Chapter 4: Tests on Mean Vectors

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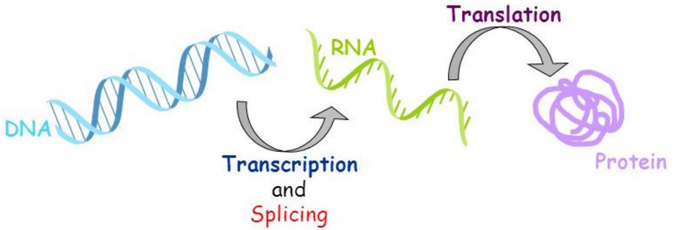
Outline

Outline

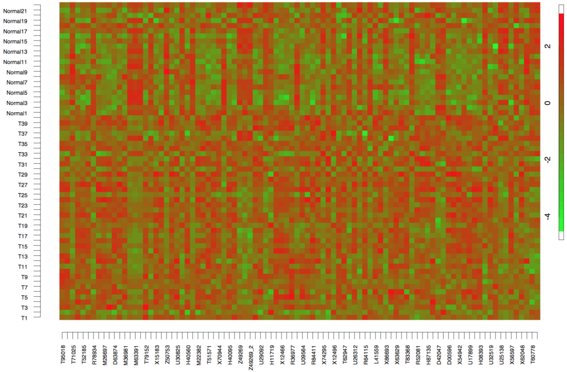
Gene Expression Examples

Example 1: Evaluate the average change of gene expression level due to the treatment. $H_0 : \boldsymbol{\mu} = \underline{0}$.



Extension: Wang, Peng and Li (2015), "A high-dimensional nonparametric multivariate test for mean vector", *JASA*.

Example 2: Groupwise comparison of gene expression between tumor tissues and normal tissues. $H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$.



Outline

Motivation for Multivariate Tests

Can we test each dimension univariately by the univariate testing procedures?

- The univariate tests inflate the Type I error rate.
- The univariate tests completely ignore the correlations among the variables.
- The multivariate test is more powerful than separately univariate tests - small individual effects may combine to significantly joint effect.

Outline

2 One-sample Multivariate Test about Mean Vector $\boldsymbol{\mu}$

- Multivariate Test on $\boldsymbol{\mu}$ with $\boldsymbol{\Sigma}$ Known
- Multivariate Test on $\boldsymbol{\mu}$ with $\boldsymbol{\Sigma}$ Unknown: Hotelling's T^2

Review: Univariate Test on μ with σ^2 Known

- Assumption:
  $y_1, \ldots, y_n$ i.i.d. from $N(\mu, \sigma^2)$ or n is large (> 30) for the central limit theorem (CLT).
- Hypotheses:
  (1) One-sided: $H_0 : \mu = \mu_0$ (or $\mu \leq \mu_0$) vs $H_1 : \mu > \mu_0$.
  (2) One-sided: $H_0 : \mu = \mu_0$ (or $\mu \geq \mu_0$) vs $H_1 : \mu < \mu_0$.
  (3) Two-sided: $H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$.
- Test statistic:

$$Z = \frac{\bar{y} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1) \text{ when } \mu = \mu_0 \text{ (under } H_0\text{)}$$

- Critical/Rejection regions:
  (1) $Z > z_\alpha$
  (2) $Z < -z_\alpha$
  (3) $|Z| > z_{\alpha/2}$

Multivariate Test for $\boldsymbol{\mu}$ with $\boldsymbol{\Sigma}$ Known

- Assumption:
  The p-variate sample $\{\mathbf{y}_1, \ldots, \mathbf{y}_n\}$ i.i.d. from $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with unknown $\boldsymbol{\mu}$ and known $\boldsymbol{\Sigma}$.
- Hypothesis:
  $$H_0: \ \boldsymbol{\mu} = \boldsymbol{\mu}_0, \text{ vs. } H_1: \ \boldsymbol{\mu} \neq \boldsymbol{\mu}_0.$$
- Test statistic:
  $$Z^2 = n(\bar{\mathbf{y}} - \boldsymbol{\mu}_0)'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{y}} - \boldsymbol{\mu}_0) \sim \chi^2(p) \text{ under } H_0.$$
- Rejection region:
  Reject H_0 if $Z^2 > \chi^2_\alpha(p)$ at significance level α.

