Non-official Spectral Impossibility of Blow-up

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1 Energy Evolution for Leray's Self-Similar Solutions with $a = 1/2$

For the normalized case where $a = 1/2$, the self-similar ansatz[Leray 34, NRS 96] simplifies to:

$$\mathbf{u}(x,t) = \frac{1}{\sqrt{T-t}}\mathbf{U}\left(\frac{x}{\sqrt{T-t}}\right) \tag{1}$$

where $T \in \mathbb{R}$ is the blow-up time, and $\mathbf{U} = (U_1, U_2, U_3)$ is defined on $\mathbb{R}^3$.

1.1 Scaling Parameter

Define the **scaling parameter**:

$$\lambda(t) = \sqrt{T-t} \tag{2}$$

so the velocity field becomes $\mathbf{u}(x,t) = \lambda^{-1}\mathbf{U}(x/\lambda)$.

1.2 Step 1: Kinetic Energy $E(t)$

The kinetic energy is $E(t) = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{u}|^2 dx$. Substituting the ansatz:

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^3} \left|\lambda^{-1}\mathbf{U}\left(\frac{x}{\lambda}\right)\right|^2 dx = \frac{1}{2\lambda^2}\int_{\mathbb{R}^3} \left|\mathbf{U}\left(\frac{x}{\lambda}\right)\right|^2 dx \tag{3}$$

Apply the change of variables $y = x/\lambda$ (so $dx = \lambda^3 dy$):

$$E(t) = \frac{1}{2\lambda^2}\int_{\mathbb{R}^3} |\mathbf{U}(y)|^2 \lambda^3 dy = \frac{\lambda}{2}\int_{\mathbb{R}^3} |\mathbf{U}(y)|^2 dy \tag{4}$$

Let $C = \int_{\mathbb{R}^3} |\mathbf{U}(y)|^2 dy$ (a **constant** independent of time). Then:

$$\boxed{E(t) = \frac{\lambda C}{2} = \frac{\sqrt{T-t}}{2}C} \tag{5}$$

1.3 Step 2: Time Derivative dE/dt

Differentiate $E(t)$ with respect to t:

$$\frac{dE}{dt} = \frac{C}{2}\frac{d\lambda}{dt}, \quad \text{where} \quad \lambda = (T-t)^{1/2} \tag{6}$$

Compute $d\lambda/dt$:

$$\frac{d\lambda}{dt} = \frac{1}{2}(T-t)^{-1/2} \cdot (-1) = -\frac{1}{2\sqrt{T-t}} = -\frac{1}{2\lambda} \tag{7}$$

Substitute into dE/dt:

$$\frac{dE}{dt} = \frac{C}{2}\left(-\frac{1}{2\lambda}\right) = -\frac{C}{4\lambda} \tag{8}$$

Since $E(t) = \lambda C/2$, we have $C = 2E(t)/\lambda$. Substituting:

$$\frac{dE}{dt} = -\frac{1}{4\lambda} \cdot \frac{2E(t)}{\lambda} = -\frac{E(t)}{2\lambda^2} \tag{9}$$

Since $\lambda^2 = T - t$:

$$\boxed{\frac{dE}{dt} = -\frac{E(t)}{2(T-t)}} \tag{10}$$

1.4 Step 3: Viscous Dissipation Term

The energy equation requires $\int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 dx$. Computing the gradient of $\mathbf{u}$:

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\lambda^{-1} U_i\left(\frac{x}{\lambda}\right)\right) = \lambda^{-2}\frac{\partial U_i}{\partial y_j} \tag{11}$$

The Frobenius norm is:

$$|\nabla \mathbf{u}|^2 = \sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j}\right)^2 = \lambda^{-4}|\nabla \mathbf{U}|^2 \tag{12}$$

Integrating over $\mathbb{R}^3$:

$$\int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 dx = \int_{\mathbb{R}^3} \lambda^{-4}|\nabla \mathbf{U}(y)|^2 \lambda^3 dy = \lambda^{-1}\int_{\mathbb{R}^3} |\nabla \mathbf{U}(y)|^2 dy \tag{13}$$

Let $D = \int_{\mathbb{R}^3} |\nabla \mathbf{U}(y)|^2 dy$ (another **constant**). Then:

$$\int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 dx = \lambda^{-1} D \tag{14}$$

1.5 Step 4: Energy Equation Constraint

The Navier-Stokes energy equation is:

$$\frac{dE}{dt} = -\nu \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 dx \tag{15}$$

Substituting the expressions from Steps 2 and 3:

$$-\frac{C}{4\lambda} = -\nu \left(\lambda^{-1} D\right) \tag{16}$$

Simplifying (multiply both sides by $-\lambda$):

$$\frac{C}{4} = \nu D \tag{17}$$

Recalling $C = \int_{\mathbb{R}^3} |\mathbf{U}|^2 dy$ and $D = \int_{\mathbb{R}^3} |\nabla \mathbf{U}|^2 dy$:

$$\boxed{\frac{1}{4} \int_{\mathbb{R}^3} |\mathbf{U}(y)|^2 dy = \nu \int_{\mathbb{R}^3} |\nabla \mathbf{U}(y)|^2 dy} \tag{18}$$

1.6 Preliminary Results

1. **Energy Evolution:**

$$E(t) = \frac{\sqrt{T-t}}{2} \int_{\mathbb{R}^3} |\mathbf{U}(y)|^2 dy \propto (T-t)^{1/2} \tag{19}$$

   The kinetic energy decays as $(T-t)^{1/2}$ approaching the potential blow-up time $t = T$.

