

# Trigonometry without $\pi$ : A Constructive Approach

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## Abstract

We present a construction of trigonometry in which angles are not real numbers, but pairs  $(x, y)$  satisfying  $x^2 + y^2 = 1$ . Several classical trigonometric formulas naturally emerge during this construction, leading to the definition of angle division by natural numbers via a convergent sequence. Notably, this approach does not rely on the prior definition of the constant  $\pi$ , which is never used. All results are rigorously proven using the Rocq proof assistant (formerly known as Coq), but no prior knowledge of Rocq is required to read this paper, as no Rocq code appears.

## 1 Introduction

The title *trigonometry without  $\pi$*  might sound a bit clickbait, and we admit it is. It’s hard to imagine how one could do trigonometry without  $\pi$ : a right angle is  $\pi/2$ , angles are often defined up to  $2k\pi$ , formulas involve things like  $\cos(x + \pi/2)$ , and even physicists write  $\hbar/2\pi$ ,  $2\pi\epsilon_0$  and all that sort of things. As soon as something rotates,  $\pi$  shows up.

This work does not aim to replace  $\pi$  with degrees or grads. It’s just a different approach, one in which  $\pi$  does not appear—not because we tried to eliminate it, but because we simply didn’t need it.

The main goal is to present a construction of trigonometry that allows us to recover some classical formulas, and that leads—this is the real point—to dividing an angle by a natural number. This turns out to be less trivial than it sounds, given the framework we adopt.

**Disclaimer.** Trigonometry has been around for centuries, if not millennia. I do not claim to have discovered anything groundbreaking. What I present here may well have been thought of before. I’m not a specialist, and a web search turns up thousands of pages that all say more or less the same thing. It’s hard to find sources that stray from the standard path.

So, this work is not necessarily original in content. What may be more original is the fact that it has been formalized it as a proof, in Rocq.

There is, however, something vaguely similar to my approach: Norman Wildberger’s *rational trigonometry* (2005) [Wil05]. But there are three key differences.

First, Wildberger avoids the word *angle* altogether; he reserves it for classical trigonometry, and instead uses the term *spread*. In contrast, I do use the word *angle*, but with a different definition.

Second, Wildberger rejects the square root, which makes sense if one wants to stick to rational numbers. He avoids expressions like  $\sqrt{2}/2$ , the cosine of  $45^\circ$  or  $\pi/4$ , and works with *quadrance*, the square of the distance:  $x^2 + y^2$  rather than  $\sqrt{x^2 + y^2}$ . We, on the other hand, not only accept square root, we use it in this approach, as we are going to see further.

Third, one of Wildberger’s goals is computational efficiency: rational arithmetic is faster than floating-point. Personally, I don’t care. I prefer doing pure math. What matters to me is that the construction is provable, proved, and holds up logically.

For those familiar with the Rocq proof assistant, I decided not to use MathComp and instead developed my own implementation, based on my own approach of mathematics.

## 2 Why trigonometry?

Why trigonometry? It is not that I am especially interested in the subject, but I needed, in one of my proofs, to show that every complex number  $x + iy$  has an  $n$ th-root, for every nonzero integer  $n$ .

How is that done? The trick is well known: one writes the complex number  $x + iy$  in polar form:

$$x + iy = \rho e^{i\theta},$$

and then the answer is immediate:

$$\sqrt[n]{x + iy} = \sqrt[n]{\rho} e^{i\theta/n}.$$

Here, the left-hand part of the right member is the  $n$ th-root of a nonnegative real number, and not of a complex number. In the context of my work, the existence of such a root can be assumed as an axiom. What remains is the right-hand part.

On the right, we have  $\theta$ , an angle. Hence, trigonometry. And this angle must be divided by an integer. At first, I thought that one had to introduce the exponential function, and, since according to Euler

$$e^{i\theta} = \cos \theta + i \sin \theta,$$

also the sine and cosine functions, all defined by power series.

