Central difference interpolation formulae

We have seen the formulae useful at beginning and end of the table of tabulated sets. We shall now discuss formulae which is useful for near middle of the table.

Gauss forward formula:

Let us consider a value y_0 at the centre which corresponds to $x = x_0$.

The Gauss forward formula is given by

$$y_p = y_0 + G_1 \Delta y_0 + G_2 \Delta^2 y_{-1} + G_3 \Delta^3 y_{-1} + G_4 \Delta^4 y_{-2} + \cdots \quad (1)$$

y_p is the interpolating polynomial and $G_1, G_2, \ldots$ etc. are to be determined.

Let us construct a difference table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_{-3}	y_{-3}						
		Δy_{-3}					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$				
		Δy_2					
x_3	y_3						

Now, LHS

$$y_p = E^p y_0$$

$$= (1+\Delta)^p y_0 \qquad (\because E = 1+\Delta)$$

$$= \left[1 + p\Delta + \frac{p(p-1)}{\lfloor 2} \Delta^2 + \cdots \right] y_0$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2} \Delta^2 y_0 + \cdots \quad (2)$$

Let us look at RHS,

$$\Delta^2 y_{-1} = \Delta^2 E^{-1} y_0$$

$$= \Delta^2 (1+\Delta)^{-1} y_0$$

$$= \Delta^2 (1 - \Delta + \Delta^2 - \Delta^3 + \cdots) y_0$$

$$= \Delta^2 y_0 - \Delta^3 y_0 + \Delta^4 y_0 - \Delta^5 y_0 + \cdots \quad (3)$$

Similarly

$$\Delta^3 y_{-1} = \Delta^3 y_0 - \Delta^4 y_0 + \cdots \quad (4)$$

$$\Delta^4 y_{-2} = \Delta^4 E^{-2} y_0$$

$$= \Delta^4 (1+\Delta)^{-2} y_0$$

$$= \Delta^4 (1 - 2\Delta + 3\Delta^2 - 4\Delta^3 + \cdots) y_0$$

$$= \Delta^4 y_0 - 2\Delta^5 y_0 + 3\Delta^6 y_0 \cdots \quad (5)$$

Putting from eqns. (2), (3), (4), (5) in eqn. (1), we get

$$y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3} \Delta^3 y_0 + \cdots$$

$$= y_0 + G_1 \Delta y_0 + G_2 (\Delta^2 y_0 - \Delta^3 y_0 + \cdots) +$$
$$G_3 (\Delta^3 y_0 - \Delta^4 y_0 + \cdots) + G_5 (\Delta^4 y_0 - 2\Delta^5 y_0 \cdots) + \cdots \quad (6)$$

Now equating coefficients in eqn. (6),

$$G_1 = p, \quad G_2 = \frac{p(p-1)}{\lfloor 2}$$

$$-G_2 + G_3 = \frac{p(p-1)(p-2)}{\lfloor 3}$$

Putting value of G_2,

or $$-\frac{p(p-1)}{\lfloor 2} + G_3 = \frac{p(p-1)(p-2)}{\lfloor 3}$$

$$\therefore \quad G_3 = \frac{(p+1)p(p-1)}{\lfloor 3}$$

Similarly

$$G_4 = \frac{(p+1)p(p-1)(p-2)}{\lfloor 4}, \text{ etc.}$$

Gauss backward interpolation

Constructing difference table

x	y	Δ	Δ^2	Δ^3	Δ^4
.	.				
.	.				
.	.				
x_{-1}	y_{-1}				
		Δy_{-1}		$\Delta^3 y_{-2}$	$\Delta^4 y_{-2}$
x_0	y_0		$\Delta^2 y_{-1}$		
		Δy_0			
x_1	y_1				
.	.				
.	.				
.	.				

Gauss backward interpolation formula

$$y_p = y_0 + G_1' \Delta y_{-1} + G_2' \Delta^2 y_{-1} + G_3' \Delta^3 y_{-2} + \cdots \quad (7)$$

$G_1', G_2', \ldots$, etc. are coefficients

Proceeding as earlier, we obtain

$$G_1' = p, \quad G_2' = \frac{p(p+1)}{\lfloor 2}$$

$$G_3' = \frac{(p+1)p(p-1)}{\lfloor 3}, \quad G_4' = \frac{(p+2)(p+1)p(p-1)}{\lfloor 4}, \text{ etc.}$$

Example: Given

x	1.00	1.05	1.10	1.15	1.20	1.25	1.30
e^x	2.7183	2.8577	3.0042	3.1582	3.3201	3.4903	3.6693

Find $e^{1.17}$ using Gauss forward formula.

