Denumerably Many Post-Complete Normal Modal Logics with Propositional Constants*

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Abstract

We show that there are denumerably many Post-complete normal modal logics in the language which includes an additional propositional constant. This contrasts with the case when there is no such constant present, for which it is well known that there are only two such logics.

1 Introduction

Say that a logic is *Post-complete* if it is consistent and has no consistent proper extension. It is well known that there are only two Post-complete normal monomodal logics—**KT!** and the smallest normal modal logic to contain $\Box\bot$—alias **KVer**.[1] This contrasts starkly with the case where we are simply considering Post-complete quasi-normal monomodal logics (i.e. Post-complete modal logics which extend **K**) of which Segerberg has shown there are non-denumerably many.[2]

As the above examples show, the phenomenon of Post-completeness is quite sensitive to the lattice of logics under investigation. When we fix upon a particular lattice of modal logics, the Post-complete logics in that lattice will be its co-atoms. Nothing about this point requires us to be concerned with modal logics, of course, as can be seen in the discussion of the lattice relativity of Post-completeness in Humberstone (2000)—the discussion in fn.10 therein providing a number of further references on the issue.[3]

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[1] We will assume the reader is familiar with modal logic, any unexplained terminology being taken from Chellas (1980). Logics here will be thought of as sets of formulas, but rather than writing $A \in \mathsf{S}$ to denote that A is a theorem of S we will instead use the more perspicuous notation $\vdash_{\mathsf{S}} A$.

[2] Segerberg has written extensively on Post-completeness in modal logics. The interested reader is referred to Segerberg (1972, 1973, 1976), as well as Makinson and Segerberg (1974) and Kohn (1977).

[3] This distinction is recorded in Williamson (1998) by referring to the logics which we will below be calling Post-complete normal modal logics, as maximal consistent normal modal logics. This seems like a mistake to the present author, making it seem as if we are applying different concepts rather than applying the same concept in different settings.

If we shift to the case where we are considering bimodal normal modal logics—modal logics containing two modal operators, both of which are normal—the situation changes again. Here we have an increase in strength from the monomodal case, but, rather than there being fewer Post-complete modal logics, we instead find that there are now non-denumerably many such logics, as shown in Williamson (1998). This is a case where we have changed the lattice of modal logics under investigation, not by changing the closure conditions we place upon logics in the lattice, but by changing the language under investigation. Here we will investigate the number of Post-complete normal modal logics when we change the language by adding a propositional constant. In §2 we will go over some formal preliminaries before, in §3 proving our main result. In §4 we will end by discussing some of the ramifications of this result.

2 Formal Preliminaries

Let L be the propositional language constructed in the usual way from denumerably many propositional variables $p_0, p_1, p_2, \ldots$ using the connectives $\rightarrow$, $\neg$, and $\Box$—the other connectives being defined in terms of these as usual. Let L_κ be a proposition language just like L except that it also contains an additional propositional constant κ. When wanting to make comparative comments about normal modal logics formulated in the two languages, we will denote logics in the language L_κ by subscripting their name with κ. So, for example, $\mathbf{K}_\kappa$ is the smallest normal modal logic in the language L_κ.

A *frame* in this setting will be a ordered triple $\langle W, R, C\rangle$ where W is a non-empty set, R a binary relation on W, and $C \subseteq W$. Throughout we will adopt the convention of identifying isomorphic frames. A *model* is a frame along with a valuation function V which maps every propositional variable p_i to a subset of W (the set of members of W at which p_i is true). We will define truth at a point w in a model $\mathcal{M} = \langle W, R, C, V\rangle$ ("$\mathcal{M} \models_w A$") inductively as follows.

$$
\begin{array}{lcl}
\mathcal{M} \models_w p_i & \iff & w \in V(p_i). \\
\mathcal{M} \models_w \neg A & \iff & \mathcal{M} \not\models_w A. \\
\mathcal{M} \models_w A \rightarrow B & \iff & \mathcal{M} \not\models_w A \text{ or } \mathcal{M} \models_w B. \\
\mathcal{M} \models_w \Box A & \iff & \forall u(Rwu \Rightarrow \mathcal{M} \models_u A). \\
\mathcal{M} \models_w \kappa & \iff & w \in C.
\end{array}
$$

