The cut operation

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Let k be a commutative ring. Consider the following morphisms in $\mathcal{LG}_k$

$$(k[x], U) \xrightarrow{(X, d_X)} (k[y], V) \xrightarrow{(Y, d_Y)} (k[z], W)$$

where $k[y] = k[y_1, \cdots, y_n]$ is a polynomial ring in n variables. Let $J_V = k[y]/(\partial_{y_1} V, \cdots, \partial_{y_n} V)$ be the Jacobi ring of the potential V.

Definition 1. The *cut* of the matrix factorisations (X, d_X) and (Y, d_Y) is the matrix factorisation $(Y|X, d_{Y|X})$ where

$$Y|X = X \otimes_{k[y]} J_V \otimes_{k[y]} Y \qquad \text{and} \qquad d_{Y|X} = d_X \otimes 1 + 1 \otimes d_Y$$

The cut operation on matrix factorisations was first defined in [Mur18]. In previous talks, the cut $(Y|X, d_{Y|X})$ arose in the proof of that that composition in $\mathcal{LG}_k$ is well-defined, specifically in when showing that $(X \otimes_{k[y]} Y, d_{X\otimes Y})$ is the direct summand of a finite rank matrix factorisation. The key idea of this proof was to show that there is a strong deformation retract

$$(Y|X, d_{Y|X}) \xleftrightarrow{\quad} (\bigwedge(k^{\oplus n}) \otimes_k X \otimes_{k[y]} Y, 1 \otimes d_{X\otimes Y}), \qquad h$$

over $k[x, z]$. The cut $(Y|X, d_{Y|X})$ is finite rank, and the module on the left-hand-side is the direct sum of copies of $X \otimes_{k[y]} Y$, some of which are shifted in degree.

The goal of the next part of this seminar is to better understand how $(X \otimes_{k[y]} Y, d_{X\otimes Y})$ sits inside the cut $(Y|X, d_{Y|X})$. This will involve producing explicit formulae for the maps in the strong deformation retract above. The key steps in producing this strong deformation retract were:

(1) We began with a strong deformation retract over k

$$(J_V, 0) \underset{\sigma}{\overset{\pi}{\leftrightarrows}} (K(\partial V), d_K), \qquad h$$

(2) We tensored both sides of this strong deformation retract by $X \otimes_{k[y]} Y$ and used the Perturbation Lemma to mix in the differential $d_{X\otimes Y}$.

(3) We showed that there is an isomorphism

$$\varphi : (\bigwedge(k^{\oplus n}) \otimes_k X \otimes_{k[y]} Y, d_K + d_{X\otimes Y}) \longrightarrow (\bigwedge(k^{\oplus n}) \otimes_k X \otimes_{k[y]} Y, d_{X\otimes Y})$$

1 The strong deformation retract

The key idea in constructing this strong deformation retract is that we can generalise the usual partial derivative maps $\frac{\partial}{\partial y_i}$ on a polynomial ring $k[y]$.

Definition 2. Let R be a k-algebra, $t = (t_1, \cdots, t_m)$ a sequence of elements of R and let $k[t] \subseteq R$ denote the k-algebra generated by $1, t_1, \cdots, t_n$. A *system of t-derivatives* are k-linear maps $\partial_{t_i} : R \to R$, $i = 1, \cdots, n$ which satisfy the following properties:

(1) Every $f \in R$ can be written uniquely in the form

$$f = \sum_{u \in \mathbb{N}^n} r_u t^u$$

where $r_u \in \bigcap_i \ker(\partial_{t_i})$.

(2) $\partial_{t_i}(t^v) = v_i t^{v-e_i}$ for all $v \in \mathbb{N}^n$ (where we understand that $0t_j^{-1} = 0$).

(3) For $f \in k[t]$ and $r \in \bigcap_i \ker(\partial_{t_i})$ we have $\partial_{t_i}(rf) = r\partial_{t_i}(f)$.

In a future talk we will show how to construct a system of t-derivatives when t is a quasi-regular sequence and R is a either a polynomial ring over a field, or a completion of a polynomial ring over an arbitrary commutative ring.

For now, we focus on the case that k is a field. Consider the potential $(k[y], V)$ defined above and let $t = (\partial_{y_1} V, \cdots, \partial_{y_n} V)$ be the sequence of partial derivatives of V. Suppose we have a system of t-derivatives $\partial_{t_1}, \cdots, \partial_{t_n} : k[y] \to k[y]$. When $V = \frac{1}{2}\sum_{i=1}^n y_i^2$ we have $t = (y_1, \cdots, y_n)$, and the usual partial derivative maps $\frac{\partial}{\partial y_i}$ form a system of t-derivatives. Since V is a potential, t is quasi-regular so later we will see how to construct the maps $\partial_{t_1}, \cdots, \partial_{t_n}$ for any potential.

