回転球面上での 2 次元順圧流体の定式化

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この文書では回転 2 次元球面上の非圧縮流体の支配方程式と渦度方程式を導出することを行う.

導出の計算の見通しは 3 次元系において渦度方程式をベクトル形式で表現し, その後に球面への拘束条件を適用し動径成分を書き下すやり方がお薦めである (理論マニュアル線形波動「ロスビー波 (2 次元非発散球面)」) が, 以下では緯度経度動径座標の各成分の式をまず書き下し, 式変形していくという手間のかかる計算を敢えて行う.

1　回転 2 次元球面上の非圧縮流体の支配方程式の導出

まず 3 次元回転系の非圧縮流体の支配方程式から回転 2 次元球面上の非圧縮流体の支配方程式を導出する. 3 次元回転系での非圧縮流体の支配方程式を緯度経度動径座標 (λ, φ, r) で書き表すと,

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + \frac{u}{r\cos\varphi}\frac{\partial u}{\partial \lambda} + \frac{v}{r}\frac{\partial u}{\partial \varphi} + w\frac{\partial u}{\partial r} + \frac{uw - uv\tan\varphi}{r} \\
&\qquad -2\Omega\sin\varphi v + 2\Omega\cos\varphi w = -\frac{1}{\rho r\cos\varphi}\frac{\partial p}{\partial \lambda} \\
&\qquad +\nu\left[\frac{1}{r^2\cos^2\varphi}\frac{\partial^2 u}{\partial \lambda^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} - \frac{\tan\varphi}{r^2}\frac{\partial u}{\partial \varphi} + \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru)\right. \\
&\qquad\quad \left. +\frac{2}{r^2\cos\varphi}\frac{\partial w}{\partial \lambda} - \frac{2\sin\varphi}{r^2\cos^2\varphi}\frac{\partial v}{\partial \lambda} - \frac{u}{r^2\cos^2\varphi},\right],
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
&\frac{\partial v}{\partial t}+\frac{u}{r\cos\varphi}\frac{\partial v}{\partial\lambda}+\frac{v}{r}\frac{\partial v}{\partial\varphi}+w\frac{\partial v}{\partial r}+\frac{vw+u^2\tan\varphi}{r}+2\Omega\sin\varphi u=-\frac{1}{\rho r}\frac{\partial p}{\partial\varphi}\\
&\quad+\nu\left[\frac{1}{r^2\cos^2\varphi}\frac{\partial^2 v}{\partial\lambda^2}+\frac{1}{r^2}\frac{\partial^2 v}{\partial\varphi^2}-\frac{\tan\varphi}{r^2}\frac{\partial v}{\partial\varphi}+\frac{1}{r}\frac{\partial^2}{\partial r^2}(rv)\right.\\
&\qquad\left.+\frac{2}{r^2}\frac{\partial w}{\partial\varphi}+\frac{2\sin\varphi}{r^2\cos^2\varphi}\frac{\partial u}{\partial\lambda}-\frac{v}{r^2\cos^2\varphi},\right],
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
&\frac{\partial w}{\partial t}+\frac{u}{r\cos\varphi}\frac{\partial w}{\partial\lambda}+\frac{v}{r}\frac{\partial w}{\partial\varphi}+w\frac{\partial w}{\partial r}-\frac{u^2+v^2}{r}-2\Omega\cos\varphi u=-\frac{1}{\rho}\frac{\partial p}{\partial r}\\
&\quad+\nu\left[\frac{1}{r^2\cos^2\varphi}\frac{\partial^2 w}{\partial\lambda^2}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\varphi^2}-\frac{\tan\varphi}{r^2}\frac{\partial w}{\partial\varphi}+\frac{1}{r}\frac{\partial^2}{\partial r^2}(rw)\right.\\
&\qquad\left.-\frac{2}{r^2}\frac{\partial v}{\partial\varphi}+\frac{2v\tan\varphi}{r^2}-\frac{2}{r^2\cos\varphi}\frac{\partial u}{\partial\lambda}-\frac{2w}{r^2},\right],
\end{aligned}
\tag{3}
$$

$$
\frac{1}{r\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{1}{r\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 w)=0.
\tag{4}
$$

ここで u, v, w はそれぞれ速度の緯度, 経度, 動径成分である. Ω は系の回転角速度でありその回転軸は北極向きである. 球面への拘束条件として, $u, v \propto r,\ w = 0$ を適用し, $r \to a$ の極限をとると, w のついた項は消え, 粘性項の $\frac{1}{r}\frac{\partial^2}{\partial r^2}(ru)=\frac{2u}{a^2}$, $\frac{1}{r}\frac{\partial^2}{\partial r^2}(rv)=\frac{2v}{a^2}$ となるので,

$$
\frac{\partial u}{\partial t}+\frac{u}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{v}{a}\frac{\partial u}{\partial\varphi}-\frac{uv\tan\varphi}{a}-2\Omega\sin\varphi v=-\frac{1}{\rho a\cos\varphi}\frac{\partial p}{\partial\lambda}+f_\lambda,
\tag{5}
$$

$$
\frac{\partial v}{\partial t}+\frac{u}{a\cos\varphi}\frac{\partial v}{\partial\lambda}+\frac{v}{a}\frac{\partial v}{\partial\varphi}+\frac{u^2\tan\varphi}{a}+2\Omega\sin\varphi u=-\frac{1}{\rho a}\frac{\partial p}{\partial\varphi}+f_\varphi,
\tag{6}
$$

$$
\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)=0.
\tag{7}
$$

ただし f_λ, f_φ は粘性項であり,

$$
f_\lambda=\nu\left[\frac{1}{a^2\cos^2\varphi}\frac{\partial^2 u}{\partial\lambda^2}+\frac{1}{a^2}\frac{\partial^2 u}{\partial\varphi^2}-\frac{\tan\varphi}{a^2}\frac{\partial u}{\partial\varphi}+\frac{2u}{a^2}-\frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial v}{\partial\lambda}-\frac{u}{a^2\cos^2\varphi},\right]
\tag{8}
$$

$$
f_\varphi=\nu\left[\frac{1}{a^2\cos^2\varphi}\frac{\partial^2 v}{\partial\lambda^2}+\frac{1}{a^2}\frac{\partial^2 v}{\partial\varphi^2}-\frac{\tan\varphi}{a^2}\frac{\partial v}{\partial\varphi}+\frac{2v}{a^2}+\frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial u}{\partial\lambda}-\frac{v}{a^2\cos^2\varphi}\right]
\tag{9}
$$

