Roger Wienaah

ME531 Problem Set 2 Solutions

I wrote on a plain white A4 sheet using a black pen. After scanning the document and inserting it into Word, I noticed the text was unclear. To improve visibility, I added a color overlay to my writing. I apologize for any inconvenience this may cause.

Roger Wienaah

ME 531 : Problem Set 2

Question 1a :

f_c = flow of cold water, f_H = flow of hot water ; $\overline{T_H}$ = Temperature in hot compartment, $\overline{T_c}$ = Temperature in cold compartment ; $\overline{T_{Hi}}$ = Temperature of hot inflow, $\overline{T_{ci}}$ = Temperature of the cold inflow ; V_H = Volume of hot water, V_c = Volume of cold water.

Temperatures evolve according to :

$$V_c \frac{d\overline{T_c}}{dt} = f_c(\overline{T_{ci}} - \overline{T_c}) + \beta(\overline{T_H} - \overline{T_c}) \quad \cdots (1)$$

$$V_H \frac{d\overline{T_H}}{dt} = f_H(\overline{T_{Hi}} - \overline{T_H}) - \beta(\overline{T_H} - \overline{T_c}) \quad \cdots (2)$$

Inputs to the system, $u_1 = \overline{T_{ci}}$, $u_2 = \overline{T_{Hi}}$; Output, $y_1 = \overline{T_c}$ and $y_2 = \overline{T_H}$

$f_c = f_H = 0.1\ (m^3/min)$, $\beta = 0.2\ (m^3/min)$ and $V_H = V_c = 1\ (m^3)$

Substituting the above into eqn (1) and (2) :

$$1\frac{dy_1}{dt} = 0.1(u_1 - y_1) + 0.2(y_2 - y_1)$$

$$1\frac{dy_2}{dt} = 0.1(u_2 - y_2) - 0.2(y_2 - y_1)$$

$$\dot{y}_1 = 0.1u_1 - 0.3y_1 + 0.2y_2$$

$$\dot{y}_2 = 0.1u_2 + 0.2y_1 - 0.3y_2$$

for zero inputs ; $u_1 = u_2 = 0$

$$\dot{y}_1 = -0.3y_1 + 0.2y_2$$

$$\dot{y}_2 = 0.2y_1 - 0.3y_2$$

In matrix form : $\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$; $\dot{x} = Ax$ or $\dot{y} = Ay$

$$A = \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix}$$

Eigenvalues of A : $\det(A - \lambda I) = 0$

$$\begin{vmatrix} (-0.3-\lambda) & 0.2 \\ 0.2 & (-0.3-\lambda) \end{vmatrix} = 0$$

$$(-0.3-\lambda)^2 - (0.2)^2 = 0$$

$$\lambda^2 + 0.6\lambda + 0.09 - 0.04 = \lambda^2 + 0.6\lambda + 0.05 = 0$$

$$\lambda = \frac{-0.6 \pm \sqrt{(0.6)^2 - 4(0.05)}}{2} = \frac{-0.6 \pm 0.4}{2}$$

$$\lambda_1 = -0.1 \text{ and } \lambda_2 = -0.5$$

Eigenvectors for each eigenvalue of A : $(A - \lambda I)v = 0$

$\lambda_1 = -0.1$: $\begin{bmatrix} -0.3-(-0.1) & 0.2 \\ 0.2 & -0.3+0.1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -0.2 & 0.2 \\ 0.2 & -0.2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-0.2v_1 + 0.2v_2 = 0$$

$v_1 = v_2 \Rightarrow$ Eigenvector for $\lambda = -0.1$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = -0.5$: $\begin{bmatrix} -0.3+0.5 & 0.2 \\ 0.2 & -0.3+0.5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 0.2 & 0.2 \\ 0.2 & 0.2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$v_1 = -v_2 \Rightarrow$ Eigenvector for $\lambda_2 = -0.5$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The unforced solution :

$$y(t) = T e^{Dt} T^{-1} y(0)$$

$$T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}, \quad D = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.5 \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} e^{\begin{bmatrix} -0.1 & 0 \\ 0 & -0.5 \end{bmatrix} t} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} y(0)$$

$$y(t) = \cancel{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0.5 \\ \\ \end{bmatrix}}$$

$$y(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-0.1t} & 0 \\ 0 & e^{-0.5t} \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} y(0)$$

$$y(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0.5e^{-0.1t} & 0.5e^{-0.1t} \\ 0.5e^{-0.5t} & -0.5e^{-0.5t} \end{bmatrix} \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}$$

$$y(t) = \frac{1}{2} \begin{bmatrix} e^{-0.1t} + e^{-0.5t} & e^{-0.1t} - e^{-0.5t} \\ e^{-0.1t} - e^{-0.5t} & e^{-0.1t} + e^{-0.5t} \end{bmatrix} \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} y_1(0)\left(e^{-0.1t} + e^{-0.5t}\right) + y_2(0)\left(e^{-0.1t} - e^{-0.5t}\right) \\ y_1(0)\left(e^{-0.1t} - e^{-0.5t}\right) + y_2(0)\left(e^{-0.1t} + e^{-0.5t}\right) \end{bmatrix}$$

$$y_1(t) = \tfrac{1}{2}\left(y_1(0)\left(e^{-0.1t} + e^{-0.5t}\right) + y_2(0)\left(e^{-0.1t} - e^{-0.5t}\right)\right)$$

$$y_2(t) = \tfrac{1}{2}\left(y_1(0)\left(e^{-0.1t} - e^{-0.5t}\right) + y_2(0)\left(e^{-0.1t} + e^{-0.5t}\right)\right)$$

The above represent the eqns to determine $y_1(t)$ and $y_2(t)$, i.e the solution to eqn ① and ②.

