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ME531 Pset 3 Solution:

Question 1:

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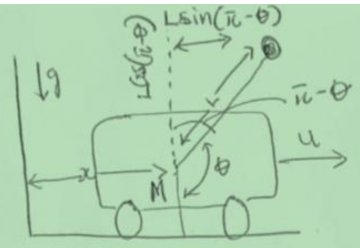


Equations of motion of the cart pole → Adapted from Lecture slide 7

$$P_c = \begin{bmatrix} x \\ 0 \end{bmatrix} \qquad P_m = \begin{bmatrix} x + L\sin\theta \\ -L\cos\theta \end{bmatrix}$$

Total kinetic energy

$$T = \tfrac{1}{2} M \dot{x}^2 + \tfrac{1}{2} m \dot{x}^2 + mL\dot{x}\dot{\theta}\cos\theta + \tfrac{1}{2} mL^2\dot{\theta}^2$$

Potential energy of the system:

$$U = mgL(1-\cos\theta)$$

The lagrangian $\nabla = T - U$

$$\nabla = \tfrac{1}{2}(M+m)\dot{x}^2 + mL\dot{x}\dot{\theta}\cos\theta + \tfrac{1}{2} mL^2\dot{\theta}^2 - mgL(1-\cos\theta)$$

$$\frac{\partial \nabla}{\partial x} = 0 \qquad \frac{\partial \nabla}{\partial \theta} = -mL\dot{x}\dot{\theta}\sin\theta - mgL\sin\theta$$

$$\frac{\partial \nabla}{\partial \dot{x}} = (M+m)\dot{x} + mL\dot{\theta}\cos\theta \qquad \frac{\partial \nabla}{\partial \dot{\theta}} = mL\dot{x}\cos\theta + mL^2\dot{\theta}$$

Damping, $D = \tfrac{1}{2} d\dot{x}^2$

Eqns of motions (1): $\dfrac{d}{dt}\left(\dfrac{\partial \nabla}{\partial \dot{x}}\right) - \dfrac{\partial \nabla}{\partial x} + \dfrac{\partial D}{\partial \dot{x}} = u(t)$

$$\Rightarrow (M+m)\ddot{x} + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta + d\dot{x} = u(t) \qquad \cdots (1)$$

and Eqn of motion: $\dfrac{d}{dt}\left(\dfrac{\partial \nabla}{\partial \dot{\theta}}\right) - \dfrac{\partial \nabla}{\partial \theta} + \dfrac{\partial D}{\partial \dot{\theta}} = 0$

$$\Rightarrow mL\ddot{x}\cos\theta + mL^2\ddot{\theta} + mgL\sin\theta = 0 \qquad \cdots (2)$$

in "Manipulator equations" form:

$$\underbrace{\begin{bmatrix} M+m & mL\cos\theta \\ mL\cos\theta & mL^2 \end{bmatrix}}_{M(q)} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \underbrace{\begin{bmatrix} -mL\dot{\theta}^2\sin\theta + d\dot{x} \\ 0 \end{bmatrix}}_{c(q,\dot{q})} + \underbrace{\begin{bmatrix} 0 \\ mgL\sin\theta \end{bmatrix}}_{g(q)} = \underbrace{\begin{bmatrix} u(t) \\ 0 \end{bmatrix}}_{\tau}$$

Nonlinear dynamics of the cart pole:

$$\ddot{x} = \frac{-m^2L^2g\cos\theta\sin\theta + mL^2(mL\dot{\theta}^2\sin\theta - d\dot{x}) + mL^2u}{mL^2(M + m(1-\cos\theta^2))}$$

$$\ddot{\theta} = \frac{(m+M)mgL\sin\theta - mL\cos\theta(mL\dot{\theta}^2\sin\theta - d\dot{x}) - mL\cos\theta\, u}{mL^2(M+m(1-\cos\theta^2))}$$

Linearizing the above at $\theta = \pi$

let $z_1 = x$, $z_2 = \dot{x}$, $z_3 = \theta$ and $z_4 = \dot{\theta}$

$$\dot{z} = f(z, u)$$

$f_1 = \dot{z}_1 = \dot{x}$; $\frac{\partial f_1}{\partial x} = 0$, $\frac{\partial f_1}{\partial \dot{x}} = 1$ $\frac{\partial f_1}{\partial \theta} = 0$ $\frac{\partial f_1}{\partial \dot{\theta}} = 0$

