Time Series Analysis

WISE&SOE, XMU

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§1.1 What is the time series?

- Definition: A time series data is a sequence of observations ordered by time, and they are the observed values of one subject at different time.
- Exercise: Are the following sequences time series data?
  (1) Hourly temperature of Chek Lap Kok, Hong Kong from August 13 to 27.
  (2) Height of Tom: 110cm (age 3), 120cm (age 5), 135cm (age 6).
  (3) Height of children: George 110cm (age 3), Jack 120cm (age 4), Bill 135 cm (age 5).

What is the time series

- All observed time series data (with regular time period) can be denoted by
$$(z_1, z_2, ..., z_T)$$
It can be considered as a realization of T random variables:
$$(Z_1, Z_2, ..., Z_T)$$
- Definition: A time series is a sequence of random variables, which are ordered by time. It is usually defined as *a doubly infinite sequence of random variables*, and denoted by $\{Z_t, t \in \mathbb{Z}\}$, or $\{Z_t\}$ in short. In this case, time series is a stochastic process with discrete times.

Time series models

- Recall that, in the linear regression model, there is a response Y and some predictors X, and the linear model is

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i.$$

  The information for predictors X are available, and we want to make inference about the response Y.

- For a time series, the available information includes:
  (1) the time index t and
  (2) the information in the past.

  Two typical types of time series models

$$Z_t = a + bt + \varepsilon_t$$

  and

$$Z_t = \theta_0 + \phi Z_{t-1} + \varepsilon_t.$$

  They correspond to the deterministic model and the stochastic model, and the focus of Chapter 3 is on the latter one.

§1.2 Means, variances and covariances of a time series

- **The mean function:** For a time series $\{Z_t, t \in \mathbb{Z}\}$, the *mean function* (or mean sequence) is defined by

$$\mu_t = \mathrm{E}(Z_t), \qquad t \in \mathbb{Z}.$$

  That is, μ_t is just the expected value of the process at time t. In general, μ_t can be different at each time t.

- **The auto-covariance function:** The *auto-covariance function* (ACVF) is defined as

$$\gamma(t,s) = \mathrm{cov}(Z_t, Z_s), \qquad t, s \in \mathbb{Z},$$

  where

$$\begin{aligned} \mathrm{cov}(Z_t, Z_s) &= \mathrm{E}[(Z_t - \mu_t)(Z_s - \mu_s)] \\ &= \mathrm{E}(Z_t Z_s) - \mu_t \mu_s. \end{aligned}$$

- **The variance function:** Especially, when $s = t$ in the above equation, we have

$$\gamma(t,t) = \mathrm{cov}(Z_t, Z_t) = \mathrm{var}(Z_t),$$

  which is the *variance function* of $\{Z_t\}$.
- **The auto-correlation function (ACF):** The *auto-correlation function* (ACF) is given by

$$\rho(t,s) = \mathrm{corr}(Z_t, Z_s), \qquad t, s \in \mathbb{Z},$$

  where

$$\mathrm{corr}(Z_t, Z_s) = \frac{\mathrm{cov}(Z_t, Z_s)}{\sqrt{\mathrm{var}(Z_t)\mathrm{var}(Z_s)}} = \frac{\gamma(t,s)}{\sqrt{\gamma(t,t)\gamma(s,s)}}.$$

- The ACVF and the ACF have the following properties:

$$
\begin{array}{lll}
1. & \gamma(t,t) = \mathrm{var}(Z_t), & \rho(t,t) = 1; \\
2. & \gamma(t,s) = \gamma(s,t), & \rho(t,s) = \rho(s,t); \\
3. & |\gamma(t,s)| \leq \sqrt{\gamma(t,t)\gamma(s,s)}, & |\rho(t,s)| \leq 1.
\end{array}
$$

  The first two results are directly from the definitions, and the third one can be derived by Cauchy-Schwarz inequality ($E|XY| \leq (EX^2)^{1/2}(EY^2)^{1/2}$).

- Some useful properties of covariance functions,

$$\text{cov}(aX, Y) = a\text{cov}(X, Y), \;\; \text{cov}(X, aY) = a\text{cov}(X, Y),$$

$$\text{cov}(X, aY + bZ) = a\text{cov}(X, Y) + b\text{cov}(X, Z),$$

$$\begin{aligned} \text{cov}(c_1 Y_1 + c_2 Y_2, d_1 Z_1 + d_2 Z_2) &= c_1 d_1 \text{cov}(Y_1, Z_1) \\ +c_2 d_1 \text{cov}(Y_2, Z_1) &+ c_1 d_2 \text{cov}(Y_1, Z_2) + c_2 d_2 \text{cov}(Y_2, Z_2), \end{aligned}$$

and

$$\text{cov}\left[\sum_{i=1}^{m} c_i Y_i, \sum_{j=1}^{n} d_j Z_j\right] = \sum_{i=1}^{m}\sum_{j=1}^{n} c_i d_j \text{cov}(Y_i, Z_j).$$

The last formula will be used frequently in this course.

