Jharkhand university of Technology

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Evolutes & Involutes

The locus of the centre of curvature

1) Find the eqn of the evolute of the ellispe $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

solⁿ $x = a\cos\theta$

$y = b\sin\theta$

$$\frac{dx}{d\theta} = -a\sin\theta \quad \& \quad \frac{dy}{d\theta} = b\cos\theta$$

$$\frac{dy}{dx} = -\frac{b}{a}\cot\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{b}{a}\cot\theta\right)$$

$$\cancel{= \frac{d}{d\theta}\left(-\frac{b}{a}\,\mathrm{cosec}^2\theta\right) \times \frac{1}{-a\sin\theta}} \quad \cancel{= -\frac{b}{a^2}\,\mathrm{cosec}^3\theta}$$

~~we know that~~

$$= \frac{d}{d\theta}\left(-\frac{b}{a}\cot\theta\right)\frac{d\theta}{dx}$$

$$= -\frac{b}{a}\left(-\mathrm{cosec}^2\theta\right) \times \frac{1}{-a\sin\theta} = -\frac{b}{a^2}\,\mathrm{cosec}^3\theta$$

$$\therefore\ \alpha = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= a\cos\theta - \frac{\left(-\frac{b}{a}\cot\theta\right)\left(1+\frac{b^2}{a^2}\cot^2\theta\right)}{-\frac{b}{a^2}\,\mathrm{cosec}^3\theta}$$

$$= a\cos\theta - \frac{\frac{b}{a}\cdot\frac{\cos\theta}{\sin\theta}\left(\frac{(a^2+b^2)}{a^2}\times\frac{\cos^2\theta}{\sin^2\theta}\right)}{-\frac{b}{a^2}\,\mathrm{cosec}^3\theta}$$

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$$= \frac{a\cos\theta - \frac{b}{a}\frac{\cos\theta}{\sin\theta}\left(1 + \frac{b^2\cos^2\theta}{a^2\sin^2\theta}\right)}{\frac{b}{a^2}\operatorname{cosec}^3\theta}$$

$$= a\cos\theta - \frac{1}{a}\cos\theta\,\frac{(a^2\sin^2\theta + b^2\cos^2\theta)}{a^2\sin^2\theta} \times \sin^3\theta \times a^2.$$

$$= a\cos\theta - \frac{1}{a}\cos\theta\,(a^2\sin^2\theta + b^2\cos^2\theta)\,\cancel{\sin\theta}$$

$$= \cos\theta\left[a - \frac{1}{a}(a^2\sin^2\theta + b^2\cos^2\theta)\right]$$

$$= \frac{1}{a}\cos\theta\left[a^2(1-\sin^2\theta) - b^2\cos^2\theta\right]$$

$$= \frac{1}{a}\cos\theta\left[a^2\cos^2\theta - b^2\cos^2\theta\right]$$

$$\alpha = \frac{1}{a}\cos^3\theta\,(a^2 - b^2)$$

$$\beta = \frac{-(a^2-b^2)}{b}\sin^3\theta.$$

Q Find the co-ordinates of the centre of curvature at the pt. $(at^2, 2at)$ on the parabola $y^2 = 4ax$

Soln Here, $x = at^2, \quad y = 2at$

$$\therefore \frac{dx}{dt} = 2at, \quad \frac{dy}{dt} = 2a$$

$$\therefore y_1 = \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$$

$$y_2 = \frac{d}{dt}\left(\frac{1}{t}\right)\cdot\frac{dt}{dx} = \frac{-1}{t^2}\cdot\frac{1}{2at} = \frac{-1}{2at^3}$$

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Let (α, β) be the co-ordinate of the centre of curvature then

$$\alpha = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= at^2 - \frac{\frac{1}{t}\left(1+\frac{1}{t^2}\right)}{\frac{-1}{2at^3}}$$

$$= at^2 + \frac{1}{\cancel{t^2}}(t^2+1) \times 2a\,\cancel{t^3}$$

$$= at^2 + 2at^2 + 2a$$

$$= 3at^2 + 2a = a(3t^2+2) \rightarrow (1)$$

$$\beta = y + \frac{1+y_1^2}{y_2}$$

$$= 2at + \frac{\left(1+\frac{1}{t^2}\right)}{\frac{-1}{2at^3}}$$

$$= 2at - \frac{(t^2+1)}{\cancel{t^2}} \times 2at^{\cancel{3}}$$

$$\cancel{= 2at}$$

$$= -2at^3$$

Hence, the co-ordinate of centre of curvature is $(\alpha, \beta) =$

$$\{a(3t^2+2), -2at^3\}$$

$\cancel{= at^2 + 2a}$

To find evolute we eliminate t from (1)

$$\alpha - 2a = 3at^2$$

$$\text{or } (\alpha - 2a)^3 = 27a^3t^6$$

$$= 27a^3 \times \frac{\beta^2}{4a^2} = \frac{27}{4}a\beta^2$$

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$\therefore$ Locus of (α, β) i.e. evolute is

$$4(x-2a)^3 = 27ay^2$$

Q Find the evolute of the cycloid

$$x = a(\theta + \sin\theta) \ \& \ y = a(1-\cos\theta)$$

Sol: Given curves are,

$$x = a(\theta + \sin\theta), \ y = a(1-\cos\theta)$$

$$\frac{dx}{d\theta} = a(1+\cos\theta), \ \frac{dy}{d\theta} = a\sin\theta$$

$$y_1 = \frac{dy}{dx} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{\cancel{2}\sin\theta/2 \ \cancel{\cos\theta/2}}{\cancel{2}\cos^{\cancel{2}}\theta/2}$$

$$= \tan\theta/2$$

$$y_2 = \frac{d}{d\theta}(\tan\theta/2) \cdot \frac{d\theta}{dx}$$

$$= \sec^2\theta/2 \times \frac{1}{2} \cdot \frac{1}{a(1+\cos\theta)} = \frac{1}{2}\sec^2\theta/2 \times \frac{1}{a\,2\cos^2\theta/2}$$

Let (α, β) be the co-ord. of the centre of curvature

then $\alpha = x - \dfrac{y_1(1+y_1^2)}{y_2}$

$$= a(\theta + \sin\theta) - \frac{\tan\theta/2\,(1+\tan^2\theta/2)}{\frac{1}{4a}\sec^4\theta/2}$$

$$= a(\theta + \sin\theta) - 4a\tan\theta/2 \times \cos^2\theta/2$$

$$= a(\theta + \sin\theta) - 4a\frac{\sin\theta/2}{\cancel{\cos\theta/2}} \times \cos^{\cancel{2}}\theta/2$$

$$= a(\theta + \sin\theta) - 4a\sin\theta/2\cos\theta/2$$

$$= a(\theta + \sin\theta) - 2a\sin\theta$$

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$\alpha = a\theta - a\sin\theta = a(\theta - \sin\theta)$

$\beta = y + \dfrac{(1+y_1^2)}{y_2}$

$= a(1-\cos\theta) + \dfrac{(1+\tan^2\theta/2)\,4a}{\sec^4\theta/2}$

$= a(1-\cos\theta) + 4a\cos^2\theta/2$

$= a\, 2\sin^2\theta/2 + 4a\cos^2\theta/2$

$= 2a\,\sin^2\theta/2 + 4a(1-\sin^2\theta/2)$

$= 4a - 2a\sin^2\theta/2$

$= 2a(2 - \sin^2\theta/2)$

$= 2a\left(2 - \dfrac{1-\cos\theta}{2}\right)$

$= \cancel{2}a\,(3+\cos\theta)$

$= \dfrac{1}{4a}\sec^4\theta/2$

Evaluate : $\int_{\alpha}^{\beta} \sqrt{(x-\alpha)(\beta-x)}\,dx$

Sol^n : Put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$\therefore dx = [-2\alpha\cos\theta.\sin\theta + 2\beta\sin\theta\cos\theta]d\theta$

$= [-\alpha\sin 2\theta + \beta\sin 2\theta)$

$= (\beta-\alpha)\sin 2\theta\, d\theta$

$x-\alpha = \alpha\cos^2\theta + \beta\sin^2\theta - \alpha$

$= \alpha(\cos^2\theta - 1) + \beta\sin^2\theta$

$= -\alpha\sin^2\theta + \beta\sin^2\theta$

$= (\beta-\alpha)\sin^2\theta$

$\beta - x = \beta - (\alpha\cos^2\theta + \beta\sin^2\theta)$

$= (\beta-\alpha)\cos^2\theta$

when $x = \alpha$, $(\beta-\alpha)\sin^2\theta = 0$

or $\sin^2\theta = 0$

or $\theta = 0$

when $x = \beta$, then $(\beta-\alpha)\cos^2\theta = 0$

or $\cos^2\theta = 0$

or $\cos\theta = 0$

$\theta = \pi/2$

$$I = \int_0^{\pi/2} \sqrt{(\beta-\alpha)\sin^2\theta\,(\beta-\alpha)\cos^2\theta}\;(\beta-\alpha)\sin 2\theta\, d\theta$$

$$= (\beta-\alpha)^2 \int_0^{\pi/2} \sin\theta.\cos\theta\ 2\sin\theta\cos\theta\, d\theta$$

$$= (\beta-\alpha)^2 \int_0^{\pi/2} 2\sin^2\theta\cos^2\theta\, d\theta$$

$$= (\beta-\alpha)^2\, \frac{1}{2} \int_0^{\pi/2} \sin^2 2\theta\, d\theta$$

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$$= \frac{1}{2}(\beta-\alpha)^2 \int_0^{\pi/2} \left(\frac{1-\cos 4\theta}{2}\right) d\theta$$

$$= \frac{1}{4}(\beta-\alpha)^2 \left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2}$$

$$= \frac{1}{4}(\beta-\alpha)^2 \left[\frac{\pi}{2} - 0 - 0 + 0\right]$$

$$= \frac{\pi(\beta-\alpha)^2}{8}$$

Q. $\displaystyle\int_2^3 \frac{1}{\sqrt{(x-2)(3-x)}}\, dx$

$$2(\beta-\alpha)^2 \frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{\Gamma 3}$$

Find the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\alpha = x - \frac{y_1(1+y_1^2)}{y_2} \qquad \beta = y + \frac{(1+y_1^2)}{y_2}$$

$$x = a\cos t \qquad y = b\sin t$$

$$\frac{dx}{dt} = -a\sin t \qquad \frac{dy}{dt} = b\cos t.$$

$$y_1 = \frac{dy}{dx} = \frac{-b}{a}\cot t$$

$$y_2 = \frac{-b}{a}(-\mathrm{cosec}^2 t)\frac{dt}{dx} = \frac{b}{a}\mathrm{cosec}^2 t \cdot \frac{1}{-a\sin t}$$

$$y_2 = -\frac{b}{a^2}\mathrm{cosec}^3 t$$

$$\alpha = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= a\cos t - \frac{(-b/a)\cot t\left(1+\frac{b^2}{a^2}\cot^2 t\right)}{\frac{-b}{a^2}\mathrm{cosec}^3 t}$$

$$\alpha = a\cos t - a\sin^2 t \cdot \frac{\cos t}{\sin t}\left[1 + \frac{b^2}{a^2}\cdot\frac{\cos^2 t}{\sin^2 t}\right]$$

$$= a\cos t - a\sin^2 t\cos t - \frac{b^2}{a}\cos^3 t$$

$$= a\cos t(1-\sin^2 t) - \frac{b^2}{a}\cos^3 t$$

$$\bar{x} = \frac{(a^2-b^2)}{a}\cos^3 t. \quad \longrightarrow (iii)$$

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$$\beta = y + \frac{(1+y_1^2)}{y_2}$$

$$= b\sin t + \frac{(1 + b^2/a^2 \cot t^2)}{\left(-\frac{b}{a^2}\right)\operatorname{cosec}^3 t}$$

$$= b\sin t - \frac{a^2}{b}\sin^3 t\left[1 + \frac{b^2}{a^2}\frac{\cos^2 t}{\sin^2 t}\right]$$

$$= b\sin t - \frac{a^2}{b}\sin^3 t - b\sin t\cos^2 t$$

$$= b\sin t(1-\cos^2 t) - \frac{a^2\sin^3 t}{b}$$

$$= b\sin^3 t - \frac{a^2}{b}\sin^3 t$$

$$\beta = \frac{(b^2-a^2)\sin^3 t}{b}$$

or $\beta \quad \frac{-(a^2-b^2)\sin^3 t}{b}$. — (IV)

eliminating t from eqn (3) & (4)

$$(\alpha)^{2/3} = \left(\frac{(a^2-b^2)\cos^3 t}{a}\right)^{2/3}$$

$$(a\alpha)^{2/3} = (a^2-b^2)^{2/3}\cos^2 t$$

$$(\beta)^{2/3} = \left(\frac{a^2-b^2}{b}\right)^{2/3}\sin^2 t$$

$$(b\beta)^{2/3} = (a^2-b^2)^{2/3}\sin^2 t$$

$$(a\alpha)^{2/3} + (b\beta)^{2/3} = (a^2-b^2)^{2/3}$$

$$\therefore (a\alpha)^{2/3} + (b\beta)^{2/3} = (a^2-b^2)^{2/3}$$ Ans

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2. Find the evolute of the parabola $y^2 = 4ax$ is $27ay^2 = 4(x-2a)^3$

$$\alpha = x - \frac{y_1(1+y_1^2)}{y_2} \qquad \beta = y + \frac{(1+y_1^2)}{y_2}$$

