

6 Sparks & Generation

6.1 *Electrostatics – when things are still*

The fundamental fact of electrostatics will be familiar to you – opposite charges attract: like charges repel. As a physicist it is not enough to know this, we also want to know how big the force is. It turns out that the equations describing the force, energy, potential and so on are very similar mathematically to the equations describing gravitational attraction.

Using the symbols F for force, U for potential energy, V for potential and E for field, we have the equations:

$$\begin{aligned} F_r &= \frac{1}{4\pi\epsilon_0} \frac{Qq}{R^2} \\ E_r &= \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} \\ U &= \frac{1}{4\pi\epsilon_0} \frac{Qq}{R} \\ V &= \frac{1}{4\pi\epsilon_0} \frac{Q}{R} \end{aligned} \tag{1}$$

Notice that the symbols are slightly different for gravity – field is now given E rather than g , and in consequence we have to use another letter for energy – hence our choice of U . Here we have put a charge Q at the origin, and we measure the quantities associated with a small ‘test’ charge q at distance R . Notice that we do not have a minus sign in front – this allows like charges to repel rather than attract (whereas in gravity, positive mass attracts positive mass [and we have never found any lumps of negative mass – if we did this would upset a lot of our thinking]). Also, in place of the G of gravity, we have the constant $1/4\pi\epsilon_0$ which is about 10^{20} times bigger. No wonder the theoreticians talk of gravity as a weaker force!

Now, you may wonder, why the factor of 4π ? This comes about, because the equations above are not the nicest way of describing electrostatic forces. They are based on the ‘Coulomb Force Law’, which is the first equation in (1) – however there is another, equivalent, way of describing the same physics, and it is called Gauss’ Law of Electrostatics. This Gauss Law is on the Olympiad syllabus, and you will find it useful because it will simplify your electrical calculations a lot.

6.1.1 Gauss Law of Electrostatics

Firstly, let us say what the law is. Then we will describe it in words, and then prove that it is equivalent to the Coulomb Law. Finally, we will show its usefulness in other calculations.

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0} \quad (2)$$

What does this mean? Firstly let us look at the individual symbols. S is a closed surface (that is what the circle on the integral sign means) – like the outer surface of an apple, a table, or a doughnut – but not the outer surface of a bowl (which ends at a rim). $d\mathbf{S}$ is a small part of the surface, with area dS , and is a vector pointing outward, perpendicular to the surface at that point. Q is the total electric charge contained inside the surface S . Finally, the vector \mathbf{E} is the electric field (in volts per metre) – where the vector points in the direction a positive charge would be pulled.

The odd looking integral tells us to integrate the dot product of \mathbf{E} with $d\mathbf{S}$ (a normal vector to the surface) around the complete surface. This may sound very foreign, strange, and difficult, but let us give some examples.

First of all, suppose S is a spherical surface of radius R , with one charge ($+Q$) at the centre of the surface. Assuming there are no other charges nearby, the field lines will be straight, and will stream out radially from the centre. Therefore \mathbf{E} will be parallel to $d\mathbf{S}$, and the dot product will simply be $E dS$ – the product of the magnitudes. Now the size of E must be the same all round the surface, by symmetry. Therefore

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \oint_S E dS = E \oint_S dS = E \times 4\pi R^2 \quad (3)$$

since the surface area of a sphere is given by $4\pi R^2$. Now by Gauss' Law, this must equal Q/ϵ_0 . Putting the two equations together gives us

$$4\pi R^2 E = \frac{Q}{\epsilon_0} \Rightarrow E = \frac{Q}{4\pi\epsilon_0 R^2} \quad (4)$$

in agreement with Coulomb's Law.

Similarly we may find the field at a distance R from a **wire** that carries a charge λ per metre – spread evenly along the wire – something which Coulomb's Law *could* do, but would need a horrendous integral to do it.

This time, our surface is a cylinder, one metre long, with the wire running down its axis. The cylinder has radius R . First, notice that the field lines will run radially out from the wire. This has the consequence that the two

flat ends of the cylinder *do not count* in the integral. Think about this for a moment, because this is important. The vector $d\mathbf{S}$ for these ends will point normal to the surface – that means parallel to the wire. The vector \mathbf{E} , on the other hand points outward. Therefore \mathbf{E} is perpendicular to $d\mathbf{S}$, and the dot product is zero.

Only the curved surface counts. Again, \mathbf{E} will have the same magnitude at all points on it because of symmetry, and \mathbf{E} will be parallel to $d\mathbf{S}$ on this surface. Once again, we have

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \oint_{\text{curved } S} E dS = E \oint_{\text{curved } S} dS = E \times 2\pi RL \quad (5)$$

where L is the length of the cylinder, and hence $2\pi RL$ is its curved surface area. By Gauss' Law, this must be equal to $Q/\epsilon_0 = \lambda L/\epsilon_0$, and so

$$E = \frac{\lambda}{2\pi\epsilon_0 R}. \quad (6)$$

We notice that for a line charge, the “inverse square” of the Coulomb Law has become an “inverse, not squared” law.

Before moving on, let us make two more points. Firstly, we have not *proved* the equivalence of Gauss' Law and Coulomb's Law – we have only shown that they agree in the case of calculating the field around a fixed, point charge. However the two can be proved equivalent – but the proof is a bit involved, and is best left to first (or second) year university courses.

Secondly, let us think about a surface S which is entirely inside the same piece of metal. \mathbf{E} will be zero within a metal, because any non-zero \mathbf{E} (i.e. voltage difference) would cause a current to flow until the \mathbf{E} were zero. Therefore a surface entirely within metal can contain no total charge!

Impossible, I hear you cry! Let us take a hollow metal sphere, with a charge $+Q$ at the centre of the cavity. How can the enclosed charge be zero – surely it's $+Q$! Oh, no it isn't. Actually the *total* enclosed charge is zero, and this enclosed charge is made up of $+Q$ at the middle of the cavity, and $-Q$ induced on the inside wall of the hollow sphere! If the sphere is electrically isolated, and began life uncharged, there must be a $+Q$ charge somewhere on the metal, and it sits on the outer surface of the sphere.

6.1.2 Capacitors

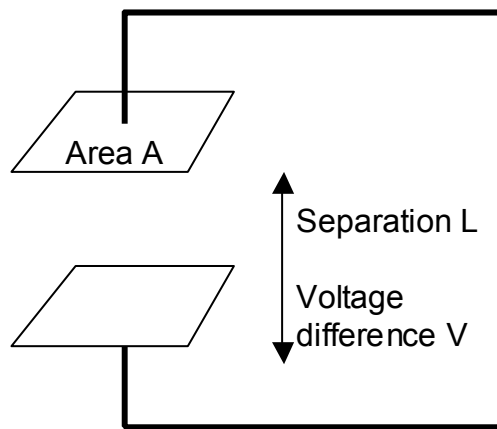
A capacitor is a device that stores charge. To be more precise, a capacitor consists of two conducting plates, with insulating space

between them. When the positive plate carries charge $+Q$, an equal amount of negative charge is stored on the other. If certain insulating materials are used to separate the plates (instead of air or vacuum), the amount of charge that can be stored increases considerably. The charge stored is proportional to the potential difference across it, and we call the constant of proportionality the capacitance.

Gauss' Law gives us a wonderful way to calculate the capacitance of simple capacitors, and we will look at the calculation for a parallel plate capacitor.

