Chapter 3 Stationary Linear Time Series Models

Time Series Analysis
WISE, XMU

Content

- This chapter introduces the theoretical properties of time series determined by models.

1. Moving average (MA) models;
2. Autoregressive (AR) models;
3. Mixed autoregressive moving average (ARMA) models;
4. Invertibility

Please read Chapter 4 of Cryer & Chan (2008).

§3.1 Moving average (MA) processes

- **Definition:** A **moving average** process of order q, and abbreviated as MA(q), is defined as:

$$Z_t = \theta_0 + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q},$$

  where $q \geq 0$ is an integer and $\{a_t\} \sim WN(0, \sigma_a^2)$.
  - A moving average process of order 0 is actually a white noise sequence if $\theta_0 = 0$.
  - The MA model was first proposed by E. Slutsky in 1927 to explain some phenomena in economic data such as period behaviors etc.

- **Example:** Consider the MA(1) process,

$$Z_t = \theta_0 + a_t - \theta a_{t-1}.$$

  - Clearly $\mu = \mathrm{E}(Z_t) = \theta_0$ and $\gamma_0 = \mathrm{var}(Z_t) = \sigma_a^2(1+\theta^2)$.
  - The auto-covariance function at lag 1 is

$$\begin{aligned} \gamma_1 = \mathrm{cov}(Z_t, Z_{t-1}) &= \mathrm{cov}(a_t - \theta a_{t-1}, a_{t-1} - \theta a_{t-2}) \\ &= \mathrm{cov}(-\theta a_{t-1}, a_{t-1}) = -\theta \sigma_a^2, \end{aligned}$$

    and then the auto-correlation function at lag 1 is

$$\rho_1 = \frac{-\theta}{1+\theta^2}.$$

- The auto-covariance function at lag 2 is

$$\gamma_2 = \mathrm{cov}(Z_t, Z_{t-2}) = \mathrm{cov}(a_t - \theta a_{t-1}, a_{t-2} - \theta a_{t-3}) = 0,$$

  since there are no a's with subscripts in common between Z_t and Z_{t-2}. Hence, $\rho_2 = 0$.
- Similarly, $\gamma_k = \mathrm{cov}(Z_t, Z_{t-k}) = 0$, and $\rho_k = 0$, whenever $k \geq 2$; that is, **the process has no correlation beyond lag 1**. This fact will be important later when we need to choose suitable models for real data (model specification).
- We can summarize these properties as follows,

$$\begin{cases} E(Z_t) = \theta_0, \\ \gamma_0 = \mathrm{var}(Z_t) = \sigma_a^2(1+\theta^2), \\ \rho_1 = \dfrac{-\theta}{1+\theta^2}, \\ \rho_k = 0, \qquad\qquad \forall\ k \geq 2. \end{cases}$$

- **Example:** Consider the MA(2) process,

$$Z_t = \theta_0 + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}.$$

  - Clearly, $\mu = \mathrm{E}(Z_t) = \theta_0$.
  - The variance is then

$$\begin{aligned}\gamma_0 &= \mathrm{var}(Z_t) = \mathrm{var}(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}) \\ &= (1 + \theta_1^2 + \theta_2^2)\sigma_a^2,\end{aligned}$$

  - The auto-covariance function at lag 1 is

$$\begin{aligned}\gamma_1 &= \mathrm{cov}(Z_t, Z_{t-1}) \\ &= \mathrm{cov}(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}, a_{t-1} - \theta_1 a_{t-2} - \theta_2 a_{t-3}) \\ &= \mathrm{cov}(-\theta_1 a_{t-1}, a_{t-1}) + \mathrm{cov}(-\theta_2 a_{t-2}, -\theta_1 a_{t-2}) \\ &= [-\theta_1 + (-\theta_1)(-\theta_2)]\sigma_a^2 = (-\theta_1 + \theta_1\theta_2)\sigma_a^2,\end{aligned}$$

- The auto-covariance function at lag 2 is

$$\begin{aligned}
\gamma_2 &= \operatorname{cov}(Z_t, Z_{t-2}) \\
&= \operatorname{cov}(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}, a_{t-2} - \theta_1 a_{t-3} - \theta_2 a_{t-4}) \\
&= \operatorname{cov}(-\theta_2 a_{t-2}, a_{t-2}) = -\theta_2 \sigma_a^2.
\end{aligned}$$

