Roger Wienaah

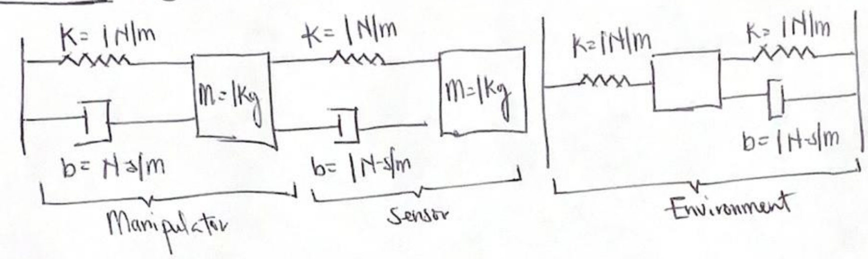
ME 531: Pset 1

Question 1a:

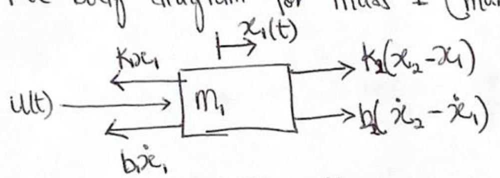
Roger Wienaah

ME 531 Problem Set 1

Question 1: (a)



Free body diagram for Mass 1 (manipulator); Assuming $x_2 > x_1$



$$\overset{+}{\rightarrow} \sum F = m_1\ddot{x}_1$$

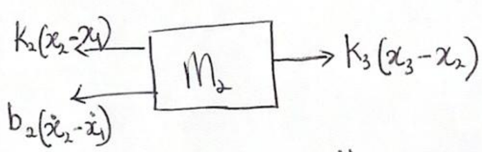
$$m_1\ddot{x}_1 = k_2(x_2-x_1) - k_1x_1 + b_2(\dot{x}_2-\dot{x}_1) - b_1\dot{x}_1 + u(t)$$

since $m_1 = 1\ kg$, $k_2 = k_1 = 1 N/m$ and $b_1 = b_2 = 1\ N\text{-}s/m$

$$\ddot{x}_1 = (x_2-x_1) - x_1 + (\dot{x}_2-\dot{x}_1) - \dot{x}_1 + u(t)$$

$$\ddot{x}_1 = -2x_1 - 2\dot{x}_1 + x_2 + \dot{x}_2 + u(t) \quad \text{--- } (1)$$

Free body diagram for Mass 2 (Sensor): Assuming $x_3 > x_2 > x_1$



$$\overset{+}{\rightarrow} \sum F = m_2\ddot{x}_2$$

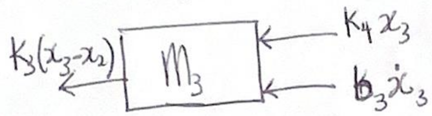
$$m_2\ddot{x}_2 = k_3(x_3-x_2) - k_2(x_2-x_1) - b_2(\dot{x}_2-\dot{x}_1)$$

but $k_3 = k_2 = 1 N/m$, $b_2 = 1 N\text{-}s/m$ and $m_2 = 1 Kg$

$$\ddot{x}_2 = x_3 - x_2 - x_2 + x_1 - \dot{x}_2 + \dot{x}_1$$

$$\ddot{x}_2 = x_1 + \dot{x}_1 - 2x_2 - \dot{x}_2 + x_3 \quad \text{--- } (2)$$

Free body diagram for mass 3 (Environment)



