| S. No. | Date | Title | Page No. | Teacher's Sign / Remarks |
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I Equations solvable for $p$ $\left(p = \frac{dy}{dx}\right)$

Exam-1 Solve: $yp^2 + (x-y)p - x = 0$ —— (1)

Sol-1 $yp^2 + xp - yp - x = 0$

$\Rightarrow yp^2 - yp + xp - x = 0$

$\Rightarrow yp(p-1) + x(p-1) = 0$

$\Rightarrow (p-1)(yp+x) = 0$

Either, $p-1 = 0$ or $yp + x = 0$

| | |
|---|---|
| i.e. $\frac{dy}{dx} - 1 = 0$ | $y\frac{dy}{dx} + x = 0$ |
| $\Rightarrow dy - dx = 0$ | $\Rightarrow ydy + xdx = 0$ |
| On Integrating, | On Integrating, |
| $y - x = c$ | $y^2 + x^2 = c$ |
| so, $y - x - c = 0$ | $\Rightarrow x^2 + y^2 - c = 0$ |

and

$\therefore (y-x-c)(x^2+y^2-c) = 0$ is a solution of eq. (1)

II Equations solvable for $y$ $\left(p = \frac{dy}{dx}\right)$

Equations solvable for $x$ $\left(p = \frac{dy}{dx}\right)$

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**Exam-2** Solve: $y + px = x^4 p^2$ —(i)

**Sol-2**

$y = -px + x^4 p^2$

diff. w.r.t. $x$,

$$\frac{dy}{dx} = -p - x\frac{dp}{dx} + 2px^4\frac{dp}{dx} + 4x^3p^2$$

$$\Rightarrow 2px^4\frac{dp}{dx} - x\frac{dp}{dx} + 4x^3p^2 - 2p = 0$$

$$\Rightarrow x\frac{dp}{dx}\left(2px^3 - 1\right) + 2p\left(2px^3 - 1\right) = 0$$

$$\Rightarrow \left(x\frac{dp}{dx} + 2p\right)\left(2px^3 - 1\right) = 0$$

Considering only $x\frac{dp}{dx} + 2p = 0$

$$\frac{dp}{p} + 2\frac{dx}{x} = 0$$

on Integrating,

$$\log p + 2\log x = \log c$$

$$\Rightarrow px^2 = c$$

$$\text{or } p = \frac{c}{x^2} \quad \text{—(ii)}$$

Eliminating $p$ from (i) & (ii), we get-

$$y + \frac{c}{x^2}\cdot x = x^4\frac{c^2}{x^4}$$

$\Rightarrow y + \frac{c}{x} = c^2$ is the required solution.

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**Exam-3** Solve: $p = \tan\left(x - \frac{p}{1+p^2}\right)$

**Sol-3**

$$x - \frac{p}{1+p^2} = \tan^{-1}p$$

$$\Rightarrow x = \tan^{-1}p + \frac{p}{1+p^2}$$

diff. w.r.t. $y$,

$$\frac{dx}{dy} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{(1+p^2)1 - p(2p)}{(1+p^2)^2}\frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{1+p^2-2p^2}{(1+p^2)^2}\frac{dp}{dy}$$

$$\Rightarrow \frac{(1+p^2)^2}{p} = 2\frac{dp}{dy}$$

$$\Rightarrow 2p\frac{dp}{dy} = (1+p^2)^2 \Rightarrow 2p\frac{dp}{dy} - (1+p^2)^2 = 0$$

$$\Rightarrow \frac{2p}{(1+p^2)^2}\frac{dp}{dy} - 1 = 0$$

$$\Rightarrow -\frac{2p}{(1+p^2)^2}\frac{dp}{dy} + 1 = 0$$

$$\Rightarrow \frac{d}{dy}\left(\frac{1}{1+p^2}\right) + 1 = 0$$

$$\Rightarrow 1 + \frac{d}{dy}\left(\frac{1}{1+p^2}\right) = 0$$

On Integrating,

$$y + \frac{1}{1+p^2} = c$$

the required soln.

(1 mark) (4 marks)
Both, General soln. as well as Singular / ~~Particular~~ Particular soln. can be determined by Clairaut's form

3 questions — Clairaut's form $y = f(x, p) = px + \phi(p)$

① Find the general and singular soln. of –
(i) $xp^2 - yp + a = 0$
(ii) $p = \log(px - y)$

Soln. (ii)
$p = \log(px - y)$
$\Rightarrow e^p = px - y$
$\Rightarrow y = px - e^p$ (Clairaut's form)
Its general soln. is $y = kx - e^k$ —(1) (Ans)
diff. (1) w.r.t $k$
$$0 = x - e^k \quad \text{—(2)}$$
On eliminating $k$ from (1) & (2), we get–
$$y = x\ln x - x$$ (Particular soln.)

Soln. (i)
$xp^2 - yp + a = 0$
$yp = xp^2 + a$
$\Rightarrow y = xp + \frac{a}{p}$
or $y = px + \frac{a}{p}$ (Clairaut's form)
Its general soln. is $y = kx + \frac{a}{k}$ —(1) (Ans)

diff. (1) w.r.t $k$, we get



$$0 = x + \left(-\frac{1}{k^2}\right)a$$
$$0 = x - \frac{a}{k^2}$$
$$x = \frac{a}{k^2}$$
$$k^2 = \frac{a}{x} \Rightarrow k = \pm\sqrt{\frac{a}{x}}$$

Putting value of $k$ in (1),
$$\cancel{y = \frac{\sqrt{a}}{\sqrt{x}}x + \frac{a\sqrt{x}}{\sqrt{a}}}$$
$$\cancel{y = \sqrt{ax} + \sqrt{ax}}$$
$$\cancel{y = 2\sqrt{ax}}$$
or
$$\cancel{y^2 = 4ax}$$

$y = \frac{\sqrt{a}}{\sqrt{x}}x + \frac{a}{\sqrt{a}}\sqrt{x}$ and $y = -\frac{\sqrt{a}}{\sqrt{x}}x - \frac{a}{\sqrt{a}}\sqrt{x}$

$y = \sqrt{ax} + \sqrt{ax}$ and $y = -\sqrt{ax} - \sqrt{ax}$
$y = 2\sqrt{ax}$ and $y = -2\sqrt{ax}$
$y^2 = 4ax$ and $y^2 = 4ax$

So, $y^2 = 4ax$ is the particular soln.

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(2) Solve: $x^2(y - px) = yp^2$

Put $x^2 = u$, $y^2 = v$

$2x\,dx = du$, $2y\,dy = dv$

$p = \frac{dy}{dx} = \frac{dv}{du}\frac{x}{y} = P.\frac{x}{y}$

So, $x^2\left(y - \frac{x}{y}Px\right) = yP^2\frac{x^2}{y^2}$

$\Rightarrow \frac{x^2}{y}(y^2 - x^2P) = P^2\frac{x^2}{y}$

$\Rightarrow v - uP = P^2$

$\Rightarrow v = uP + P^2$ (Clairaut's form)

So, the general soln. is $v = uk + k^2$

i.e. $y^2 = x^2k + k^2$ (Ans)

## Complex Analysis

If $z = x+iy$

Then, $f(z) = u(x,y) + iv(x,y)$

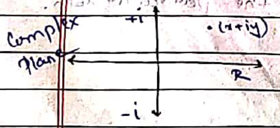
Real functions

If $u(x,y) = xy \rightarrow$ Real (Not containing $i$)

and $v(x,y) = y \rightarrow$ Real (Not containing $i$)

Then, Complex function $f(z) = xy + iy = f(x+iy) = f(x,y)$

Complex Plane




Note: Analyticity implies differentiability but differentiability may not imply analyticity.

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say, $f(z) = u(x,y) + iv(x,y)$

The equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are called Cauchy–Riemann (C-R) equations.

(1) $f(z)$ is differentiable at $(x_0, y_0) = z_0 = x_0 + iy_0$

$\Rightarrow$ $u(x,y)$, $v(x,y)$ satisfies C-R equations at $z_0 = (x_0, y_0)$

Note:- If $A \Rightarrow B$. Yes, $\sim B \Rightarrow \sim A$

then ~~$B \Rightarrow A$~~

(2) $u(x,y)$ and $v(x,y)$ has 1st and 2nd order continuous partial derivatives + $u(x,y)$, $v(x,y)$ satisfy C-R eqns.

$\Rightarrow$ $f$ is differentiable (Also ANALYTIC)

Note:- for a polynomial function, we need to check only this much that whether $u(x,y)$, $v(x,y)$ satisfies C-R eqns or not.

Polynomial function $u(x,y) = ax^m + bx^n + \dots$

i.e. POWER SHOULD BE INTEGER

$u(x,y) = \sqrt{xy} = x^{\frac{1}{2}}y^{\frac{1}{2}}$ is NOT A POLYNOMIAL.

**Exam-1** Prove that $f(z) = z + 2\bar{z}$ is not analytic anywhere in $\mathbb{C}$.

**Sol-** $f(z) = z + 2\bar{z} = (x+iy) + 2(x-iy)$
$= 3x - iy$

So, if $f(z) = u + iv$ then
$u = 3x$, $v = -y$
$u$ and $v$ are polynomial functions
Now, we need to check only that whether $u$ and $v$ satisfy CR eq$^n$ or not.

$$\frac{\partial u}{\partial x} = 3, \quad \frac{\partial v}{\partial y} = -1, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0$$

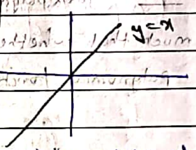
$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ for any pt of $\mathbb{C}$

~~Although~~ $\cancel{\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}}$

So, CR eq$^n$ are not satisfied at any point of $\mathbb{C}$
and $f(z)$ is not differentiable and so not analytic at any pt. of $\mathbb{C}$

Thus proved..

Random Case
Suppose $\frac{\partial u}{\partial x} = x$
and $\frac{\partial v}{\partial y} = y$




$f(z)$ is analytic along this line only and not everywhere.

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$$\lim_{h\to 0} \frac{u(x_0+h, y_0) - u(x_0, y_0)}{h}$$



**Exam-2** Prove that $f(z) = \sqrt{|xy|}$ satisfies C-R eq$^n$s at $(0,0)$ but not differentiable at $(0,0)$

**Sol-** $u = \sqrt{|xy|}$, $v = 0$ $(\because f(z) = u + iv)$

$$\left.\frac{\partial u}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\left.\frac{\partial u}{\partial y}\right|_{(0,0)} = \lim_{k\to 0} \frac{u(0,k) - u(0,0)}{k} = 0$$

As $v = 0$, $\frac{\partial v}{\partial x} = 0 = \frac{\partial v}{\partial y}$ $\forall (x,y) \in \mathbb{R}$

$$\left.\frac{\partial u}{\partial x}\right|_{(0,0)} = \left.\frac{\partial v}{\partial y}\right|_{(0,0)} \text{ and } \left.\frac{\partial u}{\partial y}\right|_{(0,0)} = -\left.\frac{\partial v}{\partial x}\right|_{(0,0)}$$

($\therefore$ CR eq$^n$s are satisfied.) (1st part is over)

Now,

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z} \quad \text{—(1)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\sqrt{|xy|}}{x+iy}$$ (homogeneous fn.)

along the straight line $y = mx$

$$\lim_{x \to 0} \frac{|x^2 m|}{x(1+im)} = \frac{|m|}{(1+im)}$$

So it has different values for different 'm' i.e. the limiting value is not unique and so limit (1) does not exist.

$\therefore f'(0)$ does not exist.

i.e. $f(z)$ is not differentiable at $(0,0)$.

Thus proved...

Exam-3

$$f(z) = \begin{cases} \frac{x^2 y^5 (x+iy)}{x^4 + y^{10}} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases}$$

Examine the analyticity of $f(z)$ in a region including origin.

