4 Vibes, Wiggles & Light

4.1 Oscillation

Any system in stable equilibrium can be persuaded to oscillate. If it is removed from the equilibrium, there will be a force (or other influence) that attempts to maintain the status quo. The size of the force will depend on the amount of the disturbance.

Suppose that the disturbance is called *x*. The restoring force can be written

$$F = -\left(Ax + Bx^2 + Cx^3 \ldots\right), \tag{1}$$

where the minus sign indicates that the force acts in the opposite direction to the disturbance. If x is small enough, x^2 and x^3 will be so small that they can be neglected. We then have a restoring force proportional to the displacement *x*.

Just because the system has a force acting to restore the equilibrium, this does not mean that it will return to x=0 immediately. All systems have some inertia, or reluctance to act quickly. For a literal force, this inertia is the mass of the system – and we know that the acceleration caused by a force (*F*) is given by *F/m*, where *m* is the mass. We can therefore work on equation (1) to find out more:

$$\begin{gathered} F = m\frac{d^2x}{dt^2} = -Ax \\ \frac{d^2x}{dt^2} = -\frac{A}{m}x \end{gathered} . \tag{2}$$

This differential equation has the solution:

$$\begin{gathered} x = x_0 \cos(\omega t + \phi) \\ \omega = \sqrt{A/m} \end{gathered} , \tag{3}$$

which is indeed an oscillation. We are using x_0 to denote the amplitude. Notice that, as we are working in radians, the cosine function needs to advance 2π to go through a whole cycle. Therefore we can work out the time period (T) and frequency (f):

$$\begin{gathered} \omega T = 2\pi \\ T = \frac{2\pi}{\omega} \\ f = \frac{1}{T} = \frac{\omega}{2\pi} \end{gathered} . \tag{4}$$

Seeing that $\omega=2\pi f$, we notice that ω is none other than the angular frequency of the oscillation, as defined in chapter 3.

These equations are perfectly general, and so whenever you come across a system with a differential equation like (2), you know the system will oscillate, and furthermore you can calculate the frequency.

4.1.1 Non-linearity

Equation (1) has left an unanswered question. What happens if x is big enough that x^2 and x^3 can't be neglected? Clearly solution (3) will no longer work. In fact the equation probably won't have a simple solution, and the system will start doing some really outrageous things. Given that it has quadratic terms in it, we say it is non-linear; and a non-linear equation will send most physics students running away, screaming for mercy.

Let me give you an example. There are very nice materials that look harmlessly transparent. However they are designed so that the non-linear terms are *very* important when light passes through them. The result – you put red laser light in, and it comes out blue (at twice the frequency). They are called 'doubling crystals' and are the sort of thing that might freak out an unsuspecting GCSE examiner.

Our world would be much less wonderful if it were purely linear – no swirls in smoke, no wave-breaking (and hence surfing), and extremely boring weather – not to mention rigid population dynamics. While the non-linear terms add to the spice of life, I for one am grateful that many phenomena can be well described using linear equations. Otherwise physics would be much more frustrating, and bridge design would be just as hard as predicting the weather.

4.1.2 Energy

Before we move from oscillations to waves, let us make one further observation. The energy involved in the oscillation is proportional to the square of the amplitude. We shall show this in two ways.

First: If the displacement is given by equation (3), we notice that the velocity is given by

$$u \equiv \dot{x} = -x_0 \omega \sin(\omega t + \phi). \qquad (5)$$

At the moment when the system passes through its equilibrium (x=0) point, all the energy is in kinetic form. Therefore the total energy is

$$E = K(x = x_0) = \tfrac{1}{2} m u^2 = \tfrac{1}{2} m \omega^2 x_0^2 \qquad (6)$$

which is indeed proportional to the amplitude squared.

Second: When the displacement is at its maximum, there is no kinetic energy. The energy will all be in potential form. We can work out the potential energy in the system at displacement *x*, by evaluating the work done to get it there:

$$E_{pot} = \int F dx = \int A x dx = \left[\tfrac{1}{2} A x^2\right] . \qquad (7)$$

Notice that we did not include the minus sign on the force. This is because when we work out the 'work done' the force involved is the force of us pulling the system. This is equal and opposite to the restoring force of the system, and as such is positive (directed in the same direction as *x*).

The total energy is given by the potential energy at the moment when *x* has its maximum (i.e. $x=x_0$). Therefore

$$E = E_{pot}\left(x = x_0\right) = \tfrac{1}{2} A x_0^2 . \qquad (8)$$

Equations (8) and (6) are in agreement. This can be shown by inserting the value of ω from equation (3) into (6).

