Chapter 9. Dynamics of Rigid Bodies

(Most of the material presented in this chapter is taken from Thornton and Marion, Chap. 11.)

9.1 Notes on Notation

In this chapter, unless otherwise stated, the following notation conventions will be used:

1. Einstein's summation convention. Whenever an index appears twice (an only twice), then a summation over this index is implied. For example,

$$x_i x_i \equiv \sum_i x_i x_i = \sum_i x_i^2. \tag{9.1}$$

2. The index i is reserved for Cartesian coordinates. For example, x_i, for $i = 1,2,3$, represents either x, y, or z depending on the value of i. Similarly, p_i can represent p_x, p_y, or p_z. This does not mean that any other indices cannot be used for Cartesian coordinates, but that the index i will only be used for Cartesian coordinates.
3. When dealing with systems containing multiple particles, the index α will be used to identify quantities associated with a given particle when using Cartesian coordinates. For example, if we are in the presence of n particles, the position vector for particle α is given by $\mathbf{r}_\alpha$, and its kinetic energy T_α by

$$T_\alpha = \frac{1}{2} m_\alpha \dot{x}_{\alpha,i} \dot{x}_{\alpha,i}, \qquad \alpha = 1,2,\ldots,n \text{ and } i = 1,2,3. \tag{9.2}$$

Take note that, according to convention 1 above, there is an implied summation on the Cartesian velocity components (the index i is used), but not on the masses since the index α appears more than twice. Correspondingly, the total kinetic energies is written as

$$T = \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha \dot{x}_{\alpha,i} \dot{x}_{\alpha,i} = \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right). \tag{9.3}$$

9.2 The Independent Coordinates of a Rigid Body

The simplest extended-body model that can be treated is that of a rigid body, one in which the distances $\mathbf{r}_i - \mathbf{r}_j$ between points are held fixed. A general rigid body will have six degrees of freedom (but not always, see below). To see how we can specify the position of *all* points in the body with only six parameters, let us first fix some point $\mathbf{r}_1$ of the body, thereafter treated as its "centre" or origin from which all other points in the body can be referenced from ($\mathbf{r}_1$ can be, but not necessarily, the centre of mass). Once the

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products of inertia. Finally, in most cases the rigid body is continuous and not made up of discrete particles as was assumed so far, but the results are easily generalized by replacing the summation by a corresponding integral in the expression for the components of the inertia tensor

$$I_{ij} = \int_V \rho(\mathbf{r})\left(\delta_{ij} x_k x_k - x_i x_j\right) dx_1 dx_2 dx_3, \tag{9.20}$$

where $\rho(\mathbf{r})$ is the mass density at the position $\mathbf{r}$, and the integral is to be performed over the whole volume V of the rigid body.

Example

Calculate the inertia tensor for a homogeneous cube of density ρ, mass M, and side length b. Let one corner be at the origin, and three adjacent edges lie along the coordinate axes (see Figure 9-1).

Solution.

We use equation (9.20) to calculate the components of the inertia tensor. Because of the symmetry of the problem, it is easy to see that the three moments of inertia I_{11}, I_{22}, and I_{33} are equal and that same holds for all of the products of inertia. So,

$$\begin{aligned}
I_{11} &= \int_0^b \int_0^b \int_0^b \rho\left(x_2^2 + x_3^2\right) dx_1 dx_2 dx_3 \\
&= \rho \int_0^b dx_3 \int_0^b dx_2 \left(x_2^2 + x_3^2\right) \int_0^b dx_1 \\
&= \rho b \int_0^b dx_3 \left(\frac{b^3}{3} + b x_3^2\right) = \rho b\left(\frac{b^4}{3} + \frac{b^4}{3}\right) \\
&= \frac{2}{3}\rho b^5 = \frac{2}{3} M b^2.
\end{aligned} \tag{9.21}$$

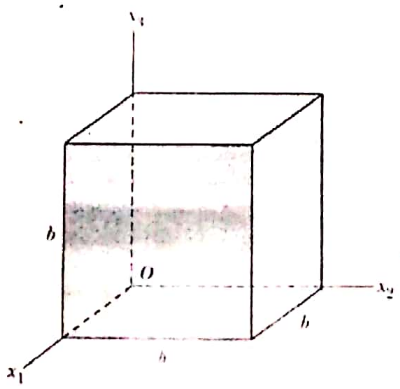

Figure 9-1 – A homogeneous cube of sides b with the origin at one corner.

