A Hypersequent Calculus for Contingent Existence

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1 The Problem with the Barcan Formula

The idea that anything which might exist does exist strikes many of us as mere superstition. Ideally our first-order modal logic should suppor this thought, and to do so it mustn't validate the *Barcan formula*:

$$\Diamond \exists x A \to \exists x \Diamond A$$

But due to Prior (1956) we know that combining the most natural rules for the quantifiers and modal operators allows us to *derive* the Barcan formula. For example, in the obvious first-order extension of the hypersequent calculus presented in Restall (2012) we have the following proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{Fa \vdash Fa}{Fa \vdash \quad | \quad \vdash \Diamond Fa}[\Diamond R]}{Fa \vdash \quad | \quad \vdash \exists x \Diamond Fx}[\exists R]}{\exists x Fx \vdash \quad | \quad \vdash \exists x \Diamond Fx}[\exists L]}{\Diamond \exists x Fx \vdash \exists x \Diamond Fx}[\Diamond L]$$

So we have a trade-off between (technical) awkwardness and (metaphysical) superstition. This technical awkwardness should be justified, though. We want to know not only that the above derivation is not available in our first-order modal logic, but also where it led us astray.

2 Kripke's Analysis

Kripke's analysis in (Kripke, 1963, p.68f) is that such proofs are question begging.

In order to prove the Barcan formula we end up appealing to that very principle, under the assumption that we understand all free-variables as being given the generality interpretation.

This means that proofs of the Barcan formula involve a scope ellision.

Unfortunately Kripke's solution requires a ban on all terms which are not variables, and so Kripke's proposed solution to this problem only works due to a "deliberate impovrishment of the formal machinery" (Prior, 1967, p.161)

3 Primitive Scope Markers

One way to make Kripke's insight here explicit is to introduce scope marking occurrences of terms into our language, making terms play a scope-indicating role in addition to their standard object-denoting role. A suggestion due to (Prior, 1967, p.167ff) and Karmo (1984).

So we draw a distinction between (i)"concerning a it is possibly F" and (ii) "possibly concerning a it is F", rending that distinction as that between:

$$(i)\, a \Diamond Fa \quad \text{and} \quad (ii) \Diamond a\, Fa$$

As some of our terms may lack a denotation at some worlds we also introduce a similar scope distinction for negation.

Definition 3.1 (Formal Language). *We define what it is for a string of symbols from L to be a pseudo-formula recursively as follows.*

- *Every variable and individual constant is a term.*
- *If $t_1, \dots, t_n$ are terms, and F an n-ary primitive predicate then $Ft_1 \dots t_n$ is a pseudo-formula.*
- *If φ and ψ are pseudo-formulas then so are $\varphi \vee \psi$, $\neg\varphi$ and $\Diamond\varphi$.*
- *If $x\ \varphi$ is a pseudo-formula then so is $\exists x \varphi$.*
- *If φ is a pseudo-formula and t is either unscoped in φ or does not occur in φ then $t\ \varphi$ is a pseudo-formula.*
- *Nothing else is a pseudo-formula.*

If φ is a pseudo-formula with unscoped terms $t_1, \dots, t_n$ then $t_1 \dots t_n \varphi$ is a formula.

4 Opaque and Transparent Rules

Our rules for the existential quantifier allow us to bind terms which have a 'wide-scope' scope-marking occurrence.

$$\dfrac{\mathcal{H} \mid X \vdash Y, u_1 \dots u_m u\, A^x_u}{\mathcal{H} \mid X \vdash Y, u_1 \dots u_m \exists x A}[\exists R]$$

$$\dfrac{\mathcal{H} \mid X, u_1 \dots u_m a\, A^x_a \vdash Y}{\mathcal{H} \mid X, u_1 \dots u_m \exists x A \vdash Y}[\exists L]\dagger$$

For all other connectives we draw a distinction between *opaque* and *transparent* insertion-srules, with transparent rules keeping all terms which are available to be bound in the premises of the rule available in the conclusion, and opaque rules making them unavailable. For example consider (for simplicities sake unary) connective #.