While we have only demonstrated that energy is proportional to amplitude squared for an oscillation, it turns out that the same is true for linked oscillators – and hence for waves. The intensity of a wave (joules of energy transmitted per second) is proportional to the amplitude squared in exactly the same way.

Intensity of a wave is also related to another wave property – its speed. The intensity is equal to the amount of energy stored on a length *u* of wave, where *u* is the speed. This is because this is the energy that will pass a point in one second (a length *u* of wave will pass in this time).

4.2 Waves & Interference

The most wonderful property of waves is that they can interfere. You can add three and four and get six, or one, or 4.567, depending on the phase relationship between the two waves. You can visualize this using either trigonometry or vectors (phasors). However, before we look at interference in detail, we analyse a general wave.

4.2.1 Wave number

Firstly, we define a useful parameter called the *wave number*. This is usually given the letter *k*, and is defined as

$$k = \frac{2\pi}{\lambda} , \qquad (13)$$

where λ is the wavelength. If we write the shape of a 'paused' wave as $y=A\cos(\phi)$, the phase ϕ of a wave is given by

$$\phi = kD. \tag{14}$$

We can see that this makes sense by combining equations (13) and (14):

$$\phi = kD = \frac{2\pi D}{\lambda}. \tag{15}$$

If the distance *D* is equal to a whole wavelength, we expect the wave to be doing the same thing as it was at $\phi=0$. And since $\cos(2\pi)=\cos(0)$, this is indeed the case.

A variation on the theme is possible. You may also see *wave vectors* **k**: these have magnitude as defined in (13), and point in the direction of energy transfer.

4.2.2 Wave equations

We are now in a position to write a general equation for the motion of a wave with angular frequency ω and wave number *k*:

$$y = A\cos(\omega t - kx + \phi).$$

We can check that this is correct, since

- if we look at a particular point (value of *x*), and watch as time passes, we will pass from one peak to the next when $\omega t = 2\pi f\, t$ has got bigger by 2π (i.e. $t=1/f$ as it should).

- if we look at a particular moment in time (value of *t*), and look at the position of adjacent peaks, they should be separated by one wavelength = $2\pi/k$. Now for adjacent peaks, the values of *kx* will differ by 2π according to the formula above, and so this is correct.

- if we follow a particular peak on the wave – say the one where $\omega t - kx + f = 0$, we notice that $x = (\omega t + f)/k = \omega t/k$ + constant, and hence the position moves to increasing *x* at a speed equal to ω/k, as indeed it should since $\omega/k = 2\pi f/(2\pi/\lambda) = f\lambda = v$.

It follows that a leftwards-travelling wave has a function which looks like

$$y = A\cos(\omega t + kx + \phi).$$

4.2.3 Standing waves

Imagine we have two waves of equal amplitude passing along a string in the two different directions. The total effect of both waves is given by adding them up:

$$\begin{aligned} y &= A\cos(\omega t - kx + \phi_1) + A\cos(\omega t + kx + \phi_2) \\ &= 2\cos\left(\omega t + \tfrac{1}{2}(\phi_1 + \phi_2)\right) \times \cos\left(kx + \tfrac{1}{2}(\phi_2 - \phi_1)\right) \end{aligned}$$

At any time, the peaks and troughs will only occur at the places where the second cosine is +1 or –1, and so the positions of the peaks and troughs do not change. This is why this kind of situation is called a standing wave. While there is motion, described by the first cosine, the positions of constructive interference between the two counter-propagating waves remain fixed (these are called antinodes), as to the positions of destructive interference (the nodes).

While there are many situations which involve counter-propagating waves, this usually is caused by the reflection of waves at boundaries (like the ends of a guitar string). Accordingly, there is nothing keeping the phase constants ϕ_1 and ϕ_2 the same, and so the standing wave doesn't develop. However if the frequency is just right, then it works, as indicated in section 4.2.7.5.

4.2.4 Trigonometric Interference

We are now in a position to look at the fundamental property of waves – namely interference. Our first method of analysis uses trigonometry. Suppose two waves arrive at the same point, and are described by $x_1 = A\cos(\omega t)$ and $x_2 = B\cos(\omega t + \phi)$ respectively. To find out the resulting sum, we add the two disturbances together.

$$\begin{aligned} X &\equiv x_1 + x_2 \\ &= A\cos\omega t + B\cos(\omega t + \phi) \\ &= A\cos\omega t + B\cos\omega t\cos\phi - B\sin\omega t\sin\phi \\ &= (A + B\cos\phi)\cos\omega t - B\sin\phi\sin\omega t \\ &= X_0(\cos\delta\cos\omega t - \sin\delta\sin\omega t) \\ &= X_0\cos(\omega_0 t + \delta) \end{aligned} \tag{9}$$

where we define

$$\begin{aligned} X_0 &= \sqrt{(A + B\cos\phi)^2 + (B\sin\phi)^2} \\ &= \sqrt{A^2 + B^2 + 2AB\cos\phi} \quad . \\ \cos\delta &= \frac{A + B\cos\phi}{X_0} \quad \sin\delta = \frac{B\sin\phi}{X_0} \end{aligned} \tag{10}$$

