Lecture

Continuing with the previous lecture. We shall be discussing some interpolation formula.

Newton's interpolation formulae

Let us assume, we have been given (n+1) set of tabulated points, (x_0,y_0), (x_1,y_1),……,(x_n,y_n).

Our aim is to find an nth degree polynomial 'y_n' so that given values of y agree with the set of tabulated points.

Our assumption is that set of tabulated points are evenly spaced i.e. $x_i=x_0+ih$, i= 0,1,,…,n.

Let us write for polynomial of n degree y_n as

$$y_n = a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+a_3(x-x_0)(x-x_1)(x-x_2)+\ldots\ldots+a_n(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})$$

Now putting $x = x_0$, we get

$y = y_0 = a_0$

If $x = x_1$, then $y_1 = a_0+a_1(x_1-x_0) = y_0+ a_1(x_1-x_0)$

Or $\frac{y_1-y_0}{x_1-x_0} = a_1$

From $x_i=x_0+ih$, putting i = 1, we get $x_1-x_0 = h$

Or $\frac{y_1-y_0}{x_1-x_0} = a_1 = \frac{y_1-y_0}{h} = \frac{\Delta y_0}{h}$

Similarly,

i=2, $x_2-x_0 = 2h$ and $x_2-x_1 = h$

Putting $x = x_2$ in above equation,

$y_2 = a_0+a_1(x_2-x_0)+a_2(x_2-x_0)(x_2-x_1)$

$$y_2 = y_0 + \frac{y_1 - y_0}{h}.2h + a_2.2h.h$$

$$y_2 - y_0 = (y_1 - y_0).2 + 2h^2 a_2$$

$$a_2 = \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2!\,h^2}$$

$a_3 = \frac{\Delta^3 y_0}{3! h^3}$ and so on.

Now setting x = x_0+ph and substituting for a_0, a_1,……,a_n in above equation, we get

$$y_n = y_0 + \frac{\Delta y_0}{h}\,.ph + \frac{\Delta^2 y_0}{2!\,h^2}.ph\{x_0 + ph - x_1\} + \cdots \ldots\ldots\ldots\ldots\ldots$$

$$y_n = y_0 + p\Delta y_0 \ + p(p-1)\frac{\Delta^2 y_0}{2!} + \cdots \ldots\ldots\ldots\ldots\ldots + p(p-1)(p-2)\ldots..(p-n+1)\frac{\Delta^n y_0}{n!}$$

This is known as Newton-Gregory interpolating polynomial or simple newton's forward difference formula. This is useful for interpolation near beginning of the table.

Similarly, we can find Newton's backward difference formula and it is given as

$$y_n(x) = y_n + p\nabla y_n \ + p(p+1)\frac{\nabla^2 yn}{2!} + \cdots \ldots + p(p+1)\ldots..(p+n-1)\frac{\nabla y_n}{n!}$$

Here, p = (x-x_n)/h

This formula is useful for interpolation near the end of the table. (Derive Newton's backward difference formula)

Example:

Given the values of the table, find y(79).

x	y	Δ	Δ^2	Δ^3
75	246			
		-44		
80	202		**-40**	
		-84		**46**
85	118		6	
		-78		
90	40			

We have to find y at x =79. Here x_0 = 75, y_0 = 246 and h=5 (difference between consecutive x values), so p = (79-75)/5= 4/5 = 0.8.

Using Newton's forward difference formula,

$$y_n = y_0 + p\Delta y_0 + p(p-1)\frac{\Delta^2 y_0}{2!} + \cdots \dots \dots \dots \dots \dots + p(p-1)(p-2) \dots \dots (p-n+1)\frac{\Delta^n y_0}{n!}$$

y(0.8) = 246+0.8(-44)+[(0.8)(-0.2)(-40)]/2+[(0.8)(-0.2)(-1.2)(46)]/6

= 246-35.2+3.2+1.472=215.472

Example:

Find a cubic polynomial for the following table.

x	y	Δ	Δ^2	Δ^3
0	1			
		1		
1	2		**-2**	
		-1		**12**
2	1		10	
		9		
3	10			

$$y_n = y_0 + p\Delta y_0 + p(p-1)\frac{\Delta^2 y_0}{2!} + p(p-1)(p-2)\frac{\Delta^3 y_0}{3!}$$

x_0= 0, h =1, y_0 = 1. p = (x-x_0) = x

y(x) = 1+(x-x_0)/1(1)+[(x-x_0)(x-x_0-1)/2](-2)+[(x-x_0)(x-x_0-1)(x-x_0-2)/6](12)

= 1+x-x(x-1)+x(x-1)(x-2)(2)

= 1+x-x^2+x+2x^3-6x^2+4x

y(x) = 2x^3-7x^2+6x+1

Problems for the students.

Prob. 1. Find the value of ‘y’ at x =2.65 from the following table.

x	-1	0	1	2	3
y	-21	6	15	12	3

Finding the value of y outside the given range of x is called extrapolation. Below is given one such problem.

Prob. 2:

x	0.10	0.15	0.20	0.25	0.30
tanx	0.1003	0.1511	0.2027	0.2553	0.3093

Find (a) tan 0.12, (b) tan0.26, (c) tan0.40 and (d) tan 0.50. Discuss the results obtained.

Shift operator:

Before proceeding further, I would like to introduce a symbolic method which can easily establish difference formulae. We have already defined few operators.

A shift operator is defined as

$Ey_i = y_{i+1}$

It means operating by shift operator shifts value of y to the next value of y.

Thus, $E^2y_i = y_{i+2}$ and so on.

The relationship between shift operator and forward difference operator can be established.

$\Delta y_0 = y_1 - y_0 = Ey_0 - y_0 = (E-1)y_0$

Or $\Delta = E-1$ or $E = 1+\Delta$

Similarly, $\Delta^2 y_0 = (E-1)^2 y_0 = (E^2-2E+1)y_0 = E^2y_0 - 2Ey_0 + y_0 = y_2 - 2y_1 + y_0$ (This result we have obtained before).

We can establish relation between Shift operator and backward difference operator.

$\nabla y_1 = y_1 - y_0 = y_1 - E^{-1}y_1 = (1-E^{-1})y_1$

So, $\nabla=(1-E^{-1})$

In the same way, we can establish $\delta = E^{1/2} - E^{-1/2}$

Using shift operator, we shall find out Gauss central difference formulae. This is given in separate sheet.