$f_2 = \dot{z}_2 = \ddot{x}$; $\frac{\partial f_2}{\partial x} = 0$, $\left.\frac{\partial f_2}{\partial \dot{x}}\right|_{\theta=\pi} = \frac{-mL^2 d}{mL^2(M+m(1-1))} = \frac{-d}{M}$, $\left.\frac{\partial f_2}{\partial \theta}\right|_{\theta=\pi} = \frac{mg}{M}$

$\frac{\partial f_2}{\partial \dot{\theta}} = 0$

$f_3 = \dot{z}_3 = \dot{\theta}$; $\frac{\partial f_3}{\partial x} = 0$, $\cancel{\frac{\partial f_3}{\partial \dot{x}} = -d/mL}$, $\cancel{\frac{\partial f_3}{\partial x}}$ $\frac{\partial f_3}{\partial \dot{x}} = 0$, $\frac{\partial f_3}{\partial \theta} = 0$

$\frac{\partial f_3}{\partial \dot{\theta}} = 1$

$f_4 = \dot{z}_4 = \ddot{\theta}$; $\frac{\partial f_4}{\partial x} = 0$, $\frac{\partial f_4}{\partial \dot{x}} = -d/mL$, $\frac{\partial f_4}{\partial \theta} = \frac{-d(m+M)g}{mL}$, $\frac{\partial f_4}{\partial \dot{\theta}} = 0$

The Linearized dynamics Matrices :

d = damping coefficient = b = 1 N-s/m

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -d/M & mg/M & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-d}{ML} & -\frac{d(m+M)g}{ML} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1/M \\ 0 \\ d/ML \end{bmatrix}$$

LQR Controller Design

Initial Conditions :
$$\begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ \pi + 0.1 \\ 0 \end{bmatrix}$$

Reference position:
$$\begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \pi \\ 0 \end{bmatrix}$$

Cost function to minimize: $J = \int [x^T Q x + u^T R u] \, dt$

Q = positive semi-definite matrix (weights the state deviations)

R = positive definite matrix (weights the control effort)

Control law, $u = -Kx$

$$K = R^{-1} B^T S$$

with S being the solution to below eqn

$$0 = SA + A^T S - S B R^{-1} B^T S + Q$$

The MATLAB Code Contains my chosen weights $\rightarrow$ Q and R and the non linear dynamics for the cart pole.

MATLAB Code for Question 1:

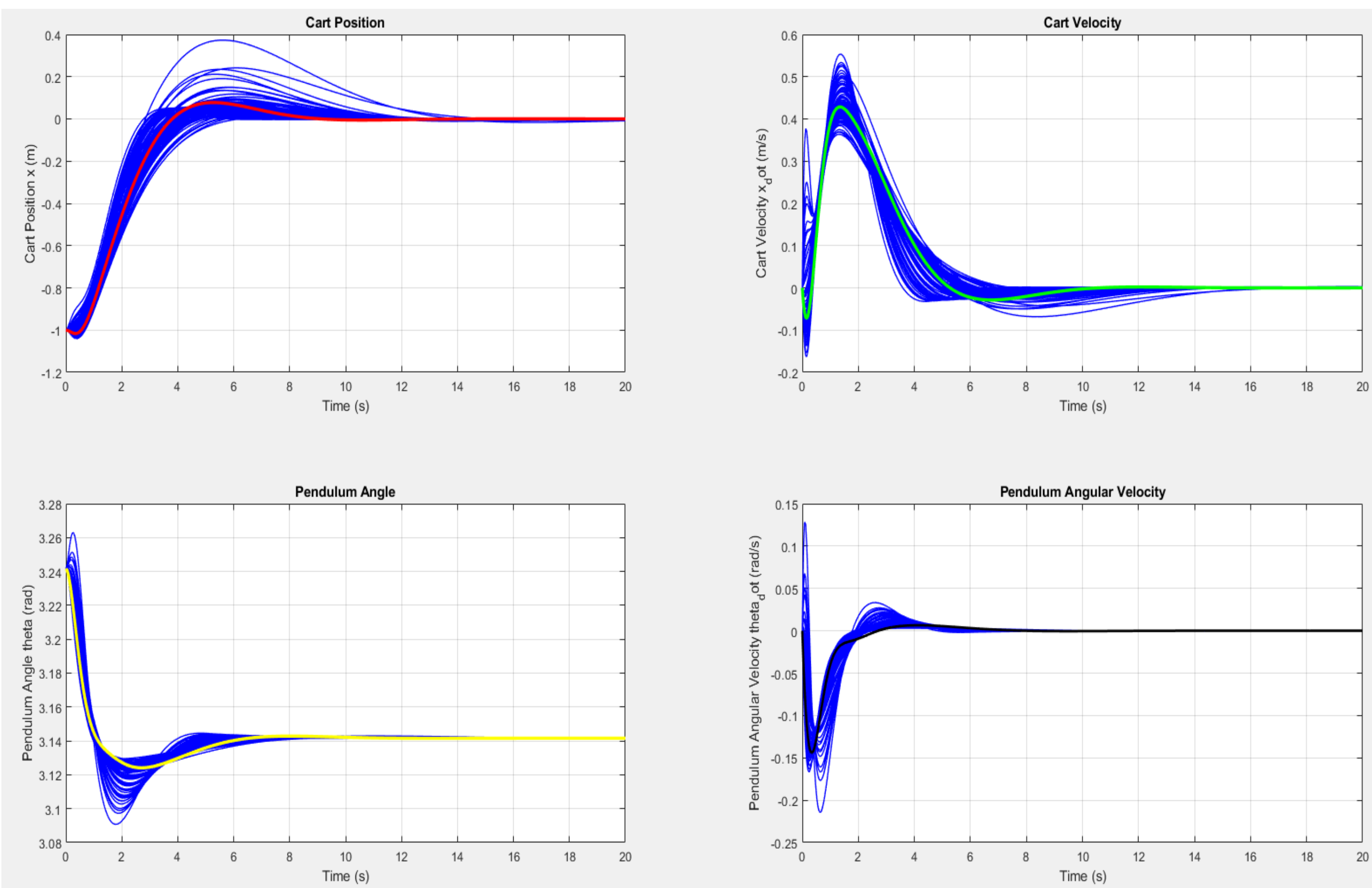
```
% Cart-Pole LQR Simulation - Question 1 of Pset3

% System Parameters
M = 5;
m = 1;
L = 2;
b = 1;
g = 9.8;


% Linearized State-Space Matrices
A = [0   1   0       0;
     0   -b/M b*m*g/M   0;
     0   0   0       1;
     0   -b/(M*L)  -b*g*(M+m)/(M*L)   0];

B = [0;
     1/M;
     0;
     1/(M*L)];


% LQR Controller Design
Q = diag([1, 1, 200, 1]);   % Weights on [x, x_dot, theta, theta_dot]
R = 0.01;                       % Weight on control effort


% LQR gain
[K, S, E] = lqr(A, B, Q, R);



% Initial Conditions
x0 = -1;
x_dot0 = 0;
theta0 = pi + 0.1;
theta_dot0 = 0;
state0 = [x0; x_dot0; theta0; theta_dot0];

% Sim time
tspan = linspace(0, 20, 500);

% Nonlinear System Dynamics
odefun_nonlinear = @(t, x) cart_pole_dynamics(x, M, m, L, b, g, K);

% Simulate system with LQR
[t_lqr, x_lqr] = ode45(odefun_nonlinear, tspan, state0);

% 100 Random Stable Eigenvalues
num_random = 100;
all_x = zeros(length(tspan), num_random + 1); % Store x for all sims
all_x_dot = zeros(length(tspan), num_random + 1);
all_theta = zeros(length(tspan), num_random + 1);
all_theta_dot = zeros(length(tspan), num_random + 1);

% keep LQR response
all_x(:, 1) = x_lqr(:, 1);
all_x_dot(:, 1) = x_lqr(:, 2);
all_theta(:, 1) = x_lqr(:, 3);
all_theta_dot(:, 1) = x_lqr(:, 4);
```

```matlab
for i = 1:num_random
    % Gen. random stable eigenvalues
    eigenvalues = -3.5 + (3.5 - 0.5) * rand(4, 1);  %  [-3.5, -0.5]
    eigenvalues = real(eigenvalues); % Ensure real eigenvalues

    % K from eigenvalues
    K_random = place(A, B, eigenvalues);

    % Simulate nonlinear system with random K
    odefun_random = @(t, x) cart_pole_dynamics(x, M, m, L, b, g, K_random);
    [~, x_random] = ode45(odefun_random, tspan, state0);

    all_x(:, i+1) = x_random(:, 1);
    all_x_dot(:, i+1) = x_random(:, 2);
    all_theta(:, i+1) = x_random(:, 3);
    all_theta_dot(:, i+1) = x_random(:, 4);
end

% --- Plots ---
figure;

%  x
subplot(2, 2, 1);
plot(tspan, all_x, 'LineWidth', 1);
xlabel('Time (s)');
ylabel('Cart Position x (m)');
title('Cart Position');
grid on;
hold on;
plot(tspan, all_x(:, 1), 'r', 'LineWidth', 2.5, 'DisplayName', 'LQR'); % LQR
hold off;
%legend;

% x_dot
subplot(2, 2, 2);
plot(tspan, all_x_dot, 'LineWidth', 1);
xlabel('Time (s)');
ylabel('Cart Velocity x_dot (m/s)');
title('Cart Velocity');
grid on;
hold on;
plot(tspan, all_x_dot(:, 1), 'g', 'LineWidth', 2.5);
hold off;


% theta
subplot(2, 2, 3);
plot(tspan, all_theta, 'LineWidth', 1);
xlabel('Time (s)');
ylabel('Pendulum Angle theta (rad)');
title('Pendulum Angle');
grid on;
hold on;
plot(tspan, all_theta(:, 1), 'b', 'LineWidth', 2.5);
hold off;


% theta_dot
subplot(2, 2, 4);
plot(tspan, all_theta_dot, 'LineWidth', 1);
xlabel('Time (s)');
ylabel('Pendulum Angular Velocity theta_dot (rad/s)');
title('Pendulum Angular Velocity');
grid on;
hold on;
plot(tspan, all_theta_dot(:, 1), 'k', 'LineWidth', 2.5);
hold off;
```

```
% nonlinear dynamics of the cart-pole - adapted from lecture slide 7

function dx_dt = cart_pole_dynamics(x, M, m, L, b, g, K)

    Sx = sin(x(3));
    Cx = cos(x(3));
    r = [0; 0; pi; 0];  %reference pos
    u = -K * (x-r);
    D = m*L*L*(M+m*(1-Cx^2));
    dx_dt = zeros(4, 1);
    dx_dt(1) = x(2);
    dx_dt(2) = (1/D)*(-m^2*L^2*g*Cx*Sx + m*L^2*(m*L*x(4)^2*Sx - b*x(2))) + m*L*L*(1/D)*u;
    dx_dt(3) = x(4);
    dx_dt(4) = (1/D)*((m+M)*m*g*L*Sx - m*L*Cx*(m*L*x(4)^2*Sx - b*x(2))) - m*L*Cx*(1/D)*u;

end
```