The amplitude of the resultant is given by X_0. Notice that if A=B, the expression simplifies:

$$\begin{aligned} X_0 &= A\sqrt{2+2\cos\phi} \\ &= A\sqrt{2(1+\cos\phi)} \\ &= A\sqrt{4\cos^2\left(\tfrac{1}{2}\phi\right)} \\ &= 2A\left|\cos\left(\tfrac{1}{2}\phi\right)\right| \end{aligned}, \qquad (11)$$

and we obtain the familiar result that if the waves are 'in phase' (ϕ=0), the amplitude doubles, and if the waves are π radians (half a cycle) 'out of phase', we have complete destructive interference.

Equation (10) can be used to provide a more general form of this statement – the minimum resultant amplitude possible is $|A-B|$, while the maximum amplitude possible is $A+B$.

This statement is reminiscent of the 'triangle inequality', where the length of one side of a triangle is limited by a similar constraint on the lengths of the other two sides. This brings us to our second method of working out interferences: by a graphical method.

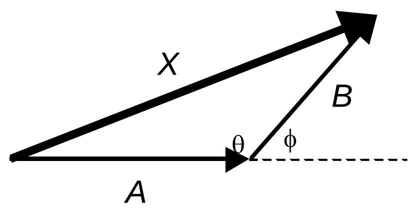
4.2.5 Graphic Interference

In the graphic method a vector represents each wave. The length of the vector gives the amplitude, and the relative orientation of two vectors indicates their phase relationship. If the phase relationship is zero, the two vectors are parallel, and the total length is equal to the sum of the individual lengths. If the two waves are π out of phase, the vectors will be antiparallel, and so will partly (or if $A=B$, completely) cancel each other out.

The diagram below shows the addition of two waves, as in the situation above. Notice that since $\theta + \phi = \pi$, $\cos\theta = -\cos\phi$. One application of the cosine rule gives

$$X_0 = \sqrt{A^2 + B^2 + 2AB\cos\phi} \qquad (12)$$

in agreement with equation (10).



4.2.6 Summary of Interference Principles

The results of the last section allow us to determine the amplitude once we know the phase difference between the two waves. Usually the two waves have come from a common source, but have travelled different distances to reach the point. Let us suppose that the difference in distances is D – this is sometimes called the *path difference*. What will ϕ be?

To find out, we use the wave number *k*. The phase difference ϕ is given by

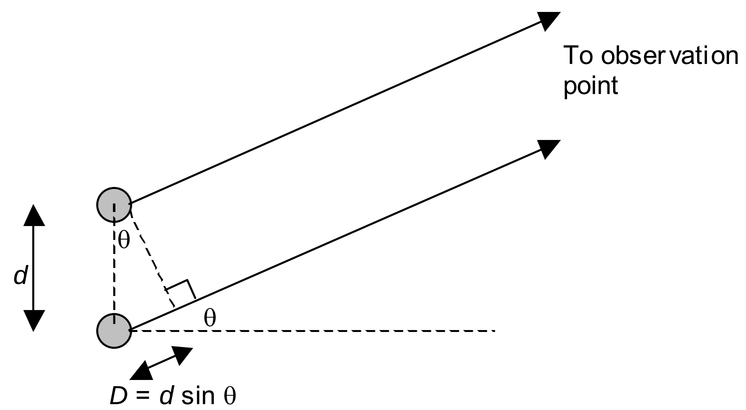
$$\phi = kD = \frac{2\pi D}{\lambda}. \tag{14}$$

If the distance *D* is equal to a whole wavelength, we expect the two waves to interfere constructively, since peak will meet peak, and trough will meet trough. In equation (15), if D=λ, then ϕ=2π, and constructive interference is indeed obtained, as can be seen from equation (12). Similarly, we find that if D is equal to λ/2, then ϕ=π, and equation (12) gives destructive interference.