- Moreover, as before,

$$\gamma_k = 0, \qquad \forall\ k \geq 3.$$

- Thus, the auto-correlation function for the MA(2) process is

$$\begin{cases}
\rho_1 = \dfrac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2}, \\
\rho_2 = \dfrac{-\theta_2}{1 + \theta_1^2 + \theta_2^2}, \\
\rho_k = 0, & \forall\ k \geq 3.
\end{cases}$$

- Consider a general MA(q) process,

$$Z_t = \theta_0 + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q}.$$

  - The mean is $\mu = \theta_0$ and the variance is

$$\gamma_0 = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\sigma_a^2.$$

  - The auto-correlation function is

$$\rho_k = \begin{cases} \dfrac{-\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \cdots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2}, \\ \qquad\qquad k = 1, 2, \cdots, q, \\ 0, \qquad\qquad k \geq q + 1. \end{cases}$$

- **Definition:** It is important to emphasize that the auto-correlation function $\rho_k \neq 0$ as $k = q$ and $\rho_k = 0$ as $k > q$. We usually say that the ACF of an MA(q) process **cuts off** after q lags. The *cut-off property* of the ACF is a special property which holds only for MA processes.

- The **back-shift operator** B on an arbitrary time series $Z = \{Z_t\}$ is defined by:

$$BZ_t = Z_{t-1}, \qquad B^k Z_t = B^{k-1}(BZ_t) = \cdots = Z_{t-k}, \qquad \forall\ k \in \mathbb{Z}.$$

  That is, $B(Z)$ is the lag-1 series of the original series Z, and $B^k(Z)$ is the lag-k series of Z. Especially, B^0 is the *identity* operator, i.e., $B^0 Z = Z$ for any series Z.
  The MA process $Z_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q}$, can be rewritten as

$$Z_t = (1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q) a_t = \theta(B) a_t,$$

  where $\theta(x) = 1 - \theta_1 x - \cdots - \theta_q x^q$ is the **MA characteristic polynomial**.

§3.2 Autoregressive (AR) processes

- **Definition:** A ***p*****th order autoregressive model** (or, for short, an AR(p) model) $\{Z_t\}$ satisfies the equation

$$Z_t = \theta_0 + \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t,$$

  where $p \geq 0$ is an integer, ϕ's are real parameters, and $\{a_t\} \sim WN(0, \sigma_a^2)$.
- The model can be rewritten as

$$\phi(B) Z_t = \theta_0 + a_t,$$

  where $\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p$ is the **AR characteristic polynomial**.
- This model was first proposed by G.U. Yule in 1927 to explain to the sunspot data.

- **Theorem:** The AR(p) model has an unique stationary solution if all roots of the corresponding **AR characteristic equation**

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0,$$

  are outside the unit circle.
    - This unique stationary solution is called the **AR**(p) **process**.
    - The condition in the above is called the **stationarity condition**.
    - For a complex value z, if $|z| > 1$, then we say that it is outside the unit circle.

Example : Consider an AR(1) model,

$$Z_t = 0.5Z_{t-1} + a_t.$$

Question: Is it stationary? Could you express Z_t as MA(∞)?
Sol: Yes as $|\phi| = 0.5 < 1$.

Method 1: Define $Z_t = \sum_{j=0}^{\infty} \psi_j a_{t-j}$. Substitute to equation above and match the coefficients. We have
$\sum_{j=0}^{\infty} \psi_j a_{t-j} = 0.5 \sum_{j=0}^{\infty} \psi_j a_{t-1-j} + a_t$. Hence $\psi_j = 0.5\psi_{j_1}$.
As $\psi_0 = 1$, we have $\psi_j = 0.5^j$, and
$Z_t = a_t + 0.5a_{t-1} + 0.5^2 a_{t-2} + \cdots$.