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coordinates of $\mathbf{r}_1$ are specified (in relation to some origin outside of the body), we have already used up three degrees of freedom. With $\mathbf{r}_1$ fixed, the position of some other point $\mathbf{r}_2$ can be specified using only two coordinates since it is constrained to move on the surface of a sphere centered on $\mathbf{r}_1$. We are now up to five degrees of freedom. If we now consider any other third point $\mathbf{r}_3$ not located on the axis joining $\mathbf{r}_1$ and $\mathbf{r}_2$, its position can be specified using one degree of freedom (or coordinate) for it can only rotate about the axis connecting $\mathbf{r}_1$ and $\mathbf{r}_2$. We thus have used up the six degrees of freedom. It is interesting to note that in the case of a linear rod, any point $\mathbf{r}_3$ must lay on the axis joining $\mathbf{r}_1$ and $\mathbf{r}_2$; hence a linear rod has only five degrees of freedom.

Usually, the six degrees of freedom are divided in two groups: three degrees for translation (to specify the position of the "centre" $\mathbf{r}_1$, and three rotation angles to specify the orientation of the rigid body (normally taken to be the so-called **Euler angles**).

9.3 The Inertia Tensor

Let's consider a rigid body composed of n particles of mass m_α, $\alpha = 1, \ldots, n$. If the body rotates with an angular velocity $\boldsymbol{\omega}$ about some point fixed with respect to the body coordinates (this "body" coordinate system is what we used to refer to as "noninertial" or "rotating" coordinate system in Chapter 8), and if this point moves linearly with a velocity $\mathbf{V}$ with respect to a fixed (i.e., inertial) coordinate system, then the velocity of the αth particle is given by equation (8.16) of derived in Chapter 8

$$\mathbf{v}_\alpha = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}_\alpha, \tag{9.4}$$

where we omitted the term

$$\mathbf{v}_{\alpha,r} \equiv \left(\frac{d\mathbf{r}_\alpha}{dt}\right)_{\text{rotating}} = 0, \tag{9.5}$$

since we are dealing with a rigid body. We have also dropped the f subscript, denoting the fixed coordinate system, as it is understood that all the non-vanishing velocities will be measured in this system; again, we are dealing with a rigid body.

The total kinetic energy of the body is given by

$$\begin{aligned}
T = \sum_\alpha T_\alpha &= \frac{1}{2}\sum_\alpha m_\alpha v_\alpha^2 \\
&= \frac{1}{2}\sum_\alpha m_\alpha \left(\mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}_\alpha\right)^2 \\
&= \frac{1}{2}\sum_\alpha m_\alpha V^2 + \sum_\alpha m_\alpha \mathbf{V} \cdot \left(\boldsymbol{\omega} \times \mathbf{r}_\alpha\right) + \frac{1}{2}\sum_\alpha m_\alpha \left(\boldsymbol{\omega} \times \mathbf{r}_\alpha\right)^2.
\end{aligned} \tag{9.6}$$

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Although this is an equation for the total kinetic energy is perfectly general, considerable simplification will result if we choose the origin of the body coordinate system to coincide with the centre of mass. With this choice, the second term on the right hand side of the last of equations (9.6) can be seen to vanish from

$$\sum_\alpha m_\alpha \mathbf{V}\cdot(\omega\times\mathbf{r}_\alpha) = \mathbf{V}\cdot\left[\omega\times\left(\sum_\alpha m_\alpha \mathbf{r}_\alpha\right)\right] = 0, \tag{9.7}$$

since the centre of mass $\mathbf{R}$ of the body, of mass M, is defined such that

$$\sum_\alpha m_\alpha \mathbf{r}_\alpha = 0. \tag{9.8}$$

The total kinetic energy can then be broken into two components: one for the translational kinetic energy and another for the rotational kinetic energy. That is,

$$T = T_{\text{trans}} + T_{\text{rot}}, \tag{9.9}$$

with

$$\begin{aligned} T_{\text{trans}} &= \frac{1}{2}\sum_\alpha m_\alpha V^2 = \frac{1}{2}MV^2 \\ T_{\text{rot}} &= \frac{1}{2}\sum_\alpha m_\alpha(\omega\times\mathbf{r}_\alpha)^2 \end{aligned} \tag{9.10}$$