- A *transparent* insertion rule for # whose premise is $u_1 \dots u_m A$ will have a conclusion $u_1 \dots u_m \# A$.
- An *opaque* insertion rule for # whose premise is $u_1 \dots u_m A$ will have $\# u_1 \dots u_m A$ as its conclusion.

So the terms which are available to be bound by a quantifier in the premises of an opaque rule are no longer available in the compound.

The rules for possibility ($\Diamond$) in this setting are opaque:

$$\dfrac{\mathcal{H} \mid X \vdash Y \quad | \quad t_1 \dots t_n A \vdash}{\mathcal{H} \mid X, \Diamond t_1 \dots t_n A \vdash Y}[\Diamond L]$$

$$\dfrac{\mathcal{H} \mid X \vdash Y \quad | \quad X' \vdash Y', t_1 \dots t_n A}{\mathcal{H} \mid X \vdash Y, \Diamond t_1 \dots t_n A \quad | \quad X' \vdash Y'}[\Diamond R]$$

The intuition here is that terms which have a denotation in the context in which they occur in the premise of a

$\Diamond$-rule may not have a a denotation in the context in which they occur in the conclusion of that rule.

In cases where we have some guarantee that this is the case we want to ensure that we can *export* that term so that it is outside the scope of $\Diamond$. This gives us the following *exportation rules* for $\Diamond$:

$$\frac{\mathcal{H} \mid X \vdash Y, \Diamond a\, A}{\mathcal{H} \mid X \vdash Y, a\, \Diamond A}\ [\Diamond\mathrm{ExpR}]^*$$

$$\frac{\mathcal{H} \mid X, \Diamond a\, A \vdash Y}{\mathcal{H} \mid X, a\, \Diamond A \vdash Y}\ [\Diamond\mathrm{ExpL}]$$

Where for $[\Diamond\mathrm{ExpR}]$ we require that there is some formula $a\,B \in X$—our rendering of Prior's intuition that for something to exist is for there to be facts about it (or in our case, concerning it).

5 Soundness, Completeness and Cut Admissibility

Relative to the semantics given on the other sheet of the handout we have the following results for the proof system given.

Theorem 5.1 (Soundness)**.** *Any sequent derivable in the system holds in all models.*

Theorem 5.2 (Completeness)**.** *Any hypersequent which is underivable in the system without* [Cut] *has a countermodel.*

Together these results also yield:

Corollary 5.3 (Cut Admissibility)**.** *Any sequent that had a derivation with* [Cut] *has a derivation without* [Cut].

6 Unrestricted Exportation & The Barcan Formula

The above proof of the Barcan formula, in the present language would look like this:

$$\begin{array}{c l}
a\,Fa \vdash a\,Fa & \\ \hline
a\,Fa \vdash \quad \mid \quad \vdash \Diamond a\,Fa & [\Diamond\mathrm{R}] \\ \hline
a\,Fa \vdash \quad \mid \quad \vdash a\,\Diamond Fa & [\mathrm{UE}] \\ \hline
a\,Fa \vdash \quad \mid \quad \vdash \exists x \Diamond Fx & [\exists\mathrm{R}] \\ \hline
\exists x Fx \vdash \quad \mid \quad \vdash \exists x \Diamond Fx & [\exists\mathrm{L}] \\ \hline
\Diamond \exists x Fx \vdash \exists x \Diamond Fx & [\Diamond\mathrm{L}]
\end{array}$$

The transition listed as [UE] ('unrestricted exportation') is not validity preserving in this system.
If we add this exportation principle to the system described then we can derive the Barcan formula (and the resulting system will be sound and complete w.r.t. constant domain versions of the models for our system).
This helps us to locate the dispute between necesssitists and contingentists in this setting in a very natural way, as being one concerning exportation principles.

7 Bibliography

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The Sequent Calculus

A *sequent* is a pair $\langle X, Y\rangle$ of multisets of formulas, which we will write as $X \vdash Y$. A *hypersequent* is a multiset of sequents, which we will write as

$$X_1 \vdash Y_1 \quad | \quad \ldots \quad | \quad X_n \vdash Y_n \quad | \quad \ldots$$

Axioms

$$\mathcal{H}[\, X, u_1 \ldots u_m F u'_1 \ldots u'_n \vdash u_1 \ldots u_m F u'_1 \ldots u'_n, Y\,]$$

where $\{u'_1 \ldots u'_n\} \subseteq \{u_1 \ldots u_m\}$ and F is an n-ary atomic predicate.

