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Master of Science in Advanced Mathematics and Mathematical Engineering

Title: Fractional Brownian motion in stochastic financial modeling

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Contents

1 Introduction

The rigorous mathematical analysis of stochastic processes is central to modern probability theory and its applications. Among the most powerful tools in this area is Malliavin calculus, a stochastic calculus of variations developed to study the regularity of probability laws associated with functionals of Brownian motion. Unlike classical Itô calculus, which is limited to adapted integrands, Malliavin calculus allows differentiation of non-adapted random variables, making it particularly suitable for analyzing anticipating processes and developing explicit representations of functionals.

The main focus of this thesis is the detailed construction and study of Malliavin calculus based on the Wiener-Itô chaos expansion, following the algebraic approach presented in [Giu08]. This formulation allows us to represent square-integrable random variables as infinite series of iterated Itô integrals, enabling a concrete and computationally accessible definition of the Malliavin derivative. Throughout the thesis, we highlight the key components of this framework, such as the Skorokhod integral and the Clark-Ocone formula.

In parallel, we introduce and study the fractional Brownian motion (fBm), a Gaussian process characterized by its Hurst parameter $H \in (0,1)$, which generalizes classical Brownian motion by incorporating memory and both long- and short-range dependence. The fBm is not a semimartingale when $H \neq \frac{1}{2}$, thus lying outside the scope of traditional Itô calculus and motivating the need for more general stochastic analysis techniques. We explore its representations, existence results, and key properties that make it a central object in modern stochastic modeling.

While financial applications such as rough volatility models provide context and motivation, the core contribution of this thesis lies in the mathematical development and exposition of Malliavin calculus and fractional Brownian motion. To illustrate the power of these tools, we revisit classical models such as Black-Scholes and contrast them with the rough volatility framework of the Rough Bergomi model in [BFa16]. In particular, we focus on the short-time behavior of at-the-money implied volatility (ATMI), where Malliavin calculus enables precise asymptotic analysis. This approach leads to decomposition formulas and convergence results that are highly sensitive to both the correlation structure and the regularity of the volatility path, emphasizing the strength of Malliavin techniques in modern stochastic analysis.

2 Malliavin calculus

Malliavin calculus, also known as the stochastic calculus of variations, is an infinite-dimensional differential calculus on the Wiener space. It was originally developed to study the regularity of the probability laws of functionals of stochastic processes. The theory was introduced by Paul Malliavin in the 1970s in connection with his probabilistic proof of Hörmander's theorem [Mal78], which concerns the smoothness of densities of diffusion processes.

Unlike classical Itô calculus, which operates under the constraint that integrands must be adapted to the filtration generated by the driving Brownian motion, Malliavin calculus enables the differentiation of non-adapted random variables with respect to the underlying noise. This extension allows for the analysis and manipulation of anticipative processes and functionals, proving particularly useful in contexts where traditional stochastic calculus falls short.

In this thesis, we adopt the Malliavin calculus framework based on the Wiener-Itô chaos expansion, as presented in [Giu08]. This formulation expresses square-integrable random variables as infinite series of iterated Itô integrals, enabling a convenient and algebraically intuitive definition of the Malliavin derivative. In particular, the derivative resembles the classical differentiation of monomials, which facilitates the derivation of key results such as the chain rule and the Clark-Ocone formula. We select and present the most relevant material, expanding upon the original proofs by providing additional context and clarification. In some cases, we also introduce omitted proofs where appropriate. Furthermore, we present original examples designed to enhance understanding and illustrate key concepts.

The theory introduces several powerful tools:

- The **Wiener-Itô chaos expansion**, which represents square-integrable random variables as infinite series of multiple stochastic integrals with respect to Brownian motion.
- The **Skorokhod integral**, an extension of the Itô integral to possibly non-adapted integrands. It is defined as the adjoint of the Malliavin derivative and plays a key role in anticipating stochastic calculus.
- The **Malliavin derivative**, a differential operator acting on random variables defined on the Wiener space, which measures their sensitivity to perturbations of the underlying Brownian path.
- The **Clark-Ocone formula**, an explicit representation of square-integrable functionals of Brownian motion as stochastic integrals, enabling the computation of optimal hedging strategies and sensitivities in financial mathematics.

While this thesis adopts the chaos expansion approach for its algebraic clarity and convenience, it is important to note that Malliavin calculus is more fundamentally developed through the Sobolev space framework in the analysis on Wiener space. This approach, as presented for example in [Nua05], forms the foundation of the modern theory and provides deeper insight into the differential structure of Gaussian functionals. It is particularly well-suited for studying the regularity of probability laws, such as the existence and smoothness of densities of solutions to stochastic differential equations. Although more abstract than the chaos expansion method, the two frameworks are mathematically equivalent under suitable conditions and offer complementary perspectives for both theoretical development and applications.

2.1 The Wiener-Itô Chaos Expansion

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Throughout this work, $W = \{W_t, t \in [0,T]\}$ will denote a standard one-dimensional Wiener process (or Brownian motion) on this space, where $T > 0$ is a fixed positive terminal time. This process is characterized by $W_0 = 0$ $\mathbb{P}$-almost surely (a.s.), continuous paths a.s., and independent increments such that $W_t - W_s$ is normally distributed with mean 0 and variance $t - s$ for $0 \leq s < t \leq T$. We associated with W its natural filtration $\{\mathcal{F}_t, t \in [0,T]\}$, which is assumed to satisfy the usual conditions of completion and right-continuity.

Our construction of the Wiener-Itô chaos expansion will rely on integrands from specific function spaces. We begin by defining the space of square-integrable functions on an n-dimensional cube.

Definition 2.1. The space $L^2([0,T]^n)$ comprises all real-valued, Borel measurable functions $g : [0,T]^n \to \mathbb{R}$ that satisfy the square-integrability condition:

$$\|g\|^2_{L^2([0,T]^n)} := \int\limits_{[0,T]^n} g^2(t_1, \ldots, t_n)\, dt_1 \cdots dt_n < \infty. \tag{2.1}$$

This norm endows $L^2([0,T]^n)$ with a Hilbert space structure.

Within $L^2([0,T]^n)$, functions exhibiting symmetry in their arguments are of particular importance for the theory of Wiener-Itô chaos expansion.

Definition 2.2. A function $g \in L^2([0,T]^n)$ is called *symmetric* if its value is invariant under any permutation of its arguments. That is, for any $(t_1, \ldots, t_n) \in [0,T]^n$ and any permutation σ of the set $\{1, 2, \ldots, n\}$,

$$g(t_{\sigma_1}, \ldots, t_{\sigma_n}) = g(t_1, \ldots, t_n).$$

The set of all such symmetric functions forms a closed subspace of $L^2([0,T]^n)$, which we will denote by $\widetilde{L}^2([0,T]^n)$.

While the theory of Wiener-Itô chaos expansion primarily utilizes symmetric integrands, we often start with arbitrary functions from $L^2([0,T]^n)$. To bridge this, we introduce the operation of symmetrization, which projects any function onto the subspace of symmetric functions. Let $f : [0,T]^n \to \mathbb{R}$ be a real-valued function. The *symmetrization* of f, denoted by $\tilde{f}$, is defined as

$$\tilde{f}(t_1, \ldots, t_n) := \frac{1}{n!} \sum_{\sigma} f(t_{\sigma_1}, \ldots, t_{\sigma_n}), \tag{2.2}$$

where the sum runs over all permutations σ of $\{1, \ldots, n\}$. This symmetrization operator is clearly linear.

The resulting function $\tilde{f}$ is indeed symmetric and if $f \in L^2([0,T]^n)$, then $\tilde{f} \in \widetilde{L}^2([0,T]^n)$.

To make the definition of symmetrization more concrete, let us consider an example for $n = 3$.

Example 2.3. Let $f(t_1, t_2, t_3) = t_1 + t_2 e^{t_3}$, $(t_1, t_2, t_3) \in [0,T]^3$, then its symmetrization is given by

$$\tilde{f}(t_1, t_2, t_3) = \frac{1}{3!}\Big[f(t_1, t_2, t_3) + f(t_1, t_3, t_2) + f(t_2, t_1, t_3) + f(t_2, t_3, t_1) + f(t_3, t_1, t_2) + f(t_3, t_2, t_1)\Big].$$

Collecting terms, this evaluates to:

$$\tilde{f}(t_1, t_2, t_3) = \frac{1}{6}\left[2(t_1 + t_2 + t_3) + t_2 e^{t_3} + t_3 e^{t_2} + t_1 e^{t_3} + t_3 e^{t_1} + t_1 e^{t_2} + t_2 e^{t_1}\right].$$

One can verify that this resulting function $\tilde{f}(t_1, t_2, t_3)$ is now symmetric with respect to any permutation of t_1, t_2, t_3.

Furthermore, if f is already symmetric, then $\tilde{f} = f$. Integrals of symmetric functions over $[0,T]^n$ can be simplified by considering their behavior on a smaller, ordered domain known as the n-simplex.

Definition 2.4. The *n-simplex* is defined as:

$$S_n = \{(t_1, \ldots, t_n) \in [0,T]^n : 0 \le t_1 \le \cdots \le t_n \le T\}. \tag{2.3}$$

On this domain, the time variables are ordered non-decreasingly.

Due to the symmetry of any function $g \in \widetilde{L}^2([0,T]^n)$, its values across the entire cube $[0,T]^n$ are determined by its values on S_n. The cube $[0,T]^n$ can be decomposed into $n!$ disjoint regions, each of which is a permutation on S_n. The integral of g^2 over each such region is identical. Consequently, the integral over the full cube is $n!$ times the integral over the simplex S_n:

$$\|g\|^2_{L^2([0,T]^n)} = n! \int\limits_{S_n} g^2(t_1, \ldots, t_n)\, dt_1 \cdots dt_n = n! \|g\|^2_{L^2(S_n)}. \tag{2.4}$$

Definition 2.5. Let f be a deterministic function defined on S_n for $n \ge 1$, such that

$$\|f\|^2_{L^2(S_n)} := \int\limits_{S_n} f^2(t_1, \ldots, t_n)\, dt_1 \cdots dt_n < \infty.$$

Then the *n-fold iterated Itô integral* of f is defined as

$$J_n(f) := \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f(t_1, \ldots, t_n)\, dW_{t_1} \cdots dW_{t_n}. \tag{2.5}$$

As a consequence of the construction of the Itô integral and the square-integrability of f over S_n, the n-fold iterated integral $J_n(f)$ is a well-defined random variable. Moreover, it belongs to $L^2(\mathbb{P})$, meaning it has finite variance and is square-integrable. Recall that the norm of $X \in L^2(\mathbb{P})$ is defined by

$$\|X\|_{L^2(\mathbb{P})} := \left(\mathbb{E}[X^2]\right)^{\frac{1}{2}} = \left(\int_{\Omega} X^2 \, d\mathbb{P}\right)^{\frac{1}{2}}. \tag{2.6}$$

A fundamental property of these iterated Itô integrals, which is essential for the structure of the Wiener-Itô chaos expansion, relates their expected products to the inner products of their deterministic integrands.

In addition to these properties, the n-fold iterated Itô integral acts as a linear operator.

Remark 2.6. By linearity of the Itô integral, the n-fold iterated Itô integral is a linear operator. That is, for any $f, g \in L^2(S_n)$ and $a, b \in \mathbb{R}$, we have

$$J_n(af + bg) = aJ_n(f) + bJ_n(g).$$

Proposition 2.7. *Let $g \in L^2(S_m)$ and $h \in L^2(S_n)$ be deterministic functions. The following relation holds for the iterated Itô integrals $J_m(g)$ and $J_n(h)$:*

$$\mathbb{E}[J_m(g)\, J_n(h)] = \begin{cases} 0, & \text{if } m \neq n, \\ (g, h)_{L^2(S_n)}, & \text{if } m = n, \end{cases} \quad \text{for all } m, n \in \mathbb{N}, \tag{2.7}$$

where

$$(g, h)_{L^2(S_n)} := \int_{S_n} g(t_1, \ldots, t_n)\, h(t_1, \ldots, t_n)\, dt_1 \cdots dt_n$$

is the inner product in $L^2(S_n)$. In particular, for $m = n$ and $g = h$, this yields the Itô isometry for n-fold iterated integrals:

$$\|J_n(h)\|_{L^2(\mathbb{P})} = \|h\|_{L^2(S_n)}. \tag{2.8}$$

Proof. The proof relies on the iterative application of the basic Itô isometry (see [**ito isometry**]), which states that for suitable integrands X_t, Y_t, $\mathbb{E}\left[\left(\int_0^T X_t\, dW_t\right)\left(\int_0^T Y_t\, dW_t\right)\right] = \mathbb{E}\left[\int_0^T X_t Y_t\, dt\right]$ and on the Fubini's theorem that, under some coniditions, allows the interchange of the expectation and the stochastic integrals yielding $\mathbb{E}\left[\int_0^T X_t Y_t\, dt\right] = \int_0^T \mathbb{E}\left[X_t Y_t\right]\, dt$. Since g and h are deterministic, the expectation over their product can be removed.

Case 1: $m = n$. In this case, $g = g(s_1, s_2, \ldots, s_m)$ and $h = h(t_1, t_2, \ldots, t_n) = h(s_1, s_2, \ldots, s_m)$. The Itô isometry is applied n times, eliminating the expectation entirely. The result reduces to the inner product of the functions f and g in $L^2(S_n)$.

$$\mathbb{E}\left[J_n(g)J_n(h)\right] = \mathbb{E}\left[\left(\int_0^T\int_0^{s_n}\cdots\int_0^{s_2} g\, dW_{s_1}\cdots dW_{s_n}\right)\cdot\left(\int_0^T\int_0^{s_n}\cdots\int_0^{s_2} h\, dW_{s_1}\cdots dW_{s_n}\right)\right]$$
$$= \int_0^T\cdots\int_0^{s_2}(g)(h)\, ds_1\cdots ds_n = (g,h)_{L^2(S_n)}.$$

The identity Eq. (2.8) follows directly by setting $g = h$:

$$\|J_n(h)\|_{L^2(\mathbb{P})} = \mathbb{E}\left[J_n(h)^2\right]^{\frac{1}{2}} = \left[(h,h)_{L^2(S_n)}\right]^{\frac{1}{2}} = \|h\|_{L^2(S_n)}.$$

Case 2: $m \neq n$. Assume $m < n$. For brevity, we write $g = g(s_1,\dots,s_{m-1},s_m)$ and $h = h(t_1,\dots,t_{n-m},\dots,t_n) = h(t_1,\dots,t_{n-m},s_1,\dots s_{m-1},s_m)$. We again apply the Itô isometry m times. Only iterated stochastic integrals belonging to J_n remain inside the expectation. By definition, these have zero mean, and therefore the entire expression vanishes:

$$\mathbb{E}\left[J_m(g)J_n(h)\right] = \mathbb{E}\left[\left(\int_0^T\int_0^{s_m}\cdots\int_0^{s_2} g\, dW_{s_1}\cdots dW_{s_m}\right)\cdot\left(\int_0^T\int_0^{t_n}\cdots\int_0^{t_2} h\, dW_{t_1}\cdots dW_{t_n}\right)\right]$$
$$= \mathbb{E}\left[\left(\int_0^T\int_0^{s_m}\cdots\int_0^{s_2} g\, dW_{s_1}\cdots dW_{s_{m-1}}dW_{s_m}\right)\cdot\right.$$
$$\left.\left(\int_0^T\int_0^{s_m}\cdots\int_0^{s_1}\int_0^{t_{n-m}}\cdots\int_0^{t_2} h\, dW_{t_1}\cdots dW_{t_{n-m}}dW_{s_1}\cdots dW_{s_{m-1}}dW_{s_m}\right)\right]$$
$$= \int_0^T \mathbb{E}\left[\left(\int_0^{s_m}\cdots\int_0^{s_2} gdW_{s_1}\cdots dW_{s_{m-1}}\right)\cdot\left(\int_0^{s_m}\cdots\int_0^{t_2} hdW_{t_1}\cdots dW_{s_{m-1}}\right)\right] ds_m = \dots$$
$$= \int_0^T\int_0^{s_m}\cdots\int_0^{s_2} g\cdot\mathbb{E}\Big[\int_0^{s_1}\cdots\int_0^{t_2} hdW_{t_1}\cdots dW_{t_{n-m}}\Big] ds_1\cdots ds_m = 0.$$

■

Definition 2.8. If $g \in \widetilde{L}^2([0,T]^n)$, we also define the *$n$-fold Itô iterated integrals* as

$$I_n(g) := \int_{[0,T]^n} g(t_1,\dots,t_n)dW_{t_1}\cdots dW_{t_n} := n!J_n(g), \tag{2.9}$$

where J_n is the n-fold iterated Itô integral in Definition 2.5.

The n-fold Itô iterated integrals $I_n(g)$ also posses the isometry and orthogonality properties.

Proposition 2.9. *Let $g \in \widetilde{L}^2(S_m)$ and $h \in \widetilde{L}^2(S_n)$ be symmetric and deterministic functions. The following relation holds for $I_m(g)$ and $I_n(h)$:*

$$\mathbb{E}[I_m(g)\, I_n(h)] = \begin{cases} 0, & \text{if } m \neq n, \\ n!(g,h)_{L^2([0,T]^n)}, & \text{if } m = n, \end{cases} \quad \text{for all } m, n \in \mathbb{N}, \tag{2.10}$$

with

$$(g,h)_{L^2([0,T]^n)} = n!(g,h)_{L^2(S_n)}. \tag{2.11}$$

Proof. We have that

$$\mathbb{E}[I_m(g) I_n(h)] = n!m!\, \mathbb{E}[J_m(g) J_n(h)],$$

so, by Proposition 2.7, the result follows. ∎

Proposition 2.10. *For all $g \in \widetilde{L}^2([0,T]^n)$, the following identity holds:*

$$\|I_n(g)\|^2_{L^2(\mathbb{P})} = n!\|g\|^2_{L^2([0,T]^n)}.$$

Proof. Making use of the identity in Eq. (2.8),

$$\begin{aligned} \|I_n(g)\|^2_{L^2(\mathbb{P})} &= \mathbb{E}[I_n^2(g)] = \mathbb{E}[(n!)^2 J_n^2(g)] = (n!)^2 \mathbb{E}[J_n^2(g)] = (n!)^2 \|J_n(g)\|^2_{L^2(\mathbb{P})} \\ &= (n!)^2 \|g\|^2_{L^2(S_n)} = (n!)^2 \frac{1}{n!} \|g\|^2_{L^2([0,T]^n)} = n!\|g\|^2_{L^2([0,T]^n)}. \end{aligned} \tag{2.12}$$

∎

Hermite polynomials are closely connected to the Wiener-Itô chaos expansion. They form an orthogonal basis in the space of square-integrable functions with respect to the Gaussian measure and appear naturally when computing iterated Itô integrals. Thanks to a result by Itô, certain multiple Wiener-Itô integrals can be expressed directly using Hermite polynomials applied to Gaussian random variables, simplifying calculations.

Definition 2.11. The *Hermite polynomials* $h_n(x)$ are defined by the explicit formula:

$$h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2}), \quad x \in \mathbb{R},\, n \in \mathbb{N}. \tag{2.13}$$

The first few Hermite polynomials are:

$$h_0(x) = 1,\ h_1(x) = x,\ h_2(x) = x^2 - 1,\ h_3(x) = x^3 - 3x,\ h_4(x) = x^4 - 6x^2 + 3.$$

A key property of the Hermite polynomials is their differentation rule:

$$h_n'(x) = n h_{n-1}(x), \quad n \geq 1. \tag{2.14}$$

We state without proof a powerful formula for computing certain iterated Itô integrals, which highlights the direct connection to Hermite polynomials. The proof is beyond the scope of this thesis and can be found in page 11 of [Giu08].

Let $g \in L^2([0,T])$. Let $g^{\otimes n}$ denote the n-th tensor product function $g^{\otimes n}(t_1, \ldots, t_n) = g(t_1)g(t_2)\cdots g(t_n)$, which is a symmetric function in $\tilde{L}^2([0,T]^n)$. The n-fold Itô iterated integrals of $g^{\otimes n}$ is given by:

$$I_n(g_n) = n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} g(t_1)g(t_2)\cdots g(t_n)dW_{t_1}\cdots dW_{t_n} = \|g\|^n h_n\left(\frac{\theta}{\|g\|}\right), \tag{2.15}$$

where $\|g\| = \|g\|_{L^2([0,T])}$ and $\theta = \int_0^T g(t)\, dW(t)$. Here, h_n denotes the n-th Hermite polynomial.

