The cut operation revisited

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In this talk we revisit the cut operation on morphisms in the bicategory of Landau-Ginzburg models. Let k be a field of characteristic zero and consider 1-morphisms

$$(k[x], U) \xrightarrow{(X, d_X)} (k[y], V) \xrightarrow{(Y, d_Y)} (k[z], W)$$

Let $k[y] = k[y_1, \ldots, y_n]$. We consider the sequence of partial derivatives $t = (\partial_{t_1} V, \ldots, \partial_{t_n})$ and the ideal $I = (t_1, \ldots, t_n)$. Since V is a potential t is Koszul-regular, hence quasi-regular. Recall that the *cut* of (X, d_X) and (Y, d_Y) is the matrix factorisation $(Y|X, d_{Y|X})$ of $W(z) - U(x)$ where

$$Y|X = X \otimes_{k[y]} J_V \otimes_{k[y]} Y \qquad \text{and} \qquad d_{Y|X} = d_X \otimes 1 + 1 \otimes d_Y$$

where $J_V = k[y]\big/I$ is the Jacobi ring. In the talk on composition we showed that the cut has the composition $(X \otimes_{k[y]Y}, d_{X\otimes Y})$ as a direct summand. Our goal for this talk is to show *how* the composition is a direct summand of the cut by producing an explicit strong deformation retract

$$(Y|X, d_{Y|X}) \longleftrightarrow (\textstyle\bigwedge(k^{\oplus n}) \otimes_k X \otimes_{k[y]} Y, 1 \otimes d_{X\otimes Y}), \qquad H$$

In the previous talk we showed how to construct a *system of* t*-derivatives* $\partial_{t_1}, \ldots, \partial_{t_n} : k[y] \to k[y]$ in the case that t was a Gröbner basis for I. We now suppose this is the case. The general case proceeds in a similar way but requires passing to the I-adic completion of $k[y]$. If we are required to pass to the completion then only approximations (in the I-adic topology) of $\partial_{t_1}, \ldots, \partial_{t_n}$ can actually be computed algorithmically, while when t is a Gröbner basis all maps involved can be computed exactly.

We consider the Koszul complex $(K(t), d_K)$, and as in previous talks we denote $K(t) = \bigwedge(\bigoplus_{i=1}^n k\,dt_i)$ where $dt_1, \ldots, dt_n$ are formal generators. Let $\nabla : K(t) \to K(t)$ be given by $\nabla(f\omega) = \sum_{i=1}^n \partial_{t_i}(f) dt_i \omega$ where $f \in k[y]$ and $\omega = dt_{i_1} \cdots dt_{i_p}$. In a previous talk we showed that we have a strong deformation retract over k

$$(J_V, 0) \underset{\sigma}{\overset{\pi}{\longleftrightarrow}} (K(t), d_K), \qquad h$$

where π is the quotient map in degree zero, $h = [d_K, \nabla]^{-1}\nabla$ and σ is uniquely determined by π and ∇. Recall that π is $k[y]$-linear while σ and h are only k-linear.

The strong deformation retract above is the starting point for defining cut operation. We tensor the deformation retract by $X \otimes_{k[y]} Y$ and mix the differential $d_{X\otimes Y}$ using the Perturbation Lemma. We fix a $k[x, z]$-basis for $X \otimes_{k[y]} Y$ of the form $\{e_a \otimes f_b\}_{a,b}$ and define a strong deformation retract

$$(Y|X, 0) \underset{\tilde{\sigma}}{\overset{\pi\otimes 1}{\longleftrightarrow}} (K(t) \otimes_{k[y]} X \otimes_{k[y]} Y, d_K \otimes 1), \qquad \tilde{h}$$

over $k[x,z]$ where $\tilde{\sigma}(e_a \otimes r \otimes f_b) = \sigma(r) \otimes e_a \otimes f_b$ and $\tilde{h}(g \otimes e_a \otimes f_b) = h(g) \otimes e_a \otimes f_b$.

