Chapter 8: Seasonal ARIMA models

Time Series Analysis
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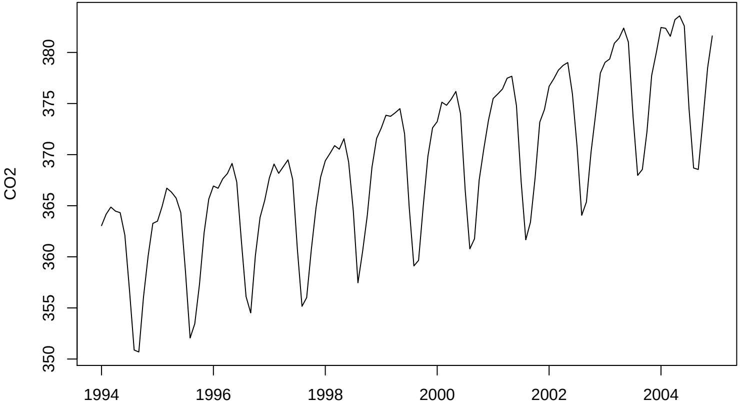
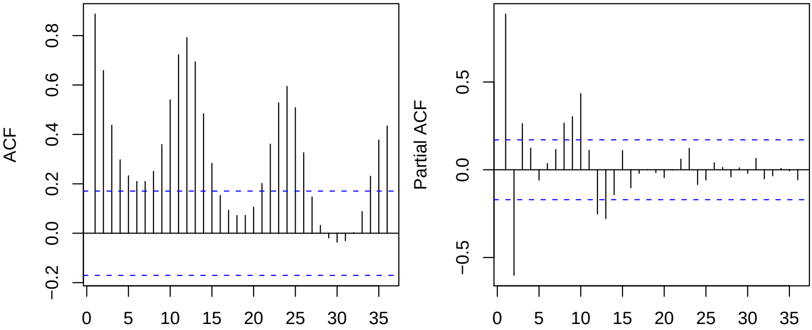
§8.1 Why seasonal ARIMA models?

- Consider the monthly CO_2 levels at Alert, Northwest Territories, Canada, from January 1994 to December 2004.



- Its sample ACF and sample PACF are as follows.



- The seasonal effect is presented in the time plot, the sample ACF, and the sample PACF. This chapter introduces some special ARIMA models to catch such feature, and these special ARIMA models are called the seasonal ARIMA models.
- We denote by s the known **seasonal period**. For example, $s = 12$ for monthly data, and $s = 4$ for quarterly data.

§8.2 Stationary seasonal ARMA models

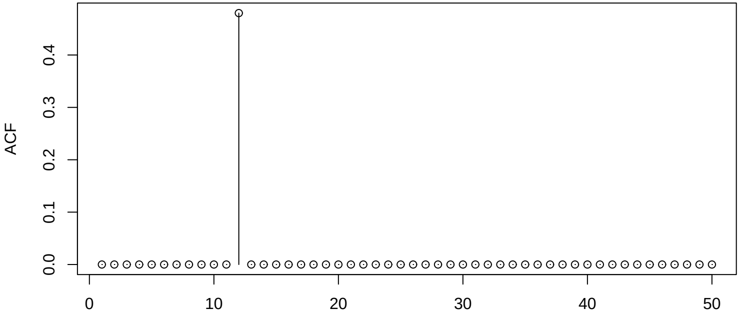
- Consider the seasonal MA(1) model,

$$Z_t = a_t - \Theta a_{t-12},$$

where $\{a_t\}$ is the white noise sequence with mean zero and variance σ_a^2.

1. Note that this is just a special MA model. It is always **stationary** just like the common MA models.
2. The variance is $\gamma_0 = (1 + \Theta^2)\sigma_a^2$.

3. Its ACF has the property that $\rho_{12} = -\Theta/(1 + \Theta^2)$ and $\rho_k = 0$ as $k \neq 12$.



- The **seasonal MA(Q) model of order P and seasonal period s** is defined as

$$Z_t = a_t - \Theta_1 a_{t-s} - \Theta_2 a_{t-2s} - \cdots - \Theta_Q a_{t-Qs},$$

where $\{a_t\} \sim WN(0, \sigma_a^2)$. The **seasonal MA characteristic polynomial** is defined as

$$\Theta(x) = 1 - \Theta_1 x^s - \Theta_2 x^{2s} - \cdots - \Theta_Q x^{Qs}.$$

1. It is useful to note that the seasonal MA(Q) model can also be viewed as a special case of an ordinary nonseasonal MA model of order $q = Qs$ but with all θ-values equal to zero except at the seasonal lags $s, 2s, \cdots, Qs$.
2. It is always stationary just like the common MA models.
3. It is invertible if all the roots of $\Theta(x) = 0$ are outside the unit circle.