A formula A is *true throughout a model* $\mathcal{M} = \langle W, R, C, V\rangle$ ("$\mathcal{M} \models A$") whenever A is true at all points $w \in W$ in $\mathcal{M}$, *valid on a frame* $\mathfrak{F} = \langle W, R, C\rangle$ ("$\mathfrak{F} \models A$") whenever A is true throughout all models on that frame, and *valid at a point* w *in a frame* $\mathfrak{F}$ ("$\mathfrak{F} \models_w A$") whenever it is true at that point in all models on that frame. Given a class of frames $\mathcal{C}$ let $Log(\mathcal{C})$ be the set of all formulas A such that A is valid on all the frames in $\mathcal{C}$. As usual we will write $Log(\mathfrak{F})$ for $Log(\{\mathfrak{F}\})$.

3 Main Result

In this section we will exhibit a denumerable collection of distinct normal modal logics in the language L_κ, each one of which is Post-complete. Consider the frames $\mathfrak{L}_n = \langle W_n, R_n, C_n \rangle$, for $n \geq 1$.

- $W_n := \{0, 1, \ldots, n, n+1\}$.
- $R_n := \{\langle n+1, 0\rangle\} \cup \{\langle i, j\rangle | j = i + 1\}$.
- $C_n := \{0\}$.

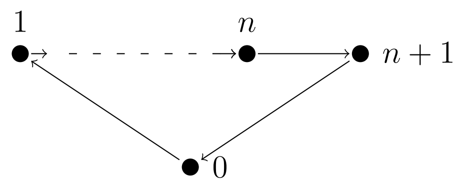

Figure 1: The Frame $\mathfrak{L}_n$.

We make the following observation without proof.

Proposition 3.1. *$\mathfrak{L}_n$ is the only (point-generated) frame for $Log(\mathfrak{L}_n)$.*

Consider, now, the formula L_n^κ:

$$L_n^\kappa : \quad \kappa \to \Diamond(\boxdot^n \neg\kappa \wedge \Diamond^{n+1}\kappa).$$

Here $\boxdot A$ is an abbreviation for $\Box A \wedge A$. Note also that, despite the fact that the logics $Log(\mathfrak{L}_n)$ are extensions of $\mathbf{KD!}_\kappa$, the use of $\Diamond$ and $\Box$ (rather than simply one or the other) is intended to be suggestive. It is easy to see that, for all n, L_n^κ is valid on the frame $\mathfrak{L}_n$. We now show that this formula is valid on none of the frames $\mathfrak{L}_m$ for which $m \neq n$.

Lemma 3.2. *For all $n, m \in Nat$ we have the following:*

$$\mathfrak{L}_n \not\models L_m^\kappa \text{ whenever } m < n.$$

Proof. Suppose that $\mathfrak{L}_n \models L_m^\kappa$. Then in particular it follows that $\mathfrak{L}_n \models (\star)$, where:

$$(\star) : \quad \kappa \to \Diamond^{m+2}\kappa.$$

Suppose, then, that $\mathfrak{L}_n \models_0 (\star)$, from which it follows that $\mathfrak{L}_n \models_0 \Diamond^{m+2}\kappa$. Thus $\mathfrak{L}_n \models_1 \Diamond^{m+1}\kappa$, and so $\mathfrak{L}_n \models_m \Diamond^2\kappa$. Now as $m + 2 < n + 2$, and $R^{n+2}(0) = \{0\}$ and the only point in C_n is 0 we thus have that $\mathfrak{L}_n \models_{R^2(m)} \kappa$ by hypothesis, and $\mathfrak{L}_n \models_{R^2(m)} \neg\kappa$ by construction, giving us a contradiction, and so the result follows. $\Box$