Test for $\boldsymbol{\mu}$ with $\boldsymbol{\Sigma}$ Known: Example

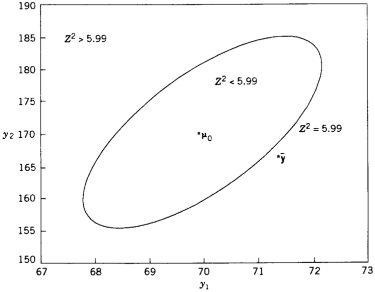
Example: Suppose the height and weight for a asample of 20 college-age males are originated from $N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\Sigma} = \begin{pmatrix} 20 & 100 \\ 100 & 1000 \end{pmatrix}$$

Objective is to test $H_0: \ \boldsymbol{\mu} = (70, 170)'$.

Person	Height x	Weight y	Person	Height x	Weight y
1	69	153	11	72	140
2	74	175	12	79	265
3	68	155	13	74	185
4	70	135	14	67	112
5	72	172	15	66	140
6	67	150	16	71	150
7	66	115	17	74	165
8	70	137	18	75	185
9	76	200	19	75	210
10	68	130	20	76	220

- Multivariate test:
  $\boldsymbol{\mu}_0 = (70, 170)'$. Since $\bar{\mathbf{y}} = (71.45, 164.7)'$.
  $Z^2 = n(\bar{\mathbf{y}} - \boldsymbol{\mu}_0)'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{y}} - \boldsymbol{\mu}_0) = 8.403 > \chi^2_{0.05}(2) = 5.99$.
  Therefore, we reject H_0.

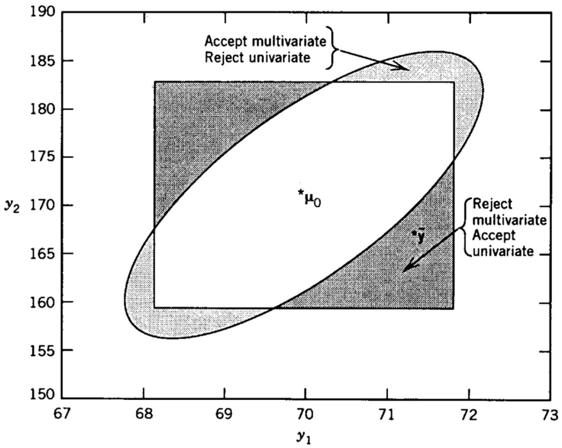


Elliptical acceptance region.

- Separate univariate test for each dimension:

$$|Z_1| = \left| \frac{\bar{y}_1 - \mu_{01}}{\sigma_1/\sqrt{n}} \right| = |1.45| < z_{0.025} = 1.96$$
$$|Z_2| = \left| \frac{\bar{y}_2 - \mu_{02}}{\sigma_2/\sqrt{n}} \right| = |-0.75| < z_{0.025} = 1.96.$$

Therefore, we cannot reject H_0.



Acceptance and rejection regions for univariate and multivariate tests.

Remarks:

- In the example, both the univariate tests fail to reject H_0, but when the positive correlation between Y_1 and Y_2 is considered, the multivariate test results in the rejection.
- Points inside the ellipse but outside the rectangle illustrate the inflation of Type I error of univariate tests.
- Points inside the rectangle but outside the ellipse illustrate that the multivariate test is more powerful.
- When multivariate tests and separately univariate tests disagree, we tend to trust the multivariate.