However, I recalled one of my previous proofs: the Banach–Tarski paradox [dR17]. In that proof, there are rotations in every direction, and rotations involve angles, sine, and cosine. I noticed, while coding that proof in Rocq, that I often ended up with definitions and theorems where the pair  $(\cos \theta, \sin \theta)$  appeared either as parameters or as hypotheses. But I realized that, in essence, I did not care that it was an angle; I did not need sine and cosine. It was much simpler to use pairs  $(x, y)$  with  $x^2 + y^2 = 1$ , thereby ignoring traditional trigonometry and all its business.

Later, in that proof I indeed had to use  $\cos$  and  $\sin$ , but for a large part of it the pairs  $(x, y)$  satisfying  $x^2 + y^2 = 1$  were sufficient.

So for this work, we decided to see if this approach could be defined systematically.

## 3 Basic Construction

### 3.1 Angles

In classical trigonometry, angles are real numbers. The set  $A$  of angles is therefore defined by

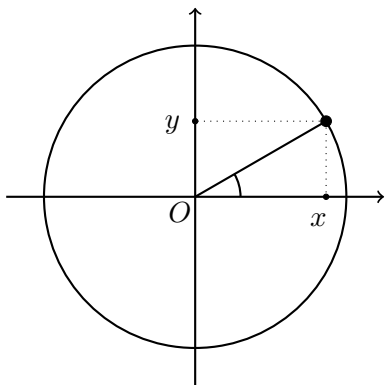
$$A ::= \mathbb{R}$$

Here, we set aside this definition and instead define:

$$A ::= \{ (x, y) \mid x^2 + y^2 = 1 \}$$

In other words, an angle is defined by a point on the unit circle. This point represents the angle between the positive  $x$ -axis and the line segment from the origin  $O$  to the point.

The angle  $(x, y)$  is illustrated below:



In our setting, an angle  $\theta$  is not a real number, but a triple of the form

$$\theta = (x, y, p),$$

where  $(x, y) \in \mathbb{R}^2$ , and  $p$  is a proof (or witness) that  $x^2 + y^2 = 1$ ; we define:

$$\cos(\theta) := x, \quad \sin(\theta) := y.$$

While in classical mathematics, cosine and sine are defined by power series, here they are just the first two components of the triplet. The theorem:

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

is true by definition: its proof is  $p$ , the third component of the triplet.

### 3.2 Notations

In the following, we sometimes represent an angle just as the pair  $(x, y)$ , the third component being implicit. Also, even if the constant  $\pi$  is not required, we use it as a notation to represent some angles, as follows:

$$\begin{array}{ll} \pi/2 & := (0, 1) \quad \text{the right angle} \\ \pi & := (-1, 0) \quad \text{the straight angle} \end{array}$$

we also write

$$\theta_1 + \theta_2 < 2\pi$$

to express

“the sum of these angles is less than one turn”

These notations are purely symbolic: they are shortcuts for angles, not expressions involving the real constant  $\pi$ .

### 3.3 Drawback

The difficulty arises from the arithmetic of angles. In classical trigonometry, this issue does not appear, since angles are treated as real numbers. Any operation that is defined on the reals—addition, subtraction, multiplication, division, and so on—is directly applicable to angles.

However, when we think of angles as points on the unit circle, we can no longer rely on the standard algebraic operations. These points, in themselves, do not carry any inherent arithmetic structure. In order to define such operations, we must construct them from scratch. We begin with addition in the next section.

## 4 Addition of angles

If we have two angles  $(x, y)$  and  $(x', y')$ , we want to define an addition, giving a third angle  $(x'', y'')$ , such that  $x''^2 + y''^2 = 1$ .

$$(x, y) + (x', y') ::= (x'', y'')$$

The solution comes from normal trigonometry. If we have two angles  $a$  and  $b$ , we know that

$$\cos(a + b) = \cos a \cdot \cos b - \sin a \cdot \sin b$$

$$\sin(a + b) = \sin a \cdot \cos b + \cos a \cdot \sin b$$

In our trigonometry, if we consider that the angle  $(x, y)$  is “ $a$ ”, and the angle  $(x', y')$  is “ $b$ ”, the angle  $(x'', y'')$  is “ $a + b$ ”. Applying the classical formulas above, we can define the addition of angles as

$$(x, y) + (x', y') := (xx' - yy', xy' + yx').$$

and we can prove that the sum of the squares of the RHS is indeed 1. This is a consequence of the fact that  $x^2 + y^2 = 1$  and  $x'^2 + y'^2 = 1$ . The two classical formulas above, about  $\cos(a + b)$  and  $\sin(a + b)$ , are now true by definition of  $+$ .