Lemma 3.3. *For all $n, m \in Nat$ we have the following:*

$$\mathfrak{L}_n \not\models L^\kappa_m \text{ whenever } m > n.$$

Proof. Suppose that $m = n + k$ for some $k > 0$. Then in **K** we have $L^\kappa_m \to (\star')$ where:

$$(\star') : \quad \kappa \to \Diamond(\boxdot^n \neg\kappa \wedge \Box^n(\boxdot^k \neg\kappa)).$$

Suppose, then, that $\mathfrak{L}_n \models L^\kappa_m$, so in particular $\mathfrak{L}_n \models (\star')$, and hence as $\mathfrak{L}_n \models_0 \kappa \wedge \Diamond\neg\kappa$ that $\mathfrak{L}_n \models_0 \Diamond(\boxdot^n \neg\kappa \wedge \Box^n(\boxdot^k \neg\kappa))$. In particular it follows that $\mathfrak{L}_n \models_1 \boxdot^n \neg\kappa \wedge \Box^n(\boxdot^k \neg\kappa)$, and so $\mathfrak{L}_n \models_n \neg\kappa \wedge \Box\boxdot^k \neg\kappa$. Consequently, $\mathfrak{L}_n \models_{n+1} \boxdot^k \neg\kappa$, and in particular $\mathfrak{L}_n \models_{n+1} \Box\neg\kappa$, from which it follows that $\mathfrak{L}_n \models_0 \kappa$ and $\mathfrak{L}_n \models_0 \neg\kappa$, giving the result. □

Theorem 3.4. *The logics $Log(\mathfrak{L}_n)$ are all distinct normal modal logics.*

Proof. Follows by Lemma 3.2 and 3.3. □

To see that each logic $Log(\mathfrak{L}_n)$ is Post-complete begin by noting that the the frames $\mathfrak{L}_n$ are *distinguishing* in the sense that for each point x there is a formula D_x such that, for all models on $\mathfrak{L}_n$, D_x is true at a point y in a model iff $x = y$. The relevant formulas are given in the following table.

$x \in W_n$	D_x
1	$\Diamond^{n+1}\kappa.$
2	$\Diamond^n\kappa.$
$\vdots$	$\vdots$
i	$\Diamond^{(n+2)-i}\kappa.$
$\vdots$	$\vdots$
n	$\Diamond^2\kappa.$
$n+1$	$\Diamond\kappa.$
0	$\kappa.$

Say that a formula is *variable-free* if it is constructed using the boolean connectives, κ and $\Box$. Further, say that a substitution σ is variable-free if for all propositional variables p_i, $\sigma(p_i) = A_i$ for some variable-free formula A_i. In what follows we will make use of the fact that the distinguishing formulas D_x above are all variable-free formulas.

Consider now the following function f from $\wp(W_n)$ to variable-free formulas.

$$f(X) = \bigvee\{D_x | x \in X\}.$$

Lemma 3.5. *For all valuations V we have the following for all formulas A and points $x \in W_n$:*

$$\langle\mathfrak{L}_n, V\rangle \models_x A \text{ if and only if } \langle\mathfrak{L}_n, V\rangle \models_x \sigma_f(A),$$

where $\sigma_f(p_i) = f(V(p_i))$.

Proof. By induction upon the complexity of A, the only case of interest being the basis case, where $A = p_i$. We show the following:

$$\langle \mathfrak{L}_n, V\rangle \models_x p_i \iff \langle \mathfrak{L}_n, V\rangle \models_x \sigma_f(p_i).$$

For the '$\Rightarrow$' direction, suppose that $\langle \mathfrak{L}_n, V\rangle \models_x p_i$. By the construction of σ_f this means that $\sigma_f(p_i) = \sigma_f(V(p_i)) = f(V(p_i) \setminus \{x\}) \vee D_x$. Since, for all y, $\langle \mathfrak{L}_n, V\rangle \models_x D_y$ iff $x = y$ it follows that $\langle \mathfrak{L}_n, V\rangle \models_x D_x$ and hence that $\langle \mathfrak{L}_n, V\rangle \models_x \sigma_f(p_i)$.

