Composition in $\mathcal{LG}$

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Throughout fix a commutative ring k. Recall that $\mathcal{LG}$ has the following data:

(1) Objects are pairs $(k[x], U)$ where $k[x] = k[x_1, \cdots, x_n]$ and $U \in k[x]$ is a *potential*. That is, U satisfies

- The sequence $\partial U = (\partial_{x_1} U, \cdots, \partial_{x_n} U)$ has a Koszul complex which is exact except in degree zero.
- The Jacobi ring $J_U = k[x]/(\partial_{x_1} U, \cdots, \partial_{x_n} U)$ is a free, finitely generated k-module.

(2) The category of 1-morphisms $(k[x], U) \to (k[y], V)$ is

$$\mathrm{hmf}(k[x, y], V(y) - U(x))^{\omega}$$

where $\mathcal{C}^{\omega}$ denotes the idempotent completion of a category $\mathcal{C}$. Equivalently the 1-morphisms are matrix factorisations of $V(y) - U(x)$ which are homotopy equivalent to a direct summand of a finite rank matrix factorisation (see [Idempotents]).

The definition of potential above differs from the definition given in [CM16, Definition 2.4]. The following result shows they are equivalent.

Lemma 1 ([Stacks, Section 15.30])**.** *For a sequence $t = (t_1, \cdots, t_n)$ of elements of a commutative ring R we have that:*

(1) If t is regular then the Koszul complex $K(t)$ is exact except in degree zero.

(2) If the Koszul complex $K(t)$ is exact except in degree zero then t is quasi-regular.

Throughout consider the following 1-morphisms in $\mathcal{LG}$:

$$(k[x], U) \xrightarrow{(X, d_X)} (k[y], V) \xrightarrow{(Y, d_Y)} (k[z], W)$$

so (X, d_X) is a finite rank matrix factorisation of $V(y) - U(x)$ over $k[x, y]$ and (Y, d_Y) is a finite rank matrix factorisation of $W(z) - V(y)$ over $k[y, z]$ and U, V and W are potentials. We define the composition of (X, d_X) and (Y, d_Y) to be the tensor product $(X \otimes_{k[y]} Y, d_X \otimes 1 + 1 \otimes d_Y)$. It is not immediate that this is well-defined. Indeed, if $X = k[x, y]^{\oplus m}$ and $Y = k[y, z]^{\oplus m'}$ then

$$X \otimes_{k[y]} Y = k[x, y]^{\oplus m} \otimes_{k[y]} k[y, z]^{\oplus m'} = k[x, y, z]^{\oplus mm'}$$

which is free, but not finite rank over $k[x, z]$. Hence our goal is to show

$$(X \otimes_{k[y]} Y, d_{X \otimes Y}) \in \mathrm{hmf}(k[x, z], W(z) - U(x))^{\omega}$$

which we do by showing $(X \otimes_{k[y]} Y, d_{X \otimes Y})$ is a direct summand of a finite rank matrix factorisation over $k[x, z]$.

1 Koszul complexes

Before continuing we need to discuss Koszul complexes and exterior algebras. Our main reference for this is [Wei94, Chapter 4.5]. Let R be a commutative ring and consider a free R-module $V = R^{\oplus n}$ with free generators $e_1, \cdots, e_n$. We define an associative bilinear formal product $(-) \wedge (-)$ on elements of V subject to the relation generated by setting

$$u \wedge v = -(v \wedge u) \qquad \text{for all } u, v \in V$$

In particular notice that $v \wedge v = 0$ for all $v \in V$. The *p-th exterior power* of V, denoted $\bigwedge^p(V)$ is defined as the set of p-fold products. One can show that this is a free R-module with generators

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p} \mid i_1 < i_2 < \cdots < i_p\}$$

and so $\dim \bigwedge^p(V) = \binom{n}{p}$. For example if $n = 3$ we have

- $\bigwedge^0(V)$ is generated by 1 (i.e. $\bigwedge^0(V) = R$).
- $\bigwedge^1(V)$ is generated by $\{e_1, e_2, e_3\}$ (i.e. $\bigwedge^1(V) = V$)
- $\bigwedge^2(V)$ is generated by $e_1 \wedge e_2$, $e_1 \wedge e_3$ and $e_2 \wedge e_3$
- $\bigwedge^3(V)$ is generated by $e_1 \wedge e_2 \wedge e_3$
- $\bigwedge^p(V)$ for $p \geq 4$ is zero.