6.1.2.1 Parallel Plate Capacitor

At its simplest, a capacitor is shown in the figure below. The two plates are square, and parallel. Each has area A and the distance between them is denoted L . Let's work out the capacitance. To do this, we first suppose that there were a charge Q stored. In other words, there is a charge $-Q$ on the top plate, and $+Q$ on the bottom plate. We can work out the electric field in the gap using Gauss' Law. We draw a rectangular box-shape surface, with one of its faces parallel to, but buried in the bottom plate, and the opposite face in the middle of the gap.



When working out the surface integral $\oint_S \mathbf{E} \cdot d\mathbf{S}$, only this face in the middle of the gap counts. The face buried in the metal of the plate has $\mathbf{E}=\mathbf{0}$, while the other four surfaces' normals are perpendicular to the field. Therefore

$$\frac{+Q}{\epsilon_0} = \oint_S \mathbf{E} \cdot d\mathbf{S} = AE. \quad (7)$$

We next work out the voltage. This is not hard, as by analogy from gravitational work (chapter 1, equation 16)

$$\text{Field} = -d(\text{Potential})/d(\text{distance})$$

$$E = -dV/dx \quad (8)$$

Here, E is constant and uniform throughout the inter-plate gap, and so $V=Ex+c$ where c is a constant of integration. Thus the voltage difference between the plates can be calculated; and from this the capacitance can be worked out.

$$\Delta V = EL = \frac{QL}{\epsilon_0 A} \Rightarrow C = \frac{Q}{\Delta V} = \frac{\epsilon_0 A}{L} \quad (9)$$

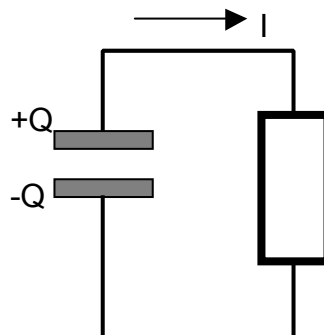
We ought to give a word of caution at this point. In a real parallel plate capacitor, the field near the edge of the plates will not point directly from one plate to the other, but will 'bow out' a bit. Therefore the equation given above is only true when these 'edge effects' are ignored. It turns out that the equation is pretty good providing that L is much smaller than both of the linear dimensions of the plates.

The equation also allows you to see the effects of wiring capacitors in series or parallel. When two identical capacitors, each of capacitance C , are connected together in parallel, the overall area A is doubled, so the capacitance of the whole arrangement is $2C$. On the other hand if the capacitors are connected in series, the result is one capacitor with twice the gap thickness L . Therefore the overall capacitance is $C/2$.

6.1.2.2 Decay of Charge on Capacitor

We next come to the case where a capacitor is charged to voltage V (that is, a voltage V across the plates), and then connected in a simple circuit with a resistor R . How long will it take to discharge?

To work this out, we need to use our characteristic equations for capacitor and resistor. For the capacitor $V=Q/C$, for the resistor $V=IR$. To solve the circuit we need to clarify the relationship between Q and I .



Here we need to take care. Depending on how the circuit has been drawn, either $I=dQ/dt$ or $I=-dQ/dt$. That is why it is essential that you

include in your circuit diagrams arrows to show the direction of current flow for $I > 0$, and which plate of the capacitor is the +ve one. Here a positive I will discharge the capacitor, so $I = -dQ/dt$.

The voltage across the capacitor is the same as that across the resistor, so

$$V = IR = -R \frac{dQ}{dt} = \frac{Q}{C} \quad (10)$$

$$\frac{dQ}{dt} = -\frac{Q}{RC}$$

This differential equation can be solved with an exponential solution:

$$Q(t) = Q_0 e^{-t/RC} \quad (11)$$

where Q_0 was the initial charge on the capacitor after it was charged up. Given that the voltage across the capacitor $V(t) = Q(t)/C$, the time dependence of the voltage obeys a similar equation. The constant RC is known as the time constant, and it is the time taken for the voltage (or charge) to fall by a factor e (approx 2.7).

6.1.2.3 Energy considerations

Next, we need to know how much energy has been stored in a capacitor, if its voltage is V and its capacitance C . The energy is actually 'stored' in the electric field between the plates – but more of this later.

To work the energy out, we charge a capacitor up from scratch (initial charge = 0), and continually measure the current flowing, and the voltage across it. The energy stored must be given by

$$U = \int P dt = \int VI dt = \int V \frac{dQ}{dt} dt = \int V \frac{d(CV)}{dt} dt \quad (12)$$

$$= \int VC \frac{dV}{dt} dt = \int CV dV = C \int V dV = C \left[\frac{1}{2} V^2 \right] = \left[\frac{1}{2} CV^2 \right]$$

Given that the energy stored must be zero when $V=0$, and there is no electric field in the gap, this fixes the constant of integration as zero, and we obtain the fact that energy stored = half the capacitance \times the square of the voltage across the gap. You could equally well say that the energy is given by half the charge multiplied by the voltage.

Before we leave this formula, let us do some conjuring tricks with this energy, assuming that the capacitor is a simple parallel plate device:

$$U = \frac{1}{2} CV^2 = \frac{\epsilon_0 A V^2}{2L} = \frac{\epsilon_0 A (EL)^2}{2L} = \frac{1}{2} \epsilon_0 E^2 \times AL \quad (13)$$

thus the energy stored per unit volume of gap is $\frac{1}{2}\epsilon_0 E^2$. Although we have only shown this to be true for a perfect parallel plate capacitor, it is possible to make any electric field look like rows and columns of parallel plate capacitors arranged like a mosaic, and from this the proof can be generalized to all electric fields.

6.1.2.4 Polarization

When we introduced capacitors, we mentioned that the capacitance can be raised by inserting insulating stuff into the gap. How does this work? Look at the diagram below. The stuff in the middle contains atoms,



which have positive and negative charges within them.

When the plates are charged, as shown, this pulls the nuclei of the stuff to the right, and the electrons of the atoms to the left. The left

plate now has a blanket of negative charge, and the right plate has a blanket of positive charge. This reduces the overall total charge on the plates, and therefore reduces the voltage across the capacitor. Of course the circuit can't remove the "polarization" charges in the 'blankets' – as that would require the chemical breakdown of the substance. So we have stored the same 'circuit charge' on the capacitor for less voltage, and so the capacitance has gone up. The ratio by which the capacitance increases is called the relative permittivity of the substance (it used to be called the dielectric constant), and is given the symbol ϵ_r . In the presence of such a material, the ϵ_0 of all the equations derived so far in this chapter needs to be multiplied by ϵ_r .

6.2 Magnetism – when things move

So far, we have just considered electric charges at rest. Our next job is surely to look at electric charges that have gone roaming, and then to study magnetism – two things still to do? No. Actually we only have one job, because magnetism *is* all about moving charges.

We ought to give one warning, though. Just because magnetism is about moving charges, we can't derive its formulae simply from Coulomb's Law and Classical mechanics. We need Relativistic mechanics! That is actually one route into relativity – it is the kind of mechanics needed if electricity and magnetism are to be described together. Put another way, your nearest piece of evidence for special relativity is not in a particle accelerator or airborne atomic clock, but in your wrist-watch (if it has hands), credit card, vacuum cleaner, fridge, CD player, printer, hard disk drive, or wherever your nearest magnet is.

We shall demonstrate this at the end of the chapter. However, for the moment, let us stick to what we need for the Olympiad, and let us carry on calling it “magnetism” as opposed to “relativistic electricity”.

