Lecture

The matrix inversion method is unsuitable for solving large systems. This is because computation of inverse of matrix by cofactors would become difficult for large matrices. Hence, other methods must be adopted which do not require the computation of cofactors.

Gauss Elimination method

In this method, the system of equations are reduced to upper-triangular matrix. Then, back substitution can be used to solve to get unknowns. Below is the method, how it is done.

Let us assume, n equations in n unknowns is given by

$A_{11} x_1 + A_{12} x_2 + A_{13} x_3 + \ldots\ldots + A_{1n} x_n = B_1$ (i)

$A_{21} x_1 + A_{22} x_2 + A_{23} x_3 + \ldots\ldots + A_{2n} x_n = B_2$ (ii)

$\ldots\ldots\ldots\ldots\ldots\ldots$ (1)

$\ldots\ldots\ldots\ldots\ldots\ldots$

$A_{n1} x_1 + A_{n2} x_2 + A_{n3} x_3 + \ldots\ldots + A_{nn} x_n = B_n$ (n)

To solve above equation (1), there are two-steps.

(i) elimination of unknowns and (ii) back substitution

Step-I

First, unknowns are eliminated to obtain upper-triangular matrix.

In order to eliminate $x_{1,}$ from second equation 1 (ii) of equation (1), we multiply equation (i) by $-A_{21}/A_{11}$. Thus, we obtain

$-A_{21}x_1 - A_{21}/A_{11}x_2(A_{12})x_2 - A_{21}/A_{11}x_2(A_{13})x_3 - \ldots\ldots - A_{21}/A_{11}x_2(A_{1n})x_n = A_{21}/A_{11}(B_1)$

Adding this equation to equation 1(i), we obtain

$$[A_{22}- A_{21}/A_{11}(A_{12})]x_2+[A_{23}- A_{21}/A_{11}(A_{13})]x_3+\ldots+[A_{2n}- A_{21}/A_{11}(A_{1n})]x_n=B_2- A_{21}/A_{11}B_1 \quad (2)$$

We will write it as

$$A_{22}'x_2+A_{23}'x_3+\ldots\ldots\ldots\ldots+A_{2n}'x_n= B_2'$$

Here, $A_{22}' = A_{22}- A_{21}/A_{11}(A_{12})$, etc.

Similarly, multiplying equation 1(i) by $–A_{31}/A_{11}$ and adding it to equation 1(iii) gives,

$$A_{32}'x_2+A_{33}'x_3+\ldots\ldots\ldots\ldots+A_{3n}'x_n= B_3'$$

We can go on and obtain

$$\begin{array}{ll}
A_{11} x_1+ A_{12} x_2+ A_{13} x_3+\ldots\ldots\ldots\ldots\ldots\ldots+ A_{1n} x_n= B_1 & \text{(i)} \\
\quad A_{22}'x_2+A_{23}'x_3+\ldots\ldots\ldots\ldots\ldots\ldots+A_{2n}'x_n= B_2' & \text{(ii)} \\
\quad A_{32}'x_2+A_{33}'x_3+\ldots\ldots\ldots\ldots\ldots\ldots+A_{3n}'x_n= B_3' & \text{(iii)} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \quad (3) \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
\quad A_{n2}'x_2+A_{n3}'x_3+\ldots\ldots\ldots\ldots\ldots\ldots+A_{nn}'x_n= B_n' & \text{(n)}
\end{array}$$

Next step is to eliminate x_2 from last (n-2) equations of equation (3).

For this we multiply equation 3 (ii) by $–A_{32}'/A_{22}'$ and add to equation 3 (iii). Repeating it gives system of equations

$$\begin{array}{ll}
A_{11} x_1+ A_{12} x_2+ A_{13} x_3+\ldots\ldots\ldots\ldots\ldots\ldots+ A_{1n} x_n= B_1 & \text{(i)} \\
\quad A_{22}'x_2+A_{23}'x_3+\ldots\ldots\ldots\ldots\ldots\ldots+A_{2n}'x_n= B_2' & \text{(ii)} \\
\qquad A_{33}''x_3+\ldots\ldots\ldots\ldots\ldots\ldots+A_{3n}''x_n= B_3'' & \text{(iii)} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \quad (3) \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &
\end{array}$$

$A_{n3}''x_3+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..+A_{nn}''x_n= B_n''$ (n)

'Double primes' means the elements have changed twice.

