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$$\frac{\partial^2 \phi(x,t)}{\partial x^2} = \frac{1}{C^2} \frac{\partial^2 \phi(x,t)}{\partial t^2} \quad \text{—— (1)}$$

$$\phi(x,t) = X(x)\,T(t)$$

So, $\frac{\partial \phi}{\partial x} = T \frac{\partial X}{\partial x}$

and, $\frac{\partial^2 \phi}{\partial x^2} = T \frac{\partial^2 X}{\partial x^2}$

Also,

$\frac{\partial^2 \phi}{\partial t^2} = X \frac{\partial^2 T}{\partial t^2}$

So, from (1), $\quad T \frac{\partial^2 X}{\partial x^2} = \frac{1}{C^2} X \frac{\partial^2 T}{\partial t^2}$

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = \frac{1}{C^2} \cdot \frac{1}{T} \frac{\partial^2 T}{\partial t^2} = \text{constant} = K$$

Then, $\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = K$ —— (2) $\Rightarrow \frac{\partial^2 X}{\partial x^2} - KX = 0$

and $\frac{1}{T} \frac{\partial^2 T}{\partial t^2} = c^2 K$ —— (3) $\Rightarrow \frac{\partial^2 T}{\partial t^2} - (c^2 K) T = 0$

ordinary second order differential equation.

DATE

578
579
~~17.23~~
17.3

Partial Differential Equation -

Let $z = f(x,y) \rightarrow z$ is a function of more than one variable.

$$\frac{\partial z}{\partial x},\ \frac{\partial z}{\partial y},\ \frac{\partial^2 z}{\partial x^2},\ \frac{\partial^2 z}{\partial y^2},\ \frac{\partial^2 z}{\partial x \partial y}$$
$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$
$$p \quad q \quad r \quad t \quad s$$

Derive a partial differential equation from the equation

arbitrary fn. with arbitrary constants 'a' and 'b'.

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad \text{--- (1)}$$

diff. partially w.r.t x, $2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \Rightarrow \frac{1}{a^2} = \frac{p}{x}$

diff. partially w.r.t y, $2\frac{\partial z}{\partial y} = \frac{2y}{b^2} \Rightarrow \frac{1}{b^2} = \frac{q}{y}$

So, from (1), $2z = px + qy$

$\boxed{z = \frac{1}{2}(px + qy)}$ is the req. PDE

Given:- $z = (x+y)\,\phi(x^2-y^2)$. Form a PDE.

$$p = \frac{\partial z}{\partial x} = (x+y)\,\phi'(x^2-y^2).2x + \phi(x^2-y^2)$$

$$q = \frac{\partial z}{\partial y} = (x+y)\,\phi'(x^2-y^2)(-2y) + \phi(x^2-y^2)$$

$$\because \frac{z}{x+y} = \phi(x^2-y^2)$$

$-\frac{\sin(2x-y)}{4} = +\frac{\sin(y-2x)}{4}$ $\cos(-\theta) = \cos\theta$

DATE ______ $-\frac{1}{4}\cos(y-2x) = -\frac{1}{4}\cos(-2x-y)$

$= -\frac{1}{4}\cos(2x-y)$

So, $p - \frac{z}{x+y} = 2x(x+y)\phi'(x^2-y^2)$

and $q - \frac{z}{x+y} = -2y(x+y)\phi'(x^2-y^2)$

$$\boxed{\frac{p-\frac{z}{x+y}}{q-\frac{z}{x+y}} = -\frac{x}{y}}$$ is the req. PDE.

$f(x,y,z,a,b) \longrightarrow$ Complete Solution

$f[x,y,z,a,\phi(a)] \longrightarrow$ General Solution

1. Solve by direct Integration –

$$\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x-y) = 0$$

Integrating w.r.t. x, keeping y constant

$$\frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 - \frac{1}{2}\cos(2x-y) = f(y)$$

Again, $\frac{\partial z}{\partial y} + 3x^3y^2 - \frac{1}{4}\sin(2x-y) = xf(y) + \phi(y)$

Now,

Integrating w.r.t. y, keeping x constant.

$$z + x^3y^3 - \frac{1}{4}\cos(2x-y) = x\underbrace{\int f(y)\,dy}_{u(y)} + \underbrace{\int \phi(y)\,dy}_{v(y)} + w(x)$$

say

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$$\boxed{z + x^3y^3 - \frac{1}{4}\cos(2x-y) = xu(y) + v(y) + w(x)}$$

2. Linear PDE

↳ Lagranges linear equation

$$\boxed{Pp + Qq = R} \quad \text{— (1)}$$

If P, Q, & R are only fn. of x, y then → eq(1) is linear

If P, Q & R are fn. of (x, y, z), then → eq(1) is quasi-linear equation.

Step1- Subsidiary Equation Formation

ie. $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Step2- The connection b/w arbitrary constants by $\phi(a,b) = 0$

Step3- Will get a form $u = a$ & $v = b$ type

and the solution will be $u = f(v)$

$\frac{y^2z}{x}p + x^2zq = y^2$

~~Comparing it with (1), we get~~

~~$P = \frac{y^2z}{x}$, $Q = xz$, $R = y^2$~~

ie. $y^2zp + x^2zq = y^2x$

So, $P = y^2z$, $Q = x^2z$ and $R = y^2x$

$\because$ P, Q, R are fn. of (x, y, z) then the given eq. is quasi-linear equation

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Step 1: $\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{xy^2}$

1st & 2nd $\Rightarrow$ $\frac{dx}{y^2} = \frac{dy}{x^2}$

$$\boxed{x^3 - y^3 = a}$$

$u = a =$ constant

1st & 3rd $\Rightarrow$ $\frac{dx}{z} = \frac{dz}{x}$

$$\cancel{x^2 - y^2} = \boxed{x^2 - z^2 = b}$$

$v = b =$ constant

so, the solution is $u = f(v)$

or $a = f(b)$

i.e. $\boxed{x^3 - y^3 = f(x^2 - z^2)}$

$\Rightarrow$ $\boxed{\phi(x^3 - y^3,\ x^2 - z^2) = 0}$

$(mz - ny)\frac{\partial z}{\partial x} + (mx - lz)\frac{\partial z}{\partial y} = ly - mx$

i.e. $(mz - ny)p + (mx - lz)q = ly - mx$

$P = mz - ny$

$Q = mx - lz$

$R = ly - mx$

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Now, $\dfrac{dx}{mz-ny} = \dfrac{dy}{mx-lz} = \dfrac{dz}{ly-mx}$

$\Rightarrow \dfrac{x\,dx}{x(mz-ny)} = \dfrac{y\,dy}{y(mx-lz)} = \dfrac{z\,dz}{z(ly-mx)}$

~~and~~ on adding these 3 terms, we get-

$\dfrac{x\,dx}{x(mz-ny)} = \dfrac{y\,dy}{y(mx-lz)} = \dfrac{z\,dz}{z(ly-mx)} = \dfrac{x\,dx+y\,dy+z\,dz}{0}$

$\Rightarrow x\,dx+y\,dy+z\,dz=0$

i.e. $x^2+y^2+z^2 = a \longrightarrow u$

$\longrightarrow l\,dx+m\,dy+n\,dz=0$

$lx+my+nz = b \longrightarrow v$

so,

the solution is $u = f(v)$

or $a = f(b)$

i.e. $\boxed{x^2+y^2+z^2 = f(lx+my+nz)}$

~~Ques~~ # $(x^2-y^2-z^2)p + 2xy\,q = 2x^2$ ← 17.9

$x^2(y-z)p + y^2(z-x)p = z^2(x-y)$ ← 17.11

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Non-Linear First-Order PDE

i.e. p and q occur other than 1st degree like p^2, q^2 type....

$f(p,q)=0 \rightarrow$ Form

Then solution: $z = ax+by+c$

$\frac{\partial z}{\partial x} = a$

$\frac{\partial z}{\partial y} = b$

$p-q=1$ — (1)

$\Rightarrow p-q-1=0$ i.e. $f(p,q)=0$

so, solution is $z = ax+by+c$

Now, $\frac{\partial z}{\partial x} = p = a$

$\frac{\partial z}{\partial y} = q = b$

so, putting in eq (1), we get $a-b=1$

$$\boxed{z = ax + (a-1)y + c}$$

$x^2p^2 + y^2q^2 = z^2$

$\left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1$

Let $\frac{dx}{x} = du$, $\frac{dy}{y} = dv$, $\frac{dz}{z} = dw$

$\log x = u$, $\log y = v$, $\log z = w$

← 17.9

← 17.11

$\frac{\partial z}{\partial x} = p$ DATE $\frac{\partial w}{\partial u} = p$ $z = ax + by + c$

so, $\left(\frac{\partial w}{\partial u}\right)^2 + \left(\frac{\partial w}{\partial v}\right)^2 = 1$

$$p^2 + q^2 = 1 \quad \text{---(1)}$$
$$\Rightarrow p^2 + q^2 - 1 = 0$$
$$\Rightarrow f(p, q) = 0$$

So, the solution is $w = au + bv + c$ ---(2)

$a = p$ & $b = q$ and putting it in (1)

we get $a^2 + b^2 = 1$

$\Rightarrow f(a, b) = 0$

so, from (2),

$$w = au \pm \sqrt{1 - a^2}\, v + c$$

ie.

$$\boxed{\log z = a \log x \pm \sqrt{1 - a^2} \log y + c}$$

Writing differential equation in the form of $f(z, p, q) = 0$

ie. not containing x and y.