Next, we denote the Koszul complex of t by $K(t) = \bigwedge(\oplus_{i=1}^n k[y]dt_i)$ where $dt_1, \cdots, dt_n$ are formal generators. This is in analogy with differential geometry: we think of the degree p part of $K(t)$ as being a module of "p-forms on affine n-space". The algebra $K(t)$ is called the algebra of *Kähler differentials*. Recall that the differential on the Koszul complex is $d_K = \sum_{i=1}^n t_i dt^* \lrcorner(-)$ where

$$dt_i^* \lrcorner(-) : dt_{i_1} \cdots dt_{i_p} \longmapsto \sum_{j=1}^p (-1)^{j+1} \delta_{i_j i} dt_{i_1} \cdots \widehat{dt_{i_j}} \cdots dt_{i_p}$$

is contraction ($\delta_{i_j i}$ is the Kronecker delta).

Definition 3. Given the system of t-derivatives we define the corresponding *connection* as the k-linear map $\nabla : K(t) \to K(t)$ given by

$$\nabla(f dt_{i_1} \cdots dt_{i_p}) = \sum_{i=1}^n \partial_{t_i}(f) dt_i dt_{i_1} \cdots dt_{i_p}$$

for $f \in k[y]$.

The connection ∇ is the same as the connection of of [Mur18, Section 3] and [DM13, Definition 2.8]. In [Mur18; DM13] a connection is first proved to exist and from this the maps $\partial_{t_1}, \cdots, \partial_{t_n}$ are extracted.

Lemma 4. $\nabla^2 = 0$.

Proof. First note that by properties (1), (2) and (3) we have $\partial_{t_i}\partial_{t_j} = \partial_{t_j} = \partial_{t_i}$ for all i, j. Now let $f dt_{i_1} \cdots dt_{i_p} \in K(t)$ where $f \in k[y]$. Then

$$\begin{aligned}
\nabla^2(f dt_{i_1} \cdots dt_{i_p}) &= \sum_{i=1}^{n} \nabla(\partial_{t_i}(f) dt_i dt_{i_1} \cdots dt_{i_p}) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \partial_{t_j}\partial_{t_i}(f) dt_j dt_i dt_{i_1} \cdots dt_{i_p} \\
&= \sum_{i<j} \partial_{t_j}\partial_{t_i}(f) dt_j dt_i dt_{i_1} \cdots dt_{i_p} + \sum_{j<i} \partial_{t_j}\partial_{t_i}(f) dt_j dt_i dt_{i_1} \cdots dt_{i_p} \\
&= \sum_{i<j} \partial_{t_j}\partial_{t_i}(f) dt_j dt_i dt_{i_1} \cdots dt_{i_p} - \sum_{i<i} \partial_{t_j}\partial_{t_i}(f) dt_j dt_i dt_{i_1} \cdots dt_{i_p} \\
&= 0
\end{aligned}$$

□

In the terminology of [DM13; Mur18], Lemma 4 shows that ∇ is *flat* [DM13, Definition 2.8].

Lemma 5. *If $\mathbb{Q} \subseteq k$ then $d_K\nabla(k[t]) = (t_1, \cdots, t_n)$.*

Proof. For $t^u = t_1^{u_1} \cdots t_n^{u_n}$ we have

$$\begin{aligned}
d_K\nabla(t^u) &= \sum_{j=1}^{n} d_K(u_j t^{u-e_j} dt_j) \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} u_j t_i t^{u-e_j} dt_i^{*} \neg (dt_j) \\
&= \sum_{j=1}^{n} u_j \cdot t^u
\end{aligned}$$

using property (2) of Definition 2. To avoid a more tedious argument in the case that some $u_j = 0$ we permit ourselves to write $0t_j^{-1} = 0$ and note that the above calculation is still correct. Hence we have $d_K\nabla(k[t]) \subseteq (t_1, \cdots, t_n)$. But the above calculation also shows that for any $t^u \in (t_1, \cdots, t_n)$ we have $t^u = d_K\nabla((\sum_{i=1}^{n} u_j)^{-1} t^u)$, where we know $\sum_{i=1}^{n} u_j$ is invertible since $\mathbb{Q} \subseteq k$ and at least one $u_j \neq 0$. Applying linearity proves the claim. □

In the terminology of [DM13; Mur18], Lemma 5 shows that when $\mathbb{Q} \subseteq k$ the connection ∇ is *standard* [DM13, Definition 8.6].