$$\xrightarrow{+} \sum F = m_3\ddot{x}_3$$

$$m_3\ddot{x}_3 = -K_3(x_3-x_2) - K_4 x_3 - b_3\dot{x}_3$$

~~but~~ Substituting the values of K_3, K_4, m_3 and b_3

$$\ddot{x}_3 = -x_3 + x_2 - x_3 - \dot{x}_3$$

$$\ddot{x}_3 = x_2 - 2x_3 - \dot{x}_3 \quad --- \ (3)$$

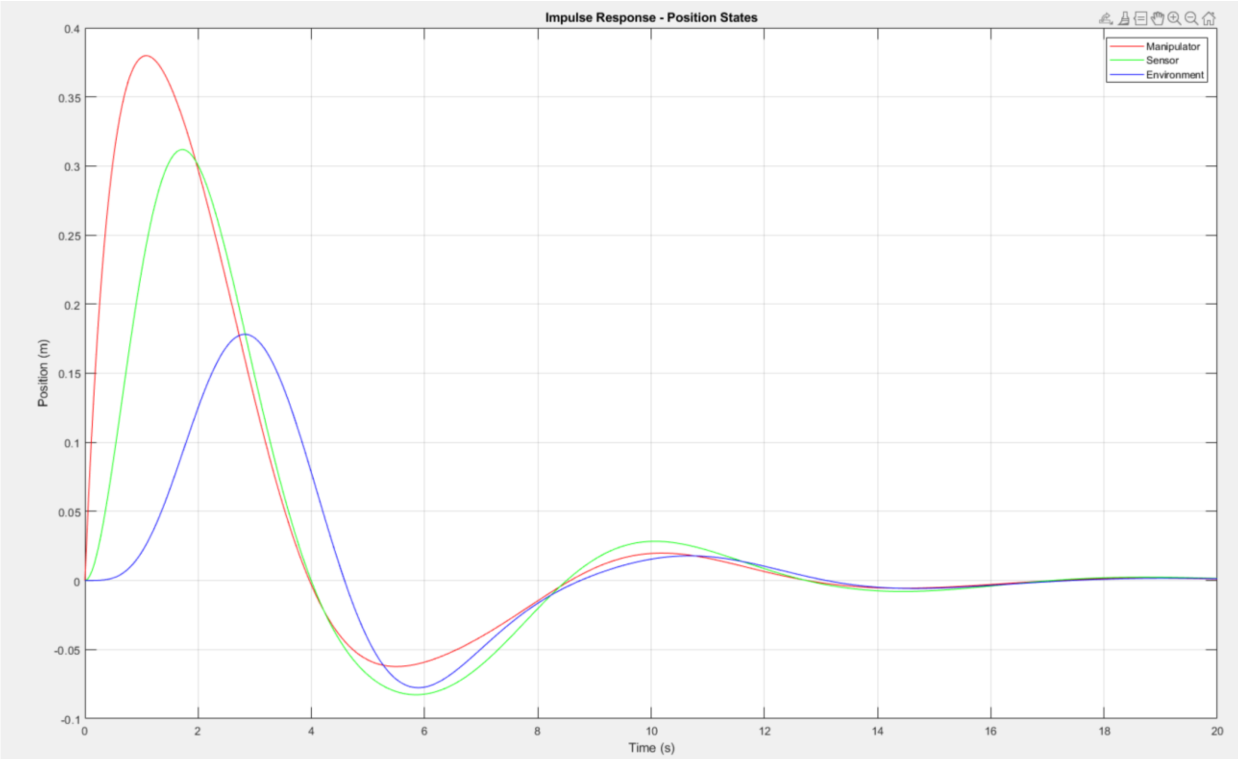
Equations (1), (2) and (3) can be written in the form

$$\dot{x} = Ax + Bu$$

$$\underbrace{\frac{d}{dt}\begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \\ x_3 \\ \dot{x}_3 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & -2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \\ x_3 \\ \dot{x}_3 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{B} \underbrace{u(t)}_{u}$$

Question 1b:

Impulse response of the system:



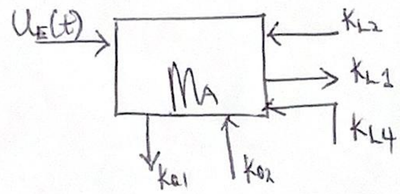
Matlab Code:

```
%% question 1b

% the state-space matrices

A = [0,   1,    0,  0,  0,  0;
    -2, -2,  1,   1,   0,  0;
    0,   0,   0,   1,   0, 0;
    1,    1,   -2, -1, 1,   0;
    0,   0,   0,  0,  0,  1;
    0,    0,   1,   0, -2, -1];

B = [0; 1; 0; 0; 0; 0];
C = eye(6);  % output matrix
D = zeros(6, 1); % feedthrough matrix

% state-space model
sys = ss(A, B, C, D);

% impulse response for 20 seconds
t = 0:0.01:20;
[y, t] = impulse(sys, t);

% Plot position states
figure;
plot(t, y(:, 1), 'r', t, y(:, 3), 'g', t, y(:, 5), 'b');
xlabel('Time (s)');
ylabel('Position (m)');
legend('Manipulator', 'Sensor', 'Environment');
title('Impulse Response - Position States');
grid on;
```

Question 2.

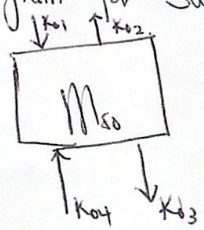
Question 2

Free body diagram for Atmosphere



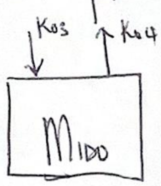
$$\frac{d M_A(t)}{dt} = U_E(t) + K_{L2} M_v(t) + K_{L4} M_s(t) + K_{o2} M_{so}(t) - (K_{o1} + K_{L1}) M_A(t) \quad \cdots (1)$$

Free body diagram for Surface ocean



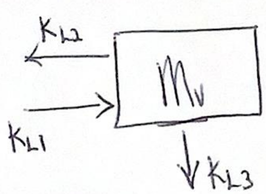
$$\frac{d}{dt} M_{so}(t) = K_{o1} M_A(t) + K_{o4} M_{IDO}(t) - (K_{o2} + K_{o3}) M_{so}(t) \quad \cdots (2)$$

Free body diagram for Intermediate and deep ocean



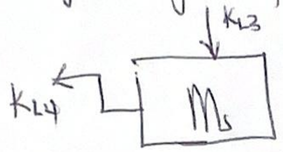
$$\frac{d}{dt} M_{IDO}(t) = K_{o3} M_{so}(t) - K_{o4} M_{IDO}(t) \quad \cdots (3)$$

Free body diagram for Vegetation



$$\frac{d}{dt} M_v(t) = K_{L1} M_A(t) - (K_{L3} + K_{L2}) M_v(t) \quad \cdots (4)$$

Free body diagram for Sod



$$\frac{d}{dt} M_s(t) = K_{L3} M_v(t) - K_{L4} M_s(t) \quad \cdots \quad (5)$$

Eqn (1), (2), (3), (4) and (5) can be written in the form

$$\dot{x} = Ax + Bu$$

$$\frac{d}{dt}\begin{bmatrix} M_A \\ M_{SO} \\ M_{IDD} \\ M_V \\ M_S \end{bmatrix} = \begin{bmatrix} -(K_{01}+K_{L1}) & K_{02} & 0 & K_{L2} & K_{L4} \\ K_{01} & -(K_{02}+K_{03}) & K_{04} & 0 & 0 \\ 0 & K_{03} & -K_{04} & 0 & 0 \\ K_{L1} & 0 & 0 & -(K_{L3}+K_{L2}) & 0 \\ 0 & 0 & K_{L3} & -K_{L4} & 0 \end{bmatrix} \begin{bmatrix} M_A \\ M_{SO} \\ M_{IDD} \\ M_V \\ M_S \end{bmatrix} +$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} U_E(t)$$

Question 3.