Outline

Review: Univariate Test on μ with σ^2 Unknown

- Assumption:
  $y_1, \ldots, y_n$ i.i.d. from $N(\mu, \sigma^2)$ or n is large (> 30) for the central limit theorem (CLT).
- Hypotheses:
  (1) One-sided: $H_0 : \mu = \mu_0$ (or $\mu \leq \mu_0$) vs $H_1 : \mu > \mu_0$.
  (2) One-sided: $H_0 : \mu = \mu_0$ (or $\mu \geq \mu_0$) vs $H_1 : \mu < \mu_0$.
  (3) Two-sided: $H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$.
- Test statistic:

$$T = \frac{\bar{y} - \mu_0}{s/\sqrt{n}} \sim t(n-1) \text{ when } \mu = \mu_0 \text{ (under } H_0)$$

- Critical/Rejection regions:
  (1) $T > t_\alpha(n-1)$
  (2) $T < -t_\alpha(n-1)$
  (3) $|T| > t_{\alpha/2}(n-1)$

Multivariate Test for $\boldsymbol{\mu}$ with $\boldsymbol{\Sigma}$ Unknown

As in the univariate case, in practice, $\boldsymbol{\Sigma}$ is seldom known. How should the test statistic be adjusted when $\boldsymbol{\Sigma}$ is unkonwn?

- Hotelling's T^2 test statistic:

$$T^2 = n(\bar{\mathbf{y}} - \boldsymbol{\mu}_0)'\mathbf{S}^{-1}(\bar{\mathbf{y}} - \boldsymbol{\mu}_0)$$

- Null distribution of T^2:
  Under H_0: $\boldsymbol{\mu} = \boldsymbol{\mu}_0$, T^2 follows Hotelling's T^2 distribution with two parameters p and $n-1$ (Hotelling, 1931).
- Rejection region:
  Reject H_0 if $T^2 > T^2_{\alpha}(p, n-1)$ obtained from the T^2 distribution table or the transformed F table.

Part of Hotelling's T^2 Distribution Table

Table A.7. Upper Percentage Points of Hotelling's T^2 Distribution

Degrees of Freedom, ν	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$	$p = 7$
					$\alpha = .05$		
2	18.513						
3	10.128	57.000					
4	7.709	25.472	114.986				
5	6.608	17.361	46.383	192.468			
6	5.987	13.887	29.661	72.937	289.446		
7	5.591	12.001	22.720	44.718	105.157	405.920	
8	5.318	10.828	19.028	33.230	62.561	143.050	541.890
9	5.117	10.033	16.766	27.202	45.453	83.202	186.622
10	4.965	9.459	15.248	23.545	36.561	59.403	106.649
11	4.844	9.026	14.163	21.108	31.205	47.123	75.088
12	4.747	8.689	13.350	19.376	27.656	39.764	58.893
13	4.667	8.418	12.719	18.086	25.145	34.911	49.232
14	4.600	8.197	12.216	17.089	23.281	31.488	42.881
15	4.543	8.012	11.806	16.296	21.845	28.955	38.415
16	4.494	7.856	11.465	15.651	20.706	27.008	35.117
17	4.451	7.722	11.177	15.117	19.782	25.467	32.588
18	4.414	7.606	10.931	14.667	19.017	24.219	30.590
19	4.381	7.504	10.719	14.283	18.375	23.189	28.975

Hotelling's T^2 Test: R Example

Example: Perspiration from 20 healthy female was analyzed. The sweat rate, sodium and potassium content were measured.

```
> sweat.data=read.table(file="T5-1.DAT", header=F, col.names=c("sweat.rate",
+ "sodium", "potassium"))
> sweat.data
   sweat.rate sodium potassium
1         3.7   48.5       9.3
2         5.7   65.1       8.0
3         3.8   47.2      10.9
4         3.2   53.2      12.0
5         3.1   55.5       9.7
6         4.6   36.1       7.9
7         2.4   24.8      14.0
8         7.2   33.1       7.6
9         6.7   47.4       8.5
10        5.4   54.1      11.3
11        3.9   36.9      12.7
12        4.5   58.8      12.3
13        3.5   27.8       9.8
14        4.5   40.2       8.4
15        1.5   13.5      10.1
16        8.5   56.4       7.1
17        4.5   71.6       8.2
18        6.5   52.8      10.9
19        4.1   44.1      11.2
20        5.5   40.9       9.4
```