Example 2.12. Let us apply the formula in the case where $n = 4$ and $g(t) = 1$ for $t \in [0,T]$. Then, $\|g\|_{L^2([0,T])} = \left(\int_0^T (1)^2\, dt\right)^{\frac{1}{2}} = \sqrt{T}$ and the Itô integral $\theta = \int_0^T (1)\, dW(t) = W(T)$. We compute $I_4(1^{\otimes 4})$:

$$\begin{aligned} I_4(1^{\otimes 4}) &= 4! \int_0^T \int_0^{t_4} \int_0^{t_3} \int_0^{t_2} dW_{t_1}dW_{t_2}dW_{t_3}dW_{t_4} = (\sqrt{T})^4 h_4\left(\frac{W(T)}{\sqrt{T}}\right) \\ &= T^2\left(\frac{W^4(T)}{T^2} - \frac{6W^2(T)}{T} + 3\right) = W^4(T) - 6TW^2(T) + 3T^2. \end{aligned}$$

We have defined the multiple Wiener-Itô integrals $I_n(f_n)$ for symmetric kernels $f_n \in \tilde{L}^2([0,T]^n)$, and established key properties such as their orthogonality in $L^2(\mathbb{P})$ and the associated isometry. These integrals play a central role in the Wiener-Itô chaos expansion, which provides a unique representation for square-integrable random variables with respect to the Wiener process.

Theorem 2.13 (The Wiener-Itô chaos expansion). *Let ξ be an $\mathcal{F}_T$-measurable random variable in $L^2(\mathbb{P})$. Then there exists a unique sequence $\{f_n\}_{n=0}^{\infty}$ of symmetric functions $f_n \in \tilde{L}^2([0,T]^n)$ such that*

$$\xi = \sum_{n=0}^{\infty} I_n(f_n), \tag{2.16}$$

where the convergence is in $L^2(\mathbb{P})$. For $n = 0$, $I_0(f_0) = f_0 = \mathbb{E}[\xi]$. Moreover, the following isometry holds:

$$\|\xi\|^2_{L^2(\mathbb{P})} = \sum_{n=0}^{\infty} n!\|f_n\|^2_{L^2([0,T]^n)}. \tag{2.17}$$

Proof. The proof is lengthy and omitted here for brevity. It relies on the Itô representation theorem, which allows a square-integrable random variable to be expressed as the sum of its expectation and an Itô integral. A full proof can be found in page 12 of [Giu08]. ∎

Proposition 2.14. *Let ξ and ζ be $\mathcal{F}_T$-measurable random variables in $L^2(\mathbb{P})$ with Wiener-Itô chaos expansions $\xi = \sum_{n=0}^{\infty} I_n(f_n)$ and $\zeta = \sum_{n=0}^{\infty} I_n(g_n)$, where $f_n, g_n \in \tilde{L}^2([0,T]^n)$ for each $n \in \mathbb{N}_0$. Then the chaos expansion of the sum $\xi + \zeta$ is given by*

$$\xi + \zeta = \sum_{n=0}^{\infty} I_n(h_n),$$

where $h_n = f_n + g_n$ for all $n \in \mathbb{N}_0$.

Proof. The sum of ξ and ζ is given by

$$\xi + \zeta = \sum_{n=0}^{\infty} I_n(f_n) + \sum_{n=0}^{\infty} I_n(g_n) = \sum_{n=0}^{\infty} \left(I_n(f_n) + I_n(g_n)\right) = \sum_{n=0}^{\infty} n! \left(J_n(f_n) + J_n(g_n)\right).$$

By the linearity of the n-fold Itô iterated integrals, as stated in Remark 2.6, we obtain:

$$\xi + \zeta = \sum_{n=0}^{\infty} n! \left(J_n(f_n) + J_n(g_n)\right) = \sum_{n=0}^{\infty} n! J_n(f_n + g_n) = \sum_{n=0}^{\infty} I_n(f_n + g_n),$$

where the sum $h_n = f_n + g_n$ is also a symmetric function in $\widetilde{L}^2([0,T]^n)$ for each $n \in \mathbb{N}_0$. ■

To better understand the Wiener-Itô chaos expansion in practice, we now compute the expansion for two specific random variables.

Example 2.15. We aim to find the chaos expansion for $\xi = W_T^2$. We start by applying Itô formula to $f(x) = x^2$ with $X_t = W_t$ (so $W_0 = 0$ and $d\langle W, W\rangle_t = dt$):

$$\begin{aligned} W_T^2 &= W_0^2 + \int_0^T \frac{\partial \xi(t, W_t)}{\partial t}\, dt + \int_0^T \frac{\partial \xi(t, W_t)}{\partial W_t}\, dW_t + \frac{1}{2}\int_0^T \frac{\partial^2 \xi(t, W_t)}{\partial W_t^2}\, d\langle W, W\rangle_t \\ &= 0 + \int_0^T 0\, dt + \int_0^T 2W_t\, dW_t + \frac{1}{2}\int_0^T 2\, dt = 2\int_0^T W_t\, dW_t + T. \end{aligned}$$

Applying the Itô's formula again to the W_t term we have that $W_t = \int_0^t dW_s$. By plugging it in inside the first expression we express the random variable in the Wiener-Itô chaos expansion setting:

$$W_T^2 = T + 2\int_0^T \int_0^{t_2} dW_{t_1} dW_{t_2} = T + 2! \int_0^T \int_0^{t_2} dW_{t_1} dW_{t_2} = I_0[f_0] + I_2[f_2],$$

with $f_0 = T$, $f_2(t_1, t_2) = 1$ and $f_n = 0$ for $n \geq 1$, $n \neq 2$.

This result can be verified using the formula connecting $I_n(f^{\otimes n})$ to Hermite polynomials (Eq. (2.15)). For $I_2(1^{\otimes 2})$ (i.e., $n = 2$ and $g(t) = 1$), we have $\|g\|_{L^2([0,T])} = \left(\int_0^T (1)^2\, dt\right)^{\frac{1}{2}} = \sqrt{T}$ and $\theta = \int_0^T (1)\, dW(t) = W(T)$:

$$I_2[1] = 2! \int_0^T \int_0^{t_2} dW_{t_1} dW_{t_2} = (\sqrt{T})^2 h_2\left(\frac{W_T}{\sqrt{T}}\right) = T\left(\frac{W_T^2}{T} - 1\right) = W_T^2 - T.$$

As a further illustration of deriving Wiener-Itô chaos expansions, we consider the random variable $\xi = W_{t_0}^3$ for a fixed time $t_0 \in [0, T]$.

Example 2.16. Let $\xi = W_{t_0}^3$ where $t_0 \in [0,T]$ is fixed. We will use Eq. (2.15) with $n = 3$ and $f(t) = \chi_{[0,t_0]}(t)$, where χ is the indicator function. The L^2-norm of f on $[0,T]$ is $\|f\| = \sqrt{t_0}$ and $\theta = W_{t_0}$. Let $f^{\otimes 3}(t_1,t_2,t_3) = f(t_1)f(t_2)f(t_3) = \chi_{[0,t_0]}(t_1)\chi_{[0,t_0]}(t_2)\chi_{[0,t_0]}(t_3)$. Then:

$$I_3[f_3] = \left(\sqrt{t_0}\right)^3 h_3\left(\frac{W_{t_0}}{\sqrt{t_0}}\right) = t_0^{\frac{3}{2}}\left(\frac{W_{t_0}^3}{t_0^{\frac{3}{2}}} - 3\frac{W_{t_0}}{\sqrt{t_0}}\right) = W_{t_0}^3 - 3t_0 W_{t_0}.$$

Rearranging this identity to express $W_{t_0}^3$, we obtain:

$$W_{t_0}^3 = 3t_0 W_{t_0} + I_3[f_3] = I_1[f_1] + I_3[f_3], \tag{2.18}$$

with $f_0 = \mathbb{E}[W_{t_0}^3] = 0$, $f_1(t_1) = 3t_0\chi_{[0,t_0]}(t_1)$,
$f_2(t_1,t_2) = 0$, $f_3(t_1,t_2,t_3) = \chi_{[0,t_0]}(t_1)\chi_{[0,t_0]}(t_2)\chi_{[0,t_0]}(t_3)$ and $f_n = 0$ for $n \geq 4$.

2.2 The Skorohod Integral

We introduce the Skorohod integral, a key concept in stochastic analysis and Malliavin calculus. This integral extends the classical Itô integral to a broader class of integrands, notably including those that are not necessarily adapted to the underlying filtration $(\mathcal{F}_t)_{t\in[0,T]}$. We will present its definition using the Itô-Wiener chaos expansion, explore some of its fundamental properties, and show how it naturally relates to the Itô integral. This approach highlights the strength of the chaos expansion framework in analyzing the construction and behavior of the Skorohod integral. We will also work through concrete examples to illustrate its computation.

Consider a measurable stochastic process $u = u(t,\omega)$ defined for $(t,\omega) \in [0,T] \times \Omega$. We assume that for almost every $t \in [0,T]$, the random variable $u(t)$ is $\mathcal{F}_T$-measurable and $u(t) \in L^2(\mathbb{P})$. Under these conditions, $u(t)$ admits a Wiener-Itô chaos expansion for each t. That is, for each $t \in [0,T]$, there exists a unique sequence of symmetric kernel functions where $f_{n,t} \in \widetilde{L}^2([0,T]^n)$, defined on $[0,T]^n$ for $n \in \mathbb{N}$, such that:

$$u(t) = \sum_{n=0}^{\infty} I_n(f_{n,t}) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t)). \tag{2.19}$$

Here, I_n denotes the n-fold Itô integral (see Definition 2.8) and $f_n(t_1,\dots,t_n,t)$ is the kernel for $u(t)$, which is symmetric in its first n variables $(t_1,\dots,t_n)$ and also depends on the parameter t.

To define the Skorohod integral of u with respect to the Wiener process, we need to construct a set of fully symmetric kernels, typically denoted $\tilde{f}_n(t_1,\dots,t_{n+1})$, from the $(n+1)$-variable functions $f_n(t_1,\dots,t_n,t_{n+1})$. The symmetrization is defined as:

$$\tilde{f}_n(t_1,\dots,t_{n+1}) = \frac{1}{n+1}\sum_{i=1}^{n+1} f_n(t_1,\dots,\widehat{t_i},\dots,t_{n+1},t_i), \tag{2.20}$$

where $f_n(t_1,\dots,t_n,t_{n+1})$ is understood to be symmetric in its first n arguments $(t_1,\dots,t_n)$ and the hat over t_i in the sum indicates that t_i is omitted from its current position among

$(t_1, \ldots, t_{n+1})$ and instead serves as the $(n+1)$-th argument (the "time parameter" slot) of f_n.

Since f_n is already symmetric in the first n arguments, there is no need to average over all $(n+1)!$ permutations as in Eq. (2.2). Instead, it suffices to consider the $n+1$ distinct ways of placing t_{n+1} into each position.

Example 2.17. Let $u(t) = W_{t_0}^3$, with $t_0 \in [0,T]$, fixed. By previous computations (see Example 2.16) $W_{t_0}^3 = I_1[f_1] + I_3[f_3]$ with $f_0 = \mathbb{E}[W_{t_0}^3] = 0$, $f_1(t_1) = 3t_0\chi_{[0,t_0]}(t_1)$, $f_2(t_1,t_2) = 0$, $f_3(t_1,t_2,t_3) = \chi_{[0,t_0]}(t_1)\chi_{[0,t_0]}(t_2)\chi_{[0,t_0]}(t_3)$ and $f_n = 0$ for $n \geq 4$. To symmetrize these functions we apply Eq. (2.20) to both f_1 and f_3:

$$\begin{aligned}\tilde{f}_1(t_1,t) &= \frac{1}{2}\left[f_1(t,t_1) + f_1(t_1,t)\right] = \frac{3t_0}{2}[\chi_{[0,t_0]}(t) + \chi_{[0,t_0]}(t_1)] = \frac{3t_0}{2}\left[\chi_{\{t<t_0\}} + \chi_{\{t_1<t_0\}}\right] \\ &= 3t_0\chi_{\{t_1,t<t_0\}} + \frac{3t_0}{2}\chi_{\{t<t_0<t_1\}} + \frac{3t_0}{2}\chi_{\{t_1<t_0<t\}},\end{aligned} \tag{2.21}$$

$$\begin{aligned}\tilde{f}_3(t_1,t_2,t_3,t) &= \frac{1}{4}\left[f_3(t,t_2,t_3,t_1) + f_3(t_1,t,t_3,t_2) + f_3(t_1,t_2,t,t_3) + f_3(t_1,t_2,t_3,t)\right] \\ &= \frac{1}{4}\left[\chi_{\{t,t_2,t_3<t_0\}} + \chi_{\{t_1,t,t_3<t_0\}} + \chi_{\{t_1,t_2,t<t_0\}} + \chi_{\{t_1,t_2,t_3<t_0\}}\right] \\ &= \chi_{\{t_1,t_2,t_3,t<t_0\}} \\ &\quad + \frac{1}{4}\left[\chi_{\{t,t_2,t_3<t_0<t_1\}} + \chi_{\{t_1,t,t_3<t_0<t_2\}} + \chi_{\{t_1,t_2,t<t_0<t_3\}} + \chi_{\{t_1,t_2,t_3<t_0<t\}}\right].\end{aligned} \tag{2.22}$$

Definition 2.18. Let $u : [0,T] \times \Omega \to \mathbb{R}$ be a measurable stochastic process such that for every $t \in [0,T]$, the random variable $u(t)$ is $\mathcal{F}_T$-measurable. Assume further that u is square-integrable in the sense that $\mathbb{E}\left[\int_0^T u^2(t)\,dt\right] < \infty$, (i.e., $u \in L^2([0,T] \times \Omega)$). Suppose that $u(t)$ admits a Wiener-Itô chaos expansion for each $t \in [0,T]$ of the form $u(t) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t))$.

The *Skorohod integral* of u, denoted $\delta(u)$, is defined by

$$\delta(u) := \int_0^T u(t)\delta W(t) := \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n), \tag{2.23}$$

provided that this series converges in $L^2(\mathbb{P})$. In this case, we say that u is Skorohod integrable and write $u \in \mathrm{Dom}(\delta)$.

The condition for the convergence of the series in Definition 2.18, and thus for u to be in the domain of δ, can be expressed in terms of the norms of the symmetrized kernels $\tilde{f}_n$.

Proposition 2.19. *A process u satisfying the conditions in Definition 2.18 is Skorohod integrable (i.e., $u \in \mathrm{Dom}(\delta)$) if and only if the series of norms of its symmetrized chaos kernels converges:*

$$\sum_{n=0}^{\infty}(n+1)!\|\tilde{f}_n\|^2_{L^2([0,T]^{n+1})} < \infty. \tag{2.24}$$

Furthermore, if $u \in \mathrm{Dom}(\delta)$, its $L^2(\mathbb{P})$-norm is given by

$$\mathbb{E}[\delta(u)^2] = \sum_{n=0}^{\infty}(n+1)!\|\tilde{f}_n\|^2_{L^2([0,T]^{n+1})}. \tag{2.25}$$

Proof. By definition, $\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n)$. We compute the second moment of this sum:

$$\begin{aligned}\mathbb{E}[\delta(u)^2] &= \mathbb{E}\left[\left(\sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n)\right)^2\right] = \mathbb{E}\left[\sum_{n=0}^{\infty} I_{n+1}^2(\tilde{f}_n) + 2\sum_{j<n}^{\infty} I_{j+1}(\tilde{f}_j) I_{n+1}(\tilde{f}_n)\right] \\ &= \sum_{n=0}^{\infty} \mathbb{E}\left[I_{n+1}^2(\tilde{f}_n)\right] + 2\sum_{j<n}^{\infty} \mathbb{E}\left[I_{j+1}(\tilde{f}_j) I_{n+1}(\tilde{f}_n)\right]\end{aligned}$$

Due to the orthogonality of multiple Wiener-Itô integrals of different orders (Proposition 2.9), the cross-terms of different orders vanish. Thus, the sum simplifies to:

$$\mathbb{E}[\delta(u)^2] = \sum_{n=0}^{\infty} \mathbb{E}\left[I_{n+1}^2(\tilde{f}_n)\right].$$

By the isometry property of iterated stochastic integrals(Proposition 2.10), we have

$$\mathbb{E}\left[I_{n+1}^2(\tilde{f}_n)\right] = (n+1)! \, \|\tilde{f}_n\|^2_{L^2([0,T]^{n+1})}.$$

Substituting this back, we obtain:

$$\mathbb{E}[\delta(u)^2] = \sum_{n=0}^{\infty} (n+1)! \, \|\tilde{f}_n\|^2_{L^2([0,T]^{n+1})}.$$

The series defining $\delta(u)$ converges in $L^2(\mathbb{P})$ if and only if $\mathbb{E}[\delta(u)^2] < \infty$. Therefore, $u \in \mathrm{Dom}(\delta)$ if and only if the sum $\sum_{n=0}^{\infty} (n+1)! \, \|\tilde{f}_n\|^2_{L^2([0,T]^{n+1})}$ is finite, and in that case, $\mathbb{E}[\delta(u)^2]$ is equal to this sum. ∎

Proposition 2.20. *For any $u \in \mathrm{Dom}(\delta)$, the Skorohod integral satisfies*

$$\mathbb{E}[\delta(u)] = 0.$$

Proof. Since $\delta(u)$ is expressed as a sum of n-fold Itô iterated integrals, each of which has zero expectation, the result follows immediately. ∎

Let us now illustrate the definition and computation of the Skorohod integral with an example.

Example 2.21. We compute the Skorohod integral of the process $u(t) = W_T$ for all $t \in [0, T]$. The chaos expansion of $u(t) = W_T$ is $I_1(f_1)$, where $f_1 = \chi_{[0,T]}(t_1) = 1$ and $f_n = 0$ for all $n \neq 1$. By definition, to compute the Skorohod integral, we need the symmetrization of f_1. In this case the symmetrization is given by $\tilde{f}_1(t_1, t) = \frac{1}{2}[f_1(t, t_1) + f_1(t_1, t)] = \frac{1}{2}(1+1) = 1$. Thus,

$$\delta(W_T) = I_2[\tilde{f}_1] = I_2[1] = 2! J_2[1] = 2! \int_0^T \int_0^{t_2} (1) \, dW_{t_1} dW_{t_2} = 2\left(\frac{W_T^2 - T}{2}\right) = W_T^2 - T.$$

In this case, the chaos expansion $I_2[1]$ can be computed using the Hermite polynomial formula as in Example 2.15.

Proposition 2.22. *The Skorohod integral is a linear operator.*

Proof. Let u and v be processes in $\mathrm{Dom}(\delta)$ with their respective Wiener-Itô chaos expansions, $u(t) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t))$, $v(t) = \sum_{n=0}^{\infty} I_n(g_n(\cdot,t))$, where $f_n(\cdot,t), g_n(\cdot,t) \in \widetilde{L}^2([0,T]^n)$ for each n and $t \in [0,T]$. Let α and β be real numbers. By the linearity of the n-fold iterated Itô integrals (Proposition 2.14), the linear combination $\alpha u(t) + \beta v(t)$ has the chaos expansion

$$\alpha u(t) + \beta v(t) = \sum_{n=0}^{\infty} I_n(\alpha f_n(\cdot,t) + \beta g_n(\cdot,t)).$$

Let $h_n(t_1,\ldots,t_n,t) := \alpha f_n(t_1,\ldots,t_n,t) + \beta g_n(t_1,\ldots,t_n,t)$. Then the symmetrized function is $\tilde{h}_n(t_1,\ldots,t_{n+1}) = \alpha \tilde{f}_n(t_1,\ldots,t_{n+1}) + \beta \tilde{g}_n(t_1,\ldots,t_{n+1})$, since symmetrization is also a linear operation.

The Skorohod integral of $\alpha u + \beta v$ is defined as:

$$\begin{aligned}
\delta(\alpha u + \beta v) &= \delta\left(\alpha \sum_{n=0}^{\infty} I_n(f_n(\cdot,t)) + \beta \sum_{n=0}^{\infty} I_n(g_n(\cdot,t))\right) = \delta\left(\sum_{n=0}^{\infty} I_n(\alpha f_n(\cdot,t) + \beta g_n(\cdot,t))\right) \\
&= \delta\left(\sum_{n=0}^{\infty} I_n(h_n(\cdot,t))\right) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{h}_n) = \sum_{n=0}^{\infty} I_{n+1}(\alpha \tilde{f}_n + \beta \tilde{g}_n) \\
&= \alpha \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n) + \beta \sum_{n=0}^{\infty} I_{n+1}(\tilde{g}_n) = \alpha\delta(u) + \beta\delta(v).
\end{aligned}$$

■

As previously discussed, the Skorohod integral generalizes the Itô integral. In particular, when the integrand u is $\mathcal{F}_t$-adapted, the Skorohod and Itô integrals agree as elements of $L^2(\mathbb{P})$. To establish this equivalence, it is necessary to characterize adaptedness in terms of the functions $f_n(\cdot,t)$, $n \geq 1$, appearing in the Wiener-Itô chaos expansion. The precise condition linking the adaptedness of a process $u(t)$ to its chaos kernels $f_n(\cdot,t)$ is given by the following lemma.