6.2.1 Magnetic Flux Density

If there is a magnetic field, there must be a measurement of the field strength, and we call this the flux density, and give it the symbol **B**. The field has a direction (from North to South), and is therefore a vector. The fundamental fact of magnetism can be stated in two ways:

1. If a wire of length **L** is carrying current **I**, and the wire is in a magnetic field **B**, it will experience a force **F**, where $\mathbf{F} = L \mathbf{I} \times \mathbf{B}$. Written without the vector cross product, this is $F = B I L \sin \theta$, where θ is the angle between the direction of the current, and the direction of the magnetic field.

2. If a charged particle, of charge **Q** is moving in a magnetic field **B**, and it has velocity **u**, it will experience a force **F**, where $\mathbf{F} = Q \mathbf{u} \times \mathbf{B}$. Written without the vector cross product, this is $F = Q u B \sin \theta$ where θ is the angle between the direction of motion, and the direction of the magnetic field.

You can show that these descriptions are equivalent, by imagining the wire containing **N** charges (each **Q** coulombs) per metre. If the wire has length **L**, the total charge is $Q_{\text{TOT}} = NQL$, and this moves when the current is flowing. If the current is **I**, this means that the charge passing a point in one second is **I**, and hence $I = NQu$, where **u** is the speed. Therefore

$$\begin{aligned} \mathbf{F} &= L \mathbf{I} \times \mathbf{B} \\ &= L (NQu) \times \mathbf{B} \\ &= (NQL) \mathbf{u} \times \mathbf{B} \\ &= Q_{\text{TOT}} \mathbf{u} \times \mathbf{B}. \end{aligned} \tag{14}$$

6.2.2 Doing the Corkscrew

Now that we have an expression for the force on a charge moving in a magnetic field, we can work out the motion if a charge is thrown into the vicinity of a magnet.

The most important fact is that the force is always at right angles to the velocity. Therefore it never does any work at all, and it never changes the kinetic energy (hence speed) of the object.

The next important fact is that if the velocity is parallel to the magnetic field, there is no force – and the particle will just carry on going: as if the magnet weren’t there at all.

This second fact is useful, because any velocity can be broken down (or resolved) into two components – one parallel to the magnetic field ($u \cos\theta$), and one perpendicular to it ($u \sin\theta$). The component parallel to the field will be unchanged by the motion – it will stay the same, just as Newton's First Law requires.

We next need to calculate what the effect of the other component will be. This will cause a force perpendicular to both the velocity and magnetic field, and we already know from classical mechanics that when a force consistently remains at right angles to the motion, a circular path is obtained. We can calculate the radius from the equations of circular motion:

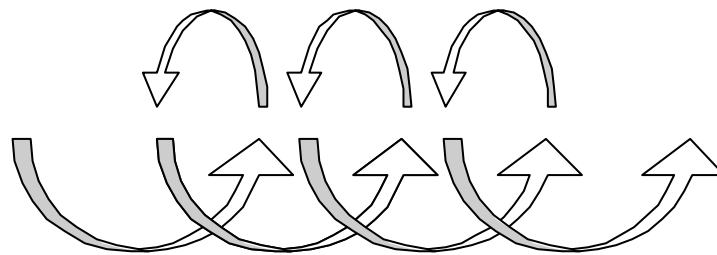
Magnetic Force = Mass \times Centripetal Acceleration

$$BQu \sin \theta = \frac{m(u \sin \theta)^2}{R} \Rightarrow R = \frac{mu \sin \theta}{BQ} \quad (15)$$

Another useful measurement is the time taken for the particle to 'go round the loop' once. Since its rotational speed is $u \sin\theta$, the time taken to go round a $2\pi R$ circumference is $T = 2\pi R / u \sin \theta$. We can also work out the angular velocity:

$$\omega = \frac{u \sin \theta}{R} = \frac{BQ}{m} \quad (16)$$

The overall motion is therefore a helix, or corkscrew shape, with the axis of the corkscrew parallel to the magnetic field. The radius of the helix is given by R in equation (15), and the 'pitch' (that is, the distance between successive revolutions), is equal to $D = T u \cos\theta$.



Helix, or corkscrew motion of an electron in a magnetic field.

Please notice that all these formulae remain valid when the particle starts going very quickly. The only correction that special relativity requires is that we use the enhanced mass $m = \gamma m_{rest}$. No further

correction is needed, because the speed remains constant, and therefore γ does not change.

6.2.3 Calculation of magnetic field strengths

So far, we have just thought about the effects of a magnetic field. However before you can study what an electron will do in such a field, you have to make the field! How do you do that?

At its simplest, magnetic fields are caused by electric currents. The bigger the current: the bigger the field. These may be 'real' currents of electrons in wires, or they can be the effective currents of electrons 'orbiting' their nuclei in atoms. The latter is responsible for the permanent magnetic property of iron (and some other metals) – however the process is quite involved and needs no further consideration for the Olympiad.

We do need to worry about the magnetism caused by wires, however, although only a brief description is necessary. The Olympiad syllabus precludes questions that involve large amounts of calculus (hence integration), and most field calculations require an integral. Therefore if you need to know how big a magnetic field is, you will probably be given the equation you need.

Nevertheless it will help to see how the calculations are done. There is a method akin to Coulomb's Law, and an alternative called Ampere's Law. We shall introduce both.

6.2.3.1 Magnetic Coulomb – The Biot Savart Law

To use this method, the wire (carrying current I) is broken down into small lengths (dL), linked head-to-tail. To work out the size of the magnetic field at point r , we sum the effects of all the current elements. Let us take one current element at point s . Its contribution to the B field at r is:

$$d\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{I} \times (\mathbf{r} - \mathbf{s})}{|\mathbf{r} - \mathbf{s}|^3} dL \quad (17)$$

where $(\mathbf{r} - \mathbf{s})$ is the vector that points from the bit of wire to the point at which we are calculating \mathbf{B} . To work out the total field $\mathbf{B}(\mathbf{r})$, we integrate the expression along the wire. In most cases, this can be put more simply as

$$dB = \frac{\mu_0}{4\pi} \frac{I \sin \theta}{d^2} dL \quad (18)$$

where d is the distance from wire element to point of measurement, and θ is the angle between the current flow and the vector from wire to measurement point.

It is possible to use this expression to calculate the magnetic field B on the axis of a coil of wire with N turns, radius R , carrying current I , if we measure the field at a distance D from the central point:

$$B = \frac{\mu_0 INR^2}{2(R^2 + D^2)^{3/2}} \quad (19)$$

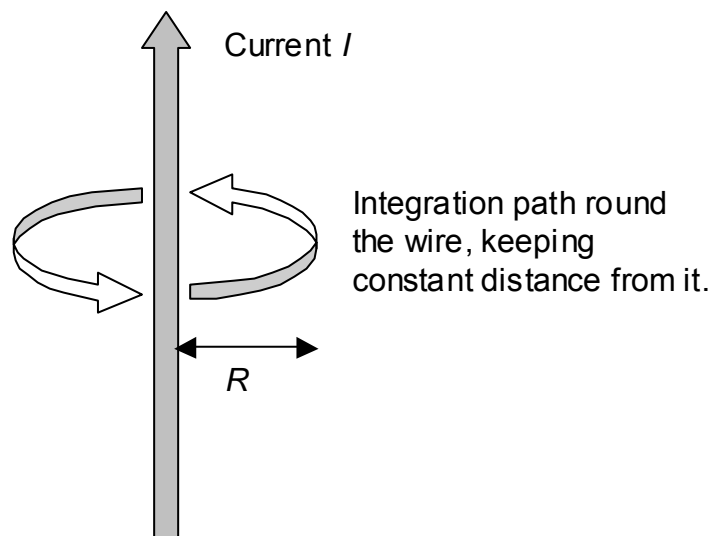
6.2.3.2 The Ampere Law

The alternative method of calculating fields is called the Ampere Law. You remember that Gauss' electrostatic law involved integrating a dot product over a surface? Well, the Ampere Law involves integrating a dot product along a line:

$$\oint_L \mathbf{B} \cdot d\mathbf{L} = \mu_0 I \quad (20)$$

In other words, if we choose a loop path, and integrate the magnetic flux density around it, we will find out the current enclosed by that loop.