4. Its ACFs will be nonzero ONLY at the seasonal lags of $s, 2s, \cdots, Qs$. In particular, for $k = 1, 2, \cdots, Q$,

$$\rho_{ks} = \frac{-\Theta_k + \Theta_1\Theta_{k+1} + \Theta_2\Theta_{k+2} + \cdots + \Theta_{Q-k}\Theta_Q}{1 + \Theta_1^2 + \Theta_2^2 + \cdots + \Theta_Q^2}.$$

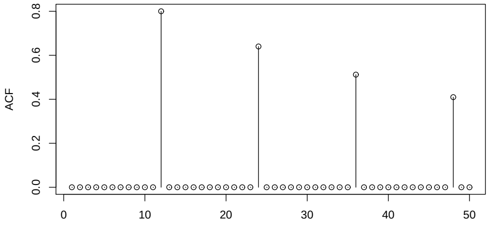
5. Its PACFs are nonzero only at lags $s, 2s, \cdots$, where it behaves like a combination of decaying exponentials and damped sine functions.

- Consider the seasonal AR(1) model,

$$Z_t = \Phi Z_{t-12} + a_t,$$

where $\{a_t\} \sim WN(0, \sigma_a^2)$.

1. It is just a special AR model. It is always invertible.
2. Its AR characteristic polynomial has the form of $\Phi(x) = 1 - \Phi x^{12}$, and the stationarity condition is $|\Phi| < 1$.
3. Its ACF has the property that $\rho_{12k} = \Phi^k$ for $k = 1, 2, \cdots$, and the values of the ACF at other lags are all zero.



4. The variance is $\gamma_0 = \sigma_a^2/(1 - \Phi^2)$.

- The **seasonal AR(P) model of order P and seasonal period s** is defined as

$$Z_t = \Phi_1 Z_{t-s} + \Phi_2 Z_{t-2s} + \cdots + \Phi_P Z_{t-Ps} + a_t,$$

where $\{a_t\} \sim WN(0, \sigma_a^2)$. The **seasonal AR characteristic polynomial** is

$$\Phi(x) = 1 - \Phi_1 x^s - \Phi_2 x^{2s} - \cdots - \Phi_P x^{Ps}.$$

1. It can be seen as a special AR(p) model of order $p = Ps$ with nonzero ϕ-coefficients only at the seasonal lags $s, 2s, \cdots, Ps$.
2. It is stationary if all the roots of $\Phi(x) = 0$ are outside the unit circle.
3. It is always invertible.
4. Its ACFs are nonzero only at lags $s, 2s, \cdots$, where it behaves like a combination of decaying exponentials and damped sine functions.

5. Its PACFs are nonzero only at lags $s, 2s, \cdots, Ps$, and *cuts off after P seasonal periods*.

Example 1: Consider an AR(p) model, $Z_t = \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} + a_t$, and a seasonal AR($p$) model, $Z_t = \phi_1 Z_{t-s} + \cdots + \phi_p Z_{t-ps} + a_t$, where $\{a_t\} \sim WN(0, \sigma_a^2)$. Show that the AR($p$) model is stationary if and only if the seasonal AR(p) model is stationary.

- We define a **multiplicative seasonal ARMA**$(p, q) \times (P, Q)_s$ **model with seasonal period** s as a model with AR characteristic polynomial $\phi(x)\Phi(x)$ and MA characteristic polynomial $\theta(x)\Theta(x)$, where

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p,$$

$$\Phi(x) = 1 - \Phi_1 x^s - \Phi_2 x^{2s} - \cdots - \Phi_P x^{Ps}$$

$$\theta(x) = 1 - \theta_1 x - \theta_2 x^2 - \cdots - \theta_q x^q,$$

and

$$\Theta(x) = 1 - \Theta_1 x^s - \Theta_2 x^{2s} - \cdots - \Theta_Q x^{Qs}.$$

1. The model may also contain a constant term θ_0.
2. It is just a special ARMA model with AR order $p + Ps$ and MA order $q + Qs$, but the coefficients are not completely general, being determined by only $p + P + q + Q$ coefficients. If $s = 12$, the value of $p + P + q + Q$ will be considerably smaller than $p + Ps + q + Qs$, and hence allow a much more parsimonious model.

- Consider the ARMA$(0,1)(0,1)_{12}$ model with the MA characteristic polynomial given by

$$(1-\theta x)(1-\Theta x^{12}) = 1-\theta x-\Theta x^{12}+\theta\Theta x^{13}.$$

