A Dialogical Account of Logical Pluralism

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My Plan

A Problem for Logical Pluralists

The Built-In Opponent Conception of Deduction

Introducing Prover-Skeptic Games

Prover-Skeptic Games: An Example

Proofs & Explanations

A Route to Pluralism

A Problem for Logical Pluralists

> *A characterisation of a pluralist position needs to explain what it is to endorse all of the various consequence relations the pluralist accepts and how they relate to an intuitive notion of logical consequence.*
>
> —*Keefe, 2014, p.1376*

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- Why are these consequence relations relations of logical consequence?

The Built-In Opponent Conception of Deduction

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- View **proofs** as grounded in semi-adversarial debates between a Prover/Proponent and a Skeptic/Opponent.
  - Prover seeks to establish the conclusion.
  - Skeptic forces Prover to make their reasoning explicit.
- Deductive proofs are then winning strategies for Prover in these debates.

> *Proponent's job is not only to 'beat Opponent'; she also seeks to persuade Opponent of the truth of the conclusion, if he has granted the truth of the premises. In fact, the goal is not only to show that the conclusion follows from the premises, but also why it does; this corresponds to the idea that deductive arguments ought to have explanatory value.*
>
> – *Dutilh Novaes, 2015, p.599f*

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Higher-Order Cooperation & Explanation

- So Prover has to *explain* why the premises follow from the conclusion.
- Similarly, Skeptic has to be clear about what would convince them, and what would count as an appropriate explanation.
- Otherwise we can end up with a ‘passive-agressive’ Skeptic such as the Tortoise, constantly moving the goalposts regarding what Achilles must do to convince them.

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- In order to beat such an idealized opponent Prover has to ensure that there are no counterexamples to the inferential steps which they employ in responding to Skeptic's queries.
- This provides a bridge between multi-agent dialogical and mono-agent inferential practices.

Introducing Prover-Skeptic Games

A Dialogue

*In the library two students, **Penelope** and **Scott** are arguing over the validity of the argument from $A \to B$ and $B \to C$ to $A \to C$. Petunia thinks that this argument is valid and is trying to convince Scott, who is skeptical.*

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- **Penelope (1)**: I reckon that $A \rightarrow C$ follows from $A \rightarrow B$ and $B \rightarrow C$
- **Scott (1)**: Yeah? Well if that's so then suppose I grant you $A \rightarrow B$ and $B \rightarrow C$ along with A, how are you mean to get C?

- **Penelope (2)**: If you grant me B I can get C from $B \to C$ (which you just granted).

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Penelope leaves the library triumphantly.

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Introduction/Elimination Rules

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Structural Rules

$$\frac{\Gamma, A, A \succ B}{\Gamma, A \succ B}\ (W) \qquad \frac{\Gamma \succ B}{\Gamma, A \succ B}\ (K)$$

A Natural Deduction System: Example & Notation

$$\dfrac{\dfrac{A \to B \succ A \to B \quad A \succ A}{A \to B, A \succ B}\,(\to E) \qquad B \to C \succ B \to C}{\dfrac{A \to B, B \to C, A \succ C}{A \to B, B \to C \succ A \to C}\,(\to I)}\,(\to E)$$

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- $[\Gamma_1 \succ A_1, \ldots, \Gamma_n \succ A_n] \Rightarrow^*_{(E)} \Delta \succ B$ iff we can derive $\Delta \succ B$ from the $\Gamma_i \succ A_i$s using $(\rightarrow E)$ and the structural rules.

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» *Prover begins by asserting the sequent under discussion.*

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2. $\mathfrak{s}_i \Rightarrow^*_{(I)} \sigma$, for some non-axiomatic sequent $\sigma \in \Sigma_i$ where $\mathfrak{s}_i$ is of the form $\Theta \succ p_i$ for some propositional atom p_i.

» *Skeptic challenges an (interesting) Prover assertion*

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3. $\Sigma_{i+1} \Rightarrow^*_{(E)} \mathfrak{s}_i$

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- We'll say that Prover *wins* a dialogue whenever Skeptic can make no further move, ...
- ... and has a winning strategy if she has a way of playing such that, no matter what moves Skeptic makes Prover can always win the dialogue.

- Note that a Skeptic move challenging a sequent $\Gamma \succ B$ will always result in a challenge

$$\beta = \Gamma, B_1, \ldots, B_n \succ p_i$$

where $B = B_1 \to (\ldots \to (B_n \to p_i))$

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- So Skeptic moves are mechanical and thus are apt for being performed 'offline' by the Prover.
- We also have the right degree of higher-order cooperation, with both parties having to agree on what counts as a challenge and what counts as an acceptable responce to a challenge.

Theorem

Prover has a winning strategy in the dialogue over (Γ, A) iff $\Gamma \succ A$ is provable.

- The 'only if' direction here is straightforward given the fact the connection between Prover and Skeptic moves and our Natural Deduction proof system.