We want to test the hypothesis H_0: $\boldsymbol{\mu} = (4, 50, 10)$ at significance level $\alpha = 0.1$.

```
> (mean.vect=apply(sweat.data, 2, mean))
sweat.rate     sodium  potassium
     4.640     45.400      9.965
> (cov.matrix=cov(sweat.data))
           sweat.rate   sodium potassium
sweat.rate   2.879368  10.0100 -1.809053
sodium      10.010000 199.7884 -5.640000
potassium   -1.809053  -5.6400  3.627658
> n=dim(sweat.data)[1]
> p=dim(sweat.data)[2]
> mu.0=c(4,50,10)
> (T.sq=n*t(mean.vect-mu.0)%*%solve(cov.matrix)%*%(mean.vect-mu.0))
         [,1]
[1,] 9.738773
> alpha=0.1
> (cut.off=(n-1)*p/(n-p)*qf(1-alpha, p, n-p))
[1] 8.172573
> (p.value=1-pf(T.sq*(n-p)/(n-1)/p, p, n-p))
           [,1]
[1,] 0.06492834
```

Thus we should reject H_0 at $\alpha = 0.1$. However, we fail to reject it at $\alpha = 0.05$.

Hotelling's T^2 Test: Remarks

- The **characteristic form** of Hotelling's T^2 statistic is

$$T^2 = (\bar{\mathbf{y}} - \boldsymbol{\mu}_0)' \left(\frac{\mathbf{S}}{n}\right)^{-1} (\bar{\mathbf{y}} - \boldsymbol{\mu}_0).$$

- We must have $n > p$ for the Hotelling's T^2 test.
- Hotelling's T^2 can be converted to F distribution by

$$\frac{n-p}{p(n-1)} T^2(p, n-1) = F(p, n-p).$$

- As n increases, Hotelling's T^2 approaches χ^2 distribution: $T^2(p, \infty) = \chi^2(p)$. But as p increases, larger values of n are required for T^2 to approach χ^2.
- The likelihood ratio test leads to the Hotelling's T^2 test for multivariate normal samples.

Outline

Univariate Two-sample Tests

About this two-sample comparison problem, we need to consider the following two scenarios:

- Independent samples: Population 1 and 2 are independent
- Paired samples: Population 1 is related to population 2

Univariate Two-sample Tests: Independent Samples

- Assumptions:
  $y_{11}, \ldots, y_{1n_1}$ i.i.d. $N(\mu_1, \sigma^2)$, $y_{21}, \ldots, y_{2n_2}$ i.i.d. $N(\mu_2, \sigma^2)$, with the common unknown variance σ^2. Furthermore, y_{1i} and $y_{2i'}$ are independent, $i = 1, \ldots, n_1$, $i' = 1, \ldots, n_2$.
- Distribution:

$$\frac{(\bar{y}_1 - \bar{y}_2) - (\mu_1 - \mu_2)}{\sqrt{s_{pl}^2(1/n_1 + 1/n_2)}} \sim t(n_1 + n_2 - 2),$$

where s_{pl}^2 is the pooled sample estimate of σ^2:

$$s_{pl}^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2} = \frac{\sum_{i=1}^{2} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2}{n_1 + n_2 - 2}.$$

- Hypotheses:
  - (1) One-sided: $H_0 : \mu_1 = \mu_2$ (or $\mu_1 \leq \mu_2$) vs $H_1 : \mu_1 > \mu_2$.
  - (2) One-sided: $H_0 : \mu_1 = \mu_2$ (or $\mu_1 \geq \mu_2$) vs $H_1 : \mu_1 < \mu_2$.
  - (3) Two-sided: $H_0 : \mu_1 = \mu_2$ vs $H_1 : \mu_1 \neq \mu_2$.
- Test statistic:

$$T = \frac{\bar{y}_1 - \bar{y}_2}{s_{pl}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t(n_1+n_2-2) \text{ when } \mu_1 = \mu_2 \text{ (under } H_0)$$

- Rejection regions:
  - (1) $T > t_\alpha(n_1 + n_2 - 2)$.
  - (2) $T < -t_\alpha(n_1 + n_2 - 2)$.
  - (3) $|T| > t_{\alpha/2}(n_1 + n_2 - 2)$.

