Lecture

Tridiagonal system and its solution

Consider a system of equations

$b_1x_1+c_1x_2 =d_1$

$a_2x_1+b_2x_2+c_2x_3=d_2$

$a_3x_2+b_3x_3+c_3x_4=d_3$

………………………… (1)

………………………....

$a_nx_{n-1}+b_nx_n=d_n$

The matrix coefficient is

(2)

This type of matrix is known as tridiagonal matrix. These types of matrices usually occurs in the solution of ordinary and partial differential equations by finite difference method. The method of factorization can be applied to solve such equations.

As an example, let us take 3x3 matrix.

As before, we can solve.

$l_{21} = a_2$, $l_{21}c_1+u_{22} = b_2$, $l_{32}u_{22} = a_3$, $l_{32}c_2+u_{33} = b_3$

Once the decomposition is complete, forward and backward substitutions give the solution.

Ill Conditioned System

As discussed in previous lecture that in some systems pivoting may not be much helpful.

An example,

Solve $x + y = 0$; $x + (401/400)\ y = 20$ using Gauss elimination method. Round off each intermediate calculation to four significant digits.

Solution:

If we apply rational arithmetic procedure in Gaussian elimination, then we get exact solution $x = -8000$ and $y = 8000$.

However, if we round of $401/400 = 1.0025$ to four significant digits, let see what happens

Hence, $y = 10{,}000$ and by back substitution $x = -y$, $x = -10{,}000$. This error is large and purely as a result of rounding off 1.0025 to 1.002 (four significant digits).

The system in above example is known as an ill-conditioned system.

We come across systems where small changes in the coefficients give large changes in the answer. In such systems, the coefficient matrix is nearly singular and these are called ill-conditioned systems. If, however, small changes in the coefficients result in small changes in the solution then such systems are called well-conditioned.

The degree of ill-conditioning of a system is measured by its **condition number**. The condition number in turn is defined by its matrix norm. We have already discussed matrix norms earlier.

To define condition number, we take product of norm of a matrix A and norm of its inverse. That is

$c(A) =$

$c(A)$ is condition number.

If condition number is small then it is a well-conditioned system. However, if condition number is large then the system is ill-conditioned.

Let us assume $A = [a_{ij}]$ and also, $s_i = [a_{i1}^2 + a_{i2}^2 + a_{i3}^2 + \ldots\ldots\ldots + a_{in}^2]^{1/2}$

Now, we define

If k is very small compared to unity, the system is ill-conditioned.

Example:

$2x + y = 2; 2x + 1.01\ y = 2.01$ (1)

The solution of this system is $x = 0.5$ and $y = 1$.

If we make small changes in coefficients,

$2x + y = 2; 2.01\ x + y = 2.05$ (2)

The solution of this system is $x = 5$ and $y = -8$.

Now, we can write equation (1) as AX=B, where

Its norm,

Therefore, c(A) = = 3.165x158.273 = 500.974

Since c(A) is large hence system is ill-conditioned.

Again, and and

So, $k = 4.468 \times 10^{-3}$

K is very small compared to unity hence system is ill-conditioned.

<u>Method to solve ill-conditioned system</u>

Let the system be

$A_{11}\ x_1 + A_{12}\ x_2 + A_{13}\ x_3 = b_1$

$A_{21}\ x_1 + A_{22}\ x_2 + A_{23}\ x_3 = b_2$ (1)

$A_{31}\ x_1 + A_{32}\ x_2 + A_{33}\ x_3 = b_3$

Let $x1^{(1)}$, $x2^{(2)}$ and $x3^{(3)}$ are approximate solutions.

Substituting these values in equation (1), we get

$$A11\ x1^{(1)} + A12\ x2^{(2)} + A13\ x3^{(3)} = b1^{(1)}$$
$$A21\ x1^{(1)} + A22\ x2^{(2)} + A23\ x3^{(3)} = b2^{(1)} \qquad (2)$$
$$A31\ x1^{(1)} + A32\ x2^{(2)} + A33\ x3^{(3)} = b3^{(1)}$$

Subtracting equation (2) from equation (1), we get

$$A11\ e1 + A12\ e2 + A13\ e3 = c1$$
$$A21\ e1 + A22\ e2 + A23\ e3 = c2 \qquad (3)$$
$$A31\ e1 + A32\ e2 + A33\ e3 = c3$$

Where, $e1 = x1-x1^{(1)}$, etc. and $c1=b1-b1^{(1)}$, etc.

Hence, we obtain $x1=x1^{(1)}+e1$, etc.

This is new approximation of x1, etc. To improve accuracy, we can repeat the procedure.

Example:

$$2x+y =2$$
$$2x+1.01y = 2.01 \qquad (4)$$

Let us assume its approximate solutions are $x^{(1)} = 1$ and $y^{(1)} = 1$

Substituting these values in the equation lead to

$$2\ x^{(1)} + y^{(1)} = 3$$
$$\text{and } 2\ x^{(1)} + 1.01\ y^{(1)} = 3.01 \qquad (5)$$

Subtracting equation (5) from equation (4)

$$2\ (x-x^{(1)}) + (y-y^{(1)}) = 2-3 = -1$$

$2(x-x^{(1)}) + 1.01(y-y^{(1)}) = 2.01-3.01 = -1$

Solving, we get

$x-x^{(1)} = -1/2$ and $y-y^{(1)} = 0$

Thus, $x = ½$ and $y = 1$.

Solution of the system of equations.