DAA HW 1

Pranav Tushar Pradhan N18401944 pp3051@nyu.edu

Udit Milind Gavasane N16545381 umg215@nyu.edu

Rohan Mahesh Gore N19332535 rmg9725@nyu.edu

Q.1.

(1) $f(n) = \Omega(g(n))$

which means $\exists$ constants $c, n_0 > 0$ s.t.

$0 \leq c \cdot g(n) \leq f(n) \quad \forall n \geq n_0$

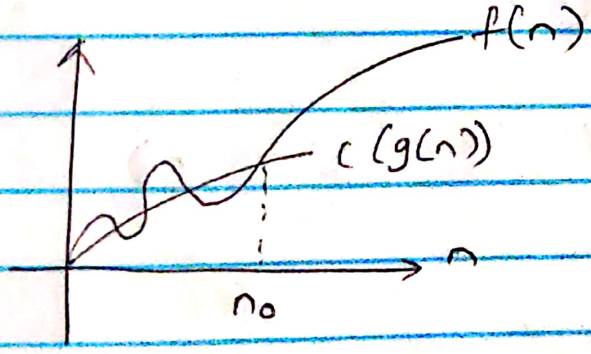
$f(n) = O(g(n))$ means

$\exists$ constants $c, n_0 > 0$ s.t

$0 \leq f(n) \leq c \cdot g(n) \quad \forall n \geq n_0$

But it's already given that $f(n) = \Omega(g(n))$

$\therefore$ According to the below graph,



$f(n)$ is lower bounded by $g(n)$ for $\exists c, n_0 > 0$ & $\forall n \geq n_0$

Hence, $f(n)$ can never be upper bounded by $g(n)$.

$\therefore f(n) = O(g(n))$ is false

$f(n) = \Omega(g(n))$ — Given

means

$0 \leq c_1 g(n) \leq f(n)$ — 1 for some $c_1 > 0$

$\forall\ n \geq N$

And $f(n) = o(g(n))$ — If

means

$0 \leq c_2 \cdot f(n) \leq g(n)$ — 2 for some $c_2 > 0$

$\therefore f(n) \leq \frac{1}{c_2} g(n)$ $\forall\ n \geq N$

By 1 and 2

$$c_1 g(n) \leq f(n) \leq \frac{1}{c_2} g(n)$$

$$\therefore c_1 g(n) \leq \frac{1}{c_2} g(n)$$

$$\therefore c_1 \times c_2\ g(n) \leq g(n)$$

We know that c_1 and $c_2 > 0$

$\therefore c_1 \times c_2\ g(n) \leq g(n)$ is false.

$\therefore f(n) = o(g(n))$ is false and

is never true

② $f(n) = \Theta(g(n))$ — Given

$\therefore$ $0 \le f(n) \le c \cdot g(n)$ $\forall\, n \geq n_0$ & $\exists\, c, n_0 > 0$.

If $f(n) = o(g(n))$

then it means $\forall\, c > 0$ $\exists\, n_0 > 0$

s.t. $0 \le f(n) \le c \cdot g(n)$

$\forall\, n \geq n_0$

Ex. $f(n) = 2(n)$ & $g = n$

$\therefore$ $2(n) \le 3(n)$ $\forall\, n \geq n_0$

$\exists\, c, n_0 > 0$ for which

This is true.

$\therefore$ $2n \le 3n$ $\forall\, n \geq 1$

Similarly, $2(n) = o(n) = o(g(n))$

If $\forall\, c > 0$ $\exists\, n_0 > 0$ s.t.

$0 \le 2(n) \le c \cdot n$

If $c = 1$

the $2n \not\le 1 \cdot n$

$\therefore$ $2(n) \neq o(n)$

$\therefore$ $2(n) \neq o(g(n))$

Hence, $f(n) = o(g(n))$ is false.