The expression for T_{rot} can be further modified, but to do so we will now resort to tensor (or index) notation. So, let's consider the following vector equation

$$(\omega\times\mathbf{r}_\alpha)^2 = (\omega\times\mathbf{r}_\alpha)\cdot(\omega\times\mathbf{r}_\alpha), \tag{9.11}$$

and rewrite it using the Levi-Civita and the Kronecker tensors

$$\begin{aligned} (\varepsilon_{ijk}\omega_j x_{\alpha,k})(\varepsilon_{imn}\omega_m x_{\alpha,n}) &= \varepsilon_{ijk}\varepsilon_{imn}\omega_j x_{\alpha,k}\omega_m x_{\alpha,n} \\ &= (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})\omega_j x_{\alpha,k}\omega_m x_{\alpha,n} \\ &= \omega_j\omega_j x_{\alpha,k}x_{\alpha,k} - \omega_j x_{\alpha,j}\omega_k x_{\alpha,k}. \end{aligned} \tag{9.12}$$

Inserting this result in the equation for T_{rot} in equation (9.10) we get

$$T_{\text{rot}} = \frac{1}{2}\sum_\alpha m_\alpha\left[\omega^2\,\mathbf{r}_\alpha{}^2 - (\omega\cdot\mathbf{r}_\alpha)^2\right]. \tag{9.13}$$

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Although this is an equation for the total kinetic energy is perfectly general, considerable simplification will result if we choose the origin of the body coordinate system to coincide with the centre of mass. With this choice, the second term on the right hand side of the last of equations (9.6) can be seen to vanish from

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$$\begin{aligned} (\varepsilon_{ijk}\omega_j x_{\alpha,k})(\varepsilon_{imn}\omega_m x_{\alpha,n}) &= \varepsilon_{ijk}\varepsilon_{imn}\omega_j x_{\alpha,k}\omega_m x_{\alpha,n} \\ &= (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})\omega_j x_{\alpha,k}\omega_m x_{\alpha,n} \\ &= \omega_j\omega_j x_{\alpha,k} x_{\alpha,k} - \omega_j x_{\alpha,j}\omega_k x_{\alpha,k}. \end{aligned} \tag{9.12}$$

Inserting this result in the equation for T_{rot} in equation (9.10) we get

$$T_{\text{rot}} = \frac{1}{2}\sum_{\alpha} m_{\alpha}\left[\omega^2 r_{\alpha}^2 - (\boldsymbol{\omega}\cdot\mathbf{r}_{\alpha})^2\right]. \tag{9.13}$$

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Alternatively, keeping with the tensor notation we have

$$\begin{aligned}T_{\text{rot}} &= \frac{1}{2}\sum_{\alpha} m_{\alpha}\left[\omega_j\omega_j x_{\alpha,k}x_{\alpha,k} - \omega_i x_{\alpha,i}\omega_j x_{\alpha,j}\right] \\ &= \frac{1}{2}\sum_{\alpha} m_{\alpha}\left[\left(\omega_i\omega_j\delta_{ij}\right)x_{\alpha,k}x_{\alpha,k} - \omega_i x_{\alpha,i}\omega_j x_{\alpha,j}\right] \\ &= \frac{1}{2}\left(\omega_i\omega_j\right)\sum_{\alpha} m_{\alpha}\left[\delta_{ij}x_{\alpha,k}x_{\alpha,k} - x_{\alpha,i}x_{\alpha,j}\right].\end{aligned} \tag{9.14}$$

We now define the components I_{ij} of the so-called inertia tensor $\{\mathbf{I}\}$ by

$$I_{ij} = \sum_{\alpha} m_{\alpha}\left[\delta_{ij}x_{\alpha,k}x_{\alpha,k} - x_{\alpha,i}x_{\alpha,j}\right] \tag{9.15}$$

and the rotational kinetic energy becomes

$$\boxed{T_{\text{rot}} = \frac{1}{2}I_{ij}\omega_i\omega_j} \tag{9.16}$$

or in vector notation

$$T_{\text{rot}} = \frac{1}{2}\boldsymbol{\omega}\cdot\{\mathbf{I}\}\cdot\boldsymbol{\omega} \tag{9.17}$$