For the '$\Leftarrow$' direction suppose that $\langle \mathfrak{L}_n, V\rangle \models_x \sigma_f(p_i)$. By the definition of σ_f this means that $\langle \mathfrak{L}_n, V\rangle \models_x f(V(p_i))$. It is easy to see, though, that $\langle \mathfrak{L}_n, V\rangle \models_x f(X)$ iff $x \in X$, for all $X \subseteq W$. Consequently it follows that $x \in V(p_i)$ and thus that $\langle \mathfrak{L}_n, V\rangle \models_x p_i$ as desired. □

In particular, as $\sigma_f(A)$ is a variable-free formula, for all formulas A, its truth or falsity at a point in a model doesn't depend upon V, giving us the following corollary.

Corollary 3.6. *For all valuations V we have the following for all formulas A and points $x \in W_n$:*

$$\langle \mathfrak{L}_n, V\rangle \models_x A \text{ if and only if } \mathfrak{L}_n \models_x \sigma_f(A),$$

where $\sigma_f(p_i) = f(V(p_i))$.

Corollary 3.7. *If A is a non-theorem of $Log(\mathfrak{L}_n)$ then for some variable-free substitution σ we have that $\sigma(A)$ is a non-theorem of $Log(\mathfrak{L}_n)$.*

Proof. Follows from Corollary 3.6 and Proposition 3.1 □

Let the formula A_n be defined as follows.

$$\bigwedge_{0\le j<n+2} \Box^j A.$$

Lemma 3.8 ([4])**.** *If A is a variable-free formula that is not valid in $\mathfrak{L}_n$ then $\neg A_n$ is valid on $\mathfrak{L}_n$.*

Proof. If $\mathfrak{L}_n \not\models A$ then there is some point $y \in W_n$ such that $\mathfrak{L}_n \not\models_y A$. Then for any $x \in W_n$ let $j < n+2$ be such that $R_n^j(x) = \{y\}$ (the existence of such a j guaranteed by the fact that R_n is functional). Then A is false at $R_n^j(x)$, and so $\mathfrak{L}_n \not\models_x \Box^j A$, and thus $\mathfrak{L}_n \not\models A_n$. As this is so for all $x \in W_n$ it follows that $\mathfrak{L}_n \models \neg A_n$. □

Theorem 3.9. *For all $n \in Nat$, $Log(\mathfrak{L}_n)$ is a Post-complete normal modal logic in the language with a single propositional constant.*

[4]The following Lemma was suggested to me by Robert Goldblatt to patch an error in a previous proof of Theorem 3.9, as well as being independently suggested to me by Lloyd Humberstone for reasons of elegance.

Proof. Suppose, for a reductio, that there is a logic S which is a consistent proper normal extension of $Log(\mathfrak{L}_n)$. Then there must be some formula A such that $\vdash_{\mathsf{S}} A$ and $\nvdash_{Log(\mathfrak{L}_n)} A$. By Corollary 3.7 it follows that there is some variable-free substitution σ such that $\nvdash_{Log(\mathfrak{L}_n)} \sigma(A)$. Thus there is a point-generated model on a frame for $Log(\mathfrak{L}_n)$ which invalidates $\sigma(A)$. By Proposition 3.1 it follows that $\mathfrak{L}_n \not\models \sigma(A)$. By Lemma 3.8 it follows that $\mathfrak{L} \models \neg\sigma(A)_n$, and thus $\vdash_{Log(\mathfrak{L}_n)} \neg\sigma(A)_n$.