Let us give an example. This formula is very useful for calculating the magnetic field in the vicinity of a long straight wire carrying current I . Suppose we take path L to be a circle of radius R , with the wire at its centre, and with the wire perpendicular to the plane of the circle, as shown in the diagram.



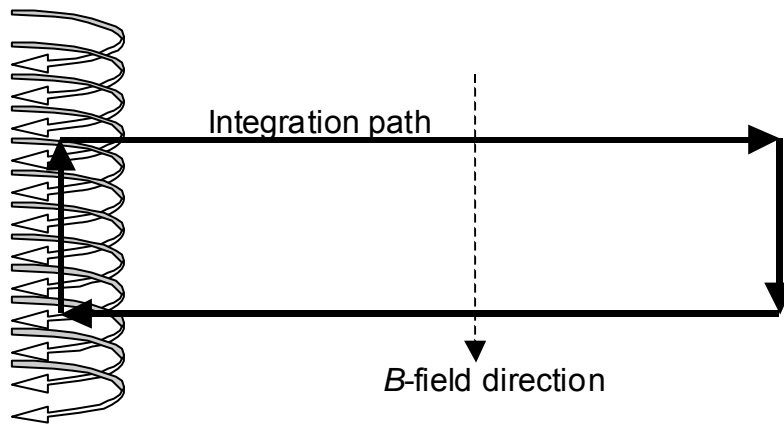
We know that \mathbf{B} points round the wire, and therefore that \mathbf{B} is parallel with $d\mathbf{L}$ ($d\mathbf{L}$ being the vector length of a small part of the path).

Furthermore, $B=|\mathbf{B}|$ must be the same all round the path by symmetry. Therefore:

$$\oint_L \mathbf{B} \cdot d\mathbf{L} = \oint_L B dL = B \oint_L dL = 2\pi R B = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi R} \quad (21)$$

Another splendid use of the Ampere Law is in calculating the magnetic field within the middle of a long solenoid with n turns per metre. This time we use a rectangular path, as shown in the diagram.



Only one of the sides of the rectangle “counts” in the integration – the one completely inside. Of the other three, one is so far away from the coil that there is no magnetic field, and the other two are perpendicular to the \mathbf{B} -field lines, so the dot product is zero. The rectangle encloses nL turns, and hence a current of nLI . Therefore, Ampere’s Law tells us:

$$BL = \mu_0 nLI$$

$$B = \mu_0 nI \quad (22)$$

Before leaving Ampere’s Law, we ought to give a word of caution. The Ampere Law is actually a simplified form of an equation called the Ampere-Maxwell Law²²; and the simplification is only valid if there are no

²² For the curious, the full Ampere-Maxwell Law states $\oint_L \mu_0^{-1} \mathbf{B} \cdot d\mathbf{L} = I + \iint_S \epsilon_0 \epsilon_{rel} \dot{\mathbf{E}} \cdot d\mathbf{S}$

where the surface S is any surface that has as its edge the loop L . This reduces to equation (20) when \mathbf{E} is constant.

changing *electric* fields in the vicinity. Therefore you would be on dodgy ground using Ampere's Law near a capacitor that is charging up!

6.2.4 Flux, Inductance & Inductors

To sum up the last section: if there is a current, there will be a magnetic field. Furthermore, the strength of the magnetic field is proportional to the size of the current. It turns out, however, to be more useful to speak of the *magnetic flux* Φ . This is the product of the field strength B and the cross sectional area of the region enclosed by the magnetic field lines. We visualize this as the total 'number of field lines' made by the magnet.

We then write

$$\Phi = LI. \quad (23)$$

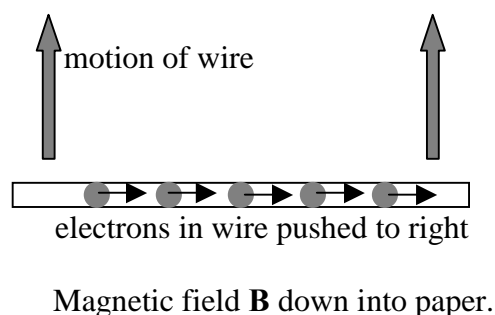
The total amount of magnetic field (the flux) is proportional to the current, and we call the constant of proportionality L – the self-inductance (or inductance for short). Any coil (or wire for that matter) will have an inductance, and this gives you an idea of how much magnetic field it will make when a current passes. You might think of an analogy with capacitors – the capacitance gives a measure of how much electric field a certain charge will cause (since $C^{-1} = V/Q$ and V is proportional to E).

Now, this magnetic field is important, because a changing current will cause a changing magnetic field, and this will generate (or induce) a voltage, and therefore upset the circuit it is in. This is something we need to understand better – but before we do so, let us remind ourselves of the laws of electromagnetic induction:

6.2.5 Generators & Induction

If a wire 'thinks' it is moving magnetically, a voltage is induced in it. It doesn't matter whether the wire is still, and the magnetic field is moving or changing; or whether the wire is moving and the magnetic field is still. However for the two situations, different equations are used. The two equations are equivalent – however it is easier to remember them both than to prove the equivalence.

6.2.5.1 Stationary field: Moving wire



Here the voltage is easy to calculate. The electrons in the wire must move with the wire, and if the wire is in a magnetic field, they will experience a force pushing them along the wire. This causes them to bunch up at one end of the wire – which in turn sets up an electric field which discourages further electrons to join the party. Assuming that the wire has length L , and is being moved at speed u through a uniform magnetic field \mathbf{B} (perpendicular to u and L – if \mathbf{B} is not so inclined, take only the component of \mathbf{B} which is), once equilibrium is established

Electric force balances magnetic force

$$\begin{aligned} qE &= quB \\ \frac{V}{L} &= uB \\ V &= LuB \end{aligned} \tag{24}$$

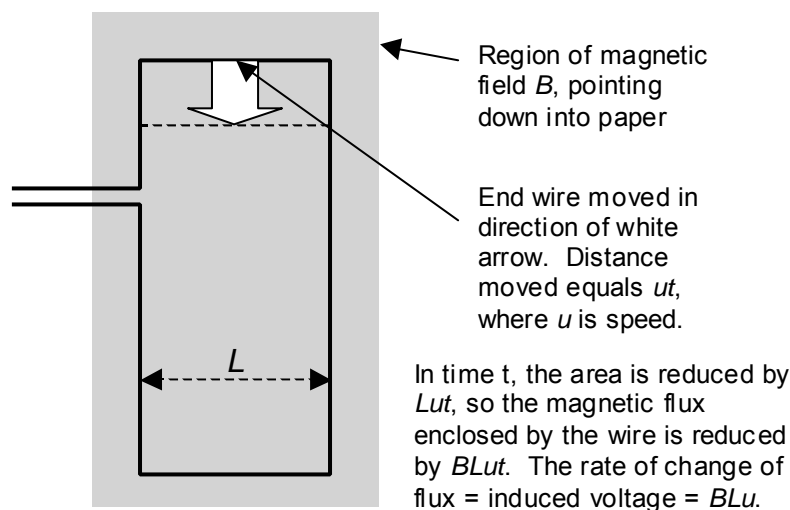
6.2.5.2 Stationary wire: Changing field

Here the voltage induced across the ends of a circular loop of wire (complete circuit, apart from the small gap) is given by

$$V = \frac{d}{dt} \iint_s \mathbf{B} \cdot d\mathbf{S} = \frac{d\Phi}{dt} \tag{25}$$

6.2.5.3 Equivalence?

The two expressions are very similar, as we shall see when we look at the first from a different perspective. Look at the diagram – we have completed a loop by using a very long wire.