## 5 Additive group

Addition of angles is trivially commutative, and associativity can also be proven. The neutral element is  $(1, 0)$ , the null angle. It says that  $\cos 0 = 1$  and  $\sin 0 = 0$ . The inverse element of  $(x, y)$  is  $(x, -y)$ , expressing that  $\cos(-\theta) = \cos \theta$  and  $\sin(-\theta) = -\sin \theta$ . All these values and formulas are what we find in normal trigonometry.

Like for all additive groups, it is possible to define external multiplication by a natural  $n$ , by adding the element  $n$  times:

$$n \theta ::= \underbrace{\theta + \theta + \dots + \theta}_{n \text{ times}}$$

### And multiplication?

What about *multiplication* of two angles? In our framework, it seems not possible. Because *addition* of angles is actually *multiplication* of complex numbers. Perhaps, then, multiplication of angles could be something that *exponentiation* of complex numbers? But this intuition seems not to work.

Moreover, as far as geometry is concerned, multiplication of angles seems to have no meaning. In this work, we didn't define it.

This has a consequence: it is not possible to define exponentiation of imaginary angles either, which we found in the famous expression  $e^{i\theta}$  of classical trigonometry, supposed to be equal to  $\cos \theta + i \sin \theta$  (Euler's formula), because exponentiation is defined by a power series holding powers of its parameter:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

i.e. its parameter  $x$  *multiplied* several times by itself. Since multiplication of angles doesn't exist, Euler's formula doesn't work, but we see further, that de Moivre's formula works in our trigonometry and we use it. Remainder: de Moivre's formula is:

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$$

But what we need, for the moment, is division by a natural. We defined above multiplication by a natural. Division is more complicated but we see in next sections that it is possible to define it.

## 6 Trying to define the division of an angle by a natural

Starting with de Moivre's formula (stated above), which is easily provable by induction to  $n$ , and supposing we already have a division of an angle by a natural, we can apply it with  $\theta/n$  and get:

$$\left(\cos\left(\frac{\theta}{n}\right) + i \sin\left(\frac{\theta}{n}\right)\right)^n = \cos \theta + i \sin \theta$$

Warning: it doesn't mean that  $\theta/n$  exists. But, if it exists, it must fit this equation. Let's rewrite the equation with simple variables. Since  $\cos(\theta/n)$  and  $\sin(\theta/n)$  are our unknowns, let's name them  $x$  and  $y$ , and since  $\cos \theta$  and  $\sin \theta$  are our data, let's name them  $u$  and  $v$ , and we have to resolve:

$$(x + i y)^n = u + i v$$

If we develop the power to  $n$ , we get a complex number whose real and imaginary part are polynomials of degree  $n$  respectively in  $x$  and in  $y$ . In general, we have no formulas to resolve these equations for all  $n$ .

But we can resolve it for  $n = 2$ , i.e. for:

$$(x + i y)^2 = u + i v$$

and it will give us the way to compute  $\theta/2$ .

## 7 Division of an angle by 2

Resolving this equation, and knowing that the angle  $(x, y)$  is actually the angle  $(u, v)/2$ , we get a realistic definition of the division of an angle by 2:

$$\frac{(u, v)}{2} ::= \left( \sigma(v) \sqrt{\frac{1+u}{2}}, \sqrt{\frac{1-u}{2}} \right)$$

where  $\sigma(v)$  is the *sign* of  $v$ . We can compute twice the right hand part, i.e. the right hand part *plus* the right hand part, using our definition of addition, and we get  $(u, v)$  back. This definition of the division of the angle by 2 is therefore reasonable.