$x = at^2$ $\qquad$ $y = 2at$

$\frac{dx}{dt} = 2at$ $\qquad$ $\frac{dy}{dt} = 2a$

$$y_1 = \frac{dy}{dx} = \frac{1}{t}$$

$$y_2 = \frac{t(0) - 1(1)}{t^2} = \frac{-1}{t^2} \times \frac{dt}{dx} = \frac{-1}{2at^3}$$

$$\alpha = at^2 - \frac{(1/\cancel{t})[1 + 1/t^2]}{-1/2at^{\cancel{3}\,2}}$$

$$= at^2 + 2at^2[1 + 1/t^2]$$

$$= at^2 + 2at^2 + 2a$$

$$\alpha = 3at^2 + 2a \longrightarrow (1)$$

~~$\alpha =$~~

$$\beta = 2at + \frac{(1 + 1/t^2)}{\frac{-1}{2at^3}}$$

$$= 2at - 2at^3(1 + 1/t^2)$$

$$= \cancel{2at} - 2at^3 - \cancel{2at}$$

$$\beta = -2at^3 \longrightarrow (2)$$

eliminate t from eqn (1) & (2)

$$\alpha = 3at^2 + 2a$$

$$(\alpha - 2a) = 3at^2$$

$$\frac{\alpha - 2a}{3a} = t^2$$

$$\left(\frac{\alpha - 2a}{3a}\right)^{3/2} = t^3$$

$$\left(\frac{\alpha - 2a}{3a}\right)^{3/2} = \left(-\frac{\beta}{2a}\right)$$

$$\alpha - 2a = 3at^2$$

$$(\alpha - 2a)^3 = 27a^3 t^6$$

$$(\alpha - 2a)^3 = \frac{27a^3 \beta^2}{4a^2}$$

$$\boxed{4(\alpha - 2a)^3 = 27a\beta^2}$$ Hence proved

Find the evolute of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\alpha = x - \frac{y_1(1+y_1^2)}{y_2} \qquad \beta = y + \frac{(1+y_1^2)}{y_2}$$

$x = a\sec t, \quad y = b\tan t$

$$\frac{dx}{dt} = a\sec t\tan t \qquad \frac{dy}{dt} = b\sec^2 t$$

$$y_1 = \frac{b\sec^2 t}{a\sec t\tan t} = \frac{b}{a}\cdot\frac{1}{\cancel{\cos t}} \times \frac{\cancel{\cos t}}{\sin t} = \frac{b}{a}\operatorname{cosec} t$$

$$y_2 = \frac{b}{a}(-\operatorname{cosec} t \cdot \cot t)\cdot\frac{dt}{dx}$$

$$= -\frac{b}{a}\operatorname{cosec} t \cdot \cot t \times \frac{1}{a\sec t\tan t}$$

$$y_2 = -\frac{b}{a^2} \times \frac{1}{\sin t} \times \cos t \times \cot t \times \cot^2$$

$$y_2 = -\frac{b}{a^2}\cot^3 t$$

$$\alpha = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= a\sec t - \frac{(b/a)\operatorname{cosec} t\left(1+\frac{b^2}{a^2}\operatorname{cosec}^2 t\right)}{\left(-\frac{b}{a^2}\right)\cot^3 t}$$

$$= a\sec t + a\frac{1}{\sin t}\frac{\sin^{3} t}{\cos^3 t}\left(1+\frac{b^2}{a^2}\frac{1}{\sin^2 t}\right)$$

$$= a\sec t + \frac{a\sin^2 t}{\cos^3 t} + \frac{b^2}{a}\times\frac{1}{\cos^3 t}.$$

$$= a\sec t + \frac{a(1-\cos^2 t)}{\cos^3 t} + \frac{b^2}{a}\sec^3 t$$

$$= \cancel{a\sec t} + a\sec^3 t - \cancel{a\sec t} + \frac{b^2}{a}\sec^3 t$$

$$\alpha = \sec^3 t\,(a + b^2/a)$$

$$\text{or } \alpha = \left(\frac{a^2+b^2}{a}\right)\sec^3 t \longrightarrow (1)$$

$$\beta = y + \frac{(1+y_1^2)}{y_2}$$

$$= b\tan t + \frac{\left(1+\frac{b^2}{a^2}\operatorname{cosec}^2 t\right)}{\left(-b/a^2\right)\cot^3 t}$$

$$= b\tan t - \frac{a^2}{b}\frac{\sin^3 t}{\cos^3 t}\left(1+\frac{b^2}{a^2}\operatorname{cosec}^2 t\right)$$

$$= b\tan t - \frac{a^2}{b}\frac{\sin^3 t}{\cos^3 t} - b\frac{\sin t}{\cos^3 t}$$

$$\beta = b\tan t - \frac{a^2}{b}\tan^3 t - b\tan t\,(1+\tan^2 t)$$

$$= \cancel{b\tan t} - \frac{a^2}{b}\tan^3 t - \cancel{b\tan t} - b\tan^3 t$$

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$$\beta = -\tan^3 t \left(\frac{a^2+b}{b}\right) \longrightarrow ②$$

$$= -\left(\frac{a^2+b^2}{b}\right) \tan^3 t \longrightarrow ②$$

eliminate t from eqn ③ & ④

$$\alpha^{2/3} = \left(\frac{a^2+b^2}{a}\right)^{2/3} \sec^2 t$$

$$(a\alpha)^{2/3} = (a^2+b^2)^{2/3} \sec^2 t$$

$$(\beta)^{2/3} = \left(\frac{a^2+b^2}{b}\right)^{2/3} \tan^2 t$$

$$(b\beta)^{2/3} = (a^2+b^2)^{2/3} \tan^2 t$$

$$(a\alpha)^{2/3} - (b\beta)^{2/3} = (a^2+b^2)^{2/3}$$

Q Find the evolute of the curve $x^{2/3} + y^{2/3} = a^{2/3}$

$x = a\cos^3 t \qquad dx/dt = a \times 3\cos^2 t \times (-\sin t)$

$y = a\sin^3 t \qquad dy/dt = a \times 3\sin^2 t \times (\cos t)$

$$y_1 = \frac{\cancel{a} \times \cancel{3}\sin^{\cancel{2}} t \,\cancel{\cos t}}{-\cancel{a} \times \cancel{3}\cos^{\cancel{2}} t \,\cancel{\sin t}} = -\tan t$$

$$y_2 = \cancel{-}\sec^2 t \times \frac{1}{\cancel{-}3a\cos^2 t \sin t} = \frac{1}{3a\sin t\cos^4 t}$$

$$\alpha = x - \frac{y_1(1+y_1^2)}{y_2} = a\cos^3 t - \frac{(-\tan t)(1+\tan^2 t)}{\frac{1}{3a\sin t\cos^4 t}}$$

$$= a\cos^3 t + 3a\sin t\cos^4 t \tan t\,(\sec^2 t)$$

$$= a\cos^3 t + 3a\sin t\cos^{\cancel{4}} t\,\frac{\sin t}{\cancel{\cos t}} \times \frac{1}{\cancel{\cos^2 t}} = a\cos^3 t + 3a(1-\cos^2 t) = a\cos^3 t + 3a - 3a\cos^2 t$$

$\alpha = a(\cos^3 t + 3\sin^2 t \cdot \cos t) \rightarrow (3)$

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$$\beta = y + \frac{(1+y_1^2)}{y_2}$$

$$= a\sin^3 t + \frac{(1+\tan^2 t)}{1/3a\sin t\cos^4 t}$$

$$= a\sin^3 t + 3a\sin t\cos^4 t\,(\sec^2 t)$$

$$= a\sin^3 t + 3a\sin t\cos^{\cancel{4}\,2} t \times \frac{1}{\cancel{\cos^2 t}}$$

$$\beta = a(\sin^3 t + 3\sin t\cos^2 t) \rightarrow (4)$$

Eliminate t from eqn (3) & (4)

$$\alpha + \beta = a(\cos^3 t + \sin^3 t + 3\cos t\sin^2 t + 3\sin t\cdot\cos^2 t)$$

$$\alpha - \beta = a(\cos^3 t - \sin^3 t + 3\cos t\cdot\sin^2 t - 3\cos^2 t\cdot\sin t)$$

$$\alpha + y = a(\cos t + \sin t)^3 \times 2/3$$

$$\alpha - y = a(\cos t - \sin t)^3$$

$$(\alpha+y)^{2/3} = a^{2/3}(\cos t + \sin t)^2$$

$$(\alpha-y)^{2/3} = a^{2/3}(\cos t - \sin t)^2$$

$$(\alpha+y)^{2/3} + (\alpha-y)^{2/3} = a^{2/3}[1 + \cancel{2\cos t\cdot\sin t} + 1 - \cancel{2\cos t\cdot\sin t}]$$

$$(\alpha+y)^{2/3} + (\alpha-y)^{2/3} = 2a^{2/3}$$

$$(\alpha+y)^{2/3} + (\alpha-y)^{2/3} = 2a^{2/3}$$

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$$\boxed{\int \frac{1}{\sqrt{ax+b}}\,dx = \frac{2}{a}\sqrt{ax+b} + C}$$

Q $\displaystyle\int_0^4 \frac{x^2}{x+1}\,dx \longrightarrow$ Improper

$$x+1\,\big)\,\cancel{x^2}\,\big(\,x-1$$
$$\cancel{x^2}+x$$
$$(-)\quad(-)$$
$$-\cancel{x}$$
$$\cancel{-x}-1$$
$$(+)\quad(+)$$
$$1$$

$= a\cos^3 t + 3a\cos t(1-\cos^2 t)$

$= a\cos^3 t + 3a\cos t - 3a\cos^3 t$

$= 3a\cos t - 2a\cos^3 t$

$$\boxed{\frac{x^2}{x+1} = (x-1) + \frac{1}{x+1}}$$

$$\int_0^a \frac{dx}{a^2+x^2} = \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^a$$

$$\int_0^2 \sqrt{4-x^2}\,dx$$

$$\boxed{\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C}$$

$$\boxed{\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx}$$

$\beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1}$ $\beta(m,n) = \int_0^{\infty} x^{m-1}(1-x)^{n-1} dx$

$\beta(m,n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx = \int \frac{x^{m-1}}{(1+x)^{m+n}}$

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Beta function

It is denoted by $\beta(m,n)$ where $(m,n>0)$ And defined by a definite integral as

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx \quad , (m>0, n>0)$$

Properties

1) $\beta(m,n) = \beta(n,m)$ (symmetry of beta function)

2) $\beta(m,n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$

3) $\beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} d\theta$

1) By definition

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \longrightarrow (1)$$

we know that $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

$$\therefore \beta(m,n) = \int_0^1 (1-x)^{m-1} [1-(1-x)]^{n-1} dx$$

$$= \int_0^1 x^{n-1} (1-x)^{m-1} dx = \beta(n,m)$$

(Hence proved)

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2) By definition

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \longrightarrow ①$$

put $x = \dfrac{1}{1+y}$ $\qquad dx = \dfrac{(-1)}{(1+y)^2}\,dy$

$\Rightarrow 1+y = \dfrac{1}{x}$ $\rightarrow$ when $x=0 \Rightarrow y \to \infty$

when $x=1 \Rightarrow y=0$

$$\beta(m,n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1}\frac{(-1)\,dy}{(1+y)^2}$$

$$= \int_0^\infty \frac{1}{(1+y)^{m-1}}\,\frac{y^{n-1}}{(1+y)^{n-1}}\,\frac{dy}{(1+y)^2}$$

$$\beta(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

$$\left[\int_a^b f(x)\,dx = \int_a^b f(y)\,dy\right]$$

— Hence proved

$$\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$[\,\beta(m,n) = \beta(n,m)\,]$$

$$\beta(n,m) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

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3) By definition

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cdot\cos\theta\cdot d\theta$

when $x=0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \pi/2$

$$\therefore \beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\, 2\sin\theta\cdot\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta)^{2m-2}(\cos\theta)^{2n-2}\sin\theta\cdot\cos\theta\, d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,d\theta$$

Hence proved

Gamma Function

The Gamma function is denoted by 'Γn' (gamma 'n') where $n>0$, and defined by a definite integral:

$$\Gamma n = \int_0^\infty e^{-x}x^{n-1}\,dx \quad (n>0)$$

gamma 'n'

$$\Gamma m = \int_0^\infty e^{-x}x^{m-1}\,dx$$

Ex

$$\Gamma 5 = \int_0^\infty e^{-x}x^{5-1}\,dx$$

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Properties of Gamma functions

1) $\Gamma 1 = 1$

2) $\Gamma n+1 = n\Gamma n = \lfloor n$

3) $\boxed{\Gamma n = z^n \int_0^\infty e^{-zx} x^{n-1}\, dx}$

4) $\boxed{\Gamma n = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy}$

5) $\boxed{\Gamma n+1 = \int_0^\infty e^{-y^{1/n}}\, dy}$

6) $\Gamma 1/2 = \sqrt{\pi}$

1) By definition

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$$

$n \to 1$

$$\Gamma 1 = \int_0^\infty e^{-x} x^{1-1}\, dx$$

$$= \int_0^\infty e^{-x}\, dx$$

$$= \left[\frac{e^{-x}}{-1}\right]_0^\infty \qquad \left[\because \lim_{x\to\infty} e^{-x} = 0\right]$$

$$= -1(0-1) = 1$$

$\Gamma n \quad n!$

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2) By definition

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx \quad \rightarrow (1)$$

$n \to n+1$

$$\Gamma n+1 = \int_0^\infty \underset{2}{e^{-x}} \underset{1}{x^n} dx$$

$$= \left[x^n \frac{e^{-x}}{(-1)} \right]_0^\infty - \int_0^\infty n x^{n-1} \frac{e^{-x}}{(-1)} dx$$

$$= -(0-0) + n \int_0^\infty x^{n-1} e^{-x} dx$$

from eqn (1)

$$\Gamma n+1 = n \Gamma n$$
$$= n(n-1)\Gamma n-1$$
$$= n(n-1)(n-2)\Gamma n-2$$
$$= n(n-1)(n-2) \ldots 3 \; 2 \; 1 \Gamma 1$$
$$\Gamma n+1 = \lfloor n$$

Hence proved

for Ex

$\Gamma 6 = \Gamma 5+1$
$= 5 \Gamma 5$
$= 5 \cdot 4 \Gamma 4$
$= 5 \, 4 \, 3 \Gamma 3$
$= 5 \, 4 \cdot 3 \, 2 \Gamma 2$
$= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \Gamma 1$
$\Gamma 6 = \lfloor 5$

3) By definition $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

Put $x = zy \Rightarrow dx = z dy$

No change in limit

$$\Gamma n = \int_0^\infty e^{-zy} (zy)^{n-1} z dy$$

$$= \int_0^\infty e^{-zy} z^{n-1} y^{n-1} z^1 dy$$

$$= z^n \int_0^\infty e^{-zy} y^{n-1} dy$$

$$\boxed{\Gamma n = z^n \int_0^\infty e^{-zx} x^{n-1} dx}$$ Hence proved.