The *exterior algebra* of V, denoted $\bigwedge(V)$, is the graded R-algebra with p-th graded component $\bigwedge^p(V)$.

Next, given a sequence of elements $t = (t_1, \cdots, t_n)$ in R we define the *Koszul complex* of t as the pair $(K(t), d_K)$ where $K(t) = \bigwedge(V)$ and $d_K : K(t) \to K(t)$ is given by

$$e_{i_1} \wedge \cdots \wedge e_{i_p} \longmapsto \sum_{j=1}^{p} (-1)^{j+1} t_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_p}$$

where "$\widehat{e_{i_j}}$" indicates that e_{i_j} is omitted from the wedge product. One can show that $d_K^2 = 0$, so $(K(t), d_K)$ is a chain complex. For example, if $t = (t_1)$ (a sequence with one element) then the Koszul complex is

$$0 \longrightarrow \underset{1}{R} \xrightarrow{t_1} \underset{0}{R} \longrightarrow 0$$

In the case of $n = 3$ we have

$$0 \longrightarrow \underset{\{e_1\wedge e_2\wedge e_3\}}{\bigwedge^3(V)} \xrightarrow{\begin{pmatrix} t_3 \\ -t_2 \\ t_1 \end{pmatrix}} \underset{\{e_1\wedge e_2, e_1\wedge e_3, e_2\wedge e_3\}}{\bigwedge^2(V)} \xrightarrow{\begin{pmatrix} -t_2 & -t_3 & 0 \\ t_1 & 0 & -t_3 \\ 0 & t_1 & t_2 \end{pmatrix}} \underset{\{e_1, e_2, e_3\}}{\bigwedge^1(V)} \xrightarrow{(t_1\ t_2\ t_3)} \underset{\{1\}}{\bigwedge^0(V)} \longrightarrow 0$$

Notice that the degree zero homology of $(K(t), d_K)$ is the quotient ring $R/(t)$. It is also useful to note that

$$K(t) = K(t_1) \otimes_R K(t_2) \otimes_R \cdots \otimes_R K(t_n)$$

where the right-hand-side is a tensor product of complexes.

2 Definitions and results from previous talks

Let S be a commutative ring and R a S-algebra. Let (L, d_L) and (M, d_M) be matrix factorisations of $f \in \varphi(S)$

Definition 2 ([Perturbation, Definition 1.1])**.** A *strong deformation retract* of (L, d_L) and (M, d_M) over S consists of S-linear maps

$$(L, d_L) \underset{i}{\overset{p}{\leftrightarrows}} (M, d_M), \qquad h$$

where $pi = 1$, $h : pi \simeq 1$, $hi = 0$, $ph = 0$ and $h^2 = 0$.

A strong deformation retract is a homotopy equivalence of matrix factorisations with additional conditions on the maps involved, the point being that strong deformation retracts can be modified using the Perturbation Lemma [Perturbation, Theorem 3.1]. In a previous talk we proved the following corollary of the Perturbation Lemma.

Lemma 3 ([Perturbation, Lemma 2.1])**.** *Suppose we have a strong deformation retract over S*

$$(L, d_L) \underset{\sigma}{\overset{\pi}{\leftrightarrows}} (M, d_M), \quad h$$

Then for any linear factorisation (Z, d_Z) of $g \in R$ where $f + g \in \varphi(S)$ there exists a deformation retract over S

$$(L \otimes_R Z, d_L \otimes 1 + 1 \otimes d_Z) \leftrightarrows (M \otimes_R Z, d_M \otimes 1 + 1 \otimes d_Z), \quad h'$$

The following result gives us a source of strong deformation retracts.

Lemma 4 ([Perturbation, Lemma 2.2])**.** *Let (P, d) be a bounded-to-the-right chain complex of projective objects in an abelian category. Suppose that (P, d) is exact except at degree zero and that $H_0(P)$ is also projective. Then we have a strong deformation retract*

$$(H(P), 0) \leftrightarrows (P, d), \quad h$$

of chain complexes, where $(H(P), 0)$ is the homology of P with zero differentials.

