

The cut operation

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Let k be a commutative ring. Consider the following morphisms in \mathcal{LG}_k

$$(k[x], U) \xrightarrow{(X, d_X)} (k[y], V) \xrightarrow{(Y, d_Y)} (k[z], W)$$

where $k[y] = k[y_1, \dots, y_n]$ is a polynomial ring in n variables. Let $J_V = k[y]/(\partial_{y_1} V, \dots, \partial_{y_n} V)$ be the Jacobi ring of the potential V .

Definition 1. The *cut* of the matrix factorisations (X, d_X) and (Y, d_Y) is the matrix factorisation $(Y|X, d_{Y|X})$ where

$$Y|X = X \otimes_{k[y]} J_V \otimes_{k[y]} Y \quad \text{and} \quad d_{Y|X} = d_X \otimes 1 + 1 \otimes d_Y$$

The cut operation on matrix factorisations was first defined in [Mur18]. In previous talks, the cut $(Y|X, d_{Y|X})$ arose in the proof of that that composition in \mathcal{LG}_k is well-defined, specifically in when showing that $(X \otimes_{k[y]} Y, d_{X \otimes Y})$ is the direct summand of a finite rank matrix factorisation. The key idea of this proof was to show that there is a strong deformation retract

$$(Y|X, d_{Y|X}) \xleftarrow{\quad} (\bigwedge(k^{\oplus n}) \otimes_k X \otimes_{k[y]} Y, 1 \otimes d_{X \otimes Y}), \quad h$$

over $k[x, z]$. The cut $(Y|X, d_{Y|X})$ is finite rank, and the module on the left-hand-side is the direct sum of copies of $X \otimes_{k[y]} Y$, some of which are shifted in degree.

The goal of the next part of this seminar is to better understand how $(X \otimes_{k[y]} Y, d_{X \otimes Y})$ sits inside the cut $(Y|X, d_{Y|X})$. This will involve producing explicit formulae for the maps in the strong deformation retract above. The key steps in producing this strong deformation retract were:

- (1) We began with a strong deformation retract over k

$$(J_V, 0) \xleftarrow[\sigma]{\pi} (K(\partial V), d_K), \quad h$$

- (2) We tensored both sides of this strong deformation retract by $X \otimes_{k[y]} Y$ and used the Perturbation Lemma to mix in the differential $d_{X \otimes Y}$.
- (3) We showed that there is an isomorphism

$$\varphi : (\bigwedge(k^{\oplus n}) \otimes_k X \otimes_{k[y]} Y, d_K + d_{X \otimes Y}) \longrightarrow (\bigwedge(k^{\oplus n}) \otimes_k X \otimes_{k[y]} Y, d_{X \otimes Y})$$

1 The strong deformation retract

The key idea in constructing this strong deformation retract is that we can generalise the usual partial derivative maps $\frac{\partial}{\partial y_i}$ on a polynomial ring $k[y]$.

Definition 2. Let R be a k -algebra, $t = (t_1, \dots, t_n)$ a sequence of elements of R and let $k[t] \subseteq R$ denote the k -algebra generated by $1, t_1, \dots, t_n$. A *system of t -derivatives* are k -linear maps $\partial_{t_i} : R \rightarrow R$, $i = 1, \dots, n$ which satisfy the following properties:

- (1) Every $f \in R$ can be written uniquely in the form

$$f = \sum_{u \in \mathbb{N}^n} r_u t^u$$

where $r_u \in \bigcap_i \ker(\partial_{t_i})$.

- (2) $\partial_{t_i}(t^v) = v_i t^{v-e_i}$ for all $v \in \mathbb{N}^n$ (where we understand that $0t_j^{-1} = 0$).

- (3) For $f \in k[t]$ and $r \in \bigcap_i \ker(\partial_{t_i})$ we have $\partial_{t_i}(rf) = r\partial_{t_i}(f)$.

- (4) The quotient map $R \rightarrow R/(t)$ restricts to an isomorphism $\bigcap_i \ker(\partial_{t_i}) \xrightarrow{\sim} R/(t)$.

In a future talk we will show how to construct a system of t -derivatives when t is a quasi-regular sequence and R is either a polynomial ring over a field, or a completion of a polynomial ring over an arbitrary commutative ring.

For now, we focus on the case that k is a field. Consider the potential $(k[y], V)$ defined above and let $t = (\partial_{y_1} V, \dots, \partial_{y_n} V)$ be the sequence of partial derivatives of V . Suppose we have a system of t -derivatives $\partial_{t_1}, \dots, \partial_{t_n} : k[y] \rightarrow k[y]$. When $V = \frac{1}{2} \sum_{i=1}^n y_i^2$ we have $t = (y_1, \dots, y_n)$, and the usual partial derivative maps $\frac{\partial}{\partial y_i}$ form a system of t -derivatives. Since V is a potential, t is quasi-regular so later we will see how to construct the maps $\partial_{t_1}, \dots, \partial_{t_n}$ for any potential.