Carrying out the operations for elimination of x_3, x_4, etc., we finally obtain

$A_{11}\,x_1+ A_{12}\,x_2+ A_{13}\,x_3+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+ A_{1n}\,x_n= B_1$ (i)

$A_{22}'x_2+A_{23}'x_3+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+A_{2n}'x_n= B_2'$ (ii)

$A_{33}''x_3+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+A_{3n}''x_n= B_3''$ (iii)

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (4)

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$A_{nn}^{(n-1)}\,x_n= B_n^{(n-1)}$ (n)

As can be seen, the system of equations have reduced to an upper-triangular matrix.

Important: When we multiplied set of equations, we assumed $A_{11}\neq 0$. The first equation is known as **pivot equation** and A_{11} is called **pivot element**. It is clear that method fails if $A_{11}= 0$. In this case, we have to adopt different approach and we will discuss it later.

<u>Step-II</u>

Next step is back substitution.

From last equation of equation 4, i.e., equation 5(n), we obtain

$x_n= B_n^{(n-1)}/ A_{nn}^{(n-1)}$

This is substituted in (n-1)th equation to obtain x_{n-1}. The process is repeated for other unknowns to obtain $x_{n-2}, x_{n-3},\ldots\ldots, x_3, x_2, x_1$. The reason for the term back substitution is clear from here.

Pivoting

As we have discussed, if pivot element is zero then the process fails. Also, if pivot element is close to zero the rounding off errors may occur. To address this point, we can use a process called pivoting.

In this process, if A_{11} is either zero or very small compared to other coefficients of the equation, we locate the largest available coefficient in the column below the pivot equation. Then, we interchange the two rows. In this way, a new pivot equation is obtained. This process is called partial pivoting. If search for largest coefficients is done both in column and rows then it is called complete pivoting. Complete pivoting is complex and rarely used.

Partial of complete pivoting may not help in all the problems. In cases where some rows have coefficients much larger in comparison to the coefficients of other rows, we might not obtain correct result by partial pivoting. In such cases, scaling could be done to get correct answer. This is called 'Scaled partial pivoting'.

Neither partial nor complete pivoting may solve all problems of rounding off error. Some linear systems, called ill-conditioned systems are extremely sensitive to such problem. An ill-conditioned system is that in which the determinant of coefficient matrix is close to zero.

Let us understand all via examples.

First, Gauss elimination method is illustrated.

Example 1: Use Gauss elimination method to solve system of linear equations

$2x+y+z = 10 \qquad (1)$

$3x+2y+3z=18 \qquad (2)$

$x+4y+9z=16 \qquad (3)$

Solution: Step-I, eliminate x from equation (2) and (3)

Multiply equation (1) by (-3/2) and add to equation (2), we get

-2x⁄3/2x-3/2y-3/2z = -15

3x   +2y +3z   = 18

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y+3z = 6   (4)

Similarly, eliminating x from equation (3), we get first by multiplying (-1/2) and adding to equation (3),

-2x1/2x-1/2/y-1/2z = -5

x   +4y   +9z = 16

7y+17z = 22   (5)

Now, eliminate y from equation (5), we obtain using equation (4)

z =5

Our system of equations become

2x+y+z = 10

y+3z=6

z = 5

Step-II, Back substitution gives us

x = 7, y= -9, z = 5

Example 2: This example illustrates how rounding off error propagates.

Evaluate the determinant of matrix

A= $\begin{bmatrix} 0.12 & 0.23 \\ 0.12 & 0.12 \end{bmatrix}$

round each intermediate calculation to two significant digits. Find the exact answer and compare these results.

Solution:

$|A| = (0.12)(0.12) - (0.12)(0.23) = 0.0144 - 0.0276$ = -0.0132 = -0.013 (2 signif. digits)

= (0.014)-(0.028) (Rounding to two significant digits)

= -0.014

Rounded solution is not correct to two significant digits, although, we have performed it correctly. Thus, rounding off leads to propagating error.

Here, rounding off has introduced an error

-0.0132-(-0.014) = 0.0008 (Rounding error)

Though it looks small but when seen in percentage error, it is

(0.0008/0.0132)x100% ≈6.1%

This is significant error and may not be useful in most calculations.

Next, is an example of partial pivoting.

Example 3: Solve system of linear equations

0.143x+0.357y+2.01z=-5.173

-1.31x+0.91y+1.99z =-5.458

11.2x – 4.30y -0.605z = 4.415

Round off each intermediate calculation to three significant digits.