But we need a division of an angle by any  $n$ , not only by 2. How to do that? To answer this question, let's see, in next section, a method to divide the natural 1 by another natural, a normal division with decimals. In that section, we temporarily forget the fact that we work with angles. We return to angles the section after that one.

## 8 Division by a natural

Let's divide 1 by  $n$ , i.e. take the inverse of  $n$ . If we do the division in binary, we get something like:

$$1/n = 0.a_1a_2a_3\dots$$

where the  $a_k$  are bits, 0 or 1, since we decided to divide in radix 2. Examples from 2 to 5:

$$1/2 = 0.10000000...$$

$$1/3 = 0.01010101...$$

$$1/4 = 0.00100000...$$

$$1/5 = 0.00110011...$$

and so on. But this representation with this decimal dot and the list of digits is actually a syntax for the mathematical expression:

$$\sum_{k=1}^{\infty} \frac{a_k}{2^k}$$

If we were in base 10, as we usually and generally do, we would have to put  $10^k$  in the denominator, instead of  $2^k$ .

So, we have:

$$1/n = 0.a_1a_2a_3... = \sum_{k=1}^{\infty} \frac{a_k}{2^k}$$

Now, we claim that:

$$a_k = \left\lfloor \frac{2^k}{n} \right\rfloor \bmod 2$$

Indeed, starting with  $1/n = 0.a_1a_2a_3...a_k a_{k+1}...$ ,

we have,  $2^k/n = a_1a_2a_3...a_k.a_{k+1}...$

Therefore its integer part,  $\lfloor 2^k/n \rfloor$ , is  $a_1a_2a_3...a_k$

and its last digit,  $a_k$ , is this value *mod 2*.

In short:

$$1/n = \sum_{k=1}^{\infty} \frac{a_k}{2^k} \quad \text{where } a_k = \left\lfloor \frac{2^k}{n} \right\rfloor \bmod 2$$

End of this parenthesis without angles. Let's return to them.

## 9 Trying to define the division of an angle by a natural

In the previous formula, if we change the natural 1 into an angle  $\theta$ , that gives a possible definition of an angle divided by a natural:

$$\frac{\theta}{n} ::= \sum_{k=1}^{\infty} \frac{a_k \theta}{2^k} \quad a_k = \left\lfloor \frac{2^k}{n} \right\rfloor \bmod 2$$

We claim that this formula is indeed composed of operations on angles that we know how to perform.

- Inside the sum, the expression  $a_k \theta$  is an angle multiplied by a natural; we said above (section 5) that we know how to do that; by the way,  $a_k$  being a bit, this is simply the null angle if  $a_k = 0$  or the angle  $\theta$  itself if  $a_k = 1$ .
- We built a way to divide an angle by 2 (section 7), so it is possible to divide it by  $2^k$ , by repeating the operation  $k$  times.

- Therefore, the current term of the summation is an angle; since we have a sum of angles, we have a summation, provided we can define limits of sequences.

The next step is to prove that this power series converges. For that, we define the sequence of this power series up to some natural  $m$ :

$$\theta_m ::= \sum_{k=1}^m \frac{a_k \theta}{2^k}$$

and prove that this sequence converges when  $m$  tends to infinity. But before doing that, using the formula for  $a_k$ , it is possible to prove that, in fact,  $\theta_m$ , the partial sum up to  $m$ , is simply equal to:

$$\theta_m ::= \left\lfloor \frac{2^m}{n} \right\rfloor \frac{\theta}{2^m}$$

By the way, if we look at this much simpler formula, we see that  $2^m$  appears in both the numerator and denominator. Of course, we cannot simplify it, but if we do so intuitively, just to see, we note that what remains is  $\theta/n$ .

Since we are in a case where there is no closed formula for the limit, we:

- proved that  $(\theta_m)$  is a Cauchy sequence;
- proved that the set of angles is complete.

