On the Value of Target Data in Transfer Learning

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Abstract

We aim to understand the value of additional labeled or unlabeled target data in transfer learning, for any given amount of source data; this is motivated by practical questions around minimizing sampling costs, whereby, target data is usually harder or costlier to acquire than source data, but can yield better accuracy.

To this aim, we establish the first minimax-rates in terms of both source and target sample sizes, and show that performance limits are captured by new notions of discrepancy between source and target, which we refer to as *transfer exponents*.

Interestingly, we find that attaining minimax performance is akin to ignoring one of the source or target samples, provided distributional parameters were known a priori. Moreover, we show that practical decisions – w.r.t. minimizing sampling costs – can be made in a minimax-optimal way *without* knowledge or estimation of distributional parameters nor of the discrepancy between source and target.

1 Introduction

The practice of transfer-learning often involves acquiring some amount of target data, and involves various practical decisions as to how to best combine source and target data; however much of the theoretical literature on transfer only addresses the setting where no target labeled data is available.

We aim to understand the value of target labels, that is, given n_P labeled data from some source distribution P, and n_Q labeled target labels from a target Q, what is the best Q error achievable by any classifier in terms of *both* n_Q and n_P, and which classifiers achieve such optimal transfer. In this first analysis, we mostly restrict ourselves to a setting, similar to the traditional *covariate-shift* assumption, where the best classifier – from a fixed VC class $\mathcal{H}$ – is the same under P and Q.

We establish the first minimax-rates, for bounded-VC classes, in terms of both source and target sample sizes n_P and n_Q, and show that performance limits are captured by new notions of discrepancy between source and target, which we refer to as *transfer exponents*.

The first notion of transfer-exponent, called ρ, is defined in terms of discrepancies in excess risk, and is most refined. Already here, our analysis reveals a surprising fact: the best possible rate (matching upper and lower-bounds) in terms of ρ and both sample sizes n_P, n_Q is - up to constants - achievable by an oracle which simply ignores the least informative of the source or target datasets. In other words, if $\hat{h}_P$ and $\hat{h}_Q$ denote the ERM on data from P, resp. from Q, one of the two achieves the optimal Q rate over any classifier having access to both P and Q datasets. However, which of $\hat{h}_P$ or $\hat{h}_Q$ is optimal is not easily decided without prior knowledge: for instance, cross-validating on a holdout target-sample would naively result in a rate of $n_Q^{-1/2}$ which can be far from optimal given large n_P. Interestingly, we show that the optimal (n_P, n_Q)-rate is achieved by a generic approach, akin to so-called *hypothesis-transfer* [1, 2], which optimizes Q-error under the constraint of low P-error, and does so without knowledge of distributional parameters such as ρ.

We then consider a related notion of *marginal* transfer-exponent, called γ, defined w.r.t. marginals P_X, Q_X. This is motivated by the fact that practical decisions in transfer often involve the use of

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cheaper unlabeled data (i.e., data drawn from P_X, Q_X). We will show that, when practical decisions are driven by observed changes in marginals P_X, Q_X, the marginal notion γ is then most suited to capture performance as it does not require knowledge (or observations) of label distribution $Q_{Y|X}$.

In particular, the marginal exponent γ helps capture performance limits in the following scenarios of current practical interest:

• **Minimizing sampling cost.** Given different costs of labeled source and target data, and a desired target excess error at most ϵ, how to use unlabeled data to decide on an optimal sampling scheme that minimizes labeling costs while achieving target error at most ϵ. (Section 6)

• **Choice of transfer.** Given two sources P_1 and P_2, each at some unknown distance from Q, given unlabeled data and some or no labeled data from Q, how to decide which of P_1, P_2 transfers best to the target Q. (Section 7.1)

• **Reweighting.** Given some amount of unlabeled data from Q, and some or no labeled Q data, how to optimally re-weight (out of a fixed set of schemes) the source P data towards best target performance. While differently motivated, this problem is quite related to the last one. (Section 7)

Although optimal decisions in the above scenarios depend tightly on unknown distributional parameters such as different label noise in source and target data, and on unknown *distance* from source to target (as captured by γ), we show that such practical decisions can be made, near optimally, with no knowledge of distributional parameters, and perhaps surprisingly, without ever estimating γ. Furthermore, the unlabeled sampling complexity can be shown to remain low. Finally, the procedures described in this work remain of a theoretical nature, but yield new insights into how various practical decisions in transfer can be made near-optimally in a data-driven fashion.

Related Work. Much of the theoretical literature on transfer can be subdivided into a few main lines of work. As mentioned above, the main distinction with the present work is in that they mostly focus on situations with no labeled target data, and consider distinct notions of discrepancy between P and Q. We contrast these various notions with the transfer-exponents ρ and γ in Section 3.1.

A first direction considers refinements of total-variation that quantify changes in error over classifiers in a fixed class $\mathcal{H}$. The most common such measures are the so-called $d_{\mathcal{A}}$-divergence [3, 4, 5] and the $\mathcal{Y}$-discrepancy [6, 7, 8]. In this line of work, the rates of transfer, largely expressed in terms of n_P alone, take the form $o_p(1) + C \cdot \text{divergence}(P, Q)$. In other words, transfer down to 0 error seems impossible whenever these divergences are non-negligible; we will carefully argue that such intuition can be overly pessimistic.

Another prominent line of work, which has led to many practical procedures, considers so-called density ratios f_Q/f_P (importance weights) as a way to capture the similarity between P and Q [9, 10]. A related line of work considers information-theoretic measures such as KL-divergence or Renyi divergence [11, 12] but has received relatively less attention. Similar to these notions, the transfer-exponents ρ and γ are *asymmetric* measures of distance, attesting to the fact that it could be easier to transfer from some P to Q than the other way around. However, a significant downside to these notions is that they do not account for the specific structure of a hypothesis class $\mathcal{H}$ as is the case with the aforementionned divergences. As a result, they can be sensitive to issues such as minor differences of support in P and Q, which may be irrelevant when learning with certain classes $\mathcal{H}$.