Then we can assume the trial solution as :-

$$u = x + ay$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{\partial z}{\partial u}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} = a \frac{\partial z}{\partial u}$$

Soln: $f(z, p, q) \rightarrow f\left(z, \frac{\partial z}{\partial u}, a\frac{\partial z}{\partial u}\right)$

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Solve: $p(1+q) = qz$

$\Rightarrow p(1+q) - qz = 0$

i.e. $f(z, p, q) = 0$

So, the trial soln. is $u = x + ay$

$$f\left(z, \frac{\partial z}{\partial u}, a\frac{\partial z}{\partial u}\right) = \frac{\partial z}{\partial u}\left(1 + a\frac{\partial z}{\partial u}\right) - az\frac{\partial z}{\partial u} = 0$$

i.e. $z = z(u)$

$$\Rightarrow \frac{dz}{du}\left(1 + a\frac{dz}{du}\right) - az\frac{dz}{du} = 0$$

$$\Rightarrow \frac{dz}{du} + a\left(\frac{dz}{du}\right)^2 - az\frac{dz}{du} = 0$$

$$\Rightarrow a\frac{dz}{du} = az - 1$$

$$\Rightarrow a\int \frac{dz}{az-1} = \int du + \cancel{\log}\, c$$

$$\Rightarrow \log(az-1) = u + \cancel{\log}\, c$$

$$\Rightarrow \boxed{\log(az-1) = x + ay + c}$$

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Solve: $z = p^2 - q^2$

$\Rightarrow z - p^2 + q^2 = 0$ —(1)

i.e. $f(z, p, q) = 0$

So, Trial soln. is $u = x + ay$

i.e. $\frac{\partial z}{\partial u} = p$

and $a\frac{\partial z}{\partial u} = q$

Putting in eq (1), we get-

$$z - \left(\frac{\partial z}{\partial u}\right)^2 + a^2\left(\frac{\partial z}{\partial u}\right)^2 = 0$$

$$\Rightarrow z = \left(\frac{\partial z}{\partial u}\right)^2 (1 - a^2)$$

$$\sqrt{z} = \frac{dz}{du}\sqrt{1 - a^2}$$

$$\int du = \int \frac{\sqrt{(1-a^2)}\, dz}{\sqrt{z}} + c$$

$$\Rightarrow u + c = 2\sqrt{1+a^2}\, z^{1/2}$$

$$x + ay + c = 2\sqrt{1+a^2}\sqrt{z}$$

$$\boxed{4z(1+a^2) = (x+ay+c)^2}$$

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Ans: $a^2z^2 = 1 + (y + ax + c)^2$

Solve: $q^2 = z^2p^2(1-p^2)$ —— (1)

i.e. $f(p, q, z) = 0$

Taking trial soln. $u = y + ax$

i.e. $a\frac{\partial z}{\partial u} = p$

and $\frac{\partial z}{\partial u} = q$

— Putting in eq (1), we get

$$\left(\frac{\partial z}{\partial u}\right)^2 = z^2a^2\left(\frac{\partial z}{\partial u}\right)^2\left[1 - a^2\left(\frac{\partial z}{\partial u}\right)^2\right]$$

$$z = \left(\frac{\partial z}{\partial u}\right)^2 - a^2\left(\frac{\partial z}{\partial u}\right)^2$$

$$z = \left(\frac{\partial z}{\partial u}\right)^2\left[1 - a^2\right]$$

$$\int \frac{\sqrt{1-a^2}}{\sqrt{z}}dz$$

$$\sqrt{1-a^2}\int \frac{dz}{\sqrt{z}}$$

$$\int z^{-\frac{1}{2}}dz$$

$$\frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$2\frac{z^{1/2}}{2}$$

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Variables in separable form

$$f_1(x,p) = f_2(y,q) \rightarrow \text{contains no } z.$$

Let $f_1(x,p) = f_2(y,q) = a$

i.e., $f_1(x,p) = a$ and $f_2(y,q) = a$

Solutions $\rightarrow$ $\Rightarrow p = F_1(x)$ $\quad q = F_2(y)$

$$z = z(x,y)$$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = p\,dx + q\,dy$$

$$\boxed{\int dz = \int F_1(x)dx + \int F_2(y)dy + c}$$

Solve: $yp + xq + pq = 0$

$$\Rightarrow yp + xq = -pq$$

$$\Rightarrow \frac{y}{q} + \frac{x}{p} = -1$$

$$\Rightarrow \frac{y}{q} = -\left(1+\frac{x}{p}\right)$$

$$f_2(y,q) = f_1(x,p)$$

Now,

$$\frac{y}{q} = -\left(1+\frac{x}{p}\right) = a \text{ (say)}$$

So, $\frac{y}{q} = a$ and $1+\frac{x}{p} = -a$

$$q = \frac{y}{a} = F_2(y) \qquad p = \frac{x}{-(1+a)} = F_1(x)$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c$$

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$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$$

So, the solution is

$$z = \int F_1(x)\,dx + \int F_2(y)\,dy + c$$

$$\Rightarrow z = -\int \frac{x}{1+a}\,dx + \int \frac{y\,dy}{a} + c$$

$$\Rightarrow \boxed{2z = -\frac{1}{(a+1)}x^2 + \frac{1}{a}y^2 + c'}$$

$$\int \sqrt{t^2}\,\frac{t\,dt}{\sqrt{t^2-a}} = \int \frac{t^2\,dt}{\sqrt{t^2-a}}$$

Solve:- $p^2+q^2 = x^2+y^2$

$$\Rightarrow p^2 - x^2 = y^2 - q^2$$

$$\Rightarrow f_1(x,p) = f_2(y,p)$$

Now, $p^2-x^2 = y^2-q^2 = a$ (say)

So, $p^2-x^2 = a$ and $y^2-q^2 = a$

$p = \sqrt{a+x^2}$ $\quad$ $q = \sqrt{y^2-a}$

$= F_1(x)$ $\quad$ $= F_2(y)$

So, solution is

$$z = \int F_1(x)\,dx + \int F_2(y)\,dy + c$$

$$\Rightarrow z = \int \sqrt{a+x^2}\,dx + \int \sqrt{y^2-a}\,dy$$

$\Rightarrow$

Ans:- $z = \frac{1}{2}x\sqrt{a+x^2} + \frac{1}{2}a\log\left[x+\sqrt{a+x^2}\right] + \frac{1}{2}y\sqrt{y^2-a} - \frac{1}{2}a\log\left[y+\sqrt{y^2-a}\right]$

$a+x^2 = t^2$

$2x\,dx = 2t\,dt$

$x\,dx = t\,dt$

$dx = \frac{t\,dt}{\sqrt{t^2-a}}$

$\cancel{a+x = t}$

$\cancel{dx = dt}$

$\cancel{\int\sqrt{a+x}\,dx = \int\sqrt{t}\,dt = \int t^{1/2}\,dt = \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{t^{3/2}}{3/2} = \frac{2}{3}t^{3/2} = \frac{2}{3}(a+x)^{3/2}}$

Equation in the form: $z = px + qy + f(p, q)$

Soln:- $z = ax + by + f(a,b)$

$$\frac{\partial z}{\partial x} = a = p$$

$$\frac{\partial z}{\partial y} = b = q$$

solve: $z = ax + by - 2\sqrt{ab}$

diff. w.r.t 'a',

$$0 = x - 2\sqrt{b} \cdot \frac{1}{2\sqrt{a}}$$

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$x = \sqrt{\frac{b}{a}}$

Now, diff. w.r.t. b, we get-

$$0 = y - 2\sqrt{a}\,\frac{1}{2\sqrt{b}}$$

$$y = \sqrt{\frac{a}{b}}$$

Then $xy = 1$

$$z = ax + by - 2\sqrt{ab}$$

Solve: $z = px + qy + c\sqrt{1+p^2+q^2}$

~~Ans~~

Complete soln. $z = ax + by + c\sqrt{1+a^2+b^2}$

Singular soln: $z = \sqrt{c^2 - x^2 - y^2}$

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Charpit's Method

If we can write the eq. in the form:

$$f(x,y,z,p,q)=0$$

then General Method to write down the subsidiary eq.