(Ques:- Check the differentiability at origin.)

Sol:
$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2 y^5 (x+iy)}{(x^4+y^{10})(x+iy)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2 y^5}{x^4+y^{10}} \quad \text{--- (1)}$$



along $y = mx$, (1) becomes

$$\lim_{x\to 0} \frac{x^2 m^5 x^5}{x^4 + m^{10} x^{10}} = \lim_{x \to 0} \frac{m^5 x^3}{1 + m^{10} x^6}$$

$$= 0$$

i.e. not solvable for $y = mx$

Let $z \to 0$ i.e. $(x,y) \to (0,0)$ along the path $x^2 = y^5$

$$\lim_{y \to 0} \frac{y^5 \cdot y^5}{y^{10} + y^{10}} = \lim_{y \to 0} \frac{y^{10}}{2y^{10}} = \frac{1}{2}$$

$\therefore$ the value of limit (1) is not unique and so limit does not exist.

Hence $f$ is not differentiable at $(0,0)$

Harmonic functions- $u(x,y) \to$ harmonic if

$$\nabla^2 u = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Laplacian Operator

Converse may not be true

Theorem: If $f(z) = u(x,y) + iv(x,y)$ is analytic then $u$ and $v$ are harmonic functions.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

CR equations + continuous partial derivative $\Rightarrow$ Analyticity

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"No C-R equations satisfy
means that no analyticity is there."

$\boxed{\text{Analytic} \Rightarrow \text{C-R eqns}}$

**Exam** Prove that $u = x^2 - y^2$ and $v = \frac{-y}{x^2+y^2}$ are harmonic functions but $u + iv$ is not analytic.

**Sol:**

$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial^2 u}{\partial x^2} = 2$

$\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial^2 u}{\partial y^2} = -2$

$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$= 2 - 2 = 0$

$\Rightarrow$ ~~u(x,y)~~ $u = x^2 - y^2$ is a harmonic function

Now,

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)(-1) - (-y)2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)^2 2y - 2xy \cdot 2(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2)^2 - 8x^2y(x^2+y^2)}{(x^2+y^2)^4}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)^2 2y - (y^2-x^2)2(x^2+y^2)2y}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2)^2 - 4y(y^2-x^2)(x^2+y^2)}{(x^2+y^2)^4}$$

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Now, $\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$

$$= \frac{2y(x^2+y^2)^2 - 8x^2y(x^2+y^2)}{(x^2+y^2)^4} + \frac{2y(x^2+y^2)^2 - 4y(y^2-x^2)(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{2y(x^4+y^4+2x^2y^2) - 8x^4y - 8x^2y^3 + 2y(x^4+y^4+2x^2y^2) - (4y^3-4x^2y)(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{x^4 2y + 2y^5 + 4x^2y^3 - 8x^4y - 8x^2y^3 + 2x^4y + 2y^5 + 4x^2y^3 - (4x^2y^3 + 4y^5 - 4x^4y - 4x^2y^3)}{(x^2+y^2)^4}$$

$$= \frac{\cancel{2x^4y} + \cancel{2y^5} + \cancel{4x^2y^3} - \cancel{8x^4y} - \cancel{8x^2y^3} + \cancel{2x^4y} + \cancel{2y^5} + \cancel{4x^2y^3} - \cancel{4x^2y^3} - \cancel{4y^5} + \cancel{4x^4y} + \cancel{4x^2y^3}}{(x^2+y^2)^4}$$

$= 0$

$\therefore \nabla^2 v = 0$

$\Rightarrow v = -\frac{y}{x^2+y^2}$ is a harmonic function.

Now

since $\frac{\partial u}{\partial x} = 2x \neq \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = 2y \neq -\frac{\partial v}{\partial x}$

$\Rightarrow$ CR eqns are not satisfied means NO ANALYTICITY.

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Exam

If $u = (x-1)^3 - 3xy^2 + 3y^2$, find the harmonic conjugate of $u$.

Sol: We need to find a function $v$ such that $u+iv$ is analytic.

$\frac{\partial u}{\partial x} = 3(x-1)^2 - 3y^2$, $\frac{\partial u}{\partial y} = -6xy + 6y$

So, we need to find $v$ such that

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Now, $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$

$= \underbrace{(6xy - 6y)dx}_{(y\ \text{constant})} + \underbrace{[3(x-1)^2 - 3y^2]\,dy}_{(\text{Neglect the } x \text{ terms})}$ — (1)

$dv = M\,dx + N\,dy$ where $M = 6xy - 6y$, $N = 3(x-1)^2 - 3y^2$

$\frac{\partial M}{\partial y} = 6x - 1 = \frac{\partial N}{\partial x} = 6(x-1)$

So, (1) is exact

Now, Integrating (1)

$v = \int (6xy - 6y)\,dx + \int (3 - 3y^2)\,dy$

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$v = 6y\frac{x^2}{2} - 6yx + 3y - \frac{3y^3}{3} + C$

$v = 3x^2y - 6xy + 3y - y^3 + C$ Ans

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$z = x + iy$

$\boxed{e^z = e^x(\cos y + i\sin y)}$

**Complex Logarithm**

$\log z = w$ is defined as $e^w = z$

Since, $e^w \neq 0 \ \forall w \in \mathbb{C}$

$\therefore \log 0$ is not defined

$z = re^{i\theta}$; $w = u + iv$

$e^w = e^{u+iv} = re^{i\theta}$

Since for $e^{u+iv}$, $e^u$ is the modulus and $v$ is the argument. so,

$e^u = r$, $v = \theta$

$\log r = u$

$\therefore \log z = w = u + iv = \log r + i\theta$, $\theta = \arg z$ —(1)

$\arg z$ has infinitely many values and so $\log z$ has also infinitely many values.

$\text{Arg}\, z$ - principle value of $\arg z$

$= \theta$, $-\pi < \theta \leq \pi$

$\arg z = \text{Arg}\, z \pm 2n\pi$, $n = 1, 2, 3, \ldots$ —(2)

So, the value of $\log z$ corresponding to $\arg z = \text{Arg}\, z$, we get principle logarithm $\text{Log}\, z = \log r + i\,\text{Arg}\, z$,

$|z| = r$

—(3)

From (1) & (2),

$$\boxed{\begin{aligned}\log z &= \log r + i\arg z = \log r + i(\operatorname{Arg} z + 2n\pi) \\ &= \log r + i\operatorname{Arg} z + 2n\pi i = \operatorname{Log} z \mp 2n\pi i\end{aligned}}$$

$\log z \rightarrow$ natural logarithm

always a single no value $\leftarrow$ $\operatorname{Log} z \rightarrow$ principle logarithm

$$e^{\log z} = z, \quad \log(e^z) = z \pm 2n\pi i$$

$$\operatorname{Log} z = \log r + i\operatorname{Arg} z$$

(i) $z \rightarrow$ +ve real axis, then $\operatorname{Log} z = \log z = \log r$, $r = |z|$

(ii) $z \rightarrow$ -ve real axis, then $\operatorname{Arg} z = \pi$, $\operatorname{Log} z = \log r + \pi i$

**Theorem** $\log z$ is not continuous for negative real number.

**Result 1:** $\log z$ is analytic for all $z$, except $z = 0$ and $z$ is on negative real axis.

**Proof:** If $z = 0$, $\log z$ is not defined and if $z$ is on -ve real axis, by Theorem $\log z$ is not cont., so not differentiable and so not analytic.

**Analytic** -

$$z = x + iy = re^{i\theta}$$

$$\log z = \log r + i(\theta \pm 2n\pi), \quad n = 1, 2, 3, \ldots$$



$$= \frac{1}{2}\log(x^2+y^2) + i\left(\tan^{-1}\frac{y}{x} \pm 2n\pi\right)$$

(with $u$ over the first term and $v$ over the second)

Now,

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}, \quad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{\partial u}{\partial y}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2+y^2} = \frac{\partial u}{\partial x}$$

$\therefore$ C R eqⁿs ~~to~~ are satisfied, Also from the function of several variables,

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \text{ are all cont.}$$

so, for any $z = x + iy$ except $z = 0$ and $z$ is on -ve real axis.

$\therefore$ $\log z$ is analytic.

**Result 2:** If $\log z$ is analytic then

$$\frac{d}{dz}(\log z) = \frac{1}{z}$$

**Proof:**

$$\frac{d}{dz}(\log z) = \frac{\partial}{\partial x}\left(\frac{1}{2}\log(x^2+y^2)\right) + i\frac{\partial}{\partial x}\left(\tan^{-1}\frac{y}{x} \pm 2n\pi\right)$$

$$= \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2} = \frac{x-iy}{x^2+y^2} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{x+iy} = \frac{1}{z}$$

$e^{-iz} = \cos z - i\sin z$

$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$

$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$

$\sec z = \frac{1}{\cos z}$, $\operatorname{cosec} z = \frac{1}{\sin z}$

As $e^z$ is entire (everywhere analytic)

so, $\cos z$, $\sin z$ are entire

but $\sec z$, $\tan z$, $\operatorname{cosec} z$, $\cot z$ are not entire.

Mobius Transformation
↓
Bilinear

$w = \frac{az+b}{cz+d}$ (where a, b, c, d are constants)

Let $z_1 \to w_1$, $z_2 \to w_2$

$w_1 - w_2 = \frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d} = \frac{(ad-bc)(z_2-z_1)}{(cz_1+d)(cz_2+d)}$

If $ad-bc=0$, $w_1 = w_2 = w_3 = w_4 \dots$

constant mapping

Suppose,

$f(z) = w = \frac{2+3z}{5+6z}$

$z_1 = 1-i$, $z_2 = 1+i$

$w_1 = \frac{2+3(1-i)}{5+6(1-i)}$, $w_2 = \frac{2+3(1+i)}{5+6(1+i)}$



so, for non-constant bilinear Transformation,

$ad-bc \neq 0$

Note: $\boxed{ad-bc \to \text{determinant of B.T.}}$ $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Conformal Mapping

magnitude & direction same

Cross Ratio:

Let $z_1, z_2, z_3, z_4$ are any four points in z-plane.
Then cross ratio of $z_1, z_2, z_3, z_4$ is

$$\frac{(z_4-z_1)(z_2-z_3)}{(z_2-z_1)(z_4-z_3)}$$

Theorem: Cross-ratio is invariant under Bilinear Transformation

If $z_1 \to w_1$, $z_3 \to w_3$

$z_2 \to w_2$, $z_4 \to w_4$

Then,

$$\frac{(z_4-z_1)(z_2-z_3)}{(z_2-z_1)(z_4-z_3)} = \frac{(w_4-w_1)(w_2-w_3)}{(w_2-w_1)(w_4-w_3)}$$

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**Ques-** Find the B.T. which maps the points $2, i, -2$ of the z-plane onto $1, i, -1$ of the w-plane resp.

**Sol-** Given: $z_1 = 2 \longrightarrow w_1 = 1$

$z_2 = i \longrightarrow w_2 = i$

$z_3 = -2 \longrightarrow w_3 = -1$

$z_4 = z \longrightarrow w_4 = w$

'i' is a fixed pt.

