Chapter 7: Forecasting

Time Series Analysis
WISE, XMU

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§7.1 What is forecasting?

- There are two components of forecasting:
  1. One of the primary objectives of building a model for a time series is to be able to **forecast** the values for that series at future times.
  2. Of equal importance is the assessment of the precision of those forecasts, measured by the **prediction limits** or confidence intervals.
- For a time series $\{Z_t\}$, we assume that the model is known **exactly**, including specific values for all parameters. Although this is not true in practice, the use of estimated parameters for large sample sizes does not seriously affect the results.

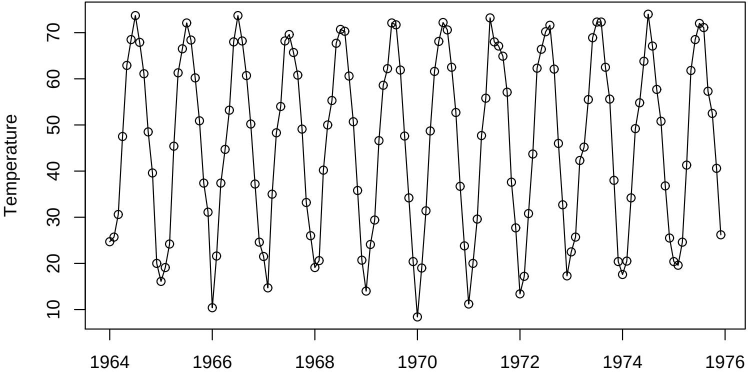
- Based on the available history of the series up to time n, namely $Z_1, ..., Z_n$, we want to forecast the value of Z_{n+l}, where n is called the **forecast origin** and l the **lead time**, and denote the forecasting value by $\widehat{Z}_n(l)$.
    1. By the criterion of the **minimum mean square error**, the forecasting value $\widehat{Z}_n(l)$ is just the conditional expectation,

    $$\widehat{Z}_n(l) = E(Z_{n+l}|Z_n, ..., Z_1).$$

    2. The **forecast error** is defined as $a_n(l) = Z_{n+l} - \widehat{Z}_n(l)$, and the **forecast error variance** is $\mathrm{var}[a_n(l)]$, which is also a conditional variance.

3. If the normality Z_t or a_t is also verified, then, conditional on $Z_n, ..., Z_1$, the random variable Z_{n+l} will follow normal distribution with mean $\widehat{Z}_n(l)$ and variance $\text{var}[a_n(l)]$.
   Then the **prediction limit** with significance level of $1 - \alpha$ will be
   $$\widehat{Z}_n(l) \pm z_{1-\alpha/2}\sqrt{\text{var}[a_n(l)]},$$
   where $z_{1-\alpha/2}$ is $1 - \alpha/2$ percentile of the standard normal distribution. Hence, there are **only two** important quantities, $\widehat{Z}_n(l)$ and $\text{var}[a_n(l)]$ (or $a_n(l)$), in time series forecasting.

- For example, consider the average monthly temperatures (in degrees Fahrenheit) in Dubuque, Iowa from January, 1964 to December 1975.



Suppose we have the cosine model, $Z_t = \mu_t + a_t$, to explain the data, where $\sigma_a^2 = 3.706^2$ and

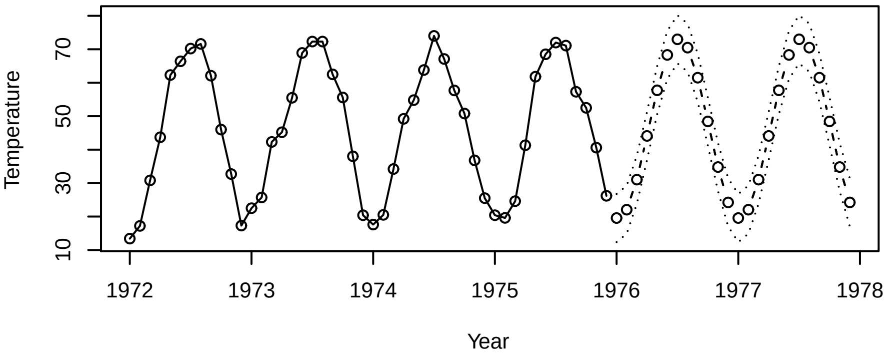
$$\mu_t = 46.2660 + (-26.7079)\cos(2\pi\frac{t-1}{12}) + (-2.1697)\sin(2\pi\frac{t-1}{12}).$$

