The Iteration Method (Fixed-point Iteration):

To find root of the equation $f(x) = 0$ ——— (1)

We write eqn (1) in the form

$$x = \phi(x) \qquad \text{——— (2)}$$

Take a simple example,

$$f(x) = x^2 - 2x - 3 = 0$$

Obviously, roots are $x = -1$ and $x = 3$.

Rearranging this to give equivalent form

$$x = \phi(x) = \sqrt{2x+3}$$

Let us start with a guess value of root $x = 4$, $x_0 = 4$

$$x_1 = \sqrt{2\times4+3} = \sqrt{11} = 3.31662$$

$$x_2 = \sqrt{9.63325} = 3.10375$$

$$x_3 = \sqrt{9.20750} = 3.03439$$

$$x_4 = \sqrt{9.06877} = 3.01144$$

$$x_5 = \sqrt{9.02288} = 3.00381$$

It appears that these approximations converging to the root $x = 3$.

Other Rearrangements $x = \phi(x) = \dfrac{3}{x-2}$

Again starting with $x_0 = 4$

$x_1 = 1.5$, $x_2 = -6$, $x_3 = -0.375$

$x_4 = -1.263158$, $x_5 = -0.919355$

$x_6 = -1.02762$, $x_7 = -0.990876$

$x_8 = -1.00305$

This converges to other root $x = -1$.

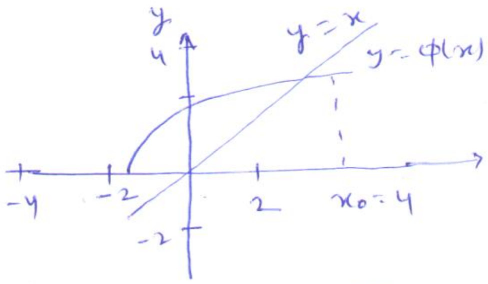
Next rearrangement $x = \phi(x) = \dfrac{x^2-3}{2}$

$x_0 = 4$, $x_1 = 6.5$, $x_2 = 19.625$, $x_3 = 191.070$

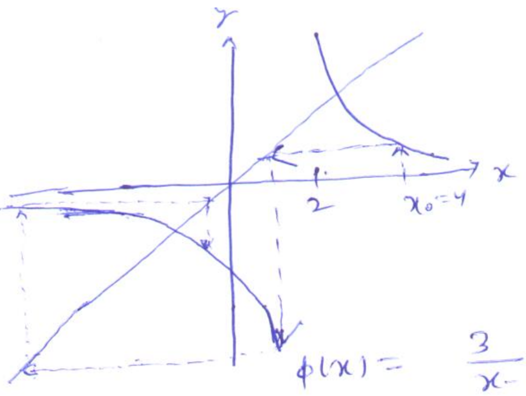
This obviously diverging.

So this arrangement will not lead to roots.

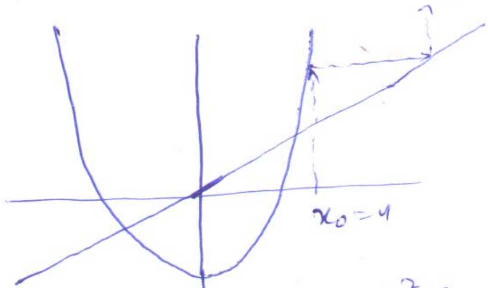
Let us look at the graphs of three cases. The fixed point of $x = \phi(x)$ is the intersection of the line $y = x$ and curve $y = \phi(x)$ plotted against x.



$\phi(x) = \sqrt{2x+3}$



$\phi(x) = \dfrac{3}{x-2}$



$\phi(x) = \dfrac{x^2-3}{2}$

What is difference between these arrangements? condition for convergence.

Here, we state conditions which are sufficient for convergence of the sequence.

Theorem: Let $x = \alpha$ be the root of $f(x) = 0$ and let I be an interval containing the point $x = \alpha$. Let $\phi(x)$ and $\phi'(x)$ be continuous in I where $\phi(x)$ is defined as $x = \phi(x)$ which is equivalent to $f(x) = 0$. Then, if $|\phi'(x)| < 1$ for all x in I, the sequence of approximations $x_0, x_1, \ldots, x_n$ defined by $x_{n+1} = \phi(x_n)$ converges to root α, provided initial approximation is chosen in I.

proof: Let α is the root of $x = \phi(x)$

$$\alpha = \phi(\alpha) \qquad \text{——— (3)}$$

$$\therefore \quad x_1 = \phi(x_0) \qquad \text{——— (4)}$$

(3) - (4)
$$\alpha - x_1 = \phi(\alpha) - \phi(x_0)$$

From mean-value theorem,

$$(\alpha - x_0)\,\phi'(\alpha_0), \quad x_0 < \alpha_0 < \alpha \qquad \text{——— (5)}$$

$$\alpha - x_1 = (\alpha - x_0)\,\phi'(\alpha_0), \quad x_0 < \alpha_0 < \alpha \qquad \text{——— (6)}$$

Similarly

$$\alpha - x_2 = (\alpha - x_1)\,\phi'(\alpha_1), \quad x_1 < \alpha_1 < \alpha \qquad \text{——— (7)}$$

$$\alpha - x_3 = (\alpha - x_2)\,\phi'(\alpha_2), \quad x_2 < \alpha_2 < \alpha \qquad \text{——— (8)}$$

$$\vdots$$

$$\alpha - x_{n+1} = (\alpha - x_n)\,\phi'(\alpha_n), \quad x_n < \alpha_n < \alpha \qquad \text{——— (9)}$$