The total flux enclosed in the loop is equal to $B \times \text{Area}$. In one second, the Area changes (decreases) by Lu , and hence the flux changes by BLu , and so we see that the first equation is a special case of the second. The second is better as it also allows us to calculate what will happen if the wire is stationary.

6.2.5.4 *Direction of Induced Voltage*

When equation (25) is written down, it is customary to put a minus sign in front of the derivative. The significance of this negation was Lenz's discovery. When the voltage is induced in a complete circuit, it will try to (and succeed in) driving a current. This current will produce a magnetic field. Lenz postulated that this 'produced' magnetic field always opposed the change being made.

Let us have an example. Imagine a large coil of wire (say, in a motor), with a decent sized current flowing in it. Now let us try and lower the current by reducing the voltage of the supply. This causes a reduction in the magnetic field, which in turn induces a voltage in the wire, which pushes a current in a desperate attempt to keep the original current going. On the other hand, if I were to try and increase the current in the motor (by increasing the supply voltage), the opposite would happen: the greater current causes the magnetic field to grow, which induces a voltage, which pushes a current to oppose this increase.

This is the origin of the phrase "back emf" to refer to the voltage induced across an inductor.

Now for a word about the minus sign. Yes, the voltage does go in opposition to the change in current, so I suppose one ought to write equation (25) with a minus sign. However if you do, please also write Ohm's law as $V = -IR$, since the voltage opposes the current in a resistor. I would prefer it, however, if you used common sense in applying your notation and were not stuck in ruts of "always" or "never" using the minus sign. We all remember which way V and I go in a resistor without being nagged about conventions, so I hope that there is no need for me to nag you when inductors come on the scene.

6.2.6 **Inductors in circuits**

Just as a capacitor requires an energy flow to change the voltage across it, an inductor requires an energy flow to change the current through it. It doesn't give in without a fight.

Let me illustrate this with a demonstration – or at least the story of one. A nasty physics teacher (yes, they do exist...) asked a pupil to come and hold one wire in his left hand, and one in his right – completing the circuit with his body. He grasped the first wire, and then the second – steeling himself for the shock which never came. The teacher then stalled him with questions, and kept him there, while he surreptitiously, slowly increased the current in the circuit, which also included an inductor.

Finally, the teacher said, “OK, now you can go back to your seat.” The unsuspecting pupil let go of the wires suddenly, and was very surprised when the inductor – indignant to have its current shut off so quickly – made its displeasure known with an arc from the wire to the pupil. It is foolish in the extreme to starve an inductor of its current. Its revenge will be short, but not sweet.

To see this, let us combine equations (23) and (25)

$$V = \frac{d\Phi}{dt} = \frac{d}{dt} LI = L \frac{dI}{dt} \quad (26)$$

Equation (26) is the definitive equation for inductors, just like $Q=CV$ was for capacitors, and $V=IR$ is for a resistor.

Again, we need to bear in mind the comments above, that the voltage is in the direction needed to oppose the change in current. You will often see (26) with a minus sign in it for that reason.

Let us now calculate how much energy is stored in the device (actually in its magnetic field)

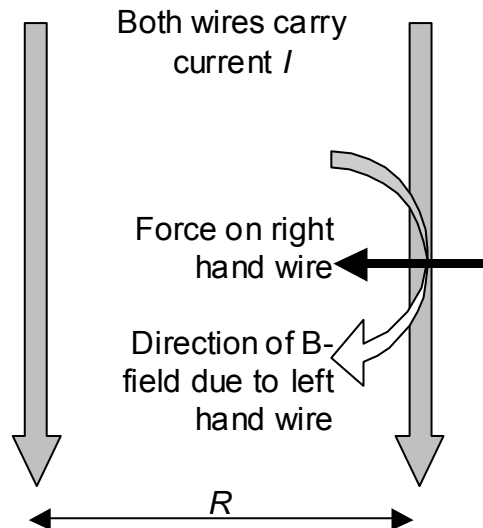
$$U = \int P dt = \int VI dt = \int L \frac{dI}{dt} I dt = L \int I dI = \left[\frac{1}{2} LI^2 \right] \quad (27)$$

Since it seems sensible for the device to hold no energy when there is no current and no field, we take the constant of integration to be zero.

6.2.7 Relativity and Magnetism

At the beginning of the chapter we stated boldly that magnetism could be derived entirely from electrostatics and special relativity. Now is the time to justify this. We shall do so by deriving the same result two ways – once using magnetism, and once using relativity.

The phenomenon we choose is the mutual attraction of two parallel wires carrying equal current in the same direction, and we shall calculate the attractive force per metre of wire.



6.2.7.1 A Classical Magnetic calculation

Equation (21) tells us that the magnetic field at a distance R from a straight infinite wire is

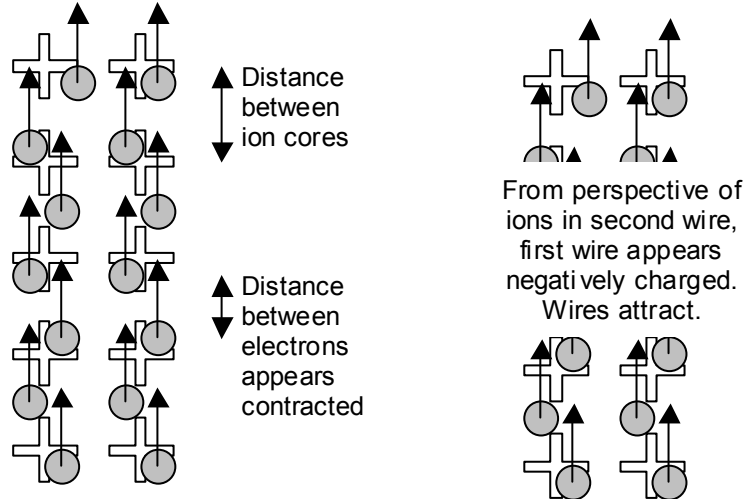
$$B = \frac{\mu_0 I}{2\pi R}$$

Therefore, the attractive force experienced by one metre of parallel conductor also carrying current I is

$$\frac{F}{L} = IB = \frac{\mu_0 I^2}{2\pi R} \quad (28)$$

6.2.7.2 A Relativistic calculation

Each wire contains positive ion cores (say Cu^+ for a copper wire), and free electrons. Let us imagine the situation in the diagram below, with conventional current flowing downwards in both wires. The ion cores are stationary, while the electrons move upwards. If the free (electron) charge per metre of cable is called λ_0 , then the current is related to the electron speed u by the equation $I = \lambda_0 u$.