A Brief Aside on Adequacy

Theorem

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- The 'only if' direction here is straightforward given the fact the connection between Prover and Skeptic moves and our Natural Deduction proof system.
- The 'if' direction is slightly more complicated: requires *normalisation* and pushing the structural rules into the E-segments of threads in the proof. Then, given that the Skeptic moves directly correspond to the I-segments of proofs, we can read a winning strategy off the threads of the proof.

Prover-Skeptic Games: An Example

Consider the following dialogue over $p \to q, q \to r \succ p \to r$.

P(1) $[p \to q, q \to r \succ p \to r]$ » *Prover Starts*

Revisiting the Previous Example

Consider the following dialogue over $p \to q, q \to r \succ p \to r$.

P(1)	$[p \to q, q \to r \succ p \to r]$
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» *Skeptic challenges the sole Prover assertion.*

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» *Prover replies, offering sequents which derive the challenge sequent.*

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S(2) $p \to q, p \succ q$

» *Skeptic challenges the only non-axiomatic sequent.*

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» *Prover replies, and wins the dialogue.*

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Proofs & Explanations

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- The notion of explanation which is most appropriate here is broadly 'pragmatic' in nature…
- … with explanations being answers to why-questions—requests for information of a particular sort
- What counts as an answer to a why-question, what information is requested, can differ from context to context.

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- For example, the most common kind of constraint that arises in these contexts concerns the 'granularity' of the responce. (Hersh)
- In teaching contexts, for example, one often needs to provide more information than in research contexts.

A Route to Pluralism

- There appear to be some contexts in which what counts as an answer to the appropriate sort of 'why does this follow' question does have a logical impact.

A Route to Pluralism – Two Examples

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- So, given the plausibility of these examples, it looks like (pace Beall & Restall) whether an argument is valid is sometimes relative to communities of inquiry.

A Route to Pluralism – Two Examples

- There appear to be some contexts in which what counts as an answer to the appropriate sort of 'why does this follow' question does have a logical impact.
- So, given the plausibility of these examples, it looks like (pace Beall & Restall) whether an argument is valid is sometimes relative to communities of inquiry.
- We will look two examples: one in the foundations of geometry, and the other concerning qualitative Hypothetico-Deductive accounts of confirmation.

Resources Consciousness in the Foundations of Geometry

- Pambuccian (2004) recounts some work in early 20th Century foundations of Geometry where we see a concern, not just for which axioms are used in proving something, but also in *how many times* such axioms are appealed to in such a proof.

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- E.g., Hessenberg notes in 1905 proof that, against the background of other axioms of projective geometry, Desargues axioms follows from a 'three-fold use' of Pappus's axiom.
- This curious situation is at least suggestive of a concern, in showing that something follows, not just with necessary truth preservation, but also with the *extent* to which certain claims are used in proofs. A concern which points towards a rejection of the structural rule of Contraction.

Hypothetico-Deductivism and Relevance

- In the 1980s Hypothetico-Deductivism was famously proclaimed by Glymour to be 'hopeless' as a theory of confirmation.
- The main issues pointed to by Glymour concerned the inability of HD-confirmation to "maintain the condition of relevance between a hypothesis and the evidence which confirms it" (Waters)
- The various attempts to deal with the issues pointed out by Glymour usually result in some variance of relevant logic á la Anderson & Belnap.
- This is suggestive of the a norm on explanation which requires that in contexts where we're concerned about confirmation we require an appropriate connection of relevance between theory, hypothesis and evidence.
- At the very least this will result in a rejection of the structural rule of Weakening.

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- To endorse a logic is to be in a context in which that logic is appropriate (local), or for that logic to be appropriate in some context (global)
- All of these notions of logical consequence arise naturally out of the built-in opponent conception of deduction—in particular out of crucial higher-order cooperation between Prover and Skeptic.

Thank You!

Adequacy of Prover-Skeptic Games

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \succ p \quad p \to q \succ p \to q}{p, p \to q \succ q}\,(\to E) \qquad \dfrac{p \to (q \to r) \succ p \to (q \to r) \quad p \succ p}{p, p \to (q \to r) \succ q \to r}\,(\to E)}{p \to (q \to r), p \to q, p, p \succ r}\,(\to E)}{p \to (q \to r), p \to q, p \succ r}\,(K)}{p \to (q \to r), p \to q \succ p \to r}\,(\to I)}{p \to (q \to r) \succ (p \to q) \to (p \to r)}\,(\to I)
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$$

