Analytic Geometry
solved exercises

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Contents

1. Introduction

This book offers a collection of analytic geometry exercises designed to strengthen skills in calculation and interpretation of figures in the cartesian plane. Through problems involving lines, parabolas, circles, ellipses, and hyperbolas, readers will deepen their understanding of equations, tangents, notable points, distances, and areas. The exercises range from determining lines passing through points or perpendicular to given lines, to studying conic

sections and their properties and analyzing plane figures
bounded by curves and segments. This path is ideal for
developing reasoning abilities and effectively preparing
for exams and assessments.
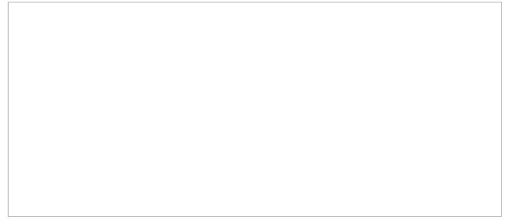
2. Exercises

2.1 Exercise 1

Determine the equation of the line whose graph passes through the points

$$P_1 = \left(5, -\frac{1}{2}\right), \quad P_2 = \left(-\frac{7}{3}, 6\right)$$

2.2 Exercise 2

Determine the vertices of the rectangular hyperbola with equation

$$xy = 5$$

and calculate their distance.

2.3 Exercise 3

Find the equation of the parabola passing through the point

$$A = (3,4)$$

and having its vertex at the point

$$V = \left(\frac{5}{3}, -\frac{4}{3}\right)$$

2.4 Exercise 4

Calculate the slope of the line r knowing that its graph passes through the points

$$P_1 = (-3,-8)$$

and

$$P_2 = (15,21)$$

2.5   Exercise 5

Let r be the line with equation

$$y = 3x - 1$$

Determine the length of the segment of r between its points with abscissas

$$x_1 = 2, \quad x_2 = 5$$

2.6   Exercise 6

Find the equation of an ellipse, knowing that it has its center at the point

$$C = (1,4)$$

passes through the point

$$A = (2,2)$$

and the area it encloses is

$$\text{Area} = 15\pi$$

2.7 Exercise 7

Find the cartesian equation, in implicit form, of the line perpendicular to the line

$$y = -\frac{2}{5}x + 8$$

and passing through the point

$$A = (1,1)$$

2.8 Exercise 8

Consider in a cartesian plane the triangle with vertices at the three points

$$P_1 = (-3,0), \quad P_2 = (5,0), \quad P_3 = \left(\frac{17}{5}, \frac{16}{5}\right)$$

Prove that the triangle is right-angled using a method involving lines in the plane.

2.9 Exercise 9

Determine the real parameter k so that the equation

$$ky = (-k+6)x - 2k^2 + 3$$

satisfies the following conditions:

1. it is the equation of a line parallel to the x-axis;

2. it is the equation of a line parallel to the y-axis.

2.10 Exercise 10

Find the equation of the tangent line to the parabola with equation

$$y = x^2 - 2x - 1$$

at its point with abscissa $x = 3$.

2.11 Exercise 11

Find the point of intersection between the lines with equations

$$y = 3x - 2$$

and

$$x + 3y = -1$$

2.12 Exercise 12

Consider the family of parabolas with equation

$$y = x^2 + bx + c$$

Determine the equation of the parabola that passes through the point $P(1,2)$ and is tangent to the line

$$y = 3x - 1$$

2.13 Exercise 13

Determine for which values of the real parameter k the equation

$$\frac{x^2}{7k+12} + \frac{y^2}{9k+3} = 1$$

represents a

- circle,
- ellipse,
- hyperbola.

2.14 Exercise 14

Determine the equation of the parabola with axis parallel to the y-axis that passes through the points

$$(1,2), \quad (3,10), \quad (5,26)$$

2.15 Exercise 15

Calculate the area of the triangle determined by the intersections of the lines

$$y = 2x + 1, \quad y = -x + 4, \quad x = 3$$

2.16 Exercise 16

Given the hyperbola with equation

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

calculate the coordinates of the vertices and the asymptotes.

2.17 Exercise 17

Find the point of tangency of the line

$$y = mx + 1$$

to the parabola

$$y = x^2 - 4x + 2$$

knowing that the line is tangent to the parabola.

2.18 Exercise 18

Given the four lines with equations

$$y = 2x + 3, \quad y = 2x - 1, \quad x = 1, \quad x = -1$$

determine the area of the parallelogram enclosed by their graphs.

2.19 Exercise 19

Find the equation of the circle with center

$$C = \left(-\frac{1}{2}, 3 \right)$$

knowing that the area it encloses is

$$\text{area} = 13\pi$$

2.20 Exercise 20

Given the quadratic function

$$f(x) = ax^2 + bx + c,$$

with $a, b, c \in \mathbb{R}$. Knowing that the vertex of the parabola lies on the line

$$y = 2x + 3,$$

and that the parabola passes through the points $(1,4)$ and $(3,0)$, determine the coefficients a, b, and c.

2.21 Exercise 21

Determine the equation of the circle that passes through the points

$$A = (2,3), \quad B = (4,5)$$

and whose center lies on the line

$$x - y + 1 = 0$$

2.22 Exercise 22

Given the parabola with implicit cartesian equation

$$5y + 3x^2 - 3 + 5x = 0$$

find its focus and vertex.

2.23 Exercise 23

Determine the equation of the line passing through the points

$$A = (2,5), \quad B = (7,-3)$$

2.24 Exercise 24

Find the midpoint of the segment with endpoints

$$P = (-1,4), \quad Q = (5,-2)$$

2.25 Exercise 25

Calculate the distance between the points

$$A = (3,7), \quad B = (-2,1)$$

2.26 Exercise 26

Determine the equation of the line perpendicular to the line

$$3x - 4y + 7 = 0$$

and passing through the point

$$A = (2,-1)$$

2.27 Exercise 27

Determine the equation of the circle with center at the point

$$C = (1, -2)$$

and radius $r = 5$.

2.28 Exercise 28

Find the coordinates of the intersection point between the lines

$$y = 2x + 1, \quad y = -x + 4$$

2.29 Exercise 29

Determine the equation of the ellipse with center at the origin, major axis along the x-axis, semi-major axis $a = 4$ and semi-minor axis $b = 3$.

2.30 Exercise 30

Calculate the area of the triangle with vertices

$$A = (1, 2), \quad B = (4, 6), \quad C = (7, 2)$$

2.31 Exercise 31

Determine the equation of the line that passes through the point

$$P = (2,3)$$

and forms equal angles with the lines

$$y = x + 1, \quad y = -x + 5$$

2.32 Exercise 32

Given the parabola with equation

$$y = x^2 - 4x + 6$$

find the equation of the tangent line to the parabola that passes through the external point

$$P = (5,1)$$

2.33 Exercise 33

Consider the triangle with vertices

$$A = (1,2), \quad B = (5,7), \quad C = (8,3).$$

Determine the equation of the angle bisector at B.

2.34 Exercise 34

Find the equation of the rectangular hyperbola with center at the origin and asymptotes

$$y = 2x, \quad y = -2x$$

that passes through the point

$$P = (3, 5)$$

2.35 Exercise 35

Calculate the area of the trapezoid formed by the lines

$$y = x + 2, \quad y = -\frac{1}{3}x + \frac{7}{3}, \quad x = 1, \quad x = 4$$

2.36 Exercise 36

Given the circle with equation

$$x^2 + y^2 - 4x + 6y - 12 = 0$$

find the equations of the tangent lines to the circle that pass through the external point

$$P = (8, 2)$$

2.37 Exercise 37

Determine the equation of the tangent line to the parabola

$$y = x^2 - 2x + 1$$

at its point with abscissa $x = 3$.

2.38 Exercise 38

Given the ellipse with equation

$$\frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} = 1$$

calculate the lengths of the axes and find the coordinates of the foci.

2.39 Exercise 39

Find the equation of the circle tangent to the x-axis at the point

$$(3,0)$$

and passing through the point

$$(6,4)$$

2.40 Exercise 40

Calculate the coordinates of the centroid of the triangle with vertices

$$A = (2,3), \quad B = (6,7), \quad C = (4,1)$$

Also verify that the centroid lies at the intersection of the medians.
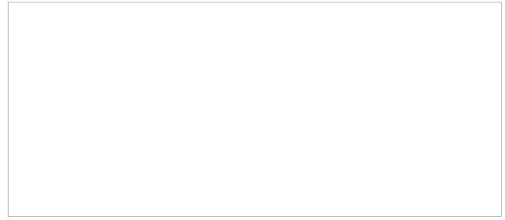
3. Results

In this chapter, we present only the results of the exercises, the complete solutions can be found in the following chapter.

3.1 Exercise 1

Text

Determine the equation of the line whose graph passes through the points

$$P_1 = \left(5, -\frac{1}{2} \right), \quad P_2 = \left(-\frac{7}{3}, 6 \right)$$

Result

The equation of the line is

$$y = -\frac{39}{44}x + \frac{173}{44}$$

3.2 Exercise 2

Text

Determine the vertices of the rectangular hyperbola with equation

$$xy = 5$$

and calculate their distance.

Result

The vertices are

$$V_1 = \left(\sqrt{5}, \sqrt{5}\right), \quad V_2 = \left(-\sqrt{5}, -\sqrt{5}\right)$$

The distance between the vertices is

$$d = 2\sqrt{10}$$

3.3 Exercise 3

Text

Find the equation of the parabola passing through the point

$$A = (3,4)$$

and having its vertex at the point

$$V = \left(\frac{5}{3}, -\frac{4}{3} \right)$$

Result

The equation of the parabola is

$$y = 3x^2 - 10x + 7$$

3.4 Exercise 4

Text

Calculate the slope of the line r knowing that its graph passes through the points

$$P_1 = (-3, -8)$$

and

$$P_2 = (15, 21)$$

Result

The slope of the line is

$$m = \frac{29}{18}$$

3.5   Exercise 5

Text

Let r be the line with equation

$$y = 3x - 1$$

Determine the length of the segment of r between its points with abscissas

$$x_1 = 2, \quad x_2 = 5$$

Result

The length of the segment is

$$d = 3\sqrt{10}$$

3.6 Exercise 6

Text

Find the equation of an ellipse, knowing that it has its center at the point

$$C = (1,4)$$

passes through the point

$$A = (2,2)$$

and the area it encloses is

$$\text{Area} = 15\pi$$

Result

The ellipses are two, with equations

$$\frac{8(x-1)^2}{15(15+\sqrt{209})} + \frac{(15+\sqrt{209})(y-4)^2}{120} = 1$$

and

$$\frac{8(x-1)^2}{15(15-\sqrt{209})} + \frac{(15-\sqrt{209})(y-4)^2}{120} = 1$$

3.7 Exercise 7

Text

Find the cartesian equation, in implicit form, of the line perpendicular to the line

$$y = -\frac{2}{5}x + 8$$

and passing through the point

$$A = (1,1)$$

Result

The implicit form of the equation is

$$2y - 5x + 3 = 0$$

3.8 Exercise 8

Text

Consider in a cartesian plane the triangle with vertices at the three points

$$P_1 = (-3,0), \quad P_2 = (5,0), \quad P_3 = \left(\frac{17}{5}, \frac{16}{5}\right)$$

Prove that the triangle is right-angled using a method involving lines in the plane.

Result

See the full demonstration for the result.

3.9 Exercise 9

Text

Determine the real parameter k so that the equation

$$ky = (-k+6)x - 2k^2 + 3$$

satisfies the following conditions:

1. it is the equation of a line parallel to the x-axis;
2. it is the equation of a line parallel to the y-axis.

Result

Horizontal line:

$$k = 6 \Rightarrow y = -\frac{23}{2}$$

Vertical line:

$$k = 0 \Rightarrow x = -\frac{1}{2}$$

3.10 Exercise 10

Text

Find the equation of the tangent line to the parabola with equation

$$y = x^2 - 2x - 1$$

at its point with abscissa $x = 3$.

Result

The equation of the tangent line is

$$y = 4x - 10$$

3.11 Exercise 11

Text

Find the point of intersection between the lines with equations

$$y = 3x - 2$$

and

$$x + 3y = -1$$

Result

The point of intersection is

$$\left(\frac{1}{2}, -\frac{1}{2}\right)$$

3.12 Exercise 12

Text

Consider the family of parabolas with equation

$$y = x^2 + bx + c$$

Determine the equation of the parabola that passes through the point $P(1,2)$ and is tangent to the line

$$y = 3x - 1$$

Result

The equation of the parabola is

$$y = x^2 + x$$

3.13   Exercise 13

Text

Determine for which values of the real parameter k the equation

$$\frac{x^2}{7k+12} + \frac{y^2}{9k+3} = 1$$

represents a

- circle,
- ellipse,
- hyperbola.

Result

The equation represents a circle for

$$k = \frac{9}{2}$$

An ellipse for

$$k > -\frac{1}{3}, \quad k \neq \frac{9}{2}$$

A hyperbola for

$$-\frac{12}{7} < k < -\frac{1}{3}$$

3.14 Exercise 14

Text

Determine the equation of the parabola with axis parallel to the y-axis that passes through the points

$$(1,2), \quad (3,10), \quad (5,26)$$

Result

The equation of the parabola is

$$y = x^2 + 1$$

3.15 Exercise 15

Text

Calculate the area of the triangle determined by the intersections of the lines

$$y = 2x + 1, \quad y = -x + 4, \quad x = 3$$

Result

The area of the triangle is

$$\text{Area} = 6$$

3.16   Exercise 16

Text

Given the hyperbola with equation

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

calculate the coordinates of the vertices and the asymptotes.

Result

The vertices are

$$V_1 = (-4,0), \quad V_2 = (4,0)$$

while the asymptotes are

$$y = \pm\frac{b}{a}x = \pm\frac{3}{4}x$$

3.17   Exercise 17

Text

Find the point of tangency of the line

$$y = mx + 1$$

to the parabola

$$y = x^2 - 4x + 2$$

knowing that the line is tangent to the parabola.

Result

The points of tangency are

- For $m = -2 \Rightarrow P_1 = (1, -1)$;
- For $m = -6 \Rightarrow P_2 = (-1, 7)$.

3.18 Exercise 18

Text

Given the four lines with equations

$$y = 2x + 3, \quad y = 2x - 1, \quad x = 1, \quad x = -1$$

determine the area of the parallelogram enclosed by their graphs.