Question 3

The time rate of change $\dot{y}_1(t)$ of $y_1(t)$ = inflow/min - Out flow/min

$$\frac{dy_1}{dt} = \frac{F}{V_2} y_2 - \frac{F}{V_1} y_1 \text{ , where } F = 2 \text{ gal/min and } V_2 = V_1 = 100 \text{ gal}$$

$$\dot{y}_1 = \frac{2}{100} y_2 - \frac{2}{100} y_1 = 0.02 y_2 - 0.02 y_1 \text{ for tank 1}$$

for tank 2

$$\dot{y}_2 = \frac{dy_2}{dt} = \frac{F}{V_1} y_1 - \frac{F}{V_2} y_2 = 0.02 y_1 - 0.02 y_2$$

$$\dot{y}_1 = -0.02 y_1 + 0.02 y_2$$
$$\dot{y}_2 = 0.02 y_1 - 0.02 y_2$$

In $\dot{x} = Ax$ form:

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Solving for the eigen values & Vectors of A.

$$\det(A - \lambda I) = \begin{vmatrix} -0.02-\lambda & 0.02 \\ 0.02 & -0.02-\lambda \end{vmatrix} = (-0.02-\lambda)^2 - (0.02)^2$$

$$= \lambda(\lambda + 0.04) = 0$$

eigenvalues, $\lambda_1 = 0$, $\lambda_2 = -0.04$

eigen vectors:

$$[A - \lambda I]x = 0$$

for $\lambda_1 = 0$

$$\begin{bmatrix} -0.02-0 & 0.02 \\ 0.02 & -0.02-0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-0.02x_1 + 0.02x_2 = 0$$
$$0.02x_1 - 0.02x_2 = 0$$
$$x_1 = x_2$$
$$x_1 = x_2$$

Thus the eigen vectors for $\lambda = 0 \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

for $\lambda_2 = -0.04$
eigen vector $= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

for a single eqn: $y = x e^{\lambda t}$ with x = eigenvectors & λ = eigenvalues,
this can be also written as: $y = C_1 x_1 e^{\lambda_1 t} + C_2 x_2 e^{\lambda_2 t}$

$$= C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{0t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t}$$

$$= C \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.04t}$$

Using the initial conditions: $y_1(0) = 0$ (fertilizer in Tank 1) and $y_2(0) = 150$
(fertilizer in Tank 2)

$$y(0) = \begin{bmatrix} C_1 + C_2 \\ C_1 - C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 150 \end{bmatrix}$$

$$C_1 + C_2 = 0 \quad --- \; ①$$
$$C_1 - C_2 = 150 \quad --- \; ②$$

Solving eqn ① & ②
$C_1 = 75$ and $C_2 = -75$

Thus, $y = 75x_1 - 75x_2 e^{-0.04t}$

$$= 75\begin{bmatrix}1\\1\end{bmatrix} - 75\begin{bmatrix}1\\-1\end{bmatrix}e^{-0.04t}$$

