# Chapter 1 Introduction

Zhenghui Feng

# Outline

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# 1.2.1 Binomial Distribution

**Bernoulli Distribution.**

A DRV $Y$ is called a Bernoulli($\pi$) ($0 < \pi < 1$) random variable if its PMF

$$f_Y(y) = \begin{cases} \pi & \text{if } y = 1, \\ 1 - \pi & \text{if } y = 0. \end{cases}$$

**Binomial Distribution.**

A DRV $Y$ is called a Binomial($n, \pi$) ($n \geq 0$ and $0 < \pi < 1$) if its PMF is

$$f_Y(y) = \left( \begin{array}{c} n \\ y \end{array} \right) \pi^y (1 - \pi)^{n-y}, \, x = 0, 1, \cdots, n$$

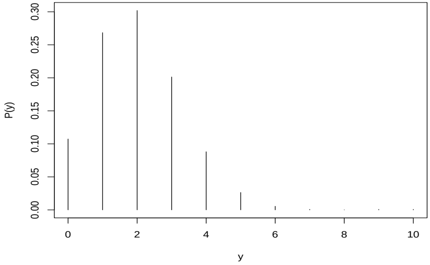
- $n$ Bernoulli trials, two possible outcomes for each (success, failure)
- $\pi = P(\text{success})$, $1 - \pi = P(\text{failure})$ for each trial
- $Y$ = number of successes out of $n$ trials
- Trials are independent

Then Y has binomial distribution, with the probability density function (pdf)

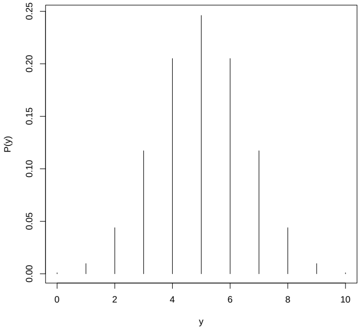
$$P(y) = \frac{n!}{y!(n-y)!}\pi^y(1-\pi)^{n-y}, \ y = 0, 1, 2, ..., n.$$

**Example (Quiz)**

Suppose a quiz has 10 multiple-choice questions, with five possible answers for each. A student who is completely unprepared randomly guesses the answer for each question. The probability of a correct response is 0.20 for a given question.




$\pi = 0.2$




$\pi = 0.5$

Figure: Binomial PMF

## Note

- $E(Y) = \mu = n\pi$, $\mathrm{Var}(Y) = n\pi(1-\pi)$, $\sigma = \sqrt{n\pi(1-\pi)}$
- $\hat{\pi} = \frac{Y}{n}$=proportion of success, $E(\hat{\pi}) = \pi$ , $\mathrm{Var}(\hat{\pi}) = \frac{\pi(1-\pi)}{n}$
- When $n$ is large, the distribution of $Y$ can be approximated by a normal distribution with $\mu = n\pi$, $\sigma = \sqrt{n\pi(1-\pi)}$
- The approximation has a guideline that $\mathbf{n}\pi$, $\mathbf{n(1-}\pi)$, should be $\geq 5$
- When each trial has $n > 2$ possible outcomes, numbers of outcomes in various categories have *multinomial distribution*.

# 1.2.2 Multinomial Distribution

**Multinomial Distribution.**

Let $c$ denote the number of outcome categories. Their probabilities are denoted by $\{\pi_1, \pi_2, \ldots, \pi_c\}$, where $\sum_j \pi_j = 1$. For $n$ independent observations, the multinomial probability that $n_1$ fall in category 1, $n_2$ fall in category 2, $\ldots$, $n_c$ fall in category $c$, where $\sum_j n_j = n$, equals

$$P(n_1, n_2, \ldots, n_c) = (\frac{n!}{n_1!n_2!\cdots n_c!})\pi_1^{n_1}\pi_2^{n2}\cdots\pi_c^{n_c}.$$

# 1.2.2 Multinomial Distribution

## Note

- The multinomial is a multivariate distribution
- The marginal distribution of the count in any particular category is binomial. For category $j$, the count $n_j$ has mean $n\pi_j$ and standard deviation $\sqrt{n\pi_j(1-\pi_j)}$

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# 1.3.1 Likelihood Function and Maximum Likelihood Estimation

**Likelihood Function**

The likelihood function is the probability of the observed data, expressed as a function of the parameter value.

