Journal of Business & Economic Statistics

ISSN: 0735-0015 (Print) 1537-2707 (Online) Journal homepage: https://amstat.tandfonline.com/loi/ubes20

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This article proposes several tests for detecting serial correlation and ARCH effect in high-dimensional data. The dimension of data $p = p(n)$ may go to infinity when the sample size $n \to \infty$. It is shown that the sample autocorrelations and the sample rank autocorrelations (Spearman's rank correlation) of the L_1-norm of data are asymptotically normal. Two portmanteau tests based, respectively, on the norm and its rank are shown to be asymptotically χ^2-distributed, and the corresponding weighted portmanteau tests are shown to be asymptotically distributed as a linear combination of independent χ^2 random variables. These tests are dimension-free, that is, independent of p, and the norm rank-based portmanteau test and its weighted counterpart can be used for heavy-tailed time series. We further discuss two standardized norm-based tests. Simulation results show that the proposed test statistics have satisfactory sizes and are powerful even for the case of small n and large p. We apply the tests to two real datasets. Supplementary materials for this article are available online.

KEY WORDS: ARCH effect; High-dimensional time series; Ljung–Box test; Rank correlation; Rank test; Serial correlation.

1. INTRODUCTION

The small n/large p problem has been extensively studied in the last decade for independent and identically distributed (iid) data. Various methods have been developed for variable selection in a large dataset with a sparse covariance matrix, such as the Lasso-type estimation procedure in Cai, Liu, and Luo (2011), the random matrix theory in Bai, Yao, and Zheng (2015) and Paul and Aue (2014), and the principal component procedure in Shen and Huang (2007). Several articles have also explored the small n/large p problem in the time series setting. For example, Song and Bickel (2011); Han and Liu (2013) and Han, Lu, and Liu (2013) proposed two different approaches to estimate the parameters of a large vector autoregressive model. Pena, Smucler, and Yohai (2017) used dynamic principle components to fit a dynamic factor model for high-dimensional time series. In this article, we focus on the detection of serial correlation and autoregressive conditional heteroscedasticity (ARCH) in a high-dimensional time series under the condition that the dimension p goes to infinity as the sample size increases.

To make the discussion clear, we first define a high-dimensional time series (HDTS) as follows.

Definition 1.1. A stochastic process $X_t = (x_{1t}, \ldots, x_{pt})'$ is called a HDTS if $X_t \to (x_{1t}, \ldots, x_{it}, \ldots)'$ as $p \to \infty$, where $t = 0, \pm 1, \pm 2, \ldots$.

We say that a HDTS $\{X_t\}$ is weakly stationary if, for any given p,

(a). $EX_t = \mu^{(p)}$,

(b). $E(X_t - \mu^{(p)})(X_{t+l} - \mu^{(p)})' = \Gamma_l^{(p)},\ l = 0, \pm 1, \pm 2, \ldots,$

where $\mu^{(p)}$ is a p-dimensional constant vector and $\Gamma_l^{(p)}$ is a $p \times p$ constant matrix independent of t. Similarly, we say that a HDTS $\{X_t\}$ is strictly stationary if it is strictly stationary for any given p. As a special case, if $\{X_t\}$ is serially independent for any given p, then $\{X_t\}$ is a HD white noise series. When $\Gamma_l^{(p)} = 0$ for all $l (\neq 0)$ and p, we say that the HDTS $\{X_t\}$ is serially uncorrelated. Let $\mathcal{F}_t$ be the σ-field generated by $\{X_s : s \leq t\}$ and denote

$$E[(X_t - \mu^{(p)})(X_t - \mu^{(p)})' | \mathcal{F}_{t-1}] = \Omega_{t-1}^{(p)}.$$

The HDTS $\{X_t\}$ is said to have no ARCH effect if, for any given p, $\Omega_t^{(p)} = \Gamma_0^{(p)}$, a constant matrix; otherwise, it has ARCH effect.

Journal of Business & Economic Statistics
XXXX, 2019, Vol. 00, No. 0
DOI: 10.1080/07350015.2019.1647844

For ease in notation, we drop the superscript of $\mu^{(p)}$ and $\Gamma_l^{(p)}$ in the sequel. Keep in mind, however, that the dimensions of μ and Γ_l are increasing with p.

Using the random matrix theory, Li et al. (2016) proposed a test for serial correlation of a HDTS. Chang, Yao, and Zhou (2017) used the maximum absolute auto- and cross-correlation to construct a test for the same purpose, but critical values of the test statistic need to be obtained via a bootstrap procedure. Recently, Tsay (2017) proposed a rank-based test using Spearman's rank correlations and the extreme value theory. The asymptotic distribution is derived under the null hypothesis and contemporaneous independence. The moment condition (i.e., $E|x_{it}|^\kappa < \infty$, $i = 1, \dots, p$, for some $\kappa > 0$) of the data is not needed and the asymptotic critical values of test statistics have a close-form solution.

Testing for the ARCH effect in a time series has become an important issue in analysis of univariate and multivariate time series since Engle (1982) proposed the ARCH model. The classical methods for detecting ARCH effect are the Lagrange multiplier test of Engle (1982) and the portmanteau test of McLeod and Li (1983), which are based on the squares of random variables. Li and Mak (1994) used the squares of residuals to construct a goodness-of-fit test for ARCH models. Li and Li (2008) considered a goodness-of-fit test by using absolute residuals. Zhu and Ling (2015) proposed a sign based test statistic. Fisher and Gallagher (2012) proposed several weighted portmanteau statistics for time series goodness of fit testing, including ARMA and GARCH models. Ling and Li (1997) extended the Li–Mak test to multivariate ARCH models by using a quadratic form of random vectors. We refer to Tsay (2014) for more discussions on the subject. To the best of our knowledge, there is no test statistic available for detecting the ARCH effect in a HDTS.

This article proposes several tests for detecting serial correlation and ARCH effect of a HDTS $\{X_t\}$, allowing its dimension $p = p(n) \to \infty$ when the sample size $n \to \infty$. It is shown that the sample autocorrelation and rank autocorrelation (Spearman's rank correlation) of the L_1-norm of data are asymptotically normal. Two portmanteau tests, respectively, based on the norm and its rank are shown to be asymptotically χ^2-distributed, and the corresponding weighted portmanteau tests are shown to be asymptotically distributed as a linear combination of independent χ^2 random variables. These tests are dimension-free, that is, independent of p, and the norm rank-based portmanteau test and its weighted counterpart can be used for heavy-tailed time series. Two standardized norm-based tests are further discussed. Simulation results show that these test statistics have satisfactory sizes and are very powerful even for small n and large p.

This article is organized as follows. Section 2 studies the L_1-norm-based portmanteau test and the weighted portmanteau test and their limiting distributions. Section 3 studies the rank-based portmanteau test and its weighted counterpart and their limiting distributions. Section 4 investigates the standardized norm-/rank-based tests. Section 5 reports some simulation results and Section 6 considers two real examples. All the proofs are in the Appendix and the online supplementary material.

2. L_1-NORM-BASED TEST STATISTIC

The usual null hypothesis for serial correlation is H_0 : $\Gamma_l = 0$ for all $l \neq 0$ and p, and the alternative hypothesis is H_a : $\Gamma_l \neq 0$ for some $l \neq 0$ and p. The classical Ljung–Box-type tests use the sample auto- and cross-correlation matrices of the data to construct the test statistics. These tests can detect only the linear dependence and may not have power for the nonlinear dependence. In the HD time series setting, Li et al. (2016) and Tsay (2017) used the norm of the sample autocorrelation-matrix and the norm of rank sample autocorrelation-matrix, respectively, to construct the test statistics. To obtain the limiting distribution of test statistics, both articles need to assume the $\{X_t\}$ is iid series. In fact, many Ljung–Box-type tests assume the time series is iid for testing ARCH effect or assume the error term is iid for a goodness-of-fit test. Under the iid assumption, it is expected that the test statistic has a power not only for linear dependence but also for nonlinear dependence, in particular, for the ARCH effect.