球面への拘束条件を適用した $u, v \propto r,\ w = 0$ は r が有限の元では正しい解にはならないことに注意されたい. 運動方程式の動径成分とつじつまをあわせることができなくなる. $r \to a$ の極限では動径方向には壁からの応力により球面へ運動が拘束されていると考えて運動方程式の動径成分は扱わない.

2　回転 2 次元球面上の非圧縮流体の渦度方程式の導出

ここでは回転する 2 次元球面上の非圧縮流体の支配方程式から渦度方程式を導出する．支配方程式は回転系での 2 次元球面上の運動方程式および連続の式 (5),(6),(7) である．渦度方程式は (5) に $-\dfrac{1}{a\cos\varphi}\dfrac{\partial}{\partial\varphi}\cos\varphi$, (6) に $\dfrac{1}{a\cos\varphi}\dfrac{\partial}{\partial\lambda}$ を作用させることにより求まる．両式の時間変化項からは

$$
\begin{aligned}
&-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\cos\varphi\frac{\partial u}{\partial t}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\frac{\partial v}{\partial t}=\frac{\partial}{\partial t}\left[-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)+\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right]\\
&=\frac{\partial\zeta}{\partial t}
\end{aligned}
$$

ここで $\zeta=-\dfrac{1}{a\cos\varphi}\dfrac{\partial}{\partial\varphi}(u\cos\varphi)+\dfrac{1}{a\cos\varphi}\dfrac{\partial v}{\partial\lambda}$ は渦度 (の動径成分) である．

非線形項からは

$$
\begin{aligned}
&-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\cos\varphi\left[\frac{u}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{v}{a}\frac{\partial u}{\partial\varphi}-\frac{uv\tan\varphi}{a}\right]\\
&+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left[\frac{u}{a\cos\varphi}\frac{\partial v}{\partial\lambda}+\frac{v}{a}\frac{\partial v}{\partial\varphi}+\frac{u^2\tan\varphi}{a}\right]
\end{aligned}
$$

渦度の移流の形と連続の式の形を意識して変形していく．第 1 項目は

$$
\begin{aligned}
&-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{u}{a\cos\varphi}\frac{\partial u}{\partial\lambda}\right)=-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{u}{a}\frac{\partial u}{\partial\lambda}\right)\\
&=-\frac{1}{a\cos\varphi}\frac{u}{a}\frac{\partial}{\partial\lambda}\left(\frac{\partial u}{\partial\varphi}\right)-\frac{1}{a^2\cos\varphi}\left(\frac{\partial u}{\partial\lambda}\frac{\partial u}{\partial\varphi}\right)\\
&=-\frac{u}{a\cos\varphi}\frac{1}{a}\frac{\partial}{\partial\lambda}\left[\frac{\partial}{\partial\varphi}\frac{u\cos\varphi}{\cos\varphi}\right]-\frac{1}{a^2\cos\varphi}\frac{\partial u}{\partial\lambda}\frac{\partial u}{\partial\varphi}\\
&=-\frac{u}{a\cos\varphi}\frac{1}{a}\frac{\partial}{\partial\lambda}\left[\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)+\frac{u\cos\varphi\sin\varphi}{\cos^2\varphi}\right]-\frac{1}{a^2\cos\varphi}\frac{\partial u}{\partial\lambda}\frac{\partial u}{\partial\varphi}\\
&=-\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]-\frac{u}{a\cos\varphi}\frac{\tan\varphi}{a}\frac{\partial u}{\partial\lambda}-\frac{1}{a^2\cos\varphi}\frac{\partial u}{\partial\lambda}\frac{\partial u}{\partial\varphi}
\end{aligned}
$$

第 2 項目は

$$
\begin{aligned}
&-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{v}{a}\frac{\partial u}{\partial\varphi}\right)=-\frac{v}{a}\frac{\partial}{\partial\varphi}\left(\frac{1}{a}\frac{\partial u}{\partial\varphi}\right)-\frac{1}{a}\frac{\partial u}{\partial\varphi}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\\
&=-\frac{v}{a}\frac{\partial}{\partial\varphi}\left[\frac{1}{a}\frac{\partial}{\partial\varphi}\left(\frac{u\cos\varphi}{\cos\varphi}\right)\right]-\frac{1}{a}\frac{\partial u}{\partial\varphi}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\\
&=-\frac{v}{a}\frac{\partial}{\partial\varphi}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)+\frac{u\cos\varphi}{a}\frac{-\sin\varphi}{-\cos\varphi^2}\right]-\frac{1}{a}\frac{\partial u}{\partial\varphi}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)
\end{aligned}
$$

$$= -\frac{v}{a}\frac{\partial}{\partial\varphi}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] - \frac{v}{a}\frac{\partial}{\partial\varphi}\left[\frac{u\tan\varphi}{a}\right] - \frac{1}{a}\frac{\partial u}{\partial\varphi}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)$$

第 3 項目は

$$-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\cos\varphi\left(-\frac{uv\tan\varphi}{a}\right) = \frac{u\tan\varphi}{a}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi) + \frac{v\cos\varphi}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{u\tan\varphi}{a}\right)$$
$$= \frac{u\tan\varphi}{a}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi) + \frac{v}{a}\frac{\partial}{\partial\varphi}\left(\frac{u\tan\varphi}{a}\right)$$