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

or $$\frac{dp}{f_x+pf_z} = \frac{dq}{f_y+qf_z} = \frac{dz}{-pf_p-qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \quad -(1)$$

Solve:- Using Charpit's Method, find the Complete solution

$$pxy+pq+qy=yz$$

$$\Rightarrow pxy+pq+qy-yz=0$$

i.e. $f(x,y,z,p,q)=0$

$$f = pxy+pq+qy-yz$$

So, by eq(1), we can write-

$$\frac{dp}{py-py} = \frac{dq}{(px+q-z)-qy} = \frac{dz}{-p(xy+q)-q(p+y)} = \frac{dx}{-(xy+q)} = \frac{dy}{-(p+y)}$$

Now,

$$\frac{dp}{0}=c$$

$$\Rightarrow dp=0$$

or $p=$ constant $=a$ (say)

$$\boxed{p=a}$$

we have, $pxy + pq + qy = yz$

so, $axy + aq + qy = yz$

$\Rightarrow q(a+y) = y(z-ax)$

$$\boxed{q = \frac{y(z-ax)}{a+y}}$$

we know that, $z = z(x,y)$

$dz = p\,dx + q\,dy$

$\Rightarrow dz = a\,dx + \frac{(z-ax)}{a+y}\,y\,dy$

$\Rightarrow dz - a\,dx = \frac{(z-ax)}{a+y}\,y\,dy$

$\Rightarrow \frac{dz - a\,dx}{(z-ax)} = \frac{y\,dy}{a+y}$

$\frac{dy}{-(p+y)} = c$ (say) $\Rightarrow \frac{d(z-ax)}{(z-ax)} = \left(1 - \frac{a}{a+y}\right)dy$

$\log(z-ax) = y - a\log(a+y) + b$

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Using Charpit's Method, find the solution of the equation: $2(z+xp+yq) = yp^2$

$$2z + 2xp + 2yq - yp^2 = 0$$

Now,

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$\Rightarrow \cancel{\frac{dp}{2p+p.2} = \frac{dq}{2q-p^2+q.2} = \frac{dz}{-p}}$$

$f_x = 2p$ $\qquad f_q = 2y$

$f_z = 2$

$f_y = 2q - p^2$

$f_p = 2x - 2yp$

Now,

$$\frac{dp}{2p+2p} = \frac{dq}{2q-p^2+2q} = \frac{dz}{-p(2x-2yp) - q2y} = \frac{dx}{-2x+2yp} = \frac{dy}{-2y}$$

$$\Rightarrow \frac{dp}{4p} = \frac{dq}{4q-p^2} = \frac{dz}{-2xp-2yq+2yp^2} = \frac{dx}{2yp-2x} = \frac{dy}{-2y}$$

$$\frac{dp}{4p} = \frac{dy}{-2y}$$

$$\Rightarrow \frac{dp}{p} + 2\frac{dy}{y} = 0$$

$\log p + 2\log y = \log a \Rightarrow py^2 = a$ (a = constant)

$p = a/y^2$

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Putting $p = a/y^2$ in eq $z + xp + yq = \frac{yp^2}{2}$, we get

$$z + \frac{ax}{y^2} + yq = \frac{y}{2} \cdot \frac{a^2}{y^4}$$

$$\boxed{q = -\frac{z}{y} - \frac{ax}{y^3} + \frac{a^2}{2y^4}}$$

Now,

$$z = z(x, y)$$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$dz = p\,dx + q\,dy$$

$$dz = \frac{a\,dx}{y^2} - \frac{z\,dy}{y} - \frac{ax\,dy}{y^3} + \frac{a^2\,dy}{2y^4}$$

or $$y\,dz = \frac{a\,dx}{y} - z\,dy - \frac{ax\,dy}{y^2} + \frac{a^2\,dy}{2y^3}$$

$$\Rightarrow (y\,dz + z\,dy) = a\left[\frac{y\,dx - x\,dy}{y^2}\right] + \frac{a^2\,dy}{2y^3}$$

$$\int d(yz) = a\int d\left(\frac{x}{y}\right) + a^2\int \frac{dy}{2y^3} + c$$

$$\Rightarrow yz = \frac{ax}{y} + \frac{a^2}{2}\frac{y^{-2}}{-2} + c$$

$$yz = \frac{ax}{y} - \frac{a^2}{4y^2} + c$$

$$\boxed{z = \frac{ax}{y^2} - \frac{a^2}{4y^3} + \frac{c}{y}}$$

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Homogeneous PDE with constant coefficient -

$p = D = \frac{\partial}{\partial x}$ $\qquad s = DD' = \frac{\partial}{\partial x \partial y}$

$q = D' = \frac{\partial}{\partial y}$

General form of Linear PDE (higher order) of nth order -

$$\left[D^n + K_1 D^{n-1} D' + K_2 D^{n-2} D'^2 + \dots\dots + K_{n-1} D D'^{n-1} + K_n D'^n\right] z = F(x,y)$$

$\boxed{F(D, D')z = F(x,y)}$ $\rightarrow$ $F(x,y) = 0 \rightarrow$ homogeneous PDE

$F(x,y) \neq 0 \rightarrow$ non-homogeneous PDE

ie. Complementary eq $F(D, D') = 0$

soln:- $z = z_c(x,y)$

and Particular Integral

$$z_{PI} = PI = \frac{F(x,y)}{F(D,D')}$$

$\boxed{z = z_c + z_{PI}}$ Complete Solution

Let us take the simplest case $\rightarrow$ 2nd order linear homogeneous PDE

$(D^2 + K_1 D \cdot D' + K_2 D'^2) z = 0$ — CE

$$\left[\left(\frac{D}{D'}\right)^2 + K_1 \frac{D}{D'} + K_2\right] z = 0$$

$\hookrightarrow$ quadratic equation in $\frac{D}{D'}$ & let the roots are m_1 and m_2.

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$$\Rightarrow \left[\left(\frac{D}{D'} - m_1\right)\left(\frac{D}{D'} - m_2\right)\right] z = 0$$

$$p - m_1 q = 0$$

$$\left[\begin{array}{l} Pq + Qq = R \\ \text{Subsidiary Equation SE: } \dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R} \end{array}\right]$$

$$\frac{dx}{1} = \frac{dy}{-m_1} = \frac{dz}{0}$$

$$dx = -\frac{dy}{m_1} \quad \& \quad dz = 0$$

$$\Downarrow$$

$$z = b \text{ (constant)}$$

$$dy + m_1 dx = 0$$

$$\boxed{y + m_1 x = a} \text{ constant}$$

soln: $z|_{m_1} = \phi(y + m_1 x)$

Similarly, $(D - m_2 D')z = 0$

soln: $z|_{m_2} = \psi(y + m_2 x)$

so, Total solution is the superposition of soln. for m_1 & m_2

$$\boxed{z = \phi(y + m_1 x) + \psi(y + m_2 x)}$$

provided $m_1 \neq m_2$

If $m_1 = m_2$,

$$\boxed{z = x\phi(y + m_1 x) + \phi(y + m_2 x)}$$

$$\left(\frac{D^4}{D'^4} - \frac{2D^3D'}{D'^4} + \frac{2DD'^3}{D'^4} - \frac{D'^4}{D'^4}\right)Z = 0$$

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Solve: $(D^4 - 2D^3D' + 2DD'^3 - D'^4)Z = 0$

Let $\frac{D}{D'} = m$

$$\Rightarrow m^4 - 2m^3 + 2m - 1 = 0$$

$$(m+1)(m-1)^3 = 0$$

$$m = 1, 1, 1, -1$$

$$z = \phi_1(y-x) + \phi_2(y+x) + x\phi_3(y+x) + x^2\phi_4(y+x)$$

Solve: $(D^4 - D'^4)Z = 0$

Let

~~f(D~~

Non-homogeneous partial differential equation -

$$f(D, D')Z = F(x,y) \neq 0$$

Here,

$$\boxed{PI = \frac{F(x)}{f(D,D')}}$$

Case 1:- If $F(x,y) = e^{(ax+by)}$

$$PI = \frac{e^{(ax+by)}}{f(D,D')}$$

$D \rightarrow a$

$D' \rightarrow b$

$$= \frac{e^{(ax+by)}}{f(a,b)}$$

provided $f(a,b) \neq 0$

$D \to m$
$D' \to 1$

$m = 1, 2, 4$

$z = \phi_1(y+x) + \phi_2(y+2x) + \phi_3(y+4x)$

DATE $m = 2, 2, 4$

$z = \phi_1(y+2x) + x\phi_2(y+2x) + \phi_3(y+4x)$

$2, 2, 2$

$z = \phi_1(y+2x) + x\phi_2(y+2x) + x^2\phi_3(y+2x)$

Case 2:- If $F(x,y) = \sin(ax+by)$ or $\cos(ax+by)$

$$PI = \frac{\sin(ax+by)}{f(D,D')}$$

$$= \frac{\sin(ax+by)}{f(-a^2, -ab, -b^2)}$$

$D^2 \to -a^2$
$D'^2 \to -b^2$
$DD' \to -ab$

Case 3:- If $F(x,y) = x^m y^n$ $m>0$, $n>0$

$$PI = \frac{x^m y^n}{f(D,D')}$$

$$= [f(D,D')]^{-1} (x^m y^n)$$

$$= P_e(D,D')\, x^m y^n$$

↳ integrate term by term.

