# Time Series Analysis
# chapter 7—Forecasting

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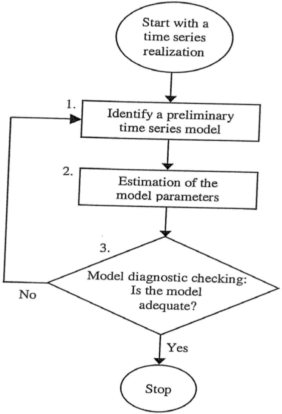
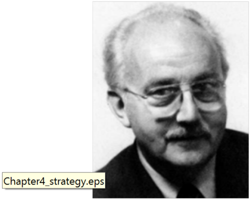
# Outline

- Model Steps
- Model Specification
- Model Estimation
- Model Diagnostic Checking
- Model Selection
- Forecasting

# Model Fitting (In-sample Fitting)

All models are wrong, but some are useful—George, E.P.Box

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# Model Forecasting (Out-of-Sample Validation/Evaluation)

- Predict future values of a time series, $y_{t+h}$, $h = 1, 2, \ldots$, based on data to the present $I_t = \{y_t, y_{t-1}, \ldots\}$.
- Let $y_{t+h|t}$ denote a forecast of $y_{t+h}$ made at time $t$, which has an associated *forecast error* or *prediction error* $\varepsilon_{t+h|t}$,

$$\varepsilon_{t+h|t} = y_{t+h} - y_{t+h|t}.$$

  Obviously, many different forecasts $y_{t+h|t}$ exists.
- Find the *optimal forecast* $y_{t+h|t}$ to minimizes the mean squared error of the forecast,

$$\text{MSE}(\varepsilon_{t+h|t}) \equiv E[\varepsilon^2_{t+h|t}] = E[(y_{t+h} - y_{t+h|t})^2].$$

- The *minimum MSE forecast* (best forecast) of $y_{t+h}$ based on $I_t$ is $E[y_{t+h}|I_t]$. (Why?)

# Model Forecasting

Proof:

$$
\begin{aligned}
E[(y_{t+h}-y_{t+h|t})^2] &= E\left\{[y_{t+h}-E(y_{t+h}|I_t)+E(y_{t+h}|I_t)-y_{t+h|t}]^2\right\} \\
&= E\left\{[y_{t+h}-E(y_{t+h}|I_t)]^2\right\}+2E\left\{[y_{t+h}-E(y_{t+h}|I_t)][E(y_{t+h}|I_t)-y_{t+h|t}]\right\} \\
&\qquad + E\left\{[E(y_{t+h}|I_t)-y_{t+h|t}]^2\right\} \\
&= E\left\{[y_{t+h}-E(y_{t+h}|I_t)]^2\right\}+E\left\{[E(y_{t+h}|I_t)-y_{t+h|t}]^2\right\}
\end{aligned}
$$

Taking the minimization with different values of $y_{t+h|t}$, we have

$$y_{t+h|t}=E(y_{t+h}|I_t).$$

Thus, the minimum MSE is $E[(y_{t+h}-y_{t+h|t})^2]=E[(y_{t+h}-E(y_{t+h}|I_t))^2]$.

# Forecasting AR($p$) model

**1-Step-Ahead Forecast:** From the AR($p$) model, we have

$$y_{t+1} = c + \phi_1 y_t + \cdots + \phi_p y_{t+1-p} + \varepsilon_{t+1}, \quad \varepsilon_t \sim \mathrm{IID}(0, \sigma^2).$$

- Under the *MSE loss function*, the *point forecast* of $y_{t+1}$ given $I_t$ is

$$y_{t+1|t} = E[y_{t+1}|I_t] = c + \phi_1 y_t + \cdots + \phi_p y_{t+1-p},$$

- The *1-step-ahead forecast error* is

$$\varepsilon_{t+1|t} = y_{t+1} - y_{t+1|t} = \varepsilon_{t+1}.$$

- The variance of *1-step-ahead forecast error* is $Var[\varepsilon_{t+1|t}] = Var(\varepsilon_{t+1}) = \sigma^2$.
- If at is *normally distributed*, then a *95% 1-step-ahead interval forecast* of $y_{t+1}$ is $y_{t+1|t} \pm 1.96\sigma$.

# Forecasting AR($p$) model

## 2-Step-Ahead Forecast:

From the AR($p$) model, we have

$$y_{t+2} = c + \phi_1 y_{t+1} + \cdots + \phi_p y_{t+2-p} + \varepsilon_{t+2}. \quad (1)$$

- Taking conditional expectation, we have

$$y_{t+2|t} = E[y_{t+2}|I_t] = c + \phi_1 y_{t+1|t} + \phi_2 y_t + \cdots + \phi_p y_{t+2-p}, \quad (2)$$

- The associated forecast error

$$\varepsilon_{t+2|t} = y_{t+2} - y_{t+2|t} = \phi_1(y_{t+1} - y_{t+1|t}) + \varepsilon_{t+2} = \varepsilon_{t+2} + \phi_1 \varepsilon_{t+1}.$$

- The variance of the forecast error is $Var[\varepsilon_{t+2|t}] = (1+\phi_1^2)\sigma^2$.
- If $\varepsilon_t$ is normally distributed, then a 95% 1-step-ahead interval forecast of $y_{t+1}$ is $y_{t+2|t} \pm 1.96\sigma\sqrt{1+\phi_1^2}$.
- It is interesting to see that $Var[\varepsilon_{t+2|t}] \geq Var[\varepsilon_{t+1|t}]$, meaning that as the forecast horizon increases the uncertainty in forecast also increases.