On the algorithmic side, many approaches assign importance weights to source data from P so as to minimize some prescribed *metric* between P and Q [13, 14]; as we will argue, *metrics*, being symmetric, can be inadequate as a measure of discrepancy given the inherent asymmetry in transfer.

The importance of unlabeled data in transfer-learning, given the cost of target labels, has always been recognized, with various approaches developed over the years [15, 16], including more recent research efforts into so-called *semisupervised* or *active* transfer, where, given unlabeled target data, the goal is to request as few target labels as possible to improve classification over using source data alone [17, 18, 19, 20, 21].

More recently, [22, 23] consider the nonparametric case (unbounded VC) under alternative performance criteria than 0-1 loss. Also recent, [24] proposed a nonparametric measure of discrepancy which successfully captures the interaction between labeled source and target under nonparametric conditions and 0-1 loss. However the measure of [24] omits the interaction with a fixed hypothesis class $\mathcal{H}$, and therefore shares similar downsides as the density-ratio approaches discussed above.

2 Setup and Definitions

We consider a classification setting where the input $X \in \mathcal{X}$, some measurable space, and the output $Y \in \{0, 1\}$. We let $\mathcal{H} \subset 2^{\mathcal{X}}$ denote a fixed hypothesis class over $\mathcal{X}$, denote $d_{\mathcal{H}}$ the VC dimension [25], and the goal is to return a classifier $h \in \mathcal{H}$ with low error $R_Q(h) \doteq \mathbb{E}_Q[h(X) \neq Y]$ under some joint distribution Q on X, Y. The learner has access to two independent labeled samples $S_P \sim P^{n_P}$ and $S_Q \sim Q^{n_Q}$, i.e., drawn from *source* distributions P and target Q, of respective sizes n_P, n_Q. Our aim is to bound the excess error, under Q, of any $\hat{h}$ learned from both samples, in terms of n_P, n_Q, and (suitable) notions of discrepancy between P and Q. We will let $P_X, Q_X, P_{Y|X}, Q_{Y|X}$ denote the corresponding marginal and conditional distributions under P and Q.

Definition 1. For $D \in \{Q, P\}$, denote $\mathcal{E}_D(h) \doteq R_D(h) - \inf_{h' \in \mathcal{H}} R_D(h')$, the **excess error** of h.

Distributional Conditions. We consider various traditional assumptions in classification and transfer. The first one is a so-called *Bernstein Class Condition* on noise [26, 27, 28, 29, 30].

(NC). *Let* $h_P^* \doteq \operatorname*{argmin}_{h \in \mathcal{H}} R_P(h)$ *and* $h_Q^* \doteq \operatorname*{argmin}_{h \in \mathcal{H}} R_Q(h)$ *exist.* $\exists \beta_P, \beta_Q \in [0, 1], c_P, c_Q > 0$ *s.t.*

$$P_X(h \neq h_P^*) \leq c_p \cdot \mathcal{E}_P^{\beta_P}(h), \quad \textit{and} \quad Q_X(h \neq h_Q^*) \leq c_q \cdot \mathcal{E}_Q^{\beta_Q}(h). \tag{1}$$

For instance, the usual Tsybakov noise condition, say on P, corresponds to the case where h_P^* is the Bayes classifier, with corresponding regression function $\eta_P(x) \doteq \mathbb{E}[Y|x]$ satisfying $P_X(|\eta_P(X) - 1/2| \leq \tau) \leq C\tau^{\beta_P/(1-\beta_P)}$. Classification is easiest w.r.t. P (or Q) when β_P (resp. β_Q) is largest. We will see that this is also the case in Transfer.

The next assumption is stronger, but can be viewed as a relaxed version of the usual *Covariate-Shift* assumption which states that $P_{Y|X} = Q_{Y|X}$.

(RCS). *Let* h_P^*, h_Q^* *as defined above. We have* $\mathcal{E}_Q(h_P^*) = \mathcal{E}_Q(h_Q^*) = 0$. *We then define* $h^* \doteq h_P^*$.

Note that the above allows $P_{Y|X} \neq Q_{Y|X}$. However, it is not strictly weaker than *Covariate-Shift*, since the latter allows $h_P^* \neq h_Q^*$ provided the Bayes $\notin \mathcal{H}$. The assumption is useful as it serves to isolate the sources of hardness in transfer beyond just shifts in h^*. We will in fact see later that it is easily removed, but at the additive (necessary) cost of $\mathcal{E}_Q(h_P^*)$.

3 Transfer-Exponents from P to Q.

We consider various notions of *discrepancy* between P and Q, which will be shown to tightly capture the complexity of transfer P to Q.

Definition 2. We call $\rho > 0$ a **transfer-exponent** from P to Q, w.r.t. $\mathcal{H}$, if there exists C_ρ such that

$$\forall h \in \mathcal{H}, \quad C_\rho \cdot \mathcal{E}_P(h) \geq \mathcal{E}_Q^\rho(h). \tag{2}$$

We are interested in the smallest such ρ with small C_ρ. We generally would think of ρ as at least 1, although there are situations – which we refer to as *super-transfer*, to be discussed, where we have $\rho < 1$; in such situations, data from P can yield faster $\mathcal{E}_Q$ rates than data from Q.

While the transfer-exponent will be seen to tightly capture the two-samples minimax rates of transfer, and can be *adapted to*, practical learning situations call for *marginal* versions that can capture the rates achievable when one has access to unlabeled Q data.