Solution: Step-I: Elimination of x, y and z.

$$\begin{bmatrix} 0.143 & 0.357 & 2.01 & -5.17 \\ -1.31 & 0.911 & 1.99 & -5.46 \\ 11.2 & -4.30 & -0.605 & 4.42 \end{bmatrix}$$

Dividing first row by 0.143

$$\begin{bmatrix} 1.00 & 2.50 & 14.1 & -36.2 \\ -1.31 & 0.911 & 1.99 & -5.46 \\ 11.2 & -4.30 & -0.605 & 4.42 \end{bmatrix}$$

Adding 1.31 times first row to second row

$$\begin{bmatrix} 1.00 & 2.50 & 14.1 & -36.2 \\ 0.00 & 4.19 & 20.5 & -52.9 \\ 11.2 & -4.30 & -0.605 & 4.42 \end{bmatrix}$$

Adding 11.2 times first row to third row

$$\begin{bmatrix} 1.00 & 2.50 & 14.1 & -36.2 \\ 0.00 & 4.19 & 20.5 & -52.9 \\ 0.00 & -32.3 & -159 & 409 \end{bmatrix}$$

Dividing second row by 4.19

$$\begin{bmatrix} 1.00 & 2.50 & 14.1 & -36.2 \\ 0.00 & 1.00 & 4.89 & -12.6 \\ 0.00 & -32.3 & -159 & 409 \end{bmatrix}$$

Adding 32.3 times second row to third row

$$\begin{bmatrix} 1.00 & 2.50 & 14.1 & -36.2 \\ 0.00 & 1.00 & 4.89 & -12.6 \\ 0.00 & 0.00 & -1.00 & 2.00 \end{bmatrix}$$

Step-II: Back substitution. Thus, z = -2.00, y= -2.82 and x = -0.900. The actual result is x =1, y= 2 and z = -3.

What is wrong here? Obviously rounding off has created error.

We can see here, the pivot element is small as compared to other elements.

Now, how pivoting helps.

$$\begin{bmatrix} 0.143 & 0.357 & 2.01 & -5.17 \\ -1.31 & 0.911 & 1.99 & -5.46 \\ 11.2 & -4.30 & -0.605 & 4.42 \end{bmatrix}$$

Largest element in first column is in 3rd row. So interchange the first row with third row.

$$\begin{bmatrix} 11.2 & -4.30 & -0.605 & 4.42 \\ -1.31 & 0.911 & 1.99 & -5.46 \\ 0.143 & 0.357 & 2.01 & -5.17 \end{bmatrix}$$ (Partial pivoting, new pivot element is 11.2)

Divide first equation by 11.2

$$\begin{bmatrix} 1.00 & -3.84 & -0.0540 & 3.95 \\ -1.31 & 0.911 & 1.99 & -5.46 \\ 0.143 & 0.357 & 2.01 & -5.17 \end{bmatrix}$$

Adding 1.31 times the first row in second row

$$\begin{bmatrix} 1.00 & -3.84 & -0.0540 & 3.95 \\ 0.00 & 0.408 & 1.92 & -4.94 \\ 0.143 & 0.357 & 2.01 & -5.17 \end{bmatrix}$$

Adding -0.143 times the first row to the third row

$$\begin{bmatrix} 1.00 & -3.84 & -0.0540 & 3.95 \\ 0.00 & 0.408 & 1.92 & -4.94 \\ 0.00 & 0.412 & 2.02 & -5.23 \end{bmatrix}$$

Now, pivot element in second column (comparing second and third row) is 0.412, so interchange these.

$$\begin{bmatrix} 1.00 & -3.84 & -0.0540 & 3.95 \\ 0.00 & 0.412 & 2.02 & -5.23 \\ 0.00 & 0.408 & 1.92 & -4.94 \end{bmatrix}$$ (interchanging second and third row)

Divide second row by 0.412, we get

$$\begin{bmatrix} 1.00 & -3.84 & -0.0540 & 3.95 \\ 0.00 & 1.00 & 4.90 & -12.7 \\ 0.00 & 0.408 & 1.92 & -4.94 \end{bmatrix}$$

Adding -0.408 times to third row

$$\begin{bmatrix} 1.00 & -3.84 & -0.0540 & 3.95 \\ 0.00 & 1.00 & 4.90 & -12.7 \\ 0.00 & 0.00 & -0.0800 & 0.240 \end{bmatrix}$$

Now, we get z = -3.00, y = 2.00 and x = 1.00. This is exact solution.

Below is example of scaled partial pivoting.

First we will see how partial pivoting might not help. Physical example of such problems may be due to widely different units in equations, like microvolt versus kilovolt, seconds versus year, etc.

Example 4:

$$\begin{bmatrix} 3 & 2 & 100 & 105 \\ -1 & 3 & 100 & 102 \\ 1 & 2 & -1 & 2 \end{bmatrix}$$

As no pivoting required since pivot element is large in comparison to other elements in the same column. If we solve it using Gauss elimination, we get

$$\begin{bmatrix} 3 & 2 & 100 & 105 \\ 0 & 3.67 & 133 & 135 \\ 0 & 0 & -82.4 & -82.6 \end{bmatrix}$$

After back substitution result we get is x = 0.939, y= 1.09, z = 1.00.

