Lecture

Diagonally dominant matrix:

Since this concept shall be used later, hence, we shall learn about it before proceeding further.

A square matrix is said to be diagonally dominant, if (i) the absolute value of each leading diagonal element is greater than or equal to sum of the absolute values of the remaining elements in that row and (ii) the greater than requirement should, at least be satisfied for one of the rows.

Let us consider a system of linear equations

$A_1 x + B_1 y + C_1 z = D_1$

$A_2 x + B_2 y + C_2 z = D_2$

$A_3 x + B_3 y + C_3 z = D_3$

which can be written in the form AX=D

For matrix A, If,

$$|A_1| \geq |B_1| + |C_1|$$

$$|B_2| \geq |A_2| + |C_2|$$

$$|C_3| \geq |A_3| + |C_3|$$

And for, at least, one of the rows: the absolute value of diagonal element is strictly > sum of the absolute values of other elements in that row.

If all rows, we have > sign then the matrix is known as strictly diagonally dominant matrix.

Example,

$$A=\begin{bmatrix} 10 & -5 & -3 \\ 4 & -12 & 7 \\ -6 & 8 & 22 \end{bmatrix}$$

This is a diagonally dominant matrix since

$$|10| > |-5| + |-3|$$

$$|-12| > |4| + |7|$$

$$|22| > |-6| + |8|$$

Note: Since, we are talking about row here so it would be more appropriate if the term row diagonally dominant matrix is used.

Iterative Methods:

The preceding methods of solving simultaneous linear equations are known as direct methods since these methods provide the solution after fixed number of computation.

Another way to solve system of linear equations by using iterative methods. In iterative methods, the solution is obtained by refining the initial approximation.

Though direct methods look lucrative to solve the linear system of equations (and should be preferred in general), it is not always possible to use them. For example, in Gauss elimination method it is not possible to refine the solution further after a fixed number of calculations. This refining becomes important, particularly, in the wake of rounding off errors. Other problems are where coefficient matrix is sparse (system where large number of zero elements are present). In dealing with such problems, iterative methods are of great use.

Two methods which we shall study here are: (i) Jacobi and (ii) Gauss-Seidel method.

Both Jacobi as well as Gauss-Seidel methods converge for any starting value if the matrix is diagonally dominant.

If the system of equations are not diagonally dominant then they must be reorder so that the coefficient matrix is diagonally dominant.

As an example,

$x + 6y - 2z = 5 \qquad (1) \qquad |1| < |6| + |-2|$

$4x + y + z = 6 \qquad (2) \qquad |1| < |4| + |1|$

$-3x + y + 7z = 15 \qquad (3) \qquad |7| > |-3| + |1|$

Clearly, here the coefficient matrix is not diagonally dominant. But if we interchange equations (1) and (2)

$4x + y + z = 6 \qquad (1) \qquad |4| > |1| + |1|$

$x + 6y - 2z = 6 \qquad (2) \qquad |6| > |1| + |-2|$

$-3x + y + 7z = 15 \qquad (3) \qquad |7| > |-3| + |1|$

The coefficient matrix is diagonally dominant now.

Jacobi iteration method:

Let us consider a system of equations

$A_1 x + B_1 y + C_1 z = D_1$

$A_2 x + B_2 y + C_2 z = D_2 \qquad (1)$

$A_3 x + B_3 y + C_3 z = D_3$

Rewriting these equations in the form

$$x = \frac{1}{A_1}(D_1 - B_1 y - C_1 z)$$

$$y = \frac{1}{B_2}(D_2 - A_2 x - C_2 z) \qquad (2)$$

$$z = \frac{1}{C_3}(D_3 - A_3 x - B_3 y)$$

Let us start with initial approximation x_0, y_0, z_0 for the values of x, y, z, respectively. Substituting for x, y, z, we get new approximations

$$x_1 = \frac{1}{A_1}(D_1 - B_1 y_0 - C_1 z_0)$$

$$y_1 = \frac{1}{B_2}(D_2 - A_2 x_0 - C_2 z_0) \qquad (3)$$

$$z_1 = \frac{1}{C_3}(D_3 - A_3 x_0 - B_3 y_0)$$

Continuing this way, we obtain successive approximations x_2, y_2, z_2, etc. The process is repeated till the values of unknowns are obtained to the required accuracy.