Definition 3. We call $\gamma > 0$ a **marginal transfer-exponent** from P to Q if $\exists C_\gamma$ such that

$$\forall h \in \mathcal{H}, \quad C_\gamma \cdot P_X(h \neq h_P^*) \geq Q_X^\gamma(h \neq h_P^*). \tag{3}$$

The following simple proposition relates γ to ρ.

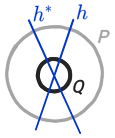
Proposition 1 (From γ to ρ)**.** *Suppose Assumptions (NC) and (RCS) hold, and that P has marginal transfer-exponent (γ, C_γ) w.r.t. Q. Then P has transfer-exponent $\rho \leq \gamma/\beta_P$, where $C_\rho = C_\gamma^{\gamma/\beta_P}$.*

Proof. $\forall h \in \mathcal{H}$, we have $\mathcal{E}_Q(h) \leq Q_X(h \neq h_P^*) \leq C_\gamma \cdot P_X(h \neq h_P^*)^{1/\gamma} \leq C_\gamma \cdot \mathcal{E}_P(h)^{\beta_P/\gamma}$. □

3.1 Examples and Relation to other notions of discrepancy.

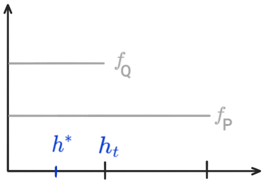
In this section, we consider various examples that highlight interesting aspects of ρ and γ, and their relations to other notions of distance $P \to Q$ considered in the literature. Though our results cover noisy cases, in all these examples we assume no noise for simplicity, and therefore $\gamma = \rho$.

Example 1. (Non-overlapping supports) This first example emphasizes the fact that, unlike in much of previous analyses of transfer, the exponents γ, ρ do not require that Q_X and P_X have overlapping support. This is a welcome property shared also by the $d_{\mathcal{A}}$ and $\mathcal{Y}$ discrepancy.



In the example shown on the right, $\mathcal{H}$ is the class of homogeneous linear separators, while P_X and Q_X are uniform on the surface of the spheres depicted (e.g., corresponding to different scalings of the data). We then have that $\gamma = \rho = 1$ with $C_\gamma = 1$, while notions such as *density-ratios*, KL-divergences, or the recent nonparameteric notion of [24], are ill-defined or diverge to ∞.

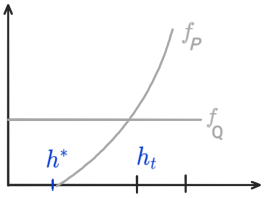
Example 2. (Large $d_{\mathcal{A}}, d_{\mathcal{Y}}$) Let $\mathcal{H}$ be the class of one-sided thresholds on the line, but now we let $P_X \doteq \mathcal{U}[0,2]$ and $Q_X \doteq \mathcal{U}[0,1]$. Let h^* be thresholded at $1/2$. We then see that for all h_t thresholded at $t \in [0,1]$, $2P_X(h_t \neq h^*) = \frac{1}{2}Q_X(h_t \neq h^*)$, where for $t > 1$, $P_X(h_t \neq h^*) = \frac{1}{2}(t - 1/2) \geq \frac{1}{2}Q_X(h_t \neq h^*) = \frac{1}{4}$. Thus, the marginal transfer exponent $\gamma = 1$ with $C_\gamma = 2$, so we have fast transfer at the same rate $1/n_P$ as if we were sampling from Q (Theorem 3).



On the other hand, recall that the $d_{\mathcal{A}}$-divergence takes the form $d_{\mathcal{A}}(P,Q) \doteq \sup_{h\in\mathcal{H}} |P_X(h \neq h^*) - Q_X(h \neq h^*)|$, while the $\mathcal{Y}$-discrepancy takes the form $d_{\mathcal{Y}}(P,Q) \doteq \sup_{h\in\mathcal{H}} |\mathcal{E}_P(h) - \mathcal{E}_Q(h)|$. The two coincide whenever there is no noise in Y.

Now, take h_t as the threshold at $t = 1/2$, and $d_{\mathcal{A}} = d_{\mathcal{Y}} = \frac{1}{4}$ which would wrongly imply that transfer is not feasible at a rate faster than $\frac{1}{4}$; we can in fact make this situation worse, i.e., let $d_{\mathcal{A}} = d_{\mathcal{Y}} \to \frac{1}{2}$ by letting h^* correspond to a threshold close to 0. A first issue is that these divergences get large in large disagreement regions; this is somewhat mitigated by *localization*, as discussed in Example 4.

Example 3. (Minimum γ, ρ, and the inherent **asymmetry** of transfer) Suppose $\mathcal{H}$ is the class of one-sided thresholds on the line, $h^* = h^*_P = h^*_Q$ is a threshold at 0. The marginal Q_X has uniform density f_Q (on an interval containing 0), while, for some $\gamma \geq 1$, P_X has density $f_P(t) \propto t^{\gamma-1}$ on $t > 0$ (and uniform on the rest of the support of Q, not shown). Consider any h_t at threshold $t > 0$, we have $P_X(h_t \neq h^*) = \int_0^t f_P \propto t^\gamma$, while $Q_X(h_t \neq h^*) \propto t$. Notice that for any fixed $\epsilon > 0$, $\lim_{t>0,\, t\to 0} \frac{Q_X(h_t \neq h^*)^{\gamma-\epsilon}}{P_X(h_t \neq h^*)} = \lim_{t>0,\, t\to 0} C\frac{t^{\gamma-\epsilon}}{t^\gamma} = \infty$.



We therefore see that γ is the smallest possible marginal transfer-exponent (similarly, $\rho = \gamma$ is the smallest possible transfer exponent). Interestingly, now consider transferring instead from Q to P: we would have $\gamma(Q \to P) = 1 \leq \gamma \doteq \gamma(P \to Q)$, i.e., it could be easier to transfer from Q to P than from P to Q, which is not captured by symmetric notions of distance ($d_{\mathcal{A}}$, Wassertein, etc ...). Finally note that the above example can be extended to more general hypothesis classes as it simply plays on how fast f_P decreases w.r.t. f_Q in regions of space.