(3) Given $f(n) = \Theta(g(n))$

$\exists$ $c_1, c_2, n_0 > 0$ s.t

$0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 g(n) \quad \forall n \geq n_0$ (1)

If $f(n) = \Omega(g(n))$ then

$\exists$ $c_3, n_1 > 0$ s.t

$0 \leq c_3 g(n) \leq f(n) \quad \forall n \geq n_1$ (2)

By 1 & 2

$$C_3 \cdot g(n) \leq f(n)$$

$$C_3 \cdot g(n) \leq C_2\, g(n)$$

$$\therefore \frac{C_3}{C_2} g(n) \leq g(n)$$

$$\therefore \cancel{g(n) \leq C_2 \cdot g(n)}$$

Multiplying Both sides by C_1

$$C_1 \cdot \frac{C_3}{C_2} g(n) \leq C_1 \cdot g(n)$$

$$\therefore \frac{C_1 C_3}{C_2} g(n) \leq C_1 g(n) \leq f(n) \text{ — By } 1$$

$\therefore \exists\ C_4 = \frac{C_1 C_2}{C_2}$ s.t

$$C_4 \cdot g(n) \leq f(n) \quad \forall n \geq n_0$$

Hence, True!

(5) $f(n) = o(g(n))$ iff $\forall\ c > 0$

$\exists\ n_0 > 0$

such that $0 \leq f(n) < c \cdot g(n)$ — (1)

$\forall\ n \geq n_0$

$f(n) = \Omega(g(n))$ iff $\exists\ c_1, n_0 > 0$

such that $0 \leq c_1 g(n) \leq f(n)$ — (2)

$\forall\ n \geq n_0$

By (2)

$c_1 \cdot g(n) \leq f(n) \quad \exists\ c_1, n_0 \geq 0$

By 1

$c \cdot g(n) > f(n) \quad \forall\ c > 0$

$\exists\ n_0 > 0$

$\angle$ As n tends to infinity

$c \cdot g(n) > f(n)$ will always be true and

$c_1 \cdot g(n) \leq f(n)$ will never occur.

$\therefore$ $f(n) = o(g(n))$ & $f(n) = \Omega(g(n))$ cannot occur at the same time and is not possible.

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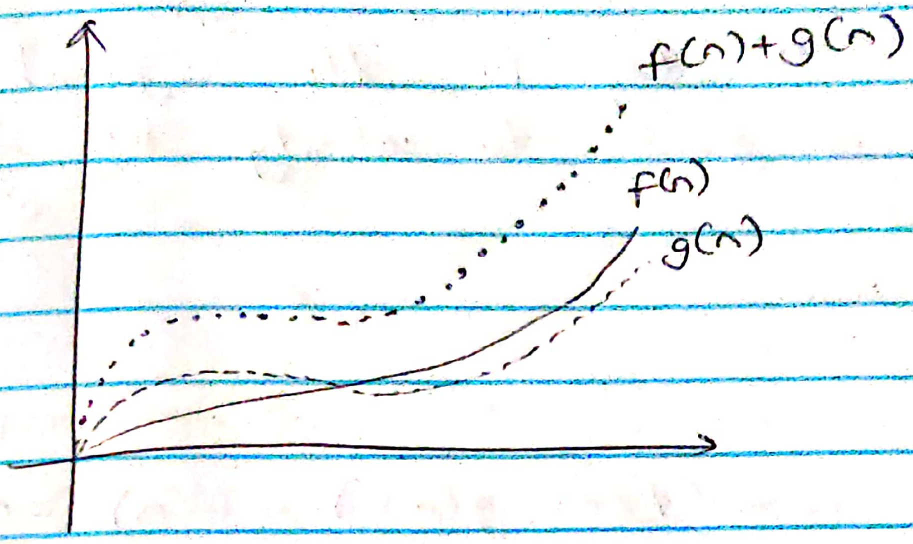


Suppose :

$f(n) = n$

$g(n) = 2n - 1$

$f(n) + g(n) = 3n - 1$

If $f(n) + g(n) = O(\min(f(n), g(n)))$

then $f(n) + g(n) = O(f(n))$

$\therefore$ $3n - 1 = O(n)$ For this to be true $\exists\ c, n_0 > 0$ s.t.