Univariate Two-sample Tests: Paired Samples

- Assumptions:
  $y_{11}, \ldots, y_{1n}$ iid with mean μ_1, and $y_{21}, \ldots, y_{2n}$ iid with mean μ_2. y_{1i} is related to/paired with y_{2i}, $i = 1, \ldots, n$.
  $y_{1i} - y_{2i}$ iid $N(\mu_d, \sigma^2)$, where $\mu_d = \mu_1 - \mu_2$.
- Statistics:
  Estimate of μ_d: $\bar{d} = \sum_{i=1}^{n} d_i/n$, where $d_i = y_{1i} - y_{2i}$;
  Estimate of σ^2: $s_d^2 = \sum_{i=1}^{n}(d_i - \bar{d})^2/(n-1)$.
- Distribution:

$$\frac{\bar{d} - \mu_d}{s_d/\sqrt{n}} \sim t(n-1)$$

- Hypotheses:
  (1) One-sided: $H_0 : \mu_d = 0$ (or $\mu_d \leq 0$) vs $H_1 : \mu_d > 0$.
  (2) One-sided: $H_0 : \mu_d = 0$ (or $\mu_d \geq 0$) vs $H_1 : \mu_d < 0$.
  (3) Two-sided: $H_0 : \mu_d = 0$ vs $H_1 : \mu_d \neq 0$.
- Test statistic:

$$T = \frac{\bar{d}}{s_d/\sqrt{n}} \sim t(n-1) \text{ when } \mu_d = 0 \text{ (under } H_0)$$

- Rejection regions:
  (1) $T > t_\alpha(n-1)$.
  (2) $T < -t_\alpha(n-1)$.
  (3) $|T| > t_{\alpha/2}(n-1)$.

Outline

3 Comparing Two Mean Vectors $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$

- Review of Univariate Two-sample Tests
- Multivariate Two-sample T^2 Test: Indepedent Samples
- Multivariate Test for Paired Samples

Multivariate Two-sample T^2 Test

- Assumption:
  The p-variate samples $\mathbf{y}_{11}, \ldots, \mathbf{y}_{1n_1}$ iid $N_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma})$ and $\mathbf{y}_{21}, \ldots, \mathbf{y}_{2n_2}$ iid $N_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma})$, with the common unknown covariance matrix $\boldsymbol{\Sigma}$. $\mathbf{y}_{1i}$ and $\mathbf{y}_{2i'}$ are independent for all $i = 1, \ldots, n_1$ and $i' = 1, \ldots, n_2$.
- Hypothesis:

$$H_0: \ \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2, \text{ vs. } H_1: \ \boldsymbol{\mu}_1 \neq \boldsymbol{\mu}_2.$$

- Test statistic:

$$T^2 = \frac{n_1 n_2}{n_1 + n_2}(\bar{\mathbf{y}}_1 - \bar{\mathbf{y}}_2)' \mathbf{S}_{pl}^{-1} (\bar{\mathbf{y}}_1 - \bar{\mathbf{y}}_2)$$

  where $\mathbf{S}_{pl} = \frac{(n_1-1)\mathbf{S}_1 + (n_2-1)\mathbf{S}_2}{n_1+n_2-2}$ is unbiased estimate of $\boldsymbol{\Sigma}$.

- Characteristic form of T^2:

$$T^2 = (\bar{\mathbf{y}}_1 - \bar{\mathbf{y}}_2)' \left[\left(\frac{1}{n_1} + \frac{1}{n_2} \right) \mathbf{S}_{pl} \right]^{-1} (\bar{\mathbf{y}}_1 - \bar{\mathbf{y}}_2)$$

  Note that $n_1 + n_2 - 2 > p$ for $\mathbf{S}_{pl}$ to be nonsingular.
- Null distribution of T^2:
  $T^2 \sim T^2(p, n_1 + n_2 - 2)$ under H_0.
- Rejection region:
  Reject H_0 if $T^2 > T^2_\alpha(p, n_1 + n_2 - 2)$.