Lemma 2.23. *Let $u = u(t)$, $t \in [0,T]$, be a measurable stochastic process such that, for all $t \in [0,T]$, the random variable $u(t)$ belongs to $L^2(\Omega)$ and is $\mathcal{F}_T$-measurable. Suppose that u admits the Wiener-Itô chaos expansion $u(t) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t))$. Then u is $\mathcal{F}_t$-adapted if and only if*

$$f_n(t_1,\ldots,t_n,t) = 0 \quad \textit{if } t < \max_{1\leq i\leq n} t_i,$$

for almost every $(t_1,\ldots,t_n) \in [0,T]^n$ with respect to Lebesgue measure.

Proof. A process u is $\mathcal{F}_t$-adapted if and only if $u(t) = \mathbb{E}[u(t)|\mathcal{F}_t]$ for all t. Applying this to the chaos expansion of $u(t)$, the following must be satisfied:

$$u(t) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t)) = \mathbb{E}\left[\sum_{n=0}^{\infty} I_n(f_n(\cdot,t))\Big|\mathcal{F}_t\right] = \sum_{n=0}^{\infty} \mathbb{E}\left[I_n(f_n(\cdot,t))\Big|\mathcal{F}_t\right].$$

Let $f \in \widetilde{L}^2([0,T]^n)$ be symmetric, then the conditional expectation of $u(t)$ is:

$$\mathbb{E}[I_n(f)|\mathcal{F}_t] = n!\,\mathbb{E}[J_n(f)|\mathcal{F}_t] = n!\mathbb{E}\left[\int_0^T\int_0^{t_n}\cdots\int_0^{t_2} f(t_1,\ldots,t_n)\,dW_{t_1}\cdots dW_{t_n}\middle|\mathcal{F}_t\right]$$
$$= n!\int_0^t\int_0^{t_n}\cdots\int_0^{t_2} f(t_1,\ldots,t_n)\,dW_{t_1}\cdots dW_{t_n} = I_n[f(t_1,\ldots,t_n)\cdot\chi_{\{\max t_i<t\}}].$$

Following the $\mathcal{F}_t$-adaptedness definition,

$$\sum_{n=0}^{\infty} I_n(f_n(\cdot,t)) = \sum_{n=0}^{\infty}\mathbb{E}[I_n(f_n(\cdot,t))|\mathcal{F}_t] = \sum_{n=0}^{\infty} I_n\Big(f_n(\cdot,t)\cdot\chi_{\{\max t_i<t\}}\Big).$$

Hence, u is adapted if and only if for each n, the functions $f_n(\cdot,t)$ satisfy

$$f_n(t_1,\ldots,t_n,t) = f_n(t_1,\ldots,t_n,t)\cdot\chi_{\{\max t_i<t\}}$$

almost everywhere. This is equivalent to $f_n(t_1,\ldots,t_n,t) = 0$ whenever $t < \max_{1\le i\le n} t_i$, as required. ■

Lemma 2.24. *Let $u(t)$ be an $\mathcal{F}_t$-adapted stochastic process with Wiener-Itô chaos expansion $u(t) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t))$, where $f_n(\cdot,t) \in \widetilde{L}([0,T]^n)$ for each $t \in [0,T]$. Let $\tilde{f}_n$ be the $(n+1)$-variable symmetrized kernel. Then, for each $n \geq 0$, the following relationship holds:*

$$\|\tilde{f}_n\|^2_{L^2([0,T]^{n+1})} = \frac{1}{n+1}\int_0^T \|f_n(\cdot,t)\|^2_{L^2([0,T]^n)}\,dt. \tag{2.26}$$

Proof. By Eq. (2.20), the symmetrization $\tilde{f}_n$ of the kernel $f_n(t_1,\ldots,t_n,t)$ of the $\mathcal{F}_t$-adapted process $u(t)$ only contains the permutation in which the last variable is the maximal variable:

$$\tilde{f}_n(t_1,\ldots,t_n,t_{n+1}) = \frac{1}{n+1} f_n(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{n+1},t_j) \tag{2.27}$$

where $j := \arg\max_{1\leq i\leq n+1} t_i$.

$$\begin{aligned}
\|\tilde{f}_n\|^2_{L^2([0,T]^{n+1})} &= (n+1)!\|\tilde{f}_n\|^2_{L^2(S_{n+1})} = (n+1)! \int\limits_{S_{n+1}} \tilde{f}_n^2(t_1,\dots,t_n,t_{n+1})dt_1\cdots dt_n dt_{n+1} \\
&= \frac{(n+1)!}{(n+1)^2} \int\limits_{S_{n+1}} f_n^2(t_1,\dots,t_n,t_{n+1})dt_1\cdots dt_n dt_{n+1} \\
&= \frac{n!}{n+1} \int\limits_0^T \int\limits_0^{t_{n+1}} \int\limits_0^{t_n} \cdots \int\limits_0^{t_2} f_n^2(t_1,\dots,t_n,t_{n+1})\, dt_1\cdots dt_n dt_{n+1} \\
&= \frac{n!}{n+1} \int\limits_0^T \int\limits_0^{t} \int\limits_0^{t_n} \cdots \int\limits_0^{t_2} f_n^2(t_1,\dots,t_n,t) dt_1\cdots dt_n dt \\
&= \frac{n!}{n+1} \int\limits_0^T \int\limits_0^{T} \int\limits_0^{t_n} \cdots \int\limits_0^{t_2} f_n^2(t_1,\dots,t_n,t) dt_1\cdots dt_n dt \\
&= \frac{1}{n+1} \int\limits_0^T \|f_n(\cdot,t)\|^2_{L^2([0,T]^n)}\, dt.
\end{aligned}$$

Eq. (2.4) is applied in the first equality and the symmetrized function $\tilde{f}_n$ is replaced by its definition in Eq. (2.27) in the third equality. In the fifth equality we perform a change of variable, setting $t := t_{n+1}$, which corresponds to the maximal time variable, consistent with the adaptedness definition. In the sixth equality, we extend the upper integration bound from t to T in the second integral. This is justified because the kernel $f_n(t_1,\dots,t_n,t)$ is typically defined with an indicator function $\chi_{[0,t]^n}(t_1,\dots,t_n)$.

■

Theorem 2.25. *Let $u = u(t)$, $t \in [0,T]$, be a measurable $\mathcal{F}_t$-adapted stochastic process such that $\mathbb{E}\left[\int_0^T u^2(t)\, dt\right] < \infty$. Then, $u \in Dom(\delta)$ and its Skorohod integral coincides with the Itô integral, that is: $\int_0^T u(t)\delta W(t) = \int_0^T u(t)dW(t)$.*

Proof. Assume u admits a chaos expansion of the form $u(t) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t))$. Using Eq. (2.25) in the first equality and applying the result from Lemma 2.24 in the second, we obtain:

$$\begin{aligned}
\mathbb{E}\left[\delta(u)^2\right] &= \sum_{n=0}^{\infty}(n+1)!\|\tilde{f}_n\|^2_{L^2([0,T]^{n+1})} = \sum_{n=0}^{\infty}(n+1)!\frac{1}{n+1}\int\limits_0^T \|f_n(\cdot,t)\|^2_{L^2([0,T]^n)}\, dt \\
&= \int\limits_0^T \sum_{n=0}^{\infty} n!\|f_n(\cdot,t)\|^2_{L^2([0,T]^n)}\, dt.
\end{aligned}$$

In the next step, we apply Eq. (2.12) to rewrite the norm of $f_n(\cdot,t)$, and use Proposition 2.9 to simplify the square of the chaos expansion. Specifically, $\left(\sum_{n=0}^{\infty} I_n(f_n)\right)^2 = \sum_{n=0}^{\infty} I_n^2(f_n) +$

$2\sum_{j<n} I_j(f_j)I_n(f_n) = \sum_{n=0}^{\infty} I_n^2(f_n)$ which eliminates all cross terms. Therefore:

$$\int_0^T \sum_{n=0}^{\infty} n!\|f_n(\cdot,t)\|^2_{L^2([0,T]^n)}\,dt = \int_0^T \sum_{n=0}^{\infty} \mathbb{E}\left[I_n^2(f_n)\right]\,dt = \int_0^T \mathbb{E}\left[\sum_{n=0}^{\infty} I_n^2(f_n)\right]\,dt$$
$$= \int_0^T \mathbb{E}\left[\left(\sum_{n=0}^{\infty} I_n(f_n)\right)^2\right]\,dt = \int_0^T \mathbb{E}\left[(u(t))^2\right]\,dt = \mathbb{E}\left[\int_0^T u^2(t)\,dt\right].$$

This completes the derivation of the identity

$$\mathbb{E}[\delta(u)^2] = \mathbb{E}\left[\int_0^T u^2(t)\,dt\right], \tag{2.28}$$

which proves the first claim. We now show that the Itô integral and Skorohod integral coincide when the integrand is adapted. As in the proof of Lemma 2.24, the adaptedness allows us to set $t_{n+1} := t$ and extend the upper integration limit from t to T.

$$\int_0^T u(t)\,\delta W_t = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n) = \sum_{n=0}^{\infty}(n+1)!J_{n+1}(\tilde{f}_n)$$
$$= \sum_{n=0}^{\infty}(n+1)!\int_0^T\int_0^{t_{n+1}}\int_0^{t_n}\cdots\int_0^{t_2} \tilde{f}_n(t_1,\ldots,t_n,t_{n+1})\,dW_{t_1}\cdots dW_{t_n}dW_{t_{n+1}}$$
$$= \sum_{n=0}^{\infty} n!\int_0^T\int_0^{t}\int_0^{t_n}\cdots\int_0^{t_2} f_n(t_1,\ldots,t_n,t)\,dW_{t_1}\cdots dW_{t_n}dW_t$$
$$= \sum_{n=0}^{\infty} n!\int_0^T\int_0^{T}\int_0^{t_n}\cdots\int_0^{t_2} f_n(t_1,\ldots,t_n,t)\,dW_{t_1}\cdots dW_{t_n}dW_t$$
$$= \sum_{n=0}^{\infty}\int_0^T\left[n!\int_0^{T}\int_0^{t_n}\cdots\int_0^{t_2} f_n(t_1,\ldots,t_n,t)\,dW_{t_1}\cdots dW_{t_n}\right]dW_t$$
$$= \sum_{n=0}^{\infty}\int_0^T I_n[f_n(\cdot,t)]dW_t = \int_0^T \sum_{n=0}^{\infty} I_n[f_n(\cdot,t)]dW_t = \int_0^T u(t)dW_t.$$

This completes the proof of the second claim. ■

As an example, we now compute two illustrative Skohorod integrals via their Itô-Wiener chaos expansions.

Example 2.26. In the following example, we compute the Skorohod integral of the adapted process $u(t) = W_t$. That is, $\delta(u(t)) = \delta(W_t) = \int_0^T W_t\delta W_t$. According to Theorem 2.25, since W_t is adapted, the Skorohod integral coincides with the classical Itô integral. Hence, the result is well known:

$$\int_0^T W_t\,\delta W_t = \int_0^T W_t\,dW_t = \frac{W_T^2 - T}{2}.$$

Nonetheless, we carry out the computation using the Skorohod integral's definition via Wiener-Itô chaos expansion, in order to explicitly verify this equivalence.

Given that $W_t = \int_0^T \chi_{[0,t]}(t_1)\, dW_{t_1}$, its Wiener-Itô chaos expansion is $W_t = I_1[f_1(\cdot, t)]$ with $f_1(t_1, t) = \chi_{[0,t]}(t_1)$. To compute the Skorohod integral, we must symmetrize this function. According to Lemma 2.23, since the integrand is adapted, all permutations vanish except for the one in which the last variable is the maximum among the arguments. Therefore, the symmetrization yields:

$$\tilde{f}_1(t_1, t) = \frac{1}{2}\left[f_1(t, t_1) + f_1(t_1, t)\right] = \frac{1}{2} f_1(t_1, t) = \frac{1}{2}\chi_{[0,t]}(t_1).$$

Now, the integral is given by

$$\begin{aligned}\delta(I_1[f_1(\cdot, t)]) = I_2[\tilde{f}_1] &= 2! \int_0^T \int_0^{t_2} \tilde{f}_1(t_1, t_2)\, dW_{t_1} dW_{t_2} \\ &= 2! \int_0^T \int_0^{t_2} \frac{1}{2}\chi_{[0,t_2]}(t_1)\, dW_{t_1} dW_{t_2} = \int_0^T W_{t_2}\, dW_{t_2} = \frac{W_T^2 - T}{2},\end{aligned}$$

and thus we observe that the result coincides with the classical Itô integral.

Example 2.27. In this example, we compute the Skorohod integral of $u(t) = W_{t_0}^3$, $t_0 \in [0, T]$. By Example 2.17, $W_{t_0}^3 = I_1[f_1] + I_3[f_3]$ with $f_0 = \mathbb{E}[W_{t_0}^3] = 0$, $f_1(t_1) = 3t_0\chi_{[0,t_0]}(t_1)$, $f_2(t_1, t_2) = 0$, $f_3(t_1, t_2, t_3) = \chi_{[0,t_0]}(t_1)\chi_{[0,t_0]}(t_2)\chi_{[0,t_0]}(t_3)$ and $f_n = 0$ for $n \geq 4$ with the following symmetrizations:

$$\tilde{f}_1(t_1, t) = 3t_0\chi_{\{t_1,t<t_0\}} + \frac{3t_0}{2}\chi_{\{t<t_0<t_1\}} + \frac{3t_0}{2}\chi_{\{t_1<t_0<t\}},$$

$$\begin{aligned}\tilde{f}_3(t_1, t_2, t_3, t) &= \chi_{\{t_1,t_2,t_3,t<t_0\}} \\ &\quad + \frac{1}{4}\left[\chi_{\{t,t_2,t_3<t_0<t_1\}} + \chi_{\{t_1,t,t_3<t_0<t_2\}} + \chi_{\{t_1,t_2,t<t_0<t_3\}} + \chi_{\{t_1,t_2,t_3<t_0<t\}}\right].\end{aligned}$$

By the linearity of the Skorohod integral (Proposition 2.22):

$$\delta(W_{t_0}^3) = \delta\left(I_1[f_1] + I_3[f_3]\right) = \delta(I_1[f_1]) + \delta(I_3[f_3]) = I_2[\tilde{f}_1] + I_4[\tilde{f}_3].$$

The first's term expansion is:

$$\begin{aligned}I_2[\tilde{f}_1] = 2!(3t_0) &\int_0^T \int_0^{t_2} \chi_{\{t_1,t_2<t_0\}}\, dW_{t_1} dW_{t_2} + 2!(3t_0) \int_0^T \int_0^{t_2} \frac{1}{2}\chi_{\{t_2<t_0<t_1\}}\, dW_{t_1} dW_{t_2} \\ + 2!(3t_0) &\int_0^T \int_0^{t_2} \frac{1}{2}\chi_{\{t_1<t_0<t_2\}}\, dW_{t_1} dW_{t_2} = I + II + III.\end{aligned}$$

The indicator function appearing in the integral II contradicts the constraints imposed by the domain of the n-fold Itô iterated integrals. Specifically, the indicator in integral

II requires that $t_1 > t_2$, which is not possible since the integration domain satisfies $0 \le t_1 \le t_2 \le T$, as outlined in Eq. (2.3). Consequently, the condition inside the indicator function cannot be fulfilled, and the integrand vanishes almost everywhere.

The first integral, $I = 2!(3t_0)\int_0^T \int_0^{t_2} \chi_{\{t_1,t_2<t_0\}}\, dW_{t_1} dW_{t_2}$, is equal to

$$2!(3t_0)\int_0^T \int_0^{t_2} \chi_{\{t_1,t_2<t_0\}}\, dW_{t_1} dW_{t_2} = (3t_0)2\int_0^{t_0} \int_0^{t_2} dW_{t_1} dW_{t_2} = 3t_0(W_{t_0}^2 - t_0),$$

and the third integral, $III = 2!(3t_0)\int_0^T \int_0^{t_2} \frac{1}{2}\chi_{\{t_1<t_0<t_2\}}\, dW_{t_1} dW_{t_2}$, is equal to

$$2!(3t_0)\int_0^T \int_0^{t_2} \frac{1}{2}\chi_{\{t_1<t_0<t_2\}}\, dW_{t_1} dW_{t_2} = (3t_0)\int_{t_0}^T \int_0^{t_0} dW_{t_1} dW_{t_2} = 3t_0(W_T - W_{t_0})(W_{t_0}).$$

Therefore:

$$I_2[\tilde{f}_1] = 3t_0\left[(W_{t_0}^2 - t_0) + ((W_T - W_{t_0})(W_{t_0}))\right] = 3t_0W_{t_0}^2 - 3t_0^2 + 3t_0W_{t_0}W_T - 3t_0W_{t_0}^2.$$

The second's term expansion is:

$$\begin{aligned}
I_4[\tilde{f}_3] = {} & 4!\int_0^T \int_0^{t_4} \int_0^{t_3} \int_0^{t_2} \chi_{\{t_1,t_2,t_3,t_4<t_0\}}\, dW_{t_1} dW_{t_2} dW_{t_3} dW_{t_4} \\
& + 4!\int_0^T \int_0^{t_4} \int_0^{t_3} \int_0^{t_2} \frac{1}{4}\chi_{\{t_4,t_2,t_3<t_0<t_1\}}\, dW_{t_1} dW_{t_2} dW_{t_3} dW_{t_4} \\
& + 4!\int_0^T \int_0^{t_4} \int_0^{t_3} \int_0^{t_2} \frac{1}{4}\chi_{\{t_1,t_4,t_3<t_0<t_2\}}\, dW_{t_1} dW_{t_2} dW_{t_3} dW_{t_4} \\
& + 4!\int_0^T \int_0^{t_4} \int_0^{t_3} \int_0^{t_2} \frac{1}{4}\chi_{\{t_1,t_2,t_4<t_0<t_3\}}\, dW_{t_1} dW_{t_2} dW_{t_3} dW_{t_4} \\
& + 4!\int_0^T \int_0^{t_4} \int_0^{t_3} \int_0^{t_2} \frac{1}{4}\chi_{\{t_1,t_2,t_3<t_0<t_4\}}\, dW_{t_1} dW_{t_2} dW_{t_3} dW_{t_4} \\
& = I + II + III + IV + V.
\end{aligned}$$

As in the first's term expansion, the indicator functions in integrals II, III, and IV contradict the domain constraints of the iterated Itô integrals (Eq. (2.3)): For instance, in II, $t_1 > t_2$ cannot occur since $0 \le t_1 \le t_2 \le T$, making the integrands vanish almost everywhere leaving integrals I and V left to compute.