Result

The area is

$$\text{Area} = 8$$

3.19 Exercise 19

Text

Find the equation of the circle with center

$$C = \left(-\frac{1}{2}, 3 \right)$$

knowing that the area it encloses is

$$\text{area} = 13\pi$$

Result

The equation of the circle is

$$x^2 + y^2 + x - 6y = \frac{15}{4}$$

3.20 Exercise 20

Text

Given the quadratic function

$$f(x) = ax^2 + bx + c,$$

with $a, b, c \in \mathbb{R}$. Knowing that the vertex of the parabola lies on the line

$$y = 2x + 3,$$

and that the parabola passes through the points $(1, 4)$ and $(3, 0)$, determine the coefficients a, b, and c.

Result

The two parabolas that meet the given conditions are

$$y = \left(\frac{-5+\sqrt{13}}{2}\right)x^2 - 2(4+\sqrt{13})x + \frac{3}{2}(-1+\sqrt{13})$$

$$y = \left(\frac{-5-\sqrt{13}}{2}\right)x^2 - 2(4-\sqrt{13})x + \frac{3}{2}(-1-\sqrt{13})$$

3.21   Exercise 21

Text

Determine the equation of the circle that passes through the points

$$A = (2,3), \quad B = (4,5)$$

and whose center lies on the line

$$x - y + 1 = 0$$

Result

The equation of the circle is

$$(x-3)^2 + (y-4)^2 = 2$$

3.22 Exercise 22

Text

Given the parabola with implicit cartesian equation

$$5y + 3x^2 - 3 + 5x = 0$$

find its focus and vertex.

Result

The vertex is

$$V = \left(-\frac{5}{6}, \frac{61}{60} \right)$$

The focus is

$$F = \left(-\frac{5}{6}, \frac{3}{5} \right)$$

3.23 Exercise 23

Text

Determine the equation of the line passing through the points

$$A = (2,5), \quad B = (7,-3)$$

Result

The equation of the line is

$$y = -\frac{8}{5}x + \frac{41}{5}$$

3.24 Exercise 24

Text

Find the midpoint of the segment with endpoints

$$P = (-1, 4), \quad Q = (5, -2)$$

Result

The midpoint is

$$M = (2, 1)$$

3.25 Exercise 25

Text

Calculate the distance between the points

$$A = (3, 7), \quad B = (-2, 1)$$

Result

The distance is

$$\overline{AB} = \sqrt{61}$$

3.26 Exercise 26

Text

Determine the equation of the line perpendicular to the line

$$3x - 4y + 7 = 0$$

and passing through the point

$$A = (2, -1)$$

Result

The equation of the line is

$$y = -\frac{4}{3}x + \frac{5}{3}$$

3.27 Exercise 27

Text

Determine the equation of the circle with center at the point

$$C = (1, -2)$$

and radius $r = 5$.

Result

The equation of the circle is

$$x^2 + y^2 - 2x + 4y - 20 = 0$$

3.28   Exercise 28

Text

Find the coordinates of the intersection point between the lines

$$y = 2x + 1, \quad y = -x + 4$$

Result

The point of intersection is

$$P = (1, 3)$$

3.29   Exercise 29

Text

Determine the equation of the ellipse with center at the origin, major axis along the x-axis, semi-major axis $a = 4$ and semi-minor axis $b = 3$.

Result

The equation of the ellipse is

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

3.30   Exercise 30

Text

Calculate the area of the triangle with vertices

$$A = (1,2), \quad B = (4,6), \quad C = (7,2)$$

Result

The area is

$$\text{Area} = 12$$

3.31   Exercise 31

Text

Determine the equation of the line that passes through the point

$$P = (2,3)$$

and forms equal angles with the lines

$$y = x + 1, \quad y = -x + 5$$

Result

There are two lines with equations

$$y = 3$$

and

$$x = 2$$

3.32 Exercise 32

Text

Given the parabola with equation

$$y = x^2 - 4x + 6$$

find the equation of the tangent line to the parabola that passes through the external point

$$P = (5, 1)$$

Result

The two tangent lines to the parabola passing through $P = (5, 1)$ are

$$y = 2(3 + \sqrt{10})x - 29 - 10\sqrt{10}$$

and

$$y = 2(3 - \sqrt{10})x - 29 + 10\sqrt{10}$$

3.33   Exercise 33

Text

Consider the triangle with vertices

$$A = (1,2), \quad B = (5,7), \quad C = (8,3).$$

Determine the equation of the angle bisector at B.

Result

The equation of the bisector is

$$y = (32 + 5\sqrt{41})x - 153 - 25\sqrt{41}$$

3.34   Exercise 34

Text

Find the equation of the rectangular hyperbola with center at the origin and asymptotes

$$y = 2x, \quad y = -2x$$

that passes through the point

$$P = (3,5)$$

Result

The equation of the hyperbola is

$$x^2 - \frac{1}{4}y^2 = \frac{11}{4}$$

3.35 Exercise 35

Text

Calculate the area of the trapezoid formed by the lines

$$y = x + 2, \quad y = -\frac{1}{3}x + \frac{7}{3}, \quad x = 1, \quad x = 4$$

Result

The area of the trapezoid is

$$\text{Area} = 9$$

3.36 Exercise 36

Text

Given the circle with equation

$$x^2 + y^2 - 4x + 6y - 12 = 0$$

find the equations of the tangent lines to the circle that pass through the external point

$$P = (8, 2)$$

Result

The tangent lines have equations

$$y = 2$$

and

$$y = \frac{60}{11}x - \frac{480}{11} + 2$$

3.37   Exercise 37

Text

Determine the equation of the tangent line to the parabola

$$y = x^2 - 2x + 1$$

at its point with abscissa $x = 3$.

Result

The equation of the tangent line is

$$y = 4x - 8$$

3.38 Exercise 38

Text

Given the ellipse with equation

$$\frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} = 1$$

calculate the lengths of the axes and find the coordinates of the foci.

Result

Length of the major semi-axis:

6

Length of the minor semi-axis:

4

The foci are

$$F_1 = (1 - \sqrt{5}, -2)$$

and

$$F_2 = (1 + \sqrt{5}, -2)$$

3.39   Exercise 39

Text

Find the equation of the circle tangent to the x-axis at the point

$$(3,0)$$

and passing through the point

$$(6,4)$$

Result

The equation of the circle is

$$(x-3)^2 + \left(y - \frac{25}{8}\right)^2 = \left(\frac{25}{8}\right)^2$$

3.40   Exercise 40

Text

Calculate the coordinates of the centroid of the triangle with vertices
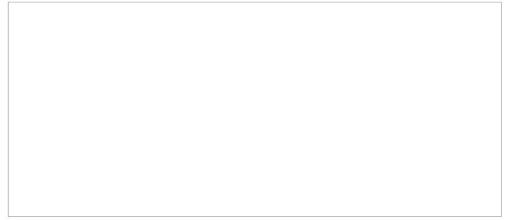
$$A = (2,3), \quad B = (6,7), \quad C = (4,1)$$

Also verify that the centroid lies at the intersection of the medians.

Result

The centroid is

$$G = \left(4, \frac{11}{3}\right)$$

4. Solutions

4.1 Exercise 1

Text

Determine the equation of the line whose graph passes through the points

$$P_1 = \left(5, -\frac{1}{2} \right), \quad P_2 = \left(-\frac{7}{3}, 6 \right)$$

Solution

To determine the equation of the required line, we must first calculate the slope m. The formula for m is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Substituting the values of the given points:

$$m = \frac{6 - \left(-\frac{1}{2}\right)}{-\frac{7}{3} - 5} = \frac{6 + \frac{1}{2}}{-\frac{7}{3} - \frac{15}{3}} = \frac{\frac{12}{2} + \frac{1}{2}}{-\frac{7}{3} - \frac{15}{3}} = \frac{\frac{13}{2}}{-\frac{22}{3}}$$

$$= \frac{13}{2} \cdot \left(-\frac{3}{22}\right) = -\frac{39}{44}$$

We use the point-slope form of the line with point P_1:

$$y - y_1 = m(x - x_1)$$

from which

$$y - \left(-\frac{1}{2}\right) = -\frac{39}{44}(x - 5), \qquad y + \frac{1}{2} = -\frac{39}{44}x + \frac{195}{44}$$

that is,

$$y = -\frac{39}{44}x + \frac{195}{44} - \frac{1}{2}, \quad y = -\frac{39}{44}x + \frac{195}{44} - \frac{22}{44}$$

$$y = -\frac{39}{44}x + \frac{173}{44}$$

Finally, the equation of the line is

$$y = -\frac{39}{44}x + \frac{173}{44}$$

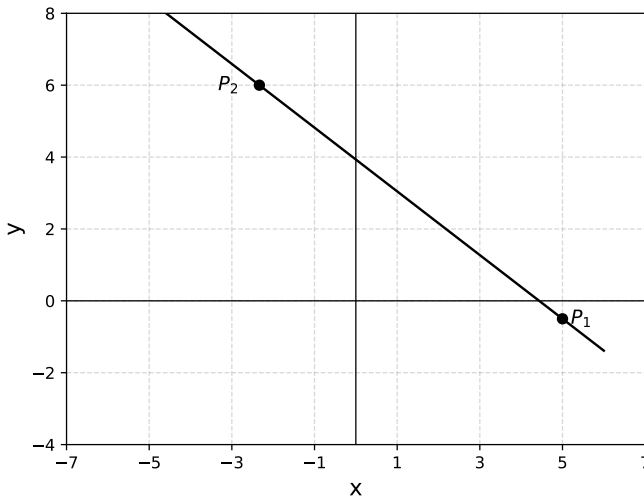and its graph is shown in Figure 4.1.1.

Figure 4.1.1: Graph of the line with equation $y = -\frac{39}{44}x + \frac{173}{44}$.

4.2 Exercise 2

Text

Determine the vertices of the rectangular hyperbola with equation

$$xy = 5$$

and calculate their distance.

Solution

The rectangular hyperbola with equation

$$xy = 5$$

can be rewritten as

$$y = \frac{5}{x}$$

This hyperbola has the coordinate axes $x = 0$ and $y = 0$ as its asymptotes. Given the general form of a rectangular hyperbola,

$$y = \frac{c}{x}$$

with c constant, the vertices are given by the formulas

$$V_1 = \left(\sqrt{c}, \sqrt{c}\right), \quad V_2 = \left(-\sqrt{c}, -\sqrt{c}\right)$$

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In our case, since $c = 5$, we obtain the vertices

$$V_1 = \left(\sqrt{5}, \sqrt{5}\right), \quad V_2 = \left(-\sqrt{5}, -\sqrt{5}\right)$$
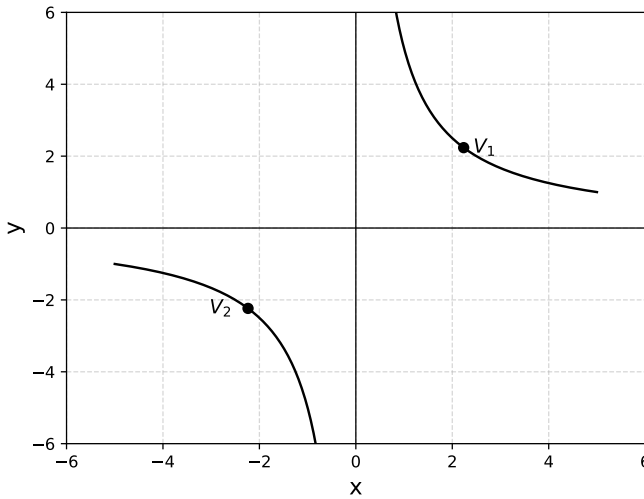
as shown in Figure 4.2.2.

Figure 4.2.2: Graph of the hyperbola with equation $y = \frac{5}{x}$.

We calculate the distance between the vertices using the formula for the distance between two points:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Substituting,

$$d = \sqrt{\left(-\sqrt{5}-\sqrt{5}\right)^2 + \left(-\sqrt{5}-\sqrt{5}\right)^2}$$
$$= \sqrt{(-2\sqrt{5})^2 + (-2\sqrt{5})^2} = \sqrt{4\cdot5+4\cdot5}$$
$$= \sqrt{20+20} = \sqrt{40} = 2\sqrt{10}$$

So the distance between the vertices is

$$d = 2\sqrt{10}$$

4.3   Exercise 3

Text

Find the equation of the parabola passing through the point

$$A = (3,4)$$

and having its vertex at the point

$$V = \left(\frac{5}{3}, -\frac{4}{3}\right)$$

Solution

The general equation of a parabola with vertex (v_x, v_y) and axis parallel to the y-axis is

$$y = a(x - v_x)^2 + v_y$$

In fact, this is a translation by vector (v_x, v_y) of the standard parabola $y = x^2$, which has vertex at the origin. In this case we have

$$y = a\left(x - \frac{5}{3}\right)^2 - \frac{4}{3}$$

We substitute the point $A = (3,4)$ to determine the coefficient a:

$$4 = a\left(3 - \frac{5}{3}\right)^2 - \frac{4}{3}, \quad 4 + \frac{4}{3} = a\left(\frac{9}{3} - \frac{5}{3}\right)^2$$

$$\frac{12}{3} + \frac{4}{3} = a\left(\frac{4}{3}\right)^2, \quad \frac{16}{3} = a \cdot \frac{16}{9}$$

from which

$$a = 3$$

So the equation of the parabola is

$$y = 3\left(x - \frac{5}{3}\right)^2 - \frac{4}{3}$$

or, expanding,

$$y = 3\left(x^2 - 2 \cdot \frac{5}{3}x + \left(\frac{5}{3}\right)^2\right) - \frac{4}{3} = 3x^2 - 2 \cdot 5x + 3 \cdot \frac{25}{9} - \frac{4}{3}$$

$$= 3x^2 - 10x + \frac{25}{3} - \frac{4}{3} = 3x^2 - 10x + 7$$

So the cartesian equation of the parabola is

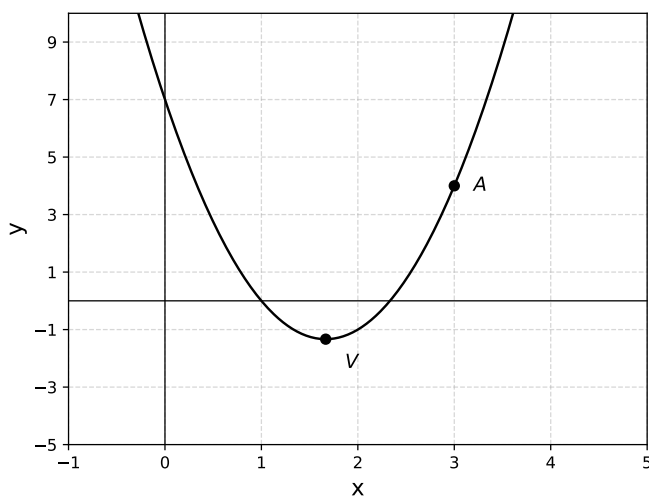$$y = 3x^2 - 10x + 7$$

and its graph is shown in Figure 4.3.3

Figure 4.3.3: Graph of the parabola with equation $y = 3x^2 - 10x + 7$.