2. **Energy Dissipation Rate:**

$$\frac{dE}{dt} = -\frac{E(t)}{2(T-t)} \tag{20}$$

   Energy dissipation accelerates as $t \to T^-$, with the rate inversely proportional to the remaining time.

3. **Self-Similar Profile Constraint:**

$$\frac{1}{4} \int_{\mathbb{R}^3} |\mathbf{U}|^2 dy = \nu \int_{\mathbb{R}^3} |\nabla \mathbf{U}|^2 dy \tag{21}$$

   Critical balance condition: For the normalized case ($a = 1/2$), exactly one-quarter of the kinetic energy norm must equal the viscous dissipation integral multiplied by the viscosity. This constraint determines whether a self-similar blow-up solution $\mathbf{U}$ can exist.

2 Non-Existence of Leray Self-Similar Solutions via Spectral Analysis

We prove that no non-trivial Leray self-similar solutions exist by showing that the energy constraint derived earlier leads to a spectral contradiction.

2.1 The Energy Constraint

For $a = 1/2$, the self-similar profile constraint is:

$$\boxed{\frac{1}{4}\int_{\mathbb{R}^3} |\mathbf{U}|^2 dy = \nu \int_{\mathbb{R}^3} |\nabla \mathbf{U}|^2 dy} \tag{22}$$

We prove this implies $\mathbf{U} \equiv 0$ via vector spherical harmonics and spectral analysis.

2.2 Step 1: Vector Spherical Harmonics Expansion

Any divergence-free vector field $\mathbf{U}$ can be expanded in toroidal ($\mathbf{Y}^{(T)}$) and poloidal ($\mathbf{Y}^{(P)}$) components:

$$\mathbf{U}(\mathbf{y}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[f_{\ell m}(r) \mathbf{Y}_{\ell m}^{(T)}(\theta, \phi) + g_{\ell m}(r) \mathbf{Y}_{\ell m}^{(P)}(\theta, \phi) \right] \tag{23}$$

where:

- $r = |\mathbf{y}|$, and (θ, ϕ) are spherical coordinates
- Vector spherical harmonics are divergence-free: $\nabla \cdot \mathbf{Y}_{\ell m}^{(T)} = \nabla \cdot \mathbf{Y}_{\ell m}^{(P)} = 0$
- Orthogonality: $\langle \mathbf{Y}_{\ell m}^{(A)}, \mathbf{Y}_{\ell' m'}^{(B)} \rangle_{L^2(S^2)} = \delta_{AA'} \delta_{\ell \ell'} \delta_{mm'}$ for $A, B \in \{T, P\}$

2.3 Step 2: Decoupling of the Constraint

By orthogonality of vector spherical harmonics, the energy constraint **decouples completely** for each mode (ℓ, m):

Toroidal modes:

$$\frac{1}{4}\int_{\mathbb{R}^3} \left| f_{\ell m} \mathbf{Y}_{\ell m}^{(T)} \right|^2 d\mathbf{y} = \nu \int_{\mathbb{R}^3} \left| \nabla \left(f_{\ell m} \mathbf{Y}_{\ell m}^{(T)} \right) \right|^2 d\mathbf{y} \tag{24}$$

Poloidal modes:

$$\frac{1}{4}\int_{\mathbb{R}^3} \left| g_{\ell m} \mathbf{Y}_{\ell m}^{(P)} \right|^2 d\mathbf{y} = \nu \int_{\mathbb{R}^3} \left| \nabla \left(g_{\ell m} \mathbf{Y}_{\ell m}^{(P)} \right) \right|^2 d\mathbf{y} \tag{25}$$

2.4 Step 3: Radial Constraint Equations

For any mode $\mathbf{V} = f(r)\mathbf{Y}_{\ell m}^{(T/P)}$, using properties of vector spherical harmonics:

L^2-norm:

$$\int_{\mathbb{R}^3} |\mathbf{V}|^2 d\mathbf{y} = \int_0^\infty |f(r)|^2 r^2 dr \tag{26}$$

Dirichlet energy:

$$\int_{\mathbb{R}^3} |\nabla \mathbf{V}|^2 d\mathbf{y} = \int_0^\infty \left[|f'(r)|^2 + \frac{\ell(\ell+1)}{r^2} |f(r)|^2 \right] r^2 dr \tag{27}$$

The constraint becomes:

$$\boxed{\frac{1}{4} \int_0^\infty |f(r)|^2 r^2 dr = \nu \int_0^\infty \left[|f'(r)|^2 + \frac{\ell(\ell+1)}{r^2} |f(r)|^2 \right] r^2 dr} \tag{28}$$

This holds identically for both toroidal and poloidal radial functions.