4.2.7 Instances of two-wave interference

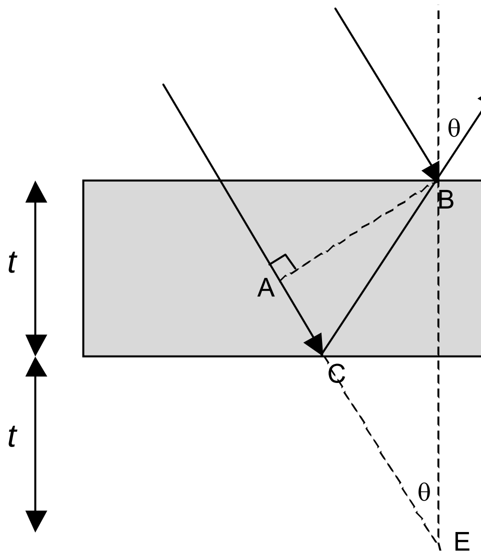
4.2.7.1 Young's "Two Slit Experiment"

Two cases need to be dealt with. The first is known as the two-slit experiment, and concerns two sources in phase, which are a distance *d* apart, as shown in the diagram below. The path difference is given by $D = d\sin\theta$, in the case that *d* is much smaller than the distance from sources to observer. Using the conditions in the last section, we see that interference will be constructive if $D = d\sin\theta = n\lambda$ where *n* is an integer.



4.2.7.2 Thin films and colours on soap bubbles

The second case is known as thin film interference, and concerns the situation in the diagram below. Here the light can take one of two routes.



The path difference is calculated:

$$\begin{aligned} D &= \mathrm{AC} + \mathrm{CB} \\ &= \mathrm{AC} + \mathrm{CE} \\ &= 2t\cos\theta \end{aligned}$$

Before we can work out the conditions required for constructive or destructive interference, there is an extra caution to be borne in mind – the reflections.

4.2.7.3 Hard & Soft Reflections

The reflection of a wave from a surface (or more accurately, the boundary between two materials) can be hard or soft.

- Hard reflections occur when, at the boundary, the wave passes into a ‘sterner’ material. At these reflections, a peak (before the reflection) becomes a trough (afterwards) and vice-versa. This is usually stated as “a π phase difference is added to the wave by the reflection.” These mean the same thing since $\cos(\theta + \pi) = -\cos\theta$.

  To visualize this – imagine that you are holding one end of a rope, and a friend sends a wave down the rope towards you. You keep your hand still. At your hand, the incoming and outgoing waves interfere, but must sum to zero (after all, your hand is not moving, so neither can the end of the rope). Therefore if the incoming wave is above the rope, the outgoing wave must be below. In this way, peak becomes trough and vice-versa.

- Soft reflections, on the other hand, are where the boundary is *from* the ‘sterner’ material. At these reflections, a peak remains a peak, and there is no phase difference to be added.

What do we mean by 'sterner'? Technically, this is a measurement of the restoring forces in the oscillations which link to produce the wave – the *A* coefficients of (1). However, the following table will help you to get a feel for 'sternness'.

Wave	**From**	**To**	**Reflection**
Light	Reflection off mirror		Hard
Light	Air	Water / Glass	Hard
Light	Water / Glass	Air	Soft
Light	Lower refractive index	Higher refractive index	Hard
Sound	Solid / liquid	Air	Soft
Sound	Air	Solid / liquid	Hard
Wave on string	Reflection off fastened end of string		Hard
Wave on string	Reflection off unsecured end of string		Soft

4.2.7.4 Film Interference Revisited

Going back to our thin film interference: sometimes both reflections will be hard; sometimes one will be hard, and the other soft.

The formulae for constructive interference are:

Both reflections hard, or both soft: $D = 2t\cos\theta = n\lambda$ (16)

One reflection hard, one soft: $D = 2t\cos\theta + \frac{1}{2}\lambda = n\lambda$ (17)

The difference comes about because of the phase change on reflection at a hard boundary.

4.2.7.5 Standing Waves

Equations (16) and (17) with θ=0 can be used to work out the wavelengths allowed for standing waves. For a standing wave, we must have constructive interference between a wave and itself (having bounced once back and forth along the length of the device). The conditions for constructive interference in a pipe, or on a string of length L (round trip total path = 2L) are

$2L = n\lambda$ soft reflections at both ends
$2L = \left(n + \tfrac{1}{2}\right)\lambda$ soft reflection at one end
$2L = (n+1)\lambda$ hard reflections at both ends.