Now set $d = 1 \otimes d_{X \otimes Y}$ and view d as a perturbation of the above strong deformation retract. Let $a = (1 - d\tilde{h})^{-1} d$, and since $d\tilde{h}$ is nilpotent since it is a degree -1 map with respect to the $\mathbb{Z}$-grading on $K(t)$. It is not hard to check (we have also shown this in previous talks) that $(1 - d\tilde{h})^{-1} = \sum_{l \geq 0} (d\tilde{h})^l$. By the Perturbation Lemma we have a strong deformation retract

$$(Y|X, d_{Y|X}) \underset{\sigma_\infty}{\overset{\pi_\infty}{\rightleftarrows}} (K \otimes_{k[y]} X \otimes_{k[y]} Y, d_K \otimes 1 + 1 \otimes d), \qquad h_\infty$$

over $k[x,y]$, where $\sigma_\infty = \tilde{\sigma} + \tilde{h} a \tilde{\sigma}$, $\pi_\infty = \pi \otimes 1 + (\pi \otimes 1) a \tilde{h}$ and $h_\infty = \tilde{h} + \tilde{h} a \tilde{h}$. In fact one can show via a direct calculation that $(\pi \otimes 1) a \tilde{h} = 0$ and so $\pi_\infty = \pi \otimes 1$. The maps σ_∞ and h_∞ can be written more conveniently as

$$\sigma_\infty = \tilde{\sigma} + \tilde{h} \sum_{l \geq 0} (d\tilde{h})^l d\tilde{\sigma} = \sum_{l \geq 0} (\tilde{h} d) \tilde{\sigma}$$

and

$$h_\infty = \tilde{h} + \tilde{h} \sum_{l \geq 0} (d\tilde{h})^l d\tilde{h} = \sum_{l \geq 0} (\tilde{h} d) \tilde{h}$$

It remains to remove the differential $d_K \otimes 1$. In a previous talk we showed that we had an isomorphism of linear factorisations

$$(K(t) \otimes_{k[y]} Z, d_K \otimes 1 + 1 \otimes d_Z) \cong K(t) \otimes_{k[y]} Z, 1 \otimes d_Z)$$

for any appropriate linear factorisation (Z, d_Z). It has now come time to explicitly state this isomorphism.

Let $(Z, d_Z) = (X \otimes_{k[y]}, d_{X \otimes Y})$. Recall that we have shown that t_i acts null-homotopically on Z, so let $\lambda_i : t_i \simeq 0$ be such a homotopy (one example is $\lambda_i = \partial_{t_i}(d_X)$). Next note that we have a canonical isomorphism $\alpha : K(t) \otimes_{k[y]} Z \to \bigwedge(\bigoplus_{i=1}^n k\theta_i) \otimes_k Z$ where $\theta_i = dt_i$. We define

$$\exp(\delta) = \sum_{m \geq 0} \frac{1}{m!} \delta^m \qquad \text{and} \qquad \exp(-\delta) = \sum_{m \geq 0} \frac{(-1)^m}{m!} \delta^m$$

where $\delta = \sum_{i=1}^n \lambda_i \theta_i^*$. This definition makes sense because δ is nilpotent: with respect to the $\mathbb{Z}$-grading on $\bigwedge(\bigoplus_{i=1}^n k\theta_i)$ we see that δ has degree -1, and $\bigwedge(\bigoplus_{i=1}^n k\theta_i)$ is zero in negative degree. The next result is [Mur18, Proposition 4.12].

Lemma 1. *The map*

$$\exp(\delta) : (\bigwedge(\bigoplus_{i=1}^n k\theta_i) \otimes_k Z, d_K + d_Z) \longrightarrow (\bigwedge(\bigoplus_{i=1}^n k\theta_i) \otimes_k Z, d_Z)$$

is an isomorphism with inverse $\exp(-\delta)$.

