DIRECT INTEGRATION OF $\ln(\sin x)$

ROHAN D. KEKATPURE

1. Introduction and Background

This note concerns the evaluation of the definite integral

$$\int_0^{\pi/2} \ln(\sin x)\, dx.$$

I encountered this integral in Paul Nahin's *Inside Interesting Integrals* book. The integral is fascinating and has been solved using various methods. The most common solution method is perhaps also the most elegant and is reproduced below. Let

$$\begin{aligned}
\mathcal{W} &= \int_0^{\pi/2} \ln(\sin x)\, dx \\
&= \int_0^{\pi/2} \ln(\cos x)\, dx \qquad \cdots \text{put } y = \pi/2 - x \text{ and simplify} \\
2\mathcal{W} &= \int_0^{\pi/2} \ln(\sin x \cos x) dx \quad \cdots \text{adding two lines above} \\
&= \int_0^{\pi/2} \ln(\sin 2x)\, dx - \int_0^{\pi/2} \ln 2\, dx \\
&= -\frac{\pi}{2}\ln 2 + \int_0^{\pi} \ln(\sin y)\, \frac{1}{2} dy \qquad \cdots \text{put } y = 2x \text{ and simplify} \\
&= -\frac{\pi}{2}\ln 2 + 2\int_0^{\pi/2} \ln(\sin y)\, \frac{dy}{2} \qquad \cdots \text{using symmetry of integral about } y = \pi/2 \\
&= -\frac{\pi}{2}\ln 2 + \mathcal{W} \\
\Rightarrow \mathcal{W} &= -\frac{\pi}{2}\ln 2
\end{aligned}$$

The indirect method, while elegant, is unsatisfactory. This is a purely personal opinion I've had since high school about methods which express the integral as an equation in terms of itself. In this note I describe a direct evaluation of this integral.[1] Note that the direct method we describe below is significantly more laborious and is perhaps not interesting for any purpose other than demonstrating the feasibility of a direct attack.

[1] My quest for direct methods for evaluating tough integrals, while often futile, is happily successful in this case.

2. Direct method

Consider the integrand

$$\ln(\sin x) = \frac{1}{2}\ln(\sin^2 x) = \frac{1}{2}\ln(1-\cos^2 x)$$

Since $\cos^2 x \leq 1$ we can expand $\ln(1-\cos^2 x)$ in a Taylor series around $x=0$:

$$-\ln(1-\cos^2 x) = \cos^2 x + \frac{\cos^4 x}{2} + \frac{\cos^6 x}{3} + \cdots = \sum_{m=1}^{\infty} \frac{\cos^{2m} x}{m}$$

Therefore,

$$\int_0^{\pi/2} \ln(\sin x)\,dx = \frac{1}{2}\int_0^{\pi/2} \ln(1-\cos^2 x)\,dx = \frac{-1}{2}\int_0^{\pi/2} \sum_{m=1}^{\infty} \frac{\cos^{2m} x}{m} \tag{1}$$

The integral $\int_0^{\pi/2} \cos^{2m} x\,dx$ is a standard integral, but we include its derivation for the sake of completeness.

$$\begin{aligned}
\cos^{2m} x &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)^{2m} \qquad \cdots \text{binomial expand and collect symmetric exponents}\\
&= 4^{-m}\left[\binom{2m}{m} + \binom{2m}{1} 2\cos 2x + \binom{2m}{2} 2\cos 4x \cdots\right]\\
\int_0^{\pi/2} \cos^{2m} x\,dx &= 4^{-m}\left[\binom{2m}{m}\int_0^{\pi/2} dx + \binom{2m}{1}\int_0^{\pi/2} 2\cos 2x\,dx + \binom{2m}{2}\int_0^{\pi/2} 2\cos 4x\,dx + \cdots\right]\\
&= 4^{-m}\binom{2m}{m}\frac{\pi}{2} \qquad \cdots \text{integrals of } \cos 2kx \text{ over 0 to } \pi/2 \text{ equal 0 for all } k \geq 1
\end{aligned}$$

Therefore our original integral becomes,

$$\begin{aligned}
\mathcal{W} = \int_0^{\pi/2} \ln(\sin x)\,dx &= \frac{1}{2}\int_0^{\pi/2} \ln(1-\cos^2 x)\,dx\\
&= \frac{-1}{2}\int_0^{\pi/2} \sum_{m=1}^{\infty} \frac{\cos^{2m} x}{m}\,dx\\
&= \frac{-1}{2}\sum_{m=1}^{\infty}\int_0^{\pi/2} \frac{\cos^{2m} x}{m}\,dx\\
&= \frac{-\pi}{4}\sum_{m=1}^{\infty} \frac{4^{-m}}{m}\binom{2m}{m}\\
&= \frac{-\pi}{4} S
\end{aligned}$$

To complete the solution we need to evaluate the sum

$$S = \sum_{m=1}^{\infty} \frac{4^{-m}}{m}\binom{2m}{m} \tag{2}$$

in a closed form. The form of the sum suggests a generating function of the form

$$g(x) = \sum_{m=1}^{\infty} \binom{2m}{m} x^m$$

If we had such a generating function, we could calculate S. Luckily, the binomial coefficient $\binom{2m}{m}$, known as the Central binomial coefficient, already has a known generating function:

$$\frac{1}{\sqrt{1-4x}} - 1 = \sum_{m=1}^{\infty} \binom{2m}{m} x^m$$

From here it is straightforward to evaluate S. Divide both sides by x to get

$$\frac{1}{x\sqrt{1-4x}} - \frac{1}{x} = \sum_{m=1}^{\infty} \binom{2m}{m} x^{m-1}$$

Now integrate with respect to x from 0 to $\frac{1}{4}$ to get

$$\int_0^{1/4} \sum_{m=1}^{\infty} \binom{2m}{m} x^{m-1}\, dx = \int_0^{1/4} \left(\frac{1}{x\sqrt{1-4x}} - \frac{1}{x} \right) dx \tag{3}$$

On the LHS (close your eyes and) swap the integration and the sum and integrate term by term. On the RHS, substitute $1 - 4x = y^2$ to get,

$$\begin{aligned} S = \sum_{m=1}^{\infty} \frac{4^{-m}}{m} \binom{2m}{m} x^m &= \int_1^0 \left(\frac{1}{\frac{1-y^2}{4} y} - \frac{1}{\frac{1-y^2}{4}} \right) \frac{-y dy}{2} \\ &= 2 \int_0^1 \frac{dy}{1+y} \\ &= 2 \ln 2 \end{aligned}$$

Which beings our final answer to

$$\int_0^{\pi/2} \ln(\sin x)\, dx = -\frac{\pi}{4} S = -\frac{\pi}{2} \ln 2$$

Email address: `rohan.kekatpure@gmail.com`