Case 4:- If $F(x,y)$

$$f(D,D') = (D - m_1 D')(D - m_2 D') \cdots (D - m_n D')$$

$$PI = \frac{1}{f(D,D')} F(x,y)$$

$$= \left[\frac{1}{(D-m_1 D')(D-m_2 D')}\right] F(x,y)$$

$y + mx = a$
$y = a - mx$

$$= \left[\frac{1}{D - m_2 D'}\right] \phi(x, a - mx)$$

$r = D^2 z$

DATE $s = DD'z$

$t = D'^2 z$

Case 5:- If $F(x,y) = e^{ax+by} \, U(x,y)$

$$PI = \frac{1}{f(D,D')} e^{(ax+by)} U(x,y)$$

$$= \frac{e^{ax+by}}{f(D+a, D-a)} U(x,y)$$

$D \to D+a$

$D' \to D-a$

Solve the differential equation:

$$f(D,D')z = (D^3 + D^2D' - DD'^2 - D'^3)z = e^x \cos 2y \qquad \frac{D}{D'} = m$$

$f(D,D') = 0$

$m^3 + m^2 - m - 1 = 0 \to$ Auxiliary Equation

or $(m-1)(m+1)^2 = 0$

$m = 1, \underbrace{-1, -1}$

$$\boxed{z_{cf}(x,y) = \phi_1(y+x) + \phi_2(y-x) + x\phi_3(y-x)}$$

Now,

$$PI = \frac{1}{f(D,D')} e^x \cos 2y \qquad \cos 2y = \frac{e^{i2y} + e^{-i2y}}{2}$$

$$= \frac{1}{2} \frac{1}{D^3 + D^2D' - DD'^2 - D'^3} e^x \left[e^{i2y} + e^{-i2y}\right]$$

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$$= \frac{1}{2}\,\frac{1}{D^3+D^2D'-DD'^2-D'^3}\left[e^{x+i2y}+e^{x-i2y}\right]$$

$$= \frac{1}{2}\,\frac{e^{x+2iy}}{1+2i+4+8i}+\frac{1}{2}\,\frac{e^{x-2iy}}{1-2i+4-8i}$$

$$= \frac{1}{2}\,\frac{e^{x+2iy}}{5(1+2i)}+\frac{1}{2}\,\frac{e^{x-2iy}}{5(1-2i)}$$

$$= \frac{e^x}{10}\left[\frac{e^{2iy}}{1+2i}+\frac{e^{-2iy}}{1-2i}\right]$$

$$= \frac{e^x}{10}\left[\frac{(1-2i)e^{2iy}}{1-4(-1)}+\frac{(1+2i)e^{-2iy}}{1-4(-1)}\right]$$

$$= \frac{e^x}{50}\left[e^{2iy}-2ie^{2iy}+e^{-2iy}+2ie^{-2iy}\right]$$

$$= \frac{e^x}{25}\left[\frac{e^{2iy}+e^{-2iy}}{2}-2i\left(\frac{e^{2iy}-e^{-2iy}}{2}\right)\right]$$

$$= \frac{e^x}{25}\left[\frac{e^{2iy}+e^{-2iy}}{2}+2\left(\frac{e^{2iy}-e^{-2iy}}{2i}\right)\right]$$

$$= \frac{e^x}{25}\left[\cos 2y+2\sin 2y\right]$$

So, complete solution is

$$z = z_{CF}+z_{PI}$$

$$= \phi_1(y+x)+\phi_2(y-x)+x\phi_3(y-x)+\frac{e^x}{25}\left[\cos 2y+2\sin 2y\right]$$

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$$D = \frac{\partial}{\partial t}$$

$$D' = \frac{\partial}{\partial x}$$

Solve: $\frac{\partial^2 y}{\partial t^2} - a^2 \frac{\partial^2 y}{\partial x^2} = E \sin \phi t$

where $y = y(x,t)$

$(D^2 - a^2 D'^2) y = E \sin \phi t$

Homogeneous Part: $(D^2 - a^2 D'^2) y = 0$

$\Rightarrow m^2 - a^2 = 0 \longrightarrow$ Auxiliary Equation

$\boxed{m = \pm a}$

$$y_{cf}(x,t) = \phi_1 (t + ax) + \phi_2 (t - ax)$$

$$PI = \frac{1}{f(D, D')} E \sin \phi t$$

$$= \frac{1}{D^2 - a^2 D'^2} E \sin pt$$

$$= \frac{E \sin pt}{-p^2}$$

$$PI = \frac{1}{f(D^2, DD', D'^2)} E \sin \phi t$$

$D^2 = -a^2$

$D'^2 = -b^2$

$DD' = -ab$

So,

$$\boxed{y(x,t) = \phi_1 (t + ax) + \phi_2 (t - ax) - \frac{E \sin pt}{p^2}}$$

$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = E \cos x \cos 2y$

$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \cos mx \cos ny$

$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6 \frac{\partial^2 z}{\partial y^2} = \cos (2x + y)$

Fourier Se

DATE $f(x+T) = f(x)$ 1

Solving D.E. (Fourier Series)

↓

deals with periodic functions

If $f(x+T) = f(x)$ Then,

↳ $f(x)$ is a periodic function

↓

least value of T is known as its period.

$\sin x \rightarrow 2\pi$ (period)

$\sin nx / \cos nx \rightarrow \frac{2\pi}{n}$ (period)

$\tan x \rightarrow \pi$ (period)

* If a function $f(x)$ can be defined in the interval $(-L, +L)$ and ~~the~~ outside the interval by this form

→: $f(x+2L) = f(x)$

Then $f(x)$ can be written in terms of sine and cosine series.

$(0 \text{ to } 2\pi) \xrightarrow{\text{Shifting of Interval}} (-\pi \text{ to } \pi)$

Fourier Series: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$

where a_0, a_n and b_n are fourier coefficients which needed to be find out.

$$a_0 = \frac{1}{L}\int_{-L}^{+L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Orthogonality of integrals involving sine & cosine series -

1. $$\int_{-L}^{+L}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = L\,\delta_{mn}$$

$\delta_{mn} \to$ Krönecker's delta

$\delta_{mn} = 1$ when $m = n$

$\delta_{mn} = 0$ when $m \neq n$

2. $$\int_{-L}^{+L}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = L\,\delta_{mn}$$

3. $$\int_{-L}^{+L}\sin\left(\frac{n\pi x}{L}\right)dx = \int_{-L}^{+L}\cos\left(\frac{n\pi x}{L}\right)dx = 0$$

when m, n are positive integers.

Proof:- $$f(x) = A + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right)$$

so, $$\int_{-L}^{+L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx = A\cancelto{0}{\int_{-L}^{+L}\cos\left(\frac{m\pi x}{L}\right)dx} + \sum_{n=1}^{\infty}\left[a_n\int_{-L}^{+L}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx\right.$$

$$\left. + b_n\cancelto{0}{\int_{-L}^{+L}\cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx}\right]$$

$$\Rightarrow \int_{-L}^{+L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = a_n L$$

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$$\text{so,}\quad \boxed{a_n = \frac{1}{L}\int_{-L}^{+L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx}$$

Again if ~~#~~ we multiply the series both sides by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate it from $-L$ to $+L$

we get -
$$\boxed{b_n = \frac{1}{L}\int_{-L}^{+L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

and
$$\boxed{a_0 = \frac{1}{L}\int_{-L}^{+L} f(x)\,dx}$$

Proof:-

$$\frac{1}{L}\int_{-L}^{+L} f(x)\,dx = \frac{A}{L}\int_{-L}^{+L} dx + 0 + 0$$

$$\Rightarrow\quad 2A = \frac{1}{L}\int_{-L}^{+L} f(x)\,dx = a_0$$

$$A = \frac{a_0}{2}$$

$(-L, L) \rightarrow$ full period

$(-L, 0)$ or $(0, L) \rightarrow$ Half period

↳ Series will be sine OR cosine.

$\cos(-x) = \cos x$

If $f(x)$ is an even function, then Half-Range series will be:

$$A = 0, \quad a_n = 0, \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

sine series
for $f(-x) = f(x)$

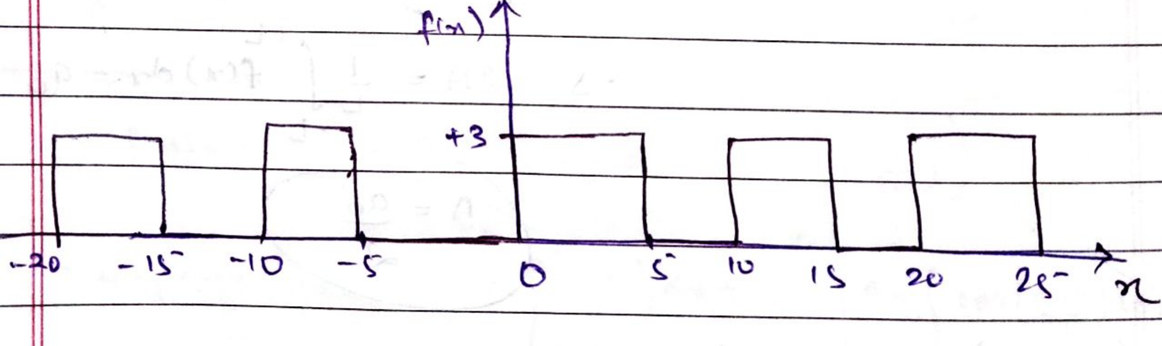
If $f(x)$ is an odd function, then Half-Range series will be.

$$A = \frac{a_0}{2}, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad b_n = 0$$

cosine series
for $f(-x) = f(x)$

Prob-1 Find the fourier coefficients of the corresponding function given by: $f(x) = \begin{cases} 0 & \text{for } -5 < x < 0 \\ 3 & \text{for } 0 < x < 5 \end{cases}$

and write down the fourier series.