4.4 Exercise 4

Text

Calculate the slope of the line r knowing that its graph passes through the points

$$P_1 = (-3, -8)$$

and

$$P_2 = (15, 21)$$

Solution

To calculate the slope m of the line r passing through the points

$$P_1 = (-3, -8), \quad P_2 = (15, 21)$$

we use the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Substituting the values,

$$m = \frac{21 - (-8)}{15 - (-3)} = \frac{21 + 8}{15 + 3} = \frac{29}{18}$$

So the slope of the line is

$$m = \frac{29}{18}$$

4.5 Exercise 5

Text

Let r be the line with equation

$$y = 3x - 1$$

Determine the length of the segment of r between its points with abscissas

$$x_1 = 2, \quad x_2 = 5$$

Solution

We compute the corresponding points on the line:

$$P_1 = (2, 3 \cdot 2 - 1) = (2, 5)$$
$$P_2 = (5, 3 \cdot 5 - 1) = (5, 14)$$
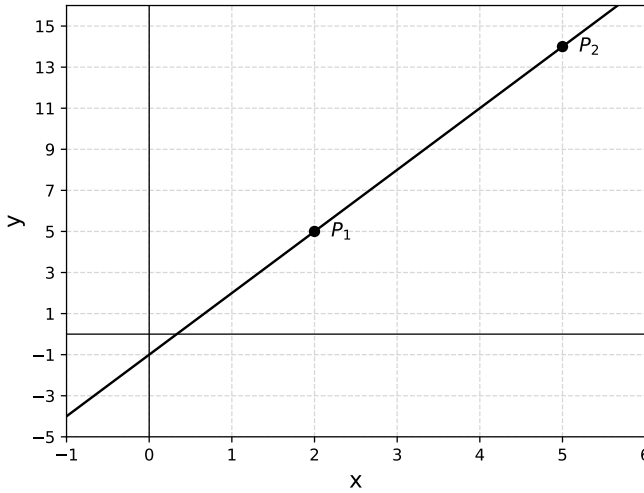
as shown in Figure 4.5.4.

Figure 4.5.4: Graph of the line with equation $y = 3x - 1$.

The length of the segment is the distance between the two points P_1 and P_2, given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We compute:

$$d = \sqrt{(5-2)^2 + (14-5)^2} = \sqrt{3^2 + 9^2} = \sqrt{9 + 81}$$
$$= \sqrt{90} = 3\sqrt{10}$$

So the length of the segment is

$$d = 3\sqrt{10}$$

4.6   Exercise 6

Text

Find the equation of an ellipse, knowing that it has its center at the point

$$C = (1,4)$$

passes through the point

$$A = (2,2)$$

and the area it encloses is

$$\text{Area} = 15\pi$$

Solution

The general equation of an ellipse with center (c_x, c_y) is

$$\frac{(x-c_x)^2}{a^2} + \frac{(y-c_y)^2}{b^2} = 1$$

In this case,

$$c_x = 1, \quad c_y = 4$$

we can therefore write

$$\frac{(x-1)^2}{a^2} + \frac{(y-4)^2}{b^2} = 1$$

The area of the ellipse is

$$\pi ab = 15\pi$$

so

$$ab = 15, \quad b = \frac{15}{a}$$

and we obtain

$$\frac{(x-1)^2}{a^2} + \frac{a^2(y-4)^2}{225} = 1$$

We define the variable

$$A = a^2$$

The previous equation becomes

$$\frac{(x-1)^2}{A} + \frac{A(y-4)^2}{225} = 1$$

with $A > 0$ (since $a \neq 0$ and $b \neq 0$ from the initial ellipse equation). The point $A = (2,2)$ belongs to the ellipse, so we impose the condition

$$\frac{(2-1)^2}{A} + \frac{A(2-4)^2}{225} = 1$$

from which

$$\frac{1}{A} + \frac{4A}{225} = 1$$

Multiplying both sides by $225A$, we get

$$225 + 4A^2 = 225A, \quad 4A^2 - 225A + 225 = 0$$

The discriminant is

$$\Delta = (-225)^2 - 4 \cdot 4 \cdot 225 = 47025$$

yielding the solutions

$$A = \frac{225 \pm \sqrt{47025}}{8} = \frac{225 \pm 15\sqrt{209}}{8}$$
$$= \frac{15(15 \pm \sqrt{209})}{8}$$

both of which are positive. The ellipses satisfying the given conditions are two, with equations

$$\frac{8(x-1)^2}{15(15+\sqrt{209})} + \frac{(15+\sqrt{209})(y-4)^2}{120} = 1$$

as shown in Figure 4.6.5 (solid curve), and

$$\frac{8(x-1)^2}{15(15-\sqrt{209})} + \frac{(15-\sqrt{209})(y-4)^2}{120} = 1$$
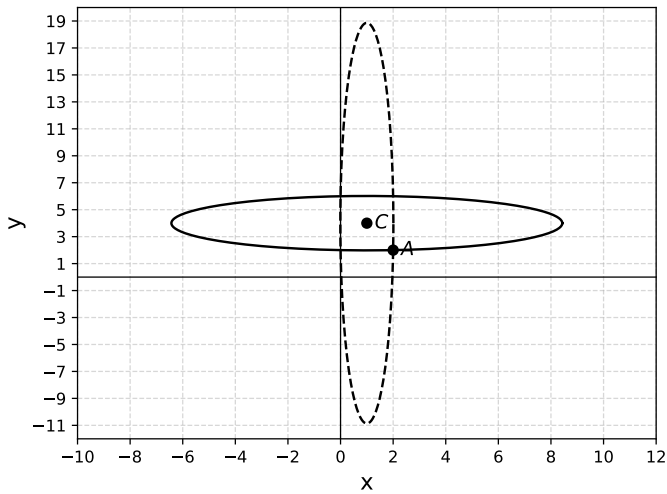
also shown in Figure 4.6.5 (dashed curve).

Figure 4.6.5: Graph of the ellipses from Exercise 6.

4.7   Exercise 7

Text

Find the cartesian equation, in implicit form, of the line perpendicular to the line

$$y = -\frac{2}{5}x + 8$$

and passing through the point

$$A = (1,1)$$

Solution

The slope of the given line is

$$m = -\frac{2}{5}$$

The perpendicular line has slope $m_\perp$ such that

$$m \cdot m_\perp = -1,$$

thus

$$m_\perp = -\frac{1}{m} = -\frac{1}{-\frac{2}{5}} = \frac{5}{2}$$

The equation of the perpendicular line passing through the point $A = (1,1)$ is

$$y - y_0 = m_\perp (x - x_0)$$

that is, in our case,

$$y - 1 = \frac{5}{2}(x - 1)$$

or equivalently,

$$y - 1 = \frac{5}{2}x - \frac{5}{2}, \quad y = \frac{5}{2}x - \frac{5}{2} + 1 = \frac{5}{2}x - \frac{3}{2}$$

To obtain the implicit form, we bring everything to the left-hand side:

$$y - \frac{5}{2}x + \frac{3}{2} = 0$$

Multiplying by 2 to eliminate denominators, we get the cartesian implicit equation of the desired line:

$$2y - 5x + 3 = 0$$
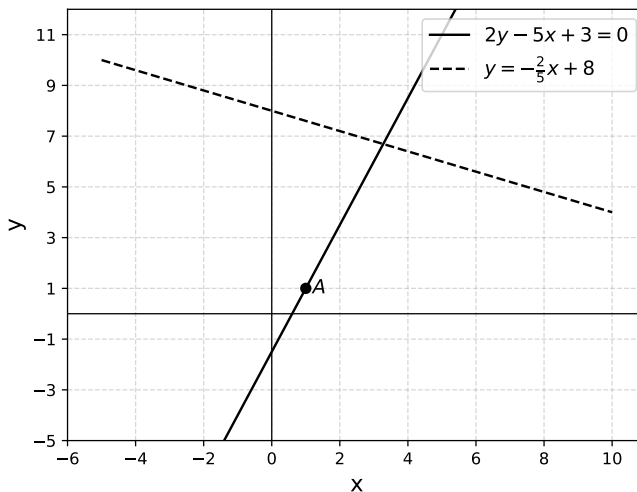
The two lines are shown in Figure 4.7.6.

Figure 4.7.6: Graph of the lines from Exercise 7.

4.8 Exercise 8

Text

Consider in a cartesian plane the triangle with vertices at the three points

$$P_1 = (-3,0), \quad P_2 = (5,0), \quad P_3 = \left(\frac{17}{5}, \frac{16}{5}\right)$$

Prove that the triangle is right-angled using a method involving lines in the plane.

Solution

We observe that P_1 and P_2 lie on the x-axis, so the line P_1P_2 has equation

$$y = 0$$

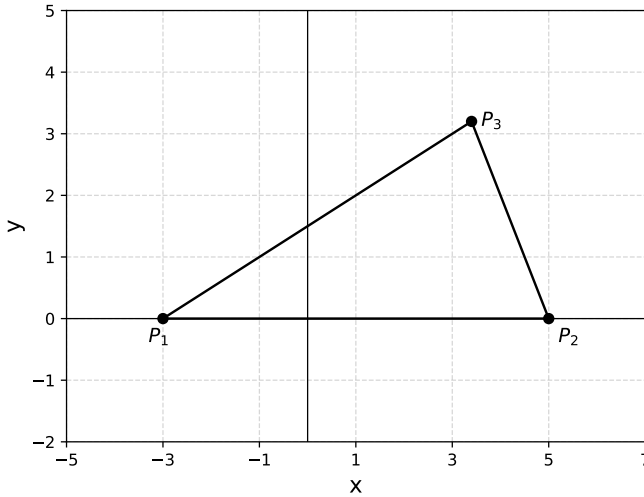The three points and the triangle are shown in Figure 4.8.7.

Figure 4.8.7: Graph of the triangle from Exercise 8.

We compute the slopes of the lines P_2P_3 and P_1P_3. For the first:

$$m_{23} = \frac{\frac{16}{5} - 0}{\frac{17}{5} - 5} = \frac{\frac{16}{5}}{\frac{17}{5} - \frac{25}{5}} = \frac{\frac{16}{5}}{-\frac{8}{5}} = -2$$

And for the line P_1P_3:

$$m_{13} = \frac{\frac{16}{5} - 0}{\frac{17}{5} - (-3)} = \frac{\frac{16}{5}}{\frac{17}{5} + \frac{15}{5}} = \frac{\frac{16}{5}}{\frac{32}{5}} = \frac{16}{5} \cdot \frac{5}{32} = \frac{16}{32} = \frac{1}{2}$$

To check if the triangle is right-angled, we verify whether two sides are perpendicular, i.e., whether the product of

their slopes is -1:

$$m_{23} \cdot m_{13} = (-2) \cdot \frac{1}{2} = -1$$

Since the product is -1, the triangle is right-angled at P_3.

4.9 Exercise 9

Text

Determine the real parameter k so that the equation

$$ky = (-k+6)x - 2k^2 + 3$$

satisfies the following conditions:

1. it is the equation of a line parallel to the x-axis;
2. it is the equation of a line parallel to the y-axis.

Solution

To be parallel to the x-axis, the line must have slope zero, hence

$$-k+6 = 0$$

from which

$$k = 6$$

With this value of k, the equation becomes

$$6 \cdot y = -2 \cdot 6^2 + 3 = -2 \cdot 36 + 3 = -72 + 3 = -69$$

that is, the line is

$$y = -\frac{69}{6}, \quad y = -\frac{23}{2}$$

which is parallel to the x-axis. Similarly, to be parallel to the y-axis, the line must have the form

$$x = k$$

with k constant. In practice, we need to eliminate the term in y in the equation. To do so, we set its coefficient to zero:

$$k = 0$$

which gives the line

$$0 = 6x + 3, \quad 6x = -3, \quad x = -\frac{1}{2}$$
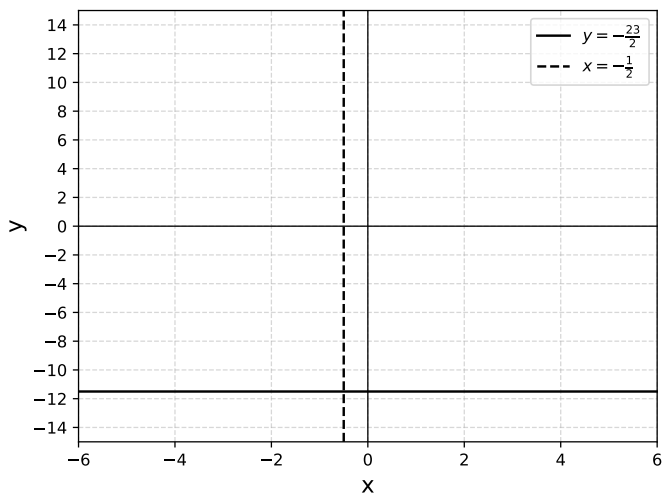
The two lines are shown in Figure 4.9.8

Figure 4.9.8: Graph of the lines from Exercise 9.

4.10   Exercise 10

Text

Find the equation of the tangent line to the parabola with equation

$$y = x^2 - 2x - 1$$

at its point with abscissa $x = 3$.