Plots

The LQR Controller is represented with a different color for each plot, while the blue color represent the plots from the randomly generated stable eigenvalues.



Question 2;

Question 2:

(a) Equations of motion of the helicopter:

$$\frac{d^2\theta}{dt^2} = -\sigma_1 \frac{d\theta}{dt} - a_1 \frac{dx}{dt} + n\delta$$

$$\frac{d^2x}{dt^2} = g\theta - a_2 \frac{d\theta}{dt} - \sigma_2 \frac{dx}{dt} + g\delta$$

State vector $x = \begin{bmatrix} \theta \\ \dot{\theta} \\ \dot{x} \end{bmatrix}$

let $x_1 = \theta$

$x_2 = \frac{d\theta}{dt} = \dot{\theta}$

$x_3 = \frac{dx}{dt} = \dot{x}$

$$\frac{dx_1}{dt} = \dot{\theta} = x_2$$

$$\frac{dx_2}{dt} = \frac{d^2\theta}{dt^2} = \ddot{\theta} = -\sigma_1 \frac{d\theta}{dt} - a_1 \frac{dx}{dt} + n\delta = -\sigma_1 x_2 - a_1 x_3 + n\delta$$

$$\frac{dx_3}{dt} = \frac{d^2x}{dt^2} = g\theta - a_2 \frac{d\theta}{dt} - \sigma_2 \frac{dx}{dt} + g\delta$$

$$= gx_1 - a_2 x_2 - \sigma_2 x_3 + g\delta$$

That is, derivative of the state variable =>

$$\frac{dx_1}{dt} = x_2 \quad \text{-- (1)}$$

$$\frac{dx_2}{dt} = -\sigma x_2 - a_1 x_3 + n\delta \quad \text{-- (2)}$$

$$\frac{dx_3}{dt} = g x_1 - a_2 x_2 - \sigma_2 x_3 + g\delta \quad \text{-- (3)}$$

writing equation ① to ③ in matrix form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\sigma_1 & -a_1 \\ g & -a_2 & -\sigma_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ n \\ g \end{bmatrix}\delta$$

