Axioms of Probability

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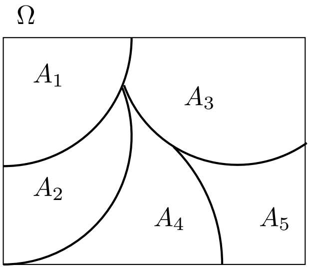
- For any event A,

$$P(A) \geq 0$$



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- If $A_1, A_2, A_3, ..., A_n$ are disjoint events (i.e., $A_i \cap A_j = \phi \quad \forall i \neq j$) then

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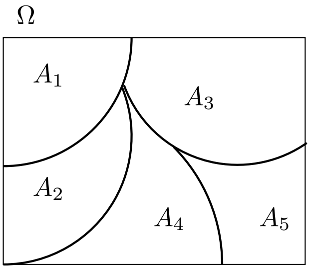

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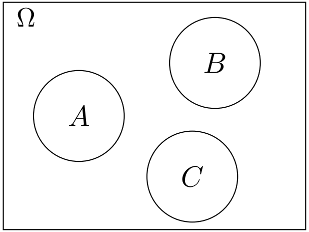
- If Ω is the universal set containing all events then

$$P(\Omega) = 1$$



Random Variable (intuition)

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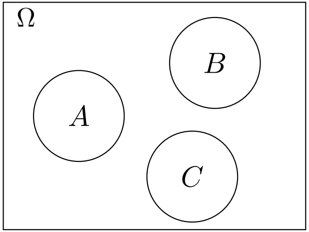

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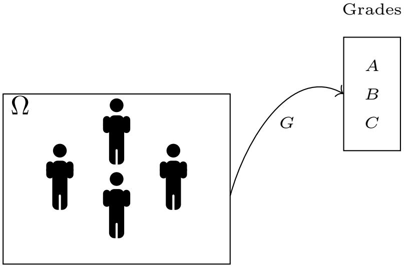

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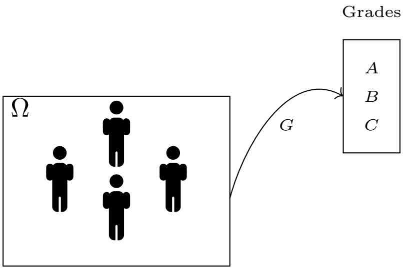

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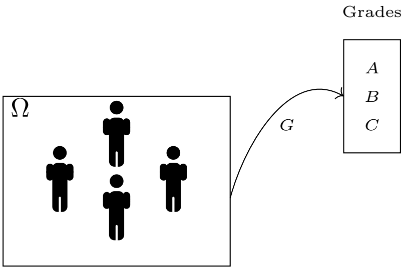

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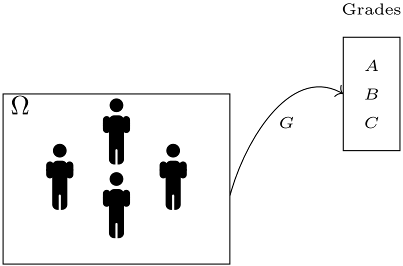
- And we are interested in $P(G = g)$ where $g \in \{A, B, C\}$
- Of course, both interpretations are conceptually equivalent



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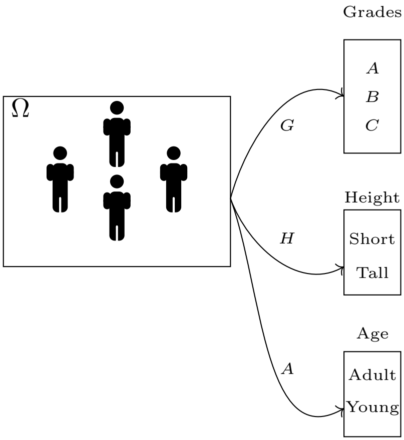

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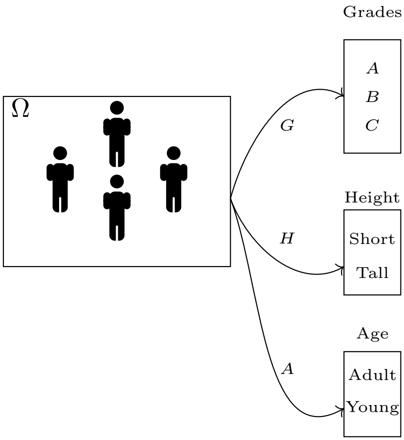