Taking cross-ratio, we get

$$\frac{(z-2)(i+2)}{(i-2)(z+2)} = \frac{(w-1)(i+1)}{(i-1)(w+1)}$$

$$\Rightarrow \frac{w-1}{w+1} = \left(\frac{z-2}{z+2}\right)\left(\frac{i+2}{i-2}\right)\left(\frac{i-1}{i+1}\right)$$

$$\Rightarrow \frac{w-1}{w+1} = \frac{4-3i}{5} \cdot \frac{z-2}{z+2}$$

$$\Rightarrow \frac{w-1+w+1}{w-1-w-1} = \frac{(4-3i)(z-2)+5(z+2)}{(4-3i)(z-2)-5(z+2)}$$

$$\Rightarrow \frac{2w}{-2} = \frac{(3z+2i)(3-i)}{(iz+6)(3-i)}$$

$$\Rightarrow w = \frac{3z+2i}{iz+6} \quad \text{(Ans)}$$

**Fixed points:-** The points which coincide with their transformation are called fixed point so if $w = f(z)$ is a BT then, $z_1$ is a fixed point if $z_1 = f(z_1)$ ie. if $z_1 = \frac{az_1+b}{cz_1+d}$

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$\boxed{w = f(z)}$

**Ques-** Find the fixed pts. of the B.T. $w = \frac{z-1}{z+1}$

**Sol-** If $z_1$ is a fixed pt. then,

$$z_1 = \frac{z_1-1}{z_1+1}$$

$$\Rightarrow z_1(z_1+1) = z_1 - 1$$

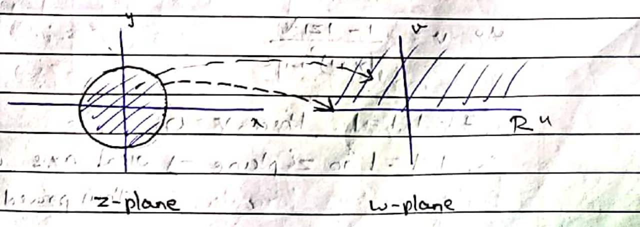
$$\Rightarrow z_1^2 + z_1 = z_1 - 1$$

$$\Rightarrow z_1^2 + 1 = 0$$

$$\Rightarrow z_1 = \pm i \quad \text{(Ans)}$$

Verification part: $z = i$, $w = \frac{i-1}{i+1} = \frac{(i-1)^2}{i^2-1} = \frac{i^2-2i+1}{-2} = \frac{-2i}{-2} = i$

**Ques-** Prove that the transformation $w = i\,\frac{1-z}{1+z}$ transforms the boundary of the circle $|z|=1$ onto the real axis of the w-plane and interior of the circle into the upper half of the w-plane.



$z = x+iy$, $w = u+iv$

To prove: $|z|=1 \rightarrow v=0$

$|z|<1 \rightarrow v>0$

Now,

$$w = i\,\frac{1-(x+iy)}{1+(x+iy)}$$

$$\Rightarrow u+iv = \frac{i - ix - i^2 y}{1+x+iy}$$

$$\Rightarrow u+iv = \frac{(1-x)i + y}{(x+1)+iy}$$

$$= \frac{[(1-x)i+y][(x+1)-iy]}{(x+1)^2+y^2}$$

$$= \frac{(1-x)(1+x)i + y(1-x) + y(1+x) - iy^2}{(x+1)^2+y^2}$$

$$= \frac{2y + i[1-x^2-y^2]}{(x+1)^2+y^2}$$

$$u+iv = \frac{2y + i(1-|z|^2)}{(x+1)^2+y^2}$$

so, $$v = \frac{1-|z|^2}{(x+1)^2+y^2}$$

If $|z|=1$, then $v=0$

So, $|z|=1$ in z-plane $\Rightarrow$ real axis of w-plane

Thus proved.



If $|z|<1$, $|z|^2<1$, $1-|z|^2>0$

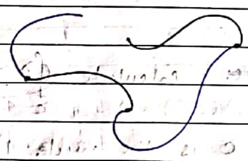
So, $v>0$

$\therefore$ $|z|<1$ in z-plane $\Rightarrow$ upper half of w-plane

Thus proved.

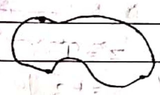
Complex Integration

Contour: Chain of finite no. of continuous arcs



Simple closed contour

Eg: circle, rectangle, ellipse



Cauchy's Theorem: If $f(z)$ is analytic within and on a simple closed contour $C$ then

$$\oint_C f(z) = 0$$

Cauchy's formula: If $f(z)$ is analytic within and on a simple closed contour $C$ and $\alpha$ is any point within $C$ then

$$f(\alpha) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\alpha}\,dz$$

derivative! $$f^n(\alpha) = \frac{\lfloor n}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^{n+1}}\,dz$$

Note :- $f(z)$ is analytic but $\frac{f(z)}{z-\alpha}$ is not analytic

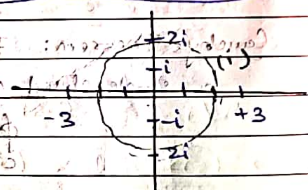
Ques- ~~Solve.~~ Calculate $\oint_C \frac{z\,dz}{(9-z^2)(z+i)}$

(i) where C is the circle $|z| = 2$

(ii) where C is the circle $|z| = \frac{1}{2}$

Sol- See denominator (when it will become zero) $z^2 = 9$, $z = \pm 3$, $z = -i$

$\boxed{z = \pm 3, -i}$ points of discontinuity




$f(z) = \frac{z}{9-z^2}$

$f(z)$ is analytic within and on C

$\alpha = -i$

Apply Cauchy's formula,



$$f(\alpha) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\alpha}\,dz$$

$$\Rightarrow f(-i) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z+i}\,dz$$

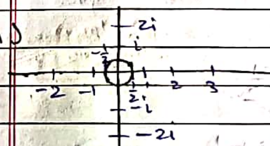
$$\Rightarrow \oint_C \frac{f(z)}{z+i}\,dz = 2\pi i\, f(-i)$$

$$= 2\pi i \,\frac{(-i)}{9-i^2}$$

$$= \frac{2\pi}{10} = \frac{\pi}{5}$$

$$\Rightarrow \oint_C \frac{z}{(9-z^2)(z+i)} = \pi/5 \quad \text{(Ans)}$$

(ii)




No pt of discontinuity

As all the pts. $z = -i, \pm 3$ are outside C : $|z| = \frac{1}{2}$

So, $g(z) = \frac{z}{(9-z^2)(z+i)}$ analytic within and on C

$\oint_C g(z)\,dz = 0$ by Cauchy's theorem. (Ans)

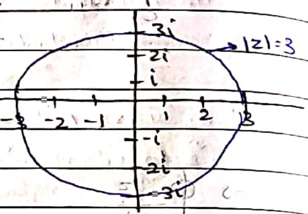
Ques- Calculate $\oint_C \frac{e^{\pi z}}{z(z^2+1)}\, dz$, where C is the circle $|z|=3$

Sol- for pts. of discontinuity,

$[z(z^2+1)=0]$

$z=0,\quad z^2=-1 \Rightarrow z=\pm i$

$z=0, \pm i$




Let $\frac{1}{z(z^2+1)} = \frac{A}{z} + \frac{B}{z+i} + \frac{C}{z-i}$

$\Rightarrow 1 = A(z+i)(z-i) + B(z-i)z + C(z+i)z \quad \text{--- (1)}$

Put $z=0, i, -i$ in (1),

$1 = A\,i(-i) \Rightarrow A=1$

$1 = C\,i(2i) \Rightarrow C = -\frac{1}{2}$

$1 = B(-i)(-2i) \Rightarrow B = -\frac{1}{2}$

$\therefore \frac{e^{\pi z}}{z(z^2+1)} = \frac{e^{\pi z}}{z} - \frac{1}{2}\frac{e^{\pi z}}{(z+i)} - \frac{1}{2}\frac{e^{\pi z}}{(z-i)}$

$\oint_C \frac{e^{\pi z}}{z}\, dz = 2\pi i f(0)$, where $f(z)=e^{\pi z}$

$= 2\pi i$

$\oint_C \frac{e^{\pi z}}{z+i} = 2\pi i f(-i) = 2\pi i e^{-\pi i}$



$\oint_C \frac{e^{\pi z}}{z-i} = 2\pi i f(i) = 2\pi i e^{\pi i}$

$\oint_C \frac{e^{\pi z}}{z(z^2+1)} = 2\pi i - \frac{1}{2} 2\pi i e^{-\pi i} - \frac{1}{2} 2\pi i e^{\pi i}$

$= 2\pi i\left[1 - \frac{1}{2}(e^{\pi i}+e^{-\pi i})\right] = \cancel{2\pi i[1-\cos\pi]}$

$= \cancel{2\pi i[1+\cos\pi]} = \cancel{2\pi i[1-1]} = \cancel{0}$

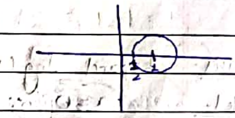
$\cancel{= 2\pi i[1+1]}$

$= 2\pi i\left[1-\cos\pi\right] = 4\pi i$

Ques. Calculate $\oint_C \frac{\log z}{(z-1)^3}\, dz$, where C is the circle $|z-1| = \frac{1}{2}$

Sol- Point of discontinuity: $(z-1)^3 = 0$

$z=1$




Let $f(z) = \log z$, $\alpha = 1$

$f''(\alpha) = \frac{\lfloor 2}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^3}\, dz$ by Cauchy's integral formula

$\Rightarrow f''(1) = \frac{\lfloor 2}{2\pi i}\oint_C \frac{f(z)}{(z-1)^3}\, dz$

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$$\Rightarrow \oint_C \frac{\log z}{(z-1)^3}\,dz = \pi i f''(1)$$

$$= \pi i(-1) = -\pi i$$

**Liouville's Theorem:** If a function $f(z)$ is analytic for all finite values of $z$ and $f(z)$ is also bounded then $f(z)$ must be constant.

**Taylor's Theorem:** If a function $f(z)$ is analytic within and on a circle $C$ with centre at $a$ and radius $R$ then for all $z$ in $C$,

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots + \frac{(z-a)^n}{n!}f^n(a) + \cdots$$

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**Ques-** Expand $\log(1+z)$ in a Taylor's series about $z=0$

**Sol-** Taylor's Series: $f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a)$

$$+ \frac{(z-a)^3}{3!}f'''(a) + \cdots$$

$\because z-a=0$

i.e. $a=0$

So, $f(z) = f(0) + zf'(0) + \frac{z^2}{2!}f''(0) + \frac{z^3}{3!}f'''(0) + \cdots \quad -(1)$

Now,

$f(z) = \log(1+z)$ so, $f(0) = \log 1$

$f'(z) = \frac{1}{1+z}$ so, $f'(0) = 1$

$f''(z) = \frac{(1+z)^2 0 - 1(0+1)}{(1+z)^2} = \frac{-1}{(1+z)^2}$ so, $f''(0) = -1$

and, $f'''(z) = \frac{(1+z)^2 0 + 1(2)(1+z)(0+1)}{(1+z)^4} = \frac{2(1+z)}{(1+z)^4} = \frac{2}{(1+z)^3}$

so, $f'''(0) = 2$

so putting these values in eq (1), we get

$$f(z) = \log(1+z) = \log 1 + z(1) + \frac{z^2}{2!}(-1) + \frac{z^3}{3!}(2) + \dots$$

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots \text{ (Ans)}$$

Ques- Expand in Taylor's Series $f(z) = \frac{1}{z^2-z-6}$ about $z=-1$

Taylor series: $f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2 f''(a)}{2!} + \frac{(z-a)^3 f'''(a)}{3!} + \dots$

$\because z = -1$

$z+1 = 0$

$z-(-1) = 0$ so, $a = -1$

$$f(z) = f(-1) + (z+1)f'(-1) + \frac{(z+1)^2 f''(-1)}{2!} + \frac{(z+1)^3 f'''(-1)}{3!} + \dots$$



$\frac{1}{64}$ $\frac{2\,4}{256}$

$\because f(z) = \frac{1}{z^2-z-6}$

$f(-1) = \frac{1}{1+1-6} = \frac{1}{-4} = -\frac{1}{4}$

Now,

$$f'(z) = \frac{(z^2-z-6)0 - 1(2z-1)}{(z^2-z-6)^2} = \frac{-(2z-1)}{(z^2-z-6)^2}$$