Example

Binomial, $n = 10$, $y = 3$.

$$P(Y = 3) = \frac{10!}{3!7!}\pi^3(1-\pi)^7 = l(\pi).$$

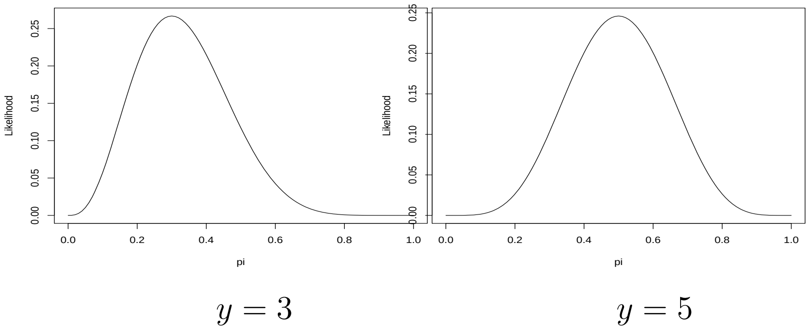



Figure: Likelihood Function for $\pi$

## Maximum Likelihood Estimate

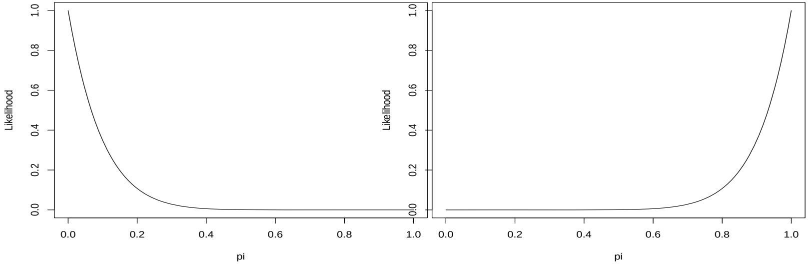
The maximum likelihood (ML) estimate is the parameter value at which the likelihood function takes its maximum.

### Example

$n = 10$, $y = 3$.

$$l(\pi) = \frac{10!}{3!7!}\pi^3(1-\pi)^7.$$

is maximized at $\hat{\pi} = 0.3$.




$y = 0$ $y = 10$

Figure: Likelihood Function for $\pi$

## Note

- For binomial, $\hat{\pi} = \frac{y}{n}$ = proportion of successes.
- If $y_1, y_2, \ldots, y_n$ are independent from normal, ML estimate $\hat{\mu} = \bar{y}$.
- In ordinary regression $Y$ ~normal, "least squares" estimates are ML.
- For large $n$ for any distribution, ML estimates are optimal (no other estimator has smaller standard error)
- For large $n$, ML estimators have approximate normal sampling distributions (under weak conditions).

# 1.3.2 Significance Test About a Binomial Proportion

### Significance Test

$H_0 : \pi = \pi_0 \qquad H_A : \pi \neq \pi_0$ (or 1-sided)

Test statistic is

$$z = \frac{p - \pi_0}{\sqrt{\frac{\pi_0(1-\pi_0)}{n}}}$$

has large-sample standard normal $N(0, 1)$ null distribution.

Question: How to do a hypothesis testing?

# 1.3.3 Example: Survey Results on Legalizing Abortion

## Example

Do a majority, or minority, of adults in the United States believe that a pregnant woman should be able to obtain an abortion? Let $\pi$ denote the proportion of the American adult population that responds "yes to the question, "Please tell me whether or not you think it should be possible for a pregnant woman to obtain a legal abortion if she is married and does not want any more children. We test $H_0 : \pi = 0.50$ against the two-sided alternative hypothesis, $H_A : \pi \neq 0.50$.

# 1.3.4 Confidence Intervals for a Binomial Proportion

Let $SE$ denote the estimated standard error of $p$. A large sample $100(1-\alpha)\%$ confidence interval for $\pi$ has the formula

$$p \pm z_{\alpha/2} SE, \qquad \text{with } SE = \sqrt{p(1-p)/n},$$

where $z_{\alpha/2}$ denotes the standard normal percentile having right-tail probability equal to $\alpha/2$.

# 1.3.4 Confidence Intervals for a Binomial Proportion

**Example**

For the attitudes about abortion example just discussed, $p = 0.448$ for $n = 893$ observations. The 95% confidence interval equals

$$0.448 \pm 1.96\sqrt{(0.448)(0.552)/893}$$

## Note

- Unless $\pi$ is close to 0.5, however, it doest not work well unless $n$ is very large. It is especially poor when $\pi$ is near 0 or 1.
- A better way: the CI contains all values $\pi_0$ for the null hypothesis that are not rejected: for given $p$ and $n$, the $\pi_0$ values are the solution to the inequality

$$\frac{|p-\pi_0|}{\sqrt{\pi_0(1-\pi_0)/n}} \leq 1.96$$

- A simple alternative approximation: add 2 to the number of successes and 2 to the number of failures (and thus 4 to $n$) and then use the ordinary formula with the estimated standard error. *Agresti-Coull confidence interval*

