Prover-Skeptic Games and Logical Pluralism

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A Problem for Logical Pluralists

The Explanatory Problem for Logical Pluralists

A characterisation of a pluralist position needs to explain what it is to endorse all of the various consequence relations the pluralist accepts and how they relate to an intuitive notion of logical consequence.

—Keefe, 2014, p.1376

Two Central Questions

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- Why are these consequence relations relations of logical consequence?

The Built-In Opponent Conception of Deduction

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- View **Proofs** as semi-adversarial debates between a Prover and a Skeptic.
- Prover seeks to establish the conclusion.
- Skeptic forces Prover to make their reasons explicit.
- Deductive proofs are winning strategies for Prover in these debates.

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- Proofs are not usually thought of as multi-agent dialogues.
- This is because Skeptic has become **Internalized** to the method, a very part of these practices.
- In effect their role is played ‘offline’ by Prover.
- This provides a bridge between Multi-Agent Dialogical and Mono-Agent Inferential Practices.

Introducing

Prover-Skeptic Games

A Dialogue

In the library two students, ***Petunia*** *and* ***Samson*** *are arguing over the validity of the argument from* $A \to B$ *and* $B \to C$ *to* $A \to C$*. Petunia thinks that this argument is valid and is trying to convince Samson, who is skeptical.*

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- **Petunia (1)**: I reckon that $A \to C$ follows from $A \to B$ and $B \to C$

A Dialogue

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- **Petunia (1)**: I reckon that $A \to C$ follows from $A \to B$ and $B \to C$
- **Samson (1)**: Yeah? Well if that's so then suppose I grant you $A \to B$ and $B \to C$ along with A, how are you mean to get C?

- **Petunia (2)**: If you grant me B I can get C from B $\rightarrow$ C (which you just granted).

A Dialogue

- **Petunia (2)**: If you grant me B I can get C from B $\rightarrow$ C (which you just granted).
- **Samson (2)**: But why should I grant you B?

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- **Petunia (3)**: Well if you were to grant me A then I could get B from $A \rightarrow$ B which you granted at the start.

A Dialogue

- **Petunia (2)**: If you grant me B I can get C from B $\rightarrow$ C (which you just granted).
- **Samson (2)**: But why should I grant you B?
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- **Samson (3)**: But why should I grant you A?
- **Petunia (4)**: Because you granted it to me at the start!

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Petunia leaves the library triumphantly.

Definition: A *dialogue* over the sequent $\Gamma \succ \varphi$ is a possibly infinite sequence $\Sigma_1, \mathfrak{s}_1, \Sigma_2, \mathfrak{s}_2, \ldots$ where:

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» *Prover Begins by asserting the sequent under discussion.*

Prover-Skeptic Games

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2. $\mathfrak{s}_i \Rightarrow^*_{(\mathrm{I})} \sigma$, for some non-axiomatic sequent $\sigma \in \Sigma_i$ where $\mathfrak{s}_i$ is of the form $\mathcal{S}_i \succ p_i$ for some propositional atom p_i.

» *Skeptic challenges an (interesting) Prover assertion*

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3. $\Sigma_{i+1} \Rightarrow^*_{(E)} \mathfrak{s}_i$

» *Prover replies, offering sequents which derive the challenge.*

Prover-Skeptic Games:

Some Examples

Revisiting the Previous Example

Consider the following dialogue over $p \to q, q \to r \succ p \to r$.

P(1) $[p \to q, q \to r \succ p \to r]$ » *Prover Starts*

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S(1) $p \to q, q \to r, p \succ r$

» *Skeptic challenges the sole Prover assertion.*

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P(2) $[q \to r \succ q \to r, p \to q, p \succ q]$

» *Prover replies, as these sequents derive the challenge sequent.*

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P(2) $[q \to r \succ q \to r, p \to q, p \succ q]$

S(2) $p \to q, p \succ q$

» *Skeptic challenges the only non-axiomatic sequent.*

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S(2) $p \to q, p \succ q$

P(3) $[p \to q \succ p \to q, p \succ p]$

» *Prover replies, and wins the dialogue.*

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P(3) $[p \rightarrow q \succ p \rightarrow q, p \succ p]$

Example: 'Prefixed' Weakening

Consider the dialogue for the sequent $p \to r \succ p \to (q \to r)$

P(1) $[p \to r \succ p \to (q \to r)]$ » *Prover Starts*

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» *Skeptic challenges, saying what would need to be shown to convince them.*

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» *From these two sequents Prover can derive the challenge by one application of* $(\to \mathsf{E})$ *and then* (K) *to weaken in* q.

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» *This wins the dialogue for Prover in the presence of* (K).

