# Logical Pluralism and Prover-Skeptic Games

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Logical Dialogue Games Workshop, Vienna • September 29, 2015

## 1 The Explanatory Problem for Logical Pluralism

- Logical Pluralists face a particular kind of explanatory problem, well put in the following quote due to Rosanna Keefe.

> A characterisation of a pluralist position needs to explain what it is to endorse all of the various consequence relations the pluralist accepts and how they relate to an intuitive notion of logical consequence. (Keefe, 2014, p.1376)

- This suggests the following two problems which any pluralist proposal must address:
  - What does it mean to endorse these logics.
  - How do they relate to an intuitive notion of logical consequence.
- Responding to this kind of challenge requires us to give some account of the nature of logical consequence (e.g. talk in Beall and Restall (2006) of the *Generalized Tarski Thesis*) and show how this allows for the 'endorsement' of multiple logics.
- Here we will tackle this problem by introducing a particular dialogical conception of logic (the 'built-in opponent' conception of deduction), and show that this gives rise to an interesting and natural kind of logical pluralism.

## 2 The Built-In Opponent Conception of Deduction

- Multi-agent dialogical account of logical consequence. Logical consequence is grounded in certain kinds of (specialized) discursive practices.
- In particular we identify *proofs* with a certain kind of semi-adversarial dialogue/debate involving two players: Prover/Proponent and Skeptic/Opponent.
- Prover seeks to establish that a certain conclusion follows from some given premises, and Skeptic challenges Prover to give the grounds for their claims.
- Deductive proofs then correspond to winning strategies in these dialogical interactions.

- Over time Skeptic has become *internalized* (or 'built-in') to the method of deductive inference, their role in effect being played 'offline', or 'simulated', by Prover.
- This allows multi-agent dialogical interactions to quite naturally give rise to our mono-agent inferential practices.

## 3 Prover-Skeptic Games: The Very Idea

- Games are played over a board of (for the time being, SET-FMLA) sequents $\Gamma \succ A$.
- A dialogue over the sequent $\Gamma \succ A$ begins with Prover asserting $\Gamma \succ A$.

- Play then alternates between Skeptic and Prover.
- In their moves Skeptic challenges a sequent

$$\Gamma \succ A_1 \to (\ldots \to (A_n \to p_i)\ldots)$$

used in the previous Prover move by issuing Prover the challenge to prove $\Gamma, A_1, \ldots, A_n \succ p_i$.
- This can be understood as Skeptic challenging the Prover to give the grounds for their assertion: "if I grant you $\Gamma, A_1, \ldots, A_n$ can you demonstrate that $p_i$".

- Prover responds to a Skeptic challenge $\Gamma \succ p_i$ by presenting sequents

$$\Gamma_1 \succ B_1, \ldots, \Gamma_n \succ B_n$$

such that
$$B_1 = B_2 \to (B_3 \to (\ldots \to (B_n \to p_i)\ldots))$$
where $\Gamma \supseteq \Gamma_1 \cup \ldots \cup \Gamma_n$.

- In giving the grounds for their assertion, Prover claims that if the Skeptic grants that each of the $\Gamma_i$ gives $B_i$ then $\Gamma$ gives $p_i$.
- Skeptic cannot challenge sequents of the form $\Gamma, A \succ A$.

## 4 Internalization

- One thing which is worth noting about the above described game is that creativity is only required in Prover moves.
- Skeptic moves are thoroughly deterministic (modulo choice of sequent to challenge).
- These moves are designed to force Prover to break down their inferences into digestible steps, showing what would need to be done to convince a tough—albeit uncreative—skeptic (sic) that the conclusion does indeed follow from the premises.

## 5 Prover-Skeptic Games: Formal Definition

- To see what is going on here it is more transparent to, following Shoham and Francez (2008), formulate these games in terms of a given natural deduction proof system (as opposed to Sørensen and Urzyczyn (2006) where everything is opaquely syntactic).
- Moreover, we will what to make the structural rules explicit so we will work in terms of multi-sets rather than sets.