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right)$$

where,

$$a_n = \frac{1}{5}\int_{-5}^{+5} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{5}\int_{-5}^{0} 0\cos\left(\frac{n\pi x}{L}\right)dx + \frac{1}{5}\int_0^5 3\cos$$

$$a_n = \frac{3}{5}\int_0^5 \cos\left(\frac{n\pi x}{L}\right)dx$$

$$a_n = \frac{3}{5}\left[\frac{\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)}\right]_0^5 = \frac{3}{5} \times \frac{L}{n\pi}[0-0] = 0$$

$$\boxed{a_n = 0}$$

$$a_0 = \frac{3}{5}\int_0^5 dx = 3 \qquad \boxed{a_0 = 3}$$

$$b_n = \frac{1}{5}\int_{-5}^{+5} f(x)\sin\left(\frac{n\pi x}{5}\right)dx = \frac{1}{5}\int_{-5}^{+5} 3\cdot\sin\left(\frac{n\pi x}{5}\right)dx$$

$$= \frac{3}{5}\left[-\frac{\cos\left(\frac{n\pi x}{5}\right)}{\left(\frac{n\pi}{5}\right)}\right]_0^5$$

$$= \frac{3}{n\pi}\left[1 - \cos n\pi\right]$$

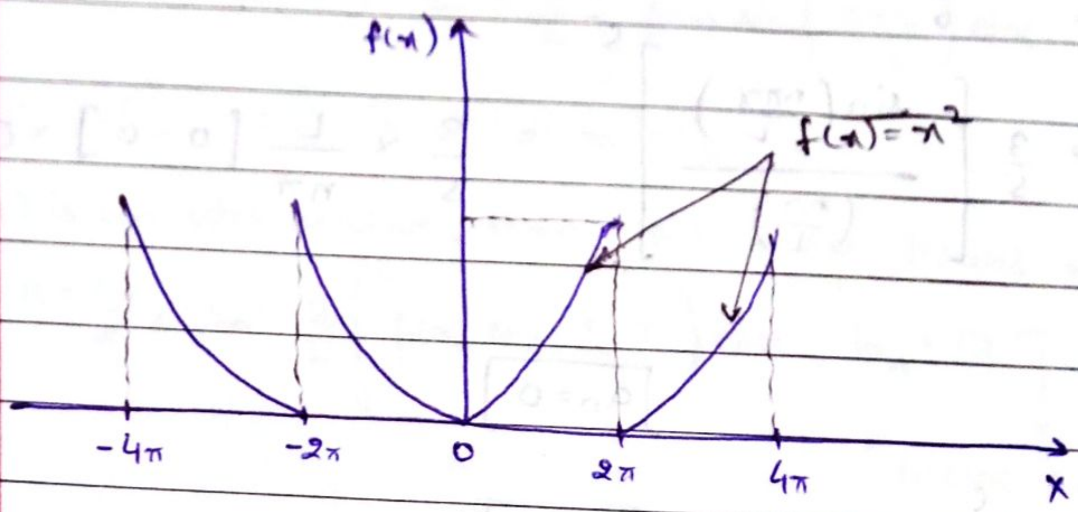
So,

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{5}\right)\cdot\frac{3}{n\pi}\left[1-\cos n\pi\right]$$

$$\boxed{f(x) = \frac{3}{2} + \frac{6}{\pi}\sin\frac{\pi x}{5} + \frac{2}{\pi}\sin\frac{3\pi x}{5} + \cdots\cdots}$$

Left margin: $f(x)$ ... ction ... $+\frac{1}{5}\int_0^5 3\cos\left(\frac{n\pi x}{L}\right)dx$

Que: Expand $f(x) = x^2$; $0 < x < 2\pi$ → period 2π



$$a_n = \frac{1}{L}\int_{-L}^{+L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{2L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{\pi}\int_0^L x^2 \cos nx \, dx = \frac{4}{n^2}$$

$$b_n = \frac{2}{2\pi}\int_0^{2\pi} x^2 \sin nx \, dx = -\frac{4\pi}{n}$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x^2 dx = \frac{8\pi^2}{3}$$

Then,

$$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

Ques: Classify each of the following functions as 'even' or 'odd' or 'neither even nor odd'.

(a) $f(x) = \begin{cases} 2 & \text{if } 0 < x < 3 \\ -2 & \text{if } -3 < x < 0 \end{cases}$ → period $(-2, +2)$

(b) $f(x) = \begin{cases} \cos x & \text{if } 0 < x < \pi \\ 0 & \text{if } \pi < x < 2\pi \end{cases}$

Ques: Show that an even function can have no sine term in its fourier series –

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$f(-x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) - b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

~~f(x) = f(x)~~

$$f(x) = f(-x)$$

$$\Rightarrow 2b_n \sin\left(\frac{n\pi x}{L}\right) = 0$$

$$\boxed{b_n = 0}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$f(x) = -f(-x)$$

$$\Downarrow$$

$$a_n = 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

even → cosine
← odd → sine } series
DATE
$a_n = 0$

Ques: Expand $f(x) = \sin x \rightarrow 0 < x < 2\pi$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin \frac{n\pi x}{\pi}\, dx = \frac{1}{\pi}$$

$$\frac{1}{L}\int_{-L}^{+L}$$

$$\frac{2}{L}\int_0^{L}$$

$$\frac{2}{\pi}\int_0^{\pi}$$

Ques: Find the soln. $U(x,t)$ of a boundary value problem

$$\frac{\partial U(x,t)}{\partial t} = 3\frac{\partial^2 U(x,t)}{\partial x^2} \qquad D_t = 3D_{xx}$$

when $t > 0,\ 0 < x < 2$

$U(0,t) = 0,\ U(2,t) = 0$

$U(x,0) = x$ for $0 < x < 2$

$$\frac{\partial U}{\partial t} = 3\frac{\partial^2 U}{\partial x^2}$$

$U(x,t) = X(x)T(t)$ (Let)

$$\frac{\partial}{\partial t}(XT) = 3\frac{\partial^2}{\partial x^2}(XT)$$

$$\Rightarrow X\frac{\partial T}{\partial t} = 3T\frac{\partial^2 X}{\partial x^2}$$

$$\Rightarrow \frac{1}{3T}\frac{\partial T}{\partial t} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \text{constant} = -\lambda^2$$

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$$\frac{dT}{dt} + 3\lambda^2 T = 0 \quad \&$$

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0$$

Soln: $T_{(t)} = c_1 e^{-3\lambda^2 t}$ and $\boxed{X(x) = A_1 \cos\lambda x + B_1 \sin\lambda x}$

Total soln: $U(x,t) = X(x)\, T(t)$

ie. $\boxed{U(x,t) = e^{-3\lambda^2 t}\left[A\cos\lambda x + B\sin\lambda x\right]}$

1) $U(0,t) = 0$ ie. for $x = 0$, $U(x,t) = 0$

$\Rightarrow \quad 0 = e^{-3\lambda^2 t}\cdot A$

ie. $\boxed{A = 0}$

Soln: $\boxed{U(x,t) = B e^{-3\lambda^2 t} \sin\lambda x}$

2) $U(2,t) = 0$ ie. for $x = 2$, $U(x,t) = 0$

$0 = Be^{-3\lambda^2 t} \sin 2\lambda$

since $B \neq 0$

ie. $\boxed{\sin 2\lambda = 0}$

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$$2\lambda = m\pi$$

$$\lambda = \frac{m\pi}{2} \quad \text{where } m = 0, \pm 1, \pm 2, \ldots$$

Soln: $U_m(x,t) = B_m e^{-\frac{3m^2\pi^2}{4}t} \sin\left(\frac{m\pi x}{2}\right)$

where $m = 0, \pm 1, \pm 2, \ldots$

By method of superposition, the General/complete soln.—

$$U(x,t) = \sum_{m=0}^{\infty} B_m e^{-\frac{3m^2\pi^2}{4}t} \sin\left(\frac{m\pi x}{2}\right)$$