$\dot{x} = Ax + Bu$ in state ~~st~~ space form with

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \\ \dot{x} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\sigma_1 & -a_1 \\ g & -a_2 & -\sigma_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ n \\ g \end{bmatrix}$$

$u = \delta$

replacing the variable with associated values:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.415 & -0.0111 \\ 9.8 & -1.43 & -0.0198 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 6.27 \\ 9.8 \end{bmatrix}$$

2b. LQR Controller Design

Initial: Pitch, $\theta = \pi/6$, $\dot{x} = 40$ m/s

Reference: $\theta = 0$, $\dot{x} = 0$

Cost function to minimize: $J = \int [x^T Q x + u^T R u]\, dt$

Q = positive semi definite matrix (weights the state deviations)

R = positive definite matrix (weights the control effort)

Control law, $u = -Kx$

$$K = R^{-1} B^T S$$

with S being the solution to below eqn

$$0 = SA + A^T S - SBR^{-1}B^T S + Q$$

MATLAB Code for Question 2:

```
% Question 2: Helicopter LQR Control

% System Parameters
sigma1 = 0.415;
alpha1 = 0.0111;
n = 6.27;
alpha2 = 1.43;
sigma2 = 0.0198;
g = 9.8;

% State-Space Matrices
A = [0    1      0;
     0   -sigma1  -alpha1;
     g   -alpha2   -sigma2];
B = [0;
     n;
     g];

% LQR Controller
Q = diag([10, 1, 0.1]);  % Weight on [theta, dtheta/dt, dx/dt]
R = 1;                   % Weight on control input delta

% LQR gain
[K, S, E] = lqr(A, B, Q, R);

% Initial conditions
theta0 = pi/6;
dtheta0_dt = 0;
dx0_dt = 40;
x0 = [theta0; dtheta0_dt; dx0_dt];

% Sim Time
tspan = linspace(0, 10, 400);


% System dynamics with LQR control
odefun = @(t, x) helicopter_dynamics(x, A, B, K);

% System simulation
[t, x] = ode45(odefun, tspan, x0);

% Control input
u = -K * x';

% Separate Plots for Each State
figure;

% Plot theta
subplot(2, 2, 1);
plot(t, x(:, 1), 'b');
xlabel('Time (s)');
ylabel('theta (rad)');
title('Helicopter Pitch Angle');
grid on;

% Plot dtheta/dt
subplot(2, 2, 2);
plot(t, x(:, 2), 'r');
xlabel('Time (s)');
ylabel('dtheta/dt (rad/s)');
title('Helicopter Pitch Rate');
grid on;
```