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Axioms

$$A \succ A$$

Introduction/Elimination Rules

$$\frac{\Gamma, A \succ B}{\Gamma \succ A \to B}\ (\to\mathrm{I}) \qquad \frac{\Gamma \succ A \quad \Delta \succ A \to B}{\Gamma, \Delta \succ B}\ (\to\mathrm{E})$$

Structural Rules

$$\frac{\Gamma, A, A \succ B}{\Gamma, A \succ B}\,(W) \qquad \frac{\Gamma \succ B}{\Gamma, A \succ B}\,(K)$$

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- This is a natural deduction system, in sequent-to-sequent style, for Intuitionistic Implicational Logic (= IIL), with the structural rules *contraction* ($(W)$) and *weakening* ($(K)$) explicit.
- Let us settle some notation. Let $\mathfrak{s}, \mathfrak{s}'$ be sequents and let $\Sigma$ be a set of sequents.
  - $\mathfrak{s} \Rightarrow^*_{(I)} \mathfrak{s}'$ if we can derive $\mathfrak{s}'$ from $\mathfrak{s}$ using only introduction rules (i.e. only using $(\to \mathrm{I})$).
  - $\Sigma \Rightarrow^*_{(E)} \mathfrak{s}$ if we can derive $\mathfrak{s}$ from $\Sigma$ using only elimination rules and structural rules (i.e. only using $(\to \mathrm{E})$, $(W)$ and $(K)$).
- Further, say that a sequent $\Gamma \succ A$ is *atom-pointed* if $A$ is a propositional atom.

**Definition 5.1** (Dialogue Game)**.** *A dialogue over $(\Gamma, A)$ is a (possibly infinite) sequence $\Sigma_1, \beta_1, \Sigma_2, \beta_2, \ldots$ where each $\Sigma_i$ is a set of sequents, and each $\beta_i$ is an atom-pointed sequent where:*

1. $\Sigma_1 = \{\Gamma \succ A\}$
2. $\beta_i \Rightarrow^*_{(I)} \sigma$ *for some non-axiomatic* $\sigma \in \Sigma_i$
3. $\Sigma_{i+1} \Rightarrow^*_{(E)} \beta_i$

- Condition 5.1.1 is just the condition that Prover starts all dialogues by asserting the sequent under discussion; 5.1.2 states that Skeptic challenges some immediately-preceeding (interesting) Prover assertion; and finally 5.1.3 states that Prover meets the most recent Skeptic challenge by providing a collection of sequents from which they can derive the challenge.

**Definition 5.2** (Winning Strategy)**.** *A winning strategy (for Prover) for the dialogue over $(\Gamma, A)$ is a labelled tree where*

- *The root note of the tree is a P-node* $(\{\Gamma \succ A\})$.
- *Each branch is a dialogue over* $(\Gamma, A)$
- *Every P-node* $(\Sigma_i)$ *has* $|\Sigma_i|$ *S-node descendants.*
- *Every S-node has a single P-node descendant.*

- Let us say that a sequent is *Dialogically valid* if Prover has a winning strategy in the dialogue over that sequent. As we will see below, dialogical validity as defined here corresponds to validity in IIL.

## 6 An Example

**Transitivity** Consider the following dialogue over $\{p \to q, q \to r\} \succ p \to r$.

| | | |
|---|---|---|
| P(1) | $[p \to q, q \to r \succ p \to r]$ | *Prover Starts* |
| S(1) | $p \to q, q \to r, p \succ r$ | *Skeptic challenges the sole Prover assertion.* |
| P(2) | $[q \to r \succ q \to r,\ p \to q, p \succ q]$ | *Prover replies, as these sequents derive the challenge sequent.* |
| S(2) | $p \to q, p \succ q$ | *Skeptic challenges the only non-axiomatic sequent.* |
| P(3) | $[p \to q \succ p \to q, p \succ p]$ | *Prover replies.* |

At this point Skeptic has no available moves, and so Prover has won this dialogue. Moreover, as Skeptic had no other moves they could have made this dialogue shows that the above sequent is valid.

## 7 Structural Rules and Norms of Explanation

- The particular kinds of discursive practices which Prover-Skeptic games are meant to model are governed by obvious norms: e.g. that the Skeptic ought to only doubt/query claims which are not obvious (hence the ban on Skeptic challenging axiomatic sequents).
- There is another normative aspect of this practice which is worth singling out, though. Namely that both sides must agree on what counts as an adequate responce to a challenge.
- A response to a challenge is meant to be *explanatory*—to show *why* it is that a given conclusion follows from given premises.
- The norms on such explanations are prima facie relative to communities of inquiry, and the particular interests of those communities (compare explaining something at the pub to explaining something in the classroom).
- These different normative standards can have logical repercussions.
- **Example**: Early 20th Century Geometry
  - Work on the foundations of Geometry at the turn of the 20th century displays a concern, not just for which axioms are used in proving a theorem, but also with the *number of times* a given axiom is used in proving that theorem.
  - E.g. G. Hessenberg in 1905 proves that Desargues axiom follows from the axioms of plane geometry from a threefold use of the Pappus axiom. As Pambuccian (2004) notes all known proofs require, and explicitly mention, this threefold use.
  - This suggests that early 20th century Geometers require that one explains, not only that a conclusion follows from given premises, but also the *extent* to which different premises are used to show this.
- **Example**: Explanatory Relevance
  - Sometimes when we explain something we want to make sure that *all* of the premises are needed in providing the explanation.
  - This is particularly important in situations where there are many premises.
  - This suggests that in such situations we ought to drop (K), as this allows us to introduce premises that were not required for deriving the conclusion.