As $\vdash_{\mathsf{S}} A$ and S is closed under uniform substitution it follows that $\vdash_{\mathsf{S}} \sigma(A)$, and hence as S is normal we have the following as a theorem of S.

$$\bigwedge_{0 \leq j < n+2} \Box^j \sigma(A).$$

As $\mathsf{S} \supseteq Log(\mathfrak{L}_n)$ it follows, though, that $\vdash_{\mathsf{S}} \neg(\sigma(A)_n)$, contradicting the supposition that S was consistent, and the result follows. $\Box$

Corollary 3.10. *There are denumerably many Post-complete normal modal logics in the language with propositional constants.*

Proof. Follows directly from Theorems 3.4 and 3.9. $\Box$

As mentioned above, all the logics $Log(\mathfrak{L}_n)$ are extension of $\mathbf{KD!}_\kappa$. This provides an interesting contrast with **KD!**, which has **KT!** as it's sole Post-complete normal extension while, as shown above, $\mathbf{KD!}_\kappa$ has denumerably many Post-complete normal extensions.

4 Conclusion

What we have shown above is that there are at least denumerably many Post-complete normal modal logics in the language with propositional constants. Rather than telling us something interesting about monomodal logics (properly speaking), though, this is better thought of as telling us something interesting about a rather odd lattice of *bimodal* logics. Typically when people talk about modal operators we implicitly restrict attention to unary operators $\Box$, but this does not have to be so. For example, in Goguadze et al. (2003)—drawing inspiration from Jónsson & Tarski's work on Boolean algebras with operators—polyadic modal operators are defined where the semantic interpretation of an n-ary modal operator $\blacksquare(p_1, \dots, p_n)$ is given in terms of an $n+1$-ary relation $R_\blacksquare$ as follows:

$$\begin{array}{l} \mathcal{M} \models_x \blacksquare(A_1, \dots, A_n) \iff \text{ for all } v_1, \dots, v_n \\ \text{if } R_\blacksquare x v_1 \dots v_n \text{ then } \mathcal{M} \models_{v_i} A_i \text{ for } 1 \leq i \leq n. \end{array}$$

In this setting a propositional constant corresponds to the limit case of a polyadic modal operator, interpreted in terms of the world of evaluation being in a given set. Now, of course, this new 0-place modal operator isn't going to be *normal*, as it isn't going to satisfy the 0-place version of necessitation—from the provability of A infer κ—as κ in the logics under consideration here is not a theorem. That simply means that we have a 0-place non-normal modal operator on our hands, not that we don't have a modal operator at all.[5]

To more easily connect with the point made above that the language with a single unary modal operator and a propositional constant used here is a bi-modal language in disguise we can think of our propositional constant as a unary modal operator which doesn't depend upon its argument. This would mean that, for example, the constant-masquerading-as-unary-modal-operator $\Box_\kappa$ would validate the schema $\Box_\kappa A \leftrightarrow \Box_\kappa B$ for all formulas A and B. Again, as mentioned above, this new operator $\Box_\kappa$ isn't going to be normal as we can have $\vdash A$ without having $\vdash \Box_\kappa A$—as this is really just saying that κ is a theorem.

We end by presenting the following open problem.

Open Problem 4.1. *Are there non-denumerably many Post-complete normal modal logics in the language L_κ?*

Update. Since the submission of this paper several people acquainted with its contents notified the author that there are indeed non-denumerably many Post-complete normal modal logics in the present language. Model-theoretic proofs of the stronger result were supplied by Robert Goldblatt and by a referee for this journal, and an algebraic proof was sketched by Tomasz Kowalski. Rather than reproducing any of their arguments here, the author has opted for simply notifying the reader of the stronger result and leaving those mentioned (and perhaps others) free to publish their proofs.

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[5] This observation lends some credence to the complaints sometimes raised against Anderson (1958) that we are smuggling in all the relevant deontic content with our constant for the sanction—(Nowell-Smith and Lemmon, 1960, p.291) making the point that the Andersonian reduction will only work if our constant has deontic content or represents a "deontic concept". As it happens, the sanction constant $\mathcal{S}$ is just a strange deontic modal operator, and so what the Anderson result shows is that we can define a unary deontic modal operator in terms of a unary alethic modal operator and a 0-place deontic modal operator. An interesting result, to be sure, but hardly a reduction of the deontic modalities to the alethic. Of course, if this kind of reduction is executed so as to avoid other problems, this objection is not so clear cut—cf. Humberstone (1981).

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