Let's imagine the situation as perceived by an ion core in the second (right hand) wire. It sees the ion cores in the other wire stationary, with charge density λ_0 and finds them repulsive. However it also sees the electrons on the other wire, and is attracted by them. The electrons are travelling, and therefore our observant ion core sees the length between adjacent electrons contracted. Therefore as far as it is concerned, the electron charge density is higher than the ion core charge density by factor $\gamma = (1 - u^2/c^2)^{-1/2}$. Therefore its overall impression is attraction – with a total effective electric field (as derived in equation 6)

Total field = Field due to ion cores + Field due to loose electrons

$$\begin{aligned}
 E &= \frac{\lambda_0}{2\pi\epsilon_0 R} - \frac{\gamma\lambda_0}{2\pi\epsilon_0 R} \\
 &= -(\gamma - 1) \frac{\lambda_0}{2\pi\epsilon_0 R} \\
 &\approx -\left(\frac{u^2}{2c^2}\right) \frac{\lambda_0}{2\pi\epsilon_0 R}
 \end{aligned} \tag{29}$$

where the final stage has made use of a Binomial expansion of $(\gamma-1)$ to first order in u/c . Now the total charge of ion cores experiencing this field per metre is of course λ_0 . Therefore the total attraction of the ion cores in the second wire to the first wire (electrons & ion cores) is

$$\begin{aligned}
 F_{ions} &= QE \approx -\left(\frac{u^2}{2c^2}\right) \frac{\lambda_0^2}{2\pi\epsilon_0 R} \\
 &= -\frac{I^2}{4c^2 \pi\epsilon_0 R}
 \end{aligned} \tag{30}$$

This is the force experienced by the ion cores in the right hand wire. By an exactly equivalent argument, the electrons in the right hand wire see their counterparts in the left wire as stationary, and see the left hand ion cores bunched up, and therefore more attractive. Therefore the electrons in the right hand wire experience an equal attraction, and the total attractive force between the wires is twice the figure in equation (30).

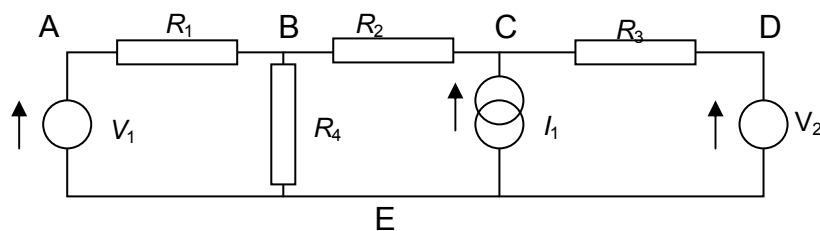
Finally, if you get a book of physical constants and a calculator, you will discover that $\mu_0 \epsilon_0 = c^{-2}$. Therefore the total force agrees exactly with our magnetic calculation in equation (28).

6.3 Circuits – putting it together

In this section, we look at combining resistors, capacitors and inductors in electrical circuits. There are two reasons for doing this. Firstly, once you have left school, you will be faced with complicated electronic networks, and you need to be able to analyse these just as well as the simple series and parallel arrangements you dealt with in the classroom. Secondly, engineers frequently use electric circuits as models or analogies for other systems (say, an oscillating bridge or the control of the nervous system over the muscles in a leg) – the better you understand electric circuits, the better you will understand any linked system.

6.3.1 Circuit Analysis

Our aim here is to be able to solve a circuit like the one below. The circles represent constant-voltage sources (a bit like cells or batteries) and the linked circles represent constant-current sources. Our aim is to find voltage difference across each component, and also to work out the current in each resistor.



In order to solve the circuit, we use two rules – the Kirchoff Laws. Kirchoff's 1st says that the total current going into a junction is equal to the total current leaving it. Therefore, at B in the circuit below, we would say that $I_{BE} = I_{AB} + I_{CB}$, where I_{BE} means the current flowing from B to E (through R_4).

Kirchoff's 2nd Law is that voltages always add up correctly. In other words, no matter which route we took from E to B, say, we would agree on the voltage difference between E and B. In symbols, if V_{BE} means

the difference in potential (as measured by a voltmeter) between B and E, then we have $V_{BE} = V_{AE} + V_{BA}$. This is basically the same thing as the law of conservation of energy. The voltage (or, more strictly, the *potential*) at B, V_B , is the energy content of one coulomb of charge at B. In travelling to E, it will lose $V_B - V_E$ joules, irrespective of the route taken.²³ In fact, we assume the truth of Kirchoff's 2nd Law whenever we say, "let's call the voltage at A ' V_A ,'" for we are assuming that the voltage of A does not depend on the route used to measure it.

Using these two rules, and the equation for the current through a resistor (for example, $V_{BA} = I_{BA} R_1$), we may write down a set of equations for the circuit. Notice that because currents are said to go from + to –, this means that if V_{AB} (the voltage of A, measured relative to B) is positive, then V_A is bigger than V_B , and hence I_{AB} will be positive too. To make the notation easier we will take the potential at E to be zero. In symbolic form, this means that we shall call V_{BE} (that is, $V_B - V_E$) V_B for short.

Kirchoff's First Law:

$$I_{EA} = I_{AB}; \quad I_{BE} = I_{AB} + I_{CB}; \quad I_1 = I_{CB} + I_{CD}; \quad I_{CD} = I_{DE}$$

Kirchoff's Second Law:

$$\begin{aligned} V_B &= I_{BE} R_4 \\ &= V_A + V_{BA} = V_1 - I_{AB} R_1 \\ &= V_D + V_{CD} + V_{BC} = V_2 + I_{CD} R_3 - I_{CB} R_2 \end{aligned}$$

After elimination, the equations reduce to two:

$$V_1 - I_{AB} R_1 = V_2 + I_1 R_3 - (R_3 + R_2) I_{CB} = (I_{AB} + I_{CB}) R_4,$$

and from these the currents I_{AB} and I_{CB} can be found (after a bit of messy algebra). After this, the remaining currents and voltages are straightforward to determine.

These principles can be used to solve any circuit. However, as networks get bigger, it is useful to find more prescriptive methods of solution, which could be used by a computer. We shall cover two methods here – for certain problems, they may be more efficient than the direct application of Kirchoff's Laws.

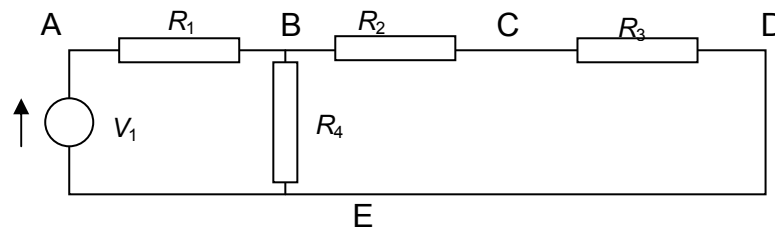
²³ To see why the Law of Conservation of Energy is involved, let us suppose that our coulomb of charge would lose 5J going from B to E via A, whereas it would lose 3J in going directly. All it would have to do is go direct from B to E, then back to B via A and it would be back where it started, having gained 2J of energy! This is not allowed.

6.3.1.1 Method of Superposition

The method of superposition relies on the fact that for a simple resistor, the current is proportional to the voltage. It follows that if current I_1 causes a voltage difference of 3V, and current I_2 causes a voltage difference of 5V, then current I_1+I_2 will cause an 8V p.d. across the component. Here is the procedure:

- Choose one of the supply components.
- Remove the other supply components from the circuit. Replace voltage sources with direct connections (short circuits), and leave breaks in the circuit where the current sources were (open circuits).
- Calculate the current in each wire, and the voltage across each component.
- Repeat the procedure for each supply component in turn.
- The current in each wire for the original (whole) circuit is equal to the sum of the currents in that wire due to each supply unit.
- The voltage across each component in the original (whole) circuit is equal to the sum of the voltages across that component due to each supply unit.