$\frac{\Gamma n}{z^n}$

4) By $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

Put $x = \log \frac{1}{y} \Rightarrow e^x = \frac{1}{y} \Rightarrow y = \frac{1}{e^x} \Rightarrow y = e^{-x}$

$dy = -e^{-x} dx$

Since $y = \frac{1}{e^x}$ when $x = 0$, $y = 1$
$x \to \infty$, $y = 0$

$$\Gamma n = \int_1^0 \cancel{e} \left(\log \frac{1}{y}\right)^{n-1} (-dy) = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy$$

(Hence proved)

5. By definition.

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx \longrightarrow (1)$$

Put $x = y^{1/n} \Rightarrow x^n = y$

$$\Rightarrow n x^{n-1} dx = dy$$

$$\Rightarrow \cancel{n} x^{n-1} dx = \frac{dy}{n}$$

(no change in limit)

$$\therefore \Gamma n = \int_0^\infty e^{-y^{1/n}} \frac{dy}{n}$$

$$\Rightarrow n\Gamma n = \int_0^\infty e^{-y^{1/n}} dy$$

property II

$$\Rightarrow \Gamma n+1 = \int_0^\infty e^{-y^{1/n}} dy$$

Relation betn Gamma & Beta function.

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

Proof $\Gamma n = \int_0^\infty e^{-zx} x^{n-1} z^n dx$

Multiply by e^{-z} & z^{m-1} on both side of eqn

$$\Gamma n \, e^{-z} z^{m-1} = \int_0^\infty e^{-zx} e^{-z} z^{m-1} z^n x^{n-1} dx$$

Integrate w.r.t z on both side betn 0 to ∞

$$\Gamma n \int_0^\infty e^{-z} z^{m-1} dz = \int_0^\infty x^{n-1} dx \int_0^\infty e^{-z(x+1)} z^{m+n-1} dz$$

$$\int_0^\infty z^{m-1} e^{-az} dz = \frac{\Gamma m}{a^m}$$

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$$\Gamma n \, \Gamma m = \int_0^\infty x^{n-1} dx \left(\frac{\Gamma m+n}{(x+1)^{m+n}} \right)$$

$$\frac{\Gamma n \, \Gamma m}{\Gamma m+n} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx = \beta(m,n)$$

(Hence Verified) Ans

Gamma function :- Questions & Solutions

$$\int_0^\infty x^{n-1} e^{-x} dx = \Gamma n$$

$$\int_0^\infty x^{n-1} e^{-ax} dx = \frac{\Gamma n}{a^n}$$

(1) $\int_0^\infty x^7 e^{-x} dx = \Gamma 8 = 7!$

(2) $\int_0^\infty x^9 e^{-x} dx = \Gamma 10 = 9!$

(1) $\int_0^\infty \cancel{x^{n-1}} x^8 e^{-4x} dx = \frac{\Gamma 9}{(4)^9} = \frac{8!}{4^9}$

Prove that

(1) $\Gamma n+1 = n \Gamma n$

$\Gamma 6 = \lfloor 5$

$\Gamma 10 = \lfloor 9$

By definition

$$\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx$$

$n \to n+1$

$$\Gamma n+1 = \int_0^\infty \underset{dv}{e^{-x}} \, \underset{u}{x^n} \, dx$$

$$= \left[-x^n e^{-x}\right]_0^\infty - \int_0^\infty n x^{n-1} (-e^{-x}) dx$$

$$= -(0-0) + n \int_0^\infty x^{n-1} e^{-x} dx$$

$\Gamma n+1 = n \Gamma n = n!$ Hence proved

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x	$\cancel{1}$	0
$\cancel{7}$	2	1

Beta function :- questions and Answers

(1) $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx \longrightarrow (1)$

question $\int_0^\infty \frac{x^2(1+x^4)}{(1+x)^{10}} dx$

$m-1=2$

$m=3$

$$\int_0^\infty \frac{x^2}{(1+x)^{10}} dx + \int_0^\infty \frac{x^6}{(1+x)^{10}} dx$$

$$= \beta(3,7) + \beta(7,3)$$

$$= 2\beta(7,3)$$

$$= 2\frac{\Gamma 7 \Gamma 3}{\Gamma 10} \text{ Ans}$$

Q. $\int_0^\infty \frac{x^2}{(1+x^2)^{7/2}} dx$

let $x^2 = t$

$2x\,dx = dt$

$dx = \frac{dt}{2x}$

$dx = \frac{dt}{2\sqrt{t}}$

$$= \int_0^\infty \frac{t}{(1+t)^{7/2}} \frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int_0^\infty \frac{t^{1/2}}{(1+t)^{7/2}} dt$$

$$= \frac{1}{2}\beta\left(\tfrac{3}{2}, \tfrac{4}{2}\right) = \frac{1}{2}\frac{\Gamma\frac{3}{2}\Gamma\frac{4}{2}}{\Gamma\frac{7}{2}} \text{ Ans}$$

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(2) $\dfrac{\beta(m,n)}{2} = \int_0^{\infty} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

Let $2m-1 = p$ & $2n-1 = q$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \frac{1}{2}\frac{\Gamma\frac{p+1}{2}\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}}$$

Q. $\int_0^{\pi/2} \sin^5\theta \, d\theta$

$$= \int_0^{\pi/2} \sin^5\theta \cdot \cos^0\theta \, d\theta$$

$$= \frac{\Gamma\frac{5+1}{2}\Gamma\frac{0+1}{2}}{\Gamma\frac{5+0+2}{2}}$$

$$= \frac{\Gamma\frac{6}{2}\Gamma\frac{1}{2}}{2\Gamma\frac{7}{2}}$$

$$= \frac{\Gamma 3\,\Gamma\frac{1}{2}}{2\Gamma\frac{7}{2}}$$

$$= \frac{\Gamma 3\,\cancel{\Gamma\frac{1}{2}}}{\cancel{2}\times\frac{5}{2}\frac{3}{2}\frac{1}{2}\cancel{\Gamma\frac{1}{2}}}$$

$$= \frac{8}{15}$$

Q $\int_0^{\pi/2} \sin^7\theta \cdot \cos^3\theta \, d\theta$

$$= \frac{\Gamma\frac{8}{2}\Gamma\frac{4}{2}}{2\Gamma\frac{12}{2}}$$

$$= \frac{\Gamma 4\,\Gamma 2}{2\Gamma 6}$$

$$= \frac{3\times 2\times 1\times 2}{2\times 6\times 5\times 4\times 3\times 2\times 1}$$

$$= \frac{\cancel{3}\times\cancel{2}\times\cancel{1}}{\cancel{2}\times 5\times 4\times\cancel{3}\times 2\times\cancel{1}} = \frac{1}{40}$$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\frac{\Gamma\frac{p+1}{2}\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}}$$

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Q $\displaystyle\int_0^{\pi/6} \cos^4 3\theta \cdot \sin^2 6\theta \, d\theta$.

let $3\theta = t$

$3\,d\theta = dt$

$d\theta = \dfrac{dt}{3}$

$$= \int_0^{\pi/2} \cos^4 t \, \sin^2 2t \, \frac{dt}{3}$$

$$= \frac{1}{3}\int_0^{\pi/2} \cos^4 t \,(2\sin t \cdot \cos t)^2 \, dt$$

$$= \frac{4}{3}\int_0^{\pi/2} \sin^2 t \, \cos^6 t \, dt.$$

$$= \frac{4}{3}\left[\frac{\Gamma\frac{3}{2}\,\Gamma\frac{7}{2}}{2\,\Gamma\frac{10}{2}}\right]$$

$$= \frac{\cancel{4}\,\cancel{2}}{3}\left[\frac{\cancel{\tfrac{1}{2}}\Gamma\tfrac{1}{2} \cdot \tfrac{5}{2}\times\tfrac{3}{2}\,\tfrac{1}{2}\Gamma\tfrac{1}{2}}{\cancel{2}\times 4!}\right]$$

$$= \frac{15\sqrt{\pi}\,\sqrt{\pi}}{24\times 8} = \frac{15\pi}{24\times 8}$$

Evaluate or prove that

Q $\displaystyle\int_0^{\pi/2} \frac{1}{\sqrt{\cos\theta}}\, d\theta \int_0^{\pi/2} \sqrt{\cos\theta}\, d\theta = \pi$

sol^n $\displaystyle\int_0^{\pi/2} \cos^{-1/2}\theta \, d\theta \int_0^{\pi/2} \cos^{1/2}\theta \, d\theta$

$$= \frac{\Gamma\frac{-\frac{1}{2}+1}{2}\,\Gamma\frac{0+1}{2}}{2\,\Gamma\left(\frac{-\frac{1}{2}+0+2}{2}\right)} \times \frac{\Gamma\frac{\frac{1}{2}+1}{2}\,\Gamma\frac{0+1}{2}}{2\,\Gamma\left(\frac{\frac{1}{2}+0+2}{2}\right)}$$

$$= \frac{\Gamma\frac{1}{4}\cdot\Gamma\frac{1}{2}}{2\,\cancel{\Gamma\frac{3}{4}}} \times \frac{\cancel{\Gamma\frac{3}{4}}\,\Gamma\frac{1}{2}}{2\,\Gamma\frac{5}{4}}$$

$$= \frac{\cancel{\Gamma\frac{1}{4}}\sqrt{\pi}\times\sqrt{\pi}}{\cancel{4}\times\cancel{\frac{1}{4}}\cancel{\Gamma\frac{1}{4}}} = \pi$$

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(3) $\Gamma n \, \Gamma 1-n = \dfrac{\pi}{\sin n\pi}$

Q $\displaystyle\int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$

$\displaystyle\int_0^{\infty} \frac{dx}{1+x^4} =$ Let $x^2 = \tan\theta$

$2x\,dx = \sec^2\theta\,d\theta$

$dx = \dfrac{\sec^2\theta\,d\theta}{2x}$

$\displaystyle\int_0^{\pi/2} \frac{1}{\cancel{(1+\tan^2\theta)}} \times \frac{\cancel{\sec^2\theta}\,d\theta}{2\sqrt{\tan\theta}}$ $\boxed{dx = \dfrac{\sec^2\theta\,d\theta}{2\sqrt{\tan\theta}}}$

$= \displaystyle\int_0^{\pi/2} \frac{d\theta}{\frac{\sin^{1/2}\theta}{\cos^{1/2}\theta}} d\theta$

$= \displaystyle\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$

$= \displaystyle\int_0^{\pi/2} \frac{\cos^{1/2}\theta}{\sin^{1/2}\theta} d\theta$

$\Gamma\frac{-1}{2} \times \Gamma\frac{-3}{2}$

$\Gamma n \, \Gamma 1-n = \dfrac{\pi}{\sin n\pi}$

put $n = -1/2$

$\Gamma -\frac{1}{2} \, \Gamma 1+\frac{1}{2} = \dfrac{\pi}{\sin(-\frac{\pi}{2})}$

$\Gamma\frac{-1}{2} \, \Gamma\frac{3}{2} = \dfrac{\pi}{-1}$

$\Gamma\frac{-1}{2} = \dfrac{\pi}{-\Gamma\frac{3}{2}}$

$= \dfrac{\pi}{-\frac{1}{2}\Gamma\frac{1}{2}}$

$= \dfrac{-2\pi}{\sqrt{\pi}}$

$\Gamma\frac{-1}{2} = -2\sqrt{\pi}$

or

$= \displaystyle\int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta \, d\theta$

$= \dfrac{\Gamma\frac{-\frac{1}{2}+1}{2} \, \Gamma\frac{\frac{1}{2}+1}{2}}{2\,\Gamma\frac{-\frac{1}{2}+1}{2} + \frac{\frac{1}{2}+1}{2}} = \dfrac{1}{2} \dfrac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{\Gamma 1}$

$= \frac{1}{2}\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}$

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Now, $\Gamma m \, \Gamma 1-m = \frac{\pi}{\sin m\pi}$

Put $m = \frac{1}{4}$

$$\Gamma \tfrac{1}{4} \, \Gamma 1-\tfrac{1}{4} = \frac{\pi}{\sin \pi/4}$$

$$\Gamma \tfrac{1}{4} \, \Gamma \tfrac{3}{4} = \frac{\pi}{1/\sqrt{2}} = \sqrt{2}\pi$$

$$\therefore \frac{1}{2} \Gamma \tfrac{1}{4} \, \Gamma \tfrac{3}{4} = \frac{1}{2} \times \sqrt{2}\pi = \frac{\pi}{\sqrt{2}}$$

Duplication Formula

$$\Gamma m \, \Gamma m+\tfrac{1}{2} = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma 2m$$

Proof

$$2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma m \, \Gamma n}{2 \, \Gamma m+n}$$

Put $2n-1 = 0 \Rightarrow n = \frac{1}{2}$

$$\int_0^{\pi/2} \sin^{2m-1} d\theta = \frac{\Gamma m \, \Gamma \tfrac{1}{2}}{2 \, \Gamma m+\tfrac{1}{2}}$$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta = \frac{\Gamma m \sqrt{\pi}}{2 \, \Gamma m+\tfrac{1}{2}}$$

Put $m = n$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1} d\theta = \frac{\Gamma m \, \Gamma m}{2 \, \Gamma m+m}$$

$$\frac{1}{2^{2m-1}}\int_0^{\pi/2}(2\sin\theta\cdot\cos\theta)^{2m-1}\,d\theta = \frac{(\sqrt{m})^2}{2\sqrt{2m}}$$

$$\frac{1}{2^{2m-1}}\int_0^{\pi/2}\sin^{2m-1}2\theta\,d\theta = \frac{(\sqrt{m})^2}{2\sqrt{2m}}$$