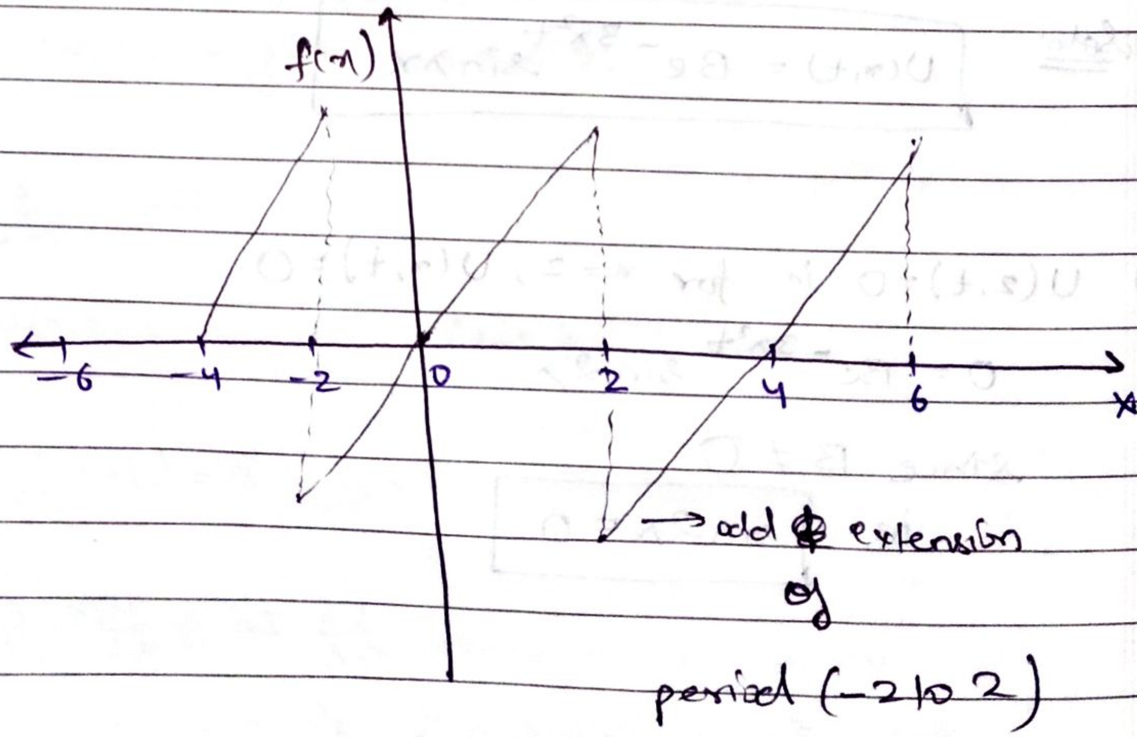
for $0 < x < 2$

unknown

Take $f(x) = x$ is periodic function in the range $0 < x < 2$

period $2L = 2$

$L = 1$



→ odd extension of period $(-2 \text{ to } 2)$

$L = 2$

$f(x) \to$ odd fn. $\to$ only sine series

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{2}\int_{0}^{2} x\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \left\{(x)\left[-\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right] - (-1)\left[\frac{-4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)\right]\right|_0^2$$

$$= -\frac{4}{n\pi}\cos n\pi$$

i.e. $b_n = -\frac{4}{n\pi}\cos n\pi$

$$b_m = -\frac{4}{m\pi}\cos m\pi$$

$$\boxed{U(x,t) = \sum_{m=1}^{\infty}\left(-\frac{4}{m\pi}\cos m\pi\right)e^{-\frac{3m^2\pi^2 t}{4}}\sin\left(\frac{m\pi x}{2}\right)}$$

L^2 $\quad$ L

$K = 3$

$L = 2$

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Dirichlet Condition (deals with the problem that - whether a particular fn. can be expanded in fourier series or not)

1) f(x) must have finite no of maxima & minima over the range of time period.

2) f(x) must have finite no. of discontinuity over the range of time period.

3) Function should be integrable over the range of time period f(x)

Two dimensional heat flow equation -

$$\left[\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right] \rightarrow \text{non-steady state condition} \quad (1)$$

For steady state condition, $\frac{\partial u}{\partial t} = 0$

so, $$\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\right] \rightarrow \text{Steady State Condition} \quad (2)$$

$u = u(x,y)$ ← Note

This is known as Laplace's equation in 2D -

solving eq-(2) } $\boxed{u_{xx} + u_{yy} = 0}$ (3) with boundary conditions

(i) $u(x,0) = 0$, $u(x,1) = x - x^2$

(ii) $u(0,y) = 0$, $u(1,0) = 0$

so, Assume $u(x,y) = X(x)\,Y(y)$

Then eq (3) can be written as $X''Y + XY'' = 0$

or $\frac{X''}{X} = -\frac{Y''}{Y} = -k^2$ (say)

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Then, $X'' + k^2 X = 0$ — (4)

& , $Y'' - k^2 Y = 0$ — (5)

eq (4) will give the solution as $X(x) = c_1 \sin kx + c_2 \cos kx$

and eq (5) will give the solution ~~$Y(y) = c_3 \sinh ky + c_4 \cosh ky$~~

$Y(y) = c_3 \sinh ky + c_4 \cosh ky$

$\therefore u(x,y) = (c_1 \sin kx + c_2 \cos kx) \cdot (c_3 \sinh ky + c_4 \cosh ky)$

Applying boundary conditions, ~~$u(x,0)=0$ and $u(1,0)=0$, we get~~

(i) $u(x,0) = 0$, ~~we get~~ $\longrightarrow Y(0) = 0$

(ii) $u(0,y) = 0 \longrightarrow X(0) = 0$

(iii) $u(1,0) = 0 \longrightarrow X(1) = 0$ or $Y(0) = 0$

we have, $X(x) = c_1 \sin kx + c_2 \cos kx$ — (6)

On applying $X(0) = 0 \Rightarrow \boxed{c_2 = 0}$

and for $X(1) = 0 \Rightarrow$ ~~$k = \frac{n\pi}{a}$~~ $\boxed{k = n\pi}$

So, eq (6) becomes as $\boxed{X(x) = c_1 \sin n\pi x}$

Similarly,

$\boxed{Y(y) = c_3 \sinh(n\pi y) + c_4 \cosh(n\pi y)}$ — (7)

Applying $Y(0) = 0$ on eq (7), we get - $c_4 = 0$

Then,

$\boxed{Y(y) = c_3 \sinh(n\pi y)}$

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Since, $u(x,y) = (C_1 \sin kx + C_2 \cos kx) \cdot (C_3 \sinh ky + C_4 \cosh ky)$

So, Now, this equation becomes as:

$u(x,y) = C_1 C_3 \sin(n\pi x) \sinh(n\pi y)$ where $n = 0, \pm 1, \pm 2, \ldots$

or $u_n(x,y) = a_n \sin(n\pi x) \sinh(n\pi y)$

Complete solution: $\left[u(x,y) = \sum_{n=0}^{\infty} a_n \sin(n\pi x) \sinh(n\pi y) \right]$ - (8)

Applying boundary condition $u(x,1) = x - x^2$ on eq (8), we get-

$$\boxed{x - x^2 = \sum_{n=0}^{\infty} B_n \sin(n\pi x)}$$

The soln. is Fourier series, expanding $f(x) = x - x^2$

$\because b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$

so, here $B_n = \frac{2}{1} \int_0^1 (x - x^2) \sin(n\pi x) dx$

$$B_n = \frac{16}{n^3 \pi^3} (1 - \cos n\pi)$$

$\because B_n = \sinh(n\pi) a_n$

so, $a_n = \frac{B_n}{\sinh(n\pi)} = \frac{16(1 - \cos n\pi)}{n^3 \pi^3 \sinh(n\pi)}$

Now eq (8) becomes as,

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$$\boxed{u(x,y) = \frac{16}{\pi^3} \sum_{n=1}^{\infty} \frac{(1-\cos n\pi)}{n^3} \sin(n\pi x) \frac{\sinh(n\pi y)}{\sinh(n\pi)}}$$

This is the solution of the steady state condition in 2D heat-flow.

Linked

In general,

$$u_{xx} + u_{yy} = 0$$

$0 < x < L_x$

$0 < y < L_y$

We have boundary conditions as:-

$u(x,0) = 0$ $\quad u(0, L_y) = 0$

$u(0,y) = 0$

So, the solution is

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L_x}\right) \frac{\sinh\left(\frac{n\pi y}{L_y}\right)}{\sinh\left(\frac{n\pi}{L_y}\right)}$$

where $b_n = \dfrac{2}{L_x} \displaystyle\int_0^{L_x} f(x) \sin\left(\frac{n\pi x}{L_x}\right) dx$

Ques:- Solve: $u_{xx} + u_{yy} = 0$, subjected to boundary conditions

$u(x,0) = 0$, $u(1,y) = y - y^2$

$u(0,y) = 0$, $u(0,1) = 0$

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On simplyfying the given boundary conditions, we have $X(0)=0$

$$Y(0)=0$$

$$Y(1)=0$$

and $u(1,y) = y-y^2$

We know that,

$$u(x,y) = X(x)Y(y)$$

Then, $\frac{X''}{X} = \frac{-Y''}{Y} = k^2 \text{ (say)}$

i.e. $X'' - k^2 X = 0$ — (1)

and $Y'' + k^2 Y = 0$ — (2)

Solutions of both eqs (1) & (2) will be respectively as.

