Lecture

Interpolation

Let us consider a continuous and single valued function in a given range $x_0 \le x \le x_n$,

$$y = f(x)$$

Using this function, we can find value of f(x) at any value of x since dependence of f(x) on x is explicitly known.

For example, if $y = f(x) = 2x+5$, then for any value of x, we can find f(x).

Consider a situation where dependence of y = f(x) is not known explicitly. For example, we have been given set of tabulated values (x_0,y_0), (x_1,y_1),…….., (x_n,y_n). In this case, nature of dependence on x is not explicitly known. Now, if we wish to find out value of 'y' at, let us say, x_0' which is within given range $x_0 \le x \le x_n$. What will be the value of 'y'?

In order to find value at x_0', we need to find nature of set of tabulated points. Whether it is linear, exponential, logarithmic, etc. If we can ascertain nature of tabulated points, we shall be able to find value. This is what interpolation is.

In our study, we shall be concerned with the polynomial interpolation i.e. the set of data points could be fit by polynomials.

The justification that an unknown function could be approximated by a polynomial is due to a theorem due to Weierstrass which states:

If f(x) is continuous in $x_0 \le x \le x_n$, the given any $\varepsilon > 0$, there exists a polynomial P(x) such that

$|f(x)-P(x)| < \varepsilon$, for $x_0 \le x \le x_n$

The interpolation finds applications in economics, business, population studies, science, engineering, etc.

Finite differences:

Let a function y = f(x) be tabulated for equally spaced values of $x = x_i$, where $x_i = x_0 + ih$ $(i=0,1,2,..,n)$. Thus giving us y_0, y_1, …….,y_n. Suppose we need to determine f(x) at some intermediate value of x or we want to determine f(x), f'(x), etc. at an x in (x_0, x_n). The following method of differences can be useful.

(A) Forward differences:

We define the forward differences as

$\Delta y_0=y_1-y_0$, $\Delta y_1=y_2-y_1$, ………… $\Delta y_n=y_n-y_{n-1}$

Δ is called forward difference operator and Δy_0, Δy_1, ………… Δy_n are called first forward differences.

The difference of first forward differences are called second forward difference and are given by

$\Delta^2 y_0=\Delta y_1-\Delta y_0$, $\Delta^2 y_1=\Delta y_2-\Delta y_1$, etc.

$\Delta^2 y_0=\Delta y_1-\Delta y_0 = (y_2-y_1)-(y_1-y_0) = y_2-2y_1+y_0$

Similarly, we can find other first forward differences.

In a similar fashion, we can find higher order differences.

Thus, we can construct a forward difference table.

x	y	Δ	Δ^2
x_0	y_0		
		Δy_0	
x_1	y_1		$\Delta^2 y_0$
		Δy_1	
x_2	y_2		

(B) Backward differences:

If we define $\nabla y_1=y_1-y_0$, $\nabla y_2=y_2-y_1$, etc. ∇ is called backward difference operator.

Similarly can define higher order backward differences ∇^2, ∇^3, etc.

So $\nabla^2 y_2 = \nabla y_2-\nabla y_0 = y_2-y_1-(y_1-y_0) = y_2-2y_1+y_0$

Likewise, we can construct backward difference table.

x	y	∇	∇^2
x_0	y_0		
x_1	y_1	∇y_1	
x_2	y_2	∇y_2	$\nabla^2 y_2$

(C) Central differences:

If we define, $y_1 - y_0 = \delta y_{1/2}$, $y_2 - y_1 = \delta y_{3/2}$, etc.

Similarly higher order central differences can be defined.

Thus, we can construct central difference table.

x	y	δ	δ^2
x_0	y_0		
		$\delta y_{1/2}$	
x_1	y_1		$\delta^2 y_1$
		$\delta y_{3/2}$	
x_2	y_2		

It is clear from table that same number occurs in the same position in all difference tables.

Thus, $\Delta y_0 = \nabla y_1 = \delta y_{1/2}$

<u>Differences of a polynomial:</u>

Here we state an important theorem which says that the nth differences of a polynomial of nth degree are constant and all higher order differences are zero.