4.2.7.6 Two waves, different frequencies

All the instances given so far have involved two waves of identical frequencies (and hence constant phase difference). What if the frequencies are different? Let us suppose that our two waves are described by $x_1 = A\cos\omega_1 t$ and $x_2 = B\cos\omega_2 t$, where we shall write $\delta \equiv \omega_2 - \omega_1$. When we add them, we get:

$$\begin{aligned} X &= x_1 + x_2 \\ &= A\cos\omega_1 t + B\cos\delta t\,\cos\omega_1 t - B\sin\delta t\,\sin\omega_1 t \\ &= \left(A + B\cos\delta t\right)\cos\omega_1 t - B\sin\delta t\sin\omega_1 t \\ &= X_0\cos\left(\omega_1 t + \alpha\right) \end{aligned}, \qquad (18)$$

where

$$\begin{aligned} X_0 &= \sqrt{\left(A + B\cos\delta t\right)^2 + \left(B\sin\delta t\right)^2} \\ &= \sqrt{A^2 + B^2 + 2AB\cos\delta t} \\ \cos\alpha &= \frac{A + B\cos\delta t}{X_0} \end{aligned}. \qquad (19)$$

We see that the effective amplitude fluctuates, with angular frequency δ. On the other hand, if the two original waves had very different frequencies, then this fluctuation may be too quick to be picked up by the detector. In this case, the resultant amplitude is the root of the sum of the squares of the original amplitudes. Put more briefly – if the frequencies are very different, the total intensity is simply given by the sum of the two constituent intensities.

These fluctuations are known as ‘beats’, and the difference f_2-f_1 is known as the beat frequency. To give an illustration: While tuning a violin, if the tuning is slightly off-key, you will hear the note pulse: loud-soft-loud-soft and so on. As you get closer to the correct note, the pulsing slows down until, when the instrument is in tune, no pulsing is heard at all because f_2-$f_{1=0}$.

4.2.8 Adding more than two waves

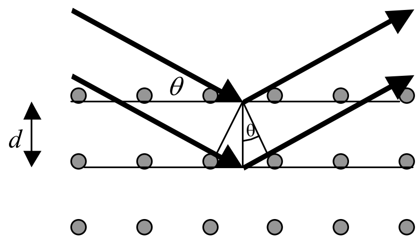
4.2.8.1 Diffraction Grating

The first case we come to with more than two waves is the diffraction grating. This is a plate with many narrow transparent regions. The light can only get through these regions. If the distance between adjacent ‘slits’ is *d*, we obtain constructive interference, as in section 4.2.4.1, when $d\sin\theta = n\lambda$ - in other words when the light from all slits is in phase.

The difference between this arrangement and the double-slit is that when $d\sin\theta \neq n\lambda$ we find that interference is more or less destructive. Therefore a given colour (or wavelength) only gets sent in particular directions. We can use the device for splitting light into its constituent colours.

4.2.8.2 Bragg Reflection

A variation on the theme of the diffraction grating allows us to measure the size of the atom.



The diagram shows a section of a crystal. Light (in this case, X-rays) is bouncing off the layers of atoms. There are certain special angles for which all the reflections are in phase, and interfere constructively.

Looking at the small triangle in the diagram, we see that the extra path travelled by the wave bouncing off the second layer of atoms is

$$D = 2d\sin\theta\,. \qquad (20)$$

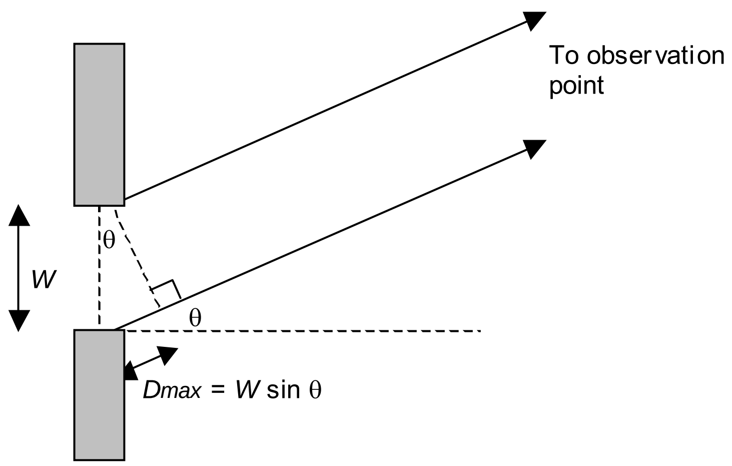
When $D=n\lambda$, we have constructive interference, and a strong reflection. There is one thing that takes great care – notice the definition of q in the diagram. It is not the angle of incidence, nor is it the angle by which the ray is deflected – it is the angle between surface and ray. This is equal to half the angle of deflection, also equal to $\pi/2 - i$.

Using this method, the spacing of atomic layers can be calculated – and this is the best measurement we have for the ‘size’ of the atom in a crystal.

4.2.8.3 Diffraction

What happens when we add a lot of waves together? There is one case we need to watch out for – when all possible phases are represented with equal strength. In this case, for each wave $\cos(\omega t + \phi)$, there will be an equally strong wave $\cos(\omega t + \phi + \pi) = -\cos(\omega t + \phi)$, which will cancel it out.