Method 2: Note that $(1 - 0.5B)Z_t = a_t$. Hence
$Z_t = \frac{1}{1-0.5B} a_t = (1 + 0.5B + 0.5^2 B^2 + \cdots)a_t$

- **Example:** Consider an AR(1) model,

$$Z_t = \theta_0 + \phi Z_{t-1} + a_t, \tag{3.1}$$

where $\{a_t\} \sim WN(0,1)$, and the stationarity condition is satisfied, i.e. $|\phi| < 1$. Find out the unique stationary solution. By Theorems, its stationary solution must has the form of

$$Z_t = \mu + \sum_{j=0}^{\infty} \psi_j a_{t-j}.$$

Substituting it into equation (3.1), we have that

$$\mu + \sum_{j=0}^{\infty} \psi_j a_{t-j} = \theta_0 + \phi\mu + \phi \sum_{j=0}^{\infty} \psi_j a_{t-1-j} + a_t,$$

or

$$
\begin{aligned}
&\mu + \psi_0 a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots \\
&= \theta_0 + \phi\mu + \phi\psi_0 a_{t-1} + \phi\psi_1 a_{t-2} + \phi\psi_2 a_{t-3} + \cdots + a_t.
\end{aligned}
$$

1. Taking expectation to both sides of the above equation, we have that

$$\mu = \theta_0 + \phi\mu \quad \Rightarrow \quad \mu = \theta_0/(1-\phi);$$

2. Taking the covariance of a_t and both sides of the above equation, then

$$\psi_0 = 1;$$

3. Taking the covariance of a_{t-1} and both sides of the above equation, then

$$\psi_1 = \phi\psi_0 \quad \Rightarrow \quad \psi_1 = \phi;$$

4. Taking the covariance of a_{t-k} and both sides of the above equation, then

$$\psi_k = \phi\psi_{k-1} \quad \Rightarrow \quad \psi_k = \phi^k \quad \text{for } k = 2, 3, \ldots;$$

Thus,

$$Z_t = \frac{\theta_0}{1-\phi} + \sum_{j=0}^{\infty} \phi^j a_{t-j}.$$

(It is noteworthy that we are just comparing the coefficients of a_{t-j}'s at both sides!)

- There is an alternative method to find out the AR process. Note that the AR model can be rewritten as

$$(1 - \phi B)Z_t = \theta_0 + a_t.$$

Then

$$\begin{aligned} Z_t &= \frac{1}{1-\phi B}\theta_0 + \frac{1}{1-\phi B} a_t \\ &= \frac{\theta_0}{1-\phi} + \sum_{j=0}^{\infty} (\phi B)^j a_t = \frac{\theta_0}{1-\phi} + \sum_{j=0}^{\infty} \phi^j a_{t-j}. \end{aligned}$$

Note: you could define $W_t = \phi W_{t-1} + a_t$, where $W_t = Z_t - \mu$. Similarly, we have $W_t = \sum_{j=0}^{\infty} \phi^j a_{t-j}$. Note that $\mu = \frac{\theta_0}{1-\phi}$. Hence $Z_t = \frac{\theta_0}{1-\phi} + \sum_{j=0}^{\infty} \phi^j a_{t-j}$

Example: Consider an AR(2) model,

$$Z_t = 0.3Z_{t-1} + 0.18Z_{t-2} + a_t.$$

(1) Is it stationary?
(2) Write down the AR process if it is stationary.

Yes, as its characteristic roots are outside unit circle.
Suppose $Z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots$.
$0.3Z_{t-1} + 0.18Z_{t-2} = 0.3(a_{t-1} + \psi_1 a_{t-2} + \psi_2 a_{t-3} + \cdots) + 0.18(a_{t-2} + \psi_1 a_{t-3} + \psi_2 a_{t-4} + \cdots)$. By matching the coefficients, $\psi_1 = 0.3$, $\psi_2 = 0.3 * \psi_1 + 0.18$, $\psi_3 = 0.3 * \psi_2 + 0.18 * \psi_1$, $\cdots$

- **Example :** Find out the AR process of an AR(2) model, $Z_t = \theta_0 + \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + a_t$, where the stationarity condition holds. Then its stationary solution has the form of

$$Z_t = \mu + \sum_{j=0}^{\infty} \psi_j a_{t-j}.$$

Substituting it into the model, we have that

$$\begin{aligned} &\mu + \psi_0 a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots \\ &= \theta_0 + \phi_1 \mu + \phi_1 \psi_0 a_{t-1} + \phi_1 \psi_1 a_{t-2} + \phi_1 \psi_2 a_{t-3} + \cdots \\ &+ \phi_2 \mu + \phi_2 \psi_0 a_{t-2} + \phi_2 \psi_1 a_{t-3} + \phi_2 \psi_2 a_{t-4} + \cdots + a_t. \end{aligned}$$

