Chapter 10: Conditional Heteroscedastic Models

Time Series Analysis

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§10.1 Features of Financial Time Series

Some common features of financial time series

- Asset prices are generally non stationary.
- Returns are usually stationary.
- Return series might show no or little autocorrelation.
- Volatility of the return series appears to be clustered.
- Serial dependence exists among the squared returns
- Normality assumption might be violated
- leverage effects may need to be considered, i.e. changes in prices negatively correlated with changes in volatility.

§10.1 Conditional Heteroscedastic Models

- We introduce ARCH(1) model to handle heteroscedasticity.
- Consider the following model:

$$a_t = \sigma_t \epsilon_t, \qquad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2,$$

where ϵ_t is Gaussian white noise with unit variance,

and $\alpha_0 > 0, \alpha_1 > 0$.

Note: In general, ϵ_t's are iid r.v. with mean 0 and var 1.

Note: Usually we require $0 < \alpha_1 < 1$.

- Another View: ARCH(1) in a_t is like AR(1) in a_t^2.

  Let $\eta_t = a_t^2 - \sigma_t^2$. Then $a_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \eta_t$.

  It can be shown that η_t is uncorrelated with mean 0.

  Hence ARCH(1) can be viewed as a AR(1) in a_t^2.

  It captures quadratic dependence.

- Property of ARCH(1):

$$E(\epsilon_t|I_{t-1}) = 0,\ var(\epsilon_t|I_{t-1}) = 1,$$
$$E(\epsilon_t) = 0,\ var(\epsilon_t) = 1,$$
$$\rho(\epsilon_i, \epsilon_j) = 0 \text{ if } i \neq j.\ \rho(\epsilon_i^2, \epsilon_j^2) = 0 \text{ if } i \neq j.$$

$$E(a_t|I_{t-1}) = 0,\ var(a_t|I_{t-1}) =: \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2,$$
$$E(a_t) = 0,\ var(r_t) = \tfrac{\alpha_0}{1-\alpha_1},$$
$$\rho(a_i, a_j) = 0 \text{ if } i \neq j.\ \rho(a_i^2, a_j^2) = \alpha_1^{|i-j|} \text{ if } i \neq j.$$

- Prediction interval under normality: $[-1.96\hat{\sigma}_{t+j}, 1.96\hat{\sigma}_{t+j}]$

  How to obtain $\hat{\sigma}_{t+j}$?

  One-step ahead: $\hat{\sigma}^2_{t+1} = \alpha_0 + \alpha_1 a_t^2$.

  multi-step: use iterative calculations and replace a^2_{t+j} is $\hat{\sigma}^2_{t+j}$.

  for example: $\hat{\sigma}^2_{t+2} = \alpha_0 + \alpha_1 \hat{\sigma}^2_{t+1}$

  Another representation:

  rewrite $\sigma^2_{t+1} - \gamma_0 = \alpha_1(a_t^2 - \gamma_0)$, where $\gamma_0 = \alpha_0/(1-\alpha_1)$.

  Hence $\hat{\sigma}^2_{t+1} = \gamma_0 + \alpha_1(a_t^2 - \gamma_0)$.

  $\hat{\sigma}^2_{t+2} = \gamma_0 + \alpha_1(\hat{\sigma}^2_{t+1} - \gamma_0) = \gamma_0 + \alpha_1^2(a_t^2 - \gamma_0)$.

  In general, $\hat{\sigma}^2_{t+j} = \gamma_0 + \alpha_1^j(a_t^2 - \gamma_0)$.

Comment: The process a_t is uncorrelated. So past info is not useful to predict the future value of a_t.

However, the process a_t is not independent as nonlinear dependence exists. ARCH model is trying to capture the quadratic dependence on a_t. Hence it could capture volatility clustering, and yield more helpful predicted intervals.

- More property: heavy tail

$$
\begin{aligned}
E(a_t^4) = E(\sigma_t^4 \epsilon_t^4) &= E[E(\sigma_t^4 \epsilon_t^4 | I_{t-1})] \\
&= E[\sigma_t^4 3] \geq 3[E(\sigma_t^2)]^2 = 3[var(a_t)]^2
\end{aligned}
$$

Therefore $kur = \frac{E(a_t^4)}{[var(a_t)]^2} \geq 3$.