Note that the observed sequence is $Z_1, ..., Z_{144}$. We want to make forecasting for the value at June, 1976, which is corresponding to the time point 150. Then,

$$Z_{144}(6) = E(Z_{150}|Z_{144}, ..., Z_1) = \mu_{150} = 68.3\ ^oF,$$

$a_{144}(6) = a_{150}$, $\text{var}[a_{144}(6)] = 3.706^2$, and the 95% confidence interval will be

$$[68.3 - 1.96 \times 3.706, 68.3 + 1.96 \times 3.706] = [61.0, 75.6].$$



Temperature
10
30
50
70
1972
1973
1974
1975
1976
1977
1978
Year

§7.2 Forecasting with ARIMA models

- Consider the AR(1) model,

$$Z_t = \phi Z_{t-1} + a_t.$$

1. $Z_{n+1} = \phi Z_n + a_t$. Hence the **one-step-ahead forecast** is

$$\begin{aligned}\widehat{Z}_n(1) &= E(Z_{n+1}|Z_n, ..., Z_1) = E(\phi Z_n + a_{n+1}|Z_n, ..., Z_1) \\ &= \phi Z_n + E(a_{n+1}|Z_n, ..., Z_1) = \phi Z_n.\end{aligned}$$

$var(a_n(1)) = var(a_t) = \sigma^2$.

2. For the l-step forecast,

$$\begin{aligned} Z_{n+l} &= \phi Z_{n+l-1} + a_{n+l} = \phi(\phi Z_{n+l-2} + a_{n+l-1}) + a_{n+1} = \phi^2 Z_{n+l-} \\ &= \phi^l Z_n + \phi^{l-1} a_{n+1} + \cdots + \phi a_{n+l-1} + a_{n+l}. \end{aligned}$$

$$var(a_n(l)) = var(\phi^{l-1} a_{n+1} + \cdots + a_{n+l}) = [\phi^{2(l-1)} + \cdots + 1]\sigma^2.$$

Remark: Since $|\phi| < 1$, for large l, we have that $\widehat{Z}_n(l) \approx 0 = E(Z_t)$ and $var(a_n(l)) \approx \sigma^2/(1-\phi^2) = var(Z_t)$.

3. In general, the forecast could be conducted as below.
   (a) Rewrite Z_{n+l} recursively until you express it using Z_t with $t \leq n$.
   (b) Then calculate $\hat{Z}_{n+l}$ and $var(a_n(l))$.

   Note that to forecast $\hat{Z}_{n+l}$, the steps above is equivalent to that we first predict Z_{n+1}, and use it to predict Z_{n+2} until we forecast Z_{n+l}.
   To derive $var(a_n(l))$, it is equivalent to using the MA representation. Say we have $Z_{n+l} = \sum_{j=0}^{\infty} \psi_j a_{n+l-j}$ Then the forecast error is

$$a_n(l) = a_{n+l} + \psi_1 a_{n+l-1} + \cdots + \psi_{l-1} a_{n+1}.$$

   The forecast error variance is

$$\text{var}[a_n(l)] = \sigma_a^2(1 + \psi_1^2 + \cdots + \psi_{l-1}^2).$$

Note that: The forecast error only depends on the form of MA representations, i.e. the results will be same for general ARMA models.

4. The **one-step-ahead forecast error** is $a_n(1) = a_{n+1}$. Hence, the white noise process $\{a_t\}$ can now be reinterpreted as a sequence of one-step-ahead forecast errors.