## 8 Prover-Skeptic Games for Sub-structural Logics

- What we have sketched above is a reason to think that what counts as a response to a challenge can be relative to a community of inquiry.
- This means we need to have some way to modify the Prover moves in our dialogues. This can easily be done, though. Note that the kinds of norms on explanation we are concerned with above concern matters of *how* given premises are used in answering the challenge; so stricter normative standards correspond to restrictions on Prover moves.
- In part we can accomplish this by limiting which structural rules we are allowed to appeal to in Prover moves (which amounts to constraints on the relationship bewteen $\Gamma$ and the $\Gamma_i$ in a Prover responce).

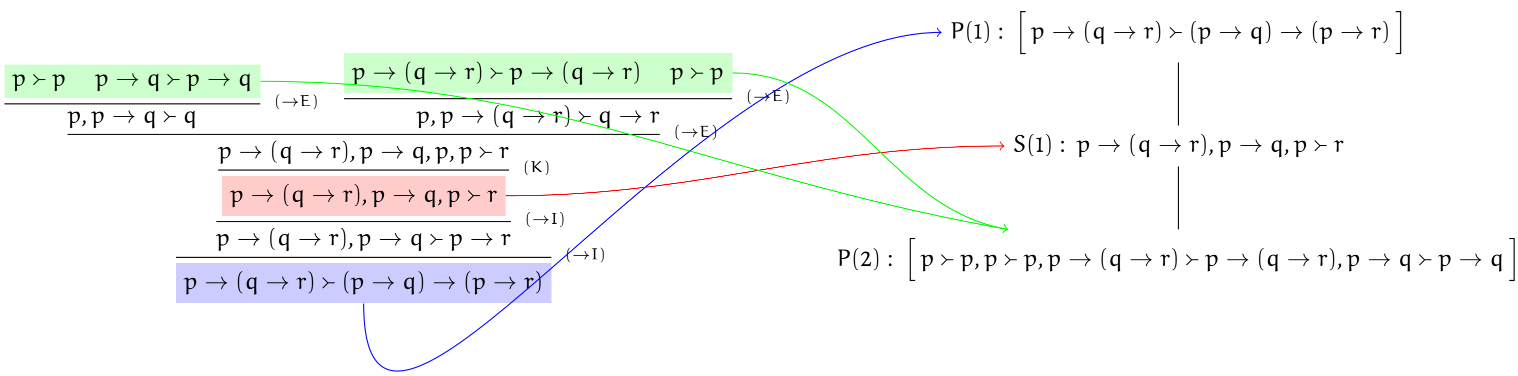
- Let $S \subseteq \{(W), (K)\}$ (i.e. a set of structural rules). Then an $S$-*dialogue* is a dialogue as in Definition 5.1, with the exception that Prover may only use structural rules from $S$ in their moves.
- We are then able to prove the following adequacy theorem

> **Theorem 8.1.** *Prover has a winning strategy in an* $S$*-dialogue over the sequent* $\Gamma \succ A$ *iff* $\vdash_S \Gamma \succ A$*, where* $\vdash_S$ *is the consequence relation determined by the proof system containing only the structural rules in* $S$*.*

- So we have a picture of logical consequence where what counts as a proof is interest relative, with the interests of a community of inquiry placing various normative requirements on what counts as a successful proof.

## 9 Coda: Adequacy of Prover-Skeptic Games, A Sketch

- Winning Strategies in Prover-Skeptic games correspond in quite a direct way to proof-sketches (and hence, modulo filling in details, to proofs).
- We can see this quite directly when we look at the connection between natural deduction proofs in the above system and Winning Strategies for Prover for the corresponding dialogue game.




- In the above figure we have both a proof of $p \to (q \to r) \succ (p \to q) \to (p \to r)$ on the left, and a winning strategy for Prover in the ((K)-)dialogue for this sequent, hightlighting the connections between different parts of the proof and the winning strategy.

- Of particular relevance is the central red block, which corresponds to the Prover challenge. This is the minimal sequent in the 'stack' of sequents which our end-sequent is on (the analogue here of Prawitz's notion of a track in a tree-style ND proof).
- Following the procedure that's implicit in the above diagram we can convert any proof (in a certain kind of normal form) into a winning strategy for Prover.

## 10 Bibliography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