Example 4. (*Super-transfer* and localization). We continue on the above Example 2. Now let $0 < \gamma < 1$, and let $f_P(t) \propto |t|^{\gamma-1}$ on $[-1,1]$, with $Q_X \doteq \mathcal{U}[-1,1]$, h^* at 0. As before, γ is a transfer-exponent $P \to Q$, and according to Theorem 3, we attain transfer rates of $\mathcal{E}_Q \lesssim n_P^{-1/\gamma}$, faster than the rates of n_Q^{-1} attainable with data from Q. We call this situation *super-transfer*, i.e., ones where the source data gets us faster to h^* as here P concentrates more mass close to h^* (such situations can also be constructed by letting $P_{Y|X}$ be less noisy than $Q_{Y|X}$ data, although it seems mostly of theoretical interest).

Now consider the following ϵ-*localization* fix to the $d_{\mathcal{A}} = d_{\mathcal{Y}}$ divergences over h's with small P error (assuming we only observe data from P): $d^*_{\mathcal{Y}} \doteq \sup_{h\in\mathcal{H}:\, \mathcal{E}_P(h)\leq\epsilon} |\mathcal{E}_P(h) - \mathcal{E}_Q(h)|$. This is no longer worst-case over all h's, yet it is still not a complete fix. To see why, consider that, given n_P data from P, the best P-excess risk attainable is n_P^{-1} so we might set $\epsilon \propto n_P^{-1}$. Now the subclass $\{h \in \mathcal{H} : \mathcal{E}_P(h) \leq \epsilon\}$ corresponds to thresholds $t \in [\pm n_P^{-1/\gamma}]$, since $\mathcal{E}_P(h_t) = P([0,t]) \propto |t|^\gamma$.

We therefore have $d^*_{\mathcal{Y}} \propto \left| n_P^{-1} - n_P^{-1/\gamma} \right| \propto n_P^{-1}$, wrongly suggesting a transfer rate $\mathcal{E}_Q \lesssim n_P^{-1}$, while the super-transfer rate $n_P^{-1/\gamma}$ is achievable as discussed above. The problem is that, even after localization, $d^*_{\mathcal{Y}}$ treats errors under P and Q symmetrically.

4 Lower-Bounds

Definition 4 ((NC) Class)**.** Let $\mathcal{F}_{\text{(NC)}}(\rho, \beta_P, \beta_Q, C)$ denote the class of pairs of distributions (P, Q) with transfer-exponent ρ, $C_\rho \leq C$, satisfying (NC) with parameters β_P, β_Q, and $c_P, c_Q \leq C$.

The following lower-bound in terms of ρ is obtained via information theoretic-arguments. In effect, given the VC class $\mathcal{H}$, we construct a set of distribution pairs $\{(P_i, Q_i)\}$ supported on $d_{\mathcal{H}}$ datapoints, which all belong to $\mathcal{F}_{\text{(NC)}}(\rho, \beta_P, \beta_Q, C)$. All the distributions share the same marginals P_X, Q_X. Any two pairs are close to each other in the sense that Π_i, Π_j, where $\Pi_i \doteq P_i^{n_P} \times Q_i^{n_Q}$, are close in KL-divergence, while, however maintaining pairs $(P_i, Q_i), (P_j, Q_j)$ far in a pseudo-distance induced by Q_X. All the proofs from this section are in Appendix A.

Theorem 1 (ρ Lower-bound)**.** *Suppose the hypothesis class $\mathcal{H}$ has VC dimension $d_{\mathcal{H}} \geq 9$. Let $\hat{h} = \hat{h}(S_P, S_Q)$ denote any (possibly improper) classifier with access to two independent labeled samples $S_P \sim P^{n_P}$ and $S_Q \sim Q^{n_Q}$. Fix any $\rho \geq 1$, $0 \leq \beta_P, \beta_Q < 1$. Suppose either n_P or n_Q is sufficiently large so that*

$$\epsilon(n_P, n_Q) \doteq \min\left\{ \left(\frac{d_{\mathcal{H}}}{n_P}\right)^{1/(2-\beta_P)\rho}, \left(\frac{d_{\mathcal{H}}}{n_Q}\right)^{1/(2-\beta_Q)} \right\} \leq 1/2.$$

Then, for any $\hat{h}$, there exists $(P, Q) \in \mathcal{F}_{\text{(NC)}}(\rho, \beta_P, \beta_Q, 1)$, and a universal constant c such that,

$$\mathop{\mathbb{P}}_{S_P, S_Q} \left(\mathcal{E}_Q(\hat{h}) > c \cdot \epsilon(n_P, n_Q) \right) \geq \frac{3 - 2\sqrt{2}}{8}.$$

As per Proposition 1 we can translate any upper-bound in terms of ρ to an upper-bound in terms of γ since $\rho \leq \gamma/\beta_P$. We investigate whether such upper-bounds in terms of γ are tight, i.e., given a class $\mathcal{F}_{\text{(NC)}}(\rho, \beta_P, \beta_Q, C)$, are there distributions with $\rho = \gamma/\beta_P$ where the rate is realized.

The proof of the next result is similar to that of Theorem 1, however with the added difficulty that we need the construction to yield two forms of rates $\epsilon_1(n_P, n_Q), \epsilon_2(n_P, n_Q)$ over the data support (again $d_{\mathcal{H}}$ points). Combining these two rates matches the desired upper-bound. In effect, we follow the intuition that, to have $\rho = \gamma/\beta_P$ achieved on some subset $\mathcal{X}_1 \subset \mathcal{X}$, we need β_Q to behave as 1 locally on $\mathcal{X}_1$, while matching the rate requires larger β_Q on the rest of the suppport (on $\mathcal{X} \setminus \mathcal{X}_1$).