As in Li et al. (2016) and in Tsay (2017), we assume that $\{X_t :\ t = 1, \dots, n\}$ is a random realization of the HD iid time series $\{X_t\}$ and $p = p(n) \to \infty$ as $n \to \infty$. In this case, any function of X_t is iid. As motivated by Li and Li (2008), we consider the L_1-norm data of $\{X_t\}$

$$\{\|X_1\|, \dots, \|X_n\|\},$$

where $\|A\| = \sum_{i=1}^p |a_i|/p$ for a vector $A = (a_1, \dots, a_p)'$. The L_1-norm can be replaced by Euclidean norm or other norms. Denote the mean and the l-lag autocovariance of $\|X_t\|$ as

$$\mu = E\|X_t\| \quad \text{and} \quad \gamma_l = \operatorname{cov}(\|X_{t-l}\|, \|X_t\|),$$

where $l \geq 0$ and γ_0 is the variance of $\|X_t\|$. The lag-l autocorrelation of $\|X_t\|$ is

$$\rho_l = \gamma_l/\gamma_0 = 0 \quad l > 0.$$

The estimators of μ and γ_l are, respectively,

$$\bar{Y} = \frac{1}{n}\sum_{t=1}^{n} \|X_t\|,$$

$$\hat{\gamma}_l = \frac{1}{n}\sum_{t=l+1}^{n} (\|X_{t-l}\| - \bar{Y})(\|X_t\| - \bar{Y})',$$

where $l \geq 0$. The autocorrelation ρ_l of $\|X_t\|$ is estimated by

$$\hat{\rho}_l = \hat{\gamma}_l/\hat{\gamma}_0,$$

where $l = 1, \dots, m$. For the components of X_t, define

$$\sigma_{ij} = \operatorname{cov}(|x_{it}|, |x_{jt}|).$$

We give the following assumption.

Assumption 2.1.

(a). $$pE(\|X_t\| - \mu)^2 = \frac{1}{p}\left(\sum_{i=1}^{p} \sigma_{ii} + \sum_{i \neq j} \sigma_{ij}\right) \to \sigma^2 \text{ as } p \to \infty.$$

(b). $$\frac{p^2}{\sqrt{n}} E(\|X_t\| - \mu)^4 \to 0.$$

Assumption 2.1(a) is reasonable. In particular, if X_t is s-dependent and identically distributed, then Assumption 2.1(a) reduces to

$$\frac{1}{p}\left(\sum_{i=1}^{p}\sigma_{ii}+\sum_{\|i-j\|\leq s}\sigma_{ij}\right)\to\sigma_{11},$$

as $p \to \infty$. Assumption 2.1(b) is satisfied generally if $p^2/\sqrt{n} \to 0$ and $\max_{1\leq i<\infty} Ex_{it}^4 < \infty$. But if X_t is s-dependent and $\max_{1\leq i<\infty} E(|x_{it}|-\mu_i)^4/\sigma_{ii}^2 < \infty$, then,

$$\begin{aligned}\frac{p^2}{\sqrt{n}}E(\|X_t\|-\mu)^4 = \frac{1}{p^2\sqrt{n}}\Bigg\{&\sum_{i=1}^{p}\sum_{j=1}^{p}E[(|x_{it}|-\mu_i)^2(|x_{jt}|-\mu_j)^2\\ &+\sum_{|i-j|\leq s, i\neq j}\ \sum_{|i_1-j_1|\leq s, i_1\neq j_1} E[(|x_{it}|-\mu_i)(|x_{jt}|-\mu_j)]\\ &(|x_{i_1t}|-\mu_i)(|x_{j_1t}|-\mu_j)\Bigg\}\\ = O\left(\frac{s^2}{p^2\sqrt{n}}\right)&\left(\sum_{i=1}^{p}\sigma_{ii}^2\right)^2 \to 0,\end{aligned}$$

as $n \to \infty$. In this case, p is allowed to be larger than n. In particular, when $X_t \sim N(0, V)$ and V is a s-sparse matrix (i.e., at most s nonzero elements in each column), Assumption 2.1 holds without any restriction on p. We now can give our first theorem as follows.

Theorem 2.1. If $\{X_t\}$ is iid and Assumption 2.1 is satisfied, then,

$$(\hat{\rho}_1, \ldots, \hat{\rho}_m)' \longrightarrow_{\mathcal{L}} N(0, I_m),$$

as $n \to \infty$, where $\longrightarrow_{\mathcal{L}}$ denotes convergence in distribution, and I_m is the $m \times m$ identity matrix.

Using Theorem 2.1, we construct the following portmanteau test, as in Box and Pierce (1970), for the null $H_0 : \rho_1 = \cdots = \rho_m = 0$.

$$T_n = n\sum_{l=1}^{m}\hat{\rho}_l^2 \sim \chi^2(m). \tag{2.1}$$

This test may be affected by the choice of m. Hong (1996) proposed a maxima of Ljung–Box tests to overcome this issue. Alternatively, following Fisher and Gallagher (2012) and Gallagher and Fisher (2015), we construct the weighted portmanteau test as follows

$$W_n = n(n+2)\sum_{l=1}^{m}\frac{m-l+1}{m}\frac{\hat{\rho}_l^2}{n-l} \sim W \equiv \sum_{l=1}^{m}\frac{m-l+1}{m}\chi_l^2, \tag{2.2}$$

where $\chi_l^2, l = 1, \ldots, m$, are iid $\chi^2(1)$ random variates. For computational ease, as in Pena and Rodrguez (2002) and Fisher and Gallagher (2012), W can be approximated by a gamma distribution, $\Gamma(\alpha, \beta)$, with

$$\alpha = \frac{3}{4}\frac{m(m+1)^2}{2m^2+3m+1} \quad \text{and} \quad \beta = \frac{2}{3}\frac{2m^2+3m+1}{m(m+1)}.$$

The simulation in Section 5 shows that W_n has a stable size and is more powerful than T_n when m is large.

The test statistics T_n and W_n are actually constructed by the summation of the absolute values and, hence, they will have some power in detecting serial correlations in absolute values. If we cannot reject the null, then we do not have sufficient evidence for serial correlation or the ARCH effect. If the null is rejected, how to model the serial correlation or ARCH effect is an open problem if we allow $p \to \infty$ (e.g., the estimated parameters may not be consistent in this case) and it needs a new approach to determine which of the many variables are autocorrelated. This is a new research topic for the future. We should mention that T_n may lose some information in certain directions. For example, if $X_t = r_t(\cos^2\theta_t, \sin^2\theta_t)'$ with $\{r_t\}$ being iid and $\{\theta_t\}$ a dependent series, then T_n has no power, because $\|X_t\|$ is an iid sequence.

3. NORM-RANK-BASED TEST STATISTIC

To relax the moment condition of X_t in Theorem 2.1, we consider the following transformed data.

$$Z_t = [g(x_{1t}), \ldots, g(x_{pt})]',$$

where $g(x)$ is a positive real function. For instance, $g(x) = \log(1+|x|)$ or $|x|^q$, depending on the moment information of X_t. Using the autocorrelation function of the L_1-norm of Z_t, we can construct a general portmanteau test or weighted portmanteau test similar to (2.1) and (2.2), respectively. If $\{X_t\}$ is serially correlated or has ARCH effect, the resulting test statistic should have power in detecting the serial correlation and the ARCH effect.

Alternatively, we can directly transform the L_1-norm itself, that is, $g(\|X_t\|)$. A special g is the distribution function F of $\|X_t\|$ and it can be estimated by the empirical distribution of $\|X_t\|$, that is,

$$F_n(x) = \frac{1}{n}\sum_{t=1}^{n}I\{\|X_t\| \leq x\|\}.$$

Let $\{R_t : t = 1, \ldots, n\}$ be the corresponding rank series of $\{\|X_1\|, \ldots, \|X_n\|\}$. Then

$$F_n(\|X_t\|) = \frac{R_t}{n}.$$

Thus, we can directly use the rank $\{R_t : t = 1, \ldots, n\}$ to test the serial correlation and ARCH effect of $\{X_t\}$. This is called the Spearman rank test. Specifically, the autocorrelation ρ_{rl} of rank R_t is defined as

$$\hat{\rho}_{rl} = \frac{12}{n(n^2-1)}\sum_{t=l+1}^{n}(R_t-\bar{R})(R_{t-l}-\bar{R})'.$$

By Theorem 1(d) in Tsay (2017), we see that, under the null hypothesis of uncorrelated white noises, $\hat{\rho}_{rl}$ and $\hat{\rho}_{rh}$ are asymptotically uncorrelated if $l \neq h$. Furthermore, extending the asymptotic normality of $\hat{\rho}_{rl}$ in Dufour and Roy (1986) to the multivariate case, we have the following theorem:

Theorem 3.1. Assume that $\{X_t\}$ are iid with a continuous p-dimensional distribution. Then

$$\sqrt{n}(\hat{\rho}_{r1}, \ldots, \hat{\rho}_{rm})' \longrightarrow_{\mathcal{L}} N(0, I_m),$$

as $n \to \infty$.