Exercise: Suppose $\mathrm{E}(X) = 2$, $\mathrm{var}(X) = 9$, $\mathrm{E}(Y) = 0$, $\mathrm{var}(Y) = 4$ and $\mathrm{cov}(X, Y) = 0.25$. Find

$$\mathrm{var}(X + Y),\ \mathrm{cov}(X, X + Y) \text{ and } \mathrm{corr}(X + Y, X - Y).$$

Solution:

$$\mathrm{var}(X \pm Y) = \mathrm{var}(X) + \mathrm{var}(Y) \pm 2\mathrm{cov}(X, Y)$$

$$\mathrm{cov}(X, X + Y) = \mathrm{var}(X) + \mathrm{cov}(X, Y)$$

$$\mathrm{cov}(X + Y, X - Y) = \mathrm{var}(X) - \mathrm{var}(Y)$$

- **Example (The random walk):** Let $\{a_t, t \in \mathbb{N}\}$ be a sequence of $i.i.d.$ random variables, each with zero mean and variance σ_a^2. The **random walk** process $\{Z_t, t \in \mathbb{N}\}$ is defined as

$$Z_t = \sum_{j=1}^{t} a_j, \qquad t \in \mathbb{N}.$$

Alternatively, we can write

$$Z_t = Z_{t-1} + a_t, \qquad t \in \mathbb{N},$$

with *initial value* $Z_0 = 0$.

1. The mean function of $\{Z_t\}$ is

$$\mu_t = \mathrm{E}(Z_t) = \mathrm{E}\left(\sum_{j=1}^{t} a_j\right) = \sum_{j=1}^{t} \mathrm{E}(a_j) = 0, \qquad t \in \mathbb{N}.$$

2. The variance function of $\{Z_t\}$ is

$$\begin{aligned}
\gamma(t,t) &= \mathrm{var}(Z_t) = \mathrm{var}\left(\sum_{j=1}^{t} a_j\right) \\
&= \sum_{j=1}^{t} \mathrm{var}(a_j) = t \cdot \sigma_a^2.
\end{aligned}$$

Notice that the process variance *increases* linearly with time.

3. For the auto-covariance function (ACVF), it is not difficult to find that for all $t \leq s$,

$$\begin{aligned}
\gamma(t,s) &= \operatorname{cov}(Z_t, Z_s) \\
&= \operatorname{cov}\left(\sum_{j=1}^{t} a_j, \sum_{j=1}^{s} a_j\right) \\
&= \operatorname{cov}\left(\sum_{j=1}^{t} a_j, \sum_{j=1}^{t} a_j + \sum_{j=t+1}^{s} a_j\right) \\
&= \operatorname{cov}\left(\sum_{j=1}^{t} a_j, \sum_{j=1}^{t} a_j\right) \\
&= \operatorname{var}\left(\sum_{j=1}^{t} a_j\right) = t \cdot \sigma_a^2.
\end{aligned}$$

4. For the auto-correlation function (ACF), by its definition, we have that

$$\begin{aligned}\rho(t,s) &= \frac{\gamma(t,s)}{\sqrt{\gamma(t,t)\gamma(s,s)}}\\ &= \frac{\sigma_a^2 t}{\sqrt{\sigma_a^2 t \cdot \sigma_a^2 s}}\\ &= \sqrt{t/s}, \qquad 1 \le t \le s.\end{aligned}$$

When $s = t + 1$,
$\rho(t, t+1) = \mathrm{corr}(Z_t, Z_{t+1}) = \sqrt{t/(t+1)} \approx 1$ as t is large.

- **Example (a moving average):** Suppose that $\{Z_t, t \in \mathbb{Z}\}$ is given by

$$Z_t = a_t - 0.5a_{t-1}, \qquad t \in \mathbb{Z},$$

where the a's are assumed to be a sequence of $i.i.d.$ random variables with zero mean and variance σ_a^2.

1. The mean function of $\{Z_t\}$ is

$$\mu_t = \mathrm{E}(Z_t) = \mathrm{E}(a_t) - 0.5\mathrm{E}(a_{t-1}) = 0, \qquad t \in \mathbb{Z}.$$

2. The variance function of $\{Z_t\}$ is

$$\mathrm{var}(Z_t) = \mathrm{var}(a_t - 0.5a_{t-1}) = \sigma_a^2 + 0.5^2\sigma_a^2 = 1.25\sigma_a^2.$$