Lemma 5. *Let $U \in k[x]$ be a potential. Consider the sequence of partial derivatives $\partial U = (\partial_{x_1} U, \cdots \partial_{x_n} U)$ and the Jacobi ring $J_U = k[x]/(\partial U)$. Then we have a strong deformation retract over k*

$$(J_U, 0) \leftrightarrows (K(\partial U), d_K), \quad h$$

Proof. By the potential hypothesis, $(K(\partial U), d_K)$ is exact in degree zero and J_U is free (hence projective) over k. The result is an application of the previous lemma. □

3 Composition in $\mathcal{LG}_k$ is well-defined

The main goal of this talk is to prove that composition in $\mathcal{LG}$ is well-defined, which we do following the strategy of [Mur18]. Consider the following 1-morphisms in $\mathcal{LG}$:

$$(k[x], U) \xrightarrow{(X, d_X)} (k[y], V) \xrightarrow{(Y, d_Y)} (k[z], W)$$

so (X, d_X) is a finite rank matrix factorisation of $V(y) - U(x)$ over $k[x, y]$ and (Y, d_Y) is a finite rank matrix factorisation of $W(z) - V(y)$ over $k[y, z]$ and U, V and W are potentials. Our goal is to show

$$(X \otimes_{k[y]} Y, d_{X \otimes Y}) \in \mathrm{hmf}(k[x, z], W(z) - U(x))^{\omega}$$

where $\mathcal{C}^{\omega}$ denotes the idempotent completion of a category $\mathcal{C}$, or equivalently (see [Idempotents]) to prove the following:

Proposition 6. *The composition of (X, d_X) and (Y, d_Y), which is $(X \otimes_{k[y]} Y, d_{X \otimes Y})$, is a direct summand of a matrix factorisation which is finite rank over $k[x, z]$.*

We do this as follows:

- We begin with a strong deformation retract over k arising from a Koszul complex of $\partial V = (\partial_{y_1} V, \cdots, \partial_{y_n} V)$ as in Lemma 5.

$$(J_V, 0) \underset{\iota}{\overset{\pi}{\leftrightarrows}} (K(\partial V), d_K), \qquad h$$

- We tensor both sides of this strong deformation retract by $(X \otimes_{k[y]} Y, d_{X \otimes Y})$ to obtain

$$(X \otimes_{k[y]} J_V \otimes_{k[y]} Y, d_Z) \underset{\tilde{\iota}}{\overset{\tilde{\pi}}{\leftrightarrows}} (\textstyle\bigwedge(k^{\oplus n}) \otimes_k X \otimes_{k[y]} Y, d_K + d_{X \otimes Y}), \qquad \tilde{h}$$

  where $Z = X \otimes_{k[y]} Y$.

Notice that on the left-hand-side we have $J_V \cong k^{\oplus m}$ for some m and so

$$X \otimes_{k[y]} J_V \otimes_{k[y]} Y \cong k[x, z]^{\oplus m}$$

Hence the left-hand-side is a finite rank matrix factorisation. Also, $\bigwedge(k^{\oplus n}) \otimes_k X \otimes_{k[y]} Y$ has $X \otimes_{k[y]} Y$ as a direct summand. The only thing which remains is to remove the differential d_K, which we do over the following several lemmas.

Lemma 7. *Let $f, g : (C, d_C) \to (D, d_D)$ be morphisms of either complexes or linear factorisations, and suppose they are homotopic via $h : f \simeq g$. Then $\mathrm{cone}(f) \cong \mathrm{cone}(g)$.*

Proof. (give as exercise) For clarity we define $\mathrm{cone}(f)$ as

$$\cdots \longrightarrow C_{n+2} \oplus D_{n+1} \xrightarrow{d_f} C_{n+1} \oplus D_n \xrightarrow{d_f} C_n \oplus D_{n-1} \longrightarrow \cdots$$

where $d_f = \left(\begin{smallmatrix} d_C & 0 \\ f & -d_D \end{smallmatrix}\right)$, and if (C, d_C) is a linear factorisation then addition in the indices is modulo 2. Likewise $\mathrm{cone}(g)$ is the graded object $C[1] \oplus D$ with differential $d_g = \left(\begin{smallmatrix} d_C & 0 \\ g & -d_D \end{smallmatrix}\right)$. Define $\varphi_n : C[1] \oplus D \to C[1] \oplus D$ as $\varphi = \left(\begin{smallmatrix} 1 & 0 \\ -h & 1 \end{smallmatrix}\right)$. We have