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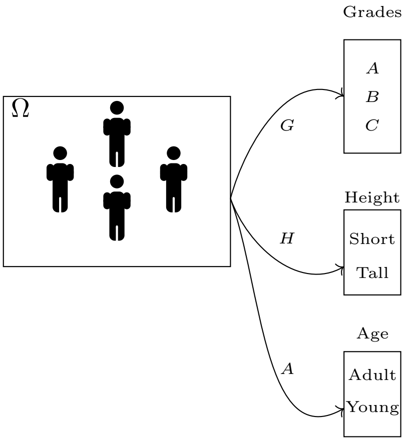

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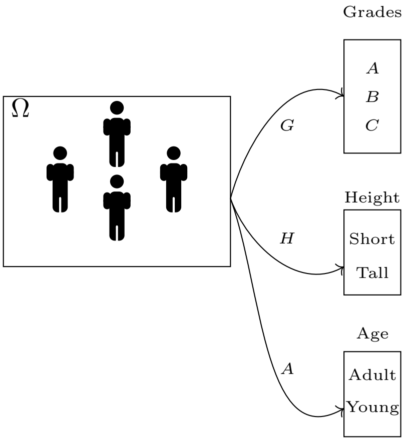
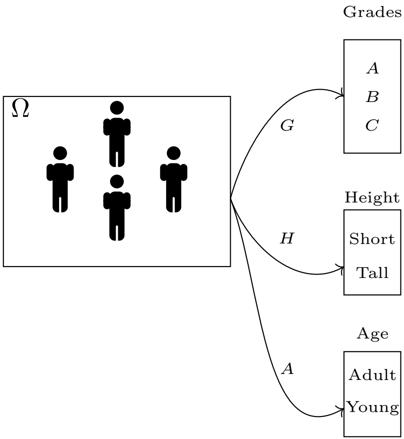
- Specially, when there are multiple attributes associated with a student (outcome) - *grade, height, age, etc.*
- We could have one random variable corresponding to each attribute
- And then ask for outcomes (or students) where $Grade = g$, $Height = h$, $Age = a$ and so on

Random Variable (formal)





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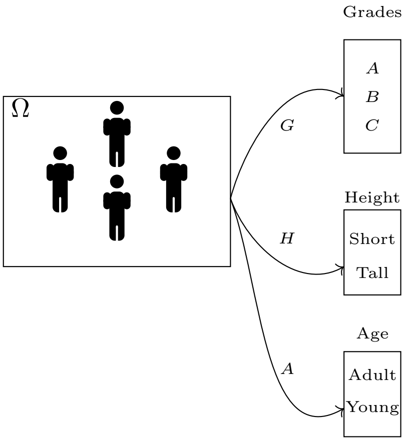
- A random variable is a ***function*** which maps each outcome in Ω to a value



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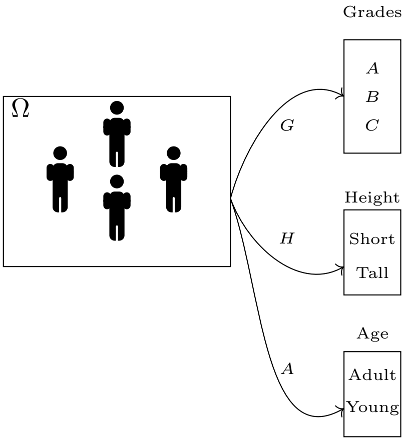
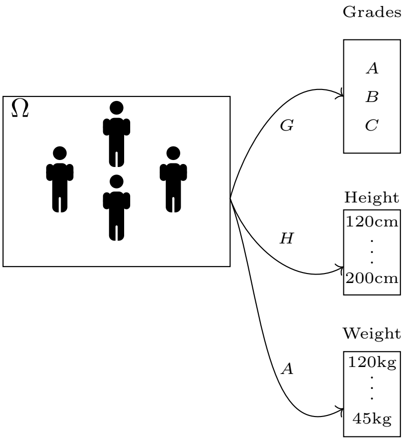