Let $f(x) = ax^n + bx^{n-1} + cx^{n-2} + \ldots\ldots\ldots\ldots\ldots + kx + l$

be nth degree polynomial.

Now, the first difference

$\Delta f(x) = f(x+h) - f(x) = a[(x+h)^n - x^n] + b[(x+h)^{n-1} - x^{n-1}] + \ldots\ldots\ldots\ldots + kh$

$= [a(x^n + {}^nC_1 x^{n-1} h + \ldots.. + h^n + \ldots\ldots\ldots\ldots$

$= anhx^{n-1} + b' x^{n-2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. + l'$

b’, c’,….., l’ are new coefficients.

Thus first difference of a polynomial of nth degree is a polynomial of (n-1) degree.

Similarly.

$\Delta^2 f(x) = \Delta f(x+h) - \Delta f(x) = an(n-1)h^2 x^{n-2} + b'' x^{n-3} + \ldots\ldots\ldots.. + l''$

b’’,….l’’ are new coefficients.

Thus second difference yield polynomial of x^{n-2} degree.

Hence, $\Delta^n f(x)= an(n-1)(n-2)\ldots\ldots\ldots\ldots..3.2.1.\ h^n = an!h^n$ (constsnt)

And all higher differences of a polynomial on nth degree are zero.

Its converse is also true. If the nth differences of a function tabulated at equally spaced intervals are constant, the function is a polynomial of nth degree.

Effect of an error in a difference table

Suppose there is an error ‘e’ in the entry y_5 of a table. As higher differences are formed this error spreads out and is magnified considerably. Let us see.

x	y	Δ	Δ^2	Δ^3	Δ^4
X_0	y_0				
		Δy_0			
X_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
X_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
X_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1+e$
		Δy_3		$\Delta^3 y_2+e$	
X_4	y_4		$\Delta^2 y_3+e$		$\Delta^4 y_2-4e$
		Δy_4+e		$\Delta^3 y_3-3e$	
X_5	y_5+e		$\Delta^2 y_4-2e$		$\Delta^4 y_3+6e$
		Δy_5-e		$\Delta^3 y_4+3e$	
X_6	y_6		$\Delta^2 y_5+e$		$\Delta^4 y_4-4e$
		Δy_6		$\Delta^3 y_5-e$	
X_7	y_7		$\Delta^2 y_6$		$\Delta^4 y_5+e$
		Δy_7		$\Delta^3 y_6$	
X_8	y_8		$\Delta^2 y_7$		
		Δy_8			
X_9	y_9				

From table, we can observe:

(i) Error increases with order of dufference.
(ii) Coefficients of ‘es’ in any column are the binomial coefficients.
(iii) Algebraic sum of errors in anydifference column is zero.
(iv) Maximum error occurs in each column opposite to the entry containing error. In this case it is y_5.

Example: One entry in the given table is wrong and y is cubic polynomial in x. Use the difference table to locate and correct the error.

x	y	Δ	Δ^2	Δ^3
0	25			
		-4		
1	21		1	
		-3		2
2	18		3	
		0		6
3	18		9	
		9		0
4	27		9	
		18		4
5	45		13	
		31		3
6	76		16	
		47		
7	123			

As y is a polynomial of degree 3 thus $\Delta^3 y$ must be constant i.e. same. (Recall the theorem we proved earlier)

The sum of third differences is 15. There are 5 entries in that column so each entry must be 15/5 =3. Only last entry is 3 so all other entries are wrong.

Starting with first entry for the third differences, the entry is 2 so error is (-1). Using the table stated before, we can write

3+(-1), 3-3(-1), 3+3(-1) and 3-(-1)

So e = -1 and so comparing with previous table, we find entry corresponding to x = 3 is wrong.

Thus, y+e = 18 or y+(-1) = 18 or y =19.

So correct entry must be y = 19.

Problem for students:

Assuming y is of degree 4 polynomial, find three entries using finite difference method.

x	0	1	2	3	4	5	6	7
y	1	-1	1	-1	1	?	?	?