

# Composition in $\mathcal{LG}$

Rohan Hitchcock

8 July 2022

Throughout fix a commutative ring  $k$ . Recall that  $\mathcal{LG}$  has the following data:

- (1) Objects are pairs  $(k[x], U)$  where  $k[x] = k[x_1, \dots, x_n]$  and  $U \in k[x]$  is a *potential*. That is,  $U$  satisfies
  - The sequence  $\partial U = (\partial_{x_1} U, \dots, \partial_{x_n} U)$  has a Koszul complex which is exact except in degree zero.
  - The Jacobi ring  $J_U = k[x]/(\partial_{x_1} U, \dots, \partial_{x_n} U)$  is a free, finitely generated  $k$ -module.

- (2) The category of 1-morphisms  $(k[x], U) \rightarrow (k[y], V)$  is

$$\text{hmf}(k[x, y], V(y) - U(x))^\omega$$

where  $\mathcal{C}^\omega$  denotes the idempotent completion of a category  $\mathcal{C}$ . Equivalently the 1-morphisms are matrix factorisations of  $V(y) - U(x)$  which are homotopy equivalent to a direct summand of a finite rank matrix factorisation (see [Idempotents]).

The definition of potential above differs from the definition given in [CM16, Definition 2.4]. The following result shows they are equivalent.

**Lemma 1** ([Stacks, Section 15.30]). *For a sequence  $t = (t_1, \dots, t_n)$  of elements of a commutative ring  $R$  we have that:*

- (1) *If  $t$  is regular then the Koszul complex  $K(t)$  is exact except in degree zero.*
- (2) *If the Koszul complex  $K(t)$  is exact except in degree zero then  $t$  is quasi-regular.*

Throughout consider the following 1-morphisms in  $\mathcal{LG}$ :

$$(k[x], U) \xrightarrow{(X, d_X)} (k[y], V) \xrightarrow{(Y, d_Y)} (k[z], W)$$

so  $(X, d_X)$  is a finite rank matrix factorisation of  $V(y) - U(x)$  over  $k[x, y]$  and  $(Y, d_Y)$  is a finite rank matrix factorisation of  $W(z) - V(y)$  over  $k[y, z]$  and  $U, V$  and  $W$  are potentials. We define the composition of  $(X, d_X)$  and  $(Y, d_Y)$  to be the tensor product  $(X \otimes_{k[y]} Y, d_X \otimes 1 + 1 \otimes d_Y)$ . It is not immediate that this is well-defined. Indeed, if  $X = k[x, y]^{\oplus m}$  and  $Y = k[y, z]^{\oplus m'}$  then

$$X \otimes_{k[y]} Y = k[x, y]^{\oplus m} \otimes_{k[y]} k[y, z]^{\oplus m'} = k[x, y, z]^{\oplus mm'}$$

which is free, but not finite rank over  $k[x, z]$ . Hence our goal is to show

$$(X \otimes_{k[y]} Y, d_{X \otimes Y}) \in \text{hmf}(k[x, z], W(z) - U(x))^\omega$$

which we do by showing  $(X \otimes_{k[y]} Y, d_{X \otimes Y})$  is a direct summand of a finite rank matrix factorisation over  $k[x, z]$ .

# 1 Koszul complexes

Before continuing we need to discuss Koszul complexes and exterior algebras. Our main reference for this is [Wei94, Chapter 4.5]. Let  $R$  be a commutative ring and consider a free  $R$ -module  $V = R^{\oplus n}$  with free generators  $e_1, \dots, e_n$ . We define an associative bilinear product  $(-) \wedge (-) : V \times V \rightarrow V$  subject to the relation generated by setting

$$u \wedge v = -(v \wedge u) \quad \text{for all } u, v \in V$$

In particular notice that  $v \wedge v = 0$  for all  $v \in V$ . The  $p$ -th exterior power of  $V$ , denoted  $\bigwedge^p(V)$  is defined as the set of  $p$ -fold products. One can show that this is a free  $R$ -module with generators

$$\{e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_p} \mid i_1 < i_2 < \dots < i_p\}$$

and so  $\dim \bigwedge^p(V) = \binom{n}{p}$ . For example if  $n = 3$  we have

- $\bigwedge^0(V)$  is generated by 1 (i.e.  $\bigwedge^0(V) = R$ ).
- $\bigwedge^1(V)$  is generated by  $\{e_1, e_2, e_3\}$  (i.e.  $\bigwedge^1(V) = V$ )
- $\bigwedge^2(V)$  is generated by  $e_1 \wedge e_2, e_1 \wedge e_3$  and  $e_2 \wedge e_3$
- $\bigwedge^3(V)$  is generated by  $e_1 \wedge e_2 \wedge e_3$
- $\bigwedge^p(V)$  for  $p \geq 4$  is zero.