Solution

We compute the derivative to find the slope of the tangent line:

$$y' = 2x - 2$$

Evaluating it at the point with abscissa $x = 3$:

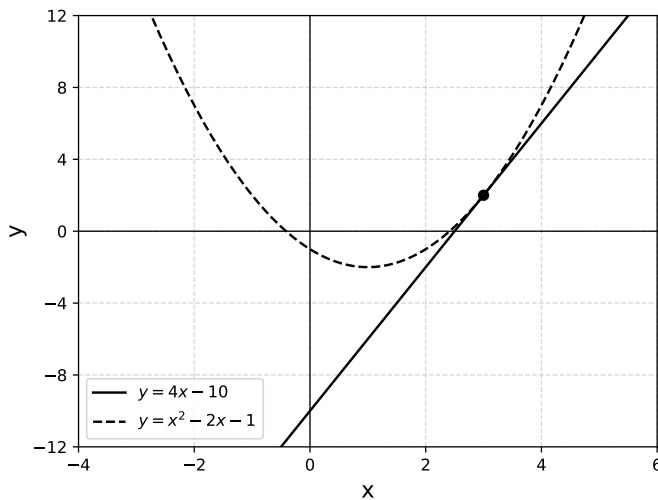$$m = y'(3) = 2 \cdot 3 - 2 = 6 - 2 = 4$$

Figure 4.10.9: Graph of the line and parabola from Exercise 10.

We compute the ordinate of the point of tangency:

$$y(3) = 3^2 - 2 \cdot 3 - 1 = 9 - 6 - 1 = 2$$

The equation of the tangent line at the point $(3,2)$ with slope $m = 4$ is:

$$y - 2 = 4(x - 3),$$

from which:

$$y - 2 = 4x - 12$$

Finally, the equation of the tangent line to the parabola at the given point is:

$$y = 4x - 10$$

and is shown, together with the parabola, in Figure 4.10.9.

4.11   Exercise 11

Text

Find the point of intersection between the lines with equations

$$y = 3x - 2$$

and

$$x + 3y = -1$$

Solution

To find the point of intersection, we solve the system:

$$\begin{cases} y = 3x - 2 \\ x + 3y = -1 \end{cases}$$

Substituting the first equation into the second:

$$x + 3(3x - 2) = -1$$

which gives:

$$x + 9x - 6 = -1, \quad 10x = 5$$

so:

$$x = \frac{1}{2}$$

Substitute this value into the first equation:

$$y = 3 \cdot \frac{1}{2} - 2 = \frac{3}{2} - 2 = -\frac{1}{2}$$

Therefore, the point of intersection is:

$$\left(\frac{1}{2}, -\frac{1}{2} \right)$$
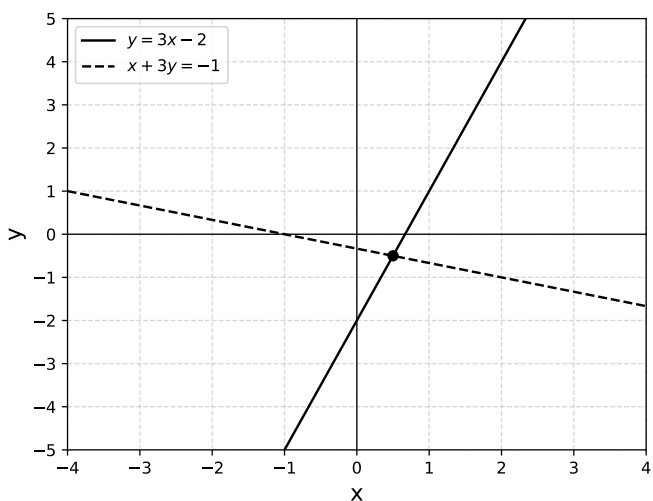
as shown in Figure 4.11.10.

Figure 4.11.10: Graph of the lines with equations $y = 3x - 2$ and $x + 3y = -1$.

4.12 Exercise 12

Text

Consider the family of parabolas with equation

$$y = x^2 + bx + c$$

Determine the equation of the parabola that passes through the point $P(1,2)$ and is tangent to the line

$$y = 3x - 1$$

Solution

We impose that the curve passes through point $P(1,2)$:

$$2 = (1)^2 + b(1) + c$$

which gives:

$$c = 1 - b$$

The parabola must be tangent to the line $y = 3x - 1$, so the system:

$$\begin{cases} y = x^2 + bx + c \\ y = 3x - 1 \end{cases}$$

must have a single solution. Setting them equal:

$$x^2 + bx + c = 3x - 1, \quad x^2 + (b-3)x + (c+1) = 0$$

This equation has a unique solution if the discriminant is zero:

$$(b-3)^2 - 4(c+1) = 0$$

Substitute $c = 1 - b$:

$$(b-3)^2 - 4(1-b+1) = 0, \quad (b-3)^2 - 4(2-b) = 0$$

Which leads to:

$$b^2 + 9 - 6b - 8 + 4b = 0, \quad b^2 - 2b + 1 = 0$$

which can be written as:

$$(b-1)^2 = 0$$

with double root $b = 1$. To find c, we substitute:

$$c = 1 - b = 1 - 1 = 0$$

The resulting parabola is:

$$y = x^2 + x$$
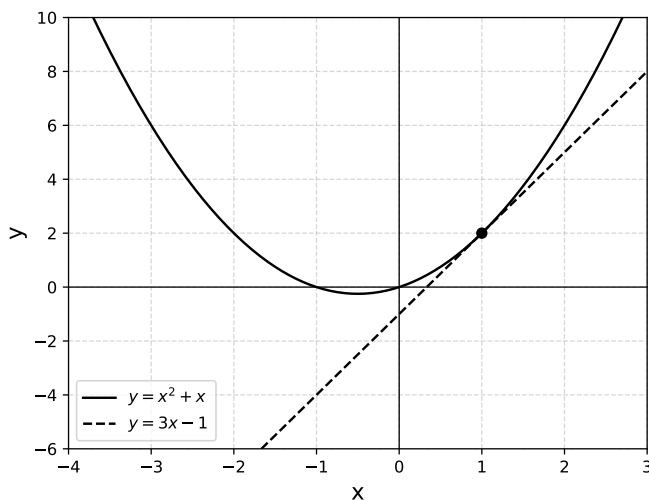
and is shown in Figure 4.12.11.

Figure 4.12.11: Graph of the parabola with equation $y = x^2 + x$.

4.13 Exercise 13

Text

Determine for which values of the real parameter k the equation

$$\frac{x^2}{7k+12} + \frac{y^2}{9k+3} = 1$$

represents a

- circle,
- ellipse,
- hyperbola.

Solution

The equation is that of an ellipse if both denominators are positive, a hyperbola if one is positive and the other negative, and a circle if the denominators are equal and positive. To obtain a circle, we set:

$$7k + 12 = 9k + 3, \quad -2k = -9 \quad k = \frac{9}{2}$$

Moreover, the denominators must be positive:

$$7k + 12 = 7 \cdot \frac{9}{2} + 12 = \frac{63}{2} + \frac{24}{2} = \frac{87}{2} > 0$$

So, for $k = \frac{9}{2}$ we have the circle with equation:

$$x^2 + y^2 = \frac{87}{2}$$

For the ellipse, both denominators must be positive and different:

$$\begin{cases} 7k + 12 > 0 \\ 9k + 3 > 0 \end{cases}$$

that is:

$$\begin{cases} k > -\frac{12}{7} \\ k > -\frac{1}{3} \end{cases}$$

Therefore, for $k > -\frac{1}{3}$ and $k \neq \frac{9}{2}$, we have an ellipse. Finally, for a hyperbola the denominators must have opposite signs. We write:

$$(7k+12)(9k+3) < 0$$

Find the zeros:

$$7k+12 = 0, \quad k = -\frac{12}{7}$$

and

$$9k+3 = 0, \quad k = -\frac{1}{3}$$

The inequality is satisfied for:

$$-\frac{12}{7} < k < -\frac{1}{3}$$

In summary:

- A circle occurs for

$$k = \frac{9}{2}$$

- An ellipse occurs for

$$k > -\frac{1}{3}, \quad k \neq \frac{9}{2}$$

- A hyperbola occurs for

$$-\frac{12}{7} < k < -\frac{1}{3}$$

4.14   Exercise 14

Text

Determine the equation of the parabola with axis parallel to the *y*-axis that passes through the points

$$(1,2), \quad (3,10), \quad (5,26)$$

Solution

The parabola has a vertical axis, so it is of the form

$$y = ax^2 + bx + c$$

we therefore impose the passage through the three points. For the first point

$$2 = a(1)^2 + b(1) + c = a + b + c$$

for the second point

$$10 = a(3)^2 + b(3) + c = 9a + 3b + c$$

finally, for the third point

$$26 = a(5)^2 + b(5) + c = 25a + 5b + c$$

We write the system

$$\begin{cases} a + b + c = 2 \\ 9a + 3b + c = 10 \\ 25a + 5b + c = 26 \end{cases}$$

we subtract the first equation from the second

$$(9a + 3b + c) - (a + b + c) = 10 - 2$$

$$8a + 2b = 8, \quad 4a + b = 4$$

we subtract the first equation from the third

$$(25a + 5b + c) - (a + b + c) = 26 - 2$$

$$24a + 4b = 24, \quad 6a + b = 6$$

now we solve

$$\begin{cases} 4a + b = 4 \\ 6a + b = 6 \end{cases}$$

We subtract the first equation from the second

$$(6a+b)-(4a+b)=6-4, \quad 2a=2, \quad a=1$$

we substitute this result into the first equation

$$4 \cdot 1+b=4, \quad b=0$$

finally, recalling the first equation of the first system

$$a+b+c=2, \quad 1+0+c=2, \quad c=1$$

therefore the parabola is
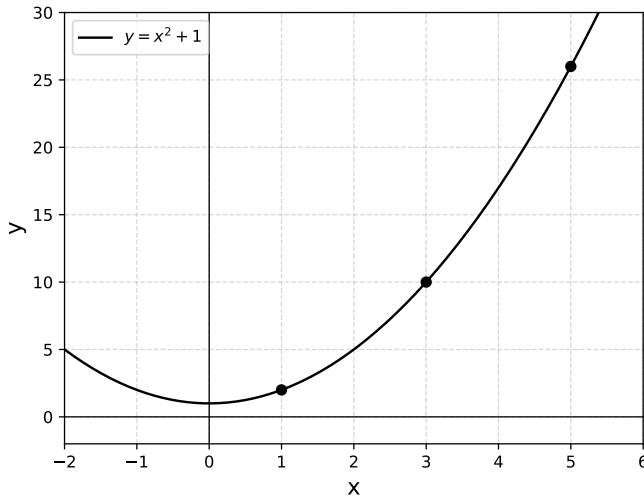
$$y=x^2+1$$

as shown in figure 4.14.12.

Figure 4.14.12: Graph of the parabola with equation $y = x^2 + 1$.

4.15 Exercise 15

Text

Calculate the area of the triangle determined by the intersections of the lines

$$y = 2x + 1, \quad y = -x + 4, \quad x = 3$$

Solution

First, we determine the intersection point between the lines $y = 2x + 1$ and $y = -x + 4$ by solving the system

$$\begin{cases} y = 2x + 1 \\ y = -x + 4 \end{cases}$$

from which

$$2x + 1 = -x + 4, \quad 3x = 3, \quad x = 1$$

substituting into the first equation of the system

$$y = 2 \cdot 1 + 1 = 3$$

and the intersection point is

$$A = (1, 3)$$

For the intersection between $y = 2x + 1$ and $x = 3$ we have

$$\begin{cases} y = 2x + 1 \\ x = 3 \end{cases}$$

from which

$$y = 2 \cdot 3 + 1 = 7$$

and the second point is

$$B = (3,7)$$

Finally, the intersection between $y = -x + 4$ and $x = 3$ gives

$$\begin{cases} y = -x + 4 \\ x = 3 \end{cases}$$

that is

$$y = -3 + 4 = 1$$

therefore

$$C = (3,1)$$

The three vertices of the triangle are $A = (1,3)$, $B = (3,7)$, $C = (3,1)$, as shown in figure 4.15.13. We apply the area formula using the determinant:

$$\text{Area} = \frac{1}{2} \left| \det \begin{pmatrix} 1 & 3 & 1 \\ 3 & 7 & 1 \\ 3 & 1 & 1 \end{pmatrix} \right| = \frac{1}{2} \left| 1(7 \cdot 1 - 1 \cdot 1) - 3(3 \cdot 1 - 1 \cdot 3) \right.$$

$$\left. + 1(3 \cdot 1 - 7 \cdot 3) \right| = \frac{1}{2} \left| 1(6) - 3(0) + 1(-18) \right|$$

$$= \frac{1}{2} \left| 6 - 0 - 18 \right| = \frac{1}{2} \cdot 12 = 6$$

Therefore, the area of the triangle is
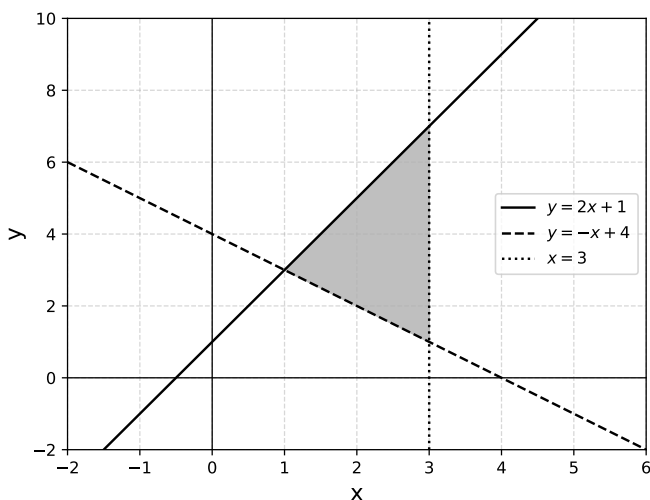
$$\text{Area} = 6$$

Figure 4.15.13: Graph of the triangle from exercise 15.

4.16 Exercise 16

Text

Given the hyperbola with equation

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

calculate the coordinates of the vertices and the asymptotes.