Using Theorem 3.1, we can define the norm-rank-based test statistic for detecting serial dependence of $\{X_t\}$ as follows,

$$T_r = n \sum_{l=1}^{m} \hat{\rho}_{rl}^2 \sim \chi^2(m). \tag{3.1}$$

Similarly to (2.2), the weighted norm-rank-based test statistic is defined as follows

$$W_r = n(n+2) \sum_{l=1}^{m} \frac{m-l+1}{m} \frac{\hat{\rho}_l^2}{n-l} \sim W, \tag{3.2}$$

where W is defined in (2.2). Both statistics can test not only for the serial correlation or ARCH effect defined in Section 1 but also for the serial dependence of heavy-tailed time series. They do not need to impose any special structural assumption on the covariance matrix of X_t as those in Li et al. (2016) and Tsay (2017). This is so because the norm is a scalar random variable. A nice feature of T_r and W_r is that they do not require any moment condition of X_t and is independent of the dimension p.

4. STANDARDIZED NORM-BASED TEST STATISTICS

The L_1-norm-based test of the previous section may encounter a scaling effect, that is, when an individual component, say x_{it}, has a relatively large variance compared with other components, the norm $\|X_t\|$ is likely to be dominated by x_{it} and the resulting test statistic could have low power in detecting serial dependence. To mitigate the effect of scaling, we consider the standardized HDTS as $\tilde{X}_t \equiv (x_{it}/\sigma_i, i = 1, \ldots, p)'$, where $\sigma_i^2 = \mathrm{var}(x_{it})$. To avoid the case with $\sigma_p \to 0$ when $p \to \infty$, we require the following assumption.

Assumption 4.1. $\min\{\sigma_i^2 : i \geq 1\} \geq \sigma_0 > 0$, where σ_0 is a constant.

We denote the mean and the lag-l auto-covariance of $\|\tilde{X}_t\|$ by

$$\mu_s = E\|\tilde{X}_t\| \quad \text{and} \quad \gamma_{sl} = \mathrm{cov}(\|\tilde{X}_{t-l}\|, \|\tilde{X}_t\|).$$

If X_t are iid, the autocorrelation function of $\|X_t\|$ is $\rho_{sl} = \gamma_{sl}/\gamma_{s0} = 0$ for $l > 0$. In practice, we estimate σ_i by the sample variance,

$$s_{in}^2 = \frac{1}{n-1} \sum_{t=1}^{n} (x_{it} - \bar{x}_i)^2,$$

where $\bar{x}_i = \sum_{t=1}^{n} x_{it}/n$. Furthermore, under Assumption 4.1, we can modify the sample variance as

$$\tilde{s}_{in} = \max\{\sigma_0^2, s_{in}^2\}.$$

Consequently, we use $\hat{X}_t \equiv (x_{it}/\tilde{s}_{in}, i = 1, \ldots, p)'$ in lieu of $\tilde{X}_t$ to estimate μ_s and γ_{sl}, respectively, by

$$\bar{Y}_s = \frac{1}{n} \sum_{t=1}^{n} \|\hat{X}_t\|,$$

$$\hat{\gamma}_{sl} = \frac{1}{n} \sum_{t=l+1}^{n} (\|\hat{X}_{t-l}\| - \bar{Y}_s)(\|\hat{X}_t\| - \bar{Y}_s)'.$$

The autocorrelation function ρ_{sl} of $\|\tilde{X}_t\|$ is estimated by

$$\hat{\rho}_{sl} = \hat{\gamma}_{sl}/\hat{\gamma}_{s0},$$

where $l = 1, \ldots, m$. Denote $\tilde{\sigma}_{ij} = \mathrm{cov}(|x_{it}|/\sigma_i, |x_{jt}|/\sigma_j)$. Since $\tilde{\sigma}_{ii} = 1$, we modify Assumption 2.1 as follows.

Assumption 4.2.

$$(a).\ \ pE(\|\tilde{X}_t\| - \mu_s)^2 = \frac{1}{p} \sum_{i \neq j} \tilde{\sigma}_{ij} \to 0, \quad as \quad \to \infty.$$

$$(b).\ \ \frac{p^2}{\sqrt{n}} E(\|\tilde{X}_t\| - \mu_s)^4 \to 0.$$

Theorem 4.1. Assume that $\{X_t\}$ are iid and Assumptions 4.1–4.2 hold. If $p/\sqrt{n} \to 0$, then

$$(\hat{\rho}_{s1}, \ldots, \hat{\rho}_{sm})' \longrightarrow_{\mathcal{L}} N(0, I_m), \quad as \quad n \to \infty.$$

The proof of Theorem 4.1 is in the supplementary appendix. Using Theorem 4.1, we can construct the portmanteau test for the null $H_0 : \rho_{s1} = \cdots = \rho_{sm} = 0$ as follows.

$$T_{sn} = n \sum_{l=1}^{m} \hat{\rho}_{sl}^2 \sim \chi^2(m). \tag{4.1}$$

Similarly to (2.2), we can also define a weighted portmanteau test by using $\hat{\rho}_{sl}$. Although we need the condition $p/\sqrt{n} \to 0$ in Theorem 4.1, the simulation results in Section 5 show that T_{sn} works well even when $p > n$.

We denote the rank series of $\{\|\tilde{X}_1\|, \ldots, \|\tilde{X}_n\|\}$ by $\{\tilde{R}_t : t = 1, \ldots, n\}$. Their sample correlation functions are as follows

$$\hat{\rho}_{rsl} = \frac{12}{n(n^2-1)} \sum_{t=l+1}^{n} (\tilde{R}_t - \bar{R}_s)(\tilde{R}_{t-l} - \bar{R})'.$$

As the T_r statistic of Section 3, the test statistic based on $\hat{\rho}_{rsl}$ is defined as

$$T_{rs} = n \sum_{l=1}^{m} \hat{\rho}_{rsl}^2. \tag{4.2}$$

Similarly to (3.2), we can further define a weighted portmanteau test by using $\hat{\rho}_{rsl}$. If

$$\sqrt{n}(\hat{\rho}_{rs1}, \ldots, \hat{\rho}_{rsm})' \longrightarrow_{\mathcal{L}} N(0, I_m), \tag{4.3}$$

then $T_{rs} \sim \chi^2(m)$. However, it seems to be challenging to verify (4.3) rigorously. On the other hand, the simulation results in Section 5 show that T_{rs} works well in term of size and power.

5. SIMULATION STUDY

In this section, we study the finite-sample performance of our tests for serial correlation and ARCH effect in high-dimensional data.