Two-sample T^2 Test: Example

Example: Four statistical tests were given to 32 men and 32 women. Part of data is shown below:

Males				Females			
y_1	y_2	y_3	y_4	y_1	y_2	y_3	y_4
15	17	24	14	13	14	12	21
17	15	32	26	14	12	14	26
15	14	29	23	12	19	21	21
13	12	10	16	12	13	10	16
20	17	26	28	11	20	16	16
15	21	26	21	12	9	14	18
15	13	26	22	10	13	18	24
13	5	22	22	10	8	13	23
14	7	30	17	12	20	19	23
17	15	30	27	11	10	11	27
17	17	26	20	12	18	25	25

$$\bar{\mathbf{y}}_1 = \begin{pmatrix} 15.97 \\ 15.91 \\ 27.19 \\ 22.75 \end{pmatrix}, \qquad \bar{\mathbf{y}}_2 = \begin{pmatrix} 12.34 \\ 13.91 \\ 16.66 \\ 21.94 \end{pmatrix},$$

$$\mathbf{S}_1 = \begin{pmatrix} 5.192 & 4.545 & 6.522 & 5.250 \\ 4.545 & 13.18 & 6.760 & 6.266 \\ 6.522 & 6.760 & 28.67 & 14.47 \\ 5.250 & 6.266 & 14.47 & 16.65 \end{pmatrix},$$

$$\mathbf{S}_2 = \begin{pmatrix} 9.136 & 7.549 & 4.864 & 4.151 \\ 7.549 & 18.60 & 10.22 & 5.446 \\ 4.864 & 10.22 & 30.04 & 13.49 \\ 4.151 & 5.446 & 13.49 & 28.00 \end{pmatrix}.$$

$$\mathbf{S}_{\mathrm{pl}} = \frac{1}{32+32-2}[(32-1)\mathbf{S}_1 + (32-1)\mathbf{S}_2]$$

$$= \begin{pmatrix} 7.164 & 6.047 & 5.693 & 4.701 \\ 6.047 & 15.89 & 8.492 & 5.856 \\ 5.693 & 8.492 & 29.36 & 13.98 \\ 4.701 & 5.856 & 13.98 & 22.32 \end{pmatrix}.$$

$$T^2 = \frac{n_1 n_2}{n_1 + n_2}(\overline{\mathbf{y}}_1 - \overline{\mathbf{y}}_2)'\mathbf{S}_{\mathrm{pl}}^{-1}(\overline{\mathbf{y}}_1 - \overline{\mathbf{y}}_2) = 97.6015.$$

According to the T^2 table, $T^2 > T^2_{0.01,4,62} = 15.373$, and we therefore reject H_0: $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$.

Two-sample T^2 Test: What's Next?

If as the previous example, $H_0: \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ is rejected, what is the next question should you consider?

- Does it mean that the univariate tests $H_0: \mu_{1j} = \mu_{2j}$ will be rejected for all $j = 1, \ldots, p$?
- Does it mean that the univariate tests $H_0: \mu_{1j} = \mu_{2j}$ will be rejected for some $j = 1, \ldots, p$?
- Which variables are significantly different between the two groups?
- How to seek for the variables (or linear combination of variables) contributing most to the separation of the two groups?

Follow-up of Two-sample T^2 Test

Possible procedures that could be used to check each variable following the rejection of $H_0: \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$:

(1) Univariate t test, one for each variable:

$$T_j = \frac{\bar{y}_{1j} - \bar{y}_{2j}}{\sqrt{[(n_1+n_2)/(n_1 n_2)]s_{jj}}},$$

where s_{jj} is the jth diagonal element of $\mathbf{S}_{pl}$. Reject $H_0: \mu_{1j} = \mu_{2j}$ if $|T_j| > t_{\alpha/2}(n_1+n_2-2)$.