By comparing the coefficients of a_{t-j}'s at both sides, we can obtain that

$$\begin{aligned}
\mu &= \theta_0/(1-\phi_1-\phi_2); \\
\psi_0 &= 1; \\
\psi_1 &= \phi_1\psi_0 = \phi_1; \\
\psi_2 &= \phi_1\psi_1 + \phi_2\psi_0 = \phi_1^2 + \phi_2; \\
\psi_3 &= \phi_1\psi_2 + \phi_2\psi_1; \\
&\cdots \\
\psi_k &= \phi_1\psi_{k-1} + \phi_2\psi_{k-2}; \cdots
\end{aligned}$$

Note that we can calculate the values of ψ_j's recursively although we may not be able to write out the explicit expression.

- Consider the general AR(p) model,

$$Z_t = \theta_0 + \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t,$$

where the stationarity condition holds. Then we can calculate the values of ψ_j's recursively,

$$\begin{cases} \mu = 1/(1 - \phi_1 - \cdots - \phi_p), \\ \psi_1 = \phi_1, \\ \psi_2 = \phi_1 \psi_1 + \phi_2, \\ \cdots\cdots \\ \psi_k = \phi_1 \psi_{k-1} + \phi_2 \psi_{k-2} + \cdots + \phi_p \psi_{k-p}, \qquad k > p. \end{cases}$$

- **The mean function:** For a stationary AR(p) model,

$$Z_t = \theta_0 + \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t,$$

  we take expectation to both sides of the above equation, and obtain

$$\mu = \theta_0 + \phi_1 \mu + \phi_2 \mu + \cdots + \phi_p \mu + 0$$

  or

$$\mu = \frac{\theta_0}{1 - \phi_1 - \phi_2 - \cdots - \phi_p}$$

  since $1 - \phi_1 - \phi_2 - \cdots - \phi_p \neq 0$.

- **The auto-correlation function:** Consider the pth-order AR process

$$Z_t = \theta_0 + \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t, \qquad (3.2)$$

  We consider $cov(Z_t, Z_{t-k})$, then divide by $\gamma_0 = var(Z_t)$, and obtain the important recursive relationship

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \cdots + \phi_p \rho_{k-p}, \qquad k \geq 1. \qquad (3.3)$$

Putting $k = 1, 2, \cdots, p$ at equation (3.3) and using $\rho_0 = 1$ and $\rho_{-k} = \rho_k$, we get the **Yule-Walker** equations (in matrix form)

$$\begin{pmatrix} 1 & \rho_1 & \cdots & \rho_{p-1} \\ \rho_1 & 1 & \cdots & \rho_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{p-1} & \rho_{p-2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_p \end{pmatrix}.$$

- Given values for ϕ_1, ϕ_2, $\cdots$, ϕ_p, the Yule-Walker equations can be solved for $\rho_1, \rho_2, \cdots, \rho_p$.
- We then use equation (3.3) to calculate ρ_k for $k = p + 1, p + 2, \cdots$.

- **The Variance function:** Note that

$$\begin{aligned} \mathrm{E}(a_t Z_t) &= \mathrm{E}[a_t(\phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t)] \\ &= \mathrm{E}(a_t^2) = \sigma_a^2. \end{aligned}$$

  We multiply equation (3.2) by Z_t, take expectations, and find

$$\gamma_0 = \phi_1\gamma_1 + \phi_2\gamma_2 + \cdots + \phi_p\gamma_p + \sigma_a^2,$$

  which, using $\rho_k = \gamma_k/\gamma_0$, can be written as

$$\gamma_0 = \frac{\sigma_a^2}{1 - \phi_1\rho_1 - \phi_2\rho_2 - \cdots - \phi_p\rho_p}.$$

- **The auto-covariance function:** It holds that, $\gamma_k = \gamma_0 * \rho_k$ for $k \geq 1$.

- There is an alternative way to calculate the mean, variance, auto-covariance and autocorrelation functions of a stationary AR(p) model

$$Z_t = \theta_0 + \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t.$$

  - We first calculate the corresponding AR process, $Z_t = \mu + \sum_{j=0}^{\infty} \psi_j a_{t-j}$;
  - then calculate these functions based on this general linear process. For example,

$$\rho_k = \frac{\mathrm{cov}(Z_t, Z_{t-k})}{\mathrm{var}(Z_t)} = \frac{\sum_{j=0}^{\infty} \psi_j \psi_{j+k}}{\sum_{j=0}^{\infty} \psi_j^2}, \quad k > 0.$$

Example : Consider an AR(2) model,

$$Z_t = 1 + 0.3Z_{t-1} + 0.18Z_{t-2} + a_t.$$

(1) Calculate $\mu = EZ_t$.