The first integral, $I = 4!\int_0^T \int_0^{t_4} \int_0^{t_3} \int_0^{t_2} \chi_{\{t_1,t_2,t_3,t_4<t_0\}}\, dW_{t_1} dW_{t_2} dW_{t_3} dW_{t_4}$, can be easily

computed using the formula in Eq. (2.15), with $\|f\| = \sqrt{t_0}$ and $\theta = W_{t_0}$:

$$4! \int_0^T \int_0^{t_4} \int_0^{t_3} \int_0^{t_2} \chi_{\{t_1,t_2,t_3,t_4<t_0\}} dW_{t_1} dW_{t_2} dW_{t_3} dW_{t_4} = \left(\sqrt{t_0}\right)^4 h_4\left(\frac{W_{t_0}}{\sqrt{t_0}}\right)$$
$$= t_0^2\left(\frac{W_{t_0}^4}{t_0^2} - 6\frac{W_{t_0}^2}{t_0} + 3\right) = W_{t_0}^4 - 6t_0W_{t_0}^2 + 3t_0^2.$$

The fifth integral, $V = 4! \int_0^T \int_0^{t_4} \int_0^{t_3} \int_0^{t_2} \frac{1}{4}\chi_{\{t_1,t_2,t_3<t_0<t_4\}}\, dW_{t_1} dW_{t_2} dW_{t_3} dW_{t_4}$, can be computed as follows:

$$4! \int_0^T \int_0^{t_4} \int_0^{t_3} \int_0^{t_2} \frac{1}{4}\chi_{\{t_1,t_2,t_3<t_0<t_4\}} dW_{t_1} dW_{t_2} dW_{t_3} dW_{t_4} = 6 \int_{t_0}^T \int_0^{t_0} \int_0^{t_3} \int_0^{t_2} dW_{t_1} dW_{t_2} dW_{t_3} dW_{t_4}$$
$$= 6 \int_{t_0}^T \int_0^{t_0} \left(\frac{W_{t_3}^2 - t_3}{2}\right) dW_{t_3} dW_{t_4} = 3 \int_{t_0}^T \left[\frac{1}{3}W_{t_0}^3 - \int_0^{t_0} W_{t_3}\, dt_3 - \int_0^{t_0} t_3\, dW_{t_3}\right] dW_{t_4}$$
$$= \int_{t_0}^T W_{t_0}^3 dW_{t_4} - 3\int_{t_0}^T \left[\int_0^{t_0} W_{t_3}\, dt_3 + \int_0^{t_0} t_3\, dW_{t_3}\right] dW_{t_4} = \int_{t_0}^T W_{t_0}^3 dW_{t_4} - 3\int_{t_0}^T t_0 W_{t_0} dW_{t_4}$$
$$= (W_T - W_{t_0})(W_{t_0}^3 - 3t_0W_{t_0}) = W_T W_{t_0}^3 - 3t_0 W_T W_{t_0} - W_{t_0}^4 + 3t_0 W_{t_0}^2.$$

In the third equality, we used the well-known result $\int_0^{t_0} W_{t_3}^2\, dW_{t_3} = \frac{1}{3}W_{t_0}^3 - \int_0^{t_0} W_{t_3}\, dt_3$, and in the fifth equality, the fact that $\int_0^{t_0} W_{t_3}\, dt_3 + \int_0^{t_0} t_3\, dW_{t_3} = t_0W_{t_0}$. This latter identity follows from applying the Itô integration by parts formula to the processes W_t and t. Specifically, since $d(W_t t) = W_t\, dt + t\, dW_t + d\langle W_t, t\rangle$, and the cross-variation term $d\langle W_t, t\rangle$ is zero, integrating both sides we obtain

$$W_{t_0}t_0 = \int_0^{t_0} W_{t_3}\, dt_3 + \int_0^{t_0} t_3\, dW_{t_3},$$

so the second term is equal to:

$$I_4[\tilde{f}_3] = I + V = W_{t_0}^4 - 6t_0W_{t_0}^2 + 3t_0^2 + (W_T - W_{t_0})(W_{t_0}^3 - 3t_0W_{t_0})$$
$$= W_{t_0}^3 W_T - 3t_0W_{t_0}^2 - 3t_0W_{t_0}W_T + 3t_0^2.$$

Adding both terms, the result of $\delta(W_{t_0}^3)$, $t_0 \in [0, T]$ is given by:

$$\delta(W_{t_0}^3) = 3t_0I_2[\tilde{f}_1] + I_4[\tilde{f}_3]$$
$$= 3t_0W_{t_0}^2 - 3t_0^2 + 3t_0W_{t_0}W_T - 3t_0W_{t_0}^2 + W_{t_0}^3W_T - 3t_0W_{t_0}^2 - 3t_0W_{t_0}W_T + 3t_0^2$$
$$= W_{t_0}^3 W_T - 3t_0W_{t_0}^2. \tag{2.29}$$

2.3 The Malliavin Derivative

In this section, we introduce the Malliavin derivative and present its fundamental properties. We begin with a precise definition based on the Wiener-Itô chaos expansion, followed by essential calculus rules such as the product and chain rules, which mirror their deterministic analogues. We then explore its connection with the Skorohod integral, which is formally the adjoint of the Malliavin derivative in $L^2(\mathbb{P})$, as captured by the duality formula. This relation serves as the basis for an integration by parts formula.

Definition 2.28. Let $F \in L^2(\mathbb{P})$ be an $\mathcal{F}_T$-measurable random variable admiting a Wiener-Itô chaos expansion of the form $F = \sum_{n=0}^{\infty} I_n(f_n)$, where each f_n is a symmetric square-integrable function on $[0,T]^n$, that is, $f_n \in \widetilde{L}^2([0,T]^n)$ for all $n \geq 1$. We say that F belongs to $\mathbb{D}_{1,2}$ if the following condition holds:

$$\|F\|^2_{\mathbb{D}_{1,2}} := \sum_{n=1}^{\infty} n n! \|f_n\|^2_{L^2([0,T]^n)} < \infty. \tag{2.30}$$

This condition ensures the existence of a derivative in the Malliavin sense. For such $F \in \mathbb{D}_{1,2}$, the *Malliavin derivative of F at time $t \in [0,T]$*, denoted $D_t F$, is defined by

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)), \quad t \in [0,T], \tag{2.31}$$

where the function $f_n(\cdot, t)$ is obtained evaluating the last argument of f_n at t, leaving the remaining $n-1$ variables to be integrated. The term $I_{n-1}(f_n(\cdot,t))$ denotes the $(n-1)$-fold Itô iterated integrals over this remaing variables. Moreover, the Malliavin derivative is linear.

Proposition 2.29. *Let λ be the Lebesgue measure, then the following relationship holds:*

$$\|D_t F\|^2_{L^2(\mathbb{P}\times\lambda)} = \|F\|^2_{\mathbb{D}_{1,2}}. \tag{2.32}$$

Proof. The proof follows similar arguments to the one in Proposition 2.19.

$$\begin{aligned}
\|D_t F\|^2_{L^2(\mathbb{P}\times\lambda)} &= \mathbb{E}\left[\int_0^T (D_t F)^2\, dt\right] = \mathbb{E}\left[\int_0^T \left(\sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot,t))\right)^2 dt\right] \\
&= \int_0^T \mathbb{E}\left[\left(\sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot,t))\right)^2\right] dt = \sum_{n=1}^{\infty} n^2 \int_0^T \mathbb{E}\left[I^2_{n-1}(f_n(\cdot,t))\right] dt \\
&= \sum_{n=1}^{\infty} n^2 \int_0^T (n-1)! \|f_n(\cdot,t)\|^2_{L^2([0,T]^{n-1})}\, dt = \sum_{n=1}^{\infty} n n! \|f_n\|^2_{L^2([0,T]^n)} = \|F\|^2_{\mathbb{D}_{1,2}}.
\end{aligned}$$

This identity establishes that the Malliavin derivative defines an isometry from $\mathbb{D}_{1,2}$ into $L^2(\mathbb{P} \times [0,T])$, thereby validating the definition of the $\mathbb{D}_{1,2}$-norm and ensuring that the derivative operator D is well defined. ■

Theorem 2.30 (Closability of the Malliavin derivative). *Suppose $F \in L^2(\mathbb{P})$ and $F_k \in \mathbb{D}_{1,2}$, $k \geq 1$ such that*

1. *$F_k \to F$, $k \to \infty$, in $L^2(\mathbb{P})$.*
2. *$\{D_t F_k\}_{k=1}^{\infty}$ converges in $L^2(\mathbb{P} \times \lambda)$.*

Then, $F \in \mathbb{D}_{1,2}$ and $D_t F_K \to D_t F$, $k \to \infty$, in $L^2(\mathbb{P} \times \lambda)$.

Proof. A complete proof can be found in page 28 of [Giu08]. ■

Example 2.31. Let $F = W_{t_0}^3 = I_1[f_1] + I_3[f_3]$ with $f_0 = \mathbb{E}[W_{t_0}^3] = 0$, $f_1(t_1) = 3t_0\chi_{[0,t_0]}(t_1)$, $f_2(t_1,t_2) = 0$, $f_3(t_1,t_2,t_3) = \chi_{[0,t_0]}(t_1)\chi_{[0,t_0]}(t_2)\chi_{[0,t_0]}(t_3)$ and $f_n = 0$ for $n \geq 4$. The Malliavin derivative of the random variable F at time t is given by

$$D_t(W_{t_0}^3) = D_t\left(I_1[f_1(t_1)]\right) + D_t(I_3[f_3(t_1,t_2,t_3)]) = I_0[f_1(t)] + 3I_2[f_3(t_1,t_2,t)].$$

The first term is equal to $I_0[f_1(t)] = 3t_0\chi_{[0,t_0]}(t)$, and the second term is equal to

$$\begin{aligned} 3I_2[f_3(t_1,t_2,t)] &= 6\int_0^T\int_0^{t_2} \chi_{[0,t_0]}(t_1)\chi_{[0,t_0]}(t_2)\chi_{[0,t_0]}(t)\,dW_{t_1}dW_{t_2} = 6\int_0^{t_0}\int_0^{t_2} dW_{t_1}dW_{t_2}\chi_{[0,t_0]}(t) \\ &= 6\left(\frac{W_{t_0}^2 - t_0}{2}\right)\chi_{[0,t_0]}(t) = 3W_{t_0}^2\chi_{[0,t_0]}(t) - 3t_0\chi_{[0,t_0]}(t). \end{aligned}$$

Putting everything together, the Malliavin derivative of the process $W_{t_0}^3$ is equal to

$$D_t(W_{t_0}^3) = 3W_{t_0}^2\chi_{[0,t_0]}(t). \tag{2.33}$$

We observe that the result resembles the classical differentiation of monomials.

Proposition 2.32. *Let $f_n = f^{\otimes n}$, for some $f \in L^2([0,T])$, that is, $f_n(t_1,\dots,t_n) = f_{t_1}\cdots f_{t_n}$. Then*

$$D_t h_n\left(\frac{\theta}{\|f\|}\right) = h_n'\left(\frac{\theta}{\|f\|}\right)\frac{f(t)}{\|f\|}, \tag{2.34}$$

where h_n is the Hermite polynomial of order n in Definition 2.11 and $\|f\| = \|f\|_{L^2([0,T])}$, $\theta = \int_0^T f_t\,dW_t$ as in Eq. (2.15).

Proof. By Eq. (2.15), we have that $I_n(f^{\otimes n}) = \|f\|^n h_n\left(\frac{\theta}{\|f\|}\right)$ or, rearranging, $\frac{I_n(f^{\otimes n})}{\|f\|^n} = h_n\left(\frac{\theta}{\|f\|}\right)$. Making use of the first equivalence in the third equality we have that:

$$D_t I_n(f_n) = nI_{n-1}(f(\cdot,t)) = nI_{n-1}(f^{\otimes(n-1)})f(t) = n\|f\|^{n-1}h_{n-1}\left(\frac{\theta}{\|f\|}\right)f(t). \tag{2.35}$$

Addressing the initial claim, using the result in Eq. (2.35) in the third equality and property Eq. (2.14) in the last equality, we have that:

$$\begin{aligned} D_t\left[h_n\left(\frac{\theta}{\|f\|}\right)\right] &= D_t\left[\frac{I_n(f^{\otimes n})}{\|f\|^n}\right] = \frac{1}{\|f\|^n}D_t[I_n(f^{\otimes n})] = \frac{1}{\|f\|^n}\left[n\|f\|^{n-1}h_{n-1}\left(\frac{\theta}{\|f\|}\right)f(t)\right] \\ &= \frac{1}{\|f\|}\left[nh_{n-1}\left(\frac{\theta}{\|f\|}\right)f(t)\right] = h_n'\left(\frac{\theta}{\|f\|}\right)\frac{f(t)}{\|f\|}. \end{aligned}$$

■

Let $\mathbb{D}_{1,2}^0$ denote the collection of square-integrable random variables whose Wiener chaos expansions consist of finitely many terms.

Theorem 2.33 (**Malliavin derivative product rule**)**.** *If $F_1, F_2 \in \mathbb{D}^0_{1,2}$, then both are elements of $\mathbb{D}_{1,2}$, and their product F_1F_2 also belongs to $\mathbb{D}_{1,2}$. Moreover, the Malliavin derivative satisfies the following product formula:*

$$D_t(F_1F_2) = F_1 D_t F_2 + F_2 D_t F_1. \tag{2.36}$$

Proof. A complete argument can be found in page 30 of [Giu08]. ■

Proposition 2.34. *Let $f \in L^2([0,T])$ and define $\theta := I_1(f) = \int_0^T f_s \, dW_s$. Then $\theta^n \in \mathbb{D}_{1,2}$ for all $n \in \mathbb{N}$, and its Malliavin derivative is given by:*

$$D_t\theta^n = n\theta^{n-1} f(t). \tag{2.37}$$

Proof. We first observe that since $f \in L^2([0,T])$, the random variable $\theta = I_1(f)$ possesses a finite Wiener chaos expansion, i.e., $\theta \in \mathbb{D}^0_{1,2}$. By repeated application of Theorem 2.33, it follows that $\theta^n \in \mathbb{D}^0_{1,2}$ for any integer $n \geq 1$. This ensures that the product rule for Malliavin derivatives is applicable to θ and its powers.

The formula in Eq. (2.37) can be established by induction on n. For $n = 1$, by the definition of the Malliavin derivative (Definition 2.28), $D_t\theta = 1 \cdot I_0(f(t)) = f(t)$. Now, suppose the claim holds for $n = k$, i.e., $D_t\theta^k = k\,\theta^{k-1} f(t)$. We aim to show the result for $n = k+1$. Using the product rule in Theorem 2.33:

$$D_t\theta^{k+1} = D_t\left(\theta \cdot \theta^k\right) = D_t\theta \cdot \theta^k + \theta \cdot D_t\theta^k.$$

Applying the induction hypothesis and the identity $D_t\theta = f(t)$, we obtain:

$$D_t\theta^{k+1} = f(t)\,\theta^k + \theta \cdot \left(k\,\theta^{k-1} f(t)\right) = (k+1)\theta^k f(t).$$

This completes the induction step, and thus the formula holds for all $n \in \mathbb{N}$. ■

Example 2.35. Consider the random variable $F = W_{t_0}^3 W_T$ for some fixed $t_0 \in [0,T]$. We aim to compute its Malliavin derivative. Observe that F can be expressed as a product $F = F_1F_2$, where $F_1 = W_{t_0}^3$ and $F_2 = W_T$. By the product rule in Theorem 2.33, we know that:

$$D_t(F) = D_t(W_T W_{t_0}^3) = W_{t_0}^3 D_t W_T + W_T D_t W_{t_0}^3.$$

To compute the respective Malliavin derivatives of the processes we can do so using the result in Proposition 2.34 with $f_s = \chi_{[0,t_0]}(t)$ and $n = 3$ for $W_{t_0}^3$ and with $f_s = \chi_{[0,T]}(t) = 1$ and $n = 1$ for W_T. That is:

$$D_t W_{t_0}^3 = 3W_{t_0}^2 \chi_{[0,T]}(t) \quad \text{and} \quad D_t W_T = 1.$$

Finally,

$$D_t(F) = W_{t_0}^3 + 3W_T W_{t_0}^2 \chi_{[0,T]}(t).$$

Theorem 2.36 (**Malliavin derivative chain rule**)**.** *Let $G \in \mathbb{D}_{1,2}$ and suppose that $g : \mathbb{R} \to \mathbb{R}$ is continuously differentiable with bounded derivative. Then the composition $g(G)$ also belongs to $\mathbb{D}_{1,2}$, and its Malliavin derivative is given by*

$$D_t g(G) = g'(G) D_t G.$$

Proof. We do not present the full proof here, as it involves technical constructions beyond the scope of this thesis. A full proof can be found in page 89 of [Giu08]. ■

Example 2.37. Let us compute the Malliavin derivative of the following random variable $F = e^{\sin\left(W_{t_0}^3 W_T\right)}$. By the previous example (Example 2.35) we know that $D_t(W_{t_0}^3 W_T) = W_{t_0}^3 + 3W_T W_{t_0}^2 \chi_{[0,T]}(t)$. Applying Theorem 2.36:

$$D_t[F] = D_t\left(e^{\sin\left(W_{t_0}^3 W_T\right)}\right) = e^{\sin\left(W_{t_0}^3 W_T\right)} \cdot \cos\left(W_{t_0}^3 W_T\right) \cdot [W_{t_0}^3 + 3W_T W_{t_0}^2 \chi_{[0,T]}(t)].$$

We now develop results on conditional expectations with respect to $\mathcal{F}_G$, the sigma-algebra reflecting information form the Wiener process on a Borel set $G \subseteq [0,T]$. These results will be instrumental for the subsequent analysis involving Malliavin derivatives and conditional expectations.

Definition 2.38. Let $G \subseteq [0,T]$ be a Borel set. We define $\mathcal{F}_G$ as the completion of the σ-algebra generated by the family of Wiener integrals

$$F = \int_0^T \chi_A(t)\, dW_t,$$

for all Borel sets $A \subseteq G$.

Lemma 2.39. *Let $u = u(t)$, defined for $t \in [0,T]$, be an $\mathcal{F}_t$-adapted stochastic process satisfying $u \in L^2(\mathbb{P} \times \lambda)$. Then, for any measurable subset $G \subseteq [0,T]$, the following identity holds:*

$$\mathbb{E}\left[\int_0^T u(t)\, dW_t \middle| \mathcal{F}_G\right] = \int_G \mathbb{E}[u(t)|\mathcal{F}_G]\, dW_t. \tag{2.38}$$

Proof. To keep the presentation concise, we do not include the proof. A detailed argument can be found in page 32 of [Giu08]. ■

Proposition 2.40. *Let $f_n \in \widetilde{L}^2([0,T]^n)$ for each $n \in \mathbb{N}$. Then, for any measurable Borel set $G \subseteq [0,T]$, the conditional expectation of the n-fold Wiener-Itô integral $I_n(f_n)$ with respect to $\mathcal{F}_G$ (Definition 2.38) satisfies*

$$\mathbb{E}[I_n(f_n) \mid \mathcal{F}_G] = I_n(f_n \cdot \chi_G^{\otimes n}), \tag{2.39}$$

where the kernel $f_n \cdot \chi_G^{\otimes n}$ is defined by

$$(f_n \cdot \chi_G^{\otimes n})(t_1, \ldots, t_n) = f_n(t_1, \ldots, t_n)\chi_G(t_1) \cdots \chi_G(t_n).$$

Proof. For $n = 1$, by Lemma 2.39:

$$\mathbb{E}[I_1(f_1) \mid \mathcal{F}_G] = \mathbb{E}\left[\int_0^T f_1(t_1)\, dW_{t_1} \middle| \mathcal{F}_G\right] = \int_G \mathbb{E}[f_1(t_1) \mid \mathcal{F}_G]\, dW_{t_1}.$$

Since f_1 is deterministic, $\mathbb{E}[f_1(t_1) \mid \mathcal{F}_G] = f_1(t_1)$, and we obtain:

$$\int_G^T \mathbb{E}[f_1(t_1) \mid \mathcal{F}_G]\, dW_{t_1} = \int_G f_1(t_1)\, dW_{t_1} = \int_0^T f_1(t_1)\chi_G(t_1)\, dW_{t_1}.$$

Let us assume the statement holds for some $n = k$, i.e., $\mathbb{E}[I_k(f_k) \mid \mathcal{F}_G] = I_k(f_k \chi_G^{\otimes k})$. Let us now consider $n = k+1$. Then:

$$\begin{aligned}
\mathbb{E}[I_{k+1}(f_{k+1}) \mid \mathcal{F}_G] &= (k+1)!\, \mathbb{E}\left[\int_0^T \int_0^{t_{k+1}} \cdots \int_0^{t_2} f_{k+1}(t_1, \ldots, t_{k+1})\, dW_{t_1} \cdots dW_{t_k} dW_{t_{k+1}} \,\middle|\, \mathcal{F}_G \right] \\
&= (k+1)! \int_0^T \mathbb{E}\left[\int_0^{t_{k+1}} \cdots \int_0^{t_2} f_{k+1}(t_1, \ldots, t_{k+1})\, dW_{t_1} \cdots dW_{t_k} \,\middle|\, \mathcal{F}_G \right] \chi_G(t_{k+1})\, dW_{t_{k+1}} \\
&= (k+1) \int_0^T \mathbb{E}[I_k(f_k)|\mathcal{F}_G]\chi_G(t_{k+1})\, dW_{t_{k+1}} = (k+1) \int_0^T I_k(f_k \chi_G^{\otimes k})\chi_G(t_{k+1})\, dW_{t_{k+1}} \\
&= I_{k+1}[f_{k+1}\chi_G^{\otimes (k+1)}]
\end{aligned}$$

Hence, the result holds for all $n \in \mathbb{N}$. ■

Proposition 2.41. *Let $F \in \mathbb{D}_{1,2}$. Then, the conditional expectation $\mathbb{E}[F|\mathcal{F}_G]$ also belongs to $\mathbb{D}_{1,2}$, and its Malliavin derivative satisfies*

$$D_t \mathbb{E}[F|\mathcal{F}_G] = \mathbb{E}[D_t F|\mathcal{F}_G] \cdot \chi_G(t), \tag{2.40}$$

where χ_G denotes the indicator function of the set $G \subset [0, T]$.