- Consider the MA(1) model,

$$Z_t = a_t - \theta a_{t-1}.$$

1. $Z_{n+1} = a_{n+1} - \theta a_n$. The one-step-ahead forecast is

   $$Z_n(1) = E(a_{n+1} - \theta a_n | Z_n, ..., Z_1) = -\theta a_n,$$

   the forecast error is $a_n(1) = Z_{n+1} - Z_n(1) = a_{n+1}$ and the forecast error variance is $\text{var}[a_n(1)] = \sigma_a^2$.
2. The l-step ($l > 1$) forecast is

   $$Z_n(l) = E(a_{n+l} - \theta a_{n+l-1} | Z_n, ..., Z_1) = 0,$$

   the forecast error is
   $a_n(l) = Z_{n+l} - Z_n(l) = a_{n+l} - \theta a_{n+l-1} = Z_{n+l}$ and the forecast error variance is $\text{var}[a_n(l)] = \sigma_a^2(1 + \theta^2)$=$\text{var}(Z_{n+l})$.

3. How to calculate a_n since it appears in the one-step-ahead forecast?
   - We can calculate them by

$$a_t = \sum_{j=0}^{\infty} \pi_j a_{t-j} = \sum_{j=0}^{t-1} \pi_j Z_{t-j}, \qquad 1 \leq t \leq n,$$

     where the initial values $Z_s = 0$ for $s \leq 0$. Note that the invertibility is needed here.
   - We may alternatively calculate them by the iterative method. Suppose that $a_0 = 0$, and then

$$a_1 = Z_1 + \theta a_0 = Z_1, \quad a_2 = Z_2 + \theta a_1, \quad \ldots, \quad a_n = Z_n + \theta a_{n-1}.$$

4. The method to forecast MA(q) models is similar.

- Consider the ARMA(1,1) model, $Z_t = \phi Z_{t-1} + a_t + \theta_1 a_{t-1}$.
  1. $Z_{n+1} = \phi Z_n + a_{n+1} + \theta_1 a_n$. For one-step forecasting,

$$
\begin{aligned}
\widehat{Z}_n(1) &= E(Z_{n+1}|Z_n, ..., Z_1) \\
&= E(\phi Z_n + a_{n+1} + \theta_1 a_n | Z_n, ..., Z_1) \\
&= \phi Z_n + \theta_1 a_n + E(a_{n+1}|Z_n, ..., Z_1) \\
&= \phi Z_n + \theta_1 a_n, \\
a_n(1) &= Z_{n+1} - \widehat{Z}_n(1) = a_{n+1}, \\
\mathrm{var}[a_n(1)] &= \sigma_a^2,
\end{aligned}
$$

2. For two-step forecasting, you could directly calculate $\widehat{Z}_n(2) = \phi \widehat{Z}_n(1) = \phi(\phi Z_n + \theta_1 a_n)$. Or we first find out that

$$
\begin{aligned}
Z_n(2) &= \phi Z_{n+1} + a_{n+2} + \theta_1 a_{n+1} \\
&= \phi(\phi Z_n + a_{n+1} + \theta_1 a_n) + a_{n+2} + \theta_1 a_{n+1} \\
&= \phi(\phi Z_n + \theta_1 a_n) + \phi a_{n+1} + a_{n+2} + \theta_1 a_{n+1} \\
\widehat{Z}_n(2) &= \phi(\phi Z_n + \theta_1 a_n) \\
a_n(2) &= a_{n+2} + (\phi + \theta_1) a_{n+1}, \\
\text{var}[a_n(2)] &= [1 + (\phi + \theta_1)^2]\sigma_a^2.
\end{aligned}
$$

3. For three-step, four-step, $\cdots$.

Example For each of the following models,

$$(i) \qquad (1-0.9B)Z_t = a_t \qquad --\text{AR}(1) \text{ model with mean } 0$$

$$(ii) \qquad Z_t = 10 + a_t - 0.9a_{t-1} \qquad --\text{MA}(1) \text{ model with mean } 10$$

where $\sigma_a^2 = 2$. Given $Z_1 = 1.2$ and $Z_2 = 0.1$, find the $l-$step ahead forecast values and forecast variances for $l = 1, 2, 3$(Suppose $a_1 = 0.3, a_2 = -1.2$).

Solution:

Reference

Please read Chapter 9 of Cryer & Chan (2008).