Lemma 6. *$[d_K, \nabla]$ is a morphism of complexes.*

Proof. First note that $[d_K, \nabla]$ has degree $\deg(d_K) + \deg(\nabla) = 0$. Then we have

$$[d_K, \nabla] d_K = d_K \nabla d_K + \nabla d_k^2 = d_K \nabla d_K + d_K^2 \nabla = d_K [d_K, \nabla]$$

where we use that $d_K^2 = 0$. □

Lemma 7. *If $\mathbb{Q} \subseteq k$ then $[d_K, \nabla] = d_K\nabla + \nabla d_K$ is invertible away from degree zero.*

Proof. Consider $t^u dt_{i_1} \cdots dt_{t_p} \in K(t)$ for $p > 0$, where $i_1 < i_2 < \cdots < i_p$ and $t^u = t_1^{u_1} \cdots t_n^{u_n}$. Let $\delta_j = 0$ if $j \in \{i_1, \cdots, i_p\}$ and $\delta_j = 1$ otherwise. Then we have

$$\begin{aligned} d_K \nabla(t^u dt_{i_1} \cdots dt_{t_p}) &= \sum_{j=1}^n d_K(u_j t^{u-e_j} dt_j dt_{i_1} \cdots dt_{i_p}) \\ &= \sum_{j=1}^n \delta_j \sum_{s=1}^n u_j t_s t^{u-e_j} dt_s^* \lrcorner (dt_j dt_{i_1} \cdots dt_{i_p}) \\ &= \sum_{j=1}^n \delta_j u_j \cdot t^u dt_{i_1} \cdots dt_{i_p} + \\ &\qquad + \sum_{(j,l)\in A} \delta_j (-1)^l u_j t^{u-e_j+e_{i_l}} dt_j dt_{i_1} \cdots \widehat{dt_{i_p}} \cdots dt_{i_p} \end{aligned}$$

where $A = \{(j,l) \mid j = 1, \cdots n,\ l = 1, \cdots, p \text{ and } j \neq i_l\}$. We also have

$$\begin{aligned} \nabla d_K(t^u dt_{i_1} \cdots dt_{i_p}) &= \sum_{l=1}^p (-1)^{l+1} \nabla(t_{i_l} t^u dt_{i_1} \cdots \widehat{dt_{i_l}} \cdots dt_{i_p}) \\ &= \sum_{l=1}^p \sum_{j=1}^n (-1)^{l+1} \partial_{t_j}(t_{i_l} t^u) dt_j dt_{i_1} \cdots \widehat{dt_{i_l}} \cdots dt_{i_p} \\ &= \sum_{l=1}^p (-1)^{l+1} (u_{i_l} + 1) t^u dt_{i_l} dt_{i_1} \cdots \widehat{dt_{i_l}} \cdots dt_{i_p} \\ &\qquad + \sum_{(j,l)\in A} \delta_j (-1)^{l+1} u_j t^{u-e_j+e_{i_l}} dt_j dt_{i_1} \cdots \widehat{dt_{i_l}} \cdots dt_{i_p} \\ &= \sum_{j=1}^n (1-\delta_j)(u_j+1) \cdot t_u dt_{i_1} \cdots dt_{i_p} \\ &\qquad - \sum_{(j,l)\in A} \delta_j (-1)^l u_j t^{u-e_j+e_{i_l}} dt_j dt_{i_1} \cdots \widehat{dt_{i_l}} \cdots dt_{i_p} \end{aligned}$$

So we have

$$(d_K \nabla + \nabla d_K)(t^u dt_{i_1} \cdots dt_{t_p}) = c \cdot t^u t_{i_1} \cdots dt_{t_p}$$

where $c = \sum_{j=1}^n (1 - \delta_j + u_j)$. Note that c is a non-zero since at least one $\delta_j = 0$. Then, using properties (1) and (3) of Definition 2 this shows that $[d_K, \nabla]$ is invertible. □

Now suppose that $\mathbb{Q} \subseteq k$ and that t is a Koszul-regular sequence. We define $H : K(t) \to K(t)$ by $H = [d_K, \nabla]^{-1}\nabla$ in all degrees where $K(t)$ is non-zero and by $H = 0$ elsewhere. Note that since ∇ is a degree $+1$ map $[d_K, \nabla]^{-1}$ exists by Lemma 7 in all degrees needed to define H.

Lemma 8. *The degree zero part of* $1 - d_K H$ *factors through* J_V.

