CS6033 Lecture 1 Slides/Notes

Introduction; Perturbation Method (Ch1, Secs. 2.1, 2.2, Ch3 (Secs. 3.1, 3,2), Notes)

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Introduction

What's an algorithm?

important aspects: terminate.
correct.
performance.

How do we measure performance?

measure # of steps. in terms of the input size n.

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Running time: $\underline{\underline{f(n)}}$.

e.g. $f(n) = 2n^2 + 5n + 6$.

Performance:

* Running time: # of computing steps.
* Memory space: size of working space

Both are in terms of the (input) problem size n.

Best-case
worst-case
average-case
} performance

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Example: Insertion Sort

2 6 3 5 8 10

$\boxed{2} \rightarrow \boxed{6}$

3

$\boxed{2} \rightarrow \boxed{3} \rightarrow \boxed{6}$

5

$\boxed{2} \rightarrow \boxed{3} \rightarrow \boxed{5} \rightarrow \boxed{6} \rightarrow \boxed{8} \rightarrow \boxed{10}$

8 10

process items one by one.
insert the current item into the sorted list maintained so far. (small $\longrightarrow$ large);

e.g.:

2 3 5 6 8 10

Worst case.

$\boxed{2} \rightarrow \boxed{3}$

$\boxed{2} \rightarrow \boxed{3} \rightarrow \boxed{5}$

comparisons

3: 1

5: 2

6: 3.

$1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2} = \frac{n^2}{2} - \frac{n}{2}$

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e.g. 10 8 6 5 3 2

$\boxed{3} \to \boxed{6} \to \boxed{8} \to \boxed{10}$

⋮

Each item only uses 1 comparison.

$\Rightarrow$ $n-1$ comparisons.

Best case:

Q: Given a sequence of n numbers, find the max of them.

Worst case: $n-1$ comparisons.

Best case: "

(same)

Ex: Selection Sort: In each iteration, select the max from the remaining items & put the selected item to the front (left) of the answer list so far.

# of comparisons in $\left\{\begin{matrix}\text{Worst case}\\ \text{Best case}\end{matrix}\right\}$ (same):

$$(n-1)+(n-2)+\cdots+1 = \frac{n(n-1)}{2} = \frac{n^2}{2} - \frac{n}{2}$$

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Review of Asymptotic notations: e.g. $f(n) = 2n^2 + 5n + 6$

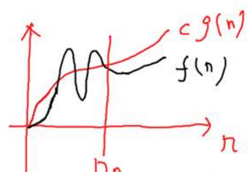
① $O(g(n)) = \{ f(n) \mid \exists$ constants $c, n_0 > 0$ s.t. $0 \le f(n) \le c\,g(n)\ \forall n \ge n_0 \}$

② $\Omega(g(n)) = \{ f(n) \mid \exists$ consts $c, n_0 > 0$. s.t. $0 \le c\,g(n) \le f(n)\ \forall n \ge n_0 \}$

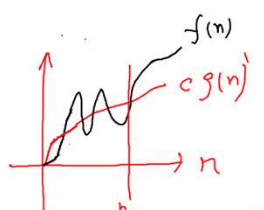
③ $\Theta(g(n)) = \{ f(n) \mid \exists$ constants $c_1, c_2, n_0 > 0$ s.t. $0 \le c_1 g(n) \le f(n) \le c_2 g(n)\ \forall n \ge n_0 \}$

④ $o(g(n))$

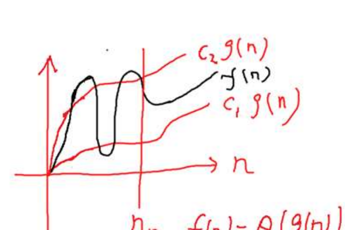
⑤ $\omega(g(n))$



$f(n) = O(g(n))$



$f(n) = \Omega(g(n))$



$f(n) = \Theta(g(n))$

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e.g. $\frac{n^2}{2} - 3n = f(n)$ $\quad$ $f(n) = \Theta(n^2)$.

We want: Find consts. $c_1, c_2, n_0 \geq 0$ s.t. $c_1 n^2 \leq \frac{n^2}{2} - 3n \leq c_2 n^2$ $\forall n \geq n_0$

sol: $c_1 n^2 \leq \frac{n^2}{2} - 3n$.

$c_1 \leq \frac{1}{2} - \frac{3}{n}$

Take $n = 7$.

$c_1 \leq \frac{1}{2} - \frac{3}{7} = \frac{7-6}{14} = \frac{1}{14}$ $\longrightarrow$ Take $c_1 = \frac{1}{14}$.

$n = 8$.

$\frac{1}{2} - \frac{3}{8} \geq \frac{1}{2} - \frac{3}{7} = \frac{1}{14} = c_1$ $\Longrightarrow$ $c_1 = \frac{1}{14} \leq \frac{1}{2} - \frac{3}{n}$ $\forall n \geq 7$

Take $n_0 = 7$

$\frac{1}{2} - \frac{3}{n} \leq c_2$.

Take $c_2 = \frac{1}{2}$.

$\left(\frac{1}{2} - \frac{3}{n} \leq \frac{1}{2} \quad \forall n \geq 1\right)$

$\boxed{\text{Take } c_1 = \frac{1}{14},\ c_2 = \frac{1}{2},\ n_0 = 7}$

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$o(g(n)) = \{ f(n) \mid \forall$ constant $c > 0$, $\exists n_0 > 0$ s.t. $0 \leq f(n) < c\, g(n)$ $\forall n \geq n_0 \}$.

Asymptotically less than $\quad \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$.