The exact result is x, y, z = 1.00. Thus, we obtain wrong result. What is wrong here?

The coefficients in third equation are much smaller than other two equations. Thus, dividing the respective rows by largest element in that row (discounting fourth row as it is on the right hand side of the equation). This is called scaling. We get

$$\begin{bmatrix} 0.0300 & 0.01 & 1.00 & 1.05 \\ -0.0100 & 0.03 & 1.00 & 1.02 \\ 0.500 & 1.00 & -0.500 & 1.00 \end{bmatrix}$$

Now, it is obvious pivot element is 0.500. So we can interchange rows one and three and carry out operations. This is scaled partial pivoting.

There is a better way to do scaled pivoting than what is described above.

The method starts with computing scaling vector (Let us call it S) whose elements are largest in magnitude in each row.

So, here scaling vector is S = [100,100,2].

Before reducing the first element, divide each element by the corresponding element of S, so we get the first column as [0.0300, -0.0100, 0.500].

Hence it is clear third equation is pivot equation in this case.

So new scale vector becomes S'=[2,100,100] (It means interchanging first and third row)

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 5 & 99 & 104 \\ 0 & -4 & 103 & 99 \end{bmatrix}$$ (Usual method of Gauss elimination, reducing first column)

To reduce column 2, we divide this column by elements of S'. We get [1, 0.0500, -0.400]. First element needs to be ignored here as we have to compare row 2 and 3. After reduction, we get

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 5 & 99 & 104 \\ 0 & 0 & 182 & 182 \end{bmatrix}$$

Hence we get z = 1.00, y =1.00 and x = 1.00.

Operational Count

How efficient is a numerical procedure depends on the number of arithmetic operations required. In Gauss elimination method, we shall count number of arithmetic operations.

In a system of n equations, we have one right-hand side element. Thus augmented matrix is n x n+1.

To reduce column below the diagonal element in column 1, we compute (n-1) multiplying factors (which takes (n-1) divides). We multiply each of these by all elements in row 1 except the first element so n multiplications.

Subtract these elements from n elements in each of the (n-1) rows below row 1. Importantly, ignoring first element since these are known to become zero, there are n x (n-1) multiplications and n x (n-1) subtractions.

In nutshell,

Divisions = (n-1)

Multiplications = n(n-1)

Subtractions = n(n-1)

In succeeding columns, we have one fewer lelemnt, so for any colun 'i',

Divisions = (n-i)

Multiplications = (n-1+i) (n-i)

Subtractions = (n-1+i)(n-i)

We add these quantities for reduction in column 1 through (n-1), we get

Divisions = $\sum_{i=1}^{n-1}(n-i) = \sum_{i=1}^{n-1} i = \frac{n^2}{2} - \frac{n}{2}$

Multiplications and subtractions each = $\sum_{i=1}^{n-1}(n-1+i)(n-i) = \sum_{i=1}^{n-1} i(i+1) = \frac{n^3}{3} - \frac{n}{3}$

Total operations in reduction = $\frac{n^2}{2} - \frac{n}{2} + \frac{n^3}{3} - \frac{n}{3} + \frac{n^3}{3} - \frac{n}{3} = \frac{2n^3}{3} + \frac{n^2}{2} - \frac{7n}{6}$

Back substitutions:

Multiplications = (n-i+1)(n-i)

Subtractions = (n-i+i)(n-i)

Divisions = n

Thus total= $\frac{2n^3}{3} + \frac{3n^2}{2} - \frac{7n}{6}$

Let us understand with an example more explicitly. This is given in separate sheet.

Gauss-Jordan Method

It is modified Gauss elimination method. The difference is that in this method unknowns are eliminated from all equations. So back substitution is not required.

As an example.

$2x+y+z= 10$ (1)

$3x+2y+3z= 18$ (2)

$x+4y+9z = 16$ (3)

Eliminating x from equation (2) and (3),

$1/2y+3/2z= 3$

$7/2y+17/2z=11$

So set of equations are

$2x+y+z= 10$ (4)

$y+3z=6$ (5)

$7y+17z=22$ (6)

Now eliminating y from equations (4) and (6)

$x-z = 5$ and $z = 5$

So, $x-z = 5$

$y+3z = 6$

$z = 5$

Hence, elimination of y and z is trivial and result is $x = 7$, $y = -9$, $z = 5$.

Gauss-Jordan method requires nearly 50% more operations as compared to Gauss elimination method. Gauss elimination requires nearly $n^3/3$ operations while Gauss-Jordan method requires nearly $n^3/2$ operations.