Cluster algorithms for the Ising model

Monte Carlo methods which overcome the problem of critical slowing down close to second order phase transitions

Contents

- Ising model
- Single spin flip metropolis
- Autocorrelation, Binning Analysis
- Wolff algorithm
- Swendsen-Wang algorithm

Ising model

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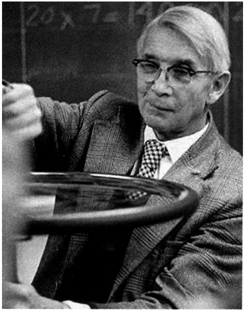

Figure : Ernst Ising (1900 - 1998)

Ising model - Definition

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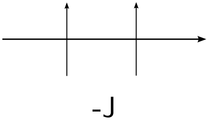

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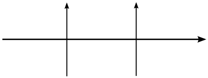

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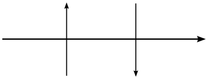

+J

Ising model - Canonical ensemble

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$$p(\sigma, T) = \frac{e^{-\beta \mathscr{H}(\sigma)}}{\mathscr{Z}(T)}, \qquad \beta = \frac{1}{k_B T}$$

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$$\langle M \rangle_T = \sum_{\sigma} M(\sigma) p(\sigma, T)$$

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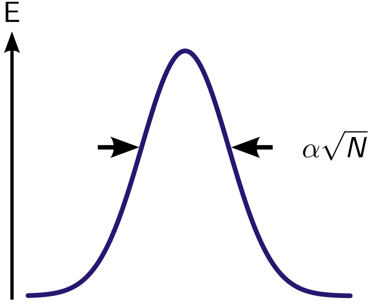

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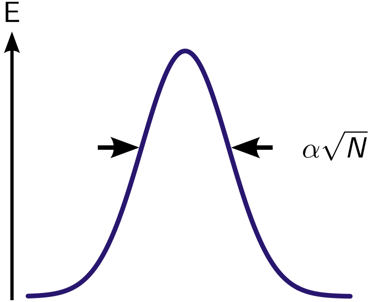

- Solution: Importance sampling using Metropolis algorithm

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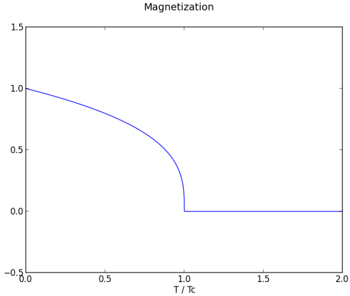

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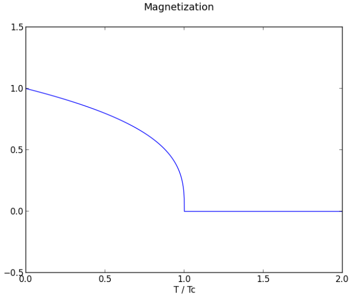
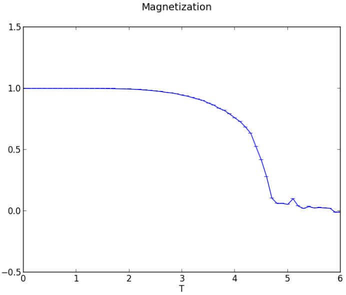


Obtained



Ising model - 2 spins

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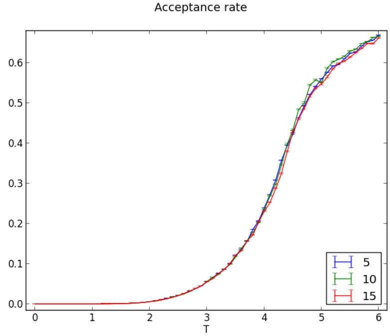

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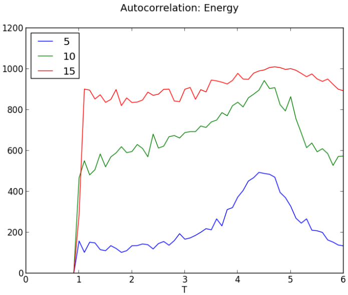

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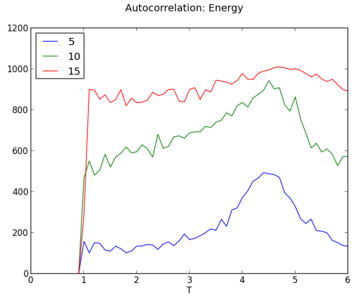

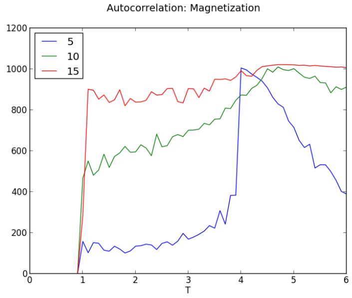

Binning analysis I

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$$\tau_A = \lim_{l\to\infty}\left(\frac{2^l \mathrm{Var}A^{(l)}}{\mathrm{Var}A^{(0)}} - 1\right)$$