The exterior algebra of  $V$ , denoted  $\bigwedge(V)$ , is the graded  $R$ -algebra with  $p$ -th graded component  $\bigwedge^p(V)$ .

Next, given a sequence of elements  $t = (t_1, \dots, t_n)$  in  $R$  we define the Koszul complex of  $t$  as the pair  $(K(t), d_K)$  where  $K(t) = \bigwedge(V)$  and  $d_K : K(t) \rightarrow K(t)$  is given by

$$e_{i_1} \wedge \dots \wedge e_{i_p} \mapsto \sum_{j=1}^p (-1)^{j+1} t_{i_j} e_{i_1} \wedge \dots \wedge \widehat{e}_{i_j} \wedge \dots \wedge e_{i_p}$$

where “ $\widehat{e}_{i_j}$ ” indicates that  $e_{i_j}$  is omitted from the wedge product. One can show that  $d_K^2 = 0$ , so  $(K(t), d_K)$  is a chain complex. For example, if  $t = (t_1)$  (a sequence with one element) then the Koszul complex is

$$0 \longrightarrow R \xrightarrow[t_1]{1} R \longrightarrow 0$$

In the case of  $n = 3$  we have

$$0 \longrightarrow \bigwedge^3(V) \xrightarrow{\begin{pmatrix} t_3 \\ -t_2 \\ t_1 \end{pmatrix}} \bigwedge^2(V) \xrightarrow{\begin{pmatrix} -t_2 & -t_3 & 0 \\ t_1 & 0 & -t_3 \\ 0 & t_1 & t_2 \end{pmatrix}} \bigwedge^1(V) \xrightarrow{(t_1 \ t_2 \ t_3)} \bigwedge^0(V) \longrightarrow 0$$

$\{e_1 \wedge e_2 \wedge e_3\}$                        $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$                        $\{e_1, e_2, e_3\}$                        $\{1\}$

Notice that the degree zero homology of  $(K(t), d_K)$  is the quotient ring  $R/(t)$ . It is also useful to note that

$$K(t) = K(t_1) \otimes_R K(t_2) \otimes_R \dots \otimes_R K(t_n)$$

where the right-hand-side is a tensor product of complexes.

## 2 Definitions and results from previous talks

Let  $S$  be a commutative ring and  $R$  a  $S$ -algebra. Let  $(L, d_L)$  and  $(M, d_M)$  be matrix factorisations of  $f \in \varphi(S)$

**Definition 2** ([Perturbation, Definition 1.1]). A *strong deformation retract* of  $(L, d_L)$  and  $(M, d_M)$  over  $S$  consists of  $S$ -linear maps

$$(L, d_L) \xrightleftharpoons[i]{p} (M, d_M), \quad h$$

where  $pi = 1$ ,  $h : pi \simeq 1$ ,  $hi = 0$ ,  $ph = 0$  and  $h^2 = 0$ .

A strong deformation retract is a homotopy equivalence of matrix factorisations with additional conditions on the maps involved, the point being that strong deformation retracts can be modified using the Perturbation Lemma [Perturbation, Theorem 3.1]. In a previous talk we proved the following corollary of the Perturbation Lemma.

**Lemma 3** ([Perturbation, Lemma 2.1]). Suppose we have a strong deformation retract over  $S$

$$(L, d_L) \xrightleftharpoons[\sigma]{\pi} (M, d_M), \quad h$$

Then for any linear factorisation  $(Z, d_Z)$  of  $g \in R$  where  $f + g \in \varphi(S)$  there exists a deformation retract over  $S$

$$(L \otimes_R Z, d_L \otimes 1 + 1 \otimes d_Z) \xrightleftharpoons{\quad} (M \otimes_R Z, d_M \otimes 1 + 1 \otimes d_Z), \quad h'$$

The following result gives us a source of strong deformation retracts.