How does this happen in practice? Look at the diagram below. Compared with the wave from the top of the gap, the path differences of the waves coming from the other parts of the gap go from zero to $D_{\max} = W \sin\theta$, where *W* is the width of the gap.



If $kD_{\max}$ is a multiple of 2π, then we will have all possible phases represented with equal strength, and overall destructive interference will result.

To summarize, destructive interference is seen for angles θ, where

$$W \sin\theta = n\lambda \,. \qquad (21)$$

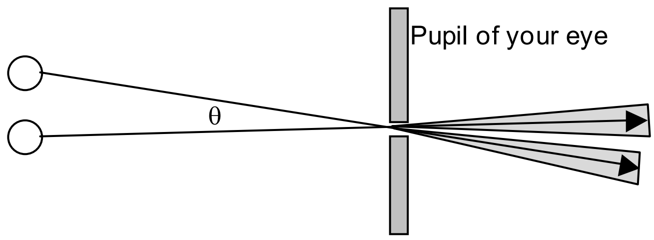
Make sure you remember that *W* is the width of the gap, and that this formula is for **destructive** interference.

This formula is only valid (as in the diagram) when the observer is so far away that the two rays drawn are effectively parallel. Alternatively the formula works perfectly when it is applied to an optical system that is focused correctly, for then the *image* is at infinity.

4.2.8.4 Resolution of two objects

How far away do you have to get from your best friend before they look like Cyclops? No offence – but how far away do you have to be before you can't tell that they've got two eyes rather than one? The results of diffraction can help us work this out. Let's call this critical distance *L*.

The rays from both eyes come into your eyeball. Let us suppose that the angle between these rays is θ, where θ is small, and that your friend's eyes are a distance *s* apart. Therefore $\tan\theta \approx \sin\theta \approx s/L$. These two rays enter your eye, and spread out (diffract) as a result of passing through the gap called your pupil. They can only just be 'resolved' – that is noticed as separate – when the first minimum of one's diffraction pattern lines up with the maximum of the other. Therefore $W \sin\theta = \lambda$ where *W* is the width of your pupil.



Putting the two formulae together gives:

$$\begin{aligned} \sin\theta &= \frac{s}{L} = \frac{\lambda}{W} \\ L &= \frac{sW}{\lambda} \end{aligned}. \qquad (22)$$

For normal light (average wavelength about 500nm), a 5mm pupil, and a 10cm distance between the eyes: you friend looks like Cyclops if you are more than 1km away! If you used a telescope instead, and the telescope had a diameter of 10cm, then your friend's two eyes can be distinguished at a distances up to 20km.

4.2.8.5 The Bandwidth Theorem

In the last section, we asked the question, "What happens when you add lots of waves together?" However we cheated in that we only considered waves of the same frequency. What happens if the waves have different frequencies?

Suppose that we have a large number of waves, with frequencies evenly spread between f and $f+\delta f$. The angular frequencies will be spread from ω to $\omega+\delta\omega$, where $\omega=2\pi f$ as in equation (4). Furthermore, imagine that we set them up so that they all agree in phase at time $t=0$. They will never agree again, because they all have different frequencies.

The phases of the waves at some later time t will range from ωt to $(\omega+\delta\omega)t$.

Initially we have complete constructive interference. After a short time δt, however, we have destructive interference. This will happen when (as stated in the last section) all phases are equally represented – when the range of phases is a whole multiple of 2π. This happens when $\delta t\times\delta\omega=2\pi$. After this, the signal will stay small, with occasional complete destructive interference.

From this you can reason (if you're imaginative or trusting) that if you need to give a time signal, which has a duration smaller than δt, you must use a collection of frequencies at least $\delta\omega=2\pi/\delta t$. This is called the bandwidth theorem. This can be stated a little differently:

$$\begin{aligned} \delta\omega\,\delta t &= 2\pi \\ \delta(2\pi f)\delta t &= 2\pi \,. \\ \delta f\,\delta t &= 1 \end{aligned} \tag{23}$$

A similar relationship between wavelength and length can be obtained, if we allow the wave to have speed *c*:

$$\begin{aligned} \delta f\,\delta\left(\frac{x}{c}\right) &= 1 \\ \delta\left(\frac{f}{c}\right)\delta x &= 1\,. \\ \delta\left(\frac{1}{\lambda}\right)\delta x &= 1 \end{aligned} \tag{24}$$

Expressed in terms of the wave number *k*, this becomes:

$$\begin{aligned} \delta\left(\frac{k}{2\pi}\right)\delta x &= 1 \\ \delta k\,\delta x &= 2\pi \end{aligned}\,. \tag{25}$$

In other words, if you want a wave to have a pulse of length *x* at most, you must have a range of *k* values of at least $2\pi/x$.