$f'(z) = \frac{-2z+1}{(z^2-z-6)^2}$, so $f'(-1) = \frac{2+1}{(1+1-6)^2} = \frac{3}{(-4)^2} = \frac{3}{16}$

Again,

$$f''(z) = \frac{(z^2-z-6)^2(-2) - (-2z+1)2(z^2-z-6)(2z-1)}{(z^2-z-6)^4}$$

$$f''(z) = \frac{-2(z^2-z-6)^2 - (-2z+1)(z^2-z-6)(4z-2)}{(z^2-z-6)^4}$$

so, $f''(-1) = \frac{-2(-4)^2 - (2+1)(-4)(-4-2)}{(-4)^4}$

i.e. $f''(-1) = \frac{-32 + 12(-6)}{256} = \frac{-32-72}{256} = \frac{\cancel{-104}}{\cancel{256}} = -\frac{13}{32}$

$\cancel{26}$ $\cancel{52}$ $\cancel{128}$ $\cancel{64}$ 32

~~we have~~ on substituting in eq (1), we get

$$f(z) = \frac{1}{z^2-z-6} = -\frac{1}{4} + (z+1)\frac{3}{16} + \frac{(z+1)^2(-13)}{2 \times 32} + \dots$$

$$\frac{1}{z^2-z-6} = -\frac{1}{4} + \frac{3}{16}(z+1) - \frac{13}{64}(z+1)^2 + \dots$$

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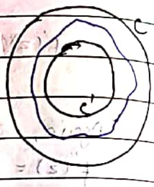
$(1+x)^{-1} = 1 - x + x^2 - x^3 \ldots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

**Laurent's series**

Let $f(z)$ is analytic in the closed ring bounded by two concentric circles $C$ and $C'$ of centre 'a' and radius 'R' and $R'$ ($R' < R$). If $z$ is any point in the annular region then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$




where, $a_n = \frac{1}{2\pi i} \oint_{\Gamma'} \frac{f(t)}{(t-a)^{n+1}} \, dt$

$b_n = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(t)}{(t-a)^{-n+1}} \, dt$

Ques. Expand the function in Laurent's series $f(z) = \frac{1}{(z-1)(z+3)}$

(i) for $1 < |z| < 3$, (ii) $|z| > 3$

Sol- $f(z) = \frac{1}{(z-1)(z+3)} = \frac{1}{4}\left[\frac{1}{z-1} - \frac{1}{z+3}\right]$

(i) $|z| > 1 \Rightarrow \frac{1}{|z|} < 1$

$|z| < 3 \Rightarrow \frac{|z|}{3} < 1$

of the type $(1+x)^{-1}$, $(1-x)^{-1}$, $|x| < 1$

Now, $f(z) = \frac{1}{4}\left[\frac{1}{z(1-\frac{1}{z})} - \frac{1}{3(1+\frac{z}{3})}\right]$



Singular point - of a function $f(z)$ is that particular pt. say $z = a$ s.t. $f(z)$ is not analytic at $z = a$

$(1+x)^{-1} = 1 - x + x^2 - x^3 \ldots$

$(1-x)^{-1} \quad 1 + x + x^2 + x^3 + \ldots$

$$= \frac{1}{4z}\left(1 - \frac{1}{z}\right)^{-1} - \frac{1}{12}\left(1 + \frac{z}{3}\right)^{-1} = \frac{1}{4z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right]$$

$$- \frac{1}{12}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \ldots\right]$$

(ii) $|z| > 3$, $\frac{3}{|z|} < 1$

$\frac{1}{|z|} < \frac{1}{3} < 1$

So, $f(z) = \frac{1}{4}\left[\frac{1}{z(1-\frac{1}{z})} - \frac{1}{z(1+\frac{3}{z})}\right]$

$$= \frac{1}{4z}\left(1 - \frac{1}{z}\right)^{-1} - \frac{1}{4z}\left(1 + \frac{3}{z}\right)^{-1}$$

$$= \frac{1}{4z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right] - \frac{1}{4z}\left[1 - \frac{3}{z} + \frac{9}{z^2} - \ldots\right]$$

**Singularities**

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_n}{(z-a)^n} + \ldots$$

($z = a$ is a singular pt.)

(i) Removable Singularity : If all $b_n$ are zero that is there is no negative powers of $(z-a)$ then $z = a$ is called removable singularity.

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$f(z) = \frac{1}{z-1} + \frac{1}{(z-1)^2}$

$z=1$ is a pole of order 2.

(ii) Pole: $0 = b_{m+1} = b_{m+2} = \dots$

but $b_m \neq 0$

then $z=a$ is called a pole of order m.

(iii) Essential singularity: If infinite no. of terms $b_n$ are non-zero that is if $\sum_{n=0}^{\infty} \frac{b_n}{(z-a)^{-n}}$ is a non-terminating series then $z=a$ is called an essential singularity.

$\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \dots$

$z=0$ is an essential singularity.

Residue at the pole $\alpha$:

Principle part of Laurent's series

$\frac{b_1}{z-\alpha} + \frac{b_2}{(z-\alpha)^2} + \dots + \frac{b_m}{(z-\alpha)^m}$

$b_1 \rightarrow$ residue of $f(z)$ at $z=\alpha$

Suppose, $f(z) = \frac{1}{z} + \frac{1}{z^2}$

$z=0$ is a pole of order 2 with residue 1.

$b_1 = \frac{1}{2\pi i}$

Cauchy's Residue Theorem: Let $f(z)$ is analytic within a closed contour C, except there are finite no. of poles, say $\alpha_1, \alpha_2, \dots, \alpha_n$ ($\alpha$'s are all distinct), then

$$\oint_C f(z)\,dz = 2\pi i \left[ \mathrm{Res}(z=\alpha_1) + \mathrm{Res}(z=\alpha_2) + \dots + \mathrm{Res}(z=\alpha_n) \right]$$

all ~~are~~ inside

Zeros of $f(z)$: The value of $z$, say $\beta$ for which $f(z)=0$ is called zero of $f(z)$.

$z=\beta$ is called a zero of order m is $(z-\beta)^m$ is a factor of $f(z)$ but $(z-\beta)^{m+1}$ is not a factor of $f(z)$

$f(\beta), f'(\beta), f''(\beta), \dots f^{m-1}(\beta) = 0$, $f^m(\beta) \neq 0$ then $\beta$ is a zero of $f(z)$ of order m.

If $\beta$ is a zero of $f(z)$, then $f(z)$ is ANALYTIC at $\beta$. Then at a neighbourhood of $\beta$, by Taylor's Theorem:

$$f(z) = f(\beta) + (z-\beta)f'(\beta) + \frac{(z-\beta)^2}{2!}f''(\beta) + \dots + \frac{(z-\beta)^m}{m!}f^m(\beta) + \dots$$

If $\beta$ is a zero of order m,

$$f(z) = \frac{(z-\beta)^m}{m!}f^m(\beta) + \dots$$

| Zero | Pole |
|---|---|
| | |

**Ques:** Find the residue of $f(z) = \frac{3z^2+z+1}{(z^2+1)(z+3)}$ at its pole inside the circle $C: |z| = 2$ and hence evaluate $\oint_C f(z)\,dz$

**Sol:** for Poles:- $(z^2+1)(z+3) = 0$

$z = \pm i$, $\boxed{z = -3}$ ($\because$ it is outside the circle $|z| = 2$, so we will not consider $z = -3$)

inside, so will be considered.

**Note:** If $\alpha$ is a simple pole of $f(z)$, then $Res(z=\alpha) = \lim_{z\to\alpha} (z-\alpha) f(z)$

If $\alpha$ is a pole of order $m$, then $Res(z=\alpha) = \frac{1}{(m-1)!} \lim_{z\to\alpha} \frac{d^{m-1}}{dz^{m-1}}$

$$Res(z=i) = \lim_{z\to i} \cancel{(z-i)} \frac{3z^2+z+1}{(z+i)\cancel{(z-i)}(z+3)}$$

$$= \lim_{z\to i} \frac{3z^2+z+1}{(z+i)(z+3)} = \frac{-2+i}{2i(3+i)}$$

and $Res(z = -i) = \frac{2+i}{2i(3-i)}$

so, by Cauchy-Residue formula,

$$\oint_C f(z)\,dz = 2\pi i \left[ Res(z=i) + Res(z=-i) \right]$$

$$= 2\pi i \left[ \quad \right]$$

**Ques:** By integrating around a unit circle, evaluate

$$\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\,d\theta$$

**Sol:** $|z| = 1$

Put $z = e^{i\theta}$

$dz = ie^{i\theta}\,d\theta$

$= iz\,d\theta$

$\Rightarrow d\theta = \frac{dz}{iz}$

$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$\cos 3\theta = \frac{1}{2}(e^{3i\theta} + e^{-3i\theta}) = \frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)$

Now, $$\int_0^{2\pi} \frac{\cos 3\theta\, d\theta}{5 - 4\cos\theta} = \oint_C \frac{\frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)}{5 - 4\frac{1}{2}\left(z + \frac{1}{z}\right)} \cdot \frac{dz}{iz}$$

$$= -\frac{1}{2i} \oint_C \underbrace{\frac{z^6+1}{z^3(2z-1)(z-2)}}_{f(z)}\,dz$$

Inside $C: |z| = 1$, the poles of $f(z)$ are

$z = 0 \longrightarrow 3$ time

$z = \frac{1}{2} \longrightarrow 1$ time

$(D^2-3D+2)y=0$

AE ⇒ $m^2-3m+2=0$

$(m-1)(m-2)=0$

$m=1,2$

CF $= C_1e^{x}+C_2e^{2x}$

$(D^3-2D^2+4D-8)y=0$

$m^3-2m^2+4m-8=0$

$m^2(m-2)+4(m-2)=0$

$(m-2)(m^2+4)=0$,

So, $\text{Res}(z=0) = \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2} \frac{z^6+1}{(2z-1)(z-2)}$

$= \frac{21}{8}$

$\text{Res}(z=\frac{1}{2}) = -\frac{65}{24}$

So, $I = -\frac{1}{2i} 2\pi i \left[\text{Res}(z=0) + \text{Res}(z=\frac{1}{2})\right]$

$= -\pi\left[\frac{21}{8} - \frac{65}{24}\right]$

Linear Higher Order Differential Equations with constant coefficients

$(D^2+k_1D+k_2)y=0$ → homogeneous equation → PI = 0

$\underbrace{(D^2+k_1D+k_2)y}_{CF} = \underbrace{\phi(x)}_{PI}$ → non homogeneous equation.

