Cluster algorithms for the Ising model

Monte Carlo methods which overcome the problem of critical slowing down close to second order phase transitions

Contents

- Ising model
- Single spin flip metropolis
- Autocorrelation, Binning Analysis
- Wolff algorithm
- Swensen-Wang algorithm

Ising model

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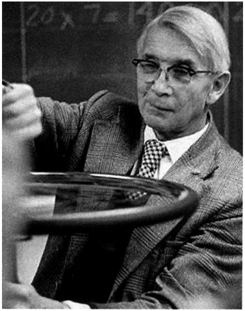

Figure : Ernst Ising (1900 - 1998)

Ising model - Definition

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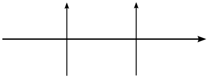

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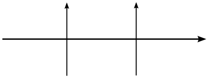

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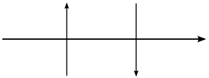

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$$\langle M \rangle_T = \sum_{\sigma} M(\sigma) p(\sigma, T)$$

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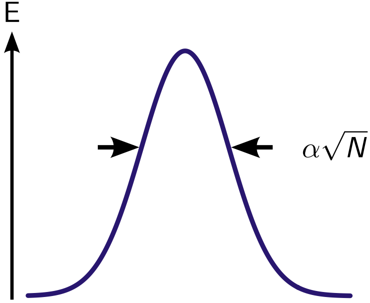

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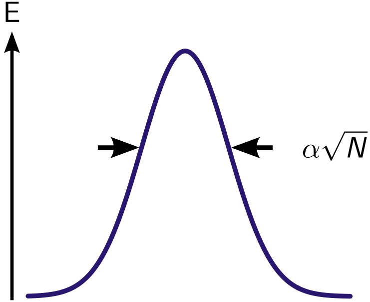

- Solution: Importance sampling using Metropolis algorithm

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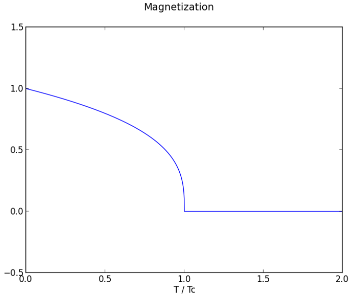

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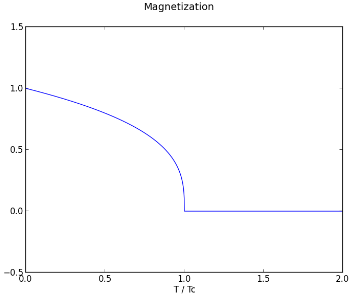
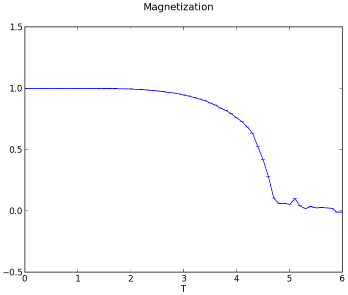


Obtained



Ising model - 2 spins

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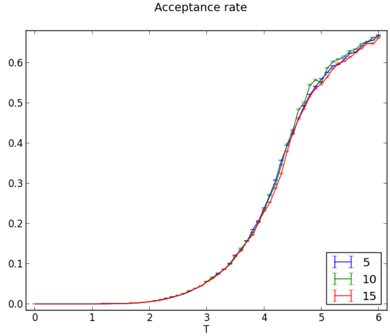

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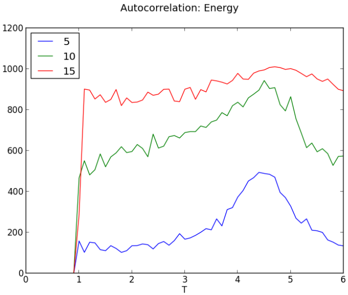

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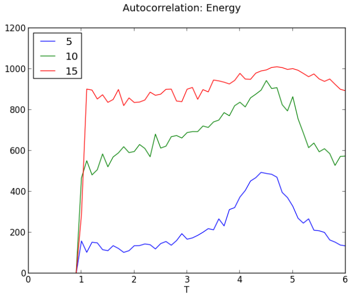

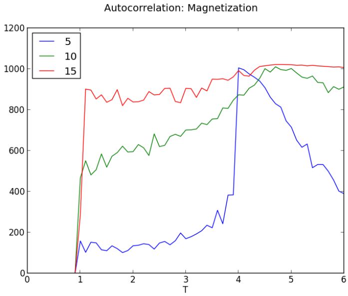

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$$\tau_A = \lim_{l\to\infty}\left(\frac{2^l\mathrm{Var}A^{(l)}}{\mathrm{Var}A^{(0)}} - 1\right)$$