Theorem 2 (γ Lower-bound)**.** *Suppose the hypothesis class $\mathcal{H}$ has VC dimension $d_{\mathcal{H}}$, $\lfloor d_{\mathcal{H}}/2 \rfloor \geq 9$. Let $\hat{h} = \hat{h}(S_P, S_Q)$ denote any (possibly improper) classifier with access to two independent labeled samples $S_P \sim P^{n_P}$ and $S_Q \sim Q^{n_Q}$. Fix any $0 < \beta_P, \beta_Q < 1$, $\rho \geq \max\{1/\beta_P, 1/\beta_Q\}$. Suppose either n_P or n_Q is sufficiently large so that*

$$\epsilon_1(n_P, n_Q) \doteq \min\left\{ \left(\frac{d_{\mathcal{H}}}{n_P}\right)^{1/(2-\beta_P)\rho \cdot \beta_Q}, \left(\frac{d_{\mathcal{H}}}{n_Q}\right)^{1/(2-\beta_Q)} \right\} \leq 1/2, \text{ and}$$

$$\epsilon_2(n_P, n_Q) \doteq \min\left\{ \left(\frac{d_{\mathcal{H}}}{n_P}\right)^{1/(2-\beta_P)\rho}, \left(\frac{d_{\mathcal{H}}}{n_Q}\right) \right\} \leq 1/2.$$

Then, for any $\hat{h}$, there exists $(P, Q) \in \mathcal{F}_{\text{(NC)}}(\rho, \beta_P, \beta_Q, 2)$, with **marginal-transfer-exponent** *$\gamma = \rho \cdot \beta_P \geq 1$, with $C_\gamma \leq 2$, and a universal constant c such that,*

$$\mathop{\mathbb{E}}_{S_P, S_Q} \mathcal{E}_Q(\hat{h}) \geq c \cdot \max\left\{\epsilon_1(n_P, n_Q), \epsilon_2(n_p, n_Q)\right\}.$$

Remark 1 (Tightness with upper-bound)**.** *Write $\epsilon_1(n_P, n_Q) = \min\{\epsilon_1(n_P), \epsilon_1(n_Q)\}$, and similarly, $\epsilon_2(n_P, n_Q) = \min\{\epsilon_2(n_P), \epsilon_2(n_Q)\}$. Define $\epsilon_L \doteq \max\{\epsilon_1(n_P, n_Q), \epsilon_2(n_P, n_Q)\}$ as in the above lower-bound of Theorem 2. Next, define $\epsilon_H \doteq \min\{\epsilon_2(n_P), \epsilon_1(n_Q)\}$. It turns out that the* *best upper-bound we can show (as a function of γ) is in terms of ϵ_H so defined. It is therefore natural to ask whether or when ϵ_H and ϵ_L are of the same order.*

Clearly, we have $\epsilon_1(n_P) \leq \epsilon_2(n_P)$ and $\epsilon_1(n_Q) \geq \epsilon_2(n_Q)$ so that $\epsilon_L \leq \epsilon_H$ (as to be expected).

Now, if $\beta_Q = 1$, we have $\epsilon_1(n_P) = \epsilon_2(n_P)$ and $\epsilon_1(n_Q) = \epsilon_2(n_Q)$, so that $\epsilon_L = \epsilon_H$. More generally, from the above inequalities, we see that $\epsilon_L = \epsilon_H$ in the two regimes where either $\epsilon_1(n_Q) \leq \epsilon_1(n_P)$ (in which case $\epsilon_L = \epsilon_H = \epsilon_1(n_Q)$), or $\epsilon_2(n_P) \leq \epsilon_2(n_Q)$ (in which case $\epsilon_L = \epsilon_H = \epsilon_2(n_P)$).

5 Upper-Bounds

The following lemma is due to [31].

Lemma 1. *Let $A_n = \frac{d_{\mathcal{H}}}{n} \log\left(\frac{\max\{n, d_{\mathcal{H}}\}}{d_{\mathcal{H}}}\right) + \frac{1}{n}\log\left(\frac{1}{\delta}\right)$. With probability at least $1 - \frac{\delta}{3}$, $\forall h, h' \in \mathcal{H}$,*

$$R(h) - R(h') \leq \hat{R}(h) - \hat{R}(h') + c\sqrt{\min\{P(h \neq h'), \hat{P}(h \neq h')\}A_n} + cA_n, \tag{4}$$

and

$$\frac{1}{2}P(h \neq h') - cA_n \leq \hat{P}(h \neq h') \leq 2P(h \neq h') + cA_n, \tag{5}$$

for a universal numerical constant $c \in (0, \infty)$.

Now consider the following algorithm. Let S_P be a sequence of n_P samples from P and S_Q a sequence of n_Q samples from Q. Also let $\hat{h}_{S_P} = \operatorname{argmin}_{h \in \mathcal{H}} \hat{R}_{S_P}(h)$ and $\hat{h}_{S_Q} = \operatorname{argmin}_{h \in \mathcal{H}} \hat{R}_{S_Q}(h)$. Choose $\hat{h}$ as the solution to the following optimization problem.