Let $2\theta = \phi$

$d\theta = \frac{d\phi}{2}$

$$\frac{1}{2^{2m-1}}\int_0^{\pi/2}\sin^{2m-1}\phi\,\frac{d\phi}{2} = \frac{(\sqrt{m})^2}{2\sqrt{2m}}$$

$a - a\sin^2\theta + b\sin^2\theta - a$

$$\frac{1}{2^{2m-1}}\;\frac{\sqrt{m}\,\sqrt{\pi}}{2\sqrt{m+\frac{1}{2}}} = \frac{(\sqrt{m})^2}{2\sqrt{2m}}$$

$$\frac{\sqrt{2m}\,\sqrt{\pi}}{2^{2m-1}} = \sqrt{m}\,\sqrt{m+\tfrac{1}{2}}$$

Q. Evaluate $\boxed{\int_0^1 \frac{dx}{\sqrt{(1-x)^n}}}$ (ii) $\int_0^{\pi/2}(x-a)^m(b-x)^n\,dx$

put $x = a\cos^2\theta + b\sin^2\theta$

$\frac{dx}{d\theta} = (-2a\sin\theta\cdot\cos\theta + 2b\sin\theta\cos\theta)$

$= 2(b-a)\sin\theta\cdot\cos\theta\,d\theta$

when $x = a \Rightarrow \theta = 0$

$x = b \Rightarrow \theta = \pi/2$

$$= \int_0^{\pi/2}(a\cos^2\theta + b\sin^2\theta - a)^m\,\{b-(a\cos^2\theta + b\sin^2\theta)\}^n\; 2(b-a)\sin\theta\cdot\cos\theta\,d\theta$$

$$= \int_0^{\pi/2}\{(b-a)\sin^2\theta\}^m\cdot\{(b-a)\cos^2\theta\}^n\; 2(b-a)\sin\theta\cdot\cos\theta\cdot d\theta$$

$$= 2\int_0^{\pi/2}(b-a)^{m+n+1}\sin^{2m+1}\theta\cdot\cos^{2m+1}\theta\cdot d\theta$$

$$\frac{\sqrt{m+1}\,\sqrt{n+1}}{2\sqrt{\frac{2(m+n)+4}{2}}}$$

$$= 2(b-a)^{m+n+1}\int_0^{\pi/2}\sin^{2m+1}\theta\cdot\cos^{2m+1}\theta\,d\theta$$

$$= 2(b-a)^{m+n+1}\;\frac{\sqrt{\frac{2m+1+1}{2}}\,\sqrt{\frac{2n+1+1}{2}}}{2\sqrt{\frac{2m+2+2n+2+2}{2}}}$$

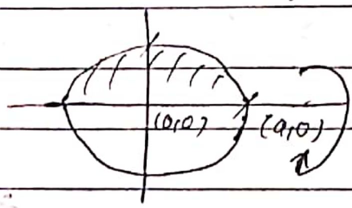
Volume of solids of Revolution

$$V = \int_{x_1}^{x_2} \pi y^2 \, dx \qquad (x\text{-axis})$$

$$V = \int_{y_1}^{y_2} \pi x^2 \, dy \qquad (y\text{-axis})$$

$$x^2 + y^2 = a^2$$

$$V = 2\int_0^a \pi y^2 \, dx$$



$$= 2\pi \int_0^a (a^2 - x^2)\, dx$$

$$= 2\pi \left[a^2 x - \frac{x^3}{3} \right]_0^a$$

$$= 2\pi \left[\frac{a^3 - a^3}{3} \right]$$

$$\boxed{V = \frac{4\pi a^3}{3}}$$

$x = a\cos\theta$, $y = a\sin\theta$

$x^2 + y^2 = a^2$

$$V = \int_{\theta_1}^{\theta_2} \pi y^2 \frac{dx}{d\theta} \, d\theta$$

$$V = 2\int_0^{\pi/2} a^2 \sin^2\theta (a \sin\theta) \, d\theta$$

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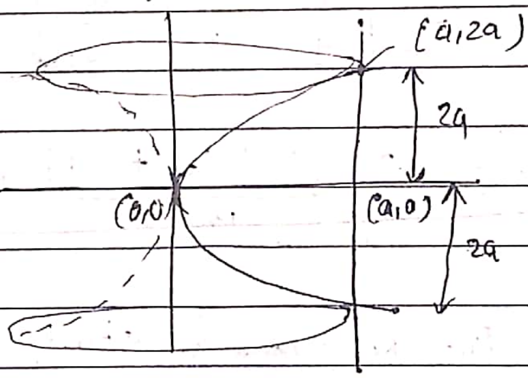
$$V = -2\pi a^3 \int_{\pi/2}^{0} \sin^3\theta \, d\theta$$

$$= -2\pi a^3 \int_{0}^{\pi/2} \sin^3\theta \cdot \cos^0\theta \, d\theta$$

$$= 2\pi a^3 \frac{\Gamma\frac{4}{2}\,\Gamma\frac{1}{2}}{2\,\Gamma\frac{5}{2}} = \frac{\cancel{2}\pi a^3 \; 1 \; \cancel{\Gamma\frac{1}{2}}}{\cancel{2}\;\frac{3}{2}\,\frac{1}{2}\,\cancel{\Gamma\frac{1}{2}}}$$

$$V = \frac{4\pi a^3}{3}$$

1) Find the volume of solid generated by revolving parabola $y^2 = 4ax$, cutoff by latus rectum about tangent at vertex.



$$V = 2\int_{y_1}^{y_2} \pi x^2 \, dy$$

$$V = 2\pi \int_{0}^{2a} \left(\frac{y^2}{4a}\right)^2 dy$$

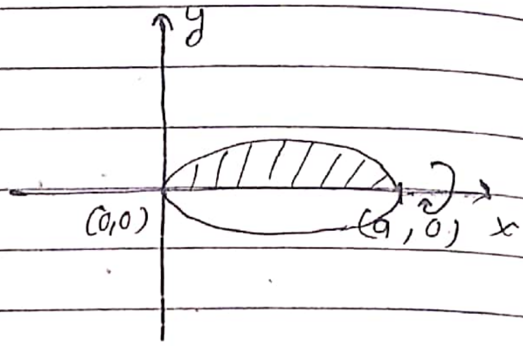
$$= \frac{2\pi}{16a^2} \left[\frac{y^5}{5}\right]_0^{2a}$$

$$= \frac{2\pi}{\cancel{16}a^2} \left[\frac{\overset{2}{\cancel{32}}a^5}{5} - 0\right]$$

$$\boxed{V = \frac{4}{5}\pi a^3}$$

Q. Find the volume of solid Generated by revolution of the loop of the curve about x-axis $3ay^2 = x(x-a)^2$

$$V = \int_{x_1}^{x_2} \pi y^2 \, dx$$



$$= \int_0^a \pi \frac{x(x-a)^2}{3a} dx$$

$$= \frac{\pi}{3a} \int_0^a x^3 - 2a x^2 + a^2 x) \, dx$$

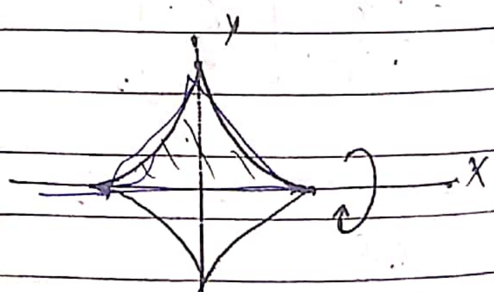
$$= \frac{\pi}{3a} \left[\frac{x^4}{4} - \frac{2a x^3}{3} + \frac{a^2 x^2}{2} \right]_0^a$$

$$= \frac{\pi}{3a} \left[\frac{a^4}{4} - \frac{2a^4}{3} + \frac{a^4}{2} \right]$$

$$= \frac{\pi}{3a} \frac{(3a^4 - 8a^4 + 6a^4)}{12} = \frac{\pi}{3a} \left(\frac{a^{4}}{12} \right) = \frac{\pi a^3}{36}$$

Q. Find the volume of spindle shaped solid Generated by revolving the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ about x-axis

$x = a\cos^3\theta$, $y = a\sin^3\theta$



$$V = \int_{\theta_1}^{\theta_2} \pi y^2 \frac{dx}{d\theta} d\theta$$

$$= 2\pi \int_0^{\pi/2} a^2 \sin^6\theta \; 3a\cos^2\theta \, (-\sin\theta) \, d\theta$$

$y^2 = 4ax$

$x = 2at^2 \quad y = 2at$

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$$V = -6\pi a^3 \int_0^{\pi/2} \sin^7\theta \cos^2\theta \, d\theta$$

$1 + \cancel{\cos\theta} + 2\cos\theta + 1 - \cancel{\cos\theta}$

$2 + 2\cos\theta$

$2(1+\cos\theta)$

$2 \cdot 2\cos^2\theta/2$

$2a\cos\theta/2$

$$= -6\pi a^3 \frac{\Gamma\frac{7+1}{2} \Gamma\frac{2+1}{2}}{2\Gamma\frac{11}{2}}$$

$$= -6\pi a^3 \frac{\Gamma 4 \, \Gamma 3/2}{2\Gamma 11/2} = -\frac{\cancel{6}\pi a^3 \cdot 3 \cdot 2 \cdot 1 \cdot \frac{1}{2}\Gamma\frac{1}{2}}{2 \times \frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\Gamma\frac{1}{2}}$$

$$\boxed{V = -\frac{32\pi a^3}{105}}$$

Find the surface ~~are~~ of the solid formed by revolving the cardiod $r = a(1+\cos\theta)$ about the initial Line.

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}$$

$$= a\sqrt{2(1+\cos\theta)} = a\sqrt{2 \cdot 2\cos^2\theta/2} = 2a\cos\theta/2$$

$$\text{volume} = \int_0^{\pi} 2\pi y \frac{ds}{d\theta} \cdot d\theta = 2\pi \int_0^{\pi} r\sin\theta \cdot 2a\cos\theta/2 \, d\theta$$

$$= 4\pi a \int_0^{\pi} a(1+\cos\theta)\sin\theta \cdot \cos\theta/2 \, d\theta = 4\pi a^2 \int_0^{\pi} 2\cos^2\theta/2 \cdot 2\sin\theta/2\cos\theta/2 \cos\theta/2 \, d\theta$$

$$= 16\pi a^2 \int_0^{\pi} \cos^4\theta/2 \cdot \sin\theta/2 \, d\theta = 16\pi a^2(-2) \int_0^{\pi} \cos^4\frac{\theta}{2} \left[-\sin\frac{\theta}{2} \cdot \frac{1}{2}\right] d\theta$$

$$= -32\pi a^2 \left|\frac{\cos^5\theta/2}{5}\right|_0^{\pi} = \frac{-32\pi a^2}{5}[0-1] = \frac{32\pi a^2}{5} \text{ Ans}$$

$\frac{\theta}{2} = \phi$

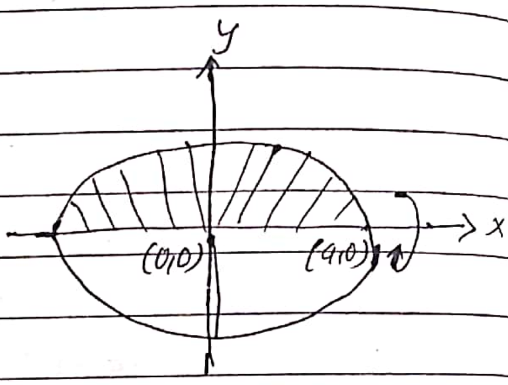
$d\theta = 2d\phi$

Surface Area of solid of Revolution

$$S = \int_{x_1}^{x_2} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad x\text{-axis}$$

$$S = \int_{y_1}^{y_2} 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \qquad y\text{-axis}$$

Q. $x^2+y^2 = a^2$



$$S = 2\int_0^a 2\pi y \sqrt{1+\frac{x^2}{y^2}}\, dx$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \cancel{\frac{-y}{x}} \; \frac{-x}{y}$$

$$S = 4\pi \int_0^a \cancel{y} \frac{\sqrt{x^2+y^2}}{\cancel{y}}\, dx$$

$$S = 4\pi \int_0^a \sqrt{a^2}\, dx$$

$$= 4\pi a \left[x\right]_0^a$$

$$\boxed{S = 4\pi a^2}$$

Q. Find the surface Area of Solid Generated by the revolution of the loop of the curve $x = t^2$, $y = t - \frac{1}{3}t^3$ about x-axis.