(2) Calculate the ACF ρ_k, $k = 1, 2, \ldots$.

(3) Calculate the autocovariance γ_k, $k = 0, 1, 2, \ldots$.

Sol: $\mu = 1/(1 - 0.3 - 0.18)$. $\rho_k = 0.3\rho_{k-1} + 0.18\rho_{k-2}$ for $k > 0$.
Let $k = 1, 2$. $\rho_1 = 0.3 + 0.18\rho_1$, $\rho_2 = 0.3 + 0.18\rho_1$.
So $\rho_1 = 0.37$, $\rho_2 = 0.29$, $\rho_3 = 0.3 * 0.29 + 0.18 * 0.37$, $\cdots$
Solve γ_0 as $\gamma_0 = 0.3\gamma_0\rho_1 + 0.18\gamma_0\rho_2 + \sigma_a^2$,
i.e. $\gamma_0 = \sigma_a^2/(1 - 0.3 * 0.37 - 0.18 * 0.29)$. Obtain $\gamma_k = \gamma_0 * \rho_k$.

§3.3 The mixed autoregressive-moving average model

- **Definition:** In general, if

$$\begin{aligned} Z_t &= \theta_0 + \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} \\ &\quad + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q}, \end{aligned}$$

  we say that $\{Z_t\}$ is a mixed *autoregressive-moving average* model of orders p and q, respectively; we abbreviate the name to ARMA(p, q).

- The model will reduce to an AR model if $q = 0$, and to an MA process if $p = 0$. Then the AR and MA models are special cases of ARMA models.
- For convenience, we may rewrite the above equation as

$$\phi(B)Z_t = \theta_0 + \theta(B)a_t,$$

where $\phi(x)$ and $\theta(x)$ are the AR and MA characteristic polynomials, respectively. That is,

$$\begin{aligned} \phi(x) &= 1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p, \\ \theta(x) &= 1 - \theta_1 x - \theta_2 x^2 - \cdots - \theta_q x^q. \end{aligned}$$

- **Theorem:** There exists an unique stationary solution to the ARMA(p, q) model if all roots of the AR characteristic equation $\phi(x) = 0$ are outside the unit circle.
  - The above condition is referred to the stationarity condition.
  - This unique stationary solution is called the ARMA(p, q) process, and has the form of

    $$Z_t = \mu + \sum_{j=0}^{\infty} \psi_j a_{t-j}$$

    in almost surely sense.

- How to find out the ARMA process?
  - Consider a stationary ARMA(1,1) model, $Z_t = \phi Z_{t-1} + a_t - \theta a_{t-1}$. By comparing the coefficients of a_{t-j}'s at both sides of

$$\begin{aligned}\psi_0 a_t + &\psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots \\ &= \phi\psi_0 a_{t-1} + \phi\psi_1 a_{t-2} + \phi\psi_2 a_{t-3} + \cdots + a_t - \theta a_{t-1},\end{aligned}$$

    we have that

$$\begin{aligned}\psi_0 &= 1; \\ \psi_1 &= \phi\psi_0 - \theta = \phi - \theta; \\ \psi_2 &= \phi\psi_1 = \phi^2 - \phi\theta; \\ &\cdots \\ \psi_k &= \phi\psi_{k-1} = \phi^k - \phi^{k-1}\theta; \\ &\cdots\end{aligned}$$

- For a stationary ARMA(p, q) process, by a similar method, we can obtain that

$$\begin{cases} \psi_0 = 1, \\ \psi_1 = -\theta_1 + \phi_1, \\ \psi_2 = -\theta_2 + \phi_2 + \phi_1\psi_1, \\ \qquad \cdots\cdots \\ \psi_j = -\theta_j + \phi_p\psi_{j-p} + \cdots + \phi_1\psi_{j-1}, \end{cases}$$

  where we take $\psi_j = 0$ for $j < 0$ and $\theta_j = 0$ for $j > q$.
- When a nonzero value of mean is involved, all ψ_j's do not change, but there is an extra constant mean $\mu = \frac{\theta_0}{1-\phi_1-\phi_2-\cdots-\phi_p}$.
  So $Z_t = \mu + \sum_j \psi_j a_{t-j}$.

