Chapter 6: Model Diagnostics

Time Series Analysis
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Content

- Residual analysis.
- Analysis of over-parameterized models.

§6.1 Residual Analysis

- Consider in particular AR(1) model with a constant term:

$$Z_t - \phi Z_{t-1} = \theta_0 + a_t.$$

Having estimated ϕ and θ_0, the **residuals** are defined as

$$\widehat{a}_t = Z_t - \widehat{\phi} Z_{t-1} - \widehat{\theta}_0.$$

1. If the model is correctly specified and the parameter estimates are reasonably close to the true values, then the residuals should have nearly the properties of white noise.
2. They should behave roughly like $i.i.d.$ normal variables with mean zero and common variances.
3. Deviations from these properties can help us discover a more appropriate model.

- How to calculate residuals?
    1. Consider a fitted ARMA(p, q) model,

    $$Z_t - \widehat{\mu} = \widehat{\phi}_1(Z_{t-1} - \widehat{\mu}) + \cdots + \widehat{\phi}_p(Z_{t-p} - \widehat{\mu}) + \widehat{a}_t - \widehat{\theta}_1 \widehat{a}_{t-1} - \cdots - \widehat{\theta}_q \widehat{a}_{t-q}.$$

    The residuals can be calculated as follows,

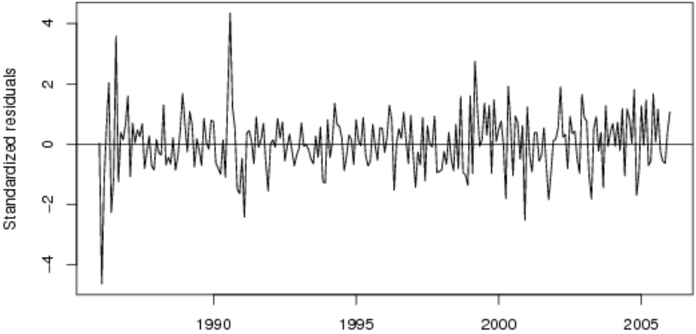
    $$\widehat{a}_t = \sum_{j=0}^{\infty} \widehat{\pi}_j (Z_{t-j} - \widehat{\mu}),$$

    where $\widehat{\pi}_j$s are functions of $\widehat{\phi}_1, \ldots, \widehat{\phi}_p$ and $\widehat{\theta}_1, \ldots, \widehat{\theta}_q$, and the initial values $Z_s = \widehat{\mu}$ for $s \leq 0$.
    Note that: The invertibility is necessary to calculate the residuals.

2. We may also consider the **standardized residuals**, $\{\widehat{a}_t/s\}$, where s^2 is the sample variance of the residual sequence.

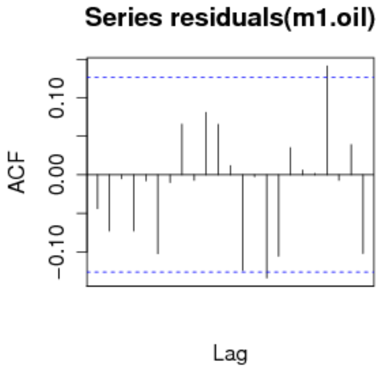
- The **residual analysis** includes
  1. to check whether or not there still exist some patterns not yet explained by the fitted model;
     (1) the time plot of the residual sequence.
  2. to check the possible normality of the residuals.
     (2) the histogram; (3) the quantile-quantile plot; and (4) some formal normality test.
  3. to check for the autocorrelations;
     (5) the correlogram, i.e. based on the ACFs individually; and
     (6) Ljung-Box test, a formal test based on all of the first K available ACFs.

- We fit an ARIMA(0,1,1) model to the logarithms of the oil prices by using maximum likelihood estimation. The resulting standardized residual sequence is as follows.



- We perform the Shapiro-Wilk normality test to the standardized residual sequence. It produces a test statistic $W = 0.9688$, which corresponds to a p-value of 3.97×10^{-5}, and then the normality is not satisfied.
  Note: the non-normality is also confirmed by the Quantile-Quantile plot.

- We draw the correlogram, i.e. the sample autocorrelation function (ACF) of the residuals.



Note that: The Bartlett's approximation is employed again to calculate the upper and lower bounds.

- **Ljung-Box test:** For a fitted ARMA model, Box & Pierce proposed a test statistic in 1970,

$$Q = n(\widehat{r}_1^2 + \widehat{r}_2^2 + \cdots + \widehat{r}_K^2) \sim \chi^2_{K-m},$$

where K is predetermined integer, $\widehat{r}_j^2$ is the sample ACF of the residuals, and $m = p + q$.
As a modified version of Box & Pierce's test statistic, the **Ljung-Box** test statistic is defined as

$$Q_* = n(n+2)(\frac{\widehat{r}_1^2}{n-1} + \frac{\widehat{r}_2^2}{n-2} + \cdots + \frac{\widehat{r}_K^2}{n-K}) \sim \chi^2_{K-m}.$$

Example 1: We have fitted an AR(1) model with intercept to a color value time series with sample size $n = 35$. The sample ACFs for the residuals are also listed here. Is there any possible correlation?

lag k	1	2	3	4	5	6
Residual ACF	-0.051	0.032	0.047	0.021	-0.017	-0.019

Solution:

$$\begin{aligned} Q_* &= 35 \times (35 + 2) \\ &\times \left(\frac{0.051^2}{35-1} + \frac{0.032^2}{35-2} + \frac{0.047^2}{35-3} + \frac{0.021^2}{35-4} + \frac{0.017^2}{35-5} + \frac{0.019^2}{35-6}\right) \\ &= 0.28 < \chi^2_{5,0.95} = 11.07 \end{aligned}$$

Hence, there is no autocorrelations.

§6.2 Analysis of over-parameterized models

- Our second basic diagnostic tool is that of **overfitting**. After specifying and fitting what we believe to be an adequate model, we fit a more general model, that is, a model that contains the original model as a special case. For example, if an AR(2) model seems appropriate, we might overfit with an AR(3) model. The original AR(2) model would be confirmed if:
  1. the estimate of the additional parameter, ϕ_3, is not significantly different from zero, and
  2. the estimates for the parameters in common, ϕ_1 and ϕ_2, (and θ_0 if it exists,) do not change significantly from their original estimates.

- To perform the analysis of over-parameterized models, we need to follow three guidelines.
  1. Specify the original model carefully. If a simpler model seems at all promising, check it out before trying a more complicated model.
  2. When overfitting, *do not* increase the orders of the AR and MA parts of the model simultaneously.
  3. Extend the model in directions suggested by an analysis of the residuals. If after fitting an MA(1) model, substantial correlation remains at lag 1 in the residuals, try an MA(2), not an ARMA(1,1).

Reference

Please read Chapter 8 of Cryer & Chan (2008).