第 4 項目は

$$\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{u}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right) = \frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right) + \frac{1}{a^2\cos^2\varphi}\frac{\partial u}{\partial\lambda}\frac{\partial v}{\partial\lambda}$$

第 5 項目は

$$\begin{aligned}
\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{v}{a}\frac{\partial v}{\partial\varphi}\right) &= \frac{v}{a}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{\partial v}{\partial\lambda}\right) + \frac{1}{a^2\cos\varphi}\frac{\partial v}{\partial\lambda}\frac{\partial v}{\partial\varphi} \\
&= \frac{v}{a}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{\cos\varphi}{\cos\varphi}\frac{\partial v}{\partial\lambda}\right) + \frac{1}{a^2\cos\varphi}\frac{\partial v}{\partial\lambda}\frac{\partial v}{\partial\varphi} \\
&= \frac{v}{a}\frac{1}{a}\frac{\partial}{\partial\varphi}\left(\frac{1}{\cos\varphi}\frac{\partial v}{\partial\lambda}\right) - \frac{v}{a}\frac{1}{a\cos\varphi}\left(\frac{\sin\varphi}{\cos\varphi}\frac{\partial v}{\partial\lambda}\right) + \frac{1}{a^2\cos\varphi}\frac{\partial v}{\partial\lambda}\frac{\partial v}{\partial\varphi} \\
&= \frac{v}{a}\frac{\partial}{\partial\varphi}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right) - \frac{v\tan\varphi}{a^2\cos\varphi}\frac{\partial v}{\partial\lambda} + \frac{1}{a^2\cos\varphi}\frac{\partial v}{\partial\lambda}\frac{\partial v}{\partial\varphi} \\
&= \frac{v}{a}\frac{\partial}{\partial\varphi}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right) + \frac{1}{a^2\cos\varphi}\frac{\partial v}{\partial\lambda}\left(\frac{\partial v}{\partial\varphi} - v\tan\varphi\right) \\
&= \frac{v}{a}\frac{\partial}{\partial\varphi}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right) + \frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)
\end{aligned}$$

第 6 項目は

$$\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{u^2\tan\varphi}{a}\right) = \frac{u\tan\varphi}{a}\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda} + \frac{u\tan\varphi}{a}\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}$$

全ての項を足しあわせると,

$$\begin{aligned}
= \quad & -\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] - \frac{u}{a\cos\varphi}\frac{\tan\varphi}{a}\frac{\partial u}{\partial\lambda} - \frac{1}{a^2\cos\varphi}\frac{\partial u}{\partial\lambda}\frac{\partial u}{\partial\varphi} \\
& -\frac{v}{a}\frac{\partial}{\partial\varphi}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] - \frac{v}{a}\frac{\partial}{\partial\varphi}\left[\frac{u\tan\varphi}{a}\right] - \frac{1}{a}\frac{\partial u}{\partial\varphi}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi) \\
& +\frac{u\tan\varphi}{a}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi) + \frac{v}{a}\frac{\partial}{\partial\varphi}\left(\frac{u\tan\varphi}{a}\right) \\
& +\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right) + \frac{1}{a^2\cos^2\varphi}\frac{\partial u}{\partial\lambda}\frac{\partial v}{\partial\lambda}
\end{aligned}$$

$$
\begin{aligned}
& +\frac{v}{a}\frac{\partial}{\partial\varphi}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right)+\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi) \\
& +\frac{u\tan\varphi}{a}\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{u\tan\varphi}{a}\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda} \\
= & -\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]-\frac{1}{a^2\cos\varphi}\frac{\partial u}{\partial\lambda}\frac{\partial u}{\partial\varphi} \\
& -\frac{v}{a}\frac{\partial}{\partial\varphi}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]-\frac{1}{a}\frac{\partial u}{\partial\varphi}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi) \\
& +\frac{u\tan\varphi}{a}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi) \\
& +\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right)+\frac{1}{a^2\cos^2\varphi}\frac{\partial u}{\partial\lambda}\frac{\partial v}{\partial\lambda} \\
& +\frac{v}{a}\frac{\partial}{\partial\varphi}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right)+\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi) \\
& +\frac{u\tan\varphi}{a}\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda} \\
= & \frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left[\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]+\frac{v}{a}\frac{\partial}{\partial\varphi}\left[\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] \\
& +\frac{u\tan\varphi}{a}\left[\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\right]-\frac{1}{a}\frac{\partial u}{\partial\varphi}\left[\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\right] \\
& +\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\left[\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\right] \\
= & \left[\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}+\frac{v}{a}\frac{\partial}{\partial\varphi}\right]\left[\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] \\
& +\left[\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}-\frac{1}{a}\frac{\partial u}{\partial\varphi}+\frac{u\tan\varphi}{a}\right]\left[\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\right] \\
= & \left[\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}+\frac{v}{a}\frac{\partial}{\partial\varphi}\right]\left[\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] \\
& +\left[\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]\left[\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\right] \\
= & \left[\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}+\frac{v}{a}\frac{\partial}{\partial\varphi}\right]\zeta+\zeta\left[\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\right] \\
= & \left[\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}+\frac{v}{a}\frac{\partial}{\partial\varphi}\right]\zeta
\end{aligned}
$$

最後に連続の式 (7) を用いた.