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Algorithm 1:
    Minimize      $\hat{R}_{S_Q}(h)$
    subject to    $\hat{R}_{S_P}(h) - \hat{R}_{S_P}(\hat{h}_{S_P}) \leq c\sqrt{\hat{P}_{S_P}(h \neq \hat{h}_{S_P})A_{n_P}} + cA_{n_P}$
                  $h \in \mathcal{H}$.
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The intuition is that, effectively, the constraint guarantees we maintain a near-optimal guarantee on $\mathcal{E}_P(\hat{h})$ in terms of n_P and the NC parameters for P, while (as we show) still allowing the algorithm to select an h with a near-minimal value of $\hat{R}_{S_Q}(h)$. The former guarantee plugs into the transfer condition to obtain a term converging in n_P, while the latter provides a term converging in n_Q, and altogether the procedure achieves a rate specified by the *min* of these two guarantees (which is in fact nearly minimax *optimal*, since it matches the lower bound up to logarithmic factors).

Formally, we have the following result for this learning rule; its proof is in Appendix B.

Theorem 3 (Minimax Upper-Bounds)**.** *Let $\hat{h}$ be the solution from Algorithm 1. Suppose $\beta_Q > 0$. For a constant C depending on $\rho, c_\rho, \beta_P, c_{\beta_P}, \beta_Q, c_{\beta_Q}$, with probability at least $1 - \delta$,*

$$\mathcal{E}_Q(\hat{h}) \leq C \min\left\{A_{n_P}^{\frac{1}{(2-\beta_P)\rho}}, A_{n_Q}^{\frac{1}{2-\beta_Q}}\right\} = \tilde{O}\left(\min\left\{\left(\frac{d_{\mathcal{H}}}{n_P}\right)^{\frac{1}{(2-\beta_P)\rho}}, \left(\frac{d_{\mathcal{H}}}{n_Q}\right)^{\frac{1}{2-\beta_Q}}\right\}\right).$$

Note that, by the lower bound of Theorem 1, this bound is optimal up to log factors.

Remark 2. *Note that, by Proposition 1, this also immediately implies a bound under the marginal transfer condition and RCS, simply taking $\rho \leq \gamma/\beta_P$. Furthermore, by the lower bound of Theorem 2, the resulting bound in terms of γ is tight in certain regimes up to log factors.*

6 Minimizing Sampling Cost

In this section and those that follow, we discuss the value of having access to *unlabeled* data from Q. The idea is that unlabeled data can be obtained much more cheaply than labeled data, so gaining access to unlabeled data can be realistic in many applications. Specifically, we begin by discussing an adaptive sampling scenario, where we are able to *draw* samples from P or Q, at different *costs*, and we are interested in optimizing the total cost of obtaining a given excess Q-risk.

Formally, consider the scenario where we have as input a value ϵ, and are tasked with producing a classifier $\hat{h}$ with $\mathcal{E}_Q(\hat{h}) \leq \epsilon$. We are then allowed to *draw* samples from either P or Q toward achieving this goal. Let us imagine samples from P have a cost $\mathfrak{c}_P$ and samples from Q have a cost $\mathfrak{c}_Q$. We are then interested in minimizing the total cost of achieving the objective $\mathcal{E}_Q(\hat{h}) \leq \epsilon$. In fact, more generally, let us suppose $\mathfrak{c}_P : \mathbb{N} \to [0, \infty)$ and $\mathfrak{c}_Q : \mathbb{N} \to [0, \infty)$ are *cost functions*, where $\mathfrak{c}_P(n)$ indicates the cost of sampling a batch of size n from P, and similarly define $\mathfrak{c}_Q(n)$. We suppose these functions are increasing, concave, and unbounded.

Definition 5. Define $n_Q^* = d_{\mathcal{H}}/\epsilon^{2-\beta_Q}$, $n_P^* = d_{\mathcal{H}}/\epsilon^{(2-\beta_P)\gamma/\beta_P}$, and $\mathfrak{c}^* = \min\{\mathfrak{c}_Q(n_Q^*), \mathfrak{c}_P(n_P^*)\}$. We call $\mathfrak{c}^* = \mathfrak{c}^*(\epsilon)$ the **minimax optimal cost** of sampling from P or Q to attain Q-error ϵ (under cost functions $\mathfrak{c}_P, \mathfrak{c}_Q$).

Note that the cost $\mathfrak{c}^*$ is effectively the smallest possible, up to log factors, in the range of parameters given in Theorem 2, i.e., Theorem 2 implies there exist scenarios where a cost of order at least $\mathfrak{c}^*$ is necessary (with *any* learner) in order to achieve the guarantee $\mathcal{E}_Q(\hat{h}) \leq \epsilon$: that is, in order to make the lower bound in Theorem 2 less than ϵ, either $n_Q = \tilde{\Omega}(n_Q^*)$ samples are needed from Q or $n_P = \tilde{\Omega}(n_P^*)$ samples are needed from P. We show that $\mathfrak{c}^*$ is nearly achievable, adaptively.

Procedure. We assume access to a large unlabeled data set U_Q sampled from Q_X. For our purposes, we will suppose this data set has size at least $\Theta(\frac{d_{\mathcal{H}}}{\epsilon} \log \frac{1}{\epsilon} + \frac{1}{\epsilon} \log \frac{1}{\delta})$.

Let $A'_n = \frac{d_{\mathcal{H}}}{n} \log(\frac{\max\{n, d_{\mathcal{H}}\}}{d_{\mathcal{H}}}) + \frac{1}{n} \log(\frac{2n^2}{\delta})$. Then for any labeled data set S, define $\hat{h}_S = \operatorname{argmin}_{h \in \mathcal{H}} \hat{R}_S(h)$, and given an additional data set U (labeled or unlabeled) define a quantity

$$\hat{\delta}(S, U) = \sup\left\{ \hat{P}_U(h \neq \hat{h}_S) : h \in \mathcal{H}, \hat{R}_S(h) - \hat{R}_S(\hat{h}_S) \leq c\sqrt{\hat{P}_S(h \neq \hat{h}_S) A'_{|S|}} + cA'_{|S|} \right\},$$