Since we have these properties, we are allowed to consider

$$\theta' = \lim_{m \rightarrow \infty} \theta_m$$

and we proved that this  $\theta'$  satisfies  $n\theta' = \theta$ , which is a reasonable property for something supposed to be  $\theta/n$ .

Now, replacing  $\theta_m$  by its value, we decided to define  $\theta/n$  as:

$$\boxed{\frac{\theta}{n} ::= \lim_{m \rightarrow \infty} \left\lfloor \frac{2^m}{n} \right\rfloor \frac{\theta}{2^m}}$$

and a proof that  $n(\theta/n) = \theta$  is available in the code.

## 10 Difficulties

We don't give the detail of all these proofs, it would be rather long, it could require a paper by itself, and they are accessible in the sources, anyway. But we met several difficulties. We cite the most two important ones.

A notable challenge comes from the fact that in our trigonometry, when adding two angles, the properties may change if this sum exceeds a turn.

## 10.1 First difficulty

We had to prove that:

$$\frac{\theta_1 + \theta_2}{2} = \frac{\theta_1}{2} + \frac{\theta_2}{2}$$

This seems obvious, and it is when angles are real numbers, but in our trigonometry, it is not. We must consider the 4 quadrants of  $\theta_1$  and the 4 quadrants of  $\theta_2$ , i.e. 16 cases. All cases are different and we didn't find common rules or lemmas, that could have made this proof easier.

Moreover, this proof is cursed by the malediction above, i.e. when the sum  $\theta_1 + \theta_2$  exceeds one turn, we must add the straight angle in the right hand side. The correct formula is actually:

$$\frac{\theta_1 + \theta_2}{2} = \begin{cases} \frac{\theta_1}{2} + \frac{\theta_2}{2} & \text{if } \theta_1 + \theta_2 < 2\pi \\ \frac{\theta_1}{2} + \frac{\theta_2}{2} + \pi & \text{otherwise} \end{cases}$$

A way to see the problem is to look at the case when  $\theta_1 = \theta_2 = \pi$ . We see that the LHS is 0 but the RHS is  $\pi$ . Adding  $\pi$  in that case returns to 0. The 16 cases of the proof therefore became 32 cases.

## 10.2 Second difficulty

We had to prove that:

$$\theta_1 + \theta_3 \leq \theta_2 + \theta_3 \Rightarrow \theta_1 \leq \theta_2$$

Three angles, this time, i.e. 64 cases, times the cases of sum of angles exceeding a turn.

# 11 Further directions

After the construction of the division of angles by a natural, we tried out to see if this approach of trigonometry could go further.

However, these developments lie outside the scope of this article. We just mention them here.

## 11.1 Derivation

We have defined a notion of limit and differentiability, allowing proofs of the classical results such as  $\cos' = -\sin$ ,  $\sin' = \cos$ , and  $\tan' = 1/\cos^2$ . For this last case, we had to previously prove the formulas of derivation of product and inverse.

Note that, generally, derivation is for functions from  $\mathbb{R}$  to  $\mathbb{R}$ . Here, our functions  $\cos$ ,  $\sin$  and  $\tan$  are from  $\mathbb{U}$  (unit circle) to  $\mathbb{R}$ . This implies a good definition of limits, continuity and derivability when the domain of the function is  $\mathbb{U}$ .

In fact, we made a general definition of limit when tending to a neighbourhood for any functions whose domain is  $A$  and codomain is  $B$  when both sets have a distance, and the set  $A$  has a relation order. So it can be used for any kind of function. This can be described in another paper.

## 11.2 Hyperbolic trigonometry

Also, we defined hyperbolic trigonometry, where hyperbolic angles are on the hyperbole, i.e. points  $(x, y)$  such that  $x^2 - y^2 = 1$ , allowing to define  $\cosh$  and  $\sinh$ . The definition must include the condition  $x > 0$ , because hyperbolic functions only work for the right branch of the hyperbole.