//

Question 1b:

Yes, the system is stable. This is because both the eigenvalues of A are real negative numbers. $\lambda_1 = -0.1$ and $\lambda_2 = -0.5$

//

Question 2a:

Van der Pol oscillator: $\ddot{y} - \mu(1-y^2)\dot{y} + y = 0$ -- ①

let $x = y$

$\dot{x} = \dot{y}$

let $v = \dot{y} = \dot{x}$; $\dot{x} = v$ -- ⓐ

$\dot{v} = \ddot{y}$

The eqn ① becomes: $\dot{v} - \mu(1-x^2)v + x = 0$

$\dot{v} = -x + \mu(1-x^2)v$ ... ⓑ

Eqn ⓐ and ⓑ in matrix form:

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & \mu(1-x^2) \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

Thus eqn ① as a system of 1st order differential equations =

$$\frac{d}{dt}\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & \mu(1-y^2) \end{bmatrix}\begin{bmatrix} y \\ \dot{y} \end{bmatrix}$$

Question 2b:

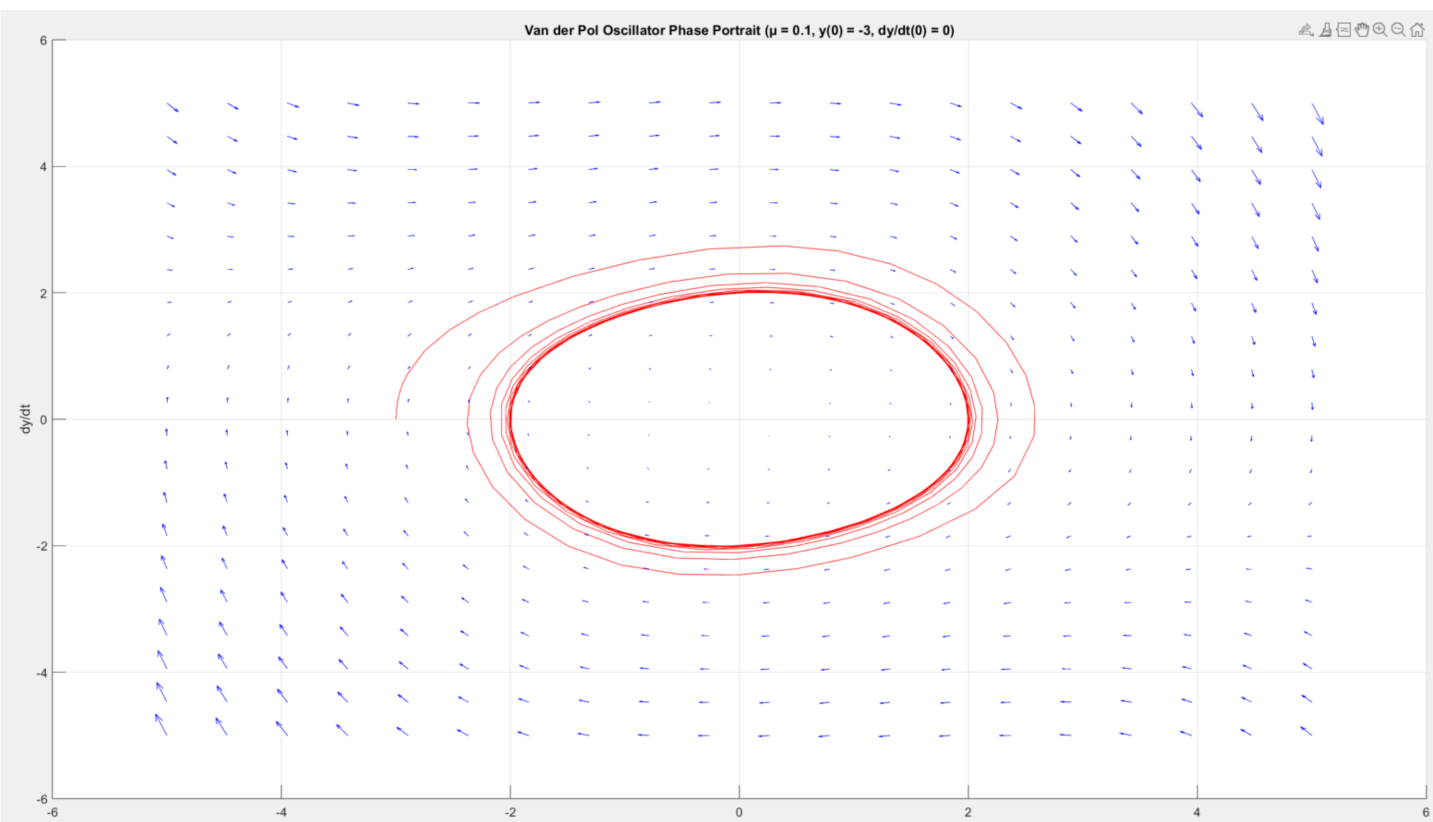
Matlab solution below

Question 2c:

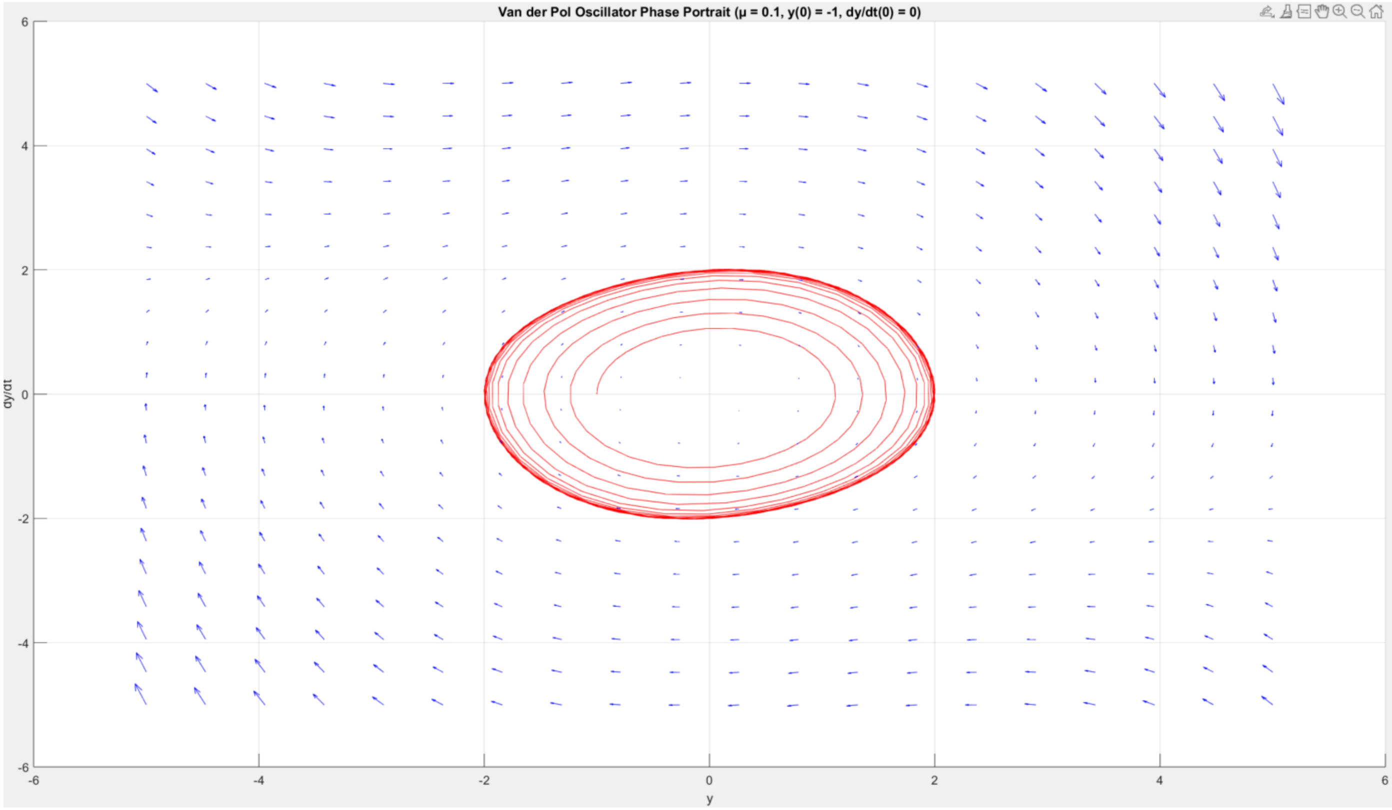
Matlab solution below

Question 2B:

First initial condition

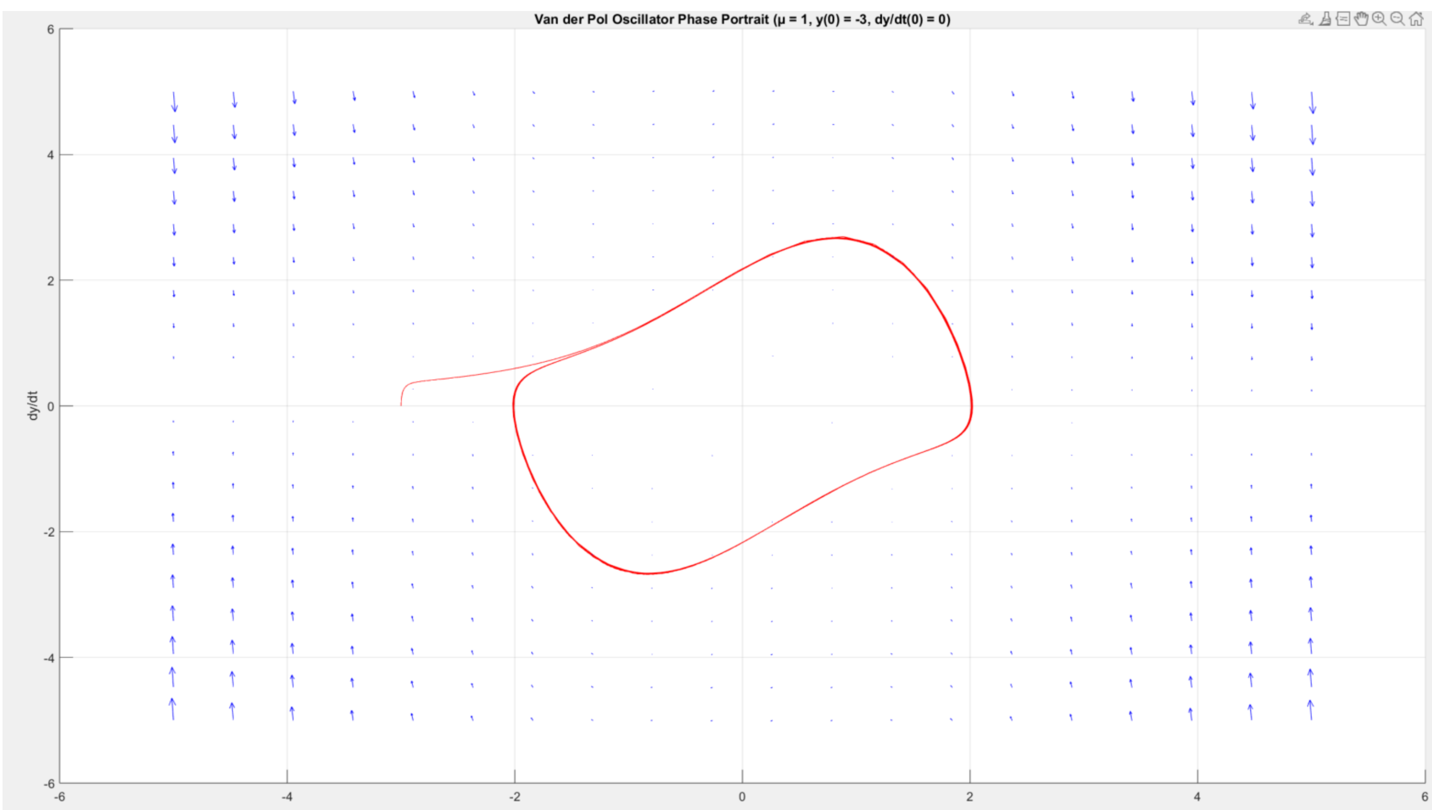


Second initial condition

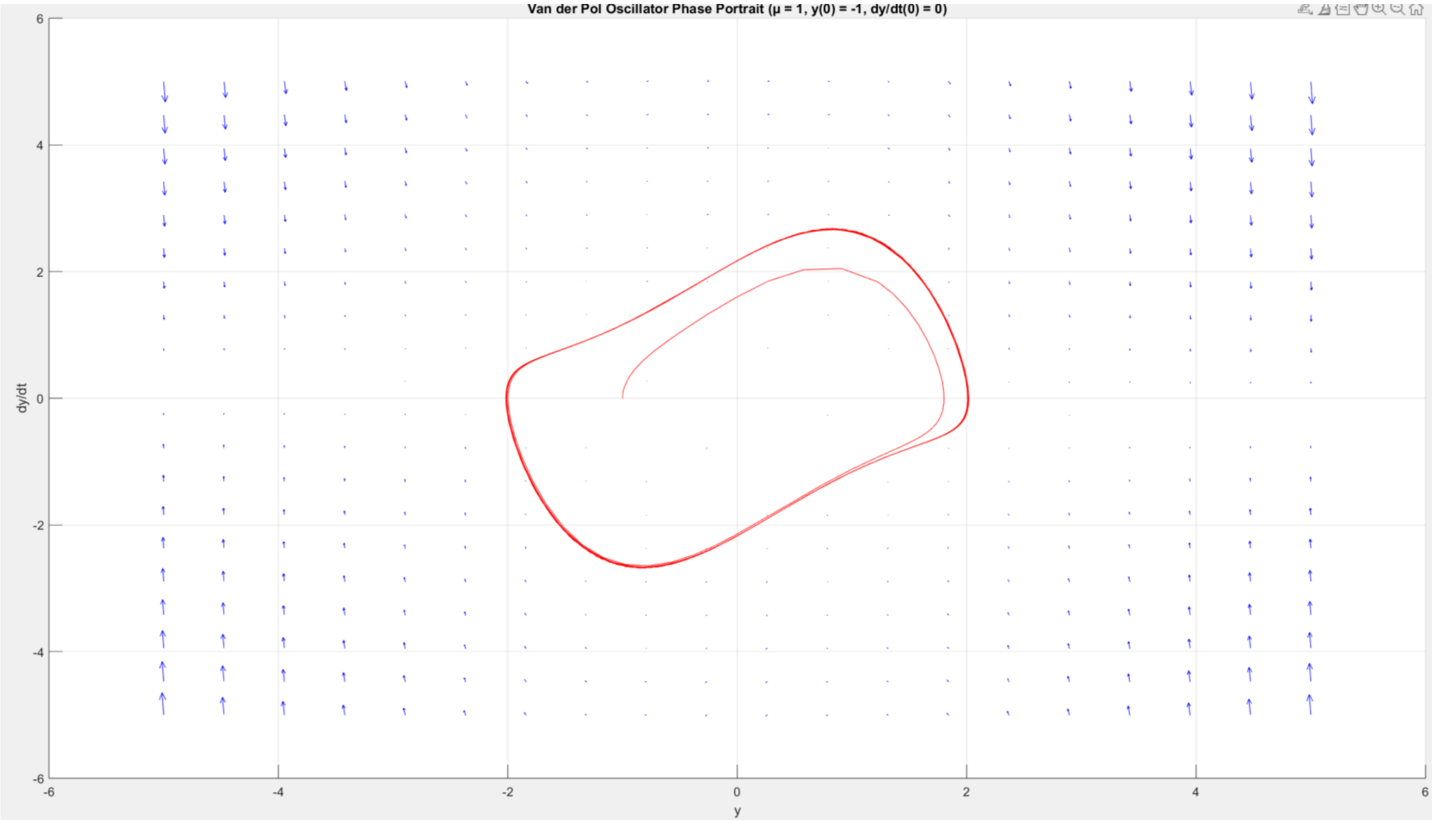


Question 2C:

First initial condition



Second initial condition



Below is the MATLAB code for the above plots:

```matlab
%% ---------------Question 2 ------------------------------------------

function dzdt = van_der_pol(t, z, mu)
  % fxn for the Van der Pol oscillator equation.
  % z = [x; v], where x = y and v = dy/dt
  dzdt = [z(2); mu * (1 - z(1)^2) * z(2) - z(1)];
end

t_span = [0, 100];

% vector field
[X, V] = meshgrid(linspace(-5, 5, 20), linspace(-5, 5, 20));

% --- Part (b): mu = 0.1 ---------------------------------
mu_b = 0.1;
initial_conditions_b = [[-3; 0], [-1; 0]];  % [[x0; v0], [x1; v1]]

% solution with ode45 and plots
for i = 1:size(initial_conditions_b, 2)
  z0 = initial_conditions_b(:, i);
  [t, z_b] = ode45(@(t, z) van_der_pol(t, z, mu_b), t_span, z0);

  figure;
  title(sprintf('Van der Pol Oscillator Phase Portrait (μ = 0.1, y(0) = %g, dy/dt(0) = %g)', z0(1), z0(2)));
  xlabel('y');
  ylabel('dy/dt');
  hold on;

  plot(z_b(:, 1), z_b(:, 2), 'r');

  dX = V;
  dV = mu_b * (1 - X.^2) .* V - X;
  quiver(X, V, dX, dV, 'b', 'AutoScaleFactor', 0.5);

  hold off;
  grid on;
end
```

```matlab
%--- Part (2c): mu = 1 ---
mu_c = 1;
initial_conditions_c = [[-3; 0], [-1; 0]];

% solution with ode45 and plots
for i = 1:size(initial_conditions_c, 2)
  z0 = initial_conditions_c(:, i);
  [t, z_c] = ode45(@(t, z) van_der_pol(t, z, mu_c), t_span, z0);

  figure;
  title(sprintf('Van der Pol Oscillator Phase Portrait (μ = 1, y(0) = %g, dy/dt(0) = %g)', z0(1), z0(2)));
  xlabel('y');
  ylabel('dy/dt');
  hold on;

  plot(z_c(:, 1), z_c(:, 2), 'r');

  dX = V;
  dV = mu_c * (1 - X.^2) .* V - X;
  quiver(X, V, dX, dV, 'b', 'AutoScaleFactor', 0.5);

  hold off;
  grid on;
end
```

Question 3a :

The nonlinear system :

$$\dot{x} = 2x - 2x^2 - xy$$
$$\dot{y} = 2y - xy - 2y^2$$

Jacobian linearization at fixed point $(2/3, 2/3)$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2x - 2x^2 - xy \\ 2y - xy - 2y^2 \end{bmatrix}$$

$$\dot{x} = f(x)$$
$$\dot{x} = \frac{Df}{Dx}(\bar{x}) \cdot \Delta x$$

Jacobian, $\frac{Df}{Dx}(\bar{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2-4x-y & -x \\ -y & 2-x-4y \end{bmatrix}$

Jacobian at fixed point $(2/3, 2/3)$:

$$= \begin{bmatrix} (2 - 4(2/3) - 2/3) & -2/3 \\ -2/3 & (2 - 2/3 - 4(2/3)) \end{bmatrix}$$

$$= \begin{bmatrix} -4/3 & -2/3 \\ -2/3 & -4/3 \end{bmatrix}$$

The linearized system is ; $\dot{x} = \frac{Df}{Dx}(\bar{x}) \cdot \Delta x$ where $\Delta x = x - \bar{x}$

Thus, $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -4/3 & -2/3 \\ -2/3 & -4/3 \end{bmatrix} \begin{bmatrix} x - 2/3 \\ y - 2/3 \end{bmatrix}$ //

Question 36:

The eigenvalues of the Jacobian Matrix $\begin{bmatrix} -4/3 & -2/3 \\ -2/3 & -4/3 \end{bmatrix}$ =

$$\begin{vmatrix} -4/3-\lambda & -2/3 \\ -2/3 & -4/3-\lambda \end{vmatrix} = 0$$

$$(-4/3-\lambda)^2 - (-2/3)^2 = 0$$

$$(\lambda + 4/3)^2 - 4/9 = 0$$

$$\lambda^2 + 8/3\lambda + 16/9 - 4/9 = 0$$

$$\lambda^2 + 8/3\lambda + 12/9 = 0$$

$$\lambda^2 + 8/3\lambda + 4/3 = 0$$

$$\lambda = \frac{-(8/3) \pm \sqrt{(8/3)^2 - 4(4/3)}}{2}$$

$$\lambda = \frac{-8/3 \pm \sqrt{16/9}}{2} = \frac{-8/3 \pm 4/3}{2}$$

$$\lambda_1 = \frac{-8/3 + 4/3}{2} = \frac{-4/3}{2} = -2/3$$

$$\lambda_2 = \frac{-8/3 - 4/3}{2} = \frac{-12/3}{2} = -2$$

The linear system is stable since all the eigenvalues ($\lambda_1 = -2/3$, $\lambda_2 = -2$) of the Jacobian matrix are real and negative.

Therefore the linearized model is stable at the fixed point $(2/3, 2/3)$.

Question 4 :

$$\dot{x} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} u(t)$$

$$\dot{x} = Ax + Bu(t)$$

The Controllability matrix, $C = [B \vdots AB]$

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$$

$$C = \left[\begin{matrix} K_1 \\ K_2 \end{matrix} \vdots \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \right]$$

$$C = \begin{bmatrix} K_1 & K_1 - K_2 \\ K_2 & -K_1 + K_2 \end{bmatrix}$$

The system is completely controllable if C has full rank, ~~the~~ and since C is 2x2 matrix, the rank of C = order of C if

$$\det(C) \neq 0$$

that is;

$$\begin{vmatrix} K_1 & K_1 - K_2 \\ K_2 & -K_1 + K_2 \end{vmatrix} \neq 0$$

$$K_1(-K_1 + K_2) - K_2(K_1 - K_2) \neq 0$$

$$-K_1^2 + \cancel{K_1 K_2} - \cancel{K_1} K_2 + K_2^2 \neq 0$$

$$-K_1^2 + K_2^2 \neq 0$$

$$K_2^2 - K_1^2 \neq 0$$

$$(K_2 - K_1)(K_2 + K_1) \neq 0$$

- $K_2 - K_1 \neq 0$, $K_2 \neq K_1$
- $K_2 + K_1 \neq 0$, $K_2 \neq -K_1$

The system is completely controllable for all values of k_1 and k_2 except when $k_1 = k_2$ or $k_1 = -k_2$

or $k_1 \neq \pm k_2$ for the system to be completely controllable.