We first compare our tests with the C and $C*$ tests of Chang, Yao, and Zhou (2017) for serial correlation in high-dimensional data and the classical test H of Hosking (1980) and $L\&M$ test of Li and McLeod (1981) for serial correlation in vector time series. The test statistic C in Chang et al. (2017) is defined as

$$C = \max_{1\le k\le l}\max_{1\le i,j\le p} |\hat{r}_{ij}(l)|,$$

where $\hat{r}_{ij}(l)$ is the (i,j)-component of

$$\hat{\Upsilon}_l \equiv \{\hat{r}_{ij}(l)\}_{1\le i,j\le p} = \mathrm{diag}\{\hat{\Gamma}_0\}^{-1/2}\hat{\Gamma}_l\mathrm{diag}\{\hat{\Gamma}_0\}^{-1/2}$$

and $\hat{\Gamma}_l = \frac{1}{n}\sum_{t=1}^{n-l} X_t X_{t+l}'$ is the sample autocovariance matrix of a zero-mean time series X_t. The test statistic C^* is defined in the same manner as C with X_t replaced by QX_t, where Q is an invertible transformation matrix obtained by the principal component analysis proposed by Chang et al. (2017) to the data $\{X_t : t = 1,\ldots,n\}$. The test H of Hosking (1980) is the portmanteau statistic

$$H = n^2\sum_{l=1}^{m} tr\{\hat{\Upsilon}_l'\hat{\Upsilon}_l\}/(n-l),$$

Table 1. Empirical sizes of various test statistics at the 5% nominal level for Case 1, where $T_n, T_r, W_n, W_r, T_{sn}$, and T_{sr} are tests proposed in this article, C and C^* are tests in Chang, Yao, and Zhou (2017), H and $L\&M$ are tests in Hosking (1980) and Li and McLeod (1981), respectively, and Q_1, Q_2, Q_3, Q_4 are used in Tsay (2014), and n and p are the sample size and dimension, respectively

p	T_n	T_r	W_n	W_r	T_{sn}	T_{sr}	C	C^*	H	$L\&M$	Q_1	Q_2	Q_3	Q_4
Normal														
$m=5$														
10	0.056	0.062	0.054	0.057	0.052	0.050	0.031	0.035	0.040	0.039	0.047	0.056	0.048	0.059
30	0.047	0.046	0.051	0.047	0.040	0.039	0.018	0.019	0.029	0.027	0.054	0.050	0.066	0.061
50	0.057	0.052	0.056	0.054	0.053	0.049	0.013	0.012	0.027	0.027	0.045	0.045	0.053	0.059
100	0.057	0.046	0.067	0.063	0.049	0.048	0.008	0.007	0.006	0.005	0.042	0.048	0.062	0.067
$m=10$														
10	0.061	0.054	0.065	0.062	0.056	0.055	0.026	0.027	0.043	0.040	0.052	0.060	0.067	0.053
30	0.045	0.046	0.051	0.055	0.049	0.042	0.013	0.014	0.036	0.036	0.061	0.053	0.083	0.057
50	0.062	0.055	0.064	0.060	0.050	0.046	0.010	0.013	0.025	0.021	0.045	0.049	0.091	0.078
100	0.048	0.048	0.056	0.049	0.060	0.052	0.005	0.010	0.014	0.013	0.050	0.054	0.132	0.145
Skewed t_3														
p	T_n	T_r	W_n	W_r	T_{sn}	T_{sr}	C	C^*	H	$L\&M$	Q_1	Q_2	Q_3	Q_4
$m=5$														
10	0.054	0.053	0.049	0.053	0.051	0.064	0.012	0.016	0.995	0.995	0.060	0.054	0.239	0.141
30	0.055	0.049	0.044	0.043	0.053	0.049	0.000	0.000	0.750	0.746	0.051	0.048	0.303	0.237
50	0.053	0.051	0.045	0.060	0.054	0.047	0.000	0.000	0.440	0.437	0.062	0.058	0.292	0.283
100	0.042	0.047	0.038	0.050	0.037	0.039	0.000	0.000	0.204	0.203	0.040	0.048	0.270	0.275
$m=10$														
10	0.051	0.049	0.051	0.057	0.056	0.054	0.007	0.012	0.998	0.998	0.068	0.052	0.238	0.153
30	0.040	0.042	0.046	0.048	0.048	0.044	0.000	0.000	0.869	0.868	0.048	0.051	0.264	0.207
50	0.055	0.048	0.049	0.043	0.046	0.049	0.001	0.001	0.606	0.602	0.054	0.056	0.331	0.269
100	0.046	0.051	0.066	0.054	0.045	0.049	0.000	0.000	0.256	0.250	0.050	0.058	0.315	0.292
Cauchy														
p	T_n	T_r	W_n	W_r	T_{sn}	T_{sr}	C	C^*	H	$L\&M$	Q_1	Q_2	Q_3	Q_4
$m=5$														
10	0.035	0.050	0.040	0.055	0.048	0.050	0.003	0.005	0.263	0.261	0.052	0.045	0.211	0.274
30	0.044	0.047	0.040	0.049	0.055	0.045	0.003	0.004	0.286	0.286	0.041	0.055	0.051	0.320
50	0.044	0.045	0.049	0.048	0.046	0.051	0.001	0.002	0.305	0.304	0.056	0.046	0.012	0.336
100	0.051	0.048	0.044	0.048	0.061	0.045	0.001	0.006	0.237	0.236	0.051	0.046	0.000	0.312
$m=10$														
10	0.052	0.063	0.050	0.070	0.059	0.069	0.006	0.005	0.256	0.255	0.072	0.064	0.182	0.267
30	0.058	0.060	0.061	0.070	0.067	0.048	0.011	0.008	0.261	0.259	0.048	0.056	0.044	0.308
50	0.066	0.047	0.052	0.050	0.075	0.050	0.002	0.006	0.279	0.279	0.059	0.055	0.009	0.333
100	0.059	0.044	0.044	0.049	0.066	0.041	0.004	0.005	0.245	0.244	0.064	0.052	0.000	0.311

where tr(A) denotes the trace of a square matrix A. The $L\&M$ test in Li and McLeod (1981) is the statistic

$$L\&M = n\sum_{l=1}^{m} tr\{\hat{\Upsilon}_l'\hat{\Upsilon}_l\} + p^2 m(m+1)/(2n).$$

We then compare our tests with the Q_i, $i = 1,2,3,4$, statistics in Tsay (2014) for ARCH effect. Specifically, Q_1 is a Ljung–Box test defined by

$$Q_1 = n(n+2)\sum_{l=1}^{m} \hat{\rho}_l^{*2}/(n-l),$$

where $\hat{\rho}_l^*$ is the lag-l sample ACF of the standardized series $\xi_t \equiv X_t'\hat{\Gamma}_0^{-1}X_t - p$. Q_2 is a rank-based test proposed by Dufour and Roy (1986). It is defined as

$$Q_2 = \sum_{l=1}^{m} \frac{[\tilde{\rho}_l - E(\tilde{\rho}_l)]^2}{\text{var}(\tilde{\rho}_l)},$$

where $\tilde{\rho}_l = \sum_{t=l+1}^{n}(R_t - \bar{R})(R_{t-l} - \bar{R})/\sum_{t=l}^{n}(R_t - \bar{R})^2$ and R_t is the rank of ξ_t and $\bar{R}$ is its sample mean. Q_3 is the well-known Ljung–Box test defined as

$$Q_3 = n^2\sum_{l=1}^{m} \frac{1}{n-l} b_l'(\tilde{\Upsilon}_0^{-1} \otimes \tilde{\Upsilon}_0^{-1})b_l,$$

Table 2. Empirical sizes of various test statistics at the 5% nominal level for Case 2, where $T_n, T_r, W_n, W_r, T_{sn}$, and T_{sr} are test statistics proposed in this article, C and C^* are tests in Chang, Yao, and Zhou (2017), H and $L\&M$ are tests in Hosking (1980) and Li and McLeod (1981), respectively, and Q_1, Q_2, Q_3, Q_4 are used in Tsay (2014), n and p are the sample size and dimension, respectively.