**Lemma 4** ([Perturbation, Lemma 2.2]). Let  $(P, d)$  be a bounded-to-the-right chain complex of projective objects in an abelian category. Suppose that  $(P, d)$  is exact except at degree zero and that  $H_0(P)$  is also projective. Then we have a strong deformation retract

$$(H(P), 0) \xrightleftharpoons{\quad} (P, d), \quad h$$

of chain complexes, where  $(H(P), 0)$  is the homology of  $P$  with zero differentials.

**Lemma 5.** Let  $U \in k[x]$  be a potential. Consider the sequence of partial derivatives  $\partial U = (\partial_{x_1} U, \dots, \partial_{x_n} U)$  and the Jacobi ring  $J_U = k[x]/(\partial U)$ . Then we have a strong deformation retract over  $k$

$$(J_U, 0) \xrightleftharpoons{\quad} (K(\partial U), d_K), \quad h$$

*Proof.* By the potential hypothesis,  $(K(\partial U), d_K)$  is exact in degree zero and  $J_U$  is free (hence projective) over  $k$ . The result is an application of the previous lemma.  $\square$

## 3 Composition in $\mathcal{LG}_k$ is well-defined

The main goal of this talk is to prove that composition in  $\mathcal{LG}$  is well-defined, which we do following the strategy of [Mur17]. Consider the following 1-morphisms in  $\mathcal{LG}$ :

$$(k[x], U) \xrightarrow{(X, d_X)} (k[y], V) \xrightarrow{(Y, d_Y)} (k[z], W)$$

so  $(X, d_X)$  is a finite rank matrix factorisation of  $V(y) - U(x)$  over  $k[x, y]$  and  $(Y, d_Y)$  is a finite rank matrix factorisation of  $W(z) - V(y)$  over  $k[y, z]$  and  $U, V$  and  $W$  are potentials. Our goal is to show

$$(X \otimes_{k[y]} Y, d_{X \otimes Y}) \in \text{hmf}(k[x, z], W(z) - U(x))^\omega$$

where  $\mathcal{C}^\omega$  denotes the idempotent completion of a category  $\mathcal{C}$ , or equivalently (see [Idempotents]) to prove the following:

**Proposition 6.** *The composition of  $(X, d_X)$  and  $(Y, d_Y)$ , which is  $(X \otimes_{k[y]} Y, d_{X \otimes Y})$ , is a direct summand of a matrix factorisation which is finite rank over  $k[x, z]$ .*

We do this as follows:

- We begin with a strong deformation retract over  $k$  arising from a Koszul complex of  $\partial V = (\partial_{y_1} V, \dots, \partial_{y_n} V)$  as in Lemma 5.

$$(J_V, 0) \xrightleftharpoons[\iota]{\pi} (K(\partial V), d_K), \quad h$$

- We tensor both sides of this strong deformation retract by  $(X \otimes_{k[y]} Y, d_{X \otimes Y})$  to obtain

$$(X \otimes_{k[y]} J_V \otimes_{k[y]} Y, d_Z) \xrightleftharpoons[\tilde{\iota}]{\tilde{\pi}} (\bigwedge(k^{\oplus n}) \otimes_k X \otimes_{k[y]} Y, d_K + d_{X \otimes Y}), \quad \tilde{h}$$

where  $Z = X \otimes_{k[y]} Y$ .

Notice that on the left-hand-side we have  $J_V \cong k^{\oplus m}$  for some  $m$  and so

$$X \otimes_{k[y]} J_V \otimes_{k[y]} Y \cong k[x, z]^{\oplus m}$$

Hence the left-hand-side is a finite rank matrix factorisation. Also,  $\bigwedge(k^{\oplus n}) \otimes_k X \otimes_{k[y]} Y$  has  $X \otimes_{k[y]} Y$  as a direct summand. The only thing which remains is to remove the differential  $d_K$ , which we do over the following several lemmas.