In this example, we treated with three unknowns, however, this could be extended to 'n' equations and 'n' unknowns.

This method is also known as "method of simultaneous displacements" since each of the equation is simultaneously changed by using recent x, y, z values (x_i, i= 0,1,…,n, in general).

In the absence of any better approximations for initial approximation, we may take each of them to be zero.

Example: Solve using Jacobi method

3 x + 4 y + 15 z = 54.8

x + 12 y + 3 z = 39.66

10 x + y -2 z = 7.74

Solution:

Writing above equations as, Ax = B

where

$$A = \begin{bmatrix} 3 & 4 & 15 \\ 1 & 12 & 3 \\ 10 & 1 & -2 \end{bmatrix}, \; x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 54.8 \\ 39.66 \\ 7.74 \end{bmatrix}$$

Clearly A is not diagonally dominant. Interchanging rows 1 and 3 gives

A= $\begin{bmatrix} 10 & 1 & -2 \\ 1 & 12 & 3 \\ 3 & 4 & 15 \end{bmatrix}$

Diagonally dominant matrix.

Thus, system of equations

10 x + y -2 z = 7.74

x + 12 y + 3 z = 39.66

3 x + 4 y + 15 z = 54.8

Now,

$$x = \frac{1}{10}(7.74 - y + 2z)$$

$$y = \frac{1}{12}(39.66 - x - 3z)$$

$$z = \frac{1}{15}(54.8 - 3\,x - 4y)$$

Let

$x_0, y_0, z_0 = 0$

So, $x_1 = 7.74/10 = 0.774$, $y_1 = 39.66/12 = 3.305$, $z_1 = 54.8/15 = 3.6533$

Second iteration,

$x_2 = \frac{1}{10}(7.74 - 3.305 + 2 \times 3.6533) = 1.1742$

$y_2 = \frac{1}{12}(39.66 - 0.774 - 3 \times 3.6533) = 2.3272$

$z_2 = \frac{1}{15}(54.8 - 3\ \times 0.774 - 4 \times 3.305) = 2.6172$

Similarly, we obtain

$x_3 = 1.0647$, $y_3 = 2.5529$, $z_3 = 2.7979$

$x_4 = 1.0783$, $y_4 = 2.5168$, $z_4 = 2.7596$

$x_5 = 1.0742$, $y_5 = 2.5252$, $z_5 = 2.7665$

$x_6 = 1.0748$, $y_6 = 2.5239$, $z_6 = 2.7651$

$x_7 = 1.0746$, $y_7 = 2.5242$, $z_7 = 2.7653$

Since, difference between 6^{th} and 7^{th} iteration is small so,

x = 1.075, y = 2.524, z = 2.765, correct to three decimal places.

Example:

$20x + y - 2z = 17$

$3x + 20y - z = -18$

$2x - 3y + 20z = 25$

Since coefficient matrix is diagonally dominant so we can proceed as before

$$x = \frac{1}{20}(17 - y + 2z)$$

$$y = \frac{1}{20}(-18 - 3x + z)$$

$$z = \frac{1}{20}(25 - 2x + 3y)$$

With $x_0, y_0, z_0 = 0$, we obtain

$x_1 = 0.85$, $y_1 = -0.9$, $z_1 = 1.25$

$x_2 = 1.02$, $y_2 = -0.965$, $z_2 = 1.03$

$x_3 = 1.00125$, $y_3 = -1.0015$, $z_3 = 1.00325$

$x_4 = 1.0004$, $y_4 = -1.000025$, $z_4 = 0.9965$

$x_5 = 0.999966$, $y_5 = -1.000078$, $z_5 = 0.999956$

$x_6 = 1.0000$, $y_6 = -0.999997$, $z_6 = 0.999992$

5^{th} and 6^{th} iterations are almost same, hence

X = 1, y = -1 and z = 1

Gauss-Seidel method:

It is modified Jacobi's method. The method is described below.

