Solution to System of Nonlinear Equations

Here, we are considering solution to simultaneous non linear equations by two methods:
(i). Method of iteration (ii). Newton - Raphson method

For simplicity we shall restrict ourselves to two equations in two unknowns.

The method of Iteration

Let the equations of be given by

$$f(x,y)=0 \quad ; \quad g(x,y)=0 \qquad \text{——(1)}$$

whose real roots are required within given accuracy. We write these equations as

$$x = F(x,y) \quad \& \quad y = G(x,y) \qquad \text{——(2)}$$

where F & G satisfy the conditions:

$$\left|\frac{\partial F}{\partial x}\right| + \left|\frac{\partial F}{\partial y}\right| < 1 \quad \& \quad \left|\frac{\partial G}{\partial x}\right| + \left|\frac{\partial G}{\partial y}\right| < 1 \qquad \text{——(3)}$$

in the neighborhood of root.

Let (x_0, y_0) be initial approximations to the root (ξ, η) of the system [eqn. (1)].

Then successive approximations

$$\left.\begin{aligned}
x_1 &= F(x_0, y_0), & y_1 &= G(x_0, y_0) \\
x_2 &= F(x_1, y_1) & y_2 &= G(x_1, y_1) \\
x_3 &= F(x_2, y_2) & y_3 &= G(x_2, y_2) \\
&\vdots & &\vdots \\
x_{n+1} &= F(x_n, y_n) & y_{n+1} &= G(x_n, y_n)
\end{aligned}\right\} \text{——(4)}$$

For faster convergence, recently computed values of x_i may be used in evaluation of y_i in eqn. (4).

If iterative process converges, then we get

$$\xi = F(\xi, \eta) \quad \& \quad \eta = G(\xi, \eta) \qquad \text{——(5)}$$

in the limit. Thus ξ, η are roots of system (2) and hence (1).

Theorem: Let $x = \xi$ and $y = \eta$ be one pair of roots of the system (2) in closed neighborhood R. If F & G and their first partial derivatives are continuous in R,

$$\left|\frac{\partial F}{\partial x}\right| + \left|\frac{\partial F}{\partial y}\right| < 1 \quad \& \quad \left|\frac{\partial G}{\partial x}\right| + \left|\frac{\partial G}{\partial y}\right| < 1 \qquad (6)$$

for all (x, y) in R, and the initial approximation (x_0, y_0) is chosen in R', the sequence of approximations given by eqn. (4) converges to the roots $x = \xi$ & $y = \eta$ of the system eqn. (2).

Ex 1. Find real root of the equations:

$$x = 0.2x^2 + 0.8 \; ; \; y = 0.3xy^2 + 0.7$$

we have

$$F(x, y) = 0.2x^2 + 0.8 \; , \; G(x, y) = 0.3xy^2 + 0.7$$

Then
$$\frac{\partial F}{\partial x} = 0.4x \; , \; \frac{\partial F}{\partial y} = 0$$

$$\frac{\partial G}{\partial x} = 0.3y^2 \; , \; \frac{\partial G}{\partial y} = 0.6xy$$

Let $x_0 = y_0 = \frac{1}{2}$, then

$$\left|\frac{\partial F}{\partial x}\right|_{(x_0,y_0)} + \left|\frac{\partial F}{\partial y}\right|_{(x_0,y_0)} = \frac{0.4}{2} + \frac{0}{2} = 0.2 < 1$$

$$\left|\frac{\partial G}{\partial x}\right|_{(x_0,y_0)} + \left|\frac{\partial G}{\partial y}\right|_{(x_0,y_0)} = 0.6 \times \tfrac{1}{2} \times \tfrac{1}{2} + 0.3\left(\tfrac{1}{4}\right)$$
$$= 0.075 + 0.15 = 0.225 < 1$$

Thus conditions (6) are satisfied, hence,

$$x_1 = F(x_0, y_0) = \frac{0.2}{4} + 0.8 = 0.85$$

$$y_1 = G(x_0, y_0) = \frac{0.3}{8} + 0.7 = 0.74$$

For second approximation, we set

$$x_2 = F(x_1, y_1) = 0.2(0.85)^2 + 0.8 = 0.9445$$

$$y_2 = G(x_1, y_1) = 0.3(0.85)(0.74)^2 + 0.7 = 0.81$$

If we carry like this convergence to roots $(1, 1)$ is obvious. For faster convergence recently computed values of x could be used, e.g.

$$y_1 = 0.3\left(\tfrac{1}{4}\right)(0.85) + 0.7 = 0.764 \; , \text{ etc.}$$

(2)

Newton - Raphson Method

Let (x_0, y_0) be initial approximation to the root of system (2). If (x_0+h) & (y_0+k) are the roots of the system, then we must have

$$f(x_0+h,\ y_0+k) = 0 \quad \& \quad g(x_0+h,\ y_0+k) = 0 \qquad \text{—— (7)}$$

Assuming f & g are sufficiently differentiable, we expand (7) by Taylor's series

$$\left.\begin{aligned} f_0 + h\frac{\partial f}{\partial x_0} + k\frac{\partial f}{\partial y_0} + \cdots &= 0 \\ g_0 + h\frac{\partial g}{\partial x_0} + k\frac{\partial g}{\partial y_0} + \cdots &= 0 \end{aligned}\right\} \text{—— (8)}$$

where

$$\frac{\partial f}{\partial x_0} = \left[\frac{\partial f}{\partial x}\right]_{x=x_0} ; \quad f_0 = f(x_0, y_0), \text{ etc.}$$