Proof. Clearly $\exp(\delta)$ and $\exp(-\delta)$ are mutually inverse isomorphisms of modules so it suffices to show that they commute with the differentials. We first show that $[d_Z, \delta^m] = m\delta^{m-1} d_Z$ for $m \geq 1$. When $m = 1$ we have

$$[d_Z, \delta] = \sum_{i=1}^m [d_Z, \lambda_i] \theta_i^* = \sum_{i=1}^n t_i \theta_i^* = d_K$$

Now consider $m > 1$. First note that

$$\begin{aligned}\sum_{i=0}^{m-1} \delta^i [d_Z, \delta] \delta^{m-i-1} &= \sum_{i=0}^{m-1} \delta^i d_Z \delta^{m-i} - \sum_{i=0}^{m-1} \delta^{i+1} d_Z \delta^{m-i-1} \\ &= \sum_{i=0}^{m-1} \delta^i d_Z \delta^{m-i} - \sum_{i=1}^{m} \delta^i d_Z \delta^{m-i} \\ &= [d_Z, \delta^m]\end{aligned}$$

Then we have

$$[d_Z, \delta^m] = \sum_{i=0}^{m-1} \delta^i [d_Z, \delta] \delta^{m-i-1} = \sum_{i=0}^{m-1} \delta^i d_K \delta^{m-i-1} = \sum_{i=0}^{m-1} \delta^{m-1} d_K = m\delta^{m-1} d_K$$

as claimed. Next we compute $[d_Z, \exp(-\delta)]$. We have

$$[d_Z, \exp(\delta)] = \sum_{m \geq 0} \frac{1}{m!} [d_Z, \delta^m] = \sum_{m \geq 1} \frac{1}{(m-1)!} \delta^{m-1} d_K = \exp(\delta) d_K$$

Rearranging this expression we find

$$\exp(\delta)(d_Z + d_K) = d_Z \exp(\delta)$$

which shows $\exp(\delta)$ is a morphism of linear factorisations as required. $\square$

Putting all this together we have constructed a strong deformation retract

$$(Y|X, d_{Y|X}) \underset{\Phi}{\overset{\Phi'}{\rightleftarrows}} (\textstyle\bigwedge(\bigoplus_{i=1}^n k\theta_i) \otimes_k X \widehat{\otimes} Y, 1 \otimes d_{X \widehat{\otimes} Y}), \qquad H$$

over $k[x, z]$, where $\theta_1, \dots, \theta_n$ are formal generators, $\Phi = \exp(\delta)\alpha\sigma_\infty$, $\Phi' = (\pi \otimes 1)\alpha^{-1} \exp(-\delta)$ and $H = \exp(\delta)\alpha h_\infty \alpha^{-1} \exp(-\delta)$.

Passing to the completion

Forget that k is a field and suppose k be a commutative ring. Consider a sequence $s = (s_1, \dots, s_m)$ in $k[y]$ and the ideal $J = (s_1, \dots, s_n)$. Let $\widehat{k[y]}$ denote the J-adic completion of $k[y]$.

Lemma 2. *Suppose s is quasi-regular and that there exists a k-linear section $\sigma : k[y]/J \to k[y]$ of the quotient map $\pi : k[y] \to R/J$. Then every $f \in \widehat{k[y]}$ can be written uniquely as a convergent series in of the form*

$$f = \sum_{u \in \mathbb{N}^n} \sigma(r_u) s^u \tag{1}$$

where $r_u \in R/J$ and $s^u = s_1^{u_1} \cdots s_n^{u_n}$.

Lemma 2 is the key result which allows us to construct a system of t-derivatives over the completion. Let t be as in the previous section. Note that we always have a k-linear section $\sigma : J_V \to k[y]$ of the quotient map since J_V is free over k; J_V is in particular projective over k so the sequence

$$0 \longrightarrow I \longrightarrow k[y] \longrightarrow J_V \longrightarrow 0$$

splits over k. Furthermore we can choose σ such that $\sigma(1) = 1$.