Let us consider a system of equations

$$\begin{aligned} A_1 x + B_1 y + C_1 z &= D_1 \\ A_2 x + B_2 y + C_2 z &= D_2 \qquad (1) \\ A_3 x + B_3 y + C_3 z &= D_3 \end{aligned}$$

As before,

$$x = \frac{1}{A_1}(D_1 - B_1 y - C_1 z)$$

$$y = \frac{1}{B_2}(D_2 - A_2 x - C_2 z)$$

$$z = \frac{1}{C_3}(D_3 - A_3 x - B_3 y)$$

With initial approximation y_0, $z_0 = 0$, we get

$$x_1 = \frac{1}{A_1}(D_1 - B_1 y_0 - C_1 z_0)$$

Now, for y_1, we use new approximation for x, i.e., x_1

$$y_1 = \frac{1}{B_2}(D_2 - A_2 x_1 - C_2 z_0)$$

For z_1, we use both x_1 and newly calculated value of $y = y_1$

$$z_1 = \frac{1}{C_3}(D_3 - A_3 x_1 - B_3 y_1)$$

In Jacobi's method, new values of 'x' were only used in the algorithm when all values of x were obtained in a particular iteration. In Gauss-Seidel method, new values are inserted as soon as they are obtained. This is the difference between two methods and for this reason Gauss-Seidel method is also known as "method of successive displacements".

Example:

$6x + 15y + 2z = 72$

$x + y + 54z = 110$

$27x + 6y - z = 85$

Since coefficient matrix is not diagonally dominant so we need to rearrange equations.

After rearrangement, we get

$27x + 6y \quad - z = 85$

$6 x + 15 y + 2z = 72$

$x + y \quad + 54z = 110$

Coefficient matrix is now diagonally dominant.

$$x = \frac{1}{27}(85 - 6y + z)$$

$$y = \frac{1}{15}(72 - 6x - 2z)$$

$$z = \frac{1}{54}(110 - x - y)$$

With y_0, $z_0 = 0$, we obtain

$x_1 = 85/27 = 3.1481$

Now for evaluating y_1, we use $x_1 = 3.1481$ and $z_0 = 0$,

$y_1 = (1/15)(72-6x3.1481) = 3.5408$

and $z_1 = (1/54)(110-3.1481-3.5408) = 1.9132$

Proceeding in this way,

$x_2 = (1/27)(85-6x3.5408+1.9132) = 2.4322$

$x = x_2$, $z = z_1$

$y_2 = (1/15)(72-6x2.4322-2x1.9132) = 3.572$

$x = x_2$, $y = y_2$

$z_2 = (1/54)(110-2.4322-3.572) = 1.9258$

Similarly,

$x_3 = 2.4257$, $y_3 = 3.5729$, $z_3 = 1.926$

$x_4 = 2.4255$, $y_4 = 3.573$, $z_4 = 1.926$

Since difference between third and fourth iteration is very small, hence,

$x = 2.426$, $y = 3.573$, $z = 1.926$

Gauss-Seidel method converges twice as fast as the Jacobi's method.

There are some instances for $Ax = B$, where coefficient matrix is not diagonally dominant but both Jacobi and Gauss-Seidel methods do converge.

(i) If the coefficient matrix A, is symmetric and positive definite[++](discussed below), the Gauss-Seidel method will converge from any starting value.

(ii) If A has diagonal elements which are all positive and off-diagonal elements all negative, in such case both Jacobi and Gauss-Seidel methods will either converge or diverge.

++A matrix is positive definite if $x^*AX>0$ for all nonzero vectors x (x^* is conjugate transpose of x). In other words a symmetric matrix with all positive eigenvalues. For more about positive definite matrix, read

https://www.math.utah.edu/~zwick/Classes/Fall2012_2270/Lectures/Lecture33_with_Examples.pdf

Problem for the students

Solve using (i) Jacobi method and (ii) Gauss-Seidel method:

$28x+4y-z = 32$

$2x+17y+4z = 35$

$x+3y+10z = 24$