Ques. Solve: $(D^2-D-2)y = \sin 2x$

Sol. Auxillary eqn. $D^2-D-2=0$

⇒ $D^2-2D+D-2=0$

⇒ $D(D-2)+1(D-2)=0$

$(D-2)(D+1)=0$

⇒ $D=-1,2$

$f(D)y=Q$

$y=\frac{Q}{f(D)}$

Complementary fn. $C_1e^{-x}+C_2e^{2x}$

$PI = \frac{1}{D^2-D-2}\sin 2x$

$(D^2-3D+2)y=0$

AE ⇒ $m^2-3m+2=0$

$(m-1)(m-2)=0$

$m=1,2$

CF $= C_1e^{x}+C_2e^{2x}$

$(D^2-4D+4)y=0$

AE ⇒ $m^2-4m+4=0$

$(m-2)(m-2)=0$

$m=2,2$

CF $=(C_1+C_2x)e^{2x}$

$m=1,1$

CF $=(C_1+C_2x)e^{x}$

$m=2$, $m=0\pm 2i$

CF $C_1e^{2x}+e^{0x}[C_2\cos 2x + C_3\sin 2x]$

$\frac{1}{D} = \int$

$D=\frac{d}{dx}$

$D^2=\frac{d^2}{dx^2}$

$D^3=\frac{d^3}{dx^3}$

$= \frac{1}{-4-D-2}\sin 2x$

$= \frac{-1}{D+6}\sin 2x = \frac{-1(D-6)}{D^2-36}\sin 2x$

$= \frac{-1(D-6)}{-4-36}\sin 2x$

$= \frac{(D-6)}{40}\sin 2x$

$= \frac{1}{40}(2\cos 2x - 6\sin 2x)$

$= \frac{1}{20}(\cos 2x - 3\sin 2x)$

$(D^2-D+4)y=0$

AE ⇒ $m^2-m+4=0$

$m=\frac{+1\pm\sqrt{1-16}}{2}$

$m=\frac{1\pm\sqrt{15}i}{2}$

$m=\frac{1}{2}\pm\frac{\sqrt{15}}{2}i$

CF $= e^{\frac{1}{2}x}\left[C_1\cos\frac{\sqrt{15}}{2}x + C_2\sin\frac{\sqrt{15}}{2}x\right]$

Now, Complete soln.

$y = C_1e^{-x}+C_2e^{2x}+\frac{1}{20}(\cos 2x - 3\sin 2x)$

Ques 2 Solve: $\frac{d^2y}{dx^2} - y = e^{x} + x^2e^{x}$

AE $D^2y - y = 0$

⇒ $D^2-1=0$

⇒ $D=\pm 1$

CF $= C_1e^{x}+C_2e^{-x}$

Soln

$$PI = \frac{1}{D^2-1}\left(e^x + x^2 e^x\right)$$

$$= \frac{1}{D^2-1} e^x + \frac{1}{D^2-1} x^2 e^x$$

$$= x \cdot \frac{1}{2} e^x + e^x \frac{1}{(D+1)^2-1} x^2$$

$$= \frac{x}{2} e^x + e^x \frac{1}{D^2+2D} x^2$$

$$= \frac{x}{2} e^x + e^x (D^2+2D)^{-1} x^2$$

$$= \frac{x}{2} e^x + e^x \frac{1}{2D}\left(1+\frac{D}{2}\right)^{-1} x^2$$

$$= \frac{x}{2} e^x + e^x \frac{1}{2D}\left(1 - \frac{D}{2} + \frac{D^2}{4} - \cdots\right) x^2$$

$$= \frac{x}{2} e^x + e^x \frac{1}{2D}\left(x^2 - x + \frac{1}{2}\right)$$

$$= \frac{x}{2} e^x + e^x \frac{1}{2}\left[\int x^2 dx - \int x\,dx + \int \frac{1}{2} dx\right]$$

3 $$= \frac{x}{2} e^x + e^x \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{2}\right]$$

Now complete soln.

$y = CF + PI$

CF → complementary function, PI → Particular Integral.

$(D^4 + 2D^3 - 3D^2) y = 0$

$m^4 + 2m^3 - 3m^2 = 0$

$m^2(m^2 + 2m - 3) = 0$

$m^2(m+3)(m-1) = 0$

$m = 0, 0, 1, -3$

$CF = (C_1 + C_2 x)e^{0x} + C_3 e^{x} + C_4 e^{-3x}$

Ques solve $(D^2 - 4D + 3)y = e^x \cos 2x$

HW → find CF

$$PI = \frac{1}{D^2 - 4D + 3} e^x \cos 2x$$

$$= \frac{1}{(D-3)(D-1)} e^x \cos 2x$$

$$= e^x \frac{1}{(D+1-3)(D+1-1)} \cos 2x$$

$$= e^x \frac{1}{D(D-2)} \cos 2x = e^x \frac{1}{D^2 - 2D} \cos 2x$$

$$= e^x \frac{1}{-4-2D} \cos 2x = \frac{e^x}{-2} \frac{1}{D+2} \cos 2x$$

$$= -\frac{e^x}{2} \frac{D-2}{D^2-4} \cos 2x$$

4 $$= -\frac{e^x}{2} \frac{1}{-8} (D-2) \cos 2x$$

$$= \frac{e^x}{16} (D-2) \cos 2x$$

$$= \frac{e^x}{16}\left[\frac{d}{dx}(\cos 2x) - 2\cos 2x\right]$$

$$= \frac{e^x}{16}\left[-2\sin 2x - 2\cos 2x\right]$$

$$= -\frac{e^x}{8}\left[\sin 2x + \cos 2x\right]$$

$(D-1)^2 (D^2+1)^2 y = 0$

$(m-1)^2 (m^2+1)^2 = 0$

$m = 1, 1, \pm i, \pm i$

$CF = (C_1 + C_2 x)e^{1x} + e^{0x}(C_3 +$

## Cauchy-Euler Equations

$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \cancel{\phi(x)}\ 0$ homogeneous

$= \phi(x)$ non-homo

coefficients are variable

$\frac{dy}{dx} \to \frac{dy}{dt}$ (Let $t = \log x$, $dt = \frac{1}{x}dx$)

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt} \Rightarrow \frac{dy}{dt} = x\frac{dy}{dx}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right)$

$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx}$

$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$

$\Rightarrow x^2\frac{d^2y}{dx^2} = -\frac{dy}{dt} + \frac{d^2y}{dt^2}$

$= (-D + D^2)y = D(D-1)y$

$D(D-1)y + Dy + k_3 y = g(t)$

**Ques.** Solve: $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \log x$

CS is $y = c_1 x^{-1} + c_2 x^{-2} + \frac{1}{2}\log x - \frac{3}{4}$

Variation of parameter

$f(D)y = \overset{A}{\tan x} \,/\, \sec x \,/\, \frac{1}{e^x}$ ?

suppose $CF = C_1y_1 + C_2y_2$ | $y_1, y_2 \to f(x)$

Wronskian of $y_1, y_2 = W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$PI = -y_1\int \frac{y_2 A}{W}dx + y_2\int\frac{y_1 A}{W}dx$

**Ques.** Solve:- $\frac{d^2y}{dx^2} + y = \tan x$

$D^2y + y = \tan x$

$(D^2+1)y = \tan x$

$f(D)\,y = Q$

AE $\to$ $m^2+1 = 0$

$m = \pm i$

$CF = C_1\cos x + C_2\sin x$

$y_1 = \cos x$

$y_2 = \sin x$

$A = \tan x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$PI = -\cos x \int \frac{\sin x \tan x \, dx}{1} + \sin x \int \frac{\cos x \tan x}{1} dx$$

$$= -\cos x \int \frac{\sin^2 x}{\cos x} dx + \sin x \int \sin x \, dx$$

$$= -\cos x \int \frac{1-\cos^2 x}{\cos x} dx + \sin x \int \sin x \, dx$$

$$= -\cos x \int (\sec x - \cos x) dx + \sin x \int \sin x \, dx$$

$$= -\cos x \left[ \log(\sec x + \tan x) - \sin x \right] + \sin x \left[ -\cos x \right]$$

CS is $y = C_1 \cos x + C_2 \sin x - \cos x \log(\sec x + \tan x)$

Ques. $y'' - 6y' + 9y = \dfrac{e^{3x}}{x^2}$

$CF = (C_1 + C_2 x) e^{3x}$

$$W = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x}$$

$$PI = -e^{3x} \int \frac{\cancel{xe^{3x}}\, \cancel{e^{3x}}}{x^2 \cancel{e^{6x}}} dx + xe^{3x} \int \frac{\cancel{e^{3x}}\, \cancel{e^{3x}}}{x^2 \cancel{e^{6x}}} dx$$

$$= -e^{3x} \int \frac{1}{x} dx + xe^{3x} \int \frac{1}{x^2} dx$$

**Power Series Solution –**

Should be of the form: $P_0(x) \dfrac{d^2y}{dx^2} + P_1(x) \dfrac{dy}{dx} + P_2(x) y = 0$

$x = 0$ ordinary pt. if $P_0(0) \neq 0$

$x = 0$ singular pt. if $P_0(0) = 0$

Example

Ques. Solve $(1+x^2) \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} - y = 0$ —— (1)

let $y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$ is a soln. of (1)

$$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1} + \dots$$

$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3 x + \dots + n(n-1) a_n x^{n-2} + \dots$$

Put $y$, $\dfrac{dy}{dx}$ & $\dfrac{d^2y}{dx^2}$ in (1), we get

$$(1+x^2)\left[ 2a_2 + 6a_3 x + \dots + n(n-1) a_n x^{n-2} + \dots \right]$$
$$+ x \left[ a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1} + \dots \right]$$
$$- \left[ a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots \right] = 0$$

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Equating the constant terms & coefficients of $x, x^2, x^3, \ldots$ from both sides of (2).

$$2a_2 - a_0 = 0 \quad \text{—(3)}$$

$$3.2a_3 + a_1 - a_1 = 0 \quad \text{—(4)}$$

$$4.3a_4 + 2a_2 + \cancel{\phantom{x}} + 2a_2 - a_2 = 0 \quad \text{—(5)}$$

$$5.4a_5 + 3.2a_3 + 3a_3 - a_3 = 0 \quad \text{—(6)}$$

Generalising

$$(n+2)(n+1)a_{n+2} + n(n-1)a_n + 2na_n - a_n = 0 \quad \text{—(7)}$$

from (4),

$$3.2a_3 = 0$$

$$a_3 = 0$$

Put $n=3$ in (7), $(3+2)(3+1)a_5 = 0$

$$a_5 = 0$$

Similarly;

$$a_7 = 0$$

ie $\cancel{a_{2n+1}}$

from (3), $2a_2 = a_0$

$$\Rightarrow a_2 = \frac{1}{2}a_0$$

from (5), $4.3a_4 = 3a_2$ $\cancel{\frac{1}{2}a_0}$

$\cancel{a_4 = \frac{a_0}{24}}$ $4a_4 = -a_2 = -\frac{1}{2}a_0$

$$a_4 = -\frac{1}{8}a_0$$

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Put $n=4$ in (7),

$$6.5a_6 + 4.3a_4 + 4a_4 - a_4 = 0$$

$$30a_6 + 15a_4 = 0$$

$$30a_6 = -15a_4$$

$$a_6 = -\frac{1}{2}a_4 = -\frac{1}{2}\left(-\frac{1}{8}a_0\right)$$

$$= \frac{1}{16}a_0$$

So, $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \ldots$

$$y = a_0 + a_1x + \frac{1}{2}a_0x^2 - \frac{1}{8}a_0x^4 + \frac{1}{16}a_0x^6 \ldots$$

$y = a_0\left(1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 + \ldots\right) + a_1x$ is a complete soln. where $a_0, a_1$ are constants.

II Series soln. when $x$ is a singular pt (Frobenious Method)

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \quad \text{—(1)}$$

$$P_0(0) = 0$$

$$P_0(x) = x^{n-1}g(x)$$

Step 1

$y = x^m(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \ldots)$ is a soln.

Step 2: Find $y$, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in (1)

Step 3: ~~Com~~ compare coefficients of $x^{m-1}$ from both sides,

This will be an quadratic eq. ~~containing~~ in m. This is called Indicial eq.

Case 1: The roots of the indicial equations are distinct and do not differ by an integer.

Case 2: The roots of the indicial eqs. are equal.

Case 3: The roots of the indicial eqs are distinct and differ by an integer.

Find values of m

Step 4: Compare other powers of $x$ from (1)

$a_2, a_3, \ldots$ in terms of $a_0, a_1$

$y_{m_1}$ at $m = m_1$

$y_{m_2}$ at $m = m_2$ $\qquad CS = C_1 y_{m_1} + C_2 y_{m_2}$

Series solution if $x=0$ is a singular point.

