Lecture

Interpolation formulae with unequal intervals:

So far various interpolation formulae been discussed where we considered evenly spaced points.

Now, we shall study some formulae which consider unequally spaced intervals.

Lagrange's formula:

Let is consider y_0, y_1, y_2,………., y_n are the values corresponding to f(x) at x_0, x_1, x_2,….., x_n. Then, an interpolating polynomial g(x) for f(x) is given by,

$$g(x) = \frac{(x-x_1)(x-x_2)\dots\dots..(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots\dots..(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\dots\dots..(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots\dots..(x_1-x_n)}y_1 + \frac{(x-x_0)(x-x_1)\dots\dots..(x-x_n)}{(x_2-x_0)(x_2-x_1)\dots\dots..(x_2-x_n)}y_2 + \cdots\dots\dots. + \frac{(x-x_1)(x-x_2)\dots\dots..(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots\dots..(x_n-x_{n-1})}y_n$$

It is simple to prove this formula.

Let us consider (n+1) points are given for f(x), so we consider a polynomial g(x) of n degree,

$$g(x) = a_0(x-x_1)(x-x_2)\dots\dots(x-x_n) + a_1(x-x_0)(x-x_2)\dots\dots(x-x_n) + \cdots\dots\dots\dots + a_n(x-x_0)(x-x_1)\dots\dots(x-x_{n-1}) \qquad (1)$$

Let x = x_0, then we get $y = y_0 = a_0(x_0-x_1)(x_0-x_2)\dots\dots(x_0-x_n)$

$$\text{So, } a_0 = \frac{y_0}{(x_0-x_1)(x_0-x_2)\dots\dots(x_0-x_n)} \qquad (2)$$

Similarly, putting x = x_1, gives

$$a_1 = \frac{y_1}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)} \tag{3}$$

Continuing, for x = x_n, we get

$$a_n = \frac{y_n}{(x_n-x_1)(x_n-x_2)\ldots\ldots(x_n-x_{n-1})} \tag{4}$$

Substituting values for a_0, a_1, a_2, a_3,……., a_n in equation (1), we get

$$g(x) = \frac{(x-x_1)(x-x_2)\ldots\ldots..(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots..(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\ldots\ldots..(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots..(x_1-x_n)}y_1 + \frac{(x-x_0)(x-x_1)\ldots\ldots..(x-x_n)}{(x_2-x_0)(x_2-x_1)\ldots\ldots..(x_2-x_n)}y_2 + \cdots\ldots\ldots. + \frac{(x-x_1)(x-x_2)\ldots\ldots..(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots\ldots..(x_n-x_{n-1})}y_n$$

The Lagrange's polynomial pass through each point. If x = x_2, then second term vanishes because numerator is zero in that case.

Example:

x	3	7	9	10
y	168	120	72	63

Find value of y at x = 6.

Solution:

As we can see that intervals are unequal in this case,

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

Here, $x_0 = 3$, $x_1 = 7$, $x_2 = 9$, $x_3 = 10$ and $y_0 = 168$, $y_1 = 120$, $y_2 = 72$, $y_3 = 63$

For x = 6, value of y is

$$y(6) = \frac{(6-7)(6-9)(6-10)}{(3-7)(3-9)(3-10)}168 + \frac{(6-3)(6-9)(6-10)}{(7-3)(7-9)(7-10)}120 + \frac{(6-3)(6-7)(6-10)}{(9-3)(9-7)(9-10)}72 + \frac{(6-3)(6-7)(6-9)}{(10-3)(10-7)(10-9)}63$$

y(6) = 12+180-72+27 = 147

Example:

Find a polynomial using Lagrange's formula for the table

x	0	1	3	4
y	-12	0	6	12

Hence find y(2).

Solution:

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

Here, $x_0 = 0$, $x_1 = 1$, $x_2 = 3$, $x_3 = 4$ and $y_0 = -12$, $y_1 = 0$, $y_2 = 6$, $y_3 = 12$

$$y(x) = \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)}(-12) + \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)}(0) + \frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)}(6) + \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)}(12)$$

$y(x) = x^3-7x^2+18x-12$

y(2) = 4

Example:

x	75	80	85	90
y	246	202	118	40

Find y(79).