- Extension: ARCH(q)
  - let ϵ_t be Gaussian white noise with unit variance
  - a_t is an ARCH(q) process if

$$a_t = \sigma_t \epsilon_t \qquad \sigma_t = \sqrt{\alpha_0 + \sum_{i=1}^{q} \alpha_i a_{t-i}^2}$$

- GARCH(p, q)
  - a_t is the GARCH(p, q) process if

$$a_t = \epsilon_t \sigma_t \qquad \sigma_t = \sqrt{\alpha_0 + \sum_{i=1}^{q} \alpha_i a_{t-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2}.$$

- Property:
  1. GARCH has no prediction power for mean value
  2. GARCH is heavy tailed and thus better handle outliers
  3. GARCH models conditional volatility by capturing quadratic dependence: $a_t \sim GARCH(m, s) \Rightarrow a_t^2 \sim ARMA(\max(m, s), s)$

     $a_t^2 = \sigma_t^2 \epsilon_t^2 = \sigma_t^2(1 + \delta_t) = \sigma_t^2 + e_t$, so $\sigma_t^2 = a_t^2 - e_t$.

$$
\begin{aligned}
a_t^2 &= \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j (a_{t-j}^2 - e_{t-j}) + e_t \\
&= \alpha_0 + \sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) a_{t-i}^2 + e_t + \sum_{j=1}^{s} (-\beta_j) e_{t-j}
\end{aligned}
$$

     AR coefficient: $\alpha_i + \beta_i$, $i = 1, \cdots, \max(m, s)$;

     MA coefficient: $-\beta_j$, $j = 1, \cdots, s$.

- Property for GARCH(1,1):

$$E(\epsilon_t|I_{t-1}) = 0,\ var(\epsilon_t|I_{t-1}) = 1,$$
$$E(\epsilon_t) = 0,\ var(\epsilon_t) = 1,$$
$$\rho(\epsilon_i, \epsilon_j) = 0 \text{ if } i \neq j.\ \rho(\epsilon_i^2, \epsilon_j^2) = 0 \text{ if } i \neq j.$$

$$E(a_t|I_{t-1}) = 0,\ var(a_t|I_{t-1}) =: \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2,$$
$$E(a_t) = 0,\ var(a_t) = \alpha_0/[1 - \textstyle\sum_{i=1}^{\max(m,s)}(\alpha_i + \beta_i)]$$
$$\rho(a_i, a_j) = 0 \text{ if } i \neq j.\ \rho_{a^2}(1) = \tfrac{(1+\phi\theta)(\phi+\theta)}{1+\theta^2+2\phi\theta} = \tfrac{\alpha_1(1-\alpha_1\beta_1-\beta_1^2)}{1-2\alpha_1\beta_1-\beta_1^2},$$
$$\rho_{a^2}(j) = (\alpha_1 + \beta_1)\rho_{a^2}(j-1) \text{ for } j > 1.$$

- Forecast

  Key idea: replace a^2_{t+j} by $\hat{\sigma}^2_{t+j}$.

  $\sigma^2_{t+1} = \alpha_0 + \sum_i \alpha_i a^2_{t+1-i} + \sum_j \beta_j \sigma^2_{t-j+1}$. So $\hat{\sigma}^2_{t+1} = \sigma^2_{t+1}$.

$$\begin{aligned}
\sigma^2_{t+2} &= \alpha_0 + \sum_{i=1}^{m} \alpha_i a^2_{t+2-i} + \sum_{j=1}^{s} \beta_j \sigma^2_{t-j+2} \\
&= \alpha_0 + \alpha_1 a^2_{t+1} + \sum_{i=2}^{m} \alpha_i a^2_{t+2-i} + \beta_1 \sigma^2_{t+1} + \sum_{j=2}^{s} \beta_j \sigma^2_{t-j+2}
\end{aligned}$$

  So

  $\hat{\sigma}^2_{t+2} = \alpha_0 + \alpha_1 \hat{\sigma}^2_{t+1} + \beta_1 \hat{\sigma}^2_{t+1} + \sum_{i=2}^{m} \alpha_i a^2_{t+2-i} + \sum_{j=2}^{s} \beta_j \sigma^2_{t-j+2}$

  As $k \to \infty$, we have

  $\hat{\sigma}^2_{t+k} \to var(a_t) = \alpha_0 / [1 - \sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i)]$

- A more general framework: $AR(1)/ARCH(1)$.
  - Suppose that

$$y_t - \mu = \phi(y_{t-1} - \mu) + a_t,$$

  - where a_t is an ARCH(1) process.

so y_t is like an AR(1) process, except that the noise term a_t is not independent white noise but rather an ARCH(1) process. Since a_t is uncorrelated, we could show that $\rho_y(k) = \phi^{|k|}$, and $\rho_{a^2}(k) = \alpha_1^{|k|}$.