4.2.8.6 *Resolution of spectra*

A spectrometer is a device that measures wavelengths. Equation (25) can be used to work out the accuracy (or resolution) of the measurement.

If you want a minimum error δk in the wavenumber, you must have a distance of at least $\delta x=2\pi/\delta k$. But what does this distance mean? It transpires that this is the maximum path difference between two rays in going through the device – and as such is proportional to the size of the spectrometer. So, the bigger the spectrometer, the better its measurements are.

4.2.9 Doppler Effect

4.2.9.1 *Classical Doppler Effect*

Suppose a bassoonist is playing a beautiful pure note with frequency *f*. Now imagine that he is practising while driving along a road. A fellow motorist hears the lugubrious sound. What frequency does the listener hear? Let us suppose that the player is moving at velocity *u*, and the listener is moving at velocity *v*. For simplicity we only consider the problem in one dimension, however velocities can still be positive or negative.

Furthermore, imagine that the distance between player and listener is L_0 at time zero, when the first wave-peak is broadcast from the bassoon. We assume that the waves travel at speed c with respect to the ground.

This peak is received at time t_1, where

$$\begin{aligned} L_0 + vt_1 &= ct_1 \\ L_0 &= (c - v)t_1 \end{aligned} \tag{26}$$

The first line is constructed like this: The travellers start a distance L_0 apart, so by the time the signal is received, the distance between them is $L_0 + vt_1$. This distance is covered by waves of speed c in time t_1 – hence the right hand side.

The next wave peak will be broadcast at time 1/f – one wave cycle later. At this time, the distance between the two musicians will be $L_0 + (v-u)T = L_0 + (v-u)/f$. This second peak will be received at time t_2, where

$$\begin{gathered} L_0 + \frac{v-u}{f} + v\left(t_2 - \frac{1}{f}\right) = c\left(t_2 - \frac{1}{f}\right). \\ L_0 + \frac{c-u}{f} = (c-v)t_2 \end{gathered} \tag{27}$$

Finally we can work out the time interval elapsed between our listener hearing the two peaks, and from this the apparent frequency is easy to determine.

$$\begin{gathered} (t_2 - t_1)(c - v) = L_0 + \frac{c-u}{f} - L_0 \\ \frac{1}{f'} = (t_2 - t_1) = \frac{c-u}{f(c-v)} \\ f' = f \times \frac{c-v}{c-u} \end{gathered} \tag{28}$$

From this we see that if $v=u$, no change is observed. If the two are approaching, the apparent frequency is high (blue-shifted). If the two are receding, the apparent frequency is low (red-shifted).

4.2.9.2 Relativistic Doppler Effect

Please note that if either u or v are appreciable fractions of the speed of light, this formula will give errors, and the relativistic calculation must be used.

For light *only*, the relativistic formula is

$$f' = f\sqrt{\frac{c+u}{c-u}}\ , \tag{29}$$

where u is the approach velocity as measured by the observer (u is negative if the source and observer are receding). The relativistic form for other waves is more complicated, and will be left for another day.

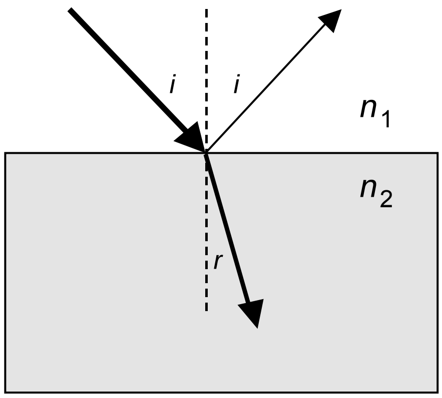
4.3 Rays

4.3.1 Reflection and Refraction

All the discussion so far has centred on the oscillatory nature of waves. We can predict some of the things waves do without worrying about the oscillations – like reflection and refraction. The diagram below shows both. We refer to a refractive index of a material, which is defined as

$$\text{Refractive Index }(n) = \frac{\text{Speed of light in vacuum}}{\text{Speed of light in the material}}\ . \tag{30}$$

Air has a refractive index of about 1.0003[14], glass has a refractive index of about 1.5, and water about 1.3.



First of all, the angle of reflection is equal to the angle of incidence (both were labelled *i* in the diagram).

Secondly, the angle of incidence is related to the angle of refraction *r* by the formula:

$$\frac{\sin i}{\sin r} = \frac{n_2}{n_1} \tag{31}$$

[14] The refractive index also gives a measure of pressure, since n-1 is proportional to pressure.