コリオリ力の項からは

$$
-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\cos\varphi(-2\Omega\sin\varphi v)+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}(2\Omega\sin\varphi u)
$$

$$
\begin{aligned}
&= 2\Omega\frac{\sin\varphi}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi) + 2\Omega\frac{v\cos\varphi}{a\cos\varphi}\frac{\partial}{\partial\varphi}(\sin\varphi) + 2\Omega\frac{\sin\varphi}{a\cos\varphi}\frac{\partial u}{\partial\lambda} \\
&= 2\Omega\sin\varphi\left[\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\right] + 2\Omega\frac{v}{a}\cos\varphi = 2\Omega\frac{v}{a}\cos\varphi.
\end{aligned}
$$

圧力項は

$$
\begin{aligned}
&-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\cos\varphi\left(\frac{1}{\rho a\cos\varphi}\frac{\partial p}{\partial\lambda}\right) + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{1}{\rho a}\frac{\partial p}{\partial\varphi}\right) \\
&= -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{1}{\rho a}\frac{\partial p}{\partial\lambda}\right) + \frac{1}{\rho a^2\cos\varphi}\frac{\partial^2 p}{\partial\lambda\partial\varphi} \\
&= -\frac{1}{\rho a^2\cos\varphi}\frac{\partial^2 p}{\partial\lambda\partial\varphi} + \frac{1}{\rho a^2\cos\varphi}\frac{\partial^2 p}{\partial\lambda\partial\varphi} = 0.
\end{aligned}
$$

粘性項は, まず f_λ の 2,3,6 項目をまとめて

$$
\begin{aligned}
&\frac{1}{a^2}\frac{\partial^2 u}{\partial\varphi^2} - \frac{\tan\varphi}{a^2}\frac{\partial u}{\partial\varphi} - \frac{u}{a^2\cos^2\varphi} = \frac{1}{a^2}\frac{\partial^2 u}{\partial\varphi^2} - \frac{1}{a^2}\frac{\partial}{\partial\varphi}(u\tan\varphi) \\
&= \frac{1}{a^2}\frac{\partial}{\partial\varphi}\left(\frac{\partial u}{\partial\varphi} - u\tan\varphi\right) = \frac{1}{a^2}\frac{\partial}{\partial\varphi}\left[\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]
\end{aligned}
$$

f_φ も同様に,

$$
\frac{1}{a^2}\frac{\partial^2 v}{\partial\varphi^2} - \frac{\tan\varphi}{a^2}\frac{\partial v}{\partial\varphi} - \frac{v}{a^2\cos^2\varphi} = \frac{1}{a^2}\frac{\partial}{\partial\varphi}\left[\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\right]
$$

したがって,

$$
\begin{aligned}
&-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(\cos\varphi f_\lambda) + \frac{1}{a\cos\varphi}\frac{\partial f_\varphi}{\partial\lambda} \\
=\ & -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\cos\varphi\left\{\frac{1}{a^2\cos^2\varphi}\frac{\partial^2 u}{\partial\lambda^2} + \frac{1}{a^2}\frac{\partial}{\partial\varphi}\left[\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] + \frac{2u}{a^2} - \frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial v}{\partial\lambda}\right\} \\
& +\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left\{\frac{1}{a^2\cos^2\varphi}\frac{\partial^2 v}{\partial\lambda^2} + \frac{1}{a^2}\frac{\partial}{\partial\varphi}\left[\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\right] + \frac{2v}{a^2} + \frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial u}{\partial\lambda}\right\}
\end{aligned}
$$

これより渦度の形を意識しながら変形していく. 第 1 項目は

$$
\begin{aligned}
&-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\cos\varphi\left\{\frac{1}{a^2\cos^2\varphi}\frac{\partial^2 u}{\partial\lambda^2}\right\} = -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left[\frac{1}{a^2\cos\varphi}\frac{\partial^2 u}{\partial\lambda^2}\right] \\
=\ & -\frac{1}{a\cos\varphi}\frac{\partial^2}{\partial\lambda^2}\frac{\partial}{\partial\varphi}\left(\frac{u}{a^2\cos\varphi}\right) = -\frac{1}{a\cos\varphi}\frac{\partial^2}{\partial\lambda^2}\frac{\partial}{\partial\varphi}\left(\frac{u\cos\varphi}{a^2\cos^2\varphi}\right) \\
=\ & -\frac{1}{a\cos\varphi}\frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{a^2\cos^2\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi) + u\cos\varphi\frac{2\sin\varphi\cos\varphi}{a^2\cos^4\varphi}\right] \\
=\ & -\frac{1}{a^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] - \frac{1}{a\cos\varphi}\frac{\partial^2}{\partial\lambda^2}\left(\frac{2u\sin\varphi}{a^2\cos^2\varphi}\right) \\
=\ & -\frac{1}{a^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] - \frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial u}{\partial\lambda}\right).
\end{aligned}
$$

第 2 項目は

$$
\begin{aligned}
& -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\cos\varphi\left\{\frac{1}{a^2}\frac{\partial}{\partial\varphi}\left[\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]\right\} \\
= & -\frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\cos\varphi\frac{\partial}{\partial\varphi}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]\right\}.
\end{aligned}
$$

第 3 項目は

$$
-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{2u}{a^2}\right) = -\frac{2}{a^2}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi).
$$

第 4 項目は

$$
-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\cos\varphi\left(-\frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial v}{\partial\lambda}\right) = \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{2\sin\varphi}{a^2\cos\varphi}\frac{\partial v}{\partial\lambda}\right).
$$

第 5 項目は

$$
\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left[\frac{1}{a^2\cos^2\varphi}\frac{\partial^2 v}{\partial\lambda^2}\right] = \frac{1}{a^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right].
$$

