9 Appendix

9.1 Multiplying Vectors

Physics is riddled with quantities which have both magnitude and direction – velocity, acceleration, displacement, force, momentum, angular velocity, torque, and electric field to name but eight. When describing these, it is very useful to use vector notation. At best this saves us writing out separate equations for each of the components. While the addition and subtraction of vectors is reasonably straightforward (you add, or subtract, the components to get the components of the result), multiplication is more tricky.

You can think of a vector as a little arrow. You can add them by stacking them nose-to-tail, or subtract them by stacking them nose-to-nose. But how do you go about multiplying them? It is not obvious!

To cut a long story short, you can't do it unambiguously. However there are two vector operators which involve multiplication and are useful in physics. Ordinary multiplication is commonly written with either the cross (×) or dot (●), so when it comes to vectors we call our two different 'multiplication' processes the dot product and cross product to distinguish them. These are the closest we get to performing multiplication with vectors.

9.1.1 The Dot product (or scalar product)

A ton of bricks is lifted a metre, then moved horizontally by 2m. How much work is done? Work is given by the product of force and distance, however only the vertical lifting (not the horizontal shuffling) involves work. In this case the work is equal to the weight (about 9.8kN) multiplied by the vertical distance (1m).

This gives us one useful way of 'multiplying' vectors – namely to multiply the magnitude of the first, by the component of the second which is parallel to the first.

If the two vectors are **A** and **B**, with magnitudes *A* and *B*, and with an angle θ between them, then the component of **B** parallel to **A** is $B\cos\theta$. Therefore the dot product is given by $AB\cos\theta$.

$$\mathbf{A} \bullet \mathbf{B} \equiv AB\cos\theta \tag{1}$$

Notice that the dot product of two vectors is itself a *scalar*. Note that when we talk about the square of a vector, we mean its scalar product with itself. Since in this case, θ=0, this is the same as the square of the vector's magnitude.

The dot product is also commutative, in other words, the order of the two vectors **A** and **B** does not matter, since **A●B**=**B●A**.

The dot product of two vectors written using Cartesian co-ordinates is particularly easy to calculate. If we use **i**, **j**, and **k** to represent the unit vectors pointing along the +*x*, +*y* and +*z* axes, then

$$\begin{aligned}(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\bullet(u\mathbf{i}+v\mathbf{j}+w\mathbf{k}) &= au\mathbf{i}\bullet\mathbf{i}+av\mathbf{i}\bullet\mathbf{j}+aw\mathbf{i}\bullet\mathbf{k}\\ &\quad+bu\mathbf{j}\bullet\mathbf{i}+bv\mathbf{j}\bullet\mathbf{j}+bw\mathbf{j}\bullet\mathbf{k}\\ &\quad+cu\mathbf{k}\bullet\mathbf{i}+cv\mathbf{k}\bullet\mathbf{j}+cw\mathbf{k}\bullet\mathbf{k}\\ &= au+bv+cw\end{aligned} \tag{2}$$

9.1.2 The Cross product (or vector product)

If the dot product produced a scalar, what are we to do if a *vector* is needed as the result of our multiplication? Answer: a cross product.

Our first dilemma is to choose the direction of the result. Given that the vectors will, in general, not be parallel or antiparallel, we can't choose the direction of one of them – that would not be fair! The two vectors will usually define a plane, so perhaps we could use a vector in this plane as the result? No, that wouldn't do either – there is still an infinite number of directions to choose from! A solution is presented if we choose the vector perpendicular to this plane. This narrows the choice down to two directions – and we use a convention to choose which.

Notice that the result of the cross product must be zero if the two vectors are parallel, since in this case we can't define a plane using the vectors. It follows that the cross product of a vector with itself is zero. This means that we aren't going to be interested in the component of the second vector which is parallel to the first when calculating the product. On the contrary, it is the perpendicular component which matters.

The cross product of two vectors is defined as the magnitude of the first, multiplied by the component of the second which is perpendicular to the first. The product is directed perpendicular to both vectors. To be more precise, imagine a screw attached to the first vector. The cross product goes in the direction the screw advances when the first vector is twisted to line up with the second. The cross product of a vector lying along the +*x* axis with one lying along the +*y* axis is one lying along the +*z* axis. The cross product of 'up' with 'forwards' is 'left'.

$$|\mathbf{A}\times\mathbf{B}| \equiv AB\sin\theta \tag{3}$$

With a definition as obtuse as this, you could be forgiven for wondering whether it had any practical use at all! However they turn out to be very useful in physics – especially when dealing with magnetic fields and rotational motion.

Notice that, where the dot product was commutative, the cross product is anticommutative. In other words, **A×B=–B×A**, so make sure you don't swap the vectors over inadvertently.