For our purposes it will be usually sufficient to treat the inertia tensor as a regular 3×3 matrix. Indeed, we can explicitly write $\{\mathbf{I}\}$ using equation (9.15) as

$$\{\mathbf{I}\} = \begin{Bmatrix} \sum_{\alpha} m_{\alpha}\left(x_{\alpha,2}^2 + x_{\alpha,3}^2\right) & -\sum_{\alpha} m_{\alpha}x_{\alpha,1}x_{\alpha,2} & -\sum_{\alpha} m_{\alpha}x_{\alpha,1}x_{\alpha,3} \\ -\sum_{\alpha} m_{\alpha}x_{\alpha,2}x_{\alpha,1} & \sum_{\alpha} m_{\alpha}\left(x_{\alpha,1}^2 + x_{\alpha,3}^2\right) & -\sum_{\alpha} m_{\alpha}x_{\alpha,2}x_{\alpha,3} \\ -\sum_{\alpha} m_{\alpha}x_{\alpha,3}x_{\alpha,1} & -\sum_{\alpha} m_{\alpha}x_{\alpha,3}x_{\alpha,2} & \sum_{\alpha} m_{\alpha}\left(x_{\alpha,1}^2 + x_{\alpha,2}^2\right) \end{Bmatrix} \tag{9.18}$$

It is easy to see from either equation (9.15) or equation (9.18) that the inertia tensor is symmetric, that is.

$$I_{ij} = I_{ji}. \tag{9.19}$$

The diagonal elements I_{11}, I_{22}, and I_{33} are called the **moments of inertia** about the x_1-, x_2-, and x_3-axes, respectively. The negatives of the off-diagonal elements are the

$\alpha_1 - x$
$\alpha_2 - y$
$\alpha_3 - z$

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products of inertia. Finally, in most cases the rigid body is continuous and not made up of discrete particles as was assumed so far, but the results are easily generalized by replacing the summation by a corresponding integral in the expression for the components of the inertia tensor

$$I_{ij} = \int_V \rho(\mathbf{r})\left(\delta_{ij}x_k x_k - x_i x_j\right)dx_1 dx_2 dx_3, \tag{9.20}$$

where $\rho(\mathbf{r})$ is the mass density at the position $\mathbf{r}$, and the integral is to be performed over the whole volume V of the rigid body.

Example

Calculate the inertia tensor for a homogeneous cube of density ρ, mass M, and side length b. Let one corner be at the origin, and three adjacent edges lie along the coordinate axes (see Figure 9-1).

Solution.

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$$\begin{aligned} I_{11} &= \int_0^b \int_0^b \int_0^b \rho\left(x_2^2 + x_3^2\right)dx_1 dx_2 dx_3 \\ &= \rho\int_0^b dx_3 \int_0^b dx_2\left(x_2^2 + x_3^2\right)\int_0^b dx_1 \\ &= \rho b\int_0^b dx_3\left(\frac{b^3}{3} + bx_3^2\right) = \rho b\left(\frac{b^4}{3} + \frac{b^4}{3}\right) \\ &= \frac{2}{3}\rho b^5 = \frac{2}{3}Mb^2. \end{aligned} \tag{9.21}$$

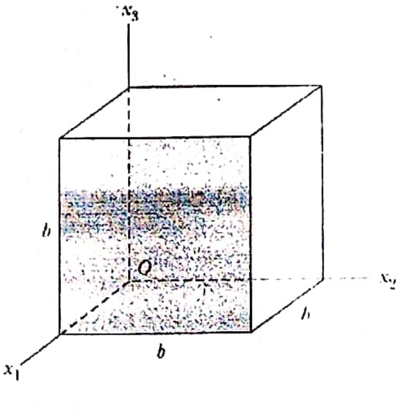

Figure 9-1 – A homogeneous cube of sides b with the origin at one corner.

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And for the negative of the products of inertia

$$
\begin{aligned}
I_{12} &= -\int_0^b \int_0^b \int_0^b \rho x_1 x_2 \, dx_1 dx_2 dx_3 \\
&= -\rho \left(\frac{b^2}{2}\right)\left(\frac{b^2}{2}\right)(b) \qquad (9.22) \\
&= -\frac{1}{4}\rho b^5 = -\frac{1}{4} M b^2 .
\end{aligned}
$$

It should be noted that in this example the origin of the coordinate system is not located at the centre of mass of the cube.

