Prover-Skeptic Games and Logical Pluralism

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A Problem for Logical Pluralists

> *A characterisation of a pluralist position needs to explain what it is to endorse all of the various consequence relations the pluralist accepts and how they relate to an intuitive notion of logical consequence.*
>
> —*Keefe, 2014, p.1376*

Two Central Questions

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- Why are these consequence relations relations of logical consequence?

The Built-In Opponent Conception of Deduction

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- View **Proofs** as semi-adversarial debates between a Prover and a Skeptic.
- Prover seeks to establish the conclusion.
- Skeptic forces Prover to make their reasons explicit.
- Deductive proofs are winning strategies for Prover in these debates.

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- This is because Skeptic has become **Internalized** to the method, a very part of these practices.
- In effect their role is played ‘offline’ by Prover.
- This provides a bridge between Multi-Agent Dialogical and Mono-Agent Inferential Practices.

Introducing

Prover-Skeptic Games

A Dialogue

In the library two students, ***Petunia*** *and* ***Samson*** *are arguing over the validity of the argument from* $A \rightarrow B$ *and* $B \rightarrow C$ *to* $A \rightarrow C$. *Petunia thinks that this argument is valid and is trying to convince Samson, who is skeptical.*

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- **Petunia (1)**: I reckon that $A \to C$ follows from $A \to B$ and $B \to C$

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- **Petunia (1)**: I reckon that $A \to C$ follows from $A \to B$ and $B \to C$
- **Samson (1)**: Yeah? Well if that's so then suppose I grant you $A \to B$ and $B \to C$ along with A, how are you mean to get C?

- **Petunia (2)**: If you grant me B I can get C from B $\rightarrow$ C (which you just granted).

A Dialogue

- **Petunia (2)**: If you grant me B I can get C from B $\rightarrow$ C (which you just granted).
- **Samson (2)**: But why should I grant you B?

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- **Samson (2)**: But why should I grant you B?
- **Petunia (3)**: Well if you were to grant me A then I could get B from $A \rightarrow B$ which you granted at the start.

- **Petunia (2)**: If you grant me B I can get C from B $\rightarrow$ C (which you just granted).
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- **Samson (3)**: But why should I grant you A?
- **Petunia (4)**: Because you granted it to me at the start!

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- **Petunia (2)**: If you grant me B I can get C from B $\rightarrow$ C (which you just granted).
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Petunia leaves the library triumphantly.

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» *Skeptic challenges an (interesting) Prover assertion*

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1. $\Sigma_1 = \{\Gamma \succ \varphi\}$
2. $\mathfrak{s}_i \Rightarrow^{*}_{(I)} \sigma$, for some non-axiomatic sequent $\sigma \in \Sigma_i$ where $\mathfrak{s}_i$ is of the form $\mathcal{S}_i \succ p_i$ for some propositional atom p_i.

» *Prover replies, offering sequents which derive the challenge.*

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3. $\Sigma_{i+1} \Rightarrow^*_{(E)} \mathfrak{s}_i$

Prover-Skeptic Games:

Some Examples

Revisiting the Previous Example

Consider the following dialogue over $p \to q, q \to r \succ p \to r$.

P(1) $[p \to q, q \to r \succ p \to r]$ » *Prover Starts*

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P(1) $[p \to q, q \to r \succ p \to r]$

S(1) $p \to q, q \to r, p \succ r$

» *Skeptic challenges the sole Prover assertion.*

Consider the following dialogue over $p \to q, q \to r \succ p \to r$.

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S(1) $p \to q, q \to r, p \succ r$

P(2) $[q \to r \succ q \to r, p \to q, p \succ q]$

» *Prover replies, as these sequents derive the challenge sequent.*

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P(2) $[q \to r \succ q \to r, p \to q, p \succ q]$

S(2) $p \to q, p \succ q$

» *Skeptic challenges the only non-axiomatic sequent.*

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S(2) $p \to q, p \succ q$

P(3) $[p \to q \succ p \to q, p \succ p]$

» *Prover replies, and wins the dialogue.*

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Example: 'Prefixed' Weakening

Consider the dialogue for the sequent $p \rightarrow r \succ p \rightarrow (q \rightarrow r)$

P(1) $[p \rightarrow r \succ p \rightarrow (q \rightarrow r)]$ » *Prover Starts*

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» *Skeptic challenges, saying what would need to be shown to convince them.*

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» *From these two sequents Prover can derive the challenge by one application of* $(\to \mathsf{E})$ *and then* (K) *to weaken in* q.

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» *This wins the dialogue for Prover in the presence of* (K).