The vector product of vectors written in Cartesian form can also be calculated:

$$\begin{aligned}(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\times(u\mathbf{i}+v\mathbf{j}+w\mathbf{k}) &= au\mathbf{i}\times\mathbf{i}+av\mathbf{i}\times\mathbf{j}+aw\mathbf{i}\times\mathbf{k}\\ &\quad +bu\mathbf{j}\times\mathbf{i}+bv\mathbf{j}\times\mathbf{j}+bw\mathbf{j}\times\mathbf{k}\\ &\quad +cu\mathbf{k}\times\mathbf{i}+cv\mathbf{k}\times\mathbf{j}+cw\mathbf{k}\times\mathbf{k}\\ &= 0+av\mathbf{k}-aw\mathbf{j}\\ &\quad -bu\mathbf{k}+0+bw\mathbf{i}\\ &\quad +cu\mathbf{j}-cv\mathbf{i}+0\\ &= (bw-cv)\mathbf{i}+(cu-aw)\mathbf{j}+(av-bu)\mathbf{k}\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a & b & c\\ u & v & w\end{vmatrix}\end{aligned} \tag{4}$$

where the most convenient way of remembering the result is as the determinant of the 3×3 matrix shown.

9.2 *Dimensional Analysis*

If you look back at the 'flow equation' (24 in section 1.3.3), you will see something interesting about the units.

Current (A) = Charge density (C/m^3) $\times$ Area (m^2) $\times$ Speed (m/s)

If we 'do algebra' with the units on the right hand side, we get

$$\frac{C}{m^3}\times m^2 \times \frac{m}{s} = \frac{C}{s} = A\,,$$

and this agrees with the units of the left hand side. Now this may all seem pretty obvious, but it gives us a useful procedure for checking whether our working is along the right lines. If, during your calculations, you find yourself adding a charge of 3C to a distance of 6m to get a result of 9N; or you multiply a speed of 13m/s by a time of 40s and get a current of 520A; then in either case you must have made a mistake!

We can also use the principle that units must balance to guess the form of an equation we do not know how to derive. For example, you may guess that the time period of a simple pendulum might depend on the length of the pendulum *L*, the strength of the local gravity *g* and the mass of the pendulum bob *m*. Now

- *L* is measured in m
- *g* is measured in N/kg or m/s^2
- *m* is measured in kg,
- and we want a time period, which will be measured in s.

The only way these measurements can be combined to make something in seconds is to take *L*, divide it by *g* (this gives something in s^2) and then take the square root. Therefore, without knowing any physics of the simple harmonic oscillator, we have shown that the time period of a pendulum is related to $\sqrt{L/g}$ and will be independent of the mass *m*.

Similarly, notice what happens if you multiply ohms by farads:

$$\Omega F = \frac{V}{A}\times\frac{C}{V} = \frac{C}{A} = s\,.$$

Yes, you get seconds. Therefore, it should come as little surprise to you that if you double the resistance of a capacitor-resistor network, it will take twice as long to charge or discharge. Furthermore, you have worked this out without recourse to calculus or the tedious electrical details of section 6.1.2.2.

Of course, one drawback of the method presented here is that some quantities have two different units. For example, gravitational field strength could be N/kg or m/s^2. Electric field strength could be N/C or V/m. How do you know which to choose? The answer is that if you restrict yourself to using the minimum number of units in your working, and express all others in terms of them, you will not have any difficulties. Usually people choose m, s, kg and A but any other combination of independent units[33] will do equally well.[34]

In books you may see folk use L, T, M, I and Θ to represent the 'dimensions' of length, time, mass, electric current and temperature. This is just a more formal way of doing what we have done here using the S.I. units. In these books, the dimension of speed would be written as

$$[\text{speed}] = \text{L T}^{-1},$$

and the dimensions of force would be written

$$[\text{force}] = \text{M L T}^{-2},$$

where the square brackets mean 'dimensions of'. Technically, this is more correct than using the S.I. units, because some quantities are dimensionally the same, but have very different meanings (and hence units). For example, torque and energy have the same dimensions, but you wouldn't want to risk confusing them by using the same unit for both. Similarly an angle in radians has no dimensions at all (being a an arc length in metres divided by a radius in metres), but we wouldn't want to confuse it with an ordinary number like 3.[35]

[33] By independent we mean that no one unit can be derived entirely from a combination of the others. For example, m, s, kg and J would be no good as a set of four since J can already be expressed in terms of the others J = kg × $(m/s)^2$, and hence we have ambiguity arising as to how we express quantities.

[34] OK, if you want to do work where there are electric currents and temperatures as well as mechanical quantities, you might need to go up to five (an extra one for temperature).

[35] Indeed, quantities with 'no units' are usually said to have the dimensions of the number one. Thus [angle] = 1. It follows that angular velocity has dimensions of [angle]÷[time] = 1÷T = T^{-1}. This comes from the property 1 has in being the 'unity' operator for multiplication.