Question 5a:

System: $\dot{x} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$

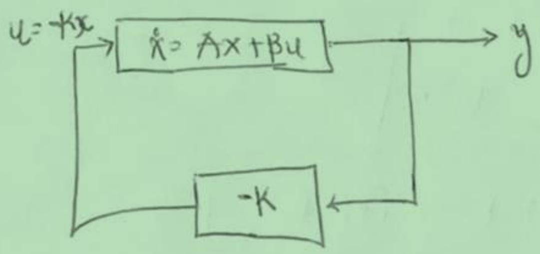
$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\lambda_1 = -5$, $\lambda_2 = -6$

Characteristic equation with desired eigenvalues:

$$(c - \lambda_1)(c - \lambda_2) = 0$$

$$(c+5)(c+6) = 0$$

$$c^2 + 11c + 30 = 0 \quad \dots \text{①}$$



$\dot{x} = Ax + Bu$ but $u = -kx$

$\dot{x} = Ax + B(-kx)$

$\dot{x} = (A - Bk)x$

let $K = [k_1 \quad k_2]$ and $(A - BK)$ = Closed loop system matrix

The closed loop system matrix $= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} [k_1 \quad k_2]$

$$= \begin{bmatrix} 1-k_1 & 1-k_2 \\ 1 & 2 \end{bmatrix}$$

The characteristic eqn of the closed loop system matrix

$$= \det\left(c\bar{I} - (A - BK)\right) = 0$$

$$= \left| \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} - \begin{bmatrix} 1-K_1 & 1-K_2 \\ 1 & 2 \end{bmatrix} \right| = 0$$

$$= \begin{vmatrix} c-(1-K_1) & -(1-K_2) \\ -1 & c-2 \end{vmatrix} = 0$$

$$= \left(c-(1-K_1)\right)(c-2) + 1\left(-(1-K_2)\right) = 0$$

$$= (c-1+K_1)(c-2) - (1-K_2) = 0$$

$$= c^2 + (K_1-3)c + (1-2K_1+K_2) = 0 \quad \cdots (2)$$

Comparing the coefficients of eqn (1) and (2)

$$K_1 - 3 = 11 \quad \Rightarrow \quad K_1 = 14$$

$$1 - 2K_1 + K_2 = 30 \quad \text{but } K_1 = 14$$

$$1 - 2(14) + K_2 = 30$$

$$-27 + K_2 = 30$$

$$K_2 = 30 + 27 = 57$$

$$\therefore K = [K_1 \;\; K_2] = [14 \;\; 57]$$

The state feedback gain matrix that places the eigenvalues at -5 and -6 is $K = [14 \;\; 57]$

Question 5b:

Matlab solution below

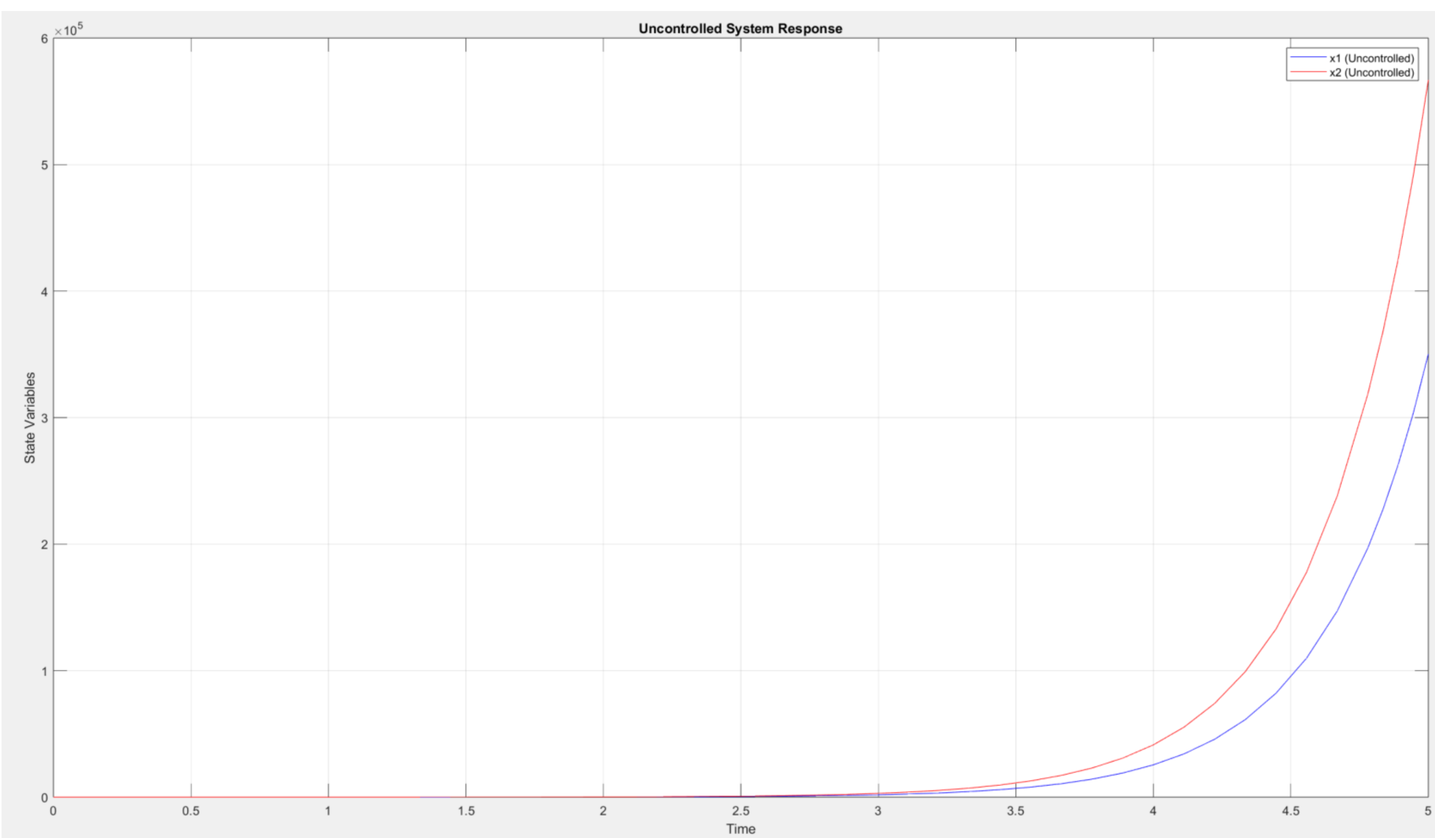
Question 5c:

Matlab solution below

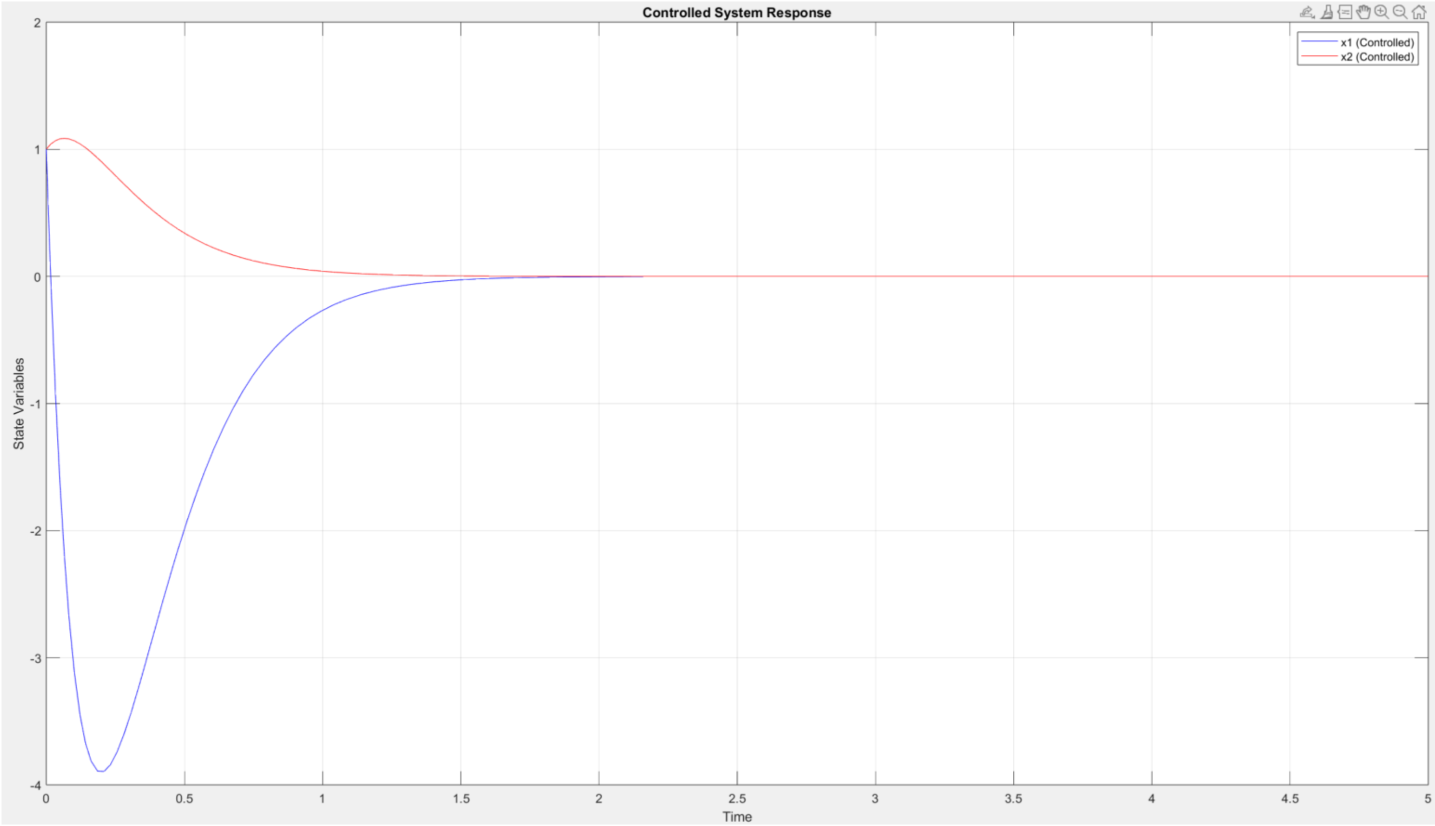
Question 5d:

Matlab solution below

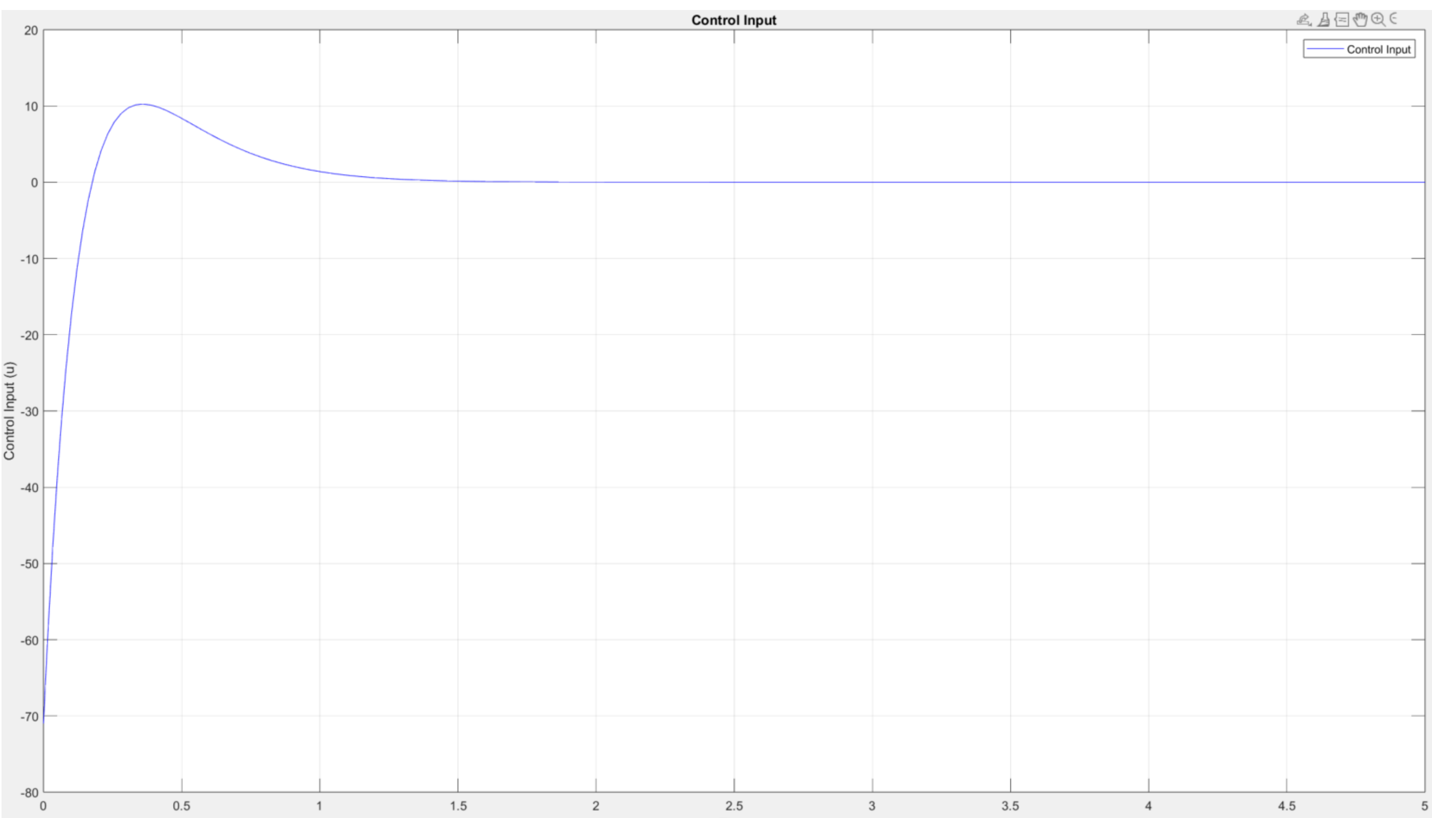
Question 5b:



Question 5c



Question 5d:



MATLAB Code for Question 5:

```
%% Question 5
% --- 5(a):State Feedback--------------
A = [1 1; 1 2];
B = [1; 0];
desired_poles = [-5; -6];
%
K = place(A, B, desired_poles);
%
% disp('State Feedback Gain K:');
% disp(K);

% --- 5(b): Uncontrolled System Response ---
%A = [1 1; 1 2];
x0 = [1; 1];
t_span = [0, 5];

[t_uncontrolled, x_uncontrolled] = ode45(@(t, x) A * x, t_span, x0);

figure;
plot(t_uncontrolled, x_uncontrolled(:, 1), 'b', 'DisplayName', 'x1 (Uncontrolled)');
hold on;
plot(t_uncontrolled, x_uncontrolled(:, 2), 'r', 'DisplayName', 'x2 (Uncontrolled)');
hold off;
xlabel('Time');
ylabel('State Variables');
title('Uncontrolled System Response');
legend;
grid on;
```

```
% --- 5(c): Controlled System Response ---
A_cl = A - B * K;
x0 = [1; 1];
t_span = [0, 5];

[t_controlled, x_controlled] = ode45(@(t, x) A_cl * x, t_span, x0);

figure;
plot(t_controlled, x_controlled(:, 1), 'b', 'DisplayName', 'x1 (Controlled)');
hold on;
plot(t_controlled, x_controlled(:, 2), 'r', 'DisplayName', 'x2 (Controlled)');
hold off;
xlabel('Time');
ylabel('State Variables');
title('Controlled System Response');
legend;
grid on;

% --- 5(d): Control Input ---
u_controlled = -K * x_controlled';
u_controlled = u_controlled';

figure;
plot(t_controlled, u_controlled, 'b', 'DisplayName', 'Control Input');
xlabel('Time');
ylabel('Control Input (u)');
title('Control Input');
legend;
grid on;
```