Solution:

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$y(6) = \frac{(79-80)(79-85)(79-90)}{(75-80)(75-85)(75-90)}246 + \frac{(79-75)(79-85)(79-90)}{(80-75)(80-85)(80-90)}202 + \frac{(79-75)(79-80)(79-90)}{(85-75)(85-80)(85-90)}118 + \frac{(79-75)(79-80)(79-85)}{(90-75)(90-80)(90-85)}40$$

$$y(6) = \frac{(-1)(-6)(-11)}{(-5)(-10)(-15)}246 + \frac{(4)(-6)(-11)}{(5)(-5)(-10)}202 + \frac{(4)(-1)(-11)}{(10)(5)(-5)}118 + \frac{(4)(-1)(-6)}{(15)(10)(5)}40$$

y(6) = 21.648+213.312-20.768+1.28 = 215.472

Important to note here that intervals in above example is same. If you can recall, we have solved this problem using Newton's formula and obtained same result.

Thus, we come to the conclusion that Lagrange's formula can be used both for equal as well as unequal intervals.

Let us suppose, we have to add or subtract a point in the table. For instance, in the above example we have found out value of y(79) and we want to include this point for the interpolation of some other point, say y(82). Once, we insert this point for the interpolation of another value we need to start calculations all over again. This is tedious job and it is a major disadvantage of this formula. We want a formula in which the newly inserted point comes merely as an additional term. This what we are going to study next.

Divided differences:

We consider (n+1) set of points (x_0,y_0), (x_1,y_1),…….,(x_n,y_n). The first divided differences are defined as:

$[x_0, x_1] = \frac{y_1-y_0}{x_1-x_0}$, $[x_1, x_2] = \frac{y_2-y_1}{x_2-x_1}$, ………..,$[x_n, x_{n-1}] = \frac{y_n-y_{n-1}}{x_n-x_{n-1}}$

The second divided differences:

$[x_0, x_1, x_2] = \frac{[x_1,x_2]-[x_0,x_1]}{x_2-x_0}$, etc.

Third divided differences,

$[x_0, x_1, x_2, x_3] = \frac{[x_1,x_2,x_3]-[x_0,x_1,x_2]}{x_3-x_0}$, and so on.

We may construct divided difference table as

x	y	$[x_i, x_{i+1}]$	$[x_i, x_{i+1}, x_{i+2}]$	$[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
x_0	y_0			
		$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$		
x_1	y_1		$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$	
		$[x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1}$		$[x_0, x_1, x_2, x_3] = \frac{[x_1,x_2,x_3]-[x_0,x_1,x_2]}{x_3-x_0}$
x_2	y_2		$[x_1, x_2, x_3] = \frac{[x_2, x_3] - [x_1, x_2]}{x_3 - x_1}$	
		$[x_2, x_3] = \frac{y_3 - y_3}{x_3 - x_2}$		
x_3	y_3			

Example:

x	y	$[x_i, x_{i+1}]$	$[x_i, x_{i+1}, x_{i+2}]$	$[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$	$[x_i, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}]$
-1	0				
		$\frac{1 - 0}{0 - (-1)} = 1$			
0	1		$\frac{4 - 1}{2 - (-1)} = 1$		
		$\frac{9 - 1}{2 - 0} = 4$		$\frac{6-1}{4-(-1)} = 1$	
2	9		$\frac{28 - 4}{4 - 0} = 6$		$\frac{1 - 1}{5 - (-1)} = 0$
		$\frac{65 - 9}{4 - 2} = 28$		$\frac{11 - 6}{5 - 0} = 1$	
4	65		$\frac{61 - 28}{5 - 2} = 11$		
		$\frac{126 - 65}{5 - 4} = 61$			
5	126				

Properties of divided differences:

1. The divided differences for equal arguments.

Proof:

Let us set, $x_1 = x_0 + \epsilon$, so that

$$[x_0, x_0] = \lim_{\epsilon \to 0} \frac{[x_0, x_0 + \varepsilon]}{\epsilon}$$

$$[x_0, x_0] = \lim_{\epsilon \to 0} \frac{y(x_0 + \epsilon) - y(x_0)}{\epsilon}$$

$[x_0, x_1] = y'(x_0)$, if differentiation of y(x) exists.