Solution

The equation of the hyperbola

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

is in canonical form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

so it has a horizontal transverse axis, by comparison we get

$$a^2 = 16, \quad a = 4$$

and

$$b^2 = 9, \quad b = 3$$

the vertices are found along the x-axis at distance a from the origin

$$V_1 = (-4,0), \quad V_2 = (4,0)$$

while the asymptotes are given by the two lines

$$y = \pm \frac{b}{a} x = \pm \frac{3}{4} x$$
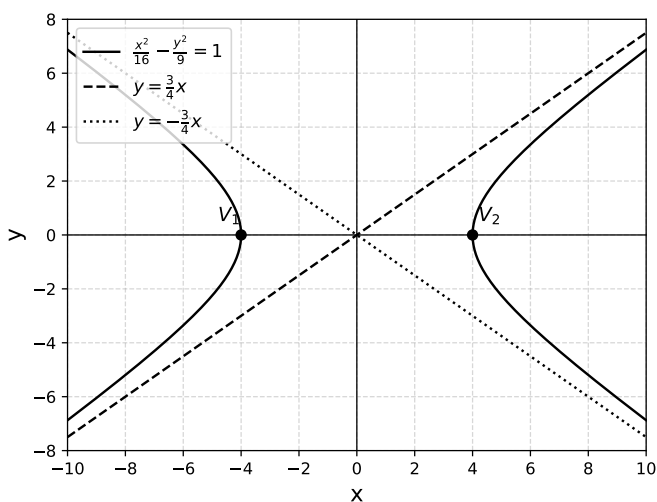
The graph is in figure 4.16.14.

Figure 4.16.14: Graph of the hyperbola with equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and its asymptotes.

4.17 Exercise 17

Text

Find the point of tangency of the line

$$y = mx + 1$$

to the parabola

$$y = x^2 - 4x + 2$$

knowing that the line is tangent to the parabola.

Solution

We write the system for the two curves

$$\begin{cases} y = mx + 1 \\ y = x^2 - 4x + 2 \end{cases}$$

from which

$$mx + 1 = x^2 - 4x + 2, \quad x^2 - (4 + m)x + 2 - 1 = 0,$$
$$x^2 - (4 + m)x + 1 = 0$$

For the line to be tangent to the parabola, the equation must have a single real solution, so the discriminant must

be zero

$$\Delta = (4+m)^2 - 4 \cdot 1 \cdot 1 = 0, \quad (4+m)^2 = 4$$

$$4 + m = \pm 2$$

from which the values of m are

$$m = -2, \quad m = -6$$

Now we find the tangent point for both values of m. For $m = -2$ the line equation is:

$$y = -2x + 1$$

intersecting with the parabola

$$x^2 - 4x + 2 = -2x + 1, \quad x^2 - 2x + 1 = 0, \quad (x-1)^2 = 0$$

that is $x_1 = 1$, with the corresponding ordinate

$$y_1 = -2x_1 + 1 = -2 + 1 = -1$$

the point is

$$P_1 = (1, -1)$$

Similarly, for $m = -6$, we write the line equation

$$y = -6x + 1$$

from which, intersecting,

$$x^2 - 4x + 2 = -6x + 1, \quad x^2 + 2x + 1 = 0, \quad (x+1)^2 = 0$$

that is $x_2 = -1$, with ordinate

$$y_2 = -6x_2 + 1 = 6 + 1 = 7$$

and the second point is

$$P_2 = (-1, 7)$$

Summarizing, the tangent points are

- For $m = -2 \Rightarrow P_1 = (1, -1)$
- For $m = -6 \Rightarrow P_2 = (-1, 7)$
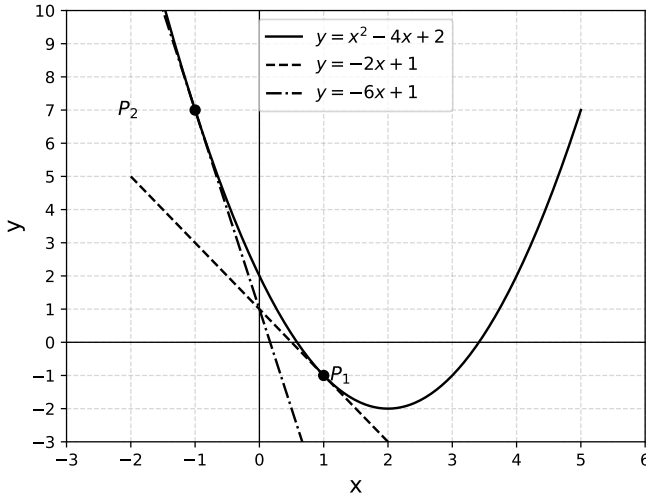
as shown in figure 4.17.15.

Figure 4.17.15: Graph of lines and parabola from exercise 17.

4.18 Exercise 18

Text

Given the four lines with equations

$$y = 2x + 3, \quad y = 2x - 1, \quad x = 1, \quad x = -1$$

determine the area of the parallelogram enclosed by their graphs.

Solution

The lines $y = 2x + 3$ and $y = 2x - 1$ are parallel to each other having the same slope equal to 2, so they form two opposite sides of the parallelogram. The lines $x = 1$ and $x = -1$ are vertical and also parallel to each other. Let us determine the intersection points, substituting $x = -1$ into $y = 2x + 3$ and $y = 2x - 1$ we get

$$y = 2 \cdot (-1) + 3 = 1, \quad y = 2 \cdot (-1) - 1 = -3$$

from which the points

$$A = (-1, 1), \quad B = (-1, -3)$$

Similarly, substituting $x = 1$ into $y = 2x + 3$ and $y = 2x - 1$ we get

$$y = 2 \cdot 1 + 3 = 5, \quad y = 2 \cdot 1 - 1 = 1$$

from which the points

$$C = (1, 5), \quad D = (1, 1)$$

The height h of the parallelogram is the vertical distance between the two lines $y = 2x + 3$ and $y = 2x - 1$, that is by subtracting the ordinates of the two lines

$$h = 2x + 3 - (2x - 1) = 4$$

The base b is obtained from the distance between $x = -1$ and $x = 1$, i.e.

$$b = 1 - (-1) = 2$$

thus the area is

$$\text{Area} = b \cdot h = 2 \cdot 4 = 8$$

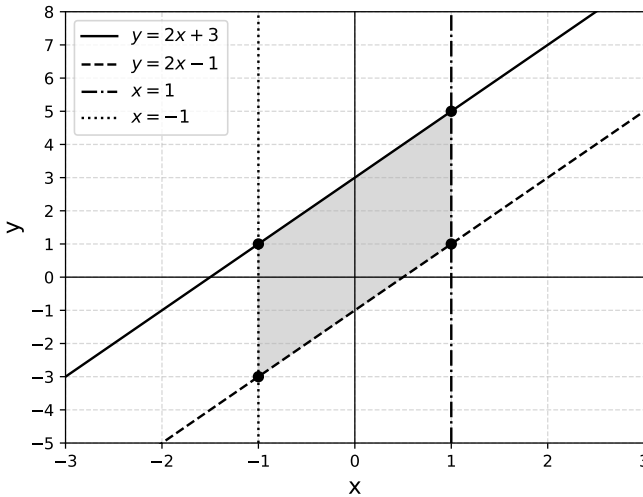The parallelogram and the lines are shown in figure 4.18.16.

Figure 4.18.16: Graph of the parallelogram and lines from exercise 18.

4.19 Exercise 19

Text

Find the equation of the circle with center

$$C = \left(-\frac{1}{2}, 3\right)$$

knowing that the area it encloses is

$$\text{area} = 13\pi$$

Solution

The area enclosed by a circle of radius r is given by πr^2, thus

$$13\pi = \pi r^2, \quad r^2 = 13, \quad r = \sqrt{13}$$

Moreover, knowing that the center is the point $C = \left(-\frac{1}{2}, 3\right)$, the equation of the circle is

$$\left(x + \frac{1}{2}\right)^2 + (y - 3)^2 = 13$$

or, in implicit form,

$$x^2 + x + \frac{1}{4} + y^2 - 6y + 9 = 13, \quad x^2 + y^2 + x - 6y + \frac{37}{4} = 13$$

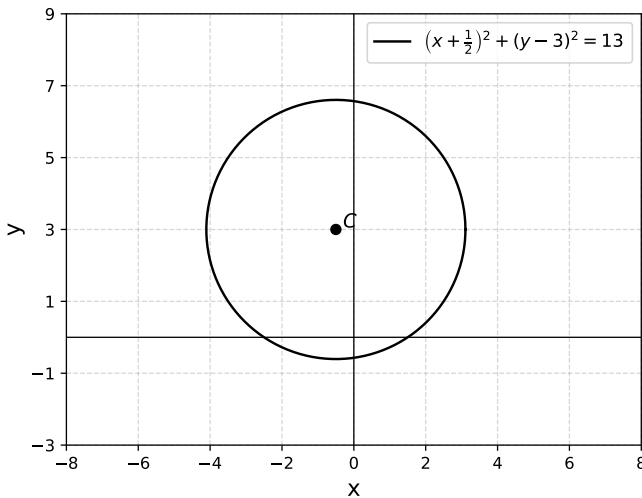$$x^2 + y^2 + x - 6y = \frac{15}{4}$$

Figure 4.19.17 shows the circle.

Figure 4.19.17: Graph of the circle from exercise 19.

4.20   Exercise 20

Text

Given the quadratic function

$$f(x) = ax^2 + bx + c,$$

with $a, b, c \in \mathbb{R}$. Knowing that the vertex of the parabola lies on the line

$$y = 2x + 3,$$

and that the parabola passes through the points $(1, 4)$ and $(3, 0)$, determine the coefficients a, b, and c.

Solution

Let $V = (x_v, y_v)$ be the vertex of the parabola, the coordinate x_v is given by

$$x_v = -\frac{b}{2a}$$

substituting into the line equation $y = 2x + 3$, we get

$$y_v = 2x_v + 3 = 2\left(-\frac{b}{2a}\right) + 3 = -\frac{b}{a} + 3$$

the point $V = \left(-\frac{b}{2a}, -\frac{b}{a} + 3\right)$ must belong to the parabola, together with the points $(1, 4)$ and $(3, 0)$, so we write the system

$$\begin{cases} a + b + c = 4 \\ 9a + 3b + c = 0 \end{cases}$$

subtracting the first equation from the second,

$$(9a + 3b + c) - (a + b + c) = 0 - 4$$

from which

$$8a + 2b = -4, \quad 4a + b = -2$$

Now consider that the vertex belongs to the parabola, thus imposing $y_v = f(x_v)$, from which

$$-\frac{b}{a} + 3 = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c$$
$$-\frac{b}{a} + 3 = \frac{b^2}{4a} - \frac{b^2}{2a} + c$$
$$-\frac{b}{a} + 3 = -\frac{b^2}{4a} + c$$

Now substitute the previously found expression, i.e. $b = -4a - 2$, obtaining

$$-\frac{-4a - 2}{a} + 3 = -\frac{(-4a - 2)^2}{4a} + c$$
$$\frac{4a + 2}{a} + 3 = -\frac{(16a^2 + 16a + 4)}{4a} + c$$
$$4 + \frac{2}{a} + 3 = -4a - 4 - \frac{1}{a} + c$$

simplifying,

$$7 + \frac{2}{a} = -4a - 4 - \frac{1}{a} + c, \quad 7 + \frac{3}{a} + 4a + 4 = c$$

$$c = 11 + 4a + \frac{3}{a}$$

Return to the equation $a + b + c = 4$ and substitute the expressions for b and c

$$a + (-4a - 2) + \left(11 + 4a + \frac{3}{a}\right) = 4, \quad a + 5 + \frac{3}{a} = 0$$

from which the quadratic equation in a is

$$a^2 + 5a + 3 = 0$$

which we solve obtaining

$$a = \frac{-5 \pm \sqrt{25 - 12}}{2} = \frac{-5 \pm \sqrt{13}}{2}$$

Calculate b from the equation $b = -4a - 2$,

$$b = -4a - 2 = -4\left(\frac{-5 \pm \sqrt{13}}{2}\right) - 2 = 2(4 \mp \sqrt{13})$$

and finally c, from the equation $c = 4 - a - b$,

$$c = 4 - a - b = 4 - \frac{-5 \pm \sqrt{13}}{2} - 2(4 \mp \sqrt{13})$$

$$= \frac{3}{2}(-1 \pm \sqrt{13})$$

The two parabolas satisfying the requirements have the following equations

$$y = \left(\frac{-5+\sqrt{13}}{2}\right)x^2 + 2(4-\sqrt{13})x + \frac{3}{2}(-1+\sqrt{13})$$

and

$$y = \left(\frac{-5-\sqrt{13}}{2}\right)x^2 + 2(4+\sqrt{13})x + \frac{3}{2}(-1-\sqrt{13})$$
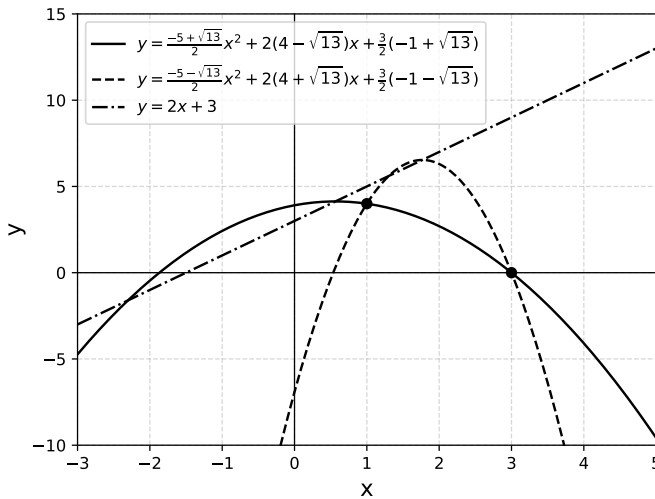
as shown in figure 4.20.18

Figure 4.20.18: Graph of the parabolas from exercise 20.