- **The mean function:** For a stationary ARMA(p, q) model,

$$Z_t = \theta_0 + \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t \\ - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q},$$

we take expectation to the both sides of the above equation, and obtain

$$\mu = \theta_0 + \phi_1 \mu + \phi_2 \mu + \cdots + \phi_p \mu + 0$$

or

$$\mu = \frac{\theta_0}{1 - \phi_1 - \phi_2 - \cdots - \phi_p}$$

since $1 - \phi_1 - \phi_2 - \cdots - \phi_p \neq 0$.

- **The second-order moment:** Not like the pure AR process, the situation for the ARMA process,

$$\begin{aligned} Z_t =& \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t \\ & - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q}, \end{aligned}$$

is a little bit complicated.
However, we can still multiply Z_{t-k} to the both side of the equation, take expectation, and obtain a Yule-Walker type equation, if $k > q$,

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} + \cdots + \phi_p \gamma_{k-p},$$

or

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \cdots + \phi_p \rho_{k-p}.$$

So we only need know how to calculate the second-order moment, including the variance function, the auto-covariance function and the auto-correlation function, for ARMA processes with lower orders.

- **Example :** Consider an ARMA(2,1), $W_t = \theta_0 + 1.3W_{t-1} - 0.4W_{t-2} + a_t + 0.7a_{t-1}$, where $\{a_t\} \sim WN(0, \sigma_a^2)$.
  (1) Is it stationary?
  (2) Calculate the mean of W_t.
  (3) Calculate the autocovariance function $\gamma_k, k = 0, 1, 2, \cdots$.
  (4) Calculate the autocorrelation function $\rho_k, k = 0, 1, 2, \cdots$.

  Sol: Check the roots of $1 - 1.3x + 0.4x^2 = 0$.
  $EW_t = \theta_0/(1 - \phi_1 - \phi_2 - \cdots - \phi_p)$.

A General idea to sub-question (3) and (4):

Step 1 We have

$$\gamma_k = 1.3\gamma_{k-1} - 0.4\gamma_{k-2}, \qquad k > 1. \tag{3.4}$$

Step 2 We calculate γ_0, γ_1. This could be done by setting three equations involving γ_0, γ_1 and γ_2. Through solving this equation group, we obtain $\gamma_0, \gamma_1, \gamma_2$.

Step 3 By equation (3.4), we get all γ_k. Then $\rho_k = \gamma_k / \gamma_0$.

Recall that $W_t = \theta_0 + 1.3W_{t-1} - 0.4W_{t-2} + a_t + 0.7a_{t-1}$.
Hence $\gamma_0 = 1.3\gamma_1 - 0.4\gamma_2 + cov(a_t, W_t) + 0.7cov(a_{t-1}, W_t)$,
$\gamma_1 = 1.3\gamma_0 - 0.4\gamma_1 + cov(at, W_{t-1}) + 0.7cov(a_{t-1}, W_{t-1})$.

First simplify $cov(a_t, W_t)$, which equals

$$1.3cov(a_t, W_{t-1}) - 0.4cov(a_t, W_{t-2}) + cov(a_t, a_t) + 0.7cov(a_t, a_{t-1})$$

Therefore, $cov(a_t, W_t) = cov(a_t, a_t) = \sigma_a^2$. Note that $cov(a_{t-1}, W_{t-1}) = \sigma_a^2$.

Next simplify $cov(a_{t-1}, W_t)$, which equals

$$1.3cov(a_{t-1}, W_{t-1}) - 0.4cov(a_{t-1}, W_{t-2}) + cov(a_{t-1}, a_t) + 0.7var(a_{t-1})$$

Therefore, $cov(a_{t-1}, W_t) = 1.3\sigma_a^2 + 0.7\sigma_a^2 = 2\sigma_a^2$.

