Prover-Skeptic Games and Logical Pluralism

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A Problem for Logical Pluralists

The Built-In Opponent Conception of Deduction

Introducing Prover-Skeptic Games

Prover-Skeptic Games: Some Examples

Proofs & Explanations

Prover-Skeptic Games for Substructural Logics

A Problem for Logical Pluralists

The Explanatory Problem for Logical Pluralists

> *A characterisation of a pluralist position needs to explain what it is to endorse all of the various consequence relations the pluralist accepts and how they relate to an intuitive notion of logical consequence.*
>
> —*Keefe, 2014, p.1376*

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- Why are these consequence relations relations of logical consequence?

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- View **proofs** as grounded in semi-adversarial debates between a Prover/Proponent and a Skeptic/Opponent.
- Prover seeks to establish the conclusion.
- Skeptic forces Prover to make their reasoning explicit.
- Deductive proofs are winning strategies for Prover in these debates.

Higher-Order Cooperation & Explanation

> *Proponent's job is not only to 'beat Opponent'; she also seeks to persuade Opponent of the truth of the conclusion, if he has granted the truth of the premises. In fact, the goal is not only to show that the conclusion follows from the premises, but also why it does; this corresponds to the idea that deductive arguments ought to have explanatory value.*
>
> *– Dutilh Novaes, forthcoming*

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- This is because Skeptic has become **internalized** to the method of giving deductive proofs.
- In effect their role is played ‘offline’ by Prover.
- This provides a bridge between multi-agent dialogical and mono-agent inferential practices.

Introducing Prover-Skeptic Games

A Dialogue

In the library two students, ***Penelope*** *and* ***Scott*** *are arguing over the validity of the argument from* $A \to B$ *and* $B \to C$ *to* $A \to C$*. Petunia thinks that this argument is valid and is trying to convince Scott, who is skeptical.*

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- **Penelope (1)**: I reckon that $A \rightarrow C$ follows from $A \rightarrow B$ and $B \rightarrow C$
- **Scott (1)**: Yeah? Well if that's so then suppose I grant you $A \rightarrow B$ and $B \rightarrow C$ along with A, how are you mean to get C?

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- **Penelope (2)**: If you grant me B I can get C from $B \rightarrow C$ (which you just granted).
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Penelope leaves the library triumphantly.

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Structural Rules

$$\frac{\Gamma, A, A \succ B}{\Gamma, A \succ B}\ (W) \qquad \frac{\Gamma \succ B}{\Gamma, A \succ B}\ (\text{K})$$

A Natural Deduction System: Example & Notation

$$\dfrac{\dfrac{A \to B \succ A \to B \quad A \succ A}{A \to B, A \succ B}\ (\to E) \qquad B \to C \succ B \to C}{\dfrac{A \to B, B \to C, A \succ C}{A \to B, B \to C \succ A \to C}\ (\to I)}\ (\to E)$$

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- Let us write $\Gamma \succ A \Rightarrow^{*}_{(I)} \Delta \succ B$ iff we can derive $\Delta \succ B$ from $\Gamma \succ A$ using $(\to I)$.

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- $[\Gamma_1 \succ A_1, \ldots, \Gamma_n \succ A_n] \Rightarrow^{*}_{(E)} \Delta \succ B$ iff we can derive $\Delta \succ B$ from the $\Gamma_i \succ A_i$s using $(\to E)$ and the structural rules.

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1. $\Sigma_1 = [\Gamma \succ A]$

» *Prover begins by asserting the sequent under discussion.*

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2. $\mathfrak{s}_i \Rightarrow^*_{(\mathrm{I})} \sigma$, for some non-axiomatic sequent $\sigma \in \Sigma_i$ where $\mathfrak{s}_i$ is of the form $\Theta \succ p_i$ for some propositional atom p_i.

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3. $\Sigma_{i+1} \Rightarrow^*_{(E)} \mathfrak{s}_i$

» *Prover replies, offering sequents which derive the challenge.*

Prover-Skeptic Games: Some Examples

Revisiting the Previous Example

Consider the following dialogue over $p \to q, q \to r \succ p \to r$.

P(1) $[p \to q, q \to r \succ p \to r]$ » *Prover Starts*

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» *Skeptic challenges the sole Prover assertion.*

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P(1)	$[p \to q, q \to r \succ p \to r]$
S(1)	$p \to q, q \to r, p \succ r$
P(2)	$[q \to r \succ q \to r, \quad p \to q, p \succ q]$

» *Prover replies, as these sequents derive the challenge sequent.*

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P(2) $[q \to r \succ q \to r, \quad p \to q, p \succ q]$

S(2) $p \to q, p \succ q$

» *Skeptic challenges the only non-axiomatic sequent.*

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S(2) $p \to q, p \succ q$

P(3) $[p \to q \succ p \to q, \quad p \succ p]$

» *Prover replies, and wins the dialogue.*

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Example: 'Prefixed' Weakening

Consider the dialogue for the sequent $p \to r \succ p \to (q \to r)$

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» *Skeptic challenges, saying what would need to be shown to convince them.*

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» *From these two sequents Prover can derive the challenge by one application of* $(\to \mathsf{E})$ *and then* (K) *to weaken in* q.

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» *This wins the dialogue for Prover in the presence of* (K).

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- For example: norms of turn taking; not questioning obvious claims.
- These are constitutive of the practice of engaging in these semi-adversarial debates.

Explanatory Value

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- What counts as an answer to a question is interest relative.
- Sometimes the relevant interests can have logical repercussions.

Resource Consciousness

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- For example: Hessenberg's 1905 proof that Desargues axiom follows from a threefold use of the Pappus axiom.
- One way of understanding this is as involving a change in what counts as a Prover response in a deductive dialogue.

Prover-Skeptic Games for Substructural Logics

S-dialogues

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- E.g. $\{(W)\}$-dialogues characterise the implicational fragment of the relevant logic R.

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- Such debates can be governed by different norms which determine what counts as an acceptable response to a Skeptic challenge.
- Differences in these norms results in different logics being correct.

Thank You!