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P(2) $[p \to r \succ p \to r, \;\; p, q \succ p]$

» *The 'best' move Prover has available without (K) is the following.*

Example: 'Prefixed' Weakening

Consider the dialogue for the sequent $p \to r \succ p \to (q \to r)$

P(1) $[p \to r \succ p \to (q \to r)]$

S(1) $p \to r, p, q \succ r$

P(2) $[p \to r \succ p \to r, \ \ p, q \succ p]$

S(2) $p, q \succ p$

» *Skeptic can only challenge the rightmost sequent.*

Example: 'Prefixed' Weakening

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P(2) $[p \to r \succ p \to r, \ \ p, q \succ p]$

S(2) $p, q \succ p$

P(3) $[q \succ p, p \succ p \to p]$

» *This is Prover's 'best' response, and things will quite quickly devolve from here...*

Example 2: Contraction

Consider the dialogue for the sequent $p \to (p \to q) \succ p \to q)$

P(1) $[p \to (p \to q) \succ p \to q]$ » *Prover starts.*

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$P(2)$ $[p \to (p \to q) \succ p \to (p \to q), p \succ p, p \succ p]$

» *Prover responds, deriving the challenge via two applications of* $(\to E)$ *and one of* (W).

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» *In the absence of (W) this is the 'best' move available to Prover.*

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» *Skeptic must challenge*

$p \to (p \to q) \succ p \to q$

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» *But this just puts us back in the position we were in at* S(1)

Structural Rules and Norms on Explanation

Prover-Skeptic Games for Sub-structural Logics

Adequacy of Prover-Skeptic Games

- Winning Strategies correspond to proof sketches.

Adequacy: A Sketch

- Winning Strategies correspond to proof sketches.
- Constructing a Winning Strategy given a proof is a relatively simple operation.

FROM PROOFS TO WINNING STRATEGIES

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \succ p \qquad p \to q \succ p \to q}{p, p \to q \succ q}\,(\to E) \qquad \dfrac{p \to (q \to r) \succ p \to (q \to r) \qquad p \succ p}{p, p \to (q \to r) \succ q \to r}\,(\to E)}{p \to (q \to r), p \to q, p, p \succ r}\,(\to E)}{p \to (q \to r), p \to q, p \succ r}\,(K)}{p \to (q \to r), p \to q \succ p \to r}\,(\to I)}{p \to (q \to r) \succ (p \to q) \to (p \to r)}\,(\to I)
$$

From Proofs to Winning Strategies

$$
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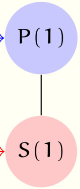
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{p \succ p \quad p \to q \succ p \to q}{p, p \to q \succ q}\,(\to E)
        \qquad
        \dfrac{p \to (q \to r) \succ p \to (q \to r) \quad p \succ p}{p, p \to (q \to r) \succ q \to r}\,(\to E)
      }{p \to (q \to r), p \to q, p, p \succ r}\,(\to E)
    }{p \to (q \to r), p \to q, p \succ r}\,(K)
  }{p \to (q \to r), p \to q \succ p \to r}\,(\to I)
}{\boxed{p \to (q \to r) \succ (p \to q) \to (p \to r)}}\,(\to I)
\qquad P(1)
$$

- $P(1) = \left[p \to (q \to r) \succ (p \to q) \to (p \to r)\right]$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \succ p \qquad p \to q \succ p \to q}{p, p \to q \succ q}(\to E) \qquad \dfrac{p \to (q \to r) \succ p \to (q \to r) \qquad p \succ p}{p, p \to (q \to r) \succ q \to r}(\to E)}{p \to (q \to r), p \to q, p, p \succ r}(\to E)}{\colorbox{pink}{$p \to (q \to r), p \to q, p \succ r$}}(K)}{p \to (q \to r), p \to q \succ p \to r}(\to I)}{\colorbox{lavender}{$p \to (q \to r) \succ (p \to q) \to (p \to r)$}}(\to I)
$$



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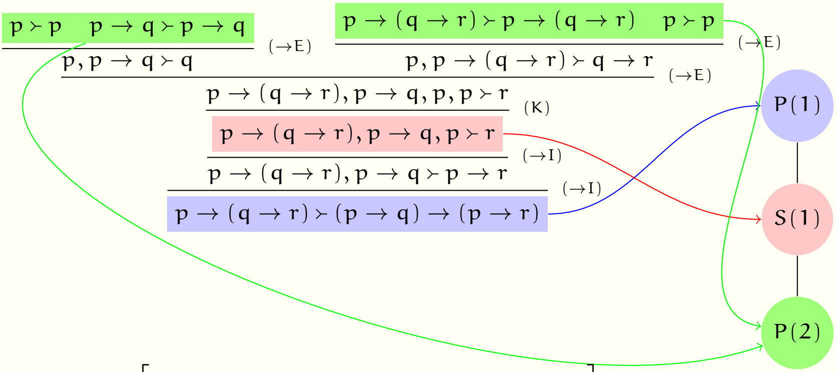
From Proofs to Winning Strategies



- $P(1) = \left[p \to (q \to r) \succ (p \to q) \to (p \to r)\right]$
- $S(1) = p \to (q \to r), p \to q, p \succ r$
- $P(2) =$

  $\left[p \to (q \to r) \succ p \to (q \to r) \quad , p \succ p \quad , p \succ p \quad , p \to q \succ p \to q\right]$

Thank You!