Next, we denote the Koszul complex of t by $K(t) = \bigwedge(\oplus_{i=1}^n k[y]dt_i)$ where dt_1, \dots, dt_n are formal generators. This is in analogy with differential geometry: we think of the degree p part of $K(t)$ as being a module of “ p -forms on affine n -space”. The algebra $K(t)$ is called the algebra of *Kähler differentials*. Recall that the differential on the Koszul complex is $d_K = \sum_{i=1}^n t_i dt^* \lrcorner (-)$ where

$$dt_i^* \lrcorner (-) : dt_{i_1} \cdots dt_{i_p} \mapsto \sum_{j=1}^p (-1)^{j+1} \delta_{i_j i} dt_{i_1} \cdots \widehat{dt_{i_j}} \cdots dt_{i_p}$$

is contraction ($\delta_{i_j i}$ is the Kronecker delta).

Definition 3. Given the system of t -derivatives we define the corresponding *connection* as the k -linear map $\nabla : K(t) \rightarrow K(t)$ given by

$$\nabla(f dt_{i_1} \cdots dt_{i_p}) = \sum_{i=1}^n \partial_{t_i}(f) dt_i dt_{i_1} \cdots dt_{i_p}$$

for $f \in k[y]$.

The connection ∇ is the same as the connection of [Mur18, Section 3] and [DM13, Definition 2.8]. In [Mur18; DM13] a connection is first proved to exist and from this the maps $\partial_{t_1}, \dots, \partial_{t_n}$ are extracted.

Lemma 4. $\nabla^2 = 0$.

Proof. First note that by properties (1), (2) and (3) we have $\partial_{t_i}\partial_{t_j} = \partial_{t_j}\partial_{t_i}$ for all i, j . Now let $f dt_{i_1} \cdots dt_{i_p} \in K(t)$ where $f \in k[y]$. Then

$$\begin{aligned} \nabla^2(f dt_{i_1} \cdots dt_{i_p}) &= \sum_{i=1}^n \nabla(\partial_{t_i}(f) dt_i dt_{i_1} \cdots dt_{i_p}) \\ &= \sum_{i=1}^n \sum_{j=1}^n \partial_{t_j} \partial_{t_i}(f) dt_j dt_i dt_{i_1} \cdots dt_{i_p} \\ &= \sum_{i < j} \partial_{t_j} \partial_{t_i}(f) dt_j dt_i dt_{i_1} \cdots dt_{i_p} + \sum_{j < i} \partial_{t_j} \partial_{t_i}(f) dt_j dt_i dt_{i_1} \cdots dt_{i_p} \\ &= \sum_{i < j} \partial_{t_j} \partial_{t_i}(f) dt_j dt_i dt_{i_1} \cdots dt_{i_p} - \sum_{i < i} \partial_{t_j} \partial_{t_i}(f) dt_j dt_i dt_{i_1} \cdots dt_{i_p} \\ &= 0 \end{aligned}$$

□

In the terminology of [DM13; Mur18], Lemma 4 shows that ∇ is *flat* [DM13, Definition 2.8].

Lemma 5. If $\mathbb{Q} \subseteq k$ then $d_K \nabla(k[t]) = (t_1, \dots, t_n)$.

Proof. For $t^u = t_1^{u_1} \cdots t_n^{u_n}$ we have

$$\begin{aligned} d_K \nabla(t^u) &= \sum_{j=1}^n d_K(u_j t^{u-e_j} dt_j) \\ &= \sum_{j=1}^n \sum_{i=1}^n u_j t_i t^{u-e_j} dt_i^* \lrcorner(dt_j) \\ &= \sum_{j=1}^n u_j \cdot t^u \end{aligned}$$

using property (2) of Definition 2. To avoid a more tedious argument in the case that some $u_j = 0$ we permit ourselves to write $0t_j^{-1} = 0$ and note that the above calculation is still correct. Hence we have $d_K \nabla(k[t]) \subseteq (t_1, \dots, t_n)$. But the above calculation also shows that for any $t^u \in (t_1, \dots, t_n)$ we have $t^u = d_K \nabla((\sum_{i=1}^n u_j)^{-1} t^u)$, where we know $\sum_{i=1}^n u_j$ is invertible since $\mathbb{Q} \subseteq k$ and at least one $u_j \neq 0$. Applying linearity proves the claim. □

In the terminology of [DM13; Mur18], Lemma 5 shows that when $\mathbb{Q} \subseteq k$ the connection ∇ is *standard* [DM13, Definition 8.6].

Lemma 6. $[d_K, \nabla]$ is a morphism of complexes.