### 11.3 Other formulas

Of course, there is the matter of proving additional trigonometric formulas. Although we have not done so because they were not needed, they should be established. In one of our works, we also proved the formula for  $\cos p + \cos q$ , which is not exactly the standard version, as it depends on whether  $p + q$  and  $p - q$  exceed a full turn or not.

### 11.4 Nth roots of unity

One can observe that dividing an angle by a natural number allows us to define the  $n$ th root of unity. However, note that one must not write  $2\pi/n$ , because in our construction  $2\pi = 0$ , yielding only the trivial root. The trick is to write  $2(\pi/n)$ , by parenthesizing  $\pi/n$  (which is important). Indeed,  $\pi/n$  is nonzero.

### 11.5 Lie groups

One may also note that the unit circle is a Lie group, which seems to be reflected in this construction. However, since my knowledge in this area is limited, I cannot fully develop the consequences of this.

### 11.6 Exponential of an imaginary angle

We have seen that it is impossible to define  $e^{i\theta}$  in our framework, since we do not have an exponential function due to the fact that we cannot define the multiplication of angles. Moreover,  $i\theta$  is not an angle; it is an “imaginary angle”. Yet, an imaginary angle resembles a hyperbolic angle. In classical hyperbolic trigonometry, we have

$$e^a = \cosh a + \sinh a,$$

which provides a definition of the exponential, and one can show that  $e^{a+b} = e^a \cdot e^b$ , the well-known property of exponentials. However, in this case, “ $a$ ” denotes a hyperbolic angle, that is, a pair  $(x, y)$  such that  $x^2 - y^2 = 1$ , which is not an exponential on the reals as one might desire. One might define the logarithm, but it would again yield a hyperbolic angle rather than a real number.

### 11.7 From normal trigonometry to trigonometry without $\pi$ and vice versa

It is important to note that switching between normal trigonometry and trigonometry without  $\pi$  is straightforward.

- If we are given an angle  $\theta \in \mathbb{R}$  in the usual sense, we can represent it as the pair  $(\cos \theta, \sin \theta)$  in our construction.
- Conversely, if we have a pair  $(x, y)$  such that  $x^2 + y^2 = 1$ , we can recover the “normal” angle by computing the arccosine:  $\theta = \sigma(y) \operatorname{acos}(x)$ , where  $\sigma(y)$  is the *sign* of  $y$ . This gives us an angle between  $-\pi$  and  $\pi$ , as expected in traditional trigonometry.

## 12 Conclusion

This paper has presented a constructive approach to trigonometry, where the notion of an angle is derived from the unit circle and its geometric properties, without the need for a predefined constant like  $\pi$ . Through the formalization in Rocq, we have shown that angle division is possible within this framework, offering a new way of understanding trigonometric relationships. While this approach

is not intended to replace traditional trigonometry, it provides an alternative perspective that is grounded in formal reasoning and geometric intuition.

The formalization of these concepts in Rocq has implications beyond the immediate scope of trigonometry. It highlights the potential for formal methods to reshape classical areas of mathematics, providing a solid foundation for further exploration in both mathematical theory and practical applications.

In this framework,  $\pi$  is no longer a prerequisite — it is a derived notation. Whether that makes trigonometry more elegant or more unsettling is left to the reader’s taste.

## 13 Acknowledgements

I would like to express my sincere gratitude to Geoffroy Couteau (CNRS), a leading expert in cryptography, for his invaluable assistance in overcoming several proof challenges during the course of this work. In particular, the formula for the convergent sequence towards  $\theta/n$  was derived during one of our discussions, and I am deeply grateful for his input. While his expertise is not directly related to the mathematical content of this paper, his insights were critical in advancing this project.

I am also thankful to my colleagues, who patiently listened to my spontaneous visits in their offices, where I would often share my formalization journey, discuss tricky proof obstacles, or simply express my excitement over a newly proven lemma. Their curiosity, support, and occasional mathematical insights created the kind of collaborative environment where ideas can truly flourish.

## 14 Source code

[https://github.com/roglo/rocq\\_trigo\\_without\\_pi](https://github.com/roglo/rocq_trigo_without_pi)

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