Proof. Let $f \in (t)$. In Lemma 5 we showed that $d_K\nabla(k[t]) = (t)$ so let $g \in k[t]$ be such that $d_K \nabla(g) = f$. Then we have

$$\begin{aligned} (1 - d_K H)(f) &= (1 - d_K H)(d_K \nabla)(g) \\ &= d_K \nabla(g) - d_K [d_K, \nabla]^{-1} \nabla d_K \nabla(g) \\ &= d_K \nabla(g) - d_K [d_K, \nabla]^{-1}([d_K, \nabla] - d_K \nabla)\nabla(g) \\ &= 0 \end{aligned}$$

using Lemma 4. □

Let $\pi_0 : K_0(t) \to J_V$ be the quotient map. By Lemma 8 we obtain a map $\sigma_0 : J_V \to K_0(t)$ satisfying $1 - d_K H_0 = \pi_0\sigma_0$. Let $(J_V, 0)$ denote the chain complex which has $R/(t)$ in degree zero and is zero elsewhere. Since t is Koszul-regular this is the homology of $K(t)$. We extend both π_0 and σ_0 to chain maps $\pi : (K(t), d_K) \to (J_V, 0)$ and $\sigma : (J_V, 0) \to (K(t), d_K)$ respectively by setting $\pi = 0$ and $\sigma = 0$ away from degree zero.

Proposition 9. *The maps π, σ and H form a strong deformation retract*

$$(J_V, 0) \underset{\sigma}{\overset{\pi}{\leftrightarrows}} (K(t), d_K), \qquad H$$

over k.

Proof. We need to show

(1) $\pi\sigma = 1$

(2) $Hd_K + d_K H = 1 - \sigma\pi$

(3) $H^2 = 0$, $\pi H = 0$ and $H\sigma = 0$

For (1), first note that away from degree zero this is clear. In degree zero we can represent any element of J_V by $\pi(f)$ where $f \in k[y]$ is such that $\partial_{t_i}(f) = 0$ for all $i = 1, \cdots, n$. Then we have

$$\pi\sigma\pi(f) = \pi(f - d_K H(f)) = \pi(f) - [d_K, \nabla]^{-1}\nabla(f) = \pi(f)$$

since $\nabla(f) = 0$. Hence we have $\pi\sigma = 1$ since π is an epimorphism.

For (2), in degree zero we have $1 - \sigma\pi_0 = d_K H_0$ by our construction of σ. For degree $p > 0$ we have

$$\begin{aligned} Hd_K + d_K H &= [d_K, \nabla]^{-1}\nabla d_K^p + d_K[d_K, \nabla]^{-1}\nabla_p \\ &= [d_K, \nabla]^{-1}\nabla d_K^p + [d_K, \nabla]^{-1} d_K \nabla_p \\ &= [d_K, \nabla]^{-1}[d_K, \nabla] \\ &= 1 \end{aligned}$$

Note we only had $d_K[d_K, \nabla]^{-1}\nabla_p = [d_K, \nabla]^{-1} d_K \nabla_p$ because $p > 0$. If $p = 0$ then the right-hand-side becomes $[d_K, \nabla]^{-1} d_K^1 \nabla_0$, and $[d_K, \nabla]^{-1}$ does not exist in degree zero.

For (3), first note that $H^2 = [d_K, \nabla]^{-1}\nabla[d_K, \nabla]^{-1}\nabla$ so for $H^2 = 0$ it suffices to show $\nabla[d_K, \nabla]^{-1}\nabla = 0$. We showed in Lemma 4 that $\nabla^2 = 0$, so we have

$$\nabla[d_K, \nabla] = \nabla d_K \nabla + \nabla^2 d_K = \nabla d_K \nabla + d_K \nabla^2 = [d_K, \nabla]\nabla$$

Then $\nabla = [d_K, \nabla]\nabla[d_K, \nabla]^{-1}$. Multiplying on the right by ∇ gives $0 = [d_K, \nabla]\nabla[d_K, \nabla]^{-1}\nabla$ which implies that $\nabla[d_K, \nabla]^{-1}\nabla = 0$ since $[d_K, \nabla]$ is invertible in this degree. Hence $H^2 = 0$. Next, we have $\pi_p H_{p-1} = 0$ in all degrees p since $\pi_p = 0$ when $p \neq 0$, and when $p = 0$ we have $H_{-1} = 0$. Finally, we show $H\sigma = 0$. Away from degree zero this is clear, and in degree zero we have

$$\begin{aligned} H\sigma\pi &= H - Hd_K H \\ &= [d_K, \nabla]^{-1}\nabla - [d_K, \nabla]^{-1}\nabla d_K [d_K, \nabla]^{-1}\nabla \\ &= [d_K, \nabla]^{-1}\nabla - [d_K, \nabla]^{-1}([d_K, \nabla] - d_K\nabla)[d_K, \nabla]^{-1}\nabla \\ &= [d_K, \nabla]^{-1} d_K \nabla [d_K, \nabla]^{-1}\nabla \\ &= 0 \end{aligned}$$

where we again use that $\nabla[d_K, \nabla]^{-1}\nabla = 0$. ∎

References

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[Mur18] Daniel Murfet. 'The cut operation on matrix factorisations'. In: *Journal of Pure and Applied Algebra* 222.7 (2018), pp. 1911–1955.