Hence we establish equation groups:

$$\gamma_0 = 1.3\gamma_1 - 0.4\gamma_2 + \sigma_a^2 + 0.7 * 2\sigma_a^2$$

$$\gamma_1 = 1.3\gamma_0 - 0.4\gamma_1 + 0.7\sigma_a^2$$

$$\gamma_2 = 1.3\gamma_1 - 0.4\gamma_0$$

Hence we have that

$$\gamma_0 = 24.11\sigma_a^2, \;\; \gamma_1 = 22.89\sigma_a^2, \;\; \gamma_2 = 20.11\sigma_a^2.$$

Since $\gamma_k = 1.3\gamma_{k-1} - 0.4\gamma_{k-2}, \quad k > 1$, we can obtain all values of γ_k's, and hence ρ_k's.

§3.4 Invertibility

- Why we need the invertibility?
  - Suppose we collected a time series with 100 observations,

    $$z_1, z_2, \ldots, z_{100}.$$

    After the complicated procedure, introduced later, we reached an AR(1) model,

    $$Z_t = 0.6Z_{t-1} + a_t.$$

    How to explain your results?
  - What happen if the model is

    $$Z_t = a_t - 0.5a_{t-1} \ \text{ or } \ Z_t = 0.3Z_{t-1} + a_t + 0.2a_{t-1}.$$

- **Definition:** A time series $\{Z_t\}$ is *invertible* if

$$a_t = c + \pi_0 Z_t + \pi_1 Z_{t-1} + \pi_2 Z_{t-2} + \cdots .$$

  - This property makes sure that we can recover the information sequence based on the past observed time series.
  - Without loss of generality, we can set $\pi_0 = 1$.
  - The AR processes are always invertible.
- **Theorem:** A general MA (or ARMA) process is invertible if all roots of its MA characteristic equation

$$\theta(x) = 1 - \theta_1 x - \theta_2 x^2 - \cdots - \theta_q x^q = 0$$

  are outside the unit root circle.
  For example, the invertibility condition of an MA(1) process, $Z_t = a_t - \theta a_{t-1}$, is $|\theta| < 1$!

- **Definition:** Suppose a time series $\{Z_t\}$ is invertible, then define

$$Z_t = c' + a_t - \pi_1 Z_{t-1} - \pi_2 Z_{t-2} - \cdots ,$$

as the *autoregressive (AR) representation*.

  - We find out the AR representations by the same method to finding out the unique stationary solution to AR or ARMA models. The key is to compare the coefficients of Z_{t-j}'s at both sides.
  - In contrast, the unique stationary solution to an AR (or ARMA) model is also called the *MA representations* of the AR (or ARMA) model.

Example : Find out the AR representation of an MA process, $Z_t = a_t - \theta a_{t-1}$ with $|\theta| < 1$.

Method 1: Define $a_t = \pi_0 Z_t + \pi_1 Z_{t-1} + \pi_2 Z_{t-2} + \cdots$
$Z_t = \pi_0 Z_t + \pi_1 Z_{t-1} + \pi_2 Z_{t-2} + \cdots - \theta(\pi_0 Z_{t-1} + \pi_1 Z_{t-2} + \cdots)$.
By coefficients matching, $\pi_0 = 1$, $\pi_1 - \theta\pi_0 = 0$, $\pi_2 - \theta\pi_1 = 0, \cdots$
Hence $\pi_1 = \theta$, $\pi_2 = \theta^2$, $\cdots$ So $a_t = Z_t + \theta Z_{t-1} + \theta^2 Z_{t-2} + \cdots$.
Correspondingly, AR representation:

$$Z_t = -\theta Z_{t-1} - \theta^2 Z_{t-2} - \cdots + a_t$$

Method 2: $Z_t = (1 - \theta B)a_t$. So
$a_t = \frac{1}{1-\theta B} Z_t = (1 + \theta B + \theta^2 B^2 + \cdots)Z_t$. Hence we also have
$a_t = Z_t + \theta Z_{t-1} + \theta^2 Z_{t-2} + \cdots$

Example : Find out the AR representation of an ARMA(1,1) process, $W_t = \phi W_{t-1} + \theta_0 + a_t - \theta a_{t-1}$ with $|\theta| < 1$.

Method 2: $(1 - \phi B)W_t = \theta_0 + (1 - \theta B)a_t$. So
$a_t = -\frac{\theta_0}{1-\theta B} + \frac{1-\phi B}{1-\theta B} W_t$. So
$a_t = -\theta_0/(1-\theta) + (1-\phi B)(1 + \theta B + \theta^2 B^2 + \cdots)W_t$.