Let's use this method to analyse the circuit above. We start by considering only source V_1 . Removing the other supply components gives us a circuit like this.



This circuit is easier to analyse as it only has one supply. Supply V_1 feeds a circuit with resistance

$$R_1 + \{R_4 // (R_2 + R_3)\}$$

$$= R_1 + \frac{R_4 (R_2 + R_3)}{R_4 + R_2 + R_3}$$

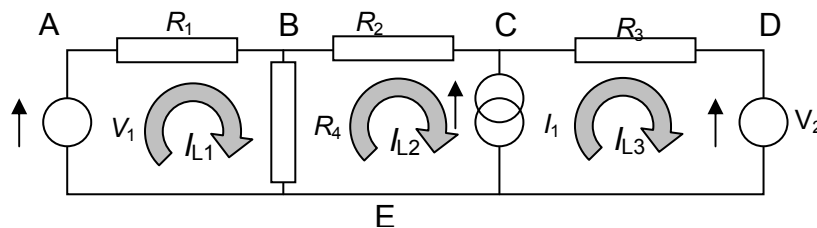
where // means 'in parallel with.' Accordingly, the current supplied by V_1 (and the current through R_1 which is in series with it) is equal to V_1 divided by this resistance. The voltages of points B, C and D can be calculated, as can the current in each wire. We make a note of the values, and add to them the results of analyses of circuits only containing I_1 and only containing V_2 .

You may find this method good in the sense that you only have to deal with one supply component at a time – and therefore all you need to know is how to combine resistors (something you’ve done before). Having said that, we end up analysing three circuits rather than one, so it is more time consuming.

Before leaving the method, you may be curious why voltage sources were replaced with short circuits, and current sources with open circuits. Here’s the reason. A voltage source does not change the voltage across its terminals, no matter what the current is ($d \text{ Voltage} / d \text{ Current} = R_{\text{equivalent}} = 0$). The only type of resistor which behaves likewise is a perfect conductor (0Ω). Similarly, a current source does not change its current, no matter what the voltage ($d \text{ Current} / d \text{ Voltage} = 1 / R_{\text{equivalent}} = 0$). The equivalent resistor in this case is a perfect insulator ($\infty\Omega$) which lets no current through ever.

6.3.1.2 Method of Loop Currents

Here we break the circuit down into the smallest loops it contains. Here there are three loops:



- E to A to B and back to E (loop 1),
- E to B to C to E (loop 2), and
- E to C to D to E (loop 3).

We call the current in loop 1 “loop current” number one (I_{L1}), with I_{L2} and I_{L3} representing the currents in the other two loops. We then express all other currents in terms of the loop currents. Clearly, $I_{AB} = I_{L1}$, since R_1 is in the first loop alone. Similarly, $I_{BC} = I_{L2}$, and $I_{CD} = I_{L3}$.

The current through R_4 is more complex, since this resistor is part of two of the loops. We write $I_{BE} = I_{L1} - I_{L2}$. Here I_{L1} is positive, since I_{BE} is in the same direction as I_{L1} , whereas I_{L2} (which goes from E to B then on to C) is in the opposite direction. These designations automatically take care of Kirchhoff’s First Law. Notice that by this method, $I_1 = I_{L3} - I_{L2}$.

Each loop now contributes one equation – Kirchhoff’s 2nd law around that loop. Clearly, if you go all the way round the loop, you must return to the voltage you started with. Taking the first loop as an example, we have:

$$0 = V_{AE} + V_{BA} + V_{EB}$$

$$\begin{aligned}
 &= V_1 - I_{AB} R_1 - I_{BE} R_4 \\
 &= V_1 - I_{L1} R_1 + (I_{L1} - I_{L2}) R_4.
 \end{aligned}$$

In a similar way, we write equations for each of the other two loops²⁴. We then have three equations in three unknowns (the three loop currents), which can be solved. The end result is the same as for a direct ‘sledgehammer’ approach with Kirchoff’s Laws – but the method is more organized.

6.3.2 Alternating Current

Having looked at circuits with resistors in them, we next turn our attention to circuits with inductors and capacitors as well. For a direct current, the situation is easy. After a brief period of settling down, there is no voltage drop across an inductor (because the current isn’t changing), and a capacitor doesn’t conduct at all.

For alternating currents the situation is more complicated. Let us suppose that the supply voltage is given by $V = V_0 \cos \omega t$. It turns out that the circuit will settle down to a steady behaviour (called the steady state). Once this has happened, the voltage across each component (and the current through each component) will also be a cosine wave with frequency ω , however it may not be in phase with the original V .

6.3.2.1 Resistor, capacitor and inductor

We start with the three simplest circuits – the lone resistor, the lone capacitor and the lone inductor, each supplied with a voltage $V = V_0 \cos \omega t$.

For the resistor, $I = V/R$, so the result is straightforward.

For the capacitor, $Q = VC$, and if we take I as positive in the direction which charges the capacitor, then

$$\begin{aligned}
 I &= \frac{dQ}{dt} = \frac{d}{dt} C V_0 \cos \omega t \\
 &= -\omega C V_0 \sin \omega t \\
 &= \omega C V_0 \cos\left(\omega t + \frac{1}{2} \pi\right) \\
 &= -\omega C V_0 \cos\left(\omega t - \frac{1}{2} \pi\right)
 \end{aligned} \tag{31}$$

For the inductor, $V = L dI/dt$, so

²⁴ It may help when writing the equations to notice the pattern: voltage sources count positively if you go through them from – to +, but negatively if you go from + to –. The voltages across resistors (e.g. $V_{BA} = I R_1$) count negatively if you go through them in the same direction as the current, and positively if you go through them the opposite way to the current.

$$\begin{aligned}
\frac{dI}{dt} &= \frac{V_0}{L} \cos \omega t \\
I &= \frac{V_0}{L\omega} \sin \omega t \\
&= \frac{V_0}{L\omega} \cos\left(\omega t - \frac{1}{2}\pi\right)
\end{aligned} \tag{32}$$

where we have taken the constant of integration to be zero. Failure to do so would lead to a non-zero mean current, which is clearly impossible as the mean supply voltage is zero.