							Normal						
p	T_n	T_r	W_n	W_r	T_{sn}	T_{sr}	C	C^*	H	$L\&M$	Q_1	Q_2	Q_3	Q_4
							$m=5$							
10	0.058	0.053	0.059	0.048	0.058	0.054	0.051	0.047	0.046	0.046	0.048	0.054	0.097	0.080
30	0.053	0.051	0.060	0.050	0.052	0.052	0.052	0.047	0.038	0.036	0.046	0.050	0.148	0.134
50	0.051	0.047	0.052	0.057	0.052	0.046	0.045	0.056	0.020	0.020	0.045	0.045	0.179	0.194
100	0.052	0.048	0.048	0.061	0.053	0.047	0.034	0.042	0.013	0.013	0.047	0.049	0.242	0.231
							$m=10$							
10	0.052	0.048	0.056	0.053	0.067	0.049	0.051	0.049	0.054	0.050	0.064	0.055	0.093	0.077
30	0.055	0.051	0.060	0.060	0.056	0.050	0.035	0.040	0.044	0.041	0.061	0.055	0.153	0.139
50	0.057	0.059	0.053	0.056	0.057	0.058	0.037	0.042	0.028	0.023	0.061	0.054	0.204	0.211
100	0.053	0.042	0.055	0.050	0.051	0.043	0.037	0.044	0.012	0.011	0.062	0.060	0.272	0.286
							Skewed t_3							
p	T_n	T_r	W_n	W_r	T_{sn}	T_{sr}	C	C^*	H	$L\&M$	Q_1	Q_2	Q_3	Q_4
							$m=5$							
10	0.040	0.057	0.039	0.058	0.039	0.062	0.997	0.999	0.996	0.997	0.058	0.044	0.245	0.191
30	0.048	0.043	0.041	0.053	0.047	0.047	0.997	0.998	0.745	0.743	0.053	0.061	0.254	0.256
50	0.051	0.039	0.064	0.052	0.049	0.039	0.996	0.998	0.427	0.421	0.046	0.052	0.307	0.300
100	0.047	0.064	0.047	0.054	0.048	0.066	0.994	0.992	0.215	0.210	0.047	0.041	0.307	0.323
							$m=10$							
10	0.062	0.059	0.047	0.062	0.050	0.050	0.995	0.994	0.999	0.999	0.069	0.052	0.247	0.170
30	0.050	0.050	0.041	0.051	0.049	0.049	0.993	0.996	0.884	0.883	0.040	0.040	0.275	0.257
50	0.040	0.044	0.067	0.064	0.040	0.045	0.991	0.997	0.557	0.544	0.047	0.050	0.296	0.283
100	0.058	0.042	0.060	0.043	0.056	0.046	0.986	0.989	0.248	0.244	0.049	0.060	0.332	0.336
							Cauchy							
p	T_n	T_r	W_n	W_r	T_{sn}	T_{sr}	C	C^*	H	$L\&M$	Q_1	Q_2	Q_3	Q_4
							$m=5$							
10	0.059	0.038	0.052	0.048	0.061	0.036	0.003	0.003	0.299	0.297	0.079	0.051	0.235	0.274
30	0.051	0.051	0.046	0.054	0.053	0.055	0.004	0.003	0.330	0.330	0.043	0.050	0.089	0.336
50	0.053	0.049	0.051	0.053	0.056	0.049	0.004	0.001	0.286	0.286	0.043	0.042	0.036	0.307
100	0.052	0.047	0.043	0.050	0.057	0.048	0.002	0.003	0.262	0.262	0.055	0.050	0.000	0.280
							$m=10$							
10	0.053	0.038	0.051	0.044	0.052	0.039	0.006	0.010	0.281	0.279	0.084	0.066	0.210	0.268
30	0.058	0.049	0.052	0.058	0.057	0.050	0.012	0.009	0.253	0.251	0.065	0.051	0.074	0.300
50	0.068	0.040	0.054	0.048	0.071	0.044	0.005	0.003	0.296	0.297	0.043	0.045	0.027	0.320
100	0.064	0.044	0.052	0.052	0.072	0.046	0.011	0.009	0.277	0.274	0.049	0.040	0.000	0.283

where $\tilde{\Upsilon}_l$ is the lag-l sample cross-correlation matrix of $X_t^2 \equiv (x_{1t}^2, \ldots, x_{pt}^2)'$ and $b_l = \mathrm{vec}(\tilde{\Upsilon}_l)$ is the column-stacking vector of $\tilde{\Upsilon}_l$. Q_4 is defined as Q_1 with the trimmed data $\{\xi_t\}$, that is, removed from $\{\xi_t\}$ those observations whose magnitudes exceed the 95% empirical quantile of $\{\xi_t\}$. These four tests are the most commonly used test statistics for detecting ARCH effect in a vector time series.

5.1. Empirical Sizes

We start with the empirical sizes of the test statistics considered in the comparison. The datasets are generated from

$$X_t = A\varepsilon_t, \tag{5.1}$$

where the components of ε_t are iid $N(0,1)$, skewed-t_3, and Cauchy distribution. The dimensions used are $p = 10, 30, 50$, and 100, and the sample size $n = 300$. We use 1000 replications. For each configuration of (p,n), we compute test statistics for $m = 5$ and 10. The significance level is $\alpha = 0.05$. The following two cases of A in Equation (5.1) are considered:

1. Case 1: $A = \mathrm{diag}(\underbrace{20,\ldots,20}_{p/2}, \underbrace{100,\ldots,100}_{p/2})$,
2. Case 2. $A = S^{1/2}$, where $S = (s_{kl})_{1\le k,l\le p}$ with $s_{kl} = 0.995^{|k-l|}$.

We report the empirical sizes of all tests in Tables 1 and 2 for Cases 1–2, respectively.

From Tables 1 and 2, we see that the proposed test statistics T_n, T_r, W_n, W_r, T_{sn}, and T_{sr} as well as the Q_1 and Q_2 statistics have empirical sizes close to the nominal level α for all cases. In particular, W_n and W_r have a relatively stable size when $m = 10$. The statistics C and C^* of Chang, Yao, and Zhou (2017) are conservative in Case 1, especially when $p = 100$, but they have good sizes close to the nominal level α in Case 2 when $\varepsilon_t \sim N(0,1)$. However, as expected, the empirical sizes of C and C^* are over conservative or distorted if the underlying distributions is $\varepsilon_t \sim$ skewed-t_3 or Cauchy. The tests H and $L\&M$ have good size only when p is small and $\varepsilon_t \sim N(0,1)$, and they become too conservative when $p = 100$. Their sizes are fully distorted for other cases. The sizes of Q_3 and Q_4 are distorted in most of cases. This finding suggests that our tests have a reliable size not only for high-dimensional data but also for nonnormal or even heavy-tailed data. The simulation results support our claim that the null distributions of our tests are insensitive to the shape of the

Table 3. Empirical powers of various test statistics for detecting serial correlation, where n is the sample size, p is dimension, and $T_n, T_r, W_n, W_r, T_{sn}$, and T_{sr} are test statistics proposed in the article, and H and $L\&M$ are, respectively, in Hosking (1980) and Li and McLeod (1981)

					$m = 5$				
n	p/n	T_n	T_r	W_n	W_r	T_{sn}	T_{sr}	H	$L\&M$
100	0.6	0.323	0.462	0.447	0.605	0.196	0.295	0.986	0.986
	0.8	0.295	0.449	0.420	0.612	0.174	0.288	0.028	0.022
	1.0	0.305	0.452	0.434	0.602	0.161	0.271	0.000	0.000
	1.2	0.295	0.453	0.416	0.614	0.174	0.267	0.000	0.000
300	0.6	0.859	0.943	0.934	0.975	0.609	0.807	0.999	0.999
	0.8	0.850	0.949	0.932	0.989	0.610	0.795	0.990	0.989
	1.0	0.879	0.959	0.955	0.985	0.659	0.808	0.000	0.000
	1.2	0.863	0.952	0.939	0.979	0.625	0.773	0.000	0.000
500	0.6	0.978	0.996	0.995	0.998	0.864	0.956	0.999	0.999
	0.8	0.990	0.999	0.999	0.999	0.892	0.957	1.00	1.00
	1.0	0.986	0.997	0.995	0.999	0.902	0.957	0.000	0.000
	1.2	0.994	1.00	0.999	1.00	0.890	0.956	0.000	0.000
					$m = 10$				
n	p/n	T_n	T_r	W_n	W_r	T_{sn}	T_{sr}	H	$L\&M$
100	0.6	0.250	0.344	0.384	0.542	0.137	0.213	0.856	0.855
	0.8	0.257	0.354	0.380	0.540	0.149	0.201	0.044	0.027
	1.0	0.262	0.361	0.397	0.541	0.160	0.236	0.000	0.000
	1.2	0.282	0.363	0.415	0.565	0.168	0.229	0.000	0.000
300	0.6	0.720	0.887	0.860	0.966	0.457	0.656	1.00	1.00
	0.8	0.763	0.915	0.918	0.973	0.498	0.694	0.756	0.748
	1.0	0.751	0.882	0.900	0.965	0.486	0.679	0.000	0.000
	1.2	0.777	0.878	0.907	0.970	0.519	0.639	0.000	0.000
500	0.6	0.958	0.991	0.994	1.00	0.776	0.916	1.00	1.00
	0.8	0.965	0.990	0.996	0.998	0.785	0.904	1.00	1.00
	1.0	0.959	0.988	0.993	1.00	0.788	0.882	0.000	0.000
	1.2	0.961	0.988	0.991	0.999	0.779	0.856	0.000	0.000

distribution of X_t so long as it is continuous. When $p = 300$ and 500, the simulation results of our six tests are available in the supplementary appendix and they support the same conclusion.