$P(1)$

- $P(1) = \left[p \to (q \to r) \succ (p \to q) \to (p \to r)\right]$

From Proofs to Winning Strategies

$$
\begin{prooftree}
\AxiomC{$p \succ p$}
\AxiomC{$p \to q \succ p \to q$}
\RightLabel{$(\to E)$}
\BinaryInfC{$p, p \to q \succ q$}
\AxiomC{$p \to (q \to r) \succ p \to (q \to r)$}
\AxiomC{$p \succ p$}
\RightLabel{$(\to E)$}
\BinaryInfC{$p, p \to (q \to r) \succ q \to r$}
\RightLabel{$(\to E)$}
\BinaryInfC{$p \to (q \to r), p \to q, p, p \succ r$}
\RightLabel{(K)}
\UnaryInfC{$p \to (q \to r), p \to q, p \succ r$}
\RightLabel{$(\to I)$}
\UnaryInfC{$p \to (q \to r), p \to q \succ p \to r$}
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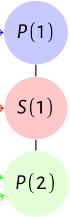
\end{prooftree}
$$

$$
\begin{tikzcd}
P(1) \arrow[d, dash] \\
S(1)
\end{tikzcd}
$$

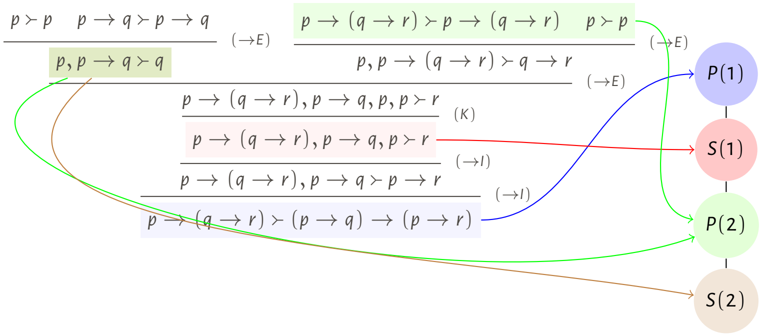
- $P(1) = \left[p \to (q \to r) \succ (p \to q) \to (p \to r)\right]$
- $S(1) = p \to (q \to r), p \to q, p \succ r$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \succ p \quad p \to q \succ p \to q}{p, p \to q \succ q}\,(\to E) \qquad \dfrac{p \to (q \to r) \succ p \to (q \to r) \quad p \succ p}{p, p \to (q \to r) \succ q \to r}\,(\to E)}{p \to (q \to r), p \to q, p, p \succ r}\,(\to E)}{p \to (q \to r), p \to q, p \succ r}\,(K)}{p \to (q \to r), p \to q \succ p \to r}\,(\to I)}{p \to (q \to r) \succ (p \to q) \to (p \to r)}\,(\to I)
$$



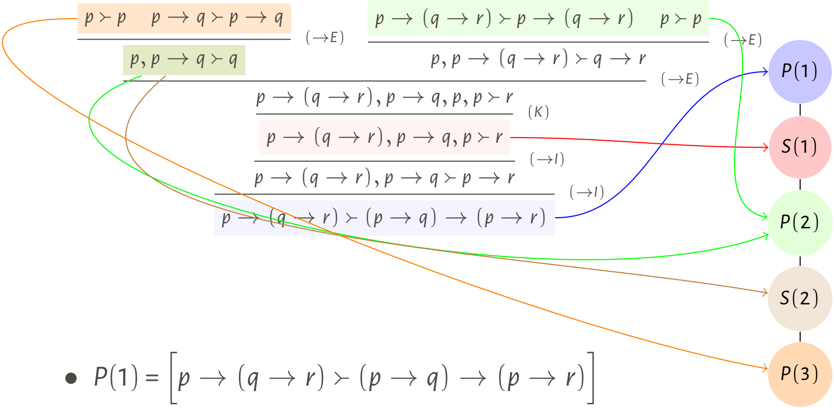
- $P(1) = \left[p \to (q \to r) \succ (p \to q) \to (p \to r)\right]$
- $S(1) = p \to (q \to r), p \to q, p \succ r$
- $P(2) = \left[p \to (q \to r) \succ p \to (q \to r) \quad , \; p \succ p \quad , \; p, p \to q \succ q\right]$

From Proofs to Winning Strategies



- $P(1) = \left[p \to (q \to r) \succ (p \to q) \to (p \to r)\right]$
- $S(1) = p \to (q \to r), p \to q, p \succ r$
- $P(2) = \left[p \to (q \to r) \succ p \to (q \to r) \quad , \ p \succ p \quad , \ p, p \to q \succ q\right]$
- $S(2) = p, p \to q \succ q$

From Proofs to Winning Strategies



- $P(1) = \left[p \to (q \to r) \succ (p \to q) \to (p \to r) \right]$
- $S(1) = p \to (q \to r), p \to q, p \succ r$
- $P(2) = \left[p \to (q \to r) \succ p \to (q \to r) \quad , \; p \succ p \quad , \; p, p \to q \succ q \right]$
- $S(2) = p, p \to q \succ q$
- $P(3) = \left[p \to q \succ p \to q \quad , \; p \succ p \right]$