9.4 Angular Momentum

Going back to the case of a rigid body composed of a discrete number of particles: we can calculate the angular momentum with respect to some point O fixed in the body coordinate system with

$$\mathbf{L} = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{p}_\alpha . \qquad (9.23)$$

Relative to the body coordinate system the linear momentum of the αth particle is

$$\mathbf{p}_\alpha = m_\alpha \mathbf{v}_\alpha = m_\alpha \boldsymbol{\omega} \times \mathbf{r}_\alpha , \qquad (9.24)$$

and the total angular momentum becomes

$$\mathbf{L} = \sum_\alpha m_\alpha \mathbf{r}_\alpha \times (\boldsymbol{\omega} \times \mathbf{r}_\alpha). \qquad (9.25)$$

Resorting one more time to tensor notation we can calculate $\mathbf{r}_\alpha \times (\boldsymbol{\omega} \times \mathbf{r}_\alpha)$ as

$$
\begin{aligned}
\varepsilon_{ijk} x_{\alpha,j} \varepsilon_{klm} \omega_l x_{\alpha,m} &= \varepsilon_{kij} \varepsilon_{klm} x_{\alpha,j} \omega_l x_{\alpha,m} \\
&= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) x_{\alpha,j} \omega_l x_{\alpha,m} \qquad (9.26) \\
&= x_{\alpha,j} x_{\alpha,j} \omega_i - x_{\alpha,j} \omega_j x_{\alpha,i} ,
\end{aligned}
$$

or alternatively in vector notation

$$\mathbf{r}_\alpha \times (\boldsymbol{\omega} \times \mathbf{r}_\alpha) = r_\alpha^2 \boldsymbol{\omega} - \mathbf{r}_\alpha (\mathbf{r}_\alpha \cdot \boldsymbol{\omega}). \qquad (9.27)$$

Then, the total angular momentum is given by

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$$\mathbf{L} = \sum_\alpha m_\alpha \left[r_\alpha^2 \boldsymbol{\omega} - \mathbf{r}_\alpha (\mathbf{r}_\alpha \cdot \boldsymbol{\omega}) \right] \tag{9.28}$$

Using the tensor notation the component of the angular momentum is

$$\begin{aligned} L_i &= \sum_\alpha m_\alpha \left(x_{\alpha,k} x_{\alpha,k} \omega_i - x_{\alpha,j} \omega_j x_{\alpha,i} \right) \\ &= \omega_j \left(\sum_\alpha m_\alpha \left(\delta_{ij} x_{\alpha,k} x_{\alpha,k} - x_{\alpha,j} x_{\alpha,i} \right) \right) \end{aligned} \tag{9.29}$$

and upon using equation (9.15) for the inertia tensor

$$L_i = I_{ij} \omega_j \tag{9.30}$$

or in tensor notation

$$\mathbf{L} = \{\mathbf{I}\} \cdot \boldsymbol{\omega} \tag{9.31}$$

Finally, we can insert equation (9.31) for the angular momentum vector into equation (9.17) for the rotational kinetic energy to obtain

$$T_{\text{rot}} = \frac{1}{2} \mathbf{L} \cdot \boldsymbol{\omega} \tag{9.32}$$

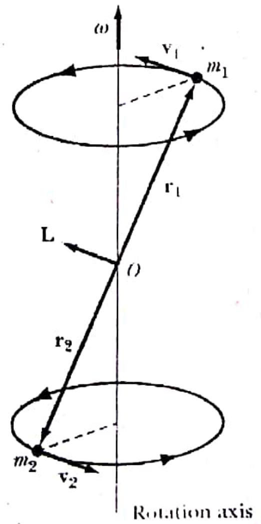

Figure 9-2 – A dumbbell connected by masses m_1 and m_2 at the ends of its shaft. Note that the angular velocity $\boldsymbol{\omega}$ is not directed along the shaft.

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Example

The dumbbell. A dumbbell is connected by two masses m_1 and m_2 located at distances r_1 and r_2 from the middle of the shaft, respectively. The shaft makes an angle θ with a vertical axis, to which it is attached at its middle (i.e., the middle of the shaft). Calculate the equation of angular motion if the system is forced to rotate about the vertical axis with a constant angular velocity ω (see Figure 9-2).