# Forecasting AR($p$) model

**Multistep-Ahead Forecast:** In general, we have

$$y_{t+h} = c + \phi_1 y_{t+h-1} + \cdots + \phi_p y_{t+h-p} + \varepsilon_{t+h}. \tag{3}$$

- Taking conditional expectation, we have

$$y_{t+h|t} = E[y_{t+h}|I_t] = c + \sum_{i=1}^{p} \phi_i y_{t+h-i|t}, \tag{4}$$

  where $y_{t+\ell|t} = y_{t+\ell}$ if $\ell \leq 0$. This forecast can be computed recursively using forecasts $y_{t+i|t}$ for $i = 1, \ldots, h-1$.
- The $h$-step-ahead forecast error is

$$\varepsilon_{t+h|t} = y_{t+h} - y_{t+h|t}.$$

- How to obtain the forecasting interval for the AR($p$) model?

# Forecasting MA($q$) models

Consider the MA($q$) model $y_t = \sum_{i=0}^{q} \theta_i \varepsilon_{t-i}$, with $\theta_0 \equiv 1$. Given the *IID* properties of $\varepsilon_t$, the *optimal forecast* is

$$y_{t+h|t} = \begin{cases} \sum_{i=h}^{q} \theta_i \varepsilon_{t+h-i}, & \text{for } h = 1, \dots, q, \\ 0, & \text{for } h > q, \end{cases}$$

whereas the corresponding forecast error follows as

$$\varepsilon_{t+h|t} = \begin{cases} \sum_{i=0}^{h-1} \theta_i \varepsilon_{t+h-i}, & \text{for } h = 1, \dots, q, \\ \sum_{i=0}^{q} \theta_i \varepsilon_{t+h-i}, & \text{for } h > q, \end{cases}$$

which can be simplified to $\varepsilon_{t+h|t} = \sum_{i=0}^{h-1} \theta_i \varepsilon_{t+h-i}$ by defining $\theta_i \equiv 0$ for $h > q$.

# Forecasting MA($q$) models

Given the assumptions on $\varepsilon_t$, it follows that

$$E[\varepsilon_{t+h|t}] = 0,$$

and for *the mean squared error (MSE) of the forecast*

$$\text{MSE}(\varepsilon_{t+h|t}) = E[\varepsilon^2_{t+h|t}] = \sigma^2 \sum_{i=0}^{h-1} \theta_i^2.$$

Assuming normality, a 95% *forecasting interval* for $y_{t+h}$ is bounded by

$$\left( y_{t+h|t} - 1.96 \cdot \text{RMSE}(\varepsilon_{t+h|t}), \quad y_{t+h|t} + 1.96 \cdot \text{RMSE}(\varepsilon_{t+h|t}) \right),$$

where $\text{RMSE}(\varepsilon_{t+h|t})$ denotes the *square root* of $\text{MSE}(\varepsilon_{t+h|t})$.

# Forecasting ARMA($p, q$) model:

The ARMA($p, q$) model for $y_t$ is

$$y_t = \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q},$$

or, in lag operator form $\phi_p(B) y_t = \theta_q(B) \varepsilon_t$.

- The true forecast value is

$$y_{t+h} = \phi_1 y_{t+h-1} + \cdots + \phi_p y_{t+h-p} + \varepsilon_{t+h} + \theta_1 \varepsilon_{t+h-1} + \cdots + \theta_q \varepsilon_{t+h-q}.$$

- The minimum MSE forecast for $y_{t+h}$ is

$$y_{t+h|t} = \phi_1 y_{t+h-1|t} + \cdots + \phi_p y_{t+h-p|t} + \theta_1 \varepsilon_{t+h-1|t} + \cdots + \theta_q \varepsilon_{t+h-q|t},$$

  where $y_{t+\ell|t} = y_{t+\ell}$ if $\ell \leq 0$; and $\varepsilon_{t+\ell|t} = 0$ if $\ell > 0$, otherwise $\varepsilon_{t+\ell|t} = \varepsilon_{t+\ell}$.

## Forecasting the ARMA($p, q$) models by MA($\infty$) representation

- It is convenient to rewrite the model as an MA($\infty$) model, that is, $y_t = \phi_p(L)^{-1}\theta_q(L)\varepsilon_t$ or

$$y_t = \varepsilon_t + \eta_1\varepsilon_{t-1} + \eta_2\varepsilon_{t-2} + \eta_3\varepsilon_{t-3} + \cdots .$$

- The minimum MSE linear forecast (best linear predictor) of $y_{t+h}$ based on $I_t$ is

$$y_{t+h|t} = \eta_h\varepsilon_t + \eta_{h+1}\varepsilon_{t-1} + \cdots , \quad \text{with } \eta_0 \equiv 1.$$

[See Hamilton (1994) page 74].

- From which it follows that the *h-step-ahead prediction error* is given by

$$\varepsilon_{t+h|t} = \varepsilon_{t+h} + \eta_1\varepsilon_{t+h-1} + \cdots + \eta_{h-1}\varepsilon_{t+1} = \sum_{i=0}^{h-1}\eta_i\varepsilon_{t+h-i},$$

and the MSE of the forecast error is

$$\text{MSE}(\varepsilon_{t+h|t}) = \sigma^2\sum^{h-1}\eta_i^2,$$