Proof. First note that $[d_K, \nabla]$ has degree $\deg(d_K) + \deg(\nabla) = 0$. Then we have

$$[d_K, \nabla]d_K = d_K \nabla d_K + \nabla d_K^2 = d_K \nabla d_K + d_K^2 \nabla = d_K [d_K, \nabla]$$

where we use that $d_K^2 = 0$. □

Lemma 7. If $\mathbb{Q} \subseteq k$ then $[d_K, \nabla] = d_K \nabla + \nabla d_K$ is invertible away from degree zero.

Proof. Consider $t^u dt_{i_1} \cdots dt_{t_p} \in K(t)$ for $p > 0$, where $i_1 < i_2 < \cdots < i_p$ and $t^u = t_1^{u_1} \cdots t_n^{u_n}$. Let $\delta_j = 0$ if $j \in \{i_1, \dots, i_p\}$ and $\delta_j = 1$ otherwise. Then we have

$$\begin{aligned} d_K \nabla(t^u dt_{i_1} \cdots dt_{t_p}) &= \sum_{j=1}^n d_K(u_j t^{u-e_j} dt_j dt_{i_1} \cdots dt_{i_p}) \\ &= \sum_{j=1}^n \delta_j \sum_{s=1}^n u_j t_s^{u-e_j} dt_s^* \lrcorner (dt_j dt_{i_1} \cdots dt_{i_p}) \\ &= \sum_{j=1}^n \delta_j u_j \cdot t^u dt_{i_1} \cdots dt_{i_p} + \\ &\quad + \sum_{(j,l) \in A} \delta_j (-1)^l u_j t^{u-e_j+e_{i_l}} dt_j dt_{i_1} \cdots \widehat{dt_{i_l}} \cdots dt_{i_p} \end{aligned}$$

where $A = \{(j, l) \mid j = 1, \dots, n, l = 1, \dots, p \text{ and } j \neq i_l\}$. We also have

$$\begin{aligned} \nabla d_K(t^u dt_{i_1} \cdots dt_{i_p}) &= \sum_{l=1}^p (-1)^{l+1} \nabla(t_{i_l} t^u dt_{i_1} \cdots \widehat{dt_{i_l}} \cdots dt_{i_p}) \\ &= \sum_{l=1}^p \sum_{j=1}^n (-1)^{l+1} \partial_{t_j}(t_{i_l} t^u) dt_j dt_{i_1} \cdots \widehat{dt_{i_l}} \cdots dt_{i_p} \\ &= \sum_{l=1}^p (-1)^{l+1} (u_{i_l} + 1) t^u dt_{i_l} dt_{i_1} \cdots \widehat{dt_{i_l}} \cdots dt_{i_p} \\ &\quad + \sum_{(j,l) \in A} \delta_j (-1)^{l+1} u_j t^{u-e_j+e_{i_l}} dt_j dt_{i_1} \cdots \widehat{dt_{i_l}} \cdots dt_{i_p} \\ &= \sum_{j=1}^n (1 - \delta_j)(u_j + 1) \cdot t^u dt_{i_1} \cdots dt_{i_p} \\ &\quad - \sum_{(j,l) \in A} \delta_j (-1)^l u_j t^{u-e_j+e_{i_l}} dt_j dt_{i_1} \cdots \widehat{dt_{i_l}} \cdots dt_{i_p} \end{aligned}$$

So we have

$$(d_K \nabla + \nabla d_K)(t^u dt_{i_1} \cdots dt_{t_p}) = c \cdot t^u dt_{i_1} \cdots dt_{t_p}$$

where $c = \sum_{j=1}^n (1 - \delta_j + u_j)$. Note that c is a non-zero since at least one $\delta_j = 0$. Then, using properties (1) and (3) of Definition 2 this shows that $[d_K, \nabla]$ is invertible. \square

Now suppose that $\mathbb{Q} \subseteq k$ and that t is a Koszul-regular sequence. We define $H : K(t) \rightarrow K(t)$ by $H = [d_K, \nabla]^{-1} \nabla$ in all degrees where $K(t)$ is non-zero and by $H = 0$ elsewhere. Note that since ∇ is a degree +1 map $[d_K, \nabla]^{-1}$ exists by Lemma 7 in all degrees needed to define H .