**Lemma 7.** *Let  $f, g : (C, d_C) \rightarrow (D, d_D)$  be morphisms of either complexes or linear factorisations, and suppose they are homotopic via  $h : f \simeq g$ . Then  $\text{cone}(f) \cong \text{cone}(g)$ .*

*Proof.* (give as exercise) For clarity we define  $\text{cone}(f)$  as

$$\cdots \longrightarrow C_{n+2} \oplus D_{n+1} \xrightarrow{d_f} C_{n+1} \oplus D_n \xrightarrow{d_f} C_n \oplus D_{n-1} \longrightarrow \cdots$$

where  $d_f = \begin{pmatrix} d_C & 0 \\ f & -d_C \end{pmatrix}$ , and if  $(C, d_C)$  is a linear factorisation then addition in the indices is modulo 2. Likewise  $\text{cone}(g)$  is the graded object  $C[1] \oplus D$  with differential  $d_g = \begin{pmatrix} d_C & 0 \\ g & -d_C \end{pmatrix}$ . Define  $\varphi : C[1] \oplus D \rightarrow C[1] \oplus D$  as  $\varphi = \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}$ . We have

$$\varphi d_f = \begin{pmatrix} d_C & 0 \\ -hd_C + f & -d_D \end{pmatrix}$$

and

$$d_g \varphi = \begin{pmatrix} d_C & 0 \\ g + d_D h & -d_D \end{pmatrix}$$

These agree since  $f - g = hd_C + d_D h$  and so  $\varphi : \text{cone}(f) \rightarrow \text{cone}(g)$  is a morphism. Likewise we define  $\psi : \text{cone}(g) \rightarrow \text{cone}(f)$  as  $\psi = \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}$ , and we have  $\varphi\psi = \text{id}$  and  $\psi\varphi = \text{id}$ .  $\square$

**Lemma 8.** *Let  $t = (t_1, \dots, t_n)$  be a sequence in  $R$  in which each  $t_i$  acts null-homotopically on a linear factorisation  $(X, d_X)$  over  $R$ . Then  $(K(t) \otimes_R X, d_K \otimes 1 + 1 \otimes d_X)$  is isomorphic to  $(K(t) \otimes_R X, 1 \otimes d_X)$ , where  $(K(t), d_K)$  is the Koszul complex of  $t$ .*

*Proof.* For each  $t_i$  consider the map  $t_i : X \rightarrow X$  which is multiplication by  $t_i$ . It is straightforward to check that  $(K(t_i), d_{K_i}) \otimes_R (X, d_X) \cong \text{cone}(t_i)$ , where  $\text{cone}(t_i) = (X \oplus X[1], \begin{pmatrix} -d_X & 0 \\ -t_i & d_X \end{pmatrix})$ .

Since we have that  $t_i$  is null-homotopic we have by Lemma 7 that  $\text{cone}(t_i)$  is homotopy equivalent to  $\text{cone}(0) = (X \oplus X[1], \begin{pmatrix} -d_X & 0 \\ 0 & d_X \end{pmatrix})$ . Note that  $X \oplus X[1] \cong (R \oplus R[1]) \otimes_R X$  and that  $K(t) = K(t_1) \otimes_R \dots \otimes_R K(t_n)$ . This gives the desired equivalence.  $\square$

**Lemma 9.** *Each partial derivative  $\partial_{y_i} V$  acts null-homotopically on  $X$  and  $Y$ .*

*Proof.* Fix a  $k$ -basis for  $X$  and let  $\partial_{y_i}(d_X)$  be the map given by differentiating the matrix of  $d_X$  entrywise. By the Leibniz rule we obtain

$$\partial_{y_i} V(y) = \partial_{y_i}(V(y) - U(x)) = \partial_{y_i}(d_X^2) = d_X \partial_{y_i}(d_X) + \partial_{y_i}(d_X) d_X$$

Likewise for  $Y$ .  $\square$

Hence we have shown that there is an isomorphism of matrix factorisations

$$\varphi : (\bigwedge(k^{\oplus n}) \otimes_k Z, d_K + d_Z) \longrightarrow (\bigwedge(k^{\oplus n}) \otimes_k Z, d_Z)$$

and completed the proof of Proposition 6.