Now, neglecting second & higher order terms, we obtain the following system of linear equations:

$$\left.\begin{aligned} h\frac{\partial f}{\partial x_0} + k\frac{\partial f}{\partial y_0} &= -f_0 \\ h\frac{\partial g}{\partial x_0} + k\frac{\partial g}{\partial y_0} &= -g_0 \end{aligned}\right\} \text{—— (9)}$$

If the Jacobian

$$J(f, g) = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} \qquad \text{—— (10)}$$

does not vanish, then linear eqns. (7) possess a unique solution given by

$$h = \frac{1}{J(f,g)}\begin{vmatrix} -f & \frac{\partial f}{\partial y} \\ -g & \frac{\partial g}{\partial y} \end{vmatrix} \quad \& \quad k = \frac{1}{J(f,g)}\begin{vmatrix} \frac{\partial f}{\partial x} & - \\ \frac{\partial g}{\partial x} & -g \end{vmatrix} \qquad \text{—— (11)}$$

The new approximations are given by

$$x_1 = x_0 + h \quad ; \quad y_1 = y_0 + k. \qquad \text{—— (12)}$$

The process is to be repeated till we obtain the roots to the desired accuracy.

Theorem is sufficient conditions for convergence to the root.

Theorem. Let (x_0, y_0) be an approximation to a root (ξ, η) of the system 2 in the closed neighborhood R containing (ξ, η). If (a) f, g and all their first & second derivatives are continuous and bounded in R, and (b) $J(f, g) \neq 0$ in R, then the sequence of approximations given by

$$x_{i+1} = x_i - \frac{1}{J(f,g)} \begin{vmatrix} f & g \\ \frac{\partial f}{\partial y} & \frac{\partial g}{\partial y} \end{vmatrix}$$ and

$$y_{i+1} = y_i - \frac{1}{J(f,g)} \begin{vmatrix} g & f \\ \frac{\partial g}{\partial x} & \frac{\partial f}{\partial x} \end{vmatrix} \quad \text{---(13)}$$

converges to the root (ξ, η) of system 2.

Example: Find a real root of the equations
$x^2 - y^2 = 3$ & $x^2 + y^2 = 13$

Soln. For starting solution let $y = x$ as our first approximation. This gives $(x^2) + (x^2) = 13$

$$x_0 = y_0 = \sqrt{6.5} = 2.54951$$

and so $f_0 = -3$ and $g_0 = 0$

where $f = x^2 - y^2 - 3$ & $g = x^2 + y^2 - 13$

Further;

$$\frac{\partial f}{\partial x_0} = 2x_0 = 5.09902 \; ; \; \frac{\partial g}{\partial x_0} = 2x_0 = 5.09902$$

$$\frac{\partial f}{\partial y_0} = -2y_0 = -5.09902 \; ; \; \frac{\partial g}{\partial y_0} = 2y_0 = 5.09902$$

Here
$$\begin{vmatrix} \frac{\partial f}{\partial x_0} & \frac{\partial f}{\partial y_0} \\ \frac{\partial g}{\partial x_0} & \frac{\partial g}{\partial y_0} \end{vmatrix} = \begin{vmatrix} 5.09902 & -5.09902 \\ 5.09902 & 5.09902 \end{vmatrix} \neq 0$$

Thus, convergence criteria is satisfied.

Contd.

Contd. (5)

We have equations;

$$\left.\begin{aligned} h\frac{\partial f}{\partial x_0} + k\frac{\partial f}{\partial y_0} &= -f_0 \\ h\frac{\partial g}{\partial x_0} + k\frac{\partial g}{\partial y_0} &= -g_0 \end{aligned}\right\}$$

$$h(5.09902) + k(-5.09902) = -(-3)$$

$$h(5.09902) + k(5.09902) = -(0)$$

$$\therefore \quad h(5.09902) + k(-5.09902) = 3$$

$$h(5.09902) + k(5.09902) = 0$$

These equations give

$$h = 0.29417 \quad \& \quad k = -0.29417$$

Hence, first approximation to the root is given by

$$x_1 = x_0 + h = 2.54951 + 0.29417 = 2.84368$$

$$y_1 = y_0 + k = 2.54951 - 0.29417 = 2.25534$$

For second approximation, we have

$$8.086516 - 5.086559 - 3$$

$$f_1 = f(x_1, y_1) = -0.000042573$$

$$g_1 = g(x_1, y_1) = 0.17307458$$

Then

$$\frac{\partial f}{\partial x_1} = 2x_1 = 5.68736; \quad \frac{\partial f}{\partial y_1} = -2y_1 = -4.51068$$

$$\frac{\partial g}{\partial x_1} = 2x_1 = 5.68736; \quad \frac{\partial g}{\partial y_1} = 2y_1 = 4.51068$$

Again

$$\begin{vmatrix} 5.68736 & -4.51068 \\ 5.68736 & 4.51068 \end{vmatrix} \neq 0$$

Condition for convergence is satisfied, so

$$h(5.68736) + k(-4.51068) = 0.000042573$$

$$\& \quad h(5.68736) + k(4.51068) = -0.17307$$

Solving them we obtain

$$h = -0.01521 \quad \& \quad k = -0.01919$$

Second approximation, thus, is

$$x_2 = \underbrace{2.84368 - 0.01521}_{x_1 + h} = 2.82847$$

$$y_2 = \underbrace{2.25534 - 0.01919}_{y_1 + k} = 2.23615$$

The true values are

$$x = \sqrt{8} = 2.82843 \quad \& \quad y = \sqrt{5} = 2.23607$$

problem:

$$\left.\begin{array}{l} x^2 - y^2 = 4, \quad x^2 + y^2 = 16 \\ x^2 + y = 11 \; ; \quad y^2 + x = 7 \end{array}\right\}$$