6.3.2.2 Reactance and Impedance

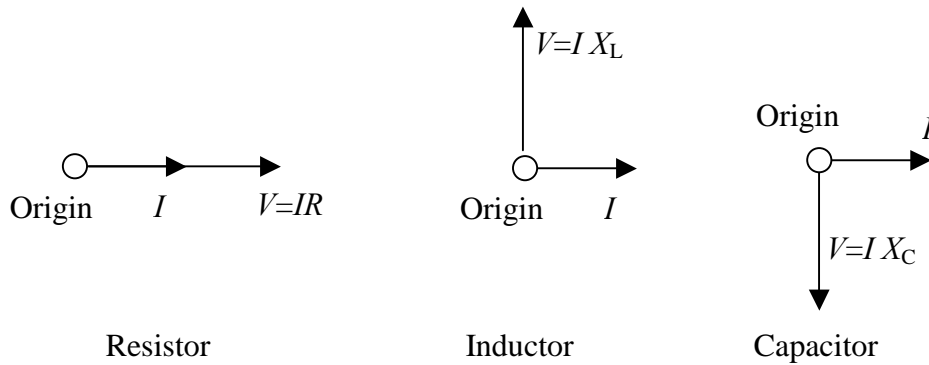
For resistors, the current and voltage are proportional, and consequently are in phase – one peaks at the same time as the other. For the other two components, this is not the case. The voltage is $\pi/2$ radians (or 90°) out of phase with respect to the current. Inductor currents peak 90° later than the voltage (the current *lags* the voltage), whereas capacitor currents peak 90° before the voltage (the current *leads* the voltage). Nevertheless, the amplitude of the voltage is still proportional to the amplitude of the current, and we call the ratio of the amplitudes the reactance (X).

$$\begin{aligned}
X_L &= \omega L \\
X_C &= -\frac{1}{\omega C}
\end{aligned} \tag{33}$$

By convention, we take reactance to be positive if the current lags the voltage by 90° , and negative if it leads by 90° . For capacitors and inductors in series, the total reactance is equal to the sum of the individual components' reactances – just as resistances add in series. Similarly, the formula for combining reactances in parallel is the same for that used for the resistance of resistors wired in parallel.

When a circuit is constructed with resistors, capacitors and inductors, then we need a way of analysing a circuit with both resistances and reactances. We visualise the situation using a 2D (phasor) diagram.

For any component or circuit, both voltage and current are represented by vectors. The length of the lines gives the amplitude, and the angle between the vectors gives the phase difference. By convention, we imagine the vectors to rotate about the origin in an anticlockwise direction (once per time period of the alternating current). The vectors for a resistor, capacitor and inductor are shown below.



As the arrows rotate anticlockwise, for the inductor, V comes before I . With the capacitor, I comes before V . This accurately represents the phase relationships between voltage and current for these components.

For a set of components in series, the current I will be the same for all of them. We usually draw the current pointing to the right. Voltages across inductors will then point up, those across resistors point right, and those across capacitors point down. By adding these voltages vectorially, we arrive at the voltage across the set of components – and can calculate its amplitude and phase relationship with respect to the current.

Similarly, for components in parallel, the voltage will be the same for each. We thus put voltage pointing to the right. Currents in capacitors now point up, currents in resistors point right, and currents in inductors point down. The total current is given by the vector sum of the individual currents.

In all cases, we call the ratio of the voltage amplitude to the current amplitude the impedance (Z) irrespective of the phase difference between the current and voltage.²⁵ In general the impedance of a component is related to resistance and reactance by $Z^2 = R^2 + X^2$.

6.3.2.3 Complex Numbers and Impedance

If you are familiar with complex numbers, there is an easier way of describing all of this, using the Argand diagram in place of 2-dimensional vectors. The impedance Z is now a complex number $Z = R + iX$, with R as its real part and X as its imaginary part.

The complex impedances of a resistor, capacitor and inductor are accordingly written as R , $-i/\omega C$ and $i\omega L$ respectively. The impedance of a set of components in series is given by the sum of the individual impedances. For a parallel network, Z^{-1} of the network is given by the

²⁵ In other words, a resistance is a special kind of impedance with zero phase difference, and a reactance is a special kind of impedance when the phase difference is 90° .

sum of Z^{-1} for each component, where ‘inverse’ (or ‘reciprocal’) is calculated in the usual way for complex numbers.

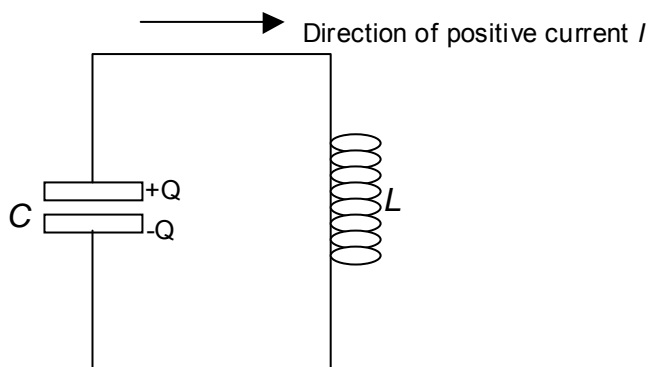
6.3.2.4 Root Mean Square values

You will also need to remember the definition of RMS voltage and current in an a.c. circuit. For a resistor, remember that the RMS supply voltage is the d.c. voltage which would supply the same mean **power** to the device.

$$\begin{aligned}\overline{P} &\equiv \frac{V_{rms}^2}{R} = \frac{\overline{V_0^2 \cos^2 \omega t}}{R} = \frac{V_0^2}{2R} \\ \Rightarrow V_{rms} &= \frac{1}{\sqrt{2}} V_0\end{aligned}\tag{34}$$

6.3.3 Resonance

One further circuit needs a mention, and that is the simple circuit of an inductor and a capacitor connected together, as shown in the diagram below. Both the voltage and current for the two components must be the same, and so with the sign conventions chosen in the diagram:



$$\begin{aligned}\frac{Q}{C} &= L \frac{dI}{dt} = -L \frac{d^2 Q}{dt^2} \\ \frac{d^2 Q}{dt^2} &= -\frac{1}{LC} Q\end{aligned}\tag{35}$$

This is an equation of ‘simple harmonic motion’ with angular frequency ω , where $\omega^2 = 1/LC$. This circuit can therefore oscillate at this frequency, and this makes it useful in radio receivers for selecting the frequency (and hence radio station) which the listener wants to detect.

6.4 Questions

1. Calculate the size of the repulsion force between two electrons 0.1nm apart.

2. In this question, you will make an estimate for the size of a hydrogen atom. Suppose an electron moves in a circular path around the proton, with radius r . Calculate, in terms of r , the potential energy of the atom (it will be negative, of course), the speed of the electron, and its kinetic energy. Now write down an expression for the total energy of the electron. Find the value of r which minimizes this total energy, and compare it to the measured radius of a hydrogen atom, which is about 5×10^{-11} m.
3. What fraction of the electrons in the solar system would have to be removed in order for the gravitational attractions to be completely cancelled out by the electrostatic repulsion?
4. A cloud of electrons is accelerated through a 20kV potential difference (so that their kinetic energy of each coulomb of electrons is 20kJ). Calculate their speed.
5. A beam of 20kV electrons is travelling horizontally. An experimenter wishes to bend their path to make them travel vertically (at the same speed) using a region with a uniform electric field. This region is square with side length 5cm. Calculate the size and direction of electric field needed to do this. What would happen to a beam of 21kV electrons passing this region?
6. A different experimenter wishes to bend the beam of 20kV electrons using a magnetic field. She chooses to bend the beam round a circular path of radius 3cm. What magnitude and direction of magnetic field is needed? What would happen to a beam of 21kV electrons passing this region?
7. I wish to make a 1T magnetic flux density inside a long coil (or solenoid) with radius 5mm. I use wire which can carry a current of 4A. How many turns per metre of coil are needed?
8. 'Clamp' ammeters used by electricians can measure the current in a wire without needing to break the wire. A metal loop encloses the wire, and the magnetic field around the wire is measured. If the loop is circular, with radius 3cm, and is centred on the wire, calculate the magnetic flux density measured when the current in the wire is 100A.
9. Calculate the impedance of a 20Ω resistor wired in series with a 3mH inductor when fed with alternating current of 50Hz. A capacitor wired in parallel with this combination causes the overall reactance of the circuit to become zero at 50Hz (in other words, the voltage is in phase with the total current). Calculate the capacitance of the capacitor.