Solution. We define the inertial and the body coordinate systems such that their respective origins are both connected at the point of junction between the vertical axis and the shaft of the dumbbell. We further define the body coordinate system as having its x_3-axis orientated along the shaft and its x_1-axis perpendicular to the shaft but located in the plane defined by the axis of rotation and the shaft. The remaining x_2-axis is perpendicular to this plane and completes the coordinate system attached to the rigid body. For the inertial system, we chose the x_3'-axis to be the vertical, and the other two axes such that the basis vectors can expressed as

$$
\begin{aligned}
\mathbf{e}_1' &= \cos(\theta)\cos(\omega t)\mathbf{e}_1 - \sin(\omega t)\mathbf{e}_2 - \sin(\theta)\cos(\omega t)\mathbf{e}_3 \\
\mathbf{e}_2' &= \cos(\theta)\sin(\omega t)\mathbf{e}_1 + \cos(\omega t)\mathbf{e}_2 - \sin(\theta)\sin(\omega t)\mathbf{e}_3 \\
\mathbf{e}_3' &= \sin(\theta)\mathbf{e}_1 + \cos(\theta)\mathbf{e}_3.
\end{aligned}
\tag{9.33}
$$

From equation (9.15), we can evaluate the components of the inertia tensor. We can in the first time identify the components that are zero (because $x_{1,1} = x_{1,2} = x_{2,1} = x_{2,2} = 0$)

$$I_{12} = I_{21} = I_{13} = I_{31} = I_{23} = I_{32} = I_{33} = 0. \tag{9.34}$$

The only two remaining components are

$$
\begin{aligned}
I_{11} = I_{22} &= m_1 x_{1,3}^2 + m_2 x_{2,3}^2 \\
&= m_1 r_1^2 + m_2 r_2^2.
\end{aligned}
\tag{9.35}
$$

The components of the angular velocity in the coordinate of the rigid body are

$$
\begin{aligned}
\omega_1 &= \omega\sin(\theta) \\
\omega_2 &= 0 \\
\omega_3 &= \omega\cos(\theta).
\end{aligned}
\tag{9.36}
$$

Inserting equations (9.35) and (9.36) in equation (9.30) we find for the L_1 component

$$L_1 = I_{1i}\omega_i = I_{11}\omega_1 = \omega\sin(\theta)\left(m_1 r_1^2 + m_2 r_2^2\right), \tag{9.37}$$

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and for the L_2 and L_3

$$L_2 = I_{2i}\omega_i = I_{22}\omega_2 = 0$$
$$L_3 = I_{3i}\omega_i = I_{33}\omega_3 = 0. \tag{9.38}$$

It should be noted from equations (9.36) and (9.37) that the angular velocity and the angular momentum do not point in the same direction. To calculate the equations of motion, we express the angular momentum with the inertial coordinates instead of the coordinates of the rigid body system. From equation (9.37) we can write

$$\mathbf{L} = \omega \sin(\theta)\left(m_1 r_1^2 + m_2 r_2^2\right)\mathbf{e}_1 = I_{11}\omega \sin(\theta)\mathbf{e}_1, \tag{9.39}$$

but we can transform the basis vector $\mathbf{e}_1$ using equation (9.33) (or its inverse)

$$\mathbf{L} = I_{11}\omega \sin(\theta)\left[\cos(\theta)\cos(\omega t)\mathbf{e}'_1 + \cos(\theta)\sin(\omega t)\mathbf{e}'_2 + \sin(\theta)\mathbf{e}'_3\right]. \tag{9.40}$$

We also know that

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}. \tag{9.41}$$

where $\mathbf{N}$ is the torque. Assuming that the angular speed is constant, we find

$$N_1 = -I_{11}\omega^2 \sin(\theta)\cos(\theta)\sin(\omega t)$$
$$N_2 = I_{11}\omega^2 \sin(\theta)\cos(\theta)\cos(\omega t) \tag{9.42}$$
$$N_3 = 0.$$

An interesting consequence of the fact that the angular momentum and angular velocity vectors are not aligned with each other is that we need to apply a torque to the dumbbell to keep it rotating at a constant angular velocity.