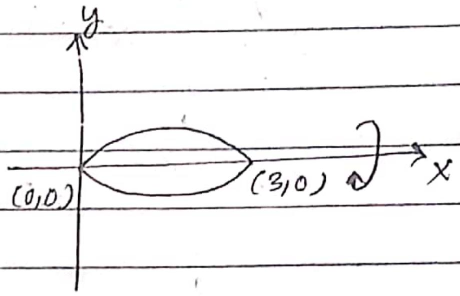
Sol:

$$y = t - \frac{1}{3}t^3$$

$$= t\left(1 - \frac{t^2}{3}\right)$$

$$y = \sqrt{x}\left(1 - \frac{x}{3}\right)$$

$$y = \frac{\sqrt{x}(3-x)}{3}$$



$$9y^2 = x(3-x)^2$$

$$\boxed{9y^2 = x(x-3)^2}$$

$$18y\frac{dy}{dx} = (x-3)^2 + 2x(x-3)$$

$$\frac{dy}{dx} = \frac{(x-3)(x-3+2x)}{18y} = \frac{(x-3)(x-1)}{6y}$$

$$S = \int_{x_1}^{x_2} 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$S = \int_0^3 2\pi y\sqrt{1 + \frac{(x-3)^2(x-1)^2}{36y^2}}\,dx$$

$$S = 2\pi\int_0^3 \cancel{y}\,\frac{\sqrt{36y^2 + (x-3)^2(x-1)^2}}{6\cancel{y}}\,dx$$

$$= \frac{\pi}{3}\int_0^3 \sqrt{\frac{36x(x-3)^2}{9} + (x-3)^2(x-1)^2}\,dx$$

$$S = \frac{\pi}{3}\int_0^3 (x-3)\sqrt{4x + x^2 - 2x + 1}\, dx$$

$$= \frac{\pi}{3}\int_0^3 (x-3)\sqrt{x^2+2x+1}\, dx$$

$$= \frac{\pi}{3}\int_0^3 (x-3)(x+1)\, dx$$

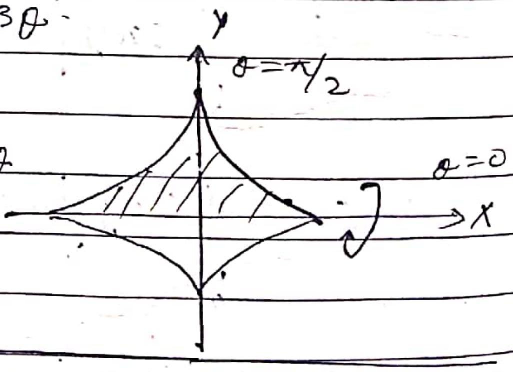
$$= \frac{\pi}{3}\int_0^3 (x^2-2x-3)\, dx$$

$$= \frac{\pi}{3}\left[\frac{x^3}{3} - x^2 - 3x\right]_0^3$$

$$S = \frac{\pi}{3}\left[9-9-9\right] = +3\pi$$

Q Find the surface area of spindle shaped solid generated by revolving Astroid $x^{2/3} + y^{2/3} = a^{2/3}$ about x-axis

Solⁿ $x = a\cos^3\theta \qquad y = a\sin^3\theta$



$$S = \int_{\theta_1}^{\theta_2} 2\pi y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= 2\int_0^{\pi/2} 2\pi a\sin^3\theta \sqrt{(-3a\cos^2\theta\sin\theta)^2 + (3a\sin^2\theta\cdot\cos\theta)^2}\, d\theta$$

$$= 4\pi a \int_0^{\pi/2} \sin^3\theta \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\, d\theta$$

$1+\cos\theta = x$

$0+\sin\theta = \frac{dx}{d\theta}$

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$$S = 4\pi a \int_0^{\pi/2} \cancel{3a\cos\theta\,\sin\theta}\;\sin^3\theta \sqrt{\cos^2\theta+\sin^2\theta}\; d\theta$$

$$= 12\pi a^2 \int_0^{\pi/2} \sin^4\theta\cos\theta\, d\theta.$$

$$= 12\pi a^2 \left[\frac{\Gamma 5/2\;\Gamma 3/2}{2\,\Gamma 7/2}\right]$$

$$= \overset{6}{\cancel{12}}\pi a^2 \left[\frac{\cancel{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}}{\cancel{2}\cdot\frac{5}{2}\cdot\cancel{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}}\right]$$

$1+\cos\theta = t$

$-\sin\theta\, d\theta = dt$

$t^{3}\,dt$

$\frac{t^{4}}{4}\,dt$

$$\boxed{S = \frac{12\pi a^2}{5}}$$

Find the volume of the solid generated by the revolution of the cardioid $r = a(1+\cos\theta)$ about the initial line.

(a) about the initial line $= \int_\alpha^\beta \frac{2\pi}{3} r^3 \sin\theta\, d\theta$..

So, Required Volume $= \int_0^\pi \frac{2\pi}{3} r^3 \sin\theta\, d\theta$

$$= \frac{2\pi}{3}\int_0^\pi a^3(1+\cos\theta)^3 \sin\theta\, d\theta = -\frac{2\pi a^3}{3}\int_0^\pi (1+\cos\theta)^3(-\sin\theta)\, d\theta$$

$$= -\frac{2\pi a^3}{3}\left|\frac{(1+\cos\theta)^4}{4}\right|_0^\pi = -\frac{\pi a^3}{6}[0-16] = \frac{8\pi a^3}{3}$$

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Sequence & series

Bounded Sequence.

Bounded below

$1, 2, 3, \dots l$

$1 < 2, \ 1 < 3, \ 1 < l$

Bounded above

$S_n = -n^2 \ \forall \ n \in N$

$\langle s_n \rangle = \{-1^2, -2^2, -3^2, \dots\}$

$= \{-1, -4, -9 \dots\}$

$= -4 < -1, \ -9 < -1$

Ex. $\langle S_n \rangle = \langle \frac{1}{n} \rangle$

$\langle S_n \rangle = \{1, \frac{1}{2}, \frac{1}{3} \dots 0\}$ — Bounded Seq.

Range $= \{0, 1\}$

Convergent Sequence

$\lim_{n \to \infty} U_n =$ finite (unique)

Ex. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\dots$

$U_n = \frac{n}{n+1}$

$\lim_{n \to \infty} U_n = \lim_{n \to \infty} \frac{n}{n+1}$

$= \lim_{n \to \infty} \frac{n}{n} \frac{1}{(1 + 1/n)} = 1$

$\boxed{\therefore \lim_{n \to \infty} U_n = 1}$

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Divergent sequence

$$\lim_{n\to\infty} u_n = +\infty \text{ or } -\infty$$

Ex - $1, 2, 3, \ldots\ldots n$

$$u_n = n$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} n = \infty$$

$$1 + 2 + 3 + \cdots \infty$$

$$u_n = 1 + 2 + 3 + \cdots\cdots n$$

$$u_n = \frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$$

Oscillating sequence

$$\lim_{n\to\infty} u_n = \text{Not unique}$$

Ex = $1, -1, 1, -1$

$$= 1 - 1 + 1 - 1 + 1 - 1 \cdots\cdots \infty$$

$$u_n = \cancel{1} - \cancel{1} + \cancel{1} - \cancel{1} + 1 - 1 \cdots\cdots n \text{ term}$$

$$\left.\begin{aligned} \lim_{n\to\infty} u_n &= 1 \quad \text{if } n \text{ is odd} \\ \lim_{n\to\infty} u_n &= 0 \quad \text{if } n \text{ is even} \end{aligned}\right\}$$

Geometric series $1+r+r^2+\cdots\infty$

1. convergent if $|r|<1$
2. Divergent if $r \geq 1$
3. oscillating if $r \leq -1$

Ex- $1+\frac{1}{2}+\frac{1}{4}+\cdots$

$r=\frac{1}{2}<1$ (convergent)

Ex $1-\frac{1}{3}+\frac{1}{3^2}-\frac{1}{3^3}+\cdots$

$r=-\frac{1}{3}$

$|r|=\frac{1}{3}<1$ (convergent)

Ex $2+4+8+\cdots\cdots$

$r=2>1$ (Divergent)

Ex- $-2+(-2)^2+(-2)^3+\cdots$

$r=-2<-1$

$\therefore$ (oscillating)

$\frac{1}{2n-1}$

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Comparision Test

If ΣU_n & ΣV_n be two series of positive term $U_n \leq V_n$

(a) ΣV_n is convergent then ΣU_n is convergent if $\lim_{n\to\infty} \frac{U_n}{V_n} = K$ (finite & positive)

(b) ΣV_n is divergent then ΣU_n is also divergent if $\lim_{n\to\infty} \frac{U_n}{V_n} = K$ (finite & positive)

P- Series Test :-

The series $\Sigma \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \cdots \frac{1}{n^p}$

① convergent if $P > 1$
② Divergent if $P \leq 1$

Ex- Test the convergence of

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots \frac{1}{2n-1}$$

Solⁿ $\Sigma U_n = \Sigma \frac{1}{2n-1}$

Let $\Sigma V_n = \Sigma \frac{1}{n}$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{1/2n-1}{1/n} = \lim_{n\to\infty} \frac{n}{2n-1}$$

$$\lim_{n\to\infty} \frac{\cancel{n}(1)}{\cancel{n}(2-1/n)} = \frac{1}{2} \text{ (finite \& positive)}$$

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But.

$$\Sigma V_n = \Sigma \frac{1}{n^1}$$

Here, $P = 1$ (by P series test)

$\therefore$ ΣV_n is divergent

By comparision test,

U_n is divergent

Q. Test the convergence of series

$$\frac{1}{1\cdot 2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \frac{5}{3\cdot 4\cdot 5} + \cdots$$

Soln $U_n = \frac{2n-1}{n(n+1)(n+2)}$ $\quad \frac{n}{n^{3\,2}}$

Let $V_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{(2n-1)\, n^2}{n(n+1)(n+2)} \quad \lim_{n\to\infty} \frac{(2 - 1/n)}{(1+1/n)(1+2/n)}$$

$= 2$ (finite & positive)

$$\Sigma V_n = \Sigma \frac{1}{n^2}.$$

Here $P = 2$ (By P series test)

ΣV_n is convergent

then By comparision Test

ΣU_n is convergent

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Q Test the convergence of the series.

$$\sum_{n=1}^{\infty}\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$$

$$u_n = \frac{\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)\times\left(\sqrt{n^4+1}+\sqrt{n^4-1}\right)}{\left(\sqrt{n^4+1}+\sqrt{n^4-1}\right)}$$

$$u_n = \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{2n^2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{2}{\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}} = 1 \text{ (finite \& positive)}$$

$$\sum v_n = \sum \frac{1}{n^2},$$

$p = 2$ (By p series test)

$\therefore$ $\sum v_n$ is convergent

By convergent test

$\sum u_n$ is convergent

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Test the convergence of series whose n th term is

$$\frac{1}{n}\sin\frac{1}{n}$$

$$u_n = \frac{1}{n}\sin\frac{1}{n} = \frac{1}{n}\left[\frac{1}{n} - \frac{1}{3!}\frac{1}{n^3} + \cdots\right]$$

Let $v_n = \frac{1}{n^2}$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\frac{1}{n}\left[\frac{1}{n} - \frac{1}{3!}\cdot\frac{1}{n^3} + \cdots\right]}{\frac{1}{n^2}}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\cancel{n^2}}{\cancel{n^2}}\left[1 - \frac{1}{3!\,n^2} + \cdots\right] = 1 \text{ (finite \& positive)}$$

$\Sigma v_n = \Sigma \frac{1}{n^2}$ Here $p = 2$ (By P-series test)

Σv_n is convergent

By comparision Σu_n is convergent

Evaluate $\Gamma\left(\frac{1}{n}\right)\cdot\Gamma\left(\frac{2}{n}\right)\cdot\Gamma\left(\frac{3}{n}\right)\cdots\cdots\Gamma\left(\frac{n-1}{n}\right)$

Let $x = \Gamma\left(\frac{1}{n}\right)\cdot\Gamma\left(\frac{2}{n}\right)\cdot\Gamma\left(\frac{3}{n}\right)\cdots\cdots\Gamma\left(\frac{n-1}{n}\right) \longrightarrow (1)$

writing in reverse order, we have

$x = \Gamma\left(\frac{n-1}{n}\right)\Gamma\left(\frac{n-2}{n}\right)\cdots\cdots\Gamma\left(\frac{3}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{1}{n}\right) \longrightarrow (2)$

Multiplying (1) & (2), we get.

$$x^2 = \left(\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{n-1}{n}\right)\right)\left(\Gamma\left(\frac{2}{n}\right)\cdot\Gamma\left(\frac{n-2}{n}\right)\right)\cdots\cdots\left(\Gamma\left(\frac{n-1}{n}\right)\cdot\Gamma\left(\frac{1}{n}\right)\right)$$

$\left\{\Gamma(m)\,\cancel{\Gamma(n-m)} = \frac{\pi}{\sin m\pi}\right\}$

$$= \left(\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)\right)\left(\Gamma\left(\frac{2}{n}\right)\cdot\Gamma\left(1-\frac{2}{n}\right)\right)\cdots\cdots\left(\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(\frac{1}{n}\right)\right)$$

$$= \frac{\pi}{\sin\frac{1}{n}\pi}\cdot\frac{\pi}{\sin\frac{2\pi}{n}}\cdot\frac{\pi}{\sin\frac{3\pi}{n}}\cdots\cdots\frac{\pi}{\sin\frac{(n-1)\pi}{n}}$$

$1-n = \frac{1}{n}$

$\cancel{1 = \frac{1}{n}+n}$

or $x^2 = \dfrac{\pi^{n-1}}{\sin\frac{\pi}{n}\,\sin\frac{2\pi}{n}\,\sin\frac{3\pi}{n}\cdots\cdots\sin\frac{(n-1)\pi}{n}}$

$1-n = 1/n$

$1-1/n = n$

$\left[\because \Gamma(m)\,\Gamma(m-1) = \frac{\pi}{\sin m\pi}\right]$ $\frac{n-1}{n} = 0$

from advanced Trig. we know that

$$\frac{\sin n\theta}{\theta} = 2^{n-1}\sin\left(\theta+\frac{\pi}{n}\right)\cdot\sin\left(\theta+\frac{2\pi}{n}\right)\cdot\sin\left(\theta+\frac{3\pi}{n}\right)\cdots\cdots\sin\left\{\theta+\frac{(n-1)\pi}{n}\right\}$$

Taking limit as $\theta\to 0$

$$n = 2^{n-1}\sin\frac{\pi}{n}\,\sin\frac{2\pi}{n}\,\sin\frac{3\pi}{n}\cdots\sin\frac{(n-1)\pi}{n}$$

$$x^2 = \frac{\pi^{n-1}}{\left(\frac{n}{2^{n-1}}\right)}$$

or $$x^2 = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}$$

Q. P.T $\Gamma\frac{1}{9}\,\Gamma\frac{2}{9}\,\Gamma\frac{3}{9}\,\cdots\cdots\,\Gamma\frac{8}{9} = \frac{16}{3}\pi^4$

Sol^n let $x = \Gamma\frac{1}{9}\,\Gamma\frac{2}{9}\,\Gamma\frac{3}{9}\,\cdots\cdots\,\Gamma\frac{8}{9} \longrightarrow (1)$

$$x = \Gamma\frac{8}{9}\cdot\Gamma\frac{7}{9}\,\cdots\cdots\,\Gamma\frac{1}{9} \longrightarrow (2)$$

$$\left(\Gamma\frac{1}{9}\,\Gamma\left(1-\frac{1}{9}\right)\right)\left(\Gamma\frac{2}{3}\,\Gamma\left(1-\frac{2}{9}\right)\right)\cdots\cdots\left(\Gamma\frac{n-1}{9}\cdot\Gamma\left(1-\frac{n}{9}\right)\right)$$

$$x = \frac{(2\pi)^{\frac{9-1}{2}}}{\sqrt{9}} = \frac{(2\pi)^4}{3} = \frac{16}{3}\pi^4$$