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x	e^x	Δ	Δ^2	Δ^3	Δ^4
1.00	2.7183				
		0.1394			
			0.0071		
1.05	2.8577			0.0004	
		0.1465			0
1.10	3.0042		0.0075		
		0.1540		0.0004	
1.15	3.1582		0.0079		0
		0.1619		0.0004	
1.20	3.3201		0.0083		0.0001
		0.1702		0.0005	
1.25	3.34903		0.0088		
		0.1790			
1.30	3.6693				

The formula is

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots$$

Now $p = \frac{x - x_0}{h}$, $h = 0.05$, $x_0 = 1.15$

$x = 1.17$

$$p = \frac{1.17 - 1.15}{0.05} = \frac{0.02}{0.05} = \frac{2}{5}$$

$$\therefore e^{1.17} = 3.1582 + \frac{2}{5}(0.1619) + \frac{2/5(2/5 - 1)}{2}(0.0079) + \frac{(2/5+1)\,2/5\,(2/5-1)}{6}(0.0004) + 0$$

$$e^{1.17} = 3.2221$$

Actual value is $e^{1.17} = 3.2219926$

As can be seen Gauss formula gives fairly accurate value.

Stirling's formula

The mean of Gauss's backward and forward formulae is

$$y_p = y_0 + \frac{\Delta y_{-1} + \Delta y_0}{2} p + \frac{p^2}{2} \Delta^2 y_{-1} + \frac{p(p^2-1)}{6} \cdot \frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}$$
$$+ \cdots$$

This is known as Stirling's formula.

Bessel's formula

An useful formula from practical application point of view.

If we construct a difference table shown below

$$\begin{array}{ccccccc}
\vdots & \vdots & & & & & \\
x_{-1} & y_{-1} & & & & & \Delta^4 y_{-2} \\
 & & & \Delta^2 y_{-1} & & & \\
x_0 & y_0 & & & \Delta^3 y_{-1} & & \\
 & & \Delta y_0 & & & & \Delta^4 y_{-1} \\
 & & & \Delta^2 y_0 & & & \\
x_1 & y_1 & & & & & \\
\vdots & \vdots & & & & &
\end{array}$$

The brackets mean average of the shown values.

The Bessel's formula can be written as

$$y_p = \frac{y_0 + y_1}{2} + B_1 \Delta y_0 + B_2 \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + B_3 \Delta^3 y_{-1}$$
$$+ B_4 \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + B_5 \Delta^5 y_{-2} + B_6 \frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}$$
$$+ \cdots \quad (1)$$

$B_1, B_2, \ldots$ are coefficients to be determined.

As before, we can write

$$y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3} \Delta^3 y_0 + \cdots$$

$$= \frac{y_0 + y_1}{2} + \frac{y_0}{2} - \frac{y_0}{2} + B_1 \Delta y_0 + B_2 \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}$$
$$+ B_3 \Delta^3 y_{-1} + \cdots$$

(Adding and subtracting $y_0/2$ on RHS)

$$= y_0 + \frac{y_1 - y_0}{2} + B_1 \Delta y_0 + B_2 \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + B_3 \Delta^3 y_{-1} + \cdots$$

$$= y_0 + \frac{\Delta y_0}{2} + B_1 \Delta y_0 + B_2 \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + B_3 \Delta^3 y_{-1} + \cdots$$

$$= y_0 + \left(B_1 + \frac{1}{2}\right) \Delta y_0 + B_2 \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + B_3 \Delta^3 y_{-1} + \cdots$$

Now equating Δy_0, etc. ... , we get

$$B_1 + \frac{1}{2} = p, \quad B_2 = \frac{p(p-1)}{\lfloor 2}, \text{ etc.} \qquad \text{—— (2)}$$

Putting B_1, B_2, etc. in eqn. (1), we get

$$y_p = y_0 + p \Delta y_0 + \frac{p(p-1)}{\lfloor 2} \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{p(p-1)(2p-1)}{\lfloor 3} \cdot \frac{\Delta^3 y_{-1}}{2} + \cdots \qquad (3)$$

While interpolating near the middle of the table, Stirling's formula is efficient for $-\frac{1}{4} \le p \le \frac{1}{4}$.

Bessel's formula is efficient for $\frac{1}{4} \le p \le \frac{3}{4}$.

Thus, choice of formula must be made out as per convenience.