4.21   Exercise 21

Text

Determine the equation of the circle that passes through the points

$$A = (2,3), \quad B = (4,5)$$

and whose center lies on the line

$$x - y + 1 = 0$$

Solution

Let

$$C = (x_c, y_c)$$

be the center of the circle. Since C lies on the line $x - y + 1 = 0$, we can write:

$$y_c = x_c + 1$$

We use the following generic form for the circle

$$(x - x_c)^2 + (y - y_c)^2 = r^2$$

which in our case becomes, since $C = (x_c, x_c + 1)$,

$$(x - x_c)^2 + (y - x_c - 1)^2 = r^2$$

that is

$$x^2 - 2x_c x + x_c^2 + y^2 + x_c^2 + 1 - 2x_c y + 2x_c - 2y = r^2$$

or better, simplifying,

$$x^2 + y^2 - 2x_c x - 2(x_c + 1)y + 2x_c^2 + 2x_c + 1 - r^2 = 0$$

Impose that A, B belong to the circle by writing the system

$$\begin{cases} 4 + 9 - 4x_c - 6(x_c + 1) + 2x_c^2 + 2x_c + 1 - r^2 = 0 \\ 16 + 25 - 8x_c - 10(x_c + 1) + 2x_c^2 + 2x_c + 1 - r^2 = 0 \end{cases}$$

First write

$$\begin{cases} 7 - 8x_c + 2x_c^2 + 1 - r^2 = 0 \\ 31 - 16x_c + 2x_c^2 + 1 - r^2 = 0 \end{cases}$$

subtract the second equation from the first

$$7 - 8x_c + 2x_c^2 + 1 - r^2 - (31 - 16x_c + 2x_c^2 + 1 - r^2) = 0$$

developing,

$$7 - 8x_c + 1 - 31 + 16x_c - 1 = 0, \quad -24 + 8x_c = 0$$

therefore

$$x_c = 3$$

substitute into the first equation of the system

$$7 - 24 + 18 + 1 - r^2 = 0, \quad r^2 = 2$$

from which the radius (being positive)

$$r = \sqrt{2}$$

finally for the ordinate of the center

$$y_c = x_c + 1 = 3 + 1 = 4$$

The equation of the circle is

$$(x - 3)^2 + (y - 4)^2 = 2$$

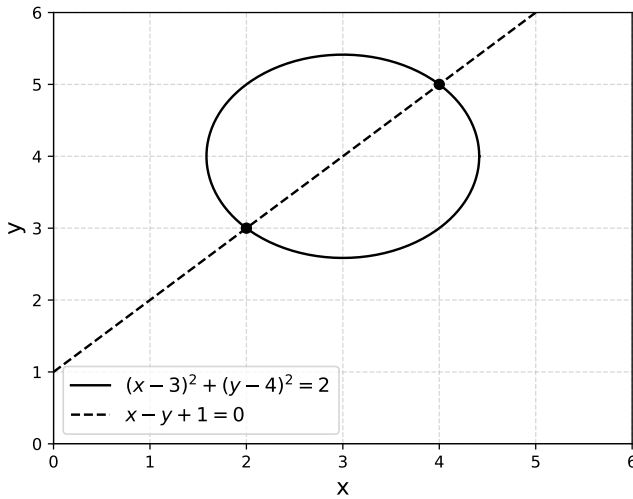and its graph is shown in figure 4.21.19.

Figure 4.21.19: Graph of the circle from exercise 21.

4.22 Exercise 22

Text

Given the parabola with implicit cartesian equation

$$5y + 3x^2 - 3 + 5x = 0$$

find its focus and vertex.

Solution

Let's rewrite the equation in explicit form

$$5y + 3x^2 + 5x - 3 = 0, \quad y = -\frac{3}{5}x^2 - x + \frac{3}{5}$$

this is a parabola with a vertical axis, the equation has the form $y = ax^2 + bx + c$, with

$$a = -\frac{3}{5}, \quad b = -1, \quad c = \frac{3}{5}$$

To find the vertex $V = (x_v, y_v)$, we use the formula

$$x_v = -\frac{b}{2a} = -\frac{-1}{2 \cdot \left(-\frac{3}{5}\right)} = \frac{1}{-\frac{6}{5}} = -\frac{5}{6}$$

substituting into the parabola equation to find y_v:

$$y_v = -\frac{3}{5}\left(-\frac{5}{6}\right)^2 - \left(-\frac{5}{6}\right) + \frac{3}{5} = -\frac{3}{5} \cdot \frac{25}{36} + \frac{5}{6} + \frac{3}{5}$$

$$= -\frac{75}{180} + \frac{150}{180} + \frac{108}{180} = \frac{183}{180} = \frac{61}{60}$$

so the vertex is

$$V = \left(-\frac{5}{6}, \frac{61}{60}\right)$$

For the focus, we can use the formula

$$F = \left(-\frac{b}{2a}, \frac{1 - (b^2 - 4ac)}{4a}\right) = \left(-\frac{b}{2a}, c - \frac{b^2}{4a} + \frac{1}{4a}\right)$$

which is related to the vertex formula by

$$V = (x_v, y_v) = \left(-\frac{b}{2a}, \frac{-(b^2 - 4ac)}{4a} \right), \quad F = \left(x_v, y_v + \frac{1}{4a} \right)$$

from which

$$F = \left(-\frac{5}{6}, \frac{61}{60} + \frac{1}{4 \cdot (-\frac{3}{5})} \right) = \left(-\frac{5}{6}, \frac{61}{60} - \frac{5}{12} \right)$$

$$= \left(-\frac{5}{6}, \frac{36}{60} \right) = \left(-\frac{5}{6}, \frac{3}{5} \right)$$
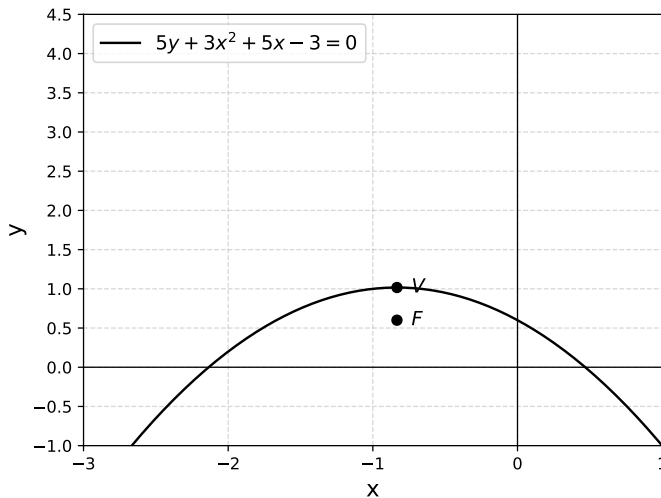
Figure 4.22.20 shows the parabola.

Figure 4.22.20: Graph of the parabola for Exercise 22.

4.23 Exercise 23

Text

Determine the equation of the line passing through the points

$$A = (2,5), \quad B = (7,-3)$$

Solution

Let's calculate the slope:

$$m = \frac{-3-5}{7-2} = \frac{-8}{5}$$

and write the equation of a generic line passing, for example, through the point $A = (2,5)$

$$y - 5 = m(x-2)$$

in this case we have

$$y - 5 = -\frac{8}{5}(x-2), \quad y = -\frac{8}{5}x + \frac{16}{5} + 5$$

thus the equation of the line is

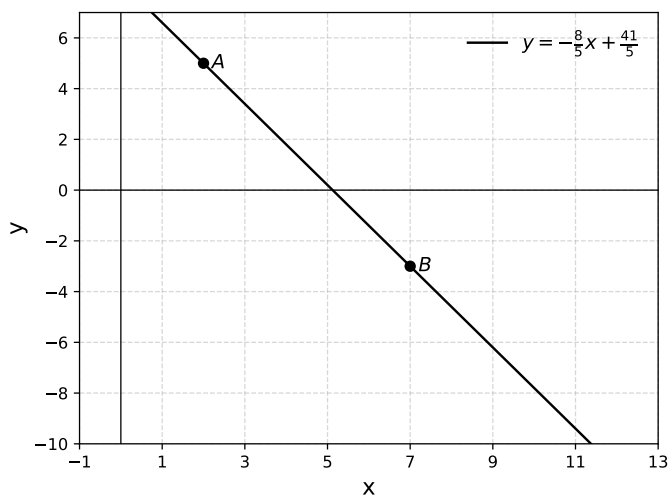$$y = -\frac{8}{5}x + \frac{41}{5}$$

as shown in figure 4.23.21.

Figure 4.23.21: Graph of the line in Exercise 23.

4.24   Exercise 24

Text

Find the midpoint of the segment with endpoints

$$P = (-1, 4), \quad Q = (5, -2)$$

Solution

The midpoint M of a segment with endpoints $A(x_1, y_1)$ and $B(x_2, y_2)$ is calculated by the formula

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Applying this formula to the given points

$$x_M = \frac{-1 + 5}{2} = \frac{4}{2} = 2$$
$$y_M = \frac{4 + (-2)}{2} = \frac{2}{2} = 1$$

therefore the midpoint is

$$M = (2, 1)$$

4.25   Exercise 25

Text

Calculate the distance between the points

$$A = (3, 7), \quad B = (-2, 1)$$

Solution

The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is calculated using the formula (from the Pythagorean

theorem)

$$\overline{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

substituting the values of the points we get

$$\overline{AB} = \sqrt{(-2-3)^2 + (1-7)^2} = \sqrt{(-5)^2 + (-6)^2}$$
$$= \sqrt{25+36} = \sqrt{61}$$

thus the distance is

$$\overline{AB} = \sqrt{61}$$

4.26 Exercise 26

Text

Determine the equation of the line perpendicular to the line

$$3x - 4y + 7 = 0$$

and passing through the point

$$A = (2, -1)$$

Solution

Let's rewrite the given line equation in explicit form

$$3x - 4y + 7 = 0, \quad y = \frac{3}{4}x + \frac{7}{4}$$

the slope is $m = \frac{3}{4}$, so the slope of the perpendicular line will be

$$m_\perp = -\frac{1}{m} = -\frac{4}{3}$$

Let's write the equation of the line passing through $A = (2, -1)$ as

$$y - (-1) = m_\perp(x - 2)$$

from which

$$y + 1 = -\frac{4}{3}(x - 2), \quad y + 1 = -\frac{4}{3}x + \frac{8}{3}$$
$$y = -\frac{4}{3}x + \frac{8}{3} - 1$$

The equation of the sought line is

$$y = -\frac{4}{3}x + \frac{5}{3}$$

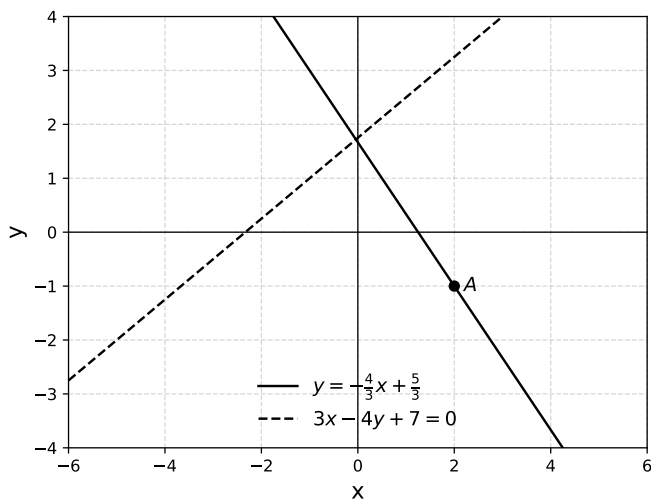and it is shown in figure 4.26.22.

Figure 4.26.22: Graph of the line with equation $y = -\frac{4}{3}x + \frac{5}{3}$.

4.27   Exercise 27

Text

Determine the equation of the circle with center at the point

$$C = (1, -2)$$

and radius $r = 5$.

Solution

The equation of the circle with center $C = (x_0, y_0)$ and radius r is given by

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

substituting the coordinates of the center and the given radius

$$(x - 1)^2 + (y + 2)^2 = 25$$

expanding the equation

$$x^2 - 2x + 1 + y^2 + 4y + 4 = 25$$

from which

$$x^2 + y^2 - 2x + 4y - 20 = 0$$

thus the equation of the circle is

$$x^2 + y^2 - 2x + 4y - 20 = 0$$

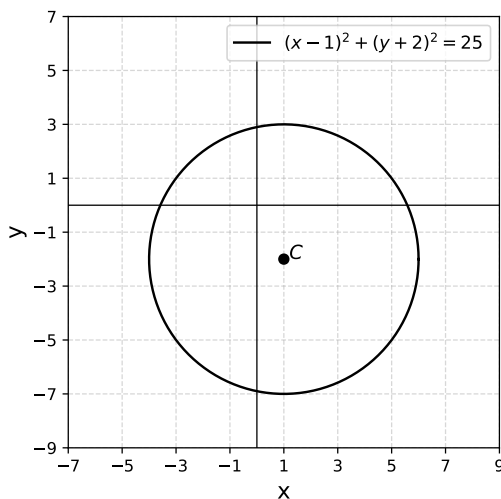and it is shown in figure 4.27.23.

Figure 4.27.23: Graph of the circle with equation $x^2 + y^2 - 2x + 4y - 20 = 0$.

4.28 Exercise 28

Text

Find the coordinates of the intersection point between the lines

$$y = 2x + 1, \quad y = -x + 4$$

Solution

To find the intersection point between the two lines, we need to solve the system

$$\begin{cases} y = 2x + 1 \\ y = -x + 4 \end{cases}$$

equating the two expressions

$$2x + 1 = -x + 4, \quad 3x = 3$$

from which

$$x = 1$$

substituting into the first equation

$$y = 2(1) + 1 = 3$$

thus the intersection point is

$$P = (1, 3)$$

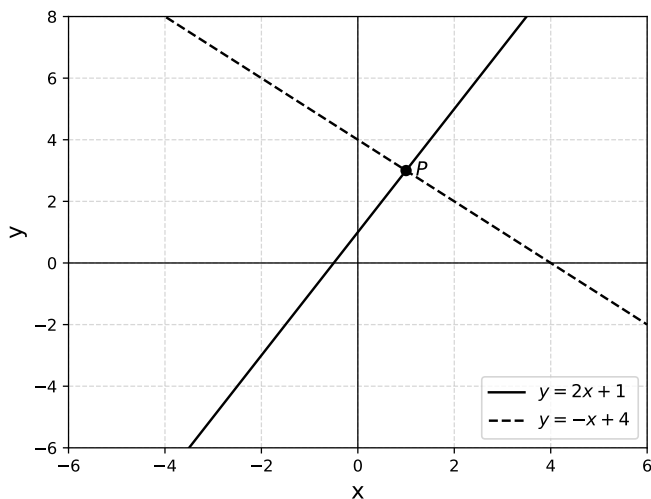The lines are shown in figure 4.28.24.

Figure 4.28.24: Graph of the lines in Exercise 28.

