Structural Reflexivity and the Semantic Paradoxes

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1 The Crime Scene

- Suppose that you're investigating a spree of murders which you are convinced have a common culprit. Various initial suspects fall off the list when they are found to be absent at the scene, until eventually you're left with only three suspects. One of them turns out to be very helpful to have around, though, so you strike him off the list and start to figure out which of the remaining two are to blame.
- This situation seems wrong, but this is precisely the situation in the literature on sub-structural solutions to the semantic paradoxes.
  + Negation is involved in the Liar, but (not obviously) present in Curry's paradox.

$$
\begin{prooftree}
\AxiomC{$T\langle\lambda\rangle \succ T\langle\lambda\rangle$}
\UnaryInfC{$\succ T\langle\lambda\rangle, \neg T\langle\lambda\rangle$}
\UnaryInfC{$\succ T\langle\lambda\rangle, \lambda$}
\UnaryInfC{$\succ T\langle\lambda\rangle, T\langle\lambda\rangle$}
\RightLabel{\textsc{Contraction}}
\UnaryInfC{$\succ T\langle\lambda\rangle$}
\AxiomC{$T\langle\lambda\rangle \succ T\langle\lambda\rangle$}
\UnaryInfC{$\neg T\langle\lambda\rangle, T\langle\lambda\rangle \succ$}
\UnaryInfC{$\lambda, T\langle\lambda\rangle \succ$}
\UnaryInfC{$T\langle\lambda\rangle, T\langle\lambda\rangle \succ$}
\UnaryInfC{$T\langle\lambda\rangle \succ$}
\RightLabel{\textsc{Cut}}
\BinaryInfC{$\succ$}
\end{prooftree}
$$

Figure 1: A Crime Scene: The Liar Paradox

  + The conditional is involved in Curry, but not in the Liar.

$$
\begin{prooftree}
\AxiomC{$T\langle\kappa\rangle \succ T\langle\kappa\rangle$}
\AxiomC{$p \succ p$}
\BinaryInfC{$T\langle\kappa\rangle \to p, T\langle\kappa\rangle \succ p$}
\UnaryInfC{$\kappa, T\langle\kappa\rangle \succ p$}
\UnaryInfC{$T\langle\kappa\rangle, T\langle\kappa\rangle \succ p$}
\RightLabel{\textsc{Contraction}}
\UnaryInfC{$T\langle\kappa\rangle \succ p$}
\UnaryInfC{$\succ T\langle\kappa\rangle \to p$}
\UnaryInfC{$\succ \kappa$}
\UnaryInfC{$\succ T\langle\kappa\rangle$}
\AxiomC{$T\langle\kappa\rangle \succ T\langle\kappa\rangle$}
\AxiomC{$p \succ p$}
\BinaryInfC{$T\langle\kappa\rangle, T\langle\kappa\rangle \to p \succ p$}
\UnaryInfC{$T\langle\kappa\rangle, \kappa \succ p$}
\UnaryInfC{$T\langle\kappa\rangle, T\langle\kappa\rangle \succ p$}
\UnaryInfC{$T\langle\kappa\rangle \succ p$}
\UnaryInfC{$T\langle\kappa\rangle \succ p$}
\RightLabel{\textsc{Cut}}
\BinaryInfC{$\succ p$}
\end{prooftree}
$$

Figure 2: A Crime Scene: Curry's Paradox

  + Neither are involved in more elaborate Hinnion-Libert style paradoxes (cf. Restall (2013))
  + So all we are left with are Cut, Contraction, and Reflexivity.
- Some (like Ripley (2012)) think that it is Cut that is to blame here, others (like Zardini (2011)) think that it is Contraction.
- My modest goal here is to show that rejecting Reflexivity is a viable and interesting responce to the paradoxes.
- Less modestly, I hope to convince you that it might even be the *correct* responce to the paradoxes.