(2) Bonferroni correction: Use $t_{\alpha/(2p)}(n_1+n_2-2)$ instead of $t_{\alpha/2}(n_1+n_2-2)$ as the critical value. (Conservative)

(3) Use $T_\alpha(p, n_1+n_2-2) = \sqrt{T_\alpha^2(p, n_1+n_2-2)}$ as the critical value. (Even more conservative than Bonferroni)

Follow-up of Two-sample T^2 Test: Fisher's Discriminant Function Approach

(4) Fisher's discriminant function approach (Chap 5)

- The univariate tests (1)-(3) can detect which variables are significantly different between the two groups.
- However, sometimes insignificantly individual effect may cooperate to produce significantly joint effect (large difference between two groups).
- Then (4) could be conducted to seek for the "best separation rule" between the two groups, i.e. the linear combination of the variables which can separate the two groups the most.

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Multivariate Paired Samples: Example

Example: To compare two types of coating for resistance to corrosion, 15 pieces of pipe were coated with each type. Two pipes, one with each type of coating, were buried together and left for the same length of time at 15 different locations. And corrosion for the first type of coating was measured by

$$y_1 = \text{maximum depth of pit in thousandths of an inch,}$$
$$y_2 = \text{number of pits,}$$

with x_1 and x_2 defined analogously for the second coating. How to compare the two types of coating?

The data and differences are given below:

	Coating 1		Coating 2		Difference	
Location	Depth y_1	Number y_2	Depth x_1	Number x_2	Depth d_1	Number d_2
1	73	31	51	35	22	−4
2	43	19	41	14	2	5
3	47	22	43	19	4	3
4	53	26	41	29	12	−3
5	58	36	47	34	11	2
6	47	30	32	26	15	4
7	52	29	24	19	28	10
8	38	36	43	37	−5	−1
9	61	34	53	24	8	10
10	56	33	52	27	4	6
11	56	19	57	14	−1	5
12	34	19	44	19	−10	0
13	55	26	57	30	−2	−4
14	65	15	40	7	25	8
15	75	18	68	13	7	5

Multivariate Test for Paired Samples

- Data Structure:

Pair Number	Treatment 1	Treatment 2	Difference $\mathbf{d}_i = \mathbf{y}_i - \mathbf{x}_i$
1	$\mathbf{y}_1$	$\mathbf{x}_1$	$\mathbf{d}_1$
2	$\mathbf{y}_2$	$\mathbf{x}_2$	$\mathbf{d}_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\mathbf{y}_n$	$\mathbf{x}_n$	$\mathbf{d}_n$

- Assumption:
  The p-variate samples $\mathbf{x}_i$ and $\mathbf{y}_i$ are paired, $i = 1, \ldots, n$.
  The differences $\mathbf{d}_i = \mathbf{y}_i - \mathbf{x}_i$ iid from $N_p(\boldsymbol{\delta}, \boldsymbol{\Sigma}_d)$, where $\boldsymbol{\delta} = \boldsymbol{\mu}_y - \boldsymbol{\mu}_x$.

- Hypothesis:

$$H_0 : \ \delta = \mathbf{0}, \ \text{vs.} \ H_1 : \ \delta \neq \mathbf{0}.$$

- Statistics:
  Estimate of δ: $\bar{\mathbf{d}} = \sum_{i=1}^{n} \mathbf{d}_i$
  Estimate of $\boldsymbol{\Sigma}_d$: $\mathbf{S}_d = \sum_{i=1}^{n}(\mathbf{d}_i - \bar{\mathbf{d}})(\mathbf{d}_i - \bar{\mathbf{d}})'/(n-1)$.
- Test statistic:

$$T^2 = n\bar{\mathbf{d}}'\mathbf{S}_d^{-1}\bar{\mathbf{d}} \sim T^2(p, n-1) \text{ under } H_0.$$

Remarks:

- When the two samples are paired, misuse of T^2 test for independent samples will often lead to loss of power.
- This paired sample T^2 test can be used for indepedent samples when the covariance matrices are not equal. But then the pairing achieves no gain in power.
- The follow-up univariate tests can be conducted using the test statistics

$$t_j = \frac{\bar{d}_j}{\sqrt{s_{d,jj}/n}}, \ j = 1, \ldots, p.$$

The critical value for t_j is $t_{\alpha/2,n-1}$.