5.2. Empirical Power of Detecting Serial Correlation

Next, we study the power of the test statistics in detecting serial correlation. We generate data from the following VAR(1) model

$$X_t = \Phi X_{t-1} + Ae_t, \tag{5.2}$$

where $X_0 = e_0$, and $\{e_t\}$ consists of independent components that are iii $\sim N(0, 1)$. The matrix A is defined in Case 2 of Equation (5.1). We set the sample size $n = 100, 300$, and 500 and the dimension p by $p/n = 0.6, 0.8, 1.0$, and 1.2. As in Tsay (2017), the $p \times p$ coefficient matrix Φ is assumed to be sparse with the number of nonzero coefficients given by $N = [p^2/n]$, where $\lfloor x \rfloor$ denotes the integer part of x. Specifically, let $\mathrm{vec}(\Psi)$ be the column-stacking vector of Φ. For each realization of X_t, we generate $\mathrm{vec}(\Psi)$ as follows:

I. Initialize $\mathrm{vec}(\Psi) = 0$.
II. Randomly select N numbers, without replacement, from 1 to p^2.
III. Randomly generate N real number from $U[-0.95, 0.95]$.
IV. Assign the N real numbers of Step III to the N locations in Step II.

We report the powers of T_n, T_r, W_n, W_r, T_{sn}, and T_{sr}, H, and $L\&M$ in Table 3 for $m = 5$ and 10 with the significance level $\alpha = 0.05$, where the number of replications is 1000. From Table 3, we can see that the powers of T_n, T_r, W_n, W_r, T_{sn}, and T_{sr} do not depend on the dimension p and increase when the sample size increases from 100 to 500. The test statistics W_n and W_r are more powerful than T_n, T_r, T_{sn}, and T_{sr}. The tests H and $L\&M$ are most powerful when p/n is small, but they do not have any power when $p/n = 0.8$, 1.0 and 1.2.

To compare the powers of our tests with those of C and C^*, we generate data from the model in Equation (5.2) with $\{e_t\}$ being iid $N(0, I_p)$. The C and C^* statistics are supposed to work well in this particular case. We set $p = 15, 50$ and 100, and $n = 500$, and use 1000 replications. Figure 1 shows the power curves of T_n, T_r, W_n, W_r, T_{sn}, and T_{sr}, C and $C*$ when $m = 2, 4, 6, 8, 10$ at the significance level $\alpha = 0.05$. We see that, as expected, C and C^* are more powerful

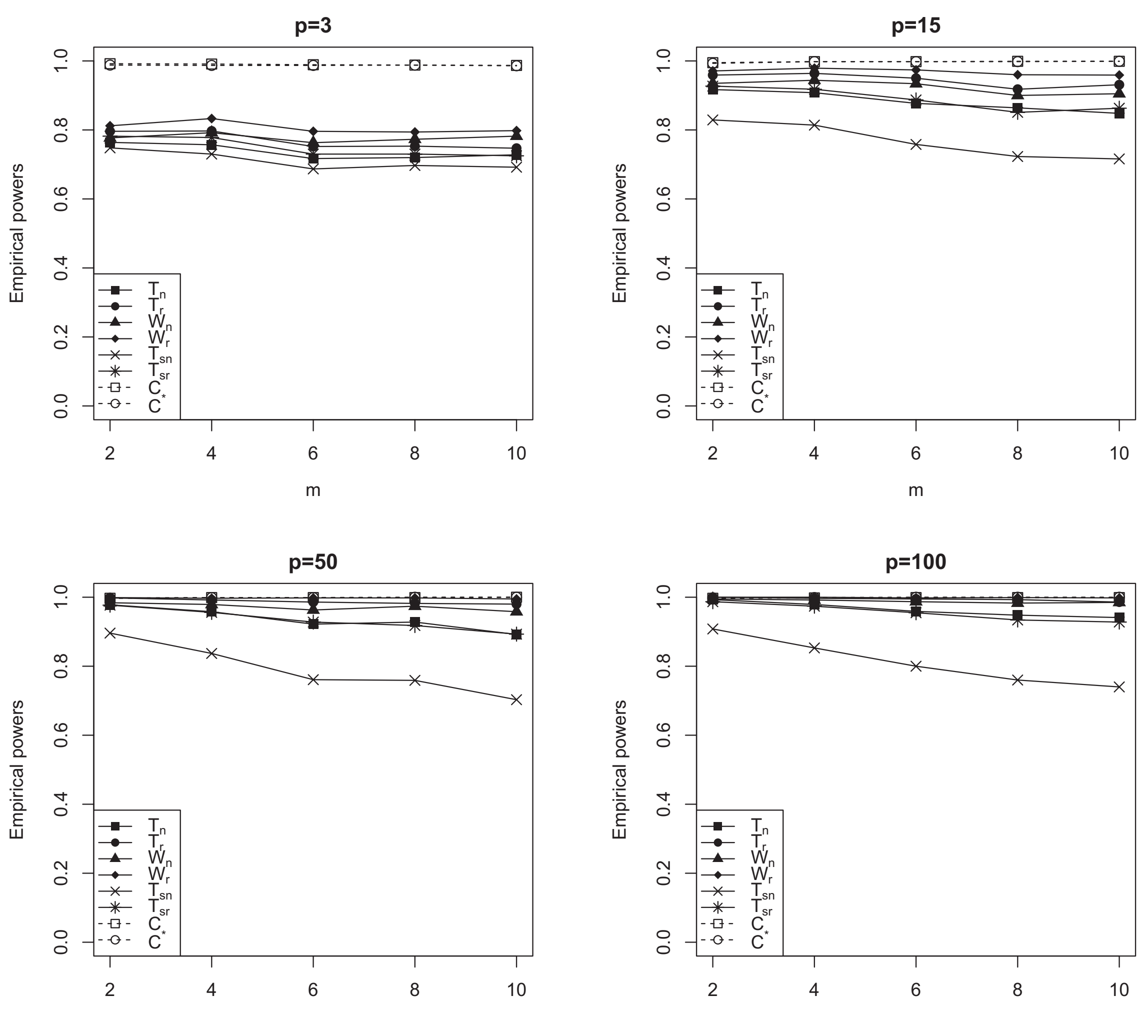

Figure 1. Empirical powers of various test statistics in detecting serial dependence, where m denotes the number of cross-correlation matrices used, the sample size is 500, and the number of iterations is 1000.

Table 4. Empirical powers of the proposed test statistics in detecting serial dependence, where n is the sample size and p the dimension