$0 \le 3n - 1 \le c \cdot n \quad \forall n \ge n_0$

$3n - 1 \le c \cdot n$

Let $c = 10$ & $n = 1$

$\therefore$ $3n - 1 \le 10 \cdot n$

$\therefore$ $3(1) - 1 \le 10$

$2 \le 10$

$\therefore$ $f(n) + g(n) = O(\min(f(n), g(n)))$ In this example.

Now, According to the graph, as n grows to infinity all functions grow.

Suppose

$f(n) = n$

$g(n) = n^2$

$\therefore \quad \min(f(n), g(n)) = f(n) = n$

But $\quad f(n) + g(n) = n^2 + n = O(n^2)$

$\therefore \quad O(n^2) \neq \Omega(n)$

$\therefore \quad f(n) + g(n) \neq O(\min(f(n) + g(n)))$

Hence, it is not always true.

Q.2] $f(n) = \omega(n)$ $\qquad\qquad$ $f(n) = o(n^2)$

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

$\forall c>0 \; \exists n_0 > 0$ $\qquad\qquad$ $\forall c>0 \; \exists n_0>0$

s.t $\qquad\qquad$ s.t

(1) $0 < c \cdot (n) < f(n) \; \forall n > n_0$ $\qquad\qquad$ (2) $0 < f(n) < c \cdot n^2 \; \forall n > n_0$

Let $f(n) = n \log n$

By (1)

$c \cdot \cancel{n} < \cancel{n} \log n$

$c < \log n$

$\forall \; c > 0$ we can find $n_0 > 0$ s.t that the above inequality is satisfied.

$\therefore \; f(n) = n \log n = \omega(n) \Longrightarrow$ (R1)

By (2)

$n \log n < c n^2$

$\log n < c n$

if $c > 1$ then $\log n < n$ for $\forall n > 1$

if $0 < c < 1$ then we can always find n_0 s.t. $\log n < cn$ $\forall n > n_0$

$\therefore \; f(n) = n \log n = o(n^2) \Longrightarrow$ (R2)

Let $f(n) = n^{1.5}$

By Another def^n of $o(n^2)$ is :

If $f(n) = o(n^2)$ then

$$\lim_{n \to \infty} \frac{f(n)}{n^2} = 0$$

$$\therefore \lim_{n \to \infty} \frac{n^{1.5}}{n^2} = 0$$

$\therefore f(n) = n^{1.5} = o(n^2) \Longrightarrow R3$

By ①

$c \cdot n < n^{1.5}$

$cn < n^{3/2}$ $\forall c > 0$ $c \cdot n < n^{3/2}$ can be achieved for some $n_0 > 0$

$\therefore f(n) = n^{1.5} = \omega(n) \Longrightarrow R4$

$\therefore f(n) = n\log n$ and $f(n) = n^{1.5}$ are 2 asymtotically different functions which are both $\omega(n)$ & $o(n^2)$ by the results R_1, R_2, R_3, and R_4.

Q.3.]

a. $A = \log^k n \quad B = n^{\varepsilon}$

$A = O(B)$? Yes

$\log^k n \le c \cdot n^{\varepsilon}$

$A = o(B)$

$\log^k n < c n^{\varepsilon}$

b. $A = n^k , \quad B = c^n$

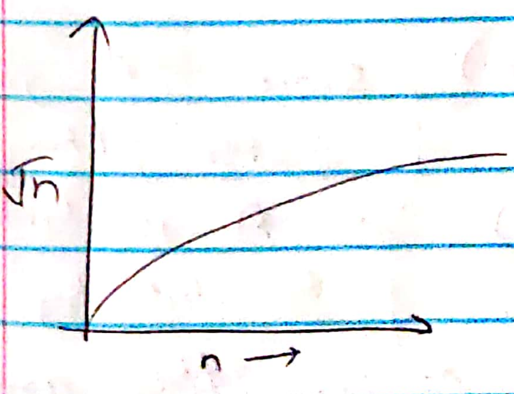
$A = O(B)$?