- ARIMA/GARCH model

  $r_t \sim ARIMA(p,d,q)$ with error term a_t.

$$\phi(B)(1-B)^d(r_t-\mu)=\theta(B)a_t,$$

  $a_t \sim GARCH(s,m)$, i.e. $a_t=\sigma_t\epsilon_t$, where

$$\sigma_t=\alpha_0+\sum_{i=1}^{m}\alpha_i a_{t-i}^2+\sum_{j=1}^{s}\beta_j\sigma_{t-j}^2$$

- Property: $\rho_r(j)$ could be derived from $ARIMA(p, d, q)$ process.

  $\rho_\epsilon(j) = 0$ as ϵ_t's are iid with mean 0 and var 1.

  $\rho_a(j) = 0$ as a_t's are WN.

  $\rho_{a^2}(j)$ could be derived from $ARMA(\max(m, s), s)$ process,

  where the AR coefficients are $\alpha_i + \beta_i$, $i = 1, \cdots, \max(m, s)$,

  and the MA coefficients are $-\beta_j$, $j = 1, \cdots, s$.

- One step Forecast when $d = 0$.

  Rewrite the model as

  $$r_t = \mu(1 - \sum_{i=1}^{p} \phi_i) + \sum_{i=1}^{p} \phi_i r_{t-i} + a_t + \sum_{j=1}^{q} \theta_j a_{t-j}$$

  Predicted Value:

  $$\hat{Y}_{t+1} = \mu(1 - \sum_{i=1}^{p} \phi_i) + \sum_{i=1}^{p} \phi_i r_{t+1-i} + \sum_{j=1}^{q} \theta_j e_{t+1-j}$$

  Assume normality. The 95% P.I. is $\hat{Y}_{t+1} \pm 1.96 * \sqrt{\sigma^2_{t+1}}$, where $\sigma^2_{t+1} = \alpha_0 + \sum_{i=1}^{m} \alpha_i a^2_{t+1-i} + \sum_{j=1}^{s} \beta_j \sigma^2_{t+1-j}$.

- How to fit and assess the data?
  - Model the mean
  - test for ARCH effect
  - Determine the order for ARCH part
  - Estimation
  - Model checking.

```
library(fGarch)

da=read.table("m-intc7308.txt",header=T)

intc=log(da$rtn+1) # log returns

acf(intc);  acf(intc^2);  pacf(intc^2)

Box.test(intc^2,lag=10,type='Ljung')

m1=garchFit(~garch(1,0),data=intc,trace=F)

summary(m1)

stdresid=m1@residuals/m1@sigma.t

acf(stdresid);  acf(stdresid^2)
```

- Disadvantage of GARCH process

  Traditional GARCH model could not handle asymmetric market response that may react more strongly to a negative return than a positive return of the same magnitude. A simple remedy is to add a dummy variable, which yields GJR-Garch:

$$\sigma_t^2 = \alpha_0 + \sum_i (\tilde{\alpha}_i a_{t-i}^2 + \tilde{\gamma}_i a_{t-i}^2 I_{a_{t-i} \le 0}) + \sum_j \beta_j \sigma_{t-j}^2.$$

  Note the above model could also be written as

$$\sigma_t^2 = \omega + \sum_i \alpha_i (|a_{t-i}| - \gamma_i a_{t-i})^2 + \sum_j \beta_j \sigma_{t-j}^2,$$

  where $\tilde{\alpha}_i = \alpha_i (1 - \gamma_i)^2$ and $\tilde{\gamma}_i = 4\alpha_i \gamma_i$.

- In general, the asymmetric power GARCH is

$$a_t = \sigma_t \epsilon_t,$$

$$\sigma_t^\delta = \omega + \sum_i \alpha_i (|a_{t-i}| - \gamma_i a_{t-i})^\delta + \sum_j \beta_j \sigma_{t-j}^\delta.$$

  - $\delta = 2, \gamma = 0 \Rightarrow$ Garch
  - $\delta = 1, \gamma = 0 \Rightarrow$ Absolute Value Garch
  - $\delta = 2 \Rightarrow$ GJR Garch
  - $\delta = 1 \Rightarrow$ TGarch

Reference

Please read Chapter 12 of Cryer & Chan (2008).