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# 1.4.1 Wald, Likelihood-Ratio, and Score Inference

## Wald Test

Let $\beta$ denote an arbitrary parameter. Consider a significance test of $H_0 : \beta = \beta_0$. Let $SE$ denote the standard error of ML estimator $\hat{\beta}$, evaluated by substituting the ML estimator for the unknown parameter in the expression for the true standard error. When $H_0$ is true, the test statistic

$$z = (\hat{\beta} - \beta_0)/SE$$

has approximately a standard normal distribution. Equivalently, $z^2$ has approximately a chi-squared distribution with $df = 1$. *This type of statistic, which uses the standard error evaluated at the ML estimate, is called a Wald statistic.* The $z$ or chi-squared test using this test statistic is called a *Wald test.*

## Likelihood Ratio Test

Under $H_0 : \beta = \beta_0$, the likelihood ratio test statistic

$$-2\log(l_0/l_1)$$

has a large-sample chi-squared distribution with $df = 1$. $l_0$ is the likelihood function calculated at $\beta_0$, and $l_1$ is the likelihood function calculated at the ML estimate $\hat{\beta}$.

## Score Test

The standard error are calculated under the assumption that the null hypothesis holds. E.g., $\sqrt{\pi_0(1-\pi_0)/n}$

# 1.4.2 Wald and Score Inference for Binomial Parameter

## Wald Statistic for Binomial Parameter

$H_0 : \pi = \pi_0$, $H_A : \pi \neq \pi_0$, Wald statistic is

$$z = \frac{p - \pi_0}{\sqrt{\frac{p(1-p)}{n}}}$$

## Wald Confidence Interval (CI) for Binomial Parameter

$$p \pm z_{\alpha/2}\sqrt{\frac{p(1-p)}{n}}$$

## Example

For $n = 20$, $y = 5$, find the 95% Wald CI.

## Score test, Score CI use null SE

Score 95% CI is the set of $\pi_0$ values for which $p$-value $> 0.05$ in testing

$$H_0 : \pi = \pi_0 \quad H_A : \pi \neq \pi_0$$

using

$$z = \frac{p - \pi_0}{\sqrt{\frac{\pi_0(1-\pi_0)}{n}}}$$

## Example

$y = 5$, $n = 20$, find the 95% Score CI.

## Likelihood-ratio test

When $H_0 : \pi = 0.5$ is true, $l_0 = [10!/9!1!](0.5)^9(0.5)^1 = 0.00977$. The likelihood-ratio test compares this to the value of the likelihood function at the ML estimate of $p = 0.9$, which is $l_1 = [10!/9!1!](0.9)^9(0.1)^1 = 0.387$. The likelihood-ratio test statistic

$$-2\log(l_0/l_1) = 7.36$$

From the $\chi^2_1$, the P-value is 0.007.

## 1.4.3 Small-Sample Binomial Inference

For small sample size, it is safer to use the binomial distribution directly (rather than a normal approximation) to calculate $P$-values. To illustrate, consider testing $H_0 : \pi = 0.50$ against $H_A : \pi > 0.50$ for the example of a clinical trial to evaluate a new treatment, when the number of successes $y = 9$ in $n = 10$ trials. The exact P-value, based on the right tail of the null binomial distribution with $\pi = 0.50$, is

$$P(Y \geq 9) = [10!/9!1!](0.50)^9(0.50)^1 + [10!/10!0!](0.50)^{10}(0.50)^0 = 0.011.$$

For the two sided alternative $H_A : \pi \neq 0.50$, the $P$-value is

$$P(Y \geq 9 \text{ or } Y \leq 1) = 2 \times P(Y \geq 9) = 0.021.$$

# 1.4.4 1.4.5 More about Small-Sample Inference

- With discrete probability distributions, small-sample inference using the ordinary P-value is *conservative*.This means that when $H_0$ is true, the P-value is $\leq 0.05$ (thus leading to rejection of $H_0$ at the 0.05 sig. level) not exactly 5% of the time, but typically less than 5% of the time.
- Mid P-value: it adds only half the probability of the observed result to the probability of the more extreme results.

# 1.4.4 1.4.5 More about Small-Sample Inference

**Example**

$H_0 : \pi = 0.5$ v.s. $H_a : \pi > 0.5$ with $y = 9, n = 10$.The ordinary P-value is

$$\text{P-value} = P(9) + P(10) = 0.011.$$

The mid P-value is $\quad P(9)/2 + P(10) = 0.006.$

# Outline

# Homework 1

1. Read Preface, Chapter 11 and Chapter 1 carefully.
2. Problems in textbook 1.2, 1.4, 1.5, 1.6, 1.12.