Cluster algorithms for the Ising model

Monte Carlo methods which overcome the problem of critical slowing down close to second order phase transitions

Contents

Ising model

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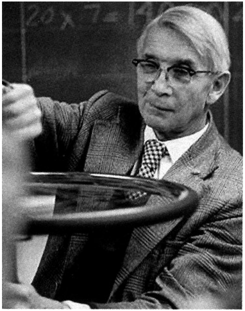

Figure : Ernst Ising (1900 - 1998)

Ising model - Definition

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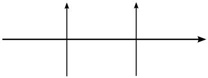

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-J +J

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$$p(\sigma, T) = \frac{e^{-\beta \mathscr{H}(\sigma)}}{\mathscr{Z}(T)}, \qquad \beta = \frac{1}{k_B T}$$

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$$\langle M \rangle_T = \sum_{\sigma} M(\sigma) p(\sigma, T)$$

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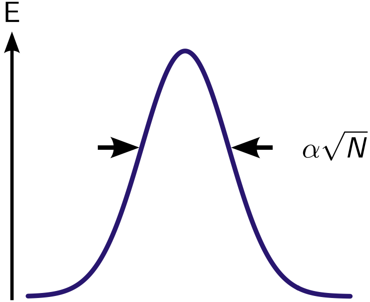

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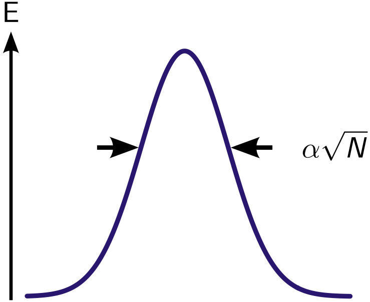

- Solution: Importance sampling using Metropolis algorithm

Single spin flip metropolis - Algorithm

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Single spin flip metropolis - Algorithm

$$
\begin{aligned}
A(X \to Y) &= \min\left(1, \frac{p(Y)}{p(X)}\right) \\
&= \min\left(1, e^{-\beta[E(Y)-E(X)]}\right)
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$$

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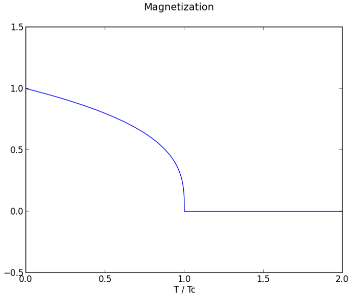

Single spin flip metropolis - Results

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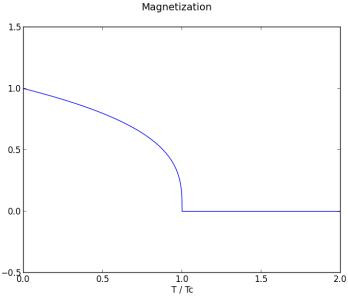
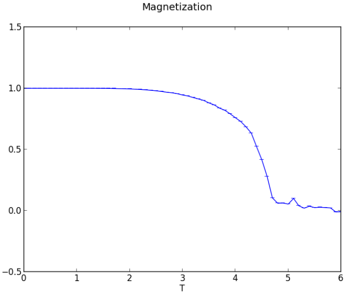


Obtained



Ising model - 2 spins

Single spin flip metropolis - To flip or not to flip

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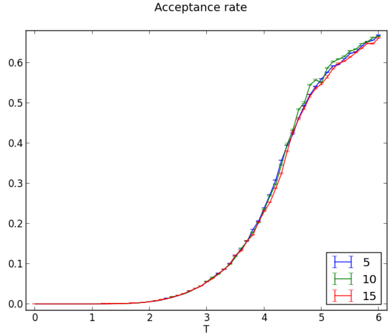

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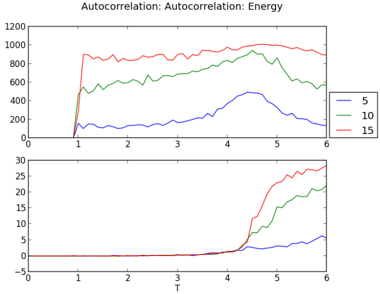

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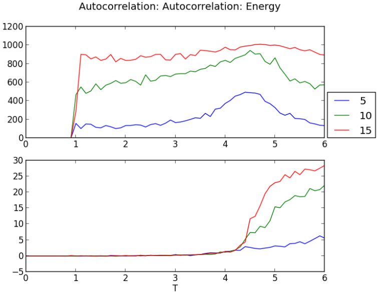