$$\varphi d_f = \begin{pmatrix} d_C & 0 \\ -h d_C + f & -d_D \end{pmatrix}$$

and

$$d_g \varphi = \begin{pmatrix} d_C & 0 \\ g + d_D h & -d_D \end{pmatrix}$$

These agree since $f - g = h d_C + d_D h$ and so $\varphi : \mathrm{cone}(f) \to \mathrm{cone}(g)$ is a morphism. Likewise we define $\psi : \mathrm{cone}(g) \to \mathrm{cone}(f)$ as $\psi = \left(\begin{smallmatrix} 1 & 0 \\ h & 1 \end{smallmatrix}\right)$, and we have $\varphi\psi = \mathrm{id}$ and $\psi\varphi = \mathrm{id}$. $\square$

Lemma 8. *Let $t = (t_1, \cdots, t_n)$ be a sequence in R in which each t_i acts null-homotopically on a linear factorisation (X, d_X) over R. Then $(K(t) \otimes_R X, d_K \otimes 1 + 1 \otimes d_X)$ is isomorphic to $(K(t) \otimes_R X, 1 \otimes d_X)$, where $(K(t), d_K)$ is the Koszul complex of t.*

Proof. For each t_i consider the map $t_i : X \to X$ which is multiplication by t_i. It is straightforward to check that $(K(t_i), d_{K_i}) \otimes_R (X, d_X) \cong \operatorname{cone}(t_i)$, where $\operatorname{cone}(t_i) = (X \oplus X[1], \left(\begin{smallmatrix} d_X & 0 \\ t_i & -d_X \end{smallmatrix}\right))$.

Since we have that t_i is null-homotopic we have by Lemma 7 that $\operatorname{cone}(t_i)$ is homotopy equivalent to $\operatorname{cone}(0) = (X \oplus X[1], \left(\begin{smallmatrix} d_X & 0 \\ 0 & -d_X \end{smallmatrix}\right))$. Note that $X \oplus X[1] \cong (R \oplus R[1]) \otimes_R X$ and that $K(t) = K(t_1) \otimes_R \cdots \otimes_R K(t_n)$. This gives the desired equivalence. $\square$

Lemma 9. *Each partial derivative $\partial_{y_i} V$ acts null-homotopically on X and Y.*

Proof. Fix a k-basis for X and let $\partial_{y_i}(d_X)$ be the map given by differentiating the matrix of d_X entrywise. By the Leibniz rule we obtain

$$\partial_{y_i} V(y) = \partial_{y_i}(V(y) - U(x)) = \partial_{y_i}(d_X^2) = d_X \partial_{y_i}(d_X) + \partial_{y_i}(d_X) d_X$$

Likewise for Y. $\square$

Hence we have shown that there is an isomorphism of matrix factorisations

$$\varphi : (\bigwedge(k^{\oplus n}) \otimes_k Z, d_K + d_Z) \longrightarrow (\bigwedge(k^{\oplus n}) \otimes_k Z, d_Z)$$

and completed the proof of Proposition 6.

3.1 Uninterrupted proof of Proposition 6

Let $k[y] = k[y_1, \cdots, y_n]$ and $t = (\partial_{y_1} V, \ldots, \partial_{y_n} V)$ be the sequence of partial derivatives in $k[y]$. Consider the Jacobi ring $J_V = k[y]/(t)$ and the Kozul complex $(K(t), d_K)$ of t. Since V is a potential J_V is a free k-module and by Lemma 5 we obtain a strong deformation retract

$$(J_V, 0) \underset{\sigma}{\overset{\pi}{\leftrightarrows}} (K(t), d_K), \qquad h$$

over k. We would like to tensor both sides of this strong deformation retract by $X \otimes_{k[y]} Y$ and mix in the differential $d_{X\otimes Y}$ along the lines of Lemma 3. However, while all the modules in the above strong deformation retract are $k[y]$-modules, the maps σ and h are only k-linear *a priori*.