### 3.1 Uninterrupted proof of Proposition 6

Let  $n$  be the number of variables of the polynomial ring  $k[y]$  and  $\partial V = (\partial_{y_1} V, \dots, \partial_{y_n} V)$  the sequence of partial derivatives. Consider the Jacobi ring  $J_V = k[y]/(\partial V)$  and the Kozul complex  $(K(\partial V), d_K)$  of  $\partial V$  as a sequence in  $k[y]$ . Since  $V$  is a potential  $J_V$  is a free  $k$ -module and by Lemma 5 we obtain a strong deformation retract

$$(J_V, 0) \xrightleftharpoons[\iota]{\pi} (K(\partial V), d_K), \quad h$$

over  $k$ . Now set  $Z = X \otimes_{k[y]} Y$  and regard  $(-) \otimes_{k[y]} Z$  as a functor from the full subcategory of  $k$ -modules which are also  $k[y]$ -modules to the category of  $k[x, z]$ -modules. Applying this to the strong deformation retract above gives a strong deformation retract over  $k[x, z]$ :

$$(X \otimes_{k[y]} J_V \otimes_{k[y]} Y, 0) \xrightleftharpoons[\iota \otimes 1]{\pi \otimes 1} (\bigwedge(k^{\oplus n}) \otimes_k Z, d_K), \quad h \otimes 1$$

noting that  $K(\partial V) \cong \bigwedge(k^{\oplus n}) \otimes_k k[k]$ . Set  $d_Z = d_X \oplus 1 + 1 \oplus d_Y$ . Using the Perturbation Lemma, and more specifically 3 we can mix in the differential  $d_Z$  to obtain a strong deformation retract

$$(X \otimes_{k[y]} J_V \otimes_{k[y]} Y, d_Z) \xrightleftharpoons[\tilde{\iota}]{\tilde{\pi}} (\bigwedge(k^{\oplus n}) \otimes_k Z, d_K + d_Z), \quad \tilde{h}$$

Next, by Lemma 8 we have an isomorphism of linear factorisations

$$\varphi : (\bigwedge(k^{\oplus n}) \otimes_k Z, d_K + d_Z) \longrightarrow (\bigwedge(k^{\oplus n}) \otimes_k Z, d_Z)$$

and so we obtain a strong deformation retract

$$(X \otimes_{k[y]} J_V \otimes_{k[y]} Y, d_Z) \xrightleftharpoons[\varphi \tilde{\iota}]{\tilde{\pi} \varphi^{-1}} (\bigwedge(k^{\oplus n}) \otimes_k Z, d_Z), \quad \varphi \tilde{h} \varphi^{-1}$$

Finally we note that  $J_V \cong k^{\oplus m}$  for some  $m$  and so

$$X \otimes_{k[y]} J_V \otimes_{k[y]} Y \cong k[x, z]^{\oplus m}$$

so the left-hand-side of the above strong deformation retract is a finite rank matrix factorisation. The right-hand-side contains  $(Z, d_Z) = (X \otimes_{k[y]} Y, d_{Y \otimes X})$  as a direct summand, proving the claim.  $\square$

## Previous Talks' Notes

- [Idempotents] *Idempotents in Categories*. URL: <https://rohanhitchcock.com/notes/idempotents.pdf>.
- [Perturbation] *The Perturbation Lemma for Linear Factorisations*. URL: <https://rohanhitchcock.com/notes/perturbation-lemma.pdf>.

## References

- [CM16] Nils Carqueville and Daniel Murfet. ‘Adjunctions and defects in Landau–Ginzburg models’. In: *Advances in Mathematics* 289 (Feb. 2016), pp. 480–566. ISSN: 00018708. DOI: [10.1016/j.aim.2015.03.033](https://doi.org/10.1016/j.aim.2015.03.033). URL: <https://linkinghub.elsevier.com/retrieve/pii/S0001870815004442> (visited on 11/09/2021).
- [Mur17] Daniel Murfet. ‘The cut operation on matrix factorisations’. In: *arXiv:1402.4541 [math]* (24 July 2017). arXiv: [1402.4541](https://arxiv.org/abs/1402.4541). URL: <http://arxiv.org/abs/1402.4541> (visited on 15/09/2021).
- [Stacks] The Stacks Project. URL: <https://stacks.math.columbia.edu/> (visited on 06/12/2021).
- [Wei94] Charles A. Weibel. *An introduction to homological algebra*. Cambridge studies in advanced mathematics 38. Cambridge [England] ; New York: Cambridge University Press, 1994. 450 pp. ISBN: 978-0-521-43500-0.