Skewed t_3													
		$m=5$						$m=10$					
n	p/n	T_n	T_r	W_n	W_r	T_{sn}	T_{sr}	T_n	T_r	W_n	W_r	T_{sn}	T_{sr}
100	0.6	0.411	0.717	0.606	0.837	0.228	0.541	0.360	0.635	0.547	0.805	0.208	0.453
	0.8	0.419	0.699	0.603	0.839	0.255	0.529	0.347	0.621	0.552	0.803	0.206	0.448
	1.0	0.472	0.736	0.645	0.860	0.260	0.567	0.333	0.622	0.518	0.811	0.196	0.439
	1.2	0.444	0.715	0.652	0.853	0.237	0.531	0.351	0.624	0.565	0.794	0.204	0.453
300	0.6	0.987	0.999	0.997	1.00	0.881	0.981	0.943	0.998	0.995	1.00	0.744	0.959
	0.8	0.987	0.998	0.997	1.00	0.890	0.983	0.951	0.996	0.992	0.999	0.775	0.960
	1.0	0.987	0.997	0.998	1.00	0.895	0.986	0.955	0.993	0.995	1.00	0.804	0.974
	1.2	0.997	1.00	0.999	1.00	0.908	0.988	0.967	0.996	0.996	0.999	0.802	0.970
500	0.6	1.00	1.00	1.00	1.00	0.996	1.00	1.00	1.00	1.00	1.00	0.988	1.00
	0.8	1.00	1.00	1.00	1.00	0.998	1.00	1.00	1.00	1.00	1.00	0.992	1.00
	1.0	1.00	1.00	1.00	1.00	0.998	1.00	0.999	1.00	1.00	1.00	0.991	0.999
	1.2	1.00	1.00	1.00	1.00	0.996	1.00	1.00	1.00	1.00	1.00	0.991	0.999
Cauchy													
		$m=5$						$m=10$					
n	p/n	T_n	T_r	W_n	W_r	T_{sn}	T_{sr}	T_n	T_r	W_n	W_r	T_{sn}	T_{sr}
100	0.6	0.672	0.988	0.898	0.996	0.392	0.972	0.380	0.980	0.765	0.998	0.197	0.940
	0.8	0.682	0.990	0.909	0.998	0.350	0.969	0.353	0.968	0.770	0.995	0.188	0.912
	1.0	0.702	0.990	0.937	0.997	0.399	0.971	0.370	0.985	0.790	0.997	0.206	0.937
	1.2	0.733	0.990	0.943	0.999	0.390	0.971	0.375	0.981	0.813	0.999	0.199	0.934
300	0.6	0.997	1.00	0.998	1.00	0.997	1.00	0.999	1.00	1.00	1.00	0.996	1.00
	0.8	0.999	1.00	0.999	1.00	0.999	1.00	0.998	1.00	0.998	1.00	0.997	1.00
	1.0	0.998	1.00	0.998	1.00	0.998	1.00	0.997	1.00	0.997	1.00	0.996	1.00
	1.2	0.998	1.00	0.999	1.00	0.998	1.00	0.997	1.00	0.998	1.00	0.994	1.00
500	0.6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	0.8	0.998	1.00	0.99	1.00	0.998	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	1.0	0.999	1.00	1.00	1.00	1.00	1.00	0.999	1.00	0.999	1.00	0.999	1.00
	1.2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

than T_n, T_r, W_n, W_r, T_{sn}, and T_{sr} in general, but the powers of all tests but T_{sr} are comparable when the dimension p becomes large, for example, 50 and 100. Figure 1 also shows that all tests are more powerful when m is smaller. This is not surprising as we employed a VAR(1) model. We only perform the simulation for smaller p as in Chang, Yao, and Zhou (2017) because the computational time of the tests C and C^* increases dramatically as the dimension p increases. In Figure 1, C test took 93.7 min, 3.8 hr, 32 hr, and 6.5 days, respectively, for $p = 3, 15, 50, 100$. On the other hand, the test statistic T_n only took 42.3 sec, 57.4 sec, 1.8 min, and 3.3 min.

To see the power of our tests for detecting serial dependence for large p and nonexponential tails or heavy-tailed time series, we generate data from the model in Equation (5.2) with $\{e_t\}$ being skewed-t_3 and Cauchy distributions, respectively. We set $n = 100, 300$, and 500 and $p/n = 0.6, 0.8, 1$, and 1.2. The results are reported in Table 4. Since the tests C, C^*, H, and $L\&M$ do not have correct empirical sizes in this situation, their powers are not reported. This table shows that all of our tests are powerful, especially when $n = 300$ and 500.

5.3. Empirical Power of Detecting ARCH Effect

In this section, we investigate the power of detecting ARCH effect. The data are generated from the first-order vector autoregressive conditional heteroscedasticity model [VARCH(1)]

$$X_t|\Im_{t-1} \sim N(0, \Omega_t) \quad \text{and} \quad \Omega_t = B'B + C'X_{t-1}X'_{t-1}C, \quad (5.3)$$

where $\Im_{t-1}$ be the σ-field generated by the past values of X_t, and B and C are $p \times p$ real-valued matrices with B being triangular. This model is referred to as the BEKK representation in Engle (1995). In the simulation, we let $B = I_p$. The coefficient matrix C is generated in the same way as that of Φ in Equation (5.2) except that Step III is modified to III$'$: randomly generate $N = 2p$ real numbers from U[0.01, 0.99]. In the simulation, we set the significance level to $\alpha = 0.05$ and the sample size to $n = 100$ and 200. As before, 1000 replications are used. The simulation results for T_n, T_r, W_n, W_r, T_{sn}, T_{sr}, Q_1, and Q_2 are summarized in Table 5. Because the test statistics Q_3 and Q_4 do not have proper empirical sizes, we do not report their powers in the table. From Table 5, our proposed tests have good power in detecting the ARCH effect, and W_n and W_r are more powerful than other test statistics.

Table 5. Empirical powers of some test statistics in detecting ARCH effect, where n and p are the sample size and dimension, respectively

	p/n	T_n	T_r	W_n	W_r	T_{sn}	T_{sr}	Q_1	Q_2
$m = 5$									
$n = 100$	0.4	0.677	0.649	0.775	0.765	0.694	0.538	0.560	0.390
	0.6	0.694	0.662	0.779	0.764	0.700	0.567	0.511	0.382
	0.8	0.706	0.638	0.805	0.755	0.717	0.521	0.289	0.314
	1.0	0.670	0.632	0.775	0.729	0.684	0.523	0.112	0.058
	1.2	0.702	0.606	0.794	0.724	0.690	0.513	0.097	0.059
$n = 200$	0.4	0.936	0.904	0.963	0.943	0.948	0.854	0.921	0.719
	0.6	0.930	0.932	0.966	0.962	0.932	0.875	0.883	0.723
	0.8	0.931	0.925	0.975	0.955	0.944	0.861	0.736	0.646
	1.0	0.944	0.911	0.970	0.957	0.954	0.839	0.150	0.060
	1.2	0.938	0.925	0.968	0.967	0.945	0.857	0.069	0.053
$m = 10$									
	p/n	T_n	T_r	W_n	W_r	T_{sn}	T_{sr}	Q_1	Q_2
$n = 100$	0.4	0.606	0.565	0.725	0.716	0.613	0.455	0.460	0.315
	0.6	0.638	0.578	0.727	0.714	0.632	0.481	0.444	0.328
	0.8	0.620	0.549	0.747	0.711	0.627	0.438	0.257	0.263
	1.0	0.583	0.544	0.729	0.687	0.600	0.426	0.094	0.062
	1.2	0.610	0.532	0.735	0.697	0.619	0.405	0.072	0.062
$n = 200$	0.4	0.902	0.853	0.948	0.928	0.902	0.788	0.878	0.632
	0.6	0.874	0.868	0.941	0.949	0.887	0.775	0.830	0.609
	0.8	0.904	0.882	0.946	0.943	0.910	0.793	0.670	0.564
	1.0	0.885	0.852	0.948	0.941	0.895	0.761	0.114	0.055
	1.2	0.899	0.857	0.957	0.941	0.909	0.774	0.055	0.048

NOTE: The results are based on 1000 iterations.

Table 6. p-values of various tests for the residuals of a VAR(1) model for the monthly log returns of 86 stocks, where m is the number of serial correlations used

Statistics	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 10$
T_n	0.5139	0.6843	0.1324	0.2284	0.3419	0.2329
T_r	0.2488	0.457	0.0878	0.1524	0.2421	0.1742
W_n	0.5111	0.6480	0.3221	0.2823	0.2950	0.2242
W_r	0.2456	0.3683	0.1929	0.1705	0.1819	0.1507
T_{sn}	0.6973	0.9004	0.0481	0.0924	0.1533	0.0779
T_{sr}	0.3032	0.5879	0.0599	0.1039	0.1711	0.0784

Furthermore, our tests are more powerful compared with the tests Q_1 and Q_2, especially when p/n increases from 0.8 to 1.2.

6. REAL EXAMPLES

6.1. Application to Monthly Asset Returns

In this section, we apply the proposed tests to monthly stock returns. Monthly prices of 86 stocks from February, 1999 to August, 2017 with 223 observations are used. These stocks consist of the components of the S&P500 index and are obtained from Yahoo Finance. Time plots of the 86 monthly stock prices and their log returns are shown in the supplementary appendix.