Q P.T $\Gamma n\,\Gamma\left(n+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma 2n}{2^{2n-1}}$

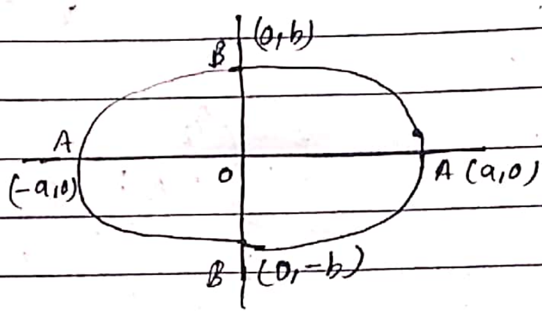
Sol^n : we know that

$$\int_0^{\pi/2} \sin^{2m-1}\theta\cdot\cos^{2n-1}\theta\,d\theta = \frac{\Gamma m\,\Gamma n}{2\,\Gamma(m+n)}$$

The volume of solid formed by the revolution about the x-axis of the area bounded by the curve $y=f(x)$ the x-axis and the two ordinates $x=a$, $x=b$ is

$$V = \int_a^b \pi y^2 \, dx$$

Q Find the volume & surface area of the solid generated about (i) x-axis (ii) y-axis of the ellipse



Given, curve is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$V = \int_{x_1}^{x_2} \pi y^2 \, dx$$

$$V = 2\pi \int_0^a b^2\left(1 - \frac{x^2}{a^2}\right) dx$$

$$= 2\pi \left[\int_0^a b^2 \, dx - \int_0^a \frac{b^2 x^2}{a^2} dx \right]$$

$$= 2\pi \left[b^2 a - \frac{b^2}{a^2} \times \frac{a^3}{3} \right]$$

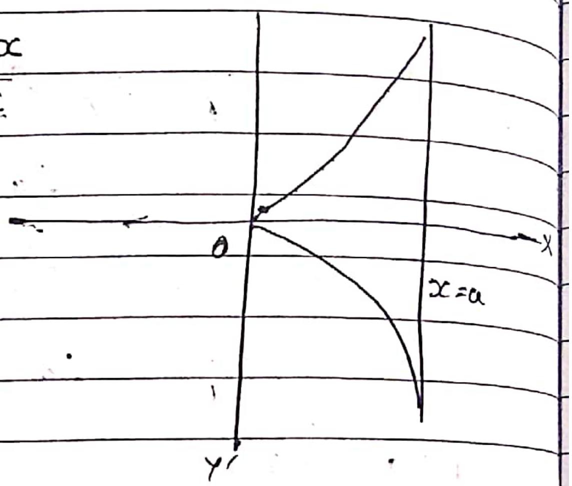
$$= 2\pi \left[\frac{2b^2 a}{3} \right]$$

$$\boxed{V = \frac{4}{3} \pi a b^2}$$

Q Find the volume of the solid formed by the revolution of the curve $y^2(a-x)=a^2x$ about its asymptote

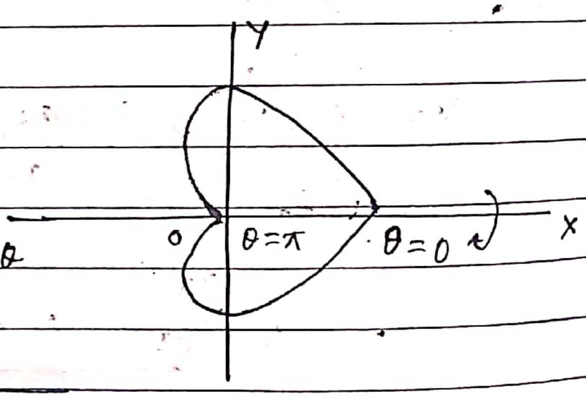
Sol^n $y^2(a-x)=a^2x$

or $y^2 = \frac{a^2x}{a-x}$



Q Find the volume of the solid generated by the revolution of cardioid $r=a(1+\cos\theta)$ about the initial line

$$V = \pi\int_{\pi}^{0} y^2\,dx$$



$$= \pi\int_{\pi}^{0} (r\sin\theta)^2 \frac{dx}{d\theta} \times d\theta$$

$$= \pi\int_{\pi}^{0} (r\sin\theta)^2 \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$= \pi\int_{\pi}^{0} a^2(1+\cos^2\theta)^2 \sqrt{a^2(1+\cos\theta)^2+a^2\sin^2\theta}$$

$$= \pi a^3\int_{\pi}^{0} 4\cos^2\theta/_2 \sqrt{4\cos^2\theta/2 + "}$$

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Directional Derivative

1) Find the directional derivatives of $\phi = 3x^2yz - 4y^2z^3$ in the direction of the vector $3\hat{i} - 4\hat{j} + 2\hat{k}$ at point $(2, -1, 3)$

Soln Directional derivative, $\dfrac{d\phi}{ds} = \hat{a} \cdot \text{grad}\,\phi$

Given, $\phi = 3x^2yz - 4y^2z^3$

and $\vec{a} = 3\hat{i} - 4\hat{j} + 2\hat{k}$ $\quad \hat{a} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{3\hat{i} - 4\hat{j} + 2\hat{k}}{\sqrt{29}}$

$|\vec{a}| = \sqrt{9+16+4}$

$|\vec{a}| = \sqrt{29}$

$$\text{grad}\,\phi = \nabla \cdot \phi = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] \cdot \left(3x^2yz - 4y^2z^3\right)$$

$$= \hat{i}(6xyz) + \hat{j}(3x^2z - 8z^3y) + \hat{k}(3x^2y - 12y^2z^2)$$

Now,

$$\frac{d\phi}{ds} = \frac{3\hat{i} - 4\hat{j} + 2\hat{k}}{\sqrt{29}} \cdot \left\{(6xyz)\hat{i} + (3x^2z - 8z^3y)\hat{j} + (3x^2y - 12y^2z^2)\hat{k}\right\}$$

$$= \frac{1}{\sqrt{29}}\left(18xyz - 12x^2z + 32z^3y + 6x^2y - 24y^2z^2\right)$$

at point $(2, -1, 3)$

$$= -\frac{1356}{\sqrt{29}} \text{ Ans}$$

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Q. What is the directional derivative of $\phi = xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of normal to the surface $x \log z - y^2 = 4$ at $(-1,2,1)$

$$\phi = xy^2 + yz^3$$

$$\text{grad}\,\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy^2 + yz^3)$$

$$= y^2\hat{i} + (2xy + z^3)\hat{j} + 3yz^2\hat{k}$$

A vector normal to the surface is

$$\vec{a} = \nabla(x\log z - y^2 - 4)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x\log z - y^2 - 4)$$

$$= (\log z)\hat{i} - 2y\hat{j} + \frac{x}{z}\hat{k}$$

at point $(-1,2,1)$

$$\vec{a} = (\log 1)\hat{i} - 2(2)\hat{j} + \hat{k}\left(\frac{-1}{1}\right)$$

$$= -4\hat{j} - k$$

$$|\vec{a}| = \sqrt{16+1} = \sqrt{17} \qquad \hat{a} = \frac{-4j - k}{\sqrt{17}}$$

Now,

$$\frac{d\phi}{ds} = \hat{a} \cdot \text{grad}\,\phi$$

$$= \frac{-4j - k}{\sqrt{17}} \cdot [y^2\hat{i} + (2xy + z3)\hat{j} + 3yz^2\hat{k}]$$

$$= \frac{1}{\sqrt{17}}(-8xy - 4z^3 - 3yz^2)$$

At pt. $(2,-1,1)$

$$\frac{d\phi}{ds} = \frac{15}{\sqrt{17}} \text{ Ans}$$

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Q Find the constant a and b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1, -1, 2)$

Let $f_1 = ax^2 - byz - (a+2)x$
& $f_2 = 4x^2y + z^3 - 4$

$$\text{grad } f_1 = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\{ax^2 - byz - (a+2)x\}$$

$$= [2ax - (a+2)]\hat{i} + (-bz)\hat{j} + (-by)k$$

at point $(1,-1,2)$

$$\text{grad } f_1 = (a-2)\hat{i} - 2b\hat{j} + b\hat{k}$$

Also, $\text{grad } f_2 = (8xy)\hat{i} + (4x^2)\hat{j} + (3z^2)\hat{k}$

At point $(1,-1,2)$

$$\text{grad } f_2 = -8\hat{i} + 4\hat{j} + 12\hat{k}$$

$\because$ The given surface cut orthogonally at $(1,-1,2)$

$\therefore$ $(\text{grad } f_1) \cdot (\text{grad } f_2) = 0$

$-8(a-2) - 8b + 12b = 0$

$-8a + 16 - 8b + 12b = 0$

$-8a + 4b = -16$

$\cancel{-}4(2a - b) = \cancel{-}16$

$\boxed{2a - b = 4}$

Also pt. $(1,-1,2)$ lies on surfaces if it lies on surface $ax^2 - byz = (a+2)x$

$a + 2b = a + 2 \Rightarrow 2b = 2 \Rightarrow \boxed{b = 1}$

$\therefore 2a = 5$

$\boxed{a = 5/2}$

D'Alembert's Ratio Test

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = K$$

$K < 1$, converge

$K > 1$, diverge

$U_n = \frac{n^2}{n!}$

Q. $\frac{1^2}{1!} + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \cdots\cdots$ Test convergence

$$U_n = \frac{n^2}{n!} = \frac{n \cdot n}{n(n-1)!} = \frac{n}{(n-1)!}$$

$$U_{n+1} = \frac{n+1}{(n+\cancel{1}-\cancel{1})!} = \frac{n+1}{n!}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{\frac{n+1}{n!}}{\frac{n}{(n-1)!}} = 0 < 1$$

$\therefore$ It is convergent

Q $\left(\frac{1}{3}\right)^2 + \left(\frac{1 \cdot 2}{3 \cdot 5}\right)^2 + \left(\frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7}\right)^2 + \cdots\cdots\cdots$

$$U_n = \left(\frac{1 \cdot 2 \cdot 3 \cdots\cdots n}{3 \cdot 5 \cdot 7 \cdots\cdots (2n+1)}\right)^2$$

$$U_{n+1} = \left(\frac{1 \cdot 2 \cdot 3 \cdots\cdots n(n+1)}{3 \cdot 5 \cdot 7 \cdots\cdots (2n+1)(2n+3)}\right)^2$$

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$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{(n+1)^2}{(2n+3)^2} \qquad \frac{\cancel{n}(n+1)}{(2n+1)(2n+3)} \times \frac{(2n+1)}{(\cancel{n})}$$

$$\lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^2}{\left(2+\frac{3}{n}\right)^2} = \frac{1}{4} < 1 \text{ (positive)}$$

$\therefore$ convergent

Q. $$\sum_{n=1}^{\infty} \frac{1}{x^n + x^{-n}} \; ; \; x > 0$$

$$U_n = \frac{1}{\frac{x^n+1}{x^n}} = \frac{x^n}{x^{2n}+1} \quad \therefore U_{n+1} = \frac{x^{n+1}}{x^{2n+2}+1}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{x(x^{2n}+1)}{x^2 \cdot x^{2n}+1} \qquad \frac{x^{n+1}}{x^{2n} \cdot x^2+1} \times \frac{x^{2n}+1}{x^n}$$

$0 < x < 1$

$$\lim_{n\to\infty} \frac{x(x^{2n}+1)}{x^2 \cdot x^{2n}+1} = \frac{x(0+1)}{x^2 \cdot (0)+1} = x < 1$$

$\therefore$ convergent series

$$x > 1 \quad \lim_{x\to\infty} \frac{x(x^{2n}+1)}{x^2 x^{2n}+1} = x \lim_{x\to\infty} \frac{\left(1+\frac{1}{x^{2n}}\right)}{\left(x^2+\frac{1}{x^{2n}}\right)}$$

$$= x\left[\frac{1+0}{x^2+0}\right] = \frac{1}{x} < 1$$

$\therefore$ convergent series.

Raabe's Test

If ΣU_n is a positive term series :

$$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) = \lambda$$

$\lambda > 1$ – converge
$\lambda < 1$ – Diverge

Q. $\left(\frac{1}{3}\right)^2 + \left(\frac{1\cdot 4}{3\cdot 6}\right)^2 + \left(\frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}\right)^2 + \cdots\cdots$

$$U_n = \left(\frac{1\cdot 4\cdot 7\cdot\cdots\cdot(3n-2)}{3\cdot 6\cdot 9\cdot\cdots\cdots 3(n)}\right)^2$$

$$U_n = \left(\frac{3n-2}{3n}\right)^2 \qquad U_{n+1} = \left(\frac{3n+1}{3n+3}\right)^2$$

$$\frac{U_{n+1}}{U_n} = \frac{(3n+1)^2}{(3n+3)^2} \times \frac{(3n)^2}{(3n-2)^2} \Rightarrow \lim_{n\to\infty}\left(\frac{3+1/n}{3+3/n}\right)^2 = 1$$

Ratio test fail

Now, Apply Rabbe's Test

$$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\left(\frac{3n+3}{3n+1}\right)^2 - 1\right)$$

$$= \lim_{n\to\infty} \frac{n(6n+4)(2)}{(3n+1)^2} = \lim_{x\to\infty} \frac{2\left(6+4/n\right)}{\left(3+1/n\right)^2} = \frac{4}{3} > 1$$

Series is convergent

$$\frac{3}{7}x + \frac{3\cdot 6}{7\cdot 10}x^2 + \frac{3\cdot 6\cdot 9\cdot x^3}{7\cdot 10\cdot 13} + \cdots \quad (x>0)$$

$$U_n = \frac{3\cdot 6\cdot 9\cdots 3n}{7\cdot 10\cdot 13\cdots(3n+4)} \cdot x^n$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{(3n+3)}{(3n+7)}\cdot x = \lim_{n\to\infty}\left(\frac{3+3/n}{3+7/n}\right)\cdot x$$

$$= x$$

Ratio Test fail.