Notice that since the sine of an angle can be no larger than one, if $n_2<n_1$, then refraction becomes impossible if $i > \sin^{-1}(n_2/n_1)$. This limiting angle is called the critical angle. For greater angles of incidence, the entire wave is reflected, and this is called total internal reflection.

When a wave passes from one material into another, the frequency remains the same (subject to the linearity provisos of section 4.1.1). Given that the speed changes, the wavelength will change too. The wavelength of light in a particular material can be evaluated:

$$\lambda = \frac{c}{f} = \frac{c_0}{nf} = \frac{\lambda_0}{n} \tag{32}$$

where c_0 (and λ_0) represent speed of light (and wavelength) in vacuum.

4.4 Fermat's Principle

Fermat's principle gives us a method of working out the route light will take in an optical system. It states that light will take the route that takes the least time. Given that the time taken in a single material is equal to

$$T = \frac{D}{c} = \frac{nD}{c_0} \tag{33}$$

where c_0 is the speed of light in vacuum; minimizing the time is the same as minimizing the product of distance and refractive index. This latter quantity (*nD*) is called the optical path. It is possible to prove the laws of reflection and refraction using this principle.[15]

4.5 Questions

1. A hole is drilled through the Earth from the U.K. to the centre of the Earth and out of the other side. All the material is sucked out of it, and a 1kg mass is dropped in at the British end. How much time passes before it momentarily comes to rest at the Australian end? (NB You may need some hints from section 1.2.4) +

2. Repeat q1 where a straight hole is drilled between any two places on Earth. Assume that the contact of the mass with the sides of the hole is frictionless. ++

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[15] To do this, imagine the plane as a sheet of graph paper, with the boundary along the x-axis. Suppose that the light starts at point (0,Y), and needs to get to (X,-Y). Now assume that the light crosses the x-axis at point (x,0). Work out the total optical path travelled along the route, and then minimize it with respect to x. You should then be able to identify sin *i* and sin *r* in the algebraic soup, and from this, you should be able to finish the proof.

3. Show that Fermat's principle allows you to 'derive' the Law of Reflection. Assume that you have a mirror along the *x*-axis. Let light start at point (X_1,Y_1) and end at point (X_2,Y_2). Show that the least-time reflected route is the one which bounces off the mirror where angles of incidence and reflection will be equal. +

4. Show that Fermat's principle allows you to 'derive' Snell's Law. Assume that you have a material with refractive index 1 for $y>0$ (that is, above the *x*-axis), and refractive index n where $y<0$. Show that the shortest time route from point (X_1,Y_1) to (X_2,Y_2), where $Y_1>0$ and $Y_2<0$, crosses the boundary at the point where $\sin i / \sin r = n$. +

5. You are the navigator for a hiking expedition in rough ground. Your company is very thirsty and tired, and your supplies have run out. There is a river running East-West which is 4km South of your current position. Your objective is to reach the base camp (which is 2km South and 6km West of your current position), stopping off at the river on the way. What is the quickest route to the camp via the river? +

6. You are the officer in charge of a food convoy attempting to reach a remote village in a famine-stricken country. On your map, you see that 50km to your East is a straight border (running North-South) between scrub land (over which you can travel at 15km/hr) and marsh (over which you can only travel at 5km/hr). The village is 141km South-East of your current position. What is the fastest route to reach the village? +

7. In an interferometer, a beam of coherent monochromatic light (with wavelength λ) is split into two parts. Both parts travel for a distance L parallel to each other. One travels in vacuum, the other in air. The beam is then re-combined. If destructive interference results, what can you say about L, λ, and n_{air}?

8. Your wind band is about to play on a pick-up truck going down a motorway at 30m/s. You want people on the bridges overhead to hear you playing 'in tune' (such that treble A is 440Hz) when you are coming directly towards them. What frequency should you tune your instruments to?

9. A police 'speed gun' uses microwaves with a wavelength of about 3cm. The 'gun' consists of a transmitter and receiver, with a small mirror which sends part of the transmission directly into the receiver. Here it interferes with the main beam which has reflected off a vehicle. The received signal strength pulsates (or beats). What will be the frequency of this pulsation if the vehicle is travelling towards you at 30mph? +

10. How far away do you need to hold a ruler from your left eye before you can no longer resolve the millimetre markings? Keep your right eye covered up during this experiment. Use equation (22) to make an

estimate for the wavelength of light based on your measurement. Remember that *W* is the width of your pupil.

11. A signal from a distant galaxy has one third of the frequency you would expect from a stationary galaxy. Calculate the galaxy's recession velocity using equation (29), and comment on your answer. (NB red-shifts this big *are* measured with very distant astronomical objects.)