Proof. Assume first that $F = I_n(f_n)$, for some $f_n \in \widetilde{L}^2([0, T]^n)$. Using Proposition 2.40 in the second equality, we have

$$\begin{aligned}
D_t \mathbb{E}[F|\mathcal{F}_G] &= D_t \mathbb{E}[I_n(f_n)|\mathcal{F}_G] = D_t I_n(f_n \chi_G^{\otimes n}) = n I_{n-1}[f_n(\cdot, t)\chi_G^{\otimes n}(t_1, \ldots, t_{n-1}, t)] \\
&= n I_{n-1}[f_n(\cdot, t)\chi_G^{\otimes (n-1)}(t_1, \ldots, t_{n-1})\chi_G(t)] = n I_{n-1}[f_n(\cdot, t)\chi_G^{\otimes (n-1)}(t_1, \ldots, t_{n-1})]\chi_G(t) \\
&= \mathbb{E}[D_t F|\mathcal{F}_G]\chi_G(t).
\end{aligned}$$

Now, let $F \in \mathbb{D}_{1,2}$ be represented by its Itô-Wiener chaos expansion, $F = \sum_{n=0}^{\infty} I_n(f_n)$. Define the approximating sequence $F_k = \sum_{n=0}^{k} I_n(f_n)$, which consists of finite chaos expansions and thus satisfies $F_k \in \mathbb{D}_{1,2}^0$. By the closability of the Malliavin derivative (Theorem 2.30), we have convergence

$$F_k \to F \quad \text{in } L^2(\Omega) \quad \text{and } D_t F_k \to D_t F \quad \text{in } L^2(\mathbb{P} \times \lambda) \quad \text{as } k \to \infty.$$

Since for each k,

$$D_t \mathbb{E}[F_k|\mathcal{F}_G] = \mathbb{E}[D_t F_k|\mathcal{F}_G]\chi_G(t),$$

passing the limit and using the continuity of conditinal expectation and Malliavin derivative in the respective L^2-spaces yields the result. ■

Proposition 2.42. *Let $u = u(s)$, for $s \in [0,T]$, be a stochastic process adapted to the filtration $\mathcal{F}$, and suppose that $u(s) \in \mathbb{D}_{1,2}$ for every $s \in [0,T]$. Then, for each fixed s, the Malliavin derivative $D_t u(s)$ defines an $\mathcal{F}$-adapted process for all t, and moreover, it vanishes for all $t > s$; that is, $D_t u(s) = 0$ whenever $t > s$.*

Proof. Making use of Proposition 2.41 and taking into account the fact that $s \in [0,T]$:

$$\chi_{[0,s]}(t) = \chi_{[0,s]}(t)\chi_{[0,T]}(s) = \chi_{\{0\le t\le s\}}\chi_{\{0\le s\le T\}} = \chi_{\{t\le s\le T\}} = \chi_{[t,T]}(s),$$

$$D_t u(s) = D_t\mathbb{E}[u(s)|\mathcal{F}_s] = \mathbb{E}[D_t u(s)|\mathcal{F}_s]\chi_{[0,s]}(t) = \mathbb{E}[D_t u(s)|\mathcal{F}_s]\chi_{[t,T]}(s). \tag{2.41}$$

■

We now introduce an important property of the Skorohod integral: it can be viewed as the adjoint of the Malliavin derivative. To make this idea precise, we first present the duality formula that connects these two operators. This will then lead us to the integration by parts formula, which plays a central role in many applications of Malliavin calculus.

Theorem 2.43 (Duality formula). *Let $F \in \mathbb{D}_{1,2}$ be $\mathcal{F}_T$-measurable and let u be a Skorohod integrable stochastic process. Then*

$$\mathbb{E}\left[F\int_0^T u(t)\,\delta W_t\right] = \mathbb{E}\left[\int_0^T u(t)D_tF\,dt\right]. \tag{2.42}$$

Proof. Let $F = \sum_{n=0}^{\infty} I_n(f_n)$ and $u(t) = \sum_{k=0}^{\infty} I_k(g_k(\cdot,t))$ denote the Wiener chaos expansions of the random variable $F \in L^2(\Omega)$ and the stochastic process $u : [0,T] \to L^2(\Omega)$, respectively. By Proposition 2.7 and Eq. (2.11) the right hand side is equal to:

$$\begin{aligned}
\mathbb{E}\left[F\int_0^T u(t)\,\delta W_t\right] &= \mathbb{E}\left[\left(\sum_{n=0}^{\infty} I_n(f_n)\right)\int_0^T\left(\sum_{k=0}^{\infty} I_k(g_k(\cdot,t))\right)\delta W_t\right]\\
&= \mathbb{E}\left[\left(\sum_{n=0}^{\infty} I_n(f_n)\right)\left(\sum_{k=0}^{\infty} I_{k+1}(\tilde g_k)\right)\right]\\
&= \mathbb{E}\left[I_0(f_0)\left(\sum_{k=0}^{\infty} I_{k+1}(\tilde g_k)\right)\right] + \mathbb{E}\left[\left(\sum_{n=0}^{\infty} I_{n+1}(f_{n+1})\right)\left(\sum_{k=0}^{\infty} I_{k+1}(\tilde g_k)\right)\right]\\
&= \sum_{k=0}^{\infty}\mathbb{E}\left[I_{k+1}(f_{k+1})I_{k+1}(\tilde g_k)\right] = \sum_{k=0}^{\infty}(k+1)!(k+1)!\mathbb{E}\left[J_{k+1}(f_{k+1})J_{k+1}(\tilde g_k)\right]\\
&= \sum_{k=0}^{\infty}(k+1)!(k+1)!\left(f_{k+1},\tilde g_k\right)_{L^2(S_{k+1})} = \sum_{k=0}^{\infty}(k+1)!\left(f_{k+1},\tilde g_k\right)_{L^2([0,T]^{k+1})}.
\end{aligned}$$

Now, applying Eq. (2.31) to F, the left hand side reads:

$$
\begin{aligned}
\mathbb{E}\left[\int_0^T u(t)D_tF\,dt\right] &= \mathbb{E}\left[\int_0^T \left(\sum_{k=0}^{\infty} I_k(g_k(\cdot,t))\right)\left(\sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot,t))\right)dt\right] \\
&= \int_0^T \mathbb{E}\left[\left(\sum_{k=0}^{\infty} I_k(g_k(\cdot,t))\right)\left(\sum_{n=0}^{\infty}(n+1)I_n(f_{n+1}(\cdot,t))\right)\right]dt \\
&= \int_0^T \sum_{k=0}^{\infty} \mathbb{E}\Big[\Big(I_k(g_k(\cdot,t))\Big)\Big((k+1)I_k(f_{k+1}(\cdot,t))\Big)\Big]\,dt \\
&= \int_0^T \sum_{k=0}^{\infty} (k+1)\,k!\,k!\,\mathbb{E}\Big[\Big(J_k(g_k(\cdot,t))\Big)\Big(J_k(f_{k+1}(\cdot,t))\Big)\Big]\,dt \\
&= \int_0^T \sum_{k=0}^{\infty} (k+1)!\,k!\,\left(g_k(\cdot,t), f_{k+1}(\cdot,t)\right)_{L^2(s_k)}\,dt \\
&= \sum_{k=0}^{\infty}(k+1)\,k!\int_0^T \left(g_k(\cdot,t), f_{k+1}(\cdot,t)\right)_{L^2([0,T]^k)}\,dt = \sum_{k=0}^{\infty}(k+1)!\,(g_k, f_{k+1})_{L^2([0,T]^{k+1})}\,.
\end{aligned}
$$

It remains to check that $(f_{k+1}, \tilde{g}_k)_{L^2([0,T]^{k+1})} = (f_{k+1}, g_k)_{L^2([0,T]^{k+1})}$. The symmetrization of $u(t)$ is given by Eq. (2.20), therefore:

$$
\begin{aligned}
(f_{k+1}, \tilde{g}_k)_{L^2([0,T]^{k+1})} &= \int_0^T \Big(f_{k+1}(\cdot,t), \tilde{g}_k(\cdot,t)\Big)_{L^2([0,T]^k)}\,dt \\
&= \int_0^T \left(f_{k+1}(\cdot,t_i), \frac{1}{k+1}\sum_{i=1}^{k+1} g_k(t_1,\dots,\widehat{t_i},\dots,t_{n+1},t_i)\right)_{L^2([0,T]^k)} dt_i \\
&= \frac{1}{k+1}\sum_{i=1}^{k+1}\int_0^T \Big(f_{k+1}(\cdot,t_i), g_k(t_1,\dots,\widehat{t_i},\dots,t_{n+1},t_i)\Big)_{L^2([0,T]^k)} dt_i \\
&= \int_0^T \Big(f_{k+1}(\cdot,t), g_k(\cdot,t)\Big)_{L^2([0,T]^k)}\,dt \\
&= (f_{k+1}, g_k)_{L^2([0,T]^{k+1})}.
\end{aligned}
$$

■

Theorem 2.44 (Integration by parts formula). *Let $u(t)$, $t \in [0,T]$, be a Skorohod integrable stochastic process and $F \in \mathbb{D}_{1,2}$ such that the product $Fu(t)$ is Skorohod integrable. Then*

$$
F\int_0^T u(t)\,\delta W_t = \int_0^T Fu(t)\,\delta W_t + \int_0^T u(t)D_tF\,dt. \tag{2.43}
$$

Proof. We first assume that $F, G \in \mathbb{D}^0_{1,2}$. Applying the duality formula in Theorem 2.43:

$$\begin{aligned}\mathbb{E}\left[G\int_0^T Fu(t)\delta W(t)\right] &= \mathbb{E}\left[\int_0^T Fu(t)D_tG\,dt\right] \\ &= \mathbb{E}\left[GF\int_0^T u(t)\delta W(t)\right] - \mathbb{E}\left[G\int_0^T u(t)D_tF\,dt\right].\end{aligned}$$

Making use of the product rule for the Malliavin derivative in Theorem 2.33 we get:

$$u(t)D_t(FG) = Fu(t)D_tG + u(t)GD_tF.$$

Since the set of all $G \in \mathbb{D}^0_{1,2}$ is dense in $L^2(\mathbb{P})$, the identity

$$F\int_0^T u(t)\,\delta W_t = \int_0^T Fu(t)\,\delta W_t + \int_0^T u(t)D_tF\,dt$$

holds $\mathbb{P}$-almost surely for all $F \in \mathbb{D}^0_{1,2}$. To extend this result to arbitrary $F \in \mathbb{D}_{1,2}$, we apply an approximation argument. Let $\{F^{(n)}, n \geq 0\} \subset \mathbb{D}^0_{1,2}$ be a sequence such that $F^{(n)} \to F$ in $L^2(\mathbb{P})$ and $D_tF^{(n)} \to D_tF$ in $L^2(\mathbb{P}\times\lambda)$ as $n \to \infty$. Then, passing to the limit in the identity for each $F^{(n)}$ yields the desired result for general $F \in \mathbb{D}_{1,2}$.

■

The integration by parts formula offers a powerful and practical tool for computing Skorohod integrals, as demonstrated in the following example.

Example 2.45. We compute the Skorohod integral of the random variable in example Example 2.27, $F = W^3_{t_0}$, for some $t_0 \in [0, T]$ fixed. Applying the integral by parts formula in Theorem 2.44 with $F = W^3_{t_0}$, $u(t) = 1$ and $D_tF = D_tW^3_{t_0} = 3W^2_{t_0}\chi_{[0,t_0]}(t)$:

$$\begin{aligned}F\int_0^T u(t)\,\delta W_t &= \int_0^T Fu(t)\,\delta W_t + \int_0^T u(t)D_tF\,dt \\ W^3_{t_0}\int_0^T (1)\,\delta W_t &= \int_0^T W^3_{t_0}(1)\,\delta W_t + \int_0^T (1)3W^2_{t_0}\chi_{[0,t_0]}(t)\,dt \\ W^3_{t_0}W_T &= \int_0^T W^3_{t_0}\,\delta W_t + 3t_0W^2_{t_0} \\ \int_0^T W^3_{t_0}\,\delta W_t &= W^3_{t_0}W_T - 3t_0W^2_{t_0}.\end{aligned}$$

This result is consistent with the earlier computation and highlights the effectiveness of the approach.

Âą We now state a fundamental result that provides an explicit formula for the Malliavin derivative of a Skorohod integral.

Theorem 2.46. *Let $u = u(s)$ belong to $\mathbb{D}_{1,2}$ for any $s \in [0,T]$, and suppose that the process $\{D_t u_s, s \in [0,T]\}$ is Skorohod integrable. Furthermore, assume that there exists a version of the process $\{\delta(D_t u_s)), t \in [0,T]\}$ belonging to $L^2(\Omega \times [0,T])$. Then $\delta(u)$ belongs to $\mathbb{D}_{1,2}$ and the following identity holds:*

$$D_t\left(\int_0^T u(s)\,\delta W_s\right) = u(t) + \int_0^T D_t u(s)\,\delta W_s. \tag{2.44}$$

Proof. See Proposition 1.3.8 in [Nua05]. ■

Example 2.47. Let $u(s) = W_T$. By Example 2.21, we know that $\delta(W_T) = W_T^2 - T$. Taking the Malliavin deritive directly gives $D_t(\delta(W_T)) = D_t(W_T^2 - T) = 2W_T\chi_{[0,T]}(t) = 2W_T$, since $t \in [0,T]$. Alternatively, we can apply Theorem 2.46:

$$D_t\left(\int_0^T W_T\,\delta W_s\right) = W_T + \int_0^T D_t W_T\,\delta W_s = W_T + \int_0^T (1)\,\delta W_s = W_T + W_T = 2W_T.$$

2.4 The Clark-Ocone formula

In this section, we present explicit representations of random variables as stochastic integrals involving their Malliavin derivatives. One of the earliest and most influential applications of Malliavin calculus is the explicit representation of martingales. Specifically, any random variable F measurable with respect ot the σ-algebra generated by a Brownian motion W can be written as a stochastic integral. The key result that facilitates this is the celebrated Clark-Ocone formula, a fundamental tool in stochastic analysis and Malliavin calculus. Beyond its theoretical importance, this formula has practical relevance, particularly in sensitivity analysis within financial mathematics. We now proceed to the precise formulation and derivation of this result.

Theorem 2.48. *Let $F \in \mathbb{D}_{1,2}$ be $\mathcal{F}_T$-measurable. Then*

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[D_t F|\mathcal{F}_t]\,dW_t. \tag{2.45}$$

Proof. Let $F = \sum_{n=0}^{\infty} I_n(f_n)$ denote the chaos expansion of F, where each $f_n \in \widetilde{L}^2([0,T]^n)$ for $n \geq 1$. We proceed by computing the right hand side of Eq. (2.45). We start applying the definition of the Malliavin derivative operator to F (see Definition 2.28):

$$\int_0^T \mathbb{E}[D_t F|\mathcal{F}_t]\,dW_t = \int_0^T \mathbb{E}\left[\sum_{n=1}^{\infty} nI_{n-1}[f_n(\cdot,t)]\middle|\mathcal{F}_t\right]\,dW_t = \int_0^T \sum_{n=1}^{\infty} n\mathbb{E}\left[I_{n-1}[f_n(\cdot,t)]\middle|\mathcal{F}_t\right]\,dW_t.$$

Proposition 2.40 allows for the computation of the conditional expectation as follows:

$$\int_0^T \sum_{n=1}^{\infty} n\mathbb{E}\left[I_{n-1}[f_n(\cdot,t)] \middle| \mathcal{F}_t \right] dW_t = \int_0^T \sum_{n=1}^{\infty} nI_{n-1}[f_n(\cdot,t)\chi_{[0,t]}^{\otimes(n-1)}]dW_t$$
$$= \int_0^T \sum_{n=1}^{\infty} n!J_{n-1}[f_n(\cdot,t)\chi_{[0,t]}^{\otimes(n-1)}]dW_t.$$

Now, all indicator functions evaluate to 1 and can be removed because the functions f_n are symmetric, and the integration domain already respects the ordering of variables:

$$\int_0^T \sum_{n=1}^{\infty} n!J_{n-1}[f_n(\cdot,t)\chi_{[0,t]}^{\otimes(n-1)}]dW_t = \sum_{n=1}^{\infty} n! \int_0^T J_{n-1}[f_n(\cdot,t)\cdot\chi_{[0,t]}^{\otimes(n-1)}]\, dW_t$$
$$= \sum_{n=1}^{\infty} n! \int_0^T J_{n-1}[f_n(\cdot,t)]\, dW_t = \sum_{n=1}^{\infty} n!J_n[f_n] = \sum_{n=1}^{\infty} I_n[f_n]$$
$$= \sum_{n=0}^{\infty} I_n[f_n] - I_0[f_0] = F - \mathbb{E}[F].$$

Rearranging the expression yields the result. ■

Example 2.49. We apply the Clark-Ocone formula to the random variable $F = W_{t_0}^3$ in Example 2.27. Since W_{t_0} is a centered Gaussian random variable, its third moment vanishes: $\mathbb{E}[W_{t_0}^3] = 0$. By the chain rule for malliavin derivatives, $D_t W_{t_0}^3 = 3W_{t_0}^2 \chi_{[0,t_0]}(t)$. The only part left to compute is the conditonal expectation of this Malliavin derivative.

The indicator function $\chi_{[0,t_0]}(t)$ is $\mathcal{F}_t$-measurable so by the properties of the conditional expectation it can be taken out of the expectation:

$$\mathbb{E}[D_t F|\mathcal{F}_t] = \mathbb{E}[3W_{t_0}^2\chi_{[0,t_0]}(t)|\mathcal{F}_t] = 3\chi_{[0,t_0]}(t)\mathbb{E}[W_{t_0}^2|\mathcal{F}_t] = 3\chi_{[0,t_0]}(t)\mathbb{E}[W_{t_0}^2|\mathcal{F}_t].$$

The remaining conditional expectation can be computed taking into account the usual properties of the conditional expectation and the Wiener process W_t.

$$\begin{aligned}
\mathbb{E}[W_{t_0}^2|\mathcal{F}_t] &= \mathbb{E}[(W_{t_0} - W_t + W_t)^2|\mathcal{F}_t] \\
&= \mathbb{E}[(W_{t_0} - W_t)^2 + W_t^2 + 2W_t(W_{t_0} - W_t)|\mathcal{F}_t] \\
&= \mathbb{E}[(W_{t_0} - W_t)^2] + \mathbb{E}[W_t^2|\mathcal{F}_t] + 2\mathbb{E}[W_t(W_{t_0} - W_t)|\mathcal{F}_t] \\
&= (t_0 - t) + W_t^2 + 2\mathbb{E}[W_t|\mathcal{F}_t]\mathbb{E}[(W_{t_0} - W_t)] \\
&= (W_t^2 + t_0 - t).
\end{aligned}$$

So $\mathbb{E}[D_t F|\mathcal{F}_t] = 3\chi_{[0,t_0]}(t)(W_t^2 + t_0 - t)$ and

$$W_{t_0}^3 = 3\int_0^T \chi_{[0,t_0]}(t)(W_t^2 + t_0 - t)\, dW_t.$$

3 Fractional Brownian motion

In this section, we introduce the fractional Brownian motion (fBm), a family of Gaussian processes indexed by a parameter parameter $H \in (0,1)$, known as the Hurst parameter, that generalizes standard Brownian motion by allowing for correlated increments. We begin by presenting the definition of the fBm and discussing its key properties, including stationary increments, self-similarity, Hölder continuity, and the fact that it is not a semimartingale for $H \neq \frac{1}{2}$, a crucial feature that prevents the use of classical Itô calculus.

We then present the Mandelbrot–Van Ness representation, verify that it defines a process with the desired properties, and use it to establish the existence of fBm via the Kolmogorov extension theorem. Additionally, we introduce the Riemann–Liouville fractional Brownian motion, valued for its simplicity and analytical tractability, and often used in applications.

The unique statistical properties of fBm, particularly its ability to capture both long- and short-range dependence and to produce sample paths with a tunable roughness, make it a foundational process in the modeling of rough volatility in mathematical finance.

3.1 fBm properties

The main goal is to provide an introduction to fractional Brownian motion (fBm), focusing on its defining characteristics that are useful for both theoretical study and numerical implementation.

Definition 3.1. A *fractional Brownian motion (fBm)* of Hurst parameter $H \in (0,1)$ is a centered Gaussian process $B^{(H)} = \{B_t^{(H)}, t \geq 0\}$ with covariance function

$$\mathbb{E}[B_t^{(H)} B_s^{(H)}] = \frac{t^{2H} + s^{2H} - |t-s|^{2H}}{2}. \tag{3.1}$$

When the Hurst parameter $H = 1/2$, the fBm coincides with the standard Brownian motion. In this regard the fBm can be considered its generalization.