Similarly for (q+1) arguments,

$$[x_0, x_0, x_0, \dots\dots, x_0] = \frac{y^q(x_0)}{q!}$$

2. The divided differences are independent of order of arguments.

Proof:

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_0 - y_1}{x_0 - x_1} = [x_1, x_0]$$

Similarly can be shown,

$$[x_0, x_1, x_2] = [x_1, x_2, x_0] = [x_2, x_0, x_1]$$

We can observe,

$$[x_0, x_1] = \frac{y_0}{x_0 - x_1} + \frac{y_1}{x_1 - x_0}$$

Now,

$$[x_0, x_1, x_2] = \frac{1}{x_2 - x_0}\left(\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}\right)$$

$$= \frac{1}{x_2 - x_0}\left(\frac{y_2}{x_2 - x_1} - y_1\left\{\frac{1}{x_2 - x_1} + \frac{1}{x_1 - x_0}\right\} + \frac{y_0}{x_1 - x_0}\right)$$

$$= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

We can similarly show for other divided differences.

3. Consider, the points are equally spaced, then $x_1-x_0 = x_2-x_1=\ldots\ldots\ldots= h$

So, the first divided difference is

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0} = \frac{1}{2h}\left(\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}\right) = \frac{\Delta^2 y_0}{2h^2} = \frac{\Delta^2 y_0}{2!\,h^2}$$

Therefore,

$$[x_0, x_1, x_2, \ldots\ldots, x_n] = \frac{[x_1, x_2, \ldots\ldots, x_n] - [x_0, x_1, \ldots\ldots, x_{n-1}]}{x_n - x_0} = \frac{\Delta^n y_0}{n!\,h^n}$$

4. nth divided differences of a polynomial of n degree are constant.

Proof:

If we take first divided difference of x^n,

$$= \frac{(x+\epsilon)^n - x^n}{\epsilon} = \frac{n\epsilon x^{n-1} + \cdots\ldots}{\epsilon}$$

We obtain a polynomial of degree (n-1). Hence, if polynomial is of n degree then its nth divided difference will be constant.

<u>Newton's divided difference formula:</u>

We have been given set of points $(x_0,f(x_0)), (x_1,f(x_1)),\ldots\ldots,(x_n,f(x_n))$.

By definition,

$$[x, x_0] = \frac{f(x) - f(x_0)}{x - x_0} \tag{1}$$

$$\therefore \quad f(x) = f(x_0) + (x - x_0)[x, x_0] \tag{2}$$

We can now write,

$$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1} \tag{3}$$

$$\therefore \quad [x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1] \tag{4}$$

Substituting equation (4) in equation (2), we get

$f(x) = f(x_0) + (x - x_0)([x_0, x_1] + (x - x_1)[x, x_0, x_1])$

Or, $f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1]$

Similarly, we can carry out to obtain

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots + (x - x_0)(x - x_1) \ldots \ldots (x - x_{n-1})[x_0, x_1, \ldots \ldots, x_n] \qquad (5)$$

The divided difference method involves less number of arithmetic operations compared to Lagrange's formula.

Example:

Find f(8), using data given.

x	f(x)				
4	48				
		52			
5	100		15		
		97		1	
7	294		21		0
		202		1		0
10	900		27		0
		310		1	
11	1210		33		
		409			
13	2028				

f(8) = 48+(8-4)[4,5]+(8-4)(8-5)[4,5,7]+(8-4)(8-5)(8-7)[4,5,7,10]+(8-4)(8-5)(8-7)(8-10)[4,5,7,10,11]+(8-4)(8-5)(8-7)(8-10)(8-11)[4,5,7,10,11,13]

f(8) = 48+208+180+12+0+0

= 448

Example:

Find equation for the given data.

x	f(x)			
4	-43			
		42		
7	83		16	
		172		1
9	327		24	
		242		
12	1053			

f(x) = -43+(x-4)[4,7]+(x-4)(x-7)[4,7,9]+(x-4)(x-7)(x-9)[4,7,9,12]

= -43+(x-4)42+(x-4)(x-7)16+(x-4)(x-7)(x-9)1

= x^3-4x^2-7x-15

For example at x = 10, f(10) = 515.