$Y(y) = C_1 \sin ky + C_2 \cos ky$ — (3)

$X(x) = C_3 \sinh kx + C_4 \cosh kx$ — (4)

Applying B.C. $Y(0)=0$ and $Y(1)=0$, we get-

$$\boxed{C_2 = 0}$$

$k = n\pi$ where $n = 0, \pm 1, \pm 2, \ldots$

Also,

for $X(0)=0$ in eq (4) $\Rightarrow$ $\boxed{C_4 = 0}$

Thus, we have the solutions as -

$Y(y) = C_1 \sin n\pi y$

and $X(x) = C_3 \sinh(n\pi x)$

~~Taco~~ Taking $C_1 C_2 = a_n$

$$u(x,y) = \sum_{n=0}^{\infty} a_n \sin(n\pi y) \sinh(n\pi x)$$

For B.C. $u(1,y) = y - y^2$,

$$y - y^2 = \sum_{n=0}^{\infty} \underbrace{a_n \sinh(n\pi)}_{A_n} \sin(n\pi y)$$

ie. $$y - y^2 = \sum_{n=0}^{\infty} A_n \sin(n\pi y)$$

On solving, we have

$$\boxed{A_n = \frac{16}{n^3\pi^3}(1 - \cos n\pi)}$$

$$A_n = a_n \sinh n\pi$$

$$\Rightarrow \boxed{u(x,y) = \frac{16}{\pi^3} \sum_{n=1}^{\infty} \frac{(1-\cos n\pi)}{n^3} \frac{\sinh(n\pi x)}{\sinh n\pi} \cdot \sin(n\pi y)}$$

In general, 2D heat-flow

$0 < x < L_x$ & $0 < y < L_y$

$u(x,0) = 0$

$u(0,y) = 0$

$u(L_x, y) = f(y)$

$u(0, L_y) = 0$

$$u(x,y) = \frac{16}{\pi^3} \sum_{n=1}^{\infty} b_n \frac{\sinh\left(\frac{n\pi x}{L_x}\right)}{\sinh\left(\frac{n\pi}{L_x}\right)} \cdot \sin\left(\frac{n\pi y}{L_y}\right)$$

where, $b_n = \frac{2}{L_y} \int_0^{L_y} f(y) \sin\left(\frac{n\pi y}{L_y}\right) dy$

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Case of 2D heat flow - non steady state condition -

$$u_t = c^2 \nabla^2 u = c^2 (u_{xx} + u_{yy})$$

$$0 < x < a \ \& \ 0 < y < b$$

with boundary conditions,

$$u(0,y,t) = u(a,y,t) = 0$$

$$u(x,0,t) = u(x,b,t) = 0$$

Initial conditions $\quad u(x,y,0) = f(x,y)$

Separation of variable, assume

$$u(x,y,t) = X(x)\,Y(y)\,T(t)$$

$$u_t = c^2 (u_{xx} + u_{yy})$$

$$XYT' = c^2 (X''YT + XY''T)$$

$$\frac{T'}{c^2 T} = \frac{X''}{X} + \frac{Y''}{Y}$$

Let $\frac{X''}{X} = -B$ and $\frac{Y''}{Y} = -C$

$$T' = c^2 T (B+C)$$

$\cancel{T' - c}$ $\boxed{T' + c^2 (B+C) T = 0}$

$\boxed{X'' - BX = 0}$ and $\boxed{Y'' - CY = 0}$

we have, $X'' + BX = 0$ with $X(0) = 0$ & $X(a) = 0$

Let $B = +\mu_m^2$

$\boxed{X'' + \mu_m' X = 0}$

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Solution: $X(x) = C_1 \sin \mu_m x + C_2 \cos \mu_m x$

Imposing B.C. $X(0) = 0$ & $X(a) = 0$

$\Rightarrow$ $C_2 = 0$ and $\mu_m = \frac{m\pi}{a}$ where $m = 0, \pm 1, \pm 2, \cdots$

$\boxed{X(x) = C_1 \sin \mu_m x}$

and $\boxed{Y(y) = C_3 \sin \nu_n y}$ where $\nu_n = \frac{n\pi}{b}$

Now,

$B = -\mu_m^2 = \frac{n^2\pi^2}{a^2}$

$C = -\nu_n^2 = \frac{n^2\pi^2}{b^2}$

we have, $T' + c^2(B+C)T = 0$

Let $\lambda_{mn} = c\sqrt{B+C}$

$$\boxed{\lambda_{mn} = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}} = c\sqrt{\mu_m^2 + \nu_n^2}$$

$T' = -\lambda_{mn}^2 T$

$T_{mn}(t) = C_5 e^{-\lambda_{mn}^2 t}$

$\boxed{T_{mn}(t) = C_5 e^{-c^2(\mu_m^2 + \nu_n^2)t}}$

$$\boxed{u_{mn}(x,y,t) = XYT = a_n \sin \mu_m x \sin \nu_n y \, e^{-c^2(\mu_m^2 + \nu_n^2)t}}$$

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$$u(x,y,t) = \sum u_{mn}(x,y,t)$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \mu_m x \sin \nu_n y \, e^{-c^2(\mu_m^2 + \nu_n^2)t}$$

where $a_n = C_1 C_3 C_5$

Initial condition $u(x,y,0) = f(x,y)$

$$f(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \cdot (1)$$

$$\frac{4}{ab}\int_0^a \int_0^b f(x,y) \sin\left(\frac{m_1 \pi x}{a}\right) \cdot \sin\left(\frac{n_1 \pi y}{b}\right) dy\, dx$$

$$= \sum_m \sum_n a_{mn} \int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{m_1 \pi x}{a}\right) \int_0^b \sin\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n_1 \pi x}{b}\right) dy$$

$$a_{mn} = \frac{4}{ab}\int_0^a \int_0^b f(x,y) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dy\, dx$$

Q. A 2x2 square plate with $c = \frac{1}{3}$ is heated in such a way that temp. in the lower half is 50 while temp. in the upper half is 0. After that, it is isolated latterally and the temp. at its edge is held at 0. Find the expression that gives you temp. in the plate at $t > 0$

Given $a = 2,\ b = 2$

$$a_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\cdot\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dy\,dx$$

$$= \frac{4}{4}\int_{x=0}^{2}\int_{y=0}^{2} f(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dy\,dx$$

$$= 50\int_{x=0}^{2}\sin\left(\frac{m\pi x}{2}\right)dx\int_{y=0}^{1}\sin\left(\frac{n\pi y}{2}\right)dy$$

$$a_{mn} = \frac{200}{\pi^2}\left[\frac{\{1+(-1)^{m+1}\}\{1-\cos\frac{m\pi}{2}\}}{mn}\right]$$

$$\therefore c^2 = \frac{1}{3^2} = \frac{1}{9}$$

$$\sqrt{\mu_m^2 + \nu_n^2} = \sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}} = \frac{1}{2}\sqrt{(m^2+n^2)}$$

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Solving differential equation by Laplace Transformation -

$$\text{ODE} \xrightarrow{\text{L.T}} \text{Algebraic equation} \xrightarrow{\text{ILT}} \text{Solution of ODE}$$

LT :- $\mathcal{L}\{F(t)\} = f(p) = \int_0^\infty e^{-pt} F(t)\,dt \quad (p > 0)$

$f(p)$ → parameter of Laplace Transformation

i.e. $f(p)$ is a LT of $F(t)$.

$$\begin{array}{ccc} & \text{LT} & \\ \left(\begin{array}{c}t\text{-domain}\\ F(t)\end{array}\right) & \underset{\mathcal{L}^{-1}\{f(p)\}}{\overset{\mathcal{L}\{F(t)\}}{\rightleftarrows}} & \left(\begin{array}{c}p\text{-domain}\\ f(p)\end{array}\right) \\ & \text{ILT} & \end{array}$$

$\mathcal{L}^{-1}\{f(p)\} = F(t)$

---

find LT -

(i) $F(t) = \text{constant} = c$

$$\mathcal{L}\{c\} = \int_0^\infty c e^{-pt}\,dt = \left[-\frac{ce^{-pt}}{p}\right]_0^\infty = \frac{c}{p}$$

$$\mathcal{L}\{F(t)\} = f(p) = \int_0^\infty e^{-pt} F(t)\,dt$$

so,

① $\mathcal{L}\{c\} = \frac{c}{p}$

② $\mathcal{L}(1) = \frac{1}{p}$

③ $\mathcal{L}^{-1}\left\{\frac{c}{p}\right\} = c$

④ $\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{p^{n+1}} = \frac{n!}{p^{n+1}}$ ← when n is a positive integer.

$\mathcal{L}\{t^2\} = \frac{2!}{p^3}$

$\mathcal{L}\{\sqrt{t}\} = \frac{\Gamma(3/2)}{p^{3/2}} = \frac{\sqrt{\pi}/2}{p^{3/2}}$

$\mathcal{L}\{c_1+c_2\} = \mathcal{L}\{c_1\} + \mathcal{L}\{c_2\}$ when c_1 and c_2 are linear.