Definition 3.2. A stochastic process $Z = \{Z_t, t \geq 0\}$ is said to have *stationary increments* if, for any $s, t \in \mathbb{R}$ with $t > s$, the distribution of $Z_t - Z_s$ depends only on the difference $t - s$.

Proposition 3.3. *The fractional Brownian motion has stationary increments.*

Proof. To specify the distribution of a Gaussian process, it suffices to specify its mean and covariance function.

Since $B_t^{(H)}$ is a centered process, we have $\mathbb{E}[B_t^{(H)}] = 0$ for all $t \in [0,T]$. Therefore, the expected value of the increment is also zero:

$$\mathbb{E}[B_t^{(H)} - B_s^{(H)}] = \mathbb{E}[B_t^{(H)}] - \mathbb{E}[B_s^{(H)}] = 0.$$

We now compute the variance of the increment:

$$\mathbb{E}[(B_t^{(H)} - B_s^{(H)})^2] = t^{2H} + s^{2H} - 2\left(\frac{t^{2H} + s^{2H} - |t-s|^{2H}}{2}\right) = |t-s|^{2H}.$$

Hence, the increment $(B_t^{(H)} - B_s^{(H)})$ is normally distributed with mean zero and variance $|t-s|^{2H}$, which depends only on the time difference $t-s$. This proves that the increments of the fBm are stationary. ■

Definition 3.4. A stochastic process $Z = \{Z_t, t \geq 0\}$ is said to be *self-similar* with parameter $b > 0$ if for all $a > 0$, the processes $\{Z_{at}, t \geq 0\}$ and $\{a^b Z_t, t \geq 0\}$ have the same finite-dimensional distributions. In other words, the process exhibits scale invariance: the statistical properties of the process are the same, up to a scaling factor, when the time is rescaled by a factor of a and the space is rescaled by a factor of a^b.

Proposition 3.5. *The fractional Brownian motion process is self-similar.*

Proof. Let $a \geq 0$. We aim to show that the processes $\{B_{at}^{(H)}, t \geq 0\}$ and $\{a^H B_t^{(H)}, t \geq 0\}$ have the same law. Since both processes are Gaussian, it suffices to verify that their means and covariances agree. Both processes are centered. Now we compare their covariance functions:

$$\mathbb{E}\left[B_{at}^{(H)} B_{as}^{(H)}\right] = \frac{(at)^{2H} + (as)^{2H} - |at - as|^{2H}}{2} = a^{2H}\frac{(t^{2H} + s^{2H} - |t-s|^{2H})}{2},$$

$$\mathbb{E}\left[a^H B_t^{(H)} \cdot a^H B_s^{(H)}\right] = a^{2H}\mathbb{E}\left[B_t^{(H)} B_s^{(H)}\right] = a^{2H}\frac{(t^{2H} + s^{2H} - |t-s|^{2H})}{2}.$$

Thus, the covariance functions coincide and therefore both processes have the same law. We conclude that the fBm is H-self-similar. ■

A fundamental characteristic distinguishing fBm from the standard Brownian motion is the presence of correlated increments. This correlation structure is intrinsically governed by the Hurst parameter $H \in (0,1)$.

Proposition 3.6. *The fractional Brownian motion process has correlated increments.*

Proof. The covariance between two increments $B_{t+h}^{(H)} - B_t^{(H)}$ and $B_{s+h}^{(H)} - B_s^{(H)}$, where $t \geq s+h$, and $t - s = nh$ is

$$\begin{aligned}
\rho_H(n) &= \mathbb{E}[(B_{t+h}^{(H)} - B_t^{(H)})(B_{s+h}^{(H)} - B_s^{(H)})] \\
&= \frac{1}{2}\left[-|t+h-s-h|^{2H} + |t+h-s|^{2H} + |t-s-h|^{2H} - |t-s|^{2H}\right] \\
&= \frac{1}{2}\left[-2|nh|^{2H} + |nh+h|^{2H} + |nh-h|^{2H}\right] \\
&= \frac{1}{2}h^{2H}\left[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right] \\
&\approx h^{2H} H(2H-1)n^{2H-2} \to 0 \quad \text{as } n \text{ tends to infinity.}
\end{aligned}$$

The following Taylor expansion has been applied:

$$(n+1)^{2H} = n^{2H} + 2Hn^{2H-1} + \frac{1}{2}2H(2H-1)n^{2H-2}.$$

$$(n-1)^{2H} = n^{2H} - 2Hn^{2H-1} + \frac{1}{2}2H(2H-1)n^{2H-2}.$$

For $H = \frac{1}{2}$, $\rho_H(n) = 0$. The fBm reduces to the standard Brownian motion, which has uncorrelated increments.

For $H \in \left(\frac{1}{2}, 1\right)$, $\rho_H(n) > 0$ and $\sum_{n=1}^{\infty} |\rho_n| \sim \sum_{n=1}^{\infty} |n|^{2H-2} = \infty$. The increments exhibit positive correlation. This implies persistence in the process: an increasing (or decreasing) movement in the past makes an increasing (or decreasing) movement in the future more probable. The process has a tendency to maintain the direction of its path, which is reflected in its smoother sample paths and long-range dependence.

For $H \in \left(0, \frac{1}{2}\right)$, $\rho_H(n) < 0$ and $\sum_{n=1}^{\infty} |\rho_n| \sim \sum_{n=1}^{\infty} |n|^{2H-2} = 0$. The increments are negatively correlated, indicating anti-persistence. In this regime, an increase is likely to be followed by a decrease and vice versa. This anti-persistent behavior gives rise to more oscillatory paths and short-range dependence.

■

Definition 3.7. Two stochastic processes $X = \{X_t\}_{t \in I}$ and $Y = \{Y_t\}_{t \in I}$ are said to be *modifications* of each other if for every $t \in I$, we have

$$\mathbb{P}(X_t = Y_t) = 1.$$

In other words, the random variables X_t and Y_t are equal almost surely for each fixed time t.

Theorem 3.8 (Kolmogorov Continuity Theorem). *Let $\{X_t, t \geq 0\}$ be a real-valued stochastic process such that there exists positive constants α, β, C satisfying*

$$\mathbb{E}[|X_t - X_s|^\alpha] \leq C|t - s|^{1+\beta},$$

for all $s, t \geq 0$. Then, X has a continuous modification which, with probability one, is locally γ-Hölder continuous with $\gamma \in \left(0, \frac{\beta}{\alpha}\right)$.

Proposition 3.9. *The fractional Brownian motion process has a continuous modification and it is locally γ-Hölder continuous.*

Proof. The fBm increments have the following Gaussian distribution: $B_t^{(H)} - B_s^{(H)} \sim N(0, |t-s|^{2H})$. Hence, we can compute the expectation by definition:

$$\mathbb{E}\left[|B_t^{(H)} - B_s^{(H)}|^p\right] = \frac{1}{\sqrt{2\pi|t-s|^{2H}}} \int_{\mathbb{R}} |x|^p e^{-\frac{1}{2}\frac{x^2}{|t-s|^{2H}}} \, dx \overset{u=\frac{x}{|t-s|^H}}{=} \frac{|t-s|^{pH}}{\sqrt{2\pi}} \int_{\mathbb{R}} |u|^p e^{-\frac{u^2}{2}} \, du.$$

To compute the integral term we can remove the absolute value by symmetry getting $\int_{\mathbb{R}} |u|^p e^{-\frac{u^2}{2}} \, du = 2 \cdot \int_0^{+\infty} u^p e^{-\frac{u^2}{2}} \, du$. Integrating twice by parts we get:

$$v = u^{p-1} \qquad v' = (p-1)u^{p-2}$$
$$w = -e^{-\frac{u^2}{2}} \qquad w' = ue^{-\frac{u^2}{2}}$$

$$\int_0^{+\infty} u^{p-1} u e^{-\frac{u^2}{2}} \, du = -u^{p-1} e^{-\frac{u^2}{2}} \Big|_0^{+\infty} + \int_0^{+\infty} (p-1) u^{p-2} e^{-\frac{u^2}{2}} \, du \quad = (p-1) \int_0^{+\infty} u^{p-2} e^{-\frac{u^2}{2}} \, du,$$

$$v = u^{p-3} \qquad v' = (p-3)u^{p-4}$$
$$w = -e^{-\frac{u^2}{2}} \qquad w' = ue^{-\frac{u^2}{2}}$$

$$\int_0^{+\infty} u^{p-3}ue^{-\frac{u^2}{2}}\,du = -u^{p-3}e^{-\frac{u^2}{2}}\Big|_0^{+\infty} + \int_0^{+\infty}(p-3)u^{p-4}e^{-\frac{u^2}{2}}\,du \;= (p-3)\int_0^{+\infty} u^{p-4}e^{-\frac{u^2}{2}}\,du.$$

At this point we have that:

$$\mathbb{E}\left[|B_t^{(H)} - B_s^{(H)}|^p\right] = |t-s|^{pH}\sqrt{\frac{2}{\pi}}(p-1)(p-3)\int_0^{+\infty} u^{p-4}e^{-\frac{u^2}{2}}\,du.$$

Integrating by parts until the u term's degree is reduced to one for the case in which p is odd and to zero for the case in which p is even we get to:

- p is odd:

$$\begin{aligned}
&|t-s|^{pH}\sqrt{\frac{2}{\pi}}(p-1)(p-3)\dots(p-(p-4))(p-(p-2))\int_0^{+\infty} ue^{-\frac{u^2}{2}}\,du \\
&= |t-s|^{pH}\sqrt{\frac{2}{\pi}}(p-1)(p-3)\dots(p-(p-4))(p-(p-2))\cdot(1) \\
&= \sqrt{\frac{2}{\pi}}(p-1)!!|t-s|^{pH}.
\end{aligned}$$

- p is even:

$$\begin{aligned}
&|t-s|^{pH}\sqrt{\frac{2}{\pi}}(p-1)(p-3)\dots(p-(p-3))(p-(p-1))\int_0^{+\infty} e^{-\frac{u^2}{2}}\,du \\
&= |t-s|^{pH}(p-1)(p-3)\dots(p-(p-3))(p-(p-1)) \\
&= (p-1)!!|t-s|^{pH}.
\end{aligned}$$

$$\mathbb{E}\left[|B_t^{(H)} - B_s^{(H)}|^p\right] = \begin{cases} \underbrace{\sqrt{\frac{2}{\pi}}(p-1)!!}_{C}\,|t-s|^{pH}, & p \text{ is odd.} \\ \underbrace{(p-1)!!}_{C}\,|t-s|^{pH}, & p \text{ is even.} \end{cases}$$

Therefore, by Theorem 3.8, we get the existence of a continuous modification and with $\alpha = p$ and $\beta = pH - 1$, we deduce that a.s. fBm sample paths are γ-Hölder continuous with exponent

$$\gamma \in \left(0, \frac{\beta}{\alpha}\right) = \left(0, \frac{pH-1}{p}\right).$$

Since this holds for all $p \in \mathbb{N}$, taking the limit as $p \to \infty$ leads to the conclusion that $\gamma \in (0, H)$.

∎

Definition 3.10. A process $X = \{X_t, t \geq 0\}$ is called a *semimartingale* if it can be decomposed as

$$X_t = M_t + A_t, \quad t \geq 0,$$

where:

- $M = \{M_t, t \geq 0\}$ is a local martingale;
- $A = \{A_t, t \geq 0\}$ is a process of finite variation, adapted, and càdlàg.

That is, a semimartingale is the sum of a local martingale and a finite variation process.

Proposition 3.11. *The fractional Brownian motion process is not a semimartingale.*

Proof. The proof of this proposition goes beyond the scope of this thesis and a complete proof can be found in [Rog97]. This proposition is of particular importance, as it implies that the Itô stochastic calculus developed for semimartingales cannot be applied to define the stochastic integral with respect to the fractional Brownian motion $B^{(H)}$. ■

3.2 fBm representations and existence

We now introduce a key integral representation that will prove essential in both theoretical analysis and simulation.

A fundamental construction of the fractional Brownian motion was introduced by Mandelbrot and Van Ness in [MN68], providing a representation that explicitly reveals its dependence on the Hurst parameter $H \in (0, 1)$. This formulation characterizes the fBm as a stochastic integral with respect to standard Brownian motion and is given by:

$$B_t^{(H)} = \frac{1}{\Gamma(H + 1/2)} \int_{\mathbb{R}} \left((t - r)_+^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}} \right) dB_r, \tag{3.2}$$

where B is a standard Brownian motion, Γ denotes the Gamma function and $r_+ := \max(r, 0)$.

We shall now establish the existence of this process and subsequently verify that it coincides with the fractional Brownian motion by showing that it is a centered Gaussian process with the same covariance structure.

The representation formula can be rewritten as

$$B_t^{(H)} = \frac{1}{\Gamma(H + 1/2)} \left[\int_{-\infty}^{0} \left((t - r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}} \right) dB_r + \int_{0}^{t} \left((t - r)^{H-\frac{1}{2}} \right) dB_r \right].$$

By the properties of the Itô integral, the process defined by the above representation is a centered Gaussian process. The covariance structure will allow us to rigorously establish the existence of the process and its equivalence to the fractional Brownian motion. For simplicity, we drop the constant term $\frac{1}{\Gamma(H+1/2)}$.

To compute the covariance function of the representation we use the following identity:

$$\mathbb{E}\left[B_t^{(H)} B_s^{(H)}\right] = \frac{\mathbb{E}\left[(B_t^{(H)})^2\right] + \mathbb{E}\left[(B_s^{(H)})^2\right] - \mathbb{E}\left[(B_t^{(H)} - B_s^{(H)})^2\right]}{2}. \tag{3.3}$$

$$\begin{aligned}
\mathbb{E}\left[(B_t^{(H)})^2\right] &= \mathbb{E}\left[\left(\int_{-\infty}^{0} \left((t-r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\right) dB_r + \int_{0}^{t} \left((t-r)^{H-\frac{1}{2}}\right) dB_r\right)^2\right] \\
&= \mathbb{E}\left[\left(\int_{-\infty}^{0} \left((t-r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\right) dB_r\right)^2\right] + \mathbb{E}\left[\left(\int_{0}^{t} \left((t-r)^{H-\frac{1}{2}}\right) dB_r\right)^2\right] \\
&\quad + 2\mathbb{E}\left[\left(\int_{-\infty}^{0} \left((t-r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\right) dB_r\right)\left(\int_{0}^{t} \left((t-r)^{H-\frac{1}{2}}\right) dB_r\right)\right].
\end{aligned}$$

The last term involves the expectation of the product of two stochastic integrals over disjoint intervals. Due to the independent increments of Brownian motion, these integrals are independent random variables. Since each stochastic integral is centered (i.e., has zero mean), the expectation of their product vanishes. As for the remaining two terms:

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{-\infty}^{0} \left((t-r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\right) dB_r\right)^2\right] + \mathbb{E}\left[\left(\int_{0}^{t} \left((t-r)^{H-\frac{1}{2}}\right) dB_r\right)^2\right] \\
&= \mathbb{E}\left[\int_{-\infty}^{0} \left((t-r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\right)^2 dr\right] + \mathbb{E}\left[\int_{0}^{t} \left((t-r)^{H-\frac{1}{2}}\right)^2 dr\right] \\
&= \int_{-\infty}^{0} \left((t-r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\right)^2 dr + \int_{0}^{t} \left((t-r)^{H-\frac{1}{2}}\right)^2 dr.
\end{aligned}$$

We have applied the Itô isometry, and then used Fubini's theorem to exchange the expectation and the integral. Since the integrands are deterministic, they can be taken outside the expectation, reducing the expression to the integral of their squared values.

We compute the first integral:

$$\begin{aligned}
\int_{-\infty}^{0} \left((t-r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\right)^2 dr &= \int_{-\infty}^{0} (t-r)^{2H-1} + (-r)^{2H-1} - 2(r^2 - tr)^{H-\frac{1}{2}} \, dr \\
&\overset{r=tu}{=} \int_{-\infty}^{0} \left[(t-tu)^{2H-1} + (-tu)^{2H-1} - 2(t^2u^2 - t^2u)^{H-\frac{1}{2}}\right] t \, du \\
&= \int_{-\infty}^{0} t^{2H-1} t \left[(1-u)^{2H-1} + (-u)^{2H-1} - 2(u^2-u)^{H-\frac{1}{2}}\right] du \\
&= t^{2H} \int_{-\infty}^{0} \left[(1-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}\right]^2 du.
\end{aligned}$$

We compute the second term:

$$\int_0^t \left((t-r)^{H-\frac{1}{2}}\right)^2 dr = \int_0^t (t-r)^{2H-1}\, dr = (-1)\frac{(t-r)^{2H}}{2H}\bigg|_0^t = \frac{t^{2H}}{2H}.$$

Adding the two results:

$$\mathbb{E}\left[(B_t^{(H)})^2\right] = t^{2H} \underbrace{\left[\int_{-\infty}^0 \left((1-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}\right)^2 du + \frac{1}{2H}\right]}_{\kappa_H} = t^{2H}\kappa_H.$$

We want to see that $t^{2H}\kappa_H < +\infty$. For the second term $\frac{1}{2H}$ we need $H > 0$ to make $\frac{1}{2H} < +\infty$. For the integral term $\int_{-\infty}^0 \left[((1-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}\right]^2 du < \infty$, we examine points in its domain that may cause it to "misbehave".

- For $u \to 0$: $[(1-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}]^2 \sim [1 - u^{H-\frac{1}{2}}]^2 \sim u^{2H-1}$ and $\int_{-\infty}^0 u^{2H-1}\, du$ converges iff $2H-1 < -1$ which is satisfied given that $H < 1$ by definition.

- For $u \to -\infty$: using $f(u) = u^{H-\frac{1}{2}} \in C^1(\mathbb{R}_+)$, we have that $\left[(1-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}\right]^2 = [f(1-u) - f(-u)]^2$ and, by the Mean Value Theorem, there exists $\xi \in (-u, 1-u)$ such that $f(1-u) - f(-u) = f'(\xi)(1-u+u) = f'(\xi)$. So $|f(1-u) - f(-u)|^2 = |f'(\xi)|^2 = \left|(\xi^{H-\frac{1}{2}})'\right|^2 = \left|(H-\frac{1}{2})\xi^{H-\frac{3}{2}}\right|^2 = (H-\frac{1}{2})^2\xi^{2H-3}$ and $\int_{-\infty}^0 u^{2H-3}\, du$ converges iff $2H-3 < -1$ which is also satisfied given that $H < 1$ by definition.

So, given that, by definition, $H \in (0,1)$, all conditions are satisfied and $\kappa_H < +\infty$ making the expression converge.

Similarly to the previous computation, this integral can be evaluated as follows:

$$\begin{aligned}
\mathbb{E}\left[(B_t^{(H)} - B_s^{(H)})^2\right] &= \mathbb{E}\left[\left(\int_{\mathbb{R}} (t-r)_+^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}} - (s-r)_+^{H-\frac{1}{2}} + (-r)_+^{H-\frac{1}{2}}\, dB_r\right)^2\right] \\
&= \int_{\mathbb{R}} \mathbb{E}\left[\left((t-r)_+^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}} - (s-r)_+^{H-\frac{1}{2}} + (-r)_+^{H-\frac{1}{2}}\right)^2\right] dr \\
&= \int_{\mathbb{R}} \left((t-r)_+^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}} - (s-r)_+^{H-\frac{1}{2}} + (-r)_+^{H-\frac{1}{2}}\right)^2 dr \\
&= \int_{\mathbb{R}} \left((t-r)_+^{H-\frac{1}{2}} - (s-r)_+^{H-\frac{1}{2}}\right)^2 dr.
\end{aligned}$$

The computation of this integral yields:

$$\begin{aligned}
&\int_{\mathbb{R}} \left((t-r)_+^{H-\frac{1}{2}} - (s-r)_+^{H-\frac{1}{2}}\right)^2 dr = \int_{\mathbb{R}} \left((t-s+s-r)_+^{H-\frac{1}{2}} - (s-r)_+^{H-\frac{1}{2}}\right)^2 dr \\
&\overset{u=r-s}{=} \int_{\mathbb{R}} \left((t-s-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}\right)^2 du \\
&\overset{u=|t-s|v}{=} \int_{\mathbb{R}} \left((t-s-|t-s|v)_+^{H-\frac{1}{2}} - (-|t-s|v)_+^{H-\frac{1}{2}}\right)^2 |t-s| dv \\
&= \int_{\mathbb{R}} \left(|t-s|^{H-\frac{1}{2}}(1-v)_+^{H-\frac{1}{2}} - |t-s|^{H-\frac{1}{2}}(-v)_+^{H-\frac{1}{2}}\right)^2 |t-s| dv \\
&= |t-s|^{2H} \int_{\mathbb{R}} \left((1-v)_+^{H-\frac{1}{2}} - (-v)_+^{H-\frac{1}{2}}\right)^2 dv \\
&= |t-s|^{2H} \int_{-\infty}^{0} \left((1-v)^{H-\frac{1}{2}} - (-v)^{H-\frac{1}{2}}\right)^2 dv + \int_{0}^{t} \left((1-v)^{H-\frac{1}{2}}\right)^2 dv \\
&= |t-s|^{2H} \underbrace{\left[\int_{-\infty}^{0} \left((1-v)^{H-\frac{1}{2}} - (-v)^{H-\frac{1}{2}}\right)^2 dv + \frac{1}{2H}\right]}_{\kappa_H} = |t-s|^{2H}\kappa_H.
\end{aligned}$$

By Eq. (3.3), we obtain a covariance function that coincides with that of the fractional Brownian motion, thus confirming the validity of the representation.

$$\mathbb{E}\left[B_t^{(H)} B_s^{(H)}\right] = \kappa_H \frac{t^{2H} + s^{2H} - |t-s|^{2H}}{2}.$$

To prove the existence of such a process we make use of the below presented proposition without proof.