Apply Rabbe's Test

$$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) = \lim_{n\to\infty}\left[\frac{3n+7}{3n+3} - 1\right]$$

$$= \lim_{n\to\infty}\left[\frac{4n}{3n+3}\right] = \frac{4}{3} > 1$$

$\therefore$ series converge at $x=1$

$$\cancel{\frac{(3n-2)(3n+1)^2}{3n\,(3n+3)^2}} \times \cancel{\frac{3n}{(3n-2)}}$$

Q Test the convergence of series

$$x^2 + \frac{2^2}{3\cdot 4} x^4 + \frac{2^2\cdot 4^2}{3\cdot 4\cdot 5\cdot 6} x^6 + \cdots$$

$$U_n = \frac{2^2\cdot 4^2 \cdots (2n)^2}{3\cdot 4\cdot 5\cdot 6 \cdot (2n+1)(2n+2)} x^{2n+2}$$

$$U_{n+1} = \frac{2^2\cdot 4^2 \cdots\cdots (2n)^2 (2n+2)^2}{3\cdot 4\cdot 5\cdot 6 \cdots\cdots (2n+1)(2n+2)(2n+3)(2n+4)} x^{2n+4}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{(2n+2)^2}{(2n+3)(2n+4)} x^2 = \lim_{n\to\infty} \frac{(2+2/n)^2}{(2+3/n)(2+4/n)} x^2$$

$$= x^2$$

By ratio test Let $x^2 = 1$

Applying Raabe's theorem,

$$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{(2n+2)^2}{(2n+3)(2n+4)} - 1\right)$$

$$= \lim_{n\to\infty} \frac{6n^2+8n}{4n^2+8n+4}$$

$$= \lim_{n\to\infty} \frac{6+8/n}{4+8/n+4/n^2} = \frac{6}{4} > 1$$

$\therefore$ Series is convergent

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Q Test the convergence of

$$1 + a + \frac{a(a+1)}{1\cdot 2} + \frac{a(a+1)(a+2)}{1\cdot 2\cdot 3} + \cdots$$

$$U_n = \frac{a(a+1)(a+2)\cdots(a+n-1)}{1\cdot 2\cdot 3\cdots n}$$

$$U_{n+1} = \frac{a(a+1)(a+2)\cdots(a+n-1)(a+n)}{1\cdot 2\cdot 3\cdots n(n+1)}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{a+n}{n+1} = \lim_{n\to\infty} \left(\frac{a/n + 1}{1 + 1/n}\right) = 1$$

$\therefore$ Ratio Test fails

Applying Raabe's theorem,

$$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) = \lim_{n\to\infty} n\left[\frac{n+1}{a+n} - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{\cancel{n}+1-a-\cancel{n}}{a+n}\right]$$

$$= \lim_{n\to\infty} n\left[\frac{-a+1}{a+n}\right]$$

$$= \lim_{n\to\infty} \frac{1-a}{a/n + 1} = 1-a$$

By Raabe's test

convergent if $1-a > 1 \Rightarrow a < 0$

divergent if $1-a < 1 \Rightarrow a > 0$

If $1-a = 1 \Rightarrow a = 0$

Series is convergent if $a \leq 0$

divergent if $a > 0$

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Logarithmic Test:

Series ΣU_n of positive terms is

(i) convergent $\lim_{n\to\infty}\left(n \log \frac{U_n}{U_{n+1}}\right) > 1$

(ii) Divergent $\lim_{n\to\infty}\left(n \log \frac{U_n}{U_{n+1}}\right) < 1$

1) Test for convergence of series

$$1 + \frac{2^1 x}{2!} + \frac{3^2 x^2}{3!} + \cdots, \; x > 0$$

Sol

$$U_n = \frac{n^{n-1} x^{n-1}}{n!} \qquad U_{n+1} = \frac{(n+1)^n x^n}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{(n+1)^n x^n}{(n+1)!} \cdot \frac{n!}{n^{n-1} \cancel{x}^{n-1}} = \frac{(n+1)^n \cancel{n!}}{(n+1)\cancel{n!}\, n^{n-1}} x$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{(n+1)^{n-1}}{n^{n-1}} x$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \left(\frac{1+n}{n}\right)^{n-1} \cdot x = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n-1} x$$

$$= e x$$

By D.A RT

Convergent if $ex < 1$ i.e $x < 1/e$

Divergent if $ex > 1$ i.e $x > 1/e$

Test fail if $ex = 1$ i.e. $x = 1/e$

$$\log \frac{u_n}{u_{n+1}} = \log \left(\frac{n}{n+1}\right)^{n-1} e$$

$$= (n-1) \log \left(\frac{1}{1+1/n}\right) + \log_e e$$

$$= (1-n) \log\left(1+1/n\right) + 1$$

Now,

$$\lim_{n \to \infty} \left(n \log \frac{u_n}{u_{n+1}}\right) = n\left[(1-n) \log\left(1+1/n\right) + 1\right]$$

$$\lim_{n \to \infty} = n(1-n)\left\{\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right) + n$$

$$\lim_{n \to \infty} = (n-n^2)\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} \cdots\right) + n$$

$$\lim_{n \to \infty} = \left(1 - \frac{1}{2n} + \cdots\right) - \cancel{1 + n} + \frac{1}{2} + \frac{1}{3n} \cdots \cancel{+n}$$

$$= \frac{3}{2} > 1$$

$\therefore$ series is convergent

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D'Alembert's Ratio test (problems)

Test the convergence of the series:

$$1^2 + 2^2x^1 + 3^2x^2 + \cdots$$

Sol:
$U_n = n^2 x^{n-1}$

$U_{n+1} = (n+1)^2 x^n$

$$\frac{U_{n+1}}{U_n} = \frac{(n+1)^2 x^n}{n^2 x^{n-1}}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^2 x\right) = x$$

By DART, ΣU_n is convergent if $x<1$
divergent if $x>1$

if $x=1$

$U_n = n^2$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} n^2 = \infty$$

at $x=1$ U_n is divergent

Hence ΣU_n is convergent if $x<1$
& divergent if $x \geq 1$

Test the convergence of $\frac{x}{1\cdot 2} + \frac{x^2}{3\cdot 4} + \cdots$

Sol: $U_n = \frac{x^n}{(2n-1)2n}$ $U_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+2)}$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(2n+1)(2n+2)} \times \frac{(2n-1)(2n)}{x^n}$$

$$= \lim_{n\to\infty} \frac{(2n-1)(2n)}{(2n+1)(2n+2)} x = \lim_{n\to\infty} \frac{(2-1/n)(2)}{(2+1/n)(2+2/n)} x$$

$$\boxed{\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = x}$$

By DART,
U_n is convergent if $x<1$
divergent if $x>1$

if $x=1$

$U_n = \frac{1}{(2n-1)2n}$ $V_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{n^2}{(2n-1)2n}$$

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$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{1}{(2-1/n)\times 2} = \frac{1}{4} < 1$$

$\therefore$ Series is convergent.

Hence ΣU_n is convergent if $x \leq 1$
divergent if $x > 1$

Test the convergent of $1 + \frac{x}{2} + \frac{x^2}{5} + \ldots \frac{x^n}{n^2+1}$.

Sol^n $U_n = \frac{x^n}{n^2+1}$, $U_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)^2+1} \times \frac{n^2+1}{x^n} = \lim_{n\to\infty} \frac{n^2+1}{n^2+2n+2} x$$

$$= \lim_{n\to\infty} \frac{1+1/n^2}{2+2/n+2/n^2} x \doteq x$$

i.e. By DART,
ΣU_n is convergent if $x < 1$
divergent if $x > 1$.

if $x = 1$

$U_n = \frac{1}{n^2+1}$ $V_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+1/n^2} = 1 \text{ (finite)}$$

$V_n = \frac{1}{n^2}$

$P = 2 > 1$

$\therefore$ convergent, By comparision test
ΣU_n is also convergent

i.e ΣU_n convergent
if $x \leq 1$
divergent
if $x > 1$

Power series

Radius of convergence :-

consider the power series $\sum_{n=0}^{\infty} a_n z^n$

The Radius of convergence R of power series is defined by (cauchy's theorem on Limits)

(i) $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}$ (ii) $\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

Q.1 $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{\lfloor n} = 1 + \frac{1}{\lfloor 1} z + \frac{1}{\lfloor 2} z^2 + \cdots\cdots + \frac{1}{\lfloor n} z^n$

$a_n = \frac{1}{\lfloor n}$; $a_{n+1} \ \frac{1}{\lfloor n+1}$

$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{\lfloor n}{\lfloor n+1}\right|$

$= \lim_{n\to\infty} \left|\frac{\cancel{\lfloor n}}{(n+1)\cancel{\lfloor n}}\right| = 0$

$\therefore \boxed{R = \infty}$

Q2 $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^n+3}$

solⁿ Here, $a_n = \frac{1}{2^n+3}$

$a_{n+1} = \frac{1}{2^{n+1}+3}$

$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{2^n+3}{2^{n+1}+3}\right| = \lim_{n\to\infty} \left|\frac{\cancel{2^n}(1+3/2^n)}{\cancel{2^n}(2+3/2^n)}\right| = \frac{1}{2}$

$\boxed{\therefore R = 2}$

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Q. $f(z) = \sum_{n=0}^{\infty} \frac{\lfloor n}{n^n} z^n$

Sol^n Here

$$a_n = \frac{\lfloor n}{n^n} \qquad a_{n+1} = \frac{\lfloor n+1}{(n+1)^{n+1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{\lfloor n+1}{(n+1)^{n+1}} \times \frac{n^n}{\lfloor n}$$

$$= \frac{\cancel{(n+1)}\,\cancel{\lfloor n}}{\cancel{(n+1)}\,(n+1)^n} \cdot \frac{n^n}{\cancel{\lfloor n}}$$

$$\frac{a_{n+1}}{a_n} = \frac{n^n}{(n+1)^n}$$

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{1}{\left(\frac{n+1}{n}\right)^n} \right|$$

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{1}{\left(1+\frac{1}{n}\right)^n} \right|$$

$$\frac{1}{R} = \frac{1}{e}$$

$$\therefore \boxed{R = e \approx 2.7}$$

Q $f(z) = \sum_{n=0}^{\infty} (5+12i)^n z^n$

$$a_n = (5+12i)^n$$
$$a_{n+1} = (5+12i)^{n+1}$$

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(5+12i)^{n+1}}{(5+12i)^n} \right|$$

$$= \lim_{n\to\infty} |5+12i|$$

$$= \sqrt{5^2+12^2} = 13$$

$$\therefore \frac{1}{R} = 13$$

$$\therefore \boxed{R = 1/13}$$

Half Range Series

Half range fourier Sine series :-

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad , \quad 0 < x < l$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx.$$

Half range fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad , \quad 0 < x < l$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

Q. Find Half Range cosine series of

$f(x) = x \quad , 0 < x < 2.$

Sol^n

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$$

$$a_0 = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}\int_0^2 x\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 = \frac{1}{2}\left[2-0\right]$$

$\therefore \boxed{a_0 = 1}$

$$a_n = \frac{2}{2}\int_0^2 x \cdot \cos\left[\frac{n\pi x}{2}\right] dx$$

$$= \left[\frac{x \sin \frac{n\pi x}{2}}{n\pi/2} + \frac{\cos(n\pi x/2)}{(n\pi/2)^2}\right]_0^2$$

$\sin n\pi = 0$

$\cos n\pi = (-1)^n$

$$a_n = \left[\frac{x \sin(n\pi)}{n\pi/2} + \frac{\cos n\pi}{(n\pi/2)^2} - 0 - \frac{1}{(n\pi/2)^2} \right]$$

$$a_n = \left[0 + \frac{(-1)^n 4}{n^2\pi^2} - \frac{4}{n^2\pi^2} \right]$$

$$a_n = \frac{4}{n^2\pi^2} \left[(-1)^n - 1 \right]$$

$$f(x) = 1 + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left((-1)^n - 1 \right) \cos\left(\frac{n\pi x}{2}\right)$$

Q Find the sine series of

$$f(x) = x \quad ; \; 0 < x < 2$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

$$b_n = \frac{2}{2} \int_0^2 \underbrace{x}_{u} \underbrace{\sin\left(\frac{n\pi x}{2}\right) dx}_{dv}$$

$$b_n = \left[\frac{-x \cos(n\pi x/2)}{(n\pi/2)} + \frac{\sin(n\pi x/2)}{n^2\pi^2/4} \right]_0^2$$

$$b_n = \left[-2 \frac{\cos n\pi}{n\pi/2} + \frac{\sin n\pi}{n^2\pi^2/4} - 0 - 0 \right]$$

$$\boxed{b_n = \frac{-2(-1)^n}{n\pi/2} = \frac{-4}{n\pi} (-1)^n}$$

$$f(x) = \frac{-4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\left(\frac{n\pi x}{2}\right)$$

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Q Find the half range sine series of

$f(x) = x(\pi - x)$ in $0 < x < \pi$

and P.T $\frac{\pi^2}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} \cdots\cdots$

soln $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\pi}\right)$

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin nx}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\, dx$$

$$b_n = \frac{2}{\pi}\left[(x\pi - x^2)\cdot\int \sin nx\, dx - \int \frac{d}{dx}(x\pi - x^2)\int \sin nx\, dx\, dx\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left[-(x\pi - x^2)\frac{\cos nx}{n} + \int (\pi - 2\pi)\frac{\cos nx}{n}\, dx\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left[-(x\pi - x^2)\frac{\cos nx}{n} + \left[-(\pi - 2x)\frac{\sin nx}{n^2} + (-2)\frac{\cos nx}{n^3}\right.\right.$$

$$b_n = \frac{2}{\pi}\left[\frac{-2(-1)^n}{n^3} + \frac{2}{n^3}\right]$$

$$b_n = \frac{4}{\pi n^3}\left[1 - (-1)^n\right] \cancel{\sin}$$

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1 - (-1)^n}{n^3}\right]\sin nx$$

$$f(x) = \frac{4}{\pi}\left[\left[\frac{2}{1^3}\right]\sin x + 0 + \frac{2}{3^3}\sin 3x + \frac{2}{5^3}\sin 5x\right]$$

$$x(\pi - x) = \frac{8}{\pi}\frac{\sin x}{1^3} + \frac{8}{\pi}\frac{\sin 3x}{3^3} + \frac{8}{\pi}\frac{\sin 5x}{5^3} + \cdots$$