第 6 項目は

$$
\begin{aligned}
& \frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left\{\frac{1}{a^2}\frac{\partial}{\partial\varphi}\left[\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\right]\right\} = \frac{1}{a^3\cos\varphi}\frac{\partial}{\partial\varphi}\left[\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial v}{\partial\lambda}\right)\right] \\
= & \frac{1}{a^3\cos\varphi}\frac{\partial}{\partial\varphi}\left[\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{\cos\varphi}{\cos\varphi^2}\frac{\partial v}{\partial\lambda}\right)\right] \\
= & \frac{1}{a^3\cos\varphi}\frac{\partial}{\partial\varphi}\left[\frac{1}{\cos\varphi}\left(\cos^2\varphi\frac{\partial}{\partial\varphi}\left(\frac{1}{\cos\varphi}\frac{\partial v}{\partial\lambda}\right) + \frac{1}{\cos\varphi}\frac{\partial v}{\partial\lambda}(-2\sin\varphi\cos\varphi)\right)\right] \\
= & \frac{1}{a^3\cos\varphi}\frac{\partial}{\partial\varphi}\left[\cos\varphi\frac{\partial}{\partial\varphi}\left(\frac{1}{\cos\varphi}\frac{\partial v}{\partial\lambda}\right) - \frac{2\sin\varphi}{\cos\varphi}\frac{\partial v}{\partial\lambda}\right] \\
= & \frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\left[\cos\varphi\frac{\partial}{\partial\varphi}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right)\right] - \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{2\sin\varphi}{a^2\cos\varphi}\frac{\partial v}{\partial\lambda}\right).
\end{aligned}
$$

第 7 項目は

$$
\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{2v}{a^2}\right) = \frac{2}{a^2}\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}.
$$

第 8 項目はそのままで

$$
\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial u}{\partial\lambda}\right).
$$

全てまとめると

$$
\begin{aligned}
& -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(\cos\varphi f_\lambda) + \frac{1}{a\cos\varphi}\frac{\partial f_\varphi}{\partial\lambda} \\
= \; & -\frac{1}{a^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] - \frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial u}{\partial\lambda}\right) \\
& -\frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\cos\varphi\frac{\partial}{\partial\varphi}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]\right\} \\
& -\frac{2}{a^2}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi) + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{2\sin\varphi}{a^2\cos\varphi}\frac{\partial v}{\partial\lambda}\right) \\
& +\frac{1}{a^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right] + \frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\left[\cos\varphi\frac{\partial}{\partial\varphi}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right)\right] \\
& -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{2\sin\varphi}{a^2\cos\varphi}\frac{\partial v}{\partial\lambda}\right) + \frac{2}{a^2}\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial u}{\partial\lambda}\right) \\
= \; & -\frac{1}{a^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] \\
& -\frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\cos\varphi\frac{\partial}{\partial\varphi}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]\right\} \\
& -\frac{2}{a^2}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi) + \frac{1}{a^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right] \\
& +\frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\left[\cos\varphi\frac{\partial}{\partial\varphi}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right)\right] + \frac{2}{a^2}\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda} \\
= \; & \frac{1}{a^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda} - \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] \\
& +\frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\left[\cos\varphi\frac{\partial}{\partial\varphi}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda} - \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right)\right] \\
& +\frac{2}{a^2}\left[\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda} - \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] \\
= \; & \frac{1}{a^2\cos^2\varphi}\frac{\partial^2\zeta}{\partial\lambda^2} + \frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial\zeta}{\partial\varphi}\right) + \frac{2\zeta}{a^2}.
\end{aligned}
$$

これより, 渦度方程式は

$$
\begin{aligned}
& \frac{\partial\zeta}{\partial t} + \left[\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\varphi}\right]\zeta + 2\Omega\frac{v}{a}\cos\varphi \\
& \qquad = \nu\left[\frac{1}{a^2\cos^2\varphi}\frac{\partial^2\zeta}{\partial\lambda^2} + \frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial\zeta}{\partial\varphi}\right) + \frac{2\zeta}{a^2}\right].
\end{aligned}
$$

ここで流線関数 ψ を

$$
u = -\frac{1}{a}\frac{\partial\psi}{\partial\varphi}, \quad v = \frac{1}{a\cos\varphi}\frac{\partial\psi}{\partial\lambda}, \tag{10}
$$

として定義すると, 連続の式 (7) を自動的に満たす. 渦度 ζ は

$$
\begin{aligned}
\zeta &= -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi) + \frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda} \\
&= \frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial\psi}{\partial\varphi}\right) + \frac{1}{a^2\cos^2\varphi}\frac{\partial^2\psi}{\partial\lambda^2} = \nabla^2\psi.
\end{aligned}
$$

ここで $\nabla^2 = \dfrac{1}{a^2\cos^2\varphi}\dfrac{\partial^2}{\partial\lambda^2} + \dfrac{1}{a^2\cos\varphi}\dfrac{\partial}{\partial\varphi}\cos\varphi\dfrac{\partial}{\partial\varphi}$ は球面上の水平ラプラシアンである. これらを渦度方程式 (10) に代入すると,

$$
\frac{\partial\nabla^2\psi}{\partial t} + \left[-\frac{1}{a}\frac{\partial\psi}{\partial\varphi}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda} + \frac{1}{a\cos\varphi}\frac{\partial\psi}{\partial\lambda}\frac{1}{a}\frac{\partial}{\partial\varphi}\right]\nabla^2\psi + \frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda} = \nu\left(\nabla^2 + \frac{2}{a^2}\right)\nabla^2\psi.
$$

さらに sin 緯度 $\mu = \sin\varphi$ を導入すると, $\dfrac{\partial}{\partial\mu} = \dfrac{1}{\cos\varphi}\dfrac{\partial}{\partial\varphi}$ であるから,

$$
\frac{\partial\nabla^2\psi}{\partial t} + \left[-\frac{1}{a^2}\frac{\partial\psi}{\partial\mu}\frac{\partial}{\partial\lambda} + \frac{1}{a^2}\frac{\partial\psi}{\partial\lambda}\frac{\partial}{\partial\mu}\right]\nabla^2\psi + \frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda} = \nu\left(\nabla^2 + \frac{2}{a^2}\right)\nabla^2\psi.
$$

よって

$$
\frac{\partial\nabla^2\psi}{\partial t} + \frac{1}{a^2}J(\psi,\nabla^2\psi) + \frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda} = \nu\left(\nabla^2 + \frac{2}{a^2}\right)\nabla^2\psi. \tag{11}
$$