Proposition 3.12. *Let $K : I \times I \to \mathbb{R}$ be a symmetric, nonnegative definite function. Then, there exists a Gaussian process $\{X_t, t \geq 0\}$ such that $\mathbb{E}[X_t] = 0$ for any $t \in I$ and $Cov(X_{t_i}, X_{t_j}) = K(t_i, t_j)$, for any t_i, $t_j \in I$.*

Proof. A complete proof, which relies on the Kolmogorov Extension Theorem, can be found in Proposition 1.1 of the lecture notes *An Introduction to Stochastic Calculus* by Marta Sanz-Solé, Universitat de Barcelona, dated January 28, 2022. ■

By Proposition 3.12, to ensure the existence of the fractional Brownian motion it suffices to check that the covariance function is symmetric and nonnegative definite on $\mathbb{R}_+$. That means, for any $t_i \geq 0$ and any real numbers a_i, $i, j = 1, \ldots, m$,

$$\sum_{i,j}^{m} a_i a_j K(t_i, t_j) \geq 0.$$

Let $K(t_i, t_j) = \kappa_H \frac{1}{2}(t_j^{2H} + t_i^{2H} - |t_j - t_i|^{2H})$ and $\Phi(t, r) = (t-r)_+^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}}$.

Then, by Eq. (3.2), $\mathbb{E}\left[B_{t_j}^{(H)} B_{t_i}^{(H)}\right] = \int_{\mathbb{R}} \Phi(t_j, r)\Phi(t_i, r)\, dr = K(t_i, t_j)$ and

$$\begin{aligned}\sum_{i,j}^{m} a_i a_j K(t_i, t_j) &= \sum_{i,j=1}^{m} a_i a_j \int_{\mathbb{R}} \Phi(t_j, r)\Phi(t_i, r)\, dr \\ &= \int_{\mathbb{R}} \sum_{i,j=1}^{m} a_i a_j \Phi(t_j, r)\Phi(t_i, r)\, dr \\ &= \int_{\mathbb{R}} \left(\sum_{j=1}^{m} a_j \Phi(t_j, r)\right)^2 dr \geq 0.\end{aligned}$$

With this result, we show that the Mandelbrot-Van Ness representation defines a centered Gaussian process with the same covariance structure as the fractional Brownian motion, and with it, the existence of the process.

Among the various constructions of fractional Brownian motion (fBm), see for example [Pic10], the Riemann-Liouville fractional Brownian motion (RLfBm), also referred to as Type II fBm, stands out due to the simplicity of its integral representation.

Definition 3.13. The *Riemann-Liouville fractional Brownian motion* (RLfBm) process, also called a *type II fractional Brownian motion* is given by

$$\hat{B_t^H} := \frac{1}{\Gamma(H + \frac{1}{2})} \int_0^t (t - r)^{H - \frac{1}{2}}\, dB_r. \tag{3.4}$$

The variance of the RLfBm can be computed using the Itô isometry and Fubini's theorem:

$$\mathbb{E}\left[\left(\hat{B_t^H}\right)^2\right] = \left(\frac{1}{\Gamma(H + \frac{1}{2})}\right)^2 \int_0^t (t - r)^{2H-1}\, dr = \left(\frac{1}{\Gamma(H + \frac{1}{2})}\right)^2 \frac{t^{2H}}{2H}.$$

Therefore, the variance of the Riemann–Liouville fractional Brownian motion (RLfBm) coincides with that of the standard fractional Brownian motion (fBm). Note that the constant factor preceding the integral in the definition of fBm only scales the process and, as such, affects the variance but not the qualitative nature of its sample paths. However, as we will see shortly, this difference in construction leads to a distinct covariance structure, which causes certain properties of the standard fBm to be lost in the RLfBm framework.

As shown in [LS95], the covariance function of the RLfBm, for $t > s > 0$, is given by

$$\mathbb{E}\left[\hat{B}_t^H \hat{B}_s^H\right] = \frac{s^{H+\frac{1}{2}} t^{H-\frac{1}{2}}}{(H + \frac{1}{2})\Gamma(H + \frac{1}{2})^2}\, {}_2F_1\left(\frac{1}{2} - H, 1, H + \frac{3}{2}, \frac{s}{t}\right), \tag{3.5}$$

where ${}_2F_1$ denotes the hypergeometric function.

While the Riemann–Liouville fractional Brownian motion (RLfBm) does not possess stationary increments or exhibit strict self-similarity like the standard (Type I) fBm, it still retains essential features such as long memory and path roughness. Owing to its simple integral representation, the RLfBm is widely used in practice, particularly in mathematical finance, where it serves as a foundational component in fractional volatility models such as the rough Bergomi model.

4 The Black-Scholes model

Consider a market consisting of one stock (risky asset) and one bond (risk-less asset).

The stochastic process of the risky asset process $\{S_t, t \in [0,T]\}$ is assumed to satisfy the following stochastic differential equation:

$$dS_t = S_t \mu_t dt + \sigma_t S_t dW_t, \tag{4.1}$$

where $W = \{W_t, t \in [0,T]\}$ is the Wiener process previously defined. The expected rate of return μ_t and the volatility process σ_t are supposed to be measurable and adapted processes satisfying the following integrability conditions

$$\int_0^T |\mu_t|\, dt < \infty, \quad \int_0^T \sigma_t^2\, dt < \infty$$

almost surely. One way to solve the equation and get the explicit price process is to apply the Itô formula to $\ln(S_t)$:

$$d(\ln(S_t)) = \frac{1}{S_t} dS_t - \frac{1}{2S_t^2} d\langle S, S\rangle_t = \frac{dS_t}{S_t} - \frac{\sigma_t^2}{2}\, dt = (\mu_t\, dt + \sigma_t\, dW_t) - \frac{\sigma_t^2}{2}\, dt,$$

$$S_t = S_0 exp\left(\int_0^t (\mu_s - \frac{\sigma_s^2}{2})\, ds + \int_0^t \sigma_s\, dW_s \right). \tag{4.2}$$

The stochastic process of the risk-less asset process $\{F_t, t \in [0,T]\}$ is assumed to satisfy the following differential equation:

$$dF_t = r_t F_t dt, \quad F_0 = 1.$$

Where r_t is the interest rate is a positive measurable and adapted process satisfying the integrability condition

$$\int_0^T r_t\, dt < \infty,$$

almost surely. Similarly, the explicit bond process is given by

$$F_t = e^{\int_0^t r_s\, ds}.$$

Definition 4.1. Let α_t be the number of non-risky assets and β_t the number of risky assets (stocks) owned by an investor at time t. The couple $\phi_t = (\alpha_t, \beta_t)$, $t \in [0,T]$ is called a *strategy* and we assume that α_t and β_t are measurable and adapted processes such that

$$\int_0^T |\beta_t \mu_t|\, dt < \infty, \quad \int_0^T \beta_t^2 \sigma_t^2\, dt < \infty, \quad \int_0^T |\alpha_t| r_t\, dt < \infty$$

almost surely.

The investor's wealth at time t is called the *value* of the portfolio and is given by

$$V_t(\phi) = \alpha_t F_t + \beta_t S_t. \tag{4.3}$$

A portfolio ϕ is said to be *self-financing* if no additional capital is injected or withdrawn over time. That is, the value at any time t equals the initial investment plus the cumulative gains

$$\alpha_t F_t + \beta_t S_t = \alpha_0 F_0 + \beta_0 S_0 + \int_0^t \alpha_s \, dF_s + \int_0^t \beta_s \, dS_s. \tag{4.4}$$

The *discount factor* is the quantity by which a future cash flow is multiplied to determine its present value. The discounted price for the risky asset are given by

$$\tilde{S}_t = \frac{S_t}{F_t} = S_0 exp\left(\int_0^t \left(\mu_s - r_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s \, dW_s\right). \tag{4.5}$$

The discounted value of a portfolio is, then, given by

$$\tilde{V}_t(\phi) = \frac{V_t(\phi)}{F_t} = \alpha_t + \beta_t \tilde{S}_t.$$

Using the Itô integration by parts formula with $d\left(\frac{1}{F_t}\right) = \frac{(-1)}{F_t} r_t dt$, $dV_t(\phi) = \alpha_t dF_t + \beta_t dS_t$ and $d\langle F, V(\phi)\rangle_t = 0$ the differential of the discounted value of a portfolio is given by:

$$\begin{aligned}
d\tilde{V}_t(\phi) &= d\left(\frac{V_t(\phi)}{F_t}\right) = V_t d\left(\frac{1}{F_t}\right) + \frac{1}{F_t} dV_t(\phi) + d\langle F, V(\phi)\rangle_t \\
&= -\frac{1}{F_t} r_t V_t(\phi) dt + \frac{1}{F_t}(\alpha_t dF_t + \beta_t dS_t) = -\frac{1}{F_t} r_t V_t(\phi) dt + \alpha_t r_t dt + \frac{1}{F_t} \beta_t dS_t \\
&= r_t \left(\alpha_t - \frac{V_t(\phi)}{F_t}\right) dt + \frac{1}{F_t} \beta_t dS_t = r_t \left(\alpha_t - \tilde{V}_t(\phi)\right) dt + \frac{1}{F_t} \beta_t dS_t \\
&= -r_t \beta_t \tilde{S}_t dt + \frac{1}{F_t} \beta_t dS_t = \beta_t(-r_t \tilde{S}_t dt + \frac{1}{F_t} dS_t) \\
&= \beta_t d\tilde{S}_t.
\end{aligned}$$

By the Itô integral by parts formula we can compute the differential of the discounted price process. Having dS_t given by Eq. (4.1) and $d\langle S, F\rangle_t = 0$:

$$\begin{aligned}
d\tilde{S}_t &= d\left(\frac{1}{F_t} S_t\right) = \frac{1}{F_t} dS_t + S_t d\left(\frac{1}{F_t}\right) + d\langle S, F\rangle_t = \frac{1}{F_t}\left(S_t \mu_t dt + \sigma_t S_t dW_t\right) + S_t \frac{(-1)}{F_t} r_t dt \\
&= \tilde{S}_t(\mu_t - r_t) dt + \tilde{S}_t \sigma_t dW_t
\end{aligned} \tag{4.6}$$

With this result we can now integrate both ends of the previous expression:

$$\tilde{V}_t(\phi) = V_0(\phi) + \int_0^t \beta_s d\tilde{S}_s = V_0(\phi) + \int_0^t \beta_s(\mu_s - r_s)\tilde{S}_s \, ds + \int_0^t \beta_s \sigma_s \tilde{S}_s \, dW_s.$$

Consequently, when the drift μ_t equals the risk-free rate r_t for all $t \in [0, T]$, the value process $V_t(\phi)$ of any self-financing portfolio follows a local martingale.

Definition 4.2. A strategy ϕ is *admissible* if it is self-financing and there exists $K > 0$ such that its discounted value $\tilde{V}_t = \alpha_t + \beta_t \tilde{S}_t \geq K$, for all t.

Definition 4.3. Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures defined on the same measurable space $(\Omega, \mathcal{F})$. We say that $\mathbb{P}$ and $\mathbb{Q}$ are *equivalent*, denoted $\mathbb{Q} \sim \mathbb{P}$, if they are mutually absolutely continuous. That is,

$$\mathbb{P}(A) = 0 \iff \mathbb{Q}(A) = 0 \quad \text{for all } A \in \mathcal{F}.$$

Definition 4.4. A probability measure $\mathbb{Q}$ on the σ-field $\mathcal{F}_T$ is called a *martingale measure* (or a risk neutral measure) if:

1. $\mathbb{Q}$ is equivalent to $\mathbb{P}$ (i.e., $\mathbb{Q} \sim \mathbb{P}$),
2. the discounted price process $\tilde{S}_t = \frac{S_t}{F_t}$ is a $\mathbb{Q}$-martingale in the probability space $(\Omega, \mathcal{F}_T, \mathbb{Q})$.

Definition 4.5. An *arbitrage* is an admissible trading strategy ϕ such that the initial value satisfies $V_0(\phi) = 0$, the terminal value satisfies $V_T(\phi) \geq 0$, and there is a strictly positive probability of gain, that is, $\mathbb{P}(V_T(\phi) > 0) > 0$.

Typically, models of the stock price process are constructed to rule out arbitrage, as the existence of arbitrage opportunities would undermine both the economic validity and the mathematical coherence of the model. From an economic perspective, arbitrage opportunities imply the possibility of generating riskless profits without investment, which is inconsistent with competitive and efficient financial markets.

The goal is then to find a risk neutral measure under which discounted prices are a martingale. Let $\sigma_t > 0$ for all $t \in [0, T]$ and $\int_0^T \theta_s^2 \, ds < \infty$ almost surely, where

$$\theta_t = \frac{\mu_t - r_t}{\sigma_t}.$$

Then we can define the process

$$Z_t = exp\left(-\int_0^t \theta_s \, dW_s - \frac{1}{2} \int_0^t \theta_s^2 \, ds \right),$$

which is a positive local martingale. This can be verified by applying the Itô formula and observing that the drift term vanishes: $dZ_t = Z_t \left(-\frac{1}{2}\theta_t^2 dt - \theta_t dW_t + \frac{1}{2}\theta_t^2 dt \right) = -\theta_t Z_t dW_t$.

By the Girsanov theorem (see Section 7.1), if $\mathbb{E}[Z_T] = 1$, then, the process Z is a martingale, and the measure $\mathbb{Q}$ defined by the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$ is a probability measure equivalent to $\mathbb{P}$. Moreover, under $\mathbb{Q}$, the process

$$\widetilde{W}_t = W_t + \int_0^t \theta_s \, ds$$

is a Brownian motion.

By previous computations we know that

$$d\tilde{S}_t = \tilde{S}_t(\mu_t - r_t)dt + \tilde{S}_t\sigma_t dW_t = \sigma_t\tilde{S}_t\left(\frac{(\mu_t - r_t)}{\sigma_t}dt + dW_t\right)$$
$$= \sigma_t\tilde{S}_t d\left(\int_0^t \frac{(\mu_s - r_s)}{\sigma_s}\,ds + W_t\right) = \sigma_t\tilde{S}_t d\widetilde{W}_t.$$

We can observe that the drift term vanishes, so: the discounted prices are a $\mathbb{Q}$-martingale if the process σ_t is uniformly bounded and measurable. The discounted risky asset process can be written as follows:

$$\tilde{S}_t = \frac{S_t}{F_t} = S_0 exp\left(\int_0^t \sigma_s\,d\widetilde{W}_s - \frac{1}{2}\int_0^t \sigma_s^2\,ds\right).$$

To verify the absence of arbitrage opportunities, we assume that $\mathbb{E}_{\mathbb{Q}}\left[\int_0^T(\sigma_s\beta_s\tilde{S}_s)^2\,ds\right] < \infty$. Under this condition, we observe that the discounted value process is a martingale, as it has no drift under $\mathbb{Q}$:

$$\tilde{V}_t(\phi) = V_0(\phi) + \int_0^t \beta_s\,d\tilde{S}_s = V_0(\phi) + \int_0^t \sigma_s\beta_s\tilde{S}_s\,d\widetilde{W}_s.$$

Using such property, we have that

$$\mathbb{E}_{\mathbb{Q}}[\tilde{V}_T(\phi)] = \mathbb{E}_{\mathbb{Q}}[\tilde{V}_T(\phi)|\mathcal{F}_0] = \tilde{V}_0(\phi) = \frac{V_0(\phi)}{F_0} = V_0(\phi) = 0,$$

which implies that $V_T(\phi) = 0$ holds $\mathbb{Q}$-almost surely. This conclusion stands in direct contradiction to the assumption that $\mathbb{P}(V_T(\phi) > 0) > 0$, thereby excluding the possibility of arbitrage.

A *financial derivative* is a contract whose value depends on, or is derived from, the value of an underlying asset, index, or interest rate. Derivatives are fundamental tools in modern financial markets, used for purposes such as hedging risk, speculating on price movements, or enhancing portfolio strategies. Among the most common types of derivatives are *options*, which grant the holder the right but not the obligation to buy or sell an asset at a predetermined price; *futures* and *forwards*, which are agreements to buy or sell an asset at a specified future date and price; and *swaps*, which involve the exchange of cash flows or other financial instruments between parties.

A *European call option* gives its holder the right, but not the obligation, to purchase the underlying asset at a predetermined *strike price* K at a specified maturity date T. Similarly, a *European put option* gives the holder the right to sell the underlying asset at the strike price K at time T. The payoff of a European call option at maturity is given by $\max(S_T - K, 0)$, where S_T denotes the price of the underlying asset at time T, while the payoff of a European put option is $\max(K - S_T, 0)$.

The goal now is to construct a self-financing portfolio with the same final value as the option, that is, $V_T(\phi) = h(S_T)$. Markets where every payoff function h can be replicated are called *complete*. If the market is complete, there exists a self-financing portfolio ϕ such that $\tilde{V}_T(\phi) = \frac{h(S_T)}{F_T}$.

Because $\tilde{S}_t$ is a martingale under the risk neutral probability $\mathbb{Q}$, $\tilde{V}(\phi)$ will also be a martingale and $\tilde{V}_t(\phi)$ will be equal to the expectation conditioned to the filtration (the flow of information) at time t, $\mathcal{F}_t$. That is,

$$\tilde{V}_t(\phi) = \frac{1}{F_T}\mathbb{E}_{\mathbb{Q}}[h(S_T)|\mathcal{F}_t] = f(t, S_t). \tag{4.7}$$

The Black-Scholes model, highly relevant in financial mathematics, revolutionized option pricing with its simplicity. This model assumes that the price process $\{S_t,\, t \in [0,T]\}$ of the risky asset satisfies Eq. (4.1) with $\sigma_t = \sigma$, $\mu_t = \mu$ and $r_t = r$ constants. Following Eq. (4.2), the risk asset process is given by:

$$S_t = S_0 exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right). \tag{4.8}$$

By the Girsanov theorem with $\theta_t = \theta = \frac{\mu - r}{\sigma}$, $\tilde{W}_t$ is a $\mathbb{Q}$-Brownian motion with

$$d\mathbb{Q} = exp\left(-\theta W_T - \frac{\theta^2}{2}T\right)d\mathbb{P}.$$

We can compute the Malliavin derivative of such process using the chain rule in Theorem 2.36:

$$D_s S_t = \sigma S_t \chi_{[0,t]}(s). \tag{4.9}$$

In the Black-Scholes model, applying the Itô formula to $f(t, S_t)$ in Eq. (4.7), gives:

$$\tilde{f}(t,\tilde{S}_t) = f(0,S_0) + \int_0^t \frac{\partial \tilde{f}(s,\tilde{S}_s)}{\partial s}ds + \int_0^t \frac{\partial \tilde{f}(s,\tilde{S}_s)}{\partial \tilde{S}_s}d\tilde{S}_s + \frac{1}{2}\int_0^t \frac{\partial^2 \tilde{f}(s,\tilde{S}_s)}{\partial \tilde{S}_s^2}d\langle \tilde{S},\tilde{S}\rangle_s. \tag{4.10}$$

This partial differential equation is solved in the Black-Scholes model in [BS73] in the case in which the payoff function is given by $h(S_T) = (S_T - K)_+$ (they payoff of a european call option) with the following result:

$$(t, S_t) = S_t\Phi(d_+) - Ke^{-e(T-t)}\Phi(d\_) \tag{4.11}$$

where $\Phi(x)$ is the standard normal distribution and

$$d_\pm = \frac{log(\frac{S_t}{K}) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

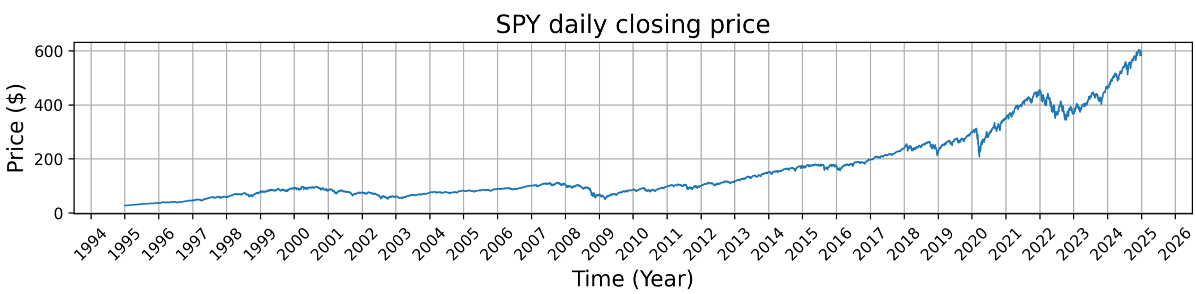
5 At the money implied volatility surface

Implied volatility is a key concept in modern option pricing and risk management. It is defined as the volatility input that, when substituted into the Black-Scholes formula, yields the observed market price of an option. In other words, instead of using the model to compute the option price from known parameters, implied volatility is obtained by reversing the procedure: the market price is taken as given, and the volatility is inferred. When computed for a range of strike prices and maturities, it forms the implied volatility surface, which captures the variation in implied volatility across different options. The surface often exhibits patterns such as skew or smile, revealing that market-implied volatilities are not constant, in contrast to the Black-Scholes assumption.