This lets us define maps $\partial_{t_1}, \ldots, \partial_{t_n} : \widehat{k[y]} \to \widehat{k[y]}$ as

$$\partial_{t_i}(f) = \sum_{u \in \mathbb{N}^n} u_i \sigma(r_u) t^{u-e_i} \qquad \text{where } f = \sum_{u \in \mathbb{N}^n} \sigma(r_u) t^u$$

These possess analogous properties to the system of t-derivatives we have constructed previously. Essentially the same results can be proved, replacing $k[y]$ with the completion $\widehat{k[y]}$.

When k is a field we can choose a section σ in such a way that the coefficients can be computed algorithmically. Let fix a monomial ordering on $k[y]$ and let g be a Gröbner basis for I. Let

$$V = \{r \in k[y] \mid \text{no term of } r \text{ is divisible by any of the } \operatorname{LT}(g_i)\}$$

Lemma 3. *The quotient map $\pi : k[y] \to k[y]/I$ restricts to an isomorphism $V \to k[y]/I$.*

Proof. For injectivity suppose $r \in V$ is such that $\pi(r) = 0$. Applying the division algorithm to divide r by g yields the remainder term r, since none of the terms in r are divisible by any of the $\operatorname{LT}(g_i)$. Since $r \in I$ and g is a Gröber basis we have $r = 0$. Note that if g is not a Gröbner basis then the restriction $\pi|_V$ will fail to be injective.

For surjectivity, consider $f \in k[y]$. Via the division algorithm we obtain an expression for f of the form

$$f = r + \sum_i q_i g_i$$

where $r \in V$. Then we have $\pi(f) = \pi(r)$, and noting that $\pi : k[y] \to k[y]/I$ is surjective proves the claim. $\square$

Lemma 4. *Any element $f \in \widehat{k[y]}$ can be uniquely expressed as a series of the form*

$$f = \sum_{u \in \mathbb{N}^n} r_u t^u$$

where $r_u \in V$.

We now consider an algorithm to generate the coefficients in the series expansion of an element $f \in k[y]$. The idea is as follows. Let $\{a_{ij}\}_{i,j}$ be the polynomials arising from Buchberger's algorithm which satisfy $g_i = \sum_{j=1}^n a_{ij} t_j$. Given $f \in k[y]$ we can divide f by g to obtain polynomials $r_0 \in C$ and $q_1, \ldots, q_{n'} \in k[y]$ satisfying

$$f = r_0 + \sum_{i=1}^{n'} q_i g_i = r_0 + \sum_{j=1}^{n} \left(\sum_{i=1}^{n'} a_{ij} q_i \right) t_j$$

Setting $p_j = \sum_{i=1}^{n'} a_{ij} g_i$, we can then divide each of the $p_1, \ldots, p_n$ by g to obtain polynomials $r_j \in C$ and $q_{1,j}, \ldots, q_{n',j} \in k[y]$ for $j = 1, \ldots, n$ satisfying

$$f = r_0 + \sum_{j=1}^{n} r_j t_j + \sum_{j=1}^{n} \sum_{i=1}^{n'} q_{i,j} g_i t_j = r_0 + \sum_{j=1}^{n} r_j t_j + \sum_{j,l=1}^{n} \left(\sum_{i=1}^{n'} q_{i,j} a_{il} \right) t_i t_l$$

The polynomials $r_0, r_1, \ldots, r_n \in C$ are the coefficients of the zeroth and first order terms in the series expansion for f. and we can continue to generate higher order coefficients in this manner. In general this algorithm will not terminate. One can show this process terminates when the $\{a_{ij}\}_{i,j}$ are all constant polynomials.

References

[Mur18] Daniel Murfet. 'The cut operation on matrix factorisations'. In: *Journal of Pure and Applied Algebra* 222.7 (2018), pp. 1911–1955. arXiv: 1402.4541 [math.AC].