1. The corresponding time series satisfies

$$Z_t = a_t-\theta a_{t-1}-\Theta a_{t-12}+\theta\Theta a_{t-13}.$$

2. Its second-order moments are as follows,

$$\gamma_0 = (1+\theta^2)(1+\Theta^2)\sigma_a^2,$$

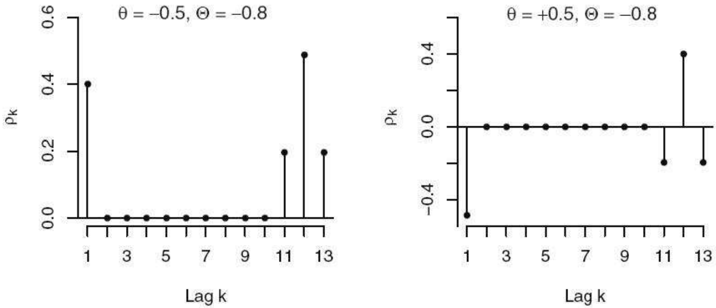
$$\rho_1 = -\frac{\theta}{1+\theta^2}, \qquad \rho_{12} = -\frac{\Theta}{1+\Theta^2},$$

and

$$\rho_{11} = \rho_{13} = \frac{\theta\Theta}{(1+\theta^2)(1+\Theta^2)}.$$

Autocorrelations at all other lags are zero.

3. Two examples are illustrated as follows.



- Consider the ARMA$(0,1)(1,0)_{12}$ model,

$$Z_t = \Phi Z_{t-12} + a_t - \theta a_{t-1}.$$

It holds that

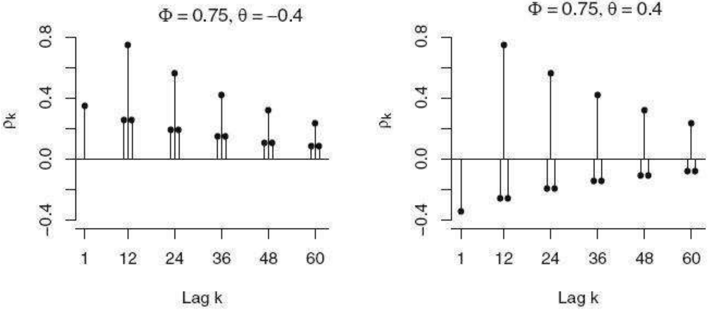
$$\gamma_1 = \Phi\gamma_{11} - \theta\sigma_a^2 \quad \text{and} \quad \gamma_k = \Phi\gamma_{k-12} \ \text{ for } \ k \geq 2.$$

After some calculations, we have that

$$\begin{aligned}
\gamma_0 &= \left(\frac{1+\theta^2}{1-\Phi^2}\right)\sigma_a^2, \\
\rho_{12k} &= \Phi^k \ \text{ for } \ k \geq 1, \\
\rho_{12k-1} = \rho_{12k+1} &= \left(-\frac{\theta}{1+\theta^2}\Phi^k\right) \ \text{ for } \ k = 0, 1, 2, \cdots,
\end{aligned}$$

and autocorrelations for all other lags equal to zero.

Two examples are illustrated as follows.



§8.3 Non-stationary seasonal ARIMA models

- An important tool in modeling non-stationary seasonal processes is the seasonal difference. The **seasonal difference** of period s for the series $\{Z_t\}$ is denoted $\nabla_s Z_t$ and is defined as

$$\nabla_s Z_t = (1 - B^s) Z_t = Z_t - Z_{t-s}.$$

  For example, for a monthly series we consider the changes from January to January, February to February, and so forth for successive years. Note that for a series of length n, the seasonal differenced series will be of length $n - s$; that is, s data values are lost due to seasonal differencing.

- A process $\{Z_t\}$ is said to be a **multiplicative seasonal ARIMA model** with nonseasonal orders p, d and q, seasonal orders P, D and Q, and seasonal period s if the differenced series

$$W_t = \nabla^d \nabla_s^D Z_t = (1 - B)^d (1 - B^s)^D Z_t$$

satisfies an ARMA$(p, q) \times (P, Q)_s$ model. We say that $\{Z_t\}$ is an **ARIMA**$(p, d, q) \times (P, D, Q)_s$ **model**. Using the backshift operator, the general ARIMA $(p, d, q) \times (P, D, Q)_s$ model can be written as

$$\phi(B)\Phi(B)\nabla^d \nabla_s^D Z_t = \theta(B)\Theta(B).$$

- Such models represent a broad, flexible class from which to select an appropriate model for a particular series. In empirical study, many series can be adequately fit by these models, usually with a small number of parameters, say, 3 or 4.

Example 2: Specify the following processes by ARIMA$(p, d, q) \times (P, D, Q)_s$:

(1) $(1 - 0.5B)(1 - 0.2B^4)Z_t = a_t - 0.3a_{t-1}$;

(2) $(1 - B)(1 - B^{12})Z_t = a_t - 0.2a_{t-12}$;

(3) $(1 - 0.2B - 0.3B^2)(1 - B^4)Z_t = a_t$

Solution: (1)ARIMA$(1, 0, 1) \times (1, 0, 0)_4$; (2)ARIMA$(0, 1, 0) \times (0, 1, 1)_{12}$; (3)ARIMA$(2, 0, 0) \times (0, 1, 0)_4$.

Reference

Please read Chapter 10 of Cryer & Chan (2008).