2.5 Step 4: Eigenvalue Problem Formulation

Rearranging as a Rayleigh quotient:

$$\mathcal{R}_\ell[f] \equiv \frac{\int_0^\infty \left[|f'|^2 + \frac{\ell(\ell+1)}{r^2} |f|^2 \right] r^2 dr}{\int_0^\infty |f|^2 r^2 dr} = \frac{1}{4\nu} \tag{29}$$

This is the Rayleigh quotient for the **radial operator:**

$$\boxed{H_\ell = -\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{\ell(\ell+1)}{r^2}} \tag{30}$$

acting on the Hilbert space $L^2(\mathbb{R}^+, r^2 dr)$.

The constraint equation $\mathcal{R}_\ell[f] = 1/(4\nu)$ implies that $1/(4\nu)$ **must be an eigenvalue** of H_ℓ.

2.6 Step 5: Spectral Analysis of H_ℓ

The operator H_ℓ is the **radial component of the Laplace-Beltrami operator** for divergence-free vector fields.

Spectral Properties:

- **Domain:** $L^2(\mathbb{R}^+, r^2 dr)$ with boundary conditions $f(0) = 0$ for $\ell \geq 1$, regularity at $r = 0$
- **For $\ell = 0$:** $H_0 = -d^2/dr^2 - (2/r)d/dr$ has continuous spectrum $[0, \infty)$
- **For $\ell \geq 1$:** The repulsive potential $\ell(\ell+1)/r^2 \to \infty$ as $r \to 0^+$

Key Result: H_ℓ has **no eigenvalues** for any $\ell \geq 0$. The spectrum is purely continuous: $\sigma(H_\ell) = [0, \infty)$.

Proof Sketch: The eigenvalue equation $H_\ell f = \lambda f$ is a Bessel-type ODE:

$$-f'' - \frac{2}{r} f' + \frac{\ell(\ell+1)}{r^2} f = \lambda f \tag{31}$$

- For $\lambda > 0$: Solutions are oscillatory but not square-integrable with weight r^2dr
- For $\lambda \leq 0$: Solutions are either singular at $r = 0$ or unbounded as $r \to \infty$
- No λ yields solutions in $L^2(\mathbb{R}^+, r^2dr)$

2.7 Step 6: Spectral Contradiction

The Fatal Contradiction:

- The energy constraint requires $1/(4\nu) > 0$ to be an eigenvalue of H_ℓ
- But H_ℓ has **no eigenvalues** in its spectrum for any $\ell \geq 0$
- Therefore, **no non-trivial radial functions $f_{\ell m}$ or $g_{\ell m}$ can satisfy the constraint**

2.8 Step 7: Special Analysis for $\ell = 0$

Toroidal modes: $\mathbf{Y}_{00}^{(T)} = 0$ (no $\ell = 0$ toroidal modes exist)

Poloidal modes: Reduce to purely radial flows:

$$\mathbf{U} = g_{00}(r)\mathbf{Y}_{00}^{(P)} \propto g_{00}(r)\frac{\mathbf{y}}{r} \tag{32}$$

Incompressibility constraint:

$$\nabla \cdot \mathbf{U} = \frac{1}{r^2}\frac{d}{dr}(r^2 g_{00}(r)) = 0 \implies g_{00}(r) \propto \frac{1}{r^2} \tag{33}$$

Integrability failure:

$$\int_0^\infty \left|\frac{1}{r^2}\right|^2 r^2dr = \int_0^\infty r^{-2}dr = \infty \tag{34}$$

Therefore $g_{00} \notin L^2(\mathbb{R}^+, r^2dr)$, so this case also fails.

2.9 Conclusion

Main Theorem: The energy constraint

$$\frac{1}{4}\int_{\mathbb{R}^3} |\mathbf{U}|^2 dy = \nu \int_{\mathbb{R}^3} |\nabla\mathbf{U}|^2 dy \tag{35}$$

has **only the trivial solution $\mathbf{U} \equiv 0$.**

Proof Summary:

1. All radial components must vanish: $f_{\ell m} = g_{\ell m} = 0$ for all ℓ, m
2. This is due to the spectral gap: no eigenvalues exist for the radial operators H_ℓ
3. Therefore $\mathbf{U} \equiv 0$ is the unique solution

Physical Implication: No non-trivial Leray self-similar blow-up solutions exist for the 3D Navier-Stokes equations. For the uniqueness of ansatz, we would need a separate derivation to be covered, if above argument stands correct.