A Hypersequent Calculus for Contingent Existence

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Motivation

The Problem with Quantified Modal Logic

- The most technically natural way of presenting quantified modal logic yields a modal logic in which we can derive the *Barcan formula*.

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$$\Diamond \exists x A \rightarrow \exists x \Diamond A$$

- But the Barcan formula has false instances according to commonsense metaphysics.

Awkwardness vs Superstition

"In both areas [quantified tense and modal logic], in fact, we have a choice between a certain amount of awkwardness and a certain amount of superstition."

–Arthur N. Prior

Past, Present and Future, p.160

Hypersequents

- Goal here is to navigate between awkwardness and superstition by looking at what's going on in proofs of the Barcan formula.

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- To do this we will use modal hypersequents in the style of Restall (2012). A *Hypersequent* is a multiset of sequents which we will write as

$$X_1 \vdash Y_1 \quad | \quad \ldots \quad | \quad X_n \vdash Y_n$$

Hypersequents: Rules

- Rules for the propositional connectives are 'local' to a sequent component $X_i \vdash Y_i$.

$$\frac{\mathcal{H} \mid X \vdash A, B, Y}{\mathcal{H} \mid X \vdash A \lor B, Y}\ [\lor R] \qquad \frac{\mathcal{H} \mid X, A \vdash Y \quad \mathcal{H} \mid X, B \vdash Y}{\mathcal{H} \mid X, A \lor B \vdash Y}\ [\lor L]$$

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- Rules for the modal perators have a more 'global' flavour

$$\frac{\mathcal{H} \mid X' \vdash Y', A \quad \mid \quad X \vdash Y}{\mathcal{H} \mid X' \vdash Y' \quad \mid \quad X \vdash Y, \Diamond A}\,[\Diamond R] \qquad \frac{\mathcal{H} \mid \ \vdash A \quad \mid \quad X \vdash Y}{\mathcal{H} \mid X, \Diamond A \vdash Y}\,[\Diamond L]$$

Derivations of the Barcan Formula

$$\dfrac{\dfrac{\dfrac{\dfrac{Fa \vdash Fa}{Fa \vdash \quad \mid \quad \vdash \Diamond Fa}\,[\Diamond R]}{Fa \vdash \quad \mid \quad \vdash \exists x \Diamond Fx}\,[\exists R]}{\exists x Fx \vdash \quad \mid \quad \vdash \exists x \Diamond Fx}\,[\exists L]}{\Diamond \exists x Fx \vdash \exists x \Diamond Fx}\,[\Diamond L]$$

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A Barcan Derivation

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- An adequate quantified modal logic needs to tell us not only that the above proof is invalid…
- … but also, by way of justifying the logical awkwardness that may be involved, where it led us astray and why.

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- ... involving tacit appeals to principles very similar to those they are meant to be proofs of.
- If we make the implicit binding of free-variables in formulas explicit then we can no longer derive the Barcan formula.

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- So his solution only works due to a "deliberate impoverishment of the formal machinery"
- Is this the only way to bring out this insight?

Primitive Sope Markers

- Let us employ variables in both their usual *object denoting* role, as well as a new *scope-indicating* role.

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  - *"concerning* α, *it is possibly* F*"*

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  and
  - *"possibly, concerning* a *it is* F*"*

$$\Diamond a \; Fa$$

- Where we can only bind a with a quantifier in the first case, but not the second.

A Sketch of a Formal Language

- Formulas in the langauge which we'll be concerned with can be constructed by taking closed sentences of the standard language of modal logic and:
  1. Removing occurrences of '$\exists$'
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- '$t\Diamond Ft$', '$\neg t\Diamond\exists xGtx$' and '$t\Diamond xRtx$' are all formulas.
- '$\Diamond Ft$' (no scope for t) and '$tRtu$' (no scope for u) are not.

