[For Divide & Conquer]

"Baby" Master Theorem: If $a>0$, $b>1$ and $d\ge 0$ are constants, the solutions of a recurrence of the form

$$T(n)=\begin{cases} a\,T(n/b)+\Theta(n^d) & n>n_0 \text{ for some const } n_0 \\ \text{constant}>0 & \text{if } n\le n_0 \end{cases}$$

is

$$T(n)=\begin{cases} \Theta(n^d) & \text{if } a/b^d<1 \\ \Theta(n^d\log n) & \text{if } a/b^d=1 \\ \Theta(n^{\log_b a}) & \text{if } a/b^d>1 \end{cases}$$

If not, n is at most a const factor away from a power of b. $\Rightarrow$ No effect on $T(n)$ in $\Theta()$

Assume: n is a power of b, i.e. $n=b^t$ for some t. Also, let $T(1)=c_1$ for some constant $c_1>0$

Sol: Let $T(n)=a\,T(\frac{n}{b})+cn^d$ for some constant $c>0$

$$\Rightarrow T(n)=a\,T\left(\tfrac{n}{b}\right)+cn^d$$

$$=a\left[a\,T\left(\tfrac{n}{b^2}\right)+c\left(\tfrac{n}{b}\right)^d\right]+cn^d=a^2\,T\left(\tfrac{n}{b^2}\right)+cn^d\left(1+\tfrac{a}{b^d}\right)$$

$$=a^2\left[a\,T\left(\tfrac{n}{b^3}\right)+c\left(\tfrac{n}{b^2}\right)^d\right]+cn^d\left(1+\tfrac{a}{b^d}\right)=a^3\,T\left(\tfrac{n}{b^3}\right)+cn^d\left[1+\tfrac{a}{b^d}+\left(\tfrac{a}{b^d}\right)^2\right]$$

$$=\cdots=a^t\,T\left(\tfrac{n}{b^t}\right)+cn^d\left[1+\tfrac{a}{b^d}+\left(\tfrac{a}{b^d}\right)^2+\cdots+\left(\tfrac{a}{b^d}\right)^{t-1}\right]$$

When $n=b^t$ i.e. $t=\log_b n$. $a^t\,T\left(\frac{n}{b^t}\right)=c_1a^t=\Theta(a^t)=\Theta(a^{\log_b n})=\Theta(n^{\log_b a})$

Case 1: $a/b^d<1$ i.e. $\log_b a<d$ $\Rightarrow a^t\,T\left(\frac{n}{b^t}\right)=\Theta(n^{\log_b a})=O(n^d)$ —①

$$cn^d\left[1+\tfrac{a}{b^d}+\left(\tfrac{a}{b^d}\right)^2+\cdots\right]\le cn^d\,\frac{1}{1-\frac{a}{b^d}}=\Theta(n^d) \quad —②$$

From ①, ② $T(n)=\Theta(n^d)$ #

Case 2: $a/b^d=1$ i.e. $\log_b a=d$ $\Rightarrow a^t\,T\left(\frac{n}{b^t}\right)=\Theta(n^{\log_b a})=\Theta(n^d)$ —①

$$cn^d\Big[\underbrace{1+\tfrac{a}{b^d}+\left(\tfrac{a}{b^d}\right)^2+\cdots+\left(\tfrac{a}{b^d}\right)^{t-1}}_{t}\Big]=\Theta(n^d\cdot t)=\Theta(n^d\log n) \quad —②$$

From ①, ② $T(n)=\Theta(n^d\log n)$ #

Case 3: $a/b^d>1$ i.e. $\log_b a>d$ $\Rightarrow a^t\,T\left(\frac{n}{b^t}\right)=\Theta(n^{\log_b a})$ —①

$$cn^d\left[1+\tfrac{a}{b^d}+\left(\tfrac{a}{b^d}\right)^2+\cdots+\left(\tfrac{a}{b^d}\right)^{t-1}\right]=cn^d\,\frac{\left(\frac{a}{b^d}\right)^t-1}{\frac{a}{b^d}-1}=\Theta\left(n^d\cdot\frac{a^t}{(b^t)^d}\right)$$

(Recall: $b^t=n$)

$$=\Theta\left(n^d\cdot\frac{a^t}{n^d}\right)=\Theta(a^t)=\Theta(a^{\log_b n})=\Theta(n^{\log_b a}) \quad —②$$

From ①, ② $T(n)=\Theta(n^{\log_b a})$ #

[Note: $a^{\log_b n}=a^{\left[\frac{\log n}{\log a}\cdot\frac{\log a}{\log b}\right]}=a^{(\log_a n)(\log_b a)}=\left(a^{\log_a n}\right)^{\log_b a}=n^{\log_b a}$ #]