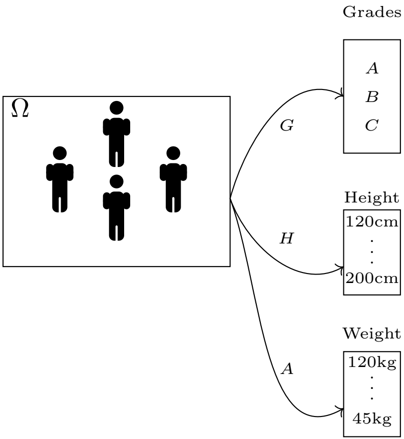
Random Variable (formal)

- A random variable is a ***function*** which maps each outcome in Ω to a value
- In the previous example, G (or f_{grade}) maps each student in Ω to a value: A, B or C
- The event $Grade = A$ is a shorthand for the event $\{\omega \in \Omega : f_{Grade} = A\}$

Random Variable (continuous v/s discrete)



- A random variable can either take continuous values (for example, $weight, height$)



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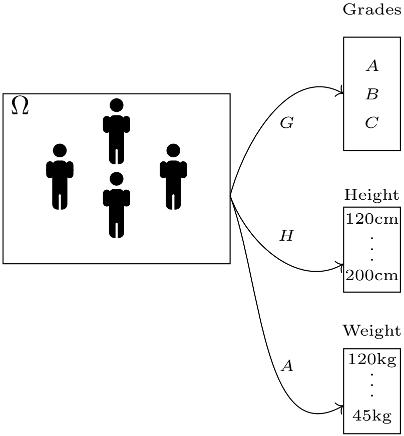

Random Variable (continuous v/s discrete)

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- For this discussion we will mainly focus on discrete random variables

Marginal Distribution

- What do we mean by ***marginal distribution*** over a random variable ?

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- Consider our random variable G for grades

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- The joint distribution over these two random variables assigns probabilities to all events involving these two random variables

$$P(G = g, I = i) \quad \forall (g, i) \in \{A, B, C\} \times \{H, L\}$$

Joint Distribution

G	I	$P(G = g, I = i)$
A	High	0.3
A	Low	0.1
B	High	0.15
B	Low	0.15
C	High	0.1
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Conditional Distribution

G	$P(G\|I=H)$
A	0.6
B	0.3
C	0.1

G	$P(G\|I=L)$
A	0.3
B	0.4
C	0.3

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Conditional Distribution

- Consider two random variable G (grade) and I (intellegence)
- Suppose we are given the value of I (say, $I = H$) then the conditional distribution $P(G|I)$ is defined as

$$P(G = g|I = H) = \frac{P(G = g, I = H)}{P(I = H)} \forall g \in \{A, B, C\}$$

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- More compactly defined as

$$P(G|I) = \frac{P(G, I)}{P(I)}$$

$$or \quad \underbrace{P(G, I)}_{joint} = \underbrace{P(G|I)}_{conditional} * \underbrace{P(I)}_{marginal}$$

Joint Distribution (n random variables)

X_1	$\ldots$	X_n	$P(X_1, X_2, \ldots, X_n)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$

$$\sum = 1$$

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for all possible values that variable X_i can take

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- If each random variable X_i can take two values then the joint distribution will assign probabilities to the 2^n possible events

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$\dots$	$\dots$	$\dots$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$

- The joint distribution over two random variables X_1 and X_2 can be written as,

$$P(X_1, X_2) = P(X_2|X_1)P(X_1) = P(X_1|X_2)P(X_2)$$

X_1	$\ldots$	X_n	$P(X_1, X_2, \ldots, X_n)$
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X_1	$\dots$	X_n	$P(X_1, X_2, \dots, X_n)$
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From Joint Distributions to Marginal Distributions

A	B	$P(A=a,B=b)$
High	High	0.3
High	Low	0.25
Low	High	0.35
Low	Low	0.1

A	$P(A=a)$
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Low	0.45

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$$= \sum_{\forall x_2, x_3, \ldots, x_n} P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$$

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- More compactly written as

$$P(X_1) = \sum_{X_2, X_3, \ldots, X_n} P(X_1, X_2, \ldots, X_n)$$

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- Recall that by Chain Rule of Probability

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Conditional Independence

- Recall that by Chain Rule of Probability

$$P(X,Y) = P(X)P(Y|X)$$

- However, if X and Y are independent, then

$$P(X,Y) = P(X)P(Y)$$

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