Condition 1: $S_0 = 0 = S_n$ (Natural spline)

$$\begin{bmatrix} 2(h_0+h_1) & h_1 & 0 & \cdots & & 0 \\ h_1 & 2(h_1+h_2) & h_2 & 0 & \cdots & 0 \\ 0 & h_2 & 2(h_2+h_3) & h_3 & & \vdots \\ \vdots & & & \ddots & & \\ 0 & \cdots & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) \end{bmatrix}$$

Condition 2: $f'(x_0) = p$ and $f'(x_n) = q$

$$\begin{bmatrix} 2h_0 & h_0 & 0 & \cdots & & 0 \\ h_0 & 2(h_0+h_1) & h_1 & & & \vdots \\ 0 & h_1 & 2(h_1+h_2) & h_2 & & \\ \vdots & & & \ddots & & \\ 0 & \cdots & & & h_{n-1} & 2h_{n-1} \end{bmatrix}$$

Condition 3: $S_0 = S_1$, $S_n = S_{n-1}$

$$\begin{bmatrix} (3h_0+2h_1) & h_1 & 0 & \cdots & & 0 \\ h_1 & 2(h_1+h_2) & h_2 & & & \\ 0 & h_2 & 2(h_2+h_3) & h_3 & & \vdots \\ \vdots & & & \ddots & & \\ 0 & \cdots & & & h_{n-2} & 2(h_{n-2}+3h_{n-1}) \end{bmatrix}$$

Condition 4:

$$\begin{bmatrix} \frac{(h_0+h_1)(h_0+2h_1)}{h_1} & \frac{h_1^2-h_0^2}{h_1} & & & \\ h_1 & 2(h_1+h_2) & h_2 & & \\ & h_2 & 2(h_2+h_3) & h_3 & \\ & & \ddots & & \\ & & & \frac{h_{n-2}^2-h_{n-1}^2}{h_{n-2}} & \frac{(h_{n-2}+h_{n-1})(h_{n-1}+2h_{n-2})}{h_{n-2}} \end{bmatrix}$$

No computation for S_0 and S_n needed for first three conditions since right hand side matrix is same.
For condition (4), S_0 and S_n need to be computed.

Once we obtain S_i, we can calculate

$$a_i = \frac{S_{i+1}-S_i}{6h_i}, \quad b_i = \frac{S_i}{2}$$

$$c_i = \frac{y_{i+1}-y_i}{h_i} - \frac{2h_iS_i + h_iS_{i+1}}{6}$$

$$d_i = y_i$$

Example: Given

x	0.0	1.0	1.5	2.25
$f(x)$	2.000	4.4366	6.7134	13.9130
	y_0	y_1	y_2	y_3

Fit using natural cubic spline curve.

Soln:

$h_0 = 1, \quad h_1 = 0.5, \quad h_2 = 0.75$

$s_0 = 0 = s_3$ for natural spline.

From Eqn. (12), $i=1$

$$h_0 s_0 + 2(h_0+h_1)s_1 + h_1 s_2 = 6\left(\frac{y_2-y_1}{h_1} - \frac{y_1-y_0}{h_0}\right)$$

$$0 + 2(1+0.5)s_1 + 0.5 s_2 = 6\left[\frac{6.7134-4.4366}{0.5} - \frac{4.4366-2.000}{1}\right]$$

$$3.0 s_1 + 0.5 s_2 = 6[4.5536 - 2.4366]$$

$$= 6(2.1170)$$

$$3.0 s_1 + 0.5 s_2 = 12.7020 \qquad \text{---(1)}$$

$i=2$

$$h_1 s_1 + 2(h_1+h_2)s_2 + h_2 s_3 = 6\left[\frac{y_3-y_2}{h_2} - \frac{y_2-y_1}{h_1}\right]$$

$$0.5 s_1 + 2(0.5+0.75)s_2 + 0 = 6\left[\frac{13.9130-6.7134}{0.75} - \frac{6.7134-4.4366}{0.5}\right]$$

$$= 6[9.5995 - 4.5536]$$

$$= 6[5.0459]$$

$$0.5 s_1 + 2.5 s_2 = 30.2754 \qquad \text{---(2)}$$

In matrix form

$$\begin{bmatrix} 3.0 & 0.5 \\ 0.5 & 2.5 \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 12.7020 \\ 30.2754 \end{bmatrix}$$

$\therefore$ $s_1 = 2.2920$ and $s_2 = 11.6518$

Now, for different intervals $[0.0, 1.0]$, $[1.0, 1.5]$ and $[1.5, 2.25]$, we can write from eqn. (1),

$$f_i(x) = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i$$

$[0.0, 1.0]$, $\quad S_0 = 0$

$$a_0 = \frac{S_1 - S_0}{6h_0}, \quad b_0 = \frac{S_0}{2}, \quad c_0 = \frac{y_1 - y_0}{h_0} - \frac{2h_0 S_0 + h_0 S_1}{6}$$

$d_0 = y_0$

$$a_0 = \frac{2.2920 - 0}{6 \times 1.0} = 0.382, \quad b_0 = 0, \quad c_0 = 2.4366 - \frac{2.2920}{6}$$
$$= 2.0546$$

$d_0 = y_0 = 2.0000$

$[1.0, 1.5]$

$$a_1 = \frac{S_2 - S_1}{6h_1}, \quad b_1 = \frac{S_1}{2}, \quad c_1 = \frac{y_2 - y_1}{h_1} - \frac{2h_1 S_1 + h_1 S_2}{6}$$

$$a_1 = \frac{11.6518 - 2.2920}{6 \times 0.5} = 3.1199, \quad b_1 = 1.146, \quad c_1 = \left[4.5536 - \left(2 \times 0.5 \times 2.2920 + 0.5 \times 11.6518\right)/6\right]$$

$$c_1 = \left[4.5536 - 1.3531\right]$$
$$= 3.2005$$

$d_1 = y_1 = 4.4366$

Similarly we can calculate for $[1.5, 2.25]$ interval

$a_2 = -2.5893, \quad b_2 = 5.8259, \quad c_2 = 6.6866, \quad d_2 = 6.7134$

Hence equations are

x_i	Interval	$f_i(x)$
x_0	$[0.0, 1.0]$	$0.3820(x-0)^3 + 0(x-0)^2 + 2.0546(x-0) + 2.0000$
x_1	$[1.0, 1.5]$	$3.1199(x-1)^3 + 1.146(x-1)^2 + 3.2005(x-1)^2 + 4.4366$
x_2	$[1.5, 2.25]$	$-2.5893(x-1.5)^3 + 5.8259(x-1.5)^2 + 6.6866(x-1.5) + 6.7134$

If, we want to calculate for $x = 0.5$, then using

f_0, $f(0.5) = 3.07505$

If $x = 1.75$, then using f_2, $f(1.75) = 8.7087$