# Lecture

## Numerical differentiation

Till last lecture we were concerned with the interpolating polynomials. In that case, given values of x and y we have found polynomials to fit the data.

In this lecture, we shall be concerned with their derivatives i.e. to find $\frac{dy}{dx}, \frac{d^2y}{d^2x}$, etc. for a given value of x in an interval ($x_0$, $x_n$).

In order to find a general method for finding derivatives, we differentiate the interpolating polynomials.

Let us consider the Newton's forward difference formula.

$$y = y_0 + p\Delta y_0 \ + p(p-1)\frac{\Delta^2 y_0}{2!} + \cdots \ldots \ldots \ldots \ldots \ldots + p(p-1)(p-2)\ldots\ldots(p-n+1)\frac{\Delta^n y_0}{n!}$$

where, $x = x_0+ph$

Now, $\frac{dy}{dx} = \frac{dy}{dp}.\frac{dp}{dx}$

Since, $p = (x-x_0)/h$ so $\frac{dp}{dx} = \frac{1}{h}$

Thus, $\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \cdots \ldots \ldots \ldots \ldots\right]$

The above formula is for non tabular points of x.

For tabular values, this formula takes simpler form.

Let $x = x_0$

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 + \cdots \dots \dots \dots \dots ..\right]$$

Further differentiation of above formula gives,

$$\left[\frac{d^2y}{d^2x}\right] = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{6p-6}{6}\Delta^3 y_0 + \frac{12p^2-36p+22}{24}\Delta^4 y_0 + \cdots \dots \dots \dots \dots ..\right]$$

For p = 0 (x = $x_0$),

$$\left[\frac{d^2y}{d^2x}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots \dots \dots \dots \dots ..\right]$$

This way higher order derivatives could be obtained.

Using Newton's backward difference formula, we get

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^2 y_n + \cdots \dots \dots \dots \dots ..\right]$$

$$\left[\frac{d^2y}{d^2x}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots \dots \dots \dots \dots ..\right]$$

Stirling's formula gives:

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\frac{\Delta y_{-1} + \Delta y_0}{2} - \frac{1}{6}\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \cdots \dots \dots \dots \dots ..\right]$$

$$\left[\frac{d^2y}{d^2x}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} + \cdots \dots \dots \dots \dots ..\right]$$

In summary,

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 + \cdots \dots\dots\dots\dots..\right] y_0$$

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta + \frac{1}{2}\Delta^2 - \frac{1}{6}\Delta^3 + \cdots \dots\dots\dots\dots..\right] y_{-1}$$

$$\left[\frac{d^2y}{d^2x}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 + \cdots \dots\dots\dots\dots..\right] y_0$$

$$\left[\frac{d^2y}{d^2x}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 - \frac{1}{12}\Delta^4 + \frac{1}{12}\Delta^5 + \cdots \dots\dots\dots\dots..\right] y_{-1}$$

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^2 + \cdots \dots\dots\dots\dots..\right] y_n$$

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla - \frac{1}{2}\nabla^2 - \frac{1}{6}\nabla^3 + \cdots \dots\dots\dots\dots..\right] y_{n+1}$$

$$\left[\frac{d^2y}{d^2x}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \cdots \dots\dots\dots\dots..\right] y_n$$

$$\left[\frac{d^2y}{d^2x}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 - \frac{1}{12}\nabla^4 - \frac{1}{12}\nabla^5 + \cdots \dots\dots\dots\dots..\right] y_{n+1}$$

Example: Given with the data in table, find $\frac{dy}{dx}$ and $\frac{d^2y}{d^2x}$ at x = 1.1.

| x | y | Δ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ | $\Delta^5$ | $\Delta^6$ |
|---|---|---|---|---|---|---|---|
| 1.0 | 7.989 | | | | | | |
| | | 0.414 | | | | | |
| 1.1 | 8.403 | | -0.036 | | | | |
| | | **0.378** | | 0.006 | | | |
| 1.2 | 8.781 | | **-0.030** | | -0.002 | | |
| | | 0.348 | | **0.004** | | 0.002 | |
| 1.3 | 9.129 | | -0.026 | | **0.000** | | **-0.001** |
| | | 0.322 | | 0.004 | | **0.001** | |
| 1.4 | 9.451 | | -0.022 | | 0.001 | | |
| | | 0.300 | | 0.005 | | | |
| 1.5 | 9.751 | | -0.017 | | | | |
| | | 0.283 | | | | | |
| 1.6 | 10.034 | | | | | | |

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0\right]$$

h = 0.1, $x_0$ = 1.1, $\Delta y_0 = 0.378$, $\Delta^2 y_0 = -0.030$, $\Delta^3 y_0 = 0.004$, $\Delta^4 y_0 = 0.000$, $\Delta^5 y_0 = 0.001$, $\Delta^6 y_0 = -0.001$

$$\left[\frac{dy}{dx}\right]_{x=1.1} = \frac{1}{0.1}\left[0.378 - \frac{1}{2}(-0.030) + \frac{1}{3}(.0.004) - \frac{1}{4}(.0.000) + \frac{1}{5}(0.001) - \frac{1}{6}(-0.001)\right]$$

= 3.9452

$$\left[\frac{d^2y}{d^2x}\right]_{x=1.1} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0\right]$$

$$\left[\frac{d^2y}{d^2x}\right]_{x=1.1} = \frac{1}{(0.1)^2}\left[(-0.30) - (0.004) + \frac{11}{12}(0.000) - \frac{5}{6}(0.001) + \frac{137}{180}(-0.001)\right]$$

= -3.556

If we wish to find out $\frac{dy}{dx}$ and $\frac{d^2y}{d^2x}$ at x = 1.6, we can use

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^2 y_n + \cdots \ldots \ldots \ldots \ldots . .\right]$$

$$\left[\frac{d^2y}{d^2x}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots \ldots \ldots \ldots \ldots . .\right]$$

With h =0.1, $x_n$ = 1.6, $\nabla y_n = 0.283$, etc.

**Maxima and minima of a tabulated function**

Differentiating Newton's forward difference formula w.r.t. p, we get

$$\frac{dy}{dp} = \left[\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \cdots \ldots \ldots \ldots \ldots . .\right]$$

Setting

$\frac{dy}{dp} = 0$ and keeping RHS to third term for simplicity,

$$\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 = 0$$

Rewriting, $\left(\frac{1}{2}\Delta^3 y_0\right)p^2 + (\Delta^2 y_0 - \Delta^3 y_0)p + \left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0\right) = 0$

Which can simply be written as $ap^2 + bp + c = 0$, where a,b and c have corresponding meanings.

Solving for p gives maxima or minima of x using x = $x_0$+ph.

Example: Find for which value of x, y will be maximum?

| x | y | $\Delta$ | $\Delta^2$ | $\Delta^3$ |
|---|---|---|---|---|
| 3 | 0.205 | | | |
| | | **0.035** | | |
| 4 | 0.240 | | **-0.016** | |
| | | 0.019 | | **0.000** |
| 5 | 0.259 | | -0.016 | |
| | | 0.003 | | 0.001 |
| 6 | 0.262 | | -0.015 | |
| | | -0.012 | | 0.001 |
| 7 | 0.250 | | -0.014 | |
| | | -0.026 | | |
| 8 | 0.224 | | | |

$\frac{dy}{dp} = \left[0.035 + \frac{2p-1}{2}(-0.016) + \frac{3p^2-6p+2}{6}(0)\right]$=0

h = 1

p = 2.6875

so x = 3+2.6875x1= 5.6875.