Solve: $4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ —(1)

Since $4x = 0$ at $x=0$ so, $x=0$ is a singular point.

Let $y = x^m(a_0 + a_1x + a_2x^2 + \cdots)$ is a solution of (1)

$$\frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \cdots$$

$$\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \cdots$$

Substituting $y$, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in (1), we get

$$4x[m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \cdots] + 2[ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \cdots] + [a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \cdots] = 0$$

Equate the coefficient of $x^{m-1}$ from both sides,

$4a_0m(m-1) + 2ma_0 = 0$

$\Rightarrow a_0 \cdot 2m[2m-2+1] = 0$

$a_0 \neq 0$ so, $2m[2m-1] = 0$

Indicial equation

$m = 0, \frac{1}{2}$

Equate the coefficient of $x^m, x^{m+1}, x^{m+2}, \ldots$

$$4m\,a_1(m+1) + 2a_1(m+1) + a_0 = 0$$

$$\Rightarrow 2a_1(m+1)[2m+1] + a_0 = 0$$

$$a_1 = -\frac{a_0}{2(m+1)(2m+1)} \quad \text{---(2)}$$

$$4(m+2)(m+1)a_2 + 2(m+2)a_2 + a_1 = 0$$

$$\Rightarrow 2(m+2)a_2[2m+3] + a_1 = 0$$

$$\Rightarrow a_2 = -\frac{a_1}{2(m+2)(2m+3)} = \frac{a_0}{2^2(m+1)(m+2)(2m+1)(2m+3)}$$

$$a_3 = -\frac{1}{2(m+3)(2m+5)}a_2$$

$\boxed{\text{For } m=0}$, $a_1 = -\frac{a_0}{2}$, $a_2 = \frac{a_0}{2^2 \cdot 2 \cdot 3}$

$$a_3 = -\frac{a_0}{2^4 \cdot 3^2 \cdot 5}$$

This gives the particular solution

$$y = x^0\left[a_0 - \frac{a_0}{2}x + \frac{a_0}{2^2 \cdot 2 \cdot 3}x^2 - \frac{a_0}{2^4 3^2 \cdot 5}x^3 + \dots\right]$$

$$y = a_0\left[1 - \frac{1}{2}x + \frac{1}{2^2 \cdot 2 \cdot 3}x^2 - \frac{1}{2^4 \cdot 3^2 \cdot 5}x^3 + \dots\right]$$

~~for~~ $\boxed{\text{For } m=\frac{1}{2}}$, $a_1 = -\frac{a_0}{2(\frac{1}{2}+1)(1+1)} = -\frac{a_0}{3 \cdot 2}$

$$a_2 = -\frac{a_1}{2(\frac{1}{2}+2)(1+3)} = -\frac{a_1}{5 \cdot 4} = \frac{a_0}{5 \cdot 4 \cdot 3 \cdot 2}$$

$$a_3 = -\frac{a_0}{\lfloor 7}$$

This gives a particular solution

$$y_2 = x^{\frac{1}{2}}\left[a_0 - \frac{a_0}{\lfloor 3}x + \frac{a_0}{\lfloor 5}x^2 - \frac{a_0}{\lfloor 7}x^3 + \dots\right]$$

$$= a_0 x^{\frac{1}{2}}\left[1 - \frac{1}{\lfloor 3}x + \frac{1}{\lfloor 5}x^2 - \frac{1}{\lfloor 7}x^3 + \dots\right]$$

$\boxed{\begin{matrix} c_1 a_0 = c_1' \\ c_2 a_0 = c_2' \end{matrix}}$

CS is ~~$y = c_1y_1 + c_2y_2$~~ $y = c_1y_1 + c_2y_2$

$$y = c_1'\left[1 - \frac{1}{2}x + \frac{1}{2^2 \cdot 2 \cdot 3}x^2 - \frac{1}{2^4 \cdot 3^2 \cdot 5}x^3 + \dots\right]$$
$$+ c_2' x^{1/2}\left[1 - \frac{1}{\lfloor 3}x + \frac{1}{\lfloor 5}x^2 - \dots\right]$$

**Bessel's equation of order 'n'**

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \quad \text{---(1)}$$

Bessel's function are the particular soln. of (1)

$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

Here, $x=0$ is a singular point of (1)

$\therefore$ let $y = x^m[a_0 + a_1 x + a_2 x^2 + \dots]$ is a soln of (1)

$m^2 - n^2 = 0$

$m = \pm n$

$J_{\frac{1}{2}}(x) = ?$

Prove that

$J_{\frac{1}{2}}(x) \stackrel{?}{=} \sqrt{\frac{2}{\pi x}} \sin x$

$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!}$

$J_{\frac{1}{2}}(x), J_{\frac{3}{2}}(x), J_{\frac{5}{2}}(x) = ?$

Bessel's function of 1st kind

$\int x^n J_n(x) =$

$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{\Gamma(n+r+1)\, r!} \left(\frac{x}{2}\right)^{n+2r}$

Prove! Recurrence Formulas for $J_n(x)$:

(i) (a) $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

(b) $\frac{d}{dx}[x^{-n} J_n(x)] = x^{-n} J_{n+1}(x)$

$J_{\frac{5}{2}}(x) =$

(ii) (a) $J_n(x) = \frac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$

(b) $J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

$nx^{n-1} J_n(x) + x^n J_n'(x) = -n x^{-n-1} J_n(x) + x^{-n} J_n'(x)$

$= x^n J_{n-1}(x) + \frac{x}{2n} J_{n+1}(x)$

$(nx^{n-1} - nx^{-n-1}) J_n(x) + (x^n + x^{-n}) J_n'(x) =$

$\frac{x}{2n}[J_{n-1}(x) +$

$n+1 / + r + 1 + 1$

$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(n+r+1)}$

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## Legendre Polynomial :-

$$P_n(x) = \frac{1}{2^n \lfloor n} \frac{d^n}{dx^n} (x^2-1)^n$$

$$\frac{1}{2^0 0!} \frac{d^0}{dx^0}(x^2-1)^0$$

So,

$P_0(x) = 1$

$\frac{1}{2^1 1!} \frac{d}{dx}(x^2-1)$

$P_1(x) = x$

$\frac{1}{2}(2x)$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{1}{2^3 \lfloor 3} \frac{d^3}{dx^3} (x^2-1)^3$

$\frac{1}{4 \times 2} \frac{d^2}{dx^2}(x^2-1)^2$

$= \frac{1}{8 \times 6} \frac{d^3}{dx^3}\left(x^6 - 1 - 3x^2(x^2-1)\right)$

$= \frac{1}{48} \frac{d^3}{dx^3}\left(x^6 - 1 - 3x^4 + 3x^2\right)$

$6x^5 - 12x^3 + 6x$

$x^4 + 1 - 2x^2$

$30x^4 - 36x^2 + 6$

$4x^3 - 4x$

$120x^3 - 72x$

$12x^2 - 4$

$\frac{1}{48}(120x^3 - 72x)$

$4(3x^2-1)$

$= \frac{5x^3 - 3x}{2} = P_3(x)$

Express $f(x) = x^3 + 2x^2 - x - 3$ in terms of Legendre polynomials.

$x^3 = \frac{1}{5}\left[2P_3(x) + 3x\right]$

$x^2 = \frac{1}{3}\left[2P_2(x) + 1\right]$

$f(x) = \frac{1}{5}\left[2P_3(x) + 3x\right] + \frac{2}{3}\left[2P_2(x) + 1\right] - x - 3$

$= \frac{2}{5}P_3(x) + \frac{4}{3}P_2(x) + \left(\frac{3}{5} - 1\right)x + \frac{2}{3} - 3$

$= \frac{2}{5}P_3(x) + \frac{4}{3}P_2(x) - \frac{2}{5}P_1(x) - \frac{7}{3}P_0(x)$

## Generating fun$^n$ formula

$$\sum_{n=0}^{\infty} P_n(x) t^n = (1 - 2xt + t^2)^{-\frac{1}{2}} \quad -(1)$$

Use:- $P_{n+1} \rightarrow P_n, P_{n-1}$

$P_n(0) = ?$ $\quad P_n(1) = ?$ $\quad P_n(-1) = ?$

Ques. Prove that (a) $P_n(-1) = (-1)^n$

(b) $P_{2r}(0) = \frac{(-1)^r \lfloor 2r}{2^{2r}(\lfloor r)^2}$

$\boxed{P_{2r+1}(0) = 0}$

Put $x=-1$ in (1)

$$\sum_{n=0}^{\infty} P_n(-1)\, t^n = (1+2t+t^2)^{-\frac{1}{2}}$$
$$= (1+t)^{-1}$$

$$\Rightarrow P_0(-1) + P_1(-1)t + \dots P_n(-1)t^n + \dots = (1 - t + t^2 - t^3 + \dots + (-1)^n t^n)$$

Comparing coeff. of $t^n$ from both sides

$$\boxed{P_n(-1) = (-1)^n}$$

(b) Put $x=0$ in (1)

$$\sum_{n=0}^{\infty} P_n(0)\, t^n = (1+t^2)^{-\frac{1}{2}} = 1$$

Recurrence formula for $P_n(x)$

$$(n+1)P_{n+1}(x) = (2n+1)x P_n(x) - nP_{n-1}(x)$$

Proof?

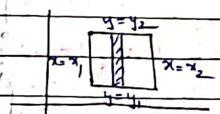
$$-\frac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(0 \cancel{-2x+2t})\ \ 2(t-x)$$

$$\frac{x-t}{(1-2xt+t^2)^{3/2}}$$

## Multiple Integral

Double Integral: $\iint_R f(x,y)\,dA \rightarrow \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dy\,dx$

Case 1:



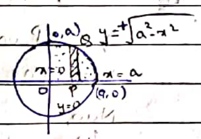

$x_1, x_2, y_1, y_2$ are constants.

Case 2: $y_1, y_2 \rightarrow$ functions of $x$ $\quad \int_{x_1}^{x_2}\left[\int_{y_1}^{y_2} f(x,y)\,dy\right]dx$

Case 3: $x_1, x_2 \rightarrow$ functions of $y$ $\quad \int_{y_1}^{y_2}\left[\int_{x_1}^{x_2} f(x,y)\,dx\right]dy$

Ques- Evaluate: $\iint_R xy\,dx\,dy$ where R is the region in the positive quadrant of the circle $x^2+y^2=a^2$.