where c is as in Lemma 1. Now we have the following procedure.

```
Algorithm 2:
0. $S_P \leftarrow \{\}$, $S_Q \leftarrow \{\}$
1. For $t = 1, 2, \ldots$
2.     Let $n_{t,P}$ be minimal such that $\mathfrak{c}_P(n_{t,P}) \geq 2^{t-1}$
3.     Sample $n_{t,P}$ samples from $P$ and add them to $S_P$
4.     Let $n_{t,Q}$ be minimal such that $\mathfrak{c}_Q(n_{t,Q}) \geq 2^{t-1}$
5.     Sample $n_{t,Q}$ samples from $Q$ and add them to $S_Q$
6.     If $c\sqrt{\hat{\delta}(S_Q, S_Q) A_{|S_Q|}} + cA_{|S_Q|} \leq \epsilon$, return $\hat{h}_{S_Q}$
7.     If $\hat{\delta}(S_P, U_Q) \leq \epsilon/4$, return $\hat{h}_{S_P}$
```

The following theorem asserts that this procedure will find a classifier $\hat{h}$ with $\mathcal{E}_Q(\hat{h}) \leq \epsilon$ while adaptively using a near-minimal cost associated with achieving this. The proof is in Appendix C.

Theorem 4 (Adapting to Sampling Costs)**.** *Define* $\tilde{n}_Q = \frac{c'}{\epsilon^{2-\beta_Q}} \left(d_{\mathcal{H}} \log \frac{1}{\epsilon} + \log \frac{1}{\delta}\right)$*, and* $\tilde{n}_P = \frac{c'}{\epsilon^{(2-\beta_P)\gamma/\beta_P}} \left(d_{\mathcal{H}} \log \frac{1}{\epsilon} + \log \frac{1}{\delta}\right)$*, where* c' *is a well-chosen constant, which depends on parameters* (c_γ, γ, c_{β_Q}, β_Q, c_{β_P}, β_P) *but not on* ϵ *or* δ*.*

Algorithm 2 outputs a classifier $\hat{h}$ *such that, with probability at least* $1 - \delta$*,* $\mathcal{E}_Q(\hat{h}) \leq \epsilon$ *and the total sampling cost incurred is at most* $\min\{\mathfrak{c}_Q(\tilde{n}_Q), \mathfrak{c}_P(\tilde{n}_P)\} = \tilde{O}(\mathfrak{c}^*)$*.*

The point here is that there are scenarios where we might end up sampling very few labeled Q samples if the P samples are cheap enough relative to the cost of Q samples. Furthermore, we achieve this without knowing (or even estimating) the relevant parameters (β_Q, β_P, γ, ...). Indeed, the algorithm is designed in a way that would also adapt to even more-favorable situations (for instance, where the noise conditions in the actual P and Q distributions might be more benign than the worst case scenarios from the lower bounds).

7 Reweighting the Source Data

In this section, we present a technique for using unlabeled data from Q to find a reweighting of the P data more suitable for transfer. This gives a technique for using the data effectively in a potentially practical way. As above, we again suppose access to the sample U_Q of unlabeled data from Q.

Additionally, we suppose we have access to a set $\mathcal{P}$ of functions $f : \mathcal{X} \to [0, \infty)$, which we interpret as unnormalized *density* functions with respect to P_X. Let P_f denote the bounded measure whose marginal on $\mathcal{X}$ has density f with respect to P_X, and the conditional $Y|X$ is the same as for P.

Now suppose $S_P = \{(x_i, y_i)\}_{i=1}^{n_P}$ is a sequence of n_P iid P-distributed samples. Continuing conventions from above $R_{P_f}(h) = \int \mathbb{1}[h(x) \neq y] f(x) \mathrm{d}P(x, y)$ is a risk with respect to P_f, but now we also write $\hat{R}_{S_P,f}(h) = \frac{1}{n_P} \sum_{(x,y)\in S_P} \mathbb{1}[h(x) \neq y] f(x)$, and additionally we will use $P_{f^2}(h \neq h') = \int \mathbb{1}[h(x) \neq h'(x)] f^2(x) \mathrm{d}P(x, y)$, and $\hat{P}_{S_P,f^2}(h \neq h') = \frac{1}{n_P} \sum_{(x,y)\in S_P} \mathbb{1}[h(x) \neq h'(x)] f^2(x)$; the reason f^2 is used instead of f is that this will represent a variance term in the bounds below. Other notations from above are defined analogously. In particular, also let $\hat{h}_{S_P,f} = \operatorname{argmin}_{h\in\mathcal{H}} \hat{R}_{S_P,f}(h)$. For simplicity, we will only present the case of $\mathcal{P}$ having finite pseudo-dimension d_p (i.e., d_p is the VC dimension of the subgraph functions $\{(x, y) \mapsto \mathbb{1}[f(x) \leq y] : f \in \mathcal{P}\}$); extensions to general bracketing or empirical covering follow similarly.

For the remaining results in this section, we suppose the condition RCS holds for *all* P_f: that is, R_{P_f} is minimized in $\mathcal{H}$ at a function $h^*_{P_f}$ having $\mathcal{E}_Q(h^*_{P_f}) = 0$. For instance, this would be the case if the Bayes optimal classifier is in the class $\mathcal{H}$.