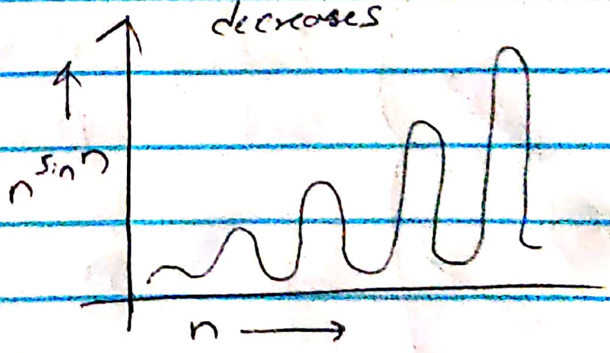
$n^k \le c_1 \cdot c^n$

Q.3.]

c. $A = \sqrt{n} , \quad B = n^{\sin n} \rightarrow$ Oscillating function

$\hookrightarrow$ but increases as $n \rightarrow \infty$ or decreases





~~$\exists\, c, n_0 > 0$ s.t.~~

~~$c \cdot \sqrt{n} \le n^{\sin n} \ \forall n \ge n_0$~~

Since $B = n^{\sin n}$ is an oscillating function it is greater or smaller than $A = \sqrt{n}$ as $n \to \infty$.

Hence, $A \neq O(B)$

Similarly, $A \neq o(B)$

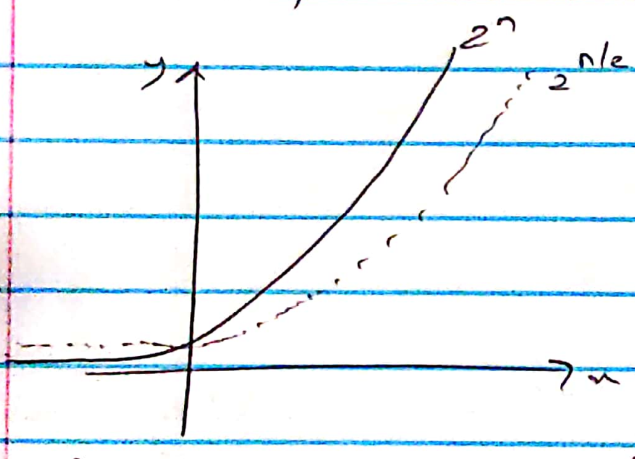
$A \neq \Omega(B)$

$A \neq \omega(B)$

$A \neq \Theta(B)$

C

d. $A = 2^n$, $B = 2^{n/2}$



By looking at the graphs, we can see that 2^n grows faster than $2^{n/2}$

$\therefore$ $A \neq O(B)$

Similarly $A \neq o(B)$

But $A = \Omega(B)$ since A grows faster tha B

and $A = \omega(B)$ as A is lower bounded by B

and $A \neq \Theta(B)$ as per the graph

d

e. $A = n^{\log c}$, $B = c^{\log n}$

According to log properties,

$a^{\log_b c} = c^{\log_b c}$

By this assuming base = 2

$n^{\log_2 c} = c^{\log_2 n}$

$\therefore$ $A = B$ —— (1)

Is $A = O(B)$

Is $n^{\log c} = O(c^{\log n})$

Does the condition,

$n^{\log c} \leq c_0 \cdot c^{\log n}$ for some constant c_0

hold true?

By 1

$$\frac{n^{\log c}}{c^{\log n}} \leq c_0$$

$$1 \leq c_0 \quad \therefore \exists \; c_0 \geq 1$$

Hence $A = O(B)$

Is $n^{\log c} < c_0 \cdot c^{\log n} \quad \forall \, c > 0$

no because $1 \nless c \quad \forall \; c > 0$

Hence, $A \neq o(B)$

~~As~~ $\cancel{c_0 \cdot c^{\log n} \le n^{\log c}}$ ~~∃ c, n_0 > 0~~

By (1) $\cancel{c_0 \le 1}$

By reflexive property,

$$f(n) = \Omega(f(n))$$

Ω holds reflexive property.