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~~9.5~~ The Principal Axes of Inertia

We now set on finding a set of body axes that will render the inertia tensor diagonal in form. That is, given equation (9.18) for $\{\mathbf{I}\}$, we want to make a change in the body basis vectors (i.e., a change of variables) that will change the form of the inertia tensor to

$$\{\mathbf{I}\} = \begin{Bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{Bmatrix}. \tag{9.43}$$

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9.9 Euler's Equations

To obtain the equations of motion of a rigid body, we can always start with the fundamental equation (see the dumbbell example on page 165)

$$\left(\frac{d\mathbf{L}}{dt}\right)_{\text{fixed}} = \mathbf{N}, \tag{9.97}$$

where $\mathbf{N}$ is the torque, and designation "fixed" is used since this equation can only applied in an inertial frame of reference. We also know from our study of noninertial frames of reference in Chapter 10 that

$$\left(\frac{d\mathbf{L}}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{L}}{dt}\right)_{\text{body}} + \boldsymbol{\omega}\times\mathbf{L}, \tag{9.98}$$

or

$$\left(\frac{d\mathbf{L}}{dt}\right)_{\text{body}} + \boldsymbol{\omega}\times\mathbf{L} = \mathbf{N}. \tag{9.99}$$

Using tensor notation we can write the components of equation (9.99) as

$$\dot{L}_i + \varepsilon_{ijk}\omega_j L_k = N_i. \tag{9.100}$$

Now, *if we chose the coordinate axes for the body frame of reference to coincide with the principal axes of the rigid body*, then we have from equations (9.44)

$$L_1 = I_1\omega_1, \quad L_2 = I_2\omega_2, \quad L_3 = I_3\omega_3. \tag{9.101}$$

Since the principal moments of inertia I_1, I_2, and I_3 are constant with time, we can combine equations (9.100) and (9.101) to get

$$\begin{aligned} I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 &= N_1 \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 &= N_2 \\ I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 &= N_3. \end{aligned} \tag{9.102}$$

Alternatively, we can combine these three equations into one using indices

$$(I_i - I_j)\omega_i\omega_j - \sum_k \varepsilon_{ijk}(I_k\dot{\omega}_k - N_k) = 0 \tag{9.103}$$

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where *no summation* is implied on the i and j indices. Equations (9.103) are the so-called **Euler equations of motion** for a rigid body.

Examples

1. The dumbbell. We return to problem of the rotating dumbbell that we solved earlier (see page 165). Referring to equation (9.39) we found that the angular momentum was given by (using the rigid body coordinate system)

$$\mathbf{L} = \omega \sin(\theta)\left(m_1 r_1^2 + m_2 r_2^2\right)\mathbf{e}_1, \tag{9.104}$$

with the principal moments of inertia from the system given by

$$\begin{aligned} I_1 &= I_2 = m_1 r_1^2 + m_2 r_2^2 \\ I_3 &= 0, \end{aligned} \tag{9.105}$$

and the angular velocity components by

$$\begin{aligned} \omega_1 &= \omega \sin(\theta) \\ \omega_2 &= 0 \\ \omega_3 &= \omega \cos(\theta). \end{aligned} \tag{9.106}$$

Since the system of axes chosen correspond to the principal axes of the dumbbell, then we can apply Euler's equations of motion (i.e., equations (9.102)). With the constraint that $\dot{\omega} = 0$, we find

$$\begin{aligned} N_1 &= 0 \\ N_2 &= I_1 \omega_3 \omega_1 = \left(m_1 r_1^2 + m_2 r_2^2\right)\omega^2 \sin(\theta)\cos(\theta) \\ N_3 &= 0. \end{aligned} \tag{9.107}$$

or

$$\mathbf{N} = \left(m_1 r_1^2 + m_2 r_2^2\right)\omega^2 \sin(\theta)\cos(\theta)\mathbf{e}_2. \tag{9.108}$$

Upon using equations (9.33) we can rewrite the torque in the inertial coordinate system (that shares a common origin with the body system) as

$$\mathbf{N} = \left(m_1 r_1^2 + m_2 r_2^2\right)\omega^2 \sin(\theta)\cos(\theta)\left[-\sin(\omega t)\mathbf{e}_1' + \cos(\omega t)\mathbf{e}_2'\right]. \tag{9.109}$$

This is the same result as what was obtained with equations (9.42) without resorting to Euler's equation.

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