$\omega(g(n)) = \{ f(n) \mid \forall$ const. $c > 0$. $\exists n_0 > 0$ st. $0 \leq c\, g(n) < f(n)$ $\forall n \geq n_0 \}$.

Asymptotically larger than $\quad \left(\lim_{n \to \infty} \frac{g(n)}{f(n)} = 0\right)$

$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$.

Summary:

$f(n) = O(g(n))$: $f(n)$ "$<=$" $g(n)$

Ω "$>=$"

Θ "$=$"

o "$<$" $\qquad$ ω: "$>$"

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Asymptotically:

$\Theta \equiv$ " $=$ "

$O \equiv$ " $\le$ "

$\Omega \equiv$ " $\ge$ "

o " $<$ "

ω " $>$ "

By (1), (2):

$\frac{n^2}{4} \le S_1(n) \le n^2$.

$\boxed{S_1(n) = \Theta(n^2)}$ (*)

e.g. (1) $S_1(n) = 1 + 2 + 3 + \cdots + n$.

$S_1(n) = \Theta(?)$

$1 + 2 + 3 + \cdots + n$

$\le \underbrace{n + n + \cdots + n}_{n} = n^2 \Longrightarrow S_1(n) \le n^2$.

(**) $\boxed{S_2(n) = \sum_{k=1}^{n} k^2 \quad \text{Similarly, } S_2(n) = \Theta(n^3)}$

(2) $1 + 2 + 3 + \cdots + n = \left(1 + 2 + \cdots + \frac{n}{2} - 1\right) + \left(\frac{n}{2} + \left(\frac{n}{2} + 1\right) + \cdots + n\right)$

$\ge 0 + \underbrace{\frac{n}{2} + \frac{n}{2} + \cdots + \frac{n}{2}}_{n/2 \text{ terms}}$

$= \frac{n}{2} \cdot \frac{n}{2} = \frac{n^2}{4}$

$\Longleftarrow \frac{n^2}{4} \le S_1(n)$

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Similar trick for $n!$

$n! = 1 \cdot 2 \cdot 3 \cdots n \le \underbrace{n \cdot n \cdots n}_{n} = n^n$. — ①

$n! = 1 \cdot 2 \cdot 3 \cdots n \ge (1 \cdot 1 \cdot 1 \cdots 1) \cdot \underbrace{\left(\frac{n}{2} \cdot \frac{n}{2} \cdots \frac{n}{2}\right)}_{n/2} = \left(\frac{n}{2}\right)^{\frac{n}{2}}$. — ②

Q: $\log(n!) = \Theta(?)$

From ①. $\log(n!) \le \log(n^n) = n \log n = O(n \log n)$

From ② $\log(n!) \ge \log\left(\left(\frac{n}{2}\right)^{\frac{n}{2}}\right) = \left(\frac{n}{2}\right) \log\left(\frac{n}{2}\right) = \Omega(n \log n)$

$\Rightarrow \underline{\log(n!) = \Theta(n \log n)}$

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Perturbation Method

Idea: write the Σ expression by leaving out the last term.

Def: $S_1(n) = \sum_{k=1}^{n} k$. $S_2(n) = \sum_{k=1}^{n} k^2$. $S_3(n) = \sum_{k=1}^{n} k^3$.

$$S_2(n+1) = \sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2 = S_2(n) + (n+1)^2 \quad \text{—— (a)}$$

$$S_2(n+1) = \sum_{k=1}^{n+1} k^2 = \sum_{k=0}^{n} (k+1)^2 = \sum_{k=0}^{n} (k^2+2k+1)$$

$$= \sum_{k=0}^{n} k^2 + 2\sum_{k=0}^{n} k + \boxed{\sum_{k=0}^{n} 1} \quad \text{—— (b)}$$

$$= S_2(n) + 2S_1(n) + (n+1)\cdot 1$$

$$(a)=(b) \qquad \cancel{S_2(n)} + (n+1)^2 = \cancel{S_2(n)} + 2S_1(n) + (n+1)$$

$$S_1(n) = \frac{1}{2}\left[(n+1)^2 - (n+1)\right] = \frac{1}{2}(n+1)\left[n+1-1\right] = \frac{n(n+1)}{2}$$

$$S_3(n) = \sum_{k=1}^{n} k^3 \qquad \left((x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \right)$$

$$S_3(n+1) = \sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3 = S_3(n) + (n+1)^3 \quad \text{—— (a)}$$

$$S_3(n+1) = \sum_{k=1}^{n+1} k^3 = \sum_{k=0}^{n} (k+1)^3 = \sum_{k=0}^{n} (k^3+3k^2+3k+1)$$

$$= \sum_{k=0}^{n} k^3 + 3\sum_{k=0}^{n} k^2 + 3\sum_{k=0}^{n} k + \sum_{k=0}^{n} 1$$

$$= S_3(n) + 3S_2(n) + 3S_1(n) + (n+1) \quad \text{—— (b)}$$

$$(a)=(b): \quad \cancel{S_3(n)} + (n+1)^3 = \cancel{S_3(n)} + 3S_2(n) + 3S_1(n) + (n+1)$$

$$\Rightarrow S_2(n) = \frac{1}{3}\left[(n+1)^3 - 3\frac{n(n+1)}{2} - (n+1)\right] = \frac{1}{3}(n+1)\left[(n+1)^2 - \frac{3n}{2} - 1\right]$$

$$= \frac{(n+1)}{3}\left(n^2+2n+\cancel{1}-\frac{3n}{2}-\cancel{1}\right) = \frac{(n+1)}{3}n\left[n+\frac{1}{2}\right] = \frac{n(n+1)(2n+1)}{6}$$