$\therefore$ $A = B = c^{\log n} = n^{\log n}$

$\therefore$ $A = \Omega(B)$

$A \neq \omega(B)$ as A is not strictly greater than B.

Similarly, by reflexive property

$$f(n) = \theta(f(n))$$

$\therefore$ $A = \theta(B)$

$$\therefore \left.\begin{array}{l} A = O(B) \\ A \neq o(B) \\ A = \Omega(B) \\ A \neq \omega(B) \\ A = \theta(B) \end{array}\right\} \text{— (e)}$$

f. $A = \log(n!)$ $B = \log(n^n)$

n^n grows extremly faster than $n!$

$\therefore$ $\log(n^n)$ grows ~~slower~~ faster than $\log(n!)$

~~$A \neq o(B)$~~

$\therefore$ $A = \log(n!) = o(\log(n^n)) = o(B)$

$A \neq o(B)$ ~~because~~ $\forall\, c > 0$

$A > B$ for $0 < c < 1$

Is $A = \Omega(B)$?

If yes then

$A \geq c \cdot B$ for some c,

Now, $A = \log(n!)$

$B = \log(n^n) = n \log n$

$\therefore$ $\log(n!) \geq c \cdot n \log n$

$\therefore$ $\dfrac{\log(n!)}{\log n} \geq c \cdot n$

as $n \to \infty$ $\dfrac{\log(n!)}{\log n} \to \infty$

$\therefore$ $\infty \geq c \cdot \infty$

$\therefore$ $1 \geq c$

$\therefore$ $\exists\, c < 1$ s.t the

$A = \Omega(B)$

$A \not> B$

$\therefore A \neq \omega(B)$

And due to similar behaviour of A and B

$$\left.\begin{array}{ll}\text{As} & A = \log(n!) \approx n\log n \\ \& & B = \log(n^n) = n\log n\end{array}\right\} \text{Similar functions}$$

$A = \Theta(B)$

$$\therefore \left.\begin{array}{l} A = O(B) \\ A \neq o(B) \\ A = \Omega(B) \\ A \neq \omega(B) \\ A = \Theta(B) \end{array}\right\} — f$$

Final Table by c, d, e, f

	A	B	O	o	Ω	ω	Θ
c	$\sqrt{n}$	$n^{\sin n}$	No	No	No	No	No
d	2^n	$2^{n/2}$	No	No	Yes	Yes	No
e	$n^{\log c}$	$c^{\log n}$	Yes	No	Yes	No	Yes
f	$\log(n!)$	$\log(n^n)$	Yes	No	Yes	No	Yes

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Q.4]

a) $S_3(n) = \Theta(?)$

$$1 + 2^3 + 3^3 + \cdots + n^3 \leq \underbrace{n^3 + n^3 + \cdots + n^3}_{n}$$

$$1 + 2^3 + \cdots + n^3 \leq n^4$$

$$\therefore \quad S_3(n) = O(n^4) \longrightarrow \text{Upper Bound}$$

$S_3(n)$

$$= 1 + 2^3 + 3^3 + \cdots + n^3 = \left[1 + 2^3 + \cdots + \left(\frac{n}{2} - 1\right)^3\right] + \left[\left(\frac{n}{2}\right)^3 + \left(\frac{n}{2}+1\right)^3 \cdots + n^3\right]$$

$$\geq 0 + \underbrace{\left(\frac{n}{2}\right)^3 + \left(\frac{n}{2}\right)^3 + \cdots + \left(\frac{n}{2}\right)^3}_{n/2 \text{ terms}}$$