4.29 Exercise 29

Text

Determine the equation of the ellipse with center at the origin, major axis along the x-axis, semi-major axis $a = 4$ and semi-minor axis $b = 3$.

Solution

The canonical equation of an ellipse with center at the origin, major axis along the x-axis, semi-major axis a and semi-minor axis b is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

substituting $a = 4,\ b = 3$

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

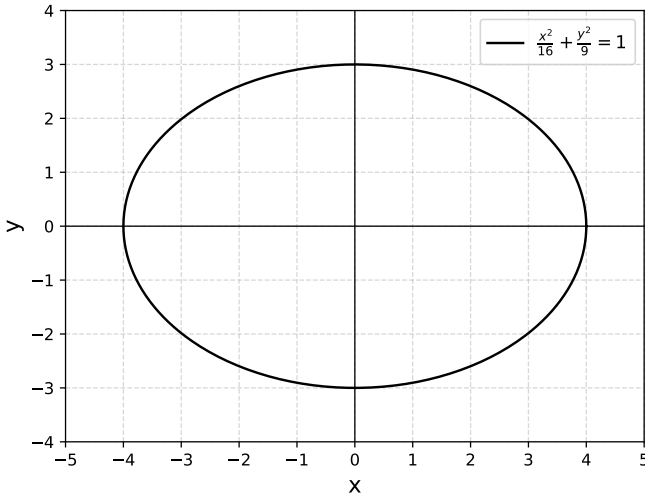which is the equation of the ellipse sought, as shown in figure 4.29.25.

Figure 4.29.25: Graph of the ellipse with equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

4.30   Exercise 30

Text

Calculate the area of the triangle with vertices

$$A = (1,2), \quad B = (4,6), \quad C = (7,2)$$

Solution

The area of a triangle given the vertices $A(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$ is calculated using the formula:

$$\text{Area} = \frac{1}{2}\left|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right|$$

applying it to the given points we get

$$\begin{aligned}\text{Area} &= \frac{1}{2}\left|1(6-2)+4(2-2)+7(2-6)\right| \\ &= \frac{1}{2}\left|1\cdot 4+4\cdot 0+7\cdot(-4)\right| \\ &= \frac{1}{2}\left|4+0-28\right| = \frac{1}{2}\cdot 24 = 12\end{aligned}$$

Thus the area is

$$\text{Area} = 12$$

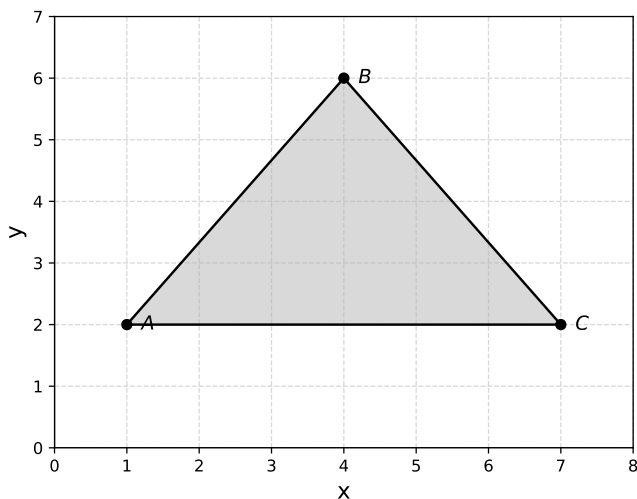The points and the triangle are shown in figure 4.30.26.

Figure 4.30.26: Graph of the triangle in Exercise 30.

4.31   Exercise 31

Text

Determine the equation of the line that passes through the point

$$P = (2,3)$$

and forms equal angles with the lines

$$y = x + 1, \quad y = -x + 5$$

Solution

A line that forms equal angles with two lines is the bisector of the angle formed by the two lines. Let's write the generic line passing through the point $P = (2,3)$

$$y - 3 = m(x - 2)$$

with

$$m = \tan \theta$$

The two lines have slopes

$$m_1 = 1, \quad m_2 = -1$$

and are the translated bisectors of the first and third quadrant and the second and fourth quadrant, respectively, therefore the sought line must be horizontal or vertical. The horizontal line passing through $P = (2,3)$ has equation

$$y = 3$$

while the vertical line passing through $P = (2,3)$ has equation

$$x = 2$$

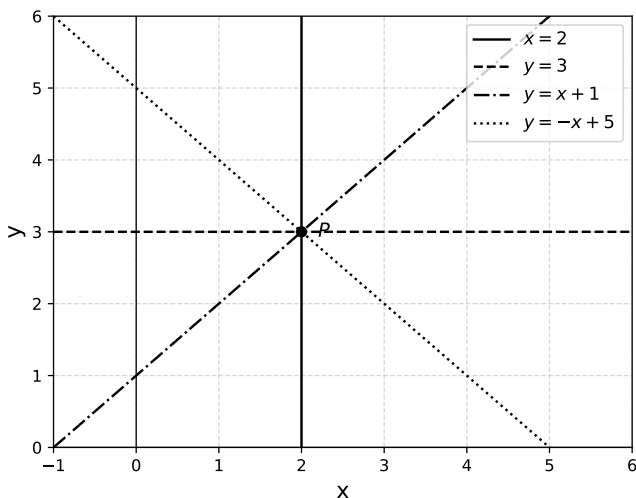The lines that satisfy the requirement are thus these two and are shown in figure 4.31.27.

Figure 4.31.27: Graph of the lines in Exercise 31.

4.32   Exercise 32

Text

Given the parabola with equation

$$y = x^2 - 4x + 6$$

find the equation of the tangent line to the parabola that passes through the external point

$$P = (5,1)$$

Solution

To find the tangent line to the parabola passing through the point $P = (5,1)$, consider a generic line

$$y = mx + q$$

impose the passage through $P = (5,1)$

$$1 = 5m + q, \quad q = 1 - 5m$$

the equation of the line becomes

$$y = mx + 1 - 5m$$

write the following system between the equations of the line and the parabola

$$\begin{cases} y = x^2 - 4x + 6 \\ y = mx + 1 - 5m \end{cases}$$

from which

$$x^2 - 4x + 6 = mx + 1 - 5m$$

bringing everything to the left side

$$x^2 - (4+m)x + 5(m+1) = 0$$

This equation must have a single solution for the tangency condition, so the discriminant must be zero

$$\Delta = (4+m)^2 - 20(m+1) = 0$$

calculate

$$\Delta = 16 + 8m + m^2 - 20m - 20$$
$$= m^2 - 12m - 4 = 0$$

The solutions of the equation $m^2 - 12m - 4 = 0$ are

$$m_{1,2} = \frac{12 \pm \sqrt{160}}{2} = 2(3 \pm \sqrt{10})$$

The two tangent lines to the given parabola passing through $P = (5,1)$ are

$$y = 2(3 + \sqrt{10})x - 29 - 10\sqrt{10}$$

and

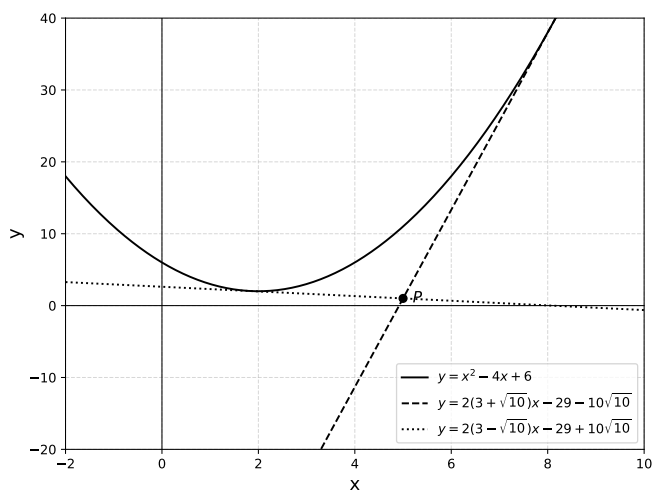$$y = 2(3 - \sqrt{10})x - 29 + 10\sqrt{10}$$

as shown in figure 4.32.28.

Figure 4.32.28: Graph of the lines in Exercise 32.

4.33   Exercise 33

Text

Consider the triangle with vertices

$$A = (1,2), \quad B = (5,7), \quad C = (8,3).$$

Determine the equation of the angle bisector at B.

Solution

To find the bisector line of the angle at B, first find the vectors $\vec{BA}$ and $\vec{BC}$:

$$\vec{BA} = (1-5, 2-7) = (-4, -5)$$
$$\vec{BC} = (8-5, 3-7) = (3, -4)$$

calculate the unit vectors (vectors with magnitude 1), first determine the magnitudes

$$\|\vec{BA}\| = \sqrt{(-4)^2 + (-5)^2} = \sqrt{41}$$
$$\|\vec{BC}\| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$$

from which the unit vectors are,

$$\vec{u}_1 = \frac{1}{\sqrt{41}}(-4, -5)$$
$$\vec{u}_2 = \frac{1}{5}(3, -4)$$

the bisector is in the direction of the sum of the unit vectors

$$\vec{u} = \vec{u}_1 + \vec{u}_2 = \left(-\frac{4}{\sqrt{41}} + \frac{3}{5}, -\frac{5}{\sqrt{41}} - \frac{4}{5}\right)$$

thus the slope of the bisector is given by the ratio

$$m = \frac{-\frac{5}{\sqrt{41}} - \frac{4}{5}}{-\frac{4}{\sqrt{41}} + \frac{3}{5}} = 32 + 5\sqrt{41}$$

The line passing through $B = (5,7)$ with slope m is given by

$$y - 7 = m(x - 5)$$

from which

$$y = (32 + 5\sqrt{41})x - 5(32 + 5\sqrt{41}) + 7$$

that is, finally,

$$y = (32 + 5\sqrt{41})x - 153 - 25\sqrt{41}$$
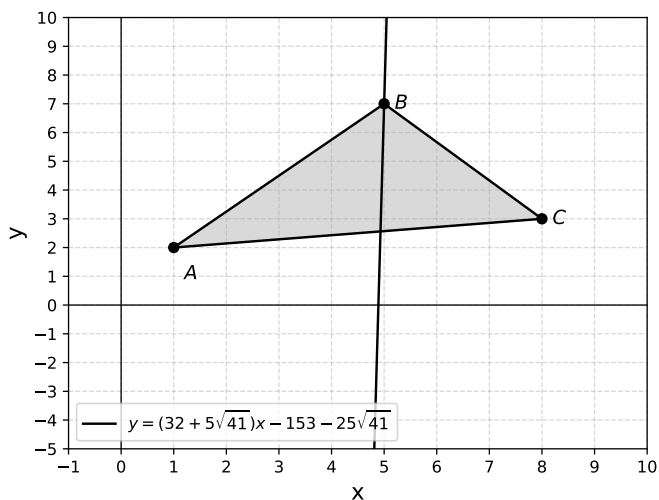
as shown in figure 4.33.29.

Figure 4.33.29: Graph of the line and triangle in Exercise 33.

4.34   Exercise 34

Text

Find the equation of the rectangular hyperbola with center at the origin and asymptotes

$$y = 2x, \quad y = -2x$$

that passes through the point

$$P = (3, 5)$$

Solution

The general equation of a rectangular hyperbola with asymptotes $y = mx$ and $y = -mx$ is

$$x^2 - \frac{1}{m^2}y^2 = a^2$$

since the asymptotes are $y = \pm 2x$, we have

$$m = 2, \quad \frac{1}{m^2} = \frac{1}{4}$$

thus the equation is

$$x^2 - \frac{1}{4}y^2 = a^2$$

Substitute the point $P = (3, 5)$

$$9 - \frac{1}{4} \cdot 25 = a^2, \quad 9 - \frac{25}{4} = a^2, \quad a^2 = \frac{11}{4}$$

the equation of the hyperbola is

$$x^2 - \frac{1}{4}y^2 = \frac{11}{4}$$
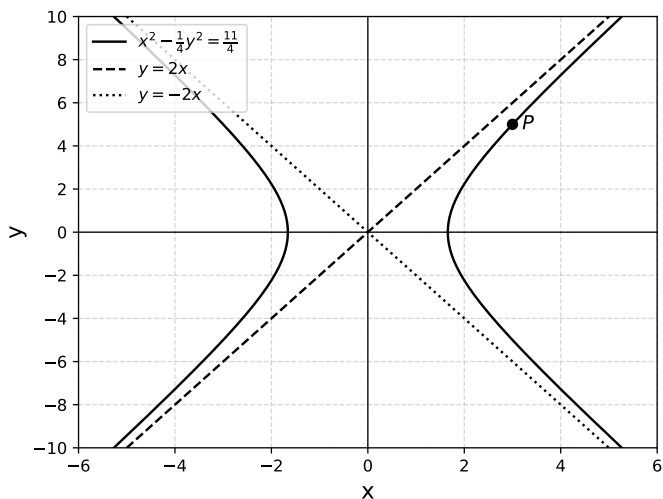
as shown in figure 4.34.30.

Figure 4.34.30: Graph of the hyperbola with equation
$x^2 - \frac{1}{4}y^2 = \frac{11}{4}$.

4.35 Exercise 35

Text

Calculate the area of the trapezoid formed by the lines

$$y = x + 2, \quad y = -\frac{1}{3}x + \frac{7}{3}, \quad x = 1, \quad x = 4$$

Solution

The lines $y = x + 2$ and $y = -\frac{1}{3}x + \frac{7}{3}$ form the oblique sides of the trapezoid, while $x = 1$ and $x = 4$ are the vertical sides. Find the intersection points of the two oblique lines with $x = 1$

$$y_1 = 1 + 2 = 3, \quad y_2 = -\frac{1}{3} \cdot 1 + \frac{7}{3} = \frac{6}{3} = 2$$

from which the points

$$A = (1,3), \quad B = (1,2)$$

and with $x = 4$

$$y_3 = 4 + 2 = 6, \quad y_4 = -\frac{1}{3} \cdot 4 + \frac{7}{3} = \frac{3}{3} = 1$$

from which

$$C = (4,6), \quad D = (4,1)$$

The trapezoid has as bases the segments AB and CD, with lengths

$$b_m = |3 - 2| = 1$$
$$b_M = |6 - 1| = 5$$

The height is the horizontal distance between the two bases given by

$$h = 4 - 1 = 3$$

the area of the trapezoid is:

$$\text{Area} = \frac{(b_M + b_m) \cdot h}{2} = \frac{6 \cdot 3}{2} = 9$$

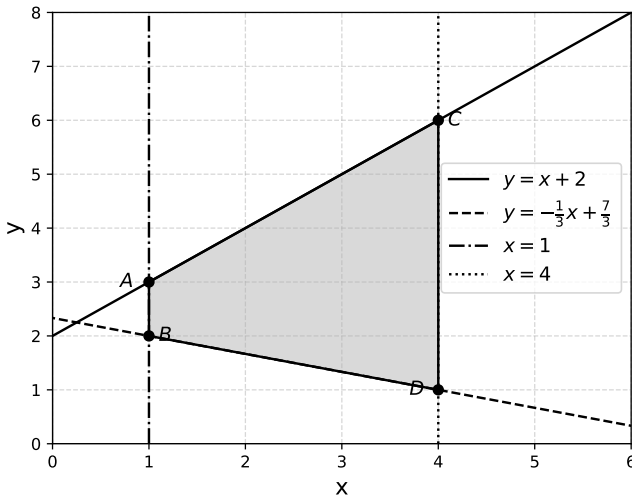The trapezoid and the lines are shown in figure 4.35.31.