ただし $J(f,g) = (\partial_\lambda f)(\partial_\mu g) - (\partial_\lambda g)(\partial_\mu f)$ はヤコビアンである. ラプラシアンを μ で表しておくと

$$
\nabla^2 = \frac{1}{a^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\cos\varphi\frac{\partial}{\partial\varphi} = \frac{1}{a^2(1-\mu^2)}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{a^2}\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu}. \tag{12}
$$

3 角運動量保存則

ψ の球面調和函数 $Y_n^m(\mu,\lambda)$ で展開した成分のうち, $n=0$ は剛体回転流を表わす. $(n,m)=(1,0)$ は極方向を回転軸とする剛体回転流, $(n,m)=(1,1),(1,-1)$ は赤道面上の軸を回転軸とする剛体回転流である. したがってこれらの成分は各々の軸に対する角運動量となっており, 非回転系では全て, 回転系では $(n,m)=(1,0)$ 成分だけ保存しているはずである. そのことを確かめてみよう.

いま流線関数が球面調和函数の足しあわせで

$$
\psi(\lambda,\mu,t) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\tilde{\psi}_{n,m}Y_n^m(\lambda,\mu) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\tilde{\psi}_{n,m}P_n^m(\lambda)e^{i\mu\lambda} \tag{13}
$$

と表わされているとする. このとき ψ が実数であることから係数には

$$\tilde{\psi}_{n,-m} = \tilde{\psi}^*_{n,m} \tag{14}$$

が成り立つ.

まず $(n,m) = (1,0)$ 成分について考えよう. 渦度方程式 (11) において, 粘性項の 1,0 成分は

$$\left(\nabla^2 + \frac{2}{a^2}\right)\nabla^2\tilde{\psi}_{1,0}Y_1^0(\lambda,\mu) = \left(\frac{-1\times 2}{a^2} + \frac{2}{a^2}\right)\frac{-1\times 2}{a^2}\tilde{\psi}_{1,0}Y_1^0(\lambda,\mu) = 0. \tag{15}$$

(11) の β 項の 1,0 成分は

$$\frac{2\Omega}{a^2}\frac{\partial}{\partial\lambda}\tilde{\psi}_{1,0}Y_1^0(\lambda,\mu) = 0. \tag{16}$$

非線形項からの寄与は, (13) をヤコビアンに代入して $Y_1^0(\lambda,\mu) = \mu$ をかけて球面上で積分することにより得られる.

$$\begin{aligned}
& \int_0^{2\pi}\int_{-1}^{1} d\lambda d\mu \frac{1}{a^2} J(\psi, \nabla^2\psi) Y_1^0(\lambda,\mu) \\
= \; & \frac{1}{a^2}\sum_{n,m}\sum_{n',m'}\int_0^{2\pi}\int_{-1}^{1}\left[im\tilde{\psi}_{n,m}\tilde{\psi}_{n',m'}\mu Y_n^m \frac{\partial Y_{n'}^{m'}}{\partial\mu} - im'\tilde{\psi}_{n,m}\tilde{\psi}_{n',m'}\mu\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}\right] d\lambda d\mu \\
= \; & \frac{1}{a^2}\sum_{n,m}\sum_{n',m'}\tilde{\psi}_{n,m}\tilde{\psi}_{n',m'}\int_0^{2\pi}\int_{-1}^{1}\left[im\mu Y_n^m \frac{\partial Y_{n'}^{m'}}{\partial\mu} - im'\mu\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}\right] d\lambda d\mu
\end{aligned}$$

ここで $m = -m'$ 成分のみ 0 でないことに注意すると

$$\begin{aligned}
& \int_0^{2\pi}\int_{-1}^{1} d\lambda d\mu \left[im\mu Y_n^m \frac{\partial Y_{n'}^{m'}}{\partial\mu} - im'\mu\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}\right] \\
= \; & \int_0^{2\pi}\left[im\mu Y_n^m Y_{n'}^{m'}\right]_{-1}^{1} d\lambda - \int_0^{2\pi}\int_{-1}^{1} imY_n^m Y_{n'}^{m'} d\lambda d\mu - \int_0^{2\pi}\int_{-1}^{1} im\mu\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'} d\lambda d\mu \\
& + \int_0^{2\pi}\int_{-1}^{1}\left[-im'\mu\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}\right] d\lambda d\mu \\
= \; & \int_0^{2\pi}\left[im\mu Y_n^m Y_{n'}^{-m}\right]_{-1}^{1} d\lambda - \int_0^{2\pi}\int_{-1}^{1} imY_n^m Y_{n'}^{-m} d\lambda d\mu - \int_0^{2\pi}\int_{-1}^{1} im\mu\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{-m} d\lambda d\mu \\
& + \int_0^{2\pi}\int_{-1}^{1} im\mu\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{-m} d\lambda d\mu \\
= \; & \int_0^{2\pi}\left[im\mu Y_n^m Y_{n'}^{-m}\right]_{-1}^{1} d\lambda - \int_0^{2\pi}\int_{-1}^{1} imY_n^m Y_{n'}^{-m} d\lambda d\mu \\
= \; & \int_0^{2\pi}\left[im\mu Y_n^m Y_{n'}^{-m}\right]_{-1}^{1} d\lambda - \int_0^{2\pi}\int_{-1}^{1} imY_n^m Y_{n}^{-m} d\lambda d\mu
\end{aligned}$$

最後に Y_n^m の直交関係を用いている. 第 1 項目の値を評価する. Y_n^m の $\mu=\pm1$ での値は $m\neq 0$ のとき 0 である. $m=0$ のときは係数 m がかかっているので結局この項は 0 となる. したがって, $n,m,\ n',m'$ 成分からは

$$-\tilde{\psi}_{n,m}\tilde{\psi}_{n,-m}\int_0^{2\pi}\int_{-1}^{1} imY_n^mY_n^{-m}d\lambda d\mu = -|\tilde{\psi}_{n,m}|^2\int_0^{2\pi}\int_{-1}^{1} imY_n^mY_n^{-m}d\lambda d\mu \tag{17}$$