$$\geq \frac{n^3}{2^3} \times \frac{n}{2}$$

$$\geq \frac{n^4}{16}$$

$$\therefore \quad S_3(n) = \Omega(n^4/16) = \Omega(n^4)$$

$$\therefore \quad S_3(n) = \Theta(n^4)$$

b) $$S_4(n) = \sum_{k=1}^{n} k^4$$

$$S_4(n+1) = \sum_{k=1}^{n+1} k^4 = \sum_{k=1}^{n} k^4 + (n+1)^4$$

$$= S_4(n) + (n+1)^4 \quad \text{--- } (1)$$

$$S_4(n+1) = \sum_{k=1}^{n+1} k^4 = \sum_{k=0}^{n} (k+1)^4$$

$$= \sum_{k=0}^{n} \left(k^4 + 4k^3 + 6k^2 + 4k + 1 \right)$$

$$= \sum k^4 + 4\sum k^3 + 6\sum k^2 + 4\sum k + \sum 1$$

$$= S_4(n) + 4S_3(n) + 6S_2(n) + 4S_1(n) + n+1 \quad (2)$$

By (1) & (2)

$$\cancel{S_4(n)} + (n+1)^4 = \cancel{S_4(n)} + 4S_3(n) + 6S_2(n) + 4S_1(n) + (n+1)$$

$$(n+1)^4 - (n+1)$$

$$(n+1)\left((n+1)^3 - 1\right) = 4S_3(n) + 6S_2(n) + 4S_1(n)$$

$$\therefore \quad S_3(n) = \frac{(n+1)\left[(n+1)^3 - 1\right] - 6S_2(n) - 4S_1(n)}{4}$$

We know,

$$S_2(n) = \frac{n(n+1)(2n+1)}{6}$$

&

$$S_1(n) = \frac{n(n+1)}{2}$$

By substituting these values we get

$$S_3 = \frac{(n+1)\left((n+1)^3-1\right) - n(n+1)(2n+1) - 2n(n+1)}{4}$$

$$= \frac{(n+1)}{4}\left[(n+1)^3 - 1 - n(2n+1) - 2n\right]$$

$$= \frac{(n+1)}{4}\left[(n+1)^3 - 1 - 2n^2 - n - 2n\right]$$

$$= \frac{(n+1)}{4}\left[(n+1)^3 - 2n^2 - 3n - 1\right]$$

$$= \frac{(n+1)}{4}\left[n^3 + 3n^2 + \cancel{3n} + \cancel{1} - 2n^2 - \cancel{3n} - \cancel{1}\right]$$

$$= \frac{(n+1)}{4}\left[n^3 + n^2\right]$$

$$= \frac{n^2(n+1)^2}{4} \quad \therefore \boxed{S_3(n) = \frac{n^2(n+1)^2}{4}}$$

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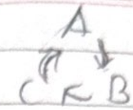
Q 5 Base case -



$n=0$

There are 0 disks

As there are 0 disks, there would be no moves

$\therefore Q_0 = 0$ and $R_0 = 0$ as no moves would be taken between A and B and B and A respectively

Hence, the statements are true

Recursive case -

$n>0$

Let there be n disks

1) Moving from A to B.

i) Moving top $(n-1)$ disks from A to C via B.
As only the top disks are moving from A to C via B it takes R_{n-1} moves.

ii) Moving the largest disk from A to B. This takes 1 move

iii) Now the remaining $(n-1)$ disks have to be moved cyclically to B from C. Therefore, from C to B $n-1$ disks take R_{n-1} moves

$\therefore$ For 3 disks

i) ~~R_2~~ 2 disks are moved from A to B in R_2 move

ii) 1 move is needed for largest disk

iii) Now the 2 disks at B have to be moved to C in R_2 moves

Hence the total steps from A to B ie Q_n is

$$Q_n = R_{n-1} + R_{n-1} + 1$$

$$\boxed{Q_n = 2R_{n-1}}$$

Hence proved.

2) Moving from B to A

i) Moving the top $(n-1)$ disks B to C via A. It takes

Q_{n-1} moves.

ii) Moving the largest disk ie n^{th} disk from B to A which takes 1 move

iii) Now moving the remaining $(n-1)$ disks from C to A. As the move takes Q_n time. &

∴ Total no. of moves $\boxed{R_n = Q_n + Q_{n-1} + 1}$ from moving disks from B to A.

Hence proved