We apply our tests T_n,T_r, W_n, W_r, T_{sn}, and T_{sr} to the return series. The asymptotic p-values of these tests are all less than 0.0001, indicating that there exists serial correlations or ARCH effect in the 86-dimensional time series. We then fit a VAR(1) model to the return series and apply our statistics for the residual series. Their p-values are in Table 6, which show that one cannot reject the null hypothesis of zero serial correlations or no ARCH effect in the residuals at the significance level 5%. The fitted VAR(1) model seems to be adequate and there is no ARCH effect in the 86 monthly stock returns. We note that, with $n = 222$ and $p = 86$, the least squares estimates of the VAR(1) model might not be consistent, rendering the applicability of the residuals questionable. However, simulation results in the supplementary appendix indicate that our tests work well for residual series in this VAR(1) model.

Table 7. p-values of various tests for the standardized residuals of daily log returns of 24 stocks

Statistics	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 10$
T_n	0.801	0.563	0.765	0.829	0.815	0.466
T_r	0.343	0.418	0.557	0.704	0.529	0.234
W_n	0.800	0.644	0.724	0.783	0.806	0.653
W_r	0.341	0.402	0.481	0.570	0.566	0.436
T_{sn}	0.793	0.564	0.766	0.833	0.820	0.471
T_{sr}	0.340	0.409	0.554	0.702	0.533	0.249

6.2. Application to Daily Returns

Next, we apply the tests to the daily log returns of 24 stocks from April 2, 2012, to December 31, 2013, for 518 observations. The 24 series consist of components of the S&P 100 index that were continuously in the index over a ten-year period and the returns are in percentages. The data are obtained from the Center for Research on Security Prices, University of Chicago. Time plots of the 24 return series are in the supplementary appendix.

We first use T_n, T_r, W_n, W_r, T_{sn}, and T_{sr} to test whether there is any serial correlation or ARCH effect in the original 24-dimensional return series $\{r_{it}, i = 1, \ldots, 24\}$. The p-values of these tests are all less than 0.0001, indicating the existence of serial correlations or ARCH effect in the data. On the other hand, we apply the order selection criteria available in the `MTS` package of R to the data and find that the selected order is zero. Therefore, the low p-values are likely to be caused by the ARCH effect, and we fit the following EGARCH (1,1) model to the individual series

$$r_{it} = \sigma_{it}\eta_{it} \quad \text{and} \quad \sigma_{it}^2 = \omega + \alpha\eta_{i,t-1} + \gamma(|\eta_{i,t-1}| - E|\eta_{i,t-1}|) + \beta \log \sigma_{i,t-1}^2, \tag{6.4}$$

where $i = 1, \ldots, 24$, $\{\eta_{it}\}$ are iid $N(0, 1)$, and it is understood that the parameters ω, α, γ, and β depend on i. Define the standardized residual as $\hat{\eta}_{it} = r_{it}/\hat{\sigma}_{it}$, where the volatility $\hat{\sigma}_{it}$ is obtained via the model in Equation (6.4) with the estimated parameters. We then apply our tests to detect if there is any serial correlation or ARCH effect in the residual series $\hat{\eta}_t \equiv (\hat{\eta}_{1t}, \ldots, \hat{\eta}_{24t})'$, $t = 1, \ldots, 518$. The p-values of these tests are reported in Table 7, and they show that the EGARCH(1,1) model is adequate for the 24-dimensional return series. Simulation results in the supplementary appendix indicate that our tests work well for both the original series and the standardized residual series. Finally, it is interesting to see that, unlike the Ljung–Box statistics, our tests do not need to adjust the degrees of freedom when they are applied to the residual series of a fitted model.

APPENDIX

Proof of Theorem 2.1. By Assumption 2.1(a), we have

$$pE(\bar{Y} - \mu)^2 = pE\left(\frac{1}{n}\sum_{t=1}^{n} \|X_t\| - \mu\right)^2 = \frac{p}{n}E(\|X_t\| - \mu)^2 = O\left(\frac{1}{n}\right).$$

By Assumption 2.1(b) and Chebychev's inequality, we have

$$P\left(\frac{p}{n}\left|\sum_{t=1}^{n}[(\|X_t\| - \mu)^2 - E(\|X_t\| - \mu)^2]\right| > C\right) \le \frac{p^2}{Cn}E(\|X_t\| - \mu)^4| \to 0, \quad n \to \infty.$$

Thus,

$$\left|\frac{p}{n}\sum_{t=1}^{n}[(\|X_t\| - \mu)^2 - \sigma^2]\right| \le \frac{p}{n}\left|\sum_{t=1}^{n}[(\|X_t\| - \mu)^2 - E(\|X_t\| - \mu)^2]\right| + \left|pE(\|X_t\| - \mu)^2 - \sigma^2\right| \to_p 0,$$

as $n \to \infty$. Furthermore, we have

$$\frac{p}{n}\sum_{t=1}^{n}[(\|X_t\| - \bar{Y})^2 = \frac{p}{n}\sum_{t=1}^{n}[(\|X_t\| - \mu)^2 + p(\bar{Y} - \mu)^2 \to_p \sigma^2, \quad n \to \infty. \tag{A.1}$$

Therefore,

$$\begin{aligned}\frac{p}{\sqrt{n}}\sum_{t=1}^{n}(\|X_t\| - \bar{Y})(\|X_{t+l}\| - \bar{Y}) &= \frac{p}{\sqrt{n}}\sum_{t=1}^{n}(\|X_t\| - \mu)(\|X_{t+l}\| - \mu) + p\sqrt{n}(\bar{Y} - \mu)^2 + o_p(1) \\ &= \frac{p}{\sqrt{n}}\sum_{t=1}^{n}(\|X_t\| - \mu)(\|X_{t+l}\| - \mu) + o_p(1).\end{aligned} \tag{A.2}$$

Denote

$$\xi_{nt}((l) = \frac{p}{\sqrt{n}}(\|X_t\| - \mu)(\|X_{t-l}\| - \mu)$$

and let $\mathcal{F}_{nt} = \sigma-$field generated by $\{\|X_t\}, \ldots, \|X_1\|\}$. Then $\{\xi_{nt}((l) : t = 1, \ldots, n\}$ is a sequence of martingale differences.

$$V_n \equiv \sum_{t=1}^{n} E(\xi_{nt}^2(l)|\mathcal{F}_{nt-1}) = \frac{p^2}{n}E(\|X_{t-l}\| - \mu)^2\sum_{t=1}^{n}(\|X_t\| - \mu)^2 \to_p (\sigma^2)^2, \tag{A.3}$$

as $n \to \infty$. Thus, we have

$$E\left\{\sum_{t=1}^{n} E(\xi_{nt}^2((l)I\{|\xi_{nt}^2(l)| > \varepsilon\}|\mathcal{F}_{nt-1})\right\} \tag{A.4}$$

$$\leq \left\{\frac{p^2 E(\|X_t\| - \mu)^4}{\sqrt{n}\varepsilon^4}\right\}^2 \to 0,$$

as $n \to \infty$. By (7.3)–(7.4) and the central limiting theorem for the martingale array,

$$\sum_{t=1}^{n} \xi_{nt} \to_{\mathcal{L}} N(0, \sigma^4),$$

as $n \to \infty$. Furthermore, by (7.1)-(7.2), we have

$$\sqrt{n}\hat{\rho}_l \to_{\mathcal{L}} N(0, 1).$$

It is straightforward to extend the previous limiting distribution from a single l to the multivariate case. This completes the proof. □

SUPPLEMENTARY MATERIALS

In the online appendix, we provide data source, plot of data, more simulation results and the proof of Theorem 1.

ACKNOWLEDGMENTS

The authors greatly appreciate the constructive suggestions and comments of three anonymous referees, the associate editor, and the editor Todd Clark. We would like to thank Jinyuan Chang and Wen Zhou for their helpful suggestions on applying their tests.

FUNDING

Shiqing Ling's research was supported by Hong Kong Research Grants Commission grants (16307516, 16500117, and 16303118, theme-based projects), NSFC (no. 11731015), and Australian Research Council. Yang's research was supported by the Fundamental Research Funds for the Central Universities (no. 20720171062) and the NSFC (no. 11801476). R. Tsay's research is supported in part by Chicago Booth.

[Received February 2018. Revised May 2019.]

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