thus $b = \frac{3.1313}{\sigma}$
  - $a = \frac{br}{2\Gamma(\frac{1}{r})} = \frac{1.2369}{\sigma}$, thus $a = \frac{1.2369}{\sigma}$
  - $p(x) = ae^{-|bx|^r}$,
- Thus $$p(x) = \frac{1.2369}{\sigma}e^{-\frac{3.1313}{\sigma}|x|^{0.7}}$$

where σ = the standard deviation of the data in the HF subband

QUANTIZATION OF HIGH-FREQUENCY SUBBANDS (5/8)
-- VERIFYING THE STATISTICAL MODEL --

LL Subband

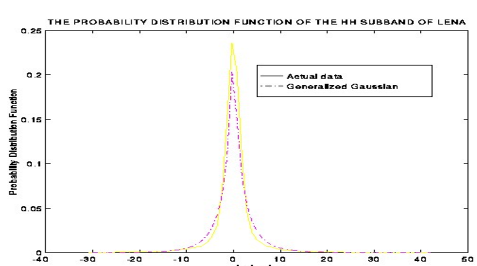
HL Subband

LH Subband

HH Subband

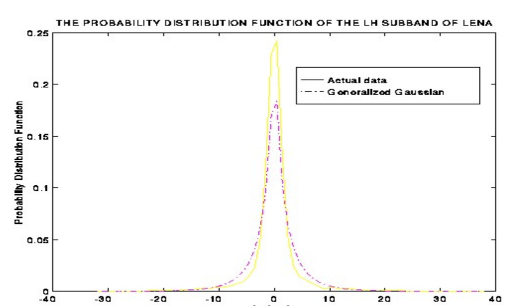
QUANTIZATION OF HIGH-FREQUENCY SUBBANDS (6/8)
-- VERIFYING THE STATISTICAL MODEL --

Observe how the actual distribution of the pixels, and the model distribution $p(x)$, are so similar



QUANTIZATION OF HIGH-FREQUENCY SUBBANDS (7/8)
-- VERIFYING THE STATISTICAL MODEL --

Observe how the actual distribution of the pixels, and the model distribution $p(x)$, are so similar



QUANTIZATION OF HIGH-FREQUENCY SUBBANDS (8/8) -- STATISTICAL MODEL OF HF SUBBANDS --

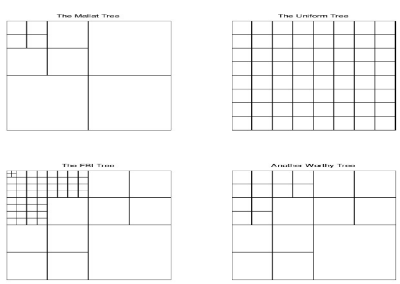
- Since the probability distribution of the pixels in any HF subband can be closely modeled by $p(x) = \frac{1.2369}{\sigma} e^{-\frac{3.1313}{\sigma}|x|^{0.7}}$, then
  - The optimal quantizer (of a certain given number n of intervals) can be fully specified by σ, the standard deviation of the data in the HF subband
  - And therefore, for dequantization, the coder need only send the σ of each HF subband to the decoder
- That way, we keep the bitrate low, while the HF subbands are quantized with optimal scalar quantizers

SUBBAND CODING ISSUES

- Filter design
- Quantization Method
- **Shape of the tree**
- Same or different filter sets per image or class of images?

SHAPE OF THE TREE (1/2)

- Shape of the tree refers to the structure of where to apply the subband coding scheme:



SHAPE OF THE TREE (2/2)

- Questions about the tree shape:
  - What is the best shape?
  - Is there a best shape for all images, or at least one best shape per class of images?
  - If not, is there an efficient way of deciding the shape of the tree on-line?
- Wavelet theory can address some of those questions
- But we won't have time to cover it
- Nevertheless:
  - The four tree shapes shown on the previous slide are good for all images
  - The uniform tree is rarely needed
  - A good dynamic way to tell, per image and per subband, whether a subband should be decomposed further, is to check if its variance > some threshold

SUBBAND CODING ISSUES

- Filter design
- Quantization Method
- Shape of the tree
- **Same or different filter sets per image or class of images?**

SAME OR DIFFERENT FILTERS FOR DIFFERENT SUBBANDS?

- In the early days of Wavelet theory (1990's), people wondered whether
  - Different subbands are best filtered with different customized filters
- Intuitively, the best filter set for a given signal is the one whose corresponding wavelet best resembles the signal in shape (i.e., in plot)
- The data in the subbands have different plots than the original data, suggesting the use for different filters than the ones applied on the original data
- However, studies have shown that, again, if you choose a good filter bank, the same filters will work really well for all subbands and all images
  - Relying on the frequency perspective