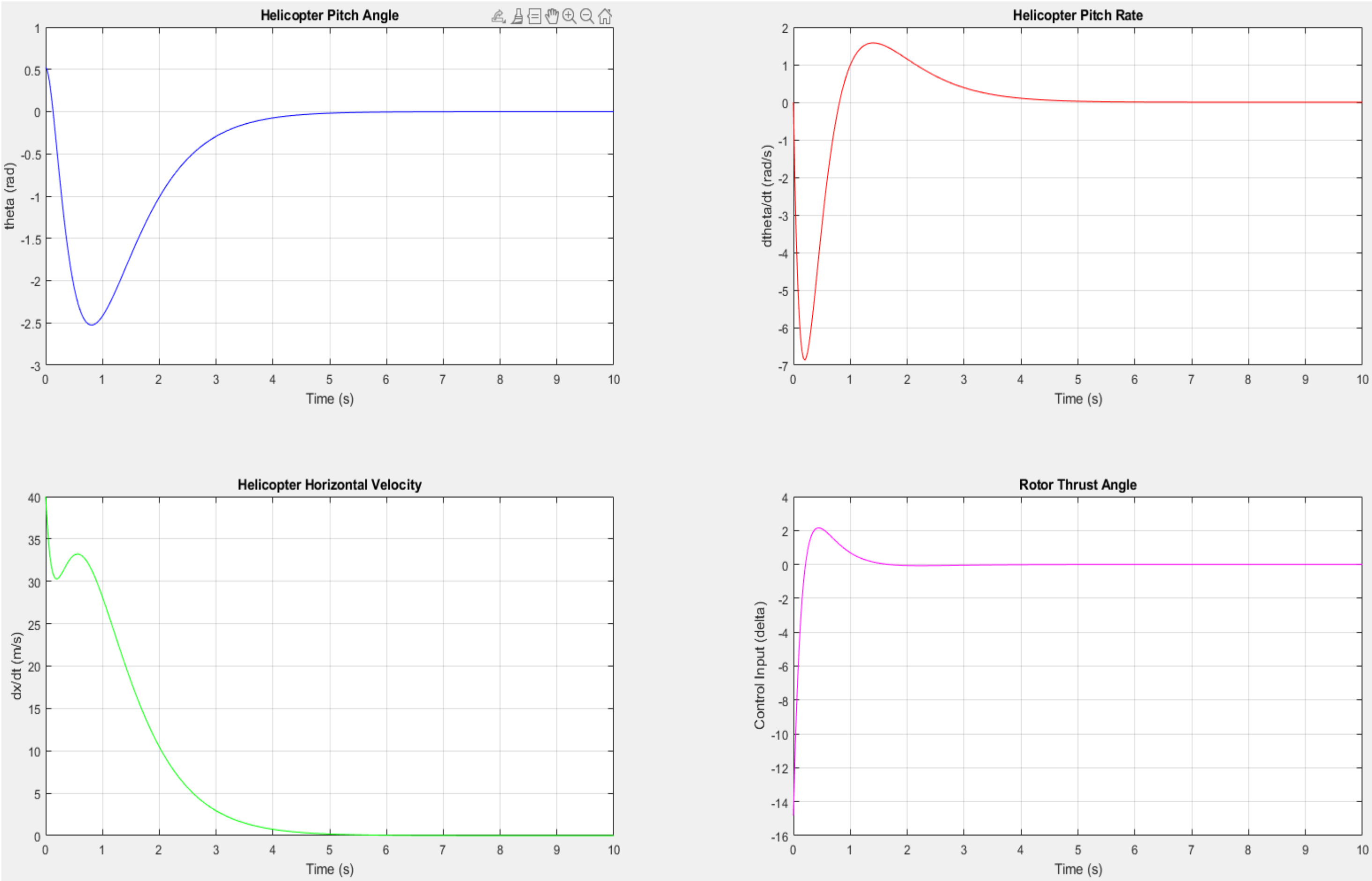
```
% Plot dx/dt
subplot(2, 2, 3);
plot(t, x(:, 3), 'g');
xlabel('Time (s)');
ylabel('dx/dt (m/s)');
title('Helicopter Horizontal Velocity');
grid on;

% Plot control input
subplot(2, 2, 4);
plot(t, u, 'm');
xlabel('Time (s)');
ylabel('Control Input (delta)');
title('Rotor Thrust Angle');
grid on;


% Helicopter dynamics with LQR control
function dx_dt = helicopter_dynamics(x, A, B, K)
    u = -K * x;
    dx_dt = A * x + B * u;
end
```

Reason for the chosen weights:

Q: I assigned a high weight of 10 to the pitch angle (θ) because it is crucial to quickly reduce the pitch angle for a smooth transition to hover and to ensure passenger comfort. The pitch rate ($d\theta/dt$) is set to 1, allowing for some angular velocity during the transition. This approach enables a more natural response without excessively restricting the helicopter's rotation. A smaller weight is placed on the horizontal velocity (dx/dt), as changes in horizontal velocity can occur more gradually.

R: R was set to 1 to achieve a smoother response with less overshoot. This leads to a conservative controller, prioritizing passenger comfort and safety in the helicopter.

Plots:


Helicopter Pitch Angle
theta (rad)
Time (s)
Helicopter Pitch Rate
dtheta/dt (rad/s)
Time (s)
Helicopter Horizontal Velocity
dx/dt (m/s)
Time (s)
Rotor Thrust Angle
Control Input (delta)
Time (s)