3. For the auto-covariance function (ACVF), it is not difficult to find that

$$\begin{aligned}\mathrm{cov}(Z_t, Z_{t-1}) &= \mathrm{cov}(a_t - 0.5a_{t-1}, a_{t-1} - 0.5a_{t-2}) \\ &= \mathrm{cov}(a_t, a_{t-1}) - 0.5\mathrm{cov}(a_t, a_{t-2}) \\ &\quad -0.5\mathrm{cov}(a_{t-1}, a_{t-1}) + 0.5^2\mathrm{cov}(a_{t-1}, a_{t-2}) \\ &= -0.5\mathrm{cov}(a_{t-1}, a_{t-1}),\end{aligned}$$

or

$$\gamma(t, t-1) = -0.5\sigma_a^2, \qquad \forall\ t \in \mathbb{Z}.$$

For any integer $k \geq 2$,

$$\mathrm{cov}(Z_t, Z_{t-k}) = \mathrm{cov}(a_t - 0.5a_{t-1}, a_{t-k} - 0.5a_{t-k-1}) = 0,$$

or $\gamma(t, t-k) = 0, \qquad \forall\ k \geq 2, t \in \mathbb{Z}.$

4. The auto-correlation function (ACF) $\rho(t,s)$ can then be calculated as follows,

$$\rho(t,s) = \frac{\gamma(t,s)}{\sqrt{\gamma(t,t)\gamma(s,s)}} = \begin{cases} -0.4, & \text{if } |t-s| = 1, \\ 0, & \text{if } |t-s| \geq 2. \end{cases}$$

§2 Stationarity

- Why we need the stationarity?
- The basic idea of stationarity is that the probability laws governing the series do not change with time — that is, the series is in *statistical equilibrium*.

- **Strict stationarity:** A time series $\{Z_t\}$ is said to be *strictly stationary* if the joint distribution of Z_{t_1}, Z_{t_2}, $\cdots$, Z_{t_n} is the same as that of Z_{t_1-k}, Z_{t_2-k}, $\cdots$, Z_{t_n-k} for all choices of natural number n, all choices of time points t_1, t_2, $\cdots$, t_n and all choices of time lag k.
  Exercise: Are the following time series $\{Z_t\}$ strictly stationary?

  $\{Z_t\} \sim i.i.d.$ Cauchy distribution with density

  $$\pi^{-1}[1+x^2]^{-1} \qquad -\infty < x < +\infty.$$

  Ans: Yes.

- **Weak stationarity:** A time series $\{Z_t\}$ is said to be **weakly** (**second-order**, or **covariance**) **stationary** if
  1. the mean function μ_t is constant over time, and
  2. $\gamma(t, t-k) = \gamma(0, k)$ for all times t and lags k.
- In this course the term *stationary* when used alone will always mean weakly stationary.
- For a weakly stationary time series, the mean function, the auto-covariance function and the auto-correlation function can be alternatively denoted by
  1. $\mu = E(Z_t)$;
  2. $\gamma_k = \mathrm{Cov}(Z_t, Z_{t-k})$; $(\gamma_{-k} = \gamma_k)$
  3. $\rho_k = \mathrm{Corr}(Z_t, Z_{t-k})$; $(\rho_{-k} = \rho_k)$

- The relationship between weak and strict stationarity is as follows,
  1. Strict stationarity + finite second moment $\Rightarrow$ weak stationarity;
  2. If the joint distributions of a time series are all multivariate normal, then the two definitions coincide;

- **Example (White noise):** A very important example of a stationary process is the so-called **white noise** process, which is defined as a sequence of $i.i.d$ random variables $\{a_t\}$ with zero mean and variance $\sigma_a^2 > 0$.
  Such a white noise is usually abbreviated as $WN(0, \sigma_a^2)$.
  1. The fact that $\{a_t\}$ is strictly stationary is obvious.
  2. For the weak stationarity, note that $\mu_t = \mathrm{E}(a_t) = 0$ is a constant and

  $$\gamma(t, t-k) = \begin{cases} \sigma_a^2, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0. \end{cases} := \gamma_k$$

  Moreover, we have

  $$\rho_k = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0. \end{cases}$$

Remark: In some books, the white noise $\{a_t\}$ is defined to be a sequence of uncorrelated random variables.

- **Example (Random walk) (Cont'd):** The random walk, $Z_t = a_1 + a_2 + \cdots + a_t$, is constructed from white noise but is *not* stationary. For example, $\mathrm{var}(Z_t) = t\sigma_a^2$ is *not* constant; furthermore, the covariance function $\gamma(t, s) = t\sigma_a^2$ for $0 \leq t \leq s$ does *not* depend only on time lags. Therefore, the random walk is a **non-stationary** time series.

  Let $\{X_t = \nabla Z_t = Z_t - Z_{t-1}\}$ be the differenced sequence of $\{Z_t\}$. Then $X_t = a_t$, i.e. $\{\nabla Z_t\}$ is stationary.

- **Example (A moving average) (Cont'd):** The moving average example, $Z_t = a_t - 0.5a_{t-1}$, is an instance of a nontrivial, stationary time series constructed from white noise. In our new notation, we have for the moving average time series that

$$\rho_k = \begin{cases} 1, & \text{for } k = 0, \\ -0.4, & \text{for } k = \pm 1, \\ 0, & \text{for } |k| \geq 2. \end{cases}$$

Reference

Please read Chapter 2 of Cryer & Chan (2008).