Put $x = \pi/2$

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$$\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots$$

Q. obtain the fourier expansion of $x \sin x$ as a cosine series in $[0, \pi]$

$$f(x) = x \sin x$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\pi}\right)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \underset{u}{x}\, \underset{dv}{\sin x}\, dx = \frac{1}{\pi}\left[x\cdot(-\cos x) - \int(-\cos x)dx\right.$$

$$= \frac{2}{\pi}\Big[x(-\cos x) + \sin x\Big]_0^{\pi}$$

$$= \frac{1}{\pi}\Big[\sin x - x\cos x\Big]_0^{\pi}$$

$$a_0 = \frac{2}{\pi}\Big[\cancel{\sin\pi}^{0} - \cancel{\sin 0}^{0} - \overbrace{\pi\cos\pi}^{(-1)} + 0\Big] = 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x\cdot\cos nx\, dx = \frac{\cancel{2}}{\cancel{2}\pi}\int_0^{\pi} x(2\cos nx\cdot\sin x)dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\left\{\sin(n+1)x - \sin(n-1)x\right\}dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \underset{u}{x}\, \underset{dv}{\sin(n+1)x}\, dx - \int_0^{\pi} x\sin(n-1)x\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(x\left\{\frac{-\cos(n+1)x}{(n+1)}\right\} + \frac{\sin(n+1)x}{(n+1)^2}\right)_0^{\pi}\right.$$

$$\left. - \left(x\left\{\frac{-\cos(n-1)x}{(n-1)}\right\} + \frac{\sin(n-1)x}{(n-1)^2}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{x\cos(n+1)x}{n+1} + \cancel{\frac{\sin(n+1)x}{(n+1)^2}}^{\,0} + \frac{x\cos(n-1)x}{n-1} - \cancel{\frac{\sin(n-1)x}{(n-1)^2}} \right]$$

$$= \frac{1}{\pi}\left[\frac{\pi\cos(n-1)\pi}{n-1} - \frac{\pi\cos(n+1)\pi}{n+1} \right]$$

$$a_n = \frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1} \quad (n \neq 1)$$

$$f(x) = 1 + \sum_{n=2}^{\infty}\left[\frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1} \right]\cos nx$$

Ans

2) Expand $f(x) = \frac{1}{4} - x$, if $0 < x < \frac{1}{2}$

$x - \frac{3}{4}$, if $\frac{1}{2} < x < 1$

as the fourier series of sine terms.

Soln $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{1}$

$$b_n = \frac{2}{1}\int_0^1 f(x)\sin n\pi x \, dx$$

$$= 2\left[\int_0^{1/2}\left(\frac{1}{4} - x\right)\sin n\pi x\, dx + \int_{1/2}^{1}\left(x - \frac{3}{4}\right)\sin n\pi x \, dx \right]$$

$$= 2\left[\left(\frac{1}{4} - x\right)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right) \right]_0^{1/2} +$$

$$\left[\left(x - \frac{3}{4}\right)\left(\frac{-\cos n\pi x}{n\pi}\right) + \frac{\sin n\pi x}{n^2\pi^2} \right]_{1/2}^{1}$$

$$= 2\left[\left| \left(x - \frac{1}{4}\right)\left(\frac{\cos n\pi x}{n\pi}\right) - \frac{\sin n\pi x}{n^2\pi^2} \right|_0^{1/2} - \right.$$

$$\left. \left| \left(x - \frac{3}{4}\right) \frac{\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2} \right|_{1/2}^{1} \right]$$

$$= 2\left[\left(\frac{1}{2} - \frac{1}{4}\right) \frac{\cos n\pi/2}{n\pi} - \left(-\frac{1}{4}\right) \frac{\cos 0}{n\pi} - \frac{\sin n\pi/2}{n^2\pi^2} + \cancel{\frac{\sin 0}{n^2\pi^2}} \rightarrow 0 \right.$$

$$- \left(1 - \frac{3}{4}\right) \frac{\cos n\pi}{n\pi} + \left(\frac{1}{2} - \frac{3}{4}\right) \frac{\cos n\pi/2}{n\pi} + \cancel{\frac{\sin n\pi}{n^2\pi^2}} \rightarrow 0$$

$$\left. - \frac{\sin n\pi/2}{n^2\pi^2} \right]$$

$$= 2\left[\cancel{\frac{\cos n\pi/2}{4n\pi}} + \frac{1}{4n\pi} - \frac{2 \sin n\pi/2}{n^2\pi^2} - \frac{\cos n\pi}{4n\pi} - \cancel{\frac{1}{4n\pi} \cos \frac{n\pi}{2}} \right]$$

$$b_n = \frac{1}{2n\pi}\left[1 - (-1)^n\right] - \frac{4 \sin n\pi/2}{n^2\pi^2}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \left\{ \frac{1}{2n\pi}\left[1 - (-1)^n\right] - \frac{4 \sin n\pi/2}{n^2\pi^2} \right\} \sin n\pi x$$

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$$\frac{2\pi^4}{5} - \frac{2\pi^4}{9} = 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{18\pi^4 - 10\pi^4}{45} = 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{8\pi^4}{45\times 16\ 2} = \sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\boxed{\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots}$$

Hence proved.

Rank-nullity theorem

$$\boxed{\text{rank}(A) + \text{nullity}(A) = n}$$

Lagarange's method of undetermined Multipliers

Q. Solve the following $f = x_1 x_2 x_3$ s.t $g = \{x_1 + x_2 + x_3 - 1 = 0\}$

$$L = f + \lambda g = x_1 x_2 x_3 + \lambda(x_1 + x_2 + x_3 - 1)$$

$$\frac{\partial L}{\partial x_1} = x_2 x_3 + \lambda = 0 \Rightarrow \lambda = -x_2 x_3$$

$$\Rightarrow x_2 \cancel{x_3} = x_1 \cancel{x_3}$$

$$\frac{\partial L}{\partial x_2} = x_1 x_3 + \lambda = 0 \Rightarrow \lambda = -x_1 x_3 \qquad \boxed{x_2 = x_3}$$

$$\boxed{x_2 = x_3}$$

$$\frac{\partial L}{\partial x_3} = x_1 x_2 + \lambda = 0 \Rightarrow \lambda = -x_1 x_2 \qquad \therefore x_1 = x_2 = x_3$$

$$\frac{\partial L}{\partial \lambda} = x_1 + x_2 + x_3 - 1 = 0$$

$$x_1 + x_1 + x_1 = 1$$

$$3x_1 = 1$$

$$x_1 = \frac{1}{3} = x_2 = x_3$$

$$L_{12} = \frac{\partial^2 L}{\partial x_1 \partial x_2}$$

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$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

$$\begin{vmatrix} L_{11}-K & L_{12} & L_{13} & g_1 \\ L_{21} & L_{22}-K & L_{23} & g_2 \\ L_{31} & L_{32} & L_{33}-K & g_3 \\ g_1 & g_2 & g_3 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 0-K & \frac{1}{3} & \frac{1}{3} & 1 \\ \frac{1}{3} & 0-K & \frac{1}{3} & 1 \\ \frac{1}{3} & \frac{1}{3} & 0-K & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = 0$$

$$0-K = \frac{1}{3}, \quad 0-K = \frac{1}{3}$$

$$K = -\frac{1}{3}, -\frac{1}{3}$$

Here K is negative so. Then maxima at $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

$\max \cancel{f} = \frac{1}{27}$

Q. obtain the extreme points for the function
$f = x_1 + x_2 + x_3$ s.t $x_1^2 + x_2^2 + x_3^2 = 1$
Find whether the extreme pt. are max or min

Sol) $f = x_1 + x_2 + x_3$ $\quad g = x_1^2 + x_2^2 + x_3^2 - 1$

$$L = f + \lambda g = x_1 + x_2 + x_3 + \lambda(x_1^2 + x_2^2 + x_3^2 - 1)$$

$$\frac{\partial L}{\partial x_1} = 1 + 2x_1\lambda = 0 \Rightarrow x_1 = -\frac{1}{2\lambda}$$

$$\frac{\partial L}{\partial x_2} = 1 + 2x_2\lambda = 0 \Rightarrow x_2 = -\frac{1}{2\lambda}$$

$$\frac{\partial L}{\partial x_3} = 1 + 2x_3\lambda = 0 \Rightarrow x_3 = -\frac{1}{2\lambda}$$

$$\frac{\partial L}{\partial \lambda} = x_1^2 + x_2^2 + x_3^2 - 1 = 0$$

$$= \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} - 1 = 0$$

$$\frac{3}{4\lambda^2} = 1 \Rightarrow \lambda^2 = \frac{3}{4}$$

$$\lambda = \pm\frac{\sqrt{3}}{2}$$

for $\lambda = +\frac{\sqrt{3}}{2}$

$$x_1 = \frac{-1}{\sqrt{3}} = x_2 = x_3$$

$$\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

for $\lambda = -\frac{\sqrt{3}}{2}$

$$x = \frac{+1}{\sqrt{3}} = x_2 = x \quad \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

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$$\begin{vmatrix} L_{11}-K & L_{12} & L_{13} & g_1 \\ L_{21} & L_{22}-K & L_{23} & g_2 \\ L_{31} & L_{32} & L_{33}-K & g_3 \\ g_1 & g_2 & g_3 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2\lambda-K & 0 & 0 & 2x_1 \\ 0 & 2\lambda-K & 0 & 2x_2 \\ 0 & 0 & 2\lambda-K & 2x_3 \\ 2x_1 & 2x_2 & 2x_3 & 0 \end{vmatrix} = 0$$

$2\lambda-K=0$, $2\lambda-K=0$

$K=2\lambda$

for $\lambda = \frac{\sqrt{3}}{2} \Rightarrow K=\sqrt{3}, \sqrt{3}$

Here there is minimal at $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$

for $\lambda = -\frac{\sqrt{3}}{2} \Rightarrow K = -\sqrt{3}; -\sqrt{3}$

There is maxima at $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

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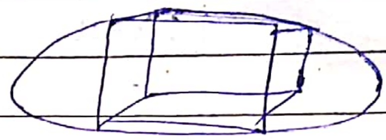
use the method of Lagrange's multipliers to find the volume of largest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$$V=(2x)(2y)(2z)=8xyz$$



$$g=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$$

$$\therefore L=V+\lambda g$$

$$L=8xyz+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$$

$$\frac{\partial L}{\partial x}=8yz+\frac{2x\lambda}{a^2}=0 \rightarrow (1)$$

$$\frac{\partial L}{\partial y}=8xz+\frac{2y\lambda}{b^2}=0 \rightarrow (2)$$

$$\frac{\partial L}{\partial z}=8xy+\frac{2z\lambda}{c^2}=0 \rightarrow (3)$$

$(1)x+(2)y+(3)z$

$$24xyz+2\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$$

$$24xyz+2\lambda=0$$

$$\lambda=-12xyz$$

$$\frac{\partial L}{\partial \lambda}=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$$

put $\lambda=12xyz$ in eq^n (1)

$$8yz+\frac{2x}{a^2}(-12xyz)=0$$

$$8yz\left(1-\frac{3x^2}{a^2}\right)=0$$

$$x=\frac{a}{\sqrt3},\ y=\frac{b}{\sqrt3},\ z=\frac{c}{\sqrt3}$$

Max V

$$=8\frac{a}{\sqrt3}\frac{b}{\sqrt3}\frac{c}{\sqrt3}$$

$$\boxed{V=\frac{8\,abc}{3\sqrt3}}$$

Max V

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Q Find the pt. upon the plane $ax+by+cz=p$ at which the function $f=x^2+y^2+z^2$ has minimum value & find minimum f.

Sol^n $f=x^2+y^2+z^2$, $g=ax+by+cz-p$

$$L=f+\lambda g$$
$$=x^2+y^2+z^2+\lambda(ax+by+cz-p)$$

$$\frac{\partial L}{\partial x}=2x+\lambda a=0 \Rightarrow x=-\frac{\lambda a}{2}$$

$$\frac{\partial L}{\partial y}=2y+\lambda b=0 \Rightarrow y=-\frac{\lambda b}{2}$$

$$\frac{\partial L}{\partial z}=2z+\lambda c=0 \Rightarrow z=-\frac{\lambda c}{2}$$

$$\frac{\partial L}{\partial \lambda}=ax+by+cz-p=0$$

$$-\frac{\lambda a^2}{2}-\frac{\lambda b^2}{2}-\frac{\lambda c^2}{2}=p$$

$$-\frac{\lambda}{2}(a^2+b^2+c^2)=p$$

$$\lambda=\frac{-2p}{a^2+b^2+c^2}$$

$$\begin{vmatrix} 2-K & 0 & 0 & a \\ 0 & 2-K & 0 & b \\ 0 & 0 & 2-K & c \\ a & b & c & 0 \end{vmatrix}=0$$

$K=2,2$

$$x=\frac{ap}{a^2+b^2+c^2},\quad y=\frac{bp}{a^2+b^2+c^2},\quad z=\frac{cp}{a^2+b^2+c^2}$$

$$\text{Min} f=\frac{a^2p^2}{(a^2+b^2+c^2)^2}+\frac{b^2p^2}{(a^2+b^2+c^2)^2}+\frac{c^2p^2}{(a^2+b^2+c^2)^2}$$

$$\min f=\frac{p^2}{a^2+b^2+c^2}$$

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