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- Our rules for the existential quantifier, then, are the following (where in [∃L] a must be an *eigenvariable*):

$$\frac{\mathcal{H} \mid X \vdash Y, u_1 \ldots u_m u A^x_u}{\mathcal{H} \mid X \vdash Y, u_1 \ldots u_m \exists x A}\ [\exists R]$$

$$\frac{\mathcal{H} \mid X, u_1 \ldots u_m a A^x_a \vdash Y}{\mathcal{H} \mid X, u_1 \ldots u_m \exists x A \vdash Y}\ [\exists L]$$

So our rules for the quantifiers only bind variables whose scope-marking occurrence has wide-scope.

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- *Transparent* insertion rules make the terms available to be bound in the premises so available in the conclusion.
- Disjunction, for example, has transparent insertion rules

$$\frac{\mathcal{H} \mid X, u_1 \ldots u_m A \vdash Y \qquad \mathcal{H} \mid X, u_1 \ldots u_m B \vdash Y}{\mathcal{H} \mid X, u_1 \ldots u_m (A \vee B) \vdash Y}\ [\vee L]$$

$$\frac{\mathcal{H} \mid X \vdash u_1 \ldots u_m A, u_1 \ldots u_m B, Y}{\mathcal{H} \mid X \vdash u_1 \ldots u_m (A \vee B), Y}\ [\vee R]$$

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$$\frac{\mathcal{H} \mid X \vdash Y \quad \mid \quad t_1 \ldots t_n A \vdash}{\mathcal{H} \mid X, \Diamond t_1 \ldots t_n A \vdash Y}\ [\Diamond L]$$

$$\frac{\mathcal{H} \mid X \vdash Y \quad \mid \quad X' \vdash Y', t_1 \ldots t_n A}{\mathcal{H} \mid X \vdash Y, \Diamond t_1 \ldots t_n A \quad \mid \quad X' \vdash Y'}\ [\Diamond R]$$

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$$\frac{\mathcal{H} \mid X \vdash Y \quad \mid \quad X' \vdash Y', t_1 \dots t_n A}{\mathcal{H} \mid X \vdash Y, \Diamond t_1 \dots t_n A \quad \mid \quad X' \vdash Y'} [\Diamond R]$$

- Similarly the rules for negation are also opaque.

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- We just want to restrict when such quantification can occurr.
- *Exportation Rules* allow for this possibility. The exportation rules for $\Diamond$ are the following, where for $[\Diamond\mathrm{ExpR}]$ we require that there be some formula $a\,B \in X$.

$$\frac{\mathcal{H} \mid X \vdash Y, \Diamond aA}{\mathcal{H} \mid X \vdash Y, a\Diamond A}\,[\Diamond\mathrm{ExpR}] \qquad \frac{\mathcal{H} \mid X, \Diamond aA \vdash Y}{\mathcal{H} \mid X, a\Diamond A \vdash Y}\,[\Diamond\mathrm{ExpL}]$$

Unrestricted Exportation & The Barcan Formula

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$$
\begin{array}{cl}
aFa \vdash aFa & \\
\hline
aFa \vdash \quad | \quad \vdash \Diamond aFa & [\Diamond R] \\
\hline
aFa \vdash \quad | \quad \vdash a\Diamond Fa & [UE] \\
\hline
aFa \vdash \quad | \quad \vdash \exists x \Diamond Fx & [\exists R] \\
\hline
\exists x Fx \vdash \quad | \quad \vdash \exists x \Diamond Fx & [\exists L] \\
\hline
\Diamond \exists x Fx \vdash \exists x \Diamond Fx & [\Diamond L]
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$$

Barcan Formula Redux

- This approach gives us a very natural explanation of what goes ‘wrong’ in Barcan derivations—they involve exportation inferences which are hidden in the standard language.

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Barcan Formula Redux

- This approach gives us a very natural explanation of what goes ‘wrong’ in Barcan derivations—they involve exportation inferences which are hidden in the standard language.
- Moreover, the only justification for those inferences appears to be question begging (e.g. all terms denote an existent in every possibility).
- So we have a system which does away with the unwanted superstition without causing *too* much formal awkwardness.

Thank You!