What really happens in the System

Cluster algorithms: Wolff algorithm

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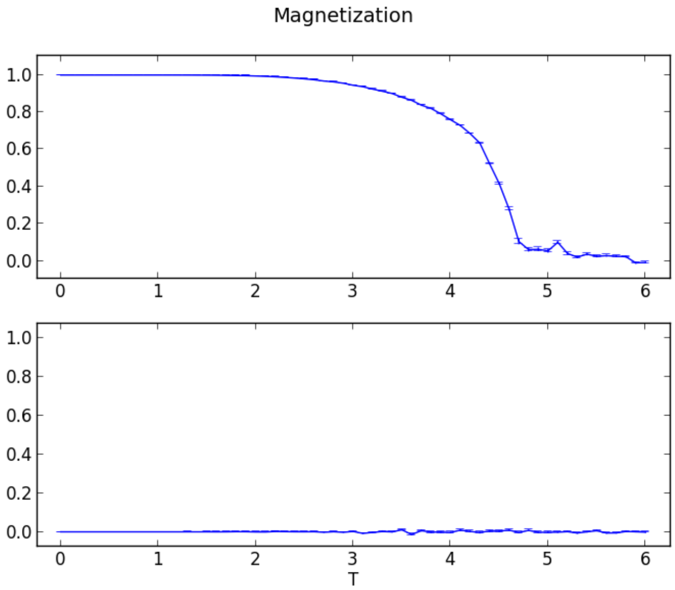
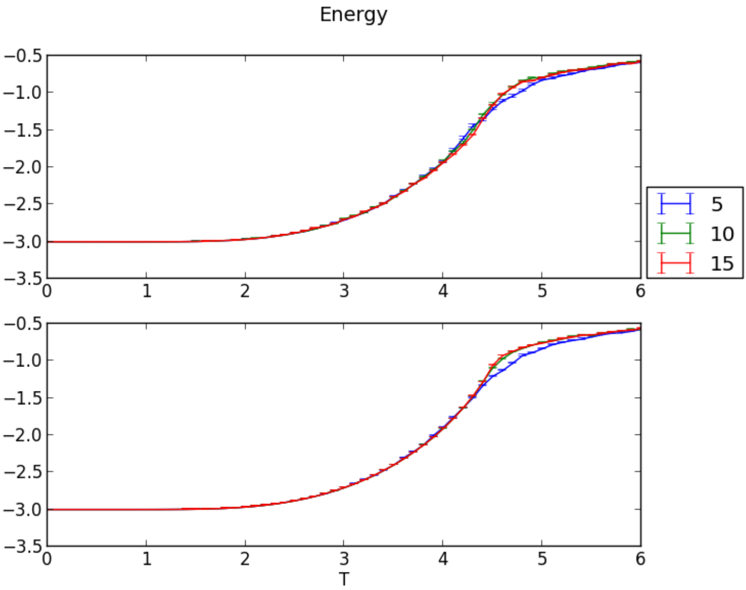
1. Choose one site (uniformly randomly)
2. Flip its spin and add it to the cluster
3. For all sites in the cluster:
   3.1 Visit every unknown neighbour, flip its spin and add it to the cluster with probability given above

Why cluster algorithms are better



Energy



Cluster size

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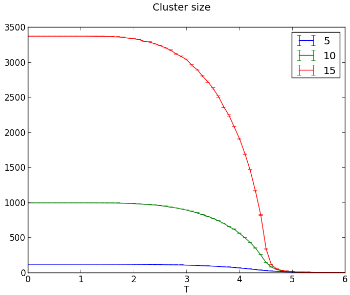