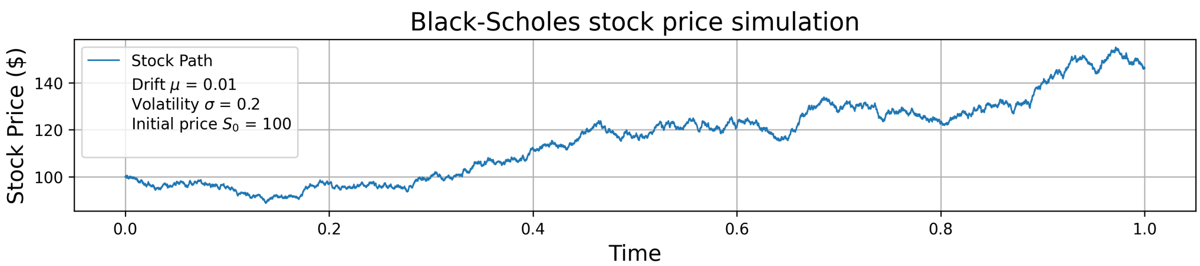
To illustrate some of the limitations of the Black-Scholes model that give rise to concepts like implied volatility, we consider the SPDR S&P 500 ETF Trust (`SPY`) as a representative underlying asset. It is an exchange-traded fund that replicates the performance of the S&P 500 Index, a theoretical benchmark comprising 500 leading publicly traded U.S. companies. Since its launch in 1993, SPY has become one of the most liquid and widely followed ETFs, offering efficient exposure to the U.S. equity market.

Understanding how assets like SPY move is crucial for financial analysis. A core assumption of the Black-Scholes model is that the underlying asset's price, such as SPY's, follows a geometric Brownian motion (see Eq. (4.8)) with constant volatility.

Figure 5.1 compares the historical closing prices of SPY from 1995 to April 2025 to a simulated price path generated under the Black-Scholes model.



(a) SPY ETF closing prices from 1995 to April 2025.

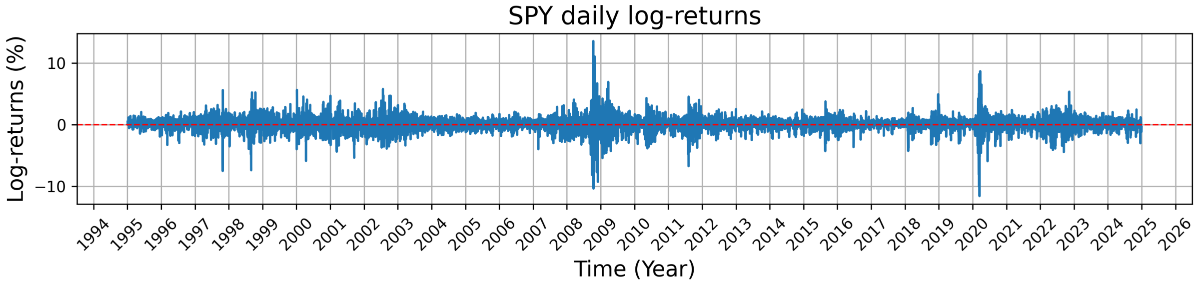


(b) Black-Scholes simulated price path with the same number of observations (7552).

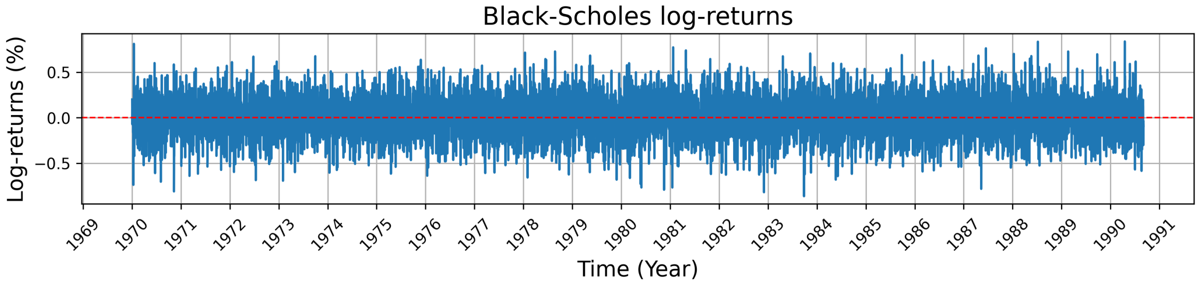
Figure 5.1: Comparison of historical SPY prices and a simulated Black-Scholes price process.

To analyze price dynamics, it is common to consider *logarithmic returns*, which provide a stationary and time-additive transformation of prices. For a discrete-time price process

$\{S_t, t \geq 0\}$, the log-return at time t is defined as $r_t = \log\left(\frac{S_t}{S_{t-1}}\right)$. Figure 5.2 displays the log-returns from the SPY historical prices and the Black-Scholes simulated path.



(a) Daily log-returns (%) of the SPY ETF from 1995 to 2025.



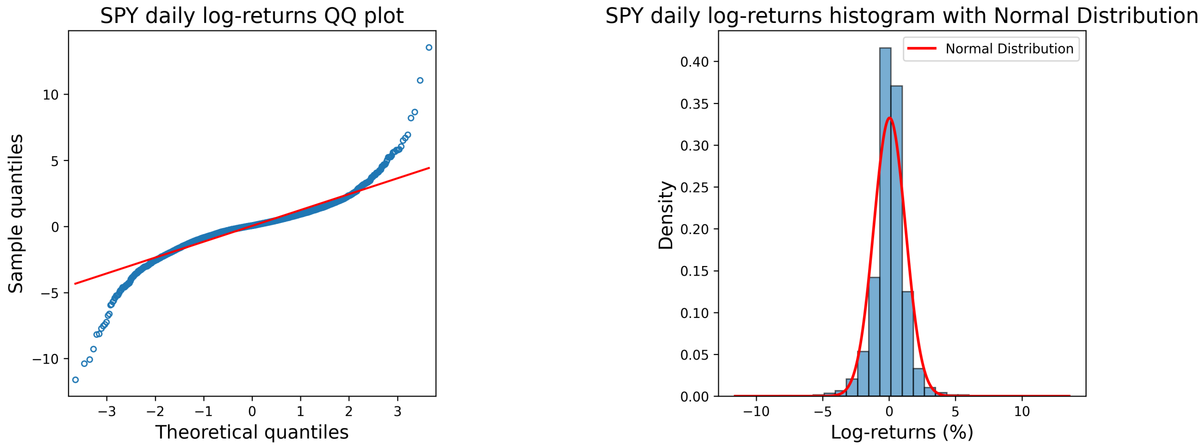
(b) Log-returns (%) of the Black-Scholes simulated price path.

Figure 5.2: Comparison of log-returns: empirical SPY data vs. Black-Scholes simulation.

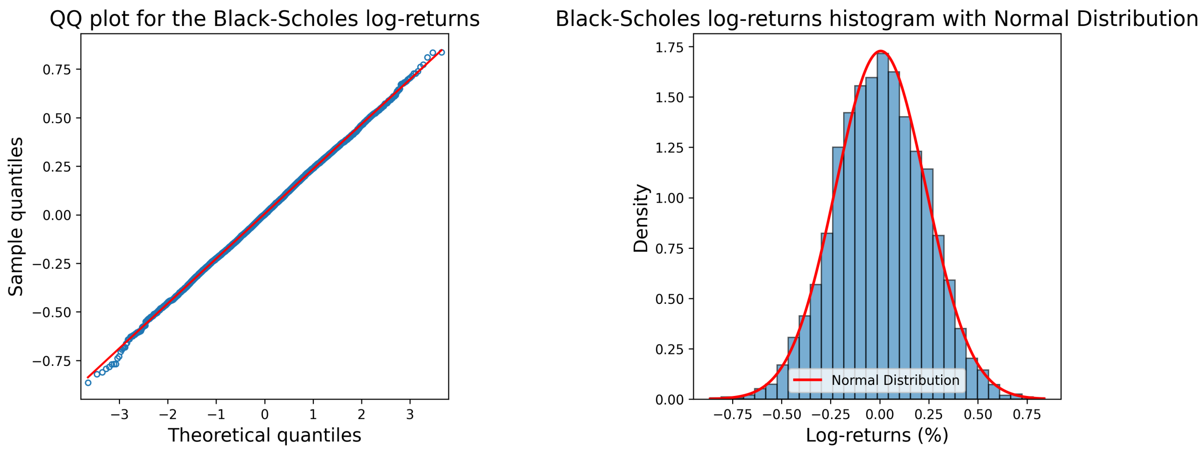
As shown in Fig. 5.2a, the SPY log-returns display pronounced *volatility clustering*, characterized by alternating periods of elevated and diminished variability. Extreme positive and negative returns also occur frequently, indicating heavy tails in the distribution. In contrast, the simulated Black-Scholes log-returns in Fig. 5.2b resemble an i.i.d. white noise process, with constant variance and no clustering, reflecting the model's assumption of constant volatility. This highlights the inadequacy of the Black-Scholes framework in capturing the heteroskedastic nature of real financial returns.

Further statistical comparison is provided in Fig. 5.3, where both QQ plots and histograms are used to assess distributional properties. In Fig. 5.3a, the QQ plot of SPY returns shows clear deviations from the diagonal reference line, especially in the tails, revealing heavy-tailed behavior. The histogram also shows a sharp peak compared to the overlaid normal distribution. These visual cues are confirmed quantitatively: the log-returns exhibit an excess kurtosis of approximately 13.65, significantly exceeding the normal value of 3, thereby indicating strong *leptokurtosis*.

In contrast, the QQ plot and histogram of the Black-Scholes returns (Fig. 5.3b) closely follow the theoretical normal distribution. There are no visible signs of fat tails or abnormal peakiness, and the empirical kurtosis is near the theoretical baseline. This further demonstrates that the Black-Scholes model lacks the flexibility to replicate the empirical features seen in financial markets.



(a) Left: QQ plot of SPY log-returns. Right: Histogram with overlaid normal density, illustrating leptokurtic behavior.



(b) Left: QQ plot of Black-Scholes log-returns. Right: Histogram showing no significant departure from normality.

Figure 5.3: Comparison of empirical vs. simulated log-returns through QQ plots and histograms.

This discrepancy motivates the use of more flexible modeling frameworks, such as stochastic volatility models (e.g., Heston, Hull-White), GARCH-type models, and more recently, rough volatility models. These approaches allow for time-varying, persistent volatility and better capture the empirical regularities observed in asset returns. The following sections will explore these models in depth.

Rough volatility models are a recent and important advancement in financial modeling. Unlike classical stochastic volatility models driven by Brownian motion, they use fractional Brownian motion (fBm) with Hurst parameter $H < \frac{1}{2}$, producing significantly rougher paths. Empirical studies (see [GJR14]), show that realized volatility exhibits this roughness, which classical models cannot capture. A key advantage is their ability to reproduce the steep implied volatility skew at short maturities, a well-known market feature. These models thus effectively capture both short- and long-term behavior of the implied volatility surface, offering a more realistic framework for option pricing.

A promiment example of a rough volatility model is the *rough Bergomi (rBergomi) model*, introduced in [BFa16]. Defined under a risk-neutral measure, the asset price S_t evolves as

$$dS_t = rS_t dt + \sigma_t S_t dW_t \tag{5.1}$$

where W_t is a Brownian motion (possibly correlated with the one driving the volatility), and the volatility process σ_t is modeled as

$$\sigma_t^2 = \sigma_0^2 exp\left(\nu\sqrt{2H}Z_t - \frac{1}{2}\nu^2 t^{2H}\right), \quad t \in [0, T], \tag{5.2}$$

where Z_t is the RLfBm defined in Definition 3.13 $\left(Z_t = \int_0^t (t-r)^{H-\frac{1}{2}}\, dB_r\right)$ and σ_0^2 and ν are some positive real values. The Malliavin derivative of the RLfBm can be computed with Proposition 2.34 yielding:

$$D_s Z_t = D_s\left(\int\limits_0^t (t-r)^{H-\frac{1}{2}}\, dB_r\right) = (t-s)^{H-\frac{1}{2}} \chi_{[0,t]}(s). \tag{5.3}$$

Thus, the Malliavin derivative of the rBergomi volatility process can be computed using the chain rule in Theorem 2.36:

$$D_s\sigma_r^2 = \sigma_r^2 D_s\left(\nu\sqrt{2H}Z_r - \frac{1}{2}\nu^2 r^{2H}\right) = \nu\sqrt{2H}\sigma_r^2 (r-s)^{H-\frac{1}{2}}\chi_{[0,r]}(s). \tag{5.4}$$

It is worth noting that when the Hurst parameter $H < \frac{1}{2}$, the Malliavin derivative $D_s\sigma_t^2$ exhibits a singular behavior, diverging as $|r-s| \to 0$. This singularity plays a crucial role in shaping the characteristics of the implied volatility surface, particularly contributing to the steepness of the short-term skew observed in market data.

The at-the-money implied volatility (ATMI) refers to the implied volatility of an option whose strike price is equal (or very close) to the current price of the underlying asset. It reflects the market's expectation of future volatility over the option's lifetime, specifically for small price deviations around the current level. ATMI plays a central role in financial modeling because it often serves as a proxy for spot volatility and is more stable and liquid than implied volatilities at other strikes.

Starting from the exact decomposition of option prices introduced in page 15 of [Alo04], which separates the uncorrelated case from the correction due to correlation, one can obtain a corresponding decomposition of the implied volatility.

Consider the following model for log-price of a risky asset process under a risk neutral probability $\mathbb{Q}$:

$$dX_t = \left(r - \frac{\sigma_t^2}{2}\right)dt + \sigma_t\left(\rho dW_t + \sqrt{1-\rho^2}B_t\right), \quad t \in [0, T], \tag{5.5}$$

for some $T > 0$ and where W and B are independent standard Wiener processes, $\rho \in [-1, 1]$, and σ is a square integrable process adapted to the filtration generated by W. According to Eq. (4.7), the price of a European call with $F_t = e^{rt}$ is given by:

$$V_t = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[\left(e^{X_T} - K\right)_+ \middle| \mathcal{F}_t\right], \tag{5.6}$$

for some strike K. We define the following operators:

$$\nu_t := \sqrt{\frac{Y_t}{T-t}}, \quad Y_t := \int_t^T \sigma_s^2 \, ds, \quad \text{denoting the integrated volatility.} \tag{5.7}$$

$$BS(t, x; \sigma) = e^x N(d_+) + K e^{-r(T-t)} N(d\_), \text{ as in Eq. (4.11).} \tag{5.8}$$

Theorem 5.1. *Assume the model in Eq. (5.5) holds. Then it follows that*

$$V_t = \mathbb{E}\left[BS(t, X_t, \nu_t) \middle| \mathcal{F}_t \right] + \frac{\rho}{2} \mathbb{E}\left[\int_t^T e^{-r(s-t)} H(s, X_s, \nu_s) \varphi_s \, ds \middle| \mathcal{F}_t \right], \tag{5.9}$$

where

$$H(t, X_t, \nu_t) := \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) BS(t, X_t, \nu_t)$$

and

$$\varphi_s := \sigma_s \int_s^T D_s^W \sigma_u^2 \, du.$$

Proof. The full proof can be found in [Alo04]. ∎

Note that for $\rho = 0$, i.e., the uncorrelated case, the price of the call is just the Black-Scholes price.

We take these results and apply them to the rBergomi model in Eq. (5.2) to further analyze the behavior of the ATMI. By Eq. (5.9), we can write:

$$\varphi_s = \sigma_s \int_s^T D_s^W \sigma_u^2 \, du = \sigma_s \int_s^T \left(\nu \sqrt{2H} \sigma_r^2 (u-s)^{H-\frac{1}{2}} \chi_{[0,r]}(s) \right) du$$

$$\begin{aligned}
V_t &= \mathbb{E}\left[BS(t, X_t, \nu_t) \middle| \mathcal{F}_t \right] \\
&\quad + \frac{\rho}{2} \mathbb{E}\left[\int_t^T e^{-r(s-t)} H(s, X_s, \nu_s) \left(\sigma_s \int_s^T \left(\nu \sqrt{2H} \sigma_u^2 (u-s)^{H-\frac{1}{2}} \chi_{[0,r]}(s) \right) du \right) ds \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}\left[BS(t, X_t, \nu_t) \middle| \mathcal{F}_t \right] \\
&\quad + \frac{\rho \nu \sqrt{2H}}{2} \mathbb{E}\left[\int_t^T e^{-r(s-t)} H(s, X_s, \nu_s) \left(\sigma_s \int_s^T \left(\sigma_u^2 (u-s)^{H-\frac{1}{2}} \chi_{[0,r]}(s) \right) du \right) ds \middle| \mathcal{F}_t \right]
\end{aligned} \tag{5.10}$$

This decomposition, culminating in Equation (5.10), provides a crucial analytical expression for option prices under the rBergomi model, paving the way for investigating its implications for implied volatility, particularly the short-end of the skew.

6 Conclusions

In this thesis, we have developed the core elements of Malliavin calculus using the Wiener-Itô chaos expansion approach, as framed in [Giu08]. This framework allowed us to rigorously define the Malliavin derivative and related operators, such as the Skorokhod integral and the Clark-Ocone formula. We also studied the fractional Brownian motion (fBm), emphasizing its unique properties and motivation for extending classical stochastic analysis beyond the semimartingale setting.

Using these tools, we investigated the short-time asymptotic behavior of at-the-money implied volatility (ATMI) within the rough Bergomi model, showing how Malliavin calculus facilitates a precise decomposition that captures the impact of correlation and the roughness of the volatility path.

7 Appendix

7.1 Girsanov's Theorem

Theorem 7.1 (**Girsanov's Theorem**)**.** *Consider a filtered probability space* $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$*, and let* $\{\theta_t, t \in [0,T]\}$ *be a* $\mathcal{F}_t$ *adapted process. Suppose that Novikov's condition holds:*

$$\mathbb{E}\left[exp\left(\frac{1}{2}\int_0^T \theta_t^2\, dt\right)\right] < \infty.$$

Define the process

$$Z_t := exp\left(\int_0^t \theta_s\, dW_s - \frac{1}{2}\int_0^t \theta_s^2\, ds\right)$$

where W_t *is an* $\mathcal{F}_t$*-Brownian motion. Then* $\{Z_t, t \in [0,T]\}$ *is a uniformly integrable martingale, and we can define a new probability measure* $\mathbb{Q}$ *on* $(\Omega, \mathcal{F})$ *by*

$$\mathbb{Q}(A) := \mathbb{E}[\mathbb{1}_A Z_T], \quad A \in \mathcal{F}.$$

Under $\mathbb{Q}$*, the process*

$$\widetilde{W}_t = W_t - \int_0^t \theta_s\, ds \quad t \in [0,T]$$

is an $\mathcal{F}_t$*-Brownian motion.*

Proof. A complete proof can be found in page 153 of [Oks03]. ■

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