for Tank 1; $y_1 = 75 - 75e^{-0.04t}$

for tank 2: $y_2 = 75 + 75e^{-0.04t}$

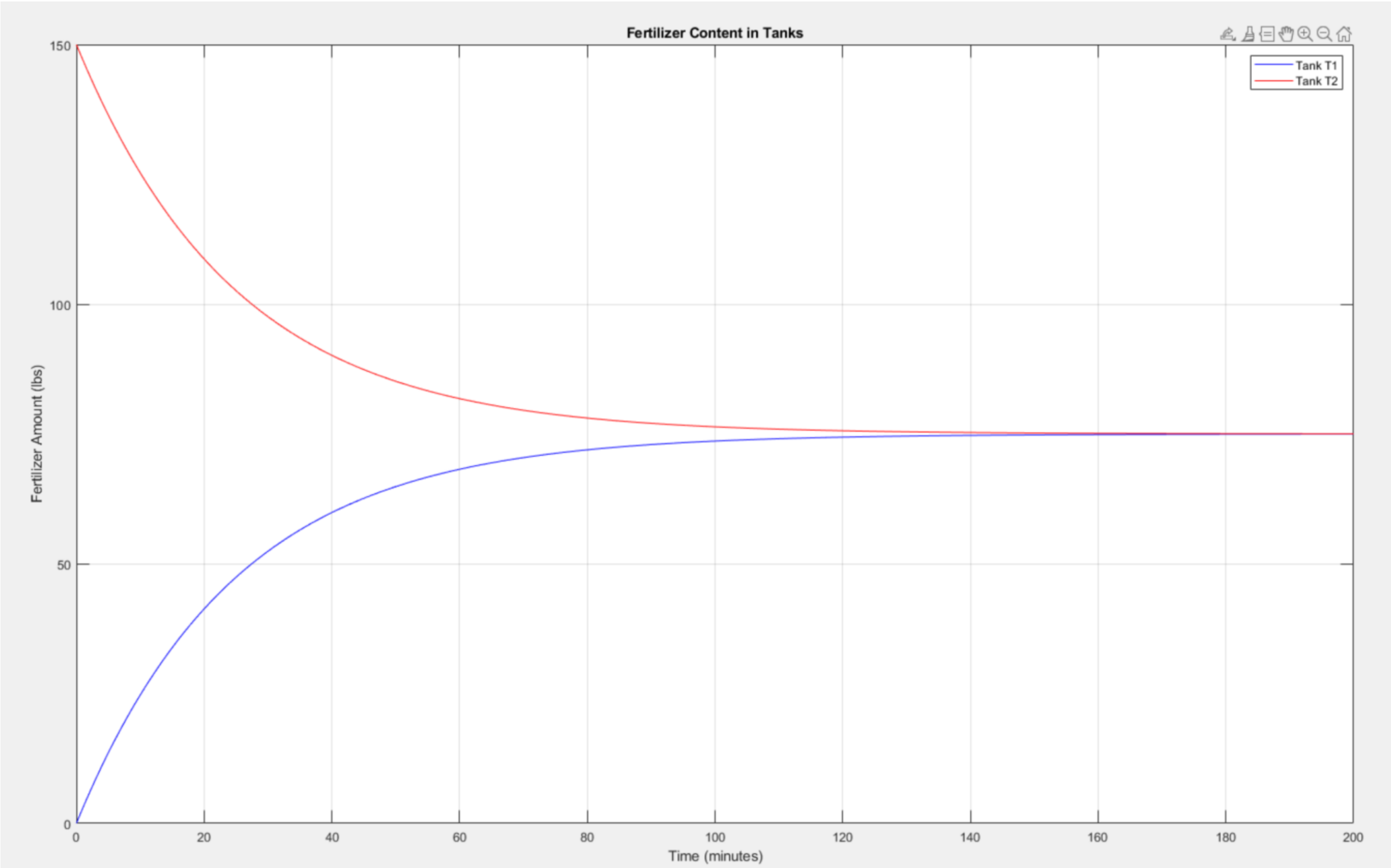
for Tank 1 to contain at least half as much as T_2

$$y_1 = 75 - 75e^{-0.04t} = 50$$

$$e^{-0.04t} = \frac{25}{75} = \frac{1}{3}$$

$$t = \ln(1/3) \,/\, 0.04 = 27.5 \text{ minutes}$$

Fertilizer content in each tank as a function of time:



Matlab Code:

```
%% question 3

V1 = 100;
V2 = 100;
F = 2;    % gal/min
y1_0 = 0;
y2_0 = 150;

% time
t = linspace(0, 200, 1000); % Time from 0 to 200 minutes

% fertilizer amounts
y1 = 75 * (1 - exp(-0.04 * t));
y2 = 75 * (1 + exp(-0.04 * t));

% Plot
plot(t, y1, 'b', t, y2, 'r');
xlabel('Time (minutes)');
ylabel('Fertilizer Amount (lbs)');
legend('Tank T1', 'Tank T2');
title('Fertilizer Content in Tanks');
grid on;
```