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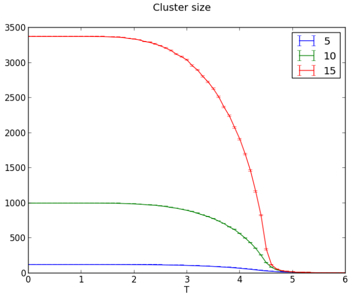
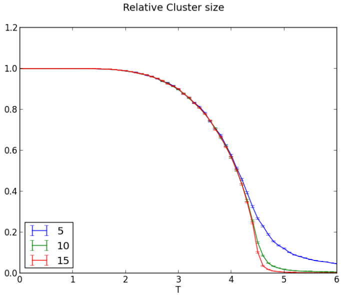
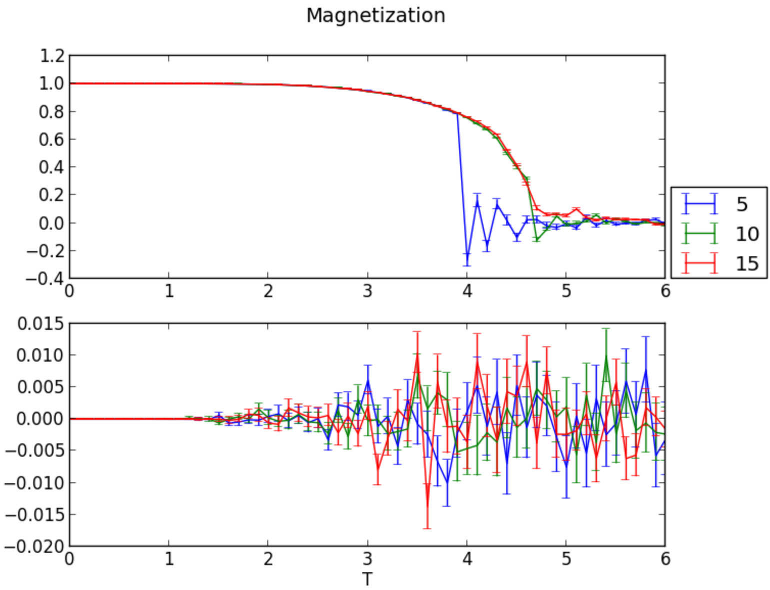
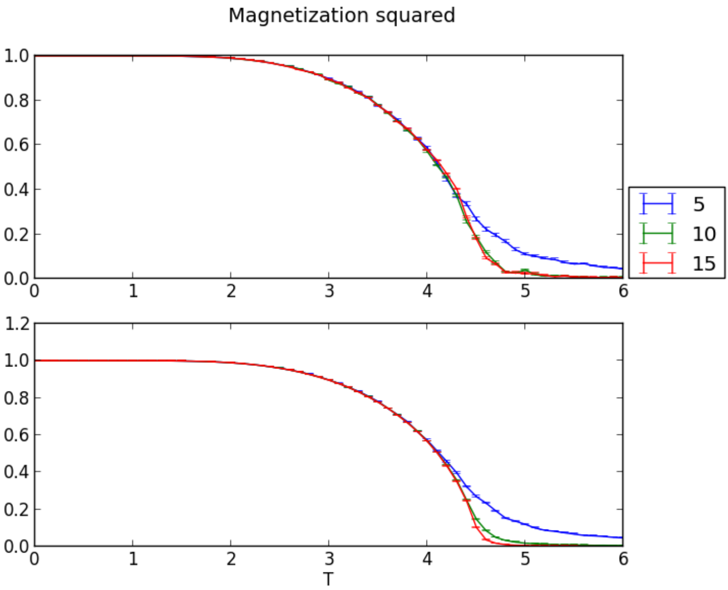




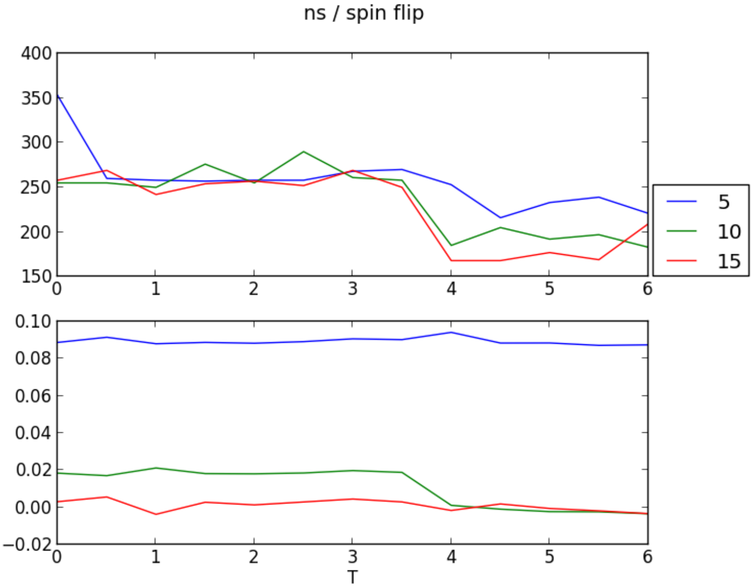
Magnetization



Magnetization squared



Computation time per spin flip



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- Partitions the whole lattice in clusters
- Randomly chooses a new (same) spin for every cluster
- Always touches every spin

Summary

References

Questions

Binning analysis in detail

$$\mathrm{Var}X := E\left[X^2\right] - E\left[X\right]^2$$

$$\begin{aligned}
(\Delta X)^2 &= \frac{1}{N^2}\sum_{i,j=1}^{N}\left(E\left[X_i X_j\right] - E\left[X\right]^2\right) \\
&= \frac{\mathrm{Var}X}{N} + \frac{1}{N^2}\sum_{i\neq j}\left(E\left[X_i X_j\right] - E\left[X\right]^2\right) \\
&= \frac{\mathrm{Var}X}{N} + \frac{2}{N^2}\sum_{i=1}^{N}\sum_{t}\left(E\left[X_i X_{i+t}\right] - E\left[X\right]^2\right) \\
&:= \frac{\mathrm{Var}X}{N}\left(1 + 2\tau_X\right)
\end{aligned}$$

Wolff or Swendsen-Wang?

Swendsen-Wang better for parallelization because it touches the whole lattice.