Define $A''_n = \frac{d_{\mathcal{H}}+d_p}{n} \log\left(\frac{\max\{n, d_{\mathcal{H}}+d_p\}}{d_{\mathcal{H}}+d_p}\right) + \frac{1}{n}\log\left(\frac{1}{\delta}\right)$. Let us also extend the definition of $\hat{\delta}$ introduced above. Specifically, define $\hat{\delta}(S_P, f, U_Q)$ as

$$\sup\left\{ \hat{P}_{U_Q}(h \neq \hat{h}_{S_P,f}) : h \in \mathcal{H}, \hat{\mathcal{E}}_{S_P,f}(h) \leq c\sqrt{\hat{P}_{S_P,f^2}(h \neq \hat{h}_{S_P,f}) A''_{n_P}} + c\|f\|_\infty A''_{n_P} \right\}.$$

Now consider the following procedure.

```
Algorithm 3:
Choose $\hat{f}$ to minimize $\hat{\delta}(S_P, f, U_Q)$ over $f \in \mathcal{P}$.
Choose $\hat{h}$ to minimize $\hat{R}_{S_Q}(h)$ among $h \in \mathcal{H}$
subject to $\hat{\mathcal{E}}_{S_P,\hat{f}}(h) \leq c\sqrt{\hat{P}_{S_P,\hat{f}^2}(h \neq \hat{h}_{S_P,\hat{f}}) A''_{n_P}} + c\|\hat{f}\|_\infty A''_{n_P}$.
```

As we establish in the proof, $\hat{f}$ is effectively being chosen to minimize an upper bound on the excess Q-risk of the resulting classifier $\hat{h}$. Toward analyzing the performance of this procedure, note that each f induces a marginal transfer exponent: that is, values $C_{\gamma,f}$, γ_f such that $\forall h \in \mathcal{H}$, $C_{\gamma,f} P_{f^2}(h \neq h^*_{P_f}) \geq Q^{\gamma_f}(h \neq h^*_{P_f})$. Similarly, each f induces a Bernstein Class Condition: there exist values $c_f > 0$, $\beta_f \in [0, 1]$ such that $P_{f^2}(h \neq h^*_{P_f}) \leq c_f \mathcal{E}^{\beta_f}_{P_f}(h)$.

The following theorem reveals that Algorithm 3 is able to perform nearly as well as applying the transfer technique from Theorem 3 directly under the measure in the family $\mathcal{P}$ that would provide the best bound. The only losses compared to doing so are a dependence on d_p and the supremum of the density (which accounts for how different that measure is from P). The proof is in Appendix D.

Theorem 5. *Suppose $\beta_Q > 0$ and that RCS holds for all P_f, $f \in \mathcal{P}$. There exist constants C_f depending on $\|f\|_\infty$, $C_{\gamma,f}$, γ_f, c_f, β_f, and a constant C depending on c_q, β_Q such that, for a sufficiently large $|U_Q|$, w.p. at least $1 - \delta$, the classifier $\hat{h}$ chosen by Algorithm 3 satisfies*

$$\begin{aligned}\mathcal{E}_Q(\hat{h}) &\leq \inf_{f\in\mathcal{P}} C \min\left\{ C_f \left(A''_{n_P}\right)^{\frac{\beta_f}{(2-\beta_f)\gamma_f}}, A_{n_Q}^{\frac{1}{2-\beta_Q}} \right\} \\ &= \tilde{O}\left(\inf_{f\in\mathcal{P}} \min\left\{ C_f \left(\frac{d_{\mathcal{H}}+d_p}{n_P}\right)^{\frac{\beta_f}{(2-\beta_f)\gamma_f}}, \left(\frac{d_{\mathcal{H}}}{n_Q}\right)^{\frac{1}{2-\beta_Q}} \right\} \right).\end{aligned}$$

The utility of this theorem will of course depend largely on the family $\mathcal{P}$ of densities. This class should contain a distribution with small γ_f marginal transfer exponent, while also small $\|f\|_\infty$ (which is captured by the C_f constant in the bound), and favorable noise conditions (i.e., large β_f).

7.1 Choice of Transfer from Multiple Sources

It is worth noting that all of the above analysis also applies to the case that, instead of a family of densities with respect to a single P, the set $\mathcal{P}$ is a set of probability measures P_i, each with its own separate iid data set S_i of some size n_i. Lemma 1 can then be applied to all of these data sets, if we simply replace δ by $\delta/|\mathcal{P}|$ to accommodate a union bound; call the corresponding quantity $A_n^{'''}$. Then, similarly to the above, we can use the following procedure.

```
Algorithm 4:
Choose $\hat{i}$ to minimize $\hat{\delta}(S_i, U_Q)$ over $P_i \in \mathcal{P}$.
Choose $\hat{h}$ to minimize $\hat{R}_{S_Q}(h)$ among $h \in \mathcal{H}$
subject to $\hat{\mathcal{E}}_{S_{\hat{i}}}(h) \le c\sqrt{\hat{P}_{S_{\hat{i}}}(h \neq \hat{h}_{S_{\hat{i}}}) A_{n_{\hat{i}}}^{'''}} + cA_{n_{\hat{i}}}^{'''}$.
```

To state a formal guarantee, let us suppose the conditions above hold for each of these distributions with respective values of $C_{\gamma,i}$, γ_i, c_i, β_i. We have the following theorem. Its proof is essentially identical to the proof of Theorem 5 (effectively just substituting notation), and is therefore omitted.

Theorem 6. *Suppose $\beta_Q > 0$ and that RCS holds for all $P_i \in \mathcal{P}$. There exist constants C_i depending on $C_{\gamma,i}$, γ_i, c_i, β_i, and a constant C depending on c_q, β_Q such that, for a sufficiently large $|U_Q|$, with probability at least $1-\delta$, the classifier $\hat{h}$ chosen by Algorithm 4 satisfies*

$$\mathcal{E}_Q(\hat{h}) \le \tilde{O}\left(\inf_{P_i \in \mathcal{P}} \min\left\{ C_i \left(\frac{d_{\mathcal{H}} + \log(|\mathcal{P}|)}{n_i} \right)^{\frac{\beta_i}{(2-\beta_i)\gamma_i}}, \left(\frac{d_{\mathcal{H}}}{n_Q} \right)^{\frac{1}{2-\beta_Q}} \right\} \right).$$

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