Figure 4.35.31: Graph of the trapezoid in Exercise 35.

4.36 Exercise 36

Text

Given the circle with equation

$$x^2 + y^2 - 4x + 6y - 12 = 0$$

find the equations of the tangent lines to the circle that pass through the external point

$$P = (8,2)$$

Solution

Let's rewrite the equation of the circle by completing the squares

$$x^2 - 4x = (x-2)^2 - 4, \quad y^2 + 6y = (y+3)^2 - 9$$

substituting

$$(x-2)^2 - 4 + (y+3)^2 - 9 - 12 = 0$$

from which

$$(x-2)^2 + (y+3)^2 = 25$$

thus the circle has center $C = (2,-3)$ and radius $r = 5$. A generic line passing through the point $P = (8,2)$ has

equation

$$y - 2 = m(x - 8), \quad y = m(x - 8) + 2$$

we can write the intersection system of the line and the circle

$$\begin{cases} y = m(x - 8) + 2 \\ (x - 2)^2 + (y + 3)^2 = 25 \end{cases}$$

and impose that the resulting equation in x, once the two equations are compared, has a single real solution (for tangency), i.e. the discriminant of the resulting equation in x is zero, or we can impose that the distance between the center $C = (2, -3)$ of the circle and the line equals the radius $r = 5$. The line equation is

$$y = m(x - 8) + 2, \quad mx - y - 8m + 2 = 0$$

the point-line distance for a generic line $ax + by + c = 0$ and a point (x_0, y_0) is given by

$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

in our case we have

$$d = \frac{|2m - (-3) - 8m + 2|}{\sqrt{m^2 + 1}} = \frac{|5 - 6m|}{\sqrt{m^2 + 1}}$$

we impose that it equals the radius $r = 5$

$$5 = \frac{|5 - 6m|}{\sqrt{m^2 + 1}}, \quad 5\sqrt{m^2 + 1} = |5 - 6m|$$

squaring both sides

$$(5 - 6m)^2 = 25(m^2 + 1), \quad 36m^2 + 25 - 60m = 25m^2 + 25$$

that is

$$11m^2 - 60m = 0, \quad m(11m - 60) = 0$$

from which the solutions

$$m_1 = 0, \quad m_2 = \frac{60}{11}$$

Recalling the expression of the generic line passing through $P(8, 2)$, $y = m(x - 8) + 2$, the tangent lines have equations

$$y = 2$$

and

$$y = \frac{60}{11}x - \frac{480}{11} + 2$$
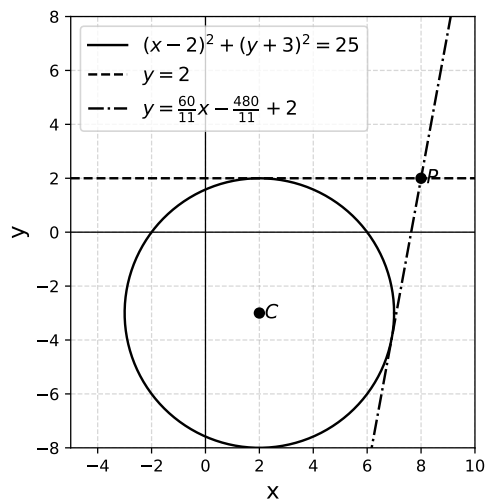
as shown in figure 4.36.32.

Figure 4.36.32: Graph of the lines in exercise 36.

4.37   Exercise 37

Text

Determine the equation of the tangent line to the parabola

$$y = x^2 - 2x + 1$$

at its point with abscissa $x = 3$.

Solution

Let's first find the point of tangency $P = (x_p, y_p)$, substituting $x = 3$ into the parabola equation

$$y_p = x_p^2 - 2x_p + 1 = 9 - 6 + 1 = 4$$

thus we have

$$P = (3, 4)$$

We derive the parabola to find the slope m of the tangent line at $x = 3$

$$y' = 2x - 2$$

therefore

$$y'(3) = 4, \quad m = 4$$

Starting from the generic line equation passing through P

$$y - y_p = m(x - x_p)$$

hence

$$y - 4 = m(x - 3)$$

substitute $m = 4$

$$y - 4 = 4(x - 3)$$

that is

$$y = 4(x - 3) + 4, \quad y = 4x - 12 + 4$$

finally the tangent line equation is

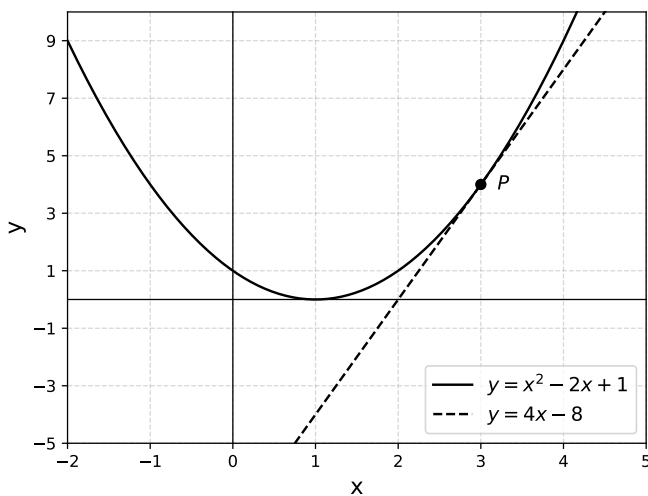$$y = 4x - 8$$

as shown in figure 4.37.33.

Figure 4.37.33: Graph of the line in exercise 37.

4.38 Exercise 38

Text

Given the ellipse with equation

$$\frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} = 1$$

calculate the lengths of the axes and find the coordinates of the foci.

Solution

The general equation of an ellipse with center (x_c, y_c) and semi-axes a and b is written as

$$\frac{(x-x_c)^2}{a^2} + \frac{(y-y_c)^2}{b^2} = 1$$

comparing, we obtain that the ellipse has center $C = (1, -2)$, major semi-axis $a = 3$ and minor semi-axis $b = 2$. Therefore, the length of the major axis is $2a = 6$, while the minor axis length is $2b = 4$. The foci are found by calculating

$$c = \sqrt{a^2 - b^2} = \sqrt{9 - 4} = \sqrt{5}$$

being the major axis parallel to the x-axis and remembering the center is $C = (1, -2)$, the foci coordinates are

$$F_1 = (1 - \sqrt{5}, -2)$$

and

$$F_2 = (1 + \sqrt{5}, -2)$$
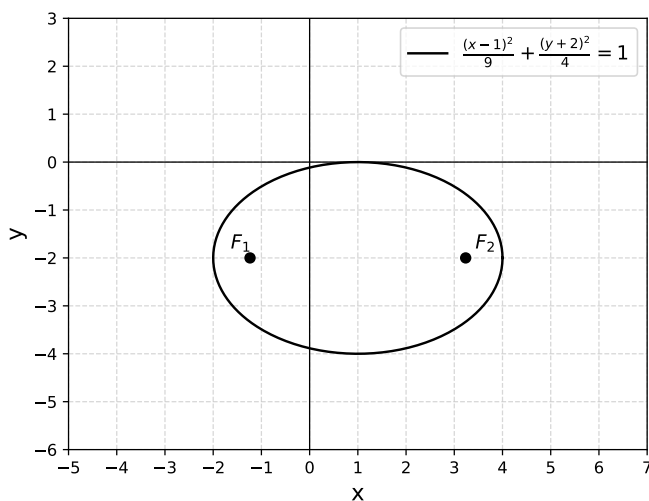
as shown in figure 4.38.34.

Figure 4.38.34: Graph of the ellipse in exercise 38.

4.39 Exercise 39

Text

Find the equation of the circle tangent to the x-axis at the point

$$(3,0)$$

and passing through the point

$$(6,4)$$

Solution

The circle is tangent to the x-axis at point $(3,0)$, so its center has abscissa 3 and ordinate equal to the radius r. Let the center be $C = (3,r)$, the equation of the circle is

$$(x-3)^2 + (y-r)^2 = r^2$$

The point $P = (6,4)$ belongs to the circle, so it must satisfy the equation

$$(6-3)^2 + (4-r)^2 = r^2$$

from which

$$9+(4-r)^2 = r^2, \quad 9+16-8r+r^2 = r^2, \quad 25-8r = 0$$

that is

$$r = \frac{25}{8}$$

The center can thus be written as

$$C = \left(3, \frac{25}{8} \right)$$

and, finally, the equation of the circle is

$$(x-3)^2 + \left(y - \frac{25}{8} \right)^2 = \left(\frac{25}{8} \right)^2$$
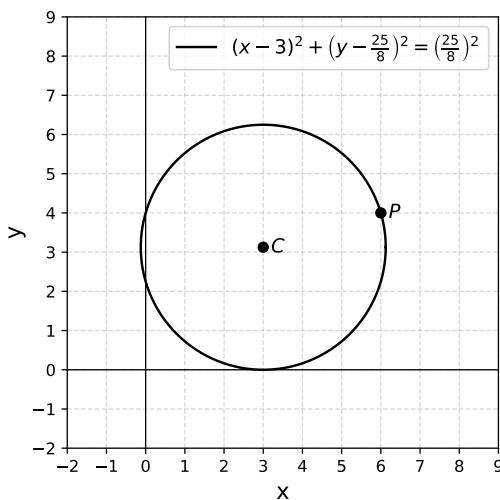
as shown in figure 4.39.35.

Figure 4.39.35: Graph of the circle in exercise 39.

4.40   Exercise 40

Text

Calculate the coordinates of the centroid of the triangle with vertices

$$A = (2,3), \quad B = (6,7), \quad C = (4,1)$$

Also verify that the centroid lies at the intersection of the medians.

Solution

The centroid G of a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ is found by averaging the coordinates

$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$$

In our case, we have

$$A = (2,3), \quad B = (6,7), \quad C = (4,1)$$

therefore

$$x_G = \frac{2+6+4}{3} = \frac{12}{3} = 4$$
$$y_G = \frac{3+7+1}{3} = \frac{11}{3}$$

from which the centroid

$$G = \left(4, \frac{11}{3}\right)$$

Let's verify that this point lies on the intersection of the medians. Consider the median from A to side BC, the midpoint of BC has coordinates

$$M_{BC} = \left(\frac{6+4}{2}, \frac{7+1}{2}\right) = (5,4)$$

the line passing through $A = (2,3)$ and $M_{BC} = (5,4)$ has slope

$$m_a = \frac{4-3}{5-2} = \frac{1}{3}$$

and the equation of the line is

$$y - 3 = \frac{1}{3}(x-2), \quad y = \frac{1}{3}x + \frac{7}{3}$$

Let's check if $G = \left(4, \frac{11}{3}\right)$ lies on this line

$$y = \frac{1}{3} \cdot 4 + \frac{7}{3} = \frac{4+7}{3} = \frac{11}{3}$$

so the centroid lies on this median. Similarly, for the median joining B to the midpoint of AC, we write

$$M_{AC} = \left(\frac{2+4}{2}, \frac{3+1}{2}\right) = (3,2)$$

the line through $B = (6,7)$ and $M_{AC} = (3,2)$ has slope

$$m_b = \frac{2-7}{3-6} = \frac{-5}{-3} = \frac{5}{3}$$

from which the equation

$$y - 7 = \frac{5}{3}(x - 6) \Rightarrow y = \frac{5}{3}x - 3$$

Check that $G = \left(4, \frac{11}{3}\right)$ lies on it

$$y = \frac{5}{3} \cdot 4 - 3 = \frac{20}{3} - 3 = \frac{11}{3}$$

the centroid also lies on this median. Finally, for the median joining C with the midpoint of AB, write

$$M_{AB} = \left(\frac{2+6}{2}, \frac{3+7}{2}\right) = (4,5)$$

the line through $C = (4,1)$ and $M_{AB} = (4,5)$, since the two points are vertically aligned, has equation

$$x = 4$$

which contains the centroid $G = \left(4, \frac{11}{3}\right)$. We have therefore shown that the intersection of the three medians coincides exactly with G. The triangle and the lines are shown in figure 4.40.36.
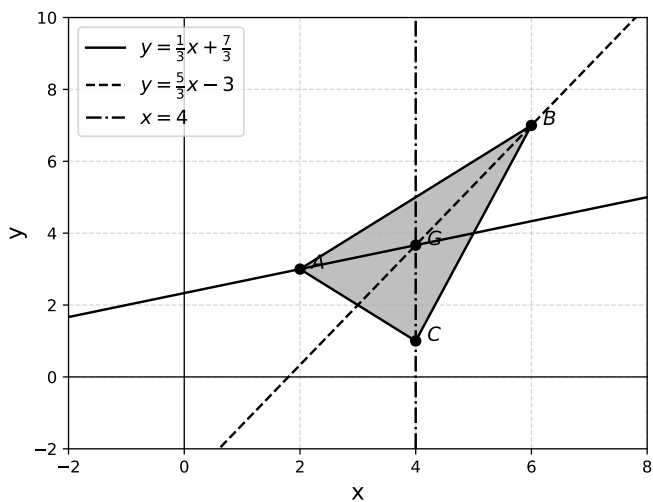
Figure 4.40.36: Graph of the triangle in exercise 40.