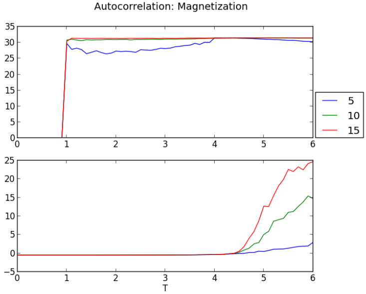

Binning analysis I

$$\mathrm{Var}X := E\left[X^2\right] - E\left[X\right]^2$$

$$\begin{aligned}
(\Delta X)^2 &= \frac{1}{N^2}\sum_{i,j=1}^{N}\left(E\left[X_i X_j\right] - E\left[X\right]^2\right)\\
&= \frac{\mathrm{Var}X}{N} + \frac{1}{N^2}\sum_{i\neq j}\left(E\left[X_i X_j\right] - E\left[X\right]^2\right)\\
&= \frac{\mathrm{Var}X}{N} + \frac{2}{N^2}\sum_{i=1}^{N}\sum_{t}\left(E\left[X_i X_{i+t}\right] - E\left[X\right]^2\right)\\
&:= \frac{\mathrm{Var}X}{N}\left(1 + 2\tau_X\right)
\end{aligned}$$

Binning analysis II

$$A_i^{(l)} = \frac{1}{2}\left(A_{2i-1}^{(l-1)} + A_{2i}^{(l-1)}\right)$$

$$\Delta^{(l)} = \sqrt{\mathrm{Var}A^{(l)}/M^{(l)}} \overset{l\to\infty}{\to} \Delta = \sqrt{(1+2\tau_A)\mathrm{Var}A/M}$$

$$\tau_A = \lim_{l\to\infty}\left(\frac{2^l\mathrm{Var}A^{(l)}}{\mathrm{Var}A^{(0)}} - 1\right)$$

What really happens in the System

Cluster algorithms: Wolff algorithm

$$P(\sigma_x, \sigma_y) = 1 - \min\left(1, e^{2\beta\sigma_x\sigma_y}\right)$$

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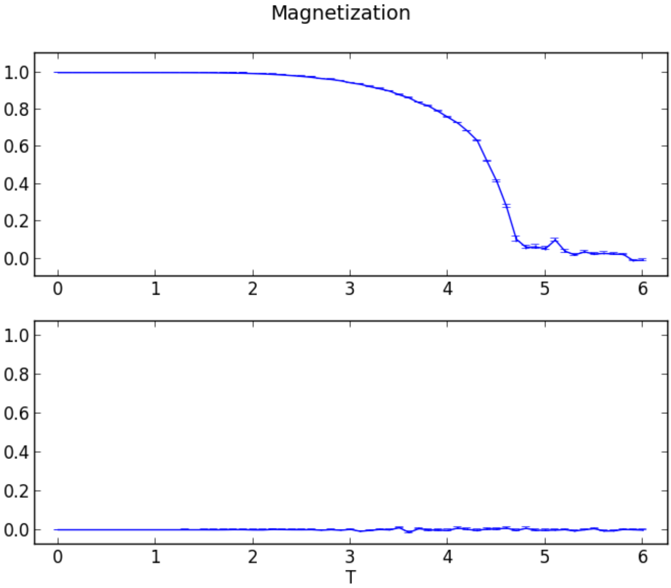
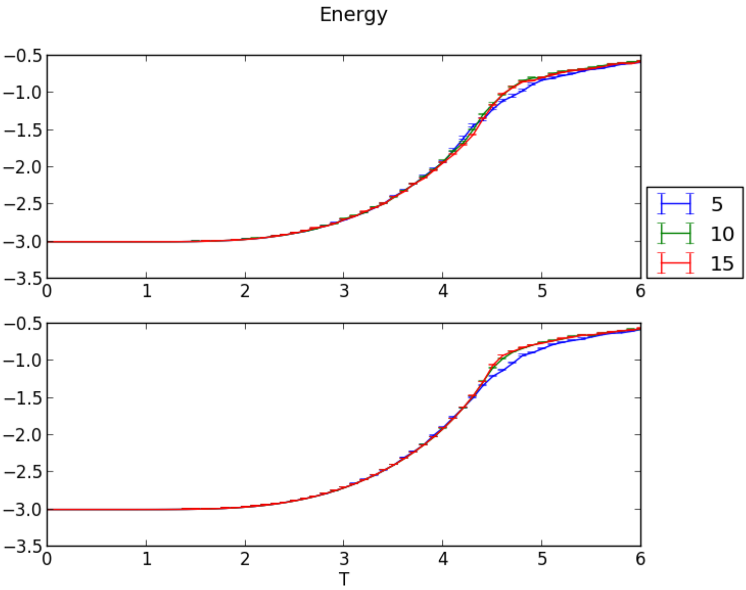
1. Choose one site (uniformly randomly)
2. Flip its spin and add it to the cluster
3. For all sites in the cluster:
   3.1 Visit every unknown neighbour, flip its spin and add it to the cluster with probability given above

Why cluster algorithms are better



Energy



Cluster size

Magnetization

Magnetization squared

Computation time per spin flip

Swendsen-Wang

Questions

Wolff or Swendsen-Wang?

Swendsen-Wang better for parallelization because it touches the whole lattice.