Lemma 8. *The degree zero part of $1 - d_K H$ factors through J_V .*

Proof. Let $f \in (t)$. In Lemma 5 we showed that $d_K \nabla(k[t]) = (t)$ so let $g \in k[t]$ be such that $d_K \nabla(g) = f$. Then we have

$$\begin{aligned} (1 - d_K H)(f) &= (1 - d_K H)(d_K \nabla)(g) \\ &= d_K \nabla(g) - d_K [d_K, \nabla]^{-1} \nabla d_K \nabla(g) \\ &= d_K \nabla(g) - d_K [d_K, \nabla]^{-1} ([d_K, \nabla] - d_K \nabla) \nabla(g) \\ &= 0 \end{aligned}$$

using Lemma 4. \square

Let $\pi_0 : K_0(t) \rightarrow J_V$ be the quotient map. By Lemma 8 we obtain a map $\sigma_0 : J_V \rightarrow K_0(t)$ satisfying $1 - d_K H_0 = \pi_0 \sigma_0$. Let $(J_V, 0)$ denote the chain complex which has $R/(t)$ in degree zero and is zero elsewhere. Since t is Koszul-regular this is the homology of $K(t)$. We extend both π_0 and σ_0 to chain maps $\pi : (K(t), d_K) \rightarrow (J_V, 0)$ and $\sigma : (J_V, 0) \rightarrow (K(t), d_K)$ respectively by setting $\pi = 0$ and $\sigma = 0$ away from degree zero.

Proposition 9. *The maps π , σ and H form a strong deformation retract*

$$(J_V, 0) \xrightleftharpoons[\sigma]{\pi} (K(t), d_K), \quad H$$

over k .

Proof. We need to show

- (1) $\pi\sigma = 1$
- (2) $Hd_K + d_K H = 1 - \sigma\pi$
- (3) $H^2 = 0$, $\pi H = 0$ and $H\sigma = 0$

For (1), first note that away from degree zero this is clear. In degree zero we can represent any element of J_V by $\pi(f)$ where $f \in k[y]$ is such that $\partial_{t_i}(f) = 0$ for all $i = 1, \dots, n$. Then we have

$$\pi\sigma\pi(f) = \pi(f - d_K H(f)) = \pi(f) - [d_K, \nabla]^{-1} \nabla(f) = \pi(f)$$

since $\nabla(f) = 0$. Hence we have $\pi\sigma = 1$ since π is an epimorphism.

For (2), in degree zero we have $1 - \sigma\pi_0 = d_K H_0$ by our construction of σ . For degree $p > 0$ we have

$$\begin{aligned} Hd_K + d_K H &= [d_K, \nabla]^{-1} \nabla d_K^p + d_K [d_K, \nabla]^{-1} \nabla_p \\ &= [d_K, \nabla]^{-1} \nabla d_K^p + [d_K, \nabla]^{-1} d_K \nabla_p \\ &= [d_K, \nabla]^{-1} [d_K, \nabla] \\ &= 1 \end{aligned}$$

Note we only had $d_K [d_K, \nabla]^{-1} \nabla_p = [d_K, \nabla]^{-1} d_K \nabla_p$ because $p > 0$. If $p = 0$ then the right-hand-side becomes $[d_K, \nabla]^{-1} d_K^1 \nabla_0$, and $[d_K, \nabla]^{-1}$ does not exist in degree zero.

For (3), first note that $H^2 = [d_K, \nabla]^{-1} \nabla [d_K, \nabla]^{-1} \nabla$ so for $H^2 = 0$ it suffices to show $\nabla [d_K, \nabla]^{-1} \nabla = 0$. We showed in Lemma 4 that $\nabla^2 = 0$, so we have

$$\nabla [d_K, \nabla] = \nabla d_K \nabla + \nabla^2 d_K = \nabla d_K \nabla + d_K \nabla^2 = [d_K, \nabla] \nabla$$

Then $\nabla = [d_K, \nabla] \nabla [d_K, \nabla]^{-1}$. Multiplying on the right by ∇ gives $0 = [d_K, \nabla] \nabla [d_K, \nabla]^{-1} \nabla$ which implies that $\nabla [d_K, \nabla]^{-1} \nabla = 0$ since $[d_K, \nabla]$ is invertible in this degree. Hence $H^2 = 0$. Next, we have $\pi_p H_{p-1} = 0$ in all degrees p since $\pi_p = 0$ when $p \neq 0$, and when $p = 0$ we have $H_{-1} = 0$. Finally, we show $H\sigma = 0$. Away from degree zero this is clear, and in degree zero we have

$$\begin{aligned} H\sigma\pi &= H - Hd_K H \\ &= [d_K, \nabla]^{-1} \nabla - [d_K, \nabla]^{-1} \nabla d_K [d_K, \nabla]^{-1} \nabla \\ &= [d_K, \nabla]^{-1} \nabla - [d_K, \nabla]^{-1} ([d_K, \nabla] - d_K \nabla) [d_K, \nabla]^{-1} \nabla \\ &= [d_K, \nabla]^{-1} d_K \nabla [d_K, \nabla]^{-1} \nabla \\ &= 0 \end{aligned}$$

where we again use that $\nabla [d_K, \nabla]^{-1} \nabla = 0$. □

References

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- [Mur18] Daniel Murfet. ‘The cut operation on matrix factorisations’. In: *Journal of Pure and Applied Algebra* 222.7 (2018), pp. 1911–1955.