Transparent Rules

$$\frac{\mathcal{H}[\, X, u_1 \ldots u_m A \vdash Y\,] \qquad \mathcal{H}[\, X, u_1 \ldots u_m B \vdash Y\,]}{\mathcal{H}[\, X, u_1 \ldots u_m (A \vee B) \vdash Y\,]}\ (\vee L)$$

$$\frac{\mathcal{H}[\, X \vdash u_1 \ldots u_m A, u_1 \ldots u_m B, Y\,]}{\mathcal{H}[\, X \vdash u_1 \ldots u_m (A \vee B), Y\,]}\ (\vee R)$$

$$\frac{\mathcal{H}[\, X \vdash Y, u_1 \ldots u_m u\, A^x_u\,]}{\mathcal{H}[\, X \vdash Y, u_1 \ldots u_m \exists x A\,]}\ (\exists R) \qquad \frac{\mathcal{H}[\, X, u_1 \ldots u_m a\, A^x_a \vdash Y\,]}{\mathcal{H}[\, X, u_1 \ldots u_m \exists x A \vdash Y\,]}\ (\exists L)_\dagger$$

Opaque Rules

$$\frac{\mathcal{H}[\, X \vdash u_1 \ldots u_m A, Y\,]}{\mathcal{H}[\, X, \neg u_1 \ldots u_m A \vdash Y\,]}\ (\neg L) \qquad \frac{\mathcal{H}[\, X, u_1 \ldots u_m A \vdash Y\,]}{\mathcal{H}[\, X \vdash \neg u_1 \ldots u_m A, Y\,]}\ (\neg R)$$

$$\frac{\mathcal{H}[\, X \vdash Y \quad | \quad t_1 \ldots t_n A \vdash\,]}{\mathcal{H}[\, X, \Diamond t_1 \ldots t_n A \vdash Y\,]}\ (\Diamond L) \qquad \frac{\mathcal{H}[\, X \vdash Y \quad | \quad X' \vdash Y', t_1 \ldots t_n A\,]}{\mathcal{H}[\, X \vdash Y, \Diamond t_1 \ldots t_n A \quad | \quad X' \vdash Y'\,]}\ (\Diamond R)$$

Exportation Rules

$$\frac{\mathcal{H}[\, X \vdash Y, \Diamond a\, A\,]}{\mathcal{H}[\, X \vdash Y, a \Diamond A\,]}\ (\Diamond ExpR)^* \qquad \frac{\mathcal{H}[\, X, \Diamond a\, A \vdash Y\,]}{\mathcal{H}[\, X, a \Diamond A \vdash Y\,]}\ (\Diamond ExpL)$$

$$\frac{\mathcal{H}[\, X \vdash u_1 \ldots u_{m-1} \neg u_m A, Y\,]}{\mathcal{H}[\, X \vdash u_1 \ldots u_m \neg A, Y\,]}\ (\neg ExpR)_* \qquad \frac{\mathcal{H}[\, X, u_1 \ldots u_{m-1} \neg u_m A \vdash Y\,]}{\mathcal{H}[\, X, u_1 \ldots u_m \neg A \vdash Y\,]}\ (\neg ExpL)$$

Structural Rules

$$\frac{\mathcal{H}[\, X, A \vdash Y\,]}{\mathcal{H}[\, X, u_m\, A \vdash Y\,]}\ (uExpL) \qquad \frac{\mathcal{H}[\, X \vdash A, Y\,]}{\mathcal{H}[\, X \vdash u_m\, A, Y\,]}\ (uExpR)_*$$

Where in (uExpL) and (uExpR) the term u_m has no scopal occurrence in A.

