Prover-Skeptic Games and Logical Pluralism

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A Problem for Logical Pluralists

A characterisation of a pluralist position needs to explain what it is to endorse all of the various consequence relations the pluralist accepts and how they relate to an intuitive notion of logical consequence.

—Keefe, 2014, p.1376

▶ What is it to 'endorse' a consequence relation?

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- Why are these consequence relations relations of logical consequence?

The Built-In Opponent Conception of Deduction

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- ► Prover seeks to establish the conclusion.

- ► Skeptic forces Prover to make their reasoning explicit.

- ► Deductive proofs are winning strategies for Prover in these debates.

Proponent's job is not only to 'beat Opponent'; she also seeks to persuade Opponent of the truth of the conclusion, if he has granted the truth of the premises. In fact, the goal is not only to show that the conclusion follows from the premises, but also why it does; this corresponds to the idea that deductive arguments ought to have explanatory value.

> – *Dutilh Novaes, forthcoming*

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- ▶ In effect their role is played 'offline' by Prover.

- ▶ This provides a bridge between multi-agent dialogical and mono-agent inferential practices.

Introducing Prover-Skeptic Games

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*In the library two students, **Penelope** and **Scott** are arguing over the validity of the argument from $A \rightarrow B$ and $B \rightarrow C$ to $A \rightarrow C$. Petunia thinks that this argument is valid and is trying to convince Scott, who is skeptical.*

- **Penelope (1)**: I reckon that $A \rightarrow C$ follows from $A \rightarrow B$ and $B \rightarrow C$

- **Scott (1)**: Yeah? Well if that's so then suppose I grant you $A \rightarrow B$ and $B \rightarrow C$ along with A, how are you mean to get C?

- **Penelope (2)**: If you grant me B I can get C from B $\to$ C (which you just granted).

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▶ **Scott (2)**: But why should I grant you B?

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- **Penelope (3)**: Well if you were to grant me A then I could get B from A → B which you granted at the start.

- **Penelope (2)**: If you grant me B I can get C from B $\rightarrow$ C (which you just granted).

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- **Penelope (4)**: You granted it to me at the start!

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A Dialogue

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Penelope leaves the library triumphantly.

A Natural Deduction System

Axioms

$$A \succ A$$

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INTRODUCTION/ELIMINATION RULES

$$\frac{\Gamma, A \succ B}{\Gamma \succ A \to B} \; (\to I) \qquad \frac{\Gamma \succ A \quad \Delta \succ A \to B}{\Gamma, \Delta \succ B} \; (\to E)$$

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Structural Rules

$$\frac{\Gamma, A, A \succ B}{\Gamma, A \succ B} \ (W) \qquad \frac{\Gamma \succ B}{\Gamma, A \succ B} \ (K)$$

$$\dfrac{\dfrac{A \to B \succ A \to B \quad A \succ A}{A \to B, A \succ B} \ _{(\to E)} \qquad B \to C \succ B \to C}{\dfrac{A \to B, B \to C, A \succ C}{A \to B, B \to C \succ A \to C} \ _{(\to I)}} \ _{(\to E)}$$

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▶ Let us write $\Gamma \succ A \Rightarrow^{*}_{(I)} \Delta \succ B$ iff we can derive $\Delta \succ B$ from $\Gamma \succ A$ using $(\to I)$.

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- $[\Gamma_1 \succ A_1, \ldots, \Gamma_n \succ A_n] \Rightarrow^{*}_{(E)} \Delta \succ B$ iff we can derive $\Delta \succ B$ from the $\Gamma_i \succ A_i$s using $(\to E)$ and the structural rules.

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> » *Prover begins by asserting the sequent under discussion.*

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2. $\mathfrak{s}_i \Rightarrow^*_{(I)} \sigma$, for some non-axiomatic sequent $\sigma \in \Sigma_i$ where $\mathfrak{s}_i$ is of the form $\Theta \succ p_i$ for some propositional atom p_i.

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3. $\Sigma_{i+1} \Rightarrow^*_{(E)} \mathfrak{s}_i$

> » *Prover replies, offering sequents which derive the challenge.*

Prover-Skeptic Games: An Examples

Consider the following dialogue over $p \to q, q \to r \succ p \to r$.

P(1) $[p \to q, q \to r \succ p \to r]$ » *Prover Starts*

Consider the following dialogue over $p \to q, q \to r \succ p \to r$.

P(1) $[p \to q, q \to r \succ p \to r]$

S(1) $p \to q, q \to r, p \succ r$

» *Skeptic challenges the sole Prover assertion.*

Consider the following dialogue over $p \rightarrow q, q \rightarrow r \succ p \rightarrow r$.

P(1) $[p \rightarrow q, q \rightarrow r \succ p \rightarrow r]$

S(1) $p \rightarrow q, q \rightarrow r, p \succ r$

P(2) $[q \rightarrow r \succ q \rightarrow r, \quad p \rightarrow q, p \succ q]$

» *Prover replies, as these sequents derive the challenge sequent.*

Consider the following dialogue over $p \to q, q \to r \succ p \to r$.

P(1) $[p \to q, q \to r \succ p \to r]$

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P(2) $[q \to r \succ q \to r, \ \ p \to q, p \succ q]$

S(2) $p \to q, p \succ q$

» *Skeptic challenges the only non-axiomatic sequent.*

Consider the following dialogue over $p \rightarrow q, q \rightarrow r \succ p \rightarrow r$.

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S(2) $p \rightarrow q, p \succ q$

P(3) $[p \rightarrow q \succ p \rightarrow q, \ \ \ p \succ p]$

» *Prover replies, and wins the dialogue.*

Consider the following dialogue over $p \rightarrow q, q \rightarrow r \succ p \rightarrow r$.

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Proofs & Explanations

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- These are constitutive of the practice of engaging in these semi-adversarial debates.

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- What counts as an answer to a question is interest relative.

- Sometimes the relevant interests can have logical repercussions.

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- For example: Hessenberg's 1905 proof that Desargues axiom follows from a threefold use of the Pappus axiom.

- One way of understanding this is as involving a change in what counts as a Prover response in a deductive dialogue.

Prover-Skeptic Games for Substructural Logics

- Let S be a subset of $\{(W), (K)\}$. Then an S-*dialogue* is a dialogue as defined above except where Prover is limited to using the structural rules in S in answering a Skeptic challenge.

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- E.g. $\{(W)\}$-dialogues characterise the implicational fragment of the relevant logic R.

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▶ Such debates can be governed by different norms which determine what counts as an acceptable response to a Skeptic challenge.

▶ Differences in these norms results in different logics being correct.

Thank You!