一方, $n,-m$ から

$$-|\tilde{\psi}_{n,-m}|^2\int_0^{2\pi}\int_{-1}^{1}(-imY_n^{-m}Y_n^m)d\lambda d\mu = |\tilde{\psi}^*_{n,m}|^2\int_0^{2\pi}\int_{-1}^{1} imY_n^{-m}Y_n^md\lambda d\mu \tag{18}$$

したがってこの項は互いにキャンセルする. 結局非線形項からの $(n,m)=(1,0)$ への寄与は 0 となる. 最後に時間変化項は

$$\frac{\partial}{\partial t}\nabla^2\tilde{\psi}_{1,0}(t)Y_1^0 = -\frac{1(1+1)}{a^2}\frac{d\tilde{\psi}_{1,0}}{dt}Y_1^0 \tag{19}$$

よって

$$\frac{d\tilde{\psi}_{1,0}}{dt}=0 \tag{20}$$

となり極方向を軸とする角運動量は保存する.

次に $(n,m)=(1,1)$ 成分について考えよう. 粘性項の $(1,1)$ 成分は $(n,m)=(1,0)$ と同様にして 0 である. (11) の β 項の 1,1 成分は

$$\frac{2\Omega}{a^2}\frac{\partial}{\partial\lambda}\tilde{\psi}_{1,1}Y_1^1(\lambda,\mu) == i\frac{2\Omega}{a^2}\tilde{\psi}_{1,1}Y_1^1(\lambda,\mu). \tag{21}$$

非線形項からの寄与は, (13) をヤコビアンに代入して $Y_1^1(\lambda,\mu)=\sqrt{1-\mu^2}e^{i\lambda}$ をかけて球面上で積分することにより得られる.

$$\begin{aligned}
&\int_0^{2\pi}\int_{-1}^{1} d\lambda d\mu\frac{1}{a^2}J(\psi,\nabla^2\psi)Y_1^1(\lambda,\mu)\\
=\ &\frac{1}{a^2}\sum_{n,m}\sum_{n',m'}\int_0^{2\pi}\int_{-1}^{1}\Bigg[im\tilde{\psi}_{n,m}\tilde{\psi}_{n',m'}\sqrt{1-\mu^2}e^{i\lambda}Y_n^m\frac{\partial Y_{n'}^{m'}}{\partial\mu}\\
&\qquad -im'\tilde{\psi}_{n,m}\tilde{\psi}_{n',m'}\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}\Bigg]d\lambda d\mu\\
=\ &\frac{1}{a^2}\sum_{n,m}\sum_{n',m'}\tilde{\psi}_{n,m}\tilde{\psi}_{n',m'}\int_0^{2\pi}\int_{-1}^{1}\left[im\sqrt{1-\mu^2}e^{i\lambda}Y_n^m\frac{\partial Y_{n'}^{m'}}{\partial\mu}-im'\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}\right]d\lambda d\mu
\end{aligned}$$

ここで $m'=-1-m$ 成分のみ 0 でないことに注意すると, $m=n$ のときは 0 になり, $0<m<n+1$ なる n,m に対して

$$\int_0^{2\pi}\int_{-1}^{1} d\lambda d\mu\left[im\sqrt{1-\mu^2}e^{i\lambda}Y_n^m\frac{\partial Y_{n'}^{m'}}{\partial\mu}-im'\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}\right]$$

$$
\begin{aligned}
&= \int_0^{2\pi}\left[im\sqrt{1-\mu^2}e^{i\lambda}Y_n^m Y_{n'}^{m'}\right]_{-1}^{1}d\lambda - \int_0^{2\pi}\int_{-1}^{1} -im\frac{\mu}{\sqrt{1-\mu^2}}e^{i\lambda}Y_n^m Y_{n'}^{m'}d\lambda d\mu \\
&\quad -\int_0^{2\pi}\int_{-1}^{1} im\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}d\lambda d\mu + \int_0^{2\pi}\int_{-1}^{1}\left[-im'\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}\right]d\lambda d\mu \\
&= \int_0^{2\pi}\int_{-1}^{1} im\frac{\mu}{\sqrt{1-\mu^2}}e^{i\lambda}Y_n^m Y_{n'}^{m'}d\lambda d\mu - i(m+m')\int_0^{2\pi}\int_{-1}^{1}\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}d\lambda d\mu \\
&= \int_0^{2\pi}\int_{-1}^{1} im\frac{\mu}{\sqrt{1-\mu^2}}e^{i\lambda}Y_n^m Y_{n'}^{m'}d\lambda d\mu + i\int_0^{2\pi}\int_{-1}^{1}\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}d\lambda d\mu \\
&= i\int_0^{2\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-\mu^2}}e^{i\lambda}\left[m\mu Y_n^m + (1-\mu^2)\frac{\partial Y_n^m}{\partial\mu}\right]Y_{n'}^{m'}d\lambda d\mu
\end{aligned}
$$

ここでルジャンドル函数の漸化式

$$
(1-\mu^2)\frac{\partial P_n^m}{\partial\mu} = \sqrt{1-\mu^2}P_n^{m+1} - m\mu P_n^m \tag{22}
$$

より

$$
\begin{aligned}
e^{i\lambda}\left[m\mu Y_n^m + (1-\mu^2)\frac{\partial Y_n^m}{\partial\mu}\right] &= e^{i(m+1)\lambda}\left[m\mu P_n^m + (1-\mu^2)\frac{\partial P_n^m}{\partial\mu}\right] \\
&= \sqrt{1-\mu^2}P_n^{m+1}e^{i(m+1)\lambda} = \sqrt{1-\mu^2}Y_n^{m+1}.
\end{aligned}
$$