Method 1:
Suppose $a_t = c + \sum_{j=0}^{\infty} \pi_j W_{t-j}$. Then

$$W_t = \phi W_{t-1} + \theta_0 + (c + \sum_{j=0}^{\infty} \pi_j W_{t-j}) - \theta(c + \sum_{j=0}^{\infty} \pi_j W_{t-1-j})$$

By coefficients matching, $c = -\theta_0/(1-\theta)$, $\pi_0 = 1$, $\pi_1 = \theta - \phi$, $\pi_2 = \theta(\theta - \phi)$, $\pi_k = \theta^{k-1}(\theta - \phi) \cdots$

General Steps

Suppose the model is $\phi(B)W_t = \theta_0 + \theta(B)a_t$.
AR representation (if exists) could be derived from
$\phi(B)/\theta(B)W_t = \theta_0/\theta(B) + a_t$.
Note that $\theta_0/\theta(B) = \theta_0/\theta(1) = \theta_0/(1 - \theta_1 - \cdots - \theta_q)$. To obtain $\phi(B)/\theta(B)$, we could instead let $a_t = c + \sum_j \pi_j W_{t-j}$ and derive π_j by matching coefficients.

Remark: Such a method could also be used to derive MA representation (if exists): $W_t = \theta_0/\phi(B) + \theta(B)/\phi(B)a_t$.
Note that $\theta_0/\phi(B) = \theta_0/\phi(1) = \theta_0/(1 - \phi_1 - \cdots - \phi_p)$. To explicitly express $\theta(B)/\phi(B)$, we could instead express $W_t = \mu + \sum_j \psi_j a_{t-j}$ and derive ψ_j by matching coefficients.

Example : Consider an ARMA(2,2) model,

$$Z_t = 0.5 + 1.1Z_{t-1} - 0.7Z_{t-2} + a_t - 0.3a_{t-1} - 0.18a_{t-2}.$$

(1) Is it Stationary? Is it invertible?

(2) Find out the MA representation if it is stationary.

(3) Find out the AR representation if it is invertible.

Sol: Stationary as the roots of $1 - 1.1x + 0.7x^2 = 0$ have modules greater than 1.

Invertible as the roots of $1 - 0.3x - 0.18x^2 = 0$ have modules greater than 1.

MA: Consider $Z_t = \mu + \sum_j \psi_j a_{t-j}$. Then
$\mu + \sum_j \psi_j a_{t-j} = 0.5 + 1.1(\mu + \sum_j \psi_j a_{t-1-j}) - 0.7(\mu + \sum_j \psi_j a_{t-2-j}) + a_t - 0.3a_{t-1} - 0.18a_{t-2}$
Then $\mu = 0.5/(1 - 1.1 + 0.7)$, $\psi_0 = 1$,
$\psi_1 = 1.1\psi_0 - 0.3 = 0.8$, $\psi_2 = 1.1\psi_1 - 0.7\psi_0 - 0.18$,
$\psi_3 = 1.1\psi_2 - 0.7\psi_1, \cdots$
Finally, $Z_t = 5/6 + \sum_{j=0}^{\infty} \psi_j a_{t-j}$.

AR: Say $a_t = c + \sum_j \pi_j Z_{t-j}$. Then
$Z_t = 0.5 + 1.1Z_{t-1} - 0.7Z_{t-2} + (c + \sum_j \pi_j Z_{t-j}) - 0.3(c + \sum_j \pi_j Z_{t-1-j}) - 0.18(c + \sum_j \pi_j Z_{t-2-j})$.
Then $c = -0.5/(1 - 0.3 - 0.18)$, $\pi_0 = 1$,
$1.1 + \pi_1 - 0.3\pi_0 = 0$, so $\pi_1 = -0.8$,
$-0.7 + \pi_2 - 0.3\pi_1 - 0.18\pi_0 = 0$, so $\pi_2 = 0.7 + 0.3\pi_1 + 0.18$,
$\ldots$
Finally, $Z_t = -c + a_t - \sum_{j=1}^{\infty} \pi_j Z_{t-j}$

Summary

1. Properties: stationarity, linearity, causality, invertibility.
2. Model definition: stationarity condition, invertibility conditon, AR representation, MA representation.
3. Probabilistic structure: mean, variance, auto-covariance, auto-correlation.