$$
\dfrac{\Gamma \succ \Delta, A \quad \Gamma, A \succ \Delta}{\Gamma \succ \Delta}\ \textsc{Cut}
$$

$$
\dfrac{\Gamma, A, A \succ \Delta}{\Gamma, A \succ \Delta} \qquad \dfrac{\Gamma \succ \Delta, A, A}{\Gamma \succ \Delta, A}\ \textsc{Contraction}
$$

$$
\dfrac{}{\Gamma, A \succ A, \Delta}\ \textsc{Reflexivity}
$$

Figure 3: Our Suspects

2 Reasons to Doubt Reflexivity

- Reflexivity is not innocent. There are numerous reasons to doubt it's validity.
- **Example 1:** Truth in Virtue of Premises Suppose that you think that for some premises to entail some conclusions requires that one of the conclusions be true *in virtue of* the truth of the premises. The 'in virtue of' relation is *irreflexive*. So on this conception we will never have $\Gamma, A \succ A, \Delta$ being valid for *any* formula/statement A.
- **Example 2:** Swyneshed on Insolubilia Consider the following inference due to Roger Swyneshed:

(1) "The conclusion of this inference is false; therefore, the conclusion of this inference is false.".

If truth-preservation is necessary for logical consequence then this instance of Reflexivity will be invalid.

3 Wither Logic?

- One might worry that resolving the semantic paradoxes by dropping Reflexivity leaves us with no logic!
- But this reaction is premature. Someone who concieved of logics as just sets of formulas might have reacted in a similar way to Strong-Kleene logic (K_3) (or any other atheorematic logic).
- But we know that while there are no K_3-valid formulas, there are K_3-valid sequents.
- Similarly non-reflexive logics have no valid sequents. But they do have valid *metasequents*: sequents composed of sequents (as opposed to sequents composed of formulas).

4 Understanding Non-Reflexive Consequence

- There are a variety of different ways of undestanding non-reflexive consequence– for example we could think of consequence in terms of the conclusion being true in virtue of the premsies (as mentioned above). This will lead to a logic in the vicinity of FDE and its neighbours.

- We will provide a different picture of logical consequence here, though. One which makes essential use of metasequents of the form

$$\Gamma_1 \succ \Delta_1, \ldots, \Gamma_n \succ \Delta_n \Rightarrow \Gamma \succ \Delta$$

- Recall the following important remark due to (Girard et al., 1989, p. 31), in which C represents the active formula in Cut (where we are concerned with the instance of Reflexivity with C on the left and right of $\succ$):

> The identity axiom [=Reflexivity] says that C (on the left) is stronger than C (on the right); this rule [= Cut] states the converse truth, i.e. C (on the right) is stronger than C (on the left).

- So we can see that rejecting Cut or Reflexivity results in a certain kind of asymmetry in how we understand statements of consequence.
- Let us understand a sequent as telling us that if we don't reject the members of Γ then we should accept some member of Δ.
- Against the background of such an understanding a failure of structural reflexivity means that we should neither accept nor reject C. (Similarly a failure of Cut would mean that we shoudl both accept and reject C.)
- This allows for a very easy reading of higher-order sequents of the kind we will be particularly concerned with. A metasequent sequent of the form

$$P_1 \succ P_1, \ldots, P_n \succ P_n \Rightarrow \Gamma \succ \Delta$$

tells us that if we either accept or reject each of the P_is then if we dont reject the members of Γ we should accept some member of Δ.

5 Recapturing Classical Reasoning

- Note that if every formula in $\Gamma \cup \Delta$ is either accepted or rejected then a sequent $\Gamma \succ \Delta$ can equivalently be read as telling us that if we accept all the members of Γ then we should accept some member of Δ.
- Consider now the case where the P_is are all the atomic formulas which appear in $\Gamma \cup \Delta$. Then if we have

$$P_1 \succ P_1, \ldots, P_n \succ P_n \Rightarrow \Gamma \succ \Delta$$

then this means that (on the assumption that we have accepted or rejected each of the P_is) that if we accept all the members of Γ then we should accept some member of Δ.

- So while the sequent $p, q \succ p \wedge q$ (alias $\emptyset \Rightarrow p, q \succ p \wedge q$) is invalid, the metasequent

$$p \succ p, q \succ q \Rightarrow p, q \succ p \wedge q$$

is valid. So we can see classical reasoning as being *enthymematic*, the suppressed assumption that the statements involved are ones which if we don't reject we ought to accept not being relevant in most (non-paradox involving) situations.

6 Diagnosis of the Paradoxes

- What paradoxical reasoning shows is that if we accept or reject sentences like λ or κ then we are forced to accept everything!
- But this just means that we shouldn't accept or reject such sentences, and should instead (echoing Wittgenstein) remain silent.

7 Bibliography

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