$$\frac{\mathcal{H}[\, X, u_1 \ldots u_m u_{m+1} \ldots u_n A \vdash Y\,]}{\mathcal{H}[\, X, u_1 \ldots u_{m+1} u_m \ldots u_n A \vdash Y\,]}\ (uPerL) \qquad \frac{\mathcal{H}[\, X \vdash u_1 \ldots u_m u_{m+1} \ldots u_n A, Y\,]}{\mathcal{H}[\, X \vdash u_1 \ldots u_{m+1} u_m \ldots u_n A, Y\,]}\ (uPerR)$$

$$\frac{\mathcal{H}[\, X, A, A \vdash Y\,]}{\mathcal{H}[\, X, A \vdash Y\,]}\ (WL) \qquad \frac{\mathcal{H}[\, X \vdash Y, A, A\,]}{\mathcal{H}[\, X \vdash Y, A\,]}\ (WR)$$

$$\frac{\mathcal{H}[\, X, A \vdash Y\,] \qquad \mathcal{H}[\, X \vdash A, Y\,]}{\mathcal{H}[\, X \vdash Y\,]}\ (Cut)$$

Side-Conditions Various of the rules presented above are required to satisfy various side-conditions, given here.
(†) : where a does not appear free in the lower sequent.
(∗) : where $u_m\, B \in X$ for some formula $u_m\, B$.

Model Theory

Models for our language will be first-order Kripke models for **S5** with variable domains. A *model* is a structure $\langle W, U, DV \rangle$ where W is a non-empty set (of 'possible worlds') D, U is a non-empty set (of objects), D is a function from W to $\wp(U)$, and V is a function from members of W and n-ary predicates to subsets of $\wp(D(w)^n)$ (i.e. the powerset of the n-fold cartesian product of $D(w)$), and from individual constants to members of U. A variable assignment s maps variables to members of U, and we will say that a variable assignment s' is an x-*variant of* a variable assignemnt s ($s' \sim_x s$) iff for all variables y, except possibly x, we have $s'(y) = s(y)$.
Given a variable assignment s and a model $\mathcal{M} = \langle W, U, D, V \rangle$ we will define the denotation function den_s as follows:

- $den_s(t) = s(t)$ if t is a variable.
- $den_s(t) = V(t)$ if t is an individual constant.

Satisfaction of a formula φ in a model $\mathcal{M} = \langle W, U, D, V \rangle$ relative to a variable assignment s and a world $w \in W$ ($\mathcal{M}, s \models_w \varphi$) is defined recursively as follows.

$\mathcal{M}, s \models_w {}_{u_1 \ldots u_m} F u'_1 \ldots u'_n$	iff	$den_s(u_1) \in D(w)$ & ... & $den_s(u_m) \in D(w)$ and $\langle den_s(u'_1), \ldots den_s(u'_n) \rangle \in V(F, w)$
$\mathcal{M}, s \models_w {}_{u_1 \ldots u_m} \neg A$	iff	$den_s(u_1) \in D(w)$ & ... & $den_s(u_m) \in D(w)$ and $\mathcal{M}, s \not\models_w {}_{u_1 \ldots u_m} A$.
$\mathcal{M}, s \models_w {}_{u_1 \ldots u_m} A \vee B$	iff	$den_s(u_1) \in D(w)$ & ... & $den_s(u_m) \in D(w)$ and either $\mathcal{M}, s \models_w {}_{u_1 \ldots u_m} A$, or $\mathcal{M}, s \models_w {}_{u_1 \ldots u_m} B$.
$\mathcal{M}, s \models_w {}_{u_1 \ldots u_m} \Diamond A$	iff	$den_s(u_1) \in D(w)$ & ... & $den_s(u_m) \in D(w)$ and there exists a $w' \in W$ s.t. $\mathcal{M}, s \models_{w'} {}_{u_1 \ldots u_m} A$.
$\mathcal{M}, s \models_w {}_{u_1 \ldots u_m} \exists x A$	iff	$den_s(u_1) \in D(w)$ & ... & $den_s(u_m) \in D(w)$ and for some $s' \sim_x s$ s.t. $s'(x) \in D(w)$ we have $\mathcal{M}, s' \models_w {}_{u_1 \ldots u_m} A$.