Example:

Using following data, find y(9).

x	5	7	11	13	17
y	150	392	1452	2366	5202

Lagrange's method:

$$y(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)}150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)}392 + \frac{(9-5)(9-7)(9-13)((9-17)}{(11-5)(11-7)(11-13)(11-17)}1452 + \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)}2366 + \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)}5202$$

y(9) = 810

x	y(x)				
5	150				
		121			
7	392		24		
		265		1	
11	1452		32		0
		457		1	
13	2366		42		
		709			
17	5202				

y(9) = 150+(9-5)121+(9-5)(9-7)24+(9-5)(9-7)(9-11)1+(9-5)(9-7)(9-11)(9-13)0

= 810

We obtain same result.

It is required to be emphasized here that Newton's divided difference formula and Lagrange's formula are identical. Lagrange's formula is useful in interpolation with computers and Newton's formula is useful in manual calculations.

Their equivalence can be proved easily

Derivation of Newton's forward difference formula from Newton's divided difference formula

Let $x_i = x_0 + ih$, where i =0,1,2,…, n,

We have earlier seen for such points that

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h} \qquad (1)$$

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0} = \frac{1}{2h}\left(\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}\right) = \frac{\Delta^2 y_0}{2h^2} = \frac{\Delta^2 y_0}{2!h^2} \qquad (2)$$

Therefore,

$$[x_0, x_1, x_2, \dots\dots, x_n] = \frac{[x_1, x_2, \dots\dots, x_n] - [x_0, x_1, \dots\dots, x_{n-1}]}{x_n - x_0} = \frac{\Delta^n y_0}{n!h^n} \qquad (3)$$

Newton's divided difference formula is

$$y(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \cdots \dots\dots\dots\dots\dots\dots\dots\dots + (x - x_0)(x - x_1)\dots\dots(x - x_{n-1})[x_0, x_1, \dots\dots, x_n] \qquad (4)$$

Substituting values from equations (1), (2), (3), etc. in equation (4), we obtain

$$y(x) = y_0 + (x - x_0)\frac{\Delta y_0}{h} + (x - x_0)(x - x_1)\frac{\Delta^2 y_0}{2!h^2} + \cdots \dots\dots\dots\dots\dots\dots\dots\dots + (x - x_0)(x - x_1)\dots\dots(x - x_{n-1})\frac{\Delta^n y_0}{n!h^n} \qquad (5)$$

Now let,

$$x = x_0 + ph \qquad \text{so} \qquad x - x_0 = ph \qquad (6)$$

Similarly, $x - x_1 = x_0 - x_1 + ph = (p - 1)h$ (7)

$x - x_2 = x_0 - x_2 + ph = (p - 2)h$ and so on. (8)

Substituting equations (6), (7), (8), etc.in equation (5), we get

$$y(x) = y_0 + (ph)\frac{\Delta y_0}{h} + (ph)(p - 1)h\frac{\Delta^2 y_0}{2!\,h^2} + \cdots \dots\dots\dots\dots\dots\dots\dots\dots + (ph)(p - 1)h\dots\dots(p - n + 1)h\frac{\Delta^n y_0}{n!\,h^n}$$

Thus,

$$y(x) = y_0 + (ph)\frac{\Delta y_0}{h} + (ph)(p-1)h\frac{\Delta^2 y_0}{2!h^2} + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots + (ph)(p-1)h \ldots \ldots (p-n+1)h\frac{\Delta^n y_0}{n!h^n} \quad (6)$$

This Newton-Gregory forward difference formula.