したがって, n, m 成分からの寄与は

$$
\begin{aligned}
&\frac{1}{a^2}\tilde{\psi}_{n,m}\tilde{\psi}_{n',m'}\int_0^{2\pi}\int_{-1}^{1}d\lambda d\mu\left[im\sqrt{1-\mu^2}e^{i\lambda}Y_n^m\frac{\partial Y_{n'}^{m'}}{\partial\mu} - im'\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}\right] \\
&= i\frac{1}{a^2}\tilde{\psi}_{n,m}\tilde{\psi}_{n',-m-1}\int_0^{2\pi}\int_{-1}^{1}Y_n^{m+1}Y_{n'}^{m'}d\lambda d\mu d\lambda d\mu \\
&= i\frac{1}{a^2}\tilde{\psi}_{n,m}\tilde{\psi}_{n,-m-1}\int_0^{2\pi}\int_{-1}^{1}Y_n^{m+1}Y_n^{-m-1}d\lambda d\mu d\lambda d\mu
\end{aligned}
$$

最後に直交関係を使って $n' = n$ とした.

一方 $m < 0$ の成分に対して, $m \to -m-1 > -n-1$ を選んで計算すると $m' = m$ の時のみ 0 でなく,

$$
\begin{aligned}
&\int_0^{2\pi}\int_{-1}^{1}d\lambda d\mu\left[im\sqrt{1-\mu^2}e^{i\lambda}Y_n^m\frac{\partial Y_{n'}^{m'}}{\partial\mu} - im'\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}\right] \\
&= \int_0^{2\pi}\int_{-1}^{1}d\lambda d\mu\left[i(-m-1)\sqrt{1-\mu^2}e^{i\lambda}Y_n^{-m-1}\frac{\partial Y_{n'}^{m'}}{\partial\mu} - im'\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^{-m-1}}{\partial\mu}Y_{n'}^{m'}\right] \\
&= \int_0^{2\pi}\int_{-1}^{1}d\lambda d\mu\left[i(-m-1)\sqrt{1-\mu^2}e^{i\lambda}Y_n^{-m-1}\frac{\partial Y_{n'}^{m}}{\partial\mu} - im\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^{-m-1}}{\partial\mu}Y_{n'}^{m}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_0^{2\pi}\int_{-1}^{1} d\lambda d\mu \left[im\sqrt{1-\mu^2}e^{i\lambda}Y_{n'}^m \frac{\partial Y_n^{-m-1}}{\partial\mu} - i(-m-1)\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_{n'}^m}{\partial\mu}Y_n^{-m-1}\right] \\
&= -i\int_0^{2\pi}\int_{-1}^{1} Y_{n'}^{m+1}Y_n^{-m-1} d\lambda d\mu d\lambda d\mu \\
&= -i\int_0^{2\pi}\int_{-1}^{1} Y_{n}^{m+1}Y_n^{-m-1} d\lambda d\mu d\lambda d\mu .
\end{aligned}
$$

最後の計算は n, m の時の式変形と同様に行った．したがって, $n, -m-1$ 成分からの寄与は

$$
\begin{aligned}
& \frac{1}{a^2}\tilde{\psi}_{n,-m-1}\tilde{\psi}_{n',m'}\int_0^{2\pi}\int_{-1}^{1} d\lambda d\mu \left[im\sqrt{1-\mu^2}e^{i\lambda}Y_n^m \frac{\partial Y_{n'}^{m'}}{\partial\mu} - im'\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}\right] \\
&= -i\frac{1}{a^2}\tilde{\psi}_{n,-m-1}\tilde{\psi}_{n',m}\int_0^{2\pi}\int_{-1}^{1} Y_n^{-m-1}Y_{n'}^{m} d\lambda d\mu d\lambda d\mu \\
&= -i\frac{1}{a^2}\tilde{\psi}_{n,-m-1}\tilde{\psi}_{n,m}\int_0^{2\pi}\int_{-1}^{1} Y_n^{-m-1}Y_{n}^{m} d\lambda d\mu d\lambda d\mu
\end{aligned}
$$

よって n, m 成分と $n, -m-1$ 成分からの寄与が打ち消しあって非線形項は 0 となる．

最後に時間変化項は $(1, 0)$ 成分と同じであり

$$
\frac{\partial}{\partial t}\nabla^2\tilde{\psi}_{1,1}(t)Y_1^1 = -\frac{1(1+1)}{a^2}\frac{d\tilde{\psi}_{1,1}}{dt}Y_1^1 \tag{23}
$$

よって $(1, 1)$ 成分の式は

$$
-\frac{2}{a^2}\frac{d\tilde{\psi}_{1,1}}{dt} + i\frac{2\Omega}{a^2}\tilde{\psi}_{1,1} = 0, \qquad \tilde{\psi}_1^1 = Ae^{i\Omega t}. \tag{24}
$$

A は任意定数であり初期値で定まる．したがって $(1, 1)$ 成分の振舞は

$$
\psi(\mu,\lambda,t) = Ae^{i(\lambda+\Omega t)}P_1^1(\mu) \tag{25}
$$

緯度方向に速度 Ω で逆向きに伝播する．この解は回転系で見れば波数 $(1, 1)$ のロスビー波である．慣性系から見て保存している極をとおる剛体回転流を, 回転系から見ると Ω で逆まわりしているように見えるのがこの解である．

$(1, -1)$ 成分は $(1, 1)$ 成分と同様に粘性項, 非線形項の寄与が 0 となる．

$$
-\frac{2}{a^2}\frac{d\tilde{\psi}_{1,-1}}{dt} - i\frac{2\Omega}{a^2}\tilde{\psi}_{1,-1} = 0, \qquad \tilde{\psi}_1^1 = Ae^{-i\Omega t}. \tag{26}
$$

したがって $(1, 1)$ 成分の振舞は

$$
\psi(\mu,\lambda,t) = Ae^{-i(\lambda+\Omega t)}P_1^1(\mu) \tag{27}
$$

緯度方向に速度 Ω で逆向きに伝播する．特に回転がなければ時間微分が 0 となり保存する．

文献

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