Sol:- $\int_0^a\left[\int_0^{\sqrt{a^2-x^2}} xy\,dy\right]dx = \int_0^a \left[\frac{xy^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$




or $\int_0^a\left[\int_0^{\sqrt{a^2-y^2}} xy\,dx\right]dy$

$$= \int_0^a \frac{x(a^2-x^2)}{2}\,dx$$

$$= \left[\frac{a^2x^2}{4} - \frac{x^4}{8}\right]_0^a = \frac{a^2a^2}{4} - \frac{a^4}{8} = \frac{a^4}{4} - \frac{a^4}{8}$$

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**Ques.** Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy$

$$= \int_0^1 \left[\int_x^{\sqrt{x}} (x^2+y^2)\,dy\right] dx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{\sqrt{x}} dx = \int_0^1 \left(x^2\sqrt{x} + \frac{x^{3/2}}{3} - x^2 x - \frac{x^3}{3}\right) dx$$

$$= \int_0^1 \left(x^{5/2} + \frac{x^{3/2}}{3} - x^3 - \frac{x^3}{3}\right) dx$$

**Change of Integration :-**

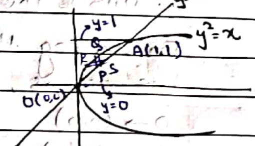
**Ques.** Evaluate the following integrals by changing the order of integration -

(a) $I = \int_0^1 \int_x^{\sqrt{x}} xy\,dy\,dx$

**Sol.** ~~Here~~ In the given case, $1^{st}$ integration will be w.r.t. $y$ from $P$ on the line $y=x$ to $Q$ on the parabola $y^2=x$ then the integration will be w.r.t. $x$ from $x=0$ to $x=1$

Now, the parabola $y^2=x$ and the line $y=x$ meet at the points $O(0,0)$ and $A(1,1)$

So, for the change of integration, we need to integrate $1^{st}$ w.r.t. $x$ from the point $R$ on the parabola $x=y^2$ to the point $Q$ on the line $x=y$. Then we integrate the resultant w.r.t. $y$ from $y=0$ to $y=1$




$$\therefore I = \int_0^1 \left[\int_{y^2}^{y} xy\,dx\right] dy$$

$$= \int_0^1 \left[\frac{x^2}{2} y\right]_{y^2}^{y} dy$$

$$= \int_0^1 \left(\frac{y^3}{2} - \frac{y^5}{2}\right) dy$$

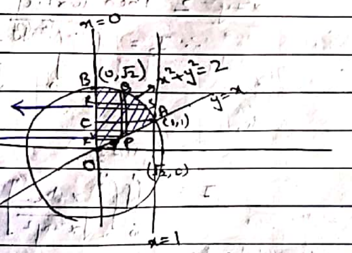
(b) $I = \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ $\qquad y = x$ to $y = \sqrt{2-x^2}$

i.e. $y^2 = 2-x^2$

i.e. $x^2+y^2=2$

$y=1$ to $y=\sqrt{2}$ $\leftarrow$ $x=0$ to $x=\sqrt{2-y^2}$

$y=0$ to $y=1$ $\leftarrow$ $x=0$ to $x=y$



The given integration is $1^{st}$ w.r.t $y$ from P on the line $y=x$ to Q on the circle $x^2+y^2=2$ then w.r.t $x$ from $x=0$ to $x=1$.

$\therefore$ OAB is the area of integration.

For changing the order, we need to break OAB into two regions, OAC and CAB.

$O \rightarrow (0,0)$

$A \rightarrow (1,1)$

$B \rightarrow (0,\sqrt{2})$

$C \rightarrow (0,1)$

For OAC, $1^{st}$ integrate w.r.t $x$ from $x=0$ to the line $x=y$, then w.r.t $y$ from $y=0$ to $y=1$.

$$\therefore I_1 = \int_0^1 \int_0^y \frac{x\,\cancel{dx\,dx}\,dy}{\sqrt{x^2+y^2}}$$

For CAB, $1^{st}$ integrate w.r.t $x$ from $x=0$ to the circle $x^2+y^2=2$ then w.r.t $y$ from $y=1$ to $y=\sqrt{2}$

$$\therefore I_2 = \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$

So,

$$I = I_1 + I_2$$

$$I = \int_0^1 \left[\int_0^y \frac{x\,dx}{\sqrt{x^2+y^2}}\right] dy + \int_1^{\sqrt{2}} \left[\int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\,dx\right] dy \quad \text{--- (i)}$$

$$\int_0^y \frac{x\,dx}{\sqrt{x^2+y^2}} = \int_y^{\sqrt{2}y} \frac{t\,dt}{\sqrt{t^2}}$$

Put $x^2+y^2=t^2$

$2x\,dx = 2t\,dt$

when $x=0$, $t=y$

when $x=y$, $t=\sqrt{2}y$

$$= \int_y^{\sqrt{2}y} dt$$

$$= [t]_y^{\sqrt{2}y} = \sqrt{2}y - y$$

Now,

$$\int_0^1 (\sqrt{2}y - y)\,dy = \left[\frac{\sqrt{2}y^2}{2} - \frac{y^2}{2}\right]_0^1$$

$$= \frac{\sqrt{2}}{2} - \frac{1}{2} = \frac{\sqrt{2}-1}{2}$$

Again,

$$\int_0^{\sqrt{2-y^2}} \frac{x\,dx}{\sqrt{x^2+y^2}}$$

Put $x^2+y^2=t^2$

$2x\,dx = 2t\,dt$

when $x=0$, $t=y$

when $x=\sqrt{2-y^2}$, $t=\sqrt{2}$

$$= \int_y^{\sqrt{2}} \frac{t\,dt}{\sqrt{t^2}} = \int_y^{\sqrt{2}} dt$$

$$= [t]_y^{\sqrt{2}} = \sqrt{2} - y$$

Now

$$\int_1^{\sqrt{2}} (\sqrt{2}-y)\,dy = \left[\sqrt{2}y - \frac{y^2}{2}\right]_1^{\sqrt{2}}$$

$$= \left(\sqrt{2}\sqrt{2} - \frac{2}{2}\right) - \left(\sqrt{2} - \frac{1}{2}\right)$$

$$= 2 - 1 - \sqrt{2} + \frac{1}{2} = 1 + \frac{1}{2} - \sqrt{2}$$

in form ①,

$$\cancel{I = \frac{\sqrt{2}}{2} - \frac{1}{2} + 1 + \frac{1}{2} - \sqrt{2}}$$

$$\cancel{I = 1} \qquad I = \frac{\sqrt{2}}{2} - \frac{1}{2} + 1 + \frac{1}{2} - \sqrt{2}$$

$$I = \frac{1}{\sqrt{2}} + 1 - \sqrt{2}$$

$$I = \frac{1}{\sqrt{2}} - \sqrt{2} + 1 = \frac{1-2}{\sqrt{2}} + 1$$

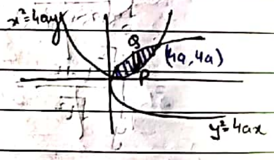
$$I = 1 - \frac{1}{\sqrt{2}}$$

**Evaluation of area by double integral:**

**Ques:** Find the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

**Sol:-**

$$\text{Area} = \int_0^{4a} \int_{x^2/4a}^{\sqrt{4ax}} 1\, dy\, dx$$




Now,

$$\int_{x^2/4a}^{\sqrt{4ax}} 1\, dy = [y]_{x^2/4a}^{\sqrt{4ax}}$$

$$= \sqrt{4ax} - \frac{x^2}{4a}$$

$$= 2\sqrt{a}x^{\frac{1}{2}} - \frac{x^2}{4a}$$

$$\int_0^{4a}\int_{x^2/4a}^{\sqrt{4ax}} 1\, dy\, dx = \int_0^{4a} \left(2\sqrt{a}x^{\frac{1}{2}} - \frac{x^2}{4a}\right) dx$$

For intersection point:

$$y^2 = 4ax$$
$$\Rightarrow y^4 = 16a^2x^2$$
$$y^4 = 16a^2 \times 4ay$$
$$y^3 = 64a^3$$
$$y = 4a$$

Now, $(4a)^2 = 4ax$

$\Rightarrow 4a = x$ i.e. $(4a, 4a)$

$$\Rightarrow \text{Area} = \left[2\sqrt{a}\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{1}{4a}\frac{x^3}{3}\right]_0^{4a}$$

$$32 \times 4 = 128$$

$$= \left[\frac{2}{3}.2\sqrt{a}.x^{3/2} - \frac{1}{4a}\frac{x^3}{3}\right]_0^{4a}$$

$$= \left[\frac{4}{3}\sqrt{a}x^{3/2} - \frac{1}{12a}x^3\right]_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}(4a)^{3/2} - \frac{1}{12a}(4a)^3$$

$$= \frac{4}{3}\sqrt{a}.8a^{3/2} - \frac{1}{12a}.64a^3$$

$$= \frac{32}{3}a^2 - \frac{64}{12}a^2 = \frac{128}{12}a^2 - \frac{64}{12}a^2$$

$$= \left(\frac{128-64}{12}\right)a^2 = \frac{64}{12}a^2$$

$$= \frac{32}{6}a^2 = \frac{16}{3}a^2$$

**Evaluation of double integral by change of variables-**

$(x, y) \rightarrow$ cartesian

$(r, \theta) \rightarrow$ polar

$x = r\cos\theta, \quad y = r\sin\theta$

$r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}\frac{y}{x}$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\, dy\, dx \rightarrow \int g(r,\theta)\, \underset{\downarrow\, r}{J}\, dr\, d\theta$$

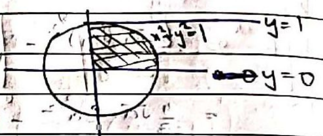
Ques: Evaluate:

(a) $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)\,dy\,dx$ } by changing to

(b) $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ } polar coordinates.

(a)

$x = 0$ to $x = \sqrt{1-y^2}$




$\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)\,dy\,dx$

$= \int_0^{\pi/2} \int_0^1 r^2\, r\,dr\,d\theta$ $\quad r = \sqrt{x^2+y^2}$

Formal writing: The region of integration being the part of the circle $x^2+y^2=1$ in 1st quadrant of the x-y plane so r varies from 0 to 1 and $\theta$ varies from 0 to $\pi/2$

$= \int_0^{\pi/2} \int_0^1 r^3\,dr\,d\theta = \int_0^{\pi/2} \left[ \int_0^1 r^3\,dr \right] d\theta$

$= \int_0^{\pi/2} \left[ \frac{r^4}{4} \right]_0^1 d\theta = \int_0^{\pi/2} \frac{1}{4}\,d\theta$

$= \left[ \frac{1}{4}\theta \right]_0^{\pi/2}$

$= \frac{1}{4}[\theta]_0^{\pi/2}$

$= \frac{1}{4}\left(\frac{\pi}{2} - 0\right) = \pi/8$

(b) $r \to 0$ to $\infty$ $\quad x = 0$ to $\infty$ $\quad$ 1 quadrant only
$\theta \to 0$ to $\pi/2$ $\quad y = 0$ to $\infty$

$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$

$= \int_0^{\pi/2} \int_0^\infty e^{-r^2}\, r\,dr\,d\theta$

Triple Integral

$\int_a^b \int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z)\,dz\,dy\,dx = ?$

($g_1(x,y)$ → limits of z)

$\int_a^b \int_c^d \int_e^f F(x,y,z)\,dz\,dy\,dx = ?$

(e → limit of z; c → limit of y; a → limit of x)

Ques: Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$

$\int_0^{x+y} e^{x+y+z}\,dz = \left[ e^{x+y+z} \right]_0^{x+y} = e^{x+y+x+y} - e^{x+y+0}$

$= e^{2x+2y} - e^{x+y}$

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$$\int_0^x \int_0^{x+y} e^{x+y+z}\,dz\,dy = \int_0^x \left(e^{2x+2y} - e^{x+y}\right) dy$$

$$\cancel{\int_0^x \left(e^{2(x+y)} - e^{x+y}\right) dy}$$

$$\cancel{\int_0^x \left(e^{2(x+y)} - e^{x+y}\right) dy}$$

$$= \left[\frac{e^{2x+2y}}{2} - e^{x+y}\right]_0^x$$

$$= \left\{\frac{e^{2x+2x}}{2} - e^{x+x}\right\} - \left\{\frac{e^{2x}}{2} - e^{x}\right\}$$

$$= \frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^{x}$$

$$\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx = \int_0^a \left(\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^{x}\right) dx$$

$$= \left[\frac{1}{2}\frac{e^{4x}}{4} - \frac{e^{2x}}{2} - \frac{1}{2}\frac{e^{2x}}{2} + e^{x}\right]_0^a$$

$$= \left\{\frac{1}{8}e^{4a} - \frac{e^{2a}}{2} - \frac{1}{4}e^{2a} + e^{a}\right\} - \left\{\frac{1}{8} - \frac{1}{2} - \frac{1}{4} + 1\right\}$$

$$= \frac{1}{8}e^{4a} - e^{2a}\left(\frac{1}{2} + \frac{1}{4}\right) + e^{a} - \left\{\frac{1 - 4 - 2 + 8}{8}\right\}$$

$$= \frac{1}{8}e^{4a} - e^{2a}\left(\frac{2+1}{4}\right) + e^{a} - \left\{\frac{9-6}{8}\right\}$$