Multivariate Test for Paired Samples: Previous Coating Example

	Coating 1		Coating 2		Difference	
Location	Depth y_1	Number y_2	Depth x_1	Number x_2	Depth d_1	Number d_2
1	73	31	51	35	22	−4
2	43	19	41	14	2	5
3	47	22	43	19	4	3
4	53	26	41	29	12	−3
5	58	36	47	34	11	2
6	47	30	32	26	15	4
7	52	29	24	19	28	10
8	38	36	43	37	−5	−1
9	61	34	53	24	8	10
10	56	33	52	27	4	6
11	56	19	57	14	−1	5
12	34	19	44	19	−10	0
13	55	26	57	30	−2	−4
14	65	15	40	7	25	8
15	75	18	68	13	7	5

Based on the 15 difference vectors,

$$\bar{\mathbf{d}} = \begin{pmatrix} 8.000 \\ 3.067 \end{pmatrix}, \qquad \mathbf{S}_d = \begin{pmatrix} 121.571 & 17.071 \\ 17.071 & 21.781 \end{pmatrix}.$$

Then

$$T^2 = (15)(8.000, 3.067) \begin{pmatrix} 121.571 & 17.071 \\ 17.071 & 21.781 \end{pmatrix}^{-1} \begin{pmatrix} 8.000 \\ 3.067 \end{pmatrix} = 10.819.$$

The critical value $T^2_{0.05}(2, 14) = 8.197$, we should reject H_0 and conclude that the two coatings differ in their effect on corrosion.

The follow-up univariate t tests can then be conducted:

$$t_1 = \frac{\bar{d}_1}{\sqrt{s_{d,11}/15}} = 2.810, \quad t_2 = \frac{\bar{d}_2}{\sqrt{s_{d,22}/15}} = 2.545.$$

Comparing with the critical value $t_{0.025}(14) = 2.145$, we should reject both of the univariate tests, and conclude that the two coatings differ in both the maximum depth of pit and the number of pits.

Outline

Other Topics Based on T^2 Statistics

The idea of T^2 statistics can also be used for

- **Partial T^2 test**: The goal is to determine whether some of the variables are redundant in the presence of other variables in terms of separating the groups - in the spirit of a full and reduced model (lack of fit) test. Refer to section 5.8 in the book.
- **Profile analysis**: When the variables $Y_1, \ldots, Y_p$ in $\mathbf{y}$ are commensurate (e.g. the statistical test data), the pattern obtained by plotting $\mu_1, \ldots, \mu_p$ against the variable index and connecting the points is called a **profile**. We may use T^2 statistics to analyse a profile (e.g. compare the means $\mu_1, \ldots, \mu_p$) or compare two or more profiles. Refer to section 5.9 in the book.

Outline

4 Other Topics

- Other Topics Based on T^2 Statistics
- Tests on Covariance Matrices

Tests on Covariance Matrices

Sometimes we need to check whether the covariance matrices satisfy certain patterns. E.g. to conduct the two-sample T^2 tests, the covariance matrices of the two population need to be assumed equal. Refer to Chapter 7 in the book for the tests on covariance matrices.

Summary and Take-home Messages

- Why do we need multivariate tests?
- How to test $H_0: \boldsymbol{\mu} = \boldsymbol{\mu}_0$ when $\boldsymbol{\Sigma}$ is known?
- What is Hotelling's T^2 test?
- How to test the equality of means of two independent populations $H_0: \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$?
- What if $H_0: \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ is rejected?
- What if the two populations are paired?