What really happens in the System

Cluster algorithms: Wolff algorithm

$$P(\sigma_x, \sigma_y) = 1 - \min\left(1, e^{2\beta\sigma_x\sigma_y}\right)$$

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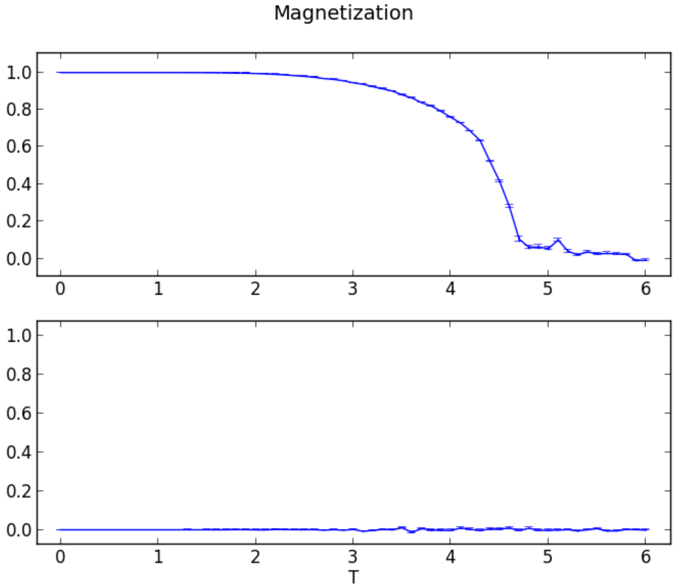
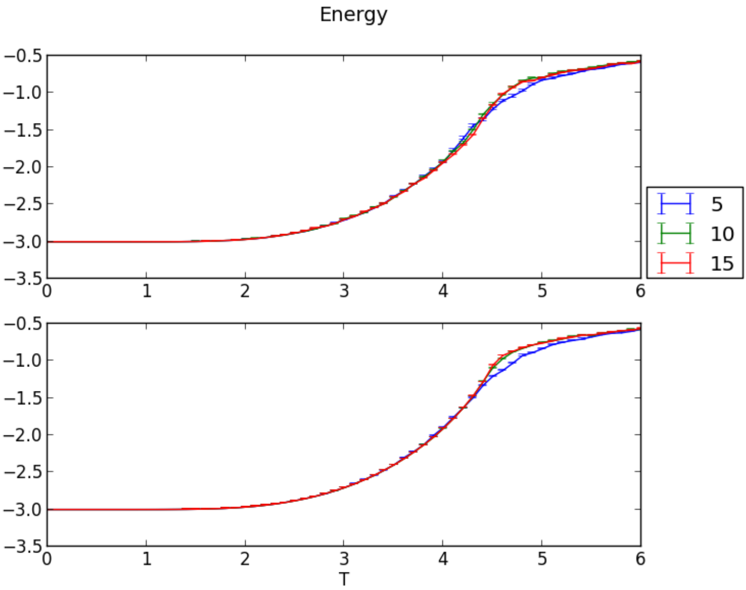
3. For all sites in the cluster:
   3.1 Visit every unknown neighbour, flip its spin and add it to the cluster with probability given above

Why cluster algorithms are better



Energy



Cluster size

Magnetization

Magnetization squared

Computation time per spin flip

Swendsen-Wang

Questions

Binning analysis in detail

$$\mathrm{Var}X := E\left[X^2\right] - E\left[X\right]^2$$

$$\begin{aligned}
(\Delta X)^2 &= \frac{1}{N^2}\sum_{i,j=1}^{N}\left(E\left[X_i X_j\right] - E\left[X\right]^2\right) \\
&= \frac{\mathrm{Var}X}{N} + \frac{1}{N^2}\sum_{i\neq j}\left(E\left[X_i X_j\right] - E\left[X\right]^2\right) \\
&= \frac{\mathrm{Var}X}{N} + \frac{2}{N^2}\sum_{i=1}^{N}\sum_{t}\left(E\left[X_i X_{i+t}\right] - E\left[X\right]^2\right) \\
&:= \frac{\mathrm{Var}X}{N}\left(1 + 2\tau_X\right)
\end{aligned}$$

Wolff or Swendsen-Wang?

Swendsen-Wang better for parallelization because it touches the whole lattice.