The solution is to fix a $k[x,z]$-basis of the form $\{e_a \otimes f_b\}_{a,b}$ for $X \otimes_{k[y]} Y$ and define $k[x,z]$-linear maps

$$(X \otimes_{k[y]} J_V \otimes_{k[y]} Y, 0) \underset{\tilde{\sigma}}{\overset{\pi\otimes 1}{\leftrightarrows}} (K(t) \otimes_{k[y]} X \otimes_{k[y]} Y, d_K \otimes 1), \qquad \tilde{h} \qquad (*)$$

as follows. Since π and d_K in the original strong deformation retract are $k[y]$-linear we obtain $\pi \otimes 1$ and $d_K \otimes 1$ by applying the functor $(-) \otimes_{k[y]} X \otimes_{k[y]} Y$. The maps $\tilde{\sigma}$ and $\tilde{h}$ are defined on the basis $\{e_a \otimes f_b\}_{a,b}$ as

$$\tilde{\sigma}(e_a \otimes r \otimes f_b) = \sigma(r) \otimes e_a \otimes f_b$$

for $r \in J_V$, and for $g \in K(t)$

$$\tilde{h}(g \otimes e_a \otimes f_b) = h(g) \otimes e_a \otimes f_b$$

where we extend $k[x,z]$-linearly. Notice that the maps $\tilde{\sigma}$ and $\tilde{h}$ depend on the choice of basis for $X \otimes_{k[y]} Y$. It is straightforward to see that $(*)$ is a strong deformation retract over $k[x,z]$.

Following [Perturbation] we now view $d = 1 \otimes d_{X\otimes Y}$ as a perturbation of the strong deformation retract $(*)$. Using that $h^2 = 0$ one shows that $(d\tilde{h})^2 = 0$ by a direct calculation, and so d is a small perturbation. Hence by the Perturbation Lemma we obtain a strong deformation retract over $k[x,z]$

$$(Y|X, d_{Y|X}) \underset{\sigma'}{\overset{\pi'}{\leftrightarrows}} (K(t) \otimes_k X \otimes_{k[y]} Y, d_K \otimes 1 + 1 \otimes d_{X\otimes Y}), \qquad h'$$

where $\sigma' = \tilde{\sigma} + \tilde{h}a\tilde{\sigma}$, $\pi' = \pi \otimes 1 + (\pi \otimes 1)a\tilde{h}$, $h' = \tilde{h} + \tilde{h}a\tilde{h}$ and $a = (1 - d\tilde{h})^{-1}d$, and we have defined $Y|X = X \otimes_{k[y]} J_V \otimes_{k[y]} Y$ and $d_{Y|X} = d_X \otimes 1 + 1 \otimes d_Y$.

In Lemma 9 we showed that the partial derivatives act null-homotopically on X and Y, and hence they also act null-homotopically on $X \otimes_{k[y]} Y$. Hence we have an isomorphism of linear factorisations

$$\varphi : (K(t) \otimes_{k[y]} X \otimes_{k[y]} Y, d_K \otimes 1 + 1 \otimes d_{X\otimes Y}) \longrightarrow (\bigwedge(k^{\oplus n}) \otimes_k X \otimes_{k[y]} Y, 1 \otimes d_{X\otimes Y})$$

and we obtain a strong deformation retract over $k[x,y]$

$$(Y|X, d_{Y|X}) \underset{\varphi\sigma'}{\overset{\pi'\varphi^{-1}}{\leftrightarrows}} (\bigwedge(k^{\oplus n}) \otimes_k X \otimes_{k[y]} Y, 1 \otimes d_{X\otimes Y}), \qquad \varphi h' \varphi^{-1}$$

Finally we note that $J_V \cong k^{\oplus m}$ for some m and so

$$X \otimes_{k[y]} J_V \otimes_{k[y]} Y \cong k[x,z]^{\oplus m}$$

so the left-hand-side of the above strong deformation retract is a finite rank matrix factorisation. The right-hand-side contains $(X \otimes_{k[y]} Y, d_{Y\otimes X})$ as a direct summand, proving the claim. $\square$

Previous Talks' Notes

[Idempotents] 'Idempotents in Categories'. URL: https://rohanhitchcock.com/notes/idempotents.pdf.

[Perturbation] 'The Perturbation Lemma for Linear Factorisations'. URL: https://rohanhitchcock.com/notes/pertubation-lemma.pdf.

References

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[Mur18] Daniel Murfet. 'The cut operation on matrix factorisations'. *Journal of Pure and Applied Algebra* 222.7 (2018), pp. 1911–1955. eprint: 1402.4541 (math.AC).

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[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics 38. Cambridge University Press, 1994.