If we assume $|\phi'(\alpha_i)| \le k < 1$ for all i — (10)

Then eqns. (5) to (9) give

$$|\alpha - x_1| \le |\alpha - x_0| \quad , \quad |\alpha - x_2| \le |\alpha - x_1| , \ldots$$

multiplying eqns. (5) to (9), we get after simplification

$$\alpha - x_{n+1} = (\alpha - x_0)\, \phi'(\alpha_0)\, \phi'(\alpha_1) \cdots \phi'(\alpha_n) \quad \text{——} (11)$$

$\because$ $|\phi'(\alpha_i)| < k$ so eqn. (11) becomes

$$|\alpha - x_{n+1}| \le k^{n+1} |\alpha - x_0| \quad \text{——} (12)$$

We see as $n \to \infty$, RHS of eqn (12) tends to zero

An example, if $k = 0.9$, $k^2 = 0.81$, $k^3 = 0.729$, etc.

So, if $k < 1$, the approximations converges to root.

Note:- (i) If $|\phi'(x)| < 1$, convergence of $x_{n+1} = \phi(x_n)$ takes place

(ii). If $|\phi'(x)| < 1$ but $\phi'(x) < 0$. The process is convergent but approximations oscillate about the root.

Acceleration of Convergence : Aitken's method

In above method convergence is slow. This can be improved as follows, known as. Aitken acceleration.

Let x_{i-1}, x_i and x_{i+1} are three successive approximations to the root 'α' of equation $x = \phi(x)$. Then

$$\alpha - x_i = k(\alpha - x_{i-1}) \quad , \quad \alpha - x_{i+1} = k(\alpha - x_i)$$

$$\therefore \quad \frac{\alpha - x_i}{\alpha - x_{i+1}} = \frac{\alpha - x_{i-1}}{\alpha - x_i}$$

Simplifying,

$$\boxed{\alpha = x_{i+1} - \frac{(x_{i+1} - x_i)^2}{x_{i+1} - 2x_i + x_{i-1}}} \quad -(13)$$

If we now define, $\Delta x_i = x_{i+1} - x_i$

$$\Delta^2 x_i = \Delta(\Delta x_i) = \Delta(x_{i+1} - x_i) = \Delta x_{i+1} - \Delta x_i$$

$$= x_{i+2} - x_{i+1} - (x_{i+1} - x_i) = x_{i+2} - 2x_{i+1} + x_i$$

$$\therefore \quad \Delta^2 x_{i-1} = x_{i+1} - 2x_i + x_{i-1}$$

So eqn. (13) can be written as

$$\boxed{\alpha = x_{i+1} - \frac{(\Delta x_i)^2}{\Delta^2 x_{i-1}}} \quad \text{——} (14)$$

Ex.1: Find a real root of the equation

$$f(x) = x^3 + x^2 - 1 = 0$$

Soln. $f(0) = -1$ & $f(1) = 1$, so clearly there is a root between 0 & 1.

Writing eqn. as $x = \frac{1}{\sqrt{x+1}}$, $\phi(x) = \frac{1}{\sqrt{x+1}}$, $\phi'(x) = -\frac{1}{2(1+x)^{3/2}}$

And so $\max_{[0,1]} |\phi'(x)| = \frac{1}{2\sqrt{8}} = k = 0.17678 < 0.2$

(Condition satisfied)

Hence, let $x_0 = 0.75$, so $x_1 = \frac{1}{\sqrt{1+0.75}} = 0.7559$

$$x_2 = \frac{1}{\sqrt{1+0.7559}} = 0.754658, \text{ etc.}$$

Ex. 2. Find a real root of the equation $\cos x = 3x - 1$, correct to three decimal places using

(i). Iteration method (ii) Aitken's method

Soln. (i). $f(x) = \cos x - 3x + 1 = 0$

$f(0) = 2 = +\text{ive}$, $f(\pi/2) = -\frac{3}{2}\pi + 1 = -\text{ive}$

Root lies between 0 & $\pi/2$.

Rewriting equation $x = \frac{1}{3}(1 + \cos x) = \phi(x)$

$|\phi'(x)| = \left|\frac{\sin x}{3}\right|$ and so $|\phi'(x)| < 1$ in $(0, \pi/2)$

Let $x_0 = 0$, $x_1 = \phi(x_0) = \frac{1}{3}(\cos 0 + 1) = 0.6667$

$x_2 = \phi(x_1) = \frac{1}{3}(\cos 0.6667 + 1) = 0.5953$

$x_3 = 0.6093$, $x_4 = 0.6067$, $x_5 = 0.6072$

$x_6 = 0.6071$

Since x_5 and x_6 being almost the same, the root is 0.607 to 3 decimal places.

(ii). We calculate x_1, x_2 and x_3

x	Δx	$\Delta^2 x$
$x_1 = 0.6667$		
	-0.0717	
$x_2 = 0.5953$		0.0857
	0.014	
$x_3 = 0.6093$		

$$x_4 = x_3 - \frac{(\Delta x_2)^2}{\Delta^2 x_1} = 0.6093 - \frac{(0.014)^2}{0.0857}$$

$$= 0.607$$