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P(2) $[p \to r \succ p \to r,\ \ p, q \succ p]$

» *The 'best' move Prover has available without* (K) *is the following.*

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Consider the dialogue for the sequent $p \to r \succ p \to (q \to r)$

P(1) $[p \to r \succ p \to (q \to r)]$

S(1) $p \to r, p, q \succ r$

P(2) $[p \to r \succ p \to r, \;\; p, q \succ p]$

S(2) $p, q \succ p$

» *Skeptic can only challenge the rightmost sequent.*

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P(2) $[p \to r \succ p \to r, \;\; p, q \succ p]$

S(2) $p, q \succ p$

P(3) $[q \succ p, p \succ p \to p]$

» *This is Prover's 'best' response, and things will quite quickly devolve from here...*

Example 2: Contraction

Consider the dialogue for the sequent $p \to (p \to q) \succ p \to q)$

P(1) $[p \to (p \to q) \succ p \to q]$ » *Prover starts.*

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» *Prover responds, deriving the challenge via two applications of* $(\to E)$ *and one of* (W).

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» *In the absence of (W) this is the 'best' move available to Prover.*

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» *Skeptic must challenge*

$p \to (p \to q) \succ p \to q$

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P(2) $[p \to (p \to q) \succ p \to q, \ \ p \succ p]$

S(2) $p \to (p \to q), p \succ q$

» *But this just puts us back in the position we were in at* S(1)

Structural Rules and Norms on Explanation

Prover-Skeptic Games for Sub-structural Logics

Adequacy of Prover-Skeptic Games

Adequacy: A Sketch

- Winning Strategies correspond to proof sketches.

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- Winning Strategies correspond to proof sketches.
- Constructing a Winning Strategy given a proof is a relatively simple operation.

From Proofs to Winning Strategies

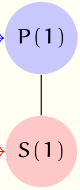
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \succ p \qquad p \to q \succ p \to q}{p, p \to q \succ q}\,(\to E) \qquad \dfrac{p \to (q \to r) \succ p \to (q \to r) \qquad p \succ p}{p, p \to (q \to r) \succ q \to r}\,(\to E)}{p \to (q \to r), p \to q, p, p \succ r}\,(\to E)}{p \to (q \to r), p \to q, p \succ r}\,(K)}{p \to (q \to r), p \to q \succ p \to r}\,(\to I)}{p \to (q \to r) \succ (p \to q) \to (p \to r)}\,(\to I)$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \succ p \quad p \to q \succ p \to q}{p, p \to q \succ q}\,(\to E) \qquad \dfrac{p \to (q \to r) \succ p \to (q \to r) \quad p \succ p}{p, p \to (q \to r) \succ q \to r}\,(\to E)}{p \to (q \to r), p \to q, p, p \succ r}\,(\to E)}{p \to (q \to r), p \to q, p \succ r}\,(K)}{p \to (q \to r), p \to q \succ p \to r}\,(\to I)}{p \to (q \to r) \succ (p \to q) \to (p \to r)}\,(\to I)
$$

P(1)

- $P(1) = \left[p \to (q \to r) \succ (p \to q) \to (p \to r)\right]$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \succ p \quad p \to q \succ p \to q}{p, p \to q \succ q}\,(\to E) \qquad \dfrac{p \to (q \to r) \succ p \to (q \to r) \quad p \succ p}{p, p \to (q \to r) \succ q \to r}\,(\to E)}{p \to (q \to r), p \to q, p, p \succ r}\,(\to E)}{\boxed{p \to (q \to r), p \to q, p \succ r}}\,(K)}{p \to (q \to r), p \to q \succ p \to r}\,(\to I)}{\boxed{p \to (q \to r) \succ (p \to q) \to (p \to r)}}\,(\to I)
$$



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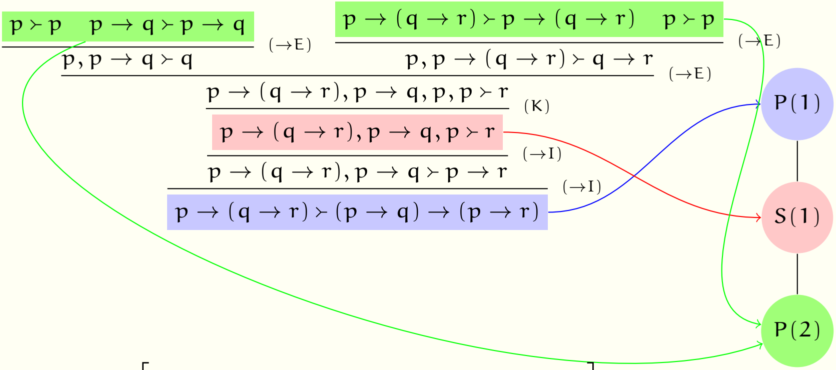
From Proofs to Winning Strategies



- $P(1) = \left[p \to (q \to r) \succ (p \to q) \to (p \to r) \right]$
- $S(1) = p \to (q \to r), p \to q, p \succ r$
- $P(2) =$

$$\left[p \to (q \to r) \succ p \to (q \to r) \quad , p \succ p \quad , p \succ p \quad , p \to q \succ p \to q \right]$$

Thank You!