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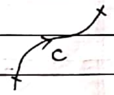
$$= \frac{1}{8}e^{4a} - \frac{3}{4}e^{2a} + e^{a} - \frac{3}{8}$$

**Line Integral and Green's Theorem:-**

$\vec{F} = f(x,y,z)\hat{i} + \phi(x,y,z)\hat{j} + \psi(x,y,z)\hat{k}$

position vector $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

so, $d\vec{R} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$




Now, Line integral of $\vec{F}$ along a curve C is

$$\int_C \vec{F}\cdot d\vec{R} = \int_C f(x,y,z)\,dx + \phi(x,y,z)\,dy + \psi(x,y,z)\,dz$$

Question Type →
1) Find line integral along a curve.
2) Calculate work done by the force field $\vec{F}$ $= \int_C \vec{F}\cdot d\vec{R}$

**Ques-** If $\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$, Evaluate $\int_C \vec{F}\cdot d\vec{R}$ along the curve C in the xy plane, $y = x^3$ from the point (1,1) to (2,8)

**Sol-**

$$\int_C \vec{F}\cdot d\vec{R} = \int_C \left[(5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}\right]\cdot\left[dx\,\hat{i} + dy\,\hat{j}\right]$$

$$= \int_C (5xy - 6x^2)\,dx + (2y - 4x)\,dy$$

$x \to 1$ to $2$
$y \to 1$ to $8$

$$= \int_1^2 (5x\cdot x^3 - 6x^2)\,dx + (2x^3 - 4x)\,3x^2\,dx$$

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**Ques:** Find the workdone in moving a particle in a force field given by $\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve C: $x = t^2+1$, $y = 2t^2$, $z = t^3$ from $t=1$ to $t=2$.

**Sol-** workdone $= \int_C \vec{F}\cdot d\vec{R}$

$$= \int_C (3xy\hat{i} - 5z\hat{j} + 10x\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_C (3xy\,dx - 5z\,dy + 10x\,dz)$$

$$= \int_{t=1}^{2} 3(t^2+1)2t^2 \cdot 2t\,dt - 5t^3 \cdot 4t\,dt + 10(t^2+1)3t^2\,dt$$

$$= \int_1^2 12t^3(t^2+1)\,dt - 20t^4\,dt + 30t^2(t^2+1)\,dt$$

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**Green's Theorem:** Let $\vec{F} = \phi(x,y)\hat{i} + \psi(x,y)\hat{j}$ be a vector function which is continuously differentiable in a region E of the xy plane bounded by a closed curve C. Then,

$$\oint_C \vec{F}(\vec{r})\cdot d\vec{R} = \int_C \phi\,dx + \psi\,dy$$

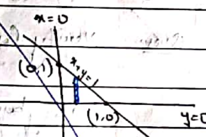
$$= \iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dy\,dx$$

Question type →
1) Verify Green's Theorem for the curve.
2) Find Line Integral by using Green's Theorem.

**Ques:** Using Green's Theorem, evaluate $\int_C (3x-8y^2)\,dx + (4y-6xy)\,dy$, where C is the boundary of the region bounded by $x=0$, $y=0$ and $x+y=1$.

**Sol-** $\phi(x,y) = 3x - 8y^2$

$\psi(x,y) = 4y - 6xy$




So, $\dfrac{\partial \phi}{\partial y} = -16y$

$\dfrac{\partial \psi}{\partial x} = -6y$

$y = 0$ to $y = 1-x$

$x = 0$ to $x = 1$

By Green's Theorem,

$$\int_C (3x-8y^2)\,dx + (4y-6xy)\,dy = \iint_E (-6y+16y)\,dy\,dx$$

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$$= \iint_E 10y \, dy \, dx$$

$$= \int_0^1 \left[ \int_0^{1-x} 10y \, dy \right] dx$$

$$= \int_0^1 \left[ \frac{10y^2}{2} \right]_0^{1-x} dx$$

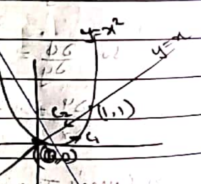
$$= \int_0^1 [5y^2]_0^{1-x} dx = \int_0^1 5(1-x)^2 \, dx$$

$$= 5\int_0^1 (x-1)^2 dx = 5\left[\frac{(x-1)^3}{3}\right]_0^1 \qquad = \cancel{\int_0^1 5(x-1)^2 dx}$$

$$= \frac{5}{3}\left[(x-1)^3\right]_0^1 \qquad \cancel{= 5\int_0^1 (x-1)^2 dx = 5\left[2(x-1)\right]_0^1}$$

$$= \frac{5}{3}\left[0+1\right] = 5/3 \qquad \cancel{= 10\left[x-1\right]_0^1}$$

**Ques.** Verify Green Theorem for $\int_C (xy+y^2)dx + x^2 dy$, where C is bounded by $y = x$ and $y = x^2$




**Sol.** For $\int_{C_1} (xy+y^2)dx + x^2 dy$ along $y = x^2$ and $x$ varies from 0 to 1

$$= \int_{x=0}^{1} (x \cdot x^2 + x^4)\,dx + x^2 \, 2x \, dx = \frac{19}{20}$$

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For $\int_{C_2} (xy+y^2)dx + x^2 dy$ along $y = x$ and $x$ varies from 1 to 0

$$= \int_{x=1}^{0} (x^2 + x^2)\,dx + x^2 dx$$

$$= \int_1^0 3x^2 dx = -1$$

$$\therefore \oint_C (xy+y^2)dx + x^2 dy = \frac{19}{20} - 1 = -\frac{1}{20}$$

Now,

$\phi = xy + y^2$, $\psi = x^2$

$\frac{\partial \phi}{\partial y} = x + 2y$, $\frac{\partial \psi}{\partial x} = 2x$

so, $$\iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dy\,dx = \iint_E (2x - x - 2y)\,dy\,dx$$

$$= \iint_E (x - 2y)\,dy\,dx$$

$$= \int_0^1 \int_{x^2}^{x} (x-2y)\,dy\,dx = -\frac{1}{20}$$

and so, Green's Theorem is verified (as both values are coming same)

## Surface Integral and Stokes Theorem:

$\vec{F}(x,y,z) = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

$S \to$ surface

$\vec{R} \to (x,y,z)$

$\hat{N} \to$ unit normal to $S$

$$\int_S \vec{F(R)} \cdot d\vec{S} = \int_S \vec{F(R)} \cdot \hat{N}\, ds$$

$$= \iint_R \vec{F(R)} \cdot \hat{N} \frac{dx\,dy}{|\hat{N}\cdot\hat{k}|}$$

projection of $S$ on $xy$ plane.

$$= \iint_R \vec{F(R)} \cdot \hat{k}\, dx\,dy$$

$$= \iint_{R'} \vec{F(R)} \cdot \hat{N} \frac{dy\,dz}{|\hat{N}\cdot\hat{i}|}$$

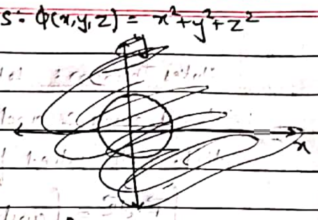
projection of $S$ on $yz$ plane.

**Ques.** Evaluate $\int_S \vec{F}\cdot\vec{N}\, dS$, where $\vec{F} = y^2z^2\hat{i} + z^2x^2\hat{j} + x^2y^2\hat{k}$, and $S$ is the part of the sphere $x^2+y^2+z^2=1$ above $xy$ plane.

**Sol.** The projection of $S$ on $xy$ plane is $R: x^2+y^2=1,\ z=0$

$$\cancel{\int_S \vec{F}\cdot\vec{N}\, ds = \iint_R 1x^2y^2 \frac{dx\,dy}{|\hat{N}\cdot\hat{k}|} = \iint_R x^2y^2 \frac{dx\,dy}{|\hat{k}\cdot\hat{k}|} = \iint_R x^2y^2\, dx\,dy}$$

$S: \phi(x,y,z) = x^2+y^2+z^2$




$$\hat{N} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}}{|\vec{\nabla}\phi|}$$

$$= \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4(x^2+y^2+z^2)}} = x\hat{i} + y\hat{j} + z\hat{k}$$

$\cancel{\int_S \vec{F}\cdot\hat{N}\, ds = \iint}$ $\vec{F}\cdot\hat{N} = xy^2z^2 + yz^2x^2 + zx^2y^2$

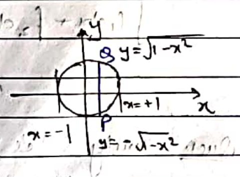
$$\int_S \vec{F}\cdot\hat{N}\, ds = \iint_R (xy^2z^2 + yz^2x^2 + zx^2y^2) \frac{dx\,dy}{z}$$

$\vec{N}\cdot\hat{k} = z$

$$= \iint_R (xy^2z + yzx^2 + x^2y^2)\, dx\,dy$$

$$= \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[\sqrt{1-x^2-y^2}\,(xy^2 + x^2y) + x^2y^2\right] dy\,dx$$

$$= 2\times 2 \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} x^2y^2\, dy\,dx$$



**Stoke's Theorem:** Let S be an open surface bounded by a closed curve C and $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be any cont. diff. vector valued function then,

$$\int_C \vec{F}\cdot d\vec{R} = \int_S \text{curl}\,\vec{F}\cdot\hat{N}\,ds$$

$\hat{N} \to$ unit external normal at any point of S.

**Green's Theorem from Stoke's:** $\vec{F} = f_1\hat{i} + f_2\hat{j}$, $\vec{R} = x\hat{i} + y\hat{j}$

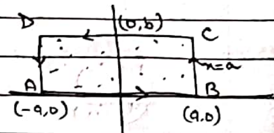
$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & 0 \end{vmatrix}$$

$$= \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial x}\right)$$

$$\int_C f_1\,dx + f_2\,dy = \int_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial x}\right) dx\,dy$$

**Ques.** Verify Stoke's Theorem for $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ taken around the rectangle bounded by the lines $x=\pm a$, $y=0$, $y=b$

**Sol.** C : ABCD




$$\int_{ABCD} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

$dy = 0$

$$\int_{AB} \vec{F}\cdot d\vec{R} = \int_{x=-a}^{a} x^2\,dx$$

$$\int_{BC} \vec{F}\cdot d\vec{R} = \int_{y=0}^{b} -2ay\,dy$$

...........

$$\int_{ABCD} \vec{F}\cdot d\vec{R} = -ab^2 + \left(-\frac{2a^3}{3} - 2ab^2\right) - ab^2 + \frac{2a^3}{3}$$

$$= -4ab^2$$

Now,

$\hat{N} = \hat{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = \hat{k}(-2y-2y) = -4y\hat{k}$$

$$\therefore \int_S \text{curl}\,\vec{F}\cdot\hat{N}\,ds = \iint_R -4y\hat{k}\cdot\hat{k}\,\frac{dx\,dy}{|\hat{k}\cdot\hat{k}|} \qquad R \to S \text{ itself}$$

$$= \int_{y=0}^{b}\int_{x=-a}^{a} -4y\,dx\,dy$$

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$\boxed{2}$

14(i)

14(ii)

12, ~~13~~ (a)(b)(c)

13 (a) " "

11(d) 4 1

$\boxed{\text{Legendre Polynomials}}$

Rodrigue's Formula (~~Proof~~?)

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$