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(5) $\mathcal{L}^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{1}{n!}t^n$

(6) $\mathcal{L}\{e^{\alpha t}\} = \frac{1}{p-\alpha}$ (when $\alpha < p$)

(7) $\mathcal{L}^{-1}\left\{\frac{1}{p-\alpha}\right\} = e^{\alpha t}$

(8) $\mathcal{L}\{\cosh at\} = \frac{p}{p^2-a^2}$

(9) $\mathcal{L}\{\sinh at\} = \frac{a}{p^2-a^2}$

(10) $\mathcal{L}^{-1}\left\{\frac{a}{p^2-a^2}\right\} = \sinh at$

(11) $\mathcal{L}^{-1}\left\{\frac{p}{p^2-a^2}\right\} = \cosh at$

(12) $\mathcal{L}\{\cos at\} = \frac{p}{p^2+a^2}$

(13) $\mathcal{L}\{\sin at\} = \frac{a}{p^2+a^2}$

(14) $\mathcal{L}^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$

(15) $\mathcal{L}^{-1}\left\{\frac{a}{p^2+a^2}\right\} = \sin at$

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Properties of LT -

① It is a linear operator.

$$\mathcal{L}\{a_1 F_1(t) + a_2 F_2(t)\} = \mathcal{L}\{a_1 F_1(t)\} + \mathcal{L}\{a_2 F_2(t)\}$$

$$\mathcal{L}^{-1}\{a_1 F_1(t) + a_2 F_2(t)\} = \mathcal{L}^{-1}\{a_1 F_1(t)\} + \mathcal{L}^{-1}\{a_2 F_2(t)\}$$

② It has a shifting property.

$$\mathcal{L}\{e^{at} F(t)\} = f(p-a)$$

③ Change in scale -

$$\mathcal{L}\{F(t)\} = f(p)$$

$$\mathcal{L}\{F(at)\} = \frac{1}{a} f\left(\frac{p}{a}\right)$$

$$\mathcal{L}^{-1}\{f\left(\frac{p}{a}\right)\} = a F(at)$$

$$\mathcal{L}^{-1}\{f(ap)\} = \frac{1}{a} F(t/a)$$

L.T. of derivatives -

$F(t) \to$ continuous

$F'(t) \to$ continuous

$$\mathcal{L}\{F'(t)\} = p\mathcal{L}\{F(t)\} - F(0)$$

$$= pf(p) - F(0)$$

Proof

$$\mathcal{L}\{F'(t)\} = \int_0^T e^{-pt} F'(t)\, dt = \left[e^{-pt} F(t)\right]_0^T + \int_0^T p e^{-pt} F(t)\, dt$$

$T \to \alpha$

Legendre
Properties of special fn.

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$$\mathcal{L}\{F'(t)\} = e^{-p\alpha}F(\alpha) - F(0) + p\int_0^\infty e^{-pt}F(t)\,dt$$

$$\boxed{\mathcal{L}\{F'(t)\} = p\mathcal{L}\{F(t)\} - F(0)}$$

$$\mathcal{L}\{F''(t)\} = p\mathcal{L}\{F'(t)\} - F'(0)$$
$$= p[p\mathcal{L}\{F(t)\} - F(0)] - F'(0)$$

$$\boxed{\mathcal{L}\{F''(t)\} = p^2\mathcal{L}\{F(t)\} - pF(0) - F'(0)}$$

$$\mathcal{L}\{F^{(n)}(t)\} = p^n\mathcal{L}\{F(t)\} - p^{n-1}F(0) - p^{n-2}F'(0) - p^{n-3}F''(0) + \cdots\cdots + pF^{(n-2)}(0) - F^{(n-1)}(0)$$

Laplace Transformation of Integration

If $F(t) \xrightarrow{LT} f(p) = \int_0^\infty e^{-pt}F(t)\,dt$

Proof!

Then, $$\boxed{\mathcal{L}\left\{\int_0^t F(z)\,dz\right\} = \frac{f(p)}{p} = \frac{1}{p}\mathcal{L}\{F(t)\}}$$

Proof $$\mathcal{L}\left\{\int_0^t F(z)\,dz\right\} = \int_0^\infty e^{-pt}\int_0^t F(z)\,dz\,dt$$
$$= \left[-\frac{e^{-pt}}{p}\int_0^t F(z)\,dz\right]_0^\infty + \frac{1}{p}\int_0^\infty F(t)e^{-pt}\,dt$$

$$= \frac{1}{p} \mathcal{L}\{F(t)\} = \frac{f(p)}{p}$$

Thus proved...

Similarly,
$$\boxed{\mathcal{L}^{-1}\left\{\int_0^t F(z)\,dz\right\} = \frac{F(t)}{t}}$$

$$\boxed{\mathcal{L}\left\{t^n F(t)\right\} = (-1)^n \frac{d^n f(p)}{dp^n}}$$

Proof

$$f(p) = \int_0^\infty e^{-pt} F(t)\,dt$$

$$f'(p) = \frac{d}{dp}\left[\int_0^\infty e^{-pt} F(t)\,dt\right]$$

$$= \int_0^\infty \frac{d}{dp}\left[e^{-pt} F(t)\,dt\right]$$

$$= \int_0^\infty \frac{d}{dp}\left[e^{-pt} F(t)\right] \cdot \frac{dt}{dp} + \int_0^\infty dt \cdot \frac{d}{dp}\left[e^{-pt} F(t)\right]$$

$$= \int_0^\infty \frac{d}{dp}\left[e^{-pt} F(t)\right] dt$$

$$= \cancel{\left[\frac{d}{dp}\left[e^{-pt} F(t)\right] \cdot t\right]_0^\infty}$$

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$$= \left[\frac{d}{dp}\left[e^{-pt}F(t)\right]\int dt\right]_0^\infty$$

$$- \int_0^\infty \left[\frac{d}{dt}\left[\frac{d}{dp}\left\{e^{-pt}F(t)\right\}\right]\int dt\right] dt$$

$$= - \int_0^\infty \frac{d}{\cancel{dt}}\left[\frac{d}{dp}\left\{e^{-pt}F(t)\right\} t\right] \cancel{dt}$$

$$= - \int_0^\infty t \cdot d\left[\frac{d}{dp}\left\{e^{-pt}F(t)\right\}\right]$$

...

Main Result]

If $\mathcal{L}\{F(t)\} = f(p)$

then

$$\mathcal{L}\{t\,F(t)\} = -f'(p)$$

$$\mathcal{L}\{t^2F(t)\} = +f''(p)$$

Similarly, $\boxed{\mathcal{L}\{t^nF(t)\} = (-1)^n f^n(p)}$

\# $\mathcal{L}\{\sin^2 at\} = ?$

$$= \frac{1}{2}\mathcal{L}\{1 - \cos 2at\}$$

$$= \frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{\cos 2at\}$$

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$$= \frac{1}{2p} - \frac{1}{2} \cdot \frac{2p}{p^2+4a^2}$$

$$= \frac{1}{2}\left[\frac{1}{p} - \frac{p}{p^2+4a^2}\right]$$

#

$$F(t) = \begin{cases} (t-1)^2, & t \geq 1 \\ 0, & 0 \leq t < 1 \end{cases}$$

$\mathcal{L}\{F(t)\} = ?$

$$= \int_0^\infty e^{-pt} F(t)\,dt$$

$$= \int_0^1 e^{-pt}(0)\,dt + \int_0^\infty e^{-pt}(t-1)^2\,dt$$

$$= \frac{2e^{-p}}{p^3} \text{ (Ans)}$$

$$\Gamma(n+1) = \int_0^\infty e^{-t} t^n\,dt$$

#

$$\mathcal{L}\{e^{-2t}(3\cos 6t - 5\sin 6t)\}$$

$$= 3\mathcal{L}\{e^{-2t}\cos 6t\} - 5\mathcal{L}\{e^{-2t}\sin 6t\}$$

Shiftings $\mathcal{L}\{e^{at}F(t)\} = f(p-a)$

$$= 3\frac{(p+2)}{(p+2)^2+36} - 5\frac{6}{(p+2)^2+36}$$

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If $\mathcal{L}\{F(t)\} = \dfrac{p^2-p+1}{(2p+1)^2(p-1)^2} = f(p)$

find $\mathcal{L}\{F(2t)\}$

If $\mathcal{L}\{F(t)\} = f(p)$

$\mathcal{L}\{F(at)\}$

$= \frac{1}{a} f(p/a)$

$$\mathcal{L}\{F(2t)\} = \frac{1}{2} f(p/2)$$

$$= \frac{1}{2} \frac{(p/2)^2 - p/2 + 1}{(2 \cdot p/2 + 1)^2 (p/2 - 1)^2} = ?$$

$\mathcal{L}^{-1}\left\{\dfrac{4}{p-2}\right\}$

$$= 4\mathcal{L}^{-1}\left\{\frac{1}{p-2}\right\} = 4e^{2t}$$

$\mathcal{L}^{-1}\left\{\dfrac{3p-2}{p^2-4p+20}\right\} = \mathcal{L}^{-1}\left\{\dfrac{3p-2}{(p-2)^2+16}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{3p}{(p-2)^2+\cancel{16}4^2}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{(p-2)^2+\cancel{16}4^2}\right\}$$

$$= 3\mathcal{L}^{-1}\left\{\frac{(p-2)+2}{(p-2)^2+4^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(p-2)^2+4^2}\right\}$$

$$= 3\mathcal{L}^{-1}\left\{\frac{p-2}{(p-2)^2+4^2}\right\} + \mathcal{L}^{-1}\left\{\frac{4}{(p-2)^2+4^2}\right\}$$

$$= 3e^{2t}\cos 4t + e^{2t}\sin 4t$$